Reduced Quasilinear Models for Energetic Particles Interaction with Alfvénic Eigenmodes

Katy Ghantous

A Dissertation
Presented to the Faculty of Princeton University in Candidacy for the Degree of Doctor of Philosophy

Recommended for Acceptance by the Program in Plasma Physics Astrophysical Sciences
Adviser: Nikolai Gorelenkov

November 2013
Abstract

The Line Broadened Quasilinear (LBQ) and the 1.5D reduced models are able to predict the effect of Alfvénic eigenmodes’ interaction with energetic particles in burning plasmas. This interaction can result in energetic-particle losses that can damage the first wall, deteriorate the plasma performance, and even prevent ignition.

The 1.5D model assumes a broad spectrum of overlapping modes and, based on analytic expressions for the growth and damping rates, calculates the pressure profiles that the energetic particles relax to upon interacting with the modes. 1.5D is validated with DIII-D experiments and predicted neutron losses consistent with observation [1, 2]. The model is employed to predict alpha-particle fusion-product losses in a large-scale operational parameter-space for burning plasmas.

The LBQ model captures the interaction both in the regime of isolated modes as well as in the conventional regime of overlapping modes. Rules were established [3] that allow quasilinear equations to replicate the expected steady-state saturation levels of isolated modes. The fitting formula is improved and the model is benchmarked with a Vlasov code, BOT [4]. The saturation levels are accurately predicted and the mode evolution is well-replicated in the case of steady-state evolution where the collisions are high enough that coherent structures do not form. When the collisionality is low, oscillatory behavior can occur. LBQ can also exhibit non-steady behavior, but the onset of oscillations occurs for much higher collisional rates in BOT than in LBQ. For certain parameters of low collisionality, hole-clump creation and frequency chirping can occur which are not captured by the LBQ model. Also, there are cases of non-steady evolution without chirping which is possible for LBQ to study. However the results are inconclusive since the periods and amplitudes of the oscillations in the mode evolution are not well-replicated.
If multiple modes exist, they can grow to the point of overlap which results in two-dimensional diffusion with cross terms. A diffusion scheme is proposed and validated to resolve this dynamics in \((P_\phi, E)\) phase-space.
First and foremost, I would like to thank my advisors, Herbert Berk and Nikolai Gorelenkov for making this work possible. Thank you Nikolai for making me feel like a colleague while all the same being the mentor I have learned so much from. I have acquired valuable skills that make me a better researcher. You have always emphasized communication and I have learned so much from your comments and advice on everything I have produced, written, and presented. You were so patient with me as I learned to run NOVA and were readily available at all times to discuss physics. I appreciate all the opportunities you have provided to attend meetings and present at conferences, and for your support for traveling to Austin to work with Herb. Thank you for pushing me in the directions needed for this work to be completed.

Herb, thank you for hosting me at UT Austin as your student and making it feel like my own university. My frequent visits to Austin were the bursts of productivity without which this dissertations would not have been possible. I have learned more than I thought possible to absorb in the meetings that would go for hours every day for weeks. Although we are in different parts of the country for most of the time, you have always been available to talk physics at any time of the day and on all days. It felt like I have learned more physics over the phone than in classrooms.

My first year in Princeton, I was strongly encouraged to pick up an experimental project despite my resistance to the idea. When I told the program director my interests, he recommended I talk to Sam Cohen. After a five minute chat, I knew he will be my advisor to this project. I have come to be less fearful of machines and managed to not break anything, but more importantly I learned so much from Sam. Thank you Sam for trusting me with the FRC. Working on it with you has allowed me to understand the experimental aspect of things in ways I never expected I could. You were always encouraging and your faith in your students takes them
far. You have been a pivotal point in my journey. I particularly would like to thank you for introducing me to Sasha Landsman and for giving me the flexibility to work on whatever problem I choose. This led me to work out, with Sasha, a model for the particle interaction with the RMF whose basics is revisited in this dissertation. Sasha, it was a pleasure working with you and thank you for spending the time talking with me about Hamiltonians, the discussions are certainly at the roots of most of this work.

I would like to thank my readers, John Krommes and Eric Fredrickson for giving such insightful and useful suggestions for the dissertation. It would not have been in the form it is if not for your many comments.

PPPL would not be the same without our graduate administrator Barbara Sarfaty who from the very first day took us in and made everything seem so easy and manageable. Thank you Barbara for making PPPL a friendly space that felt like home.

I tend to be late on everything, and this does not exclude books I rent out. Thank you Judy for always being there to help out. It has been fun chatting with you over the years.

Some mentors are mentors for life. I would like to thank my friend and mentor Jihad Touma from whom I have learned the beauty in physics and being human and continue to be inspired from. Throughout the years, your encouraging and wise advise have given me the confidence and motivation to finalize this work. You build such an elegant picture of the world that equipped me with the tools to maneuver around the many obstacles life likes to throw at us.

Princeton would not have been the experience it is if not for the friendships I have built and fostered.

Loubna, you have come to know me so well as the years go by that I find myself
going to you for rewiring my head every time fuses pop. You are always there, be it at Princeton or Beirut, for talking about the world and humans and helping me make sense of things often difficult to make sense of. Your honestly and clarity have been at many times a much needed refuge from a world drenched in pretense, expectations, conventions, and propriety. I am so glad to have met you at Princeton and to have such a valuable human being in my life.

Tanya, Lyosha, Nick and Daya, there was a time I felt that I didn’t want to finish my PhD so I don’t lose the world we have created for ourselves. As neighbors and friends, we were able to build a community and a life style that I would have loved to keep forever. We have shared so many rich experiences, summer BBQs, and long nights hanging out on the front yard. You have made Princeton truly feel like home. Lyosha thank you for always being there when needed, you have been a true friend over the years.

I am grateful for making close friendships with Kerstin and Tanya. At non-overlapping periods during my stay in Princeton, each of you have been my closest friend with whom I have shared many crazy and fun times as well as long walks and talks. I have learned from both of you discipline and kindness, and you reminded me of what is important in life.

I am grateful for the new friendships I made with Chantal and Sandra that were kept alive long after they graduated, and the old friendships with Naima and Sara that have been strengthened in Princeton.

Some of the greatest benefits of being in a place like Princeton are the interactions and connections we make with people. Many have touched my life in one way or another and I have come out of my experience here with not just a PhD, but a stronger mind set, a higher appreciation of all that surrounds, and beautiful memories of long nights talking and laughing. Princeton and its people have made me feel empowered and have provided me with a sense that anything is possible. That theory
was validated when, with the support of friends at Princeton, I ended up earning my pilot’s license at Princeton airport that have added a new dimension (literally and metaphorically) to my life.

They are an ocean apart, but my family remains a relentless support without which the ebbs and flows of life could sometimes be crippling. Thank you Edward and Gina for giving me the perfect balance and for building a solid ground upon which my life unravelled. Thank you for providing a life so sweet and simple that made me, Tarek, Mariam and Joseph truly happy people, and thank you for making us believe that we can pursue anything we dream up. I couldn’t even imagine parents that can be more perfect for me. If I trace back anything I have accomplished in my life to its roots, I find you. This dissertation is no different, but you are not only at the roots. You stood by me throughout the whole six years, helped me trim the branches, cut the weed, and waited patiently for the blossoms. You did all that with no expectations, instead you have always emphasized the importance of my happiness at the expense of everything else. Without this kind of support, I could not imagine having competed this work.

Tarek, thank you for showing by example what hard work and perseverance can lead to, for being the friend that distance and time could not shake, for being the one person I shared my childhood with; and in adulthood, for never failing to remind me of who I am, what I value, and what I have always wanted. Joseph, my only teenage friend, you have made life so sweet since the day you were born. Thank you for making my visits to Amyoun so much fun and hanging out with me while everyone else was at work. Thank you for being genuinely interested in what I am working on and actually wanting to learn about it. I can’t wait to see the man you grow up to be. Your pride in me means more to me than you think.

Thank you Mariam for being my best friend and confidant, and making me smile
every time I remember your beautiful face. You were there to tolerate my venting and my excitement, you were always and still are a strong support to lean on. You believe in me in ways I can not believe in myself. You make me feel that I can put a cape on and fly. I am very fortunate to have a friend-for-life like you. We have done so much together and shared so many stories and philosophies that I can not imagine how life would be have you not been an ever-present part of it.

Dan, my family resides across the ocean, but you were my family here. You have like no other brought out the person I am. You accepted me with all my crazy, sporadic whims and cherished it. You have comforted me during the tough times and made the good times exuberant. I can not be more grateful for all the support you have given throughout the years trying to complete this work. Thank you for hearing out all my talks rehearsals and proofreading my work. Thank you for your unwavering faith in my abilities. Your creativity and appreciation of all things simple and beautiful colored my days and made things worthwhile. I can not imagine Princeton have you not been a huge part of my journey.
To Edward and Gina
# Line Broadened Quasilinear Model

## Introduction

## Line Broadening Model

## Width of the Broadening

### Ideal Plasma

### Collisionality

### Modeling the Width $\Delta \Omega$

## Parametric Dependencies of $\Delta \Omega$

### Analytic Results

### LBQ1D

### LBQ2D

## LBQ Dynamics

### Steady-State Solutions (High Collisionality)

### Non Steady-State Solutions (Low Collisionality)

## Benchmarking LBQ1D with Vlasov Codes

### Vlasov’s Codes

### Results of the Benchmark

### BOT vs LBQ parameter space

## Regime of Applicability of LBQ

## Conclusion

# Numerical Schemes for LBQ

## Introduction

## LBQ1D

### Diffusion Equation

### Mode Evolution

### Conservation Properties

## LBQ2D

## Conclusion
List of Figures

1.1 (a) A gap forming in the Alfvénic frequency continuum as a result of coupling to remove the degeneracy, (b) the poloidal harmonics. Adapted from C. Z. Cheng et al. Ann. Phys., 161(21), 1985.

3.1 Depiction of the resonant and non-resonant regions of \((v_\parallel, v_\perp)\) phase space for finding the contribution of the velocity to the redistribution of energetic particles.

3.2 An illustration for the integration scheme of 1.5D. (a) The values for the original gradient in pressure profile, (blue) and the critical gradient, (red). \([r_-, r_+]\) is the original region of instability where \(\partial \beta / \partial r > \partial \beta / \partial r|_{\text{crit.}}\). (b) The original pressure profile (blue) and the relaxed profile (red). The redistribution has extended into the region \([r_1, r_2]\), which is larger than the originally unstable region \([r_1, r_2]\).

3.3 Radial profile of ECE radiometer power spectrum depicting the observed TAE and RSAE activity in DIII-D discharge # 122117 at \(t = 410\) ms. The solid line is the safety factor profile, \(q\). These measurements depict the existence of AEs around the minimum of \(q\) and extending to \(r/a = 0.8\). Adapted from M. A. Van Zeeland, Phys. Rev. Lett., 97, 135001-2, 2006. Copyright (2006) American Physical Society.
3.4 A depiction of the unperturbed profiles from TRANSP in dashed lines of different colors for the times, $t = 360\, ms$, $t = 780\, ms$, and $t = 1200\, ms$. The profiles are modified due to interaction with TAE activity depicted in Fig. 3.3 and the FIDA measurements are presented as dots while the MSE fit is in solid lines. Adapted from W. W. Heidbrink et al., Nucl. Fusion, 48, 6 (2008). Reprinted with permission of IOP Publishing Ltd.

3.5 (a) FILD activity signifying coherent NBI loss peaking around $t = 500\, ms$ and disappearing after $t = 800\, ms$. (b) AE activity measured throughout the discharge in the spectrum of Mirnov and CO$_2$ interferometer data. Adapted from M. A. Van Zeeland, Phys. Plasmas, 18, 056114-5, (2011). Copyright (2011) American Institute of Physics.

3.6 The structure of modes localized around $r = 0.32$ and $r = 0.78$ for shot #122117 at $t = 360\, ms$ as computed by NOVA for $n = 3$. The modes are computed as a function of $\sqrt{\psi/\psi_0}$, the square root of the normalized poloidal field flux. The dotted line is the safety factor profile.

3.7 The structure of modes localized around $r = 0.25$ and $r = 0.75$ for shot #142111 at $t = 675\, ms$ as computed by NOVA for $n = 4$. The dotted line is the safety factor profile.

3.8 Redistribution of energetic particles at $t = 360\, ms$ results in significant EP losses and associated neutron losses. At later times, AE activity results in the redistribution of particles without EP losses.
3.9 The time evolution of the ratio of the measured neutron emission to TRANSP computed classical rate is represented by a solid line, adapted from W. W. Heidbrink et al., Nucl. Fusion, 48, 6, (2008). Reprinted with permission of IOP Publishing Ltd. Juxtaposing the experimental results, the emission rates calculated by the 1.5D model are represented with colored crosses at the times of interest, $t = 360$ ms (red), 780 ms (green) and 1200 ms (blue).  

3.10 Relaxation of EP distribution for different time slots. EP losses peak at around $t = 525$ ms while AE activity is evident throughout. At $t = 425$ ms and at $t = 975$ ms, TAE activity only results in redistribution of EP without resulting losses.  

3.11 TRANSP predicted (red) and experimentally measured (blue) neutron emission rate as a function of time adapted from M. A. Van Zeeland, Phys. Plasmas, 18, 056114, (2011). Copyright (2011) American Institute of Physics. Green dots are added to the figure as markers for the 1.5D results for neutron rates at $t = 425, 525, 675, 725, 800$, and $975$ ms.  

3.12 Comparison of 1.5D results to the DIII-D measurements at six different times of the discharge. The solid lines are the initial profiles and the dashed lines are the resulting fast-ion profiles that the 1.5D model calculates at the different times.  

3.13 The mode structure of TAE modes are computed by NOVA code as a function of $XG = \sqrt{\psi/\psi_0}$ where $\psi$ is the normalized poloidal flux. The left figure is the mode with $(n = 9, \omega = 0.89)$ localized around $r/a = 0.4$ and to the right, the mode $(n = 5, \omega = 0.89)$ localized around $r/a = 0.6$.  

xvii
3.14 A depiction of the loss fraction in $(\beta_p, T_i)$ space. The light green region of parameter space bounded by a black line is where TAE modes are stable. The space depicted in cyan is where TAE modes are destabilized, however $\alpha$ particles are only redistributed without being lost. Beyond the yellow line, losses start incurring, where the percentage of lost particles corresponds to that in the color bar.

4.1 (a) The resulting distribution function $f(\zeta, \Omega)$ in $(\zeta, \Omega(I))$ phase space as a result in interacting with the mode resonant at $\Omega(I_0) = \omega$. (b) The resulting distribution function $f(\Omega)$ averaged over the angle overlaid with the resulting distribution function under LBQ diffusion with a choice $\sigma = 1.5$.

4.2 A schematic for the resonant curve in $(P_\phi, E)$ space in solid blue line. The direction of diffusion along $E - \omega P_\phi/n = \text{cst}$ is represented in a dotted black line (−) and a few resonant islands in red. Note that the width of the island, $3.3 \omega_b$, varies along the resonant curve since $\omega_b$ depends on the Fourier component, $V_l(J)$.

4.3 The resulting saturated $\omega_b(\gamma_d/\gamma_{L0})^{1/3}/\nu_{\text{eff}}$ for 1D counterpart and the analytically expected behavior in the limiting cases of near marginal stability (NMS), − − −, and far form marginal stability (FMS), − − −, as derived from 1D.

4.4 The saturation levels, −, as predicted by Eq. (4.36) and the results of the code, *, for various values of $\gamma_d/\gamma_L$. A choice of $\nu_{\text{eff}}/\gamma_{L0} = 3$ is used to guarantee steady-state saturation.

4.5 The change in time evolution of $\omega_b/\gamma_{L0}$ as $\gamma_d/\gamma_{L0}$ is lowered for $\nu_{\text{eff}}/\gamma_{L0} = 0.18$. The onset of relaxation oscillation can be seen.

xviii
4.6 The evolution of the mode amplitude as compared to the evolution of the slope in the distribution function. Both are scaled to their maximum value to best depict the predator-prey behavior. The figure is zoomed into the region where the oscillations have been long established since the initial setup is not interesting.

4.7 The mode evolution for four values of $\gamma_d$ with $\nu_{\text{eff}} = 0.18 \gamma_L$ that depict the transition to different behaviors. Values of $\gamma_d/\gamma_L$ 0.3 and 0.17 are relaxation oscillations. However, for lower damping rates another bifurcation occurs where the mode amplitude is quiescent for some time before it oscillates again more harmonically such as the evolution for $\gamma_d/\gamma_L = 0.035$ presented. For even lower $\gamma_d/\gamma_L$, the LBQ equations undergo another bifurcation where the mode again evolves to a steady state such as the behavior for $\gamma_d/\gamma_L = 0.025$ presented above.

4.8 For a choice of $\nu_{\text{eff}}/\gamma_{L0} = 0.18$, LBQ is run for a range of $\gamma_d/\gamma_{L0}$ to study the variation in amplitude and period of oscillations. The top left figure depicts the period of the oscillations for the different runs and the top right figure depicts their respective amplitudes.

4.9 The resulting amplitude of the bursts in the evolution of $\omega_b/\gamma_L$ in the $(\gamma_d/\gamma_{L0}, \nu_{\text{eff}}/\gamma_{L0})$ parameter space. As $\nu_{\text{eff}}/\gamma_{L0}$ decreases for a given $\gamma_d/\gamma_{L0}$ the steady state become oscillatory. However for $\gamma_d/\gamma_{L0} < 0.1$, smaller values of $\nu_{\text{eff}}/\gamma_{L0}$ is required to maintain bursts. Otherwise, the amplitude will saturate at a steady level after some initial overshoot.

4.10 The resulting saturated $\omega_b(\gamma_d/\gamma_{L0})^{1/3}/\nu_{\text{eff}}$ for choice of $p = 2$, $p = 4$ as compared to BOT simulations and interpolation formula used by Petviashvili.

4.11 The saturation levels as predicted by BOT and LBQ1D for different choices of $p = 4$ in the broadening $\Delta \Omega = [(2 \sigma \omega_b)^p + (2 \lambda \nu_{\text{eff}})^p]^{1/p}$.
4.12 Oscillatory behavior of LBQ as compared with BOT in some examples
where the mode frequency does not chirp. .......................... 93

4.13 The parameter space for the oscillations resulting from Eq. (4.37)
solved by BOT is different from that resulting from Eq. (4.4) solved by
LBQ. The BOT regions of oscillations is for all parameters below the
solid blue line, while LBQ region of oscillations is below the green line
with crosses. ................................................................. 95

5.1 At every time step, one first evaluates the distribution function at
\( t + dt \) using the current diffusion coefficients, then evolve the mode amplitude
using the current growth rate. The instantaneous linear growth rate
\( \gamma \) is evaluated to compute the broadening width using the updated \( \omega_b \)
and \( \gamma \). This completes the cycle where all the updated values are used
to evaluate the diffusion coefficients used in the first step to evolve the
distribution function. ................................................. 100

5.2 The figure to the left depicts the solution of \( f \) at \( t = 100\Delta t, 200\Delta t, 
300\Delta t, \) and \( 400\Delta t \) using the diffusion scheme vs the analytical results.
To the right, the difference between these two results normalized to
the value of \( f \) at \( x = 0 \) is depicted. The parameters \( D = 1, dx = 10^{-2}, 
\) and \( \Delta t = 10^{-4} \) are chosen. .................................................. 103

5.3 The a top view of the mesh in \( (P_\phi, E) \) space for a rotation of slope
\( S = 1. \) ................................................................. 107

5.4 (a) An arbitrary distribution function in \( (P_\phi, E) \) space with a section
along \( E' = E - \omega_0 P_\phi / n_0 = 0 \) singled out. (b) \( f_{i,j} \) along the \( E' = 0 \) as
a function of \( P_i \). ......................................................... 108

5.5 (a) The difference scheme used to solve a fully two-dimensional diffu-
sion problem, and (b) the associated banded matrix that needs to be
inverted. ................................................................. 108
5.6 As the time step is taken smaller the time evolution of the mode converges. For any further reduction in the time step, the resulting time evolution is indistinguishable from that for $dt = 0.1/\gamma_L$.

5.7 The distribution function $f(P_\phi)$ at $E = 0$ when the mode reaches saturation for three different choice of number of grids. The solution converges as the grid number increases; beyond $N = 60$ the change is negligible.

5.8 The number of iterations needed to achieve convergence for a given slope of $S = 0.4$.

5.9 The number of iterations for $S = 0.4$ is compared to $S = 0.3$, $S = 0.2$, and $S = 0.1$.

5.10 The diffusion coefficient computed for the two choices of mesh. The figure to the left depicts the diffusion coefficients as computed by the mesh aligned with the diffusion path (blue) and the cartesian mesh at which the diffusion path is at an angle (cyan). The figure to the right depicts the area of nonzero $F$ that is outlined by the envelope of the broadening. One sees the effect of the slope since the function $F$ is symmetric with respect to the resonant curve, $P = 0.5$, along the path of diffusion.

5.11 A comparison of the results of the fully implicit scheme to that of the IMP-EXP scheme that solves the two-dimensional diffusion equation. The modes have a diffusion path of slope $S = 0.4$ in the $(P, E)$ frame. The results are shown at different angles. The red dots present the result for the fully implicit scheme using a slanted mesh while the blue dots are that of the IMP-EXP scheme using a Cartesian mesh.
5.12 The evolution of the bounce frequency using the IMP-EXP scheme (blue) vs the fully implicit scheme (red) made possible by rotating the mesh into the direction of diffusion. 118

5.13 The evolution of the distribution function as it flattens due to its interaction with the modes. The solution is at times $\tau = 2.7$, $\tau = 5.4$, $\tau = 8.1$, and $\tau = 10.7$. 119

5.14 The evolution of the amplitude of the two modes when they are overlapped (solid) as compared to their evolution in case they do not overlap. The discrepancy is accentuated after the modes overlap at $\tau = 3$. 119

5.15 The region of nonzero $F$ for both modes are depicted and one can see the effect of the slope on the shape of the broadening. The figure is superimposed with lines that present the paths of diffusion. 120

5.16 The evolution of the distribution function as the modes overlap briefly before resorting to being isolated. 121

5.17 Comparison of the resulting evolution of the mode amplitude in the case of the modes overlapping (solid) or if the same modes are isolated such that the other mode doesn’t exist (dash). 122
Chapter 1

Introduction

For over 50 years, scientists have been researching ways to harness fusion power to provide a clean source of energy for civilian use. It started with the discovery made by Atkinson and Houterman in the late 1920s that two hydrogen atoms are heavier than a helium atom. It was soon understood that fusion is what powers the sun and stars [5].

In 1934, Rutherford, Oliphant and Harteck succeeded in fusing two deuterium atoms to produce either $^3\text{He}$ and a neutron or tritium and a proton and releasing energy in either case. This marks the beginning of research in fusion. While it started out as classified work under the Manhattan project for developing thermonuclear weapons, scientists soon became interested in using fusion as a source of energy that powers up cities. By 1951 the first fusion device was claimed (albeit falsely) by the Argentinian dictator Juan Peron. However it catapulted Lyman Spitzer’s studies and subsequently led him to establish the Princeton Plasma Physics Laboratory that pioneered research in the new science of plasma physics and in concepts of magnetic confinement devices for fusion.

Fusion occurs when atomic nuclei gain energy enough to overcome their nuclear repelling potential in order to fuse and give rise to a new element. For this to occur, a gas needs to be heated to very high temperatures of the order of millions of degrees
Celsius. This is necessary to first separate the electrons from the nuclei of the atoms creating an ionized mixture called a plasma. Once fully ionized, a high enough density of the plasma needs to be confined long enough for the possibility of a collision to occur that would result in a fusion reaction. The energy confinement time $\tau_E$, which is the ratio of the energy stored to the rate of energy loss, is required to satisfy the Lawson \cite{6} criterion

$$n\tau_E > 1.5 \times 10^{20} \text{ m}^{-3}\text{s} \quad (1.1)$$

where $n$ is the density of the plasma.

Fusion is the mechanism that powers stars. While the gravitational force of stars confines the plasma long enough for fusion to occur, alternative approaches are needed in the laboratory. Two main approaches have been the inertial and magnetic confinement. In magnetically confined plasmas, the system studied in this thesis, magnetic fields confine the plasma long enough for the nuclear fusion reaction to take place.

In 1951, I. Y. Tamm and A. Sakharov proposed what is known as a tokamak, which utilizes toroidal closed field lines with a weak poloidal field to confine the plasma. Other magnetic confinement concepts have also been studied theoretically and experimentally such as the compact spherical tokamak, stellarator, FRC, etc., that are built in nationally funded labs (such as PPPL, LANL, Lawrence Livermore, ORNL, Culham) and privately funded labs (such as TriAlpha). While there have been significant advances in all these devices towards understanding the physics of the plasmas and resolving some of the issues, the most promising results to date were observed on tokamaks.

At this stage of the fusion research program, it is time for a burning plasma experiment. It will be a test bed for studying the necessary physics to build the next step demonstration plant, DEMO, which will be connected to the power grid and is projected to be in the construction stage by 2050 \cite{7}. A burning (or ignited) plasma is predominantly heated by its own fusion products to perpetuate the reaction.
In comparison to other concepts, the maturity of research on tokamaks and the promising results reported in JET, TFTR, and JT-60U, led the fusion community to decide on the tokamak design for the next-generation burning plasma experiment, ITER.

ITER is an internationally funded experiment that began construction in Cadarache France in 2010 and is expected to have first plasma in 2020. It will operate a burning plasma, in a self-sustained manner where the alpha particles fusion products will be the main source of heating. ITER is expected to produce up to 500 MW of power from 50 MW input for periods of 300 to 500 seconds and ultimately operate in a steady state thousands of seconds long.

ITER will use a deuterium-tritium (DT) fuel since it is the most efficient for nuclear reactions in tokamaks. A DT reaction is

\[
D + T \rightarrow 4\text{He}(3.52 \text{ MeV}) + n(14.06 \text{ MeV})
\]  

which produces alpha particles and neutrons. Compared to other fuel choices, DT fuel is optimal for burning plasmas since it has a higher reaction rate that peaks at a relatively lower temperature. Additionally, deuterium is abundant and while tritium is not as abundant as deuterium, it can be bred from the neutron reaction with lithium walls.

While fusion research so far has been focused on the confinement of thermal particles, future research in burning plasmas also needs to contend with the confinement of energetic particles whose transport may be subject to different processes.

1.1 Energetic Particles

Since one of the main objectives of ITER is to ignite a plasma, fusion-product alphas need to be confined long enough for their energy to be collisionally transferred to the
background plasma. A burning plasma needs to be thermally stable where the heating provided by the alphas and external sources are balanced by the heat conduction losses. Therefore the power from alpha production should not exceed heat conduction as temperature increases. Although this is guaranteed for many parameters of the plasma density, temperature, and external power source, controlling the plasma evolution to maintain this equilibrium will be one of the challenges of the burning plasmas experiments [10].

The other big challenge concerns alpha particles confinement, which is necessary for achieving the ignited state. While the classical confinement of energetic particles is well understood, alpha population in future devices could undergo anomalous transport, which needs to be better understood, modeled, and predicted. Alpha loss degrades the plasma performance, and could result in a loss of ignition or difficulty in achieving ignition at all. Moreover, anything beyond a few percent of alpha losses will gravely damage the first wall, setting back the experiment, and significantly adding to the overall cost.

Present-day experiments [11, 12] can operate within a parameter space where neutral beam injection (NBI) can destabilize plasma modes resulting in particle losses. The results of these experiments cannot be extrapolated to ITER whose dimensionless parameters are very different and where alphas have isotropic distributions and significantly contribute to the total plasma pressure. However, they are useful for validating models and testing theories that can be applied to ITER.

Fast ions can undergo anomalous transport due to many mechanisms, most notably, due to the field ripples resulting from the finite number of coils generating the toroidal magnetic field, the breaking of axisymmetry, and the plasma instabilities that resonantly interact with the particles.

Energetic particles in fusion devices will generally have velocities higher than the Alfvén velocity resulting in their resonant interaction with the shear Alfvén modes.
Free energy from the alpha radial gradient can destabilize the modes, which in turn transport the alpha particles and possibility jeopardize their confinement.

Next-generation burning plasmas will have both alpha particles and beams that could interact with plasma modes.

1.1.1 Alpha Particles

The fusion-product alpha particles are born at $E_0 = 3.5$ MeV with a distribution approximated by $f = S_0 \exp[-(E - E_0)^2/\Delta E] \approx S_0 \delta(E - E_0)$ where $\Delta E$ is the spread in energy of the alpha particles born and $S_0 = n_D n_T \langle \sigma_D v \rangle$ dictates the radial profile of the alpha particles. The cross section for the reaction rate $\langle \sigma v \rangle$ depends on the fuel used and the background plasma temperature and pressure. Due to higher temperature in the core, which results in higher reaction rates, the alpha particles have a peaked profile that tapers off towards the plasma edge. This radial gradient is the free energy source behind alpha-driven instabilities.

All alpha particles are born with energies about $E_0 = 3.5$ MeV. Shortly after being produced, the particles start undergoing collisions with the background plasma. The Fokker-Plank equation describes the evolution of the distribution function with collisions taken into account. The resulting steady-state solution, $\partial f/\partial t = 0$ is the slowing-down distribution function [14]

$$f(E) = \frac{C}{E^{3/2} + E_{\text{crit}}^{3/2}} \text{Erfc} \left( \frac{E - E_0}{\Delta E} \right) \approx \frac{C}{E^{3/2} + E_{\text{crit}}^{3/2}} H(E - E_0). \quad (1.3)$$

$C \propto S_0 \tau_s$. The alphas slow down primarily through collisions with electrons, where the slowing-down time [15]

$$\tau_s = \frac{3 \sqrt{2} m_e T_e^{3/2}}{8 n_e \sqrt{m_e \ln \Lambda_e} Z^2 Z_e^2 \sqrt{\pi}} \approx \frac{10^6 T_e^{3/2}[\text{keV}]}{n_e[\text{m}^3]} \quad (1.4)$$
is the time it takes for the distribution function to settle into the slowing-down form of Eq. (1.3).

Erfc is the complimentary error function and $H$ is the heaviside function. $E_{\text{crit}}$ is the critical energy for which the power transferred to the ions is balanced by that transferred to the electrons

$$E_{\text{crit}} = 14.8 A_f T_e \left( \frac{\sum_i n_i (Z_i^2 / A_i) \ln \Lambda_i}{n_e \ln \Lambda_e} \right)^{2/3}. \quad (1.5)$$

$A$ and $Z$ are the atomic mass and number respectively. The subscripts $f$, $i$, and $e$ are for the fast ions, thermal ions and electrons respectively. $\Lambda$ is the ratio of the Debye length to the closest distance between two particles scattering under classical Coulomb collisions. For alpha particles with energies below the critical energy $E < E_{\text{crit}}$, collisions with ions are dominant while particles whose $E > E_{\text{crit}}$ are dominated by collisions on electrons. Measurements [16–18] in JET and TFTR plasmas confirm the slowing-down evolution of the fusion born particles.

### 1.1.2 NBI and RF heating

Energetic particles, which are a product of auxiliary heating mechanisms such as neutral beam injection (NBI) and Radio Frequency (RF) heating, are also expected to play a role in future fusion devices. Since these particles can be as high as MeVs in ITER [19], making them super-Alfvénic, they also are prone to resonant interaction with Alfvénic modes. The main difference from alpha particles is that they have strongly anisotropic distribution functions where NBI is predominantly along the magnetic field and RF perpendicular to it. Their energy distribution function is also a balance between the heat source and electron drag [20].
1.2 Interaction with Alfvénic Modes

The Alfvénic eigenmodes (AE) are thought to be stable due to efficient continuum damping that is enough to damp out any drive mechanisms. However, gaps are formed in the Alfvénic continuum due to toroidal symmetry, and within these gaps exist discrete Toroidal Alfvénic Eigenmodes (TAE)’s that are not effectively damped by the continuum and, therefore, are prone to being driven unstable. Since fast ions in burning plasmas are expected to be super Alfvénic, they are at risk of resonating with the modes that can be driven unstable by the free energy of the particles [21].

1.2.1 TAE Modes

The following is a summary of the derivation of TAE modes as presented by Mikhailovskii [22]. Starting with the current closure equation, \( \nabla \cdot j = 0 \), it is assumed that the perturbation is an electromagnetic ideal small scale perturbation, which is flute-like along the magnetic field, \( B_0 \) and of small scale across it. If one neglects the plasma temperature, viscosity, and equilibrium plasma velocity, the current closure equation can be expressed as a function of the scalar potential \( \phi \),

\[
\hat{L}\phi \equiv (B_0 \cdot \nabla) \left( \frac{1}{B_0} \nabla^2 (B_0 \cdot \nabla) \phi \right) + \frac{\omega^2}{v_A^2} \nabla^2 \phi = 0, \tag{1.6}
\]

where \( v_A \) is the Alfvén velocity \( v_A^2 = B_0^2/4\pi \rho_0 \). Due to poloidal and toroidal periodicity, \( \phi \) can be expressed as a Fourier sum over poloidal harmonics

\[
\phi = e^{i(n\phi - \omega t)} \sum_m \phi_m(r)e^{-im\theta}. \tag{1.7}
\]

In toroidal geometry, the variables \( \theta \) and \( \phi \) are coupled via the safety factor, \( q(r) = (\dot{\phi}/R)/(\dot{\theta}/r) \equiv (rB_\phi)/(RB_\theta) \). Therefore, \( n \) and \( m \) are not good quantum numbers anymore, and \( k_{||m} = (nq - m)/qR \) is used instead of \( n \).
This reduces Eq. (1.7) to the following set of eigenvalue equations, Eq. (1.8), not separable in $\phi$ due to the field’s dependence on poloidal variable, $\theta$:

$$
(\hat{L}\phi)_m = L_m^0 \phi_m + \sum_{m' \neq m} L_{m'}^1 \phi_{m'}.
$$

Assuming circular tokamaks, where $B \propto e^{i\theta}$, only nearest neighbor poloidal harmonics, $m$ and $(m \pm 1)$, couple. This reduces the sum over $L_{m'}^1$ in Eq. (1.8) to $L_{m \pm 1}^1 \phi_{m \pm 1}$.

Denoting the sum by $Q$ allows the set of equations to be represented in a compact form:

$$
\begin{pmatrix}
P_m & Q \\
Q & P_{m+1}
\end{pmatrix}
\begin{pmatrix}
\phi_m \\
\phi_{m+1}
\end{pmatrix} = 0,
$$

where

$$
P_m \equiv \frac{d}{dr} r^3 \left( \frac{\omega^2}{v_A^2} - k_{||m}^2 \right) \frac{d}{dr} - (m^2 - 1) r \left( \frac{\omega^2}{v_A^2} - k_{||m}^2 \right) + \left( \frac{\omega^2}{v_A^2} \right)' r^3;
$$

$$
Q \equiv \frac{5 \epsilon}{2} \frac{d}{dr} \left( \frac{\omega^2}{v_A^2} \right) \frac{d}{dr} \frac{r^4}{a} \frac{d}{dr},
$$

where $\epsilon \equiv r/R$ is considered small in the large aspect ratio limit that is assumed.

In the cylindrical limit, $\epsilon = 0$, the Alfvénic dispersion relation is $\omega = \pm k_{||m}(r)v_A(r)$. However, due to the coupling term, $Q$, the degeneracies are lifted and a gap in $\omega$ is formed at $k_{||m} = k_{||m+1}$. This occurs at radial positions $r_0$ where

$$
q(r_0) = \frac{m + 1/2}{n}.
$$

Within these gaps exist the discrete TAE modes that can be approximated analytically by using asymptotic analysis [23]. Assuming small shear, expand Eq. (1.9) around $r = r_0$, to find the inner structure of the TAE mode within the gap. Away from $r = r_0$, the coupling term can be neglected and the outer structure is computed. Solving Eq. (1.9) separately in the inner and outer regions and matching the solutions
gives rise to the structure of the TAE modes and their eigenvalue

\[ \omega_{\text{TAE}} = \frac{v_A}{2qR} \left( 1 - \frac{\epsilon}{2} \left( 1 - \pi^2 S^2/8 \right) \right), \quad (1.11) \]

where S is the shear, \( S \equiv rq'/q \).

An illustration of the gap that forms in Alfvénic continuum and the poloidal harmonics is presented in the following figure.

![Figure 1.1](image)

Figure 1.1: (a) A gap forming in the Alfvénic frequency continuum as a result of coupling to remove the degeneracy, (b) the poloidal harmonics. Adapted from C. Z. Cheng et al. Ann. Phys., 161(21), 1985.

### 1.2.2 Destabilization of TAE modes

The energetic particles' free energy in the spatial gradient can drive TAE modes unstable if there are not enough damping mechanisms in the plasma. An equilibrium energetic particle distribution function can be written in terms of the constants of motion. The distribution function is represented in terms of \( P_{\phi}, E, \) and \( \mu \) where \( E = Mv^2/2 \) is the energy of the fast ions. \( M \) is mass and \( v \) is velocity, \( \mu \) the magnetic
moment, $P_\phi$ the canonical angular momentum

$$
P_\phi(r) = MRv_\phi - \frac{Z_fe}{c} \int_0^r dr' \frac{r'B(q(r'))}{q(r')}.
$$

(1.12)

e the elementary charge, $Z_f$ the atomic number, $c$ the speed of light, $q$ the safety factor, $r$ the radius, and $B$ the toroidal magnetic field.

The centrally peaked radial profile with $\partial f/\partial r < 0$ has a profile where $\partial f/\partial P_\phi > 0$. Therefore the mode gains energy from the resonant particles in a mechanism akin to the inverse Landau damping. The energy transfer between particles and the mode is proportional to the gradients in the distribution function along $E' = E - \omega P_\phi/n$ at the resonant curve given by $\omega = n\bar{\omega}_\phi(P_\phi, E) - (m+l)\bar{\omega}_\theta(P_\phi, E)$.

$$
\gamma \propto \omega \frac{\partial f_{ep}}{\partial E} + n \frac{\partial f_{ep}}{\partial P_\phi}.
$$

(1.13)

$\omega$, $n$, and $m$ are the TAE frequency, toroidal, and poloidal mode number. $\bar{\omega}_{\phi,\theta}$ are the average toroidal and poloidal frequency of the resonant particles. The free energy from the radial profile $\partial f_{ep}/\partial P_\phi > 0$ drives the mode while the gradient in energy $\partial f_{ep}/\partial E < 0$ is stabilizing.

The growth rate $\gamma$ depends on both the driving and the stabilizing mechanism of energetic particles. The plasma diamagnetic frequency $\omega_*$ is a measure of the relation of the radial gradient in the energetic particle profiles to the velocity gradient,

$$
\omega_* \propto n \frac{\partial f/\partial P_\phi}{\partial f/\partial E}.
$$

(1.14)

A distribution function with $\omega_* > \omega_{TAE}$ at the resonant curve would be prone to destabilizing TAE modes if the background damping is not strong enough. In the case of TAE modes, the growth rate is an integral over the resonant curve in phase space. Note therefore that even with $\omega_* < \omega$ in parts of phase space, the mode could
be marginally destabilized due to $\omega_\ast > \omega$ for other parts, which could still result in $\gamma_L > \gamma_d$.

The possibility of resonant interaction of the energetic particles with Alfvénic modes was first recognized by Mikhailovski [24] and Rosenbluth [25]. While in some cases interaction results in local flattening, in others it can result in large scale transport. Although this interaction has been a subject of many studies [21, 26–28], it is still not fully understood and more research is needed on all levels. For the viability of fusion reactors, it is necessary to understand and be able to predict what will happen if the slowing-down distribution is not formed due to the energetic particle interaction with Alfvénic modes.

1.2.3 Need for Reduced Models for Next-Generation Fusion Devices

The problems of energetic particles interacting with modes is critical for building next-generation fusion devices. With ITER on the horizon, this interaction is in immediate need of extensive analysis and study. The existence of a multitude of different time scales makes it difficult if not impossible to perform a brute force numerical simulation for many inverse growth time steps. While many codes exist to simulate short time scales, reduced models are needed to make general predictions and provide a framework for understanding, interpreting, and extrapolating from the results of these existing first-principle codes. Two reduced models based on the quasilinear theory were developed for this thesis, the 1.5D model and the Line Broadened Quasilinear (LBQ) model. The models assess the effect of the Alfvénic Eigenmodes on the redistribution and/or loss of energetic particles. *The ultimate goal of these models is to find the operational space resulting in minimum alpha losses while keeping their production high enough to guarantee self-heating.*
1.3 Overview of the Dissertation

The 1.5D reduced QL model and the LBQ model are developed to study the interaction of TAE modes with energetic particles in toroidally shaped fusion devices. A presentation of the QL theory is critical for a self-contained understanding of the proposed models. Chapter 2 is an overview of the conventional quasilinear theory that both models are based on. Chapters 3, 4, and 5 introduce the models, validate them, and present their results.

1.3.1 1.5D Reduced QL Model

The 1.5D reduced QL model [27, 29] utilizes analytic forms of the damping and growth rate to calculate the redistributed distribution function of the energetic particles as a result of interacting with the mode. The model, which is based on a reduction of the full quasilinear theory, assumes that the radial gradient in energetic particle distribution function is modified such that the modes, that the particles interacts with, are marginally stabilized. This requires for each mode that the growth rate, due to the free energy of the energetic particles, becomes equal to or less than the sum of the damping rates. For more realistic results, NOVA-K [30] is used to normalize the rates, and TRANSP [31] is used to obtain the plasma profiles that are needed. In Chapter 3 the model is described and validated with DIII-D measurements. The code is developed with full documentation and is available for use by the fusion community.

Within the regimes of applicability that are discussed in the thesis, the model has made very good agreement with present-day experiments [1, 2] and can be used to predict losses on ITER and ARIES as demonstrated in Refs. [32] and [33].

In the 1.5D model, one postulates a steady state with a present source and that the modes are overlapped in order to predict the QL diffusion coefficient. Strong collisionality is needed for this model to be valid. However, the quasilinear diffusion
coefficient required to maintain the marginal stable condition may not be high enough to justify the mode overlap condition. Even if overlap occurs, it might be transient if there are not enough collisions, which makes it difficult to maintain the steady state \[34\]. Therefore one needs to use a non-overlap model, which is what motivates the LBQ model. The LBQ model can be a stand-alone model, or a model that justifies the 1.5D reduced QL model. Although LBQ is a numerically efficient model, capable itself of parameter scale scans, 1.5D reduced QL is orders of magnitude faster and is readily applicable with the capacity of normalization with NOVA and NOVA-K. At the moment it is being further validated on DIII-D and NSTX in relevant regimes.

1.3.2 Revisiting the Line Broadened Model

A Line Broadened Quasilinear model was proposed by Berk et al. \[3, 35\] in 1995 and implemented for the case of the bump-on-tail by Fitzpatrick in his thesis \[36\]. While conventional quasilinear theory applies only to multiple modes overlapping, LBQ is developed to also simulate the dynamics of particle interaction with isolated modes. This is done by using the same structure of the quasilinear equations but with broadening the delta function into a function whose parametric dependancies are modeled to reproduce the expected behavior of isolated modes.

The LBQ model is revisited in Chapter \[4\]. The parametric dependencies of the broadening width are modified to better predict expected saturation levels. The model is benchmarked with existing analytic and numerical results and the regime of applicability is presented. An extension of the model, LBQ2D, is implemented in the two-dimensional \((P_\phi, E)\) space to study the Alfvénic eigenmodes interacting with energetic particles in toroidal plasmas. The quasilinear diffusion of a particle is along paths of constant \(E' = E - \omega_{\text{TAE}} P_\phi / n\), which depends on the TAE mode parameters: the frequency \(\omega_{\text{TAE}}\) and toroidal mode number \(n\). The schemes used to solve the LBQ equations in the full two-dimensional space are tested and validated.
It is often found that unstable modes have $\gg \omega_{TAE}$ where $\omega_*$ is the fast-ion diamagnetic frequency. This results in diffusion approximately along $P_\phi$, which reduces the two-dimensional diffusion equations to a set of one-dimensional equation also discussed in Fitzpatrick’s thesis [36]. However, modes that have $\omega_* \gtrsim \omega_{TAE}$ can also be stabilized, especially modes with low $n$. The effect of these modes on energetic particles cannot be determined without a code that captures diffusion in the full $(P_\phi, E)$ space. If the modes are isolated, the conservation of $E' = E - \omega_{TAE} P_\phi / n$ during the interaction makes the dynamics fundamentally one-dimensional. An appropriate rotation in space can reduce the diffusion equation in the LBQ model to a set of one-dimensional equations that are straightforward to solve. However, when modes overlap the dynamics is stochastic in the regions of phase space affected by the overlapping modes and the LBQ diffusion equation is two-dimensional with cross terms. Since the introduced cross terms makes it costly to resolve the diffusion using conventional schemes, a semi-implicit scheme denoted by IMP-EXP is implemented. IMP-EXP is inspired by the Alternating Direction Implicit (ADI) method, but instead of half time steps in each direction, only the cross terms and the term for diffusion in $E$ are treated explicitly. The source term (or terms treated explicitly) is updated iteratively until the solution converges to that resulting from a fully implicit scheme.

The number of iterations needed for convergence depends on the time step taken and the relative strength of the terms dealt with explicitly.

Chapter 5 is a discussion on the numerical implementation of the LBQ model and the schemes used. The schemes are validated and the model is applied to a test case of multi-mode overlap with different $\omega/n$.

The 1.5D and the LBQ models are valuable predictive tools needed at the design stages of burning plasma experiments. They give physical insight into one of the most critical mechanisms that need to be understood for building a viable fusion power plant.
Chapter 2

Conventional Quasilinear Theory

2.1 Introduction

Quasilinear theory (QL) was first developed in the 1960’s by Vedenov, Valikov, and Sagedev [37] and Drummond and Pines [38]. Initial treatments were focused on the one-dimensional interacting of particles with electrostatic modes [39]. It was soon extended for the treatment of electromagnetic modes in multiple dimensions [40, 41] and the inclusion of collisional terms and sources and sink [42, 44].

QL theory provides a powerful tool to deal with the relaxation of collision-less plasmas interacting with modes. QL codes have the advantage over particle orbit codes for not needing the follow the exact particle orbits to assess the slow time evolution of the averaged distribution function of resonant particles that interact with a broad spectrum of particle wave resonances. QL has been implemented in many codes in the fusion community. QL theory has been validated with numerically intensive first-principle codes [45]. It has been especially useful in the study of ICRF heating, which lies in the appropriate limits for the theory [46, 50].

Underlying this elegant theory of seeming simplicity is a lot of subtlety especially in lieu of the assumptions taken to derive its equations. The validity, limitations, and
restrictions of the theory have been the subject of decades of research. Refs. [51– 53] are a few of the many references dedicated to studying the contradictions and applicability of CQL.

This manuscript examines an extension of CQL theory using a line broadened quasilinear model (LBQ) for a TAE interaction with energetic particles in toroidal geometry. The present work has its roots in the work of Berk and Fitzpatrick [3, 35, 36].

The LBQ2D code that has been developed here is mainly an extension of the work of Fitzpatrick’s LBQ code in that it incorporates diffusion in both angular momentum and energy (Fitzpatrick’s work only allowed diffusion in momentum). The LBQ2D formulation enables a QL description of isolated wave-particle resonances as well as multiple overlapping modes. Conventional quasilinear theory (CQL) is only applicable when resonance are overlapping, while the present formulation allows for a QL description for both isolated, overlapping, as well as intermediate cases.

To present the LBQ model clearly, a thorough account of the CQL theory is discussed. Parallel to Kaufman’s [40] treatment, a derivation of the CQL equations is expressed in the ($P_\phi, E$) phase space variables that will use to develop the code for LBQ2D.

2.2 Conventional Quasilinear Diffusion

The following study is of the dynamics of particles confined in electromagnetic field ($E, B$) with a weak perturbation ($\delta E, \delta B$). The motion of the confined particles is assumed regular in the unperturbed system. This allows one to write the Hamiltonian of the full system as a sum of the integrable part and the perturbation $H = H_0 + \delta H$, where $\delta H = \sum_\alpha \delta H_\alpha$ is the perturbed interaction Hamiltonian with $\alpha$ denoting a partial Hamiltonian that has only single resonance, as will show below.
To derive the quasilinear equations describing the system’s response to the weak perturbation due to these modes, one starts with Vlasov’s equation. Since the motion of the unperturbed Hamiltonian is integrable, it is most efficient to write the equations in terms of the action-angle variables. The continuity equation in action-angle variables is

\[
\frac{\partial f}{\partial t} + \{f, H_0\} + \sum_\alpha \{f, \delta H_\alpha\} = 0, \quad (2.1)
\]

where \(\{\}\) is the Poisson bracket. Action-angle variables are used, and \(\{f, H_0\}\) is reduced to one term as follows:

\[
\{f, H_0\} = \frac{\partial H_0}{\partial J} \frac{\partial f}{\partial \xi} - \frac{\partial H_0}{\partial \xi} \frac{\partial f}{\partial J} \rightarrow \Omega \cdot \frac{\partial f}{\partial \xi} \quad (2.2)
\]

since \(\partial H_0/\partial \xi = 0\) because \(\xi\) is a cyclic variable. From Hamilton’s equations \(\dot{\xi} = \Omega = \partial H_0/\partial J\), one uses \(\Omega\), the frequency at which \(\xi\) gyrates, to get the expression in Eq. (2.2).

Since the trajectories are linearized, the derivation between having one or many modes is essentially identical. Hence, for simplification of the presentation we focus on one mode and drop the sum in \(\alpha\) over the modes, though the sum over individual resonances remain. At the end of the derivation one can add in a straightforward manner the sum of all resonances from many modes. Note however that a single mode has many resonances, and these resonances are retained in the derivation.

The Hamiltonian, in \((r, p)\) space, of particles in electromagnetic field is of the form

\[
H = \frac{1}{2} [p - qA(r, t)]^2 + q\phi(r, t), \quad (2.3)
\]

where \(A\) is the vector potential and \(\phi\) is the electric potential. If one chooses the radiation gauge \(\delta \phi \equiv 0\), the Hamiltonian of the perturbation becomes

\[
\delta H(r, t) = -qv \cdot \delta A = -\frac{i}{\omega}qv \cdot \delta E, \quad (2.4)
\]
where \( v = \dot{r} \) is the drift velocity of the particle. The expressions for the electric field of the perturbation of the form:

\[
\delta E = C(t)e(r)e^{-i\omega t} + c. c.,
\]  

where \( C(t) \) is the amplitude of the mode, \( e(r) \) is the eigenfunction, and the mode frequency \( \omega \) is the eigenvalue solution to the wave equation. The perturbation is therefore of the form:

\[
\delta H(r, t) = iqC(t)\frac{v \cdot e(r)}{\omega}e^{-i\omega t} + c. c.
\]  

\( \delta H \) is written in terms as of the action-angle variables \((\xi, J)\) of the integrable Hamiltonian \( H_0 \) and represented as a Fourier sum over the angles \( \xi \)

\[
\delta H(\xi, J) = \sum_l H_l e^{il\cdot\xi-\omega t} + c. c. = 2C(t) \sum_l V_l \cos(l \cdot \xi - \omega t),
\]  

where \( V_l \) are the Fourier coefficients

\[
V_l = \frac{iq}{\omega} \int \frac{d\xi}{(2\pi)^3} v(r(\xi)) \cdot e(r(\xi))e^{-il\cdot\xi} \equiv \frac{i}{\omega} \int d^3r \langle j(r|J) \cdot e(r, \omega) \rangle_l e^{-l \cdot \xi},
\]  

and \( \langle j(r|J) \cdot e(r, \omega) \rangle_l \) are the Fourier components of the perturbed current density dotted with the eigenfunction of the electric field. \( V_l \) is also represented in Eq. (2.8) as a function of \( \langle j(r|J) \cdot e(r, \omega) \rangle_l \) since it will be used later in deriving the QL equations.

The Poisson bracket for the lowest-order perturbed part of the distribution function in terms of \((\xi, J)\) is

\[
\{f_0, \delta H\} = \frac{\partial \delta H}{\partial J} \cdot \frac{\partial f_0}{\partial \xi} - \frac{\partial \delta H}{\partial \xi} \cdot \frac{\partial f_0}{\partial J} = \sum_l 2C(t)V_l l \cdot \frac{\partial f_0}{\partial J} \sin(l \cdot \xi - \omega t).
\]  

18
Vlasov’s equation in \((\xi, J)\) becomes

\[
\frac{\partial f}{\partial t} + \Omega \cdot \frac{\partial f}{\partial \xi} + 2C(t) \sum_l V_l \sin (l \cdot \xi - \omega t) l \cdot \frac{\partial f_0}{\partial J} + \{\delta f, \delta H\} = 0,
\]

where \(f = f_0 + \delta f\) with \(f_0\) being the unperturbed distribution function.

### 2.2.1 Linearization

Linearizing the equation for the perturbation in the distribution function, the first-order in perturbation is written as \(\delta f = \sum_l \delta f_l e^{i(l \cdot \xi - \omega t)}\). To leading order for each of the Fourier components \(\delta f_l\) one gets

\[
-\omega \delta f_l + l \cdot \Omega \delta f_l = 2C(t) V_l \sin (l \cdot \xi - \omega t) l \cdot \frac{\partial f_0}{\partial J},
\]

resulting in the Fourier component

\[
\delta f_l = \frac{2C(t) V_l \sin (l \cdot \xi - \omega t)}{(l \cdot \Omega - \omega)} l \cdot \frac{\partial f_0}{\partial J}.
\]

These linearized trajectories are then used to derive the CQL equation.

### 2.2.2 Quasilinear Form

To find the effect of the perturbation on the distribution function a second-order correction is needed. Taking the ensemble average over Vlasov’s equation, which is equivalent to averaging over the angles \(\langle \rangle \equiv \int d^3\xi/(2\pi)^3\), one gets

\[
\frac{\partial \langle f \rangle}{\partial t} + \langle \Omega \cdot \frac{\partial f}{\partial \xi} \rangle + \sum_l \langle \frac{\partial (H_l e^{i(l \cdot \xi - \omega t)} + c. c.)}{\partial \xi} \cdot \frac{\partial f_0}{\partial J} \rangle + \langle \{\delta f + f_2, \delta H\} \rangle = 0,
\]

where \(f = f_0 + \delta f + f_2\). The ensemble average leads to the vanishing of the first-order terms and the terms \(\Omega \cdot \partial f/\partial \xi, \{\delta f, H_0\}, \) and \(\{f_0, \delta H\}\). For the second-order terms...
one finds with a little manipulation of the Poisson bracket
\[
\langle \delta f, \delta H \rangle = \left\langle \frac{\partial \delta H}{\partial J} \cdot \frac{\partial \delta f}{\partial \xi} - \frac{\partial \delta H}{\partial \xi} \cdot \frac{\partial \delta f}{\partial J} \right\rangle \tag{2.14}
\]
that the first term vanishes because it is a total derivative in one of the action-angles.

Then up to second-order in the perturbation, the equation becomes
\[
\frac{\partial \langle f \rangle}{\partial t} - \left\langle \frac{\partial \delta H}{\partial \xi} \cdot \frac{\partial \delta f}{\partial J} \right\rangle = 0. \tag{2.15}
\]

Rewriting the averaged function \( \langle f \rangle \) by a slowly varying zeroth-order function \( f \) and using Eq. (2.12) and Eq. (2.7) one finds that
\[
\left\langle \frac{\partial \delta H}{\partial \xi} \cdot \frac{\partial \delta f}{\partial J} \right\rangle = \int \frac{d^3 \xi}{(2\pi)^3} \sum_l i2C(t)V_l \sin (l \cdot \xi - \omega t) l \times \sum_{l'} \frac{\partial}{\partial J} \left( \frac{2C(t)V_{l'}}{l' \cdot \Omega - \omega} \sin (l' \cdot \xi - \omega t) l' \cdot \frac{\partial f}{\partial J} \right). \tag{2.16}
\]

Noting that
\[
\int \frac{d^3 \xi}{(2\pi)^3} \sin (l \cdot \xi - \omega t) \sin (l' \cdot \xi - \omega t) \rightarrow -\frac{1}{2} \delta_{l,l'} \tag{2.18}
\]
after performing the integral, one reduces the double sum with the use of the delta function, and Eq. (2.16) reduces to
\[
\left\langle \frac{\partial \delta H}{\partial \xi} \cdot \frac{\partial \delta f}{\partial J} \right\rangle \rightarrow -\frac{1}{2} \sum_l il \cdot \frac{\partial}{\partial J} \frac{[2C(t)V_l]^2}{l \cdot \Omega - \omega} l \cdot \frac{\partial f}{\partial J}. \tag{2.19}
\]

Substitute this result into Eq. (2.15). one obtains
\[
\frac{\partial f}{\partial t} + \frac{\pi}{2} \sum_l l \cdot \frac{\partial}{\partial J} [2C(t)V_l]^2 \delta (l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J} = 0, \tag{2.20}
\]
where the Plemelj formula has been used and the resonant part is kept while the nonresonant part (the principal value part of the Plemelj formula) vanishes due to symmetry. This results in

\[
\frac{1}{l \cdot \Omega - \omega} \rightarrow i\pi \delta(l \cdot \Omega - \omega).
\] (2.21)

where the delta function is over the region in phase space satisfying the resonant condition \( l \cdot \Omega = \omega \). Therefore, conventional quasilinear theory requires multiple modes existing and overlapping for stochastic diffusion across phase space to take place.

When there are multiple frequencies present, the quasilinear equation remains the same, but with a sum over the resonances of each unstable frequency. For each mode there are multiple resonant conditions for each \( l \) and the quasilinear equation remains the same form, where the sum over the resonants interactions of each frequency is still in place for multiple modes with their respective resonant surfaces.

### 2.2.3 Wave Equation

As discussed previously, the perturbation in the electric field is of the form of Eq. (2.5), which satisfies the linearized Maxwell’s equation. The equations include the response from the plasma, \( 4\pi \delta \langle j \rangle \) which is the average current density, resulting from the perturbation \( \delta f \).

The plasma response is written as a function of the perturbation in the field \( \delta E \) through the conductivity kernel \( \sigma \), \( \delta \langle j \rangle = \int d^3r' \sigma(r', r; \omega) \cdot \delta E(r, \omega) \):

\[
\sigma(r, r'; \omega) = \frac{i}{\omega} \int d^3J \dot{j}^*(r, \omega) \dot{j}(r', \omega) \frac{l}{l \cdot \Omega - \omega} \cdot \frac{\partial f}{\partial J}.
\] (2.22)

The derivation of the expression of the conductivity kernel can be found in Ref. [41].
Eq. (2.22) can be used to write the linearized Maxwell’s equations as

$$K(\omega) \cdot \delta E(r, \omega) = -\delta E(r, \omega),$$  \hspace{1cm} (2.23)

where the operator $K$ is

$$K(\omega) \cdot () = -\omega^2 \nabla \times (\nabla \times) + \frac{4\pi i}{\omega} \int d^3 r' \sigma(r, r'; \omega) \cdot ().$$ \hspace{1cm} (2.24)

The solution to the eigenvalue problem $\Omega_\alpha = \omega_\alpha + i\gamma_\alpha$ is almost real with $\gamma_\alpha \ll \omega_\alpha$ since $K$ is nearly Hermitian. This allows for the problem to be treated using perturbation theory, where the operator is split into a Hermitian and an anti-Hermitian part $K = K' + iK''$. Using Eq. (2.5) for $\delta E$, the growth rate $\gamma_\alpha$ becomes

$$\gamma_\alpha = \frac{1}{4\pi} \int d^3 r e^*(r) \cdot K''(\omega_\alpha) \cdot e(r) \frac{\partial K'}{\partial \omega}|_{\omega_\alpha} \cdot e(r),$$ \hspace{1cm} (2.25)

where the mode frequency $\omega_\alpha$ and the wave function $e(r)$ are the eigenvalue and eigenfunction solutions to the Hermitian part $K'(\omega)$.

$W_\alpha$ is introduced such that the wave energy is written as

$$W_\alpha = \frac{1}{4\pi} \int d^3 r C(t)e^*(r) \cdot \omega_\alpha \frac{\partial K'}{\partial \omega}|_{\omega_\alpha} \cdot C(t)e(r) = W_\alpha C^2.$$ \hspace{1cm} (2.26)

Multiplying the numerator and denominator in Eq. (2.25) by $\omega_\alpha$, one can use Eqs. (2.22) and (2.26) to rewrite Eq. (2.25) as

$$\gamma_\alpha = -\int \frac{\pi d^3 J}{\omega_\alpha W_\alpha} \sum_l \int d^3 r d^3 r' \langle e^*(r) \cdot J^*(r', \omega) | t \langle J(r', \omega) \cdot e(r) \rangle t \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}$$

$$= \frac{\pi \omega_\alpha}{W_\alpha} \int d^3 J \sum_l \left| \int d^3 r \frac{i}{\omega_\alpha} \langle J(r, \omega) \cdot e(r) \rangle t \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J} \right|^2.$$ \hspace{1cm} (2.27)
Using Eq. (2.8), the numerator and denominator are multiplied by 4 to write $\gamma_\alpha$ consistent with expression for the particles in Eq. (2.20). One gets

$$\gamma_\alpha = \frac{\pi}{4W_\alpha} \int d^3 J \sum l \omega_\alpha|2V_l|^2 \delta(l \cdot \Omega - \omega)l \cdot \frac{\partial f}{\partial J}.$$  \hspace{1cm} (2.28)

Differentiating Eq. (2.26) with respect to time and averaging over the fast oscillations with rate $\omega_\alpha$, the slow evolution of the mode energy becomes

$$\frac{d}{dt} W_\alpha C^2 = 2\gamma_\alpha W_\alpha C^2.$$  \hspace{1cm} (2.29)

The growth rate, $\gamma$, is the imaginary part of the complex frequency, $\Omega_{mode}$, which is the solution of the eigenvalue problem for the perturbation $\delta E$.

The coupled equations for the distribution function and the mode amplitude are solved for self-consistently. The growth rate is the instantaneous growth rate that depends on the slowly changing distribution function $f$. The self-consistent coupled Quasilinear equation for the particles $f$ and each of the $\alpha$ modes are

$$\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_\alpha \sum_l l \cdot \frac{\partial}{\partial J} [2C_\alpha(t)V_l]^2 \delta(\Omega_l - \omega)l \cdot \frac{\partial f}{\partial J},$$  \hspace{1cm} (2.30a)

$$\frac{d C^2_\alpha}{dt} = 2\gamma_\alpha C^2_\alpha,$$  \hspace{1cm} (2.30b)

where $\gamma_\alpha$ is calculated using Eq. (2.28).

### 2.2.4 Conservation of Energy and Momentum

Quasilinear theory conserves energy (momentum) between the wave energy (momentum) density and the resonant particles. The wave energy density as defined by Eq. (2.26) includes both the non resonant particle kinetic energy $E_{\text{kin}}^{\text{nR}}$ and the electric field energy, $E_{\text{field}}^{\text{field}}$. Although one can balance the total particle energy $E^{\text{R}} + E^{\text{nR}}$ and
the field energy \( E^{\text{field}} \), here the total wave density energy, \( E^{\text{nR}} + E^{\text{field}} \) with the energy from the resonant particles \( E_R \) is chosen to be balanced. The same will be done for the momentum balance.

Taking the energy moment of Vlasov’s equation, the total kinetic energy of the resonant particles becomes

\[
\frac{d W}{dt} = \frac{d}{dt} \left( \sum_\alpha W_\alpha + \int dJ H f \right) \equiv \sum_\alpha 2\gamma_\alpha W_\alpha + \int dJ H \frac{\partial f}{\partial t},
\]

(2.31)

where \( H \) is the Hamiltonian of the particles and \( f \) is the distribution function. Equation (2.30a) is used for \( \frac{\partial f}{\partial t} \), where the Plemelj decomposition has been included to isolate the resonant particles contribution. The change in energy of the resonant particles is

\[
\frac{d}{dt} \int dJ H f = \frac{\pi}{2} \sum_\alpha \int dJ \sum_l H l \cdot \frac{\partial}{\partial J} [2C(t)V_l]^2 \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}
\]

\[
= -\frac{\pi}{2} \sum_\alpha \int dJ \sum_l l \cdot \frac{\partial H}{\partial J} [2C(t)V_l]^2 \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}
\]

\[
= -\frac{\pi}{2} \sum_\alpha \int dJ \sum_l [2C(t)V_l]^2 \omega \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}.
\]

(2.32)

Integration by parts is used to go from the first to the second equation. Using Hamilton’s equation \( \partial H/\partial J = \Omega \) and the identity \( l \cdot \Omega = \omega \), the expression is simplified to its final form.

Eq. (2.28) is used in Eq. (2.30b) to find the change in energy of the modes

\[
\sum_\alpha \frac{d}{dt} W_\alpha = 2\bar{W}_\alpha C^2 \sum_\alpha \frac{\pi}{4\bar{W}_\alpha} \int dJ \omega \sum_l [2V_l]^2 \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}
\]

\[
= \frac{\pi}{2} \sum_\alpha \int dJ \omega \sum_l [2C(t)V_l]^2 \delta(l \cdot \Omega - \omega) l \cdot \frac{\partial f}{\partial J}.
\]

(2.33)
Combining Eq. (2.32) and Eq. (2.33), one gets

\[
\frac{d}{dt} W = \frac{d}{dt} \int dJ \ H f + \sum_\alpha \frac{dW_\alpha}{dt} = 0.
\]  

(2.34)

This demonstrates that the total energy \( W \) of the wave-particle system is conserved.

As for the conservation of momentum, following the same method, one notices that the momentum of the modes, \( \mathcal{M} \), is related to the energy, \( \mathcal{W} \) through

\[
\mathcal{W}_\alpha = \frac{\mathbf{l}}{\omega} \mathcal{M}_\alpha.
\]  

(2.35)

The equation for momentum conservation becomes

\[
\frac{d}{dt} W = \frac{d}{dt} \left( \sum_\alpha \mathcal{M}_\alpha + \int dJ \ J f \right) \equiv \sum_\alpha 2\gamma_\alpha \mathcal{M}_\alpha + \int dJ \ J \frac{\partial f}{\partial t}.
\]  

(2.36)

The change in particle momentum is

\[
\frac{d}{dt} \int dJ \ f = \frac{\pi}{2} \sum_\alpha \int dJ \ \sum_l Hl \cdot \frac{\partial}{\partial J} [2C(t) \mathbf{V}_l]^2 \delta (l \cdot \Omega - \omega) \mathbf{l} \cdot \frac{\partial f}{\partial J}
\]

\[
= -\frac{\pi}{2} \sum_\alpha \int dJ \ \sum_l l [2C(t) \mathbf{V}_l]^2 \delta (l \cdot \Omega - \omega) \mathbf{l} \cdot \frac{\partial f}{\partial J},
\]  

(2.37)

where again one uses the integration by parts and the identity \( \frac{\partial J}{\partial J} = 1 \) where \( \mathbf{1} \) is a second-rank tensor. The change in the mode’s momentum becomes

\[
\sum_\alpha \frac{d}{dt} \mathcal{M}_\alpha = 2\mathcal{M}_\alpha C^2 \sum_\alpha \frac{\pi}{4W_\alpha} \int dJ \ \omega \sum_l [2\mathbf{V}_l]^2 \delta (l \cdot \Omega - \omega) \mathbf{l} \cdot \frac{\partial f}{\partial J}
\]

\[
= \frac{\pi}{2} \sum_\alpha \int dJ \ l \sum_l [2C(t) \mathbf{V}_l]^2 \delta (l \cdot \Omega - \omega) \mathbf{l} \cdot \frac{\partial f}{\partial J},
\]  

(2.38)
where Eq. (2.35) is used to replace $\omega M_\alpha / W_\alpha$ by $l$. Once again, combining Eq. (2.37) and Eq. (2.38), one gets

$$
\frac{d}{dt} M = \frac{d}{dt} \int dJ J f + \sum_\alpha \frac{d}{dt} M_\alpha = 0 \quad (2.39)
$$

to show that the momentum is conserved between the resonant particles and the mode.

### 2.3 Bounce Frequency of Normal Modes

The quasilinear equations can be rewritten in terms of the local angle and action variables, which for each $l$ reduces the expression to one dimension. For each $l$ treated separately, the angle variable $\zeta = l \cdot \xi$ and its conjugate action $I$ are chosen where

$$
\dot{\zeta} = \frac{\partial H_0}{\partial I} \rightarrow l \cdot \frac{\partial H_0}{\partial J},
$$

which would reduce the Hamiltonian near an isolated resonance into a one-dimensional Hamiltonian with $\partial / \partial I \equiv l \cdot \partial / \partial I$ for the conjugate action. The frequency for $\zeta$ is such that $\dot{\zeta} = \Omega$, which is equivalent to saying $l \cdot \dot{\xi} = l \cdot \Omega$ resulting in the one-dimensional generalized frequency

$$
\Omega \equiv l \cdot \Omega. \quad (2.41)
$$

The Hamiltonian in the rotated frame ($\zeta, \Omega$) is reduced to a one-dimensional Hamiltonian

$$
H = H_0(I) + 2C(t)V_\ell \cos(\zeta - \omega t). \quad (2.42)
$$
Using Hamilton’s equations

\[
\dot{\zeta} = \frac{\partial H}{\partial I} = \frac{\partial H_0}{\partial I} = \Omega, \quad (2.43a)
\]

\[
\dot{I} = -\frac{\partial H}{\partial \zeta} = 2C(t)V_I \sin(\zeta - \omega t), \quad (2.43b)
\]

where \(V_I\), the matrix-elements function, is a function of \(I_r\) along the resonant surface and \(\Omega(I_r) = \omega\) is not a function of the variable \(I\).

To write the equation of motion near the resonance, the second derivative of \(\zeta\) is taken. The chain rule is used on Eq. (2.43a) to get

\[
\ddot{\zeta} = \frac{\partial \Omega}{\partial I} \dot{I} + \frac{\partial \Omega}{\partial \zeta} \dot{\zeta}. \quad (2.44)
\]

Since \(\zeta\) and \(I\) are action-angles of the zeroth-order Hamiltonian \(H_0\), \(\Omega\) is the frequency at which \(\zeta\) gyrates and is a function of \(I\) only. Therefore \(\partial \Omega/\partial \zeta = 0\) and the equation of motion for \(\zeta\) becomes

\[
\ddot{\zeta} = -2C(t) V(I) \sin(\zeta - \omega t + \zeta_0) \frac{\partial^2 H_0}{\partial I^2}, \quad (2.45)
\]

where \(\zeta_0\) is a phase. This is the equation for the pendulum

\[
\ddot{\zeta} + \omega_b^2 \sin(\zeta - \omega t + \zeta_0) = 0, \quad (2.46)
\]

where

\[
\omega_b = \left| 2C(t) V(I) \frac{\partial \Omega}{\partial I} \right|^{1/2} \quad (2.47)
\]

is the bounce frequency of the deeply trapped particles.
2.3.1 CQL in Terms of $\omega_b$

The quasilinear equation, Eq. (2.20), is expressed in a simplified form that utilizes the expression for $\omega_b$. Equation (2.47) holds true for the frequency of the deepest trapped particles if each of the Fourier components is considered separately:

$$\omega_{b,l} = \left| 2C(t)V_I \frac{\partial \Omega_I}{\partial I} \right|^{1/2}. \quad (2.48)$$

To rewrite Eq. (2.20) in terms of $\omega_{b,l}^l$, each of the components in terms of $(\zeta_l, I_l)$ is expressed in terms of the generalized frequency $\Omega_l = l \cdot \Omega$ using the chain rule. This results in

$$\frac{\partial f}{\partial t} + \pi \sum_l \frac{\partial}{\partial \Omega_l} \omega_{b,l}^4 \delta(\Omega_l - \omega) \frac{\partial f}{\partial \Omega_l} = 0. \quad (2.49)$$

2.4 Writing QL equations in $P_\phi, E$ Phase Space

In this manuscript the LBQ2D model is developed and implemented in $(P_\phi, E)$ phase space. The choice of these variables is motivated by the simplicity in treating on equal footing all the various particle orbits (banana, passing, stagnant, etc.) arising in toroidal geometry. As will be elaborated later, diffusion of particles from one part of phase space to another simply modifies the orbit but is still represented by the same phase-space point. Since the CQL equations are used to write the LBQ2D equations, the CQL are written in terms of these variables.

2.4.1 Operator $\partial/\partial I$ in $(P_\phi, E)$ Phase Space

Using the chain rule to write the differential in $I$ as a sum over the differentials in $J_i$ where $I$ is a function of $J, I(J_1, J_2, J_3)$, one gets

$$\frac{\partial}{\partial I} = \sum_i \frac{\partial J_i}{\partial I} \frac{\partial}{\partial J_i} \rightarrow l \cdot \frac{\partial}{\partial J} \quad (2.50)$$
since $J_i$ are actions and $\partial J_i/\partial J_j = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker function. The actions for the Hamiltonian are $J_1 = P_\theta$, $J_2 = P_\phi$, and $J_3 = \mu mc/q$. However, the choice of the variables $E, P_\phi$ is made to represent the dynamics where $E$ is the energy (Hamiltonian) of the unperturbed motion and $P_\phi$ is an action itself. Since $E$ is a function of the actions, $E(J_1, J_2, J_3)$, this transformation is readily made. The differential is written as

$$\frac{\partial}{\partial I} = \frac{\partial E}{\partial I} \frac{\partial}{\partial E} + l_2 \frac{\partial}{\partial J_2} + l_3 \frac{\partial}{\partial J_3},$$

(2.51)

where $l_2$ is $n$ and $J_2$ is $P_\phi$ and $l_3$ must be taken as zero since Alfvénic modes have frequencies that are much lower than the gyrofrequency.

Having $\omega_i = \partial H_0 / \partial J_i$ results [54] in

$$\frac{\partial E}{\partial I} = l_1 \frac{\partial E}{\partial J_1} + l_2 \frac{\partial E}{\partial J_2} + l_3 \frac{\partial E}{\partial J_3} \rightarrow l_1 \omega_1 + l_2 \omega_2 + l_3 \omega_3,$$

(2.52)

and the resonant condition is $\Omega(I_r) = l_1 \omega_1 + l_2 \omega_2 + l_3 \omega_3 = \omega$ which results in

$$\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial E} + n \frac{\partial}{\partial P_\phi}.$$  

(2.53)

Noticing that the derivative with respect to $P_\phi$ along the path $E' = E - \omega P_\phi/n$ can be written as

$$\frac{\partial f(P_\phi, E(P_\phi, E'))}{\partial P_\phi} \bigg|_{E'} = \frac{\partial f(P_\phi, E(P_\phi, E'))}{\partial P_\phi} \bigg|_{E'} + \frac{\partial f(P_\phi, E(P_\phi, E'))}{\partial E} \frac{\partial E(P_\phi, E')}{\partial P_\phi} \bigg|_{E'}$$

$$= \frac{\partial f(P_\phi, E)}{\partial P_\phi} + \omega \frac{\partial f(P_\phi, E)}{n \partial E},$$

(2.54)
where \( E' \) is used to mean the derivative is taken such that \( E' \) is held constant. This results in writing the differential as

\[
\omega \frac{\partial}{\partial E} + n \frac{\partial}{\partial P_\phi} = n \frac{\partial}{\partial P_\phi} \bigg|_{E'}.
\]

This way the operator for QL equations is written as

\[
\frac{\partial}{\partial I} = \omega \frac{\partial}{\partial E} + n \frac{\partial}{\partial P_\phi} = n \frac{\partial}{\partial P_\phi} \bigg|_{E'}.
\]

Since the same angles are used where the transformation is made on the action alone, the Jacobian for transforming from \((P_\phi, P_\theta)\) to \((P_\phi, E)\) is simply \(\partial P_\theta / \partial E\) the integral thus becomes.

\[
\int dP_\phi dP_\theta = \int dP_\phi dE \frac{\partial P_\theta}{\partial E} \rightarrow \int dP_\phi dE \tau_\theta,
\]

where \(\tau_\theta = 1/\bar{\omega}_\theta\) is a result of Hamilton’s equations

\[
\bar{\omega}_\theta = \frac{d\xi_\theta}{dt} = \frac{\partial H}{\partial P_\theta},
\]

where \(\bar{\omega}_\theta\) is the frequency of gyration of the angle \(\xi_\theta\), which is constant since \((\xi_\theta, P_\theta)\) are action-angle variable of \(H_0\). This is the same frequency see in the resonant condition

\[
l \cdot \Omega = n\bar{\omega}_\phi + m\bar{\omega}_\theta = \omega,
\]

where \(n\) is a component of \(l\).

### 2.4.2 CQL in Terms of \(P_\phi\)

Using Eq. (2.56) it becomes straightforward to write Eq. (2.20) in the phase-space variables \((P_\phi, E)\). The growth rate is normalized with the mode momentum, \(M_\alpha\) instead of the energy and the relation \(M_\alpha = nW/\omega\) is used. This results in the
following coupled equations:

\[
\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_l \frac{\partial}{\partial P_\phi} \bigg|_{E'} (2nC(t)V_l)^2 \delta(P_\phi - P_{\phi_0}) \frac{\partial \Omega_l}{\partial P_\phi} \bigg|_{E'} \frac{\partial}{\partial P_\phi} \bigg|_{E'} f 
\]

\[
\gamma = \frac{nP/\omega}{2M_\alpha C^2} = \frac{\pi}{4M_\alpha} \int dP_\phi \tau_0 dE \sum_l (2nV_l)^2 \delta(P_\phi - P_{\phi_0}) \frac{\partial \Omega_l}{\partial P_\phi} \bigg|_{E'} \frac{\partial f}{\partial P_\phi} \bigg|_{E'},
\]

where \(P_{\phi_0}\) is the position of the resonance for each \(E'\) satisfying the resonant condition \(\Omega_l(P_{\phi_0}, E(E', P_{\phi_0}), \mu) = \omega\). Each different \(l\) has a different slope for the directions \(E'\) since \(l\) has \(n\) as one of its components. This makes the diffusion two-dimensional.

### 2.5 Assumptions and Validity Limits for CQL

Conventional quasilinear theory is a direct application of mean-field theory for the evolution of the particles resonantly interacting with modes. It assumes the particle distribution function can be considered as two parts: a rapidly oscillating part, which is averaged over (taken the mean square of) to study the part that is slowly evolving with time, itself an ensemble average. The CQL equations are derived with a multitude of assumptions regarding regimes of applicability and amplitude of mode.

- **Amplitude of Mode:**
  The theory is developed to treat the regime of weak turbulence where the plasma oscillations are much less than the total thermal energy of the particles but also much larger than the thermal noise energy of plasma waves \[39\]:

\[
nT \gg W \gg \frac{nT}{N_D},
\]

where \(W\) is the total energy of the mode, \(T\) is the plasma temperature, and \(N_D\) is the number of particles in a Debye sphere.
• Regime:

The theory only holds when there is full stochasticity resulting in irreversible diffusion. This is fulfilled when the islands in phase space overlap. Therefore the separation of the resonances need to be less than the island width so that the Chirikov condition is met.

• Time Scales:

The derivation of the quasilinear equations requires the use of the linearized trajectory of the unperturbed motion. This is only true as long as the dynamics does not become coherent. This requires the autocorrelation time scale $\tau_c$ for the modes to be less than the bounce time, $\tau_b$, which is the inverse of the bounce frequency, Eq. (2.47). This is equivalent to saying that the mode should have a broad spectrum.

Also, in the transition from Vlasov’s equation to Eq. (2.20), the closure of the equation is only meaningful if the distribution function $\langle f \rangle$ relaxes at a time scale $\tau_{rel}$ much longer than the time scales for the mode growth $1/\gamma_L$ or the autocorrelation time scales $\tau_{ac}$. Noting that the bounce frequency is comparable to the growth rate, one gets $1/\tau_b \approx \gamma_L$. Putting all this together the following condition for time scales [55, 56] for the applicability of CQL is found:

\[
\frac{1}{\tau_{rel}} < \gamma_L \approx \frac{1}{\tau_b} < \frac{1}{\tau_c}. \tag{2.62}
\]

These are some of the assumptions implicit in the derivation of CQL and the limitations thereafter it creates on the application of the theory. Efforts have made to alleviate some of these assumptions. For example, Dupree [57] has modified the diffusion operator to study the strong turbulence with quasilinear theory, which has been adopted and further developed [58] but also has been criticized [59, 60].
2.6 Conclusion

I have given a thorough account of the conventional quasilinear theory describing weak turbulence, the derivation of the equations, and the limitations and applicability of the theory. This is necessary to set the foundations for the 1.5D and LBQ2D model that are developed in this manuscript.

However, this chapter has been an introduction to the conventional quasilinear theory as first developed without presenting results of the extensive research over the following decades. The 1.5D and LBQ2D are both based on CQL but are reduced models that are meant to give estimates of the effect of TAE modes on Energetic Particles in Tokamaks. The following chapters examine the regimes in which the reduced models are applicable. While the limitations and regimes of validity for 1.5D and LBQ2D are different from that of CQL, we wanted to give the reader a complete presentation of the CQL and its validity/limitations.
Chapter 3

1.5D Reduced Quasilinear Theory

3.1 Introduction

This chapter examines the 1.5D reduced quasilinear model that computes the effect of the destabilization of Alfvénic Eignemodes on the fast-ion profiles. The 1.5-dimensional term is introduced to account for the real space modification of the density profile and an added 0.5 for the modification of the velocity profile of fast ions that are in resonance.

To study the energetic particle interaction with Alfvénic modes, present-day machines operate in regimes where the Alfvénic velocity $v_A = B/\sqrt{4\pi n_i m_i}$ is comparable to that of the injected beams. Alfvénic modes can be strongly damped by the continuum. However, due to toroidicity, gaps in the continuum arise that allow discrete modes, such as Toroidicity induced Alfvén Eigenmodes (TAE), to exist and these waves have relatively small intrinsic damping when the driving energetic particles are absent. These discrete modes can be driven unstable by the free energy source of the energetic particles posing, therefore, a threat to energetic ion confinement.

Many present-day experiments observe TAE activity with accompanying ion losses and profile relaxation [01]. Under some circumstances, a few modes expel fast ions
from the plasma in bursts of activity. This was the case, for example, in the initial TAE experiments on DIII-D [62] and TFTR [63]. In these cases, fast-ion transport occurs in a burst cycle reminiscent of predator-prey models [64] that cannot be described by quasilinear models. In contrast, some contemporary experiments contain a large number of relatively small amplitude Alfvén instabilities. Examples include the TAE and RSAE activity observed during beam-heated reversed shear discharges in DIII-D [65] and the TAE and “tornado” modes observed during ion cyclotron heating on JT-60U [66], JET [67], or TFTR [68]. In the reversed shear plasmas, the hundreds of wave-particle resonances associated with the Alfvén instabilities flatten the fast-ion profile [69]. Modeling of these plasmas finds stochastic, diffusive transport over the affected region [70]. Under these circumstances, which may also govern fast-ion transport in ITER, a quasilinear model is expected to be reasonable. The predictions of this proposed 1.5D quasilinear model are compared to the measured losses in DIII-D’s well-diagnosed discharges #142111 [11] and #122117 [12]. 1.5D achieves good correlation with the experimental results.

3.2 1.5D model

This section describes the assumptions and approximations made to allow for reduction of the quasilinear theory of diffusion to a 1.5-dimensional model. Then, a detailed description of the 1.5D model is given and the method for integrating the relaxed profiles is described. Finally, an account of the analytic growth/damping rates is represented.

3.2.1 Assumptions and Approximations

- **Radial redistribution only.** The assumption $\omega_{EP} \gg \omega_{TAE}$, where $\omega_{EP}$ is the mean drift frequency of the energetic particles, is taken everywhere. This entails
that significant diffusion takes place only in $P_\phi$ or the radial direction and transport in $E$ or $v$ direction is neglected. The 1.5D model is developed to find the resulting relaxation of the EP pressure profile in the radial direction. In full dimensionality, the diffusion coefficient $D_k$ is a function of all the phase-space variables. Since it is the diffusion in the radial direction that is of concern, $D_k$ is denoted by $D(r)$ in what follows.

- **Accounting for phase-space relaxation.** Since the pressure profiles are obtained by integrating the distribution function over the velocity, Kolesnichenko’s calculations [71] is used to find the ratio of particles, $\eta$, that are in resonance with the TAEs. It is roughly estimated for two cases, alphas and beams, as follows:

\[
\eta = \frac{(v_{a0} - v_{||})v_{||}}{v_{a0}^2} \quad \text{for isotropic distributions,} \quad (3.1)
\]
\[
\eta = \frac{v_{||}^2}{v_{b0}^2} \quad \text{for anisotropic distributions.} \quad (3.2)
\]

This means that only an $\eta$ fraction of fast ions relax, while $1 - \eta$ of them remain unchanged, where $\eta \leq 1$ since $v_{||} \leq v_{a0,b0}$. Figure 3.1 illustrates the resonant region part in phase space.

- **Local growth and damping rates.** Local expressions for the growth and damping rates of TAEs are used, while the modes are, in actuality, global. Therefore, the use of a comprehensive code such as NOVA-K [30] is required. Then the values of the damping and growth rate computations are used to normalize the rates. Having the damping and growth rates normalized in this manner, their analytically computed dependence on the plasma parameters is maintained. This way the 1.5D quasilinear model is expected to describe the functional dependencies correctly. In addition, the rates are averaged over the linear eigenmode width $\Delta_m$ to account
Figure 3.1: Depiction of the resonant and non-resonant regions of \((v_{\parallel}, v_{\perp})\) phase space for finding the contribution of the velocity to the redistribution of energetic particles.

for the spatial breadth of the mode.

- **Instant radial transport.** There is no consideration for the transition time period in which the mode and the distribution both evolve with time. Instead, it is assumed that the quasilinear equations will bring the mode to a marginal stability state and a near-steady-state distribution function and fields will be achieved to bring about a steady state, \(D(r) \to 0 \Rightarrow \partial f(r, t)/\partial t = 0\). This happens as a result of fast-ion diffusion relaxation in all unstable regions of phase space, which diminishes the gradient in the EP pressure profile until the plasma state \(\gamma_L \leq \gamma_d\) is achieved everywhere.

### 3.2.2 Integrating the Relaxed Profile

Assuming that the background plasma remains unaffected, the local damping rates are unchanged. The estimated local critical EP pressure gradient is calculated by balanc-
ing the TAE drive to the most important damping mechanisms. TRANSP \cite{transp} provides the pressure profiles, established by classical relaxation processes (which when unstable will relax to marginal stability profiles in accord with 1.5D assumptions) as well as any other equilibrium plasma quantity needed in the 1.5D calculation.

The analytic expressions for growth \cite{growth, growth2} and damping \cite{damping, damping2, damping3, damping4} rates assume the existence of a multitude of TAEs centered at all radial positions. Each of these radially localized modes have rates that depend on the mode numbers $n$, $m$, and other local plasma quantities. This will be detailed in Sec. 3.2.3.

Many modes can exist at every radial location, but the growth rate of the most unstable mode is used. This is theoretically predicted \cite{theory} to be around $\rho_{\text{EP}} k_\perp \approx 1$ where $\rho_{\text{EP}}$ is the particle orbit width due to cross-field drift.

It is shown that there is a plateau \cite{plateau} in the growth rate as a function of toroidal mode number $n$ for which $\gamma_L$ is maximized at a given radial location,

$$\frac{r^2 \omega_c}{R q^2 v_A} \approx n_{\text{min}} < n < n_{\text{max}} \approx \frac{r \omega_c}{q^2 v_A},$$

(3.3)

where $\omega_c$ is the EP cyclotron frequency, $q$ is the safety factor, and $v_A$ the Alfvénic velocity. The growth rate chosen at a given radius is that of the mode whose $n$ number satisfies Eq. (3.3). The poloidal mode number $m$ is related to $n$ through $q(r) = (m + 1/2)/n$. This specifies the mode for which the rates are computed at a given radial location.

For each mode, the absolute value of the local linear damping rate $\gamma_d$ is assumed fixed since it depends on the unaffected background plasma. However, the mode’s growth rate $\gamma_L$ is a function of the gradient of the EP pressure, which undergoes relaxation in the presence of TAEs. In regions where the mode is unstable, $\gamma_L > \gamma_d$, the gradient diminishes as energetic particles diffuse due to their resonant interaction with TAE modes. Assuming that the transport is strongest in the $P_\phi$, i.e., radial
direction, as would be the case for $\omega_s E \gg \omega$, any transport in the E direction is neglected. Therefore, the growth rate is expressed as a function of the radial gradient in EP pressure profile, $\gamma_L = \gamma'_L \partial \beta / \partial r$ where $\gamma'_L$ is independent of EP profiles.

At marginal stability, the gradient assumes a critical value such that the growth rate equals the damping rate, $\gamma_d = \gamma'_L (\partial \beta / \partial r)_{\text{crit}}$. The unperturbed plasma profiles are computed using equilibrium codes such as TRANSP, from which $\gamma_d$ is calculated everywhere and $\gamma'_L$ is computed. This is used to calculate the critical value of the EP gradient

$$\frac{\partial \beta_{\text{EP}}}{\partial r} |_{\text{crit}} = \frac{\gamma'_L}{\gamma_d}.$$  \hspace{1cm} (3.4)

Note how the right-hand side of Eq. (3.4) is independent of $\beta_{\text{EP}}$ and only depends on the unchanged background plasma.

In the 1.5-dimensional quasilinear diffusion equation $D(r)$ grows at a rate proportional to $\gamma_L - \gamma_d$. This results in diffusion of particles in the unstable region, where $D(r) > 0$. The changing radial profile leads to an increased instability drive that extends the unstable region into neighboring regions that were originally stable. Particles continue to undergo diffusion in the unstable region, until $\partial \beta_{\text{EP}} / \partial r$ decreases to the point where $\gamma_L \leq \gamma_d$. The relaxed $\beta$ profiles are integrated assuming the critical value of the gradient in beta $\partial \beta / \partial r |_{\text{crit}}$ in regions of TAE instability, $[r_1, r_2]$ where $\partial \beta_{\text{EP}} / \partial r > \partial \beta_{\text{EP}} / \partial r |_{\text{crit}}$, and maintain the original value in regions that are stable. To insure the continuity and the conservation of particles, the boundaries of the unstable region are extended to $[r_-, r_+]$ iteratively until both conditions, continuity and particle conservation, are satisfied. As a result, the pressure profile is modified over a region larger than the initially unstable region. This is a feature that was realized in the pioneering work in quasilinear theory [39, 79].

Figure 3.2 is an illustration of a redistribution resulting in a profile $f(r)$ with its corresponding critical gradient $\partial f / \partial r |_{\text{crit}}$ everywhere. The calculation for the illustration is done using the 1.5D model integration scheme described above for an
arbitrary \( f(r) = (1 - r^2)^{5/2} \) and arbitrary critical gradient \( \partial f(r)/\partial r|_{\text{crit}} = 20(2r - 0.8)(r - 0.6) + 0.6 \) where \( r = [0, 1] \).

![Graph](image)

Figure 3.2: An illustration for the integration scheme of 1.5D. (a) The values for the original gradient in pressure profile, (blue) and the critical gradient, (red). \([r_-, r_+]\) is the original region of instability where \( \partial \beta / \partial r > \partial \beta / \partial r|_{\text{crit}} \). (b) The original pressure profile (blue) and the relaxed profile (red). The redistribution has extended into the region \([r_1, r_2]\), which is larger than the originally unstable region \([r_1, r_2]\).

This mechanism agrees with the quasilinear theory of diffusion in which regions of instability extend into originally stable regions as particles diffuse into them. A similar effect is known to be significant in the plasma turbulence area [80].

One can see how, in 1.5D, this behavior is captured without the detailed calculation of the perturbed fields that produce this diffusion coefficient. Therefore 1.5D is capable of quickly and efficiently predicting an estimate of the amount of redistribution and transport of energetic particles.

### 3.2.3 Growth and Damping Rates

For TAE modes to be excited, the mode’s growth rate must be larger than the sum of all damping rates. NOVA simulations indicate that the different damping mechanisms are of varying importance depending on the fusion device and the plasma parameters. It has been found for typical parameters projected for ITER [3, 27, 76, 81–83] that ion Landau damping and radiative damping are predominant with the electron collisional
damping negligible. On the other hand, in the DIII-D experiments investigated here, radiative and electron collisional damping are the main damping mechanisms. Unlike the rest of the damping mechanisms, radiative damping is non-perturbative.

**Radiative Damping**

Radiative damping results from the coupling of short wavelength kinetic Alfvén waves (KAW) to the TAE mode. It should be computed in the two-fluid MHD-like models, where the finite Larmor radius effect (FLR) and electron pressure perturbations are used to extend the second-order differential equation structure to the fourth order. The dissipation on short wavelength KAW structures leads to finite damping of TAE modes. Although radiative damping is non-perturbative, NOVA-K computes it perturbatively \[84-86\] following the formulation in Ref. \[87\], which limits the range of applicability to the cases of low shear and low damping.

The expression \[76\] used for the analytic calculation of radiative damping is

\[
\frac{\gamma_{\text{rad}}}{\omega} = -3 \left| \frac{\sqrt{\rho^2} sm(m + 1)}{r \sqrt{2}} \right|^{0.67},
\]

(3.5)

where \(s\) is the local shear at the location of the mode, \(m\) is the poloidal mode number and \(\rho = v_s/\omega\) with \(v_s\) the speed of sound and \(\omega\) is the ion gyrofrequency.

The following damping mechanisms are derived perturbatively from linear theory.

**Ion Landau Damping**

The following expression \[88\] derived for kinetic response due to the collisionless interaction of the background ions is

\[
\frac{\gamma_{\text{iL}}}{\omega} = -\frac{q^2 \sigma \sqrt{\pi}}{18(1 + \sigma/4)} \beta_i^3 e^{-\beta_i^2},
\]

(3.6)
where $\sigma$ is the plasma ion depletion factor $\sigma = n_i/n_e$ and $x_i = v_A/3v_i$.

**Electron Landau Damping**

The analytic expression for the damping rate resulting from the collisionless interaction of the background electrons is derived by Candy [74] in the limit of low beta, large aspect ratio. The expression is

$$\frac{\gamma_e L}{\omega} = -\frac{\pi^{3/2}}{6} q^2 \beta e v_e \left( \frac{5}{2} \right)^{1/2} G(\hat{\epsilon}) e^{1/2};$$

with $G(\hat{\epsilon}) \approx 4.47 - 0.42\hat{\epsilon} + 0.02\hat{\epsilon}^2$, $s$ is the local shear, $\beta e$ is the electron $\beta$, and $\hat{\epsilon} = 2\epsilon/(1 - \epsilon)$.

**Electron Collisional Damping**

The electron kinetic response [75] affects the TAE modes through the perturbation in their distribution function resulting from collisional damping with electrons. Since $v_{Te} \gg v_{Ta}$, only electrons with velocities $v_\parallel \ll v_{Te}$ would interact with the Alfvén modes. Therefore the main contribution comes from trapped particles where $v_\parallel \ll v_\perp$. The expression [27] for the resulting damping rate is

$$\frac{\gamma_{eColl}}{\omega} = -\frac{1}{4} \left( I_1 \left( \frac{8snq\rho_s}{5r\epsilon} \right)^2 + I_2 q^2 \frac{8\beta_{pc}}{1 + \sigma} \right) \sqrt{\frac{\nu}{\omega}} \ln \left( 16 \sqrt{\frac{\omega \epsilon}{\nu}} \right)^{-3/2},$$

where $\beta_{pc}$ is the core plasma $\beta$, $I_1 = 0.43Z_{eff} + 1.06$ and $I_2 = 1.03Z_{eff} + 2.3$, with $Z_{eff} = 6 - 5\sigma$ and the electron collision frequency

$$\frac{\nu}{\omega} = \frac{4\pi n_e e^4 \ln \Lambda_e}{\omega m_e^2 v_{Te}^3}.$$  

**Growth Rate**

The main mechanism for the TAE drive is provided by the free energy from the gradient in the energetic particle distribution. Using the linear theory, one can find
the perturbed EP distribution function \( \tilde{f} \) from which the current density is evaluated:
\[ \tilde{j} = \int d^3v \, v \, \tilde{f}. \]
The energetic particle kinetic response is thus calculated and different expressions can be derived for various limits, such as passing and trapped or whether \( \Delta_m \ll \Delta_b \) or vice-versa. The expression is only valid in the limit where \( \rho_\alpha \ll \Delta_m \), \( \rho_\alpha \) being the fast-ion gyroradius.

To find the kinetic response of the energetic particles, the drift kinetic equation is used to write the analytic form \[73\] for the growth rate.

### 3.2.4 Normalizing with NOVA-K

The above analytic expressions are used for computing the local growth and damping rates of the TAE modes at all radial positions. For a quantitatively accurate representation, the values are calibrated to those computed by the established code, NOVA-K. This procedure allows for more accurate application of the 1.5D model on experiments such as DIII-D.

The NOVA-K code computes exactly the TAE growth rate, including effects coming from TAE mode structure, finite Larmor radius, and drift orbit radial width. It also computes the damping rates. This is an area being successfully investigated by the ITPA working group \[89\]. A specific procedure is developed in order to include the finite orbit width effect in the drive. First, NOVA code is used to find several mode structures localized at different radii. To simplify the NOVA study focus only one toroidal mode number is used, which is taken at a value approximately corresponding to the maximum (plateau region) value of the growth rate as a function of the ratio \( \Delta_m / \Delta_b \). The goal here is to evaluate the growth rate in the plateau regime, which would give the required approximation for the TAE-like mode growth rate for the 1.5D model developed here. However, because of the finite value of \( n \), it is unlikely that the maximum of the growth rate plateau is hit. Instead of going through the different choices of \( n \) numbers \( z_{EP} \), the particle charge is what is changed.
Note that in the expression for the drive the particle’s charge enters only through the combination \( n/z_{\text{EP}} \). Thus, with the fixed \( n \) value one can find the plateau regime by changing \( z_{\text{EP}} \) and finding the growth rate of the most unstable mode. This procedure is robust since the plateau regime is found regardless of whether \( n \) or \( n + 1 \) is used and \( z_{\text{EP}} \) is varied in the search.

For the application of 1.5D on DIII-D results, a two-point normalization scheme is used, where the two most unstable modes (localized at two radial points) are found. One mode is usually relatively close to the center of the plasma \( r_1 \approx 0.3a \) and the other is towards the edge \( r_2 \approx 0.7a \) where \( a \) is the minor radius. The analytically computed rates are modified by a constant multiple \( g_1 = \gamma_{1(\text{NOVA})}/\gamma_{\text{anlt}}(r_1) \) for \( r < r_1 \) so that it coincides with the value of the NOVA-K computed one at \( r_1 \). The procedure is identical for \( r > r_2 \), which is multiplied by \( g_2 = \gamma_{2(\text{NOVA})}/\gamma_{\text{anlt}}(r_2) \) where \( \gamma_{1,2(\text{NOVA})} \) are the NOVA computed rates of the modes around \( r_{1,2} \). For \( r_1 < r < r_2 \) the rate is multiplied by a linear function \( ar + b \) such that \( ar_1 + b = g_1 \) and \( ar_2 + b = g_2 \).

For robustness, the code is built such that the input for normalization can be in two forms. On one hand, the input is the normalization constant at two given radial positions from which \( g(r) \) is directly known and multiplies the analytic rates. The same normalizations for multiple times of interest for the same discharge, such as in the following case of #127111 for \( t > 600 \) ms. This method can be useful in producing PopCon figures where a scan over many parameters is performed as will be discussed in Sec. 3.5.1. Alternatively, the input can be NOVA-K values for the rates at two given radial positions from which the normalization constants are then computed and in turn used on the analytic profiles. This is done when analyzing specific times distinctly as in #142111 and for #122117 at \( t < 600 \) ms as will be described later.
3.2.5 Computing Losses

To account for the losses of particles, the condition for the conservation of particles is lifted once the region of instability grows large enough to result in particle loss at the edge. While integrating for the relaxed profiles according to the method discussed in Sec. 3.2.2, fast ions are lost when the boundary $r_+ \rightarrow r_b$ reaches the loss boundary at $r_b = a - \Delta f/2 + \rho_f$ where $a$ is the minor radius $\Delta f = q\rho_f$ is the estimate for the fast-ion orbit width, and $\rho_f$ is the fast-ion Larmor radius. The profile is then integrated with $r_+ = r_b$ and the fraction of lost particles is computed as follows:

$$L = \frac{\int_0^a dr [\beta_{\text{rlx}}(r) - \beta_{\text{ini}}(r)]}{\int_0^a dr \beta_{\text{ini}}(r)},$$  \hspace{1cm} (3.10)$$

where $\beta_i$ is the ion pressure, $\beta_{\text{ini}}$ is the initial beam pressure, and $\beta_{\text{rlx}}$ is the pressure of the relaxed beam.

To compare with the experimental observation of the neutron rate depletion, we calculate the fraction of the neutrons expected with the relaxed profile to that expected from the initial profile using the cross section $\langle \sigma v \rangle$ as derived from TRANSP. The following is the expression used

$$L_n = \frac{\int_0^a dr \langle \sigma v \rangle \beta_i \beta_{\text{rlx}}}{\int_0^a dr \beta_i \beta_{\text{ini}}};$$  \hspace{1cm} (3.11)$$

Even if energetic particles are not lost to the wall, the mere modification in their $\beta$ profile due to redistribution can result in a change in the neutron rate, albeit less significantly.

3.3 Observed DIII-D flattening

A series of DIII-D experiments \cite{62, 65, 90} has been conducted to investigate TAE activity and AE induced fast-ion transport. It has been observed that a rich spec-
trum of TAE and Reverse Shear Eigenmodes (RSAE), (also called Alfvén Cascade modes) are produced when high energy neutral beams are injected into reversed shear plasmas. Two discharges have been extensively diagnosed and analyzed, discharge #122117 and discharge #142111, where AE activity has been associated with significantly flattened fast-ion pressure profiles and losses.

The first discharge shows the presence of strong TAE and RSAE modes and resulting losses that are captured by 1.5D. The second discharge is of special interest, since during the time span of the discharge, TAE/RSAE activity is significantly reduced until the instability becomes marginal at some point. This provides a sensitive test for theoretical models, such as the 1.5D model developed here.

### 3.3.1 Diagnostics

Alfvénic activity is detected with a suite of fluctuation diagnostics that are sensitive to Alfvén instability oscillations. Electron temperature is measured with electron cyclotron emission (ECE). CO$_2$ interferometer, beam-emission spectroscopy, and reflectometers are used for measuring the density fluctuations, and Mirnov coils are used to measure the magnetic fluctuations.

As for the fast-ion diagnostics, there are four main diagnostics used.

- **A Plastic Scintillator** is used to measure the volume-averaged neutron emission rate that is predominantly due to plasma-beam interaction making it a direct function of the energetic particle pressure profiles.

- **FIDA** is a charge exchange recombination spectrometer technique that uses the Doppler-shifted Balmer-$\alpha$ light emitted by fast ions. This is used to measure the fast-ion profile.

- **MSE** is a diagnostic that depends on the motional Stark effect to find the internal magnetic fields. The total pressure is computed using MHD equilibria obtained
from the measured MSE magnetic fields. Subtracting the plasma pressure (computed using measured temperature and density profiles) from this total pressure results in the fast-ion pressure profile.

- FILD is a scintillator-based Fast Ion Loss Detector \cite{91} inserted just outside the last closed flux surface to detect fast ions over a large region of the phase space, \((E, \mu)\) where \(E\) is the energy and \(\mu\) the pitch-angle.

3.3.2 Experimental Observation

I present an overview of the experimental results for discharges #122117 and #142111 that will be used to validate the 1.5D model. The results presented here on experiments are adapted from works of Van Zeeland \cite{11} and Heidbrink \cite{12}.

**Discharge # 122117**

In the baseline shot #122117, a deuterium neutral beam is injected with \(P_{\text{NBI}} = 4.6 \text{ MW} \), \(E_{00} = 80 \text{ keV} \), at a tangency radius of \(1.15 \text{ m}\). The peak of the energetic-particle distribution is around \(v_{||}/v = 0.68\). AE activity is detected using the diagnostics described in Sec. 3.3.1 and depicted in Fig. 3.3 below. The EP detectors reveal flattening in the profiles around the center at all times of interest. Neutron deficits like those observed in #122117 can be a result of fast-ion redistribution to larger radii as well as fast-ion loss. My simulations (which will be discussed in the next section) show that early during the discharge, there is a combination of EP losses and redistribution in phase space, while later during the current ramp, the EP are predominately redistributed without losses resulting from particle transport beyond the last closed flux surface.

**Discharge #142111**

Another well-diagnosed discharge is #142111 \cite{11}. The on-axis magnetic field for this discharge is \(B_T = 2 \text{ T} \), and the current increases at a rate 0.8 MA/s until it reaches 1.2 MA at \(t = 1000\). 80 keV neutral beams are injected at \(t = 300 \text{ ms}\) with a
Figure 3.3: Radial profile of ECE radiometer power spectrum depicting the observed TAE and RSAE activity in DIII-D discharge # 122117 at $t = 410$ ms. The solid line is the safety factor profile, $q$. These measurements depict the existence of AEs around the minimum of $q$ and extending to $r/a = 0.8$. Adapted from M. A. Van Zeeland, Phys. Rev. Lett., 97, 135001-2, 2006. Copyright (2006) American Physical Society.

Figure 3.4: A depiction of the unperturbed profiles from TRANSP in dashed lines of different colors for the times, $t = 360$ ms, $t = 780$ ms, and $t = 1200$ ms. The profiles are modified due to interaction with TAE activity depicted in Fig. 3.3 and the FIDA measurements are presented as dots while the MSE fit is in solid lines. Adapted from W. W. Heidbrink et al., Nucl. Fusion, 48, 6 (2008). Reprinted with permission of IOP Publishing Ltd.
total power of 6.8 MW. Figure 3.5 shows the AE activity as a function of time and the associated EP coherent losses detected by FILD. Note that the FILD-detected coherent losses correspond primarily to TAE frequencies.

Figure 3.5: (a) FILD activity signifying coherent NBI loss peaking around $t = 500$ ms and disappearing after $t = 800$ ms. (b) AE activity measured throughout the discharge in the spectrum of Mirnov and CO$_2$ interferometer data. Adapted from M. A. Van Zeeland, Phys. Plasmas, 18, 056114-5, (2011). Copyright (2011) American Institute of Physics.

3.4 Comparison of 1.5D with DIII-D measurements

The extensive DIII-D experiments that are dedicated to the study of AEs and their interaction with energetic particles readily lend themselves for validation of EP trans-
port models and numerical simulations. The most notable discharges, #142111 and #122117, are used here to validate the proposed 1.5D model.

TRANSP provides the machine and plasma parameters such as the major radius $R$, minor radius $a$, and on-axis magnetic field $B$, as well as the plasma and beam profiles. The profiles used for this 1.5D model are the ion temperature $T_i(r)$, electron temperature $T_e(r)$, plasma pressure $\beta_{pc}(r)$, beam pressure $\beta_{bm}(r)$, electron pressure $\beta_e(r)$, ion density $n_i(r)$, and the safety factor $q(r)$. Once these parameters and profiles are accessible, the analytic linear growth and damping rates, as discussed in Sec. 3.2.3, are easily calculated for the most unstable mode at each radial position. These rates are then calibrated to NOVA-K computed linear rates using a normalization scheme described in Sec. 3.2.4.

### 3.4.1 NOVA-K Simulations

Due to its crucial role in determining the results of 1.5D on experimental data, the NOVA-K computed eigenmodes are presented along with their linear growth and damping rates that are used to normalize the analytically computed rates of the two discharges analyzed. A two-point normalization scheme is used, which entails finding two modes and their rates for each case analyzed. The procedure is discussed in detail in Sec. 3.2.4.

**Discharge #122117**

Results of NOVA-K simulations are used for the rates at two radial positions to normalize the analytically computed rates as a function of radius. The safety factor measurements have been optimized for $t < 600$ ms, so NOVA-K values at $t = 360$ ms and $t = 410$ ms are chosen to compare the model’s predictions for neutron losses with the experimentally detected ones. However, since FIDA results for later times, $t = 780$ ms and $t = 1200$ ms, also exist, the rates computed by NOVA-K at $t = 360$ ms are used to normalize later times where the TAE-EP interaction is also modeled.
using 1.5D. Figure 3.6 presents the mode structures of the TAE modes computed by NOVA for discharge #122117 at $t = 360$ ms. However, we present the rates computed for both $t = 360$ ms and $t = 410$ ms for which NOVA-K was used.

Figure 3.6: The structure of modes localized around $r = 0.32$ and $r = 0.78$ for shot #122117 at $t = 360$ ms as computed by NOVA for $n = 3$. The modes are computed as a function of $\sqrt{\psi/\psi_0}$, the square root of the normalized poloidal field flux. The dotted line is the safety factor profile.

The following tables are the linear growth and damping rates that NOVA-K calculates for the two TAE modes, whose structures calculated by NOVA, are presented in Fig. 3.6.

For $t = 360$

<table>
<thead>
<tr>
<th>mode location</th>
<th>$\gamma_{\text{growth}}/\omega$</th>
<th>$\gamma_{\text{eColl}}/\omega$</th>
<th>$\gamma_{\text{rad}}/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r/a = 0.32$</td>
<td>11%</td>
<td>2%</td>
<td>0.3%</td>
</tr>
<tr>
<td>$r/a = 0.78$</td>
<td>1.7%</td>
<td>3%</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

For $t = 410$

<table>
<thead>
<tr>
<th>mode location</th>
<th>$\gamma_{\text{growth}}/\omega$</th>
<th>$\gamma_{\text{eColl}}/\omega$</th>
<th>$\gamma_{\text{rad}}/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r/a = 0.23$</td>
<td>26%</td>
<td>0.1%</td>
<td>1.4%</td>
</tr>
<tr>
<td>$r/a = 0.75$</td>
<td>4.1%</td>
<td>0.09%</td>
<td>8.8%</td>
</tr>
</tbody>
</table>

51
When there is continuum damping, it is added to the radiative damping. This was the case for $t = 410 \text{ ms}$.

NOVA is a perturbative ideal MHD code \cite{30} that solves in matrix form a reduced set of second-order differential equations in $\xi(\rho)$, the vector of amplitudes of poloidal harmonics of the radial displacement as a function of $\rho \equiv \sqrt{\psi/\psi_0}$, the square root of the normalized poloidal field flux. However, it is possible that the numerical solution propagates into the continuum and still has finite Fourier components. It is an approximation but it is consistent with expectations from ideal MHD. This results in a jump in the imaginary part of the eigenmode structure giving rise to the continuum damping. However, since this procedure only provides the real part of the eigenvalue for the singular TAE mode, the flux function $\hat{C}(d\xi/d\rho)$, which is continuous around the resonance with the continuum, is introduced \cite{84} to find the effect of the interaction with the continuum. The MHD equations written in terms of flux function are not singular and one can use the perturbative technique to find the complex correction resulting from the interaction of the mode with the continuum. The details are found in Gorelenkov’s treatment \cite{84}.

**Discharge #142111**

For this discharge, the safety-factor measurements have been optimized for $t = 425, 525, 675, 725, 800, \text{ and } 975 \text{ ms}$. Therefore, NOVA-K simulations are made at all these times to compare the model expectations with the experimental results. To avoid redundancy, only the mode structures of the $t = 675 \text{ ms}$ for discharge #142111 is shown while the rates of the most unstable modes are presented at two given radii for all the times analyzed. The computed NOVA-K linear growth and damping rates are:
Figure 3.7: The structure of modes localized around \( r = 0.25 \) and \( r = 0.75 \) for shot \#142111 at \( t = 675 \) ms as computed by NOVA for \( n = 4 \). The dotted line is the safety factor profile.

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>( \gamma_{\text{growth}}/\omega )</th>
<th>( \gamma_{\text{eColl}}/\omega )</th>
<th>( \gamma_{\text{rad}}/\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r/a = 3 )</td>
<td>8.23%</td>
<td>0.6%</td>
<td>1.8%</td>
</tr>
<tr>
<td>( r/a = 0.82 )</td>
<td>0.37%</td>
<td>2.2%</td>
<td>0.733%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>( \gamma_{\text{growth}}/\omega )</th>
<th>( \gamma_{\text{eColl}}/\omega )</th>
<th>( \gamma_{\text{rad}}/\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r/a = 0.23 )</td>
<td>11%</td>
<td>0.16%</td>
<td>1.8%</td>
</tr>
<tr>
<td>( r/a = 0.83 )</td>
<td>4.9%</td>
<td>0.29%</td>
<td>0.67%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>( \gamma_{\text{growth}}/\omega )</th>
<th>( \gamma_{\text{eColl}}/\omega )</th>
<th>( \gamma_{\text{rad}}/\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r/a = 0.25 )</td>
<td>22%</td>
<td>0.15%</td>
<td>3.5%</td>
</tr>
<tr>
<td>( r/a = 0.75 )</td>
<td>4.9%</td>
<td>0.16%</td>
<td>2.7%</td>
</tr>
</tbody>
</table>
For $t = 725$

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>$\gamma_{\text{growth}}/\omega$</th>
<th>$\gamma_{\text{coll}}/\omega$</th>
<th>$\gamma_{\text{rad}}/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r/a = 0.3$</td>
<td>56%</td>
<td>0.11%</td>
<td>1.3%</td>
</tr>
<tr>
<td>$r/a = 0.85$</td>
<td>3.1%</td>
<td>2.7%</td>
<td>1.3%</td>
</tr>
</tbody>
</table>

For $t = 800$

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>$\gamma_{\text{growth}}/\omega$</th>
<th>$\gamma_{\text{coll}}/\omega$</th>
<th>$\gamma_{\text{rad}}/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r/a = 0.2$</td>
<td>12.5%</td>
<td>0.21%</td>
<td>2.9%</td>
</tr>
<tr>
<td>$r/a = 0.8$</td>
<td>3%</td>
<td>0.11%</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

For $t = 975$

<table>
<thead>
<tr>
<th>Mode Location</th>
<th>$\gamma_{\text{growth}}/\omega$</th>
<th>$\gamma_{\text{coll}}/\omega$</th>
<th>$\gamma_{\text{rad}}/\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r/a = 0.2$</td>
<td>1.8%</td>
<td>0.8%</td>
<td>0.052%</td>
</tr>
<tr>
<td>$r/a = 0.8$</td>
<td>2.2%</td>
<td>18%</td>
<td>0.44%</td>
</tr>
</tbody>
</table>

3.4.2 1.5D results

To implement the 1.5D model on the DIII-D shots of interest, the analytic expressions for the growth rate are used in the limit of highly anisotropic passing particles with consideration for the finite orbit width. The expression therefore depends on the ratio $\Delta_b/\Delta_m$, where $\Delta_b = q\rho_b$ is the beam ion orbit width with $q$ the safety factor, $\rho_b$ is the EP Larmor radius, and $\Delta_m \approx r_m/m$ is the mode width with $r_m$ the location of the TAE mode and $m$ the poloidal mode number. Using local expressions for the case of the two discharges being analyzed, the value of $\Delta_b$ varies between 5 and 20 cm, while $\Delta_m$ is around 2 cm. Therefore, the following analytic expression [72, 73] is used for the growth rate expanded in the limit of $\Delta_b \geq \Delta_m$:

$$\frac{\gamma}{\omega} = -q_m^2 \beta_b^p \frac{\omega_p}{\omega_r} (1 + \chi_0^2)^2 \times 24 \frac{\Delta_m^3}{\xi^3} \sum_{s=\pm 1} y_s \theta(1 - y_s),$$

(3.12)

where $\theta$ is the step function, $q$ the safety factor where the mode is, $\omega_r$ the local diamagnetic frequency, $\chi_0$ the pitch-angle, $\xi = (1+\chi_0^2)q\nu_0/(\chi_0 \omega_r)$, and $y_s = v_A/(|2s-1|\chi_0 \nu_0)$
The critical value of the beam pressure profiles of discharge #122117 that would result in marginally stable AEs is calculated. Following the procedure described in Sec. 3.2.2 for calculating the functional form of the rates, Eq. (3.12) is used first to compute the growth rates, and Eqs. (3.5)–(3.8) to compute the damping rates. Then the results from NOVA-K simulations, as given in Sec. 3.4.1, are used to calibrate the analytical growth rate profiles using the two-point normalization scheme described in Sec. 3.2.4. Figure 3.8 depicts the 1.5D computed relaxed profiles. For $t < 600$ ms, NOVA-K is run for the two cases separately to calibrate the analytic expressions, however, for the two cases where $t > 600$ ms whose safety factors measurements are not optimized, the normalization rates calculated for $t = 360$ ms are used to compute the resulting distribution function and compare it to FIDA results. To avoid redundancy, only the resulting profile redistribution for $t = 360$ ms is shown as a representative of the times $t < 600$ ms where losses occur. The 1.5D predicted losses for these cases are provided in Fig. 3.9. The simulated losses and flattening for $t = 410$ ms, are similar to $t = 360$ ms. The 43% neutron losses for $t = 410$ ms further supports the results of the represented $t = 360$ ms case.

The predicted neutron losses from the 1.5D model are compared to the experimentally measured neutron deficits for DIII-D discharge #122117. The ratio of the neutron emission rate to the classical rate is computed using Eq. (3.11) and is represented in Fig. 3.9 as crosses in juxtaposition with the experimentally measured values. The predicted emission rates are quite close to the measured neutron loss rates.
Figure 3.8: Redistribution of energetic particles at $t = 360$ ms results in significant EP losses and associated neutron losses. At later times, AE activity results in the redistribution of particles without EP losses.

Discharge #142111

As discussed in Sec. 3.3.2, there are measurements of AE induced EP losses that peak around $t = 525$ ms. Although AE activity persists throughout most of the shot, the associated losses diminish after $t = 800$ ms.
In this section the results of 1.5D applied to \#142111 are presented. As shown in Fig. 3.10, AE activity around \( t = 425 \) ms causes redistribution of energetic particles contained within the last closed flux surface without resulting in losses. Modification of the EP pressure profile, however, results in some neutron losses in accordance with Eq. (3.11). TAE activity increases at \( t = 525 \) ms and EP redistribution now extends all the way to the last closed flux surface, resulting in their loss.

As time evolves, losses continue to occur up to \( t \approx 800 \) ms where the system reverts to transport and redistribution of EP without any losses. This is depicted in Fig. 3.10 for \( t = 925 \) ms, where the particles are redistributed due to the unstable central modes but the redistribution is contained within the last closed flux surface.

The results reported by Van Zeeland [11] on the experimental measurements of the neutron emission rate in comparison to the TRANSP expectations are used to validate the predictions of 1.5D. Figure 3.11 shows the model’s results in juxtaposition with the measurements of Van Zeeland.
Figure 3.10: Relaxation of EP distribution for different time slots. EP losses peak at around $t = 525$ ms while AE activity is evident throughout. At $t = 425$ ms and at $t = 975$ ms, TAE activity only results in redistribution of EP without resulting losses.

Figure 3.11: TRANSP predicted (red) and experimentally measured (blue) neutron emission rate as a function of time adapted from M. A. Van Zeeland, Phys. Plasmas, 18, 056114, (2011). Copyright (2011) American Institute of Physics. Green dots are added to the figure as markers for the 1.5D results for neutron rates at $t = 425, 525, 675, 725, 800$, and $975$ ms.
3.4.3 Discussion Regarding Benchmarking with #112117 and #142111

Good agreement is found between the 1.5D model with experimental measurements of DIII-D discharges #122117 and #142111.

AE activity does not need to be associated with losses of energetic particles. This is an important feature that 1.5D captures by modeling the quasilinear diffusion of particles as described in Sec. 3.2.2. The basic idea is that particles undergo diffusion and are redistributed in phase space such that the pressure gradient is maintained below the critical value that destabilizes the modes.

The critical value of the gradient in EP pressure is proportional to the ratio of the damping to the growth rate of TAE modes. Therefore, when the ratio is high, particles can stack up and maintain a steep gradient without destabilizing the modes, i.e, when there is significant damping, the free energy from the gradient in EP pressure profiles is not enough to destabilize the modes. However, if this ratio is low, the free energy can be strong enough to destabilize the mode. The associated low critical gradient models a high quasilinear diffusion coefficient, which continues to redistribute the particles until the gradient reduces to or below the critical value at which the mode is stable.

In cases where the critical gradient in EP pressure is lower than the original gradient around the center but significantly higher at the edge, energetic particles undergo diffusion that will transport them from the center towards the edge. However, due to the high critical gradient threshold at the edge, particles can accumulate without being transported beyond the last closed flux surface. This results in redistribution and central flattening without any losses to the wall as AE modes saturate.

On the other hand, in cases where the critical value of the gradient is not sufficiently high towards the edge, diffusion results in particles being transported beyond the last closed surface and consequently results in particle losses.
I also note the discrepancy in the neutron loss measured to that predicted by the 1.5D model for $t \approx 975$ and $t \approx 425$ in discharge #142111. There are other modes prevalent in these cases whose effects are not captured by the model. Most notably, the EGAM might cause neutron deficit at 425 ms while BAAEs result in EP transport and neutron deficit at 925 ms. EGAMs are symmetric modes ($n = 0$) that result in EP transport in velocity space only. However, this affects the cross section for neutron emission rates, which could lead to neutron deficit. The discrepancy might have been caused by a deficiency in input to TRANSP and thus NOVA. Therefore, the stability calculations performed by NOVA-K may be questionable, especially some damping rates that are very sensitive to plasma profiles such as the continuum damping. Overall these discrepancies reflect a deficiency of the stability calculation. Due to the sensitivity of the 1.5D model results on the NOVA-K calculations used for normalization, discrepancy arises if certain growth and/or damping mechanisms are excluded.

**Further Application on DIII-D**

The 1.5D model is again applied to the DIII-D discharge [2] for further validations. Multiple-point normalization was used at each time. The model was used to compute the resulting flattening in the distribution function and losses in neutrons. In Fig. 3.12 the results of the distribution function modification are presented in the top figure, and the losses compared to that measured in the bottom figure. Remarkable agreement is achieved on the measured losses as compared to the losses predicted by the 1.5D model. Although this is not a quantitative confirmation of the model, it does suggest that the assumptions for applicability of the 1.5D model are indeed met in these plasmas. In a way, these results further the confirmation of the linear theory that the model is so heavily based on since NOVA-K was used extensively in these runs where multiple points were used for normalization.
Figure 3.12: Comparison of 1.5D results to the DIII-D measurements at six different times of the discharge. The solid lines are the initial profiles and the dashed lines are the resulting fast-ion profiles that the 1.5D model calculates at the different times.

3.5 Summary and Applications

In this work the established routines, NOVA and NOVA-K are combined with the newly developed 1.5D. The code NOVA calculates the eigenmode structure while NOVA-K determines the contributions of growth and damping rates. NOVA is used to find two eigenmodes, one tending to be localized in the center of the plasma and the other towards the edge. The new code 1.5D uses the NOVA-K computed rates
for these two eigenmodes to normalize the rates that are analytically computed for all radial positions. 

Then, in 1.5D, the profiles are presumed to relax to a locally marginal state at every radial position where the local drive exceeds the local damping. With the constraint that the energetic particle number be conserved during the profile relaxation, providing the redistribution region does not reach the loss region, the change of profile can then be computed. If the instability region reaches the plasma edge, then the constraint on the conservation of particles is released and the profile is integrated using the critical value of the gradient for which the mode is marginally stable. This results in a profile where \( \left( \int dr \beta_{\text{ini}}(r) - \int dr \beta_{\text{diss}}(r) \right) / \int dr \beta_{\text{ini}}(r) > 0 \) is the fraction of EP loss.

The well-diagnosed DIII-D discharges #142111 and #122117 have shown significant TAE activity and correlated neutral beam losses and profile redistributions. These discharges are used to validate the proposed 1.5D model. The model predicted losses that agree with the experimentally measured ones, and the relaxed profiles computed by 1.5D depicts the flattening measured by FIDA.

The strength of the model is in predicting the relaxed profiles in an efficient and quick way without detailed computations of the fields and wave-particle interactions. The 1.5D model can be used as an initial check for EP losses in a large parameter space. This then can be followed by more detailed simulations in a specific parametric subspace of interest.

### 3.5.1 Application to Burning Plasmas

The model is being developed for the purpose of predicting the effect of AE modes on energetic particles in ITER and other burning plasmas where alpha confinement is crucial. The success of the model in predicting the redistribution of EP in DIII-D and making ballpark estimates of the neutron losses provides the confidence in applying
it to burning plasma experiments. The model is yet to be expanded to include two species, isotropic and anisotropic particle distribution functions, which is necessary for ITER applications, especially when an additional energetic particle source from neutral beam injection is present.

Since designs of next-generation fusion devices cover a range of parameters, 1.5D QL model lends itself as a valuable tool for the analysis of the effect of TAE interaction with fusion-product alpha particles over these large parameter spaces. A code is developed for producing the PopCon figures depicting the loss of alphas for different core values of plasma temperature $T_0$ and pressure $\beta_{p0}$. To demonstrate the code, ARIES ACT-1 is used to show the different features.

**Analytic or TRANSP**

The developed code performs a parameter scan using either TRANSP profiles or analytic expressions. In the case it reads TRANSP files, the code uses the profiles for an equilibrium with a given $T_0$ and $\beta_{p0}$ and the profiles is multiplied by the appropriate ratio for the rest of the scan in $T_0, \beta_{p0}$ space.

In the case of analytic expressions, the code uses quadratic approximations for the plasma temperature and pressure profiles, $T(r) = T_0(1 - r^2)$, and $\beta_p(r) = \beta_{p0}(1 - r^2)$ respectively. Knowing the temperature and pressure profiles allow for approximating the resulting alpha-particle fusion-product beta profiles

$$\frac{\beta_\alpha}{\beta_p} = \frac{8n_Dn_T \langle \sigma_{DTv}\rangle n_e \tau_{\text{sc}} \xi_{\alpha0}}{n_e^2(1 + \sigma) 12T}.$$ (3.13)

where $\sigma$ is the quasi-neutrality coefficient. The expression from Ref. [92] is used for $\xi(T) \equiv \langle \sigma_{DTv}\rangle$, the cross section of the thermonuclear D-T reaction as a function of
where the constants are approximated for the various temperature ranges. $T < 50$ keV is the range used in this case.

$\tau_{se}$ is the energy slowing-down time of alpha particles on the background plasma due to coulomb collisions with electrons and ions. Since $v_{iT} < v_\alpha < v_{Te}$ the slowing-down time is approximated \cite{27, 92} by $n_e \tau_s \approx 4 \times 10^{12} T^{2/3}$ where $T$ is expressed in keV.

Using $n_{iD} \approx n_{iT}$ the expression for $\beta_\alpha$ is then simplified to

$$\beta_\alpha = \beta_p \frac{2 \times 10^{12} \sigma^2}{3T(1+\sigma)} \xi(T)T^{3/2} \mathcal{E}_{\alpha 0}.$$  \hfill (3.15)

Since a test using arbitrary TRANSP run for ARIES has shown good agreement between TRANSP profiles and the analytic profiles \cite{32}, the analytic equations can be used to reliably compute the incurred losses over a vast parameter space.

For evaluating the growth and damping rates, the ion density profiles are needed to evaluate $x_i = v_A/v_i$ and $x_A = v_A/v_{\alpha 0}$. However, the approximation $x_{iD} \approx \sqrt{(1+\sigma)/(9(1+\sigma/4)\beta_p)}$ is used resulting in $x_{iT} \approx x_{iD} \sqrt{3/2}$ and $x_A \approx 0.00226 x_{iD} \sqrt{T}$ where $T$ is in eV \cite{27}.

**Normalization with NOVA-K**

There is flexibility in either running the PopCon using growth and damping rates as analytically computed or using the rates normalized with NOVA-K. Since the scan is made over many parameters, the normalization appropriate in this case is of the first kind described in Sec. 3.2.4, where the ratio of the rates for one case is computed and used for the rest. This requires running NOVA-K for a given case to
find the normalization constants. Since the PopCon produced is using ARIES ACT-1 discharge 1000A53, the NOVA-K results for the two-point normalization of the first kind is done for one point in the PopCon to compare the results with the analytic unnormalized growth rates used in the PopCon.

Two modes located at two different radial positions are found. The mode structures computed by NOVA, Fig. 3.13, are of toroidal mode number \( n = 9 \) at \( r = 0.4a \) and \( n = 5 \) at \( r = 0.6a \) with frequencies \( \omega = 1.06 \) and \( \omega = 0.89 \) respectively.

![Figure 3.13: The mode structure of TAE modes are computed by NOVA code as a function of \( XG = \sqrt{\psi/\psi_0} \) where \( \psi \) is the normalized poloidal flux. The left figure is the mode with \( (n = 9, \omega = 0.89) \) localized around \( r/a = 0.4 \) and to the right, the mode \( (n = 5, \omega = 0.89) \) localized around \( r/a = 0.6 \)](image)

The resulting growth and damping rates are:

<table>
<thead>
<tr>
<th>mode location</th>
<th>( \gamma_{\text{growth}}/\omega )</th>
<th>( \gamma_{iD}/\omega + \gamma_{iT}/\omega )</th>
<th>( \gamma_{\text{rad}}/\omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r/a = 0.4 )</td>
<td>19%</td>
<td>5%</td>
<td>6%</td>
</tr>
<tr>
<td>( r/a = 0.6 )</td>
<td>10%</td>
<td>9%</td>
<td>2%</td>
</tr>
</tbody>
</table>

The 1.5D reduced QL model is now used for the various profiles of \( T_0 \) and \( \beta_{\rho0} \) without normalization to 1000A53 NOVA-K results to make the PopCon in Fig. 3.14. To compare to the result using the NOVA/NOVA-K simulations, the case study is
presented as a separate dot. While the analytic results predict around 15% for that case, with normalization, 1.5D predicts 9%.

Note that there could be cases where the TAE modes are linearly unstable and relaxation of the profile can occur without any losses. That is the case in which relaxation occurs such that the relaxation region $r_{rel}$ is not extended beyond the last closed flux surface.

Figure 3.14: A depiction of the loss fraction in $(\beta_p, T_i)$ space. The light green region of parameter space bounded by a black line is where TAE modes are stable. The space depicted in cyan is where TAE modes are destabilized, however $\alpha$ particles are only redistributed without being lost. Beyond the yellow line, losses start incurring, where the percentage of lost particles corresponds to that in the color bar.

The PopCon figures that the 1.5D reduced QL model produces can provide valuable ballpark estimates of the AE modes interaction with energetic particles in future burning plasma machines. While there are many linear codes that are validated such as the NOVA-K [84] code used here, they do not provide predictions regarding the nonlinear time evolution of the possibly destabilized modes. The 1.5D reduced QL model can use those linear calculations to make, under certain assumptions, predictions of the resulting relaxation of energetic particle profiles. Application of this model needs to be made within the validity limits: First, that there exist numerous
resonances with small amplitude modes and second, that the steady-state solution persists, which requires significant collisions. While these conditions apply to many machines, especially next-generation burning plasmas, the line broadened quasilinear model, LBQ, is developed to apply to the wave-particle evolution whether modes are overlapped or isolated. The LBQ model is discussed in the following chapter.
Chapter 4

Line Broadened Quasilinear Model

4.1 Introduction

The Conventional Quasilinear theory (CQL) introduced in Chapter 2 is the basis of the LBQ2D model discussed here. CQL is a well established theory that applies to weak turbulence where modes are fully overlapped. The 1.5D model discussed in the previous chapter is also applicable under the same conditions of multiple modes overlapping. However, it is important to account for the regime where modes do not necessarily grow to the point of overlap, which is what motivates the development of LBQ. The ability to solve for isolated as well as overlapped modes allows for distinguishing between the cases where resulting cascades can incur large-scale losses and the cases where modes locally flatten the distribution function without much losses. Therefore, LBQ serves as an extension of CQL that bridges the gap between the two disparate regimes of isolated non-overlapping and fully overlapped modes.

LBQ1D code is developed as a proof of principle and to benchmark the model. However, the main interest is in modeling the dynamics of TAE modes interacting with energetic particles in phase space. This dynamics occur in the two-dimensional $(P_\phi, E)$ space where the quasilinear diffusion of particle is along paths of constant
\( E' = E - \omega P_\phi/n \), which depends on the mode parameters \( \omega \) and \( n \). For most unstable modes, one has the condition \( \omega_* \geq \omega_{\text{TAE}} \) where \( \omega_* \) is the diamagnetic frequency and \( \omega_{\text{TAE}} \) is the mode frequency. This results in diffusion approximately along \( P_\phi \), which reduces the two-dimensional diffusion equations to a set of one-dimensional equation as solved by Fitzpatrick [36]. However, it is necessary to develop the full code, LBQ2D, that also studies modes near marginal stability where \( \omega_* \approx \omega_{\text{TAE}} \) and that gives an understanding of the effect of diffusion in \( E \). Therefore, LBQ2D can be used to determine optimized scenarios for next-generation fusion devices and scan whole parameter spaces without reservation. Additionally, including diffusion along \( E \) allows for LBQ2D the possibility of examining problems such as ash removal and alpha channeling. These applications are left as future work while the focus of this manuscript is on the development and validation of the model and the numerical scheme used.

In this chapter, the line broadening model is discussed and the code developed for the one-dimensional case, LBQ1D, with collisions. Modeling the isolated modes with collisions results in myriad behaviors depending on how the time scales \( \nu_{\text{eff}}, \gamma_L \), and \( \gamma_d \) compare. LBQ1D is benchmarked with the code BOT [93] and is in excellent agreement for the evolution and saturation of isolated modes in the regimes of high collisionality.

In addition, the LBQ2D model is discussed and the derivation for the expression of the saturation of isolated modes is presented. The resulting curve \( C^2/\nu_{\text{eff}} \) as a function of \( \gamma_d/\gamma_L \) is calculated for a given choice of distribution function and matrix elements \( V_i \).

Finally, some numerical solutions of LBQ1D and LBQ2D codes are presented although the detailed description of the numerical schemes is left for Chapter 5.
4.2 Line Broadening Model

The LBQ model has been proposed \cite{3,35,36} to compute the flattening of the distribution function resulting from isolated modes and the subsequent saturation of the modes using the same structure and diffusion coefficients of the equations of CQL theory. This simplifies the calculations immensely because no phase information is required to be resolved in LBQ. Since the same equations for CQL are used, the model is readily applicable to the case of multiple modes overlapping. The focus therefore is on developing LBQ to model the dynamics of isolated modes.

Recall that CQL assumes that the resonant manifold is of infinitesimal width in phase space, $\delta(\Omega(I_0) - \omega)$ which requires many overlapping modes for large scale diffusion across phase space to take place. The action $I_0$ satisfies the resonant condition $\Omega(I_0) - \omega = 0$ with $\omega$ being the mode frequency and $\Omega(I_0) = \mathbf{l} \cdot \Omega$ where $\Omega = \partial H/\partial I_0$. In CQL, transport across phase space only occurs when there is a broad spectrum of densely packed modes to allow for large scale diffusion. The dynamics of the particles is made stochastic due to the random kicks in phase-space resulting from the $\delta$ functions. However, the mathematical $\delta$ is an idealization that is not physical, since the particles trapped in the wave undergo libation as described in Sec. 2.3. In the limit of CQL validity, the detailed phase-space dynamics of the single particles trapped in the modes does not need to be resolved. This is because in the regime of CQL validity, the particles are kicked out of their resonant island to a nearby island and so on and so forth before they advance significantly in the island to retain any coherence. However, the interest is in modeling the evolution of the system in the case of isolated modes where the existence of collisions justifies quasilinearity. The broadening of the resonant manifold is modeled using physically motivated arguments whose parametric dependencies are calculated such that the expected saturation levels and dynamics of the mode evolution are reproduced.
The resonant singularity is modeled to be broadened over a window of width $\Delta \Omega(J)$,

$$\delta(\Omega - \omega) \rightarrow F(\Omega - \omega),$$

such that $\int d\Omega F(\Omega - \omega) = 1$ and

$$F(\Omega - \omega_m) = \begin{cases} 
g(\Omega - \omega) & \text{if } |\Omega - \omega| < \Delta \Omega/2, \\
0 & \text{if } |\Omega - \omega| > \Delta \Omega/2.
\end{cases}$$

$g(\Omega - \omega_m)$ can be of any form as long as it integrates to one over the window width. If chosen to be a square well, then $g(\Omega - \omega) = 1/\Delta \Omega$. If a quadratic window is chosen for instance, then

$$g(\Omega - \omega) = \frac{- (\Omega - \omega)^2 + (\Delta \Omega/2)^2}{\Delta \Omega^3/6}.$$  

The shape of the window $g$ and the functional dependency of the width of the window $\Delta \Omega$ on the dynamical features of the system are the main elements in the line broadened model.

It is justified to use the CQL diffusion equations, which are intrinsically irreversible, when the particle motion is randomized at scales faster then the scales for nonlinearity to take effect. There are two mechanisms [50] to randomize particle motion, either intrinsically due to Hamiltonian stochasticity [94] resulting from phase-space islands overlapping, or extrinsically such as coulomb collisions [95]. When the modes are isolated, the first condition cannot be met, but the existence of collisions motivates the LBQ model.

The form for the collision operator depends on the dominant processes for the wave-particle problem being modeled. For the example of energetic particles interacting with TAE modes, the fast-ion velocities $v_f$ satisfies $v_i \ll v_f \ll v_e$ where $v_i$ and $v_e$ are the background ion and electron thermal velocities. In that limit, Coulomb collisions are a combination of ion pitch-angle scattering and ion energy scattering,
while electron drag [96, 97] is neglected under the assumption that a particle diffuses out of the resonance region faster than drag could transport the particle through the resonance region. The scattering is represented by a diffusive operator as investigated in Refs. [97], [98], and [54]; the details are discussed in Appendix A. In developing the model the collisions are included in the LBQ model as a diffusive term $\nu_{\text{eff}}^3 \partial^2 f/\partial \Omega^2$, where $\nu_{\text{eff}}^3$ depends on the parameters of the system.

Using the expression for the bounce frequency $\omega_{b,l} = (2nCV_l \partial \Omega/\partial P_\phi)^{1/2}$ from Eq. (2.47) in Chapter 2, the self-consistent LBQ equations are written with the Fokker-Plank operator for collisions

$$\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_n \sum_l \frac{\partial}{\partial \Omega_l} \omega_{b,l}^{(n)4} F \frac{\partial}{\partial \Omega_l} + \nu_{\text{eff}}^3 \frac{\partial^2}{\partial \Omega^2} (f - f_0)$$ (4.4a)

$$\frac{d}{dt} \omega_{b,l}^{(n)4} = 2\gamma^{(n)} \omega_{b,l}^{(n)4},$$ (4.4b)

where the growth rate is

$$\gamma^{(n)} = \frac{\pi}{4} \sum_l \int \tau_g dE' \left(2nV_l \frac{\partial \Omega}{\partial P_\phi} \right)^2 F \frac{\partial f}{\partial \Omega}. \quad (4.5)$$

### 4.3 Width of the Broadening

In LBQ2D, the width of the broadening around $\Omega(I_0)$ is modeled as some combination of $\omega_b$, $\nu_{\text{eff}}$, and $\gamma_L$, which are the main time scales of interest. This section examines the physical motivation behind the choice of modeling the broadening as such.

#### 4.3.1 Ideal Plasma

Based on single-particle dynamics, the broadening in $\Omega$ is proportional to the separatrix width $\Delta \Omega_{\text{sep}} \approx 4 \omega_b$. This can be justified by understanding the ideal plasma interaction with an isolated mode. Within the LBQ model, the modes are assumed
weak and the particle wave interaction can be treated using canonical perturbation theory as discussed in Sec. 2.3. According to Jean’s theorem, the solution to Vlasov’s equation in the unperturbed case can be presented as a function of the constants of motion $I$, independent of the angles $\xi$. The particles that are trapped in the growing mode undergo phase mixing that results in flattening of the distribution function within the separatrix. Fig. 4.1a represents the resulting distribution in the full phase space.

In quasilinear theory, the dynamics is integrated over the angle and the width of the diffusion broadening is modeled to be proportional to the separatrix width: $\Delta \Omega \propto 2\sigma \omega_b$. For comparing the LBQ results to the expected flattening due to phase mixing, $f(\zeta, \Omega)$ is integrated over the angles, and $f_{PM}(\Omega) = \int d\zeta f(\zeta, \Omega)/2\pi$ is compared to $f_{LBQ}(\Omega)$. In the ideal case, LBQ diffusion would result in total flattening within the broadened width $\Delta \Omega = 2\sigma \omega_b$ for all angles. Figure 4.1b is an illustration of the resulting distribution function $f_{PM}$ as it flattens within the separatrix due to phase mixing (solid blue) compared to $f_{LBQ}$ resulting in flattening due to diffusion within $\Delta \Omega$.

![Figure 4.1](image_url)

**Figure 4.1**: (a) The resulting distribution function $f(\zeta, \Omega)$ in $(\zeta, \Omega(I))$ phase space as a result in interacting with the mode resonant at $\Omega(I_0) = \omega$. (b) The resulting distribution function $f(\Omega)$ averaged over the angle overlaid with the resulting distribution function under LBQ diffusion with a choice $\sigma = 1.5$.

Getting the expected saturation levels of the modes is an important element of the model. The saturation levels are reproduced with a choice of $\sigma$ such that the
momentum exchanged as the distribution flattens due to LBQ diffusion,

$$\Delta M_{p}^{LBQ} = \int_{\Omega=-\Delta\Omega/2}^{\Omega=+\Delta\Omega/2} d\Omega \Omega (f^f - f^i) = \int_{\Omega=-\Delta\Omega/2}^{\Omega=+\Delta\Omega/2} d\Omega \Omega^2 \frac{\partial f_0}{\partial \Omega} = -\frac{2\sigma^3 \omega_b^3}{3} \frac{\partial f_0}{\partial \Omega}, \quad (4.6)$$

is equivalent to that exchanged by the flattening of particles in the separatrix due to canonical perturbation theory,

$$\Delta M_{p}^{CP} = \int \frac{d\zeta}{2\pi} d\Omega \Omega (f^f - f^i) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d\zeta \int_{-\omega_b\sqrt{2(1+\cos(\zeta))}}^{\omega_b\sqrt{2(1+\cos(\zeta))}} d\Omega \Omega^2 \frac{\partial f_0(\xi, \Omega)}{\partial \Omega}$$

$$= -\frac{64}{9\pi} \omega_b^3 \frac{\partial f_0(\xi, \Omega)}{\partial \Omega}. \quad (4.7)$$

$f^f$ is assumed to flatten completely and have a value $f_0$, which is the initial value of the distribution function at the resonance. The initial distribution function is Taylor expanded around the resonance to get $f^i = f_0 + \Omega \frac{\partial f_0}{\partial \Omega}$. Equating Eq. (4.6) and Eq. (4.7) results in $\sigma = 1.5$. This reproduces the saturation levels predicted by adiabatic saturation. However, the saturation levels seen in experiments and simulations, $\omega_b = 3.2\gamma_L$, is somewhere between the adiabatic calculation, $\omega_b = 2.9\gamma_L$, and the saturation level calculated assuming instantaneous flattening, $\omega_b = 3.5\gamma_L$. Using momentum conservation arguments, it is calculated that a choice of $\sigma = 1.55$ reproduces the expected saturated values for $\omega_b$.

The calculation of the saturation levels is readily extended to two-dimensional $(P_{\phi}, E)$ phase space where the same logic applies at each point along the resonant manifold. In the adiabatic limit, the separatrix remains intact and the actions of the trapped particles are conserved as the mode amplitude grows in an assumed monotonic manner thereby trapping more particles until the saturation value is reached. The phase-space dependence of the matrix elements $V_l$ is assumed to vary slowly over the width of the separatrix. Therefore, $V_l$ are taken as constant in the resonant region. Since the structure of the mode is assumed unchanged, the matrix elements and the
resonant curve remain unaffected as the mode evolves. Figure 4.2 is a schematic in the 2D phase space of the islands whose widths $3.3\omega_b$ are dependent on the matrix element $V_l$ that varies along the resonant curve. The equations are the same as the

Figure 4.2: A schematic for the resonant curve in $(P_\phi, E)$ space in solid blue line (—). The direction of diffusion along $E - \omega P_\phi/n = \text{cst}$ is represented in a dotted black line (…) and a few resonant islands in red. Note that the width of the island, $3.3\omega_b$, varies along the resonant curve since $\omega_b$ depends on the Fourier component, $V_l(J)$.

one-dimensional equations except for an integral over $E'$ for all phase space.

4.3.2 Collisionality

The arguments in Sec. 4.3.1 were made for illustrating the motivation behind choosing the parametric dependance of $\Delta\Omega$ on $\omega_b$. The ideal case is not applicable to LBQ and is rarely a regime of interest in fusion plasmas where there exist damping mechanisms, source, sink and collisional effects taking place. For larger values of $\nu_{\text{eff}}$, particles are redistributed at a faster rate into and out of the separatrix. This would result in a larger region of phase space affected by the resonance; this is modeled by making the width also proportional to the effective collisional rate: $\Delta\Omega \propto \nu_{\text{eff}}$. 

75
4.3.3 Modeling the Width $\Delta \Omega$

Putting it all together, the width is assumed to have the form of a weighted sum:

$$\Delta \Omega = \{ [2\sigma \omega_b]^p + [2\lambda (\nu_{\text{eff}} + |\gamma|)]^p \}^{1/p}$$

(4.8)

Introducing $\sigma$, $\lambda$, and $p$ as arbitrary coefficients in the broadening width is a critical step in developing the model. In Sec. 4.4, the parameters are calculated to model the evolution and saturation of the mode to best fit the expected behavior of isolated modes. $\lambda$ and $\sigma$ are chosen by using the analytic results of the saturation levels in the two opposite limits $\gamma_d/\gamma_L \ll 1$ and $\gamma_d/\gamma_L \gg 1$. However, in the wide range in-between there are no analytic expressions to rely on. Instead simulations are used to best fit the LBQ results for the appropriate choice of $p$.

4.4 Parametric Dependencies of $\Delta \Omega$

Analytically known results for the saturation of modes are used to deduce the value of $\sigma$ and $\lambda$ necessary to reproduce these results in LBQ. In this section, the analytic results are presented in the two opposing limits, very far from marginal stability, $\gamma_d/\gamma_L \ll 1$, and very near marginal stability, $\gamma_d/\gamma_L \approx 1$. Additionally, the expected saturation level is calculated using LBQ theory in these two limits and the values of the parameters are found. All these calculations are made in one dimension. Finally, the saturation levels that LBQ predicts in the two-dimensional case is presented.

4.4.1 Analytic Results

Perturbation theory is used to calculate the saturation levels in the limits of near and far from marginal stability. The derivation, which is omitted here, can be found in Petviashvilli’s thesis [99]. Near marginal stability, Petviashvilli follows the derivation...
in Ref. [54]. The saturation level for the bounce frequency is

\[ \omega_b = 1.18 \nu_{\text{eff}} \left( \frac{\gamma_{L0} - \gamma_d}{\gamma_d} \right)^{1/4}. \]  \hspace{1cm} (4.9)

Far from marginal stability, Petviashvilli’s derivation follows Refs. [100–102]. The saturation level for the bounce frequency in this case is

\[ \omega_b = 1.2 \nu_{\text{eff}} \left( \frac{\gamma_{L0} - \gamma_d}{\gamma_d} \right)^{1/3}. \]  \hspace{1cm} (4.10)

These analytic results are used to determine the appropriate coefficients in the LBQ model to reproduce the expected dynamics.

### 4.4.2 LBQ1D

The LBQ equation for the time evolution of the distribution function is

\[ \frac{\partial f}{\partial t} = \frac{\pi}{2} \frac{\partial}{\partial \Omega} \omega_b^4 \mathcal{F} \frac{\partial}{\partial \Omega} f + \frac{\partial}{\partial \Omega} \nu_{\text{eff}}^3 \frac{\partial}{\partial \Omega} (f - f_0). \]  \hspace{1cm} (4.11)

Using a function form \( \mathcal{F} = 1/\Delta \Omega \), one has \( \partial f/\partial t = 0 \) at steady-state, therefore

\[ \left( \nu_{\text{eff}}^3 + \frac{\pi}{2} \frac{\omega_b^4}{\Delta \Omega} \right) \frac{\partial f}{\partial \Omega} = \nu_{\text{eff}}^3 \frac{\partial f_0}{\partial \Omega}, \]  \hspace{1cm} (4.12)

and at saturation one has \( \gamma_d = \gamma_\alpha \), which results in

\[ \gamma_d = \gamma_{L0} \frac{\partial f}{\partial \Omega} / f_0 / \partial \Omega. \]  \hspace{1cm} (4.13)

Therefore

\[ \Delta \Omega = \frac{\pi}{2} \omega_b^4 \left( \frac{\gamma_d}{\nu_{\text{eff}} (\gamma_{L0} - \gamma_d)} \right). \]  \hspace{1cm} (4.14)
Recall that in the LBQ model one has $\Delta \Omega = [(2\sigma \omega_b)^p + (2\lambda \nu_{\text{eff}})^p]^{1/p}$. The parameters $\sigma$ and $\lambda$ are synchronized to give the analytic values of Sec. 4.4.1. In the limit very near marginal stability, the relation $\nu_{\text{eff}} \gg \omega_b$ holds true and to good approximation $\Delta \Omega = 2\lambda \nu_{\text{eff}}$. Using this width in Eq. (4.14), the saturation level is equated to the analytically expected result, Eq. (4.9), to get

$$\left(\frac{4}{\pi}\lambda\right)^{1/4} = \omega_b \left(\frac{\gamma_d}{\nu_{\text{eff}}(\gamma L_0 - \gamma_d)}\right)^{1/4} = 1.18,$$

which results in $\lambda = 1.25$.

In the opposite limit of very far from marginal stability, one has $\omega_b \gg \nu_{\text{eff}}$ and to good approximation $\Delta \Omega = 2\sigma \omega_b$. Using this in Eq. (4.14) and Eq. (4.10), one gets

$$\left(\frac{4}{\pi}\sigma\right)^{1/3} = \omega_b \left(\frac{\gamma_d}{\nu_{\text{eff}}(\gamma L_0 - \gamma_d)}\right)^{1/3} = 1.2,$$

which results in $\sigma = 1.35$.

**Arbitrary Shape for the Window**

There is flexibility in choosing the shape for the window $g$ in Eq. (4.2). However, the shape changes the calculation of the momentum exchanged. Therefore the value of the coefficients need to be recalculated specific to the window shape in order to get the expected saturation levels. The previous example was for a top-hat window. To illustrate how the window may change the result, an example is provided using the quadratic window from Eq. (4.3).

One has

$$F(\Omega) = \frac{-\Omega^2 + (\Delta \Omega/2)^2}{(\Delta \Omega)^3/6}$$
for $|\Omega - \omega| < \Delta\Omega/2$ instead of $\mathcal{F}(\Omega) = 1/(\Delta\Omega)$ for the square window. For the saturation calculation in Eq. (4.12), one gets

$$\gamma_d + \frac{\pi}{2} \omega_b^4 \left[ \frac{\pi}{4} \int_{-\Delta\Omega/2}^{\Delta\Omega/2} \left( -\Omega^2 + (\Delta\Omega/2)^2 \right)^2 \frac{\partial f}{\partial \Omega} \right] = \gamma_L. \tag{4.18}$$

Since

$$\int_{-\Delta\Omega/2}^{\Delta\Omega/2} \left( -\Omega^2 + (\Delta\Omega/2)^2 \right)^2 \frac{\Delta\Omega^3}{6} = \frac{6}{5\Delta\Omega}, \tag{4.19}$$

the equation for saturation becomes

$$\omega_b^4 = \frac{10\Delta\Omega}{6\pi} \nu_{\text{eff}}^3 \left( \frac{\gamma_L - \gamma_d}{\gamma_d} \right). \tag{4.20}$$

In the two limiting cases, the coefficients $\sigma$ and $\lambda$ are therefore modified by a factor 6/5 if the quadratic window is to be used. This results in the synchronized values $\sigma = 1.62$ and $\lambda = 1.5$.

### 4.4.3 LBQ2D

Akin to the 1D case, one starts with Eq. (4.12), which holds true for each slice in E. We present the derivations for the saturation levels in the two limits of near and far from marginal stability and demonstrate how they revert to the analytic results, Sec. 4.4.1, of the one-dimensional case.

**Near Marginal Stability**

The island width is $\Delta\Omega \approx 2\lambda\nu_{\text{eff}}$ and the saturation for a given slice would be such that $\omega_b \ll \nu_{\text{eff}}$. Throughout the treatment of the two-dimensional case, the partial derivative $\partial/\partial P_\phi$ is understood to be $\partial/\partial P_\phi|_{E'}$, the derivative in $P_\phi$ where $E' = E + \omega P_\phi/n$ is held constant. Each side of Eq. (4.12) is multiplied by $\pi/(4M)(2nV \partial\Omega/\partial P_\phi)^2$ and an integral over all E is made. $\gamma_d = \pi/(4M) \int \tau_{\theta} dE (2nV \partial\Omega/\partial P_\phi)^2 \partial f/\partial\Omega$ is used,
since $\gamma_d = \gamma_L$ at saturation, to get:

$$\gamma_d = \frac{\pi}{4M} \int \tau_\theta dE \frac{(2nV \frac{\partial \Omega}{\partial P_\phi})^2}{1 + \pi \frac{\omega_b^4}{4\lambda \nu_{\text{eff}}} \frac{\partial f_0}{\partial \Omega}}. \quad (4.21)$$

Since $\omega_b \ll \nu_{\text{eff}}$ at saturation the binomial expansion is used

$$\left(1 + \frac{\pi \omega_b^4}{4\lambda \nu_{\text{eff}}}\right)^{-1} \rightarrow 1 - \frac{\pi \omega_b^4}{4\lambda \nu_{\text{eff}}}$$

(4.22)

to get:

$$\nu_{\text{eff}}^4 \gamma_d \rightarrow \frac{\pi}{4M} \int \tau_\theta dE \left(2nV \frac{\partial \Omega}{\partial P_\phi}\right)^2 \frac{\partial f_0}{\partial \Omega} - \frac{\pi \omega_b^4}{4\lambda \nu_{\text{eff}}} \frac{\partial f_0}{\partial \Omega}. \quad (4.23)$$

With

$$\gamma_{L0} = \frac{\pi}{4M} \int \left(2nV \frac{\partial \Omega}{\partial P_\phi}\right)^2 \frac{\partial f_0}{\partial \Omega} \quad (4.24)$$

one gets

$$\frac{\pi^2}{16\lambda M} \int \tau_\theta dE \left(2nV \frac{\partial \Omega}{\partial P_\phi}\right)^2 \omega_b^4 \frac{\partial f_0}{\partial \Omega} = \nu_{\text{eff}}^4 (\gamma_{L0} - \gamma_d), \quad (4.25)$$

which results in the saturated amplitude

$$C = \nu_{\text{eff}} \left(\frac{\pi^2}{16\lambda M} \int \tau_\theta dE \left(2nV \frac{\partial \Omega}{\partial P_\phi}\right)^2 \frac{\partial f_0}{\partial \Omega} \right)^{1/4}. \quad (4.26)$$

If $V$ is chosen to be a constant as a function of $E$, this condition can revert back to the 1D case by multiplying both sides by $V$ and replacing the resulting integral in the denominator $\int \tau_\theta dE \left(2nCV \frac{\partial \Omega}{\partial P_\phi}\right)^2 \frac{\partial f_0}{\partial \Omega} \rightarrow \gamma_{L0}$, $\omega_b = (2nCV \frac{\partial \Omega}{\partial P_\phi})^{1/2}$ and the appropriately synchronized $\lambda = 1.25$ are used for 1D to get

$$\omega_b = 1.18 \nu_{\text{eff}} \left(\frac{\gamma_{L0} - \gamma_d}{\gamma_{L0}}\right)^{1/4}. \quad (4.27)$$
Far from Marginal Stability

The island width is $\Delta \Omega \approx 2\sigma \omega_b$ since the saturated level $\omega_b \gg \nu_{\text{eff}}$. This results in

$$\nu_{\text{eff}}^3 \frac{\partial f_0}{\partial \Omega} = \left( \nu_{\text{eff}}^3 + \frac{\pi}{4\sigma \omega_b} \right) \frac{\partial f}{\partial \Omega} \quad (4.28)$$

for each $E$. Multiply both sides by $\pi/4 (2nV \partial \Omega/\partial \phi)^2$ and integrating over all $E$, one gets

$$\frac{\pi}{4} \int \tau \delta E \nu_{\text{eff}}^3 \left( 2nV \frac{\partial \Omega}{\partial \phi} \right)^{1/2} \frac{\partial f_0}{\partial \Omega} = C^3 \frac{\pi^2}{16\sigma} \int \tau \delta E \left( 2nV \frac{\partial \Omega}{\partial \phi} \right)^2 \frac{\partial f}{\partial \Omega}, \quad (4.29)$$

where the first term on the right-hand side of Eq. (4.28) is neglected since $\nu_{\text{eff}} \ll \omega_b$. This results in an expression for the saturated amplitude

$$C = \nu_{\text{eff}} \left( \frac{\pi^2}{16\sigma} \int \delta E \left( 2nV \frac{\partial \Omega}{\partial \phi} \right)^2 \frac{\partial f_0}{\partial \Omega} \right)^{1/3} \quad (4.30)$$

Again, this can revert to the 1D result. In case $V$ is independent of $E$ one has $\omega_b = (2nCV \partial \Omega/\partial \phi)^{1/2}$. Multiplying Eq. (4.30) by $(2nV \partial \Omega/\partial \phi)^{1/2}$ on both sides and using the appropriately synchronized value for $\sigma = 1.35$ one gets

$$\omega_b = 1.2 \nu_{\text{eff}} \left( \frac{2nV \partial \Omega}{\partial \phi} \right) \frac{\partial f_0}{\partial \Omega} \left( \frac{\gamma_{L \theta}}{\gamma_d} \right)^{1/3} \quad (4.31)$$

Since $\gamma_{L \theta} \gg \gamma_d$ this is approximately equivalent to

$$\omega_b = 1.2 \nu_{\text{eff}} \left( \frac{\gamma_{L \theta} - \gamma_d}{\gamma_d} \right)^{1/3} \quad (4.32)$$
4.5 LBQ Dynamics

Although the detailed discussion of the numerical scheme is deferred to Chapter 5, the developed numerical codes are used in this section to study the nonlinear behavior of the LBQ equations. The focus here is on the dynamics of isolated modes since that is the nontrivial part of the model, which needs to be verified and benchmarked. Solutions to the self-consistent set of equations, Eq. (4.4a) and Eq. (4.4b), for an isolated mode exhibits a variety of behaviors depending on the regime of interest. For high collisionality $\nu_{\text{eff}}/\gamma_L \gg 1$ the equations result in steady-state evolution, which is presented in Sec. 4.5.1. For lower collisionality, the steady-state evolution is not guaranteed and various behaviors arise depending on the damping rate $\gamma_d/\gamma_L$. The resulting dynamics is presented in Sec. 4.5.2 and an account of the oscillatory behavior is given.

4.5.1 Steady-State Solutions (High Collisionality)

Now that the values of the parameters $\sigma$ and $\lambda$ are determined, the steady-state solution is calculated for the saturation levels for all $\gamma_d/\gamma_L$ ranging between the two limits. The window width is $\Delta \Omega = [(2\sigma \omega_b)^p + (2\lambda \nu_{\text{eff}})^p]^{1/p}$ and the equation satisfying each cross section in $E'$ becomes

$$\nu_{\text{eff}}^3 \frac{\partial f_0}{\partial \Omega} = \left( \nu_{\text{eff}}^3 + \frac{\pi}{2} \frac{\omega_b^4}{[(2\sigma \omega_b)^p + (2\lambda \nu_{\text{eff}})^p]^{1/p}} \right) \frac{\partial f}{\partial \Omega}.$$

(4.33)
Since $\gamma_d = \gamma_L$ at saturation, the above equation can be written using $\omega_b = (2nCV \partial \Omega/\partial P_\phi)^{1/2}$ as

$$G(C) \equiv \frac{\pi}{4M} \int \tau_\theta dE \left( 2nV \frac{\partial \Omega}{\partial P_\phi} \right)^2 \frac{\nu_{\text{eff}}^3}{(2nCV \frac{\partial \Omega}{\partial P_\phi})^2} \frac{\partial f_0}{\partial \Omega} \nu_{\text{eff}}^3 + \frac{\pi}{2} \left\{ \frac{(2\sigma \nu_{\text{eff}} V \frac{\partial \Omega}{\partial P_\phi})^{1/2}[2\lambda \nu_{\text{eff}}]^p}{\nu_{\text{eff}}^3 + (2\lambda \nu_{\text{eff}})^p} \right\}^{1/p} \partial f_0 \partial \Omega = \gamma_d. \quad (4.34)$$

Dividing the numerator and denominator of $G(C)$ by $\nu_{\text{eff}}^3$, one gets

$$G(C) = \frac{\pi}{4M} \int \tau_\theta dE \left( 2nV \frac{\partial \Omega}{\partial P_\phi} \right)^2 \frac{1}{(2nCV \frac{\partial \Omega}{\partial P_\phi})^2} \frac{\partial f_0}{\partial \Omega}. \quad (4.35)$$

Note that $G(C)$ depends on $\nu_{\text{eff}}$ through the combination $C^{1/2}/\nu_{\text{eff}}$. The mode saturates at an amplitude where $C$ is the solution of $G(C) - \gamma_d = 0$. The Newton secant method is used on Eq. (4.35) to solve $C$ for a given distribution function $f_0$ and a given resonance (which specifies the values of $V$ and $\partial \Omega/\partial P_\phi$).

**One-dimensional Counterpart**

The above result, Eq. (4.35), is reduced to the one-dimensional case by assuming that $2nV \partial \Omega/\partial P_\phi$ is a constant and $\partial f_0/\partial \Omega$ is independent of $E$. This makes the integral trivial. $\gamma_{L0} = (\pi/4M) \int \tau_\theta dE (2nV \partial \Omega/\partial P_\phi)^2 \partial f_0/\partial \Omega$ and a choice of $2nV \partial \Omega/\partial P_\phi = 1$ are used for simplicity of notation. This results in $\omega_b$ equivalent to $C^{1/2}$ and the relation for the saturated $\omega_b$

$$\frac{\gamma_{L0}}{\gamma_d} = 1 + \frac{\pi}{2} \left( \frac{\omega_b}{\nu_{\text{eff}}} \right)^4 \left[ (2\sigma \omega_b \nu_{\text{eff}})^p + (2\lambda \nu_{\text{eff}})^p \right]^{1/p}. \quad (4.36)$$
Equation (4.36) is solved for the saturation levels $\omega_b/\nu_{\text{eff}}$ as a function of $\gamma_d/\gamma_L$. However, the parameter $p$ is yet to be determined. For purpose of illustration, $p = 2$ is chosen and Fig. 4.3 represents the resulting one-dimensional saturation levels and the two approximations for near MS and far from MS. Fig. 4.4 depicts the saturation levels as calculated numerically in LBQ1D in the regime of steady-state evolution and that predicted by Eq. (4.36). The results are also confirmed in the two-dimensional case, but the figure is omitted for brevity.

Figure 4.3: The resulting saturated $\omega_b(\gamma_d/\gamma_{L0})^{1/3}/\nu_{\text{eff}}$ for 1D counterpart and the analytically expected behavior in the limiting cases of near marginal stability (NMS), and far form marginal stability (FMS), as derived from 1D.

Figure 4.4: The saturation levels, $\ast$, as predicted by Eq. (4.36) and the results of the code, $\ast$, for various values of $\gamma_d/\gamma_L$. A choice of $\nu_{\text{eff}}/\gamma_{L0} = 3$ is used to guarantee steady-state saturation.
4.5.2 Non Steady-State Solutions (Low Collisionality)

For values of $\nu_{\text{eff}}$ small, various possible behaviors arise from the LBQ equations. For simplicity and without loss of generality, the study is focused on the 1D case. For a given $\nu_{\text{eff}} \ll \gamma_L$, there are values of $\gamma_d/\gamma_L$ for which the mode does not evolve to a steady state smoothly. Instead, the mode amplitude can oscillate initially, then saturate. Alternatively, it can grow and diminish before it gets reconstituted again, then diminishes and continue this relaxation oscillation cycle akin to the predator-prey dynamics. Also, it can diminish after the initial rise for a long time before it gets reconstituted enough to rise again and either continue to oscillate around a certain value or saturate. As $\gamma_d/\gamma_L$ decreases, the behavior changes from steady-state saturation to relaxation oscillation. This transition is depicted in Fig. 4.5 for a choice of $\nu_{\text{eff}}/\gamma_L = 0.18$. This depicts the first bifurcation in the set of Eq. (4.4), where the dynamics is changed from steady state to relaxation oscillation. For a given $\gamma_d/\gamma_L$ where the relaxation oscillations are well established, Fig. 4.6 depicts the evolution of $\omega_b$ as well as the evolution of the slope of the distribution function $\partial f/\partial \Omega$ to illustrate

\[ \gamma_d/\gamma_L \rightarrow 0.4 \]

Figure 4.5: The change in time evolution of $\omega_b/\gamma_{L0}$ as $\gamma_d/\gamma_{L0}$ is lowered for $\nu_{\text{eff}}/\gamma_{L0} = 0.18$. The onset of relaxation oscillation can be seen.
the predator-prey behavior underlying the equations. However, as the damping rate

![Graph of normalized quantities: \( \omega_b \) and \( \partial f/\partial \Omega \) against time.](image)

\( \gamma_d = 0.4 \gamma_L \)

\( \nu_{\text{eff}} = 0.176(\gamma_L - \gamma_d) \)

Figure 4.6: The evolution of the mode amplitude as compared to the evolution of the slope in the distribution function. Both are scaled to their maximum value to best depict the predator-prey behavior. The figure is zoomed into the region where the oscillations have been long established since the initial setup is not interesting.

diminishes further, the dynamics undergoes yet another bifurcation where the mode amplitude slowly rises again to either oscillate by a small amount around a given value or, for smaller \( \gamma_d/\gamma_L \), get to steady-state saturation. Figure 4.7 depicts the different dynamics of the mode evolution for values of \( \gamma_d/\gamma_L \) varying from 0.3 to 0.02 for \( \nu_{\text{eff}}/\gamma_L = 0.18 \).

Figure 4.8 represents as a function of \( \gamma_d/\gamma_L \) the resulting amplitude and period of the evolution in the mode oscillations for \( \nu_{\text{eff}}/\gamma_L = 0.18 \). Figure 4.9 depicts the amplitude of oscillation in the mode for different values in the \( \nu_{\text{eff}}/\gamma_L, \gamma_d/\gamma_L \) parameter space. For modes that saturate without oscillations, the amplitude is zero.

## 4.6 Benchmarking LBQ1D with Vlasov Codes

The LBQ model is developed to solve the evolution of the wave-particle system without resolving the dynamics of the angles by using the second-order approximation of Vlasov’s equations and averaging over the angle. As discussed in Chapter 2 this approach has been implemented and validated with CQL models. However, CQL only
Figure 4.7: The mode evolution for four values of $\gamma_d$ with $\nu_{\text{eff}} = 0.18\gamma_L$ that depict the transition to different behaviors. Values of $\gamma_d/\gamma_L$ 0.3 and 0.17 are relaxation oscillations. However, for lower damping rates another bifurcation occurs where the mode amplitude is quiescent for some time before it oscillates again more harmonically such as the evolution for $\gamma_d/\gamma_L = 0.035$ presented. For even lower $\gamma_d/\gamma_L$, the LBQ equations undergo another bifurcation where the mode again evolves to a steady state such as the behavior for $\gamma_d/\gamma_L = 0.025$ presented above.

Figure 4.8: For a choice of $\nu_{\text{eff}}/\gamma_{L0} = 0.18$, LBQ is run for a range of $\gamma_d/\gamma_{L0}$ to study the variation in amplitude and period of oscillations. The top left figure depicts the period of the oscillations for the different runs and the top right figure depicts their respective amplitudes.

Applies when modes overlap completely and cannot resolve isolated modes and their saturation. The LBQ model is developed to solve for the evolution of isolated modes and their interaction with the distribution function without resolving the dynamics of angle. It is expected to retrieve the results of the full Vlasov equation and predict
Figure 4.9: The resulting amplitude of the bursts in the evolution of $\omega_b/\gamma_L$ in the $(\gamma_d/\gamma_{L0}, \nu_{\text{eff}}/\gamma_{L0})$ parameter space. As $\nu_{\text{eff}}/\gamma_{L0}$ decreases for a given $\gamma_d/\gamma_{L0}$ the steady state become oscillatory. However for $\gamma_d/\gamma_{L0} < 0.1$, smaller values of $\nu_{\text{eff}}/\gamma_{L0}$ is required to maintain bursts. Otherwise, the amplitude will saturate at a steady level after some initial overshoot.

the correct saturation levels. In this section, the LBQ model is benchmarked with Vlasov solvers [4, 99] in the one-dimensional limit.

4.6.1 Vlasov’s Codes

Vlasov codes solve Eq. (4.37a), the one-dimensional Vlasov equations for the distribution function, and Eq. (4.37b), the one-dimensional counterpart of the linear wave equation derived in Sec. 2.2.3.

\[
\frac{\partial f}{\partial t} = -\Omega \frac{\partial f}{\partial \xi} + \frac{1}{2} \left( C e^{i\xi} + \text{c.c.} \right) \frac{\partial f}{\partial \Omega} + \frac{\partial}{\partial \Omega} \nu_{\text{eff}}^3 \frac{\partial (f - f_0)}{\partial \Omega},
\]

(4.37a)

\[
\frac{d C}{dt} = -\gamma_d C + 2 \int_0^{2\pi} \text{d}\xi \int \text{d}\Omega \, C e^{i\xi} f,
\]

(4.37b)
where $\gamma_d$ and $\nu_{\text{eff}}$ are both normalized to $\gamma_L$. Since the equations are periodic in $\xi$, both Lilley and Petviashvili solve the set of equation by expanding $f$ in Fourier series over $\xi$, and they expand $f$ as an integral Fourier transform in $\Omega$. They then solve the resulting algebraic equations iteratively. This results in codes that are very fast, especially compared to PIC simulations. They are also noise free, which allows for best comparison with LBQ1D.

**Resulting Dynamics**

The set of Eqs. (4.37) can exhibit either steady-state solutions or oscillations and chirping. In the case of high collisionality the mode evolves to a steady state without any bursts or chirping since the de-trapping of particles due to collisions randomizes the motion faster than it takes for nonlinearities to arise.

However, for low collisionality the particles can remain trapped in the separatrix long enough for coherent behavior to incur. Therefore, spontaneous holes and clumps are created, which results in chirping of the mode frequency and oscillatory behavior in the mode amplitude. This has been extensively studied and the interested reader can refer to Refs. [4, 103–108].

**Lilley’s BOT**

BOT is a code that Lilley [93] has made available for scientific and educational purposes. The results of LBQ1D are compared to the BOT simulations for the case of isolated modes. BOT is a Matlab code where the damping rates and effective collisions are input parameters. BOT also has the capacity to include drag, but there is no drag in the LBQ code.

BOT uses a spectral method to solve the 1D Vlasov equations (4.37a) and the trapezoid method for the mode evolution, Eq. (4.37b). The major difference between BOT and LBQ is the phase information, which for certain parameters exhibit coherent
behavior not captures by LBQ. These are the regimes where a discrepancy between the results of BOT and LBQ is expected.

**Petviashvili’s Interpolation Formula**

The numerical scheme Lilley uses in solving the Vlasov equation in BOT is based on a scheme developed by Breizman and Petviashvili to resolve the fine phase-space structures in Fourier space. While BOT is available to run for various choices of parameters, Petviashviali provides an interpolation formula, Eq. (4.38) (Eq. (2.41) in his thesis [99]), to fit his resulting numerical values for the saturation levels. The interpolation formula is

\[
\frac{\gamma_d}{\gamma_L} = \frac{1}{1 + \frac{0.57(\omega_b/\nu_{\text{eff}})^3}{\left(1 + \frac{1.45}{(\omega_b/\nu_{\text{eff}})^3}\right)^{1/3}}}. \tag{4.38}
\]

This interpolation formula is used as well for the numerical results from running BOT to Benchmark the LBQ1D simulations. It is worth noting here that this expression is used in NOVA-K code [109].

### 4.6.2 Results of the Benchmark

**Steady-State Evolution (high collisionality)**

The saturation levels resulting from LBQ1D numerical simulations are compared to the results of BOT and Petviashvilli’s interpolation, Eq. (4.38). A large enough \( \nu_{\text{eff}} \) is chosen to guarantee steady-state saturation and lack of coherent structures. The expected saturation level for LBQ1D are computed and BOT is run for the appropriate parameters of collisionality and damping. In Fig. 4.10 the results of LBQ1D, the formula used by Petviashvili and the results of the BOT simulations are presented. There is very good agreement between the results of LBQ equations and the Vlasov equations for the steady-state saturation amplitude. The choice of
Figure 4.10: The resulting saturated $\omega_b(\gamma_d/\gamma_{L0})^{1/3}/\nu_{\text{eff}}$ for choice of $p = 2$, $p = 4$ as compared to BOT simulations and interpolation formula used by Petviashvili.

the parameter $p$ determines how closely the results are to the analytic values at the limits.

It is also important to compare the time evolution of the mode to get a measure of the time scales and how realistic it is to model the evolution with quasilinear diffusion coefficients over a broadened window whose parameters are calculated. In Fig. 4.11 the time evolution of the mode amplitude resulting from LBQ1D is compared to that simulated in BOT. There is very good agreement between BOT and LBQD for the time evolution of the mode in the case of steady-state saturation. This provides confidence in using the LBQ model to study the self-consistent dynamics of the distribution function with TAE modes whether they are isolated or overlapped in the case of high collisionality. The efficiency and speed of the LBQ model in calculating the effect of modes on the energetic particles is very useful for making large parameter scans to find suitable regimes for sustaining a burning plasmas.

**Oscillatory Behavior (low collisionality)**

As mentioned in Sec. 4.6.1 the wave-particle system either evolves to a steady state or exhibits bursts and chirping. Intrinsic to LBQ is the assumption that the mode
Comparing saturation in LBQ1D and BOT

\[
\gamma_d = 0.2\gamma_L
\]

\[
\gamma_d = 0.4\gamma_L
\]

\[
\gamma_d = 0.6\gamma_L
\]

\[
\gamma_d = 0.8\gamma_L
\]

Figure 4.11: The saturation levels as predicted by BOT and LBQ1D for different choices of \( p = 4 \) in the broadening \( \Delta \Omega = [(2\sigma\omega_b)^p + (2\lambda\nu_{\text{eff}})^p]^{1/p} \).

structure and the resonance are unchanged with time, therefore LBQ is not expected to capture any of the behavior in the case of mode chirping.

However, in certain cases in the Vlasov treatment, the mode exhibits oscillations without mode chirping. This gives one the impression that it is possible to use the LBQ model to obtain some of the characteristics of the non-steady response.

Indeed, it is possible to reproduce with LBQ some of the behavior similar to BOT’s with certain choices of the coefficients for the broadening width that are different from the ones calculated assuming steady-state saturation. Figure 4.12 depicts an example of such a result where \( \lambda \) and \( \sigma \) are changed to best fit the periods of oscillations and
the amplitude of the initial burst. However, the results are not exactly reproduced

![Comparing LBQ1D to BOT](image)

\[ \frac{\omega_b}{\nu_{\text{eff}}} = 0.25 \]

\[ \frac{\gamma_d}{\gamma_L} = 0.2 \]

**Figure 4.12:** Oscillatory behavior of LBQ as compared with BOT in some examples where the mode frequency does not chirp.

and the parameters have been chosen by trial and error. Although there might be a systematic way of allowing LBQ equations to reproduce the same oscillations, the results are reported as inconclusive and LBQ is considered not applicable in these regimes.

### 4.6.3 BOT vs LBQ parameter space

Lilley [107] presented an analysis of the nonlinear behavior resulting from Eq. (4.37) for different values of \( \nu_{\text{eff}} \) (diffusive collisions) and \( \nu_a \) (drag). Drag strongly increases the coherent behavior, which results in chirping and for any value of \( \nu_a \) higher values of \( \nu_{\text{eff}} \) is needed to have LBQ validated. For that reason, drag is not treated in the LBQ model since its existence puts the system in a regime where LBQ is not applicable.

Using the ratio of diffusivity to drag as deduced from Eq. (9) of Ref. [107], one needs to have

\[
\frac{\nu_{\text{eff}}}{\nu_a} \propto \frac{T_e^{3/4}}{n_e^{1/6}} \frac{B_0^{5/6}}{B_0^{9/6}} \gg 1
\]

(4.39)
for the effect of drag to be neglected. This clearly poses a limitation of the quasilinear model, which only applies for large enough collisionality. For typical ITER parameters, the ratio is 1.4 [107]. Although the dynamics is dominated by diffusion, drag is not insignificant and care is needed in applying LBQ to scenarios where the ratio is marginal. However, there are many scenarios for ITER and future burning plasma devices where diffusion can be strongly dominating.

In Fig. 1 of Ref. [107], Lilley shows that for $\nu_a = 0$ the BOT equations result in a steady state for values of $\nu_{\text{eff}}/(\gamma_L - \gamma_d) > 2$, which is the curve $\nu_{\text{eff}}/\gamma_L > 2(1 - \gamma_d/\gamma_L)$ in the $(\nu_{\text{eff}}/\gamma_L, \gamma_d/\gamma_L)$ parameter space.

In Fig. 4.13 this curve is overlaid with the curve resulting from Fig. 4.9 to find the different parameter spaces for steady-state evolution and oscillation in BOT and LBQ. $\nu_{\text{eff}}/\gamma_L$ needs to be significantly low ($\nu_{\text{eff}} < 0.3\gamma_L$) to get oscillations in LBQ, while $\nu_{\text{eff}}$ as large as $2\gamma_L$ can result in oscillations in BOT for low values of $\gamma_d$. Figure 4.13 shows the different regions in $(\gamma_d/\gamma_L, \nu_{\text{eff}}/\gamma_L)$ parameter space where oscillations occur for BOT and LBQ. We run BOT for 10 values of $\gamma_d/\gamma_L$ while varying $\nu_{\text{eff}}/\gamma_L$ until oscillations start. The values of $(\gamma_d/\gamma_L, \nu_{\text{eff}}/\gamma_L)$ for the onset of oscillations are represented with red circles. We fit the points with a curve depicted as a solid blue line in Fig. 4.13. The upper region (above the solid blue line) is accurately captured by LBQ. The middle region marked with blue crosses is elusive to LBQ. However, in parts of the lower region (below the green line) oscillations can occur but without the creation of holes and clumps. As mentioned in Sec. 4.6.2 it is possible for LBQ to capture the dynamics in these cases, but it is left as work in progress without further analysis.
Figure 4.13: The parameter space for the oscillations resulting from Eq. (4.37) solved by BOT is different from that resulting from Eq. (4.4) solved by LBQ. The BOT regions of oscillations is for all parameters below the solid blue line, while LBQ region of oscillations is below the green line with crosses.

4.7 Regime of Applicability of LBQ

The existence of collisions is necessary to justify the LBQ model, which requires that the motion of the particle be linearized [110]. Collisions insure that the particle leaves the resonant region in a time shorter than it takes to complete a bounce in the resonant island. This makes the linearized approximation reasonable. This limitation requires that the bounce time \( \tau_b = \frac{2\pi}{\omega_b} \) be greater than the de-trapping time \( \tau_{det} \), which is the time required for the collisions to change the momentum of the particle by an amount \( \Delta v = \Delta v_{sep} \) sufficient to kick it out of the separatrix. Using the random walk argument, one has \( \nu_{coll} \approx \left( \frac{\Delta v_{sep}}{v_{th}} \right)^2 / \tau_{det} \). With this condition satisfied, the particle that is trapped is released from the island and is replaced by a new (previously untapped) particle with a different phase. The randomization of phase prevents the nonlinear effects such as the creation of holes and clumps and frequency chirping. Since LBQ modeling assumed that the mode structure (matrix elements, \( V_{l,m} \)) and
resonant manifolds \((1 \cdot \Omega = \omega_{\text{TAE}})\) are not modified with time, these nonlinear effects cannot be captured by LBQ. The condition, \(\nu_{\text{eff}}/\gamma_L > 2(1 - \gamma_d/\gamma_L)\), for which coherent phase-space structures are not created, sets a bound to insure that \(\tau_{\text{det}} < \tau_b\).

### 4.8 Conclusion

I presented and discussed the LBQ model based on the quasilinear theory of diffusion, which is capable of capturing the wave-particle interaction in the case where modes are isolated as well as fully overlapped. TAE modes interacting with energetic particles is the problem of interest that is treated by LBQ2D, in the two-dimensional phase space \((P_\phi, E)\). Since TAE modes are driven by the free energy of the pressure gradient, it is expected that they are mostly destabilized for \(\omega_* \gg \omega\) where \(\omega_*\) is the diamagnetic frequency. This results in diffusion, which can be approximated to be along \(P_\phi\), which thus reduce LBQ2D to a one-dimensional problem. However, the full code is developed to give the flexibility of studying a wide range of modes including those very near marginal stability \(\omega_* \approx \omega\), which are strongly affected by the diffusion along \(E\). The numerical schemes to solve the LBQ2D equations are left to discuss in Chapter 5.

Since an extensive amount of work in the literature has been devoted to the one-dimensional bump-on tail problem, the LBQ1D code is developed to analyze the model and benchmark it with existing codes. In the case of isolated modes, the problem of TAE modes can be reduced to one dimension with the appropriate rotation in phase space. Therefore, the conclusions arrived at with LBQ1D are readily applicable on LBQ2D in the case of isolated modes.

As a result of the benchmarking, one concludes that the proposed LBQ model is applicable in the case of steady-state evolution when there is high collisionality, while the results are still inconclusive in the case of low collisionality. Although oscillatory
behavior occurs in the solution of Vlasov’s equations, the LBQ resulting oscillations do not reproduce the same dynamics. There is a region of parameter space, marked by blue plus signs in Fig. 4.13, where non-stationary solutions occur. Towards the lower region, holes and clumps can be created in the non-steady-state solution, which results in chirping. The dynamics in that region cannot be captured by LBQ. However, in the region where Vlasov equations result in oscillations without the creation of holes and clumps, the applicability of LBQ is still under investigation. It was possible to find cases where the oscillations can be made to match BOT results in certain ways if the parameters of the model are modified. However, that was a result of trial and error and more detailed analysis is needed if LBQ is to be used in that region of phase space.

I discussed, in Sec. 4.7, the conditions for the validity of LBQ. These conditions are a result of mathematical and physical considerations used in developing the model and can be reliably determined for a given system. While the conditions can be limiting, they are satisfied in a large range of experiments in fusion plasmas, especially next-generation burning plasmas where it is expected that diffusive collisions will play a significant role \[111\]. The efficiency of the model allows for scans a large plasma parameter space to find optimized scenarios where energetic particle losses are minimized to maintain the burning plasma and protect the first wall.
Chapter 5

Numerical Schemes for LBQ

5.1 Introduction

Modeling the Alfvén mode interaction with energetic particles in toroidal plasmas is critical for next-generation fusion power plants. Since burning plasma experiments are still in the design stages, it is valuable to have models that capture underlying physical mechanisms and can be simulated with speed and efficiency to make parameter space scans with predictive capability. The Line Broadened Quasilinear (LBQ) model described in Chapter 4 is proposed to serve this purpose. The canonical toroidal momentum $P_\phi$, the energy $E$, and the magnetic moment $\mu$ are the constants of motions used for the phase space. Since the Alfvén frequency is much slower than the gyrofrequency, diffusion only occurs in the $P_\phi, E$ space, which is why the code is denoted by LBQ2D. The main challenge is in resolving the diffusion in two-dimensional phase space where overlapping modes have different values of frequency, $\omega$, and toroidal mode number, $n$, which introduces cross terms to the diffusion equation.

In this chapter, we present, test, and validate the schemes developed to solve the LBQ equations. Section 5.2 examines the code, LBQ1D, in the one-dimensional case of the bump on tail. LBQ1D has the essential elements for building the two-
dimensional code, LBQ2D, which differs mainly in the IMP-EXP scheme developed to solve the quasilinear diffusion in \((P_\phi, E)\) space. Section 5.3 examines the code LBQ2D, where the IMP-EXP scheme is described and validated. Some examples of overlapping modes are presented.

5.2 LBQ1D

By treating the one-dimensional case, most of the core concepts are introduced and the general algorithm which is applicable to the two-dimensional code LBQ2D is presented. The main difference between one and two-dimensional LBQ is the scheme for solving the diffusion equation.

In the one-dimensional limit, Eqs. (4.4) are solved. They are written as

\[
\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_n \sum_l \frac{\partial}{\partial x} C_n^2 \mathcal{F}_{l,n} \frac{\partial}{\partial x} f + \nu_{\text{eff}}^3 \frac{\partial^2}{\partial x^2} (f - f_0),
\]

\[
\frac{d C_n^2}{dt} = 2(\gamma_L - \gamma_d) C_n^2,
\]

where the growth rate is the integral

\[
\gamma_{L,n} = \frac{\pi}{4} \sum_l \int dx \mathcal{F}_{l,n} \frac{\partial f}{\partial x}.
\]

\(\gamma_d\) is the damping rate, \(\nu_{\text{eff}}^3\) is the rate of collisionality, \(\nu_{\text{eff}} \frac{\partial^2 f_0}{\partial x^2}\) is the source term with \(f_0\) being the initial distribution function. More details on the collisional term are found in Appendix A.

This is a self-consistent set of equations solved in the sequence illustrated in Fig. 5.1. At the beginning of every time step \(t\), the current diffusion coefficients \(D(t)\), which is updated from the previous time step, are used to evaluate the distribution function at \(t + dt\) using Eq. (5.1a). The mode amplitude evolves according
to Eq. (5.1b) using $\gamma^{(t)}$. Then the instantaneous linear growth rate $\gamma$ is computed using $f^{(t+1)}$ according to Eq. (5.2). Now that the values of $\omega_b$ and $\gamma$ are computed at $t + dt$ the broadening width $\Delta \Omega$ is computed and the diffusion coefficients at $t + dt$ are evaluated. This resets the cycle to the first step where the updated diffusion coefficient is used to evolve the distribution function.

Figure 5.1: At every time step, one first evaluates the distribution function at $t + dt$ using the current diffusion coefficients, then evolve the mode amplitude using the current growth rate. The instantaneous linear growth rate $\gamma$ is evaluated to compute the broadening width using the updated $\omega_b$ and $\gamma$. This completes the cycle where all the updated values are used to evaluate the diffusion coefficients used in the first step to evolve the distribution function.

Section 5.2.1 presents the scheme used for evolving the distribution function and validate it with analytically known results. Section 5.2.2 describes the difference equation used for evolving the mode amplitude and the discretization for the growth rate. It is noted in Sec. 5.2.3 that, unless the mode grows to a point where the broadening width reaches the boundary, the equations conserve particles, and conserve momentum in the ideal case where $\nu_{\text{eff}} = 0$ and $\gamma_d = 0$. The scheme described and validated is used to get the results in Sec. 4.6.
5.2.1 Diffusion Equation

The diffusion equation is solved using an implicit scheme with discretization that is centered in space and backward in time. Allowing for variable grid size, one has

\[
\frac{1}{\Delta t} \left( f_i^{(t+1)} - f_i^{(t)} \right) = \nu_{\text{eff}}^3 \left( f_{i+1}^{(t+1)} - f_i^{(t+1)} \right) \left( f_i^{(t+1)} - f_{i-1}^{(t+1)} \right) \left( x_{i+1} - x_i \right) - \frac{f_i^{(t+1)} - f_i^{(t+1)}}{x_i - x_{i-1}} \left( x_{i+1/2} - x_{i-1/2} \right) + \pi \sum_n C_n^2 \left( f_i^{(t+1)} - f_i^{(t+1)} \right) \left( x_{i+1/2} - x_{i-1/2} \right) \left( x_{i+1/2} - x_{i-1/2} \right)
\]

holding true for each grid in space. The boundary condition can be either Neumann conditions with the gradients at the boundaries \( f_0 \) and \( f_N \) fixed:

\[
f_i^t = f_i^{0} - f_i^{0} - f_i^{0} + f_i^{t-1},
\]

or Dirichlet conditions with the values of \( f \) at the boundary fixed: \( f_N^t = f_N^{0} \) and \( f_0^t = f_0^{0} \). The boundary conditions are required in order to invert the matrix. In this treatment Neumann conditions are used to allow for particle losses if the mode grows beyond the plasma’s last closed flux surface.

I write the equations using a tridiagonal matrix as \( \mathbf{D} \mathbf{f}^{(t+1)} = \mathbf{s} \), which is inverted using Gaussian elimination to solve for \( \mathbf{f}^{(t+1)} \). The boundary conditions are included by modifying the first and last row of the matrix \( \mathbf{D} \) and adding a term to the first and last element of the source vector \( \mathbf{s} \) as follows:
where

\[
B_0 = f_0^{(t)} + (f_1^0 - f_0^0),
\]

\[
B_N = f_N^{(t)} + (f_N^0 - f_{N-1}^0),
\]

are the modified elements that impose the boundary conditions and the diffusion is summed over all modes \(m\), written as

\[
D_{-1} = -\frac{\Delta t}{\Delta x^2} \nu_{\text{eff}}^3 \sum_m \Delta t \frac{D_{i-1/2}^{(m)}}{\Delta x^2}, \tag{5.4}
\]

\[
D_0 = +2 \frac{\Delta t}{\Delta x^2} \nu_{\text{eff}}^3 + \sum_m \Delta t \left( \frac{D_{i-1/2}^{(m)}}{\Delta x^2} + \frac{D_{i+1/2}^{(m)}}{\Delta x^2} \right), \tag{5.5}
\]

\[
D_1 = -\frac{\Delta t}{\Delta x^2} \nu_{\text{eff}}^3 \sum_m \Delta t \frac{D_{i+1/2}^{(m)}}{\Delta x^2}. \tag{5.6}
\]

and \(D^{(m)} = \frac{1}{2} \pi C_{(m)}^2 F\).

In the case of multiple modes, each mode is evolved separately according to Eq. (5.1b) and then \(F_n^i\) is computed accordingly and the diffusion coefficient is the sum over all the modes.
Testing the Scheme

The diffusion scheme is validated by solving the heat equation with a point source initial condition, which has known analytic solutions. The diffusion equation is

\[
\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}
\]

with initial value \( f_0 = \delta(x) \). The analytic solution is

\[
f(t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}
\]

with either Dirichlet or Neumann conditions. Figure 5.2 represents the results of the diffusion scheme as compared to the analytic results at four different times. This validates the numerical scheme used to invert the matrix and solve for the distribution function at each time step.

Figure 5.2: The figure to the left depicts the solution of \( f \) at \( t = 100\Delta t, 200\Delta t, 300\Delta t, \) and \( 400\Delta t \) using the diffusion scheme vs the analytical results. To the right, the difference between these two results normalized to the value of \( f \) at \( x = 0 \) is depicted. The parameters \( D = 1, dx = 10^{-2}, \) and \( \Delta t = 10^{-4} \) are chosen.

This validates the numerical scheme used to invert the matrix and solve for the distribution function at each time step.
5.2.2 Mode Evolution

Once the distribution function is evolved at a given time step, the modes need to be updated. Equation \((5.1b)\) is solved using a forward-in-time explicit difference scheme

\[
C^2|_{t+1}^{n+1} = 2(\gamma^t - \gamma_d)C^2|_n^t + C^2|_n^t,
\]

where the growth rate, Eq. \((5.2)\), of each mode is computed from the sum

\[
\gamma^t_n = \frac{\pi}{4} \sum_i F^t_{n,i+1/2} \frac{f^t_{i+1} - f^t_{i-1}}{x_{i+1} - x_i}.
\]

The broadening width is then evolved accordingly and results in an updated \(F_n\) to be used in the diffusion equation for the following time step.

5.2.3 Conservation Properties

When the mode width is confined within the boundaries of the plasma, the number of particles is conserved: \(\frac{d}{dt}\sum_i f_i = 0\). Additionally, in the ideal case \(\nu_{\text{eff}} = 0\) and \(\gamma_d = 0\) the LBQ equations conserve momentum. Although it is a regime that is not applicable within the LBQ model, it acts as a good test for the self-consistency of the schemes. Following the LBQ equations, it is expected that the total momentum is conserved: \(M^{(t+1)} - M^{(t)} = 0\), where \(M^{(t)} = M C^2(t) + \sum_i x_i f_i^{(t)}\). The detailed calculations starting with the difference equations, \((5.3)\), \((5.9)\), and \((5.10)\) are straightforward albeit algebraically involved, so the calculation is omitted in this manuscript. The flow outlined in Fig. 5.1 is essential for the conservation of momentum. Following that procedure, both the amplitude \(C^{2(t+1)}\) and \(f^{(t+1)}\) are evaluated using the same \(F^{(t)}\) and therefore guarantee that the same momentum is transferred from the particles to the mode or vice versa. The schemes developed are numerically verified to indeed conserve the total momentum up to numerical errors. The plot, which simply shows
that the $M^{(t+1)} - M^{(t)}$ is a random number of the order $10^{-15}$ for all $t$, is omitted. While momentum is exactly conserved with errors that are floating-point errors, the energy is not exactly conserved, where the errors are a result of truncation errors.

5.3 LBQ2D

The proposed LBQ model is motivated by the problem of Alfvén mode interaction with energetic particles in two-dimensional phase space. In Sec. 5.2, the algorithm presented for the one-dimensional code, Fig. 5.1, is readily applicable to the TAE-Energetic particles interaction. The focus in this section is on the form of the diffusion equation, the scheme used to solve it, and the validation of this scheme.

5.3.1 TAE-EP Interaction in LBQ Model

The energetic particle distribution is in the three-dimensional phase space $(P_{\phi}, E, \mu)$. Since the problem of interest is the Alfvén mode interaction with these particles, motion in $\mu$ remains invariant under this interaction. Although integration over the whole phase space is needed at each time step to evaluate the growth rate, the particle diffusion in each slice in $\mu$ can be treated separately since there is no particle transport across $\mu$.

The main differences between the slices in $\mu$ are the resonant curves, matrix elements, and the loss boundaries of energetic particles. For a given mode, the resonant manifold satisfies the condition $\mathbf{l} \cdot \Omega(P_{\phi}, E, \mu) = \omega$ that results in different curves in $(P_{\phi}, E)$ space for different values of $\mu$. The matrix elements depend on phase space and therefore would be different for different slices. As for the loss boundaries, energetic particles confined in a tokamak can assume different orbits in phase space (co and counter passing orbits, trapped banana or potato orbits, stagnant orbits, etc.). These orbits occupy regions of phase space with clear boundaries. As particles diffuse,
the changing values of $P_\phi$ and $E$ can transform the orbit from a passing to a slightly trapped banana orbit with very large orbit width that is promptly lost to the edge. The loss boundaries in phase space depend therefore on the type of orbits for the different $P_\phi$ and $E$ values depending on $\mu$. For $\mu = 0$, given values of $P_\phi$ and $E$ could result in a co-passing orbit, while for $\mu \neq 0$, the same values of $P_\phi$ and $E$ could result in a trapped orbit that intercepts the wall and is lost. Therefore, the loss boundary conditions in $(P_\phi, E)$ space are different for different sections in $\mu$.

If $\mu = 0$ particles are all passing and the loss boundary is simply the values of $P_\phi$ and $E$ that satisfies $r(P_\phi, E) = a$, where $a$ is the last closed flux surface and $r$ is the equation for the radial location depending on the phase-space variables. However, for values of $\mu$ where particles can be either in trapped orbits or passing orbits, the loss boundary is still $r = a$ for the passing orbits, while it is the curve bounding the trapped orbits whose orbit width is smaller than what would intercept with the wall.

LBQ2D equations in $(P_\phi, E)$ are

$$\frac{\partial f}{\partial t} = \frac{\pi}{2} \sum_n \sum_l \frac{\partial}{\partial P_\phi} \left. \frac{(2C_nV_{l,n})^2}{\partial \Omega/\partial P_\phi|_{E_n}} F_{n,l} \frac{\partial}{\partial E_n} \right|_{E_n} f + \nu[f], \quad (5.11a)$$

$$\frac{dC^2}{dt} = 2(\gamma_{L,n} - \gamma_d)C^2, \quad (5.11b)$$

where $\nu[\cdot]$ is the collision operator that is further discussed in Appendix $A$. Equation (5.11a) is true for all sections in $\mu$, and the growth rate for each mode is

$$\gamma_{L,n} = \frac{\pi}{4} \sum_l \int d\mu \int \tau \Gamma dP_\phi (2V_{l,n})^2 F_{n,l} \frac{\partial}{\partial P_\phi} \right|_{E_n} f. \quad (5.12)$$

These are the set of equations that need to be solved at each time step to evolve the particle mode system. The prompt loss boundary is calculated initially for a given value of $\mu$. Neumann condition is imposed at these boundaries in setting up the difference scheme.
5.3.2 Equation of Particle Diffusion

This discussion focuses on one given value for $\mu$ to treat the particle evolution, Eq. (5.11a), in $(P_\phi, E)$ space and describe the diffusion scheme used. Before evaluating $\gamma_L$, the diffusion procedure described in this section is repeated for all values of $\mu$.

The diffusion term in Eq. (5.11a) is cast relative to a direction that is decided on, preferably the diffusion path of the strongest mode, $(n_0, \omega_0)$. This is done by rewriting the differential for a given mode, $(n, \omega)$ as follows:

$$
\left. \frac{\partial}{\partial P_\phi} \right|_{E'_n, \omega_0} = \left( \frac{\partial}{\partial P_\phi} + \frac{\omega}{n} \frac{\partial}{\partial E} \right) \rightarrow \left( \frac{\partial}{\partial P_\phi} + \frac{\omega}{n} \frac{\partial}{\partial E} + \frac{\omega_0}{n_0} \frac{\partial}{\partial E} - \frac{\omega_0}{n_0} \frac{\partial}{\partial E} \right)
= \left. \frac{\partial}{\partial P_\phi} \right|_{E'_n, \omega_0} + \left( \frac{\omega}{n} - \frac{\omega_0}{n_0} \right) \frac{\partial}{\partial \omega}.
$$

(5.13)

Notice that a choice of $\omega_0/n_0 = 0$ reverts the equations to their original form. A choice of $\omega_0/n_0$ is akin to a rotation of some sort where the distribution function $f_{i,j}$ is discretized along the axis where $E' = E - \omega_0 P_\phi/n_0$ and along $E$ as shown in Fig. 5.3. The procedure is akin to taking sections of the distribution function along

![Figure 5.3: The a top view of the mesh in $(P_\phi, E)$ space for a rotation of slope $S = 1$.](image)

a slant as illustrated in Fig. 5.4 for a choice of slope $S = 1$. 

107
Figure 5.4: (a) An arbitrary distribution function in $(P_\phi, E)$ space with a section along $E' = E - \omega_0 P_\phi/n_0 = 0$ singled out. (b) $f_{i,j}$ along the $E' = 0$ as a function of $P_i$.

Using a central difference scheme as described in Sec. 5.2 the problem can be solved implicitly. This has the logic shown in Fig. 5.5a, where a banded matrix illustrated in Fig. 5.5b needs to be inverted.

Figure 5.5: (a) The difference scheme used to solve a fully two-dimensional diffusion problem, and (b) the associated banded matrix that needs to be inverted.

Inverting the full matrix is numerically a very expensive procedure, $O(N^3)$, except in the limit of isolated modes where a rotation into the direction of diffusion can be made to reduce the equations to one dimension. This special case is treated in Sec. 5.3.3 then the full two-dimensional problem is treated in Sec. 5.3.4.

Before the numerical scheme is described, the notation is simplified by implying that $\partial/\partial P$ is $\partial/\partial P_\phi$ with $E'$ held constant and writing the slope with respect to the
chosen reference direction as \( S_m = (\omega/n - \omega_0/n_0) \). Equation (5.11a) becomes

\[
\frac{\partial f}{\partial t} = \sum_m \left( \frac{\partial}{\partial P} + S_m \frac{\partial}{\partial E} \right) D^{(m)} \left( \frac{\partial}{\partial P} + S_m \frac{\partial}{\partial E} \right) f + \nu_{\text{eff}}^3 \frac{\partial^2}{\partial P^2} (f - f_0),
\]

(5.14)

where the sum over the different values of \( l \) of the different modes are all considered under one sum and \( D^{(m)} = \frac{1}{2 \pi} (2C_n V_{n,l})^2 F_{n,l} / (\partial \Omega / \partial P |_{E'}) \).

As for the collision operator \( \nu[f] \), a diffusive operator is chosen \( \nu_{\text{eff}}^3 \frac{\partial f}{\partial P} \), where \( \nu_{\text{eff}}^3 \) is the projection of the collision operator along \( E' = E - \omega_0 P \phi / n_0 \). Details of the collision operator are in Appendix A.

5.3.3 Isolated modes

In the case of isolated modes, a rotation is made into the direction of diffusion by choosing \( \omega_0/n_0 = \omega/n \), making \( S_m = 0 \). This results in an LBQ set of equations

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f + \nu_{\text{eff}}^3 \frac{\partial^2}{\partial P^2} (f - f_0),
\]

(5.15a)

\[
\frac{d C^2}{d t} = 2(\gamma_L - \gamma_d) C^2,
\]

(5.15b)

where the linear growth rate for the isolated mode is

\[
\gamma_L = \frac{1}{2MM} \int dE \int dP \frac{D^{(m)}}{C^2} \frac{\partial f}{\partial P}.
\]

(5.16)

Equation (5.14) is thus reduced from two dimensions to a set of one-dimensional diffusion equations with Neumann boundary conditions. The difference equation of the particle diffusion, Eq. (5.15a), for a given \( j \), is equivalent to Eq. (5.3) where \( f_i \) is replaced with \( f_{i,j} \). The same solver as described and validated in Sec. 5.2.1 is used to solve the \( N \) one-dimensional equations.

In a grid with \( NP \) points in \( P \) and \( NE \) points in \( E \), one ends up with \( NE \) matrices of size \( NP \times NP \) to invert in the same manner as Sec. 5.2. This results in an algorithm
of $O(N)$ complexity vs the full problem of complexity $O(N^3)$ if the rotation has not been made.

### 5.3.4 Multiple Modes

In general there exist multiple modes with different directions of diffusion. This results in a two-dimensional diffusion equation with cross terms, Eq. (5.14).

The first step is to calculate the diffusion coefficient, $D^{(m)}$, that results from each mode. $D^{(m)}$ depend on the calculated width of the broadening for each mode and has a form $F$, Eq. (4.2) in Chapter 4, along the path of diffusion.

To calculate the diffusion coefficient for the mesh points from all the modes, one needs to know which island the mesh point, $(P,E)$, belongs to and where along the island it is for each mode.

Given a mode with a discrete set of points along the resonant curve denoted by $R^{(m)}$, and a slope $S^{(m)}$ for the diffusion path of the mode, one needs to find the intersection between the line passing through the given mesh point $(P,E)$ with slope $S^{(m)}$ and the line passing through the two consecutive points $R^{(m)}_i$ and $R^{(m)}_{i+1}$ on the resonant curve. If the two lines intersect at a point $R_{P,E}$ within the segment $[R^{(m)}_i, R^{(m)}_{i+1}]$, the mesh belongs to the island centered at $R_{P,E}$ in that segment whose matrix element and width are the weighted average of those of $R^{(m)}_i$ and $R^{(m)}_{i+1}$.

The diffusion coefficient is crucial for setting up the equations. Once the coefficients of each mode are computed at each point in phase space, the following diffusion equation is solved:

$$
\frac{\partial f}{\partial t} = \sum_m \left( \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f \right) + \left( S_m^2 \frac{\partial}{\partial E} D^{(m)} \frac{\partial}{\partial E} f \right) + \left( S_m \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f \right) + \left( S_m \frac{\partial}{\partial E} D^{(m)} \frac{\partial}{\partial E} f \right) + \nu_{\text{el}}^3 \frac{\partial^2}{\partial P^2} (f - f_0). \tag{5.17}
$$
A mixture of explicit and implicit schemes is used as discussed in Sec. 5.3.5, where the term for diffusion in P is treated implicitly while the terms for diffusion in E, the cross terms, and the collisions are treated as explicitly as a source term. The value of the source term is used at the time step \( t + 1 \) and is iteratively updated.

### 5.3.5 Implicit-Explicit Scheme

Solving the problem using explicit equations demands very small time steps to satisfy the Courant condition \( \Delta t < \Delta x^2/4D \), where \( D \) is the maximum value of the diffusion coefficient. On the other hand, solving it using an implicit scheme is also very time consuming since it requires inverting a 9-banded \( NP^2 \times NE^2 \) matrix at each time step. NP and NE are the number of grid points in the \( P \) and \( E \) direction respectively.

The proposed scheme IMP-EXP utilizes aspects from both schemes in an attempt to minimize the computational time. The two-dimensional diffusion equation is written as an NE set of difference equations with a tridiagonal matrix to invert. The cross terms and diffusion in \( E \) are treated as a source term, denoted by \( Ex \). After using the \( f^{(t)} \) for the source term in the first iteration, \( Ex(f^{t+1}) \) is used and an iteration is made until \( f^{(t+1)} \) converges to what is expected with a fully implicit scheme.

Starting with Eq. (5.17) the term

\[
\sum_m \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f
\]

is treated implicitly, while the terms

\[
\sum_m S^2_m \frac{\partial}{\partial E} D^{(m)} \frac{\partial}{\partial E} f + S_m \frac{\partial}{\partial E} D^{(m)} \frac{\partial}{\partial P} f + S_m \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial E} f + \nu_{eff}^3 \frac{\partial^2}{\partial P^2} (f - f_0)
\]

are treated explicitly.

The exact difference equations used in LBQ2D are presented in Appendix B. Assume a grid with NP points in \( P \) and NE points in \( E \). Treating \( P \) implicitly and \( E \)
explicitly allows one to reduce the two-dimensional diffusion equation to a set of NE
one-dimensional equations coupled through, Ex, the explicit term that is Eq. (5.19)
evaluated at $t+1$ and iteratively updated. At each iteration, the NE set of equations:

$$\frac{d f_{E}^{t+1(k+1)}}{dt} = \frac{\partial}{\partial P} \sum_{n} D \frac{\partial f_{E}^{t}}{\partial P} + Ex^{k}$$

(5.20)

are solved for each slice in $E$, where $n$ is the number of modes. The explicit term,
$Ex$, is computed using $f^{t+1}$ form the previous iteration. The procedure is outlined in
Algorithm 1. Once the error is below a given threshold, the operation is terminated

\begin{algorithm}
\begin{algorithmic}
1: while $err > \epsilon$ do
2: for $E = 1 : 1 : NE$ do
3: Solve Eq. (5.20) to get the $k^{th}$ iteration for $f_{E}^{t+1}$
4: end for
5: Compute $Ex^{k+1}$ using the $f^{k+1}$ solved for.
6: Set $err = (Ex^{k+1} - Ex^{k})/Ex^{k}$
7: end while
\end{algorithmic}
\end{algorithm}

and the resulting distribution function is used as a starting point for the following
time step.

In what follows, the convergence of the numerical scheme is studied by systemati-
cally decreasing the time step and grid size. The convergence of the iterative process
is also presented and found to depend on the time step and the slope of the mesh.
Finally, the IMP-EXP scheme is validated by comparing its results for isolated modes
whose $S \neq 0$ in a cartesian mesh to the results of the fully implicit code for the right
choice of mesh.

Convergence in Time Step

It is shown that the scheme converges as the time step is reduced. Figure 5.6 presents
the results as the time step is reduced from $\Delta t = 2/\gamma L$ to $\Delta t = 0.1/\gamma L$. The chosen
mode has slope $S = 0.3$, $\gamma d = 0.3\gamma L$, and $\nu_{eff} = 3\gamma L$. 
Figure 5.6: As the time step is taken smaller the time evolution of the mode converges. For any further reduction in the time step, the resulting time evolution is indistinguishable from that for $dt = 0.1/\gamma L$.

**Convergence in Grid Size**

For assessing the discretization error, the code is run with decreasing grid size and the resulting distribution functions are compared. Figure 5.7 depicts the distribution function $f(P_{\phi})$ at $E = 0$ at saturation for three choices of grid point numbers ($N = 90, 60, \text{and } 40$). The distribution function is represented at $E = 0$, where the matrix element peaks, since a comparison in two-dimensional phase space is difficult to depict. The solution converges as the grid size is taken to be 1.5 times smaller for each run. Part of the discrepancy arises from the mismatch between the window edges and the grid points. Therefore, some of the error can be alleviated by using the same grid size but with adaptive grids points. The code is developed for a non-uniform grid in $P$; however, once the grid points are specified, they cannot change with time since it would require significant code development while only marginally reducing the error.
Figure 5.7: The distribution function $f(P_\phi)$ at $E = 0$ when the mode reaches saturation for three different choice of number of grids. The solution converges as the grid number increases; beyond $N = 60$ the change is negligible.

**Number of Iterations**

The other main convergence concern is the number of iterations needed to resolve the cross term contribution. Ideally, the proposed scheme would converge onto the solution resulting from inverting the 9-banded $(N^2 \times N^2)$ matrix of the fully implicit difference scheme. The larger the explicit contribution, the more iterations are needed to converge to the solution. The explicit contribution increases with increasing slope or with larger time step, which results in more iterations required to resolve the dynamics. Figure 5.8 shows that for a given slope $S = 0.4$, a time step $\Delta t > 0.4/\gamma_L$ will not converge, and that an increasing number of iterations is needed as $\delta t$ increases. The time step at which the scheme converges depends on the grid size. If all else is equal, a decreasing grid size would require a smaller time step for the scheme to converge.

To study the effect of the slope on convergence of the scheme, the code is run for modes with different choice of the mesh slope while keeping the same mode. Figure 5.9 shows the resulting number of iterations required for convergence of the scheme when
Figure 5.8: The number of iterations needed to achieve convergence for a given slope of $S = 0.4$.

the mode has slope $S = 0.4$ as compared to $S = 0.3, 0.2,$ and $0.1$. Indeed for a given time step small enough to guarantee convergence for $S = 0.4$, the number of iterations decreases as the slope decreases with all else kept the same. The case of $S = 0$ results in no iterations since the problem becomes fully implicit.

Figure 5.9: The number of iterations for $S = 0.4$ is compared to $S = 0.3$, $S = 0.2$, and $S = 0.1$. 

115
Validation

In the case when modes are isolated, it is shown in Sec. 5.3.3 how one can choose the slope of the mesh to align with the direction of diffusion to reduce the problem to a set of one-dimensional diffusion equations. However, a grid with $S = 0$ can be chosen to use the IMP-EXP scheme to solve for the same isolated mode. This allows one to compare the results of the proposed IMP-EXP scheme to the standard fully implicit scheme, which is itself verified in Sec. 5.2.1.

As an example, a mode with $\nu_{\text{eff}} = 3\gamma L_0$ and $\gamma_d = 0.2\gamma L_0$ is chosen. The slope in phase space is taken to be $S^{(m)} = 0.4$.

Before validating the diffusion scheme, Fig. 5.10a depicts that the diffusion coefficients computed using the mesh along the diffusion path are indeed equivalent to those computed using a cartesian mesh. In general, the resonant curve is a function of $P$ and $E$, but here, and without loss of generality, the resonance is at $P = P_0 = 0.5$.

![Figure 5.10: The diffusion coefficient computed for the two choices of mesh. The figure to the left depicts the diffusion coefficients as computed by the mesh aligned with the diffusion path (blue) and the cartesian mesh at which the diffusion path is at an angle (cyan). The figure to the right depicts the area of nonzero $F$ that is outlined by the envelope of the broadening. One sees the effect of the slope since the function $F$ is symmetric with respect to the resonant curve, $P = 0.5$, along the path of diffusion.](image)

Now that it is verified that the calculation of the diffusion coefficient is consistent, one can validate the diffusion scheme. Figure 5.11 presents the resulting distribution function using the fully implicit scheme vs the IMP-EXP scheme that is proposed.
Since the choice of mesh is different for the two cases, comparing the exact value requires interpolation that introduces errors. The important features however are captured in the figures, and it is clear that the scheme works closely to the fully implicit one.

![Graphs showing comparison between fully implicit scheme and IMP-EXP scheme](image)

Figure 5.11: A comparison of the results of the fully implicit scheme to that of the IMP-EXP scheme that solves the two-dimensional diffusion equation. The modes have a diffusion path of slope \( S = 0.4 \) in the \((P, E)\) frame. The results are shown at different angles. The red dots present the result for the fully implicit scheme using a slanted mesh while the blue dots are that of the IMP-EXP scheme using a Cartesian mesh.

Fig. 5.12 presents the comparison between the evolution of the bounce frequency resulting from fully implicit vs IMP-EXP scheme.

Since the implicit scheme is already well validated, comparing the IMP-EXP with the implicit scheme validates the IMP-EXP scheme that is needed to resolve the
Figure 5.12: The evolution of the bounce frequency using the IMP-EXP scheme (blue) vs the fully implicit scheme (red) made possible by rotating the mesh into the direction of diffusion.

dynamics when there are overlapped multiple modes with different $S$. The following section presents an example to illustrate mode overlapping.

### 5.3.6 Mode Overlapping

To illustrate the IMP-EXP scheme resolving the dynamics of multiple modes overlapping, two modes are chosen with resonant curves $P = 0.4$ and $P = 0.6$, matrix elements approximated with gaussians, and slopes of the direction of diffusion $S = 0$ and $S = 0.6$. Both modes have $\gamma_d = 0.2$. The effective collision rate is taken to be $\nu_{\text{eff}} = 3\gamma_L$.

The resulting modes evolve self-consistently as the distribution function is flattened within the broadened regions in phase space. Figure 5.13 shows the distribution function at four different times that best represents the evolution.

The overlap of modes results in a release of energy more than what would have been released have the modes evolved without overlap. To elucidate this, the code is run with each mode isolated while keeping all else equal. In Fig. 5.14, the mode
Figure 5.13: The evolution of the distribution function as it flattens due to its interaction with the modes. The solution is at times $\tau = 2.7$, $\tau = 5.4$, $\tau = 8.1$, and $\tau = 10.7$.

amplitude evolution for the case the modes are separately evolved is compared to the evolution of modes when they co-exist and overlap.

Figure 5.14: The evolution of the amplitude of the two modes when they are overlapped (solid) as compared to their evolution in case they do not overlap. The discrepancy is accentuated after the modes overlap at $\tau = 3$. 

119
Figure 5.15 depicts the envelope of the mode broadening function at saturation. Lines are drawn in the direction of the mode diffusion paths to illustrate the difference in diffusion paths.

The additional energy released from two overlapping modes can sometimes be enough to result in overlap with an otherwise isolated mode. This in turn can release more energy to overlap with yet another mode and so on and so forth. This domino effect can be detrimental in case many modes overlapping induces transport in large regions of phase space, which results in significant losses of energetic particles.

Burst of Energy

The two scenarios of mode evolution are when modes evolve until they overlap or when the modes flatten the distribution locally. However, for certain parameters it is possible for modes to overshoot the expected saturation level and slowly relax back to the saturated state as $\gamma_L$ approaches $\gamma_d$. The following is a study of what happens
in this case if the modes are close enough to overlap at their peak but not overlap at their saturated level.

The parameters and resonant conditions are chosen to ensure that the modes overlap briefly during their time evolution before resorting to being isolated. This case is demonstrated in Fig. 5.16 which depicts the distribution function at four different time steps representing the transition of the mode from being isolated to overlapped and back to being isolated.

Figure 5.16: The evolution of the distribution function as the modes overlap briefly before resorting to being isolated.

The code is run for each mode considered separately. Figure 5.17 represents the evolution of the two modes’ amplitude in the case they coexisted as compared to their amplitude in the case the modes are isolated.

There is a 2.5 times increase in mode energy due to the overlap. Even if the saturated amplitudes of the two modes are not enough for them to overlap, the brief overlap at the peak results in the expected extra energy and momentum transfer.
resulting in a gain in amplitude before settling back into being isolated. This case is important to capture since the gain in mode amplitude can be enough for particles to be transported beyond the last closed flux surface. This would result in particle expulsion, which would not be captured if the saturated amplitude is assumed and the modes considered non-overlapping. Additionally, if there were more modes present, the observed increase in energy due to the overlap, albeit brief, could lead to a domino effect where more modes overlapping results in large-scale transport and subsequent losses of energetic particles.

5.4 Summary and Discussion

Studying energetic particles interaction with Alfvén Eigenmodes in toroidal geometry is very important for future burning plasmas. The LBQ model discussed in Chapter 4 captures the effect of this interaction on the particle confinement and mode evolution in certain regimes of applicability. In this chapter, the code used to solve the equations
of the model is described and the schemes validated. In the limit of $\omega_\ast \geq \omega$ the interaction can be reduced to a trivial one-dimensional case where diffusion is assumed along $P_\phi$ only as implemented in Ref. [36]. However there may exist modes [112] where this condition does not hold, in which case the diffusion in $E$ needs to be resolved. The LBQ2D code is developed to capture the interaction in $(P_\phi, E)$ space for all $\omega/n$. In the $(P_\phi, E)$ space, the equations can be reduced to a set of one-dimensional equations if modes are isolated; however, when modes with different $\omega/n$ overlap, the quasilinear diffusion is fully two-dimensional with cross terms. This problem can be numerically intensive if solved with known schemes such as an implicit or explicit scheme. A fully implicit scheme would require inverting a 9-banded $N^2 \times N^2$ matrix at each time step. A fully explicit scheme requires a very small time step to satisfy the Courant condition $\Delta t < \Delta x^2/4D$, where $D$ is the maximum value of the diffusion coefficient. I described and validated the scheme, IMP-EXP, that resolves the diffusion by striking a balance between the implicit and the explicit schemes. The diffusion equation is written as a set of difference equations with a tridiagonal matrix to invert. The cross terms and diffusion in $E$ are treated as a source term evaluated at $t + 1$ and are iteratively updated. The iterative scheme converges to the solution of a fully implicit scheme since the source term is evaluated using $f(t+1)$ from the previous iteration. There is still a constraint on the required time step since the cross terms are treated as a source term, but the time steps applicable are still much larger than $N_{itr} \Delta t_{exp}$ where $N_{itr}$ is the number of iterations required and $\Delta t_{exp}$ is the time step required of explicit schemes. This results in IMP-EXP scheme being much faster than a fully implicit or a fully explicit scheme. LBQ2D is developed as a tandem module that relies on calculations of equilibrium codes such as TRANSP to provide the distribution function, and linear codes such as NOVA to provide the TAE mode structure for the matrix elements and the resonant curves and NOVA-K to provide the damping rates and $\nu_{eff}$. 
5.4.1 Work in Progress

With the IMP-EXP scheme validated and demonstrated to work, the LBQ2D code is being further developed to incorporate two major additions: introducing an adaptive time step, and parallelization.

Figures 5.8 and 5.9 show how the number of iterations vary during the mode evolution since more iterations are needed as $D_{QL}$ increases. This implies that an adaptive time step would speed up the simulation. The code is being developed to have a time step that changes such that a fixed number of iterations $N_{itr}$ are needed for convergence of the IMP-EXP scheme at every time step. Currently, a fixed time step $\Delta t$ is chosen such that it guarantees convergence at all times. The choice of $\Delta t$ is small enough such that the maximum number of iterations is around 20 or 30. This results in very few iterations at certain times during the evolution and a moderate number at others. On the other hand, in the case of an adaptive time step, for certain times, a much larger $\Delta t$ could result in convergence with the number of iterations $N_{itr}$ predetermined. This would both speed up the code overall and automatically ensure convergence at every time step. The choice of the optimal $N_{itr}$ is a point of investigation and is expected to depend on the slope, grid size, and time scales in the system such as $\gamma_L$, $\nu_{eff}$, the matrix elements, etc.

The other advantage of the IMP-EXP scheme is that the dimensions are decoupled. The diffusion equation for the distribution function in $NP \times NE$ sized mesh is reduced to $NE$ difference equations where all the cross terms are treated as a source term. This makes the code strongly parallelizable since at each iteration of every time step the NE equations are solved separately.

Making the code parallel and introducing a variable time step significantly adds to the efficiency and speed of LBQ2D.
Appendix A

Collision Operator for Fast Ions

The Fokker-Plank equation for fast ions is

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m} E \cdot \frac{\partial f}{\partial v} = \nu[f], \quad (A.1)$$

where Vlasov equation from which the CQL equations are derived is the special case of $\nu[f] = 0$. However, to study the effect of collisions and the form of the equilibrium distribution function of fast ions, the form of $\nu[f]$ needs to be known.

Fast ions (tail ions) are assumed to interact with a Maxwellian bulk plasma resulting in fast-ion pitch-angle scattering and collisions with ions and electrons result in fast-ion diffusion and drag. In the velocity limit of $v_{Ti} \ll v \ll v_{Te}$, simplifications can be made to the collisional parameters form to get the following operator

$$\nu[f] = \frac{\nu^{i^F}_{ai}}{v^3} \mathcal{L}(f) + \frac{\nu^{E}_{ai}}{v^2} \frac{\partial}{\partial v} \left[ \left( 1 + \frac{v^3}{v^3_c} \right) \left( 1 + \frac{T}{m_{ei} v \frac{\partial}{\partial v}} \right) f \right], \quad (A.2)$$

where $\mathcal{L}$ is the Lorentz pitch-angle scattering operator and $\nu^{F,E}_{ai}$ are the collision frequencies. In the LBQ model, only the ion collisional terms are considered while
the electron drag terms are neglected. This results in the diffusive collisional operator

\[ \nu[f] = \frac{\bar{v}^E_{ai}}{v^2} \frac{\partial}{\partial v} \left[ \frac{T}{m} \left( 1 + \frac{v^3}{v_c^3} \right) \frac{\partial}{\partial v} f \right]. \]  

(A.3)

A.1 Collision Operator in \( P_\phi, E \) space

The operator is written as a function of \( P_{E'} \) where

\[
\frac{\partial}{\partial v} = \frac{\partial E}{\partial v} \frac{\partial}{\partial E} + \frac{\partial P}{\partial v} \frac{\partial}{\partial P} = \frac{\partial P}{\partial v} \left[ \frac{\partial}{\partial P} + \frac{\partial E/\partial v}{\partial P/\partial v} \frac{\partial}{\partial E} \right]
\]

\[
\rightarrow \frac{\partial P}{\partial v} \left[ \frac{\partial}{\partial P} \bigg|_{E'} + \left( \frac{\partial E/\partial v}{\partial P/\partial v} - \frac{\omega}{n} \right) \frac{\partial}{\partial E} \right] \]  

(A.4)

and

\[
\frac{\partial}{\partial P} \bigg|_{E'} = \left( \frac{\partial}{\partial P} + \frac{\omega}{n} \frac{\partial}{\partial E} \right). \]  

(A.5)
Appendix B

Difference Scheme

A forward time and center in space discretization is used to set up the implicit explicit finite-difference scheme to solve the diffusion equation. The implicit part of the difference scheme is written as

\[ \sum_m \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f = \sum_m \]

\[ f^j_{i+2} \frac{D^{(m)}_{i+1}}{(P^j_{i+2} - P^j_i)(P^j_{i+1} - P^j_{i-1})} + f^j_{i-2} \frac{D^{(m)}_{i-1}}{(P^j_{i-2} - P^j_i)(P^j_{i+1} - P^j_{i-1})} - \]

\[ f^j_i \left[ \frac{D^{(m)}_{i+1}}{(P^j_{i+2} - P^j_i)(P^j_{i+1} - P^j_{i-1})} + \frac{D^{(m)}_{i-1}}{(P^j_{i-2} - P^j_i)(P^j_{i+1} - P^j_{i-1})} \right], \]

(B.1)

while the explicit term is composed of three parts:
\[ \sum_m S_m^2 \frac{\partial}{\partial E} D^{(m)} \frac{\partial}{\partial E} f = \sum_m S_m^2 \]

\[ f_{i+1}^{j+1} = \frac{D_{i+1}^{j+1(m)}}{(E_{i+1} - E_i^{j+1})(P_{i+1}^j - P_i^{j+1})} - f_{i-1}^{j+1} \frac{D_{i-1}^{j+1(m)}}{(E_{i-1} - E_i^{j+1})(P_{i+1}^j - P_i^{j+1})} \]

\[ f_{i-1}^{j-1} = \frac{D_{i-1}^{j-1(m)}}{(E_{i-1} - E_i^{j-1})(P_{i+1}^j - P_i^{j-1})} - f_{i+1}^{j-1} \frac{D_{i+1}^{j-1(m)}}{(E_{i+1} - E_i^{j-1})(P_{i+1}^j - P_i^{j-1})} \]

\[ \sum_m S_m \frac{\partial}{\partial P} D^{(m)} \frac{\partial}{\partial P} f = \sum_m S_m \]

\[ f_{i+1}^{j+1} = \frac{D_{i+1}^{j(m)}}{(E_{i+1} - E_i^{j+1})(P_{i+1}^j - P_i^{j+1})} - f_{i-1}^{j+1} \frac{D_{i-1}^{j(m)}}{(E_{i-1} - E_i^{j+1})(P_{i+1}^j - P_i^{j+1})} + \]

\[ f_{i-1}^{j-1} = \frac{D_{i-1}^{j(m)}}{(E_{i-1} - E_i^{j-1})(P_{i+1}^j - P_i^{j-1})} - f_{i+1}^{j-1} \frac{D_{i+1}^{j(m)}}{(E_{i+1} - E_i^{j-1})(P_{i+1}^j - P_i^{j-1})} \]

and the collision operator

\[ \nu_\text{eff}^2 \frac{\partial^2 f}{\partial P^2} = \frac{\nu_\text{eff}^2}{P_{i+1/2} - P_{i-1/2}} \left( \frac{f_{i+1}^j - f_i^j}{P_{i+1} - P_i} - \frac{f_i^j - f_{i-1}^j}{P_i - P_{i-1}} \right) \]
Bibliography


[33] C. E. Kessel et al. The physics basis for an advanced physics and advanced technology tokamak power plant configuration, ARIES ACT-1. in progress, 2013.


