BERRY PHASE IN COMPOSITE FERMI LIQUIDS

Jie Wang

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
PHYSICS
ADVISER: F. D. M. HALDANE

JUNE 2019
Abstract

Two-dimensional electron gas in high magnetic field exhibits a wide variety of interesting physical properties. Perhaps most notable of these is the quantum Hall effect, which is a classic example of a topological phase. Another interesting phase occurs at even denominator filling fraction is the so-called “composite Fermi liquid”. Such compressible phase is traditionally thought of as a Fermi liquid of “composite fermions” due to B. I. Halperin, P. A. Lee and N. Read (HLR).

Composite Fermi liquid has gained renewed interest recently due to the particle-hole symmetry and Berry phase: when the lowest Landau level is half filled, the effective Hamiltonian is particle-hole symmetric. However, it is unclear how the HLR description realizes this symmetry. A key ingredient that was missing in HLR’s treatment seems to be a PI Fermi sea Berry phase associated with transporting a composite fermion around the Fermi surface. Motivated by the symmetry and Berry phase, recently D. T. Son conjectured that composite fermions are relativistic Dirac particles. In Son’s theory, particle-hole symmetry acts in a way akin to time reversal on Dirac fermions, and the PI Berry phase is a curvature singularity at Dirac node.

A direct measurement of this PI Berry phase is one of the main results in this dissertation. We examined a model wavefunction that explicitly exhibits a Fermi surface, and has been shown to give good agreement with states found in exact diagonalization studies. We then formulated a many-body version of Berry phase for transporting a single composite fermion around a path in momentum space, and evaluated the Berry phase. To study the property of model wavefunction and Berry phase on larger system sizes, we developed “lattice Monte Carlo” technique based on a mathematically exact discretized formulation of holomorphic quantum Hall states on torus. Besides half filling, the Berry phase at 1/4 was found to be remarkably interesting: it suggests the emergence of Dirac fermion at generic filling fraction.
Motived by this, an effective theory, dubbed as “flux attached Dirac fermion theory”, which generalizes Son’s theory and covers all filling fractions was proposed.
Acknowledgements

This thesis was made possible due to the masterly guidance of my advisor Prof. F. D. M. Haldane. I would never forget the countless hours he spent on meeting and guiding me since 2014. I consider myself to be incredibly privileged to have worked so closely with a great physicist like him. Prof. Haldane initiated me into the world of modern condensed matter physics, and brought me into the field of fractional quantum Hall effect. Most importantly, his enthusiasm for doing physics, insightful thinking, brilliant physical intuition, as well as the patience to figure out every single detail in research have deeply influenced me.

I want to thank collaborators Dr. Scott Geraedts and Prof. E. H. Rezayi, without them this thesis is impossible. To me, Scott is a young mentor and a good friend. I won’t forget the time when we met almost everyday at Jadwin third floor, and the exciting moment when progress were made. I would also like to thank Yu Shen. The discussions with Yu always made my understanding much better and clearer. I also thank other group members including Bo Yang, Yeje Park, Zhao Liu, Wei Zhu, Yi Li, Junyi Zhang, without whom my research would not have gone so smoothly.

I would also like to express my acknowledgment to many enlightening discussions with mentors and friends outside Princeton: Prof. D. T. Son, T. Senthil, Mike Zaletel, Andrey Gromov, Yi Zhang, Xin Wan, Yahui Zhang and Peng Ye. I would especially thank Prof. Son for discussions and comments on my unpublished work during the workshop at Vietnam at 2018 summer.

My life at Princeton would not be complete without my friends: Zhaoqi Leng, Biao Lian, Jinyu Luo, Zheng Ma, Juven Wang, Siwei Wang, Fang Xie, Zhenbin Yang, Yizhi You and many others. I especially thank Yunqin Zheng and Huan He, the condensed matter colleagues as well as 10 year-long classmates, for numerous discussions.
I would like to thank Prof. Robert Austin for the summer support at 2014 during which I finished my experimental project. I thank Prof. Shivaji Sondhi for reading this thesis. And thank Prof. David Huse, Daniel Marlow for attending the defense.

Lastly, thank you to Lu Liao, for all her support and understandings.

This acknowledge would be incomplete without expressing my deepest gratitude to my parents.
To my parents.
Contents

Abstract ................................................................. iii
Acknowledgements ...................................................... v
List of Tables .......................................................... xi
List of Figures .......................................................... xiii

1 Introduction to Quantum Hall Physics 1
  1.1 The phenomena Hall effect ........................................ 1
      1.1.1 Classical Hall effect ........................................ 1
      1.1.2 Integer quantum Hall effect ................................ 2
      1.1.3 Fractional quantum Hall effect ............................. 7
  1.2 Formalism for translational invariant clean quantum Hall problems . 9
      1.2.1 Non-commutativity and effective Hamiltonian ............ 9
      1.2.2 Laughlin wavefunction and its reinterpretation: quantum metric 13

2 Composite Fermi Liquids (CFLs) 16
  2.1 The physics of CFL: dipole, Fermi sea and Berry phase ........... 16
  2.2 A standard theory by B. I. Halperin, P. A. Lee and N. Read ....... 20
  2.3 Particle-hole symmetry in a half filled Landau level ............... 23
  2.4 An alternative effective theory: Dirac Fermion Theory .......... 25

3 Quantum Hall States on Torus 31
  3.1 Translation group and Heinsberg algebra ....................... 31
3.1.1 Torus and single body translation ........................................ 31
3.1.2 Many body translation ....................................................... 33
3.2 Model Wavefunctions on Torus ................................................. 34
3.2.1 Weistrass sigma functions .................................................. 34
3.2.2 Holomorphic representation of Heisenberg algebra .................... 35
3.2.3 Single body wavefunction ................................................... 38
3.2.4 Center-of-mass wavefunction .............................................. 40
3.2.5 Model wavefunctions: Laughlin, CFL and others ....................... 41
3.3 Lattice Representation .......................................................... 42
3.3.1 Intuition for Lattice Representation ..................................... 42
3.3.2 A Proof for Lattice Representation ..................................... 46
3.4 Lattice Monte Carlo ............................................................. 49
3.4.1 Consistency check I: Coulomb energy on torus ........................... 50
3.4.2 Consistency check II: transporting a hole in Laughlin state .......... 52
3.4.3 Structure factor and a “fast Lattice Monte Carlo” algorithm .......... 54
3.4.4 Particle hole symmetry from Monte Carlo ................................ 63
4 Many Body Berry Phase in a half filled Landau level ...................... 68
4.1 Definition of many body Berry phase ....................................... 68
4.2 Property of $\nu = 1/2$ CFL: symmetry, Berry phase and the “singularity” 71
4.3 $\pi$ Berry curvature singularity at $\nu = \frac{1}{2}$ .............................. 73
5 Berry Phase and effective theories at generic $\nu$ ........................... 81
5.1 From half filling to generic filling .......................................... 81
5.2 Smoking gun of Dirac fermion: $\pi$ Berry phase at $\nu = \frac{1}{4}$ ............ 85
5.3 “Luttinger’s Hypothesis” ....................................................... 95
5.4 Generalization of Son’s theory: flux-attached Dirac fermion theory .... 98
6 Conclusion ............................................................................. 102

ix
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Comparison of exact and Monte Carlo energies for the Laughlin wave function at $\nu=1/3$. The agreement between the two, limited only by statistical error, is a confirmation that our lattice Monte Carlo is correct. For $n&gt;1$ the statistical error is large. The cause and solution of this problem are described in Sec. 3.4.3</td>
</tr>
<tr>
<td>3.2</td>
<td>By separating the energy calculation into short- and long-ranged parts and approximating the long-ranged part, we can dramatically reduce our statistical error. In this table we compute the Coulomb energy of the Laughlin wave function in $n = 2$. We used the same number of Monte Carlo steps as we did in Table 3.1, but find that our statistical error is reduced by up to two orders of magnitude even at these relatively small sizes. We also did test on larger systems, e.g. $N_e = 11, 12$</td>
</tr>
<tr>
<td>3.3</td>
<td>The pair amplitude calculated for $N_e = 6$ particles in Laughlin $\nu = 1/3$ state. The $\delta E_M$ and $\delta E_S$ are Monte Carlo and systematic errors, respectively. The total error $\delta E_{tot} = \sqrt{\delta E_M^2 + \delta E_S^2}$</td>
</tr>
</tbody>
</table>
5.1 Paths $\Gamma$ and phases $\Phi_\Gamma$ for $N=37$ and 69 Fermi sea. See Eq. (5.6) for definitions of $K$ and $K'$. For example, in $\Gamma_1$ of $N=37$, the Berry phase is computed as follows:

$$\Phi_{\Gamma_1} = -i \ln \text{Tr} \langle \Psi_{(1,4)} | \rho | \Psi_{(-1,4)} \rangle - \langle \Psi_{(-1,4)} | \rho | \Psi_{(-2,3)} \rangle \cdots \langle \Psi_{(2,3)} | \rho | \Psi_{(1,4)} \rangle.$$ 

Note that the extra dipole is in fact transported clock-wisely.

5.2 Same as Table 5.1 but for $N=69$ dipoles.
List of Figures

1.1 Electron moving in 2 dimension in the presence of electric field, magnetic field etc. ........................................... 2
1.2 Typical shape and cross section of a GaAsAl$_x$Ga$_{1-x}$As heterostructure used for quantum Hall effect measurements. ................................................ 3
1.3 Integer quantum Hall effect: the longitudinal Hall resistance vanishes while the transversal resistance forms plateaus. .......................................................... 4
1.4 Laughlin’s gauge invariance argument. ................................................. 7
1.5 Fractional quantum Hall effects: longitudinal and transversal resistance measured at various filling fractions. .......................................................... 8
1.6 Illustration of Landau level quantization of a quadratic band. In strong magnetic field, electron’s cyclotron energy is much larger than the electron-electron interaction energy scale. The effective Hamiltonian describes the physics in a partially filled Landau level. ........................................... 10
1.7 ........................................................................................................ 14
2.1 A cartoon picture to explain the formation of composite fermion Fermi sea at half filling. .................................................... 17
2.2 Dipole and dipole-momentum locking. .................................................. 17
2.3 Landau level quantization of massless Dirac fermion. The Landau level energy scales with $\sqrt{|n|}$. With chemical potential set to be zero, the zero mode ($n = 0$) is half filled. .................................................... 26
2.4 The composite Fermi liquid in Son’s theory. The composite fermions are massless Dirac fermion. Particle hole symmetry acts in a way akin to time reversal symmetry on Dirac fermions. Hence finite chemical potential is consist with particle hole symmetry.

3.1 Fermi sea of dipoles.

3.2 $N_e = 50, \nu = 1/q$. Laughlin-hole state Berry phase ($\Phi$) v.s. area the loop enclosed ($A$). The dot and the square show Monte-Carlo data for $q = 3$ and $q = 5$ respectively. The Monte-Carlo error is bounded by the line width. This demonstrates that $\Phi = A/q$. The inset shows the overlap $|\langle \psi(0)\psi(\Delta x) \rangle|$ v.s. $\Delta x$ which allows us to take step steplength to be $\Delta x = 0.02$ i.e. the quantum distance between consecutive steps is small.

3.3 The guiding center structure factor for $N_e = 50$ electrons in the Laughlin $\nu = 1/3$ state. The subfigure a) shows its plot together with error bars. The Gaussian function $e^{-\frac{1}{4}q^2l_B^2}$ limits us to see $S(q)$ only within a window. In subfigure b), we check the long-wavelength expansion ($c_4$ and $c_6$) in (3.82) by comparing it to the Monte Carlo data, where $c_4, c_6$ is given by (3.83) and all other $c_i = 0$. It can be seen that the long-wavelength behavior of $S(q)$ is correctly described by (3.83).

3.4 CFL $S(q)$ and dipole configuration. The peak of the structure factor and composite fermion Fermi surface have the same shape. The radius of the former is twice of the latter.

3.5 Overlaps between a wave function and its PH conjugate. The red points from come from doing an exact second quantization of the model wave function as in [9], while the blue comes from a Monte Carlo calculation. For each $N_e$, the configuration of $d’s$ with the largest overlap was used.
4.1 The energy, energy variance, overlap with exact eigenstates, and overlap with $\mathcal{PH}$ conjugates for a number of model states, with the $\{d\}$ configurations shown. We see that clustered $\{d\}$ configurations have low energies, good overlap with exact eigenstates, and are $\mathcal{PH}$ symmetric, while less clustered configurations lose these properties.

4.2 The density operator in Eq. (4.10) adds an additional phase to our Berry phase calculation, with is imaginary due to particle-hole symmetry. An anti-unitary reflection symmetry (present in the thermodynamic limit, as well as in the square lattice considered here) introduces a relative $-1$ between clockwise and counterclockwise hopping, while also forbidding hopping in a direction normal to the Fermi surface.

4.3 Berry phases observed for a variety of paths around the Fermi surface. The solid symbols represent the locations of composite fermions, and for each step on the path we remove a composite fermion at the location of the empty symbols (i.e. we are moving a composite hole around the Fermi surface). There are a number of effects which arise from the insertion of a density operator in Eq. (4.10). These differences lead to additional phases summarized in Eq. (4.20). We compute these Berry phase $\tilde{\Phi}_{\text{exact}}$ for the exact ED states, and $\tilde{\Phi}_{\text{model}}$ for the model wavefunction of Eq. (3.56). In both cases the results, combined with Eq. (4.20), indicate a Berry phase of $\Phi = \pi$ when the center of the Fermi surface is encircled.
4.4 \( N_c = 13 \) CFL Berry phase. Cross mark “x” represents the “composite fermion” we are moving. This is a consistency check with the same calculation done in Ref. [9] (but using a different numerical approach). This data can be interpreted through Eq. (4.21), which shows that in addition to a \( \mathbb{Z}_2 \) piece there is a piece depending on the direction of motion around the Fermi surface. When this is accounted for we find a residual “−1” from the \( \mathbb{Z}_2 \) part whenever the composite fermion encloses the Fermi sea.

4.5 \( N_c = 69 \) CFL Berry Phase. Cross mark “x” represents the “composite fermions” we are moving. The results are again consistent with Eq. (4.21).

5.1 A half filled composite Fermi liquid state is particle hole symmetric and consequently all Berry phases are pushed into Fermi sea center as a Berry curvature singularity. At generic filling fraction, while symmetry is absent, Berry phase might remain as an important characterization of the composite Fermi liquid phase.

5.2 Berry curvature distribution [right] obtained from \( \Psi_{n=1}^{1/4} \) model wave function by a linear regression on a Fermi sea [left] consisting of \( N=37 \) dipoles. The red dashed line represents the path, which can be interpreted as Fermi sea boundary, along which \textit{anti-clock-wisely} transporting a single composite fermion has \(-2\pi\nu\) Berry phase. The area enclosed by the red dashed line contains 46 grids. The Berry curvature has a peak of around \(-0.25+0.011=-0.239\) [in units of \( \pi \)], while the rest values are around \(-(2/4-1)/46=0.011\). It suggests an interesting Berry curvature distribution for CFLs at a generic filling fraction in the thermodynamic limit: a \(-\pi\) singularity at center and \(-2\pi\nu-\pi\) uniformly distributed over Fermi sea.
5.3 From half filling to 1/4 or 3/4 filling, we expect Dirac fermion perceives magnetic field. If a Fermi sea forms in the end, we expect magnetic field can be attached to Dirac fermions. Furthermore, Fermi sea sizes are supposed to shrink, due to the Luttinger’s theorem.

5.4 Variational energies [red dots for $\Psi_{n=1}^{1/4}$, blue dots for $\Psi_{n=2}^{1/4}$] and exactly diagonalized Coulomb energies [dashed lines] as a function of many body momentum $\mathbf{K}_1, \mathbf{K}_2$ for $N=10$ electrons for the $\nu=1/4$ filled LLL on a square torus. Energies are plotted in units of $e^2/\epsilon l_B$. For each $K_1, K_2$ is chosen to match the momentum of the lowest energy state. Due to inversion symmetry, only $K_1 \in [0, 5]$ are plotted.

5.5 Comparison of the Berry phases $\Phi_\Gamma$ associated with various clock-wised paths $\{\Gamma\} = \Gamma_1, ..., \Gamma_6$ on a Fermi sea of $N=37$ dipoles [see FIG. 5.2 for Fermi sea] computed from $\Psi_{n=1}^{1/4}$ [red dots] and from the formula $\Phi_\Gamma = \delta_\Gamma \cdot \pi + (2\pi \nu - \pi) A_\Gamma/A_{FS}$ [black lines], where $\delta_\Gamma$ is the winding number of $\Gamma$ relative to the Fermi sea center, $A_\Gamma$ and $A_{FS}$ are the $k$-space areas enclosed by the path $\Gamma$ and Fermi sea area respectively. See appendix for details about paths and more examples including $N=69$ Fermi sea and $\nu=1/3$.

5.6 The Fermi sea and discretized Berry curvature $\phi$. The left and right Fermi sea has $N=37$ and 69 dipoles respectively, on square toruses. The Berry curvatures related by rotation and inversion are not represented. The red dashed line represents the Fermi surface boundary along where transporting a single dipole has $\Phi_\Gamma = 2\pi \nu$ Berry phase.

5.7 Illustration of LLL projection.

5.8 The Berry phases associated with paths $\Gamma_1, ..., \Gamma_6$ listed in Table 5.1 for $N=37$ Fermi sea computed from $\nu=1/4$ [upper panel] and $\nu=1/3$ [lower panel] CFL model wavefunction.
5.9 Same as FIG. 5.8, but for $N=69$ Fermi sea. In FIG. 5.8 and FIG. 5.9, black dashed lines are Berry phases according to Eq. (5.8) and the red dots stand for Monte Carlo values which can be found in Table 5.1.

5.10 Illustration of Fermi sea form by composite-fermions at $\nu = 1/4$ and by composite-holes at $\nu = 3/4$ in HLR’s theory. The green area are filled $k$-space areas. Fermi sea sizes are the same.

5.11 Guiding center structure factor $S(k)$ as a function of $k_x$ along $k_y=0$ axes computed from $\Psi_{n=1}^{1/4}$. The plot is obtained after a finite size scaling for model wavefunctions of $\sqrt{N} \times \sqrt{N}$ square Fermi seas where $N$ is the number of electrons. The red lines are $2k_Fl_B=\sqrt{2\pi\nu}$, a value of twice of the Fermi wave vector obtained by applying Luttinger theorem on a square Fermi sea. The fact that $S(k)$ plots fit into one curve and the numerical singularities match with the analytical value implies that Luttinger theorem is true for CFLs.

5.12 Illustrations of Fermi sea, particles, band structure in Son’s half filled Fermi sea theory [middle] and flux-attached Dirac fermion theory. PH acts like time reversal, thus flipping the fluxes [arrows] attached to the Dirac fermions [black dots]. Fermi sea sizes of PH conjugate states are the same, fixed by Luttinger theorem.
Chapter 1

Introduction to Quantum Hall Physics

1.1 The phenomena Hall effect

1.1.1 Classical Hall effect

The Hall effect describes how electrons move in two dimension in the presence of electric and magnetic fields. With $E$ and $B$ being the in-plane electric and out-of-plane magnetic fields, the simple Drude model for electron’s motion is described as follows,

$$m \frac{dv}{dt} = -eE - ev \times b - \frac{mv}{\tau}. \quad (1.1)$$

where $m$ is the electron mass, $e$ is the charge, $v$ is the velocity. $\tau$ is the life-time, or the mean time between two adjacent scattering events. The conductivity relates the
current density $J = ne\mathbf{v}$ where $n$ is the particle density, to the electric field,

$$
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix} =
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}.
$$

(1.2)

The resistivity tensor is the inverse of the conductivity tensor,

$$\rho = \sigma^{-1}.$$  

(1.3)

The simple Drude model predicts that the longitudinal resistivity and transversal resistivity are respectively,

$$\rho_{xx} = \frac{m}{ne^2\tau},$$

$$\rho_{xy} = \frac{B}{ne}.$$  

(1.4)

Note that $\rho_{xy}$ is linear to magnetic field strength $B$.

### 1.1.2 Integer quantum Hall effect

In stronger magnetic field and at lower temperature, quantum effects become more and more important. Perhaps one of the most fantastic phenomena in modern quantum condensed matter physics is the quantum Hall effect. The first time quantum
Figure 1.2: Typical shape and cross section of a $GaAs_{1-x}Ga_{x}As$ heterostructure used for quantum Hall effect measurements.

effect was discovered in Hall system dates back to 1980 when Klaus von Klitzing et.al. [30] reported the high-accuracy quantized Hall conductivity measurement for quasi-two-dimensional electrons gas in the inversion layer of a MOSFET (metal-oxide-semiconductor field-effect transistor).

Because these resistivity plateau happened at integer numbers, such phenomena was called “integer quantum Hall effects”. Integer quantum Hall effects can be explained by single body quantum mechanics. Disorder and gauge invariance played an important role to explain the formation of the plateaus. The simple Hamiltonian for a two-dimensional quantum electron moving in magnetic field is,

$$H_0 = (1/M)^{ab} \pi_a \pi_b, \quad \pi_a = p_a - eA_a.$$  \hspace{1cm} (1.5)

where $M$ is the effective band mass tensor, $\pi_a$ is the gauge-invariant momentum, $A_a$ is the vector potential whose curl is the magnetic field $B = \nabla \times A$. Without loosing any generality, we will choose the magnetic field to point in the negative $\hat{z}$ direction,
so $\hat{z} \cdot \mathbf{B} = B < 0$ while $eB > 0$. The commutator of $\pi_a$ is nontrivial,

$$
\begin{align*}
[\pi_a, \pi_b] &= \left[ -i\hbar \partial_a - eA_a, -i\hbar \partial_b - eA_b \right], \\
&= ie\hbar \partial_a A_b - ie\hbar \partial_b A_a, \\
&= ie\epsilon_{ab}B, \\
&= i\hbar \epsilon_{ab}l_B^2.
\end{align*}
$$

(1.6)

where $\epsilon_{xy} = -\epsilon_{yx} = 1$ is the two-dimensional anti-symmetric epsilon symbol. A characteristic length called magnetic length is defined,

$$
l_B = \sqrt{\frac{\hbar}{eB}}.
$$

(1.7)
Because of Lorentz force, electrons process cyclotron motions with a coordinate of \( \vec{R}^a \),

\[
\vec{R}^a = -l_B^2 \epsilon^{ab} \pi_b.
\]  

(1.8)

The Landau orbit coordinates \( \vec{R}^a \) do not commute with themselves,

\[
[\vec{R}^a, \vec{R}^b] = il_B^2 \epsilon^{ab}.
\]  

(1.9)

The center of the cyclotron motion is called the guiding center coordinate \( R^a \),

\[
R^a = r^a - \vec{R}^a = r^a + l_B^2 \epsilon^{ab} \pi_b.
\]  

(1.10)

\[
[R^a, R^b] = -il_B^2 \epsilon^{ab}.
\]  

(1.11)

It commutates with the Hamiltonian \( H_0 \) and hence is a constant of motion. We now see that in magnetic field, electron’s coordinates \( r^a \) are factorized into two sets of non-commutative coordinates, Landau orbit \( \vec{R}^a \), and guiding centers \( R^a \).

As promised, the integer quantum Hall effects can be understood in terms of single body physics. In many cases, the effective mass tensor is diagonal \( \mathcal{M}^{ab} \sim \delta^{ab} \), a set of ladder operators can be defined,

\[
\bar{a} = \bar{w}_a \vec{R}^a.
\]  

(1.12)

\[
\bar{a}^\dagger = \bar{w}_a^* \vec{R}^a.
\]  

(1.13)

where \( \bar{w}_x = \frac{1}{\sqrt{2}} \) and \( \bar{w}_y = \frac{i}{\sqrt{2}} \). In the generic case, the effective tensor is not diagonal, but Eqn. (1.12) and Eqn. (1.13) are still true but with a different choice of \( \bar{w}_a \), i.e. ladder operator can still be defined. The \( \bar{w}_a \) is called complex-structure, and will be explained in more details later. So, pretty generically, the quantum Hamiltonian
describing two dimensional electron in magnetic fields is as follows,

\[ H_0 = \hbar \omega_c (\bar{a} \dagger \bar{a} + \frac{1}{2}). \]  

(1.14)

The energy levels are called “Landau level”, typically the energy spacing between Landau levels is proportional to magnetic field strength, \( i.e. \) cyclotron frequency \( \omega_c \sim B \). The integer value of Hall resistivity corresponds to the number of Landau level that is filled.

The integer quantum Hall effects were observed in various experimental samples, all shown robust Hall plateau and vanishing longitudinal resistivity. This means that they can not be explained solely from the simple Hamiltonian Eqn. (1.15). Naively, one would expect increasing the magnetic field strength increases the Landau orbital number and there will be more places electrons can scatter to, leading to a different value of \( \rho_{xx} \). The robust experimental phenomena was kinds of opposite to this intuition.

In fact, disorder and gauge invariance played an important role in explaining the integer quantum Hall effect. The experimental samples are always dirty, so many electrons are indeed localized. Tuning magnetic fields or electron density merely changes the occupation of these localized states. Besides localized electrons, delocalized electrons also exist, and contribute to the transversal part of resistivity. The precise quantization of Hall plateau was first explained by Laughlin based on a gauge invariance \[31\].

What Laughlin considered is a toy model where electrons were put on a loop (cylinder). According to Maxwell equation, the uniform electric field is generated by threading (linear to time) a flux quanta through the cylinder. After threading an integer flux quanta into the cylinder, the Hamiltonian remains invariant, but delocalized states are pumped from one edge to the other by one unit: this corresponds
to the quantized Hall conductivity! Such gauge invariance derivation or argument is
general, and thus independent on the microscopic details like disorder, sample width
etc.

1.1.3 Fractional quantum Hall effect

At lower temperature and stronger magnetic field, the effects of electron-electron
interaction become more and more important. The first observation of the fractional
quantum Hall effect was made by Daniel Tsui et.al. [47] at filling $\nu = \frac{1}{3}$ (a plateau
of $\rho_{xy}$ and a vanishing small value of $\rho_{xx}$). In contrast to the integer quantum Hall
phenomena, in general the fractional quantum Hall effect occurs at fractional filling
fraction,

$$\nu = \frac{N_e}{N_\phi} = \frac{p}{q}. \quad (1.15)$$

where $N$ and $N_\phi$ are number of electrons and number of flux quanta respectively. The
reason why topological robust incompressible state occurs at such filling fractions lies
in a highly nontrivial way in the role played by electron-electron interactions.
The fractional quantum Hall effect, by its name, happened in a partially filled Landau level. Without interaction, a partially filled Landau level has multiple degeneracies. The reason why and how interaction selects a unique or few ground states is highly non-trivial. Moreover, fractional quantum Hall effects have much richer physics than integer quantum Hall effects, such as fractionalized charge, anyon, entanglement, etc. For this reason, I would like to review the physics of fractional quantum Hall effect in a separated section.
1.2 Formalism for translational invariant clean quantum Hall problems

1.2.1 Non-commutativity and effective Hamiltonian

The generic Hamiltonian describing interacting 2D electrons gases (2DEG) in a high magnetic field contains a kinetic term $H_0$ and an interaction term $V$,

$$ H = H_0 + \sum_{i<j} V(r_i - r_j), \quad H_0 = \sum_i \epsilon(\pi_i). $$

where $\epsilon(p)$ is the single body band dispersion and $\pi_i = p_i - eA(r_i)$ is the gauge invariant dynamical momentum. In the proceeding section, with effective mass tensor we described the single body energy as a quadratic function of momentum. Here, for the sake of generality, we just write it as $\epsilon(p)$. As we described, the gauge invariant momentums are non-commutating,

$$ [\pi_{i,a}, \pi_{j,b}] = i\hbar \delta_{i,j} \epsilon_{ab} l_B^{-2}. $$

where subscripts $i, j$ label different electrons while spatial indices $a, b$ label directions.
Figure 1.6: Illustration of Landau level quantization of a quadratic band. In strong magnetic field, electron’s cyclotron energy is much larger than the electron-electron interaction energy scale. The effective Hamiltonian describes the physics in a partially filled Landau level.

There are two energy scales in the quantum Hall problem. One is the energy spacing of Landau levels, which is typically proportional to $B$. The other is the Coulomb energy scale. The Coulomb energy scale can be estimated by the magnetic length, since the magnetic length measures the size of an orbital $2\pi l_B^2$, i.e. the separation of electrons. The ratio of Coulomb interaction to Landau level energy gap is,

$$\frac{\Delta_{ee}}{\Delta_{LL}} \sim \frac{|e|^2/l_B}{B} \sim \frac{m}{\sqrt{B}}. \quad (1.18)$$

In strong field limit $B \to \infty$, all physics happens in a single partially Landau level. The Landau level mixing effects (electrons hop inter Landau level) are negligible.

We also have mentioned that the electron positions (displacements from an arbitrary origin on the 2D plane) $\{r_i\} = \{r_i^a e_a\}$ can be reorganized to two independent sets,

$$r_i^a = R_i^a + \tilde{R}_i^a. \quad (1.19)$$
where $\tilde{R}_i^a \equiv -l_B^2 e^{ab} \pi_{i,b}$ describes the electrons orbital motion and $R_i^a$ is the guiding center coordinate which is the center of classical cyclotron motion (note that our formalism distinguishes components of displacements, which have upper indices, from those of derivatives, which have lower indices, as the Euclidean metric $\delta_{ab} = e_a \cdot e_b$ has no role in the physics other than to define a Cartesian coordinate system, and the common assumption that introduces it by postulating a Newtonian form $\varepsilon(\pi) = \frac{1}{2}m^{-1}\delta^{ab}\pi_a\pi_b$ will not be made here; in general there is no place for the Euclidean metric in the physics of electrons moving in a fixed crystalline background.)

Projecting the original problem into a single Landau level is called Landau level projection, which we denote as $P_{LLL}$. After Landau level projection, the electron coordinates become guiding center coordinates $R_i^a$ which are no longer commute with each other. The algebras of them are summarized as follows,

\begin{align}
[\tilde{R}_i^a, \tilde{R}_j^b] &= i l_B^2 e^{ab} \delta_{i,j}, \quad (1.20) \\
[R_i^a, R_j^b] &= -i l_B^2 e^{ab} \delta_{i,j}, \quad (1.21) \\
[\tilde{R}_i^a, R_j^b] &= 0. \quad (1.22)
\end{align}

We can define two sets of harmonic oscillator operators by,

\begin{align}
a &= w_a^* R^a, \quad a^\dagger \equiv w_a R^a, \quad (1.23) \\
\bar{a} &= \bar{w}_a^* \bar{R}^a, \quad \bar{a}^\dagger \equiv \bar{w}_a \bar{R}^a. \quad (1.24)
\end{align}

where $\{w_a, w_a^*\}$ and $\{\bar{w}_a, \bar{w}_a^*\}$ are two sets of “complex structures” which describes a mapping from the Euclidean plane to the complex plane. The complex structures are
obtainable from metric. Two sets of metric are defined by,

\[ g_{ab} = w_a^* w_b + w_a w_b^*, \]
\[ i\epsilon_{ab} = w_a^* w_b - w_a w_b^*. \] (1.25)

and

\[ \bar{g}_{ab} = \bar{w}_a^* \bar{w}_b + \bar{w}_a \bar{w}_b^*, \]
\[ i\bar{\epsilon}_{ab} = \bar{w}_a^* \bar{w}_b - \bar{w}_a \bar{w}_b^*. \] (1.26)

The physical meaning of the Landau orbit metric \( \bar{g}_{ab} \) and guiding center metric \( g_{ab} \) are the inverse of the effective mass tensor, and the shape of the correlation hole (shape of composite boson) respectively.

In the limit where the Landau level energy-splitting is much larger than the interaction energy, the wave function can be written as an \textit{unentangled product} of the Landau orbit and the guiding center parts:

\[ |\psi_n\rangle = |\psi_{n}^{LO}\rangle \otimes |\psi^{GC}\rangle. \] (1.27)

Here, \( |\psi_{n}^{LO}\rangle \) is the Landau orbit part, where \( n \) indicates that the system is in the \( n \)th LL. \( |\psi^{GC}\rangle \) is the guiding center part of the wave function. The LL part of the wave function can be projected out, leaving the problem essentially a degenerate perturbation problem within a specific LL described by a set of non-commutative interacting guiding center coordinates \( R_i^a \) [18],

\[ H = \sum_{i<j} V(R_i - R_j). \] (1.28)
Here the “guiding-center interaction potential” $V(r)$ depends not only on the “bare” Coulomb interaction, but also on the “form factor” of the LL into which it is projected. It is an extremely smooth function, with a rapidly decaying Fourier transform due to the Gaussian-like form factor, which means that the expansion of $V(r + \delta r)$ in powers of $\delta r^a$ is absolutely convergent for all $r$. This property ensures that, as a function of non-commuting variables, $V(R_i - R_j)$ is well-defined.

1.2.2 Laughlin wavefunction and its reinterpretation: quantum metric

One of the most famous wavefunction for fractional quantum Hall states perhaps is the Laughlin wavefunction,

$$\Psi(\{x\}) = \prod_{i<j} (z_i - z_j)^3 \prod_i e^{-\frac{1}{2}|z_i|^2}.$$  \hspace{1cm} (1.29)

The above wavefunction describes the correlated states happened at filling fraction $\nu = \frac{1}{3}$. The $z_i$ is the electron coordinate. Usually, in the presence of rotational invariance, complex coordinate is written as $z = (x + iy)/\sqrt{2}$. If one extra flux quanta is inserted into the two dimensional plane, due to Maxwell equation, circular electric is generated, see FIG. 1.7. Because it is a $\nu = \frac{1}{3}$ quantum Hall state, current is generated by the $E$ field. The result of inserting an extra flux quanta is to remove $\frac{1}{3}$ charge from the position of flux quanta inserted (vortex). Hence, the vortex carries a $-\frac{2}{3}$ fractional charge, i.e. one quasi-hole. This perhaps is a most intuitive hand-waving argument for the charge factorization in fractional quantum Hall effects.
Let us proceed with a formal question. In more general sense, \( z \) in Eqn. (1.29) should be written with the help of a complex structure. But what complex structure \( w_a \) or \( \bar{w}_a \) should we use? To answer this question, it makes more sense to write the Laughlin wavefunction as a Heisenberg state. As mentioned in Eqn. (1.27), in the limit of vanishing Landau level mixing, the wavefunction can be written as a direct product of Landau orbit and guiding center part. The \( \vert \psi^{LO}_n \rangle \) then will be constructed solely by Landau orbit ladder operators and Landau orbit metric, while \( \vert \psi^{GC} \rangle \) will be described solely by guiding center ladder operators and guiding center metric. Following this logic [18], the Laughlin state is written as,

\[
\vert \Psi(g) \rangle = \prod_{i<j} (a_i^\dagger - a_j^\dagger)^3 \vert 0_{GC} \rangle. \tag{1.30}
\]

where \( \vert 0_{GC} \rangle \) is the vacuum of guiding center ladder operators (the zero momentum state \( a \vert 0_{GC} \rangle = 0 \)). To define the ladder operators, a metric \( g \) (guiding center metric) is needed. This is why the Laughlin Heisenberg state dependents on a parameter \( g \). Its mean value is determined by minimizing the energy of Eqn. (1.28). We will shortly see that \( g^{ab} \) is a nontrivial parameter: it has quantum dynamics which means that once one formulate the fractional quantum Hall problems in terms of path integral, the \( g_{ab} \) field has functional integral rather than being a dummy parameter.
This formal rewriting of Laughlin’s wavefunction actually initialized a huge step towards a deeper understanding of the correlated state \[18\]. The fractional quantum Hall effect in principle should be understood solely based on the guiding center Hamiltonian Eqn. (1.28), \textit{i.e.} any theoretical explanation or understanding relying on Landau orbit parts \(\{\bar{R}, \bar{g}_{ab}, |\psi_n^{LO}\rangle\}\) are then regarded as “non-generic” (dependent on Landau orbit index and thus details of Landau levels). All essence should lie in the non-commutativity of Eqn. (1.28).

One mystery that we have not yet explained enough is the guiding center metric, although we have formally defined them in Eqn. (1.25). To better understand the physics of \(g_{ab}\), let us define an operator,

\[
\Lambda^{ab} \equiv \frac{1}{4l_B^2} \sum_i \{R^a_i, R^b_i\}.
\]  

(1.31)

which satisfies the \(SL(2, R)\) algebra,

\[
[\Lambda^{ab}, \Lambda^{cd}] = -\frac{i}{2} (\epsilon^{ac} \Lambda_{bd} + \epsilon^{bd} \Lambda_{ac} + a \leftrightarrow b).
\]  

(1.32)

They are generators for area-preserving deformation on the non-commutative plane. The Casimir for their algebra is \(C_2 = -\frac{1}{2} \det \Lambda\). This \(\Lambda^{ab}\) is also called “quantum-metric”, because its expectation value under the wavefunction gives the guiding center metric. After some simple algebras, it can be shown that,

\[
\langle \psi(g)|\Lambda^{ab}|\psi(g)\rangle = \frac{1}{2} g^{ab}.
\]  

(1.33)

where \(g^{ab}\) is the inverse of the guiding center metric. And Eqn. (1.32) and Eqn. (1.33) indicates that the guiding center metric has \textit{quantum dynamics}. 

15
Chapter 2

Composite Fermi Liquids (CFLs)

2.1 The physics of CFL: dipole, Fermi sea and Berry phase

The gapped fractional quantum Hall states have rich physics. The focus of this dissertation, however, are gapless states called “composite Fermi liquid” states. Such compressible states can in principle occur at filling fractions \( \nu = \frac{1}{2m} \), e.g. \( \frac{1}{2}, \frac{1}{4} \) and their particle-hole conjugate fillings \( \frac{3}{4} \) and so on. They were first explained by B. I. Halperin, P. A. Lee and N. Read in terms of a Fermi sea of composite fermions.

To explain Halperin-Lee-Read’s idea, suppose we have three electrons in six flux quanta. The total filling fraction is one half. A composite fermion is formed by attaching two flux quanta to each electron. When exchanging two composite fermions, the total phase factor (AB phase and Fermi-Dirac phase) is \(-1\). For such composite fermions, the net magnetic field is zero, since all fluxes are used to attach to electrons. Therefore, a composite fermion Fermi sea can form, and this Fermi sea would be the experimentally observed gapless phase.

The Halperin-Lee-Read theory is successful in many aspects. However recently it has been challenged by the issue of particle-hole symmetry (PH symmetry)
Figure 2.1: A cartoon picture to explain the formation of composite fermion Fermi sea at half filling.

Figure 2.2: Dipole and dipole-momentum locking.

and **Berry phase**. The PH symmetry and Berry phase are two key words of this dissertation. They will be discussed in more details in the next few sub-sections.

Microscopically, composite fermions are not only one electron plus two flux quanta, but has an important structure: dipole. Physically, dipole origins from the mismatch of flux attachment center and electron position. Moreover, according to electromagnetism (force balance), for any dipolar electron, its dipole direction is locked to the kinetic momentum as shown in Eqn. (2.1). The dipole-momentum locking is illustrated in FIG. 2.2.

\[ \mathbf{d}^a = \epsilon^{ab} k_b l_B^2. \]  

In fact, the Fermi sea formed by composite fermions is supposed to be a nontrivial Fermi sea: a Fermi sea with non-zero Berry curvature. To explain the Berry curvature, let us adiabatically transport a composite fermion in \( k \)-space. Due to the dipole-momentum locking, dipole (and hence charge) rotates in the real space, picking up an Aharonov-Bohm phase. Such process is supposed to induce a Berry phase. The
Berry phase will have nontrivial impact to the property of the Fermi sea, for example breaking time reversal symmetry and inducing non-zero Hall conductivity.

Perhaps the relation of Berry phase and Hall conductivity is best illustrated through the anomalous Hall effect theory \[17\]. Consider the semiclassical equations of motion of Bloch electrons in weak electric and magnetic fields, which reads,

\[ \hbar \frac{d \mathbf{k}_a}{dt} = e \mathbf{E}_a(\mathbf{x}) + e F_{ab}(\mathbf{x}) \frac{dx^b}{dt}, \]  
\[ \frac{dx^a}{dt} = \frac{1}{\hbar} \nabla^a \epsilon_n(\mathbf{k}) + \mathcal{F}^{ab}_n(\mathbf{k}) \frac{dk^b}{dt}. \]

where \( F_{ab} = \nabla_a A_b - \nabla_b A_a = \epsilon_{abc} B^c \) is the magnetic flux density, \( \epsilon_n(\mathbf{k}) \) is the energy of a Bloch electron in band \( n \), \( \nabla^a_k \equiv \partial / \partial k_a \) and \( \mathcal{F}^{ab}_n \) is the Berry curvature tensor in \( k \)-space. The anomalous velocity is the \( k \)-space dual of the Lorentz force. The above equation of motion can also be thought of as the equation of motion for the composite fermions in two dimensional plane.

The current response to a uniform electric field \( \mathbf{E} \) (with \( \mathbf{B} = 0 \)) is,

\[ J^a_e = \frac{e}{\hbar \Omega N} \sum_{kn} \nabla^a_k \epsilon_n(\mathbf{k}) \delta n_{kn} + \sigma^a_{0b}(\mu) E_b, \]

where \( \delta n_{kn} \) is the electron’s occupation deviated from the mean value \( n^0_n(\mathbf{k}) \). \( N \) is the number of primitive unit cells, which have volume \( \Omega \). The first part corresponds to the dissipation, while the second part is the dissipationless intrinsic anomalous Hall conductivity, whose expression is:

\[ \sigma^{ab}_0(\mu) = \frac{e^2}{\hbar \Omega N} \sum_{kn} \mathcal{F}^{ab}_n n^0_n(\mathbf{k}, \mu). \]
The Berry curvature is obtained from a vector potential derived from the Bloch states $|\psi_n(k)\rangle$:

$$\mathcal{A}_n^a(k) = -i \langle \psi_n(k) | \nabla_k^a \psi_n(k) \rangle,$$  \hspace{1cm} (2.6)

$$\tilde{F}_{n}^{ab}(k) = \nabla_k^a \mathcal{A}_n^b(k) - \nabla_k^b \mathcal{A}_n^a(k).$$  \hspace{1cm} (2.7)

After a few trivial algebra, it is easy to show that Fermi surface integral gives to the non-quantized part of the intrinsic anomalous Hall conductivity,

$$\sigma_{0}^{ab} = e^{ab} \nu \frac{e^2}{h},$$  \hspace{1cm} (2.8)

$$\nu = \frac{1}{2\pi} \oint \mathcal{A}_n^a(k_F) d k_{F_a} = \frac{\phi_F}{2\pi}. $$  \hspace{1cm} (2.9)

To conclude, the dipole-momentum locking mechanism is supposed to assign the Fermi sea with Berry phase. The Berry phase picked up by transporting a composite fermion around the Fermi sea $\Phi_{FS}$ equals to the composite fermion’s Hall conductivity,

$$\sigma_{CF}^H = \Phi_{FS} \frac{e^2}{2\pi} \frac{e}{h}. $$  \hspace{1cm} (2.10)

Because each composite fermion consists of $2m$ flux quanta and one electron. To work out how composite fermion Hall conductivity $\sigma_{CH}^CF$ is related to the electron Hall conductivity $\sigma_{H}$, let us consider the following model,

$$\mathcal{L} = \mathcal{L}_0(\psi, a) + \frac{1}{2m} \frac{1}{4\pi} (A - a) d(A - a). $$  \hspace{1cm} (2.11)

where $\mathcal{L}_0(\psi, a)$ is the composite fermion action. The Chern-Simons terms is doing nothing but attaching $2m$ flux quanta. The electron and composite fermion response
function are denoted as $\Pi^{\mu\nu}$ and $\bar{\Pi}^{\mu\nu}$ respectively.

\[
\mathcal{L}(\psi, a, A) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{1}{2} A_\mu^*(q) \Pi^{\mu\nu}(q) A_\nu(q). \tag{2.12}
\]

\[
\mathcal{L}_0(\psi, a) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{1}{2} a_\mu^*(q) \bar{\Pi}^{\mu\nu}(q) a_\nu(q). \tag{2.13}
\]

$a_\mu$ can be trivially integrated out, since it is quadratic. The electron’s Hall conductivity $\sigma_H(q)$ and composite fermion Hall conductivity $\sigma^{CF}_H(q)$ are found to be related by,

\[
2\pi \sigma_H(q) \equiv -\frac{\pi}{i\omega} \epsilon_{ab} \Pi^{ab}(q), \tag{2.14}
\]

\[
= \left( \frac{1}{2m} \right)^2 \left( \frac{\omega}{2\pi} \right)^2 2\pi \sigma^{CF}_H(q) + \frac{1}{2m} + \frac{1}{2m},
\]

where $\bar{\Pi}^{ab}(q) = \bar{\Pi}^{ab}(q) - \frac{1}{2m} \frac{i\omega}{2\pi} \epsilon^{ab}$. In principle, $\det \bar{\Pi}^{ab}(q)$ includes the effects of disorder and interactions. Hence the Hall conductivity gets a non-universal correction from composite fermions unless $2\pi \sigma^{CF}_H = -\frac{1}{2m}$. We will see that at one half filling, particle-hole symmetry constrains the Hall conductivity to be $\frac{1}{2}$, which implies composite fermion Hall conductivity to be $-\frac{1}{2}$. The $-\pi$ Fermi sea Berry phase appears to be an important ingredient for composite fermions. At a generic filling, the Fermi sea Berry phase will be $-2\pi \nu$. We will revisit it in more details in Chapter. In the following subsections, we will discuss HLR theory and Son’s Dirac fermion theory, with an emphasis on symmetry and Berry phase.

### 2.2 A standard theory by B. I. Halperin, P. A. Lee and N. Read

As we mentioned, a celebrated theory for the half filled Landau level was first proposed by B. I. Halperin, P. A. Lee and N. Read. In this subsection, we discuss in more details
The starting point in HLR theory is still the interacting electron Hamiltonian Eq. (1.16). Theoretically, the lowest Landau level limit is equivalently achieved by taking electron’s bare mass to zero,

\[ LLL : m_e \rightarrow 0. \] (2.15)

Then, HLR transformed the electron description to composite fermion description, through the so-called singular gauge transformation which is exact mathematically. The Hamiltonian after such transformation is written in terms of composite fermion fields, plus a Chern-Simons term. The Chern-Simons term does flux attachment transformation:

\[ \psi_e^\dagger(r) \rightarrow \psi^\dagger(r) = \psi_e^\dagger(r) \exp[-i\tilde{\phi} \int d^2 r' \text{arg}(r - r') \rho(r')]. \] (2.16)

where \( \text{arg}(r - r') \) is the angle the vector \( (r - r') \) forms with the x axis. Note that the density remains the same,

\[ \rho(r) \equiv \psi_e^\dagger(r)\psi_e(r) = \psi^\dagger(r)\psi(r). \] (2.17)

Written in terms of composite fermion fields, the Hamiltonian is,

\[ \mathcal{L} = \frac{1}{2m_e} \int d^2 r \psi^\dagger(r)[-i\nabla + eA(r) - a(r)]^2 \psi(r). \] (2.18)

where

\[ a(r) \equiv \tilde{\phi} \int d^2 r' g(r - r') \rho(r'). \] (2.19)

The singular gauge transformation that maps electron field to composite fermion field is a mathematically exact transformation. This also means that the Hamiltonian
is hard to solve exactly. To make progress, HLR had to make approximations. One of them is the replacement of the bare electron mass $m_e$ with the so-called renormalized electron mass $\mathcal{M}$, which they argued to contain the effects of renormalization to obtain a low energy effective theory. However, till now, the precise connection between $m_e$ and $\mathcal{M}$ is still unestablished. In the lowest Landau level limit, although $m_e \to 0$, $\mathcal{M}$ is argued to remain finite. It is also shown that due to the presence of the internal gauge field $a_\mu$, $\mathcal{M}$ diverges logarithmically at the Fermi surface due to Coulomb interactions. Perhaps one of the most crucial criticisms is from the particle hole symmetry as will be mentioned in the next section. Nevertheless, low energy properties of CFLs at $\nu = \frac{1}{2m}$ in HLR theory are supposed to be captured by the following effective theory by starting with a mean field Fermi sea and by adding in effects of electron interaction and gauge field fluctuations perturbatively,

$$
\mathcal{L} = i\psi\dagger(\partial_0 - ia_0)\psi - (\mathcal{M}^*)^{-1}_{ab} D_a\psi\dagger D_b\psi, \\
- \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{2}\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda ... \quad (2.20)
$$

$$
D_a \equiv \partial_a - ia_a. \quad (2.21)
$$

Despite the criticism above, HLR theory was successful in many aspects. For example, it explained the surface acoustic wave propagation experiments very well [5]. Many other experimental findings supporting HLR theory can be found in [5, 14, 28].

The HLR’s composite fermion idea not only motivated an effective theory, but helped a lot to write down a model wavefunction. The commonly used model wavefunction for CFLs are the following,

$$
\Psi(\{r\}, \{k\}) = P_{LLL} \left( \prod_{ij} \left( e^{ik_j - r_j} \right) \prod_{k<l} (z_k - z_l)^2 \right). \quad (2.22)
$$
where \( \mathbf{r}_i \) is the \( i_{th} \) electron’s coordinate, \( z = w_\alpha r^\alpha \). The set \( \{ \mathbf{k} \} \) parameterizes the momentums of composite fermions. Due to the dipole-momentum feature, \( \mathbf{k} \) is strength proportional and perpendicular to the dipole of composite fermion. The key ingredients of the above model wavefunction includes:

- A Jastrow factor \( \prod_{i<j} (z_i - z_j)^2 \) that implements flux attachment transformation
- A determinant which is the wavefunction for a Fermi sea.
- Landau level projection operator \( P_{LLL} \) which is defined as \( P_{LLL} \equiv \sum_{\{n\}} |\{n\}\rangle \langle \{n\}| \)
  where \(|\{n\}\rangle\) represents the full set of basis in a single Landau level.

In Sec. 3.2.5, we will replace the Landau level projection operator with a determinant to obtain a computational efficient model wavefunction on torus, from which we carried detailed study of the half filled Landau level state and CFLs at various filling fractions.

### 2.3 Particle-hole symmetry in a half filled Landau level

The particle hole transformation is an anti-unitary transformation that takes electrons to holes. Let \( c_m^\dagger \) be the operator that creates an electron with angular momentum \( m \). Under the particle-hole transformation,

\[
\begin{align*}
  c_m & \leftrightarrow c_m^\dagger, \\
  i & \leftrightarrow -i.
\end{align*}
\]

It is anti-unitary such that the guiding center algebra is preserved,

\[
[R^a, R^b] = -i\epsilon^{ab} \hbar/(eB).
\]
We will now prove that a generic translational invariant two-body interacting Hamiltonian projected in a single Landau level is particle-hole symmetric. Consider the Hamiltonian,

$$H = \sum_{m,n,p,q} V_{m,n;p,q} \delta_{m+n,p+q} c_m^\dagger c_n^\dagger c_p c_q. \quad (2.25)$$

where $V_{m,n;p,q} = -V_{n,m;p,q} = (V_{p,q;m,n})^*$ to ensure the Hermitian. Under particle-hole transformation,

$$H \rightarrow \sum_{m,n,p,q} V^*_{m,n,p,q} \delta_{m+n,p+q} c_m^\dagger c_n^\dagger c_p^\dagger c_q^\dagger,$$

$$= H + 4V_{m,n,m,n} \left( c_n^\dagger c_n - \frac{1}{2} \right). \quad (2.26)$$

Translational invariance guarantees that $V_{mnmn}$ is a constant. At half filling, the Hamiltonian then is invariant under particle-hole transformation,

$$H \rightarrow H. \quad (2.27)$$

Hence particle-hole is a symmetry for the Hilbert space of translational invariant two-body interaction in a single Landau level. Any low energy theories must obey this symmetry if there is no spontaneous symmetry breaking. However it is not clear how HLR’s theory is consistent with particle hole symmetry, since HLR attaches fluxes to electrons rather than holes. Naively one would expect the particle conjugate of HLR theory will be a Fermi sea of composite-holes, and therefore distinct from the Fermi sea of composite-fermions.

Moreover, the particle-hole symmetry of HLR theory has been criticized because of the Hall conductivity. As first pointed out by S. A. Kivelson et.al. in [29], it is found that when disorders are statistically PH symmetric, the Hall conductivity
in a half filled Landau level must strictly be $\frac{1}{2}\frac{e^2}{h}$. In the RPA approximation of HLR theory, the resistivity tensor of electrons is related to resistivity of composite fermions $\rho_{xy} = \rho^{CF}_{xy} + 2h/e^2$. These two implies the composite fermion resistivity is $-\frac{1}{2}$. However at mean field, composite fermions perceive zero net magnetic field, hence how can it carries $-\frac{1}{2}\frac{e^2}{h}$ Hall conductivity? As we mentioned, the solution to this puzzle lies in the nontrivial Berry phase on the Fermi sea, as pointed out by Duncan Haldane.

2.4 An alternative effective theory: Dirac Fermion Theory

Recently, Son proposed an effective theory as an alternative to the “standard model” Halperin-Lee-Read theory. Son’s theory is explicitly particle-hole symmetric and has $\pi$ Berry phase. In this subsection, we review the Son-Dirac theory, starting from considering a massless Dirac fermion in a magnetic field. The Lagrangian description is,

$$\mathcal{L} = i\tilde{\Psi}(\partial_\mu - iA_\mu)\Psi.$$ (2.28)

where $\Psi$ is a two-component spinor. We choose the representation,

$$\gamma^0 = \sigma^3, \gamma^a = i\sigma^2, \gamma^2 = -i\sigma^1.$$ (2.29)

Quantization of massless Dirac fermions in magnetic fields produces Landau levels with energy $E = \pm \sqrt{2|n|B}$ where $n$ is the Landau level index, see FIG (2.3). The
Figure 2.3: Landau level quantization of massless Dirac fermion. The Landau level energy scales with $\sqrt{|n|}$. With chemical potential set to be zero, the zero mode ($n = 0$) is half filled.

eigenstates are,

$$|\psi_{n>0}\rangle = \begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix}. \quad (2.30)$$

$$|\psi_{n=0}\rangle = \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix}. \quad (2.31)$$

where $\phi_n$ is the wavefunction of $n_{th}$ nonrelativistic Landau level. We see that the zero mode wavefunction is essentially identical to that of a lowest Landau level nonrelativistic electron. If we take interactions into account, which leads to Landau level mixing, the spinor representing the zero mode can be denoted as,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (2.32)$$

where $\chi \sim e^2$ which vanishes if Landau level mixing is absent.
Besides the zero modes, the Dirac fermion has an infinitely-deep Dirac sea, which is fully occupied. Since each filled Landau level contributes $B/(2\pi)$ to the Dirac density, the Dirac density contribution from the Dirac sea is,

$$\left(\sum_{N=-1}^{\infty} 1\right)\frac{B}{2\pi} = \zeta(0)\frac{B}{2\pi} = -\frac{1}{2}\frac{B}{2\pi}. \quad (2.33)$$

where we used the Zeta function regularization,

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s) = -\frac{1}{2}. \quad (2.34)$$

The Dirac sea not only contributes to the density, but also contributes to the Hall conductivity. The effect of the Dirac sea is described by an extra-$\frac{1}{2}$ leveled Chern-Simons term,

$$-\frac{1}{2}\frac{1}{4\pi} A dA. \quad (2.35)$$

Hence according to Son, the physics of the lowest level Landau level in the limit of small Landau level mixing can be equivalently described as relativistic fermions or non-relativistic fermion,

$$\mathcal{L}^D(\Psi^D, A_\mu) + \frac{1}{2}\frac{1}{4\pi} A dA \leftrightarrow \mathcal{L}^{NR}(\psi, A_\mu). \quad (2.36)$$

The advantage of formulating the whole problem in terms of Dirac fermions is precisely the particle-hole symmetry. The left hand side of Eq. (2.36) is invariant under particle-hole transformation. A key conjecture made by Son is the duality between Dirac fermions and Dirac composite fermions,

$$\mathcal{L}_{QED3} \leftrightarrow \mathcal{L}. \quad (2.37)$$
where

\begin{align}
\mathcal{L}_{QED3} &= i\bar{\psi}\gamma^\mu (\partial_\mu - iA_\mu) \psi, \\
\mathcal{L} &= i\bar{\psi}\gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{2} \frac{1}{2\pi} adA.
\end{align}

Based on Eq. (2.36) and the conjecture Eq. (2.39), Son proposed the following effective action as the low energy description of the composite fermion liquids at half filling in the lowest Landau level,

\begin{equation}
\mathcal{L} = i\bar{\psi}\gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{2} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + \frac{1}{2} \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \ldots
\end{equation}

where \( \psi \) is a Dirac field describing the composite fermion, \( a_\mu \) is the emergent gauge field, and \( \ldots \) stands for other terms including Maxwell term for \( a_\mu, A_\mu \) and interaction terms.

In Son’s convention, the particle hole symmetry is the \( \mathcal{CT} \) symmetry. One of the keys in Son’s theory is that: \( \mathcal{CT} \) acts on the Dirac field in a way akin to time-reversal symmetry, see Eq. (2.42). Therefore, Dirac fermion action Eq. (2.40) is
invariant under $\mathcal{CT}$, and a finite chemical potential is still consistent with particle-hole symmetry. See FIG. 2.4 for an illustration.

$$\mathcal{CT} c_m (\mathcal{CT})^{-1} = c_m^\dagger, \quad \mathcal{CT} i (\mathcal{CT})^{-1} = -i.$$  \hfill (2.41)

$$\mathcal{CT} \psi(t, x) \mathcal{CT}^{-1} = - i \sigma^2 \psi(-t, x).$$  \hfill (2.42)

The electron density in the action Eq. (2.40) is given by,

$$\rho_e \equiv \frac{\delta L}{\delta A_0} = \frac{1}{2} \frac{B}{2 \pi} - \frac{1}{2} \frac{b}{2 \pi}.$$  \hfill (2.43)

where $b = e^{ab} \partial_a a_b$. Since $a_0$ is the Lagrangian multiplier of the theory. Differentiating the action with $a_0$, one finds a relation between Dirac composite fermion density and magnetic field,

$$\rho_\psi = \langle \bar{\psi} \gamma^0 \psi \rangle = \frac{1}{2} \frac{B}{2 \pi}.$$  \hfill (2.44)

From Eq. (2.43) and Eq. (2.44), one finds that the role of charge and flux are swapped: this is a feature of particle-vortex duality. In Dirac fermion theory, only at half filling the density of Dirac fermions equals to the density of electrons. Away from half filling, the Dirac composite fermion density is still half the magnetic flux density, but they see finite magnetic fields. This is unlike HLR theory where the composite fermion density always equal to the electron density, because of the Chern-Simons flux attachment. To summarize,

- In Son’s theory, the composite fermion is a Dirac fermion. The theory is explicitly particle-hole symmetric, because Dirac fermion transforms in a way akin to time-reversal under electron’s particle-hole transformation.

- The density of Dirac fermion is determined by half the electron’s magnetic flux density.
• The dipole-momentum locking is represented as the spin-orbit locking of Dirac fermion. Transporting a single Dirac fermion around the Fermi surface has $\pi$ Berry phase. This $\pi$ Berry phase origins from the spin-half nature of Dirac fermion. Equivalently, it is attributed to the gapless Dirac node at the center of Fermi sea. The $\pi$ Berry phase in fact is a Berry curvature singularity localized at the Fermi sea center.

In the following section, we will discuss lattice representation and model wavefunctions. We will have a composite fermion liquid model wavefunction on torus. And prove that any continuous integration on torus can be replaced by a discrete lattice summation. With this technique, we can study the particle hole symmetry and Berry phase of composite Fermi liquid in a large system size. We will test whether the Fermi sea Berry phase is $-2\pi \nu$ as predicted from the theory of two dimensional anomalous Hall effect. And we will test how the Berry phase is distributed on the Fermi sea: it is indeed predicted as Son’s theory that there is a $\pi$ peak in the Fermi sea center, or distributed in some other ways?
Chapter 3

Quantum Hall States on Torus

3.1 Translation group and Heinsberg algebra

3.1.1 Torus and single body translation

In the following we will work on a torus with primary translations $L_1$ and $L_2$, which contains flux $2\pi N_\phi = |L_1 \times L_2|$.

Model wave functions on the torus contain an implicit “complex structure” which describes a mapping between the torus and the complex plane: $z \equiv w_\alpha x^\alpha$. A complex structure is defined by a unimodular (unit determinant) Euclidean-signature metric.

![Figure 3.1: Fermi sea of dipoles.](image)

\[
\text{area} = 2\pi N_\phi l_B^2
\]

$N_\phi$ : flux quanta
through $g_{ab} = w_a^* w_b + w_a w_b^*$ and $i \epsilon_{ab} = w_a^* w_b - w_a w_b^*$. The complex lattice is then

$$
\mathbb{L} \equiv \{mL_1 + nL_2 | m, n \in \mathbb{Z}\}. \quad (3.1)
$$

$$
L_i = w_a L_i^a. \quad (3.2)
$$

and the quantization condition translates to

$$
L_1^* L_2 - L_1 L_2^* = 2\pi i N\phi. \quad (3.3)
$$

The metric $g_{ab}$ is a continuously variable parameter of the model wave function that generally parameterizes the shape of “flux attachment” or a correlation hole around each particle. If the interaction $V(r)$ has a rotational symmetry, so $V(r) = f(g_{ab} r^a r^b)$, the natural choice for $g_{ab}$ is $g_{ab}^0$, but otherwise $g_{ab}$ should be chosen to minimize the variational energy of the model wave function.

The symmetry group on a torus in a magnetic field is the magnetic translation group, whose group elements are

$$
t(d) \equiv e^{i d \times R}. \quad (3.4)
$$

where $d$ is a vector in real space. The $t(d)$ satisfy the Heisenberg algebra

$$
t(d + d') = t(d)t(d') e^{\frac{i}{2} d \times d'}. \quad (3.5)
$$

A periodic translation must leave the wave function invariant up to a phase:

$$
t(L) \psi(z) = \eta_L N \psi(z), \quad (3.6)
$$
where

\[ \eta_L = \begin{cases} 
1 & \text{if } \frac{1}{2}L \in \mathbb{L}, \\
-1 & \text{if otherwise.} 
\end{cases} \]  

(3.7)

The phase \( \phi_L \) is a boundary condition parameter, that has no significance if the system is translationally invariant, when it can conveniently be set to zero. Translating the wave function by a non-integer multiple of \( L/N_\phi \) would change the boundary condition and this imposes the condition. Requiring,

\[ [t(d), t(L)] = 0, \quad d \equiv w_a d^n. \]  

(3.8)

the allowed translation is quantized with discrete values,

\[ d \in \left\{ \frac{mL_1}{N_\phi} + \frac{nL_2}{N_\phi} | m, n \in \mathbb{Z} \right\} \]  

(3.9)

### 3.1.2 Many body translation

For many body translational invariant Hamiltonian, translating all electrons by a same distance leaves \( H \) invariant. We define such an operation as many body translation:

\[ T(d) \equiv \prod_{i=1}^{N} t_i(d). \]  

(3.10)

where \( t_i(d) \) is the single body translation operator associated to \( i_{th} \) electron. The many-body translation form Heinsberg algebra too,

\[ T(d_1)T(d_2) = e^{i \frac{N}{2} d_1 \times d_2} T(d_1 + d_2) = e^{iN d_1 \times d_2} T(d_2) T(d_1). \]  

(3.11)
Like in the single body translation, allowed $d$ are quantized on $1/N_\phi$ lattice,

\[ d \in \{m \frac{L_1}{N_\phi} + n \frac{L_2}{N_\phi} | m, n \in \mathbb{Z}\}. \tag{3.12} \]

Besides the filling $\nu = p/q$ of Eq. (1.15), lets also define,

\[ \nu = \frac{p_0}{q_0}. \tag{3.13} \]

where $p_0$ and $q_0$ are prime to each other. For example, in Laughlin state $p = p_0 = 1,$ $q = q_0 = \nu^{-1}$, while in Moore-Read Pfaffian state $p = 2,$ $q = 4,$ $p_0 = 1,$ $q_0 = 2.$ The minimal commuting sets is then,

\[ \{H, T^{q_0} \left( \frac{L_1}{N_\phi} \right), T \left( \frac{L_2}{N_\phi} \right) \}. \tag{3.14} \]

### 3.2 Model Wavefunctions on Torus

#### 3.2.1 Weistrass sigma functions

Since wave functions on a torus are quasiperiodic, they are naturally expressed in terms of various elliptic functions. Usually such elliptic functions are chosen as Jacobi-theta functions \[24, 16\]. The downside of Jacobi-theta functions is that they are not modular invariant. Here we instead use “modified Weistrass sigma function” \[?\] to write down quantum Hall wavefunctions on torus. Let’s first review the “standard Weierstrass sigma function” and explain the modification.

**Definition 3.2.1** (Standard Weierstrass sigma function, zeta function, almost modular form). The “standard Weierstrass sigma” function $\tilde{\sigma}(z)$ is defined as,

\[ \tilde{\sigma}(z) = z \prod_{m,n} \left(1 - \frac{z}{\Omega_{mn}}\right)e^{z \Omega_{mn} + \frac{z^2}{2\Omega_{mn}}}, \quad \Omega_{mn} \equiv mL_1 + nL_2. \tag{3.15} \]
It has zero at \( z = 0 \). In addition, it satisfies:

\[
\tilde{\sigma}(z + L_i) = -e^{2\tilde{\eta}_i(z+L_i/2)}\tilde{\sigma}(z)
\]  

(3.16)

where \( \tilde{\eta}_i \) is the zeta function evaluated at half period, which is related to the Eisenstein series \( G_2(L_i) \), \( \tilde{\eta}_i = G_2(L_i)L_i/2 \). An almost modular form is a modular invariant quantity that,

\[
\tilde{G}(L) = G_2(L_i) - \frac{1}{N_0} \frac{L_i^5}{L_i}.
\]  

(3.17)

**Definition 3.2.2** (Modified Weierstrass sigma function). The modified Weierstrass sigma function is defined as,

\[
\sigma(z) \equiv \tilde{\sigma}(z)e^{\frac{1}{2}\tilde{G}(L)z^2}.
\]  

(3.18)

It is a holomorphic function, has zero at \( z = 0 \) modular \( L \), and is modular invariant.

It is not difficult to check the quasi-periodicity of \( \sigma(z) \):

\[
\sigma(z + L_i) = -e^{\frac{1}{N_0}(z+L_i/2)}\sigma(z).
\]  

(3.19)

### 3.2.2 Holomorphic representation of Heisenberg algebra

Single body Heinsberg algebra fixed the form of single body wavefunction, and in the same way, the \( T(d) \) will fix the many body wavefunction uniquely. Therefore I will discuss in general how Heinsberg algebra constraints the wavefunction.

The common character for \( t(d) \) and \( T(d) \) can be summarized as follows. Con-
sider a \( \mathcal{M} = \frac{N_\phi}{N} \)-dimensional Heinsberg group, whose group elements \( \mathcal{T}(d) \) satisfies

\[
\mathcal{T}(d_1 + d_2) = \mathcal{T}(d_1)\mathcal{T}(d_2) e^{\frac{i}{2}N_\phi d_1 \times d_2}.
\] (3.20)

The center of the group (commuting sets) is,

\[
\{ \mathcal{T}^\mathcal{M}(d) \}, \quad \mathcal{M} = \frac{N_\phi}{N}, \quad d \in \{ m \frac{L_1}{N_\phi} + n \frac{L_2}{N_\phi} | m, n \in \mathbb{Z} \}. \tag{3.21}
\]

A set of ladder operators are associated with the Heisenberg algebra,

\[
\begin{align*}
\mathcal{T}(d) &= e^{d^*A^\dagger - dA} = e^{d^*A^\dagger} e^{-\frac{N}{2}dd^*}. \\
[A, A^\dagger] &= \mathcal{N}. \\
A|0\rangle &= 0.
\end{align*}
\] (3.22, 3.23, 3.24)

where the lowest weight state \(|0\rangle\) is the kernel of \( A \).

Boundary condition is defined as \((d_L \equiv L/N_\phi)\),

\[
\mathcal{T}^\mathcal{M}(d_L)|\psi\rangle = \mathcal{T}(\frac{L}{N_\phi}) = \eta^\mathcal{M}_L e^{i\phi L}|\psi\rangle. \tag{3.25}
\]

Holomorphic representation (let \( h(\mathcal{Z}) \) be a holomorphic function),

\[
\begin{align*}
A &\rightarrow \mathcal{N}\partial_{\mathcal{Z}}. \\
A^\dagger &\rightarrow \mathcal{Z}. \\
\mathcal{T}(d)h(\mathcal{Z}) &= e^{d^*A^\dagger - dA} = e^{d^*A^\dagger} e^{-\frac{N}{2}dd^*} h(\mathcal{Z}), \\
&= e^{d^*(\mathcal{Z} - \frac{N}{2}d)} h(\mathcal{Z} - \mathcal{N}d). \tag{3.28}
\end{align*}
\]
For single body wavefunction $\langle z|0 \rangle = \exp(-\frac{1}{2}zz^*)$, where $z$ is electron’s holomorphic coordinate; for center-of-mass case $\langle Z|0 \rangle = \exp(-\sum_i \frac{1}{2}z_i z_i^*)$ with $z_i$ being the $i_{th}$ electron’s coordinate. The boundary condition then translates to,

$$h(Z - L) = \eta_L^M e^{-i\phi_L} e^{-\frac{L_i^*}{N}(Z - \frac{L_i}{2})} h(Z).$$

(3.29)

Such representation can be constructed by using the modified Weierstrass sigma function with $\mathcal{M}$ zeros $\{\alpha\}$,

$$h(Z) = \prod_{k=1}^{\mathcal{M}} \sigma(Z - \alpha_k) e^{\frac{1}{N\phi} Z \alpha_k^*},$$

(3.30)

$$\mathcal{M} = \frac{N\phi}{N}.$$  

(3.31)

with quasi-periodicity (with $L_i$ being $L_1$ or $L_2$),

$$h(Z - L_i) = \prod_{k=1}^{\mathcal{M}} \sigma(Z - \alpha_k - L_i) e^{\frac{1}{N\phi} (Z - L_i) \alpha_k^*},$$

$$= (-1)^{\mathcal{M}} h(Z) e^{-\frac{L_i^*}{N}(Z - L_i)} e^{L_i^* \alpha - L_i \bar{\alpha}^*},$$

(3.32)

$$\bar{\alpha} \equiv \frac{1}{N\phi} \sum_{k=1}^{\mathcal{M}} \alpha_k.$$  

(3.33)

where I used the quasi-periodicity of modified Weierstrass sigma function Eq. (3.19),

$$\sigma(z - L_i) = -e^{-\frac{L_i^*}{N\phi} (z - L_i/2)} \sigma(z).$$  

(3.34)

Hence the average of zeros $\bar{\alpha}$ sets the boundary condition phase factor,

$$e^{-i\phi_L} = e^{L_i^* \alpha - L_i \bar{\alpha}^*}.$$  

(3.35)

Now, we are ready to write down single body and many body wavefunctions.

37
3.2.3 Single body wavefunction

For single body wavefunctions $N = 1$, the dimension of the Heisenberg group is $M = N_\phi$. A single body wavefunction is a holomorphic function times a Gaussian factor $e^{-\frac{1}{2}zz^*}$. Note that the transformation of Gaussian factor under $z \rightarrow z - L_i$ is,

$$e^{-\frac{1}{2}zz^*} \rightarrow e^{-\frac{1}{2}zz^*^e} e^{\frac{1}{2}(zL_i^* + z^*L_i)} e^{-\frac{1}{2}L_i^*L_i}.$$  \hspace{1cm} (3.36)

Define,

$$\psi(z, \{w\}) \equiv h(z, \{w\}) e^{-\frac{1}{2}zz^*}. \hspace{1cm} (3.37)$$

There is,

$$\psi(z - L_i, \{w\}) = (-1)^{N_\phi} e^{-\frac{1}{2}(L_i^*z - L_i z^*)} e^{L_i^*\tilde{\alpha} - L_i \tilde{\alpha}^*} \psi(z, \{w\}). \hspace{1cm} (3.38)$$

To summarize,

**Theorem 3.2.3 (Generic single body wavefunction).** A holomorphic wavefunction is uniquely determined by it’s zeros $\{w_i\}$. There are $N_\phi$ zeros in total, $i = 1, ..., N_\phi$. A general wavefunction looks like$^1$

$$\psi(z, \{w\}) = \prod_i^{N_\phi} g(z; w_i) \hspace{1cm} (3.39)$$

$$g(z; w) = f(z - w) e^\frac{1}{2N_\phi} (w^* z - wz^*) \hspace{1cm} (3.40)$$

$$f(z) = \sigma(z) e^{-\frac{1}{2N_\phi} zz^*}. \hspace{1cm} (3.41)$$

$^1$Strictly speaking, $\psi(z)$, $g(z)$ and $f(z)$ are not holomorphic functions although I write them as function of $z$. But $\psi(z)$ is indeed a holomorphic function provided exponential factor $e^{-\frac{1}{2}zz^*}$ is divided.
where $\sigma(z) = \tilde{\sigma}(z)e^{-\frac{1}{2}G(L)z^2}$ where $\tilde{\sigma}(z)$ is the standard weierstrass sigma function and $G(L)$ is the almost modular form associated to the torus.

To check Eq. (3.41) is indeed the wavefunction, what we need to do is simply to check the boundary condition. To do this, we need to recall the quasi-periodicity of weierstrass sigma function. The quasi-periodicity of $f(z)$ and $g(z, w)$ function is as follows,

$$f(z + L) = \eta_L f(z)e^{\frac{1}{N\phi}(L^*z - Lz^*)}. \quad (3.42)$$

$$g(z + L; w) = \eta_L g(z; w)e^{\frac{1}{N\phi}(L^*z - Lz^*)}e^{-\frac{1}{N\phi}(L^*w - Lw^*)}. \quad (3.43)$$

where $\eta_L = 1$ if $L \in \{mL_1 + nL_2 | m, n \in \mathbb{Z}\}$. Then, it’s easy to check that,

$$t(L)\psi(z, \{w\}) = \eta_N e^{L^*\bar{w} - L\bar{w}^*}\psi(z, \{w\}) \quad (3.44)$$

$$\psi(z + L, \{w\}) = \eta_N e^{\frac{1}{2}(L^*z - Lz^*)} e^{-\frac{1}{2}(L^*\bar{w} - L\bar{w}^*)}\psi(z, \{w\}). \quad (3.45)$$

**Corollary 3.2.3.1** (Boundary Condition and Zeros). *From the translation property of $\psi(z, \{w\})$ above, it’s found that boundary condition is determined by average of zeros,*

$$\bar{w} = \frac{1}{N\phi} \sum_i w_i = -\frac{\phi_2}{2\pi N\phi} L_1 + \frac{\phi_1}{2\pi N\phi} L_2. \quad (3.46)$$

*If two $\bar{w} = \frac{1}{N\phi} \sum_i w_i$ are on the same lattice $\{\frac{mL_1 + nL_2}{N\phi}\}$, then the two wavefunction have the same boundary condition.*

Without loss of generality, I will choose periodic boundary condition, s.t. sum of zeros equals to zero modular $L$, throughout the note.
Corollary 3.2.3.2 (Landau Basis). The Hilbert space includes $N_\phi$ linearly independent states, called Landau basis. For the $k$th state, it’s $N_\phi$ zeros are as follows,

$$w_i^{(k)} = \frac{2i - (N_\phi + 1)}{2N_\phi} L_1 + \frac{k}{N_\phi} L_2, \quad i = 1, \ldots, N_\phi. \quad (3.47)$$

A set of “Landau basis” is defined in the following. In many body case, they are topological sectors that are resolved by translation.

$$t_1 f_k(z) = -e^{-2\pi ik/m} f_k(z),$$
$$t_2 f_k(z) = f_{k-1}(z),$$
$$f_{k+m}(z) = (-1)^m f_k(z). \quad (3.48)$$

### 3.2.4 Center-of-mass wavefunction

For center-of-mass wavefunction, $\mathcal{N} = N$ the number of electrons. The dimension of Heisenberg group is $\mathcal{M} = N_\phi/N$. The holomorphic coordinates are sum of electrons’ coordinates $Z \equiv \sum_i^N z_i$. Define the following as center-of-mass wavefunction,

$$\Phi(Z) \equiv h(Z, \{\alpha\}) e^{-\frac{1}{2} \sum_i z_i z_i^*}. \quad (3.49)$$

where

$$h(Z, \{\alpha\}) = \prod_{k=1}^M \sigma(Z - \alpha_k) e^{\frac{1}{N_\phi} Z \alpha_k^*}. \quad (3.50)$$

Note that Gaussian factor transforms as,

$$e^{-\frac{1}{2} \sum_i (z_i - L)(z_i - L)^*} = e^{-\frac{1}{2} \sum_i z_i z_i^*} e^{\frac{1}{2} (L_i^* Z + L_i Z^*)} e^{-\frac{N}{2} LL^*}. \quad (3.51)$$
3.2.5 Model wavefunctions: Laughlin, CFL and others

Laughlin Wavefunction

The Laughlin wavefunction is:

\[
\Psi(\{\alpha\}) = \prod_{i<j} f^m(z_i - z_j) \prod_{k=1}^{m} f(Z - \alpha_k) \times e^{\frac{1}{2N_f} (a^*Z - aZ^*)}
\]  (3.52)

where \(\{\alpha_k\}\) is the set of com zero that specifies ground state. For each \(z_i\), its zeros sum to \(\sum_k \alpha_k = m\tilde{\alpha}\). Thus, the boundary condition is determined by

\[
\tilde{w} = \alpha/N_\phi, \quad \alpha = \sum_k \alpha_k.
\]  (3.53)

\[
t_i(L) \sim N_\phi e^{-L^*\tilde{w} - L\tilde{w}^*}.
\]  (3.54)

Therefore, 

*Periodic boundary condition sets \(\alpha = 0\), modular \(L\).*

CFL Wavefunction

The model wavefunction of composite Fermi liquids are obtained motivated by Halperin-Lee-Read’s flux attachment idea [41, 42], as described in Sec. (2.2). Key ingredients of the MWFs at \(\nu=1/2m\) include flux attachment represented by the Jastrow factor and the lowest Landau level (LLL) projection \(P_{LLL}\) operator,

\[
|\Psi_{CF}(\{k_i\})\rangle = P_{LLL}\{\det \prod_{i<j} (z_i - z_j)^{2m}\}.
\]  (3.55)

where \(\{k_i\}\) are distinct and clustered to form a *compact FS*. Holomorphic determinant MWFs are obtained after approximating \([26, 39, 38]\) \(P_{LLL}\) by creating dipoles \(\{d_i\}\), in accordance with the dipole-momentum locking which is a fundamental property of
composite fermions in a LLL. With $\sigma(z)$ as the modified Weierstrass sigma function \cite{1}, \{\alpha_k\} as the center of mass zeros which set the topological sector, MWF at $\nu=1/2m$ reads \cite{23 43},

$$\Psi_{n}^{1/2m} = \det_{ia} M_{ia} \prod_{i<j}^{N} \sigma^{2(m-n)}(z_i - z_j) \times \prod_{k}^{2m} \sigma \left( \sum_{i}^{N} z_i - \alpha_k \right) \prod_{i}^{N} e^{-\frac{1}{2}z_{i}z_{i}^{*}}. \quad (3.56)$$

where

$$M_{ia} = e^{\frac{m}{2m}z_{i}d_{a}^{*}} \prod_{k\neq i}^{N} \sigma^{n} \left( z_i - z_k - d_{a} + \bar{d} \right). \quad (3.57)$$

The $\bar{d}$ is a free parameter, i.e. changing $\bar{d}$ only renormalizes the MWF \cite{9}. $n$ represents a scheme of flux attachment: $2n$ out of the total $2m$ flux quanta are shifted from electron’s position to form a dipole. Such as momentum quantization, dipoles \{d_i\} are quantized by periodic boundary condition \cite{24 16} to take discrete values $d \in \{L/(nN)\}$ where $L$ is the 2D periodic lattice defining the torus.

### 3.3 Lattice Representation

#### 3.3.1 Intuition for Lattice Representation

A number of useful calculations (e.g., overlaps, operator expectation values) can be made by integrating over the positions of all electrons in a model wave function. As shown in \cite{22}, overlap of two holomorphic torus wave functions can be replaced by an exact lattice sum. In this short section, we start with an introduction to the “lattice representation” with an emphasis on its operator form, and then describe how these calculations can be performed using the Metropolis-Hastings algorithm \cite{34 25}. We
will focus on two-body operators since these are used in calculations of important quantities such as the energy or the structure factor.

The key advantage of our method lies in the fact that continuous integrations can be replaced with lattice sums on the torus in an exact way [22]. We are interested in knowing the mean value of a translationally invariant two-body operator \( \sum_{i<j} O(x_i - x_j) \), averaged by states \( |\psi_{n,1}\rangle \) and \( |\psi_{n,2}\rangle \) in the \( n \)-th LL, which by definition is given by continuous integration,

\[
\langle \psi_{n,1} | \hat{O} | \psi_{n,2} \rangle \equiv \prod_k \int d^2 x_k \ \psi^*_{n,1}(\{x\}) \psi_{n,2}(\{x\}) \sum_{i<j} O(x_i - x_j).
\]

where we use \( \hat{O} \) to represent an operator and \( O(x) \) to denote its coordinate representation.

In fact, such calculation can be replaced by a lattice summation for operator \( \hat{O}^{\text{Lat}} \), which we call as the “lattice representation” of \( \hat{O} \),

\[
\langle \psi_{n,1} | \hat{O} | \psi_{n,2} \rangle \equiv C \langle \psi_{0,1} | \hat{O}^{\text{Lat}} | \psi_{0,2} \rangle_{\text{Lat}}.
\]

where the symbol \( \langle |...| \rangle_{\text{Lat}} \) means lattice summation, defined as follows:

\[
\langle \psi_{0,1} | \hat{O}^{\text{Lat}} | \psi_{0,2} \rangle_{\text{Lat}} \equiv \prod_k \sum_{x_k} \psi^*_{0,1}(\{x\}) \psi_{0,2}(\{x\}) \sum_{i<j} O^{\text{Lat}}(x_i - x_j).
\]

In the above, \( \sum'_{x} \) means summing over the \( N_{q} \times N_{q} \) evenly spaced lattice \( x \in \{(mL_1 + nL_2) / N_{q} \mid m, n \in \mathbb{Z}\} \). The constant \( C \) is fixed once the \( N_{q} \times N_{q} \) lattice is chosen, and it is not important since it is always canceled out by wave function normalization factors. In the following, we want to derive the expression of \( O^{\text{Lat}}(x) \equiv \langle x | \hat{O}^{\text{Lat}} | x \rangle \).
Note that in Eq. (3.59) we wrote states on the right side with LL index \( n = 0 \). By doing this, we mean that the physical problem in an arbitrary LL can be solved by using the lowest LL wave functions as a technical device.

The translation group plays the central role in the derivation of the lattice representation. To see this, we start by finding the effective interaction potential for guiding centers. First, do a Fourier expansion for \( \hat{O} \), yielding,

\[
\langle x_{ij}|\hat{O}|x_{ij}\rangle \equiv O(x_i - x_j) = \frac{1}{2\pi N\phi} \sum_q O(q)e^{iq\cdot(x_i-x_j)}.
\]

where the unprimed sum \( \sum_q \) sums all discrete \( q \) allowed by boundary condition.

Now we split up the coordinate and wave function into “Landau orbit” and “guiding center” parts using Eqs. (1.19) and (1.27). This allows us to write it as,

\[
\langle \psi_{n,1}|\hat{O}|\psi_{n,2}\rangle \equiv \frac{1}{2\pi N\phi} \sum_q \sum_{i<j} O(q)f_1^n(q)\langle \psi_{GC,1}^{|e^{iq\cdot(R_i-R_j)}|\psi_{GC,2}^}\rangle.
\]

where \( f_n(q) \) is the LL “form factor” defined as:

\[
f_n(q) \equiv \langle \psi_n^{LO}|e^{iq\cdot\vec{R}}|\psi_n^{LO}\rangle = L_n(\frac{1}{2}q^2l_B^2)e^{-\frac{1}{4}q^2l_B^2}.
\]

A key observation is that \( e^{iq\cdot\vec{R}_i} \) in Eq. (3.61) is nothing but the magnetic translation operators, which satisfy the Heisenberg algebra Eq. (3.5). Note that the periodic translation \( (q \in \mathbb{L}) \) leaves the state invariant up to a phase factor Eq. (3.6). We thus can break up the sum over \( q \) into a sum over the first Brillouin zone [indicated by the prime on the sum \( \sum' \)], and the sum over the rest of \( q \)--space included in \( O^{GC}(q) \).
Now Eq. (3.61) becomes,

\[
\langle \psi_{n,1} | \hat{O} | \psi_{n,2} \rangle \equiv \frac{1}{2\pi N_{\phi}} \sum_q \sum_{i<j} O^{GC}(q) \langle \psi_{1,GC}^{i} | e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} | \psi_{2,GC}^{j} \rangle.
\]

where \(O^{GC}(q)\) is the interaction defined only in the first Brillouin zone but it includes all other short range interactions exactly. Because of the exponential tail of the form factor, the numerical value of \(O^{GC}(q)\) is dominated by the part of \(O(q)\) in the first Brillouin zone,

\[
O^{GC}(q) = [O(q) f_2^2(q)]_c \tag{3.64}
\]

\[
\equiv \sum_{q'} O(q + q' N_{\phi}) f_2^2(q + q' N_{\phi}).
\]

Eqs. (3.63) and (3.64) are close to the final expression of lattice representation \(O^{Lat}(x)\) that satisfies Eq. (3.59). As the central result in this section, it is,

\[
O^{Lat}(x) = \frac{1}{2\pi N_{\phi}} \sum_q \sum_{q'} \frac{O^{GC}(q)}{|f_0(q)|_{N_{\phi}}} e^{i\mathbf{q} \cdot \mathbf{x}}. \tag{3.65}
\]

We give more details of the derivation in Appendix. They symbol \([...]\) used in Eq. (3.65) denotes a “compactification”, defined as the following for arbitrary operator \(\chi(q)\),

\[
[\chi(q_{mn})]_{N_{\phi}} = \sum_{k,l} \chi(q_{mn} + q'_{kl} N_{\phi}) e^{i(k\phi_1 + l\phi_2)}
\]

\[
\times (-1)^{mL-nk+N_{\phi}(kl+k+l)}. \tag{3.66}
\]

where \(\phi_{1,2}\) is the boundary condition \(t(L_{1,2})|\psi\rangle = -e^{i\phi_{1,2}}|\psi\rangle\), \(q_{mn} \equiv (mL_1 + nL_2)/(N_{phi}^2)\). Due to the exponential decay of the form factor, the compactification
used in Eq. (3.65) is effectively constrained in the first Brillouin zone. Note that when \( O(x) \) is identity, we recover the central result in Ref. [22]: the expression of the overlap of wave functions can be replaced by a lattice summation.

At this stage, we make some comments on the result. The emergence of the \( q \)-space Brillouin zone in Eq. (3.63) is purely a consequence of the translation group, and this indeed implies a real space lattice structure: states and operators can be formulated on the lattice. Since we worked out the whole problem in the guiding center space, the lattice representation is generic to any LL; the lowest LL wave function in Eq. (3.59) is not special, but serves just as a technical device to solve the problem in a generic LL. Furthermore, in some cases when the two body operator \( O(q) \) is divergent if it is put on the infinite plane, their lattice representations are convergent. As can be seen from Eq. (3.65), the numerator and denominator are regularized by the Gaussian factor first and are compactified separately, making the potential convergent. This is not surprising, since the lattice provides a natural regularization. We will see this again when working on the high LL Coulomb energy and pair amplitude.

### 3.3.2 A Proof for Lattice Representation

In this section, we derive the lattice representation of translationally invariant two-body operators, which is used above in Eq. (3.65). We will start with finding the lattice representation for one-body operators.

We will start by answering this question: given a periodic single body operator \( O(x) = O(x + L) \), what is the corresponding lattice operator \( O^{Lat}(x \in \mathbb{L}/N_\phi) \) such that the continuous integral equals to the lattice summation (up to some constant
\[ \langle \psi_1 | \hat{O} | \psi_2 \rangle = C \langle \psi_1 | \hat{O}^{\text{Lat}} | \psi_2 \rangle_{\text{Lat}} \]  

(3.67)

where \( \langle \ldots \rangle \) and \( \langle \ldots \rangle_{\text{Lat}} \) represent the integral and lattice summations, respectively. Since \( O(x) \) is periodic, its Fourier transform is,

\[ O(x) = \frac{1}{2\pi N_\phi} \sum_q O(q)e^{iq \cdot x}, \]

\[ O(q) = \int d^2x \ O(x)e^{-iq \cdot x}. \]  

(3.68)

The \( O^{\text{Lat}}(x) \) is not only periodic but is defined on lattice. Its Fourier transform is,

\[ O^{\text{Lat}}(x) = \frac{1}{2\pi N_\phi} \sum_q O^{\text{Lat}}(q)e^{iq \cdot x}, \]

\[ O^{\text{Lat}}(q) = \frac{2\pi}{N_\phi} \sum_x O^{\text{Lat}}(x)e^{-iq \cdot x}. \]  

(3.69)

where \( \sum' \) sums \( q \) in the first Brillouin zone. The resolutions of identity are:

\[ \int d^2xe^{iq \cdot x} = 2\pi \delta_{q,0}, \quad \sum_q e^{iq \cdot x} = 2\pi N_\phi \delta(x), \quad \text{and} \quad \frac{1}{N_\phi} \sum_q e^{iq \cdot x} = \delta_{q,0}, \quad \text{where} \quad \delta_{q,0} = N_\phi \text{ if} \quad q = 0, \quad \delta_{q,0} = 0 \text{ if otherwise}. \]

\[ \int d^2x \delta(x) = 1. \]

In the same spirit as Sec. 3.3, we can write the expectation value as follows:

\[ \langle \psi_1 | \hat{O} | \psi_2 \rangle = \frac{1}{2\pi N_\phi} \sum_q O'(q)\langle \psi_1^{\text{GC}} | e^{iq \cdot R} | \psi_2^{\text{GC}} \rangle. \]  

(3.70)

where \( O'(q) \) is the operator defined only in the first Brillouin zone but it includes all other interactions exactly,

\[ O'(q) = \sum_{q'} O(q + q'N_\phi)f_0(q + q'N_\phi). \]  

(3.71)
where \( f_0(q) \) is the lowest LL form factor Eq. (3.62). Since the form factor decays rapidly, when the system size is large \( O'(q) \) is almost the same as \( O(q) \) except at the Brillouin zone. Now take \( O(x) = e^{i q \cdot x} \). Eq. (3.70) and Eq. (3.68) then lead to the following equation,

\[
\int d^2x \, \psi_1^*(x) \psi_2(x) e^{i q \cdot x} = f_0(q) \langle \psi_{GC1} | e^{i q \cdot R} | \psi_{GC2} \rangle.
\]

By Eq. (3.68), its inverse Fourier transformation is,

\[
2\pi N_\phi \psi_1^*(x) \psi_2(x) = \sum_q f_0(q) e^{-i q \cdot x} \langle \psi_{GC1} | e^{i q \cdot R} | \psi_{GC2} \rangle.
\]

Then, by using the same method that leads to Eq. (3.71), Eq. (3.72) can be compactified to the first Brillouin zone. However, this time, we need to take into account the effect of the phase factor, which leads to a similar but different notion of compactification. The expression for this new compactification \([...]|_{N_\phi}\) can be found in Eq. (3.66),

\[
2\pi N_\phi \psi_1^*(x) \psi_2(x) = \sum_q' [f_0(q)]_{N_\phi} e^{-i q \cdot x} \langle \psi_{GC1} | e^{i q \cdot R} | \psi_{GC2} \rangle.
\]

Eq. (3.73) is the key identity. Based on it, and by using Eq. (3.69), we can easily derive,

\[
\frac{2\pi}{N_\phi} \sum_x' \psi_1^*(x) \psi_2(x) e^{i q \cdot x} = [f_0(q)]_{N_\phi} \langle \psi_{GC1} | e^{i q \cdot R} | \psi_{GC2} \rangle.
\]
Taking \( q = 0 \), in the limit \( N_\phi \to \infty, [f_0(0)]_{N_\phi} \to 1 \), the lattice sum is replaced by the integration \( \frac{2\pi}{N_\phi} \sum'_{x} \to \int d^2x \) and we recover the orthogonality of wave functions [?]. The lattice inner product is then found to be,

\[
\sum'_{x} \psi^*_1(x)\psi_2(x)O^{\text{Lat}}(x) = \frac{1}{(2\pi)^2} \sum'_{q} O^{\text{Lat}}(q)[f_0(q)]_{N_\phi} \langle \psi_{1}^{GC} | e^{iq \cdot R} | \psi_2^{GC} \rangle.
\]

By comparing this to Eq. (3.70), the lattice representation for a single body operator is easily found to be \( O^{\text{Lat}}(q) = \frac{2\pi}{N_\phi} \frac{O(q)}{[f_0(q)]_{N_\phi}} \). The two-body operator is a straightforward generalization since,

\[
\sum'_{q, x_1, x_2} \psi^*_1(x_1)\psi^*_2(x_2)O^{\text{Lat}}(q)e^{i\mathbf{q} \cdot (\mathbf{x}_2 - \mathbf{x}_1)}\psi_3(x_2)\psi_4(x_1)
= \sum'_{q, x_1, x_2} \psi^*_1(x_1)\psi_4(x_1)e^{i\mathbf{q} \cdot \mathbf{x}_1}[\psi^*_2(x_2)\psi_2(x_2)O^{\text{Lat}}(-\mathbf{q})e^{i\mathbf{q} \cdot \mathbf{x}_2}]^*.
\]

Applying Eq. (3.73) or Eq. (3.74) twice, and comparing it with Eq. (3.63), we can solve the lattice representation for two-body operators, which is shown in Eq. (3.65).

### 3.4 Lattice Monte Carlo

To set up the Metropolis algorithm, using the lattice representation, we rewrite expectation value as:

\[
\frac{\langle \psi_1 | \hat{O} | \psi_2 \rangle}{\sqrt{\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle}} = \frac{\sum' |\psi_1(x)|^2 \cdot O^{\text{Lat}}(x) \cdot \psi_2(x)/\psi_1(x)/\sum' |\psi_1(x)|^2}{\sqrt{\sum' |\psi_1(x)|^2 \cdot |\psi_2(x)/\psi_1(x)|^2/\sum' |\psi_1(x)|^2}}.
\]
We obtained this equation by writing the overlaps as sums over all positions of the coordinates, and then multiplying the numerator and denominator by $|\psi_1(x)|^2/[|\psi_1(x)|^2 \sum_{x'} |\psi_1(x')|^2]$. In the above, $\sum'$ sums over $x = \{x_1, ... x_N\}$ which represents a point in the many body coordinate space. All $x_i$ live on the lattice, therefore $x$ is $N^2N_e$ dimensional. Writing the overlap in this way makes it clear that both the numerator and denominator can be computed using a Monte Carlo algorithm with Metropolis weight $|\psi_1|^2$.

3.4.1 Consistency check I: Coulomb energy on torus

In Table 3.1 we test our Monte Carlo method by computing the Coulomb energy, $O(x) \rightarrow V(x) = 1/|x|$, for the Laughlin wave function at $\nu = 1/3$ in the first few LLs. The tables shows the exact energies and those determined by Monte Carlo, for a few different system sizes. The energy includes the “Madelung energy” [51, 4], which is due to an electron’s interaction with periodic copies of itself. The fact that our results agree to several digits (limited only by the statistical error of the Monte Carlo) is a confirmation that our lattice Monte Carlo does give correct results. To the best of our knowledge, the first Monte Carlo calculation of Laughlin $\nu = 1/3$ lowest LL Coulomb energy was done in [35] for up to 144 electrons. Using the lattice method presented here, and by virtue of a modern computer, we can do much larger sizes (e.g. for 200 electrons the energy $-0.40969 \pm 0.00002$ can be computed within 100 CPU hours). For higher LLs $n > 1$, we find very large statistical errors which prevent us from obtaining the energy directly through the Monte Carlo. The cause of this problem (and a solution which improves the Monte Carlo efficiency significantly) are provided in Sec. 3.4.3.
Table 3.1: Comparison of exact and Monte Carlo energies for the Laughlin wave function at $\nu=1/3$. The agreement between the two, limited only by statistical error, is a confirmation that our lattice Monte Carlo is correct. For $n>1$ the statistical error is large. The cause and solution of this problem are described in Sec. 3.4.3.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$N_e$</th>
<th>Exact</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>$-0.414171$</td>
<td>$-0.414172 \pm 0.000001$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$-0.412399$</td>
<td>$-0.412397 \pm 0.000001$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$-0.411583$</td>
<td>$-0.411585 \pm 0.000001$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$-0.339105$</td>
<td>$-0.33907 \pm 0.00005$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$-0.334207$</td>
<td>$-0.33421 \pm 0.00004$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$-0.331879$</td>
<td>$-0.33190 \pm 0.00007$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$-0.280537$</td>
<td>$-0.278 \pm 0.004$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$-0.278052$</td>
<td>$-0.280 \pm 0.005$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$-0.274849$</td>
<td>$-0.26 \pm 0.01$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>$-0.257681$</td>
<td>$-0.26 \pm 0.09$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$-0.254155$</td>
<td>$-0.4 \pm 0.2$</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$-0.251042$</td>
<td>$-0.7 \pm 0.9$</td>
</tr>
</tbody>
</table>
3.4.2 Consistency check II: transporting a hole in Laughlin state

As a second example we move one quasi-hole in the $\nu = 1/q$. In this section, we use $q$ as the inverse of the filling fraction. Laughlin state around an area $A$ in which no other quasiholes are present, and the particle density is uniform. Since the quasi-hole is charged and there is a magnetic field passing through the system, the quasi-hole should pick up a Berry phase of $2\pi A/q$. Before doing the Berry phase calculation, we first review the Laughlin and Laughlin-hole wave function on the torus.

On the infinite plane, Laughlin’s $\nu = 1/q$ wave function \cite{Laughlin81} is given by $\prod_{i<j}(z_i - z_j)^q e^{-\frac{1}{\pi B} \sum_i z_i z_i^*}$. The torus generalization of it \cite{Thouless82, Nelson82} is,

$$
\Psi(\{\alpha\}) = \prod_{i<j}^N [f(z_i - z_j)]^q \prod_{k=1}^q f(Z - \alpha_k). \quad (3.77)
$$

where $Z = \sum_i^N z_i$ is the center-of-mass coordinate. The first term of Eq. (3.77) is the usual Vandermonde factor on a torus, while the second term places $q$ center-of-mass zeros at positions $\{\alpha_k\}$, $k=1,\ldots,q$. From now on we will enforce periodic boundary conditions by requiring that $\sum_k^q \alpha_k = 0 \mod L$. As already discussed in Sec. 3.2.3, the $f(z)$ is the “modified Weierstrass sigma” function times Gaussian: $f(z) = \sigma(z) \exp -\frac{1}{2N_\sigma} z z^*$. All model wave functions presented in this section are factorizable into a holomorphic function times a Gaussian factor $\prod_i \exp(-\frac{1}{2} z_i z_i^*)$.

Inserting additional $N_h$ fluxes in the $\nu = 1/q$ Laughlin wave function creates quasi-hole excitations. The wave function with $\{w\}$ representing the positions of the quasiholes is,

$$
\Psi(\{\alpha\}, \{w\}) = \prod_{i<j}^N f^q(z_i - z_j) \prod_{i,a}^{N_e-N_h} f(z_i - w_a) \prod_{k=1}^q f(Z + \frac{W}{q} - \alpha_k). \quad (3.78)
$$
In the following, we will use the Monte Carlo to calculate this Berry phase $\Phi$. We take the one-hole model wave function $N_h = 1$, and move it around a path $w_0 \to w_1 \to \ldots \to w_{n-1} \to w_0$. At each step, we compute the overlap between the wave functions with $w = w_n$ and $w_{n+1}$. To compute the Berry phase, we take the product of these overlaps:

$$\langle \psi(w_0) | \psi(w_1) \rangle \ldots \langle \psi(w_{n-1}) | \psi(w_n) \rangle = |D| e^{i\Phi}.$$ 

Since our numerics turns the continuous motion of the quasi-hole into a series of discrete steps, the amplitude $|D|$ will be smaller than one. The system has probability of $1 - |D|$ jumping to the excited state and scrambling the phase. Therefore it is important to keep the step length $|w_i - w_{i+1}|$ small so that $|D|$ is close to one.

The numerical results for Laughlin $q = 3$ and $q = 5$ states are represented in (Fig. 3.2). We see that our observed values are what we expect them to be.

Figure 3.2: $N_e = 50$, $\nu = 1/q$. Laughlin-hole state Berry phase ($\Phi$) v.s. area the loop enclosed ($A$). The dot and the square show Monte-Carlo data for $q = 3$ and $q = 5$ respectively. The Monte-Carlo error is bounded by the line width. This demonstrates that $\Phi = A/q$. The inset shows the overlap $|\langle \psi(0) | \psi(\Delta x) \rangle|$ v.s. $\Delta x$ which allows us to take step steplength to be $\Delta x = 0.02$ i.e. the quantum distance between consecutive steps is small.
One can also do braiding of holes, or even more exotic anyons in other topological states \cite{16,2}. Here we just use a Laughlin hole as a trivial example to illustrate the Berry phase calculation.

### 3.4.3 Structure factor and a “fast Lattice Monte Carlo” algorithm

Another application of the Monte Carlo technique is the (static) guiding center structure factor $S(q)$ which plays an important role in the FQH.

In the “single-mode approximation” first introduced by Feynman in superfluid helium-4 \cite{6} and then adopted by Girvin, MacDonald and Platzman in the FQH \cite{12,13}, the structure factor provides a variational upper bound of the neutral excitation gap. In particular, the $|q/l_B| \to 0$ behavior of $S(q)$ is closely related to the collective modes in the system, and gives a criterion for the system to be gapped or gapless at long wavelength. For example, in superfluid Helium-4, $S(q) \sim |q/l_B|^2$, corresponds to the gapless phonon mode, while in FQH $S(q) \sim |q/l_B|^4$ corresponds to the gapped “graviton” mode \cite{18}. For the Laughlin wave function, the forth- and sixth-order expansion coefficients of $S(q)$ are predicted in Ref. \cite{27}. The larger sizes accessible using our Monte Carlo method allow us to test these predictions.

Additionally, for the gapless CFL state, the peak in the structure factor can be used to identify the composite fermion Fermi surface, and identify its symmetry properties \cite{10}. We can observe this physics in our Monte Carlo data. Lastly, from the structure factor, we found a method to greatly improve the Monte Carlo efficiency.

The guiding center (static) structure factor by definition is the density-density correlation function,

$$S(q) \equiv \frac{1}{2N_{\phi}} \langle \psi^{GC}|\{\delta \rho(q), \delta \rho(-q)\}|\psi^{GC}\rangle.$$  

(3.79)
where $\delta \rho(q)$ is the fluctuation of density operator $\rho(q)$ relative to the background 
\[ \langle \rho(q) \rangle / N_\phi = 2\pi l_B^2 v \delta^2(q), \]

\begin{align*}
\rho(q) &= \sum_i e^{iq_i R_i}, \\
\delta \rho(q) &= \rho(q) - \langle \rho(q) \rangle.
\end{align*}

(3.80)

Note that both $\delta \rho(q)$ and $\rho(q)$ satisfy the GMP algebra. Several properties of $S(q)$ are worth mentioning [19]. First, the large-$|q| l_B$ asymptotic value is determined by the filling fraction $S(\infty) = \nu (1 + \xi \nu)$, where $\xi = -1$ if the underlying particles are fermions, $= 1$ if bosons. Second, $S(q)$ is self-dual under a Fourier transformation [19],

\[ S(q) - S(\infty) = \xi \int \frac{d^2 q'}{2\pi} e^{i q' \times q} l_B^2 (S(q') - S(\infty)). \]

(3.81)

Third, the coefficients of the small-$|q| l_B$ expansion

\[ S(|q|) = c_2 |q| l_B^2 + c_4 |q| l_B^4 + c_6 |q| l_B^6 + ..., \]

(3.82)

contains useful information. For a gapped system, $c_2 = 0$. For a gapless system, $S(|q|)$ goes to zero more slowly than $|q| l_B^4$. For a Laughlin $\nu = 1/q$ state, $c_2 = 0$, and predictions exist for $c_4$ and $c_6$ [27]:

\begin{align*}
c_4 &= \frac{\nu |s|}{4}, \\
c_6 &= \frac{\nu |s|}{8} (s - \frac{c - \nu}{12} \frac{1}{\nu s}).
\end{align*}

(3.83)

where $s = -\frac{1}{2} (q - 1)$ is the guiding center spin [18], $c$ is the central charge. Our Monte Carlo method allows us to test these predictions (Fig. 3.3). Another way to
write Eq. (3.79) is the following,

\[ S(q) = \frac{1}{N_\phi} \sum_{i,j} \langle \psi^G | e^{i q (R_i - R_j)} | \psi^G \rangle - \frac{1}{N_\phi} \langle \rho(q) \rangle \langle \rho(-q) \rangle. \]  

(3.84)

Writing \( S(q) \) in this way reveals a challenge when computing it with our Monte Carlo method, which computes expectation values relative to the real-space coordinates \( r \) and Schrödinger wave functions rather than the guiding center versions. What our Monte Carlo calculates is the “full structure factor” (per flux), defined as:

\[ S^{\text{full}}(q) = \frac{1}{N_\phi} \sum_{i,j} \langle \psi_0 | e^{i q (r_i - r_j)} | \psi_0 \rangle_{\text{Lat}} - \frac{1}{N_\phi} \langle \rho(q) \rangle \langle \rho(-q) \rangle. \]  

(3.85)

We can relate these two quantities by using the form factor to simplify \( S^{\text{full}}(q) \). This shows that \( S^{\text{full}}(q) \) is related to the guiding center structure factor \( S(q) \) via,

\[ S^{\text{full}}(q) - \nu = \left| [f_0(q)]_{N_\phi} \right|^2 \cdot [S(q) - \nu]. \]  

(3.86)

The \( \nu \) in the above equation comes from the terms in the sum where \( i = j \). Because of the Gaussian function \( f_0(q) = e^{-\frac{1}{4} q^2 l_B^2} \), the Monte Carlo error in \( S(q) \) is amplified greatly when \( |q l_B| \) is large. This limits us to see the \( S(q) \) within a window of small \( |q l_B| \). For Laughlin, \( |q_{\text{max}}| \approx 3 l_B^{-1} \) (Fig. 3.3).
Figure 3.3: The guiding center structure factor for \( N_e = 50 \) electrons in the Laughlin \( \nu = 1/3 \) state. The subfigure a) shows its plot together with error bars. The Gaussian function \( e^{-\frac{1}{4}q^2 l_B^2} \) limits us to see \( S(q) \) only within a window. In subfigure b), we check the long-wavelength expansion \((c_4, c_6)\) in (3.82) by comparing it to the Monte Carlo data, where \( c_4, c_6 \) is given by (3.83) and all other \( c_i = 0 \). It can be seen that the long-wavelength behavior of \( S(q) \) is correctly described by (3.83).

For the CFL states, the shape of the Fermi surface can be seen from the peak of structure factors. And the radius of the latter should be twice as large as that of the former. Here we plot the structure factor for model wave functions with different dipole-moment configurations for \( N_e = 37 \) electrons (Fig. 3.4).
In Sec. 3.3 we showed how to calculate any two-body expectation value $O(\mathbf{x}_i - \mathbf{x}_j)$, in any LL, and to demonstrate our method we computed the Coulomb energy of the Laughlin state in the first two LLs. However we found that for higher LLs ($n>1$), our method was subject to large Monte Carlo errors. In this section we will use our insights about the structure factor to understand and ameliorate these errors. The algorithm discussed in this section applies to other translational invariant two body interactions, like pair amplitude. We will use the notation of Sec. 3.3.
The first step in this process is to find out the effective potential acting on the guiding centers,

\[
\langle \hat{O} \rangle = \frac{1}{2\pi N_\phi} \sum_q \sum_{i<j} O^{GC}(q) \langle e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} \rangle = \frac{1}{4\pi N_\phi} \sum_q O^{GC}(q) \cdot [S(q) - \nu]. \tag{3.87}
\]

In the problem of high LLs with index \( n \), the Coulomb energy is,

\[
O^{GC}(q) = [O(q) f_n^2(q)]_c, \\
O(q) = \frac{2\pi}{|ql_B|}, \quad |ql_B| \neq 0. \tag{3.88}
\]

where the symbol \([\ldots]^{GC}\) is defined in Eq. (3.64), not to be confused with \([\ldots]_{N_\phi}\) defined in Eq. (3.66). Eq. (3.87) tells us that, at least in principle, the guiding center structure factor allows us to calculate any two-point expectation value. However, we found in the previous section that \( S(q) \) determined from our Monte Carlo procedure has very large errors as at large \( |ql_B| \). The reason these errors do not completely ruin our calculation is that Eq. (3.87) also contains a form factor \( f_n(q) \), which decays to 0 exponentially at large \( |ql_B| \), thus suppressing the errors. Unfortunately, the decay of \( f_n(q) \) gets weaker as the LL index \( n \) is increased. This is why we had difficulty calculating Coulomb energies for \( n > 2 \) in Sec. 3.4.1.

We can conclude that the large \( |ql_B| \) modes contribute a tiny amount to the mean value we want, and merely introduce large Monte Carlo error. Fortunately, Eq. (3.87) allows us to see a way to efficiently and accurately approximate \( \langle \hat{O} \rangle \) since \( S(\infty) = \nu(1 - \nu) \) when \( |ql_B| \to \infty \) \cite{37, 19}. We thus do the following trick: introduce a cutoff \( Q \), and separate Eq. (3.87) into short-ranged (\( |q| > Q \)) and long-ranged (\( |q| < Q \)) parts. Only for the long-ranged part, we calculate by Monte Carlo by using...
the lattice representation Eq. (3.65). For the short-ranged part, we simply replace $S(q)$ with $S(\infty)$ and calculate directly.

Assuming that $S(q)$ is saturated when $|q| > Q$ introduces systematic error $\delta E_S$. Although we do not know the short wavelength oscillation behavior of $S(q)$, we are still able to give an upper bound of $|\delta E_S|$, which could be calculated analytically. Note that $S(q)$ is positive and is bounded from above by its maximum value $S_{\text{max}}$, the oscillation must be less than $\min\{S_{\text{max}} - S(\infty), S(\infty)\}$. Hence, an upper bound of the systematic error is given by the following,

$$|\delta E_S| < \frac{1}{4\pi N_{\phi}} \sum_{|q| > Q} |O^{G\phi}(q)| \cdot \delta S$$

$$\delta S = \min\{S_{\text{max}} - S(\infty), S(\infty)\}.$$

From the plot of the structure factor Fig (3.3) and Fig (3.4), we empirically set $S_{\text{max}} \approx 0.5$ for the Laughlin $\nu = 1/3$ state, $S_{\text{max}} \approx 0.8$ for CFL $\nu = 1/2$ state.

This systematic error $\delta E_S$ must be included, together with the Monte Carlo error $\delta E_M$, into the total uncertainty $\delta E_{\text{tot}} = \sqrt{\delta E_M^2 + \delta E_S^2}$. Increasing the cutoff $Ql_B$ decreases $\delta E_S$ but makes $\delta E_M$ larger. The best value of $Ql_B$ is taken as the one for which $\delta E_M$ and $\delta E_S$ are of the same order.

Table 3.2 uses this approach to recalculate the Coulomb energies which were originally calculated in Table 3.1. We can see that by cutting-off and approximating the large $q$ contribution, we can significantly decrease the statistical error, and obtain improved estimates for the energy.

The self duality relation in Eq. (3.81) implies the $S(q)$ can be expanded in terms of Laguerre polynomials (multiplied by Gaussians), which form a complete basis of polynomials that are self-dual under Fourier transformations. The expansion coefficients in this basis are known as “pair amplitudes”. Such pair amplitudes appear in
Table 3.2: By separating the energy calculation into short- and long-ranged parts and approximating the long-ranged part, we can dramatically reduce our statistical error. In this table we compute the Coulomb energy of the Laughlin wave function in $n = 2$. We used the same number of Monte Carlo steps as we did in Table 3.1 but find that our statistical error is reduced by up to two orders of magnitude even at these relatively small sizes. We also did test on larger systems, e.g. $N_e = 11, 12$.

Before defining pair amplitudes on the torus, let us first look at the infinite plane geometry, where the pair amplitude is better understood. The infinite plane has rotational symmetry, and angular momentum is well defined. A projector that projects a two-particle pair into a given state with relative angular momentum $m$ is,

$$P_{ij}^m = 2 \int \frac{d^2 \mathbf{q}^2}{2\pi} L_m(2l_B^2) e^{-\frac{1}{4}q^2 l_B^2} e^{iq \cdot (\mathbf{R}_i - \mathbf{R}_j)}.$$  

(3.90)

Here the $P_{ij}^m$ are orthogonal projectors that satisfy,

$$P_{ij}^m P_{ij}^m = P_{ij}^m, \quad (3.91)$$

$$P_{ij}^m P_{ij}^n = 0, \quad \text{if } m \neq n. \quad (3.92)$$

The $m^{th}$ pair amplitude $\xi_m$ is the probability of finding a pair of particles with relative angular momentum $m$,

$$\xi_m \equiv \langle \sum_{i < j} P_{ij}^m \rangle.$$  

(3.93)
Table 3.3: The pair amplitude calculated for $N_e = 6$ particles in Laughlin $\nu = 1/3$ state. The $\delta E_M$ and $\delta E_S$ are Monte Carlo and systematic errors, respectively. The total error $\delta E_{\text{tot}} = \sqrt{\delta E_M^2 + \delta E_S^2}$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>ED</th>
<th>MC Value</th>
<th>$Ql_B$</th>
<th>$\delta E_M$</th>
<th>$\delta E_S$</th>
<th>$\delta E_{\text{tot}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1$</td>
<td>0.</td>
<td>1e-3</td>
<td>5.00</td>
<td>1e-3</td>
<td>5e-4</td>
<td>1e-3</td>
</tr>
<tr>
<td>$\xi_3$</td>
<td>5.928056</td>
<td>5.84</td>
<td>5.00</td>
<td>0.08</td>
<td>0.04</td>
<td>0.09</td>
</tr>
<tr>
<td>$\xi_5$</td>
<td>4.441078</td>
<td>3.75</td>
<td>5.00</td>
<td>0.8</td>
<td>0.7</td>
<td>1.08</td>
</tr>
</tbody>
</table>

On the torus, $P^{ij}_m$ is defined similarly as in Eq. (3.90), but with the integral over a continuum of momenta replaced by a discrete sum over all points in the reciprocal space. Since the torus does not have continuous rotation symmetry, the $m$ does not have the meaning of “relative angular momentum” any more, and $P^{ij}_m$ are no longer orthogonal: (3.92) does not hold, while (3.91) is modified

$$P^{ij}_m P^{ij}_m = C_m P^{ij}_m,$$

(3.94)

where $C_m$ is a number that is slightly larger than one. The fact that (3.94) does not introduce any projectors with $m \neq n$ ensures that the torus Laughlin wave function is still the exact ground state of the pseudo-potential Hamiltonian. Although the torus has only discrete rotation symmetry, the continuous rotation symmetry is restored and the $P^{ij}_m$ become orthogonal ($C_m \rightarrow 1$) in the limit $N_\phi \rightarrow \infty$.

The calculation of pair amplitudes is similar in spirit to that of the Coulomb energy in high-index LLs. In the problem of calculating the pair amplitude, we simply replace Eq. (3.88) with Eq. (3.95). The error analysis follows the same algorithm as discussed in the last section. This time,

$$O^{GC}(q) = 4\pi L_m(q^2 l_B^2) e^{-\frac{1}{2} q^2 l_B^2}$$

(3.95)

In Table 3.3 we showed several orders of calculated pair amplitudes for Laughlin $\nu = 1/3$ state for $N_e = 6$ particles.
3.4.4 Particle hole symmetry from Monte Carlo

It is interesting to ask how close wave functions such as Eq. (3.57) are to having particle-hole symmetry. In Ref. [9], we have addressed this question by numerically second quantizing these wave functions, and then implementing particle-hole symmetry in the second quantized basis by exchanging the filled and empty orbitals. Since we have now developed a tool for rapid calculations in the Schrödinger representation, it is natural to ask whether we can evaluate particle-hole symmetry in this representation.

According to Ref. [11], if we have some wave function $\Psi_1$, we can compute its particle-hole conjugate as follows:

$$
\Psi^{PH}_1(\tilde{z}_j) = \int N_e \prod_{i=1}^{N_c} dz_i \Psi_1(z_i) \Psi^*_{LL}(z_i, \tilde{z}_j) \tag{3.96}
$$

where $\Psi_{LL}$ is the wave function for a filled LL. Using this definition of particle-hole conjugation we can compute the quantity $$\langle \Psi^{\beta}_{CFL}(\{d\})|PH\Psi^{\beta'}_{CFL}(-\{d\}) \rangle$$, which is the overlap between the CFL state and its particle-hole conjugate. Here $\beta$ indicates which center-of-mass sector the wave function is in, while $\{d\}$ represents the set of composite fermion dipole-moments of the wave function. Particle-hole symmetry on its own changes the momentum of a wave function, so when we write $PH$ we really mean particle-hole symmetry combined with a rotation by $\pi$, an operation which preserves the symmetry [9]. The $\pi$-rotation reverses the center-of-mass sector (so we will need $\beta \neq \beta'$), and also takes $d \rightarrow -d$. Equivalently to reversing the $d$’s, we can instead reverse all the coordinates $z$, which is what we will do from now on. Using
Eq. (3.96), we can write the particle-hole overlap as follows:

\[ \langle \Psi^\beta_{CFL}(\{d\})|PH|\Psi'^\beta_{CFL}(-\{d\}) \rangle = \int \prod_{j=N_e+1}^{N_\phi} d\bar{z}_j \Psi^\beta_{CFL}(\{\bar{z}_j\}) \times \int \prod_{i=1}^{N_e} dz_i \Psi'^\beta'_{CFL}(-\{z_i\})\Psi^*_LL(\{z_i\},\{\bar{z}_j\}) = \langle \psi_1(x)|\psi_2(x) \rangle \]

\[ \psi_1(x) = \Psi^\beta_{CFL}(\{z_j\})\Psi'^\beta_{CFL}(-\{z_i\}) \]  \hspace{1cm} (3.98)

\[ \psi_2(x) = \Psi^*_{LL}(\{z_i\},\{\bar{z}_j\}) \]  \hspace{1cm} (3.99)

In the above equation we have stopped explicitly writing the variational parameters \( \{d\} \), and as in Sec. (3.3) we use \( x \) as a shorthand for all the coordinates \( \{z_i\},\{\bar{z}_j\} \).

In Sec. (3.3) [specifically Eq. (3.76)], we were calculating the overlap \( \langle \psi_1|\psi_2 \rangle \) and we manipulated the wave functions in such a way that \( |\psi_1|^2 \) could be used as a Metropolis weight. However, we could in principle use any real, non-negative function as a weight. This inspires a more general version of Eq. (3.76):

\[ \frac{\langle \psi_1|\psi_2 \rangle}{\sqrt{\langle \psi_1|\psi_1 \rangle \langle \psi_2|\psi_2 \rangle}} = \frac{\sum' O_{12}(x)p(x)}{\sqrt{\sum' O_{11}(x)p(x)\sum' O_{22}(x)p(x)}}. \]  \hspace{1cm} (3.100)

where \( O_{ij}(x) \equiv \frac{\psi^*_i(x)\psi_j(x)}{p(x)} \). Here \( p(x) \) is the statistical weight, so it must be real and non-negative. \( |\psi_1|^2 \) is a good choice for \( p(x) \) when \( \psi_1 \) and \( \psi_2 \) are very similar, because it means that \( O(x) \) will be of order one, and this is necessary for efficient importance sampling. If \( O(x) \) can vary widely, then we will no longer be doing importance sampling (i.e. configurations where \( O(x)p(x) \) is large will not be sampled frequently) and the algorithm will be inefficient. If \( p(x) = 0 \) when \( \psi_1(x) \) [or \( \psi_2(x) \)] is non-zero the Monte Carlo will give incorrect results since \( O_{11}(x) \) [or \( O_{22}(x) \)] is infinite.
The $\psi_1$ and $\psi_2$ defined in Eqs. (3.98-3.99) are not very similar, and in fact one can have zeros where the other one is large. A simple way to see this is that whenever $z_i = \tilde{z}_j$ for any $i,j$ in Eq. (3.97), $\psi_2$ will vanish but $\psi_1$ does not have to. Therefore simply using $|\psi_1|^2$ or $|\psi_2|^2$ for $p(x)$ will not work. In this work we make the following choice for $p(x)$:

$$p(x) = (\alpha |\psi_1(x)| + |\psi_2(x)|)^2. \quad (3.101)$$

The virtue of this choice is that $p(x)$ will be large when either $\psi_1$ or $\psi_2$ is large. The CFL wave functions are not normalized so the parameter $\alpha$ is included to make the two terms in the sum have approximately equal size. Using a fixed value of $\alpha$ (e.g. $\alpha = 1$) will give correct results but tuning $\alpha$ for a given system size can dramatically improve the performance of the Monte Carlo. We find for the wave functions used in this paper that $|\psi_1(x)|^2$ is roughly two orders larger than that of $p(x)$. Note that other choices of $p(x)$ are possible so long as it is large whenever either wave function is large, it may be possible to further improve performance with a better choice of $p(x)$.

A final obstacle to computing the particle-hole overlap is that the wave functions produced by Eq. (3.96) are not normalized, even if the wave functions on the right-hand side of that equation are normalized. In order to obtain a normalized wave function (and therefore a sensible overlap) we need to multiply Eqs. (3.96) and (3.97) by a normalization constant $\sqrt{C}$, where

$$C = \begin{pmatrix} N_\Phi \\ N_e \end{pmatrix}. \quad (3.102)$$

The value of this constant can be explained by thinking about the overlap we are calculating as an overlap of the wave functions $\psi_1, \psi_2$ defined in Eqs. (3.98) and (3.99). If the two CFL wave functions were particle-hole symmetric, this overlap
would be 1. But wave function $\psi_1$ is completely antisymmetric under interchanging coordinates $z_i$ (which appears in one CFL wave function) and $\tilde{z}_j$ (which appears in the other wave function). In order for the overlap to be 1, $\psi_2$ must therefore also have this symmetry, but it clearly does not. Therefore to get sensible results we must antisymmetrize Eq. (3.99). Each term in such an antisymmetrization will be exactly the same once all positions are summed over, but in order to stay normalized we must divide by the square root of the number of terms in the antisymmetrization, which is exactly $\sqrt{C}$.

This normalization constant means both the values produced by numerically computing Eq. (3.100) and their statistical errors must be multiplied by $\sqrt{C}$. Therefore to keep the statistical errors constant in system size, the number of Monte Carlo steps requires scales as $\propto C$. This is the same algorithmic complexity as numerically second quantizing the wave function, as in [9]. Therefore there is no benefit to using our Monte Carlo method to compute particle-hole overlaps. Nevertheless the algorithm does work, as can be seen in Fig. 3.5 where we show the particle-hole overlaps for a few values of $N_e$, and compare them to the results of numerical second quantization. The $N_e = 9$ data in Fig. 3.5 took 600 CPU hours, while doing the exact second quantizing algorithm takes around ten minutes. Therefore though using Monte Carlo does give correct results, it is not a practical method to evaluate the particle-hole symmetry of model wave functions.
Figure 3.5: Overlaps between a wave function and its PH conjugate. The red points from come from doing an exact second quantization of the model wave function as in [9], while the blue comes from a Monte Carlo calculation. For each $N_e$, the configuration of $d$’s with the largest overlap was used.
Chapter 4

Many Body Berry Phase in a half filled Landau level

4.1 Definition of many body Berry phase

In this and the following chapters, we will perform numerical computation for the Berry phase acquired by the composite fermions when it is transported around the Fermi surface. The first thing we will do is to define what we mean by the composite fermion’s Berry phase. Let us first recall what is the Berry phase in Bloch bands.

The Hamiltonian describing the non-interacting electrons in a periodic crystal is $H(\hat{p})$. Its eigenstates $\Psi_{n,k}(r)$ satisfies the Bloch theorem,

$$\Psi_{n,k}(r + a) = e^{ik\cdot a} \Psi_{n,k}(r). \quad (4.1)$$

where $a$ is the lattice constant of the crystal, $n$ is the band index. A periodic wave-function is defined via,

$$u_{n,k}(r) \equiv e^{-ik\cdot r} \Psi_{n,k}(r). \quad (4.2)$$
The periodic states \( u_{n,k}(r) \) as parameterized by base manifold \( k \) form principle fiber bundle. To compare states with different \( k \), a notion of parallel transport must be introduced. The vector connection is defined as,

\[
A^a(k) \equiv -i \langle u_{n,k}| \partial_{k_a} u_{n,k} \rangle = -i \int d^d r u^*_n(r) (\partial_{k_a} u_n)(r).
\]

(4.3)

and curvature defined as,

\[
F^{ab}(k) \equiv \partial_a A^b(k) - \partial_b A^a(k).
\]

(4.4)

The Berry phase factor is the phase acquired when a quantum state \(|u_{n,k}\rangle\) is adiabatically evolved around a closed path \( \Gamma \) in parameter space \( \{k\} \),

\[
|D| e^{i \Phi_\Gamma} \equiv \int_{\Gamma} A^a(k) k_a.
\]

(4.5)

By Stocks theorem, when \( \Gamma \) is closed, the Berry phase can be represented as a surface integration of curvature and thus is gauge invariant,

\[
\int_{\Gamma} A^a(k) k_a = \int\int_{S_\Gamma} F^{ab} d k_a \wedge k_b.
\]

(4.6)

When the parameter space is discretized, which is the case when the system size is finite, a discretized version of Berry phase is defined,

\[
|D| e^{i \Phi_\Gamma} = \prod_{i \in \Gamma} \langle u_{k_i} | u_{k_{i+1}} \rangle.
\]

(4.7)
by taking the continuum limit, the discretized Berry phase agrees with the integration definition. Now, we reinterpret the overlap $\langle u_{k_1} | u_{k_2} \rangle$ as the matrix element,

$$\langle u_{k_1} | u_{k_2} \rangle = \langle \psi_{k_1} | \rho(k_1 - k_2) | \psi_{k_2} \rangle$$

(4.8)

where $\rho(q)$ is the Fourier-transformed density operator,

$$\rho(q) = \exp(iq \cdot r)$$

(4.9)

This helps us to generalize the momentum space Berry phase to a many-body system in the following way:

$$e^{i\tilde{\Phi}_\Gamma} = \prod_{\text{path}} \langle \Psi(K_i) | \rho(q) | \Psi(K_{i+1}) \rangle,$$

(4.10)

$$\rho(q) = \sum_{i=1}^{N} e^{iq \cdot R_i}.$$

(4.11)

where instead of single body ingredients, many body wavefunction $|\Psi(K)\rangle$, many body momentum $K$ and projected density operator $\rho(q)$ are used. Their discussions can be found in Chapter. \[3\] Note that the density operator are non-commutative now, and satisfies the celebrated Girvin-Macdonald-Plazman algebra,

$$[\rho(q_1), \rho(q_2)] = 2i \sin \frac{q_1 \times q_2}{2} \rho(q_1 + q_2).$$

(4.12)
4.2 Property of $\nu = 1/2$ CFL: symmetry, Berry phase and the “singularity”

In this section, we will discuss in more detail about the particle hole symmetry, and numerically study the model wavefunction for the half filled Landau level Fermi liquid state.

For reader’s convenience, we copy the wavefunction from Eq. (3.56) as follows.

$$\Psi_{CFL} = e^{\frac{1}{2N_e}[(\alpha-d)^*Z-(\alpha-d)Z^*]} det \tilde{M}_{ij}^{N_e} \prod_{i<j} f^{m-2}(z_i-z_j) \prod_{k=1}^{m} f(Z-\alpha_k).$$  \hspace{2cm} (4.13)

$$\tilde{M}_{ij} = e^{\frac{1}{2m}(z_id_j^* - z_j^*d_i)} \prod_{k \neq i}^{N_e} f(z_i-z_k-d_j+d).$$  \hspace{2cm} (4.14)

$$\sum_{i=1}^{N} d_i = \sum_{k=1}^{m} \alpha_k.$$  \hspace{2cm} (4.15)

The Coulomb interaction ground state is known to favor the compressible Fermi liquid state in the lowest Landau level when the Landau level filling is $\nu = \frac{1}{2}$. Earliest work can be found in [41, 42]. To apply Eq. (4.10) we need to show that the states produced by ED can indeed be identified with the analogs of Slater determinants of composite fermions, and assign a configuration $\{k\}$ to each ED state in the sequence. We accomplish this by comparing the ED states to a model wavefunction [8, 49, 7].

There are several ways to numerically investigate the particle hole symmetry of a given model wavefunction. One way is to work in the first quantized formalism. According to [11], given a wavefunction $\Psi(\{z\})$, its particle hole conjugate can be obtained by projecting into the integer quantum Hall wavefunction $\Psi^{IQH}(\{z\}, \{w\})$,

$$\Psi^{PH}(\{z\}) \sim \int (\Psi^{IQH}(\{z\}, \{w\}))^* \Psi(\{w\})$$  \hspace{2cm} (4.16)
In the following section, we will discuss the possibility of computing the particle hole symmetry of wavefunction in this fashion by Monte Carlo method. A second method of studying particle hole property is by projecting a first quantized model wavefunction into second quantized basis, and then interchange the filled and empty states,

$$\Psi(\{z\}) = \sum_{\alpha} C_\alpha n_\alpha. \quad (4.17)$$

where $n_\alpha$ denotes a basis of the many body Hilbert space. The particle hole conjugate is,

$$\Psi^{PH}(\{z\}) = \sum_{\alpha} C_\alpha^* n^{PH}_\alpha. \quad (4.18)$$

For example, the particle-hole conjugate of $n_\alpha = |1000111\rangle$ state is $n^{PH}_\alpha = |0111000\rangle$. Note that particle hole transformations maps many body momentum $K$ to $-K$. Therefore the combination of particle hole and inversion leaves the total momentum unchanged. So we define the particle hole symmetry as the following quantity,

$$\langle \Psi | \mathcal{P}\mathcal{H} \ R_\pi | \Psi \rangle. \quad (4.19)$$

and expect that it is 1 for $\mathcal{P}\mathcal{H}$ symmetric states. In Fig. (4.1) we give the values for this quantity for various $\{d\}$ configurations, for both exact and model states. We see that the model states in which the composite fermions are clustered to form a Fermi sea are indeed nearly particle-hole symmetric.
To conclude this section, we discussed particle hole symmetry and overlap of exact state for the model wavefunction. The motivated model wavefunction is motivated from Halperin-Lee-Read flux attachment idea, and has been numerically found to be very close to the exact Coulomb ground state which is known to be a Fermi liquid state, and has very high particle hole symmetry, as long as the dipoles form a clustered Fermi sea.

4.3 $\pi$ Berry curvature singularity at $\nu = \frac{1}{2}$

We now measure the Berry phase. For a given ED state, we use the overlap with the model wavefunction to determine whether that state can be well-described in terms of composite fermions, and also which composite fermions are filled for that state. We prepare a sequence states which consist of a filled “Fermi surface” plus or minus one composite fermion. We can compute Eq. (4.10) for a sequence of states where this “extra” composite fermion moves around some closed path.
One subtlety when computing the Berry phase around the center of the Fermi sea is that the center of the Fermi sea is not gauge invariant: we can translate all the composite fermions by one lattice spacing without changing any physics. Therefore the notion of transporting a composite fermion “around the center of the Fermi sea” is only well-defined when a compact Fermi sea is present. Furthermore in order for a path around the Fermi surface to be defined we also have to exclude non-trivial paths around the torus. Therefore in what follows we must restrict ourself to the case where a composite Fermi sea exists, and the composite fermion we are moving is not too far from this Fermi sea (so that it cannot wind around the torus). The results of the previous section show that such states are the only states which are well described by a composite fermion picture.

Another issue when computing the “many-body Berry-like phase” defined in Eq. (4.10) is that we expect that the phase should behave smoothly in the thermodynamic limit as the individual steps in the path become infinitesimal and the the path in momentum space becomes smooth and continuous. However we did observe an additional geometric phase factor associated with the the projected GMP density operator $\rho(q)$ in each segment of the path. Fortunately we can use the symmetries of the problem to determine what this phase is, and we summarize these results in Fig. (4.2). First consider the combination of particle hole and inversion symmetry discussed in Eq. (4.19). This symmetry takes $\rho(q) \rightarrow -\rho(q)$ and $i \rightarrow -i$. A simple overlap behaves like $\langle \Psi_1 | \Psi_2 \rangle \rightarrow \langle \Psi_1 | \Psi_2 \rangle^*$ under this symmetry, while the matrix element in Eq. (4.10) behaves like $\langle \Psi_1 | \rho | \Psi_2 \rangle \rightarrow -\langle \Psi_1 | \rho | \Psi_2 \rangle^*$. From this we deduce that the extra phase contributed by the $\rho(q)$ is purely imaginary.

Empirically, we observed that the expression for the phase factor defined by Eq. (4.10) appears to be
\[ e^{i\tilde{\Phi} \Gamma} = (i)^{N_+ - N_-} e^{i\Phi \Gamma}. \]
Figure 4.2: The density operator in Eq. (4.10) adds an additional phase to our Berry phase calculation, with is imaginary due to particle-hole symmetry. An anti-unitary reflection symmetry (present in the thermodynamic limit, as well as in the square lattice considered here) introduces a relative \(-1\) between clockwise and counterclockwise hopping, while also forbidding hopping in a direction normal to the Fermi surface.

where \(N_+\) are the number of discrete steps around the Fermi surface that are in the positive (anticlockwise) sense and \(N_-\) is the number in the negative (clockwise) sense, and \(e^{i\Phi}\) is the “true” Berry phase factor which is

\[
e^{i\Phi} = (-1)^W. \tag{4.21}
\]

if the path stays close to the Fermi surface with winding number \(W\).

Another constraint on the phase associated with \(\rho(q)\) can be found from an anti-unitary reflection symmetry about the (e.g.) \(x\)-axis, which takes \(i \rightarrow -i\) and \(k_x \rightarrow -k_x\). Such a symmetry exists for a torus with square or hexagonal boundary conditions, and it also exists in the thermodynamic limit. For a given composite fermion momentum \(\mathbf{k}\), we can define an angle \(\theta_k\) relative to the center of the Fermi sea. Such a definition requires that a compact Fermi sea exists, but we have seen in the previous section that this is true for composite Fermi liquid states. Given a composite fermion momentum \(\mathbf{k}_i\) and a momentum change \(\mathbf{q}\), we can then define \(d\theta_q \equiv \theta_{k+q} - \theta_k\). The reflection symmetry takes \(d\theta_q \rightarrow -d\theta_q\). It takes \(\langle \Psi_1 | \Psi_2 \rangle \rightarrow \langle \Psi_1 | \Psi_2 \rangle^*\) and (since we already established that \(\rho(q)\) contributes a purely imaginary phase) \(\langle \Psi_1 | \rho(d\theta_q) | \Psi_2 \rangle \rightarrow -\langle \Psi_1 | \rho(-d\theta_q) | \Psi_2 \rangle^*\). From this we know that the phase contributed by the \(\rho(d\theta_q)\) changes sign when the sign of \(d\theta_q\) changes sign.
Putting reflection and particle-hole symmetry together gives precisely the relation in Eq. (4.20). Note that for odd \(N_{\text{steps}}\) one can change the sign of the Berry phase by going around the path \(\Gamma\) in the opposite direction, to avoid dealing with this ambiguity we restrict to paths with even \(N_{\text{steps}}\). Also note that steps with \(d\theta_q = 0\) (perpendicular to the Fermi surface, see Fig. 4.2) are forbidden by the reflection symmetry.

Figure 4.3: Berry phases observed for a variety of paths around the Fermi surface. The solid symbols represent the locations of composite fermions, and for each step on the path we remove a composite fermion at the location of the empty symbols (i.e. we are moving a composite hole around the Fermi surface). There are a number of effects which arise from the insertion of a density operator in Eq. (4.10). These differences lead to additional phases summarized in Eq. (4.20). We compute these Berry phase \(\tilde{\Phi}_{\text{exact}}\) for the exact ED states, and \(\tilde{\Phi}_{\text{model}}\) for the model wavefunction of Eq. (3.56). In both cases the results, combined with Eq. (4.20), indicate a Berry phase of \(\Phi = \pi\) when the center of the Fermi surface is encircled.

The results of measuring the Berry phase can be seen in Fig. 4.3, where we show the \(\Phi_{\Gamma}\) extracted from Eq. (4.10) for a variety of system sizes, and paths taken by the composite fermion. In addition to computing these phases using exact wavefunction obtained from diagonalization of the Hamiltonian (\(\tilde{\Phi}_{\text{exact}}\)), we can compute them purely from the model wavefunction (\(\tilde{\Phi}_{\text{model}}\)), providing another way of estimating how close to exact the model wavefunction is. By comparing Figs. 4.3(a-c) we observe the sign structure predicted from the above symmetry analysis. Using Eq. (4.20) we find that the true Berry phase \(\Phi_{\Gamma} = \pi\) for paths encircling the center.
of the Fermi surface. Note also that our Berry phase is always 0 or \( \pi \), there is no component related to the area enclosed by the path, consistent with the composite fermions seeing no external field.

The Monte Carlo calculation enables us to look at the Berry phase at much larger sizes, up to \( N_e = 69 \), and lets us check the Berry phase in a more convincing way. The following (Fig. 4.4) is done for \( N_e = 13 \), and (Fig. 4.5) is for \( N_e = 69 \). The results agree with Eq. (4.21), confirming that a \( Z_2 \) phase is indeed obtained when composite fermions encircle the origin. Note that the model wave function Eq. (3.57) is not exactly particle-hole symmetric, therefore the observed Berry phase obtained using it is not exact, but is close to \( Z_2 \) phase.
Figure 4.4: $N_e = 13$ CFL Berry phase. Cross mark “x” represents the “composite fermion” we are moving. This is a consistency check with the same calculation done in Ref. [9] (but using a different numerical approach). This data can be interpreted through Eq. (4.21), which shows that in addition to a $\mathbb{Z}_2$ piece there is a piece depending on the direction of motion around the Fermi surface. When this is accounted for we find a residual “−1” from the $\mathbb{Z}_2$ part whenever the composite fermion encloses the Fermi sea.
To conclude this section, we found the many body Berry phase for half filled Fermi liquid state is,

$$e^{i\Phi} = (i)^{N_+ - N_-} e^{i\Phi_{\Gamma}}$$  \hspace{1cm} (4.22)

where $N_+$ and $N_-$ are anti-clock wised and clock wised path defined relative to the Fermi sea. The $\Phi_{\Gamma}$ is $\pi$ as long as path $\Gamma$ encloses the Fermi sea center while is 0 is otherwise. This numerical finding is consistent with the prediction from Son-Dirac theory.
To conclude this chapter, we defined the many body Berry phase as

$$e^{i\Phi_{\Gamma}} = \prod_{\text{path}} \langle \Psi(K_i)|\rho(q)|\Psi(K_{i+1}) \rangle. \tag{4.23}$$

where

$$\rho(q) = \sum_{i=1}^{N} e^{iqR_i}. \tag{4.24}$$

is the projected density operator. From exact diagonalization and from model wavefunctions, we performed the numerical computation of the Berry phase acquired by transporting the composite fermion around the Fermi surface and found that the Berry phase has the following structure,

$$e^{i\Phi_{\Gamma}} = (i)^{N_+ - N_-} e^{i\Phi_{\Gamma}} \tag{4.25}$$

where $N_+$ and $N_-$ are anti-clockwised and clockwised paths relative to the Fermi sea center. The $\Phi_{\Gamma}$ is a $Z_2$ phase: it is $\pi$ for all paths enclosing the Fermi sea center and 0 if otherwise. The $Z_2$ phase indicates that the Berry phase is a Fermi sea center singularity, and is a supporting evidence for Son-Dirac theory. Although we don’t have a full understanding of the path dependent phase, we argue it origins from the non-commutativity of the projected density which is intrinsic to our definition of the many-body Berry phase. In the following chapter, we consider the Berry phase distribution on $\nu = \frac{1}{4}$ model wavefunction, and will conjecture a Dirac type effective theory for all filling fractions that composite Fermi liquids can occur.
Chapter 5

Berry Phase and effective theories at generic $\nu$

5.1 From half filling to generic filling

Son’s half filled Dirac fermion theory predicts a $\pi$ Berry phase, which in fact is a $\pi$ Berry curvature singularity located at Fermi sea center. In Dirac fermion picture, this origins from the spin orbit locking and the spin half nature of Dirac fermion. In fact, the Fermi sea Berry phase $\Phi_{FS}$ and Hall conductivity $\sigma_H$ is predicted to be more general [17, 45]:

$$\sigma_H^{CF} = -\sigma_H = \frac{e^2}{h} \frac{\Phi_{FS}}{2\pi}, \quad \Phi_{FS} = -2\pi \nu. \quad (5.1)$$

The motivation to study Berry phase at generic filling is precisely due to the lack of symmetry. We will ask, besides symmetry, can any other quantities tell you some useful information about the composite Fermi liquids. For example, whether the Fermi sea is a “Landau Fermi liquid” or “Dirac Fermi liquid”. Fortunately, despite of the lack of symmetry at generic filling $\nu \neq \frac{1}{2}$, Berry phase remains a well defined quantity. It is thus natural to ask what is the Berry curvature at $\nu \neq 1/2$, and whether
or not it can be described by alternative low energy theories. See FIG. (5.1) for an illustration.

![Figure 5.1: A half filled composite Fermi liquid state is particle hole symmetric and consequently all Berry phases are pushed into Fermi sea center as a Berry curvature singularity. At generic filling fraction, while symmetry is absent, Berry phase might remain as an important characterization of the composite Fermi liquid phase.]

In principle, CFLs can occur as long as the HLR flux attached particles are fermions; whether or not they occur depends on the details of interaction. When the underlying physical particles are fermions, the filling fractions of CFLs can be grouped into two classes: \( \nu = 1/2m \) if Fermi seas are formed by composite fermions [we denote them as fCFLs], and \( \nu = 1 - 1/2m \) if by composite holes [anti-fCFLs]. The filling fraction for fCFLs can be

\[
\nu_{fCFL} = \frac{1}{2m}
\]

(5.2)

and the fillings for the anti-fCFLs is

\[
\nu_{anti-fCFL} = 1 - \frac{1}{2m}
\]

(5.3)

throughout the following sections, we use \( m \) as a positive integer.

In this chapter, we numerically study the Berry curvature of \( \nu = 1/4 \) model wave function. The numerical results seems very interesting,
• Fermi sea contains $-2\pi\nu$ Berry phase in agreement with Eq. (5.1).

• The $-2\pi\nu$ Fermi sea Berry phase consists of a $-\pi$ peak located at Fermi sea center, and $-2\pi(\nu - \frac{1}{2})$ phase uniformly distributed over Fermi sea.

The presence of $\pi$ Berry curvature at Fermi sea center even at $\nu \neq \frac{1}{2}$ strongly suggests that a Dirac type low energy effective theory is possible as a description for fCFLs and anti-fCFLs at generic filling fractions. Motived by the Berry curvature plot, we proposed an effective theory for fCFLs and anti-fCFLs at all filling fractions that they can occur.

\[
\mathcal{L} = i\bar{\psi}\gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{2m} \frac{1}{2\pi} adA - \eta \left( \frac{1}{2} - \frac{1}{2m} \right) \frac{1}{4\pi} ada + \left( \frac{1}{2} - \frac{\eta m - 1}{m} \right) \frac{1}{4\pi} AdA. \tag{5.4}
\]

Figure 5.2: Berry curvature distribution [right] obtained from $\Psi^{1/4}_{n=1}$ model wave function by a linear regression on a Fermi sea [left] consisting of $N=37$ dipoles. The red dashed line represents the path, which can be interpreted as Fermi sea boundary, along which anti-clock-wise transporting a single composite fermion has $-2\pi\nu$ Berry phase. The area enclosed by the red dashed line contains 46 grids. The Berry curvature has a peak of around $-0.25+0.011=-0.239$ [in units of $\pi$], while the rest values are around $-(2/4-1)/46=0.011$. It suggests an interesting Berry curvature distribution for CFLs at a generic filling fraction in the thermodynamic limit: a $-\pi$ singularity at center and $-(2\pi\nu-\pi)$ uniformly distributed over Fermi sea.

Before going into more detailed discussions, let us brief discuss what happens if we start from a half filled Landau level and decrease the electron density all the
way down to $\frac{1}{2m}$. See FIG. (5.3) for illustration. In Dirac fermion picture, based on particle-vortex duality, we expect the following phenomena happens,

- Dirac fermions will see nonzero magnetic field, since electron density maps to the magnetic field strength.

If indeed a Fermi liquid can form in the end, we further expect,

- The magnetic fields will be “canceled” by some mechanism. For example, flux attachment.

- Fermi sea size will be reduced. This is a consequence of Luttinger’s theorem, which we will discuss later.

Figure 5.3: From half filling to $1/4$ or $3/4$ filling, we expect Dirac fermion perceives magnetic field. If a Fermi sea forms in the end, we expect magnetic field can be attached to Dirac fermions. Furthermore, Fermi sea sizes are supposed to shrink, due to the Luttinger’s theorem.

In the following section, we will first discuss numerics followed by discussion of the effective theory.
5.2 Smoking gun of Dirac fermion: $\pi$ Berry phase at $\nu = \frac{1}{4}$

Model wave functions

The CFL model wavefunction at generic $\nu = \frac{1}{2m}$ filling fraction is,

$$\Psi_{\frac{1}{2m}}^{n} = \det \, M_{ia} \prod_{i<j}^{N} \sigma^{2(m-n)}(z_i - z_j)$$

$$\times \prod_{k}^{2m} \sigma \left( \sum_{i}^{N} z_i - \alpha_k \right) \prod_{i}^{N} e^{-\frac{1}{2} z_i \bar{z}_i^*}. \quad (5.5)$$

where dipoles $\{d\}$, center of mass zeros $\{\alpha\}$ can be found in Sec. (3.2.5). In the above model wavefunction, we only consider model wavefunctions with $2m \geq 2n$ because otherwise they are 0/0 indeterminate forms when multiple particles collide onto the same site. The analytical values of these indeterminate forms are well defined, but numerically difficult to evaluate.

We adopt the lattice Monte Carlo method [50] to study the Berry phase. We consider $\nu = 1/4$ model wavefunctions with $m \geq n$: $\Psi_{n=1}^{1/4}$ and $\Psi_{n=2}^{1/4}$. They are found to have large overlaps with each other for all dipole configurations, e.g. $|\langle \Psi_{n=1}^{1/4} | \Psi_{n=2}^{1/4} \rangle| \geq 97\%$ for $N=69$ dipoles. This means that observables computed from either of them are almost identical.
In a half filled LLL, Coulomb interaction low energy states were found to have a remarkably large overlap [9] with the cluster-like ansatz of Eq. (5.5). At one quarter, second quantizing a model wavefunction to compute overlap becomes difficult for large system sizes. Instead, as shown in FIG. 5.4, we present the energy spectrum of LLL Coulomb interaction and the variational energy of model wavefunction for \( N = 10 \) electrons on a square torus. The variational energies of model wavefunctions and exact diagonalization energies are close, but slightly worse compared to one half states. As pointed out in [42], at half filling in the lowest two Landau levels, varying short range interactions induce a first-order phase transition from striped phase to a strongly paired Moore-Read state, followed by a possible crossover to a weak paring phase. The exact diagonalization states we obtained at Coulomb point at one quarter, presumably, are weakly paired states; tuning \( v_{1,3} \) pseudo-potentials might help improve the overlaps.
We found good agreement except at (5, 0) sector. Similar phenomena appears at 1/2. At half filling for 10 particles in the (5, 0) sector, the lowest three Coulomb energies are $-0.4649$, $-0.4634$, $-0.4567$ [in units of $e^2/\epsilon_B$]. The lowest variational energy of the model wavefunction, which corresponds to the most compact Fermi sea consisting of 10 dipoles on a square torus, is $-0.4625 \pm 0.0006$. Same as $\nu=1/4$, this variational energy is close to the next lowest rather than the lowest energy. We hence believe that it is due to the difficulty of packing 10 dipoles into a compact Fermi sea in the (5, 0) sector on a square torus.

**Berry curvature**

We next turn to the numerical investigation of the Berry curvature as shown in FIG. 5.2. Within the composite Fermi liquid (CFL) phase, low energy exact diagonalization states can be identified with model wave functions whenever dipoles form clustered configurations. Model wavefunction with a non-compact Fermi sea is hard to be identified as a single exact diagonalization state; instead it might be a linear combination of low energy and excited states. For this reason, we did not consider measuring the Berry phase using a composite hole located deep inside the Fermi sea, nor a composite fermion excited too far away. Instead, Berry phases were computed on clock-wised paths close to the Fermi surface, after which the Berry curvatures were mapped out by a linear regression. The Berry curvature distribution was found as described in FIG. 5.2. A consistency check is shown in FIG. 5.3. The agreement indicates FIG. 5.2 makes sense.
Figure 5.5: Comparison of the Berry phases $\Phi_{\Gamma}$ associated with various clock-wised paths $\{\Gamma\}=\Gamma_1, ..., \Gamma_6$ on a Fermi sea of $N=37$ dipoles [see FIG. 5.2 for Fermi sea] computed from $\Psi_n^{1/4}$ [red dots] and from the formula $\Phi_{\Gamma}=\delta_{\Gamma}\cdot\pi+(2\pi\nu-\pi)A_{\Gamma}/A_{FS}$ [black lines], where $\delta_{\Gamma}$ is the winding number of $\Gamma$ relative to the Fermi sea center, $A_{\Gamma}$ and $A_{FS}$ are the $k$-space areas enclosed by the path $\Gamma$ and Fermi sea area respectively. See appendix for details about paths and more examples including $N=69$ Fermi sea and $\nu=1/3$.

More numerical details about computing Berry curvature, as well more case studies including $N=69$ and $\nu=1/3$ states whose underlying particles are bosons are presented in the following. Fermi sea configurations are shown in FIG. 5.6. The red lines represent the Fermi sea boundary, along which transporting a composite fermion has $2\pi\nu$ Berry phase within Monte Carlo precision. The $\phi$ represents the Berry phase on the corresponding grid, i.e. the discretized Berry curvature.
Figure 5.6: The Fermi sea and discretized Berry curvature $\phi$. The left and right Fermi sea has $N=37$ and 69 dipoles respectively, on square toruses. The Berry curvatures related by rotation and inversion are not represented. The red dashed line represents the Fermi surface boundary along where transporting a single dipole has $\Phi_{\Gamma}=2\pi\nu$ Berry phase.

The determinant with Jastrow factors of power two vanishes identically when $\bar{d}$ approaches Fermi sea center, if $N$ is even and $\{d\}$ is inversion symmetric [50]. This gives some hints that two fluxes attachment turns an electron into a Fermi sea. Our intuition for the Berry curvature distribution from a model wavefunction point of view is as follows: the determinant [which has 2 Jastrow factors] of model wavefunction corresponds to the $\pi$ singularity at Fermi sea center, while the rest $2m-2$ Jastrow factors corresponds to the rest $-(2\pi/2m - \pi)$ Berry phase. From effective theory point of view, the former is attributed to the gapless Dirac node, while the latter is attributed to the Chern-Simons term. From the dipole picture, 2 fluxes turns electron into Fermi sea, while the rest $(2m-2)$ fluxes are attached to Fermi seas, which after LLL projection, form residual dipole-momentum lockings. As argued in the main text, these pictures are consistent with each other.
In the numerical Berry phase part, we adopt the lattice Monte Carlo \[50, 21, 22\] to extract the many body Berry phase $\tilde{\Phi}_\Gamma$, which is defined as \[9\],

$$
|D| \ e^{i \tilde{\Phi}_\Gamma} = \text{Tr} \prod_{i \in \Gamma} \langle \Psi(K'_i) | \rho(K'_i - K_i) | \Psi(K_i) \rangle,
$$

(5.6)

$$
e^{i \tilde{\Phi}_\Gamma} = (i)^{N_+ - N_-} \ e^{i \Phi_\Gamma}.
$$

(5.7)

where $|D|$ is a real number representing the many body amplitude. $|\Psi(K)\rangle$ is a many body model wavefunction, Eq. (5.5).

Same results were found even for bosonic $\nu=1/3$ states. From wave function point of view, the Berry curvature feature can be argued as follows \[15\]: the determinant and the Jastrow factor are implementing respectively the $\pi$ Berry phase and the $U(1)$ gauge field. We believe that the Berry curvature feature we observed on $\nu=1/4$ and $\nu=1/3$ model states applies to other filling fractions as well since model wavefunctions at different Landau level fillings essentially differ only by a different power of Jastrow factors.

The $\Phi_\Gamma$ is an area weighted sum of the discretized curvature $\phi$. The numerical values of $\Phi_\Gamma$ computed on the corresponding paths $\Gamma$ from $\nu=1/4$ and $1/3$ model wavefunctions are listed in TABLE 5.1. We found an empirical formula for the Berry phase, which is the key result in the numerical part of this work, as follows [transport composite fermion clock-wisely],

$$
\Phi_\Gamma = \delta_\Gamma \cdot \pi + (2\pi \nu - \pi) \cdot A_\Gamma / A_{FS}.
$$

(5.8)
where $\delta_\Gamma$ is the winding number of the path $\Gamma$ relative to the Fermi sea center: it is $+1$ if $\Gamma$ encloses Fermi sea center once and 0 if not. $A_\Gamma$ and $A_{FS}$ are the momentum space area enclosed by path $\Gamma$ and Fermi sea area respectively. The importance of Eq. (5.8) has already been emphasized in the main text: it implies a uniform Berry curvature together with a $\pi$ singularity at Fermi sea center. Motivated by Eq. (5.8), a flux-attached Dirac fermion theory is proposed in the main text. In FIG. 5.8 and FIG. 5.9 we compare the $\Phi_\Gamma$ value computed from numerics and the analytical value calculated from Eq. (5.8) for various paths on the $N=37$ and 69 Fermi seas. The excellent agreement indicates that Eq. (5.8) makes sense.

To better visualize the Berry curvatures, we did a linear regression and mapped out Berry curvatures on $N=37$ Fermi sea. The linear relations of $\Phi_\Gamma$ and $\phi$ can be found in the following Eq. (5.9).

\[
\begin{align*}
\Phi_{\Gamma 1} & = 2\phi_4 + \phi_7 \\
\Phi_{\Gamma 2} & = 2\phi_3 + 2\phi_4 + 2\phi_6 + \phi_7 + \phi_8 \\
\Phi_{\Gamma 3} & = 2\phi_2 + 2\phi_3 + 2\phi_4 + 2\phi_5 + 4\phi_6 + 2\phi_7 + \phi_8 \\
\Phi_{\Gamma 4} & = 2\phi_3 + 2\phi_6 + \phi_8 \\
\Phi_{\Gamma 5} & = 2\phi_2 + 2\phi_3 + 2\phi_5 + 4\phi_6 + \phi_7 + \phi_8 \\
\Phi_{\Gamma 6} & = 4\phi_1 + 4\phi_2 + 4\phi_3 + 4\phi_4 \\
\Phi_{\Gamma 7} & = \phi_3 + 2\phi_4 + \phi_6 + \phi_5/2 \\
\Phi_{\Gamma 8} & = \phi_2 + 3\phi_3 + 4\phi_4 + \phi_5 + \phi_6 \\
\Phi_{\Gamma 9} & = 4\phi_1 + 6\phi_2 + 2\phi_3 + 2\phi_5 + 4\phi_6 + 2\phi_7 + 2\phi_8 + 2\phi_9 \\
\Phi_{\Gamma 10} & = 4\phi_1 + 6\phi_2 + 89\phi_3/15 + 11\phi_4/3 + 2\phi_5 + 41\phi_6/15 \
& + 3\phi_7/5 + \phi_8/15. \\
\Phi_{\Gamma 11} & = 4\phi_1 + 8\phi_2 + 8\phi_3 + 8\phi_4 + 4\phi_5 + 8\phi_6 \
& + 4\phi_7 + 2\phi_8.
\end{align*}
\]
Figure 5.8: The Berry phases associated with paths $\Gamma_1, \ldots$ listed in Table 5.1 for $N=37$ Fermi sea computed from $\nu=1/4$ [upper panel] and $\nu=1/3$ [lower panel] CFL model wavefunction.
Figure 5.9: Same as FIG. 5.8 but for $N=69$ Fermi sea. In FIG. 5.8 and FIG. 5.9, black dashed lines are Berry phases according to Eq. (5.8) and the red dots stand for Monte Carlo values which can be found in Table 5.1.
Table 5.1: Paths $\Gamma$ and phases $\Phi_{\Gamma}$ for $N=37$ and 69 Fermi sea. See Eq. (5.6) for definitions of $K$ and $K'$. For example, in $\Gamma_1$ of $N=37$, the Berry phase is computed as follows: $\Phi_{\Gamma_1} = -i \ln \text{Tr} \langle \Psi_{(1,4)} | \rho | \Psi_{(-1,4)} \rangle \langle \Psi_{(-1,4)} | \rho | \Psi_{(-2,3)} \rangle \ldots \langle \Psi_{(2,3)} | \rho | \Psi_{(1,4)} \rangle$. Note that the extra dipole is in fact transported clock-wisely.

<table>
<thead>
<tr>
<th>$\Gamma_i$</th>
<th>Path $\Gamma_i$</th>
<th>$\Phi_{\Gamma,n=1}/\pi$</th>
<th>$\Phi_{\Gamma,n=1}/\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>$(1,4) \rightarrow (-1,4) \rightarrow (-2,3)$</td>
<td>0.011 ± 0.015</td>
<td>−0.004 ± 0.021</td>
</tr>
<tr>
<td></td>
<td>$(2,3) \rightarrow (1,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>$(1,4) \rightarrow (-1,4) \rightarrow (-3,2)$</td>
<td>−0.006 ± 0.010</td>
<td>−0.067 ± 0.023</td>
</tr>
<tr>
<td></td>
<td>$(3,2) \rightarrow (1,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>$(1,4) \rightarrow (-1,4) \rightarrow (-4,1)$</td>
<td>−0.077 ± 0.013</td>
<td>−0.150 ± 0.018</td>
</tr>
<tr>
<td></td>
<td>$(4,1) \rightarrow (1,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>$(2,3) \rightarrow (-2,3) \rightarrow (-3,2)$</td>
<td>−0.015 ± 0.014</td>
<td>−0.056 ± 0.023</td>
</tr>
<tr>
<td></td>
<td>$(3,2) \rightarrow (2,3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>$(2,3) \rightarrow (-2,3) \rightarrow (-4,1)$</td>
<td>−0.081 ± 0.014</td>
<td>−0.152 ± 0.022</td>
</tr>
<tr>
<td></td>
<td>$(4,1) \rightarrow (2,3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_6$</td>
<td>$(4,1) \rightarrow (-4,1) \rightarrow (-4,-1)$</td>
<td>0.905 ± 0.017</td>
<td>0.832 ± 0.018</td>
</tr>
<tr>
<td></td>
<td>$(4,-1) \rightarrow (4,1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_7$</td>
<td>$(0,4) \rightarrow (-4,0) \rightarrow (-4,-1)$</td>
<td>−0.002 ± 0.013</td>
<td>−0.029 ± 0.019</td>
</tr>
<tr>
<td></td>
<td>$(1,4) \rightarrow (0,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_8$</td>
<td>$(0,4) \rightarrow (-4,0) \rightarrow (-4,-2)$</td>
<td>−0.046 ± 0.014</td>
<td>−0.098 ± 0.021</td>
</tr>
<tr>
<td></td>
<td>$(2,4) \rightarrow (0,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_9$</td>
<td>$(2,4) \rightarrow (-4,-2) \rightarrow (-2,4)$</td>
<td>0.848 ± 0.018</td>
<td>0.741 ± 0.021</td>
</tr>
<tr>
<td></td>
<td>$(4,2) \rightarrow (2,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_{10}$</td>
<td>$(1,4) \rightarrow (-1,4) \rightarrow (-4,1)$</td>
<td>0.809 ± 0.011</td>
<td>0.688 ± 0.019</td>
</tr>
<tr>
<td></td>
<td>$(4,1) \rightarrow (1,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Gamma_{11}$</td>
<td>$(1,4) \rightarrow (-1,4) \rightarrow (-4,1)$</td>
<td>0.748 ± 0.015</td>
<td>0.535 ± 0.017</td>
</tr>
<tr>
<td></td>
<td>$(4,1) \rightarrow (1,4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N = 69$, path $\Gamma (K_i' \to K_i)$</td>
<td>$\Phi_{\nu,n=1}^{\frac{1}{2}}/\pi$</td>
<td>$\Phi_{\nu,n=1}^{\frac{1}{2}}/\pi$</td>
<td></td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_1$</td>
<td>(2,5) $\to$ (-2,5) $\to$ (3,4) $\to$ (2,5)</td>
<td>0.016 ± 0.026</td>
<td>-0.002 ± 0.028</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>(2,5) $\to$ (-2,5) $\to$ (-4,3) $\to$ (2,5)</td>
<td>-0.042 ± 0.024</td>
<td>-0.049 ± 0.033</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>(2,5) $\to$ (-2,5) $\to$ (5,2) $\to$ (2,5)</td>
<td>-0.036 ± 0.027</td>
<td>-0.098 ± 0.035</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>(5,2) $\to$ (-5,2) $\to$ (5,1) $\to$ (5,2)</td>
<td>-0.024 ± 0.022</td>
<td>-0.074 ± 0.031</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>(5,1) $\to$ (-5,1) $\to$ (5,1) $\to$ (5,1)</td>
<td>0.919 ± 0.026</td>
<td>0.849 ± 0.028</td>
</tr>
<tr>
<td>$\Gamma_6$</td>
<td>(5,2) $\to$ (-5,2) $\to$ (-5,-2) $\to$ (5,2)</td>
<td>0.820 ± 0.022</td>
<td>0.719 ± 0.026</td>
</tr>
<tr>
<td>$\Gamma_7$</td>
<td>(2,5) $\to$ (-2,5) $\to$ (5,2) $\to$ (2,5)</td>
<td>0.749 ± 0.021</td>
<td>0.547 ± 0.038</td>
</tr>
</tbody>
</table>

Table 5.2: Same as Table. 5.1 but for $N = 69$ dipoles.

### 5.3 “Luttinger’s Hypothesis”

In this section, we will discuss the Luttinger theorem. The Fermi sea size of $\nu = 1/2m$ and $\nu = 1 - 1/2m$ are supposed to be the same.

Some knowledge about the Fermi sea can be obtained before an interpretation [whether non-relativistic HLR fermions or relativistic Fermi seas] of the particles that compose the Fermi sea is assigned. The first requirement for being a Fermi liquid is no net magnetic fields on the Fermi sea: $\langle B_\psi \rangle \equiv \langle \epsilon^{ab} \partial_a a_b \rangle = 0$ where $\langle ... \rangle$ denotes mean field expectation value.

Second, in Landau Fermi liquid theory, the Fermi sea volume is determined by the charge density, known as Luttinger theorem. For spinless fermion, the Fermi surface is defined as the zeros of the time-ordered single particle Green function at
zero frequency. Particle number \( n \) is determined by the Fermi sea size,

\[
n = \sum_k \Theta (\Re G(k, \omega = 0)).
\]  

\( G(r, t) = \langle \mathcal{T} \psi(r, t)\psi^\dagger(0, 0) \rangle. \)

where \( \Theta \) is the Heaviside step function, \( \mathcal{T} \) is time-ordering operator, \( \langle \ldots \rangle \) is ground state average.

It has been conjectured \[44, 10, 1\] that CFLs satisfy Luttinger theorem too, \textit{i.e.} composite Fermi wave vector is determined by electrons’ filling factor. The Fermi sea volume \( A_{FS} \) and total particle numbers \( N \) are supposed to be related by,

\[
\frac{A_{FS}}{(2\pi)^2} = N.
\]  

where torus area \( A = 2\pi N_\phi l_B^2 \). For square Fermi sea \( A_{FS} = 4k_F^2 \), for disk Fermi sea \( A_{FS} = \pi k_F^2 \) where \( k_F \) is the Fermi wavevector. Hence,

\[
A_{FS} = 2\pi \nu l_B^{-2} = \begin{cases} 
(2k_F)^2, & \text{square Fermi sea,} \\
\pi k_F^2, & \text{circular Fermi sea.}
\end{cases}
\]  

which leads to

\[
k_F l_B = \begin{cases} 
\sqrt{\pi\nu/2}, & \text{square Fermi sea,} \\
\sqrt{2\nu}, & \text{circular Fermi sea.}
\end{cases}
\]

In FIG. [5.11], we investigated Luttinger theorem for CFLs by computing \( S(k) \) of a \( \nu=1/4 \) model wavefunction. As a hallmark of CFL, there are peaks in the guiding center structure factor \( S(k) \) and the peak positions are tied to \( k=2k_F \), twice the Fermi wave vector. The measured \( k_F \) agrees with the value predicted from Luttinger
theorem, suggesting that Luttinger theorem applies to CFLs \cite{1, 44, 10}. We will then assume Luttinger theorem for CFLs and use it as a constraint to derive the effective action.

The PH conjugate of a CFL is supposed to have same Fermi sea size \cite{3, 1, 36, 33}. In HLR picture, this is because Fermi seas of fCFLs and anti-fCFLs are formed by composite-fermions composite-holes respectively whose Fermi level are the same. See FIG. 5.10 for an illustration. Dirac fermion theory interprets the Fermi sea as formed by Dirac fermions, which fill Dirac fermion bands up to the Fermi level determined by electrons’ filling factor through Luttinger theorem. As a result, Dirac fermion density must be,

\[
\rho_\psi = \frac{1}{2m} \frac{1}{2\pi l_B^2}, \quad \nu = \begin{cases} 
\frac{1}{2m} \text{ or,} \\
1 - \frac{1}{2m}.
\end{cases}
\]  

(5.15)

\[\text{v=1/4} \quad \text{v=3/4}\]

Figure 5.10: Illustration of Fermi sea form by composite-fermions at \(\nu = 1/4\) and by composite-holes at \(\nu = 3/4\) in HLR’s theory. The green area are filled \(k\)-space areas. Fermi sea sizes are the same.
Figure 5.11: Guiding center structure factor \( S(k) \) as a function of \( k_x \) along \( k_y = 0 \) axes computed from \( \Psi_{n=1}^{1/4} \). The plot is obtained after a finite size scaling for model wavefunctions of \( \sqrt{N} \times \sqrt{N} \) square Fermi seas where \( N \) is the number of electrons. The red lines are \( 2k_F l_B = \sqrt{2} \pi \nu \), a value of twice of the Fermi wave vector obtained by applying Luttinger theorem on a square Fermi sea. The fact that \( S(k) \) plots fit into one curve and the numerical singularities match with the analytical value implies that Luttinger theorem is true for CFLs.

5.4 Generalization of Son’s theory: flux-attached Dirac fermion theory

The presence of \( \pi \) Berry curvature singularity at \( \nu \neq 1/2 \) strongly suggests the emergence of Fermi seas at low energy at generic filling fractions. In this section, we justify our proposed flux-attached Fermi sea theory Eq. (5.4) by starting from a Dirac type effective action with undetermined coefficients, Eq. (5.16). We will then fix \( C_{1,2,3} \) by physical requirements: Luttinger theorem, Fermi sea Berry phase and Hall conductivity, and argue the Berry curvature distribution is consistent with the prediction from flux-attached Fermi sea picture.

\[
\mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{C_1}{2\pi} adA - \frac{C_2}{4\pi} ada + \frac{C_3}{4\pi} AdA. \tag{5.16}
\]
Taking variation of $a_0$ and $A_0$ for Eq. (5.16), the Fermi sea density and electron density at mean field level are found to be:

$$\langle \rho_{\psi} \rangle = C_1 \frac{B_A}{2\pi} \quad (5.17)$$

$$\langle \rho_A \rangle = C_3 \frac{B_A}{2\pi}. \quad (5.18)$$

Hence, Luttinger theorem and Hall conductivity determine $C_1$ to be $1/2m$ and $C_3$ to be $\frac{1}{2} - \frac{\eta}{2} \frac{m-1}{m}$ at mean field where $\eta=+1$ for fCFLs and $=-1$ for anti-fCFLs. Based on the observation of a $\pi$ peak concentrated at Fermi sea center, we conjecture Fermi sea is massless. Unlike Son’s theory, the presence of Chern-Simons (CS) term $\frac{C_2}{4\pi} ada$ has a non-universal contribution to $\sigma_H$, and induces nonzero $U(1)_A$ charge to Fermi seas [40] [48]. The effects of CS term are canceled provided Fermi sea carries $2\pi C_2$ Berry phase; in other words, CS term assigns the Fermi sea with extra $2\pi C_2$ Berry phase. This $2\pi C_2$ phase, together with the $-\pi$ Berry curvature singularity located at Fermi sea center, comprises the total $-2\pi \nu$ Fermi sea berry phase as observed. Based on the fact that Berry phase is odd under PH transformation, we set

$$C_2 = \eta \left( \frac{1}{2} - \frac{1}{2m} \right). \quad (5.19)$$

Thus we determined Eq. (5.4) by Luttinger theorem, Hall conductivity and Fermi sea Berry phase.

For reader’s convenience, we copy the effective action Eqn. (5.4) here, with an emphasis on the meaning of each term,

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}(\partial_{\mu} - ia_{\mu})\psi - \frac{1}{2m} \frac{1}{2\pi} adA - \eta \left( \frac{1}{2} - \frac{1}{2m} \right) \frac{1}{4\pi} ada + \left( \frac{1}{2} - \frac{\eta}{2} \frac{m-1}{m} \right) \frac{1}{4\pi} AdA.$$  

Luttinger theorem    Fermi sea Berry phase    Hall conductivity
Conceptually, at $\nu = \frac{1}{2m}$, composite fermion consists of one electron and $2m$ flux quanta. Now two of the fluxes turn electron into Dirac fermion, the rest $2m - 2$ flux quanta are attached to the Dirac fermion. In this sense, the Fermi sea can be viewed as the Fermi sea formed by flux-attached Dirac fermions. The Fermi sea of filling $\nu = 1 - \frac{1}{2m}$ states are then the Fermi sea of Dirac fermion attached with opposite $2m - 2$ fluxes. We illustrated this idea in FIG. 5.12

Figure 5.12: Illustrations of Fermi sea, particles, band structure in Son’s half filled Fermi sea theory [middle] and flux-attached Dirac fermion theory. PH acts like time reversal, thus flipping the fluxes [arrows] attached to the Dirac fermions [black dots]. Fermi sea sizes of PH conjugate states are the same, fixed by Luttinger theorem.

We will then argue for the Berry curvature distribution presented in FIG. 5.2 based on a flux-attached Fermi sea picture. The flux attached Dirac fermion has both spin-orbit locking and residual dipole-momentum locking. The residual dipole-momentum locking is formed by shifting the $\pm(2m-2)$ fluxes away from the Dirac fermion location. The residual dipole-momentum locking of Fermi sea has a nontrivial impact on the Berry phase associated with transporting a composite fermion [dipolar Fermi sea] in the momentum space [$k$-space]. The Berry curvature distribution is predicted to be: (I) $k$-space uniform except at the Fermi sea center point $k=0$, (II) where there is an additional $\pi$ Berry phase. The argument goes as follows. The dipole-momentum locking provides a nature mapping from the real space to the $k$-space. The motion of a dipolar Dirac fermion in $k$-space induces the rotation of the residual dipoles in real space. Since the real-space density is uniform, and since $k$-space area is
proportional to real space area, (I) is a manifestation of the real space Aharonov-Bohm effect. The contribution to the Fermi sea Berry phase $\Phi_{FS}$ from (I) should be $-(2\pi \nu - \pi)$ in accordance with the fact that it vanishes at half filling. Then, (II) origins from being massless spin-half Fermi seas. Finite mass tilts the Dirac fermion’s spin away from the 2D plane hence mass term $|M|$ represents how much the Fermi sea Berry phase $\Phi_{FS}$ deviates from $-2\pi \nu$. We thus conjecture that Dirac fermion mass $|M|=0$ which we emphasis is not protected by symmetry but instead constrained to take this value by the Fermi sea Berry phase Eq. (5.1).
Chapter 6

Conclusion

In this dissertation, we studied the “composite Fermi liquid” states numerically and theoretically.

In Chapter 1, we reviewed the history of quantum Hall physics.

In Chapter 2, we reviewed the basic concepts of composite Fermi liquid, including composite fermion, dipole and dipole-momentum locking. We also reviewed the Halperin-Lee-Read theory and Son-Dirac theory. The theory of Halperin-Lee-Read was successful in many aspects, but has been challenged by the seemingly lack of particle-hole symmetry and Berry phase. The Halperin-Lee-Read theory at $\nu = \frac{1}{2m}$ is,

$$\mathcal{L} = i \psi \hat{\partial}_0 i a \psi - (\mathcal{M})^{-1}_{ab} D_a \psi \hat{D}_b \psi - \frac{1}{2m} \frac{1}{4\pi} ad a + \frac{1}{2m} \frac{1}{2\pi} ad A + \frac{1}{2m} \frac{1}{4\pi} A d A.$$

(6.1)

where $D_a = \partial_a - ia_a$. The Chern-Simons term does flux attachment transformation, $\mathcal{M}$ is the effective mass tensor.

A half filled Landau level without Landau level mixing must be particle-hole invariant, for any translational invariant two-body interactions. Unfortunately Halperin-Lee-Read theory does not explicitly has this symmetry.
It was first pointed out by Haldane that the Fermi sea of the composite Fermi liquid in fact is a nontrivial Fermi sea: has Berry phase. This Berry phase origins from the dipole structure of composite fermions. The rotation of composite fermion around Fermi surface induces dipole to rotate due to dipole-momentum locking and induces a Berry phase. At half filling, the Fermi sea is supposed to have a $\pi$ Berry phase, which gives raise to the right Hall conductivity required by particle-hole symmetry. The Berry phase and particle-hole symmetry are the two key words of this dissertation.

Recently, Son conjectured that the particle forming the Fermi sea at half filling can be a relativistic Dirac fermion. His key point is that when particle-hole transformation is acted on electrons, the composite fermion see a time-reversal transformation. Hence a Dirac fermion Fermi sea with finite chemical potential is consistent with particle-hole symmetry. His action reads,

$$L = i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{2}\frac{1}{2\pi}adA + \frac{1}{2}\frac{1}{4\pi}AdA. \quad (6.2)$$

where $\psi$ is the Dirac fermion field which describes a composite fermion. His theory is explicitly particle-hole symmetric and has the right $\pi$ Berry phase. The dipole is represented as the spin of Dirac fermion. Dipole-momentum locking is represented as spin-orbit locking. The $\pi$ Berry phase origins from the gapless Dirac node in the center of the Fermi sea.

In Chapter 3, we developed a lattice formalism to study torus quantum Hall problems. We proved that any continuous integration for quantum Hall wavefunctions on torus can be replaced by a discretized lattice summation. We also provided a determinant model wavefunction for composite Fermi liquids,

$$\Psi(\{d\}, \{\alpha\}, \{z\}) = \det_{ia} M_{ia} \prod_{i<j}^{N} \sigma^{2(m-n)}(z_i - z_j) \prod_{k=1}^{2m} \sigma \left( \sim_{i}^{N} z_i - \alpha_k \right) \prod_{i}^{N} e^{-\frac{1}{2}|z_i|^2}. \quad (6.3)$$
where

\[ M_{ia} = e^{\frac{2\pi}{m}} c^{d_i} \prod_{k \neq i}^{N} \sigma^n (z_i - z_k - d_a + \bar{d}). \] (6.4)

This model wavefunction is for composite Fermi liquid at filling fraction \( \nu = \frac{1}{2m} \). The \{d\} is the dipole configuration, the discretized variational parameter and the object that forms the Fermi sea.

In Chapter 4, we defined the concept called “many body Berry phase”, and carried the numerics study of the symmetry and Berry phase at half filled Landau level. The many body Berry phase is defined as follows,

\[ e^{i\tilde{\Phi}_\Gamma} = \prod_{\text{path}} \langle \Psi(K_i) | \rho(K_i - K_{i+1}) | \Psi(K_{i+1}) \rangle. \] (6.5)

where \( \rho(q) = \sum_i^{N} e^{iq \cdot R_i} \) is the projected density operator. \( |\Psi(K)\rangle \) are many body wavefunctions with many body momentum \( K \). We performed the numerics of transporting the composite fermion around the Fermi surface and observed a \( Z_2 \) Berry phase at half filling, supporting Son-Dirac theory.

In Chapter 5, we considered the Berry phase and the emergent Dirac fermion at generic filling fractions \( \nu \neq \frac{1}{2} \). The composite Fermi liquid can in principle occur at \( \frac{1}{2m} \) and \( 1 - \frac{1}{2m} \) filling fractions. Only at half filling \( m = 1 \) there is particle hole symmetry. So what is the nature of composite fermions at generic filling fractions? Can they still be Dirac fermions?

To answer this question, we studied the \( \nu = \frac{1}{4} \) model wavefunction, and found that the Fermi sea contains \(-2\pi \nu \) Berry phase. Moreover, there is a \( \pi \) peak at the Fermi sea center with the rest \(-2\pi \nu - \pi \) Berry phase uniformly distributed on the Fermi sea. The \( \pi \) Berry phase is a smoking gun for the emergence of Dirac fermions. Based on this numerical observation, we proposed a flux-attached Dirac
fermion theory that unifying all composite Fermi liquids in one Dirac type effective action. The flux-attached Dirac fermion effective action we proposed is the following,

\[
\mathcal{L} = \bar{\psi} \gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{2m} \frac{1}{2\pi} adA - \eta \left( \frac{1}{2} - \frac{1}{2m} \right) ada + \left( \frac{1}{2} - \frac{\eta m - 1}{2} \right) \frac{1}{4\pi} AdA.
\]

(6.6)

This theory described the composite Fermi liquids at filling fraction \( \nu = \frac{1}{2} - \frac{\eta m - 1}{2m} \), which is \( \frac{1}{2m} \) if \( \eta = +1 \) and its particle-hole conjugate \( 1 - \frac{1}{2m} \) if \( \eta = -1 \). Conceptually, the Fermi sea of composite Fermi liquids at generic filling fractions can be viewed as flux-attached Dirac fermions.

The dissertation is based on the publications:

- **Lattice Monte Carlo for Quantum Hall States on a Torus,**
  

- **Berry phase and model wavefunction in the half-filled Landau Level,**
  
  Scott D. Geraedts, Jie Wang, E. H. Rezayi and F. D. M. Haldane.

- **A Dirac Fermion Hierarchy of Composite Fermi Liquids,**
  *arXiv: 1808.07529 (2018, under review)*
  
  Jie Wang.
Bibliography


108


