UNDERSTANDING TURBULENCE IN COMpressING PLASMAS

AND ITS EXPLOITATION OR PREVENTION

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Abstract

Unprecedented densities and temperatures are now achieved in compressions of plasma, by lasers and by pulsed power, in major experimental facilities. These compressions, carried out at the largest scale at the National Ignition Facility and at the Z Pulsed Power Facility, have important applications, including fusion, X-ray production, and materials research. Several experimental and simulation results suggest that the plasma in some of these compressions is turbulent. In fact, measurements suggest that in certain laboratory plasma compressions the turbulent energy is a dominant energy component. Similarly, turbulence is dominant in some compressing astrophysical plasmas, such as in molecular clouds. Turbulence need not be dominant to be important; even small quantities could greatly influence experiments that are sensitive to mixing of non-fuel into fuel, such as compressions seeking fusion ignition.

Despite its important role in major settings, bulk plasma turbulence under compression is insufficiently understood to answer or even to pose some of the most fundamental questions about it. This thesis both identifies and answers key questions in compressing turbulent motion, while providing a description of the behavior of three-dimensional, isotropic, compressions of homogeneous turbulence with a plasma viscosity. This description includes a simple, but successful, new model for the turbulent energy of plasma undergoing compression. The unique features of compressing turbulence with a plasma viscosity are shown, including the sensitivity of the turbulence to plasma ionization, and a “sudden viscous dissipation” effect which rapidly converts plasma turbulent energy into thermal energy.

This thesis then examines turbulence in both laboratory compression experiments and molecular clouds. It importantly shows: the possibility of exploiting turbulence to make fusion or X-ray production more efficient; conditions under which hot-spot turbulence can be prevented; and a lower bound on the growth of turbulence in molecular clouds. This bound raises questions about the level of dissipation in existing molecular cloud models.
Finally, the observations originally motivating the thesis, Z-pinch measurements suggesting dominant turbulent energy, are reexamined by self-consistently accounting for the impact of the turbulence on the spectroscopic analysis. This is found to strengthen the evidence that the multiple observations describe a highly turbulent plasma state.
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Chapter 1

Introduction

This thesis focuses on the behavior of compressing plasma turbulence. This plasma turbulence is modeled essentially as a neutral fluid, with one “plasma” feature, which will be shown to have substantial impact on the behavior of the compressing turbulence. This feature is the plasma viscosity, which depends strongly on temperature and plasma charge state. Both the temperature and plasma charge state may change substantially in a plasma undergoing compression, so that the viscosity, in turn, can grow or shrink enormously. In general, the turbulence treatment is restricted to the simplest case; that of homogeneous, isotropic, turbulence. Only compressions for which the turbulence maintains these properties are considered.

Each chapter contains introductory and background material for the topic at hand. This includes citations to previous studies of turbulence undergoing compression, on which this work builds. Previous studies have largely looked at compressing turbulence in two contexts. First, there are many studies of neutral gas turbulence undergoing compression, often with applications such as internal combustion engines and aerodynamic flows in mind. Second, there is an interest in understanding the behavior of compressing turbulence in certain astrophysical cases, in particular in molecular clouds. These clouds, which are supersonically turbulent, undergo compression by self-gravity or pressure. The properties of this
compressing supersonic turbulence are then related to important questions, such as the rate of star formation. Although most of this thesis focuses on compressing turbulence in the context of inertial confinement fusion (ICF) and Z-pinches, it also presents results in the context of molecular cloud turbulence.

1.1 Motivation

This brings us to the primary motivation for the studies of compressing plasma turbulence in this thesis. Major experimental facilities, notably the National Ignition Facility (NIF) and the Z Pulsed Power Facility (Z machine), regularly conduct extreme compressions of plasmas, in pursuit of fusion, X-ray generation, materials research, and for laboratory astrophysics. These compressions can reduce the initial plasma volume by a factor of $10^3$, far exceeding many previously imagined compressions of turbulence, particularly in neutral gas applications (typical internal combustion engines compress the volume by a factor around 10). Although measurements of any turbulence in these compressions is highly challenging, simulations suggest they may be turbulent to varying degrees (Thomas and Kares, 2012; Weber et al., 2014). Smaller scale experiments offer more diagnostic opportunities, and this thesis work was particularly motivated by highly detailed measurements on gas puff Z-pinch experiments at the Weizmann Institute. These measurements showed that, at stagnation, the plasma in such experiments is dominated by non-radial hydrodynamic motion (Kroupp et al., 2007a, 2011; Maron et al., 2013). That is, hydrodynamic motion beyond the radially inward motion associated with compression. Given the high Reynolds numbers at stagnation ($\sim 10^5$), there is a strong possibility any such hydrodynamic motion would be turbulent. It is unclear as yet whether this non-radial hydrodynamic motion is generated at stagnation, or carried along and compressed during the implosion, although it is likely a combination of both possibilities.
The motivation for studying compressing turbulence or hydrodynamic motion is clear, when it is dominant in the energy and pressure balance, as in the Weizmann gas puff Z-pinches or in astrophysical molecular clouds. Even when turbulence is only a small part of the plasma energy in a compression, its effects, particularly on heat transport and mixing, can be substantial. Thus, understanding the expected behavior of turbulence in ICF hot spots during compression is important, where even modest cooling or mixing of capsule shell materials can have large impact. Present understanding of compressing plasma turbulence is insufficient for either the case of modest hot spot turbulence, or for the case where turbulence is a dominant energy component. A number of results in this thesis can be used in efforts to reduce the quantity of turbulence in a plasma compression, and therefore to reduce its possible deleterious effects. However, it is also interesting to consider the possibilities for exploiting turbulence during a compression.

The prospect for exploiting turbulence during a plasma compression comes about primarily from the fact that the loss mechanisms for turbulent energy are different from those for thermal energy. For example, high temperature plasma radiates strongly, causing the loss of thermal energy, while energy in hydrodynamic motions such as turbulence does not cause radiation and is not directly lost to it. An extreme example of this can occur in the aforementioned molecular clouds: a radiative equilibrium keeps them at a constant temperature as they compress, but the turbulent energy can be enhanced by the compression, causing them to have an increasing proportion of energy in the turbulence. The end goal of plasma compression experiments seeking fusion or X-ray generation is to have a plasma at high density and a maximal possible temperature. The most efficient means of reaching this end state may involve storing energy in a form other than thermal energy, with lower losses, during the compression. This non-thermal energy must then be converted to thermal energy later in the compression. One such non-thermal storage possibility is turbulence.

Viewed in this context, the present work is related to other examinations of the advantages of storing non-thermal energy during compression. Non-thermal energy may be
stored in organized hydrodynamic motion, such as flow with the velocity profile of solid body rotation. Having energy in such organized flow during a compression gives a number of interesting and possibly exploitable effects (Geyko and Fisch, 2013, 2016, 2017). It is worthwhile noting that, at high Reynolds numbers, organized flow profiles need not be stable under compression (Mansour and Lundgren, 1990; Leblanc and Le Penven, 1999; Boree et al., 2002), meaning one may be stuck with trying to exploit turbulent flow, at least in certain parameter regimes. Plasma waves are another possible place for storing and exploiting non-thermal energy during compression (Schmit et al., 2010, 2011a,c,b; Schmit and Fisch, 2012; Schmit et al., 2013).

1.2 Highlights

This section gives an overview of work conducted in this thesis, with a focus on introducing the most important accomplishments. These accomplishments can be roughly divided into two categories: one set related to understanding the basic phenomenology of compressing turbulence with a plasma viscosity; and a second set related to applying this understanding in the various “application areas” identified above, astrophysics, Z-pinches, and ICF. The first three chapters following the introduction, chapters 2, 3 and 4, primarily deal with the fundamental behaviors of compressing turbulence with a plasma viscosity, while the last three chapters before the conclusion, chapters 5, 6, and 7, primarily deal with the applications.

Importance of viscosity — At the most fundamental level, a key message of this thesis is that, for extreme compressions of plasma turbulence or hydrodynamic motion, viscosity matters. This statement may sound counterintuitive, as it is often safe in the study of turbulence to neglect the details of the viscosity. However, the plasma viscosity is particularly sensitive to temperature and ionization state (especially compared to the neutral gas viscosity). This fact, coupled with the large changes typical in these variables under compression,
means that in many cases the viscosity behavior during compression must be accounted for in order to accurately compute the turbulence behavior. This point is demonstrated in Ch. 3, perhaps most clearly in Fig. 3.3, where it is shown that changing only the viscosity scaling under compression leads to drastically different results for the turbulent kinetic energy (TKE). The importance of considering viscosity for turbulent behavior in ICF has been emphasized before, in a much more limited context, by Weber et al. (2014). The influence of the viscosity variation on the behavior of compressing turbulence has also been noted in the neutral gas case, see Coleman and Mansour (1991) and references therein.

Even in cases where the viscosity is not expected to play a role, considering it can be useful. This fact arises because there exists a transformation and rescaling of the fluid equations governing compressing turbulence that can simplify them by shifting time dependence between terms. The time-dependence can be shifted to the viscous dissipation term, even in cases where it originally has no time-dependence. Although this transformation and rescaling is not new, it is sometimes forgotten; this thesis makes extensive use of it, showing its application both to simplify the numerics of simulations (e.g. in Ch. 2 and Ch. 3) and to arrive at new insights (e.g. in Ch. 5, and to aid in the model development of Ch. 4). The general result of the scaling is first presented in Sec. 3.6.2.

The utility of shifting time-dependence to the viscous term is demonstrated in Ch. 5, where this technique is used to derive a lower bound on the growth of turbulence in contracting molecular clouds. Molecular clouds have high Reynolds numbers and are often treated as isothermal, so that the viscosity will not change under compression, remaining small. Nevertheless, considering a time-dependent viscosity through the transformation is useful in this case. The importance of the lower bound derived this way deserves separate mention, and will be discussed further, later in this section.

Sensitivity of turbulence to ionization — In the end, the parameter regimes the compressing plasma passes through will determine the importance (or not) of accounting for the viscosity; many Z-pinch or ICF plasmas will not stay at high enough Reynolds numbers
throughout the compression to neglect the viscosity scaling, owing to the aforementioned
viscosity sensitivity. The fact that part of this viscosity sensitivity comes from the plasma
charge state yields another contribution of this thesis worth highlighting: turbulence be-

havior in plasma compressions can be quite sensitive to the plasma material, through the
range of achievable charge states. While fusion experiments compress D–T (with \( Z = 1 \)),
Z-pinches seeking to generate X-rays often compress high-Z elements while reaching com-
parable \(~\text{keV}\) temperatures. It is shown here (particularly in chapters 3 and 4) that the
increasing charge state in these compressing Z-pinch plasmas could exacerbate turbulence;
this may contribute to the observation of substantial TKE in the Weizmann gas-puff Z-
pinches. While it is currently thought that NIF fusion hot spots are viscous enough to damp
turbulence (Weber et al., 2014), the work here suggests that even modest contamination by
higher-Z mix could change this.

**Sudden viscous dissipation and a novel fast-ignition scheme** — One of the key con-
tributions of this thesis is the identification and demonstration of a sudden viscous dissipa-
tion effect, which causes a rapid conversion of plasma TKE into thermal energy. This effect
is caused by the viscosity’s sensitivity to temperature, and the viscosity must be accounted
for in order to capture it. Chapter 2 introduces sudden viscous dissipation, but important
aspects of it are touched on in other chapters, including chapters 3, 4, and 7.

The essence of this sudden dissipation effect is as follows. In a rapid compression, the
plasma TKE can grow substantially, with most of this energy stored in the largest spatial
scales. In the meantime, the plasma viscosity can steadily grow because of a rising temper-
ature caused by the compression. At first the viscosity acts at the smallest scales, however,
eventually the rising viscosity will start affecting the energy containing scales, and a rapid
conversion of the TKE into temperature can ensue. The rapidity of this conversion can be
enhanced if the TKE is similar to or larger than the thermal energy, because then, if the
viscosity causes the dissipation of a small amount of TKE, the temperature will increase
in response. This increasing temperature, in turn, increases the viscosity, causing yet more
dissipation of the TKE. This feedback then can lead to a rapid transition from a plasma with nearly all TKE to nearly all thermal energy. Results without the feedback are shown in various figures in chapters 2, 3, and 4. The feedback is included in a model in Ch. 7, in particular see Fig. 7.1. Note that these calculations ignore the energy in the background flow that generates the compression, see the discussion in Ch. 2 and also in Sec. 4.5.

Our identification of the sudden viscous dissipation effect now allows us to make novel proposals for inertial confinement fusion and X-ray generation by compressing plasma. Thus, beyond its academic interest, the practical importance of this effect lies in the fact that it significantly enhances the potential for exploiting plasma turbulence by temporarily storing compressive work in TKE. It provides a route for converting this stored TKE into thermal energy at a later time during the compression. This opens the possibility for new fast-ignition fusion or new X-ray burst generation schemes, based upon exploiting the sudden viscous dissipation of turbulence. It is worthwhile to note that the sudden viscous dissipation phenomenon does not rely on the flow being turbulent; it may be equally possible to exploit it in non-turbulent flows, which may have other properties that are preferable to turbulent ones. However, the flow energy is most easily amplified by compression when viscous effects are initially small, and in such cases non-turbulent flow need not be stable under compression, as previously mentioned.

The sudden viscous dissipation mechanism is explored in Ch. 2, and its possible exploitation is touched on there. For X-ray generation, ionization effects need to be included, and the mechanism is explored further in Ch. 3 with them included. A model that successfully captures the sudden dissipation effect is presented in Ch. 4. The prospects for sudden viscous dissipation occurring in typical hot-spot compression scenarios are touched on in Ch. 7, which explores hot-spot turbulence dynamics. The model and analysis of hot-spot turbulence are contributions that warrant their own explanation, see below.

**TKE for energy storage and loss reduction** — In plasma compressions where radiation is the dominant loss mechanism, it is clear how advantage could possibly be gained
by storing energy in TKE rather than in thermal energy. As far as radiation is concerned, a plasma with half the thermal energy instead in TKE will have substantially lower instantaneous radiated power than a plasma with the same total energy entirely in the thermal component. Some compressing plasmas will be dominated by (electron) thermal conduction losses. This is the case, at times, for ICF hot spots. It is natural to ask, then, what the effect on total conductive heat loss is when one shifts some proportion of the plasma thermal energy to TKE.

Chapter 7 explores hot-spot dynamics with turbulence included. As part of this, it contains (simple) models both for electron thermal conduction and for turbulent thermal conduction model. These models suggest that, at least in certain regimes, it is possible to reduce the total heat conduction losses by shifting most of the thermal energy to TKE (yielding a comparatively cold, but highly turbulent, plasma). Considering an extreme case, one can imagine that the plasma with the energy shifted to TKE is then colder than the boundary. Instead of losing thermal energy to the boundary the flow of thermal energy would reverse, with the plasma gaining thermal energy instead. This rather remarkable finding is described in more detail in Sec. 7.4.1.

**Modeling the impact of viscosity** — It is desirable to have a model that can capture the range of possible TKE behaviors for compressing turbulence with a plasma viscosity. Although a number of models for the TKE behavior of compressing neutral gas have been developed, they generally fail when confronted with the range of viscosity behaviors possible in a plasma. Chapter 4 develops a relatively simple model that successfully captures the TKE behavior of a compressing plasma under the influence of different temperature and/or ionization behavior. This includes modeling the sudden viscous dissipation effect. The model does not require any “free” parameters.

**Energy partition, total energy injection** — A key question that immediately confronts us when considering a plasma with energy split between the TKE and thermal energy is: what will be the energy partition between these two components after a certain amount of
compression, given the initial partition? Whereas in the absence of TKE the temperature is a state function of compression, the TKE is not. Further, when dissipated TKE shows up in the temperature, the temperature itself will cease to be a state function of compression. If the amount of TKE dissipated is small compared to thermal energy at all points, the temperature can still be approximately a state function.

The answer to the partition question requires a model for the TKE behavior under compression. Ideally this model would be applicable to both the case of subsonic and supersonic turbulence, since a plasma with a very large fraction of energy in the TKE compared to the thermal energy can be supersonic (the fraction of TKE at which the turbulence becomes supersonic will depend on the plasma charge state). Nevertheless, as a first answer to the question of energy partition, one can use the model developed in Ch. 4, which, strictly speaking, is for the subsonic case.

A second key question regarding the compression of a plasma with energy split between TKE and thermal energy is: how much total energy will be put into the plasma after a certain amount of compression? This question, as well as the question of energy partition, are discussed further in Sec. 4.5. It is shown that, for the isotropic three-dimensional compressions primarily handled in this work, the total plasma energy (thermal energy plus TKE) will be a state function of compression while the individual components are not. Then the total energy injected is determined by the initial condition and the total compression only. This is only true when the total energy is conserved (but energy can still transfer between the thermal and TKE components), as soon as loss mechanisms, such as radiation or conduction, are added, the total energy will cease to be a state function, and the time of the compression will matter. The model in this thesis can be used to predict the total energy required in such cases. In non-isotropic compressions, the total energy will not necessarily be a state function, even in the case without loss mechanisms; for the most part, such compressions are outside the scope of this thesis. It must be noted that the total energy discussed here neglects the energy associated with the background flow responsible
for the compression. This energy component is constant, but necessarily substantial in fast compressions.

The framing and clarification of these two key questions (energy partition and total energy injection) should be regarded as a significant contribution of this thesis, independent of the initial steps taken to resolve them.

**Bound** — Chapter 5 derives a lower bound on the amount of growth that turbulence compressing under the influence of gravity can undergo. Applying this bound to (supersonic) molecular cloud turbulence, it is shown that an existing model for such turbulence may be too dissipative. The existing model has found use in various works on molecular cloud dynamics and star formation (e.g. Murray and Chang (2015)); if it is indeed too dissipative these works may need to be revisited.

A possible explanation for the apparent discrepancy between the bound and the model is that the turbulent length scale in the model and accompanying simulations is saturated. The largest scale in turbulence typically grows in time, but if the turbulence is contained (e.g. by physical walls, or simulation resolution), the length scale can saturate. If this explains the discrepancy, then the discrepancy becomes a physical question. In a frame moving with the compressing molecular cloud turbulence (where the turbulence no longer compresses), should the outer length scale be restricted from growing? This question should be considered, since if the answer is no, it is more likely the model needs to be reconsidered.

The technique used to derive the bound should be useful beyond its application to the particular case of molecular cloud turbulence.

**Reconsideration of Z-pinch experiments** — Insights gained in the course of completing this thesis have led to a reconsideration of experimental results. In Ch. 6, experimental observations of a gas puff Z-pinch stagnation are re-analyzed. The previous analysis assumed a uniform stagnating plasma. However, there are indications the stagnating plasma may be supersonically turbulent, in which case it must be highly nonuniform. The new analysis accounts for the nonuniform density distribution of supersonic turbulence; it is
consistent with the observations and, in fact, improves the agreement with them. The improved agreement, however, suggests that the stagnating plasma density is lower by a factor of two, which can be anticipated to be significant for understanding the power balance in the implosion stage. This more precise analysis supports the picture that the stagnating plasma in such Z-pinches is highly turbulent. As such, these Z-pinches represent a new possible setup for laboratory astrophysics experiments with bearing on important astrophysical problems (such as the molecular clouds examined in Ch. 5), an idea discussed further in Ch. 6.

This first analysis of Z-pinch measurements including the impact of turbulence, in addition to furthering our understanding of such Z-pinches and opening up a new possible laboratory astrophysics setup, should also serve as an example for when and how to include the impact of turbulence for other Z-pinch or ICF experiments.

**Hot-spot stability and turbulence saturation levels** — Chapter 7 analyzes the behavior of hot spots with turbulence included, by coupling the TKE model developed in this thesis with a prior model for ICF hot-spot dynamics (Lindl, 1995). This coupled model allows for tracking hot-spot evolution under compression, following the path traced out in $\rho R - T$ space by a hot spot that starts from a given initial density, radius, temperature, and TKE. Part of this analysis involves looking at ways in which hot-spot turbulence might be exploited, as mentioned above, by storing energy in TKE to reduce losses and then later making use of the sudden viscous dissipation effect to convert the TKE to thermal energy. On top of examining these effects, there are two results in Ch. 7 worth calling attention to.

First, Sec. 7.3 shows a condition for whether hot-spot turbulence should grow or decrease along a trajectory, along with a map of this condition in $\rho R - T$ space for a typical ICF compression velocity (see Fig. 7.2). Of note is that the $\rho R - T$ space where the hot spot gains energy in the model (where heating terms outweigh loss terms) is almost entirely in the region that permits growing hot-spot turbulence. In addition to depending on where in $\rho R - T$ space a trajectory is, the condition depends on the rate of temperature
growth with compression for the trajectory. This is a result of the importance of the viscosity behavior (which depends sensitively on the temperature) in separating which hot-spot trajectories will have growing TKE versus decreasing TKE. Since the temperature growth with compression is the slope of the trajectory in $\rho R - T$ space, this condition can be used to “eyeball” a rough sense of expected turbulent behavior.

Second, and going hand-in-hand with the first result, Sec. 7.3 gives a derivation of a saturation level for hot-spot turbulence in mechanically heated hot spots, which depends only on the hot-spot mass and the compression velocity. Here, mechanically heated refers to those hot spots where the temperature scales as $T \sim (\rho R)^{0.4}$; this is the scaling obtained by balancing $PdV$ compressive work against electron thermal conduction in the hot-spot model of Lindl (1995). It is shown that the saturated TKE can in principle be on order of the thermal energy at least up to $\sim$keV temperatures for typical compression velocities. However, the saturation level can be regarded as more of an upper-limit on hot-spot turbulence for such trajectories, and the turbulence is found to often not reach the saturation level. This is in part because, if the TKE grows to become a significant fraction of the thermal energy, its dissipation will begin to heat the hot spot, which can change the heating rate and slope enough to reduce the turbulent growth, preventing the turbulence from reaching saturation.

Although these results are for an idealized situation, their resulting simplicity gives easy insight into the expected dynamics of hot-spot turbulence. By providing simple criteria for when hot-spot turbulence will grow, and to what possible level, they allow for a general sense of hot-spot turbulence that was essentially entirely lacking before.

**Final remarks** — Turbulent effects in ICF and Z-pinches have been touched on to a limited degree previously, but typically in such complicated integrated situations that it is difficult to have any sense of the general impacts of compressing plasma turbulence. The work in this thesis begins building a systematized understanding of these effects, and lays out a framework in which to consider them. It also shows a number of theoretical and
practical applications of the resulting improved understanding, which have been outlined above.

There is much more work to be done; this future work will undoubtedly be done to some degree in the context of ICF and Z-pinches, and to some degree in the context of astrophysics. In addition to any particular results, it is hoped the publication of the work in this thesis helps to tie together communities studying compressing turbulence in neutral fluids, plasma physics, and astrophysics. The work here has taken inspiration from all three communities, and one hope is this thesis serves to cross-pollinate ideas from each field to the others.

1.3 Organization

This thesis begins, in Ch. 2, with a presentation of the sudden viscous dissipation effect and the basics of the novel fast-ignition scheme based on it. Chapter 3 more broadly examines the effect on compressing turbulence of including a plasma viscosity. This includes both the temperature influence, which leads to sudden viscous dissipation, and ionization effects. The following chapter, Ch. 4, presents a relatively simple, but successful, model for the energy behavior of compressing plasma turbulence. These three chapters form the “base” of understanding compressing turbulence with a plasma viscosity. Following these chapters are “applications”. First, Ch. 5, derives a lower bound on the amount of growth that supersonic turbulence in a gravitationally contracting plasma can undergo, a result applicable in the astrophysical context. Next, Ch. 6 reconsiders experimental measurements of a gas puff Z-pinch stagnation in light of the fact the stagnating plasma may be supersonically turbulent. Finally, in Ch. 7, the new model developed in Ch. 4 is used to include turbulence effects in a prior model for ICF hot-spot dynamics. This combined model is used to conduct a first systematic examination of hot-spot turbulence.
With some alterations, the work in certain chapters is published. The references are as follows: Ch. 2 (Davidovits and Fisch, 2016b), Ch. 3 (Davidovits and Fisch, 2016a), Ch. 5 (Davidovits and Fisch, 2017). Note that the discussion section of Davidovits and Fisch (2017) has been significantly enhanced in Ch. 5, in light of new understanding gained from the modeling work in Ch. 4. The work presented in Appendix A, on kinetic effects in ICF and Z-pinch hot spots, is published in Davidovits and Fisch (2014). The work in Ch. 6, to which substantial contributions were made by collaborators at the Weizmann Institute (see the Acknowledgments), has been submitted.
Chapter 2

Sudden viscous dissipation

2.1 Introduction

Unprecedented densities and temperatures are now reached by compressing plasma using lasers or magnetic fields, with the objective of reaching nuclear fusion, prodigious x-ray production, or new regimes of materials. The plasma motion in these compressions can be turbulent, whether magnetically driven (Kroup et al., 2007a,b, 2011; Maron et al., 2013) or laser driven (Thomas and Kares, 2012; Weber et al., 2014). However, rapid compression of this turbulent plasma, where the viscosity is highly sensitive to temperature, is demonstrated here to exhibit unusual behavior, where the turbulent kinetic energy (TKE) abruptly switches from growing to rapidly dissipating. This behavior occurs in plasma, but is not predicted by studies of neutral gas compression (Wu et al., 1985; Coleman and Mansour, 1991; Cambon et al., 1992; Durbin and Zeman, 1992; Coleman and Mansour, 1993; Cambon et al., 1993; Hamlington and Ihme, 2014). In fact, it was observations of the dominant effect of the TKE in Z-pinches, both in pressure balance (Maron et al., 2013) and in radiation balance (Kroup et al., 2007b; Maron et al., 2013), that stimulated the present study.

Compression is rapid if the rate of compression is fast compared to the turbulent timescale $\tau = k/\epsilon$ with $k$ the TKE and $\epsilon$ the viscous dissipation rate. In the initially rapid
plasma compressions here, the viscous dissipation eventually grows such that the turbulent timescale $\tau$ shortens, and the plasma TKE suddenly dissipates. This dissipation is sudden in that it occurs over a time interval that is small compared to the overall compression time.

This sudden dissipation mechanism now suggests a new fast ignition paradigm. Imagine an initially turbulent fusion fuel plasma where the majority of the energy is in the turbulent motion. This plasma is then rapidly compressed, causing both the TKE and thermal energy to grow, while the TKE retains most of the energy, as observed in certain Z-pinch experiments (Kroupp et al., 2007a,b, 2011; Maron et al., 2013). Since radiation losses (both synchrotron and bremsstrahlung) and nuclear fusion are dependent on thermal energy but not the TKE, those processes are minimized under such compression, since the plasma stays comparatively cool. However, late in the compression, the TKE suddenly dissipates viscously into thermal energy, thereby igniting the plasma without having undergone large radiation losses.

In neutral gas, upon rapid compression, the TKE grows. In an isotropic 3D compression, it grows as $1/L^2$, where $L$ is the (time dependent) side length of a box that is compressing with the mean flow along each axis. This is true for both the zero Mach case (Wu et al., 1985), where the TKE is solenoidal, as well as in the finite Mach case, where the TKE has both solenoidal and dilational components, which each grow in energy as $1/L^2$ (Cambon et al., 1993). This is the same rate at which the temperature increases for an adiabatic ideal gas compression in 3D. Thus, in an idealized rapid compression the ratio of TKE to thermal energy stays constant, making the initial conditions very important for the energy dynamics while the compression is rapid.

In plasma, as in neutral gases, the TKE similarly grows under compression, but the greater sensitivity of viscosity to temperature now has a telling effect. The rapid compression causes the TKE to grow, with most of the TKE contained in the large scale eddies.
The energy equation for the viscous dissipation of TKE, $E_{\text{TKE}}$, is

$$\frac{dE_{\text{TKE}}}{dt} = \frac{\mu(T)}{\rho} \langle \mathbf{u}_l \cdot \nabla^2 \mathbf{u}_l \rangle \sim -LT^{5/2}E_{\text{TKE}},$$

where the last scaling reflects the viscosity dependence in plasma going as $\mu(T) \sim T^{5/2}$. This contrasts with the weaker scalings used for compressing gases, such as $\sim T^{3/4}$ (e.g. (Wu et al., 1985; Coleman and Mansour, 1993)). The compression forces the energy containing eddies to smaller and smaller scales; $\nabla^2 \sim 1/L^2$, where $L$ is the shrinking domain (and largest eddy) scale. The density scales as $\rho \sim 1/L^3$. The temperature increases during the compression, going as (for a 3D adiabatic compression of monatomic gas) $T \sim 1/L^2$, so that the total dissipation scales as $1/L^4$. The increasing viscosity dissipates the smallest scales first, then works up to the large energy-containing scales. The sensitive dependency of the viscosity to the temperature now means that the viscous dissipation of the large-scale TKE occurs rapidly as a function of $L$. Thus, under rapid compression, the TKE first grows with decreasing $L$, and then suddenly dissipates essentially at constant $L$.

This sudden dissipation effect is demonstrated here numerically in the limit of small initial TKE, so that the increase in temperature is due only to the direct compression, and not the self-consistent transfer of TKE to thermal energy. For turbulence where a substantial fraction of the energy is contained in the TKE, the dissipation should be even more sudden, in fact, explosive, because there is a feedback mechanism for the viscous dissipation. To see this, consider that the viscous dissipation drives the temperature according to: $dT/dt \sim LT^{5/2}E_{\text{TKE}}$. If most of the energy is in the TKE, then the viscous dissipation will rapidly raise the temperature. This in turn, will rapidly raise the viscosity, since $\mu(T) \sim T^{5/2}$, accelerating the conversion of TKE into temperature. The result is an explosive instability, until the TKE is depleted. Thus, while we simulate sudden viscous dissipation for small initial TKE, it is with the understanding that the effect may be even more...
An initially turbulent flow field, with TKE normalized to 1, is compressed on times ($\tau_c = 1/2V_b$) slower than ($V_b = 0.1$), comparable to ($V_b = 1$) and faster than ($V_b = 10, 100$) the initial turbulence timescale $\tau_0 = (k/\epsilon)_0 \sim 1/2$. The initial domain is a box of size $1^3$, time progresses right to left ($t = (1 - L)/(2V_b)$) as the compression shrinks the domain. When the compression is slow, the TKE damps, albeit at a slower rate than it would with no compression. When it is comparable, the energy stays relatively constant before damping. When it is faster, the TKE grows substantially, before being suddenly dissipated over a small range of $L$. Also shown is the theoretical rapid distortion theory (RDT) solution, which is exact while the compression is extremely fast.

RIGHT: Flowfield slices showing the magnitude of the turbulent flow velocity in the lab frame for $V_b = 10$. The fields progress in time from $A \rightarrow B \rightarrow C \rightarrow D$, and are marked on the left graph to show the amount of compression and the total TKE of the flow. As time increases the flow is increasingly contained in the largest structures as the smaller structures run into the viscous scales; see also Fig. 2.2. All plots are normalized to side length 1, in the lab frame the absolute size of all structures decreases in time.

sudden for large initial TKE. The sudden dissipation in any event underlies the proposed new paradigm for fast ignition.

### 2.2 Model

To model the compression of a turbulent plasma in the zero Mach limit, we follow previous work on compressing fluids (Wu et al., 1985; Blaisdell, 1991; Coleman and Mansour, 1991;
Figure 2.2: Fourier mode distribution of the turbulent kinetic energy (TKE) for turbulence compressed at a rate that is initially fast compared to the turbulent timescales ($V_b = 10$). The total TKE as a function of $L$ (time) is shown as the solid cyan line in Fig. 2.1. The initial spectrum is $L = 1$ (blue, solid). After moderate compression ($L = 0.32$, dotted green), but before the peak TKE, the energy in the highest modes has damped substantially, but energy has gone into the large scale flow. Near the peak TKE ($L = 0.15$, dashed red) this trend has continued, with viscosity working its way towards larger scales, and the largest scales continuing to gain energy. When the viscosity hits the largest scales, the energy is suddenly dissipated, after which all modes have been damped ($L = 0.05$, dash-dotted cyan). The four spectra approximately correspond in $L$ (time) to the four fields in Fig. 2.1.

Cambon et al., 1992; Hamlington and Ihme, 2014). The total (subscript $t$) plasma density, pressure, and flow are decomposed into mean (overbar) and fluctuating parts (subscript $l$),

\[
\rho_t = \bar{\rho} + \rho_l, \quad \text{(2.2a)}
\]
\[
p_t = \bar{p} + p_l, \quad \text{(2.2b)}
\]
\[
\mathbf{u}_t = \bar{\mathbf{u}} + \mathbf{u}_l. \quad \text{(2.2c)}
\]

The mean flow, $\bar{\mathbf{u}}$ in Eq. (2.2c) is taken to be an externally enforced background flow of the form

\[
\bar{\mathbf{u}} = A(t) \mathbf{x}_l = (\dot{L}/L) I \mathbf{x}_l, \quad \text{(2.3)}
\]
where I is the identity matrix, the subscript \( l \) on the coordinate \( \mathbf{x}_l \) is to indicate it is a stationary lab coordinate, and \( L(t) \) is the length along each side of a box that is advected with the background flow. The compression is chosen to be isotropic, so that the matrix \( A \) can be rewritten with \( I \), and \( \dot{L} \) is taken to be constant.

In the low Mach number limit, the density fluctuations \( \rho_l \) are neglected, and the turbulent velocity is divergence free \( \mathbf{u}_l \) (Blaisdell, 1991; Cambon et al., 1992). The mean density depends only on time,

\[
\bar{\rho} = \rho_0 \left( \frac{L_0}{L(t)} \right)^3,
\tag{2.4}
\]

with \( \rho_0 \) and \( L_0 \) the initial density and box length respectively. Under these conditions, the momentum equation for the plasma has the form (Cambon et al., 1992),

\[
\frac{\partial \mathbf{u}_l}{\partial t} + \mathbf{u}_l \cdot \nabla_l \mathbf{u}_l + A \mathbf{x}_l \cdot \nabla_l \mathbf{u}_l + A \mathbf{u}_l = -\frac{1}{\bar{\rho}} \nabla_l p_l + \frac{\mu(T)}{\bar{\rho}} \nabla_l^2 \mathbf{u}_l.
\tag{2.5}
\]

The viscosity is taken as a function of temperature,

\[
\mu(T(t)) = \mu_0 \left( \frac{T(t)}{T_0} \right)^{\beta} = \mu_0 \left( \frac{L_0}{L(t)} \right)^{2\beta}.
\tag{2.6}
\]

Here \( T_0 \) and \( \mu_0 \) are the initial temperature and the initial (dynamic) viscosity respectively. The compression is assumed adiabatic, and the gas is ideal and monatomic, so that the temperature increases as \( T(t) = T_0 \left( \frac{L_0}{L(t)} \right)^2 \). As previously stated, feedback from the viscous dissipation of kinetic energy into temperature is neglected.

Working in a frame that moves with the background flow \( \bar{\mathbf{u}} \) eliminates the explicit \( \mathbf{x}_l \) dependence in Eq. (2.5), allowing for periodic boundary conditions. This is achieved with the coordinate transformation \( \mathbf{x}_l = L(t) \mathbf{x} \). The turbulent velocity is transformed as

\[
\mathbf{u}_l(\mathbf{x}_l, t) = \mathbf{u}(\mathbf{x}, t).
\tag{2.7}
\]

and the pressure is transformed similarly, \( p_l(\mathbf{x}_l, t) = p(\mathbf{x}, t) \).
After transformation, the momentum equation for \( u \) is

\[
\frac{\partial u}{\partial t} + \frac{1}{L} u \cdot \nabla u + \frac{\dot{L}}{L} u = -L^2 \nabla p + \frac{1}{\text{Re}_0} L \mu(T) \nabla^2 u
\]  

(2.8)

where the standard nondimensionalization has been used, and \( L \) is understood to be normalized to \( L_0 \). The Reynolds number is subscripted with a 0 because one may view the effective Reynolds number as changing due to the compression.

The effects of the compression in the moving frame appear as time dependent coefficients on the nonlinear, pressure and dissipation terms, and as a new term, proportional to \( u \). This new term may be viewed as a forcing or damping, depending on its sign (as written, a negative sign is forcing). Indeed, a (constant coefficient) term proportional to \( u \) has been used to force turbulence for turbulence codes working in real space, where spectral forcing methods may be difficult to implement (Lundgren, 2003; Rosales and Meneveau, 2005; Carroll and Blanquart, 2013).

The variables in the momentum equation in the moving frame, Eq. (2.8) can be rescaled so that the forcing term is eliminated, and one is left with regular Navier-Stokes with a time dependent viscosity (Cambon et al., 1992). We choose instead to eliminate the time dependence from all terms except the forcing term, by using the scalings,

\[
\begin{align*}
    u &= L^{(2-2\beta)} \hat{u}, \\
    p &= L^{(1-4\beta)} \hat{p}, \\
    dt' &= L^{(1-2\beta)} dt.
\end{align*}
\]

(2.9a)  
(2.9b)  
(2.9c)

Doing so speeds up our simulations, by eliminating time dependence from the stiff viscous term. The momentum equation, Eq. (2.8), with the scalings in Eqs. (2.9) is,

\[
\frac{\partial \hat{u}}{\partial t'} + \hat{u} \cdot \nabla \hat{u} + (3 - 2\beta) \frac{\dot{L}}{L^{2-2\beta}} \hat{u} = -\nabla \hat{p} + \frac{1}{\text{Re}_0} \nabla^2 \hat{u}.
\]

(2.10)
The scaled momentum equation in the moving frame, Eq. (2.10), is incompressible Navier-Stokes with an extra time dependent forcing (or damping) term proportional to $\hat{u}$.

### 2.3 Sudden viscous dissipation

We consider constant velocity compressions, with $L = 1 - 2V_b t$, and use the plasma viscosity, $\beta = 5/2$. The forcing term in Eq. (2.10) acts as a damping term, the strength of which decays in time, asymptoting to 0 as $t' \to \infty (L \to 0)$. Thus, in the moving frame and scaled variables, the turbulence simply decays, at a rate that is faster than through viscosity alone. The damping term, in Fourier (wavenumber) space, has no wavenumber dependence and damps all modes proportional to their strength, so that the overall spectral dynamics is different than decaying turbulence.

While in this frame and variables the turbulence decays, in the lab frame, it may grow. Equation (2.9a) combined with Eq. (2.7) show the lab flow is amplified by the factor $1/L^3$ compared to the moving frame flow. Thus, even though the flow field $\hat{u}$ decays in the moving frame, it may increase in the lab frame if the decrease in $L$ is rapid enough that the $1/L^3$ growth overcomes the decay.

We simulate Eq. (2.10) with periodic boundary conditions using the spectral code Dedalus (ded). An initial turbulent flow field is generated using the forcing method of Lundgren (Lundgren, 2003; Rosales and Meneveau, 2005), with a coefficient of 1 so that the initial turnover time is approximately $1/2$, and $\text{Re}_0$ is taken as 3000. We use a $128^3$ Fourier grid and $3/2$ dealiasing. The simulations are initially under-resolved, but quickly become resolved as the compression progresses. Our main focus is to show the qualitative behavior of the sudden dissipation. For the same initial flow field, we simulate compression at various compression velocities $V_b$. The compression is continued until the turbulent kinetic energy dissipates. Different initial flow fields with similar but non-identical energy
spectra show similar behavior to the results presented. Key results are shown in Figs. 2.1 and 2.2 and discussed in the captions.

To understand this sudden dissipation, consider the TKE equation in the lab frame. Because \( u_l = u \), Eq. (2.7), the lab TKE can be put as \( k(t) = \langle u^2/2 \rangle \), where, for the homogeneous, isotropic case considered here, the angle brackets denote a spatial average over the domain. The equation for \( dk/dt \) is

\[
\frac{dk}{dt} = -2\frac{L}{L} k - L^{1-2\beta} \epsilon.
\] (2.11)

Here, \( \epsilon \) is the viscous dissipation, given by \( \epsilon = -\langle u \cdot \nabla^2 u \rangle / Re_0 \), and the first term on the right hand side in Eq (2.11) represents energy increase due to the compressive forcing. The compression is rapid when this energy increase term is much larger than the viscous dissipation term. Then the dissipation term in Eq. (2.11) can be dropped and the kinetic energy in the lab frame is found to be \( k(L) = k_0/L^2 \). This is the basic rapid distortion theory solution (Wu et al., 1985) which is shown for comparison in Fig. 2.1. When \( \beta = 5/2 \), the dissipation term in Eq. (2.11) has a prefactor of \( 1/L^4 \), which grows strongly as \( L \) decreases. It is apparent from Fig. 2.1 that under rapid compression the turbulent kinetic energy initially grows, but at some point the growth of the viscous dissipation overcomes this growth and causes it to damp.

### 2.4 Discussion and Caveats

For an illustration of a parameter regime where the sudden dissipation effect could possibly appear, consider the parameters of a magnetized liner inertial fusion (MagLIF) point design (Slutz et al., 2010); after laser pre-heating, but before compression, the cylindrical liner with radius 2.7 mm contains a 50/50 mix of deuterium (D) and tritium (T) at 250 eV, and densities of \( \sim 3.5 \times 10^{20}/\text{cm}^3 \). The gas has a kinematic viscosity of \( \nu \sim 7.8 \times 10^{-3} \text{ m}^2/\text{s} \). Suppose the gas had an initial Reynolds number of 3000, as used for the results in Fig. 2.1.
This would mean large scale flows on order of 8.6 km/s existed inside the capsule, which would be much smaller than the D and T thermal velocities (∼ 100 km/s), making the turbulence low Mach. The Z machine compresses the gas with a velocity around 50 km/s, slower than the thermal speeds, but much faster than the flow speed. Approximating it as 10 times faster, making crude accounting for the fact the compression is 2D rather than 3D, and comparing to the cyan curve on Fig. 2.1, it is estimated that the flow energy would peak and begin suddenly dissipating after a factor of 10 compression, around a radius of 0.27 mm. The TKE would grow up by a factor of approximately 10 before the dissipation, while remaining a minor fraction of the total energy. A more complete plasma model should be used before predictions are made for any specific experiment, MagLIF included.

There are, of course, a number of important caveats for the present work. On one hand, very precise measurements show that compression on the WIS Z-pinch leads to ion motion that is dominated by TKE in the imploding plasma (Kroupp et al., 2007a). On the other hand, alternative explanations for this effect have not been exhausted (Giuliani et al., 2014). Also, the proposed fast ignition paradigm contemplates supersonic turbulence, not the zero Mach number turbulence limit simulated here. A flow with much more energy in the turbulence than in temperature is necessarily supersonic. While the present work provides good motivation for plasma with supersonic TKE also to exhibit a sudden viscous dissipation effect, it remains to demonstrate it. As noted in the introduction, in the small Mach number turbulence limit, rapid 3D compression causes the TKE to grow as $1/L^2$, as does the temperature. Therefore the turbulence in such compressions stays low Mach. The initial state of supersonic turbulence, which decays on a timescale long enough to be rapidly compressed, must be generated in some fashion. A staged implosion, where a first stage generates supersonic turbulence, and a second stage rapidly compresses it, is one possibility.
The low Mach number in the compressions treated here means that even at late times, when the Reynolds number is small, the Knudsen number is small \((\text{Kn} \sim M/\text{Re})\), so that the fluid description should remain appropriate.

Rapid compressions are necessary for the TKE to grow under compression. In these rapid compressions, the background flow, Eq. (2.3), will necessarily have a large quantity of energy, which is a constant for a consistent box of compressing plasma \(E_{bg} = \rho_0 V^2_b L^3_0/2\). Of particular note is that the energy associated with this background flow will be initially much larger than the TKE, and generally at least comparable to it. Then, in the case where this background flow stagnates perfectly, converting to thermal energy, one may expect that any impact of the TKE on the thermal energy is secondary. On the other hand, the imperfect stagnation of this background flow may actually generate more turbulence; this may be the case in the experiments discussed in Ch. 6. In either event, sudden dissipation of the TKE could trigger dissipation also of the background flow, were the background flow accounted for self-consistently.

Note that the constant velocity compression considered here requires an ever increasing compressive force. This may occur naturally for some period of time (not indefinitely), especially on Z-pinches where the constriction of the current causes the magnetic pressure to continually increase. The consideration of alternate compression histories is straightforward. Boundary effects, such as viscous drag on the fluid motion, were ignored here through the use of periodic boundaries. They may be important in practical applications, especially as the imploding fuel becomes smaller. In the late stages of the implosion a new equation of state and viscosity dependence on temperature may alter the results. Finally, although the virtuous features of TKE were exploited here, namely the reduced radiation and the deferral of fusion reactions, there could be deleterious features of most of the energy residing in TKE during the compression phase, such as by increasing heat transport or contributing to mixing (Wilson et al., 2003; Thomas and Kares, 2012; Weber et al., 2014).
These caveats notwithstanding, what remains is a tantalizing, if speculative, new paradigm, namely using rapid compression to increase the energy in turbulent structures incapable of radiating much energy away during the compression, nor prematurely igniting the plasma through fusion reactions, with the energy in these turbulent structures then explosively converting to heat as the viscosity grows, thereby creating the deferred ignition conditions without radiative loss. The strong dependency of viscosity on temperature in plasma facilitates the sudden dissipation, as confirmed by the simulations presented here in the subsonic limit.
Chapter 3

Compressing turbulence and sudden viscous dissipation with compression-dependent ionization state

3.1 Introduction

In Ch. 2 simulations of compressing turbulent plasma demonstrated a sudden dissipation mechanism, which may enable a new paradigm for fast ignition inertial fusion. A plasma with initial (turbulent) flow is compressed on a timescale that is much faster than the dissipation time of the flow. This amplifies the turbulent kinetic energy (TKE) in the flow, for an ideal gas with subsonic flows. In a very rapid three dimensional adiabatic compression the energy in the flow scales at the same rate as the temperature. As the temperature increases, the plasma viscosity, which starts small, grows, because it scales as $\mu \sim T^{5/2}$. The viscosity first dissipates the smaller scales in the flow, which do not contain much energy. Eventually, as the compression continues, the energy-containing (largest) scales become viscous, and at this time all the TKE very suddenly dissipates into temperature. By initially putting most of the plasma energy in TKE, it may be possible to keep the plasma comparatively...
cool up until the sudden dissipation event, at which point it would ignite fusion or produce a burst of X-rays.

However, in addition to the temperature, the plasma charge state, $Z$, factors strongly into the viscosity, $\mu \sim T^{5/2}/Z^4$. Laser and magnetically driven fusion experiments typically compress deuterium and tritium, with $Z = 1$, so that, ignoring contaminants from the shell, the charge state is constant during the compression. In contrast, compression experiments designed to produce X-rays use a variety of higher $Z$ materials, which increase in ionization state during the compression.

This increase in ionization state has the effect of slowing the viscosity growth. Consider, for example, a neon gas-puff Z-pinch (Kroup et al., 2011) that starts with $T \sim 13$ eV and $Z \sim 3$, and finishes with $T \sim 200$ eV and $Z \sim 9$. The temperature increase causes a growth in the viscosity by a factor of $\sim 900$, while the (mean) ionization state growth reduces the viscosity by a factor of $\sim 80$, drastically cutting the overall viscosity increase.

In this chapter, as in Ch. 2, we consider a plasma temperature that increases due to the 3D adiabatic compression of an ideal gas in a box of side length $L$, going as $T = T_0 (L_0/L)^2$. The (mean) ionization state, $Z$, is treated as having some dependence on $L$ (i.e., the amount of compression) as well. This dependence is treated as fittable with some power, $Z = Z_0 (L_0/L)^\zeta$. Then, defining $\beta = (5 - 4\zeta)/2$, the viscosity can be written

$$\mu = \mu_0 (L_0/L)^{2\beta} = \mu_0 (T/T_0)^\beta.$$  \hspace{1cm} (3.1)

In this model, regarding the ionization state as a function of $L$ is equivalent to regarding it as a function of $T$ ($Z = Z_0 (T/T_0)^{\zeta/2}$), because $T \propto 1/L^2$.

Ionization processes in Z-pinch (Foord et al., 1994) and laser driven plasmas are not simply temperature dependent, depending on density and more complex processes (e.g. shock dynamics). However, if the (mean) ionization state for a given experiment can be reasonably fit to $L$ as described, then the net effect in the present model is that the overall
temperature dependence of the viscosity can be treated as some power other than $5/2$. We expect $\beta \leq 5/2$, reflecting the assumption that the charge state increases under increasing compression (or temperature). For a rough estimate of a possible value for $\zeta$ and therefore $\beta$, consider that the first 26 ionization states of krypton (covering 13 eV - 1200 eV) can be fit with $Z \sim T^{0.59}$. This corresponds to $\zeta = 1.18$, and $\beta = 0.14$. Since the ionization state can be higher at a given temperature than one would predict purely based on comparing the temperature to the ionization energies, one expects based on this example that a wide range of $\beta$ is possible in experiments, possibly including negative values.

Note that, if the adiabatic index of the compression is smaller than the value of $5/3$ assumed here, this also weakens the scaling of the viscosity with compression, effectively lowering $\beta$ ($\beta$ is defined so that $\mu \propto 1/L^{2\beta}$).

We consider initially turbulent plasma undergoing rapid, constant velocity, 3D isotropic compression and described by the same model as in Davidovits and Fisch (2016b), but with general $\beta$ rather than $\beta = 5/2$. This model is described briefly in Sec. 3.2, and a derivation is given in Sec. 3.6.1. We show that there will be an eventual sudden dissipation when $\beta > 1$. For identical initial condition, starting viscosity, and compression velocity, lower $\beta$ cases show larger TKE growth and later sudden dissipation (when $\beta$ is still $> 1$). Additionally, lower $\beta$ cases can show TKE growth under compression rates that would lead to the TKE damping in higher $\beta$ cases. For $\beta = 1$, the TKE reaches a statistical steady state under constant velocity compression, for any compression rate above a threshold which we determine. When $\beta < 1$, it seems there is no sudden dissipation, with the TKE increasing indefinitely instead.

There are a number of implications of these results. The plasma in magnetically driven (Kroupp et al., 2007a,b, 2011; Maron et al., 2013) or laser driven (Thomas and Kares, 2012; Weber et al., 2014) compressions can be turbulent. There can be substantial reduction in viscosity growth due to increasing ionization state for a gas-puff Z-pinch. To the extent the turbulence generation mechanism(s) of a given compression approach is
Insensitive to $Z$, the present results show that, for a fixed rapid compression rate, a larger increase in $Z$ (weaker viscosity growth) is expected to correspond to larger TKE growth. Furthermore, increases in $Z$ can make the difference between growing or decaying TKE.

Note that while these gas-puff Z-pinches appear to have substantial non-radial TKE even at stagnation (Maron et al., 2013), turbulence in the hot spot of ignition shots at the National Ignition Facility (NIF) is expected to be dissipated by high viscosity (Weber et al., 2014). The much higher temperatures in these hot spots create this high viscosity, but they are assisted by fuel of $Z = 1$, to the extent it is not contaminated by mix. Our results demonstrate that even moderate reductions in the effective power $\beta$ from 5/2 can cause large differences in TKE growth, and can determine whether for a given amount of compression ($L_{\text{final}}/L_0$) one can reach the dissipation regime.

The analysis in this work is carried out for 3D compressions, and as such is not strictly applicable to 2D compressions such as those in Z-pinches. In a 2D compression, the relative scaling of the TKE with compression, compared to the temperature is different (if the temperature growth is still assumed to be adiabatic and isotropic, the latter now being a larger assumption, since the plasma is driven anisotropically). Nevertheless, the intuition developed here may still be useful; all else being equal, more ionization enhances TKE growth under rapid compression by weakening the viscosity growth. The present work also neglects magnetic field effects, in line with other studies of turbulence in 3D compressions (Thomas and Kares, 2012; Weber et al., 2014). This, too, limits the applicability to Z-pinch compressions, though there are also many instances in which the magnetic field need not dominate the dynamics in a Z-pinch (Maron et al., 2013).

The evolving ionization state during compression may be exploitable to optimize TKE growth before sudden dissipation, and to control the timing of the dissipation. If the ions in the compression become maximally ionized, then the viscosity change reverts to being dominated by temperature, while TKE growth up to this point will be larger than without ionization. Mixes of ion species open up a wide range of control possibilities for the viscos-
ity dependence on compression, but also introduce other complications (e.g. species separation), and are beyond the scope of the present work. However, the prospect of controlling ion charge state and thereby viscosity appears to enlarge considerably the parameter space of opportunities for optimizing both the energy and pulse length in a sudden dissipation resulting in X-ray emission.

The structure of this chapter is as follows. Section 3.2 gives a brief description of the model, and discusses the energy equation for the turbulence, which is used in Sec. 3.3 to show some analytic results and to describe the general phenomenology. To go along with this analysis, the results from numerical simulations of compressing turbulence with ionization are displayed in Figs. 3.1, 3.2 and 3.3 and discussed in the captions and Sec. 3.4. Section 3.5 discusses implications of the results and caveats associated with them. Some secondary calculations associated with Secs. 3.2 and 3.3 are contained in Sec. 3.6, and referenced at the appropriate point.

### 3.2 Model and energy equation

#### 3.2.1 Model

The model used here follows previous work by Wu et al. (1985) and others (Coleman and Mansour, 1991; Blaisdell, 1991; Cambon et al., 1992; Hamlington and Ihme, 2014), and is the same as that in Ch. 2, allowing for a general power $\beta$ for the viscosity dependence on temperature. For completeness a derivation is given in the Sec. 3.6.1.

The essence of the model is as follows. It describes the 3D, isotropic compression of homogeneous turbulence in the limit where the turbulence Mach number goes to zero. Compression is achieved through an imposed background flowfield. The effect of the flow is that a cube, of initial side length $L_0$, will shrink in time but remain a cube. The side
length of the box as a function of time will be

\[ L(t) = L_0 - 2U_b t, \quad (3.2) \]

where \( U_b \) is the (constant) velocity of each side of the cube. In the low Mach limit, density fluctuations are ignored, and the density increases in time as one would expect for the compression,

\[ \rho_0(t) = \rho_0(0) \left( \frac{L_0}{L(t)} \right)^3. \quad (3.3) \]

The temperature of the compressing plasma is that for adiabatic compression of an ideal gas,

\[ T(t) = T_0 \left( \frac{L_0}{L(t)} \right)^2. \quad (3.4) \]

The viscosity dependence on \( L \) (alternatively, \( T \)) is given by Eq. (3.1).

The evolution of the initially turbulent flow is solved in a frame that moves along with the background flow, on a domain that extends from \([-L_0/2, L_0/2]\) in each dimension and has periodic boundary conditions. The energy in the turbulence in this frame is the same as in the lab frame. In this frame, after using Eqs. (3.1,3.3,3.4) to write the density, temperature and viscosity dependence in terms of \( L \), the Navier-Stokes equation for the turbulence is

\[ \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{L} \mathbf{V} \cdot \nabla \mathbf{V} - \frac{2U_b}{L} \mathbf{V} + \frac{L^2}{\rho_0(0)} \nabla P = \nu_0 \left( \frac{1}{L} \right)^{2\beta-1} \nabla^2 \mathbf{V}. \quad (3.5) \]

The initial kinematic viscosity is \( \nu_0 = \mu_0/\rho_0(0) \), and \( \bar{L} = L/L_0 \).

### 3.2.2 Energy equation

The energy density in the fluctuating flow, calculated in the moving frame is \( E = \rho_0(0) \mathbf{V}^2/2 \). The total energy is then \( E^T = \int_{-L_0/2}^{L_0/2} dx E \). Since \( \mathbf{v}' = \mathbf{V} \) (see Sec. 3.6.1), this total energy is the same as the total energy in the lab frame (in the lab frame, the density increases, but the volume to be integrated decreases in a manner that balances it). The time
The evolution of the energy density is

\[ \frac{\partial E}{\partial t} = \rho_0(0) \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t}. \]  

Equation (3.5) is used to write this energy equation explicitly. In Fourier \((k, \text{wavenumber})\) space, since the flow is assumed to be homogeneous and isotropic, it is

\[ \frac{\partial E(k,t)}{\partial t} = \frac{T(k,t)}{L} + \frac{4U_b}{L} E(k,t) - 2\nu_0 \tilde{L}^{1-2\beta} k^2 E(k,t), \]  

with \(T(k,t)\) a nonlinear term that includes the effects of the pressure and \(\mathbf{V} \cdot \nabla \mathbf{V}\) terms (see, e.g. McComb (1990)). The effect of \(T(k,t)\) is to transfer energy between wavenumbers (modes), conservatively. Integrated over the whole of \(k\) space, it vanishes. The total energy is

\[ E^T(t) = \int_{k_{\text{min}}}^{\infty} dk E(k,t). \]  

In the moving frame, \(k_{\text{min}} = 2\pi/L_0\) is fixed, given by the initial size of compressing system, e.g. capsule (although the current model uses periodic boundaries). In principle structures can be arbitrary small, so \(k_{\text{max}} = \infty\), but practically \(E(k,t)\) will be zero above some \(k\). The evolution of the total energy is,

\[ \frac{dE^T(t)}{dt} = \int_{k_{\text{min}}}^{\infty} dk \left( \frac{4U_b}{L} - 2\nu_0 \tilde{L}^{1-2\beta} k^2 \right) E(k,t). \]  

### 3.3 Analysis

Since \(E(k,t) \geq 0\), the energy is guaranteed to decrease if the coefficient of \(E(k,t)\) in Eq. (3.9) is negative for all \(k \in [k_{\text{min}}, \infty]\); conversely, it is guaranteed to increase if the coefficient is positive for all \(k\) where \(E(k,t) \neq 0\). (However, this latter condition is difficult to work with, since for \(k \to \infty\) there is always damping and as the energy increases \(T(k,t)\)}
Figure 3.1: Turbulent kinetic energy (TKE) during compression at various rates, for two different effective viscosity dependencies on temperature, $\mu(T) \sim T^\beta$, representing ionization effects (see Eq. (3.1) and the surrounding discussion). On the left, $\beta = 2.5$, the plasma case with no ionization effects. On the right, $\beta = 1.5$. An initial flow field, with TKE normalized to 1, is compressed with velocity $U_b$ on times equal to ($U_b = 1$) and faster than ($U_b = 5, 10$) the initial turbulence decay time (compression times $L_0/(2U_b)$ are normalized to the initial turbulent decay time). The initial domain is a box of size $L^3 = 1^3$, time progresses right to left ($t = (1 - L)/(2U_b)$) as the compression shrinks the domain. The same initial flow field is used for all compressions, so that the only difference is $\beta$. All cases show an eventual sudden dissipation of the TKE. For a given compression velocity, the lower $\beta$ case shows stronger TKE growth and a later, more sudden, dissipation. For this initial Reynolds number (600), the change from $\beta = 2.5$ to $\beta = 1.5$ pushes the dissipation $L$ for $U_b = 5, 10$ from $\sim 0.1$ to $< 0.01$, which could make the difference between dissipating during a compression or not. Similarly, for $U_b = 1$, the change from $\beta = 2.5$ to $\beta = 1.5$ greatly increases the compression needed to reach the point where the TKE dissipates. The theoretical rapid distortion theory (Durbin and Reif, 2010; Savill, 1987; Hunt and Carruthers, 1990) (RDT) solution is shown for comparison (in this case, it is the solution to Eq. (3.9), neglecting the dissipation, see also Wu et al. (1985)). It gives the theoretical maximum growth of the TKE with the compression.
Figure 3.2: Same as Fig. 3.1, but for $\beta = 1.0$, at a lower Reynolds number, and with a logarithmic scale for $L$. No eventual sudden dissipation is observed, even after extreme amounts of compression. After an initial growth phase, the turbulent kinetic energy (TKE) saturates and fluctuates around the mean level predicted by Eq. (3.13). This theoretically predicted mean level of the TKE is shown as a dotted line for each compression velocity. Note Eq. (3.13) must be written in the same velocity normalization as the figure before being applied.
Figure 3.3: Similar to Figs. 3.1, 3.2; the turbulent kinetic energy (TKE) for the same initial condition compressed at two different rates and different values of $\beta$, showing the effect of varying the amount of ionization during compression ($\beta$). The red, solid lines use a compression time that is half the initial turbulent decay time, while the blue, dashed lines use a compression time that is the same as the initial turbulent decay time. For a given compression rate ($U_b = 1$ or $U_b = 2$), the TKE is larger at every stage of the compression when $\beta$ is lower (when there is more ionization during compression). For the case when $U_b = 1$, the TKE purely decays when $\beta = 2.5$ (the plasma case with no ionization). Ionization during compression can cause this to no longer be the case; when $\beta$ decreases to 1.5 or 1.0, the TKE either grows before dissipating, or grows without dissipating.
will tend to move energy to higher $k$). These conditions are sufficient, but not necessary. The guaranteed decrease condition requires that for every mode

$$\frac{2U_b/L_0}{\nu_0 k^2} < \bar{L}^{2-2\beta}. \tag{3.10}$$

The left hand side is largest for $k = k_{\text{min}}$, and trends to 0 as $k \to \infty$.

### 3.3.1 $\beta > 1$

When $\beta > 1$, the right hand side of Eq. (3.10) starts at 1 at $t = 0$ and increases towards $\infty$ as $L \to 0$. At some time the condition will be satisfied for all $k$, when

$$\frac{2U_b/L_0}{\nu_0 k_{\min}^2} = \frac{1}{\bar{L}^{2\beta-2}} \tag{3.11}$$

Thus, the energy will always decay eventually for fixed $\nu_0, U_b, k_{\min}$ when $\beta > 1$. This is not to say that the energy cannot decrease before this condition is satisfied.

### 3.3.2 $\beta = 1$

When $\beta = 1$, the right hand side of Eq. (3.10) is 1. In this case there is no time dependence in the condition for guaranteed energy decrease. If Eq. (3.10) is initially satisfied for all $k$, the energy will purely decay, with no initial growth phase.

Otherwise, a fixed range of wavenumbers have a net positive coefficient for $E(k, t)$ in Eq. (3.9), while the rest have a net negative coefficient (ignoring the nonlinearity). The wavenumber cutoff between these two regions is given by equality in Eq. (3.10),

$$k_{\text{cutoff}} = \left( \frac{4U_b/L_0}{2 \nu_0} \right)^{1/2}. \tag{3.12}$$

The width of wavenumbers with a net forcing (linearly) is $\Delta k_{\text{forced}} = k_{\text{cutoff}} - k_{\min}$. Since the range of net forced wavenumbers is fixed, it might be expected that the energy will
reach a (statistical) steady state. This is the case, and it can be shown (see Sec. 3.6.3) that the statistically steady state energy is

\[ E_{\text{steady}}^T = (1.9) \, \rho_0(0) \, U_b^2. \]  

Further, the spectrum itself, \( E(k, t) \), converges to a statistical steady state \( E(k) \) in simulations (Rosales and Meneveau, 2005). While the steady state energy is independent of the viscosity \( \nu_0 \) (alternatively, the initial Reynolds number), the details of the energy spectrum of the saturated turbulence will not be. Also, as already mentioned, if the initial viscosity is too large, there is no steady state and the energy will purely decay.

This steady state energy can be rewritten in terms of \( \Delta k_{\text{forced}} \) by using Eq. (3.12),

\[ E_{\text{steady}}^T = \frac{(1.9)}{4} \, \rho_0(0) \, \nu_0^2 L_0^2 \left( \Delta k_{\text{forced}} + \frac{2\pi}{L_0} \right)^4 \]  

Once the sign of the coefficient of \( E(k, t) \) at a given \( k \) in Eq. (3.9) is being considered (rather than the sign of all coefficients), the nonlinearity cannot be ignored. Thus, \( k_{\text{cutoff}} \) is not necessarily a true (statistically steady state) cutoff between net forced and damped modes, but rather the linear cutoff.

### 3.3.3 \( \beta < 1 \)

*(Note: this original discussion is revised and expanded upon in follow-up work in Sec. 4.6)*

When \( \beta < 1 \), the right hand side of Eq. (3.10) trends to 0 as time increases, and an increasing number of shorter wavelength modes will have a net forcing (ignoring the nonlinearity). This means that \( \Delta k_{\text{forced}} \) trends to infinity as \( L \to 0 \). With the rather large caveats that in this case the problem is not an equilibrium one, and that the nonlinearity has been ignored in looking at the number of modes with a net forcing, the result from the equilibrium case that the steady state energy is proportional to the number of linearly forced modes suggests
that the energy for $\beta < 1$ continually increases for late times (after any initial transients are erased) under constant compression.

Note that neutral gas, compared to plasma with no ionization, has a weak dependence of viscosity on temperature, with studies of compressing gas turbulence using values that fall in the $\beta < 1$ case (e.g. $\beta = 3/4$ (Wu et al., 1985; Coleman and Mansour, 1991, 1993)). Turbulence closure models in these works, which include the evolution of the TKE in a neutral gas under compression, give a continually increasing TKE when evaluated for an initially rapid, constant velocity, 3D compression, consistent the suggestion here.

### 3.4 Simulations

In Section 3.3 we showed that; for $\beta > 1$, the TKE should always eventually damp, even with continued constant velocity compression (which represents an ever increasing compressive force); and when $\beta = 1$, the TKE will either purely decay or reach a steady state under continued compression. We also suggested that the energy always increases under continued constant velocity compression when $\beta < 1$ (if the compression is initially rapid, if not, the energy may decrease for some period before eventually increasing). These represent different regimes of the viscosity dependence on compression - with little to no ionization during compression, $\beta$ will be near the ionization free value, $\beta = 5/2$, and the sudden viscous dissipation phenomenon will still be possible. If substantial ionization occurs during a phase of the compression, then $\beta$ may be significantly reduced from $5/2$, and the viscous dissipation of the TKE will be prevented.

In order to get a better sense of the effect of decreasing $\beta$, we perform direct numerical simulation of compressing turbulence for a few values of $\beta$. The scaled form of the momentum equation, Eq. (3.47), is simulated with periodic boundary conditions using the spectral code Dedalus (ded). Results are then translated back into the lab frame using the appropriate rescaling. Initial conditions are generated using the forcing method of Lund-
gren (Lundgren, 2003; Rosales and Meneveau, 2005). All simulations are carried out on a 192$^3$ Fourier grid, which is dealiased to 128$^3$.

Simulations are done for three different values of $\beta$, 5/2, 3/2, and 1. Of note is that for $\beta = 3/2$, the forcing term drops out of Eq. (3.47), and it is simply the usual Navier-Stokes equation. This means that a single decaying turbulence simulation can give results for all compression velocities (at one initial Reynolds number).

Figures 3.1, 3.2 and 3.3 and their captions describe results from these simulations. The simulations in Fig. 3.1 are carried out with an initial Reynolds number of 600. Those for Fig. 3.2 are carried out with an initial Reynolds number of 100, which is necessary so that the turbulence remains fully resolved at saturation. Those in Fig. 3.3 also use an initial Reynolds number of 100, again to keep the $\beta = 1.0$ case fully resolved at saturation.

### 3.5 Discussion

The present model ignores many effects that do or may play an important role during compression of plasmas. The suggested $\beta = 1$ cutoff between eventually dissipating and saturating (or perpetually growing) TKE need not hold true in a more complete model. Non-ideal equation of state effects are neglected. It should be noted that only constant velocity compressions were considered; compressions with time-dependent velocities would also change the cutoff. Boundary effects, which are ignored, would be expected to become increasingly important as the amount of compression increased. The manner in which the ionization is accounted for neglects, among other effects, the energy required to achieve the ionization. If this energy comes at the expense of the temperature, and the true rate of temperature increase is less than $\sim 1/L^2$, this would alter $\beta$, but the general idea remains the same.

Subsonic compressions have been assumed, which is not necessarily the case for current compression experiments, nor is it the regime in which schemes utilizing the sudden
dissipation effect would likely be operated. Because the compressions are subsonic, the feedback of the dissipated TKE into the temperature is also neglected, which is expected to only make the sudden dissipation, once it happens, even more sudden.

As previously discussed, magnetic effects have also been neglected. Although this may be reasonable for 3D compressions, and certain regimes of 2D Z-pinch compressions, to expand the study for general 2D compression (Z-pinch) applicability will require the inclusion of magnetic effects. The inclusion of magnetic fields through a magnetohydrodynamic (MHD) model will introduce a number of new considerations. If there is a strong background magnetic field, the plasma conditions can be highly anisotropic, and the intuition from the present discussion may be difficult to apply. With magnetic fields included, turbulent energy can be stored in fluctuations of the magnetic field, and turbulent dissipation can occur through the plasma diffusivity, $\eta$. Continuing to assume the incompressible limit, then, in addition to the Reynolds number, the magnetic Reynolds number $Re_m = U L / \eta$ and the magnetic Prandtl number $Pr_m = Re_m / Re = \nu / \eta$ are also important for characterizing any turbulence.

The plasma magnetic diffusivity scales with ion charge state and plasma temperature as $\eta_{Spitzer} \sim Z / T^{3/2}$. Then, the magnetic Prandtl number has a temperature and charge state scaling of $Pr_m \sim T^4 / Z^5$. The behavior of MHD turbulence is influenced by the relative values of $Re$, $Re_m$, and $Pr_m$; they affect whether magnetic fluctuations will grow (see e.g. (Schekochihin et al., 2007; Iskakov et al., 2007)), the saturated ratio of turbulent magnetic energy compared to kinetic energy, and the steady state ratio of viscous dissipation to dissipation through the magnetic diffusivity (see e.g. (Brandenburg, 2014)). From the considerations in the present work, it is clear that, depending on the amount of ionization during compression, a range of behaviors for the dimensionless quantities is possible. Considering the limit of no ionization, and assuming the scalings $T \sim 1 / L^2, \mathcal{L} \sim L, U \sim 1 / L$, one has that: the viscosity increases with compression (and the Reynolds number decreases), the magnetic diffusivity decreases (and the magnetic Reynolds number increases), and the
magnetic Prandtl number increases. At high magnetic Prandtl number and large magnetic Reynolds number (but assuming the Reynolds number is still large enough for turbulent flow), the small scale dynamo is effective, so that magnetic perturbations can grow up quickly and saturate, while the ratio of kinetic dissipation to magnetic dissipation appears to grow large (Brandenburg, 2014). Investigations of these effects are a subject of current research and debate, and are typically carried out in steady state, whereas the sudden dissipation effect relies on dynamics far from steady state. As such, specific investigations of MHD effects on sudden dissipation are needed before making predictions.

A simple model of the impact of radiation, the effects of which have been neglected in the preceding discussion, is included in Sec. 3.6.4. This model consists of a temperature equation that includes mechanical heating and radiative cooling due to optically thin electron bremsstrahlung. With no radiation, the mechanical heating gives the $T \sim 1/L^2$ adiabatic temperature scaling. When the bremsstrahlung is included, it is shown that the temperature can still track closely with the adiabatic result for a large amount of the compression, provided the initial ratio of the radiation term to the mechanical heating term is very small. Then, the results in the present work will not be significantly modified. The ratio of radiation term to mechanical heating term, $R$, can be written as $R \sim (\tau_c/3.01 \times 10^{-9})(\rho g_{cc}/A_i T_{keV}^{1/2})(2Z^3/(1 + Z))$. For details, see Sec. 3.6.4. Here $\tau_c$ is the compression time, $\rho$ the density, $T$ the temperature, $A_i$ the ion mass number, and $Z$ the ion charge state. These considerations are a subset of the usual power balance requirements for inertial confinement experiments (see, for example, Lindl (1995)). In both cases, it is desirable to operate in parameter regimes where the temperature increases under compression when radiation effects are included. From the perspective of radiation, the presence and quantity of the hydrodynamic motion does not modify the potential operating regimes as compared to compression schemes without hydrodynamic motion. The same will be true with the inclusion of line-radiation, important for high-Z plasmas.
However, the operating regimes where the temperature increases under compression will be modified by the turbulence in (at least) two ways that are neglected in this work. First, before any sudden dissipation event, there will be some level of viscous dissipation of hydrodynamic motion into temperature. When the hydrodynamic energy is large compared to the thermal energy, this heating may somewhat relax the operating regimes where the temperature increases under compression. On the other hand, a second neglected effect, turbulent heat transport, represents a cooling effect that opposes this heating effect.

Once the sudden dissipation event is triggered, in the supersonic case with the feedback of dissipated TKE into temperature included, the temperature should rise rapidly. Taking into account radiation will then be important for modeling the sudden dissipation event itself, which occurs over a small time interval so that the plasma volume hardly changes.

Despite these deficiencies, the present work serves to highlight the sensitivity of TKE growth under compression to changes in the viscosity scaling with compression, in which ionization can play a strong role. Thus, even modest contamination of a low-Z plasma with higher-Z constituents may have substantial hydrodynamic implications, as, say, atomic mix in an ICF hotspot. Finally, this sensitivity to the ionization state suggests that the possibilities for control of TKE growth and sudden dissipation for X-ray production are now significantly expanded. This expansion of possibilities comes in part from the prospect of considering a wide range of ion-species mixes. Although outside the scope of the present work, it can be anticipated that using a variety of mixtures could enable detailed and controlled shaping of the X-ray emission pulse.

3.6 Derivations

3.6.1 Model derivation

Although essentially identical models have been discussed elsewhere (Wu et al., 1985; Coleman and Mansour, 1991; Blaisdell, 1991; Cambon et al., 1992; Hamlington and Ihme, 1992;...
for the sake of completeness, and to explain some details, we present a derivation here. Start with the continuity and momentum equations for compressible Navier-Stokes,

$$\frac{\partial}{\partial t} (\rho) + \frac{\partial}{\partial x_i} (\rho v_i) = 0,$$  \hspace{1cm} (3.15)

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) + \frac{\partial}{\partial x_j} (\delta_{ij} p) = D_i,$$  \hspace{1cm} (3.16)

$$\frac{\partial}{\partial x_j} \left( \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right) \right) = D_i.$$  \hspace{1cm} (3.17)

The Stokes’ hypothesis has been used, that the second viscosity coefficient, often denoted $\lambda$, is $\lambda = -\frac{2}{3} \mu$. This form of the rate of strain tensor is consistent with the Braginskii result (Braginskii, 1965). The unknowns are rewritten as two parts,

$$v_i(x,t) = v_i^0(x,t) + v'_i(x,t)$$  \hspace{1cm} (3.18)

$$\rho(x,t) = \rho_0(x,t) + \rho'(x,t)$$  \hspace{1cm} (3.19)

$$p(x,t) = p_0(x,t) + p'(x,t)$$  \hspace{1cm} (3.20)

where $v_i^0$ is given, and the subscript 0 indicates ensemble averaged quantities, while prime quantities have 0 ensemble average. The prime quantities are assumed to be statistically homogeneous, and ultimately the equations governing their evolution will have no explicit spatial dependence, allowing the use of periodic boundary conditions. For the prime quantities to be homogeneous, it can be shown (see, e.g. Blaisdell (1991)) that the flow $v_i^0$ must be of the form,

$$v_i^0(x,t) = A_{ij}(t) x_j.$$  \hspace{1cm} (3.21)

For this work, only pure (no shear), isotropic compressions are considered, so that

$$A_{ij}(t) = a(t) \delta_{ij}$$  \hspace{1cm} (3.22)
with $\delta_{ij}$ the Kronecker delta. When $a(t) < 0$ this enforced, “background”, flow is compressive.

With these assumptions, the continuity equation is

$$\frac{\partial}{\partial t} (\rho_0 + \rho') + \frac{\partial}{\partial x_i} \left( (\rho_0 + \rho') a(t) x_i + \rho_0 v'_i + \rho' v'_i \right) = 0. \quad (3.23)$$

Taking an ensemble average gives an equation for $\rho_0$. Denoting the average as $\langle \rangle$, then by definition $\langle \rho' \rangle = \langle v'_i \rangle = 0$. Also, $\partial \langle \rho' v'_i \rangle / \partial x_i = 0$ because ensemble averages, such as $\langle \rho' v'_i \rangle$, are assumed to be homogeneous. The equation for $\rho_0$ is then,

$$\frac{\partial \rho_0}{\partial t} + a(t) x_i \frac{\partial \rho_0}{\partial x_i} + 3a(t) \rho_0 = 0. \quad (3.24)$$

It can be shown (Blaisdell, 1991) that only for $\rho_0(x, t) = \rho_0(t)$ can the homogeneous turbulence constraint be satisfied. Dropping the second term in Eq. (3.24) accordingly, the density is

$$\rho_0(x, t) = \rho_0(t) = \rho_0(0) \exp \left[ -3 \int_0^t a(t') \, dt' \right]. \quad (3.25)$$

The fluctuating density is determined by Eq. (3.23), which can be simplified by canceling the terms that sum to 0 according to Eq. (3.24). It is

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i} \left( \rho' a(t) x_i + \rho_0 v'_i + \rho' v'_i \right) = 0. \quad (3.26)$$

For incompressible fluctuating (non-background) flow, we assume that the flow $v_i$ is low Mach, so that sound waves can be neglected and the density perturbation $\rho'$ can be ignored. Then, the fluctuating continuity equation reduces to the divergence free constraint on the prime velocity,

$$\rho_0 \frac{\partial}{\partial x_i} (v'_i) = 0. \quad (3.27)$$
With \( v_{i0} \) as given, \( \rho' \to 0 \), and \( \rho_0 \) depending only on time, the momentum equation is

\[
\rho_0 \left( \frac{\partial v'_i}{\partial t} + v'_j \frac{\partial v'_i}{\partial x_j} + \left( a^2 + \dot{a} \right) x_i + a x_j \frac{\partial v'_i}{\partial x_j} + a v'_i \right) + \frac{\partial}{\partial x_j} \left( \delta_{ij} \left( p_0 + p' \right) \right) = D_i. \tag{3.28}
\]

The ensemble averaged momentum equation is

\[
\rho_0 \left( a^2 + \dot{a} \right) x_i + \frac{\partial}{\partial x_j} \left( \delta_{ij} p_0 \right) = 0. \tag{3.29}
\]

In arriving at \( \langle D_i \rangle = 0 \), the viscosity \( \mu \) is assumed to be independent of space. The mean momentum equation, Eq. (3.29), says \( p_0 \) is quadratic in \( x \), unless \( a^2 + \dot{a} = 0 \), in which case \( p_0 \) is independent of \( x \). Since \( a \) sets the time dependence of the background flow (the rate of compression), this means only for one particular background flow can \( p_0 \) be independent of \( x \). For the purposes of this work, we consider temperature dependent viscosity, \( \mu = \mu(T) \).

The equation of state relates the pressure, density and temperature, \( p = \rho RT \). This is \( p_0 + p' = \rho_0 RT \), which becomes, after taking the ensemble average,

\[
p_0 = \rho_0(t) \, R \langle T \rangle. \tag{3.30}
\]

In order to have \( T = T(t) \), so that \( \mu(T) \) is independent of space, we must take

\[
a^2 + \dot{a} = 0, \tag{3.31}
\]

so that \( p_0 = p_0(t) \). Then, Eqs. (3.25,3.30,3.31) and the condition for an adiabatic compression together determine \( T \) and \( p_0 \).

Subtracting Eq. (3.29) from Eq. (3.28) gives the equation governing the fluctuating flow,

\[
\rho_0 \left( \frac{\partial v'_i}{\partial t} + v'_j \frac{\partial v'_i}{\partial x_j} + a x_j \frac{\partial v'_i}{\partial x_j} + a v'_i \right) = -\frac{\partial}{\partial x_j} \left( \delta_{ij} p' \right) + \mu(T) \frac{\partial^2 v'_i}{\partial x_j \partial x_j}. \tag{3.32}
\]
The explicit spatial dependence can be removed by transforming coordinates. Transforming as

\[ x_i = \alpha(t) X_i, \quad (3.33) \]

\[ v'_i(x, t) = V_i(X, t), \quad (3.34) \]

\[ p'(x, t) = P(X, t), \quad (3.35) \]

yields,

\[
\rho_0 \left( \frac{\partial V_i}{\partial t} + \frac{1}{\alpha} V_j \frac{\partial V_i}{\partial X_j} + \left( a - \frac{\dot{\alpha}}{\alpha} \right) X_j \frac{\partial V_i}{\partial X_j} + aV_i \right) = -\frac{1}{\alpha} \frac{\partial}{\partial X_j} (\delta_{ij} P) + \frac{\mu(T)}{\alpha^2} \frac{\partial^2 V_i}{\partial X_j \partial X_j}. \quad (3.36)
\]

Then, if the condition

\[ a - \dot{\alpha}/\alpha = 0 \quad (3.37) \]

is satisfied, the explicit spatial dependence is removed from the moving frame momentum equation, Eq. (3.36) and it becomes,

\[
\rho_0 \left( \frac{\partial V_i}{\partial t} + \frac{1}{\alpha} V_j \frac{\partial V_i}{\partial X_j} + aV_i \right) = -\frac{1}{\alpha} \frac{\partial}{\partial X_j} (\delta_{ij} P) + \frac{\mu(T)}{\alpha^2} \frac{\partial^2 V_i}{\partial X_j \partial X_j}. \quad (3.38)
\]

Together, the conditions Eq. (3.37) and Eq. (3.31) say that \( \dot{\alpha}(t) = 0 \). Consistent with this, define

\[ \alpha(t) = \left( L_0 - 2U_b t \right) / L_0 = L(t) / L_0, \quad (3.39) \]

\[ L(t) = L_0 - 2U_b t. \quad (3.40) \]

Then

\[ a = L/L_0, \quad (3.41) \]
and the background flow \((v_0)\) is such that a cube of initial side length \(L_0\), placed in the flow at \(t_0 = 0\), will remain a cube and shrink in time at a constant rate while having a side length of \(L(t)\). Using \(a\) from Eq. (3.41) in Eq. (3.25) gives the expected density dependence, Eq. (3.3). Using the viscosity, density, and temperature solutions, Eqs. (3.1,3.3,3.4), in the moving frame momentum equation, Eq. (3.38) gives the model equation Eq. (3.5).

### 3.6.2 Scaled momentum equation

The independent variables in Eq. (3.5) can be rescaled, and some time dependent coefficients eliminated. This is useful for simulations, and can be an aid in analysis. Using the scalings,

\[
V_i = \bar{L}^\delta \hat{V}_i, \quad (3.42)
\]
\[
P = \bar{L}^\eta \hat{P}, \quad (3.43)
\]
\[
d\hat{t} = \bar{L}^\tau dt, \quad (3.44)
\]
in Eq. (3.5) gives,

\[
\frac{\partial \hat{V}}{\partial \hat{t}} + \bar{L}^{\delta - 1 - \tau} \hat{V} \cdot \nabla \hat{V} - 2U_b \bar{L}^{-\tau - 1} (1 + \delta) \hat{V} = -\bar{L}^{2 + \eta - \delta - \tau} \hat{P} + \frac{1}{Re_0} \bar{L}^{-2\beta - \tau + 1} \nabla^2 \hat{V}. \quad (3.45)
\]

The standard nondimensionalization has been used, so that \(Re_0 = L_0V_0/\nu_0\). Equation 3.45 has four independent powers of \(\bar{L}\), and three undetermined scaling factors, \(\delta, \eta, \) and \(\tau\), so that the time dependence can be eliminated from all but one term. One specific choice takes \(\delta = -1\) to eliminate the forcing term (with \(U_b\) in the coefficient), and then the time dependence of two other terms can be eliminated. The choice where the forcing term and all time dependence but the viscosity’s are eliminated has been discussed by Cambon et al. (Cambon et al., 1992)). Choosing to eliminate the time dependence of all but the
forcing term, by selecting,

\[ \tau = 1 - 2\beta, \]
\[ \delta = 2 - 2\beta, \]
\[ \eta = 1 - 4\beta, \]

(3.46a, 3.46b, 3.46c)

gives

\[ \frac{\partial \hat{V}}{\partial t} + \hat{V} \cdot \nabla \hat{V} - 2 \bar{U}_b \bar{L}^{2\beta - 2} (3 - 2\beta) \hat{V} = -\nabla \hat{P} + \frac{1}{\text{Re}_0} \nabla^2 \hat{V}. \]

(3.47)

### 3.6.3 \( \beta = 1 \) steady state energy

A steady state solution (\( \frac{dE^T}{dt} = 0 \)) to the total energy equation, Eq. (3.9), when \( \beta = 1 \), would mean

\[ E^T_{\text{steady}} = \frac{L_0}{4\bar{U}_b} \epsilon_{\text{steady}}, \]

(3.48)

where \( \epsilon_{\text{steady}} = 2\nu_0 \int_{k_{\text{min}}}^{\infty} dk k^2 E(k) \) is the mean dissipation in steady state. When \( \beta = 1 \), the scaled momentum equation in the moving frame, Eq. (3.47) is the usual Navier-Stokes equation with a time independent forcing. This equation has been studied in the context of a forcing scheme for isotropic fluid turbulence, where the term \( 2 \bar{U}_b \hat{V} \) is added as an alternative to band-limited wavenumber space forcings (Lundgren, 2003; Rosales and Meneveau, 2005; Carroll and Blanquart, 2013). Numerical simulations by Rosales and Meneveau (Rosales and Meneveau, 2005) show that, in steady state, solutions have a characteristic length scale, \( l = u_{\text{rms}}^3 / \epsilon = 0.19 L \), where \( L \) is the domain size. Accounting for definitions and the scalings in Eqs. (3.46), this relationship between \( \epsilon_{\text{steady}}, L_0 \), and \( E^T \propto u_{\text{rms}}^2 \) allows us to solve for \( \epsilon_{\text{steady}}, \)

\[ \epsilon_{\text{steady}} = \left( \frac{2E^T_{\text{steady}}}{3\rho_0(0)} \right)^{3/2} \frac{1}{0.19 L_0}. \]

(3.49)

Then Eqs. (3.48) and (3.49) can be solved for \( E^T_{\text{steady}} \), yielding Eq. (3.13) in section 3.3.2.
3.6.4 Temperature equation including bremsstrahlung

Given here is a simple accounting of the effects of radiation without straying far from the present model. An optically thin plasma, with a single ion species of a single (time dependent) charge state $Z$ is assumed. The power density of electron bremsstrahlung emitted from an optically thin plasma, assuming $T_i = T_e = T$ and $n_e = Zn_i = Zn$, is

$$\text{P}_{\text{Br}} \left[ \text{W/m}^3 \right] = C_B \left( T \text{[eV]} \right)^{1/2} n^2 Z^3. \quad (3.50)$$

The bremsstrahlung constant is $C_B = 1.69101 \times 10^{-38} \text{W} \times \text{m}^3 / \sqrt{\text{eV}}$. The internal energy equation for the isotropically compressed plasma, including the mechanical work and Bremsstrahlung terms only, and continuing to assume that the adiabatic index $\gamma = 5/3$, is

$$\frac{\partial}{\partial t} \left( \frac{3}{2} n_T k_B T \right) = -\frac{5}{2} n_T k_B T \left( \frac{\dot{L}}{\bar{L}} \right) - C_B T^{1/2} n^2 Z^3. \quad (3.51)$$

Here $k_B$ is the Boltzmann constant, $\dot{L}$ can be found from Eq. (3.40), and $n_T$ is the total number density, $n_T = n_i + n_e = (1 + Z) n$. Consistent with the spirit of the model described in Sec. 3.2 and Sec. 3.6.1, the density is taken to be $n = n_0 / \bar{L}^3$ (see Eq. (3.3)). Then, if the bremsstrahlung term in Eq. (3.51) is ignored, the solution is $T = T_0 / \bar{L}^2$, as in Eq. (3.4).

Rewriting Eq. (3.51) as an equation for the normalized temperature, $\bar{T} = T / T_0$, as a function of the compression, while assuming that the charge state $Z$ is a function of temperature, gives

$$\frac{\partial \bar{T}}{\partial \bar{L}} = -2 \frac{\bar{T}}{\bar{L}} + 2 \frac{\tau_e}{\tau_{e,0}} \frac{2 Z^3}{1 + Z} \bar{L}^{-3} \bar{T}^{1/2} - \frac{\bar{T}}{1 + Z} \frac{\partial Z}{\partial \bar{T}} \frac{\partial \bar{T}}{\partial \bar{L}}. \quad (3.52)$$

The first term in Eq. (3.52) gives the mechanical heating (adiabatic heating when taken alone), while the second term represents bremsstrahlung cooling. The last term is associated with the energy needed to bring newly ionized electrons to the temperature $T$. If the charge state increases with temperature, and the temperature increases with compression
(with decreasing \( \bar{L} \)) then it is a cooling term (acts to decrease the temperature). It should not, however, be taken as an accurate accounting of this energy. Our primary focus is comparing the radiation and adiabatic compression terms. The relative size of the radiation term is set by the compression time,

\[
\tau_c = \frac{L_0}{2U_b}
\]  

(3.53)

and the initial radiation time,

\[
\tau_r [s] = 3.01 \times 10^{-9} \frac{A_i T_{1/2}^{1/2}}{\rho_{g/cc}}
\]  

(3.54)

The ratio \( \tau_c/\tau_r \) multiplied by the charge state coefficient \( 2Z^3/(1 + Z) \), gives the ratio of the bremsstrahlung cooling to the mechanical heating for any set of density, temperature, charge state, and ion mass number \( A_i \). To solve for the temperature evolution as a function of compression, one evaluates the ratio at the initial temperature and density, as in Eq. (3.52), and solves that equation. For an arbitrary function \( Z(T) \), the temperature will have some dependence on \( L \), which can be used instead of the adiabatic relation \( \bar{T} = \bar{L}^{-2} \) in the model described in Secs. 3.2, 3.6.1. Generally this will break the ability to reach a nicely scaled equation for the sake of simulation, Eq. (3.47).

To give a simple example, consider the case where the charge state takes a simple power law relation with the temperature,

\[
Z = Z_0 \bar{T}^\phi.
\]  

(3.55)

Approximating \( 1 + Z \sim Z \), Eq. (3.52) can be reduced to

\[
(1 + \phi) \frac{\partial \bar{T}}{\partial \bar{L}} = -2 \frac{\bar{T}}{L} + 2 \frac{2\tau_c}{\tau_r,0} \frac{\bar{T}^{2\phi+1/2}}{\bar{L}^3},
\]  

(3.56)

where the prefactor is due to the last term in Eq. (3.52) (the energy required to bring newly ionized electrons to temperature \( \bar{T} \)), and will result in an effective lower adiabatic index.
However, this is not a radiation effect, and will be ignored for the discussion here. Equation 3.56 can be solved analytically, and the solution takes a particularly simple form for $\phi = 1/4$, which has behavior that is qualitatively similar to the solutions for other $\phi$. When $\phi = 1/4$, and ignoring the prefactor on the derivative, the solution to Eq. (3.56) is

$$\bar{T}_{\phi=1/4} = \frac{1}{\bar{L}^2} \exp \left[ \frac{2\tau_c}{\tau_{r,0}} \left( 1 - \bar{L}^{-2} \right) \right]$$

(3.57)

For small initial compression time to radiation time ($\tau_c/\tau_{r,0} \ll 1$), the temperature tracks very closely with $1/\bar{L}^2$, up until the radiation becomes important – $\bar{L} \sim \sqrt{2\tau_c/\tau_r}$ for this $\phi = 1/4$ case. Then, the model for turbulence behavior with ionization discussed in this work will be unmodified up until the point where the radiation becomes important. Provided that one starts the compression with a small initial $\tau_c/\tau_r$, this can hold for large compression ratios. Note that in this case, the temperature is no longer a state-function of compression, since it depends also on the compression rate. When the temperature tracks closely with $1/\bar{L}^2$, $\phi = 1/4$ corresponds to the $\beta = 1.5$ case, for which simulation results are included in Figs. 3.1 and 3.3. Radiation considerations are discussed further in Sec. 3.5.
Chapter 4

Modeling turbulent energy behavior and sudden viscous dissipation in compressing plasma turbulence

4.1 Introduction

We present a simple model for the turbulent kinetic energy behavior of plasma turbulence undergoing isotropic three-dimensional compression, such as may exist in various inertial confinement fusion experiments or astrophysical settings. The plasma viscosity depends on both the temperature and the ionization state, for which many possible scalings with compression are possible. For example, in an adiabatic compression the temperature scales as $1/L^2$, with $L$ the linear compression ratio, but if thermal energy loss mechanisms are accounted for, the temperature scaling may be weaker. As such, the viscosity has a wide range of net dependencies on the compression. The model presented here, with no parameter changes, agrees well with numerical simulations for a range of these dependencies.

A variety of turbulent, or possibly turbulent, plasmas undergo compression. Experimental results in gas-puff Z-pinchess suggest that the plasma contains substantial non-radial
hydrodynamic motion at stagnation (Kroup et al., 2011; Maron et al., 2013). The Reynolds number at stagnation is large, such that this hydrodynamic motion could be turbulent. Although the source of the hydrodynamic motion is unclear, it may be carried along (and compressed) during the implosion. Detailed simulations of inertial fusion implosions suggest the presence of turbulence to varying degrees (Thomas and Kares, 2012; Weber et al., 2014). In the context of astrophysics, turbulence is ubiquitous in interstellar gas (Elmegreen and Scalo, 2004), including in molecular clouds, where the impact of gravitational compression on turbulence is studied (Robertson and Goldreich, 2012; Davidovits and Fisch, 2017), see also Ch. 5.

Although there have been a range of modeling efforts for compressing turbulence in neutral gases (Morel and Mansour, 1982; Wu et al., 1985; Coleman and Mansour, 1991; Blaisdell, 1991; Speziale and Sarkar, 1991; Durbin and Zeman, 1992; Cambon et al., 1992; Coleman and Mansour, 1993; Blaisdell et al., 1996; Hamlington and Ihme, 2014; Grigoriev et al., 2016), the viscosity growth with compression in neutral gas is more restricted than for plasma, necessitating new models capable of capturing plasma phenomenon, such as sudden viscous dissipation as discussed in Ch. 2 and Ch. 3. We present one such model here. The model aims to capture the turbulent kinetic energy (TKE) behavior of compressing plasma turbulence. To the extent it successfully does so, it can be used to give a partial answer to a key question for plasma turbulence undergoing compression. What is the partition of input energy between thermal energy and TKE for a given compression? This point is addressed in Sec. 4.5.

The present work, as in much previous work on compressing plasma turbulence or compressing neutral gas turbulence, considers isotropic, three-dimensional (3D), constant velocity compression of homogeneous turbulence. The plasma is modeled as a fluid, but with a plasma viscosity. The plasma viscosity depends sensitively on temperature and charge state, \( \mu \sim T^{5/2}/Z^4 \), with \( T \) the temperature and \( Z \) the charge state. Either (or both) of \( Z \) or \( T \) may change during the compression. The amount of compression is indicated by
\( \bar{L} = L/L_0 \), which is the linear compression ratio, so that, for example, \( \bar{L} = 10 \) corresponds to a reduction in volume by a factor \( 1/\bar{L}^3 = 1/1000 \). Similarly to Ch. 3, we assume that the net effect of changes to \( Z \) and \( T \) during compression is that the viscosity can be written as a function of \( L \),

\[
\mu = \mu_0 \bar{L}^{-2\beta}.
\]  (4.1)

As a “base” plasma case we consider no ionization \((Z = 1)\), and a temperature dependence on compression of \( T \sim \bar{L}^{-2} \). This is the temperature dependence for adiabatic, 3D, compression of a monatomic ideal gas. With \( \mu \sim T^{5/2}/Z^4 \), this gives \( \mu \sim \bar{L}^{-5} \), or \( \beta = 5/2 \) for the “base” case. Achieving \( \beta > 5/2 \) would require a stronger temperature growth with compression than \( T \sim \bar{L}^{-2} \); this could be achieved by, say, having a heating source in addition to the compression, but still no thermal losses. If ionization is included, so that \( Z \) can increase during the compression, \( \beta \) could be greatly reduced. Similarly if thermal losses are included.

The model given in this work matches reasonably well with simulations over the range \( \beta \in [1, 5/2] \), with tests at \( \beta = 1, 3/2, 5/2 \). It is unclear whether it is correct when \( \beta < 1 \) (see Sec. 4.6), where the present model predicts a saturation of the TKE under continuing compression. There is at least one possible case of interest with \( \beta < 1 \): isothermal turbulence, where both \( T \) and \( Z \) are constant, corresponding to \( \beta = 0 \). Compressing isothermal turbulence is of interest for understanding molecular clouds (Robertson and Goldreich, 2012; Davidovits and Fisch, 2017), where supersonic turbulence is compressed by gravity. The present model applies, strictly speaking, only to subsonic turbulence. However, the decay of supersonic turbulence is still mediated through a cascade not too dissimilar from subsonic turbulence Aluie (2011); Aluie et al. (2012); Aluie (2013). Given this, and that the saturation predicted by the current model for the \( \beta = 0 \) case is so different from current astrophysical models, this work suggests that a closer examination of the astrophysical case would be worthwhile. See the discussion Sec. 4.6 as well as Ch. 5, particularly Sec. 5.5, for more on this point.
Other cases of interest will be included in the validity range. As an example, Lindl (1995) showed that for a hot spot where the temperature is determined by the balance of mechanical heating and thermal conduction, $T \sim (\rho \bar{L})^{2/5}$. With $\rho \sim \bar{L}^{-3}$, this means $T \sim \bar{L}^{-4/5}$, which corresponds to the case $\beta = 1$, when $Z$ is constant, as in a deuterium-tritium hot spot with no mix.

The model may be valid for $\beta > 5/2$, but we have not tested it in this regime. The inertial fusion applications we have in mind typically do not have temperature growth with compression strong enough to bring them in this range, once thermal energy loss mechanisms like conduction and radiation are taken into account. Two examples of heating mechanisms that could potentially push the temperature growth with compression into this range are fusion heating, and heating from the sudden dissipation of TKE, if the dissipated TKE is fed back self-consistently into temperature. Fusion typically happens when the compression is essentially over.

The structure of this chapter is as follows. Section 4.2 describes the system of equations governing the compressing turbulence. The following section, Sec. 4.3 describes the model for the TKE of the system. Section 4.4 discusses the setting of model constants and shows comparisons between the model and numerical simulations. The way in which the model addresses the partition of input compression energy between thermal energy and TKE is discussed in Sec. 4.5. Finally, Sec. 4.6 provides additional discussion and caveats.

## 4.2 Governing equations

The system of equations governing the compressing turbulence is the same as that derived in Sec. 3.6.1, which in turn is very similar to previous work in neutral gases (e.g. (Wu et al., 1985; Coleman and Mansour, 1991; Blaisdell, 1991; Cambon et al., 1992; Hamlington and Ihme, 2014)). There are also similarities with previous work in astrophysics (e.g.
(Robertson and Goldreich, 2012) and Peebles (1980), Section 9), although here we consider only subsonic turbulence.

The system is the Navier-Stokes (NS) equations, with the flow, $v_i$, broken into an imposed background flow and an unknown (turbulent) component, $v_i = v_{i0} + v'_i$. The imposed background flow, $v_{i0}$, has the form, $v_{i0}(x, t) = (\dot{L}/L)x_i$, with the overdot the time derivative. When $L$ is defined as

$$L(t) = L_0 - 2U_b t,$$

(4.2)

the effect of the background flow is as follows; a homogeneous initial turbulent field remains homogeneous, and a 3D box of initial side length $L_0$, advected by the background flow, will remain a box and have a side length given by $L(t)$. That is, the sides of this compressing box move inward at a constant velocity $U_b$.

The turbulent field itself, $v'_i$, is treated in the low-Mach (incompressible) limit. This means density fluctuations can be ignored, and the continuity equation gives the expected density dependence for a 3D compression,

$$\rho_0(t) = \rho_0(0) \left( \frac{L_0}{L(t)} \right)^3 = \rho_0(0) \bar{L}^{-3}.$$  

(4.3)

Here, $\rho_0(0)$ is the initial (uniform) density.

Under these conditions, and working in a coordinate system that is co-moving with the background flow, the NS momentum equation for the turbulence is,

$$\frac{\partial \mathbf{V}}{\partial t} + \frac{1}{L} \mathbf{V} \cdot \nabla \mathbf{V} - \frac{2U_b}{L} \mathbf{V} + \frac{\bar{L}^2}{\rho_0(0)} \nabla P = \nu_0 \bar{L}^{1-2\beta} \nabla^2 \mathbf{V}.$$  

(4.4)

The third term is a forcing term associated with the compression. The initial kinematic viscosity is $\nu_0 = \mu_0/\rho_0(0)$, and Eq. (4.1) has been used for the viscosity dependence on $\bar{L}$. The turbulent velocity field $\mathbf{V}$ is simply $\mathbf{v}'$ rewritten in the moving coordinates, $\mathbf{V}(\mathbf{X}, t) = \mathbf{v}'(\mathbf{x}, t)$, where $x_i = \bar{L}X_i$. This means that the TKE behavior determined in the moving
Turbulent KE

\[ \beta = 2.5 \]

\[ \beta = 1.5 \]

Figure 4.1: A comparison of simulation and model turbulent kinetic energy (TKE) during compression at two rates, for two different net viscosity dependencies on compression, \( \mu(\bar{L}) \sim \bar{L}^{-2\beta} \); see Eq. (4.1) and the surrounding discussion. On the left, \( \beta = 2.5 \), the “base” plasma case with adiabatic plasma heating and no ionization. On the right, \( \beta = 1.5 \), representing either weaker heating, and/or some ionization during compression. An initially turbulent flow (Taylor-Reynolds number \( Re_\lambda \approx 82 \)) is compressed at a velocity \( U_b \) on times equal to \( (U_b = 1) \) or faster than \( (U_b = 10) \) the initial turbulent decay time. The x-axes are the linear compression ratio; the domain is a box of initial side length 1, with the side length shrinking as the compression progresses from right to left in the graphs. For each simulation, we also plot the result of the model with the same \( U_b \), \( \beta \), \( \nu_0 \) and initial TKE, Eq. (4.13), rescaled to the lab frame using Eq. (4.10). The results show reasonable agreement, with no “free” parameters used between the two different \( \beta \) cases. There is an apparent tendency for the model to suddenly dissipate somewhat early (at larger \( L \)).

coordinates is the same as the laboratory frame TKE behavior. We consider Eq. (4.4) on a cubic domain with sides extending from \(-L_0/2\) to \(L_0/2\), and periodic boundary conditions.

The TKE, \( E \), is defined as,

\[ E = \frac{V^2}{2}. \]  

(4.5)

The total energy in the domain is then \( E_{\text{tot}} = \rho_0(0)V^2L_0^3/2 \).
Figure 4.2: Same as Fig. 4.1, but for $\beta = 1.0$, at a lower initial Reynolds number ($\text{Re}_{\lambda,0} \approx 14$, and with a logarithmic scale for $L$. The low initial Taylor-Reynolds number and modest compression speeds are used to ensure the turbulence is resolved at saturation. The model correctly predicts that the turbulent kinetic energy saturates, although at a level below the actual saturation for these cases. This difference will disappear at higher initial Reynolds numbers or for faster compressions. One can observe the saturation level is closer to the true value for $U_b = 2$ than for $U_b = 1$. At the slowest compression speed, $U_b = 0.5$, the model predicts that the energy decays, while it actually saturates at a finite level. The simulation in this case is near the boundary in parameter space where the TKE decays under compression; the model has the location of this boundary somewhat wrong. For more discussion of the $\beta = 1$ case, see Sec. 4.4.
4.3 TKE model

We seek a model that will give the TKE behavior of the compressing turbulence described by Eq. (4.4). By rescaling \( \tilde{V}, \tilde{P}, \tilde{t} \) by appropriate powers of \( \bar{L} \), this equation can be rewritten as the usual, unforced, NS equation, but with a time-varying viscosity. In the rescaled equation the turbulence will decay. Given a model for the TKE behavior during this decay, one can translate the model back from the rescaled variables, yielding a model for the compressing turbulence TKE behavior. This observation has been used by Cambon et al. (1992) to aid modeling efforts for compressing turbulence previously, but with the viscosity variation assumed to be negligible. Here, we expand on this work by including the viscosity variation and low Reynolds number effects. These effects are found to have a substantial impact on the TKE behavior in the regime of interest.

Using the rescalings,

\[
\begin{align*}
\tilde{V} &= \bar{L}^{-1} \hat{V}, \\
\tilde{P} &= \bar{L}^{-5} \hat{P}, \\
\tilde{t} &= \bar{L}^{-2} \hat{t},
\end{align*}
\]

in Eq. (4.4) gives,

\[
\frac{\partial \hat{V}}{\partial \tilde{t}} + \hat{V} \cdot \nabla \hat{V} = -\frac{1}{\rho(0)} \nabla \hat{P} + v_0 \bar{L}^{3-2\beta} \nabla^2 \hat{V}.
\]

The rescaled TKE is \( \hat{E} = \hat{V}^2/2 \), and it will be related to the laboratory frame TKE by,

\[
E = \frac{V^2}{2} = \bar{L}^{-2} \hat{V}^2/2 = \bar{L}^{-2} \hat{E}.
\]

When the viscosity coefficient in Eq. (4.9) has no time dependence (\( \beta = 3/2 \)), this scaled equation is the usual NS momentum equation. At high Reynolds numbers the decay of NS turbulence closely follows a power law in time that is independent of the viscosity, \( \hat{E} \sim \hat{t}^{-n} \) (see Sinhuber et al. (2015) and references therein). Neglecting viscosity, the
power $n$ depends on whether the energy containing turbulent scale is free to grow ($n \sim 1.2$) or not ($n \sim 2$) (Skrbek and Stalp, 2000). One could translate this high Reynolds number expression for $\hat{E}(\hat{t})$ into one for the lab-frame energy $E(t)$, using Eqs. (4.8,4.10). However, using this power law decay to capture the behavior of $\hat{E}$ neglects viscous effects; the decay is determined solely by the rate of energy cascade from the large scales. It has been shown, however, that varying only $\beta$, and therefore only the viscous coefficient time dependence, can make a huge difference in the TKE behavior of compressing turbulence (see Ch. 3, in particular Fig. 3.3). Thus, to model this TKE behavior, we must account somehow for the viscous effects.

The present approach to do this accounting is to use the high to low Reynolds number decay model of Lohse (1994), with the additional step of assuming the viscosity in that model is time dependent. This assumption is convenient, rather than rigorous, but yields reasonable results for the cases examined. Lohse’s model for the energy behavior of the decaying NS equation Eq. (4.9), but with time dependent viscosity, is

$$\frac{(12/3b)^{3/2}}{L_{outer}} \left[ \nu_n(\hat{t}) + \sqrt{\hat{E}(\hat{t}) + (\nu_n(\hat{t}))^2} \right] \dot{\hat{E}}(\hat{t}) = -\frac{d}{dt} \dot{\hat{E}}(\hat{t}) . \quad (4.11)$$

The model is completed with the definition,

$$\nu_n(\hat{t}) = \frac{3}{4} b^{3/2} \frac{L_{outer}}{\nu_0} \nu(\hat{t}) = B \frac{v(\hat{t})}{\nu_0} . \quad (4.12)$$

Here $b$ is the Kolmogorov constant, and $L_{outer}$ is a (constant) length scale associated with the large scales of the turbulence (as in Lohse (1994)). Lohse’s model is recovered by setting $\nu_n(\hat{t}) = B$. It is convenient to work in terms of $\bar{L}$, rather than $\hat{t}$. This is achieved by making use of Eqs. (4.2,4.8). Doing so gives,

$$A \left[ B\bar{L}^{3-2\beta} + \sqrt{\hat{E} (\bar{L}) + (B\bar{L}^{3-2\beta})^2} \right] \left( \frac{\hat{E}(\bar{L})}{\bar{L}^2} \right) = \frac{d}{d\bar{L}} \hat{E}(\bar{L}) . \quad (4.13)$$
We have also substituted for the appropriate viscosity, which is the coefficient of the last term in Eq. (4.9), \( \nu = \nu_0 \tilde{L}^{3-2\beta} \). The constant \( A \) is defined,

\[
A = 8b^{-3/2} \frac{L_0}{2U_b} \frac{1}{L_{\text{outer}}}.
\]

(4.14)

The domain of the model, Eq. (4.13), is \( \tilde{L} \in (0, 1] \), with the initial condition (the starting energy) being set at \( \tilde{L} = 1 \). After setting values of the constants \( b \) and \( L_{\text{outer}} \), the model, Eq. (4.13) can be solved for \( \hat{E} \) for the set of physical parameters, \( \{ \beta, U_b, \nu_0, L_0 \} \), of interest. Having solved for \( \hat{E}(\tilde{L}) \), the untransformed energy is obtained by rescaling by \( \tilde{L}^{-2} \), as per Eq. (4.10).

### 4.4 Setting constants and simulation comparison

Here we compare the solutions of the model, Eq. (4.13), to simulations of compressing turbulence as described by Eq. (4.4). To set values of \( b \) and \( L_{\text{outer}} \), we use the following procedure. When \( \beta = 1 \), compressing turbulence described by Eq. (4.4) will reach a steady state energy as \( \tilde{L} \to 0 \) (see Sec. 3.3.2). This steady state energy is \( E = 1.9U_b^2 \). The characteristic large scale associated with the steady state is \( L_{\text{outer}} \approx 0.19L_0 \). We chose this to be the value of \( L_{\text{outer}} \). In the limit \( L \to 0 \), when \( \beta = 1 \), the model, Eq. (4.13), also predicts that \( E \) reaches a steady state (that is, it predicts \( \hat{E} \sim \tilde{L}^2 \)). The model gives for \( E \),

\[
E(\tilde{L} \to 0) = \tilde{L}^{-2} \hat{E}(\tilde{L} \to 0) = \frac{4}{A^2} (1 - BA) .
\]

(4.15)

Setting \( b \approx 5.949 \) makes \( 4/A^2 \approx 1.9U_b^2 \). Then the leading term in Eq. (4.15) matches the expected result. Further, this value of \( b \) is not far off from experimental values of the Kolmogorov constant (see Lohse (1994) for more discussion on values of \( b \)). The term \( BA \) itself has no dependence on \( b \); this term can be rewritten in terms of the ratio \( \tau_c / \tau_v \) of the compression timescale, \( \tau_c = L_0/2U_b \), to the viscous timescale, \( \tau_v = L_{\text{outer}}^2/\nu_0 \). For rapid
compressions, or compressions with small initial viscosity, this will be very small, so that
$1 - BA \approx 1$, and the energy saturation predicted by the model will match the expected
result.

Figures 4.1 and 4.2 show comparisons between the model with $b$ and $L_{outer}$ set as just
described, and direct numerical simulations of compressing turbulence carried out in the
spectral code Dedalus ded. For the initial condition we generate a turbulent flow field using
Lundgren’s method Lundgren (2003); Rosales and Meneveau (2005). This turbulence is
then compressed, evolving according to Eq. (4.4) (the simulations actually use a rescaled
version of Eq. (4.4), and then the results are appropriately scaled back, see Sec. 3.4). All
simulations use a $192^3$ Fourier grid, dealiased to $128^3$, and periodic boundary conditions.
For each comparison, $U_b, E(L = 1), \nu_0, \beta$, and $L_0$ are set in the model to the simulation
values. Thus, having set $b$ and $L_{outer}$ once, there are no “free” parameters used in the
comparisons.

The figure captions contain more discussion of the comparison. Here we comment
on the disagreement between the model and simulations for the $\beta = 1, U_b = 0.5$ case in
Fig. 4.2. One can show that, if the ratio $U_b/\nu_0$ is too small, the energy will purely decay
in the $\beta = 1$ case, rather than reach the saturated state (Sec. 3.3.2). The model predicts a
similar breakdown, if $1 - BA < 0$. However, with the present choices of $b$ and $L_{outer}$, the
value of the ratio $U_b/\nu_0$ for which pure decay occurs in the model is not quite correct. It
can be made correct, while simultaneously keeping the agreement with the $\beta = 1$ steady
state energy, with a different choice for $b$ and $L_{outer}$. We find this requires a value for $b$ far
outside the reasonable range, and, while improving the fit for the $\beta = 1$ case, worsens the fit
in the other cases. Note that the simulations in Fig. 4.2 are carried out with relatively high
initial viscosity (low Reynolds number) to keep them resolved at saturation for our modest
resolution. It is only at these low initial Reynolds numbers and for slow compressions that
the pure decay occurs (in either the model or simulations), which are not the cases most of
interest.
4.5 Energy partition

4.5.1 Coupled model

By coupling the model developed here to a temperature equation, we can examine how the energy injected by the compression is partitioned between thermal energy and turbulent energy. To do this for the case of the plasma viscosity, we revise Eq. (4.1). Instead of assuming $T \sim \bar{L}^{-2}$, we treat $T$ as an unknown, and use the plasma viscosity dependence on temperature, $\mu = \mu_0 T^{5/2}$. Making this substitution yields for the TKE model equation, instead of Eq. (4.13),

\[
\frac{d}{d\bar{L}} \bar{E}(\bar{L}) = -2 \frac{\bar{E}}{\bar{L}} + \bar{L}^{-4} \bar{\epsilon}_{\text{model}}, \tag{4.16}
\]

where,

\[
\bar{\epsilon}_{\text{model}}(\bar{L}) \equiv 2 \frac{\tau_c}{\tau_v} \left[ \bar{T}^{5/2} \bar{L}^3 + \sqrt{\frac{1}{2} \frac{V_0^2}{\gamma^2} \bar{L}^2 \bar{E}(\bar{L}) + \bar{T}^5 \bar{L}^6} \right] \bar{L}^2 \bar{E}(\bar{L}). \tag{4.17}
\]

At present we work with a normalized laboratory TKE, $\bar{E}$, which has been normalized using an initial turbulent velocity, $V_0$. It can be related to the laboratory TKE, $E$, as

\[
E(\bar{L}) = \frac{V_0^2}{2} \bar{E}(\bar{L}). \tag{4.18}
\]

Because we have normalized the TKE, it is convenient to convert $A$ and $B$ to a new set of constants. These constants are

\[
\tau_c = -\frac{1}{\bar{L}} = \frac{L_0}{2 U_b}, \tag{4.19}
\]

\[
\tau_v = \frac{\alpha^2 L_0^2}{3 \nu_0}, \tag{4.20}
\]

\[
\gamma = \left( \frac{9 b^3}{16} \right)^{3/2} \frac{\nu_0}{\alpha L_0}. \tag{4.21}
\]
Here $\tau_c$ is the compression time, $\tau_\nu$ is an initial viscous time, and $\gamma$ can be thought of as a “viscous velocity”. Note that $\tau_c$ and $\tau_\nu$ only enter the model as a ratio (this is why we have gone from 2 constants to 3).

This model TKE equation, Eq. (4.16), must then be coupled to a temperature equation in order to look at the question of energy partition. The simplest consistent coupled system includes only the mechanical compression and the viscous dissipation in the temperature equation, taking $\bar{\mathcal{T}}$ to be,

$$\frac{d \bar{T}}{d \bar{L}} = -2 \frac{\bar{T}}{\bar{L}} - E_{r0} \bar{L}^{-4} \epsilon_{\text{model}}.$$

The temperature is normalized to the initial temperature, $T_0$, and $E_{r0}$ is the initial ratio of the energy density of TKE to thermal energy,

$$E_{r0} = \frac{E^T_0}{E^\text{th}_0} = \frac{\rho_0 V_0^2}{2} \frac{1}{3 n_0 k_B T_0}.$$

The thermal energy is written assuming a plasma with $Z = 1$, and assuming equal temperature ions and electrons. The $Z = 1$ assumption can be relaxed in principle, and a possibly changing ionization state included in the viscosity used in the model.

### 4.5.2 Energy partition discussion

The coupled system, Eqs. (4.16), (4.22), can be used to predict the proportion of thermal energy and turbulent energy after any amount of compression at a given rate. This represents one key component to understanding the behavior of plasma turbulence under compression. Because the present model is, strictly speaking, only for subsonic compressions, it may not be accurate if $E_{r0}$ is too large.

Of note is that, for the coupled system, the total energy density, which is proportional to $E_{r0} \bar{E} + \bar{T}$, grows as $\bar{L}^{-2}$ from its initial value of $1 + E_{r0}$,

$$\frac{d}{d \bar{L}} \left( E_{r0} \bar{E} + \bar{T} \right) = -2 \frac{2}{\bar{L}} \left( E_{r0} \bar{E} + \bar{T} \right).$$
This means that, while the thermal energy and the turbulent energy are no longer state functions of the compression, the total energy is a state function, depending only on the initial value and the amount of compression. In the event that loss mechanisms, such as radiation, are added to the system (as will be done in Ch. 7), the total energy will also cease to be a state function of the compression. However, one could imagine that, even in the lossless case presently examined, one might have found that the total energy injected by the compression is not a state function.

The fact that the total energy for the lossless system is a state function is apparently due to a combination of the manner of compression, and the fact that the compression is isotropic in three-dimensions. Chapter 3, particularly Sec. 3.6.1, describes the compression technique in detail. The compression is caused by a background flow. This compression gives rise to the forcing term in Eq. (4.4), $2U_b \mathbf{V}/L$. Dotting the momentum equation with $\mathbf{V}$ to obtain the energy equation, gives an energy forcing term, $2U_b \mathbf{V}^2/L$. This energy forcing term, with proper normalizations, yields the first term in Eq. (4.16). Its dependence is only on the total TKE in the system, a property it shares with the temperature under mechanical compression. This property then carries over to the total energy in the coupled system.

If one assumes a compression-generating background flow that is not isotropic (i.e. $A_{ij}$ in Eq. (3.21) diagonal but with unequal entries), this state-function property for the total energy can be lost. In the case of two-dimensional compression (letting $A_{33} = 0$, $A_{11} = A_{22}$), the behavior of the total energy can be written,

$$\frac{d}{dL} \left( \mathcal{E}_{t0} \hat{E} + \hat{T} \right) = -\frac{4}{3} \hat{L} \left( \mathcal{E}_{t0} \hat{E} \left[ \frac{3 \nu_\parallel^2}{2 \nu^2} \right] + \hat{T} \right).$$

(4.25)

Here, $\nu_\parallel^2 = \nu_x^2 + \nu_y^2$, if the 2D compression is along the $x$ and $y$ directions. If the turbulent energy is split evenly between the three velocity components, then $\nu_\parallel^2/\nu^2 = 2/3$, and the total energy will behave as a state function, growing as $\hat{L}^{-4/3}$. However, there is no need for the energy to be split in such an equilibrium manner during a compression, and the amount
of total energy change in general will depend on the split. The 2D case and this split is not addressed in the present model, or elsewhere in this thesis.

Note that the total energy considered here neglects the (constant) energy associated with the background flow, Eq. (3.21). This energy, calculated for the usual domain considered in this thesis (with initial side length $L_0$), is $E_{bg} = \rho_0 U_b^2 L_0^3/2$. For fast compressions in the subsonic picture, the energy in the background flow is necessarily substantial compared to the turbulent energy. While one expects intuitively that the supersonic case will be similar in the mean, the density – background velocity correlation could instantaneously play a role, since density perturbations in the supersonic case can be substantial (such density perturbations are discussed in Ch. 6, and are present in the compressing astrophysical turbulence discussed in Ch. 5).

### 4.6 Discussion

With fixed values of the constants $b$ and $L_{outer}$, the model shows reasonable agreement with simulations over a range of compression speeds and viscosity dependencies on compression ($\beta$s). At the same time, it is a relatively simple model to calculate, involving the solution of a single differential equation, Eq. (4.13). It is in some respects similar in spirit to two or three (differential) equation $k - \epsilon$ or $k - \epsilon - \tau$ models, some of which have been developed for compressing fluid turbulence (e.g. (Wu et al., 1985; Coleman and Mansour, 1991)). Such models typically include multiple constants that are determined by fitting to simulations. Straightforward applications of these existing models to the present cases show unsatisfactory results.

For $\beta < 1$, the model predicts that the TKE saturates under continuing compression, at $E(\tilde{L} \to 0) = 4/A^2 = 1.9U_b^2$. The final equality holds for the choice of $b$ and $L_{outer}$ used here. It is unclear whether this should be the case. Section 3.3.2 showed that for $\beta = 1$, the number of linearly forced modes (in Fourier space) is a constant in time (equivalently in $\tilde{L}$),
in which case the TKE reaching a steady state is unsurprising. When $\beta > 1$, the number of linearly forced modes always eventually reaches zero, so that all modes are damped beyond some time, and thus the TKE will eventually decay. If $\beta < 1$, the number of linearly forced modes grows indefinitely in time. Naively, one could then expect the TKE to grow indefinitely, disagreeing with the model prediction. However, the compression forces each mode proportional to its TKE content. Assuming a TKE spectrum that decays with increasing mode number (decreasing wavelength), each additional forced mode contributes less forcing than the previous one, perhaps yielding a finite total forcing, and a TKE that reaches a steady state under continuing compression. This possibility is hinted at by the fact that the saturated energy in the $\beta = 1$ case doesn’t depend on viscosity, but the number of linearly forced modes does. Thus, two cases with the same compression velocity, but different viscosities, reach the same steady state energy ($E = 1.9U_b^2$) while having different (but constant) numbers of linearly forced modes.

In Lohse’s model, the outer length scale is fixed, not free to grow. This is reflected in the zero-viscosity-limit decay given by the model, $\dot{E} \sim \dot{t}^{-2}$. One could imagine compressing turbulence with an outer length scale that is either fixed or free to grow, depending on the initial length scale, time of compression, and means of compression. If the outer length scale is free to grow during a compression, the present model may not predict the results well.

The present modeling has used a fixed value for $\alpha = L_{outer}/L_0$, chosen based on the steady-state outer length scale for the $\beta = 1$ case. In principle, it may be more appropriate to create a ($\beta$ dependent) model for the outer length scale. That is, the value of $\alpha$ may be different from 0.19 when the compressing turbulence is not in a steady state.

These considerations surrounding $\beta < 1$ behavior and a fixed or free outer length scale may have some bearing on the problem of astrophysical molecular cloud turbulence. This turbulence is typically treated as isothermal, so that $\beta = 0$. Although the molecular cloud turbulence is supersonic, and Lohse’s model is for subsonic turbulence, an existing model
for molecular cloud turbulence under contraction does not predict a saturation of the TKE (Robertson and Goldreich, 2012). This is despite the fact that the length scale in the model is apparently saturated, see Ch. 5 for evidence of this, and further discussion.

We mention that the rapid distortion theory (RDT) (Durbin and Reif, 2010; Savill, 1987; Hunt and Carruthers, 1990) solution to Eq. (4.4), with the viscous dissipation term included, can also give reasonable results when $\beta = 5/2$, the compression is fast, and the initial Reynolds number is not that high, as in the simulations presented here. In this case, the viscous RDT solution captures the initial growth well, and the decay well, but overshoots at the peak of the TKE. This overshoot decreases the more rapid the compression. Initially, the linear compressive forcing dominates the solution for a rapid compression. During the sudden dissipation, the linear viscosity term dominates the solution. Apparently, for $\beta = 5/2$, a modest Reynolds number, and a rapid compression, there is only a brief window for nonlinear effects between these two linear regimes. Note that, by $\beta = 3/2$, the nonlinear effects are very important, since this case corresponds to regular decaying NS turbulence in the rescaled equation, Eq. (4.9).

While we have discussed, in Sec. 4.5, the coupling of dissipated TKE to the temperature evolution, this treatment of the feedback only occurs in that section. In particular, the comparison to simulations is done for fixed $\beta$, while in the feedback case $\beta$ will effectively change as the rate of temperature growth with compression can change. The model should still be useful in this case, but it may not be as accurate as in the fixed $\beta$ cases where the comparison is carried out. Future work should compare the model to simulations that include the feedback of dissipated TKE to temperature (or, a variable $\beta$ that mimics it).

We have presented a simple model for the TKE behavior of compressing turbulence, which agrees with simulations over a range of viscosity dependencies on compression. In this regard, it represents a substantial improvement over previously existing models. It is hoped this model will be useful for evaluating the prospects of preventing or exploiting plasma turbulence in plasma compression experiments.
Chapter 5

A lower bound on adiabatic heating of compressed turbulence for simulation and model validation

5.1 Introduction

The energy in turbulent flow can be amplified by compression, when the compression occurs on a timescale shorter than the turbulent dissipation time. This mechanism may play a part in sustaining turbulence in various astrophysical systems, including molecular clouds. The amount of turbulent amplification depends on the net effect of the compressive forcing and turbulent dissipation. By giving an argument for a bound on this dissipation, we give a lower bound for the scaling of the turbulent velocity with the compression ratio in compressed turbulence. That is, turbulence undergoing compression will be enhanced at least as much as the bound given here, subject to a set of caveats that will be outlined. Used as a validation check, this lower bound suggests that some models of compressing astrophysical turbulence are too dissipative. The technique used highlights the relationship between compressed turbulence and decaying turbulence.
Turbulence undergoing mean compression, also called compressed turbulence, is of interest in a variety of disciplines. A number of studies, ranging from investigations of its essential behavior to detailed application studies, have been conducted with an eye towards internal combustion engines and aerodynamic flows. These include studies focusing on the zero-mach-number limit (e.g. Morel and Mansour (1982); Wu et al. (1985); Coleman and Mansour (1991); Cambon et al. (1992); Guntsch and Friedrich (1996); Liu and Haworth (2010); Hamlington and Ihme (2014)), and those focusing on the finite-mach-number limit (e.g. Blaisdell (1991); Speziale and Sarkar (1991); Durbin and Zeman (1992); Coleman and Mansour (1993); Cambon et al. (1993); Blaisdell et al. (1996); Grigoriev et al. (2016)).

Other contexts where compressed turbulence is of interest include plasma physics and inertial fusion (Davidovits and Fisch, 2016b,a; Thomas and Kares, 2012; Weber et al., 2014; Kroup et al., 2011, 2007a; Maron et al., 2013; Kroup et al., 2007b), and astrophysics (Robertson and Goldreich, 2012). In the astrophysics context, the turbulence undergoing compression is typically supersonic, and the present work focuses on compressed turbulence in this context.

Turbulence is ubiquitous in interstellar gas (Elmegreen and Scalo, 2004), and the properties of supersonic turbulence have been related to important astrophysical questions such as the core mass and stellar initial mass functions (Padoan and Nordlund, 2002; Ballesteros-Paredes et al., 2006; Hennebelle and Chabrier, 2008), star formation efficiency (Elmegreen, 2008), and the origin of Larson’s laws (Kritsuk et al., 2013a). As such, supersonic turbulence has been the subject of numerous investigations in the context of astrophysics (e.g. Mac Low et al. (1998); Mac Low (1999); Kritsuk et al. (2007); Federrath et al. (2008); Kritsuk et al. (2013b); Federrath (2013); Banerjee and Galtier (2014)). This astrophysical turbulence is often undergoing contraction or expansion under the influence of gravity or pressure. Robertson and Goldreich (2012) pointed out that little work has been done on compressed turbulence in astrophysics, although intuition and some results from prior work on compressed turbulence in other contexts should be expected to carry over. Since
contraction (or expansion) influences the behavior of the turbulence, and the turbulence plays a role in many problems related to interstellar gas dynamics, it is desirable to better understand how exactly contraction influences turbulent behavior.

At the most basic level, the first classifying parameter for turbulence undergoing compression is the ratio, \( S = \tau_d/\tau_c \), of the turbulent dissipation time, \( \tau_d \), to the compression time, \( \tau_c \). When the compression is very slow, \( S \ll 1 \), and the compression has little effect. If the compression is very fast, \( S \gg 1 \), the turbulence is essentially “frozen” and its behavior can be treated with rapid distortion theory (RDT) (Savill, 1987; Hunt and Carruthers, 1990; Durbin and Reif, 2010). For three-dimensional rapid isotropic compressions, one finds that the root mean square (rms) turbulent velocity \( v_{rms} \sim v_{rms,0}/\bar{L} \), where \( \bar{L} \) is the contraction factor along each axis, \( \bar{L} = L/L_0 \) (see e.g. Wu et al. (1985) for a zero-mach RDT treatment, or Cambon et al. (1993) for a finite-mach RDT treatment; a similar result is given by Peebles (1980) in section 90). In actuality, the turbulence will not be completely “frozen”, and there will be turbulent dissipation, the quantity of which depends (in part) on the rapidity of the contraction. This dissipation reduces the rms turbulent velocity scaling with compression below the \( 1/\bar{L} \) “adiabatic” result.

Here we present an argument for an upper bound on the amount of this turbulent dissipation, thereby providing a lower bound on the amount of adiabatic heating that turbulence in a contracting gas can undergo. This argument rests on the following assumption. Consider as a base case the rate of decay for unforced Navier-Stokes (NS) turbulence with a constant viscosity. We assume that when the viscosity is a shrinking function of time, with the same initial value as the base case, the rate of decay is not larger than that for the base case. If this physically reasonable assumption holds, the bound follows directly. Then the bound can be used as a check on models and simulations of compressing turbulence, or as a model itself. We carry out an initial comparison with some previous work, which suggests that at least some approaches to simulating or modeling compressed high-mach turbulence are too dissipative. They will give, for example, asymptotic scaling (in \( \bar{L} \to 0 \)) of the tur-
bulent velocity for a gravitational contraction that is below the minimum predicted by the bound. Since similar approaches are used in many astrophysical simulations, this apparent disagreement with the bound may have implications for other work as well.

The focus of the current work is to present the bound and an initial comparison against some previous work, thereby motivating future work to determine if the key assumption made in arriving at the bound holds. We note that even if some approaches to simulating and/or modeling compressed high-mach turbulence are in fact too dissipative, a separate determination needs to be made as to whether this affects the results of interest. While physically reasonable, the assumption is not rigorous. In the surprising event the assumption is violated, so that the decay rate of NS turbulence is increased if the viscosity shrinks in time, there will likely still be implications for turbulence in astrophysical settings. Of course, the assumption (and bound) may hold in some situations and not in others, depending on the mechanism(s) by which the assumption is violated, if it is.

In the process of arriving at the bound, the sometimes forgotten relationship (Cambon et al., 1992) between turbulence forced by contraction and decaying turbulence is highlighted. Beyond its use in the argument for a lower bound, which is the focus of the present work, this connection may be helpful for understanding the influence of contraction on astrophysical processes, since it gives a means of translating quantities (e.g. correlation functions) between compressing and decaying cases. The relationship can also be useful for simplifying simulations of compressing turbulence (e.g. as used in Ch. 2).

Although the bound presented here has a number of caveats associated with it, the approach used to arrive at it should be adaptable to create new bounds with different applicability. The bound is given in terms of $\bar{L}$, the Hubble parameter, $H = \dot{L}/L$ (with the overdot the time derivative), and the decay time constant $t_0$ and power $\alpha$ (in the spirit of Mac Low et al. (1998); Mac Low (1999)) for the rms velocity in decaying supersonic turbulence. The bound is

$$\frac{v_{rms}}{v_{rms,0}} \geq \frac{1}{L} \left( 1 + \frac{1}{t_0} \int_1^L (L')^{-3} \frac{dL'}{H'} \right)^{-\alpha/2}.$$  (5.1)
As will be shown later, this form of the bound follows once a fit for the decay of $v_{rms}$ in unforced NS turbulence (with a regular, constant viscosity) is chosen. If these fits are refined, the bound will be as well.

This chapter is organized as follows. The model, essentially the NS equations in coordinates comoving with the contraction (or expansion), is described in Sec. 5.2. Section 5.3 shows the use of a time-dependent variable rescaling to change the NS equations forced by contraction into NS equations for decaying turbulence, with extra time-dependent coefficients. An argument for the bound, Eq. (5.1), is given in Sec. 5.4, using the rescaled NS equations. Section 5.5 compares the bound to some previous results on compressing supersonic turbulence and discusses the caveats and implications of the bound and rescaling.

5.2 Model

The model is the NS equations written in contracting (or expanding) coordinates. These coordinates, $x$, are defined in terms of the proper coordinates, $r$, as

$$x = r/L. \quad (5.2)$$

The proper velocity, $u$, written in terms of the peculiar velocity $v$ and the contracting coordinates, is

$$u = \dot{L}x + v(x, t) \quad (5.3)$$

Beginning with the NS equations for $u$ and the density $\rho$ in the proper coordinates, and rewriting in terms of $x$ and $v$, gives

$$\frac{\partial \rho}{\partial t} + \frac{1}{L} \nabla \cdot (\rho v) + 3 \frac{\dot{L}}{L} \rho = 0 \quad (5.4)$$

$$\frac{\partial v}{\partial t} + \frac{1}{L} (v \cdot \nabla) v + \frac{1}{\rho L} \nabla p + \frac{1}{L} \nabla \Phi + \dot{\Phi} \frac{\dot{L}}{L} x + \frac{\dot{L}}{L} v = \frac{D}{L^2} \quad (5.5)$$

$$\frac{1}{\rho} \left( \mu \nabla^2 v + (\mu + \lambda) \nabla (\nabla \cdot v) \right) = D. \quad (5.6)$$
Here, $p$ is the pressure, $\Phi$ is the gravitational potential, and $D$ is the usual dissipation term in the momentum equation, which is given in Eq. (5.6). It has been assumed that the dynamic and bulk viscosities, $\mu$ and $\lambda$ respectively, are constants. A derivation of these equations, with the exception of the dissipation term, can be found in Peebles (1980), section 9. Essentially identical equations, based on contractions identical to those dictated by $u$ in Eq. (5.3), but without the gravitational potential, underlie studies of compressing turbulence in other contexts (Blaisdell, 1991; Cambon et al., 1993; Coleman and Mansour, 1993).

Besides giving spatial derivatives time-dependent coefficients (powers of $\bar{L}$), the effect of the contraction is to add forcing (or dissipation) to both the density and momentum equations. In the density equation, Eq. (5.4), the third term is a forcing term when $\dot{L}$ is negative. This, in part, causes the mean density to increase as expected for the contraction.

In the momentum equation, Eq. (5.5), the first term to the left of the equals sign is similarly a forcing term when $\dot{L}$ is negative. In fact, a similar term has been used as a way to add real space forcing for turbulence simulations (Lundgren, 2003; Rosales and Meneveau, 2005; Petersen and Livescu, 2010). It is this term that, taken alone, will lead to the “adiabatic” increase of turbulent velocity $v_{rms} \sim 1/\bar{L}$.

The second term to the left of the equality in Eq. (5.5) is related to the acceleration of the contraction, $\ddot{L}$. It depends on $\mathbf{x}$, and can cause the turbulence to be inhomogeneous (see Blaisdell (1991), Section 2.4, for a thorough discussion). In the case where the contraction (the time dependence of $L$) is determined by gravity, this acceleration term can be removed from the momentum equation by the gravitational field of the mean density (see Peebles (1980)).

For the present work, we will treat this as the case, and we will also choose to ignore the gravitational effects associated with density fluctuations (as in Robertson and Goldreich...
The pressure is taken to obey a polytropic law,

\[ p = K \rho^\gamma, \]  

(5.7)

with \( K \) and \( \gamma \) constants. Together, these choices give the model momentum equation,

\[ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{L} (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{K}{\rho L} \nabla \rho^\gamma + \frac{\dot{L}}{L} \mathbf{v} = \frac{\mathbf{D}}{L^2}. \]  

(5.8)

### 5.3 Rescaling

Substituting rescaled values of the density, velocity, and time,

\[ \rho = \bar{L}^\phi \hat{\rho}, \]  

(5.9)

\[ \mathbf{v} = \bar{L}^\delta \hat{\mathbf{v}}, \]  

(5.10)

\[ d\hat{t} = \bar{L}^\tau dt, \]  

(5.11)

in the density and momentum equations, Eqs (5.4,5.8), gives

\[ \frac{\partial \hat{\rho}}{\partial \hat{t}} = -\bar{L}^{\delta-\tau-1} \nabla \cdot (\hat{\rho} \hat{\mathbf{v}}) - \bar{L}^{-\tau} (3 + \phi) H \hat{\rho} \]  

(5.12)

\[ \frac{\partial \hat{\mathbf{v}}}{\partial \hat{t}} = -\bar{L}^{\delta-\tau-1} \hat{\mathbf{v}} \cdot \nabla \hat{\mathbf{v}} - \bar{L}^{-\delta-\tau+\phi(\gamma-1)-1} \frac{K}{\hat{\rho}} \nabla \hat{\rho}^\gamma \]  

\[ = (1 + \delta)\bar{L}^{-\tau} H \hat{\mathbf{v}} + \bar{L}^{-\phi-\tau-2} \hat{\mathbf{D}}. \]  

(5.13)

The Hubble parameter \( H = \dot{L}/L \), and the dissipation \( \hat{\mathbf{D}} \) is the same as in Eq. (5.6), but with \( \hat{\rho}, \hat{\mathbf{v}} \) in place of \( \rho, \mathbf{v} \).

By choosing \( \phi = -3 \) and \( \delta = -1 \), the forcing terms can be eliminated from the density and momentum equations. Then, choosing \( \tau = -2 \) removes the time-dependent coefficient from the divergence term in the density equation, and also removes it from the nonlinear term in the momentum equation. For these choices of \( \phi, \delta, \tau \), the incompressible case of this
transformation has been discussed by Cambon et al. (1992). A different choice was made in Ch. 2 and Ch. 3, for the convenience of simulations. Various similarity transformations (e.g. Nishitani and Ishii (1985); Nishitani (1991); Davis and Peebles (1977)) are related.

We will also take the polytropic index $\gamma = 5/3$. Then, the rescaled NS equations become,

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} = - \nabla \cdot (\hat{\rho} \hat{v})$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} = \hat{v} \cdot \nabla \hat{v} - \frac{K}{\hat{\rho}} \nabla \hat{\rho}^{5/3} + \bar{L}^3 \hat{D}. \tag{5.15}$$

Up to the $\bar{L}^3$ scaling on the dissipation term, these are the usual, unforced, NS equations for a gas with polytropic index $\gamma = 5/3$. Note that there is no separate energy equation because the system was closed with the assumption of a polytropic pressure, Eq. (5.7). An energy equation can be derived as usual from the momentum equation, Eq. (5.15), but it does give “new” information, in the sense that the system is closed without it.

### 5.4 Bound

Turbulence governed by the rescaled equations, Eqs. (5.14,5.15), will decay, as it is unforced in these variables. The usual compressible NS equations are recovered by setting $\bar{L} = 1$. For contraction $\bar{L}(\hat{t}) \leq 1$ is a strictly decreasing function of time (the equality holds at $\hat{t} = 0$). Since the viscous dissipation in Eq. (5.15) is multiplied by $\bar{L}^3$, it has a smaller coefficient at all times after $\hat{t} = 0$ in the compressing case than in the $\bar{L} = 1$ usual case. Then it is reasonable to expect that the turbulent decay rate for the system Eqs. (5.14,5.15) will be slower than (or equal to) the decay for the usual compressible NS equations ($\bar{L} = 1$). This is the key assumption on which the bound rests.

If this assumption is true, then the rms turbulent velocity for the system, Eqs. (5.14,5.15), will be at least as great as that given by the usual power-law decay for the system when
\[ \dot{\bar{L}} = 1, \]

where \[ \dot{v}_{rms} \geq \dot{v}_{rms,0} \left(1 + \dot{t}/t_0\right)^{-\alpha/2}. \] (5.16)

Here, \( \alpha \) and \( t_0 \) are to be determined for turbulent decay in the non-compressing \((\bar{L} = 1)\) case. Then, arriving at the bound, Eq. (5.1) requires using Eqs. (5.10,5.11) to transform Eq. (5.16) into the unscaled (non-hat) variables. One could instead write a comparable bound for the turbulent kinetic energy (TKE), \( \langle \rho v^2/2 \rangle \), under compression. We use the turbulent velocity in keeping with previous work (Robertson and Goldreich, 2012). If a decay law of a different form than that given by Eq. (5.16) is more appropriate, there will still be an equivalent bound, derived once again by transforming the decay law back into the unscaled variables.

Although we are not aware of work determining \( t_0 \) and \( \alpha \) for the rms velocity decay of supersonic turbulence with \( \gamma = 5/3 \), we can estimate these values from closely related work. The bound will then be only a guide. Mac Low (1999) found that for supersonic (initially mach 5) isothermal decaying turbulence, \( t_0 \) is the initial turnover time for the turbulence (at the driving scale). Mac Low et al. (1998) found that the TKE in supersonic (mach 5) turbulence with \( \gamma = 7/5 \) decays with power \( \alpha \sim 1.2 \). In the isothermal case (\( \gamma = 1 \)), they found \( \alpha \sim 1 \), suggesting some slight dependence on \( \gamma \), at least within this modest range. Smith et al. (2000) found that for the decay of hypersonic (mach 50) isothermal turbulence, the decay power \( \alpha \sim 1.5 \), after an initial transient period. While these results suggest a single value of \( \alpha \) will not suffice for all situations, we may expect that for \( \gamma = 5/3 \), \( \alpha \) is roughly in the range \( 1 \sim 1.5 \), depending on the initial mach number.

Note that these decay rates are for the TKE, not the rms velocity. Using them for the decay of the rms velocity discounts density-velocity correlations. Mac Low (1999) found these correlations make for a 10% – 15% difference between the TKE calculated from the rms velocity, \( m v_{rms}^2 / 2 \), and the TKE calculated directly, \( \langle \dot{\rho} \dot{v}^2 / 2 \rangle \). Again, this result is for mach 5 turbulence, and may change with mach number.
5.5 Discussion

5.5.1 Comparison with previous results

The bound, Eq. (5.1), can be used as a validation tool. For example, let us compare it with the compressing turbulence model and matching simulations of Robertson and Goldreich (2012). That model is

\[
\frac{dv_{rms}}{d\bar{L}} = -\left[ 1 + \eta \frac{v_{rms}}{H \bar{L} L_0} \right] \frac{v_{rms}}{\bar{L}}.
\]  

(5.17)

This model for \(v_{rms}\) includes two components: the forcing due to the contraction (the first term left of the equals sign in Eq. (5.5)), and the dissipation of \(v_{rms}\) calculated from the equilibrium dissipation rate for forcing at a given scale, as found by Mac Low (1999). The forcing scale is taken to decrease in time as determined by the contraction, \(\bar{L}\). Robertson and Goldreich (2012) found that \(\eta = 1.2\) creates a good match between the model and their simulation results, which were carried out for isothermal turbulence. The model is nominally independent of \(\gamma\), although to the extent the turbulent dissipation rate depends on \(\gamma\), one may expect that \(\eta\) could change.

Assuming equality in the bound, Eq. (5.1) and differentiating with respect to \(\bar{L}\), one can write an expression for \(v_{rms}\) that is comparable to Eq. (5.17):

\[
\frac{dv_{rms}}{d\bar{L}} = -\left[ 1 + \frac{\alpha}{2 t_0} \frac{\bar{L}^{2/\alpha-2}}{H} \left( \frac{v_{rms}}{v_{rms,0}} \right)^{2/\alpha} \right] \frac{v_{rms}}{\bar{L}}.
\]  

(5.18)

For \(\alpha = 2, \eta = 1\), and taking the initial turnover time as \(t_0 = L_0/v_{rms,0}\), the two expressions, Eq. (5.17) and Eq. (5.18), are equal. The bound and the model of Robertson and Goldreich (2012) are calculated for different values of \(\gamma\), complicating a direct comparison. To compare, consider the \(\gamma = 5/3\) case, for which the bound is calculated. First, assume that the model, Eq. (5.17), is still valid for \(\gamma \neq 1\), with a possibly different value of \(\eta\) (as suggested by Robertson and Goldreich (2012)). Then, the comparison hinges on the value of the decay power \(\alpha\) of \(v_{rms}\) for supersonic turbulence when \(\gamma = 5/3\).
To see this, consider the asymptotic scaling of $v_{rms}$ with $\bar{L} \to 0$, for the case of a gravitational-like contraction with $H = -H_0\bar{L}^{-3/2}$. The prediction of the model, Eq. (5.17), is

$$v_{rms} / v_{rms,0} \to (H_0L_0/2v_{rms,0}\eta)\bar{L}^{-1/2}. \quad (5.19)$$

Using Eq. (5.18) with $t_0 = L_0/v_{rms,0}$, the same contraction gives for $\bar{L} \to 0$ that

$$v_{rms} / v_{rms,0} \to (H_0L_0/2v_{rms,0})^{\alpha/2}\bar{L}^{\alpha/4-1}. \quad (5.20)$$

Now, note that, unless $\alpha \geq 2$, the model Eq. (5.17) will cross below the bound as $\bar{L} \to 0$ for any $\eta$. Apparently, assuming the bound holds, either $\alpha \geq 2$ for the decay of $v_{rms}$ when $\gamma = 5/3$, or the model, Eq. (5.17) will be too dissipative when applied to the $\gamma = 5/3$ case.

We are not aware of available decay rates for $v_{rms}$ in turbulence with $\gamma = 5/3$ and initial mach numbers $M \sim 6$. However, available decay rates in the literature, for various values of $\gamma$ and initial mach number suggest $\alpha \geq 2$ would be an outlier. In the low-mach case, decay rates this large are associated with bounded turbulence (Skrbek and Stalp, 2000; Thornber, 2016). Mac Low et al. (1998) found for initially mach 5 turbulence a TKE decay rate $\alpha \sim 1$ for $\gamma = 1$ and $\alpha \sim 1.2$ for $\gamma = 7/5$. For initially mach-20 turbulence with a complicated equation of state, Pavlovski et al. (2002, 2006) find for the TKE that $\alpha \sim 1.34$. These decay rates are calculated for the TKE. As noted in Sec. 5.4, using them for $v_{rms}$ neglects density-velocity correlations. While the size of these correlations will likely change when $\gamma$ changes, they have a relatively small impact when $\gamma = 1$. The decay rate of $v_{rms}$ has been found directly, using a number of different simulation algorithms, by Kitsionas et al. (2009). They found for isothermal, mach 4 turbulence, $\alpha \sim 1$, which is very similar to the decay rate inferred from the TKE. Given the available results, it seems unlikely the bound, Eq. (5.1) and the model, Eq. (5.17) will be consistent for the $\gamma = 5/3$ case, with the model being too dissipative. As the methods used to arrive at both the model and the bound appear reasonable, reconciling this difference requires more detailed consideration.
One possibility for reconciling the model and bound is to conclude that the simulations in Robertson and Goldreich (2012) have a saturated length scale, and that a saturated length scale gives supersonic turbulence a decay rate $\alpha \gtrsim 2$, as in the subsonic case. If length scale saturation is the explanation of the discrepancy, then it must be determined whether such saturation is physical or not. That is, should one expect that, in a frame moving with the mean gravitational compression, the maximum length scale of the cloud is bounded and not free to grow. In simulations with periodic boundary conditions the outer scale will saturate at some point; it is not obvious that this should be the case for a molecular cloud.

The modeling results of Ch. 4 assume a saturated length scale, but this is reasonable for, say, hot spots in inertial fusion experiments, since the hot spot is bounded by a shell. The model of Ch. 4 predicts that the TKE saturates under compression for weak viscosity growth ($\beta \leq 1$); the present case corresponds to $\beta = 0$. However, the time history of compression, $\bar{L}(t)$, for the present case of gravitational contraction, is not identical to the constant velocity compressions considered in Ch. 4 and elsewhere in this thesis. This complicates any comparison between the model in Ch. 4 and the results here. Nonetheless, it is worthwhile to consider whether, even supposing the length scale saturation is physical in the present case (with $\alpha = 2$), the asymptotic behavior of $v_{rms}$ for $\bar{L} \to 0$ must be $v_{rms} \propto \bar{L}^{-1/2}$.

The bound hints at how the asymptotic behavior in the saturated length scale case could be different from $v_{rms} \propto \bar{L}^{-1/2}$. In writing the scaled bound, Eq. (5.16), from which the bound is derived, a constant power-law decay has been assumed. It is well known in subsonic turbulence that at late times, the decay power increases (see, e.g., Sagaut and Cambon (2008)). Since the compressing system can be written as a decaying one, Eqs. (5.14), (5.15), this late time increase in the decay rate may need to be considered to write a bound that is valid for the asymptotic behavior. If the late time decay power is $\alpha \geq 4$, then the asymptotic $v_{rms}$ predicted by Eq. (5.20) will be a constant (for equality) or decreasing (for
greater than). Of course, in the subsonic case, $\alpha = 2$ already apparently represents the asymptotic behavior.

### 5.5.2 Bound discussion

Assuming equality in the bound, Eq. (5.1), is equivalent to asserting that the time-dependent pre-factor ($\bar{L}^3$) of the viscous dissipation, $\dot{D}$, in Eq. (5.15) does not decrease the dissipation rate of the turbulence, despite the fact that the coefficient decreases in time. That is, that the dissipation rate (and therefore energy behavior) of the turbulence is independent of time dependence in the viscous coefficient. For various subsonic compressing turbulence studies, this has not been found to be the case (Coleman and Mansour, 1991; Cambon et al., 1992), see also Ch. 3. Since dissipation in decaying supersonic (isothermal) turbulence is primarily in shocks (Smith et al., 2000), it is conceivable that the situation changes between subsonic and supersonic turbulence. Perhaps more importantly, in the previously studied subsonic cases, the viscous coefficient was generally increasing in time, rather than decreasing as in the present situation.

If the shrinking-in-time dissipation coefficient did have no impact on the dissipation rate, then Eq. (5.1), with equality assumed, would be a model for $v_{rms}$, rather than a bound. Furthermore, in this case, for a given initial condition, a single simulation of Eqs. (5.14,5.15) would be sufficient for all compression histories $\bar{L}(t)$ (or Hubble parameters, $H(t)$, alternatively). This is because the Eqs. (5.14,5.15) would no longer have any dependence on $\bar{L}$.

If the shrinking dissipation coefficient counter-intuitively led to more dissipation than in the case where the coefficient is constant, the lower bound would be invalid. If this effect were consistent, it would instead represent an upper bound (with $\geq$ in Eq. (5.1) switching to $\leq$).

The bound depends to some degree on the choice of physical model for the dissipation process. If the relevant dissipation process is not captured by the NS viscous dissipation,
Eq. (5.6), the bound may change. This is because the time-dependent coefficient of the dissipation in the rescaled momentum equation, Eq. (5.15), results from transformation and rescaling of the dissipation. For a different dissipation form, one could imagine that the coefficient after rescaling is different from the $\bar{L}^3$ coefficient found here. This could alter the bound, particularly if the coefficient were no longer shrinking in time. Additionally, the form of the NS dissipation, $\mathbf{D}$, does not change under transformation to the moving frame and rescaled variables, allowing the analogy between the compressing case and the uncompressing case.

This will not necessarily be true for all imaginable dissipation forms. For example, if the physically correct dissipation for the fluid equations took the form of the artificial viscosity commonly used for shock-capturing (see e.g. VonNeumann and Richtmyer (1950); Stone and Norman (1992)), the bound would need to be reconsidered. Note that, for numerical simulations in the moving frame (solving Eqs. (5.4,5.5)), the form of the dissipation may need to be considered explicitly, as done here, so that its transformation can be accounted for.

We now turn to the dependence of the bound on the adiabatic index. When $\gamma \neq 5/3$, the scaled momentum equation, Eq. (5.15), will pick up additional time dependence, as a coefficient for the pressure gradient term. This worsens the analogy between the scaled momentum equation and regular NS, but need not necessarily dramatically alter the bound. The impact on the bound depends on the effect of the pressure term (through the pressure-dilatation) on the energy dissipation in supersonic turbulence. As an example, consider the isothermal scenario ($\gamma = 1$). In this case, the scaled momentum equation becomes

$$\frac{\partial \mathbf{\hat{v}}}{\partial t} = \mathbf{\hat{v}} \cdot \nabla \mathbf{\hat{v}} - \bar{L}^2 \frac{K}{\rho} \nabla \bar{\rho} + \bar{L}^3 \mathbf{\hat{D}}.$$  \hspace{1cm} (5.21)

The decay rate of compressible turbulence is the result of the net effect of the viscous dissipation, $\propto \mathbf{\hat{v}} \cdot \mathbf{\hat{D}}$, and the pressure-dilatation, which comes from the dot product of the
pressure gradient term with \( \hat{v} \). In the high-Reynolds-number limit, it can be shown that the mean pressure-dilatation acts primarily on the largest scales, with its impact on small scales averaging out (Aluie, 2011, 2013; Aluie et al., 2012). To the extent that the pressure-dilatation enhances the decay rate, the bound should be insensitive to the \( \bar{L}^2 \) scaling. This is because the bound comes about by considering \( \bar{L} = 1 \) to be a more dissipative case than when \( \bar{L} \) shrinks in time, which would remain the case. There is some evidence the pressure-dilatation does in fact increase the dissipation in decaying turbulence, at least in the subsonic case (Sarkar, 1992; Samtaney et al., 2001). Even without this, the bound will be approximately preserved so long as the net effect on the decay of the pressure-dilatation term with the \( \bar{L}^2 \) coefficient is small compared to that of the viscous dissipation term with the \( \bar{L}^3 \) coefficient. For the isothermal case the pressure term scales as the sound speed squared, \( C_s^2 \), which becomes small in the high-mach limit. However, for very large compressions (reaching very small \( \bar{L} \)), the weaker decrease on the pressure term may relatively enhance its contribution even if it would normally be small.

Overall, even for \( \gamma = 5/3 \), the bound can only be universal to the extent that the decay of supersonic turbulence is (Federrath, 2013). To the extent the mix of compressible and solenoidal modes in the initial condition affects the decay rate, this must be accounted for in the value of \( \alpha \). Similarly with the impact of changing \( \gamma \) and changing initial mach number.

As noted in Sec. 5.2, the present treatment considers contractions where the time dependence of \( \bar{L} \) is determined by the gravitational attraction of the mean density. Strictly speaking, if \( \bar{L} \) is taken to have a different form, one must consider the effect of an acceleration term \( \ddot{\bar{L}}x \) in the momentum equation, Eq. (5.5). This may or may not have a significant impact on the bound. As also noted in Sec. 5.2, gravitational effects from the density fluctuations have been neglected. In many astrophysical problems of interest, there is forcing besides for the contraction which acts on the turbulence, which is neglected here.

These various assumptions and restrictions, if limiting to the generality of the bound, should be replicable for simulations. Then the bound provides a relatively simple, high-
level check on the simulations, particularly on the degree of dissipation. The initial application of the bound in this manner suggests a commonly used model (and matching simulations) may be too dissipative. Note that, even if a simulation or model is too dissipative, it may still be useful, depending on the physics under consideration.

The implications of the rescaled equations, Eqs. (5.14,5.15), apart from the bound, deserve mention. These equations are reached because forcing of the type generated by contraction can be scaled out of the NS equations. The only difference between the rescaled equations and compressible NS equations is in the dissipation term. However, many turbulent quantities, for example, inertial range properties, are not influenced by the dissipation properties of the turbulence. Therefore, we may expect that already known results for decaying supersonic turbulence can be translated by undoing the scaling, and applied to turbulence undergoing compression. This task is made simpler by the fact that the rescaling, Eqs. (5.9,5.10,5.11) is purely time dependent, so that, for example, spatial correlation functions are translatable.

In conclusion, we have suggested a lower bound on the increase in turbulent velocity associated with compression of turbulence. This lower bound follows directly once one assumes that a decreasing-in-time coefficient of viscosity in the NS equations does not increase the rate of dissipation for turbulence. This assumption, while physically reasonable, should be verified or disproved, since the bound represents a useful means of checking models or simulations of compressing turbulence, and an initial application of the bound in this capacity indicates some previous work may be too dissipative.
Chapter 6

Implications of turbulent motion during stagnation of Z pinch plasma

The work in this chapter contains substantial writing and calculation contributions from Evgeny Stambulchik at the Weizmann Institute, as well as conceptual input from collaborators, primarily at the Weizmann Institute. See the Acknowledgments for the full list of contributors.

6.1 Introduction

Evolution of the ion kinetic energy in a stagnating plasma was previously determined (Kroupp et al., 2011) from Doppler-dominated lineshapes augmented by measurements of plasma properties and assuming a uniform-plasma model. Notably, the energy was found to be dominantly stored in hydrodynamic flow. The Reynolds and Mach numbers are such that this motion could be supersonically turbulent, implying a non-uniform distribution of the plasma density. Indeed, here we show that re-analyzing the data under this hypothesis results in a consistent physical description while improving agreement of the model with observations. In addition, the inferred mean density is found to be substantially lower.

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In implosions of a cylindrical z-pinch plasma, hydrodynamic kinetic motion is ultimately transferred to thermal motion of plasma particles—electrons and ions—through a cascade of atomic and thermodynamic processes (Ryutov et al., 2000; Slutz and Vesey, 2012; Herrmann, 2014; Sinars et al., 2016). These processes culminate at the stagnation phase, producing high-energy-density plasmas and generating powerful x-ray and neutron radiation (Giuliani and Safronova, 2016).

Based on x-ray spectroscopic analysis of pinch plasmas previously reported (Kroupp et al., 2011; Maron et al., 2013), it was found that the ion kinetic energy at the stagnation phase was dominantly stored in the form of non-thermal hydrodynamic motion, while the plasma appeared largely uniform at spatial and temporal scales down to at least 100 µm and ~1 ns, respectively. An explanation of this phenomenon has been offered (Giuliani et al., 2014), where simulations showed steep radial velocity gradients in the stagnation region. On the other hand, the Reynolds number in the stagnating plasma is initially high (~10^5), making turbulence an obvious candidate for such significant small-scale hydrodynamic motion (the 2D simulations in Giuliani et al. (2014) would not be expected to reproduce turbulent behavior). The inferred Mach numbers, \( M \), at stagnation are initially supersonic, see Table 6.1 for Re and \( M \). Were supersonic turbulence present, it would imply substantial nonuniformity in quantities such as the density (see, e.g., Fig. 2 in Federrath, C. et al. (2010)). However, the previous analysis (Kroupp et al., 2011) assumed a uniform plasma.

This study re-analyzes the experimental data (Kroupp et al., 2011) without assuming uniformity, using, instead, a modeled turbulent density distribution (Hopkins, 2013). We find full (actually, improved) consistency with the observations, but a significantly (about two-fold) lower average density. This reinforces the picture of a stagnating plasma dominated by turbulent hydromotion. The results are believed to be relevant also for large-scale z-pinch devices and in various astrophysical contexts.
6.2 Short description of the previous study

In brief, a 9-mm-long neon-puff z-pinch was imploded in 500 ns under a current rising to 500 kA at the stagnation time. Experimental diagnostics included high-resolution (\(\sim 200 \mu m\)) gated x-ray filtered-pinhole imaging, a spectrometer recording Ne He-like dielectronic satellites with a resolving power of 6700, and a photo-conductive detector (PCD) sensitive to \(\hbar \omega \gtrsim 700\) eV radiation. All the data were simultaneously acquired over the stagnation period, about \(\pm 5\) ns around the peak of the PCD signal with a time resolution of \(\sim 1\) ns. A plasma segment at \(z = 5 \pm 1\) mm along the pinch axis was used for the analysis.

The modeling assumed a uniform-cylinder plasma with a prescribed (within experimental uncertainties) time evolution of \(T_e, n_e\), and plasma radius \(r_{pl}\). The experimental data and uniform model parameters are shown in Table 6.1. Assuming uniformity, the electron density \(n_e\) was determined based on the satellite-intensity ratio (Seely, 1979; Kroupp et al., 2007a); it is \(n_e^0\) in the table. A separately measured time-integrated continuum slope (Alumot, 2007) was found to agree with the \(T_e(t)\) assumed. The x-ray images give \(r_{pl}\) to within the \(\sim 200 \mu m\) resolution, \(r_{min}, r_{max}\) in Table 6.1. The self-consistency of the uniform-model time dependencies for \(n_e, T_e\) and \(r_{pl}\) was verified using the additional measurement of the absolutely calibrated PCD signal, which is sensitive to all three quantities. With \(T_e\) fixed, it was found that either \(n_e\) or \(r_{pl}\) could be taken at the center of its measured value range (i.e. uncertainty), with the other quantity then within one or two standard deviations of its independently measured value. With \(n_e\) the more important quantity, it was chosen to let \(r_{pl}\) vary outside one deviation; this gave \(r_{pl}^0\) in Table 6.1.

6.3 New model

Although the uniform plasma analysis was reasonably consistent with the data, the uniform density assumption will not be physically sound if the plasma is highly turbulent, as expected for the measured \(Re\) and \(M\). Therefore, we use a (non-uniform) turbulent
plasma model. Within such a model, all plasma properties—density $\rho$, electron $T_e$ and ion $T_i$ temperatures, and the non-thermal ion velocity $v_{\text{flow}}$—have certain distributions, with possible correlations between them. This work analyzes the simplest case, of isothermal turbulence, where $T_e$ and $T_i$ are uniform (see the discussion below), whereas $\rho$ (and $n_e$) are not. Then, the previous spectral fits and $T_e$ analysis are still valid because: the correlation between turbulent velocity and density is very weak for isothermal turbulence (Federrath, C. et al., 2010); and the turbulent velocity distribution is well approximated by a Gaussian (see, e.g. Smith et al. (2000); Porter et al. (2002); Kitsionas et al. (2009)), which was also the assumed non-thermal velocity distribution used in the original study (Kroup et al., 2011). This leaves $T_i^{\text{eff}}$ inferred from Doppler broadening (Kroup et al., 2007a) unaffected.

We now show that, using a turbulent density probability distribution function (PDF) that is consistent with the measured $\text{Re}$ and $M$, the inferred density is substantially reduced, which allows the inferred plasma radius at each time to be larger, while staying consistent with the PCD signal. This larger radius now agrees well with $[r_{\text{min}}, r_{\text{max}}]$ (except at the first measurement time, which has the weakest signal). Thus, the new turbulent model is an improvement both because it is physically sound and gives an improved match to the observations.

We work with electron density $n_e$ instead of mass density $\rho$, since the atomic experimental data are sensitive to $n_e$. The two are related, $\rho = \langle Z_i \rangle^{-1} m_i n_e$, where $\langle Z_i \rangle$ is the mean ion charge and $m_i$ is the ion mass. In principle, $\langle Z_i \rangle$ is a function of $T_e$ and $n_e$, but for the ranges of plasma parameters of interest, it varies very weakly (Kroup et al., 2011; Giuliani et al., 2014), so we assume $\rho \propto n_e$.

For each measurement the density has a PDF, $P(n_e)$. The previous data analysis (Kroup et al., 2011) corresponds to $P(n_e) \equiv \delta(n_e - n_e^0)$. $P(n_e)$’s are different at different times and $z$-positions, i.e., $P(t, z; n_e)$; for brevity, these $t, z$ labels will be omitted.
Let us switch to dimensionless quantity

\[ \xi \equiv n_e/n_e^0; \int P(\xi) \, d\xi = 1. \quad (6.1) \]

The average density is \( \langle n_e \rangle = n_e^0 \int \xi \, P(\xi) \, d\xi \). It is important to note that \( \langle n_e \rangle \) is not the same as \( n_e^0 \). The nonuniform density affects two of the previous measurements: \( n_e \) from line ratios and the absolutely calibrated PCD signal, from which one can infer the radiating mass (product of \( n_e \) and \( r_{pl}^2 \)), for a given \( T_e \). These measurements give two constraints on the turbulent PDF, \( P(\xi) \), which thus determine the new mean density.

Assuming the collisional-radiative equilibrium is established much faster than the characteristic hydromotion time, the intensity of a discrete spectral line or continuum radiation in a turbulent plasma can be obtained in the static approximation (Stamm et al., 2017), viz.,

\[ \langle I \rangle = \int \alpha(\vec{r}) \, d^3r = \pi r_{pl}^2 \ell \int \alpha(\xi) \, P(\xi) \, d\xi. \quad (6.2) \]

Here, \( \alpha \) is the local plasma emissivity, approximately scaling as \( \propto \xi^2 \) if the density does not vary too much, \( \ell \) is the length (in the \( z \) direction) of the plasma segment being analyzed, and we assumed that density variations are independent of \( r \). In particular, the PCD signal is

\[ I_{PCD} \propto \pi r_{pl}^2 \ell \int \xi^2 \, P(\xi) \, d\xi. \quad (6.3) \]

Using this, and the fact that the previous model described \( I_{PCD} \) self-consistently (within the errors bars \( \delta I_{PCD} \)) by assuming \( r_{pl}^0 \), we can get a first constraint on \( P(\xi) \), bounding \( I_{PCD} \) with \( r_{min} \) and \( r_{max} \) and \( \pm \delta I_{PCD} \),

\[ \left( 1 - \frac{\delta I_{PCD}}{I_{PCD}} \right) \left( \frac{r_{pl}^0}{r_{max}} \right)^2 \leq \int \xi^2 \, P(\xi) \, d\xi \leq \left( 1 + \frac{\delta I_{PCD}}{I_{PCD}} \right) \left( \frac{r_{pl}^0}{r_{min}} \right)^2. \quad (6.4) \]

Some of the autoionizing dielectronic satellites have even stronger density dependence than \( \propto \xi^2 \)—which is why the intensity ratio of such a satellite to another line (in our case—
another close-by dielectronic satellite) allows for inferring the density (Seely, 1979). Both dependencies are complex, but around the density point of interest ($\sim 5 \times 10^{20} \text{ cm}^{-3}$), their ratio is rather close to a linear form, $R \approx R^0 + a_R (n_e/n_0 e - 1)$, in a steady-state optically thin plasma (see Fig. 6.1). Hence, if $n_e$ does not vary too wildly, (say, within a factor $\times 2$ in each direction),

$$\langle R \rangle = R^0 + a_R \int \frac{(\xi - 1)\xi^2 P(\xi) \, d\xi}{\int \xi^2 P(\xi) \, d\xi}. \quad (6.5)$$

The measured quantity $R_{\text{expt}}$ is known within its error bars, i.e., $\langle R \rangle = R_{\text{expt}} = R^0 \pm \delta R$. Therefore, Eq. (6.5) gives a second constraint on $P(\xi)$,

$$1 - \frac{\delta R}{a_R} \leq \frac{\int \xi^3 P(\xi) \, d\xi}{\int \xi^2 P(\xi) \, d\xi} \leq 1 + \frac{\delta R}{a_R}. \quad (6.6)$$

To model the density PDF that would result from turbulence in the stagnating plasma, we use the PDF of Hopkins (2013). Since the model assumes the average density is known, it is convenient to introduce dimensionless $volumetric$ density by normalizing to $\langle n_e \rangle$, i.e.,
\[ \xi_V \equiv n_e/\langle n_e \rangle. \] Evidently, \( \xi/\xi_V = \langle n_e \rangle/n_e^0 \). In terms of \( \xi_V \), the (volumetric) PDF is

\[
P_V(\xi_V) \, d\xi_V = I_1(2\sqrt{\lambda \omega(\xi_V)}) \times \\
\exp[-(\lambda + \omega(\xi_V))] \sqrt{\frac{\lambda}{\theta^2 \omega(\xi_V) \xi_V}} \, d\xi_V, \tag{6.7}
\]

where \( \lambda \equiv \sigma_{s,V}^2/2\theta^2 \), and \( \omega(\xi_V) \equiv \lambda/(1 + \theta) - \ln(\xi_V)/\theta \), and \( I_1 \) is the modified Bessel function of the first kind. This two-parameter PDF depends on a variance, \( \sigma_{s,V}^2 \), and a measure of intermittency, \( \theta \). As \( \theta \to 0 \), the PDF becomes lognormal. This PDF fits well for simulations conducted at a wide range of Mach numbers (Hopkins, 2013). Although we presently treat the turbulence as isothermal, this PDF has been shown to fit for simulations of non-isothermal turbulence (Federrath and Banerjee, 2015). In general, the values of, \( \sigma_{s,V}^2 \), \( \theta \), depend on the turbulence properties; they are typically modeled as depending on the turbulent Mach number, the mix of compressive and solenoidal forcing, and, in the non-isothermal case, the polytropic gamma (Hopkins, 2013; Federrath and Banerjee, 2015). As such, the turbulence model does not introduce any “free” parameters, since its parameters vary only as a direct consequence of the variation of measured or inferred plasma properties.

For the value of \( \theta \), we use the fit to simulation data (Hopkins, 2013), which is \( \theta \approx 0.05M_c \). Here \( M_c \) is the compressive Mach number, also written \( M_c = bM \) (Konstandin et al., 2012, 2016), and \( b \) is related to the mix of solenoidal and compressive modes (Federrath et al., 2008; Konstandin et al., 2012, 2016). For the density variance, \( \sigma_{s,V}^2 \), we combine the usual isothermal logarithmic density variance (see, e.g., Padoan et al. (1997); Passot and Vázquez-Semadeni (1998); Padoan and Nordlund (2011); Molina et al. (2012)), \( \sigma_s^2 \approx \ln[1 + b^2 M^2] \), with the relationships \( \sigma_{s,V}^2 = (1 + \theta)^3 \sigma_{s,M}^2 \) (Hopkins, 2013) and \( \sigma_s^2 = \sigma_{s,V} \sigma_{s,M} \) (Federrath and Banerjee, 2015). This yields \( \sigma_{s,V}^2 = (1 + \theta)^{3/2} \ln[1 + b^2 M^2] \). Here we take \( b = 0.4 \); see the discussion below for more on this choice, and caveats associated with the turbulence model.
The Mach number at each time is calculated using the data in Table 6.1; $M = \frac{v_{\text{flow}}}{c_s}$, where
\[ v_{\text{flow}} = \sqrt{3(T_{\text{eff}} - T_i)/m_i} \] (6.8)
and $c_s = \sqrt[2]{\gamma (T_{\text{ene}} + T_{\text{ini}})/(n_{\text{ini}} + n_{\text{ene}})}$, where $\gamma = 1$ is used, assuming isothermality (discussed below).

### 6.4 Results and discussion

We now use the constraints (6.4) and (6.6) to examine theoretical predictions of turbulent density fluctuations. The density PDF of the turbulence model, Eq. (6.7), in addition to satisfying the usual normalization condition, Eq. (6.1), also conserves the average density, $\int \xi V P_V(\xi V) d\xi V = 1$. However, experimentally the average density is unknown; in order to use the volumetric PDF and its moments, we connect $\xi$ and $\xi V$ with a free parameter $\beta$,

\[ \xi = \beta \xi V. \] (6.9)

Once the turbulence PDF satisfying the experimental data within the constraints (6.4) and (6.6) is determined, $\beta n_e^0$ will give the new mean density, corrected for the presence of turbulence; more generally, $\langle \xi^k \rangle = \beta^k \langle \xi_V^k \rangle$. With this in mind, Eqs. (6.4) and (6.6) become a set of inequalities on $\beta$,

\[ \sqrt{1 - \frac{\delta \rho_{\text{PCD}}}{\rho_{\text{PCD}}} \frac{r_0^0}{r_{\max}}} \leq \beta \leq \sqrt{1 + \frac{\delta \rho_{\text{PCD}}}{\rho_{\text{PCD}}} \frac{r_0^0}{r_{\min}}} \] (6.10)

\[ \left(1 - \frac{\delta R}{a_R} \right) \langle \xi_V^2 \rangle \leq \beta \leq \left(1 + \frac{\delta R}{a_R} \right) \langle \xi_V^3 \rangle, \] (6.11)

shown graphically in Fig. 6.2a. The new model predicts a significantly (about two-fold) lower average density. With $\beta$ chosen, the plasma radius needs to be corrected, accounting for the turbulence-modified average emissivity. Using Eq. (6.3), it follows that $r_{\text{pl}}^{\text{turb}} = \ldots$
Figure 6.2: a) Limits of the double inequalities (6.11) and (6.10) are visualized by the black and red dashed lines, respectively. The ranges of $\beta$ satisfying both inequalities are designated by the gray filled area, with the tentative values used to correct the uniform-model parameters indicated by the solid line. b) $r_{pl}^{turb}$ (the solid line, with the grey area denoting uncertainties) shows an improved agreement with the experimental data (symbols with error bars). $r_{pl}^0$ of the uniform-plasma model is given by the dashed line.

$$r_{pl}^0/\sqrt{\langle \xi^2 \rangle} = r_{pl}^0/\left(\beta \sqrt{\langle \xi^2 \rangle} \right).$$ Notably, $r_{pl}$’s in the present model (listed as $r_{pl}^{turb}$ in Table 6.1) fit the measured values better than the original model (Kroup et al., 2011), as shown in Fig. 6.2b.

For clarity, we have presented results in Fig. 6.2 with only experimental uncertainty. There are also uncertainties associated with the turbulence model. Changes in the results due to most of these uncertainties are primarily expected to be quantitative, with the picture of reduced mean density remaining. One uncertainty comes from the possibly non-equilibrium nature of any turbulence at stagnation. The turbulent velocity decreases in time during stagnation, as evidenced by the decreasing non-thermal energy excess per-ion ($T_i^{eff} - T_i$) in Table 6.1. However, contrary to the turbulence simulations usually considered for modeling (e.g. in Hopkins (2013)), the total mass is not constant in time: at least
Initially, plasma continues to flow into the stagnation region. Using the isothermal turbulence $r_{pl}$ and $n_e$ in Table 6.1 (that is, $r_{pl}^{turb}$, $n_e^{turb}$) along with a turbulent energy per particle of $T_i^{eff} - T_i$, yields a total turbulent energy in the stagnation region that remains relatively constant from $t = -3.4$ ns to $t = 0$ ns, then falls. Although the present density PDF model works in a variety of cases, it has typically been tested in situations with equilibrium forcing, which may not be the best analog for the present case.

Assuming the model applies, there are still uncertainties. One is the degree to which turbulence in this stagnating plasma would be isothermal. To estimate the isothermality, we compare the large scale turbulent velocity, $v_{flow}$, to a thermal conduction velocity (derived following (Zeldovich and Raizer, 1967)),

$$
\frac{L_h}{\tau_{cond}} \approx 4 \times 10^{21} \frac{\zeta(\langle Z_i \rangle)}{\langle \langle Z_i \rangle \rangle + 1) \alpha_{el} n_e L_h} T^{5/2} (6.12)
$$

where $L_h$ is a length scale, $\alpha_{el}$ is the Coulomb logarithm, and $\zeta(8.5) \approx 2.7$; $T$ is in units of eV, $n_e$ in cm$^{-3}$, and $L_h$ in cm. When the ratio $v_{cond}/v_{flow} \gg 1$, isothermality is expected. Since this ratio is a minimum for $L_h \sim r_{pl}$, and will increase as the lengthscale being considered shrinks, considering its value for $L_h \sim r_{pl}$ is sufficient. At present, $v_{cond}/v_{flow} \sim 2$ for $L_h \sim r_{pl}$ at $t = -3.4$ ns, and becomes $\sim 6$ for later times. Evidently, an accurate determination of the degree of isothermality would require detailed simulations, as in other topic areas (Pavlovski et al., 2002, 2006). The turbulence model used here applies in the non-

<table>
<thead>
<tr>
<th>$t$ (ns)</th>
<th>$\delta R$</th>
<th>$I_{CD}$ (GW)</th>
<th>$v_{min}$</th>
<th>$v_{max}$</th>
<th>$T_i^{eff}$</th>
<th>$r_{pl}^{turb}$</th>
<th>$n_e^{turb}$</th>
<th>$T_i$</th>
<th>$T_e$</th>
<th>$M$</th>
<th>$Re$</th>
<th>$\theta$</th>
<th>$\sigma_{x,y}^{2}$</th>
<th>$\langle \xi_r \rangle$</th>
<th>$\langle \xi_r \rangle$</th>
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<th>$r_{n}^{turb}$</th>
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</thead>
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<tr>
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<td>0.15</td>
<td>2.0 ± 1.0</td>
<td>0.25</td>
<td>0.47</td>
<td>2100</td>
<td>0.29</td>
<td>6.0</td>
<td>175</td>
<td>250</td>
<td>1.7</td>
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<td>0.048</td>
<td>0.70</td>
<td>1.84</td>
<td>5.77</td>
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<td>1.9</td>
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<tr>
<td>-1.2</td>
<td>0.15</td>
<td>3.8 ± 1.1</td>
<td>0.36</td>
<td>0.52</td>
<td>1800</td>
<td>0.31</td>
<td>6.0</td>
<td>190</td>
<td>210</td>
<td>1.6</td>
<td>$T_i$</td>
<td>0.034</td>
<td>0.40</td>
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<tr>
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<td>6.5 ± 0.7</td>
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<td>0.68</td>
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<td>3.7</td>
<td>0.30</td>
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</table>

Table 6.1: The experimental data (Kroupp et al., 2011) relevant for the analysis presented; the plasma parameters assumed for ($r_{pl}^{0}$, $n_e^{0}$, $T_e$) and inferred from ($T_i$, $M$, $Re$) the uniform plasma modeling; the calculated isothermal turbulence parameters, volumetric density factor $\beta$ and respectively corrected plasma electron density and radius. Units are as follows: all radii are in mm, all temperatures are in eV, and densities are in $10^{20}$ cm$^{-3}$.
isothermal case, with different $\theta$, $\sigma^2_{s,V}$ (Federrath and Banerjee, 2015). At this level, non-isothermality is expected to only modestly change the parameters in Table 6.1, although then the inferred $T_e$ and $M$ will also need to be reconsidered, because non-isothermality would have a pronounced effect on the local plasma emissivity (it depends rather strongly on $T_e$), requiring modifications to Eqs. (6.2) and (6.5)—and therefore also to (6.10) and (6.11).

Any magnetic fields in the stagnating region could alter the values of $\theta$ and $\sigma^2_{s,V}$ (Padoan and Nordlund, 2011; Molina et al., 2012; Hopkins, 2013), although the form of the PDF remains valid. These corrections should be small because the plasma pressure is much higher than the magnetic pressure in the stagnation region ($\beta_{\text{magnetic}} \gtrsim 20$) (Rosenzweig et al., 2014).

Even within the isothermal turbulence PDF model, there are uncertainties. Simulations show substantial spread in values of the PDF parameters around the expressions for $\theta$ and $\sigma^2_{s,V}$, see, e.g., Hopkins (2013). Apart from modeling errors, spread in these values can be physical, due to the fluctuations of turbulence (Lemaster and Stone, 2008). The correct value of $b$, presently taken to be $b = 0.4$, is uncertain. Generally, $b \in [1/3, 1]$ (Federrath et al., 2008; Federrath, C. et al., 2010), with $b = 1/3$ occurring for solenoidal (divergence free) forcing (Federrath et al., 2008) and $b = 1$ occurring for compressive (curl-free) forcing. Equal parts solenoidal and compressive forcing gives $b \approx 0.4$ (Federrath, C. et al., 2010). For a z-pinch, one might expect the forcing to be largely compressive. Given this uncertainty, one could calculate the range of $\beta$ in Fig. 6.2a including also uncertainty in $b$. A larger $b$ yields a lower range for $\beta$, while a smaller $b$ yields a higher range.

The ion temperatures in Table 6.1 are inferred through a calculation involving the electron–ion temperature equilibration time (Kroupa et al., 2011). Since this time is density dependent, it will be affected by density fluctuations. The equilibration timescale is faster in the high density regions, which dominate the measurements, thus, the ion temperature may be driven slightly closer to the electron temperature. Since the electron–ion temper-
ature equilibration timescale is already very fast, this is expected to cause $T_i$ to be a few percent lower than in Kroup et al. (2011).

The underlying atomic model used for the present analysis is the same as in the previous study (Kroup et al., 2011) and, therefore, no additional uncertainties have been introduced. In fact, the associated inaccuracy may be surprisingly low, as the Monte-Carlo analysis of uncertainty propagation in collisional-radiative models indicates (Ralchenko, 2016). So far, we have neglected possible opacity effects. Fortunately, the satellites used have a negligible optical thickness. The bound–free and free–free (bremsstrahlung) radiation that contributes to the PCD signal is also optically thin, however strong bound–bound transitions are not. This requires a modification of Eq. (6.2) which cannot be represented analytically. However, the plasma absorption coefficients, similar to the emission ones, for these transitions scale as $n_e^2$. Therefore, the difference from the uniform-plasma model (in which the opacity was properly accounted for numerically) should vanish in the lowest order.

The mechanism generating the (non-radial) hydrodynamic motion is unclear; while energy is dumped in the hydrodynamic motion in the process of stagnation (Maron et al., 2013), this hydrodynamic motion could be seeded by turbulence generated and carried along during the compression itself, or could be generated entirely at stagnation. In either event, there are important implications, both for z-pinches, and more broadly.

If the (turbulent) hydromotion is generated and carried along during the compression, these z-pinches represent a test bed for the properties of plasma turbulence undergoing compression. These properties are relevant for a proposed novel fast ignition or x-ray burst generation scheme (Davidovits and Fisch, 2016b,a). Of particular interest is that the present hydromotion is supersonic, the regime in which these schemes would operate. Further, the behavior of compressing supersonic turbulence is of critical interest in astrophysics, particularly for molecular cloud dynamics (Robertson and Goldreich, 2012; Davidovits and Fisch, 2017). Supersonic turbulence behavior has been related to the star formation effi-
ciency (Elmegreen, 2008), the core mass/stellar initial mass functions (Padoan and Nordlund, 2002; Ballesteros-Paredes et al., 2006; Hennebelle and Chabrier, 2008), and Larson’s laws (Kritsuk et al., 2013a).

If the hydrodynamic motion is generated at stagnation, and then decays, its properties could still be of astrophysical interest (see, e.g., Mac Low et al. (1998); Mac Low (1999); Smith et al. (2000); Ostriker et al. (2001); Pavlovski et al. (2002, 2006); Lemaster and Stone (2008); Kitsionas et al. (2009); Davidovits and Fisch (2017)). To the extent generation and/or decay of the hydrodynamic motion at stagnation can be observed, studies of supersonic turbulence in z-pinches could serve as a new and important area for laboratory astrophysics. Indeed, in the present study, not all values of the turbulent PDF parameters, $\theta, \sigma_s^2, V$, will be consistent with the observations; more measurements could help to constrain turbulent properties. Z-pinches such as the present may yield other cross-over opportunities with astrophysics, for example, in turbulent density PDF measurement techniques (e.g. Ostriker et al. (2001); Brunt et al. (2010)), or in mechanisms for turbulent generation and forcing in complex plasma environments (e.g. Gritschneder et al. (2009)).

The present analysis is likely relevant to high-current implosions, like z-pinch experiments on the Z machine (Jones et al., 2006). Indeed, based on the plasma parameters given in (Maron et al., 2013), Re is also high ($\sim 10^4$), and M is similar to the experiments analyzed here. Interestingly, in those experiments, where $T_e$ and $\langle Z_i \rangle$ reach higher values, the ratio $v_{\text{cond}}/v_{\text{flow}} \gg 1$, thus the assumption of turbulence isothermality should be fully justified.

In summary, a new analysis of stagnating pinch data, replacing the assumption of uniform plasma with density variations consistent with a stagnating plasma that is turbulent, shows that the picture of a preponderance of turbulent energy remains intact. This picture is not only consistent with the observations, it improves the agreement with them. We find the mean plasma density is reduced by a factor $\sim 2$. While there is uncertainty in the precise value of the density reduction, the general picture, of a data analysis in the presence
of highly turbulent stagnating plasma reducing the inferred stagnation density compared to
the uniform case, is believed to be robust. Beyond aiding our understanding of z-pinches, it
is hoped this study has highlighted fertile ground for relation to problems of astrophysical
interest.
Chapter 7

Turbulence in mechanically heated hot-spots

7.1 Introduction

Recent work (Davidovits and Fisch, 2016b,a) has presented a “sudden viscous dissipation” mechanism and suggested this mechanism may be of use in a new type of fast ignition scheme. The essence of this new fast ignition scheme is as follows. An initially turbulent plasma, with a substantial portion of its energy in the turbulent kinetic energy (TKE), is rapidly compressed. The rapid compression causes the TKE to grow, along with the temperature. With the temperature growth comes an increasing plasma viscosity. Once this viscosity begins to act on the larger, energy containing scales of the turbulence, a very rapid dissipation of the TKE into thermal energy (temperature) is initiated. By correctly timing this rapid conversion of TKE to thermal energy, a burst of fusion or X-rays may be achieved. To the extent that the loss mechanisms of TKE differ from those for thermal energy (e.g. in their impact on radiation or conduction), it may be advantageous to keep energy in TKE during the compression. More broadly than presenting the sudden viscous
dissipation mechanism, this previous work investigated the behavior of plasma turbulence under compression.

The primary purpose of the present work is to investigate in more detail the behavior of turbulence and the prospects for exploiting it in a specific circumstance. This circumstance is a mechanically-heated hot-spot ignition scenario for a three-dimensional (3D) compression, as in ignition shots at the National Ignition Facility (NIF). For this first approach to modeling turbulent hot-spot dynamics we couple two simple analytic models; one for the hot-spot dynamics (Lindl, 1995) and one for the turbulence behavior (Ch. 4). We use these coupled models to follow hot-spot trajectories in areal-density ($\rho R$) versus temperature space, and to compute quantities of interest, such as the relative heating contributions of mechanical work ($p dV$) and viscous dissipation of turbulence. Throughout this work, we refer to the model solutions as hot-spot trajectories (imagining them plotted in $\rho R - T$ space).

Although the simple analytic model of hot-spot dynamics does not correctly give actual experimental trajectories, or even simulated trajectories using more complete models, it is instructive for understanding the effects of changing parameters (e.g. initial conditions, compression velocity) and of various sources of heating and cooling. Thus, it is hoped that the current work, by comparing the hot-spot model with and without turbulence, helps to illuminate the new possibilities opened up by a turbulent hot-spot, while not being an exact accounting.

As part of examining the exploitability of hot-spot turbulence, we also show a condition for preventing hot-spot turbulence during compressions. It is shown that the “attractor” solution of Lindl (1995), describing hot-spot behavior dominated by the balance of electron-conduction and $PdV$, is generally in the unstable region of $\rho R - T$ space. However, simulated hot-spot trajectories such as those present in Lindl (1995), may cross between stable and unstable regions. Beyond predicting stability or instability, we predict the saturation level of turbulence for certain unstable trajectories. Note that the condition
for avoiding hot-spot turbulence under compression is for turbulence in the bulk (volume), assuming homogeneity. It does not include the common interfacial instabilities affecting inertial confinement fusion (ICF) experiments (e.g. Rayleigh-Taylor instability).

This chapter is organized as follows. The following section, Sec. 7.2, shows the result of coupling the TKE model with a temperature equation containing only mechanical heating from compression and heating from viscous dissipation of the TKE. This is the simplest self-consistent coupled system of interest for the sudden viscous dissipation mechanism. Previous work on sudden viscous dissipation (Davidovits and Fisch, 2016b,a), lacked the feedback of viscous dissipation in the temperature equation. With it now included, the system in Sec. 7.2 will have dissipated TKE properly appear in the temperature, and will therefore model the more sudden dissipation possible with feedback (see (Davidovits and Fisch, 2016b)). This system helps introduce some the issues at hand and also shows that the coupled model does in fact contain sudden viscous dissipation.

Next, Sec. 7.3, explains the condition for preventing hot-spot turbulence, and discusses the saturation of turbulence in unstable hotspots. Section 7.4 presents the full coupled TKE—temperature model. The full model builds on the model in Sec. 7.2 by adding to the temperature equation terms for: bremsstrahlung radiation, electron thermal conduction, turbulent conduction, and deuterium-tritium (D-T) fusion. This full model is then used in Sec. 7.5 to look at the exploitability of turbulence in model hot-spots. Finally, Sec. 7.6, discusses caveats associated with the analysis and concludes.

### 7.2 Sudden viscous dissipation with feedback

In this section we demonstrate that the sudden viscous dissipation effect is captured in the present model system, including feedback of dissipated TKE into thermal energy. The same coupled system discussed here was used to examine energy partition between thermal energy and TKE in Sec. 4.5. Accounting only for mechanical heating and viscous dissi-
pation of TKE, the temperature evolution of a plasma undergoing a constant velocity, 3D, compression, is

\[
\frac{d\bar{T}}{d\bar{L}} = -2\frac{\bar{T}}{\bar{L}} - \bar{E}_r0\bar{L}^{-4}\bar{\epsilon}_{\text{model}}. \tag{7.1}
\]

Here \(\bar{\epsilon}_{\text{model}}\) is the (normalized) model viscous dissipation rate, and \(\bar{E}_r0\) is the starting ratio of TKE to thermal energy,

\[
\bar{E}_r0 = \frac{\rho_0 V_0^2}{2} \frac{1}{3n_0 k_B T_0} \tag{7.2}
\]

The initial mass density, number density, and temperature are, respectively, \(\rho_0, n_0,\) and \(T_0\), and \(k_B\) is the Boltzmann constant. The velocity \(V_0\) is an initial mean turbulent velocity, so that the initial turbulent energy density, \(E_{0T}\), is \(E_{0T} = \rho_0 V_0^2/2\). It has been assumed that the plasma charge state \(Z = 1\), and that the plasma has an ideal equation of state. The linear compression ratio is given by \(\bar{L}\), which is defined,

\[
\bar{L} = \frac{L}{L_0} = 1 - \frac{2V_b t}{L_0}, \tag{7.3}
\]

where \(V_b\) is a compression velocity. In the absence of dissipated TKE (that is, for \(\bar{\epsilon}_{\text{model}} = 0\)), Eq. (7.1) gives a temperature solution \(\bar{T} = 1/\bar{L}^2\). This is the pure adiabatic compression solution. The viscous dissipation rate \(\bar{\epsilon}_{\text{model}} \geq 0\), so that the power scaling of temperature with compression in this model will be at least as strong as the pure adiabatic compression solution.

To calculate \(\bar{\epsilon}_{\text{model}}\), we use the compressing TKE model described in Ch. 4, with one change. There, the viscosity, \(\mu\), was taken as a known function of \(L\), \(\mu = \mu_0 \bar{L}^{-2\beta}\). For the current case, we use \(\mu = \mu_0 \bar{T}^{5/2}\), reflecting the plasma viscosity when \(Z = 1\). This change is because in the present model the temperature dependence on \(\bar{L}\) is being solved for self-consistently, whereas in previous work an effective net temperature and charge state dependence on compression was assumed (the plasma viscosity going as \(\mu \sim T^{5/2}/Z^4\)).
Letting the viscosity depend explicitly on $\bar{T}$, the model for the $\bar{\epsilon}_{\text{model}}$ is,

$$
\bar{\epsilon}_{\text{model}}(\bar{L}) = 2 \frac{\tau_c}{\tau_\nu} \left\{ \bar{T}^{5/2} \bar{L}^3 + \sqrt{\frac{1}{2 \, \gamma^2} \hat{E}(\bar{L}) + \bar{T}^5 \bar{L}^6} \right\} \hat{E}(\bar{L}) .
$$

(7.4)

Here $\hat{E}$ is a normalized (unitless), transformed (see Ch. 4) TKE, which is related to the lab TKE by,

$$
E(\bar{L}) = \frac{1}{L^2} \frac{V_0^2}{2} \hat{E}(\bar{L}) .
$$

(7.5)

Note that this TKE is a velocity squared, and excludes the density, whereas the TKE normalization used in $\bar{\epsilon}_{\text{e0}}$ includes the initial mass density $\rho_0$. A complete system for coupling $\bar{T}(\bar{L})$ and $\hat{E}(\bar{L})$ is obtained by the relation of $\bar{\epsilon}_{\text{model}}$ to the dissipation of TKE,

$$
\frac{d}{d \bar{L}} \hat{E}(\bar{L}) = \bar{L}^{-2} \bar{\epsilon}_{\text{model}} .
$$

(7.6)

Then, the complete system is given by Eqs. (7.1),(7.4),(7.6). In addition to the quantities already given, we have also defined,

$$
\tau_c = -\frac{1}{\bar{L}} = \frac{L_0}{2U_b} ,
$$

(7.7)

$$
\tau_\nu = \frac{\alpha^2 L_0^2}{3 \nu_0} ,
$$

(7.8)

$$
\gamma = \left(\frac{9b^3}{16}\right)^{3/2} \frac{\nu_0}{\alpha L_0} .
$$

(7.9)

The quantities $\alpha \approx 0.19$ and $b \approx 5.949$ are related to the turbulence model (see Ch. 4), and $\nu_0$ is the initial kinematic viscosity, for which we will use the Braginskii result (Braginskii, 1965). The compression timescale and initial viscous timescale are $\tau_c$ and $\tau_\nu$, respectively. The quantity $\gamma$ is an initial “viscous velocity”, which can be seen to enter the equations in a ratio with the initial mean turbulent velocity, $V_0$. This ratio, $V_0^2/(2\gamma^2)$, affects both whether the turbulence is in a high or low Reynolds number limit and the initial quantity.
of dissipation. The latter is also impacted by $\tau_c/\tau_v$. This is evident by examining Eq. (7.4). Initially, $\bar{T} = 1, \bar{L} = 1$, so that the first and third terms inside the bracket can be ignored if $V_0^2/(2\gamma^2) \gg 1$. This is the high-Reynolds limit. In this limit, however, the magnitude of the dissipation ($\epsilon_{\text{model}}$) depends on the product of $\tau_c/\tau_v$ and $\sqrt{V_0^2/(2\gamma^2)}$, with a large $V_0^2/(2\gamma^2)$ tending to increase the dissipation rate.

Suppose that the initial turbulent energy is normalized to 1, $E_T^0 = 1$. Then, by Eq. (7.2), the initial thermal energy, $3n_0k_BT_0 = 1/\epsilon_{\text{r0}}$. The total initial energy is $1 + 1/\epsilon_{\text{r0}}$. In the absence of dissipation ($\epsilon_{\text{model}} = 0$), both the TKE and the thermal energy grow with compression as $1/\bar{L}^2$. When the dissipation is nonzero, their growth will be different from $1/\bar{L}^2$, but the growth of the total energy (TKE + thermal) will still be $1/\bar{L}^2$, because any dissipated TKE shows up in temperature. In other words, the total energy, $E_{\text{total}}$, will be $E_{\text{total}}(\bar{L}) = (1 + 1/\epsilon_{\text{r0}})/\bar{L}^2$.

Figure 7.1 shows the solution to the system, Eqs. (7.1),(7.4),(7.6), when $\epsilon_{\text{r0}} = 100$ and $E_T^0 = 1$, for a particular set of the parameters, $\tau_c/\tau_v$ and $V_0^2/\gamma^2$. See the figure caption for more description. The sudden viscous dissipation effect, with feedback included, is clearly displayed. The temperature is kept low (as a fraction of the total energy) up until the dissipation event. In a real compression, this lower temperature could possibly reduce overall energy losses, for example by reducing losses to electron conduction or radiation. One motivation of the present study is to attempt to achieve such a useful conversion of TKE to thermal energy for “realistic” hot-spot trajectories.

### 7.3 Turbulence saturation and prevention in mechanically heated hot-spots

In Davidovits and Fisch (2016a), Ch. 3 Sec. 3.3, it was shown that turbulence undergoing compression would decay under a certain condition, assuming the viscosity dependence on compression fell in a certain range of behaviors. The condition for turbulent decay was
Figure 7.1: Solution to the system, Eqs. (7.1),(7.4),(7.6), when $\mathcal{E}_{r0} = 100$, showing sudden viscous dissipation including feedback of the TKE (E, blue solid line) to thermal energy (T, green solid line). The red dashed line shows the total energy, which increases as $1/\bar{L}^2$ under the compression. The compression (time) progresses from right ($\bar{L} = 1$) to left, with $\bar{L}$ the linear compression ratio. The initial condition has 100 times the turbulent energy (TKE) as thermal energy. Once viscous dissipation begins to convert some TKE into thermal energy, the rise in thermal energy increases the viscosity, increasing the dissipation. This feedback in the model system causes a very rapid conversion of TKE into thermal energy at some point during the compression. Before this conversion, the plasma is much colder than it would be if all the energy were thermal (by a factor $\sim \mathcal{E}_{r0}$). The goal of the present study is to achieve such a conversion in a parameter regime relevant for hot-spot ignition. In this example, $\tau_c/\tau_\nu \approx 6.7 \times 10^{-6}$ and $V_0^2/(2\gamma^2) \approx 3.3 \times 10^6$. The small $\tau_c/\tau_\nu$ indicates a “rapid” compression where the compression time is much shorter than the initial turbulent dissipation time. The large $V_0^2/(2\gamma^2)$ means low-Reynolds-number corrections are initially negligible.
found to be
\[
\frac{2U_b}{L_0 \nu_0 k_{\text{min}}^2} < 1. \quad (7.10)
\]
Here \( k_{\text{min}} \) is the smallest wavenumber (longest wavelength) possible in the cubic domain for which the condition was derived. This is \( k_{\text{min}} = 2\pi/L_0 \). The condition, Eq. (7.10), is valid when the net viscosity dependence of the compression is \( \mu = \mu_0 \bar{L}^{-2\beta} \) with \( \beta \geq 1 \) (also assuming the other conditions of the model are satisfied, e.g. an isotropic, constant velocity compression of the type considered in Ch. 2 and Ch. 3).

For ICF plasmas with \( Z = 1 \), such as the D-T hotspots used in ignition shots, this viscosity dependence condition becomes a condition on the temperature behavior during compression, at least in the regimes in which the viscosity is modeled by the Braginskii form. Since this viscosity has a temperature dependence \( \mu \sim T^{5/2} \), the condition \( \beta \geq 1 \) will be met for \( T \sim \bar{L}^{-\theta} \) when \( \theta \geq 4/5 \). For an adiabatic ideal gas compression, \( \theta = 2 \), which satisfies this condition. When the hot-spot temperature is determined by the balance of electron conduction and \( PdV \) work, it can be shown that implosion trajectories are attracted to a solution where the temperature dependence goes as \( \theta = 4/5 \) (see Eq. (32) and the surrounding discussion in Lindl (1995)). This “attractor” solution then also satisfies the needed viscosity dependence for the condition for turbulent decay, Eq. (7.10), to apply.

Equation (7.10) can be rewritten in terms of \( T, \rho R, U_b, A_i \), and the Coulomb logarithm, \( \ln \Lambda \), as

\[
6.0 \times 10^4 \frac{(\rho R)_0 U_b \ln \Lambda}{T_{0,\text{eV}}^{5/2} A_i^{1/2}} < 2\pi^2 \quad (7.11)
\]

The substitution \( L_0 = 2R_0 \) has been made. Note that the condition is derived in a cubic domain, but hot-spots are typically spherical, which may introduce some small error. The implosion velocity and \( \rho R \) in Eq. (7.11) are in cgs units, while the temperature is in electron volts. For a 50-50 D-T hotspot \( A_i = 2.5 \). The stability boundary, equality in Eq. (7.11), can be written as

\[
T_{0,\text{keV}} = 26.6 \left( \frac{(\rho R)_0 U_b}{3 \times 10^3} \right)^{0.4}, \quad (7.12)
\]
Figure 7.2: Gain and loss regions for the hot-spot model presented in Sec. 7.4, ignoring any contributions from TKE; only mechanical heating, Bremsstrahlung radiation, electron thermal conduction, and D-T fusion production terms in Eq. (7.16) are included. The filled contour areas are the energy gain (cream color) and energy loss (blue) regions when the implosion velocity is $U_b = 3 \times 10^7$ cm/s, and the Coulomb logarithm is assumed to be $\ln \Lambda = 2$. Energy gain occurs when the net contribution of the four terms (mechanical heating, radiation, conduction, fusion) increases the hot-spot energy; a negative net contribution occurs in the loss regions. The orange dotted line is the “attractor” solution of Lindl (1995), Eq. (7.13), to which trajectories tend (until fusion kicks in). The green line is a stability boundary, Eq. (7.12), which divides $\rho R - T$ space into regions where turbulence will grow up and saturate versus decay, at least for trajectories with certain slopes, see Sec. 7.3. The dotted cyan line shows the temperature for which the saturated TKE equals the thermal energy for unstable trajectories where the TKE reaches saturation. This is Eq. (7.15) when the left hand side is 1 ($E_{\text{th}} = E_{\text{sat}}^T$), and $U_b = 3 \times 10^7$ cm/s. Below this line the saturated TKE will be more energetic than the thermal energy, while above it the thermal energy will be dominant.
which can then be easily compared to the “attractor” solution (Lindl, 1995), which is

\[ T_{keV} = 7.8 \left( \frac{\rho R}{3 \times 10^7} \right)^{0.4}. \]  

Both Eq. (7.12) and Eq. (7.13) assume \( A_i = 2.5 \) and \( \ln \Lambda = 2 \). In this form, it is readily seen that the “attractor” solution falls below the stability boundary, in the unstable region. Since the “attractor” solution has a temperature dependence on compression that corresponds to \( \beta = 1 \), we can say a hot-spot following this trajectory (or any parallel trajectory below Eq. (7.12) in \( \rho R - T \) space) will be unstable to turbulence (meaning the turbulence will not decay towards 0).

The applicability of the stability condition can be summarized as follows. For a given compression velocity, trajectories going through points in \( \rho R - T \) space above the line Eq. (7.12) will be stable (experience decaying turbulence) if the viscosity growth on the trajectory corresponds to \( \beta \geq 1 \). For \( Z = 1 \) (e.g. D-T) and Braginskii viscosity, this condition on \( \beta \) is equivalent to saying that the slope of the trajectory in \( \rho R - T \) space is at least as steep as that of the boundary (or, equivalently, of the “attractor” solution, Eq. (7.13)). Conversely, trajectories going through a point in \( \rho R - T \) space below the line Eq. (7.12) will be unstable (turbulence will not decay towards 0) if the slope of the trajectory is no steeper than that of the boundary. In unstable cases where the trajectory slope is parallel to the boundary or “attractor” solution, the turbulence will saturate at a predictable level (see Sec. 3.3.2), discussed further below. The present turbulence model predicts that this saturation level applies not only for trajectories parallel in slope to the boundary or “attractor” solutions, but also for shallower slopes (see the discussion about \( \beta < 1 \) in Sec. 4.6).

Note that any point along a trajectory can be considered the “starting point” for the sake of the stability analysis. Strictly speaking, one must know the amount of TKE present in order to predict whether the TKE will grow or not in the unstable region. This is because if
the TKE were initially above the saturation level, it could actually decrease in the unstable region, to the saturation level. Such a scenario could occur if say, a mechanism outside the model generated the TKE for the initial condition, or if the compression velocity switched from a higher value to a lower value partway through.

Figure 7.2 shows, for $U_b = 3 \times 10^7$ cm/s, Eqs. (7.12) and (7.13) plotted on top of the (thermal) energy gain and loss regions for the hot-spot model used here neglecting any TKE in calculating gain and loss regions. This means the gain and loss regions represent the net impact of $PdV$ (mechanical) heating, electron thermal conduction, Bremsstrahlung radiation, and D-T fusion. The model for each of these terms is described further in Sec. 7.4. See also Fig. 7.3, where the dominant gain and loss term in each region of $\rho R - T$ space is indicated.

For unstable trajectories with viscosity growth corresponding to $\beta = 1$ (with slopes equal to the attractor trajectory slope), a result from Davidovits and Fisch (2016a) can be used to predict the saturation level of the turbulence. The saturated turbulent energy density for such trajectories will be

$$E^T_{\text{sat}} = 1.9 \rho U_b^2.$$  \hspace{1cm} (7.14)

Note that, given that density is assumed to be conserved in the hot-spot for the present modeling, the saturated total TKE is constant along a trajectory for a fixed implosion velocity. This is because the total TKE $\propto r^3 E^T_{\text{sat}} \propto r^3 \rho$, i.e. the hot-spot mass. As an example, an implosion with velocity $U_b = 3 \times 10^7$ cm/s and hot-spot mass of 800 ng, gives a saturated TKE of around 140 J.

The saturated TKE density in Eq. (7.14) can be compared to the thermal energy density. Assuming $Z = 1$, the total ion and electron thermal energy density is $E_{\text{th}} = 3nk_bT$. For 50-50 DT ($A_i = 2.5$), the ratio of thermal energy to saturated turbulent energy can then be written

$$\frac{E_{\text{th}}}{E^T_{\text{sat}}} = 0.67 \left( \frac{3 \times 10^7}{U_b} \right)^2 T_{\text{keV}}.$$  \hspace{1cm} (7.15)
For typical compression velocities, the saturated hot-spot TKE can be similar to or larger than the hot-spot thermal energy, at least for temperatures at or below 1 keV. For a compression velocity of \( U_b = 3 \times 10^7 \text{ cm/s} \), the ratio of thermal energy to saturated TKE reaches 1 at \( T \sim 1.5 \text{ keV} \). This “break even” line is plotted in Fig. 7.2. Again, this analysis is for volumetric turbulence of the type treated in (Davidovits and Fisch, 2016a). In practice, we find the TKE in our model typically does not reach \( E_{\text{sat}}^T \). This can be understood as a consequence of the feedback of dissipated TKE to thermal energy. Once the TKE becomes a substantial fraction of the thermal energy, its dissipation contributes to the temperature behavior of the trajectory. Thus a trajectory following the “attractor” solution, which is capable of reaching the saturated TKE, may in fact experience a somewhat steeper temperature increase with compression than that of Eq. (7.13) once the TKE becomes substantial as a fraction of the total energy. This stronger increase in temperature with compression then means the TKE will not reach \( E_{\text{sat}}^T \).

### 7.4 Turbulent hot-spot model

To model the temperature evolution of the hot-spot in the presence of TKE, we couple a hot-spot model similar to that presented by Lindl (1995) and Atzeni and Meyer-ter-Vehn (2004), Sec. 4.1, with the turbulence model from Ch. 4. Section 7.2 covers this coupling when only the mechanical heating term and TKE viscous dissipation term are included in the temperature equation. In this section, we add terms for Bremsstrahlung radiation, electron conduction, D-T fusion, and an additional loss term to model the turbulent conduction. With these terms added, the new temperature equation, replacing Eq. (7.1), is

\[
\frac{d \bar{T}}{d \bar{L}} = -2 \frac{\bar{T}}{\bar{L}} - \frac{\bar{E}}{\bar{L}^4} \bar{\epsilon}_{\text{model}} + \frac{2 \tau_c}{\tau_r} \bar{L}^{-1} \bar{T}^{1/2} + \frac{2 \tau_c}{\tau_e} \bar{L}^{7/2} - 2 \frac{\tau_c}{\tau_f} \bar{L}^{-3} \langle \bar{\sigma} \bar{v} \rangle (\bar{L}, \bar{T}) f_d (\bar{L}, \bar{T}) + 2 \frac{\tau_c}{\tau_f \tau_r} \frac{\bar{E}^2}{\bar{\epsilon}_{\text{model}}} \bar{T} \bar{L}^{-2}. \tag{7.16}
\]
Figure 7.3: Gain and loss regions for the hot-spot model, Eq. (7.16), ignoring TKE terms, as for the contours in Fig. 7.2; see that caption. Contours for increasingly slower implosions are indicated by thin solid black lines. Below a certain velocity, the gain region separates into two regions, with the high density, high temperature fusion region “cut off” from the gain region for mechanical heating. The orange dotted line separates $\rho R - T$ space into two regions, one where fusion is the dominant heating mechanism, and one where mechanical heating ($PdV$ work) is the dominant heating mechanism. This separation depends somewhat on the implosion velocity, and is shown for $U_b = 3 \times 10^7$ cm/s. The magenta dotted line also separates $\rho R - T$ space into two regions, by the dominant loss mechanism, either electron thermal conduction or Bremsstrahlung radiation.
Figure 7.4: Two example trajectories representing time (compression) dependent solutions of the hot-spot model, Eq. (7.16), when there is no TKE, $\mathcal{E}_{t0} = 0$. Trajectory 1, the green dotted line, starts with $\rho R = 5 \times 10^{-5} \text{ g/cm}^2$ and $T = 1$ keV. As the compression progresses (steadily increasing $\rho R$), this hot-spot initially cools, then follows the “attractor” solution shown in Fig. (7.2) until it reaches the $\rho R$ and $T$ where fusion heating kicks in (as indicated by the “Fusion” label in Fig. 7.3). The second trajectory, the dotted orange line labeled 2, starts at the same $\rho R$ but with $T = 0.01$ keV. Under compression, it initially heats essentially adiabatically, until it reaches the “attractor” solution, at which point this trajectory follows the same path in $\rho R - T$ space as trajectory 1. The thin gray line is plotted to show the slope of adiabatic heating in $\rho R - T$ space. In this model, capsules below the “attractor” in $\rho R - T$ space can heat nearly adiabatically under mechanical work until they reach the attractor (or the fusion gain zone or right hand loss region, depending on the starting point).
The TKE equation, unchanged, is still Eq. (7.4) with the definition Eq. (7.6). This means the only loss mechanism for TKE included in the present work is bulk viscous dissipation. Any boundary effects are ignored. The first two terms on the right hand side (RHS) of Eq. (7.16), the mechanical heating and viscous dissipation terms respectively, are unchanged from Sec. 7.2. The third through sixth terms on the RHS of Eq. (7.16) represent, respectively, Bremsstrahlung radiation losses, electron conduction losses, D-T fusion energy production, and turbulent conduction losses. Note that, because the time variable is the linear compression ratio, \( \bar{L} \), energy input terms enter with negative signs on the RHS, while energy loss terms enter with positive signs. In addition to the timescale constants \( \tau_c \) and \( \tau_r \), previously defined in Eqs. (7.7) and (7.8), we have defined the timescales,

\[
\tau_r = 9.51 \times 10^{-11} \frac{A_i \rho_0/\text{eV}}{\rho_{0,\text{g/cc}}},
\]

(7.17)

\[
\tau_e = 63.7 \frac{\ln \Lambda L_0^2 \rho_{0,\text{g/cc}}}{c_e T_{0,\text{eV}}}^{5/2},
\]

(7.18)

\[
\tau_f = 4.46 \times 10^{-16} \frac{T_{0,\text{eV}}}{\rho_{0,\text{g/cc}}},
\]

(7.19)

\[
\tau_t = 7.7 \times 10^{-16} \frac{\Pr A_i^{3/2} T_{0,\text{eV}}^{3/2}}{C_D E_{0,\text{eV}} \rho_{0,\text{g/cc}} \ln \Lambda}.
\]

(7.20)

The constant \( c_e \) in \( \tau_e \), Eq. (7.18), is a numerical coefficient (see (Atzeni and Meyer-ter-Vehn, 2004) Sec. 4.1.4) which is taken to be unity in the present calculations. The turbulent conduction time, Eq. (7.20), requires the Prandtl number, \( \Pr \), and a constant \( C_D \). The present calculations use \( \Pr = 1 \) and \( C_D = 0.09 \).

The presence of \( C_D \) is a result of the (standard) eddy viscosity model (Lauder and Sharma, 1974; Pope, 2000) used to calculate the turbulent thermal transport. This transport is calculated in a similar manner to the electron thermal conduction, but with an eddy viscosity used for the diffusive coefficient. The turbulent transport is discussed further in Sec. 7.4.1.
For convenience, we present $\tau_n$, Eq. (7.8), and the ratio $V_0^2/2\gamma^2$ in an easily computed form. They are,

\[
\tau_n = 3.6 \times 10^2 \frac{L_0^2 \rho_{0,8/cc} \ln \Lambda}{T_{0,eV} A_i^{1/2}},
\]

(7.21)

\[
\frac{1}{2} \frac{V_0^2}{\gamma^2} = 7.9 \times 10^{17} \frac{E_{r0} L_0^2 \rho_{0,8/cc}^2 (\ln \Lambda)^2}{T_{0,eV} A_i^2}.
\]

(7.22)

The remaining component of the model to be described is the D-T fusion energy production term. Here we again follow Atzeni and Meyer-ter-Vehn (2004); the alpha particle deposition fraction, $f_\alpha$ is,

\[
f_\alpha(L, \tilde{T}) = \begin{cases} 
\frac{3}{2} \tau_\alpha - \frac{4}{5} \tau_\alpha^2 & \tau_\alpha \leq 1/2 \\
1 - \frac{1}{4 \tau_\alpha} + \frac{1}{160 \tau_\alpha^3} & \tau_\alpha \geq 1/2
\end{cases}
\]

with $\tau_\alpha(L, \tilde{T})$ calculated for the present case as

\[
\tau_\alpha(L, \tilde{T}) = L^{-2} \tilde{T}^{-3/2} \times 7.12 \times 10^5 \frac{\ln \Lambda_{ae} \rho_{0,8/cc} L_0}{5 T_{0,eV}^{3/2}},
\]

(7.23)

with $\ln \Lambda_{ae}$ the alpha particle - electron collision Coulomb logarithm. For the normalized fusion cross-section, $\langle \sigma \tilde{v} \rangle$, we use the D-T fit from Bosch and Hale (1992), with the constant $C_1$ pulled out; $C_1$ has been included in $\tau_f$, Eq. (7.19). That is $\langle \sigma \tilde{v} \rangle = \langle \sigma v \rangle_{BH}/C_{1,BH}$, where the subscript BH indicates expressions from Bosch and Hale (1992). Apparent from this modeling of the fusion energy production is that we assume no energy deposition to the hot-spot from the neutrons produced by D-T fusion.

Figures 7.3 and 7.4 and their captions explore the results of this model when there is assumed to be no TKE, $E_{r0} = 0$. The following section, Sec. 7.5, explores the behavior of the hot-spot model in the presence of TKE, $E \neq 0$. 

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7.4.1 Turbulent conduction

To model the turbulent heat conduction, we use an eddy viscosity model, which is similar in spirit to the length-scale argument used for the electron thermal conduction (Atzeni and Meyer-ter-Vehn, 2004). In the electron case, the power flow through a unit surface is proportional to $\chi_e \nabla T_e$, where $\chi_e$ is the electron conductivity, which is calculated using the Spitzer expression. The temperature gradient length scale is taken to be the capsule radius, $R_h = L/2$, so that $\nabla T_e \sim T_e / R_h$.

For the turbulent conduction, the temperature gradient is treated the same way, $\nabla T \sim T / R_h$. The present model has a single temperature, $T = T_i = T_e$, meaning it should be most appropriate when the electron-ion thermal equilibration timescale is small compared to the timescales of the other processes (e.g. radiation, electron thermal conduction, turbulent thermal conduction). The turbulent conduction coefficient is taken to be the eddy viscosity, $\nu_{\text{eddy}} = C_D E^2 / \epsilon$. In the present case this eddy viscosity can be written $\nu_{\text{eddy}} = C_D \tau_{\text{edd}} (V_0^2 / 2) \hat{E}^2 / \epsilon_{\text{model}}$.

This eddy viscosity conduction coefficient has units of $\text{length}^2 / \text{time}$, and can be viewed in the usual diffusive transport picture. The step size, $\Delta x = V_0 \tau_{\text{eddy}}$, is given by the (large scale) turbulent velocity, $V_0 \propto \sqrt{E}$ times the eddy turnover time $\tau_{\text{eddy}}$. The timescale is the eddy turnover time. This turnover time can be written as $\tau_{\text{eddy}} = E / \epsilon$. Together, one finds $\nu_{\text{eddy}} = C_D E^2 / \epsilon$, with $C_D$ a numerical coefficient.

Using this model, we find the turbulent conduction is generally small compared to the electron thermal conduction for the trajectories examined here. As such, in the present model there are regimes where it is possible to reduce initial energy losses for a trajectory by shifting thermal energy to turbulent energy, because the increase in turbulent conduction does not offset the decrease in electron thermal conduction. This effect is shown in the beginning of the comparison between trajectories 3 and 4 in Fig. 7.8. Another effect of having weaker turbulent conduction than electron thermal conduction is that capsules with some energy shifted to turbulence can achieve higher temperatures along given trajectories.
than their counterparts with no turbulence. This effect is apparent in the middle and later stages of the comparison between trajectories 3 and 4 in Fig. 7.8.

In the most extreme case, one can imagine actually reversing the flow of heat. The boundary in the present model is treated as having no temperature, so that thermal losses always . However a real boundary will exist at a finite temperature. If the plasma under consideration is hotter than the boundary when all its energy is in the thermal component, this could cease to be the case upon shifting much of the thermal energy into TKE. Where previously heat would be lost from the plasma to the boundary, in the case where most of plasma energy is in the form of TKE the boundary could be made to heat the plasma.

These possibilities are rather remarkable. It is possible that, in more a complete and self-consistent transport model they will disappear for any number of reasons, or become more restricted in applicable parameters (possible reasons for disappearance could include turbulent mixing of material from the cold boundary, or that boundary damping of TKE is a loss that outweighs any gains in conduction reduction). Nonetheless, the prospect of “optimizing” conduction losses by shifting energy away from the dominant electron conduction loss channel, and into flow, is a tantalizing idea raised by the results here.

7.5 Scenarios

The goal in this section is to determine whether, within the confines of the present model, we may gain advantage by starting with $E_{r0} \neq 0$. In general we will compare cases with equal starting energy but different allocation to TKE and thermal energy. The “base case” has all the of starting energy as thermal energy, while a case of comparison could have, say, half of this same starting energy in thermal energy and half in the TKE, making $E_{r0} = 1$.

The figures and results from Secs. 7.3 and 7.4 are useful for gaining intuition about the sorts of trajectories that are likely to be exploitable in the present model. Trajectories in the current model are restricted to within the gain region (with possible modest devia-
Figure 7.5: Contours for the quantity $\frac{\tau_c}{\tau_\nu}$ plotted on the gain and loss regions for a compression with implosion velocity $U_b = 3 \times 10^7$ and assuming $\ln \Lambda = 2$. This quantity is the ratio of the compression timescale, $\tau_c$ (Eq. (7.7)), to the viscous timescale, $\tau_\nu$ (Eq. (7.8) or (7.21)). As the ratio shrinks, the compression becomes more rapid for the turbulence. All else being equal, more rapid compression leads to less turbulent dissipation (and therefore more turbulent growth). This quantity is one of two that enter into the turbulence model, Eqs. (7.4) and (7.6). Contours for the other quantity, $\frac{V_0^2}{(2\gamma^2)}$ are shown in Fig. 7.6. Note that the contours depend on the compression velocity, through $\tau_c$. If the compression velocity increases, compressions at a fixed starting $(\rho R)_0T_0$ will become more rapid (the ratio $\tau_c/\tau_\nu$ will shrink).
Figure 7.6: Contours for the quantity $V_0^2/(2\gamma^2)$, Eq. (7.22), plotted for an initial ratio of turbulent to thermal energy of $E_{r0} = 10$. These contours are plotted over the gain and loss regions for a compression with implosion velocity $U_b = 3 \times 10^7$ and assuming $\ln \Lambda = 2$. This quantity is one of two that enter into the turbulence model, Eqs. (7.4) and (7.6). Contours for the other quantity, $\tau_c/\tau_v$ are shown in Fig. 7.5. When $V_0^2/(2\gamma^2)$ is large, the turbulence is initially in a high Reynolds number regime. Increasing $E_{r0}$ increases $V_0$, so that the quantity $V_0^2/(2\gamma^2)$ increases at a fixed $\rho R$.

Since this region is almost entirely in the “unstable” zone where turbulence can be amplified (Fig. 7.2), the stability bound plays little roll in determining exploitable trajectories. We will see that it is difficult to achieve a sudden viscous dissipation as drastic as that in Fig. 7.1 for typical hot-spot parameters in the current simple model. The previous sections’ results give two important constraints on finding trajectories where sudden viscous dissipation plays a role. Despite the difficulties in achieving sudden viscous dissipation for the present hot-spots with typical compression velocities, we will see that hot-spot turbulence still presents interesting possible advantages.

First, in order to trigger sudden dissipation, the viscosity must grow sufficiently with contraction (we require $\beta > 1$, $\mu = \mu_0 L^{-2\beta}$). For D-T, with $Z = 1$ and the Braginskii
viscosity, this requirement is equivalent to a requirement on the slope in \( \rho R - T \) space. For a trajectory to possibly trigger sudden viscous dissipation of its TKE, its slope in \( \rho R - T \) space must be steeper than the “attractor” slope. Solutions that asymptote to the “attractor” solution at some point then must either have the TKE dissipate before reaching the attractor, or else the TKE will be present when fusion kicks in. The heating from fusion in the model is sufficient to increase the viscosity rapidly and dissipate TKE; note the steep slopes (even steeper than adiabatic mechanical heating) of trajectories 1 and 2 in the fusion region in Fig. 7.4. While there may be some utility in dissipating the TKE during the time of fusion reactions, the sudden viscous dissipation scenario previously imagined uses TKE to help reach the conditions necessary for fusion. In this case, it seems we require trajectories that at least near the fusion gain region on a “steeper” course. Within the confines of the present model, this apparently requires colder and/or denser initial states (for example, starting points along the gray line indicating the slope of adiabatic heating in Fig. 7.4 could conceivably reach the fusion gain region without substantially running into the “attractor” trajectory).

Second, if the dissipation of the TKE is to give a substantial change to the thermal energy content of the hot-spot, the energy ratio \( \mathcal{E}_r \) at the time of the sudden dissipation should not be too small. For trajectories with \( \beta = 1 \) (the slope of the “attractor”), the TKE saturates, as discussed in Sec. 7.3. When \( \beta > 1 \), the viscosity increase is stronger, and it stands to reason that for the same compression speed, the TKE will not reach a level larger than the saturation level for a corresponding \( \beta = 1 \) case. Thus, we can use Eq. (7.15) to estimate the maximum impact of a sudden dissipation for a given trajectory. The ratio of thermal energy to TKE for steeper trajectories should be at least as great as that for the saturated state represented by Eq. (7.15). Consider the following example. At a compression speed of \( U_b = 3 \times 10^7 \) cm/s, we have \( \mathcal{E}_{r,\text{sat}} = 1 \) at \( T = 1.5 \) keV (shown in Fig. 7.2). If a trajectory had saturated TKE and triggered a sudden dissipation event right when \( T = 1.5 \) keV, the maximal temperature ignoring any energy loss would be \( T = 3 \) keV.
In reality the temperature after the sudden dissipation will be lower for two reasons. First, because the actual amount of TKE will be below the saturation level if the trajectory is steep enough for sudden dissipation, so that $\mathcal{E}_r < 1$ for the present example. Second, there will be some time elapse during the dissipation, and the loss mechanisms will continue to act, reducing the final temperature. If the sudden dissipation event in the present example occurs at a temperature above 1.5 keV, we necessarily have $\mathcal{E}_r < 1$, because the thermal energy will be larger but the TKE can be no greater than the saturated level.

If the TKE starts above the saturation level for a given compression velocity, the TKE should decrease under compression. This means that, for a given compression velocity there is a maximal amount of TKE that can be “stored” during the compression, and that “extra” starting TKE will be dissipated into thermal energy rather than be available for later use. Faster compressions can reach larger absolute quantities of TKE, and thus can delay having the quantity of TKE overtaken by the thermal energy.

We note that these two constraints are useful for gaining intuition but all parts are not necessarily exact, even within the present model; the presence of the TKE can modify the gain/loss regions and the “attractor” solution, because the TKE represents an additional source of heating. Thus a trajectory with substantial TKE that reaches the “attractor” may not exactly follow it, having extra heating from the dissipation of the TKE.

The initial conditions of a trajectory set the value of two quantities that enter into the turbulence model, $\tau_c/\tau_\nu$ and $V_0^2/(2\gamma^2)$. Figures 7.5 and 7.6 show contours of these quantities; see the figure captions for more description. The overall trend for both quantities is similar; as the starting $T$ decreases and/or the starting $\rho R$ increases, a compression at a fixed velocity becomes more rapid and low-Reynolds-number effects become smaller. In the simple coupled model, Eqs. (7.1),(7.4),(7.6), these two quantities control the timing of the dissipation of TKE into thermal energy, as well as how sudden the dissipation is (over how long of a compression interval, $\Delta L$ it occurs). There is interplay between the parameters, but as a guiding picture, varying one parameter at time yields the following: more
Figure 7.7: Comparison of two compressions (trajectories) starting from the same initial condition and an initial ratio of TKE to thermal energy of $E_{r0} = 10$. On the left, for the simple model, Eq. (7.1), including only mechanical heating and turbulence. On the right, for the full model, Eq. (7.16), additionally including electron thermal conduction, Bremsstrahlung radiation, D-T fusion, and a model for turbulent heat conduction. The top plots show the evolution of the (normalized) TKE ($E$, blue line), thermal energy ($T$, green line), and sum of TKE and thermal energy ($E + T$, dotted red line). The thin red line shows the lossless evolution of the total energy, which matches the total energy evolution for the simple model and does not match for the full model, as expected. The bottom plots show the hot-spot path in $\rho R - T$ space, plotted on top of the gain and loss regions for the compression velocity used, $U_b = 3 \times 10^7 \text{ cm/s}$. The thin gray line shows the slope of adiabatic mechanical heating. The coloration on the trajectory indicates the ratio of TKE to thermal energy ($E_r$): the trajectory is red when $E_r \geq 1$, it is pink when $0.1 < E_r < 1$, and it is white when $E_r < 0.1$. Thus, the crossover between the E and T curves on the top graphs correspond to the transition between red and pink on the bottom trajectory plots. In the simple model, the TKE dissipates shortly after the crossover, after $E_r < 1$, making the feedback of thermal energy into TKE less important than in Fig. 7.1. In the full model, the trajectory asymptotes to the “attractor” solution (see Fig. 7.2), which prevents the dissipation of the TKE. The TKE only dissipates much later, after the temperature starts to rise from fusion reactions. In the full model, thermal energy losses mean the total energy falls below the lossless evolution. Also of note is that the temperature initially rises faster than the adiabatic heating solution, due to dissipation of TKE to thermal energy which is non-negligible from the start for these initial conditions and compression speed. The impact of this heating from TKE dissipation is evident by comparison to trajectory 2 in Fig. 7.4, which starts from similar $\rho R$. $T$ and initially heats adiabatically.
Figure 7.8: Comparison of two trajectories using the full model, Eq. (7.16), for the compression velocity \( U_b = 3 \times 10^7 \) cm/s. Both have the same total starting energy, but apportioned differently. The left plot shows two trajectories, labeled 3 and 4, in \( \rho R - T \) space, while the right plot shows the ratios of two quantities for trajectory 3 to the same quantity for trajectory 4. These quantities are the total energy (turbulent plus thermal), in dashed blue and the temperature, in solid red. Trajectory 3, starting in red on the left plot, has \( E_{t0,3} = 10 \), while trajectory 4, dashed green in the left plot, has essentially all the initial energy in the thermal component, \( E_{t0,4} = 0.001 \). In this case the TKE does not have a substantial impact on the trajectory. The trajectories overlap (have identical \( \rho R, T \)) twice, once for \( \rho R \approx 10^{-3}, \bar{L} \approx 0.27 \) and near the end (\( \rho R \approx 1, \bar{L} \approx 0.01 \), after the trajectory 3 has its TKE dissipated by the increase in temperature caused by fusion reactions. At the first crossing point, mechanical heating of 185 J has heated the hot-spot 4 from \( \sim 87 \) eV to \( \sim 0.64 \) keV, while a total heating of 173 J has heated the hot-spot 3 from \( \sim 7.9 \) eV to the same temperature, with approximately \( 1/3 \) of the heating contribution coming from turbulent dissipation. The much more efficient temperature increase up to this point for trajectory 3 is a result of much lower (by about a factor of 2) electron thermal conduction loss, which dominates the other losses for both trajectories in the present model (see also Sec. 7.4.1). After the first crossing point, trajectory 3 is seen to consistently maintain a higher temperature until the TKE is dissipated after fusion kicks in. On the right plot, it is apparent this temperature is 10% - 20% higher, while the difference in total hot-spot energy is larger and never below 1.0. This latter fact indicates that the total hot-spot pressure in the turbulent trajectory is generally greater than the pressure in the non-turbulent trajectory at any given \( \rho R \). The final dissipation of the TKE is visible as a spike in the temperature ratio, since only trajectory 3 has substantial TKE when fusion kicks in. After the TKE is dissipated and the trajectories overlap, and the conduction, radiation, and turbulent conduction losses for trajectory 3 are all larger than those for trajectory 4, as a result of the higher temperature for trajectory 3 during the middle and later compression phases. The fusion gains are likewise larger for trajectory 3, for the same reason.
Figure 7.9: Similar to Fig. 7.8, a comparison of two trajectories with the same total starting energy divided differently between TKE and thermal energy. Trajectory 5 has $\mathcal{E}_{r0} = 12$, while trajectory 6 has $\mathcal{E}_{r0} = 0.01$ but the same total starting energy, so that the starting temperature is essentially 13 times higher. The initially turbulent trajectory, 5, reaches the fusion gain region (see Fig. 7.3) and eventually dissipates its TKE, while the non-turbulent trajectory, 6, fails to reach the fusion gain region. This difference is a result of the reduced losses associated with storing the energy in the turbulence, and the sustained heating provided by turbulent dissipation. Trajectory coloration is described in the caption of Fig. 7.7.

Rapid compressions (decreasing $\tau_c/\tau_v$) shifts the dissipation later (to smaller $\bar{L}$) and makes it more rapid, while increasing $V_0^2/(2\gamma^2)$ gives earlier and more gradual dissipation of the TKE.

The example shown in Fig. 7.1, using the simplified $T$ equation, Eq. (7.1), has both an initially rapid compression ($\tau_c/\tau_v \approx 6.7 \times 10^{-6} \ll 1$) and small initial viscous corrections ($V_0^2/(2\gamma^2) \approx 3.3 \times 10^6 \gg 1$). When $\mathcal{E}_{r0} = 10$ and $U_b = 3 \times 10^7$, it is evident from examining Figs. 7.5 and 7.6 that there is no set of $(\rho R)_0, T_0$ within the plotted range that will give this combination of $\tau_c/\tau_v$ and $V_0^2/(2\gamma^2)$. This becomes even more true for $\mathcal{E}_{r0} = 100$, as in
Fig. 7.1, because increasing $E_{r0}$ increases $V_0$ and therefore the value of $V_0^2/(2\gamma^2)$ at a fixed $(\rho R)_0, T_0$.

As such, we first show, in Fig. 7.7, a comparison of a simple model trajectory, Eq. (7.1), and a full model trajectory, Eq. (7.16), that have achievable $\tau_c/\tau_v$ and $V_0^2/(2\gamma^2)$ when $U_b = 3 \times 10^7$ cm/s. The TKE dissipation in the simple model is less sudden than that in Fig. 7.1, and occurs at a point when the thermal energy has already grown larger than the TKE, meaning the dissipation of TKE does not cause as drastic a change in the thermal energy as is the case for the trajectory in Fig. 7.1. This is essentially because the compression is not rapid enough starting from this region of $\rho R - T$ space with a compression velocity of $U_b = 3 \times 10^7$. The full model trajectory has a more fundamental problem, discussed above; the trajectory reaches the “attractor” solution before the TKE dissipates, and the TKE dissipation is therefore delayed until fusion heating kicks in. When the dissipation does occur, it is sudden, but it does not help the trajectory reach the fusion regime in the sense that the fusion heating must kick in before the dissipation occurs.

Despite this problem, it is worthwhile to ask whether any advantages may be realized from the presence of TKE for full model trajectories like that shown on the right side of Fig. 7.7. The assess this, we compare this trajectory to another. Suppose that instead of starting with 10 parts TKE and 1 part thermal energy ($E_{r0} = 10$), we started a trajectory from the same initial $\rho R$ but with all the energy in thermal energy, $E_{r0} = 0$, making the starting temperature higher. How will the energy losses to conduction, radiation, and turbulent conduction, compare for these compressions with the same initial energy content stored in different components? This comparison supposes that, in the process of initiating the compression we are not able to choose the total energy deposited into the hot-spot, but that we can control its partition. This comparison is shown in Fig. 7.8 and described in the caption.

Another comparison of two trajectories that differ only by the allocation of the initial energy content between TKE and thermal energy is shown in Fig. 7.9 and described in the
caption. This comparison shows that changing the allocation between TKE and thermal energy can make the difference between whether or not a trajectory reaches the fusion gain region. In this case, the reduced losses associated with storing energy in the TKE means the trajectory with that starts cold but with substantial TKE reaches the fusion region, while the trajectory with the same initial energy purely in the thermal component fails to reach the fusion region.

7.6 Discussion

The present work has integrated a model for bulk hot-spot turbulence into a previously existing hot-spot model to carry out a first systematic examination of the behavior of turbulence in ICF hot-spots. Prior work on hot-spot turbulence has typically investigated a single compression, e.g. (Thomas and Kares, 2012; Weber et al., 2014). Here, it was shown that most of the $\rho R - T$ space region where the hot-spot gains energy is also a region where hot-spot turbulence will grow under compression, if the trajectory parallels the solution obtained by balancing mechanical heating and electron conduction losses. A saturation level for hot-spot turbulence was shown. In the present coupled model, it is found that turbulence typically doesn’t reach this saturation level, although it can reach substantial fractions (e.g. 25% - 50%) of it.

The sudden viscous dissipation mechanism was demonstrated in the model, but dissipation of the turbulence in achievable trajectories for typical compression velocities ($\sim 3 \times 10^7$ cm/s) and parameters was generally found to either be gradual, or sudden only after fusion heating begins to drive the hot-spot temperature. Nonetheless, we showed ways in which the presence of hot-spot turbulence can enhance trajectories or open up new possible trajectories. More generally, we demonstrated ways in which hot-spot turbulence impacts the development of trajectories in $\rho R - T$ space.
The trajectories available in the present hot-spot model are somewhat limited. Simulated trajectories, such as those in plotted on the gain/loss region by Lindl (1995) can have more complicated paths, and can travel outside the model gain region. As has we have shown, the turbulent growth and dissipation are sensitive to the viscosity growth with compression. For $Z = 1$ plasmas such as a D-T hot-spot, this translates to a sensitive dependence on the growth of temperature with compression, at least in regimes where the Braginskii viscosity is an accurate description. Since real hot-spots can follow more complicated paths in $\rho R - T$ space, additional possibilities (or complications) may arise in a more complete hot-spot description than the simple model used here.

The viscosity in the present model has been assumed to be Braginskii over the entirety of all the trajectories. We have ignored fusion neutron deposition power. Both of these assumptions can be checked a posteriori for an individual trajectory. The turbulence model used assumes that the compression forces the turbulence in a very specific, volumetric way (see Ch. 3 and Ch. 4), and that turbulent energy is only lost through bulk viscous dissipation. In particular, the possibility of boundary drag is ignored. The energy in the forcing (compression causing) flow, which is constant in time, actually represents a substantial component of the plasma energy, which never dissipates in the trajectories considered here. More discussion about this energy can be found in the discussion section of Ch. 2 and at the end of Sec. 4.5. In general, a drawback of the present hot-spot model is that the hot-spot mass is fixed, which does not reflect the reality for hot-spots surrounded by a cold D-T layer.

More work is required to examine the regimes of accuracy for the current turbulence treatment. The TKE model used was found to match simulations reasonably well for a range of fixed rates of temperature growth with compression (corresponding to constant slope trajectories), but is deployed here even for trajectories where the rate of temperature growth varies during a compression. Further, the simulations it was tested against were
of subsonic turbulence, but when $E_r \gtrsim 5/9$, as it is at times here, the turbulence will be supersonic (assuming the adiabatic index $\gamma = 5/3$ and 50-50 D-T).

Despite these limitations, it is hoped the present work has clarified some of the principles expected to guide the behavior of hot-spot turbulence.
Motivated by recent experimental and computational results indicating the possibility for substantial turbulence in compressing plasmas, this thesis has initiated an exploration of the properties of plasma turbulence undergoing compression. This thesis has first formed an understanding of the essential impact of plasma viscosity on compressing turbulence, and has then used that understanding to make contributions in a variety of important subject areas, including theoretical and experimental high-energy-density physics and astrophysics. This chapter recaps work completed in the thesis, and discusses some possible follow-on work. An effort was made in the introduction to give a thorough placement of the results in context and a thorough explanation of their importance, and that is discussion is not repeated here. As such, readers who have made it this far are also encouraged to revisit the introduction, Ch. 1.

8.1 Primary results

First, a new “sudden viscous dissipation” phenomenon was demonstrated. Turbulent kinetic energy (TKE), having grown up during a compression, is suddenly dissipated into temperature by viscosity. This mechanism opens up the potential for a new design basis for compression experiments, based around the storage of energy gained during compres-
sion in turbulence. This turbulence can then be converted, through the sudden dissipation mechanism, into thermal energy at a desired time. To the extent that this reduces energy losses during the compression, such as losses due to radiation or electron heat transport, it may be of great benefit. This sudden dissipation is of particular importance in plasmas.

Next, this thesis examined the impact of ionization (increases in the plasma charge state) during compression on turbulence behavior. It was shown that an increasing charge state during compression can greatly influence TKE behavior, including making the difference between whether the TKE increases or decreases. Too much increase in the charge state can prevent the sudden viscous dissipation mechanism from kicking in. A modest increase in the charge state can mean much larger TKE growth before the sudden dissipation. These findings have implications both for the prevention and exploitation of turbulence in plasma compressions. While hot spots of targets compressed for fusion at the NIF are believed to be viscous at stagnation, suppressing turbulence, even modest contamination (mix) with high Z material could make this no longer true. This is especially a concern because of the difficulty of modeling mix. At the same time, in light of our results, it is less surprising that Z-pinches like those at Weizmann may be highly turbulent. These Z-pinches (and others) commonly compress materials other than the hydrogen (Z = 1), which undergo large increases in charge state during the compression.

The sensitive dependence of the TKE during compression on the charge state was used to suggest a means of “tuning” experiments designed to take advantage of the sudden viscous dissipation mechanism. If one desires later, or earlier, sudden dissipation for a fixed set of parameters, it may be possible to achieve this by adding slight amounts of impurities with a different Z.

A model of the TKE behavior for compressing turbulence was developed. This model improves on previous compressing turbulence models by taking into account viscous (low Reynolds numbers) effects. Unlike previously existing models, the new model gives the correct qualitative TKE behavior for compressions where the viscosity increases in time.
It correctly predicts the TKE saturation for constant velocity compression when $\beta = 1$, and it gives reasonable quantitative agreement over a range of viscosity dependencies on compression. The model should be useful for estimating the expected TKE behavior in three dimensional compressions where the compression velocity is reasonably constant over a period of the compression. This is not uncommon. The energy partition and total energy injection in compressing turbulence of the type studied here were covered.

A lower bound on the growth of the turbulent velocity in supersonic turbulence undergoing compression was given. The particular lower bound applies for gravitationally compressing supersonic turbulence such as that in astrophysical molecular clouds. However, the technique used to arrive at the bound can be applied in other situations as well. This lower bound suggests that a previous model of compressing supersonic turbulence is too dissipative, at least when applied to the case when the polytropic index is $5/3$. The simulations accompanying the previous model may also be too dissipative, which could have broader implications for other work looking at compressing astrophysical turbulence. It is suggested that this discrepancy in dissipation levels is likely tied to whether the turbulent length scale is allowed to saturate or not. This raises the question of whether a saturated turbulent length scale is physical or not for problems of astrophysical interest.

In conjunction with collaborators at the Weizmann Institute, experimental results on a K-shell-radiation optimized gas-puff Z-pinch were re-examined. Whereas previous analysis of the experimental results assumed a uniform density, the re-examination accounted for the fact the density at stagnation is likely to be highly non-uniform as a result of supersonic turbulence. Using a model density PDF, this updated analysis was shown to be both consistent with the observations, and to improve the agreement with them compared to the previous uniform density analysis. This improved agreement resulted from a factor of 2 reduction in the inferred plasma density at stagnation, which led to an inferred plasma radius that better matched the observed stagnation radius. This reduced density, while consistent with the observations, is likely to substantially impact our understanding
of the power balance during the compression phase of such Z-pinches. Further, the analysis overall reinforces the picture of a Z-pinch dominated by turbulent motion; this is at once a new paradigm for plasma compression experiments as well as a potential new platform for laboratory astrophysics problems.

The final body chapter of the thesis examined turbulence in mechanically heated hot-spots by coupling the compressing plasma TKE model developed earlier with a model for the hot-spot temperature. The $\rho R - T$ space was divided into two regions, based on the expected turbulence behavior of trajectories. The turbulence behavior in each region was shown to depend on the slope of the trajectory in $\rho R - T$ space, which corresponds to the rate of temperature (and therefore viscosity) growth under compression. The marginal case is that of the “attractor” solution, where the temperature growth is governed by the balance of $PdV$ (mechanical) work and electron thermal conduction. In the “stable” region, trajectories with a slope at least as steep as the “attractor” were shown to experience decreasing TKE, while in the “unstable” region all trajectories can experience growing TKE. A saturation level for hot-spot turbulence was given, showing that, in principle, hot-spot turbulence can reach levels comparable to the thermal energy up until the hot-spot reaches multi-keV temperatures, for typical compression velocities. It was shown that compared to trajectories starting solely with thermal energy, those starting with a the same energy mostly allocated to the TKE could exhibit possibly advantageous properties. These turbulent trajectories can reduce early losses by remaining cooler, and also can maintain higher temperatures late in the compression. The turbulent trajectories may also reach the fusion gain region even in cases where their non-turbulent counterparts will fail to reach fusion conditions.

Overall, it is hoped the publication of the work presented in this thesis gives those working on plasma compression experiments a better handle on the expected behavior of turbulence undergoing compression. An effort has been made to further tie together turbulence compression literatures, particularly those of neutral fluids and astrophysics.
8.2 Future work

Given the relatively sparse amount of work on the behavior of compressing plasma turbulence, there are a wide variety of future avenues for investigation. In the immediate future there are plans to better understand the apparent discrepancy between the lower bound of Ch. 5 and previous work. If possible, extending the lower bound to include turbulence with magnetic fields would broaden its applicability and its utility as a check on simulations and/or models.

Future plans also include further collaboration to analyze the Z-pinch experiments at the Weizmann Institute. Two problems of interest are to determine the process by which the compression energy becomes turbulent energy at stagnation, and to determine whether the turbulence is present during the compression phase.

A number of assumptions were made for most of the work in this thesis; relaxing these various assumptions may yield new or more realistic behavior. These assumptions include those of constant velocity compressions, periodic boundary conditions and homogeneity. Subsonic turbulence was assumed for the present work on sudden viscous dissipation; a true assessment of the utility of this dissipation mechanism requires working with supersonic turbulence. The model presented here should then be improved if necessary to capture the behavior of compressing supersonic turbulence.

This thesis has only dealt with isotropic, three-dimensional (3D) compressions. The behavior of two-dimensional (2D) compressions is relevant for Z-pinches, and may have quite different properties than 3D compressions. First, the temperature scaling with compression ratio in 2D compressions will generally be different than for 3D compressions. This will impact the viscosity behavior under compression. Perhaps more importantly, at least linearly, a 2D compression supplies energy to only the hydrodynamical components parallel to the compression. While it is to be expected that nonlinear effects will act to spread this energy into the third component, an initially rapid compression could experi-
ence a time with a large disparity in hydrodynamic energy parallel versus transverse to the compression.

Electric and magnetic fields have been ignored for the turbulence studies in this thesis. This may be reasonable in three dimensional compressions with no initial field. However, the fields may be important in various Z-pinch setups. Studies of 2D compressions of turbulence with varying levels and geometries of initial fields are needed. Substantial crossover with astrophysics problems can be anticipated for studies of compressing turbulence with magnetic fields, as magnetic fields can play an important role in molecular clouds.

The compressions studied in this work were “volumetric”, acting to simultaneously and continuously compress the whole domain. In plasma compression experiments some of the compression is caused by shocks. Although shock-turbulence interaction has is an active area of research, there are opportunities for examining shock compression of turbulence in the context of effects identified here, such as the sudden viscous dissipation phenomenon.
Appendix A

Fusion utility in the Knudsen layer

A.1 Introduction

In inertial confinement fusion, the loss of fast ions from the edge of the fusing hot-spot region reduces the reactivity below its Maxwellian value. The loss of fast ions may be pronounced because of the long mean free paths of fast ions, compared to those of thermal ions. We introduce a fusion utility function to demonstrate essential features of this Knudsen layer effect, in both magnetized and unmagnetized cases. The fusion utility concept is also used to evaluate restoring the reactivity in the Knudsen layer by manipulating fast ions in phase space using waves.

Knudsen layer reduction of fusion reactivity occurs when some fusion fuel ions stream out of the fusing region of the capsule before having a chance to fuse. The fast fuel ions, which typically have the best chances of fusing, also have very long mean free paths compared to thermal ions. Thus, even when the bulk of the plasma is collisionally confined over the time of the fusion burn, the highly effective fast particles may be lost before fusing, substantially decreasing the fusion reactivity.

The possibility for Knudsen layer reduction of the fusion reactivity was initially explored by Henderson (1974) and Petschek and Henderson (1979). Molvig et al. (2012)
formulated an asymptotic, steady state theory of the effect, and applied it in radiation-hydrodynamic simulations of OMEGA implosions. They found that including the Knudsen layer model significantly improved agreement in calculated D-T fusion neutron yield between the simulations and experiments. The treatment of the boundary in this model was subsequently improved by Albright et al. (2013). This past work found the effect to be pronounced only in capsules with small fuel $\rho r$. Tang et al. (2014c) considered larger capsules that may still have Knudsen losses due to hydrodynamic mix, and the recovery of some portion of the losses due to lost fast ions fusing in the surrounding cold fuel. Tang et al. (2014a,b) and McDevitt et al. (2014) studied a hierarchy of reduced Fokker-Planck operators to capture the essentials of the Knudsen layer effect and to compute the tail distributions at hot-cold plasma interfaces.

In the case where the implosion is magnetized, the picture of fast ion trajectories changes, since ions are limited in traveling in the direction perpendicular to the magnetic field. This fundamentally changes the length and velocity scalings. Schmit et al. (2013) considered the effect of magnetization on the Knudsen layer reduction of fusion reactivity, giving heuristic conditions for the reactivity to be largely restored, and showing the applicability of these conditions by numerically generating the steady state fast ion distribution function in cylindrical and spherical magnetized geometries.

The present work describes the Knudsen layer phenomenon in a way that is complementary to previous work. By identifying a fusion utility function for fast ions, we address the question, “How much fusion energy is an ion starting at position $x_0$ with velocity $v_0$ expected to produce over its lifetime?” After deriving the fusion utility function we show how it can be used to consider both the unmagnetized and magnetized Knudsen layer problems, while allowing for spatial density dependence.

In the unmagnetized case we find that it is the total density between a fast ion and the boundary that determines its lifetime fusion utility, not the absolute distance to the boundary. The fusion utility increases as the amount of density to the boundary increases.
until the fast ion is far from the boundary. For the magnetized case, the fusion utility is independent of the density.

The fusion utility function is a particularly powerful construct for evaluating incremental effects. For example, waves can be used to locally change the fast ion distribution function, both by increasing ion energy and by pushing ions away from the Knudsen layer. The utility function gives the change in lifetime fusion energy production that occurs on moving an ion from \( x_0, v_0 \) to \( x_1, v_1 \), thereby giving the effect on fusion energy production of such waves.

The utility function approach, in general, has been useful in considering incremental or differential effects of external perturbations in plasma. It has been particularly useful in resonant rf interactions with plasma, particularly in the case of wave-driven electrical current (Fisch, 1987). The rf waves diffuse particles along well-constrained diffusion paths, so that, essentially, the rf removes particles from one phase space location, and inserts those particles in an adjacent phase space location, with the phase space residing in the 6D space of velocity and position. By associating with each point in phase space a utility, the differential utility, as well as the energy cost, can be calculated under any rearrangement of the phase space by wave excitation. Thus, the current-carrying utility of a superthermal electron at an initial position in the 6D phase space may be used to calculate the current drive efficiency (Fisch and Boozer, 1980). Similarly, a runaway probability can be associated with each initial position of an electron in the 6D phase space (Karney and Fisch, 1986). In both cases, the differential effect relates the rf power dissipation to either the generation of current or the production of runaways.

Here the fusion utility function gives the total extra fusion energy at the cost of moving the ion in phase space. The utility function may thus be used to answer whether it is useful to expend rf power to move particles away from the boundary, say if the rf power is simply applied from an external antenna. If, however, the power were supplied from tapping the alpha particle energy, say through an instability driven by the alpha particles themselves,
then there would be a number of added benefits. Among the added benefits, for example, would be to avoid direct electron heating, thereby obtaining a hot ion mode, where the ion temperature exceeds the electron temperature (Fisch and Herrmann, 1994). Another benefit is that, if the rf wave is generated by the alpha particles, then the alpha particles may be transported toward the boundary. The present analysis considers only the direct utility of extra fusion energy, rather than these added benefits which depend on whether the rf power is internally generated or externally supplied. It is also beyond the scope of this work to propose specific waves that might be destabilized by the alpha particles specifically near the Knudsen layer boundary.

This appendix is organized as follows. In Sec. A.2, we describe the basic idea of the utility function and the scheme for mitigating Knudsen layer fusion reactivity losses. Next, Sec. A.3 describes the fusion utility function more formally. Section A.4 shows example calculations of an unmagnetized utility function and a magnetized utility function. In Sec. A.5 we find the theoretical fusion energy production gains from phase space manipulation in the Knudsen layer. Finally, Sec. A.6 discusses caveats for the reactivity restoration scheme, and possible improvements and generalizations of our work.

A.2 Utility function

Consider tracking a fast ion moving through a plasma as it pitch angle scatters and slows down due to drag. The quantity of interest is the total expected fusion energy generated by the ion over its lifetime in the plasma. This lifetime is defined by following the fast ion until it slows down to thermal speed (at which point its chance for fusion is negligible) or until it leaves the plasma by exiting at a boundary. The boundary might be unreacting liner surrounding the ICF implosion hotspot.

We write the expected fusion energy generated by the fast ion over its lifetime as $\mathcal{E}(x_0, v_0)$, where $x_0$ and $v_0$ are the fast ion’s initial position and velocity. In the limit
where the fast ion starts very far (in mean free paths) from the boundary, the chance it leaves the plasma before slowing down to thermal speed is negligible, and the expected lifetime fusion energy will tend to depend only on the initial velocity, $E(x_0, v_0) \rightarrow E_0(v_0)$. As $x_0$ gets closer and closer to the boundary, the fast ion is more and more likely to leave the plasma before slowing down completely, decreasing the expected fusion yield. This region of decreased yield coincides with the Knudsen layer.

It is possible in some circumstances to use plasma waves to change the velocity and position of particles, for example, in alpha channeling in tokamaks (Fisch and Rax, 1992). If a fast ion that starts near to the boundary is pushed in position away from the boundary by $\Delta x$ while being heated in energy by $\Delta \epsilon$, the expected lifetime fusion energy yield (the utility) $E$ will increase,

$$E \rightarrow E + \frac{\partial E}{\partial x} \cdot \Delta x + \frac{\partial E}{\partial \epsilon} \Delta \epsilon.$$  \hspace{1cm} (A.1)

The gain, $g$, from such pushing will be the incremental fusion energy produced divided by the energy required to do such pushing, represented by the change in the fast ion’s energy in the push, $\Delta \epsilon$

$$g = \left( \frac{\partial E}{\partial x} \cdot \Delta x + \frac{\partial E}{\partial \epsilon} \Delta \epsilon \right) / \Delta \epsilon.$$  \hspace{1cm} (A.2)

In certain cases (Fisch and Rax, 1992), the spatial push $\Delta x$ can be proportional to the energy push $\Delta \epsilon$ - a larger push in energy yields a larger spatial push. Moreover, the direction of this push can be arranged through the wave polarizations. In regions where the fusion yield is lost most rapidly $|\partial E/\partial x|$ is large, so that a small spatial push can give significant gains. Thus, the regions of rapid yield loss are also those where yield can be regained with lowest energy cost. Indeed, we will show that in some circumstances, the gain may be high enough to consider such a mitigation strategy for counteracting Knudsen layer losses of fusion yield.
A.3 Approach

To write a fusion utility function as described in Sec. A.2, consider a function $g(x, v, t; x', v', t')$ that gives the probability an ion initialized at phase space point $x', v'$ at time $t'$ is found later at time $t$ at phase space point $x, v$. The instantaneous expected fusion production at time $t$ for this ion is

$$E(t; x', v', t') = \int dx \int dv \ g(x, v, t; x', v', t') \ W(x, v) \quad (A.3)$$

where $W(x, v)$ is the fusion energy production rate for a fast ion located at point $x, v$. Here $W$ is taken as,

$$W(x, v) = E_f \langle \sigma v \rangle \quad (A.4)$$

the Maxwellian averaged fusion reactivity multiplied by the energy from fusion, $E_f$, with $\sigma$ the velocity dependent fusion cross-section. The function $W$ may depend on position through its dependence on the density. One could generalize this to account for multiple species reacting, but for simplicity we consider the fast ion reacting with only one species here. The integrals in Eq. (A.3) are carried out over the domain of $g$ in the unprimed variables - anywhere the ion may exist at time $t$. An integration of $g$ in the unprimed variables may give a total probability less than 1 if the ion can be lost, say through a boundary.

The fusion utility is the integral of Eq. (A.3) over all time,

$$\mathbb{E}_{t'}(x', v') = \int_{t'}^{\infty} dt \ E(t; x', v', t') . \quad (A.5)$$

This gives a more precise definition of the utility function appearing in Eqs. (A.1),(A.2). Finding the function $g$ and then integrating it to find $\mathbb{E}$ is the Langevin approach to finding the fusion utility.
There is another approach, based on an adjoint formalism. This approach gives a more
direct way of solving for the utility, and shows that the utility is the function that connects
forcings on the distribution function to changes in fusion energy production. The adjoint
approach is outlined here; it is discussed more completely elsewhere (Fisch, 1986).

Consider the general kinetic equation for a single plasma species, with a collision op-
erator that includes all relevant collisions (e.g. electron-ion, electron-electron, for a two
species plasma),

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + F \cdot \frac{\partial f}{\partial v} - C[f] = \frac{\partial}{\partial v} \cdot \Gamma_v - \frac{\partial}{\partial x} \cdot \Gamma_x. \tag{A.6}
\]

Here \( \Gamma_{x,v} \) are wave induced fluxes in space and velocity respectively, which will be useful
when we consider wave manipulation of ions.

Expand the distribution function \( f \) as

\[
f = f_M (1 + \chi) \tag{A.7}
\]

with \( \chi \) assumed to be a small correction, induced in our case by an external perturbation
such as waves or boundary effects. Plugging Eq. (A.7) into Eq. (A.6), assuming that the
background non-drifting Maxwellian quantities (density, temperature) have no time depen-
dence, and linearizing the collision operator, yields

\[
\frac{\partial}{\partial t} (f_M \chi) + v \cdot \frac{\partial}{\partial x} (f_M \chi) + F \cdot \frac{\partial}{\partial v} (f_M \chi) - \hat{C} (\chi) = \\
- v \cdot \frac{\partial f_M}{\partial x} - F \cdot \frac{\partial f_M}{\partial v} - \frac{\partial}{\partial v} \cdot \Gamma_v - \frac{\partial}{\partial x} \cdot \Gamma_x. \tag{A.8}
\]

Equation A.8 has the form

\[
\hat{L} \chi = s \tag{A.9}
\]

where \( \hat{L} \) is a linear operator and \( s \) is a source term.
In the present application, our interest is in the change in fusion energy production in the plasma when it is perturbed from Maxwellian, not the full solution of Eq. (A.8). That is to say, the relevant quantity is a moment of the distribution function \( \chi \), rather than \( \chi \) itself. In this case, it is natural to use an adjoint formulation, which allows us to write an equation for a general moment \( M \) of the distribution,

\[
M(t) = \int dx \int dv W(x, v) f_M \chi(x, v, t).
\]  

(A.10)

For purposes of calculating the fusion energy production \( W \) is given by Eq. (A.4).

The Green’s function solution to an equation of the form of Eq. (A.9) is

\[
\chi(x, v, t) = \int dt' \int dx' \int dv' g(x, v, t; x', v', t') s(x', v') - \int dt' \int d\sigma' \cdot J[\chi(x', v', t'), g(x, v, t; x', v', t')]
\]  

(A.11)

with the Green’s function \( g \) solving

\[
\hat{L}g = \delta(x - x') \delta(v - v') \delta(t - t') .
\]  

(A.12)

Integrals in \( x', v' \) are carried out over the interior of a general, possibly bounded domain, while the integral in \( \sigma' \) is over the \( x', v' \) domain boundary. Time integrals are carried out over an appropriate time domain (e.g. \([0, \infty]\)). The operator \( \hat{J} \) is defined through the relation

\[
\int dt \left( \{ \psi, \hat{L} \chi \} - \{ \chi, \hat{L}^\dagger \psi \} \right) = \int dt \int d\sigma \cdot \hat{J}[\chi, \psi] .
\]  

(A.13)

This relation also serves to define the adjoint operator \( \hat{L}^\dagger \). The inner product is defined as

\[
\{ \psi, \hat{L} \chi \} = \int dx \int dv \psi \hat{L} \chi .
\]  

(A.14)
Given an operator \( \hat{L} \), one uses Eq. (A.13) to find \( \hat{L}^\dagger \) and \( J \). Substituting \( \chi \) from Eq. (A.11) into \( M \), Eq. (A.10), it is possible to write an equation for \( M \) and the moment of \( \chi \) over the domain boundary. Carrying out this procedure for the current \( \hat{L} \), and specializing for homogeneous boundary conditions on \( \psi \) gives an equation for \( M \),

\[
\int dx \int dv \ W(x, v) f_M \chi(x, v, t) = \int dt' \int dx' \int dv' \left( \frac{\partial \psi}{\partial x'} \cdot S_x + \frac{\partial \psi}{\partial v'} \cdot S_v \right) \tag{A.15}
\]

with the fluxes

\[
S_x = v' f_M + \Gamma_x, \tag{A.16a}
\]
\[
S_v = F f_M + \Gamma_v. \tag{A.16b}
\]

With \( W \) given by Eq. (A.4), Eq. (A.15) gives the volume averaged fusion energy production of the perturbed distribution \( \chi \).

The function \( \psi \) is defined as

\[
\psi(t; x', v', t') = \int dx \int dv \ g(x, v; x', v', t') \ W(x, v). \tag{A.17}
\]

It obeys the adjoint equation

\[
\hat{L}^\dagger \psi = -f_M \frac{\partial \psi}{\partial t'} - f_M v' \cdot \frac{\partial \psi}{\partial x'} - f_M F \cdot \frac{\partial \psi}{\partial v'} - \hat{C}[\psi] = 0 \tag{A.18}
\]

with an initial condition given by \( W \), and homogeneous boundary conditions for the present work. The function \( \psi \) in Eq. (A.17) is the same as the fusion energy production rate, Eq. (A.3).
In order to get the utility, $E, \psi$ must be integrated in $t$,

$$E_{t'}(x', v') = \int_{t'}^{\infty} dt' \psi(t'; x', v', t').$$  \hfill (A.19)

Thus, in the case that the ion obeys a linear equation, the utility can be found by integrating in time the solution to the adjoint equation with initial condition $W$. Furthermore, the function $g$ becomes the Green’s function for the kinetic equation.

In the applications considered here $\psi$ will only depend on the time difference $t - t'$, and an integration in $t$ corresponds to integrating the adjoint equation in $t'$ from $-\infty$ to an initial condition at $t$.

Since $\psi$ is the fusion production rate, Eq. (A.15) shows that the instantaneous fusion energy production of the perturbation $\chi$ can be written in terms of fluxes and this fusion production rate. The same relationship will hold after time integration; the total fusion energy production of the perturbation can be written in terms of fluxes and the utility.

### A.4 Example utility calculations

We make a number of simplifying assumptions in our example calculations for both the unmagnetized and magnetized cases, but the adjoint formulation is also applicable to more complicated scenarios. The unmagnetized case contains no magnetic fields, while the magnetized case has a constant $z$ directed field. We assume that the plasma region of interest has no electric fields, and allow spatial dependence in only one direction, the $z$ direction in the unmagnetized case, and the $x$ direction in the magnetized case. This dependence is on the half line $z, x \in [0, \infty]$, with an absorbing boundary at $z, x = 0$, so that we can isolate the effects of the boundary.

Additionally, we use the high velocity limit of the collision operator. In this limit, the fast ion only undergoes velocity drag and pitch angle scattering, with frequencies $v_E$ and
\( \nu, \mu \), respectively,

\[
\hat{\mathcal{C}}[\phi] = \frac{1}{2} \nu \mu \frac{V^3_T}{v^3} \frac{\partial}{\partial \mu'} \left( 1 - \mu'^2 \right) \frac{\partial \phi}{\partial \mu'} + \nu E \frac{V^3_T}{v^2} \frac{\partial \phi}{\partial v'}.
\]  

(A.20)

The lack of dependence on the thermal velocity is made clear by writing

\[
V^3_T \nu_E = C_E n(z'),
\]

(A.21a)

\[
V^3_T \nu_\mu = C_\mu n(z'),
\]

(A.21b)

where \( C_E \) and \( C_\mu \) are constants independent of the thermal velocity and density (and any coordinates), and \( n(z) \) is the density, which we allow to vary in the \( z \) direction. While the collisional dynamics are somewhat more complicated for the magnetized case, they can still be expressed in terms of these frequencies. In factoring out the density we assume that collision partner species all have the same functional form of dependence in \( z \), although they need not appear in equal amounts. In other words, consider

\[
V^3_T \nu_\mu = \frac{e^4 Z_a^2 \ln(\Lambda)}{4\pi m_a^2 \varepsilon_0^2} \sum_b Z_b^2 n_b(z')
\]

(A.22)

where we have ignored dependence inside the Coulomb logarithm, \( Z_{a,b} \) are the charge numbers of the fast ion and collision partner species, \( m_a \) is the fast ion mass, and \( \varepsilon_0 \) is the permittivity of free space. Each \( n_b(z') \) is assumed to have the same functional form, \( n_b(z') = n_{b0} n(z') \), so that the functional dependence can be factored out of the sum,

\[
\sum_b Z_b^2 n_b(z') = n(z') \sum_b Z_b^2 n_{b0}.
\]

(A.23)

The coefficients \( n_{b0} \) are dimensionless. Thus \( C_\mu \) is defined by Eqs. (A.21b), (A.22), and (A.23). The same process gives \( C_E \),

\[
C_E = \frac{e^4 Z_a^2 \ln(\Lambda)}{4\pi m_a^2 \varepsilon_0^2} \sum_b Z_b^2 \frac{m_a}{m_b} n_{b0}.
\]

(A.24)
As previously mentioned, in the unmagnetized case, we allow \( z \) dependence, for the magnetized case, the dependence is in the \( x \) direction, and so \( z \) should be replaced with \( x \) in the preceding expressions.

**A.4.1 Unmagnetized utility**

When the pitch angle scattering frequency is much larger than the rate at which the fast ion slows down, the particle motion is diffusive on scales longer than the mean free path. This is shown formally through an expansion and averaging in \( \mu \), see, for example Kirk et al. (1994), or Albright et al. (2013). Such an approximation would be most valid for, say, protons in proton-boron fusion where the high \( Z \) and mass of boron make pitch angle scattering occur significantly faster than slowing down for protons. Its application to D-T fusion has also been discussed (Molvig et al., 2012; Albright et al., 2013). The presence of impurities in the plasma can also make the approximation more valid. To derive the fusion utility in this limit, the kinetic equation

\[
\frac{\partial f}{\partial t} + v \mu \frac{\partial f}{\partial z} = \frac{C \mu n(z)}{2v^3} \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial f}{\partial \mu} + \frac{C En(z)}{v^2} \frac{\partial^2 f}{\partial v^2} \tag{A.25}
\]

is rewritten in terms of the variable \( Z \),

\[
Z(z) = \int_0^z n(\tilde{z}) \, d\tilde{z}. \tag{A.26}
\]

Performing the expansion and averaging in \( \mu \) gives the diffusive kinetic equation

\[
\frac{1}{C En(Z)} \frac{\partial f}{\partial t} = \frac{v^5}{3C_\mu C} \frac{\partial^2 f}{\partial Z^2} + \frac{1}{v^2} \frac{\partial f}{\partial v}. \tag{A.27}
\]

The adjoint equation, as defined by Eq. A.13, rewritten in primed variables, is

\[
- \frac{1}{C En(Z')} \frac{\partial \psi}{\partial t'} = \frac{v'^5}{3C_\mu C} \frac{\partial^2 \psi}{\partial Z'^2} - \frac{1}{v'^2} \frac{\partial \psi}{\partial v'}. \tag{A.28}
\]
To get the utility $\mathcal{E}$, Eq. (A.19), integrate this equation in $t'$ from $-\infty$ to an initial condition of $W$ at $t$, per the discussion at the end of Sec. A.3,

$$\frac{C_E}{\nu'^2} \frac{\partial \mathcal{E}}{\partial \nu'} = \frac{1}{3} C_2 \frac{\partial Z}{\partial \nu'} \frac{W(Z', \nu')}{n(Z')}.$$  \hfill (A.29)

The dependence of $W$ on $Z$ is only through the density. To see this, consider, for simplicity,

$$W(Z, \nu) = E_f n_f n(Z) \nu \sigma(\nu)$$  \hfill (A.30)

which approximates the collision partners in the Maxwellian average to all be stationary with respect to the fast ion. The density of collision scattering centers is $n_f = n_f n(Z)$, typically representing a single species from the sum in Eq. (A.22). Then the solution of Eq. (A.29) on the domain $z \in [0, \infty]$ is

$$\mathcal{E}(\bar{v}', \bar{z}') = \mathcal{E}_0 \int_0^{\bar{v}'} d\bar{v} \bar{\sigma}(\bar{v}) \bar{v}^3 \text{Erf} \left( \frac{\sqrt{6}}{\sqrt{\bar{v}'^8 - \bar{v}^8}} \int_0^{\bar{z}'} \bar{n}(\bar{z}) d\bar{z} \right)$$  \hfill (A.31)

in variables where the velocity is normalized to the velocity at the peak of the fusion cross-section, the density is normalized to a mean density, the cross section to the peak cross section, and distance is normalized to a hybrid mean free path that naturally appears,

$$\bar{v} = \frac{\nu}{\nu_G},$$  \hfill (A.32a)

$$\bar{n} = \frac{n}{n_0}, \quad \bar{\sigma} = \frac{\sigma}{\sigma_0},$$  \hfill (A.32b)

$$\bar{z} = \frac{z}{\lambda_*} = \frac{z}{\nu_G/\sqrt{\nu_G \nu_E^G}}.$$  \hfill (A.32c)

The constants are grouped into $\mathcal{E}_0 = E_f \nu_G^4 \sigma_0 n_f / C_E$, which has units of energy. The superscript $G$ on the collision frequencies indicates the collision frequency at velocity $\nu_G$ and mean density $n_0$. Note that $n_f$ will also appear in the sum in $C_E$ Eq. (A.24), so that $\mathcal{E}_0$ does not scale linearly with it. The contours of the fusion utility $\mathcal{E}$, as given in

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Eq. (A.31), normalized to $E_f$ are shown in Fig. (A.1). Normalizing to $E_f$ effectively turns the utility into the lifetime probability of the particle fusing, assuming no removal of the particle after a fusion event. The limit $Z \to \infty$ (equivalently $z \to \infty$ for non-vanishing density profiles) gives the utility in the absence of a boundary. Horizontal contours in Fig. A.1 indicate the region where fast ions are fully utilized; moving the ion spatially in a region of horizontal contours has no effect on the utility. In other words, the ion does not feel the boundary. Figure A.1 shows that as $z$ approaches the boundary, there is increasing underutilization of fast ions that is characteristic of Knudsen layer effects. The
utility in Fig. A.1 is plotted using the D-T cross section (Bosch and Hale, 1992) for $\sigma(v)$ and assuming a 50/50 D-T plasma, but would have a similar structure for any cross section in these normalized units, assuming it is qualitatively similar in being peaked, and that the background plasma temperature is well below the peak so that the scattering centers can be approximated as stationary. Using a fusion reaction other than D-T would scale the utility by affecting various factors in $\mathcal{E}_0$, especially $E_f$ and $\sigma_0$. Subsequent figures also use the D-T cross section and a 50/50 D-T plasma. The fact the plasma is 50/50 D-T, instead of some other ratio, affects the distance scale through the collision frequencies in the hybrid mean free path, Eq. (A.32c), and the utility scaling, $\mathcal{E}_0$, through $n_{f0}$ and $C_E$, but not the overall structure of the utility contours. The density dependence of Eq. (A.31) makes clear an earlier assertion – that it is the total density between the fast ion’s starting position and the edge that matters in determining its utility, not the absolute distance. Furthermore, as a result of the error function, there are initially large gains in utility for adding density between a particle and the edge, which then quickly become diminishing.

Figure A.1 shows, for example, that a particle at $v' = 1$ (deuteron energy of 108 keV) has a normalized utility of 0.01, so that it is expected to produce 176 keV of energy in its lifetime as a fast particle. On the other hand, a particle that is $\sim 300$ keV hotter, and it is located at least 8 $\lambda_*$ from the boundary, has a normalized utility of 0.09, so that it produces nine times the energy, 1.58 MeV, while having approximately only four times the energy. Far from a boundary, the ratio of utility to particle energy increases rapidly up to a peak around $v' = 2$ (deuteron energy of 433 keV) and then falls off gradually. Note that, infinitely far from a boundary, the utility itself will be a strictly increasing function of velocity, which can be seen from Eq. (A.31), since the cross section is positive and the error function evaluates to one.
A.4.2 Magnetized utility

To treat the magnetized case we use a guiding center Fokker-Planck collision operator, specializing for simplicity to a uniform magnetic field in the $z$ direction (Brizard, 2004; Hirvijoki et al., 2013). In order for the fast ions to be treated by this collision operator, their cyclotron frequency $\omega_c$ must be greater than the collision frequency for fast particles, $\omega_c \gg \nu^G_\mu$. The length scale of the density variation allowed must also be larger than the gyroradius scale.

$$\frac{\partial f}{\partial t} = -v_\mu \frac{\partial f}{\partial z} + \frac{C_\mu n(x)}{2v^3} \frac{\partial}{\partial \mu} \left( 1 - \mu^2 \right) \frac{\partial f}{\partial \mu} + \frac{\mu^2 C_\mu}{v} \frac{\partial^2}{\partial x^2} [n(x)f] + \frac{C_E n(x)}{v^2} \frac{\partial f}{\partial v}. \quad (A.33)$$

The first and second terms after the equal sign are precisely those that lead to diffusive transport in the unmagnetized case, and will lead to similar transport along the magnetic field in this case. To isolate the cross field effects, we ignore the $z$ dynamics. We also average over a Maxwellian in pitch angle. This assumption of uniformity in pitch angle breaks down when near a boundary. After dropping terms and averaging out $\mu$, the kinetic equation is

$$\frac{\partial f}{\partial t} = \frac{C_\mu}{6v^2 \omega_c^2} \frac{\partial^2}{\partial x^2} (n(x)f) + \frac{C_E n(x)}{v^2} \frac{\partial f}{\partial v}. \quad (A.34)$$

for which the adjoint must be found and then integrated in time, as in Sec. A.4.1. The adjoint equation is

$$-\frac{\partial \psi}{\partial t'} = \frac{C_\mu n(x')}{6v' \omega_c^2} \frac{\partial^2 \psi}{\partial x'^2} - \frac{C_E n(x')}{v'^2} \frac{\partial \psi}{\partial v'}. \quad (A.35)$$

Integrating in time gives the lifetime utility equation

$$\frac{C_E}{v'^2} \frac{\partial \mathcal{E}}{\partial v'} = \frac{C_\mu}{6v' \omega_c^2} \frac{\partial^2 \mathcal{E}}{\partial x'^2} + \frac{W(x',v')}{n(x')} \frac{\partial f}{\partial v'}. \quad (A.36)$$
Solving this equation gives the lifetime utility function

$$\mathbb{E}(\bar{v}', \bar{z}') = \mathbb{E}_0 \int_0^{\bar{v}'} d\bar{v}' \bar{\sigma}(\bar{v}) \bar{v}'^3 \text{Erf}\left(\frac{\sqrt{3}\bar{v}'}{\sqrt{\bar{v}'^2 - \bar{v}^2}}\right).$$  \hfill (A.37)

In this case the distance coordinate is normalized to a modified fast particle gyro-radius,

$$\bar{x} = x / \rho_*$$  \hfill (A.38)

where

$$\rho_* = \sqrt{R \left( \frac{v_G}{\omega_c} \right)}$$  \hfill (A.39)

and $R$ is the ratio of collision frequencies, $R = C_\mu / C_E$. Figure A.2 shows this utility function on similar (normalized) axes as Fig. A.1 for the unmagnetized case. As might be expected, given the effects included, we can see the weaker penetration of the Knudsen layer effect, as well as the altered scaling of utility with increased velocity. In the unmagnetized case, fast ion utility decreases beyond a certain velocity, due to much higher edge loss probability outcompeting gains in fusion production. In the magnetized case, this is no longer true, and the utility increases with increasing velocity, albeit at a much slower rate near the edge than would occur with no boundary. Note that the $z$ scales in the magnetized and unmagnetized cases are very different. The observation of this decreased Knudsen penetration with magnetization is consistent with the work of Schmit et al. (2013). With the present approximations, the magnetized utility is independent of density. The much higher maximum normalized utility values in the magnetized case compared to the unmagnetized one result from full fast ion utilization in the highest velocity phase space region shown. In other words, at $3 \rho_*$ and $v' \sim 2.75$, the normalized utility of 0.15 in the magnetized case is the same value as would be achieved in the absence of a boundary. The unmagnetized case will reach this same normalized utility, at the same velocity (as it must), at a distance much greater than the maximum distance shown in Fig. A.1.
Figure A.2: Same as Fig. A.1, the normalized utility function $E_f/E_f$, but in the magnetized case. The horizontal axis scale is now $\rho_*$ instead of $\lambda_*$.

### A.5 Incremental utility calculation and reactivity loss mitigation

Figures A.1 and A.2 show that the energy produced by a fast ion in the edge region can be increased substantially by moving the ion away from the boundary. For example, Fig. A.1 shows that moving an ion with velocity $v' = 1.5$ by one $\lambda_*$ away from the boundary, from $Z' \sim 1$ to $Z' \sim 2$, increases the expected fusion probability by more than 50% (from $\sim 0.03$ to $\sim 0.05$), and therefore also the expected energy production. All the while, the particle energy remains the same. While the numbers are different for the magnetized case, the effect is clear. In the magnetized case, the magnetic field can link wave pushes in energy to
pushes in space, making it possible to move hot ions away from the boundary and increase their utility. Here we calculate the fusion gains possible from such pushing.

Using the magnetized fusion utility, Eq. (A.37), and simplifying the moment Eq. (A.15) under the same set of assumptions, we can write the expected change in fusion power production as a result of wave induced fluxes in space and velocity,

$$\mathcal{E} = \int d\mathbf{v} \int d\mathbf{x} \ W(\mathbf{v}, \mathbf{x}) f_M(x, \mathbf{v}, \mathbf{x}) = \int d\mathbf{v}' \int d\mathbf{x}' \frac{\partial \mathcal{E}}{\partial \mathbf{x}' \cdot \mathbf{v}} + \frac{\partial \mathcal{E}}{\partial \mathbf{v}' \cdot \mathbf{v}} \cdot \Gamma. \quad \text{(A.40)}$$

To isolate the impact of the waves in phase space, consider localized fluxes

$$\Gamma_{x,v} = \Gamma_{x,v0} \delta(x' - x_0) \delta(v' - v_0). \quad \text{(A.41)}$$

If pushing the fast ion to a new point in phase space produces more net fusion energy over its lifetime than energy required to push, there will be a net gain in energy. This gain is defined by

$$g(x_0, v_0) = \frac{\mathcal{E}}{\epsilon_\Delta} \quad \text{(A.42)}$$

where $\epsilon_\Delta$ is the energy absorbed by the ion during the push,

$$\epsilon_\Delta = \int dx' \int dv' \Gamma_{v0} \cdot \frac{d(mv'^2/2)}{dv'}. \quad \text{(A.43)}$$

The gain for the magnetized case is then

$$g(z_0, v_0) = \left( \frac{\partial \mathcal{E}}{\partial x_0} \Gamma_{x0} + \frac{\partial \mathcal{E}}{\partial v_0} \Gamma_{v0} \right) / (mv_0) = \frac{E_0}{mv_0^3} \left( \bar{\sigma} \bar{v}^3_0 \right) \right.$$

$$+ 2 \sqrt{\frac{3}{\pi}} \int_0^{v_0} \frac{d\bar{v}}{\sqrt{\bar{v}^2_0 - \bar{v}^2}} \exp \left(- \frac{3\bar{z}^2_0}{\bar{v}^2_0 - \bar{v}^2} \right) \left[ \frac{d\bar{z}_0}{d\bar{v}_0} - \frac{\bar{x}_0 \bar{v}_0}{\bar{v}^2_0 - \bar{v}^2} \right]. \quad \text{(A.44)}$$
The first and third terms in Eq. (A.44) occur due to changes in utility with changing velocity, the second term (first in the square brackets) occurs due to changes in utility with changing position. In writing this expression, we have made the replacement

$$\Gamma_{x0} = \frac{dx_0}{dv_0} \Gamma_{v0}$$

(A.45)

without loss of generality. The flux $\Gamma_{v0}$ is set to 1 so that $g$ represents the single particle gain, which is useful to see for gaining intuition. The amount of gain in the edge region depends heavily on the factor $d\bar{x}_0/d\bar{v}_0$, which represents the amount of change in spatial position a wave can impart for a given velocity change. For an ion gyro-orbiting a $z$ directed magnetic field, and a wave directed in the $y$ direction, we can write a simple resonance condition as $\omega - kyv_y = 0$. If the wave imparts a velocity kick $dv_y$ and a corresponding energy change $mv_ydv_y$, then the change in guiding center for the particle is

$$\frac{d\bar{x}_{gc}}{dv_0} = -\frac{v_0ky}{\omega}.$$  

(A.46)

Equation A.46 shows that the amount of change in position for a given velocity change is in large part determined by the wave properties. Figure A.3 shows the gain, Eq. (A.44), plotted for $d\bar{x}_0/d\bar{v}_0 = 2$. In this case, the gains may be quite high in the region where Knudsen effects are prominent. For ions pushed over a non-infinitesimal path through phase space, the gain would be averaged along the path. In Fig. A.3 this path is constrained to be a line of slope $1/2$, since the gain contours are calculated assuming $d\bar{x}_0/d\bar{v}_0 = 2$.

As before, regions of horizontal contours indicate where Knudsen effects cease to have an impact. Larger values of $d\bar{x}_0/d\bar{v}_0$ will result in even larger gains, but these gains won’t extend into regions of horizontal contours. For example, Knudsen impacted gains for particles less than $1.5v_G$ (or equivalently 244 keV deuteron energy in the D-T plasma considered here) have a maximum extent of approximately $1\rho_s$ from the boundary. Far from the boundary, pushing in space has no effect, so that the gains are due purely to a baseline
gain from pushing in velocity. This baseline is given by the first term in Eq. (A.44). In the present case, the baseline has a maximum gain of approximately $49.2/\ln(\Lambda)$ for fast ions near the peak of the fusion cross section. Gains here are calculated using the full 17.6 MeV fusion energy for $E_f$, and must be scaled down accordingly if one is only interested in the 3.5 MeV alpha particle energy. Figure A.3 uses $\ln(\Lambda) = 8$. A different Coulomb logarithm value would, again, affect the scaling of the figure but not the structure. Having large gains requires being able to find a wave with the right properties (e.g. phase velocity, wavenumber) in the edge region of the ICF plasma.

The gains calculated here assume that the only increase in fusion energy production as a result of the injected energy is that generated by increases in the ion chance of fusion. However, the effective gains may be increased by the fact that the some portion of the injected energy will be transferred by collisions from the ion to other plasma particles, heating them. Energy transferred from the fast ion that helps generate other fast ions would most increase the gain in fusion reactions, but any energy going into ion heating is useful.

### A.6 Discussion

Ultimately, the usefulness of the scheme for restoring fusion reactivity lost to the deleterious effects of the Knudsen layer depends on two factors: the efficiency with which reactivity can be restored, and the total amount of reactivity that can be restored compared to the entire hot spot reactivity. With the right wave, the theoretical single particle efficiency may be high. Pushing many fast particles from the edge region towards the interior may result in a lower individual efficiency, since such pushing requires diffusion paths in phase space - once a particle has been pushed inwards to a new position in the phase space, it raises the phase space density there, eventually making it infeasible to push to the same location. Since the distribution function drops off quite rapidly as a function of velocity in the region of velocity space occupied by fast particles, pushing particles more in velocity
Figure A.3: Gain function $g$ in near boundary region, giving the multiplier between input energy and increase of expected fusion energy output, when a fast ion at $x_0, v_0$ is pushed incrementally in space and velocity. Axis scales are the same as discussed in Fig. A.1, but with a horizontal scale of $\rho_*$ instead of $\lambda_*$. Gain values are the same for both deuterons and tritons.

for a given spatial push opens up more phase space, but lowers the efficiency. Tackling the global efficiency of the scheme in a dynamic situation is a challenging problem.

The second factor depends largely on the design of magnetized ICF experiments. The larger the fraction of the burning plasma volume that is subject to the depletion of fast ions due to edge loss, the more theoretically useful the mitigation scheme. The point design for magnetized liner experiments and recent magnetized OMEGA implosions are not expected to suffer substantial Knudsen related losses (Schmit et al., 2013). However, it is possible that future magnetized ICF experiments may be in a regime where there is some level of magnetized Knudsen edge loss. Unforeseen kinetic or dynamical effects may also cause more ion loss than currently expected. The mitigation strategy presented here should re-
main relevant for more inclusive physics models of edge ion loss, so long as the loss is kinetic in nature. While no mitigation may be needed, it is reassuring that the more necessary it is the more theoretically efficient it may become - when the utility decreases rapidly near the edge, large restorations can be had for small spatial pushes.

Note that neither utility function derived here is expected to be accurate immediately near the boundary. This is due to the breakdown of underlying assumptions in each model at the boundary, particularly the lack of dependence of the distribution function on $\mu$, which is not sensible for an absorbing boundary but underlies the diffusive approximation. This is a well known problem. Albright et al. (2013) have demonstrated the implementation of an improved boundary condition for the unmagnetized diffusive model used here in the Knudsen layer context. Improvements for the unmagnetized case, beyond the diffusive model used here, have been discussed by Tang et al. (2014a) and McDevitt et al. (2014). For simplicity of demonstrating the technique and ideas, we have used a zero boundary condition and diffusive approximations.

The linearization in Eq. (A.7) is not strictly valid near boundaries, where past work (Molvig et al., 2012; Albright et al., 2013; Schmit et al., 2013) has indicated the excursion from Maxwellian in the tail of the distribution function can be rather large. This affects the validity of equations involving $\chi$, like the adjoint moment equation for the differential fusion energy production, Eq. (A.15). However, the utility, given by Eq. (A.19), is relatively insensitive to the background distribution so that it is still well defined and valid within the approximations used in its calculation. The utility is in essence a single particle calculation that helps determine what we can say without calculating the actual distribution function.

The high velocity approximation discussed in Sec. A.4 can be accurate for the calculation of the utility of very fast particles because the vast majority of the expected lifetime fusion energy created by a particle that starts fast will occur while it is fast. Even if we fail to accurately capture the particle dynamics when it starts getting closer to thermal speed
(for example, velocity diffusion which starts to kick in), the contribution to the fusion utility is negligible there, so we will still get a reasonable estimate of the expected fusion energy production. Note that this is true since we are considering a finite ‘lifetime’, i.e., there is some velocity, say the thermal velocity or some substantial fraction of it, below which we stop tracking the particle. This means we are not treating the circumstance when a fast ion has slowed down to nearly zero velocity and is then jostled back into being a fast ion. This circumstance does not matter when calculating the incremental energy production due to an initial energy or spatial push, since the ion loses memory of the push after it slows down. However, the finite lifetime may limit the applicability of the utility functions given here for other problems. This is not a fundamental limitation of the utility function formulation, but rather of the present approximations.

For the approximately 10 keV operating temperature targeted in typical ICF experiments, the Gamow peak in a 50-50 DT plasma is located at approximately 3 times the thermal energy (less than $2V_T$). The Gamow peak gives the particle energy value where the maximum fusion production occurs, when both the fusion cross section and the number of particles at each energy in a Maxwellian distribution are taken into account. The high velocity approximation means utility values for these particles will not be quantitatively accurate, although trends in the utility at these lower velocities can still be correct. As the temperature considered decreases, the broad Gamow peak will start to contain more and more high (normalized to $V_T$) velocity particles. Then the high velocity approximation will yield increasingly accurate utility results. (For reference, far from the boundary, Monte Carlo simulations indicate that the high velocity utility for a particle starting at $4V_T$ is off by $\sim 10\%$ compared to a utility calculated with the next order velocity diffusion term included.) Of course, a more accurate calculation of the utility function could be obtained by avoiding the high velocity approximation and solving instead the complete adjoint equation.
The utility function and adjoint approach could be applied to calculate a total reactivity reduction, which past Knudsen layer work has focused on (Molvig et al., 2012; Albright et al., 2013; Schmit et al., 2013; Tang et al., 2014c,a). However, a full consideration of the relative benefits of different approaches for calculating reactivity reduction is beyond the scope of this present work.

The adjoint approach for the utility function can be systematically generalized to increasingly complex situations. More general moment equations than Eq. (A.15) can be written, allowing for more complicated boundary conditions. One could include other effects not considered here, such as electric fields or more complex collisional dynamics. The adjoint formulation can also be expanded to include a time evolving background (Fisch, 1986).

While the examples given in this work could be made quantitatively more accurate, the approach should be useful as more inclusive and accurate pictures of ion kinetic physics in ICF implosions are developed. With a simple application, it has given us insight into the density dependence of magnetized and unmagnetized Knudsen dynamics, as well as a basic evaluation of a scheme for combating Knudsen losses.
Bibliography

http://dedalus-project.org/.


