Quantum Cohomology of Hypertoric Varieties and Geometric Representations of Yangians

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Abstract

This thesis compares two geometric constructions of a Yangian, due to Varagnolo and Nakajima on the one hand and Maulik and Okounkov on the other. It also, separately, computes the quantum cohomology of smooth hypertoric varieties, and finds a mirror formula for their quantum connection. It contains brief introductions to the background material for both problems.
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I was supported by an NSERC PGS D scholarship for four of my years at Princeton.
To my parents.
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Chapter 1

Introduction

This thesis concerns two seemingly separate subjects.

The first part compares two geometric constructions of a Yangian, one by Nakajima and Varagnolo, the other by Maulik and Okounkov. Chapter 2 is motivational, and discusses spin chains, their integrals of motion and their applications. Chapter 3 explains how Yangians arise from the symmetries of spin chains, and defines them for general simply laced lie algebras. In chapter 4 we recall the construction of Yangian representations in the cohomology of quiver varieties, due to Nakajima and Varagnolo. In chapter 5 we review a construction of a different class of Yangians, also using quiver varieties, due to Maulik and Okounkov. Chapter 6 contains the original result of part one: we show that the two constructions are closely related for an ADE quiver.

The second part of this thesis computes the quantum cohomology of a family of varieties called hypertoric varieties, and a mirror formula for their quantum connection. Quantum cohomology and the quantum connection are defined in chapter 7 and hypertoric varieties are introduced in chapter 8. Their quantum cohomology and mirror formulae are computed in chapter 9.

In fact, the two parts of this thesis are not unrelated. The quantum cohomology of quiver varieties has been shown in [19], following work by Nekrasov and Shaatsvili
to describe the spectrum of certain large commutative subalgebras of the Yangian on its representations. Computing these spectra is in many cases an open problem. There is an ongoing project to accomplish this by relating the quantum cohomology of a quiver to that of a related hypertoric variety called its abelianization: we discuss this further in chapter A.

1.1 Original Work vs Exposition

Chapters 6 and 9 present original work; chapter 9 is joint with Daniel Shenfeld. All errors are mine.
Chapter 2

Spin Chains

2.1 The $XXX_{1/2}$ spin chain

Consider a circular chain of $L$ evenly-spaced spin-1/2 particles. The Hilbert space for this system is given by

$$\mathbb{H} = V_1 \otimes V_2 \otimes ... \otimes V_L$$

where each $V_i$ is a copy of the fundamental representation of $sl_2(\mathbb{C})$. We consider the Hamiltonian

$$H = \sum_{i=1}^{L} P_{i,i+1} - \frac{L}{2}$$  \hspace{1cm} (2.1)

where $P_{i,j}$ permutes $V_i$ and $V_j$ and we identify $P_{L,L+1}$ with $P_{L,1}$.

In the next section we’ll construct a family of operators commuting with the Hamiltonian and making the spin chain into a ‘quantum integrable system’.

2.2 Finding integrals of motion

To each factor $V_i$ along the spin chain we attach a number $u_i \in \mathbb{C}$. We define a representation of the braid group on $L$ letters $\Psi_L^{(u_i)} : B_L \rightarrow End(\mathbb{H})$ sending the
crossing 2.2 to the ‘R-matrix’ \( \hat{R}_{ij}(u_i - u_j) \in \text{End}(V_i \otimes V_j) \subset \text{End}(\mathbb{H}) \) given by

\[
\hat{R}(u) = P_{ij} - \frac{1}{u} id
\]  

(2.2)

where \( P_{ij} \) permutes \( V_i \) and \( V_j \). One can think of \( R(u - v) \) as a scattering matrix for a two-particle interaction with momenta \( u \) and \( v \). It satisfies the Quantum Yang-Baxter Equation (QYBE)

\[
\hat{R}_{ij}(u_i - u_j) \hat{R}_{ik}(u_i - u_k) \hat{R}_{jk}(u_j - u_k) = R_{jk}(u_j - u_k) \hat{R}_{ik}(u_i - u_k) \hat{R}_{ij}(u_i - u_j).
\]  

(2.3)

The QYBE states that the two diagrams in figure 2.2 have the same image under \( \Psi_{\{u_i\}} \), thus ensuring that \( \Psi_{\{u_i\}} \) is well defined on \( B_L \).

Now introduce an auxiliary particle labeled 0 with corresponding space \( V_0 = \mathbb{C}^2 \). Write \( \nu \) for the braid depicted in figure 2.2 Define the transfer matrix by

\[
T(u) = \Psi_{L+1}^{\{u_i\}}(\nu) \in \text{End}(V_0 \otimes \mathbb{H}).
\]

where we fix \( u_0 = u \) and \( u_i = 0, i \neq 0 \). The Yang-Baxter equation 2.3 implies

\[
\hat{R}(u - v)T(u)T(v) = T(v)T(u)\hat{R}(u - v).
\]  

(2.4)
Figure 2.2: Two diagrams whose images under $\Psi_L\{u_i\}$ are identical.

Figure 2.3: Braid $\nu$ crossing $L$ strands.

To recover the spin chain Hamiltonian, we first expand $T(u)$ at $u = \infty$:

$$T(u) = \prod_{i=1}^{L} P_{0i} - \frac{1}{u} \sum_{j=1}^{L} \prod_{i=1, i \neq j}^{L} P_{0i} + O\left(\frac{1}{u^2}\right)$$

Taking traces and rewriting, we get:

$$trT(u) = \prod_{i=1}^{L} P_{i,i+1} - \frac{1}{u} \sum_{j=1}^{L} \prod_{i=1, i \neq j}^{L} P_{i,i+1} + O\left(\frac{1}{u^2}\right)$$

Hence we have

$$H = \frac{1}{2} \frac{d}{du} \left(\ln(trT(u))\right)\bigg|_{u=\infty} - \frac{L}{2}. \quad (2.5)$$
where $H$ is the spin chain Hamiltonian $2.1$. 

Now consider the $u$-coefficients of $tr(T(u))$. The relation $2.4$ shows that they form a commutative algebra, and by $2.5$ they contain $H$. We thus have a family of quantum integrals of motion (QIM), making the $XXX\frac{1}{2}$ spin chain a quantum integrable system. 

### 2.2.1 Applications

Perhaps one does not expect to find one-dimensional chains of particles laying about. Spin chains are nevertheless important physical systems: aside from their value as a toy model and testing ground for general properties of integrable systems, they have important continuum limits [5], they appear in sectors of SUSY gauge theory [3], they have been suggested as candidates for quantum information transport [4] and one can even find actual materials whose atoms are organized into bundles of spin chains, such as $KCuF_3$ [25].

For these and other reasons, the description of spin chain spectra, their energy eigenstates and their correlation functions is an ongoing area of research: see [18] for a few places to start.

---

1The idea of scattering a particle or wave packet off a system and using the scattering data to rewrite the system’s Hamiltonian and produce integrals of motion comes from the ‘inverse scattering method’, introduced by Clifford S. Gardner, John M. Greene, and Martin D. Kruskal et al. (1967, 1974) for the Kortewegde Vries equation.

2To get a full family of integrals of motion, one must also take one of the total spin components, but we won’t worry about that here.
Chapter 3

Yangian

3.1 Introduction

The study of the $XXX_\frac{1}{2}$ system leads to the definition of an important Hopf algebra $\mathcal{Y}(gl_2)$ called the Yangian, which acts on its Hilbert space and contains in its image the quantum integrals of motion.

In this chapter we give two definitions of the Yangian for a general finite dimensional semi-simple simply-laced lie algebra, and discuss its commutative subalgebras and representations. As both definitions are somewhat intimidating, we start by defining the much simpler $\mathcal{Y}(gl_2)$, guided by the calculations from the previous chapter. We follow [6] and [8], to which we refer the interested reader.

3.2 The Yangian of $gl_2$

Equation 2.4 inspired the following:

**Definition 3.2.1** $\mathcal{Y}(gl_2)$ is the algebra generated by the coefficients of

$$T(u) = \sum_{i,j=1}^{2} \sum_{r=0}^{\infty} T_{ij}^{(r)} u^{-r}$$
subject to relations

\[ \check{R}(u - v) T(u) T(v) = T(v) T(u) \check{R}(u - v) \]

where \( \check{R}(u - v) = P - \frac{1}{u - v} \text{id} \).

It is a deformation of \( \mathcal{U}(gl_2[u]) \), where \( gl_2[u] \) is the Lie algebra of polynomial maps from \( \mathbb{C} \) into \( gl_2 \).

Given a complex number \( a \), one can define a representation of \( Y(gl_2) \) on \( \mathbb{C}^2 \), denoted \( \mathbb{C}^2(a) \), by sending \( T(u) \) to \( T(a) \). It is a deformation of the evaluation representation of \( gl_2[u] \) at \( u = a \).

\( Y(gl_2) \) has a natural coproduct, given by

\[ \Delta(T_{ij}(u)) = T_{ik}(u)T_{kj}(u), \]

and antipode given by

\[ S(T(u)) = T(u)^{-1}. \]

Together with the unit 1 and counit \( \epsilon(T(u)) = 1 \), they make \( Y(gl_2) \) into a Hopf algebra.

Given an (ordered!) \( L \)-tuple of complex numbers \( a_1, ..., a_L \) we thus have a representation \( \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_L) \). The \( XXX_\frac{1}{2} \) system corresponds to the representation \( \mathbb{C}^2(0) \otimes \cdots \otimes \mathbb{C}^2(0) \); other values of \( a_i \) correspond to ‘inhomogeneous’ spin chains. The image of \( Y(gl_2) \) clearly contains the QIM from the previous section. In fact, of its many uses is provide a natural setting for diagonalizing the QIM and computing their eigenvalues; this is called the algebraic Bethe ansatz. We will not discuss it further here.

**Remark 3.2.2** One can similarly define \( Y(gl_n) \), for any \( n \).
3.3 Yangians of other Lie algebras

In this section we extend the definition of the Yangian to a Lie algebra \( g \) of type ADE (some of the constructions below work in greater generality). Write \( I \) for a set of simple roots, and \( C \) for the cartan matrix of \( g \).

**Definition 3.3.1** Let \( \{ x_\lambda \} \) be an orthonormal basis of \( g \) with respect to an invariant bilinear form \((\ ,\ )\). Given elements \( z_1, z_2, z_3 \) of an associative algebra, set

\[
\{z_1, z_2, z_3\} = \frac{1}{24} \sum_{\sigma \in S_3} z_{\sigma(1)} z_{\sigma(2)} z_{\sigma(3)}.
\]

The Yangian \( Y(g) \) is generated by elements \( x, J(x) \) for \( x \in g \), with relations:

\[
[x, y]_{Y(g)} = [x, y]_g
\]

\[
J(ax + by) = aJ(x) + bJ(y)
\]

\[
[x, J(y)] = J([x, y])
\]

\[
[J(x), J([y, z])] + [J(z), J([x, y])] + J(y), J([z, x])
\]

\[
= \hbar^2 \sum_{\lambda, \mu, \nu} ([x, x_\lambda], [[y, x_\mu], [z, x_\nu]]) \{x_\lambda, x_\mu, x_\nu\}
\]

\[
[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]
\]

\[
= \hbar^2 \sum_{\lambda, \mu, \nu} ([x, x_\lambda], [[y, x_\mu], [[z, w], x_\nu]]) \{x_\lambda, x_\mu, J(x_\nu)\}
\]
for all \(x, y, z \in g, a, b \in \mathbb{C}\). The Hopf structure is given by

\[
\Delta x = x \otimes 1 + 1 \otimes x
\]
\[
\Delta J(x) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{\hbar}{2}[x \otimes 1, t]
\]
\[
S(x) = -x
\]
\[
S(J(x)) = -J(x) + \frac{1}{4}cx
\]
\[
\epsilon(x) = \epsilon(J(x)) = 0
\]

where \(t\) is the Casimir for \(g\), viewed as an element of \(U(g) \otimes 2\), and \(c\) is the eigenvalue of the Casimir in the adjoint representation of \(g\).

\(Y(g)\) is endowed with a family of ‘translation’ automorphisms \(\tau_\lambda \in Aut(Y(g)), \lambda \in \mathbb{C}\), defined by \(\tau_\lambda(x) = x, \tau_\lambda(J(x)) = J(x) + \lambda x\). One has the following construction, generalizing the \(R\)-matrix from 2.2 up to a permutation of the tensor factors:

**Theorem 3.3.2** There exists a unique formal power series

\[
R(\lambda) = 1 \otimes 1 + \frac{t}{\lambda} + \sum_{r=1}^{\infty} R_r \lambda^{-r-1} \in (Y(g) \otimes Y(g))[[\lambda^{-1}]]
\]

such that

\[
(\Delta \otimes id)(R(\lambda)) = R_{13}(\lambda)R_{23}(\lambda), \quad (id \otimes \Delta)R(\lambda) = R_{13}(\lambda)R_{12}(\lambda)
\]

\[
(\tau_\lambda \otimes id)(\Delta^{op}(a)) = R(\lambda)((\tau_\lambda \otimes id)(\Delta(a))R(\lambda)^{-1}.
\]

Furthermore, \(R(\lambda)\) satisfies the Yang-Baxter equation 2.3 and the triangularity condition

\[
R_{21}(-\lambda) = R_{12}(\lambda)^{-1}.
\]

\(R(\lambda)\) is known as the pseudo-universal \(R\)-matrix.
We will need a second presentation, which makes sense for any Cartan data $C_{ij}$, when we consider Varagnolo’s construction in chapter [1].

**Theorem 3.3.3** \( Y(g) \) is isomorphic to the algebra generated by elements $H_{i,r}, X_{i,r}^\pm$ for $i \in I$, with relations:

\[
[H_{i,r}, H_{i,r}] = 0 \\
[H_{i,0}, X_{i,r}^\pm] = \pm d_i C_{ij} X_{j,r}^\pm\\
[H_{i,r+1}, X_{j,s}^\pm] - [H_{i,r}, X_{j,s+1}^\pm] = \frac{\hbar}{2} d_i C_{ij} (X_{i,r}^\pm H_{j,s} + X_{j,s}^\pm H_{i,r}) \\
[X_{i,r}^+, X_{j,r}^-] = \delta_{i,j} H_{i,r+s} \\
[X_{i,r+1}^+, X_{j,s}^-] - [X_{i,r}^+, X_{j,s+1}^-] = \frac{\hbar}{2} d_i C_{ij} (X_{i,r}^\pm X_{j,s}^\pm + X_{j,s}^\pm X_{i,r}^\pm) \\
\sum_{\sigma \in S_m} [X_{i,r(\sigma(1))}^\pm, [X_{i,r(\sigma(2))}^\pm, ..., [X_{i,r(\sigma(m))}^\pm, X_{j,s}] \cdot \cdot \cdot ]] = 0
\]

for all non-negative sequences of integers $r_1, ..., r_m$ where $m = 1 - C_{ij}$.

The coproduct does not have a known closed expression in this presentation. The isomorphism from the first to the second presentation is specified by:

\[
\varphi(H_i) = H_{i,0} \\
\varphi(X_i^\pm) = X_{i,0}^\pm \\
\varphi(J(H_i) - \hbar \nu_i) = H_{i,1} \\
\varphi(J(X_i^\pm) - \hbar \mu_i) = X_{i,1}^\pm
\]

where

\[
\nu_i = \frac{1}{4} \sum_{\beta \in \Delta^+} (\alpha_i, \beta)(X_\beta^+ X_\beta^- + X_\beta^- X_\beta^+) - \frac{H_i^2}{2}\\
\mu_i = \pm \frac{1}{4} \sum_{\beta \in \Delta^+} ([X_i^+, X_\beta^+] X_\beta^+ + X_\beta^+ [X_i^+, X_\beta^+]) - \frac{1}{4} (X_i^+ H_i + H_i X_i^+).
\]
3.4 Representations of $Y(g)$

$Y(g)$ has a family of finite dimensional representations $F_w(a)$ indexed by a fundamental weight $w$ of $g$ and a complex number $a$. When $g = sl_2$, these are the representations $C^2(a)$ mentioned earlier. We write $Rep(Y(g))$ for the braided monoidal abelian category generated by the $F_w(a)$.

3.5 Bethe subalgebras

Given $F_1$ and $F_2$ in $Rep(Y(g))$, one can take the image

$$R_{F_1,F_2}(\lambda) \in \text{End}(F_1,F_2)[[\lambda^{-1}]]$$

of the pseudo-universal $R$-matrix. Fix semisimple element $q \in G$, where $G$ is the simply connected algebraic group associated to $g$.

**Definition 3.5.1** We define the Bethe subalgebra $B_q$ by its image in any given $F \in Rep(Y(g))$: it is generated by the $\lambda$-coefficients of

$$Tr_{F_w(0)}\left((q \otimes \text{id})R_{F_w(0),F}(\lambda)\right) \in \text{End}(F)$$

as $w$ varies over all $g$-weights.

The QYBE for $R(\lambda)$ implies that $B_q$ is commutative. One can use the action of $B_q$ on $F_{w_1}(a_1) \otimes \cdots \otimes F_{w_L}(a_L)$ to define various integrable ‘higher spin chains’; $B_q$ provides the Hamiltonian and higher quantum integrals of motion. In fact, while the Yangian was discovered through the study of the $XXX_1$ spin chain, the integrable Hamiltonians for the $XXX_a$ chain were discovered through the Yangian.
Chapter 4

Quivers

4.1 Introduction

In this chapter we introduce symplectic resolutions, along with a large class of examples called quiver varieties. We describe an action of Yangians on the cohomology of quiver varieties. References for this chapter are [20] and [26].

4.2 Symplectic Resolutions

A symplectic resolution is a holomorphic symplectic variety $X$, such that the canonical map

$$\pi : X \to X_0 = \text{Spec}(H^0(X, O_X))$$

is projective and birational. We assume that $X$ admits an action by a group $G = G \times \mathbb{C}^*$, where $G$ is reductive, satisfying the following conditions:

1. The $G$ action is Hamiltonian;

2. The $\mathbb{C}^*$ action scales the symplectic form by a nontrivial character $\hbar$;

3. The fixed point locus $X^g$ is proper for some $g \in G$. 
The simplest examples are cotangent bundles to homogeneous spaces. We will encounter two other classes of examples in this thesis: quiver varieties and hypertoric varieties.

**Deformations**

The deformations of $(X, \omega)$ are classified by the period map, namely the image of the symplectic form $\omega$ in $H^2(X, \mathbb{C})$:

\[
\begin{array}{c}
X \searrow \\
\downarrow \\
\tilde{X} \searrow \\
\downarrow \\
[\omega] \longrightarrow H^2(X, \mathbb{C})
\end{array}
\]

(4.1)

The fibers of $\phi$ are symplectic resolutions, and the generic fiber is affine. We call primitive effective curve classes $\beta \in H_2(X, \mathbb{Z})$ *primitive coroots*. The pairing $\omega(\beta)$ vanishes along a hyperplane $K_\beta \subset H^2(X, \mathbb{C})$ called a *root hyperplane*; $\beta$ is only effective along $K_\beta$. We call the union $\Delta$ of all $K_\beta$ the *discriminant locus*. Note that the fiber $X_\lambda$ above a generic point $\lambda \in K_\beta$ contains a unique primitive effective curve class $\beta$.

**Steinberg correspondences**

Define the *Steinberg variety* as the fiber product

\[ Z = X \times_{X_0} X. \]

By a result of Kaledin [15], the irreducible components of $Z$ are lagrangian subvarieties of $X \times X$, called *Steinberg correspondences*. The class $L$ of such a correspondence
defines an endomorphism of $H^\bullet_G(X, \mathbb{C})$ by

$$L(\gamma) = p_1^*(L \cap p_2^*(\gamma)).$$

where $p_1, p_2$ are the projections of $X \times X$ onto the factors.

### 4.3 Hyperkähler Reduction

Let $G$ be a reductive algebraic group acting on a hyperkähler space $X$ and preserving a holomorphic symplectic form $\omega$. Pick a linearization $\theta$ of the action, in the sense of GIT. $\omega$ descends to a holomorphic two-form on the GIT quotient $X/\!/_{\theta}G$, but the result will typically be quite degenerate. However, suppose the $G$ action is hamiltonian, i.e. there exists a ‘moment map’ $\mu : X \to \mathfrak{g}^*$ such that for all $\zeta \in \mathfrak{g}$, we have $d\mu(\zeta) = \Omega(X_\zeta, -)$ where $X_\zeta$ is the vector field on $X$ obtained by differentiating the $G$-action along $\zeta$. $\mu$ will automatically be $G$-invariant, and given $\lambda \in \mathfrak{g}^*$, we can consider $Y_{\theta,\lambda} = \mu^{-1}(\lambda)/\!/_{\theta}G$. If one chooses $\lambda$ and $\theta$ appropriately, $Y_{\theta,\lambda}$ will often be smooth, in which case $\omega$ descends to a non-degenerate two form on $Y_{\theta,\lambda}$.

We call this procedure ‘hyperkähler reduction’. It is a way to produce new hyperkähler spaces from old; we will use it to define symplectic resolutions such as quiver varieties and hypertoric varieties.

### 4.4 Quiver varieties

A quiver $Q$ is an oriented multigraph. Write $I$ for the set of vertices and $E$ for the edges. Each edge $e \in E$ has a source $s(e) \in I$ and a target $t(e) \in I$.

A representation of a quiver assigns a vector space $V_i$ to each vertex $i \in I$ and a linear map $X_e : V_{s(e)} \to V_{t(e)}$ to each arrow $e \in E$. Given $Q$, $\overrightarrow{Q}$ is the quiver obtained from $Q$ by adding a new ‘framing’ vertex $i'$ for each old vertex $i \in I$, and
an edge from \( i \) to \( i' \). When discussing representations of \( \overrightarrow{Q} \), we write \( W_i \) rather than \( V_{i'} \). Given \( v, w \in \mathbb{N}^I \), we write \( \text{Rep}^{\overrightarrow{Q}}(v, w) \) for the space of representations of \( \overrightarrow{Q} \) with \( \dim(V_i) = v_i \) and \( \dim(W_i) = w_i \).

Consider the space \( T^*\text{Rep}^{\overrightarrow{Q}}(v, w) \), with coordinates \((X, Y, I, J)\) where \( X \) denotes the edge maps of \( Q \), \( I \) the maps from \( i \) to \( i' \), and \( Y, J \) their respective cotangents. \( T^*\text{Rep}^{\overrightarrow{Q}}(v, w) \) carries a canonical hyperkähler form \( \Omega \). There is an action of the group \( G = \prod_{i \in I} \text{Gl}(V_i) \) on \( \text{Rep}^{\overrightarrow{Q}} \) by reparametrization\(^1\) which induces a hamiltonian action on \( T^*\text{Rep}^{\overrightarrow{Q}}(v, w) \) with moment map \( \mu : T^*\text{Rep}^{\overrightarrow{Q}}(v, w) \to \mathfrak{g}^* \). A linearization of this action is specified by a choice of character \( \chi : G \to \mathbb{C}^* \), i.e. a vector \( \theta \in \mathbb{Z}^I \). Given a \( G \)-invariant moment value \( \lambda \in (\mathfrak{g}^*)^G = \mathbb{C}^I \), we define the associated quiver variety as

\[
\mathcal{M}_{\theta,\lambda}(v, w) := \mu^{-1}(\lambda) \mid_{G}.
\]

For \( \theta \) avoiding a set of ‘root hyperplanes’ \( K_\beta \subset \mathbb{R}^I \), \( \mathcal{M}_{\theta,\lambda}(v, w) \) will be smooth (but possibly empty). For \( \lambda \) avoiding the complexification of \( K_\beta \), \( \mathcal{M}_{\theta,\lambda}(v, w) \) will be smooth and affine. In all cases, the affinization map

\[
\pi : \mathcal{M}_{\theta,\lambda}(v, w) \to \mathcal{M}_{0,\lambda}(v, w) = \text{Spec}(\mathcal{O}(\mathcal{M}_{\theta,\lambda}(v, w)))
\]

is projective and birational onto its image.

### 4.4.1 Group actions

\( \mathcal{M}_{\theta,\lambda}(v, w) \) admits a free symplectic action of

\[
G_w = \prod_{i \in I} \text{Gl}(W_i)
\]

\(^1\)Note that \( G \) acts only at the old vertices; the framing vertices remain ‘parametrized’.
induced from the natural action on $T^*\text{Rep}_Q(v, w)$. We will choose a maximal torus $A \subset G_w$ in what follows. We will sometimes specify the action of a cocharacter $\sigma$ of $A$ by writing $W = W^{(1)} \oplus zW^{(2)}$, meaning $\sigma$ scales $W^{(2)}$ by $z$.

When $\lambda = 0$, the action of $\mathbb{C}^*$ dilating the cotangent fibers of $T^*\text{Rep}_Q(v, w)$ by a character $h$ descends to $\mathcal{M}_{\theta, 0}(v, w)$ and scales the symplectic form by $h$; we write $A := A \times \mathbb{C}^*$.

**Remark 4.4.1** When distinct quiver varieties appear in the same expression, the symbol $A$ may denote distinct tori, as in $H^*_A(\mathcal{M}(v_1, w_1)) \otimes H^*_A(\mathcal{M}(v_2, w_2))$; we hope this does not cause too much confusion.

When $\theta$ is generic, $\mathcal{M}_{\theta, 0}(v, w)$ is thus a symplectic resolution if it is non-empty. For this reason, we’ll mostly focus on the case $\lambda = 0$. We write $\mathcal{M}(v, w) := \mathcal{M}(v, w)_{\theta, 0}$ where $\theta \in \mathbb{N}^I$ is sufficiently generic.

**Example 4.4.2** Let $Q$ be the quiver with one vertex and no edges. Then

$$T^*\text{Rep}_Q(v, w) = \text{Hom}(V, W) \oplus \text{Hom}(W, V)$$

where $V = \mathbb{C}^v$ and $W = \mathbb{C}^w$. The reader can convince themselves that

$$\mathcal{M}_{\theta>0, 0}(v, w) = T^*\text{Gr}(v, w).$$

whereas

$$\mathcal{M}_{\theta<0, 0}(v, w) = T^*\text{Gr}(w - v, w).$$

**Example 4.4.3** Let $Q$ be the $A_n$ Dynkin diagram. Choose $w, v$ as shown in figure 4.4.3 for $n = 5$. Then for appriate choice of $\theta$,

$$\mathcal{M}_{\theta, 0}(v, w) = T^*\text{SL}_{n+1}/B.$$
Figure 4.1: Type $A_5$ quiver with dimension vector, corresponding to $\mathcal{M}(v, w) = T^*SL_6/B$

Example 4.4.4 Let $Q$ have a single vertex with one loop. Then

$$\mathcal{M}_{\theta,0}(n, 1) = \text{Hilb}_n \mathbb{C}^2.$$ 

4.5 Kac-Moody algebra Associated to a Quiver

Quivers without loops are in bijection with symmetric generalized Cartan matrices: $Q$ corresponds to the $|I| \times |I|$ matrix $C_{ij} = 2 - Q_{ij}$ where $Q_{ij}$ is the number of edges between vertices $i$ and $j$. To $C$ (and hence $Q$) we can associate a possibly infinite dimensional Lie algebra $g$, generated by $H_i$, $X_i^\pm$, $i \in I$, subject to the following relations [6]:

$$[H_i, H_j] = 0,$$
$$[H_i, X_j^\pm] = C_{ij}X_j^\pm,$$
$$[X_i^+, X_j^-] = \delta_{ij}H_i,$$
$$(ad_{X_i^\pm})^{1-a_{ij}}(X_j^\pm) = 0, i \neq j.$$
4.6 Action of $g$ on $H^*_A(\mathcal{M}(w))$

Pick a vertex $i \in I$. Write $\delta_i \in \mathbb{N}^I$ for the delta function at $i$.

The affinizations of $\mathcal{M}(v, w)$ and $\mathcal{M}(v', w)$ can be naturally embedded in that of $\mathcal{M}(v + v', w)$, allowing us to define $Z(v, v', w) = \mathcal{M}(v, w) \times \mathcal{M}(v', w)$; its top dimensional irreducible components are Lagrangian in the product. There is map

$$H^*_A(Z(v, v', w)) \to \text{Hom}(H^*_A(\mathcal{M}(v, w)), H^*_A(\mathcal{M}(v', w))) \quad (4.2)$$

given by convolution. Set $Z(w) = \bigcup_{v,v'} Z(v, v', w)$ and $\mathcal{M}(w) := \bigcup_{v \in \mathbb{N}^I} \mathcal{M}(v, w)$. Then we have a corresponding map

$$H^*_A(Z(w)) \to \text{End}(H^*_A(\mathcal{M}(w))).$$

We will often identify elements on the LHS with their image on the RHS.

Consider the correspondence $C_i^+ \subset Z(v, v + \delta_i, w)$ given by all pairs

$$\{(X_1, Y_1, I_1, J_1), (X_2, Y_2, I_2, J_2)\}$$

such that $X_2|_{V^1} = X_1, Y_2|_{V^1} = Y_1, I_2 = I_1, J_2|_{V^1} = J_1$. Here $V^1$ denotes the space $\bigoplus_{i \in I}(V_i \oplus W_i)$ attached to the LHS representation, thought of as a subspace of the RHS representation.

Set $C_i^- = \tau(C_i^+) \subset Z(v, v + \delta_i, w)$, where $\tau : \mathcal{M}(v, w) \times \mathcal{M}(v + \delta_i, w) \to \mathcal{M}(v + \delta_i, w) \times \mathcal{M}(v, w)$ permutes the factors.
Let \( \Delta^+ : C^+_i \to Z(w) \) denote the natural embedding, and likewise for \( \Delta^- : C^-_i \to Z(w) \). Define

\[
x^+_i = \Delta^+_i [C^+_i] \\
x^-_i = \Delta^-_i [C^-_i] \\
h_i = C_{ji}v_j - w_i
\]

**Theorem 4.6.1 (Nakajima)** The operators \( x^+_i, h_i, i \in I \) on \( H^*_A(\mathcal{M}(v,w)) \) satisfy the relations of the corresponding generators of \( g \).

### 4.7 Yangian action

The action of \( g \) from the previous section can fact be extended to \( Y(g) \).

The spaces \( V_i \) and \( W_i \) descend to vector bundles \( \mathcal{V}_i \) and \( \mathcal{W}_i \) on \( \mathcal{M}(v,w) \) with natural \( A \)-equivariant structures. Note that \( \mathcal{W}_i \) is topologically trivial. Upgrade the Cartan matrix \( C \) to a \( K_A(pt)[\sqrt{h}] \)-valued matrix \( \mathcal{C} = 1 + h - \sqrt{h}Q_{jk} \).

Now set

\[
1 + \sum_{r=0}^{\infty} h_{i,r} \frac{h}{z^{r+1}} = \frac{c_{-\frac{1}{2}}(\mathcal{H}_i \otimes (\mathcal{W}_i - \mathcal{C} \cdot \mathcal{V}_i))}{c_{-\frac{1}{2}}(\mathcal{W}_i - \mathcal{C} \cdot \mathcal{V}_i)}. \tag{4.3}
\]

Here \( c_{-\frac{1}{2}}(V) = 1 - c_1(V) \frac{V}{Z} + c_2(V) \frac{V^2}{2Z^2} - \ldots \).

There is a natural line bundle on \( C^+_i \) defined by \( \mathcal{L}^+_k = V^2_k / V^1_k \), and a corresponding bundle \( \mathcal{L}^-_k = \tau_* \mathcal{L}^+_k \) on \( C^-_i \). Define

\[
x^\pm_{i,r} = \sum_{v_2} (-1)^{(\delta_i | v_2)} \Delta^\pm_s(c_1(\mathcal{L}^\pm_k)^r)
\]

Following a similar theorem of Nakajima in the quantum affine case, Varagnolo proved:

---

2 This is a different lift from the one used in [19].
Theorem 4.7.1 The map

\[ X^\pm_{i,r} \rightarrow x^\pm_{i,r} \]
\[ H^\pm_{i,r} \rightarrow h_{i,r} \]

defines a representation of \( Y(g) \) on \( H^\lambda_\Lambda(\mathcal{M}(w)) \).
Chapter 5

Geometric Coproduct

5.1 Introduction

In the previous chapter we saw that \( Y(g) \) acts on \( H^*_{\Lambda}(M(w)) \) for ADE quivers. For a fundamental weight \( \delta_i \), this in fact identifies \( H^*_{\Lambda}(M(\delta_i)) \) with \( F_{\delta_i}(a) \), where \( a \) is the equivariant parameter of \( A \subset A \). For two fundamental weights \( w_1, w_2 \) and generic equivariant parameters \( a_1, a_2 \), one knows from looking at characters that \( H^*_{\Lambda}(M(\omega_1 + \omega_2)) \cong F_{\omega_1}(a_1) \otimes F_{\omega_2}(a_2) \cong F_{\omega_2}(a_2) \otimes F_{\omega_1}(a_1) \). Constructing these isomorphisms was a challenge; see for instance [21].

In [19], the authors reverse the problem and start by defining isomorphisms

\[
\text{Stab}_{12}, \text{Stab}_{21} : H^*_{\Lambda}(M(\omega_1)) \otimes H^*_{\Lambda}(M(\omega_2)) \cong H^*_{\Lambda}(M(\omega_1 + \omega_2))
\]

for any quiver variety, called stable maps. These are used to construct an \( R \)-matrix satisfying the QYBE, and hence a version \( \mathcal{Y} \) of the Yangian in the spirit of [3.2.1]. We will show in the next chapter that for ADE quivers, \( \mathcal{Y} \) contains \( Y(g) \). The results in this chapter are almost exclusively from [19], to which we refer the reader for details and proofs.
5.2 Stable envelopes in symplectic resolutions

let $T = T \times \mathbb{C}^*$ be an algebraic torus acting on a symplectic resolution $X$, such that $T$ acts symplectically and $\mathbb{C}^*$ scales the symplectic form by $\hbar$. Write $t$ for the Lie algebra of $T$ and $\mathfrak{t}$ for the Lie algebra of $T$. We assume that the fixed point locus $X^T$ is proper, and denote the set of its fixed components by $F_T(X)$. Restriction to $X^T$ gives a map in equivariant cohomology:

$$res : H^*_T(X) \rightarrow H^*_T(X^T).$$

The idea of the stable basis is to construct an integral map in the other direction. In order to define the stable basis, we first need a few preliminaries.

5.2.1 Chamber decomposition

The cocharacters $\sigma : \mathbb{C}^* \rightarrow T$ form a full rank integral lattice in $t$. Denote $t_\mathbb{R} = \text{Cochar}(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

The normal bundle to any component of $X^T$ splits as a sum of $T$-equivariant line bundles.

**Definition 5.2.1** The roots of the $T$ action on $X$ are the $T$-weights appearing in the normal bundle to $X^T$.

Each root $\alpha$ defines a hyperplane in $t_\mathbb{R}$. Their union over all fixed loci divides $t_\mathbb{R}$ into open chambers $\mathcal{C}_i$. If a cocharacter $\sigma$ lands in a chamber, then $X^\sigma = X^T$. 
5.2.2 Stable leaves

Fix a chamber $\mathcal{C}$, a cocharacter $\sigma$ landing in it, and $Z \in F_T(X)$. The leaf of $Z$ is defined to be its attractive locus:

$$leaf(Z) = \{ x \in X : \lim_{z \to 0} \sigma(z)x \in Z \}.$$

There is a partial order on $F_T(X)$, defined as the transitive closure of the relation

$$\overline{leaf(Z)} \cap Z' \neq \emptyset \Rightarrow Z \leq Z'.$$

If $X$ is affine, this ordering is trivial: $leaf(Z)$ is already closed.

We define the slope of $Z$ as

$$slope(Z) = \bigcup_{Z' \geq Z} leaf(Z).$$

The slope is closed in $X$. We say that a class $\gamma \in H^\bullet_T(X)$ is supported on $slope(Z)$ if $\gamma|_{X \setminus slope(Z)} = 0$ in $H^\bullet_T(X \setminus slope(Z))$. Here and in what follows we use the restriction symbol to denote the natural restriction maps in equivariant cohomology.

5.2.3 Polarization

For $Z \in F_T(X)$ as above, we have a $T$-invariant decomposition of $N_X Z$, the normal bundle to $Z$ in $X$ as

$$N_X Z = N_+ \oplus N_-$$

into the $T$-weights on which $\sigma$ is positive or negative respectively. The symplectic form yields an identification

$$N_+ = N_-^\vee \otimes \mathcal{h}$$
so that the $T$-equivariant Euler class $e(N_X Z)$ is a perfect square. A choice of $\varepsilon_Z$ for each $Z$ such that $\varepsilon_Z^2 = e(N_X Z)$ is called a polarization of the symplectic resolution. Note that

$$\text{leaf}(Z)|_Z = \pm \varepsilon_Z.$$

### 5.2.4 Definition of stable basis

Fix a chamber $\mathcal{C}$ and polarization $\varepsilon$ as before.

**Theorem 5.2.2 ([19], 3.3.4)** There exists a unique map of $H^*_T(pt)$ modules

$$\text{Stab}_\varepsilon : H^*_T(X^T) \to H^*_T(X),$$

called the stable map so that for any $Z \in F_T(X)$ and any $\gamma \in H^*_T(Z)$, $\text{Stab}_\varepsilon(\gamma)$ satisfies

- (support) $\text{Stab}_\varepsilon(\gamma)$ is supported on $\text{slope}(Z)$;
- (normalization) $\text{Stab}_\varepsilon(\gamma)|_Z = \varepsilon_Z \cup \gamma$;
- (degree) $\text{Stab}_\varepsilon(\gamma)|_{Z'}$ is divisible by $\hbar$ for all $Z' > Z$.

Moreover,

$$\text{Stab}_\varepsilon : H^*_T(X^T) \otimes \mathbb{C}(t) \to H^*_T(X)(t)$$

is an isomorphism.

The construction of the stable basis is done by induction along the downslope fixed components.
5.3 A Geometric R-matrix

Pick two chambers $C_1$ and $C_2$. Since $\text{Stab}_{C_2}$ becomes an isomorphism after tensoring with $\mathbb{Q}(t')$, we can define the ‘$R$-matrix’

$$R_{C_1, C_2} = \text{Stab}_{C_2}^{-1} \circ \text{Stab}_{C_1} \in \text{End}(H^\bullet_T(X^T)) \otimes \mathbb{Q}(t).$$

We will see that it shares many of the properties of the previously mentioned $R$-matrices. It is enough to understand $R_{C_1, C_2}$ for $C_1, C_2$ separated by a wall $\alpha$. Let $t_\alpha \subset t$ be the kernel of $\alpha$, and let $T_\alpha \subset T$ be the corresponding subtorus. $T/T_\alpha$ acts on $\mathcal{M}(w)^{T_\alpha}$ with two chambers, given by $\alpha > 0$ and $\alpha < 0$. Write

$$R_\alpha \in \text{End}(H^\bullet_T(X^{T_\alpha})) \otimes \mathbb{Q}(t/t_\alpha)$$

for the corresponding $R$-matrix. We have $R_{C_1, C_2} = R_\alpha$; we call it a root $R$-matrix. It has an expansion

$$R_\alpha = 1 + \frac{hr_\alpha}{\alpha} + O\left(\frac{1}{\alpha^2}\right)$$

where $r_\alpha$ is a Steinberg operator called the ‘classical $r$-matrix’.

**Example 5.3.1** Let $X = T^*\mathbb{P}^1$ with the action of $T = \mathbb{C}^*$ induced from $\mathbb{P}^1$. Then $X^T = \{0, \infty\}$. Let $u$ be the $T$-weight of $T_0\mathbb{P}^1$ and let $\epsilon$ be the polarization coming from the fibers. Let $C_\bullet^*$ act on the fibers with weight $h$.

$t_\mathbb{R} = \mathbb{R}$ has two chambers corresponding to $u > 0$ and $u < 0$. We leave it as an exercise (see [19]) to check that

$$R_{u<0, u>0}(u) = \frac{1 - \frac{h}{u}P}{1 - \frac{h}{u}}$$

where $P$ permutes $0, \infty$. 

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5.3.1 Vacuum elements of the R-matrix

Pick a fixed component $Z \subset X^T$ such that $Z' \cap leaf(Z) = 0$ for all other components $Z'$. We call this a vacuum component. Let $\gamma_1, \gamma_2 \in H_A^\bullet(Z)$. Then

$$ (\gamma_2, R(u) \gamma_1) = \int_Z \gamma_1 \cup \gamma_2 \cup \frac{eu(N^-)}{eu(N^- \otimes \hbar)} \quad (5.1) $$

5.4 Stable envelopes for quiver varieties

We now set $X$ to be a quiver variety $\mathcal{M}(w)$, where $w = \{w_i\} \in \mathbb{N}^I$. Recall the torus $A \subset G_w$ acting on $\mathcal{M}(w)$. We have

$$ \mathcal{M}(w)^A = \prod_{i \in I} \mathcal{M}(\delta_i)^{w_i}. $$

Set

$$ \mathcal{k} = H^\bullet_C(pt) = \mathbb{C}[\hbar] $$

$$ F_i = H^\bullet_C(\mathcal{M}(\delta_i)) $$

$$ F_i(a) = F_i \otimes \mathcal{k}[a] $$

One can identify $F_i(a)$ with $H^\bullet_A(\mathcal{M}(\delta_i))$. The roots of $A$ are simply the roots of $Sl(\sum_{i \in I} w_i)$. Write $a_{is}, s = 1, ..., w_i$ for the equivariant chern roots of $W_i$. Given a chamber $\mathcal{C} \subset \mathfrak{a}$, i.e. an ordering of the $a_{is}$ where $i$ ranges over $I$, we have a corresponding map

$$ Stab_{\mathcal{C}} : \bigotimes_{i \in I} F_i(a_{i1}) \otimes F_i(a_{i2}) \otimes \cdots \otimes F_i(a_{iw_i}) \rightarrow H^\bullet_A(\mathcal{M}(w)). $$
5.5 R-matrices for quivers

Consider a splitting $W_i = W_i^{(1)} \oplus W_i^{(2)}$, and a subtorus $(\mathbb{C}^*)^2 \subset A$ scaling $W_i^{(k)}$ by $a_k$. Write

$$H_w = H^*_A(M(w))$$

We then get stable maps which we denote

$$\text{Stab}_{12}, \text{Stab}_{21} : H_{w^{(1)}} \otimes H_{w^{(2)}} \rightarrow H_w$$

and we write the corresponding R-matrix

$$R_{w^{(1)},w^{(2)} + w^{(3)}}(a_1 - a_2) = Stab_{21}^{-1} \circ Stab_{12} \in \text{End}(H_{w^{(1)}} \otimes H_{w^{(2)}}).$$

5.5.1 Yang-Baxter equation

Given $(\mathbb{C}^*)^3$ acting on $W^{(1)} \oplus W^{(2)} \oplus W^{(3)}$, we have three R-matrices $R_{ij}(a_i - a_j) \in \text{End}\left(H_{w^{(i)}} \otimes H_{w^{(j)}}\right)$. One easily sees from the definitions that they satisfy the Yang-Baxter equation

$$R_{12}(a_1 - a_2)R_{13}(a_1 - a_3)R_{23}(a_2 - a_3) = R_{23}(a_2 - a_3)R_{13}(a_1 - a_3)R_{12}(a_1 - a_2).$$

5.5.2 Factorization

Let $\mathbb{C}^* < (\mathbb{C}^*)^3$ act on $W^{(1)} \oplus W^{(2)} \oplus W^{(3)}$ with weights $(u, 0, 0)$ resp $(u, a_2, a_3)$. We have

$$R_{w^{(1)},w^{(2)} + w^{(3)}}(u) \in \text{End}(H_{w^{(1)}}, H_{w^{(2)} + w^{(3)}})$$

and an isomorphism

$$\text{Stab}_{23} : H_{w^{(2)}} \otimes H_{w^{(3)}} \otimes \mathbb{C}(h, a_1, a_2) \cong H_{w^{(2)} + w^{(3)}} \otimes \mathbb{C}(h, a_1, a_2).$$
One can prove

\[ Stab_{23}^{-1} \circ R_{w(1),w(2)+w(3)}(u) \circ Stab_{23} = R_{w(1),w(3)}(u - a_1)R_{w(1),w(2)}(u - a_1). \] (5.2)

### 5.6 The Maulik-Okounkov Yangian

We now define a Hopf algebra \( Y \) acting on all tensors \( F_{w(1)}(u_1) \otimes \cdots \otimes F_{w(n)}(u_n) \). It will generalize the Hopf algebra \( Y(g) \) discussed in chapter 3.

Pick \( F_{w_0} \) and \( m \in \text{End}(F_{w_0})(u) \). Set

\[ E(m(u)) = -\frac{1}{\hbar} \text{Res}_{u=\infty} \text{tr}_{F_{w_0}(u)} R_{w_0,w} \in \text{End}(F_w) \]

**Definition 5.6.1** \( Y \) is the algebra generated by all \( E(m(u)) \) as we vary \( m(u) \) and \( w_0 \).

Note that this is very close to the definition 3.2.1. We define a coproduct

\[ \tilde{\Delta} : Y \to \hat{Y} \otimes Y \]

as follows. Let \( K \in Y \) and write \( K_w \) for the image of \( K \) in \( \text{End}(H_w) \). Similarly, for \( L \in \hat{Y} \otimes Y \) write \( L_{w_1,w_2} \) for its image in \( \text{End}(H_{w_1} \otimes H_{w_2}) \). Then\(^1\)

\[ \tilde{\Delta}(K)_{w_1,w_2} = Stab_{12}^{-1} \circ K_{w_1+w_2} \circ Stab_{12}. \]

\(^1\)The reader may object that we have only defined \( \tilde{\Delta} \) at the level of representations, but it is shown in [19] that \( \tilde{\Delta} \) lifts to a genuine coproduct on \( Y \).
5.7 \( gQ \)

The operators \( E(m) \), where \( m \) is a constant polynomial in \( u \), form a lie subalgebra \( gQ \subset Y \) over \( k \). They are the components of the classical \( r \)-matrix, i.e. the \( \frac{\hbar}{u} \) coefficient of \( R(u) \). It follows that for \( g \in gQ \), \( \tilde{\Delta}g = g \otimes 1 + 1 \otimes g \).

We will later show that for ADE quivers, \( gQ \) is a central extension of the corresponding ADE lie algebra. We now review a few facts which we will need for the proof; for more details see [19]. Define

\[
\overline{Q} = \begin{bmatrix}
Q + Q^T & id \\
0 & id
\end{bmatrix}
\]

and

\[
\overline{C} = \begin{bmatrix}
-C & -id \\
id & 0
\end{bmatrix}
\]

Let \( h_Q \) be the space of linear polynomials in \( v_i \) and \( w_i \) (the dimensions of the quiver representation at the vertices). Then \( h_Q \subset gQ \) and is a maximal commutative subalgebra. \( \overline{Q}^{-1} \) defines a non-degenerate bilinear form on \( h_Q \), which extends to a non-degenerate invariant bilinear form \( (\ , \ )_Q \) on \( gQ \).

The roots of \( gQ \) with respect to \( h_Q \) lie in \( \mathbb{Z}^I \). We call a root \( \eta \) positive if \( \eta \in \mathbb{N}^I \), negative if \( -\eta \) is positive. All roots are either positive or negative, and all root-spaces are finite dimensional.

Given a root \( \eta \), define the coroot \( h_\eta = \overline{C}\eta \in h_Q \). Write \( g_\eta \) for the root space attached to \( \eta \). Then \( [\ , \ ] : g_\eta \otimes g_{-\eta} \rightarrow h_Q \) is a perfect pairing with image \( \mathbb{C}h_{\eta} \).

Finally, let \( F^\eta_w \) be the subspace of weight \( w + \eta \). Let \( \eta > 0 \) and suppose \( w(h_\eta) \neq 0 \). Then the \( gQ \) action defines an injection \( g_\eta \rightarrow F^\eta_w \).
5.8 Characteristic classes

Consider $z \in \mathbb{C}^*$ acting on $\mathcal{M}(v, w + \delta_i)$ by $W_i \oplus z\mathbb{C}$. The fixed component $\mathcal{M}(0, w_i) \times \mathcal{M}(v, w) = \mathcal{M}(v, w)$ is called a vacuum component, as in 5.3.1 Formula 5.3 specializes to

\[(\gamma_1, R_{w_i, w_1+w_2}(u)\gamma_2) = \int_{\mathcal{M}(v, w)} \gamma_1 \cup \gamma_2 \cup \frac{eu(V_i)}{eu(V_i \otimes h)} \]  \hspace{0.5cm} (5.3)

One can show using such considerations that all the characteristic classes of $V_i$ and $W_i$ are contained in $\mathcal{Y}$. In fact:

**Theorem 5.8.1** $\mathcal{Y}$ is generated by $gQ$ and the characteristic classes of $V_i$ and $W_i$. 


Chapter 6

\( \mathcal{Y}(g) \subset \mathcal{Y} \) for ADE quivers

6.1 Introduction

For finite type quivers corresponding to the ADE Dynkin diagrams, we now have two different constructions of a ‘Yangian’: one due to Varagnolo and Nakajima, the other due to Maulik and Okounkov. In this chapter we show that the former is contained in the latter, and this inclusion preserves the coproduct. The calculation is quite straightforward, and I’m grateful to the authors of \([19]\) for the opportunity to carry it out.

6.2 Comparing \( g_Q \) and \( g \)

In this section we show that when \( Q \) is of type \( ADE \), the lie algebra \( g_Q \) constructed in \([19]\) is a central extension of the semisimple Lie algebra \( g \) with Dynkin diagram \( Q \).

Let \( \eta \) be a positive root, and let \( h_\eta = \bar{C} \eta \). Recall that if \( w(h_\eta) \neq 0 \), then \( g_\eta \) injects into \( F_w^\eta \). Since \( \mathcal{M}(w) \) is of finite type, \( F_w^\eta = H^*_A(\mathcal{M}(\eta, w)) = 0 \) for \( \eta \gg 0 \). Hence the sum of the positive root spaces is finite dimensional. Since all roots are either positive or negative, \( g_Q \) is finite dimensional.
Recall that $\delta_i \in \mathbb{N}'$ is the delta function at $i$. Consider $F_{\delta_i} = H^\bullet(\mathcal{M}(\delta, \delta)) = \mathbb{C}$. By the above, $g_{\delta_i}$ is at most one-dimensional. The same reasoning for $m > 1$ shows that $g_{m\delta_i} = 0$.

We will now show that $r_{-\delta_i, \delta_i} = x_i^- \otimes x_i^+$. Since $g_{\delta_i}$ is 1-dimensional, it is sufficient to check for $r_{-\delta_i, \delta_i}$ acting on $F_i(a_i) \otimes F_j(a_j)$, and in fact on $F_i(a_i)[1] \otimes F_j(a_j)$ where $F_i(a_i)[1]$ is the weight $C\delta_i - \delta_i$ subspace.

Consider $Z_1 = \mathcal{M}(\delta_i, \delta_i) \times \mathcal{M}(v, \delta_j) \subset \mathcal{M}(\delta_i + v, \delta_i + \delta_j)^A$, and pick a cocharacter acting by $zW_i + W_j$ on the framing.

The slope of $Z_1$ intersects only one other fixed component: $Z_0 = \mathcal{M}(0, \delta_i) \times \mathcal{M}(\delta_i + v, \delta_j)$. Let $L_{Z_1} \subset Z_1 \times \mathcal{M}(\delta_i + v, \delta_i + \delta_j)$ be the component of the stable correspondence attached to $Z_1$. We now characterize

$$L_{Z_1} \cap (Z_1 \times Z_0)$$

The lagrangian components of this intersection give the action of $r_{\delta_i, -\delta_i}$ on $F_i(a_i)[1] \otimes F_j(a_j)$.

A point $\rho \in \mathcal{M}(\delta_i + v, \delta_i + \delta_j)$ satisfies $\lim_{z \to 0} z\rho \in \mathcal{M}(\delta_i, \delta_i) \times \mathcal{M}(v, \delta_j)$ if there is a splitting $V = V' \oplus \mathbb{C}$ such that $X, Y, I, J$ preserve $V' \oplus W_j$, and $J(\mathbb{C})$ has nonzero component in $W_i$. We have $\lim_{z \to \infty} z\rho \in \mathcal{M}(0, \delta_i) \times \mathcal{M}(\delta_i + v, \delta_j)$ if moreover part of $J(\mathbb{C})$ lies in $W_j$.

We conclude that

$$L_{Z_1} \cap (Z_1 \times Z_0) = C_i^+$$

as defined in [4.6] proving

$$r_{-\delta_i, \delta_i} = x_i^- \otimes x_i^+.$$ 

The $g_{\pm \delta_i}$ therefore generate the subalgebra $g$ constructed by Nakajima. We have $g \cap h_Q = \text{span}\{h_{\delta_i}\}$.
Specializing \( h \) to a generic value \( \epsilon \), \( g_Q \otimes \mathbb{C}_\epsilon \) becomes a finite dimensional Lie algebra over \( \mathbb{C} \) with a non-degenerate invariant bilinear form and maximal torus \( h_Q \). Since \( h_Q = \text{span}\{h_\delta\} \oplus \text{span}\{w_i\} \), where the latter summand is central in \( g_Q \), we conclude that the map
\[
g \oplus \text{span}\{w_i\} \rightarrow g_Q \otimes \mathbb{C}_\epsilon
\]
is surjective, hence so is the corresponding map over \( k \). Hence

**Theorem 6.2.1**
\[
g_Q = g \oplus \text{span}\{w_i\}.
\]

We can now write

**Corollary 6.2.2**
\[
\mathbf{r} = \sum_{\beta \in \Delta^+} x^+_\beta \otimes x^-_\beta + x^-_\beta \otimes x^+_\beta + w_i \otimes v_i + v_i \otimes w_i - C_{ij}v_i \otimes v_j.
\]

where \( x^\pm_\beta \) are dual elements in the root spaces \( g_{\pm\beta} \).

Varagnolo’s Yangian is generated by \( g \) and \( H^{(1)}_k \). We conclude that

**Corollary 6.2.3** As algebras, \( Y(g) \subset Y \).

### 6.3 Comparing coproducts

In this section we check that the coproduct \( \tilde{\Delta} : Y \rightarrow Y \otimes Y \) preserves \( Y(g) \) and matches the coproduct \( \Delta \).

Since \( Y(g) \) is generated by \( g \) and \( H_{i,1} \) and \( g \) is primitive, it is enough to understand \( \tilde{\Delta}h_{i,1} \). Recall that by Varagnolo’s prescription,
\[
\frac{c_{-\frac{1}{2}} (h^{-1} \otimes (W_i - C \cdot V_i))}{c_{-\frac{1}{2}} (W_i - C \cdot V_i)} = 1 + \frac{h}{z} h_{i,0} + \frac{h^2}{z^2} h_{i,1} + O\left(\frac{1}{z^3}\right).
\]

(6.1)
For any virtual bundle $E$, we have
\begin{equation}
\frac{c_{-\frac{1}{2}}(h^{-1} \otimes E)}{c_{-\frac{1}{2}}(E)} = 1 - \frac{h}{z}v_kE + \frac{hc_1(E) + \frac{h^2}{2}vrkE(1 - vrkE)}{z^2} + O\left(\frac{1}{z^3}\right) \tag{6.2}
\end{equation}

We conclude
\begin{align*}
h_{i,0} &= C_{ji}v_j - w_i \\
h_{i,1} &= c_1(W_i) - C_{ji}c_1(V_j) + \frac{h}{2}h_i^2 + \ldots
\end{align*}

where the dots stand for linear terms in $v_i, w_i$. The only troublesome term above is $c_1(V_j)$; we compute its coproduct as follows. Write $Z_i = H^\bullet(M(0, w_i)) \subset H^\bullet(M(w_i)) = F_{w_i}$ for the highest weight subspace. Given an operator $K \in \text{End}(F_{w_i} \otimes F_{w_1+w_2})(u)$, write $K^{\text{vac}}$ for the part of $K$ mapping $Z_i \otimes F(u)$ to itself. We will often abuse notation and treat $K^{\text{vac}}$ as an element of $\text{End}(F_{w_1+w_2})(u)$. We have
\begin{equation}
R_{w_i,w_1+w_2}(u)^{\text{vac}} = 1 + \frac{hv_i}{u} + \frac{hc_1(V_i) + \frac{1}{2}h^2v_i(v_i + 1)}{u^2} + O\left(\frac{1}{u^3}\right)
\end{equation}

Moreover, recall that
\begin{equation}
\text{Stab}_{12}^{-1} \circ R_{w_1,w_1+w_2}(u) \circ \text{Stab}_{12} = R_{w_1,w_2}(u - a_2)R_{w_1,w_1}(u - a_1).
\end{equation}
Hence, writing $L^{(1)}$ resp. $L^{(2)}$ for $L \otimes id$ resp $id \otimes L$:

$$\text{Stab}_{12}^{-1} \circ \left( 1 + \frac{\hbar v_i}{u} + \frac{hc_1(V_i) + \frac{1}{2} \hbar^2 v_i(v_i + 1)}{u^2} \right) \circ \text{Stab}_{12}$$

$$= \left( R_{w_1,w_2}(u - a_2)R_{w_1,w_1}(u - a_1) \right)^{\text{vac}} + O \left( \frac{1}{u^3} \right)$$

$$= 1 + \frac{\hbar}{u} (v_i^{(1)} + v_i^{(2)}) + \frac{\hbar}{u^2} \left( \hbar(c_1(V_i)^{(1)} + c_1(V_i)^{(2)} + a_1 v_i^{(1)} + a_2 v_i^{(2)}) + \hbar(r_{02} r_{01})^{\text{vac}} + \frac{\hbar}{2} (v_i^{(1)}(v_i^{(1)} + 1) + v_i^{(2)}(v_i^{(1)} + 1)) \right)$$

$$+ O \left( \frac{1}{u^3} \right)$$

We have

$$(r_{02} r_{01})^{\text{vac}} = v_i \otimes v_i + C_{ij}^{-1}(\alpha_j, \beta) x_\beta^+ \otimes x_\beta^-.$$

The computation is direct, using only the equality $[X_{\beta}^- \otimes 1 + 1 \otimes X_{\beta}^-, \sum g^u \otimes g_a] = 0$

where $g_a, g^u$ are the dual bases of $g$ given by $X_\beta^\pm, H_i$. We deduce

$$\text{Stab}_{12}^{-1} \circ c_1(V_i) \circ \text{Stab}_{12} = c_1(V_i)^{(1)} + c_1(V_i)^{(2)} + a_1 v_i^{(1)} + a_2 v_i^{(2)} + \hbar (r_{13} r_{12})^{\text{vac}} + \frac{\hbar}{2} (v_i^{(2)}(v_i^{(2)} + 1)$$

$$+ v_i^{(1)}(v_i^{(1)} + 1)) - \frac{1}{2} \hbar(v_i^{(1)} + v_i^{(2)})(v_i^{(1)} + v_i^{(2)} + 1)$$

$$= c_1(V_i)^{(1)} + c_1(V_i)^{(2)} + \hbar (r_{13} r_{12})^{\text{vac}} - v_i^{(1)} v_i^{(2)}) + a_1 v_i^{(1)} + a_2 v_i^{(2)}$$

$$= c_1(V_i)^{(1)} + c_1(V_i)^{(2)} + \hbar \sum_{\beta \in \Delta^+} C_{ij}^{-1}(\alpha_j, \beta) x_\beta^- \otimes x_\beta^+ + a_1 v_i^{(1)} + a_2 v_i^{(2)}$$

It follows that

$$\tilde{\Delta} h_{i,1} = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + \hbar \left( h_{i,0} \otimes h_{i,0} - \sum_{\beta \in \Delta^+} (\alpha_i, \beta) x_\beta^- \otimes x_\beta^+ \right) + a_1 v_i \otimes 1 + 1 \otimes a_2 v_i \quad (6.3)$$
Now we compute $\tau_{a_1} \otimes \tau_{a_2}(\Delta H_{i,1})$. We use the explicit coproduct in the presentation \[3.3.1\], together with the isomorphism \[3.3\].

\[
\Delta(J(H_i) - \hbar \nu_i)
\]
\[
= J(H_i) \otimes 1 + 1 \otimes J(H_i) + \frac{\hbar}{2}[H_i \otimes 1, t] + \frac{\hbar}{2}(H_i \otimes 1 + 1 \otimes H_i)^2
\]
\[
- \frac{\hbar}{4} \sum_{\beta \in \Delta^+} (\alpha_i, \beta) \left((X^+_\beta \otimes 1 + 1 \otimes X^+_\beta)(X^-_\beta \otimes 1 + 1 \otimes X^-_\beta) + (X^-_\beta \otimes 1 + 1 \otimes X^-_\beta)(X^+_\beta \otimes 1 + 1 \otimes X^+_\beta)\right)
\]
\[
= (J(H_i) - \hbar \nu_i) \otimes 1 + 1 \otimes (J(H_i) - \hbar \nu_i) + \hbar \left(H_i \otimes H_i - \sum_{\beta \in \Delta^+} (\alpha_i, \beta)(X^+_\beta \otimes X^-_\beta)\right).
\]

Hence

\[
\tau_{a_1} \otimes \tau_{a_2}(\Delta H_{i,1}) = H_{i,1} \otimes 1 + 1 \otimes H_{i,1} + a_1 H_i \otimes 1 + 1 \otimes a_2 H_2 + \hbar \left(H_i \otimes H_i - \sum_{\beta \in \Delta^+} (\alpha_i, \beta)(X^+_\beta \otimes X^-_\beta)\right).
\]

Comparing with \[6.3\], we conclude that the coproducts match on $h_{i,1}$, hence on all of $Y(g)$. 
Chapter 7

Quantum cohomology

7.1 Introduction

Let $X$ be a complex variety. The cup product makes its cohomology $H^\bullet(X, \mathbb{C})$ into a graded commutative ring; Poincare dually, the intersection product counts intersections of cycles. For spaces such as Grassmanians or flag varieties, this ring contains much representation theoretic information. Quantum cohomology is a graded commutative deformation of $H^\bullet(X, \mathbb{C})$ whose structure constants count complex curves intersecting the given cycles, weighed by the area of the curves. It was originally imported from string theory, where it describes the correlation functions of a certain topological sigma model called the $A$-model, but it now has relations to many areas of mathematics.

7.2 Stable maps

Let $X$ be a smooth proper algebraic variety. We are interested in maps $f : C \to X$ with domain a nodal rational (genus 0) curve with $n$ marked points. We ask that the marked points be distinct, and avoid the nodes. Moreover, we require that the map have a finite number of automorphisms, i.e. any contracted component must
contain at least three marked or nodal points. We call such things ‘stable maps’.

Fixing \([f(C)] = \beta \in H_2(X, \mathbb{Z})\), there is a Deligne-Mumford moduli stack \(\overline{M}_{0,n}(X, \beta)\) of stable maps. It is a compactification of the stack \(M_{0,n}(X, \beta)\) of smooth maps. The subscript 0 refers to the genus: one can also define such stacks for higher genus curves, but we won’t need them in this thesis.

\(\overline{M}_{0,n}(X, \beta)\) carries evaluation maps \(ev_1, ..., ev_n: \overline{M}_{0,n}(X, \beta) \to X\) sending a marked point to its image.

### 7.3 Virtual curve counting

Given classes \(\gamma_1, ..., \gamma_n \in H^\bullet(X, \mathbb{C})\), one wants to count the number of curves in \(X\) of given class \(\beta\) intersecting their Poincare duals. Formally, such a count would look like

\[
\int_{[\overline{M}_{0,n}(X, \beta)]} \prod_{k=1}^{n} ev_k^* \gamma_k,
\]

However, this integral is often badly behaved and unsuitable for algebraic constructions. Heuristically, the moduli of stable maps is cut out by certain equations in the space of all (possibly non-holomorphic) maps, and it sometimes happens that these equations are non-generic and produce an unexpectedly large moduli, which must be cut down to size.

One therefore defines a ‘virtual fundamental class’

\[
[\overline{M}_{0,n}(X, \beta)]^{vir} \in H^{c_1(\beta) + \dim X + n - 3}(\overline{M}_{0,n}(X, \beta)),
\]

which behaves much better than the actual fundamental class, and coincides with it in ideal situations such as when \(X\) is a homogeneous space. We then define

\[
\langle \gamma_1, ..., \gamma_n \rangle^{X}_{0,n, \beta} = \int_{[\overline{M}_{0,n}(X, \beta)]^{vir}} \prod_{k=1}^{n} ev_k^* \gamma_k,
\]
Such numbers are called Gromov-Witten (GW) invariants.

**Remark 7.3.1** Although GW invariants are defined in terms of the complex structure on $X$, they are invariant under deformations of the complex structure.

### 7.4 Equivariant GW invariants of a non-proper target space

Given an algebraic group $G$ acting on $X$, there is an induced action of $G$ on $\overline{\mathcal{M}}_{0,n}(X,\beta)$, and one can define $[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\text{vir}} \in H^c_G(\beta + \operatorname{dim}X + n - 3)(\overline{\mathcal{M}}_{0,n}(X,\beta))$. Then $(\gamma_1,\ldots,\gamma_n)_{0,n,\beta}^{X}$ takes values in $H^*_G(pt)$. One can in particular apply equivariant localization to the result; see [13]. If $X$ is not proper but the $G$ action has proper fixed locus, one can still define $(\gamma_1,\ldots,\gamma_n)_{0,n,\beta}^{X}$ using equivariant localization; it will take values in the fraction field $H^*_G(pt)_{\text{loc}}$ of $H^*_G(pt)$. In what follows, we will assume $X$ is equivariantly formal, and the Poincaré pairing $\langle \cdot, \cdot \rangle$ on $H^*_G(X)$ is perfect.

### 7.5 Quantum cohomology

Consider the torus $\mathfrak{H} = H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$. We consider the following partial compactification: let $Eff(X) \subset H_2(X,\mathbb{Z})$ be the cone of effective curve classes on $X$. Let $\mathbb{C}[q^\beta]$ be the commutative algebra generated by symbols $q^\beta, \beta \in Eff(X)$ with relations $q^\beta q^{\beta'} = q^{\beta + \beta'}$. Let $\overline{\mathfrak{H}} = \text{Spec}(\mathbb{C}[q^\beta])$ be the associated toric variety; then $\mathfrak{H} \subset \overline{\mathfrak{H}}$ is the open orbit. Write $\Lambda$ for the ring of formal power series at the point $o \in \overline{\mathfrak{H}}: q^\beta = 0, \beta \neq 0$.

**Definition 7.5.1** The equivariant quantum cohomology ring $QH_G(X)$ of $X$ is the associative, graded-commutative deformation of $H^*_G(X) \otimes H^*_G(X)_{\text{loc}}$ over $\Lambda$, defined
by
\[ \langle \gamma_1 \ast \gamma_2, \gamma_3 \rangle = \sum_{\beta > 0} \langle \gamma_1, \gamma_2, \gamma_3 \rangle^X_{0,3,\beta} \cdot q^\beta, \tag{7.3} \]
where the sum is taken over all effective curve classes.

In the examples of interest to us, the deformation will in fact extend over an open \( \mathcal{U} \subset \mathcal{F} \) containing \( o \) and will be defined over \( H_G^*(X) \). For simplicity, we will restrict to this case for the rest of this chapter.

**Remark 7.5.2** Commutativity is clear, but associativity is less obvious.

**Remark 7.5.3** GW invariants are invariant under deformations of the complex structure, so the quantum cohomology ring should properly be thought of as an invariant of a symplectic manifold. However, there are often analytic difficulties in defining symplectic Gromov-Witten invariants on badly behaved spaces.

### 7.6 Quantum connection

**Definition 7.6.1** We have a surjective map \( H^2_G(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \), yielding a surjective map \( \chi : \mathcal{S}_G = H^2_G(X, \mathbb{C})/H^2_G(X, \mathbb{Z}) \to \mathcal{S} \).

Let \( E \) be the trivial bundle with base \( \mathcal{S}_G \cap \chi^{-1}(\mathcal{U}) := \mathcal{S}_G^\circ \) and fiber \( H^*_G(X, \mathbb{C}) \). Identify \( T_q \mathcal{S}_G^\circ = H^2_G(X, \mathbb{C}) \) for all \( q \in \mathcal{S}_G^\circ \). The quantum connection is defined by
\[ \nabla_u = \frac{\partial}{\partial u} + u \ast \tag{7.4} \]

Here \( \ast \) is the quantum product evaluated at \( q \).

The quantum connection is flat, which follows from (and is equivalent to) the associativity of quantum multiplication. Quantum connections of homogeneous spaces have been known to relate to integrable systems such as the Toda lattice since the work of Givental and Kim [12].
7.7 Mirror symmetry

Suppose one has a family \( Y \to B \) of smooth complex varieties; write \( Y_b \) for the fiber over \( b \). The cohomology \( H^\bullet(Y_b, \mathbb{C}) \) forms a bundle \( E \) over \( B \), containing the lattice bundle \( E_{\mathbb{Z}} \) with fiber \( H^\bullet(Y_b, \mathbb{Z}) \).

**Definition 7.7.1** The Gauss-Manin connection is the flat connection on the cohomology bundle \( E \) obtained by decreeing that all sections with image in \( E_{\mathbb{Z}} \) are flat.

Expressing a connection \( \nabla \) over a base \( B \) as the Gauss-Manin connection of a family \( Y \) over \( B \) provides integral representations for the flat sections of \( \nabla \), and possibly explicit formulae for its monodromy. *Mirror symmetry*, a principle coming from string theory, predicts the existence of such Gauss-Manin representations for the quantum connection. The ‘mirror family’ \( Y \) is obtained, however, by a rather complex geometric procedure, still not fully understood or computable for examples such as quiver varieties.

Mirror symmetry in fact predicts more: the symplectic geometry of \( X \) should be related to the complex geometry of \( Y \) [17]. Since a lot of representation theory is encoded in the symplectic geometry of symplectic resolutions, a construction of their mirrors would be valuable. One may look for candidates for the mirror family \( Y \) by directly finding Gauss-Manin representations of the quantum connection; this is what we do for hypertoric varieties in chapter 9.

7.8 Quantum multiplication in symplectic resolutions

**Basic setting**

We briefly review some of the results on \( QH_G(X) \) presented in [5], where \( X \) is a symplectic resolution and \( G \) is the group introduced in 4.2.
Quantum product from the deformation

For a divisor $D$ and $\beta \neq 0$, we have

$$\langle \gamma_1, D, \gamma_2 \rangle_0^X = (D, \beta) \langle \gamma_1, \gamma_2 \rangle_0^X \beta$$

by the divisor equation (see eg. [2]). If cohomology is generated by divisors, as is the case for hypertoric spaces, the quantum cohomology is thus determined by the two-point invariants.

One can rewrite them as follows. We have

$$(ev_1 \times ev_2)_* [\mathcal{M}_{0,2}(X, \beta)]^{vir} = h L_\beta; \quad L_\beta \in H^{BM,G}_{2\dim X}(X \times X, \mathbb{C}).$$

Each curve lies in a fiber of the affinization map, hence $L_\beta$ lies in the Steinberg variety. It follows from degree considerations that $L_\beta$ is a sum of fundamental classes of components. We have

$$\langle \gamma_1, \gamma_2 \rangle_0^X = h \langle L_\beta(\gamma_1), \gamma_2 \rangle.$$

The following is shown in [5]:

**Proposition 7.8.1** Only multiples of coroots contribute to quantum multiplication by a divisor. In other words, with notation as above, the operator of quantum multiplication by a divisor $u \in H^2_G(X)$ is

$$u * \equiv u \cup + h \sum_{\beta, m \geq 1} (u, m\beta) q^{m\beta} L_{m\beta}(-). \quad (7.5)$$

where $\beta$ ranges over the coroots of the resolution.

In fact, it is enough to understand the two point invariants of a generic fiber $X_\lambda$ for $\lambda \in K_\beta$, as in (4.1). Denote the cycle $(ev_1 \times ev_2)_* [\mathcal{M}_{0,2}(X_\lambda, \beta)]^{vir}$ in $X_\lambda \times X_\lambda$.
by $L_\lambda$. As a non-equivariant cycle, it corresponds to a unique linear combination of fundamental classes of Steinberg correspondences on the central fiber, which we write $Spec(L_\lambda)$. The following is implicit in [5]:

**Proposition 7.8.2**

$$L_\beta = Spec(L_\lambda)$$

where we choose the natural lift of fundamental classes to $T^d \times \mathbb{C}^*$ equivariant correspondences.
Chapter 8

Hypertoric varieties

Hypertoric varieties are the second large class of symplectic resolutions considered in this thesis. They are hyperkahler cousins of toric varieties, and aside from their intrinsic interest they provide a convenient laboratory for the study of more general symplectic resolutions. The reader may find a fuller treatment in [23] and the references within.

8.0.1 Definitions

Consider the torus $T^n = (\mathbb{C}^*)^n$ acting symplectically on $T^*\mathbb{C}^n$. Setting $t^n = \text{Lie}(T^n)$, the moment map $\mu_n : T^*\mathbb{C}^n \to (t^n)^*$ is given by

$$\mu_n(z, w) = (z_1w_1, ..., z_nw_n).$$

Let $T^k \leq T^n$ be an algebraic subtorus, which is not contained in any coordinate hypersurface $T^{n-1} \leq T^n$. Set $T^d = T^n/T^k$ ($d = n - k$), and let $t^k, t^d$ be their respective Lie algebras. We have the exact sequence

$$0 \to t^k \to t^n \to t^d \to 0$$
and, dualizing,

\[ 0 \to (t^d)^* \xrightarrow{a^*_i} (t^n)^* \xrightarrow{\iota^*_n} (t^k)^* \to 0. \]

Throughout this paper, we will often identify elements of \( t^k \) with their images in \( t^n \). Taking \( \mu_k = \iota^* \circ \mu_n \) we obtain a moment map for the \( T^k \) action on \( T^* C^n \). Fix a character \( \theta \) of \( T^k \) and a level \( \lambda \in (t^k)^* \). We define the associated hypertoric variety by

\[ M_{\theta,\lambda} = \mu_k^{-1}(\lambda) / \theta T^k, \]

where we take the GIT quotient with respect to the linearization determined by \( \theta \).

The induced \( T^d \) action on \( M_{\theta,\lambda} \) preserves the holomorphic symplectic form. There is a further action of \( \mathbb{C}^* \) dilating the fibers of \( T^* C^n \), which scales the symplectic form by \( h \). This also preserves \( \mu_k^{-1}(0) \), and visibly descends to an action of \( \mathbb{C}^* \) on \( M_{\theta,0} \) commuting with the \( T^d \) action.

### 8.0.2 Hyperplane arrangements

The geometry of hypertoric varieties can be described by means of a hyperplane arrangement. The Lie algebras \( t^k, t^n \) and \( t^d \) inherit integral structures from the associated tori. Let \( (t^k)^*_\mathbb{R} = (t^k)^*_\mathbb{Z} \otimes \mathbb{R} \), and define \( (t^n)^*_\mathbb{R} \) and \( (t^d)^*_\mathbb{R} \) analogously.

Write \( e_i \) for the standard generators of \( (t^n)^*_\mathbb{Z} \) and \( a_i \) for their images in \( (t^d)^*_\mathbb{Z} \). We will assume that Choose a lift \( \hat{\theta} \) of \( \theta \) to \( (t^n)^* \), with coordinates \( \hat{\theta}_i \). Define hyperplanes \( H_1, ..., H_n \) in \( (t^d)^* \) by

\[ H_i = \left\{ x \in (t^d)^*_\mathbb{R} : a_i \cdot x + \hat{\theta}_i = 0 \right\}. \]
These are the intersections of \((t^d)^* + \tilde{\theta}\) with the coordinate hyperplanes of \((t^n)^*\). Call the collection of oriented affine hyperplanes \(\mathcal{A} = \{H_i\}_{i=1}^n\) a hyperplane arrangement. The arrangement \(\mathcal{A}\) is called

- **Simple** if every subset of \(m\) hyperplanes with nonempty intersection intersects in codimension \(m\);

- **Unimodular** if every collection of \(d\) independent vectors in \(\{a_1, ..., a_n\}\) spans \(t^d\) over \(\mathbb{Z}\);

- **Smooth** if it is simple and unimodular.

The associated hypertoric variety is smooth if and only if the arrangement is smooth. The affinization map is the canonical GIT map \(\mathcal{M}_{\theta,0} \rightarrow \mathcal{M}_{0,0}\), and it is birational. In particular, smooth hypertoric varieties are symplectic resolutions. We assume from now on that \(\mathcal{M}_{\theta,0}\) is smooth. We will also assume that

To reduce clutter, in the sequel we fix \(\theta\) and write \(\mathcal{M}\) for \(\mathcal{M}_{\theta,0}\).

**Example 8.0.3** The hypertoric variety \(T^*\mathbb{P}^n\) is obtained as the quotient of \(T^*\mathbb{C}^{n+1}\) by the action of the diagonal torus. The corresponding hyperplane arrangement is composed of \(n + 1\) hyperplanes bounding a simplex in \(\mathbb{R}^n\). Dividing instead by the complementary torus \(\{((\zeta_1, ..., \zeta_{n+1}) \in (\mathbb{C}^*)^{n+1} | \prod \zeta_i = 1\}\) we obtain a crepant resolution of the singular surface \(xy = z^{n+1}\). The corresponding arrangement is simply \(n + 1\) points on a line.

### 8.0.3 Cohomology

Consider the character of \(T^n\) given by \(\text{diag}(\zeta_1, ..., \zeta_n) \mapsto \zeta_i\). Restricting to \(T^k\), we obtain an induced \(T^d \times \mathbb{C}^*\)-equivariant line bundle on \(\mathcal{M}\), with \(T^d \times \mathbb{C}^*\)-equivariant Euler class \(u_i\), corresponding to the divisor \(z_i = 0\). Recall that \(\hbar\) is the weight of the

\footnote{At this stage, we are not excluding that some of the \(H_i\) may coincide.}
symplectic form under the $\mathbb{C}^*$ action; the divisor $w_i = 0$ thus corresponds to the class $h - u_i$.

**Definition 8.0.4** A circuit $S \subseteq A$ is a minimal subset satisfying $\cap_{i \in S} H_i = \emptyset$. Alternatively, $S$ corresponds to relation in $t^d$

$$\sum_{i \in S^+} a_i - \sum_{i \in S^- \setminus S^+} a_i = 0$$

containing a minimal set of terms. We fix the splitting $S = S^+ \sqcup S^-$ so that if we set

$$\beta_S = \sum_{i \in S^+} e_i - \sum_{i \in S^-} e_i$$

then $\hat{\theta}(\beta_S) > 0$. We can view $\beta_S$ as an element of $t^d_\mathbb{Z} = H_2(\mathcal{M}, \mathbb{Z})$; we will later see that it is a coroot.

**Theorem 8.0.5** [14]

$$H^*_{T^d \times \mathbb{C}^*}(\mathcal{M}) \cong \mathbb{Z}[u_1, \ldots, u_n, h]/I$$

where the ideal $I$ is generated by the relations

$$\prod_{i \in S^+} u_i \prod_{i \in S^-} (h - u_i)$$

for all circuits $S$.

Specializing the equivariant parameters also gives linear relations in $H^*(\mathcal{M}, \mathbb{C})$, as follows. Fix a basis $b_j$ of $t^d$ and let $a_{ij}$ be the corresponding coefficients of $a_i$. Fix dual equivariant parameters $c_j$ for $T^d$. Then

$$c_j = \sum_{i \in A} a_{ij} u_i.$$
Chapter 9

Quantum Cohomology of Hypertoric Varieties

9.1 Introduction

In this chapter we describe the quantum cohomology ring of hypertoric varieties. We will prove:

Theorem 9.1.1 The $T^d \times \mathbb{C}^*$-equivariant quantum cohomology of a smooth hypertoric variety $\mathcal{M}$ is defined over $\mathbb{C} \left[ q^\beta, \frac{1}{1-(-1)^{|S|} q^\beta_S} \right]$, where $S$ runs over all circuits. It is generated by $u_1, \ldots, u_n, h$, subject to the relations

$$\prod_{i \in S^+} u_i \prod_{i \in S^-} (h - u_i) = q^{\beta_S} \prod_{i \in S^+} (h - u_i) \prod_{i \in S^-} u_i$$

(9.1)

for each circuit $S$.

Setting $q^{\beta_S} = 0$ one recovers the relations in classical equivariant cohomology, which were described in [14], although our proof relies on their result.

In line with the general philosophy of [5], we start by studying deformations of $\mathcal{M}$ obtained by varying the level of the moment map. One can find such a deformation
where all effective curve classes are multiples of $\beta_S$, and we show that this curve is contained in a projective bundle over an affine base, where all necessary computations are straightforward. Finally, we deduce the relations (9.1) by specializing to the central fiber.

In section 9.4 we construct a ‘mirror family’ over $(T^k)^\vee$ of complex manifolds $M_q$ equipped with a local system $L_{\hbar,c}$, and prove the following mirror formula:

**Theorem 9.1.2** For generic equivariant parameters, the Gauss-Manin connection on $H^d(M_q, L_{\hbar,c})$ over $(T^k)^\vee$ can be identified with the quantum connection on $H^\cdot_{T^d \times \mathbb{C}^*}(M, \mathbb{C})$ over the same base.

The novel feature of our mirror formula is its extension to the non-symplectic action of the torus $\mathbb{C}^*$, without which the Gromov-Witten invariants of $M$ would be trivial due to its hyperkähler structure. This $\mathbb{C}^*$ action is shared by other hyperkähler spaces like quiver varieties and Hitchin systems, for which a similar situation may hold.

### 9.2 Quantum multiplication by a divisor

In 9.2.1 we recall the following:

**Proposition 9.2.1** Root hyperplanes (see 4.2) are indexed by circuits $S \subset \mathcal{A}$.

To each circuit $S$ corresponds a primitive coroot $\beta_S$. In what follows, it will be convenient to use the following modified parameter:

$$q^S = (-1)^{|S|} q^{\beta_S}. \tag{9.2}$$

In the language of [19], this is the shift by the canonical theta characteristic, as will be clear from the argument below.
Theorem 9.2.2 The operator of quantum multiplication by a divisor $u$ is given by the following formula:

$$u \ast \cdot = u \cup \cdot + \hbar \cdot \sum_S \frac{q^S}{1 - q^S} (u, \beta_S) L_S (\) \) (9.3)$$

where $L_S$ is the specialization of a certain explicit Steinberg correspondence.

The rest of this section is devoted to the proof of the theorem.

9.2.1 Deformation of hypertoric varieties

In [16], Konno identifies the periods of $\omega$ with the level of the moment map. In particular, in the diagram \([4.1]\) the base of the universal family $H^2(\mathfrak{M}, \mathbb{C})$ is isomorphic to $(t^k)^*$, and its fiber over $\lambda \in (t^k)^*$ is the hypertoric variety $\mathfrak{M}_\lambda = \mu_k^{-1}(\lambda)//T^k$.

Our study of the variation of $\lambda$ closely follows Konno’s study of the variation of the stability parameter $\theta$ (equivalently, the level of the real component of the hyperkähler moment map) in loc. cit.

**Discriminant locus**

Let $\{e_i^\vee\}$ be the dual basis to $\{e_i\}$ in $(t^n)^*$.  

**Proposition 9.2.3 ([16])** $\lambda \in (t^k)^*$ is in the discriminant locus iff it lies in a codimension 1-hyperplane spanned by a subset of $\{e_i^\vee\}$.  

Let $\lambda$ be sub-regular, i.e. it lies on a unique root hyperplane

$$K_S = \text{span}(e_i^\vee : i \notin S),$$

for some $S \subset A$. It is an easy exercise to check that $S$ is a circuit. Let $\beta_S$ be the corresponding element of $t^k_\beta$. For simplicity, in the rest of this section we will assume $S = \{1, 2, ..., |S|\}$.  

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Structure of subregular deformations

Proposition 9.2.4 ([16],5.10) Let \((z,w) \in \mu_k^{-1}(\lambda)\). Then \((z,w)\) is \(\theta\)-stable iff either of the following conditions hold:

1. \(z_i \neq 0\) for some \(i \in S^+\);
2. \(w_i \neq 0\) for some \(i \in S^-\).

Proposition 9.2.5 \(\mathcal{M}_\lambda\) contains a codimension \(|S| - 1\) subvariety \(\mathcal{P}^S\), which is a \(\mathbb{P}^{|S|-1}\) bundle over an affine hypertoric variety \(\mathcal{P}^S_0\). All positive dimensional projective subvarieties in \(\mathcal{M}_\lambda\) are contained in \(\mathcal{P}^S\).

Define the codimension \(|S|\) subspace

\[ P^S = \{ w_i = 0 : i \in S^+; z_i = 0 : i \in S^- \} \subset \mathbb{T}^* \mathbb{C}^n, \]

and set

\[ \mathcal{P}^S = (P^S \cap \mu_k^{-1}(\lambda))//_\theta T^k. \]

To construct \(\mathcal{P}^S_0\), let \(p: t^n \to \mathbb{C}^{n-|S|}\) denote the projection onto the last \(n - |S|\) coordinates. Then:

1. \(\ker p|_k = \mathbb{C} \beta_S\);
2. \(p(t^k)^*\), i.e. the dual of the subspace \(p(t^k)\), is canonically identified with \(K_S \subseteq (t^k)^*\);
3. \(\lambda \in p(t^k)^*\).

By abuse of notation we also denote by \(p\) the corresponding map \(T^n \to (\mathbb{C}^*)^{n-|S|}\) and the projection \(T^* \mathbb{C}^n \to T^* \mathbb{C}^{n-|S|}\) given by \((z_i, w_i)_{i=1}^n \mapsto (z_i, w_i)_{i \notin S}\); in particular \(p(T^k)\) acts on \(T^* \mathbb{C}^{n-|S|}\) with moment map \(\mu_{n-|S|}\) landing in \(K_S\). We obtain a hypertoric
variety

\[ \mathcal{P}_0^S = \mu_{n-|S|}^{-1}(\lambda)/\theta p(T^k). \]

Since \( \lambda \) is regular as an element of \( K_S \), \( \mathcal{P}_0^S \) is affine and the stability parameter \( \theta \) is immaterial.

Note that if \((z, w) \in P^S \cap \mu_{n-|S|}^{-1}(\lambda)\), then \( p(z, w) \in \mu_{n-|S|}^{-1}(\lambda) \). Hence we have a map \( \mathcal{P}^S \to \mathcal{P}_0^S \), whose fiber is isomorphic to the quotient of \( C^{|S|} = \{ z_i : i \in S^+, w_i : i \in S^- \} \) by \( C^* = \ker(p) : T^k \to p(T^k) \). By the definition (8.2) of \( S^+ \), \( S^- \) and \( \beta_S \), this quotient is \( \mathbb{P}^{|S|-1} \).

Any point in a positive-dimensional projective subvariety of \( \mathcal{M}_\lambda \) must correspond to a \( T^k \) orbit in \( T^*\mathbb{C}^n \) whose closure intersects the unstable locus. All such orbits are clearly contained in \( P^S \), hence all positive dimensional projective subvarieties are contained in \( \mathcal{P}^S \).

![Figure 9.1: Sample hyperplane arrangement corresponding to a complex 4 dimensional hypertoric variety. There are two circuits of order 2: (1, 2) and (3, 4), corresponding to \( \mathbb{P}^1 \) fibrations. The circuits of order 3 are (1, 3, 5), (1, 4, 5), (2, 3, 5) and (2, 4, 5), and correspond to embedded copies of \( \mathbb{P}^2 \). Note that each circuit encloses a union of (possibly noncompact) chambers corresponding to the moment polytope of the corresponding \( \mathbb{P}^{|S|-1} \) fibration.](image)
9.2.2 Quantum cohomology of $T^*\mathbb{P}^n$

By proposition (9.2.5) all effective curve classes in $\mathcal{M}_\lambda$ are contained in $\mathcal{P}^S$. Further, since the latter fibers over an affine base, any curve is actually contained in a fiber. Since the base $\mathcal{P}_0^S$ is symplectic and the fibers are isotropic, the normal bundle along a fiber is identified with its cotangent bundle. This reduces the computation of Gromov-Witten invariants to the equivariant invariants of $T^*\mathbb{P}^{[S]-1}$.

This is a special case of the computation for cotangent bundles to Grassmannians, worked out in detail in [19] (note: their $\hbar$ is the negative of ours). In the notation of section 7.8 putting $X = T^*\mathbb{P}^{[S]-1}$ and incorporating the shift (9.2), there is a unique effective primitive curve class $\beta$, and we have

$$L_{m\beta} = \left(-\frac{1}{m}\right)^{|S|} [\mathbb{P}^{[S]-1} \times \mathbb{P}^{[S]-1}].$$

We conclude that for a primitive coroot $\beta_S$, on the generic fiber $\mathcal{M}_\lambda$, $\lambda \in K_S$ we have

$$L_{m\beta_S} = \left(-\frac{1}{m}\right)^{|S|} [\mathbb{P}^S \times \mathbb{P}^S]_S.$$  

The correspondence for $\mathcal{M}_{\lambda=0}$ is obtained by specialization, as in (7.8.2). Plugging this into (7.5), the proof of theorem (9.2.2) is concluded.

9.3 Generators and relations for the quantum cohomology of a hypertoric space

In this section we prove that the following relations hold in quantum cohomology, where $S$ runs over the set of circuits of the arrangement, and the notation is as in Theorem 8.0.5

$$\prod_{i \in S_+}^* u_i \prod_{j \in S_-}^* (\hbar - u_i) = q^{\beta_S} \prod_{i \in S_+}^* (\hbar - u_i) \prod_{j \in S_-}^* u_j, \quad (9.4)$$

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We will always decorate quantum products with a star to distinguish them from their drab classical cousins. Note that there is no shift in the deformation parameter.

We begin with a vanishing lemma:

**Lemma 9.3.1** Consider a circuit $S$ and a subset $M \subset A$ such that if $i \in S, i \notin M$, then $M \cup i$ contains no circuits. Choose any splitting $M = M^+ \cup M^-$. Then

\[
L_S \left( \prod_{i \in M^+} u_i \cdot \prod_{i \in M^-} h - u_i \right) = 0.
\]  

(9.5)

Let $\mu_d : \mathcal{M} \to (t^d)^*$ be the moment map for the action of $T^d$ on $\mathcal{M}$. Recall the embedding $a^* : (t^d)^* \to (t^n)^*$. Each $i \in A$ thus determines a hyperplane $F_i$ through the origin of $(t^d)^*$ by restricting the corresponding linear form. Since $\mu$ maps to an affine space, Steinberg correspondences act fiberwise: in fact, the correspondence $L_S$ is supported over the intersection $F_S = \cap_{i \in S} F_i$.

Let $u_M$ be the argument of $L_S$ in (9.5). It is naturally represented by a cycle supported above $F_M$. Since $M$ contains no circuits, $F_M$ has codimension $|M|$ in $(t^d)^*$. Suppose $\text{codim}(F_S \cap F_M) = \text{codim}(F_M) = |M|$. Then $\text{Span}(a_i)_{i \in S} \subset \text{Span}(a_i)_{i \in M}$. Hence given any $i \in S$, $i \cup M$ contains a circuit, contradicting our hypothesis. It follows that $\text{codim}(F_S \cap F_M) > |M|$. Since $L_S$ acts fiberwise, $L_S(u_M)$ is supported above $F_S \cap F_M$. Since $L_S$ is degree preserving and $u_M$ has degree $|M|$, $L_S(u_M) = 0$.

[Proof of 9.1.1] We claim that in the products on either side of (9.4), only the last factor can carry a quantum modification. More precisely, let

\[
v_i = \begin{cases} 
  u_i, & \text{if } i \in S_+ \\
  h - u_i, & \text{if } i \in S_-
\end{cases}
\]

We have $(v_i, \beta_S) = 1$. Choosing $i_0 \in S$, theorem 9.2.2 applied to one of the $v_i$-s and lemma (9.3.1) imply

\[
\prod_{i \in S, i \neq i_0}^* v_i = \prod_{i \in S, i \neq i_0} v_i.
\]  

(9.6)
and

\[ v_{i_0} \ast \prod_{i \in S, i \neq i_0} v_i = \prod_{i \in S} v_i + \frac{h(q^S)}{1 - q^S} L_S \left( \prod_{i \in S, i \neq i_0} v_i \right) \]  \hspace{1cm} (9.7)

and likewise for \( h - v_i \). To see this, note that the factor \( (u_i, \beta_S) \) in Equation 9.2.2 vanishes unless \( i \in S \). Thus Lemma 9.3.1 applies to all quantum corrections except the one appearing in (9.7). Using the classical relations, we can therefore rewrite the relation (9.4) as

\[
\frac{hq^S}{1 - q^S} L_S \left( \prod_{i \in S, i \neq i_0} v_i \right) = (-1)^{|S|} q^S \left( \prod_{i \in S} (h - v_i) - \frac{hq^S}{1 - q^S} L_S \left( \prod_{i \in S, i \neq i_0} h - v_i \right) \right).
\]  \hspace{1cm} (9.8)

We begin by showing

**Lemma 9.3.2**

\[
h L_{\lambda} \left( \prod_{i \in S, i \neq i_0} v_i \right) = (-1)^{|S|} \prod_{i \in S} h - v_i. \]  \hspace{1cm} (9.9)

Choose a generic line \( V \subset K_\beta \) through the origin. Let \( \lambda \in V \setminus \{0\} \). Since all relevant intersections are transverse, it follows from the construction of \( \mathfrak{P}^S \) and the definition of \( v_i \) that in the \( T^d \) equivariant cohomology of \( \mathfrak{M}_\lambda \) we have

\[
L_{\lambda} \left( \prod_{i \in S, i \neq i_0} v_i \right) = (-1)^{|S|} [\mathfrak{P}^S].
\]

Denote the restriction of the family (4.1) to \( V \) by \( \tilde{\mathfrak{M}}_S \). The total space of \( \tilde{\mathfrak{M}}_S \) carries a fiberwise action of \( T^d \); we denote the \( T^d \times \mathbb{C}^* \)-invariant submanifold \( \tilde{\mathfrak{M}}_S \setminus \mathfrak{M} \) by \( \tilde{\mathfrak{M}}_S^\circ \). Let \( \tilde{v}_i \) and \( \tilde{\mathfrak{P}}^S \) be the natural extensions to \( \tilde{\mathfrak{M}}_S \). In the \( T^d \times \mathbb{C}^* \) equivariant cohomology of \( \tilde{\mathfrak{M}}_S^\circ \) we have

\[
h[\tilde{\mathfrak{P}}^S] = \prod_{i \in S} h - \tilde{v}_i.
\]
\(L_S\) similarly extends over \(\widetilde{\mathcal{M}}_S\), and specializing to the central fiber we obtain that the equation

\[
L_S \left( \prod_{i \in S, i \neq i_0} v_i \right) = (-1)^{|S|} \frac{1}{\hbar} \prod_{i \in S} \hbar - v_i.
\]

holds up to a class divisible by \(\hbar\). But in the notation of 9.3.1, such a class must be supported on \(\mu^{-1}_d(F_S)\), which is a subvariety of codimension \(|S| - 1\). Since the class has degree \(|S| - 1\) and is divisible by \(\hbar\), it must vanish. Essentially the same proof shows

**Lemma 9.3.3**

\[
hL_S \left( \prod_{i \in S, i \neq i_0} \hbar - v_i \right) = - \prod_{i \in S} \hbar - v_i. \tag{9.10}
\]

The combination of 9.3.2 and 9.3.3 proves 9.8 hence 9.4. To prove 9.1.1, we must now show that there are no further relations. A basis for \(H^\bullet_{T^d \times \mathbb{C}^*}(\mathcal{M}, \mathbb{C})\) is given by monomials in \(u_i\) containing no circuits, with coefficients in \(\mathbb{C}[\hbar]\) (where a monomial is defined using the classical product). One can use our quantum relations to write any quantum monomial in terms of monomials without circuits, hence the dimension of the algebra defined by our relations is no greater than that of \(H^\bullet_{T^d \times \mathbb{C}^*}(\mathcal{M}, \mathbb{C})\). It follows that the dimensions must be equal. This concludes the proof of Theorem 9.1.1.

### 9.4 Mirror symmetry for hypertoric spaces

In this section we give a mirror formula for the quantum connection of \(\mathcal{M}\).

#### 9.4.1 Quantum Connection

We view \(\hat{\theta} \in H^2_{T^d}(\mathcal{M}, \mathbb{C}) = (t^n)^*\) as a \(T^d\)-equivariant complexified Kähler class.
Definition 9.4.1 Let $E$ be the trivial bundle with base $(T^n)^\vee = Hom(t^n_Z, \mathbb{C}^*)$ and fiber $H_{T^n \times \mathbb{C}^*}(\mathcal{M}, \mathbb{C})$. The basis $e_i$ of $t^n_Z$ defines coordinates $q_i = e^{2\pi i (e_i, \hat{\theta})}$ on $(T^n)^\vee$. The quantum connection is the connection on $E$ defined by

$$\nabla_i = q_i \frac{\partial}{\partial q_i} + u_i *$$

Here $*$ is the quantum product evaluated at $q$. We see from (9.2.2) that the connection is singular exactly along $e^\Delta := \text{Exp}(\Delta) \subset (T^n)^\vee$, where $\Delta$ is the discriminant locus described in (9.2.1), or rather its preimage in $(t^n)^*$ under $\iota^*$. 

9.4.2 Mirror formula

Consider the torus

$$(T^d)^\vee = Hom(t^d_Z, \mathbb{C}^*)$$

Choose a basis $b_j$ of $(t^d_Z)$ and let $t_j$ be the corresponding coordinates on $(T^d)^\vee$. Given $q$ such that

$$q \in (T^n)^\vee \setminus e^\Delta,$$

define complex multiplicative analogues of the hyperplanes from (8.1) by

$$\mathcal{H}_i = \{ t \in (T^d)^\vee \text{ s.t. } q_it^{a_i} = -1 \} \quad (9.11)$$

and define the mirror family

$$\mathcal{M}_q = (T^d)^\vee \setminus \{ \mathcal{H}_i \}_{i \in A}.$$
Let $T^d$ have equivariant parameters $c_j$ dual to the basis $b_j$, and recall that $\mathbb{C}^*$ has parameter $\hbar$. Define a local system $\mathcal{L}_{h,c}$ on $\mathcal{M}$ with monodromy $\hbar$ around the hyperplanes $H_i$ and $-c_j$ around $t_j = 0$. The space $H_d(\mathcal{M}, \mathcal{L}_{h,c})$ is spanned over $\mathbb{C}$ by the lattice of integral cycles, and dually $H^d(\mathcal{M}, \mathcal{L}_{h,c})$ is spanned by a lattice of integral classes. Hence a homotopy class of paths from $q_1$ to $q_2$ avoiding $e^{\Delta}$ yields an identification $H^d(\mathcal{M}_{q_1}, \mathcal{L}_{h,c})$ with $H^d(\mathcal{M}_{q_2}, \mathcal{L}_{h,c})$; this is called the Gauss-Manin connection.

**Theorem 9.4.2** For generic $\hbar$ and $c_j$, there is an isomorphism

$$H^d(\mathcal{M}, \mathcal{L}_{h,c}) \rightarrow H^*_{T^d \times \mathbb{C}^*}(\mathcal{M}, \mathbb{C}) \otimes \mathbb{C}_{h,c_j}$$

taking the Gauss-Manin connection to the quantum connection, where $\mathbb{C}_{h,c_j}$ is the one dimensional $H^*_{T^d \times \mathbb{C}^*}(pt)$ module with parameters $h, c_j$.

We can reformulate Theorem 9.4.2 in terms of a certain differential equation.

**Definition 9.4.3** Let $[\mathcal{M}]$ be the fundamental class viewed as a constant section of $E$. We define the quantum differential equation or QDE as the set of differential relations $P$ satisfied by $[\mathcal{M}]$:

$$P(\nabla_i, q_i)[\mathcal{M}] = 0$$

Since the quantum cohomology of $\mathcal{M}$ is generated by divisors, it is easy to see that knowing the QDE is equivalent to knowing the quantum connection. Now define $\Omega \in H^d(\mathcal{M}, \mathcal{L}_{h,c})$ by

$$\Omega = \prod_{i \in A} (1 + q_i t^{a_i})^h \prod_{j=1}^d t_j^{-c_j} dt_j t_j$$

(9.12)
Choosing $\gamma \in H_d(M_q, \mathcal{L}_{h,c})$ and identifying the homology of nearby fibers using the Gauss-Manin connection, we see that the period

$$J_\gamma(q) = \int_{\gamma \subset M_q} \Omega,$$  \hspace{1cm} (9.13)

is a multivalued function of $q$.

**Theorem 9.4.4** For generic equivariant parameters $c_j$ and $h$, the periods (9.13) form a full set of solutions to the quantum differential equation.

We begin by proving (9.4.4), from which we deduce (9.4.2).

### 9.4.3 QDE of a hypertoric space

Write $a_{ij}$ for the coordinates of $a_i$ in the basis $b_j$.

**Proposition 9.4.5** The QDE of $\mathfrak{M}$ contains the following differential relations:

For all $1 \leq j \leq d$:

$$\sum_{i=1}^{n} a_{ij} \nabla u_i = c_j$$

For all circuits $S$:

$$\left( \prod_{i \in S^+} \nabla u_i \prod_{j \in S^-} (h - \nabla u_j) - q^{\beta S} \prod_{i \in S^+} (h - \nabla u_i) \prod_{j \in S^-} \nabla u_j \right) [\mathfrak{M}] = 0$$

The first equation follows from the linear relations in $H^2_{T_d}(\mathfrak{M})$. The second follows from

$$\left( \prod_{i \in S} \nabla v_i \right) [\mathfrak{M}] = \prod_{i \in S}^* v_i$$

and

$$\left( \prod_{i \in S} h - \nabla v_i \right) [\mathfrak{M}] = \prod_{i \in S}^* h - v_i,$$

which in turn follow immediately from the absence of quantum corrections up till the last factor (9.3.1). In fact these generate all the relations, since their symbols

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generate the quantum relations. This is an example of a GKZ system, as defined in [9]. In the next section we rewrite the above as Picard-Fuchs equations, following [10].

### 9.4.4 Picard-Fuchs equations for $\int_\gamma \Omega$

By partial integration,

$$
\int \frac{\partial}{\partial t'} \left( \prod_{i \in A} (1 + q_i t^{a_i}) \right) \prod_{k=1}^d t_k^{-c_k} \frac{dt_k}{t_k} + \int \prod_{i \in A} (1 + q_i t^{a_i})^h \frac{\partial}{\partial t} \left( \prod_{k=1}^d t_k^{-c_k-1} \right) dt_k = 0.
$$

(9.14)

Set $E_i = q_i \frac{\partial}{\partial q_i}$. Then

$$
E_i \Omega = \hbar \frac{q_i t^{a_i}}{(1 + q_i t^{a_i})} \Omega.
$$

(9.15)

By (9.14) we have

$$
\left( \sum_i a_{ij} E_i - c_j \right) \int_{\gamma \subset M_q} \Omega = 0.
$$

(9.16)

Now let $S$ be a circuit corresponding to a relation $\sum_{i \in S^+} a_i - \sum_{i \in S^-} a_i = 0$. Then by direct calculation,

$$
\left( \prod_{i \in S^+} E_i \prod_{i \in S^-} (h - E_i) - q_\beta S \prod_{i \in S^-} E_i \prod_{i \in S^+} (h - E_i) \right) \Omega = 0.
$$

(9.17)

Equations (9.16) and (9.17) show that $J_\gamma(q)$ satisfies the GKZ system under the correspondence $E_i \rightarrow \nabla_i$. The system is called **non-resonant** [10] if $J_\gamma(q)$ satisfies no other relations; we prove that our system is non-resonant for generic $(h, c_j)$ in appendix 1. For such a non-resonant system, the integrals $J_\gamma(q)$ for $\gamma \in H_*(M_q, L_{h,c})$ span exactly the solution space, thus concluding the proof of Theorem 9.4.4. Theorem 9.4.2 follows by identifying $P(E_i, q)\Omega$ and $P(\nabla_i, q)[M]$ for all polynomials $P$.

**Remark 9.4.6** The ‘mirror space’ $M_q$ is half the dimension of $\mathcal{M}$. One can view it as the target of a ‘multiplicative’ moment map [1] arising from a hyperkähler
action of $T^d$ on a multiplicative analogue of $\mathcal{M}$, of the same dimension $[27]$. The affine subtori which we remove from $(T^d)^\vee$ are simply the locus where the moment fibers degenerate.

**Remark 9.4.7** Writing $\Omega = \text{Exp}(Y_q) \prod_j d\log(t_j)$, where the ‘superpotential’ $Y_q$ is a multi-valued function on $\mathcal{M}_q$, we can rephrase the above result as a presentation of the equivariant quantum cohomology of $\mathcal{M}$ as the spectrum of the critical locus of $Y_q$, in the spirit of [11].

### 1 Resonant parameters of a GKZ system

References for this section are [10] and [24]. For certain values of the parameters $(\hbar, c_j)$, the space of periods of $\Omega$ does not surject onto the space of solutions. One can guarantee a surjection by choosing a ‘non-resonant’ parameter; we now define these parameters and show they are generic.

Let $\mathcal{A} = \{1, 2, ..., n\}$ (resp. $\mathcal{A}^* = \{1^*, ..., n^*\}$) index the classes $u_i$ (resp. $h - u_i$). Given a split circuit $S = S^+ \cup S^-$ as in the quantum relation (9.4), one obtains a pair $S^L, S^R \subset \mathcal{A} \cup \mathcal{A}^*$, $S^L = \{i \in S^+\} \cup \{i^* \in S^-\}$, $S^R = \{i \in S^-\} \cup \{i^* \in S^+\}$ corresponding to the factors on the left (resp. right) of the quantum relation.

**Definition 1.1** We call a collection $Q \subset \mathcal{A} \cup \mathcal{A}^*$ saturated if for every $S$, either $Q \cap S^L = Q \cap S^R = \emptyset$ or both intersections are nonempty. We call $Q$ minimal saturated if it is non-empty and minimal with respect to this property.\footnote{In the set-up of [10], such $Q$ correspond to toric divisors in the support of the Fourier transform of the GKZ D-module.}

Given $Q$, let $Q^c = \mathcal{A} \cup \mathcal{A}^* \setminus Q$ and let $\text{Lin}(Q^c)$ be the linear span in $\mathbb{C}^n \oplus \mathbb{C}^d$ of $\{e_i \oplus a_i : i \in Q^c\} \cup \{e_i \oplus 0 : i^* \in Q^c\}$. Given a parameter $(\hbar, c_j)$, set $v_{\hbar, \alpha} = (\hbar, h, ..., h, c_j) \in \mathbb{C}^n \oplus \mathbb{C}^d$. This is the usual GKZ parameter for our system; it lies in the subspace $V_n \subset \mathbb{C}^n \oplus \mathbb{C}^d$ whose first $n$ coordinates are identical.
Definition 1.2 \((h, c_j)\) is non-resonant if for each minimal saturated \(Q\), we have \(v_{h,\alpha} \notin \text{Lin}(Q^c) + \mathbb{Z}^{d+n}\).

Theorem 1.3 [10] For non-resonant parameters, the space of Euler integrals \((9.13)\) spans the space of solutions to the GKZ system \((9.4.5)\).

We now show that non-resonant parameters are generic. We will show that for each minimal saturated \(Q\), \(\text{Lin}(Q^c)\) intersects \(V_n\) in a strict subspace.

Suppose this fails for some \(Q\). \(Q^c\) must contain a collection of pairs \(\{i, i^*\}_{i \in I \subseteq A}\) such that \(\{a_i\}_{i \in I}\) span \(t^d\). \(Q^c\) also clearly contains either \(i\) or \(i^*\) for all \(i \in A\). However, since \(Q\) is non-empty, for some \(i_0\) we have either \(i_0 \in Q, i^*_0 \in Q^c\) or \(i^*_0 \in Q, i_0 \in Q^c\); suppose the former case holds. Since the \(\{a_i\}_{i \in I}\) are a spanning set, a non-empty subset of them appear alongside \(i_0\) as the indices of some circuit \(S\). Since \(Q\) is saturated, \(Q\) must also contain some \(i\) or \(i^* : i \in I\). This is a contradiction. The same reasoning holds for the latter case. We have proved

Lemma 1.4 There is a generic set of non-resonant parameters \((h, c_j)\) for the GKZ system \((9.4.5)\).

Theorem \((9.4.4)\) follows immediately.
Appendix A

Conclusion

A.1 Future Work

Any quiver variety $\mathcal{M}(v, w)$ admits a hypertoric ‘abelianization’ $\mathcal{M}(v, w)_{ab}$; when constructing $\mathcal{M}(v, w)$, simply take the hyperkähler reduction by a maximal torus of $\prod_{i \in I} GL(V_i)$ instead. One may hope to relate the quantum cohomology of $\mathcal{M}(v, w)$ to that of $\mathcal{M}(v, w)_{ab}$, thus providing a generators and relations presentation for a general quiver. This would yield Bethe-type equations for the spectrum of more general ‘integrable spin chains’.

One may also hope to prove a version of homological mirror symmetry for hypertoric varieties, using our mirror formulae for the quantum connection as a guide; we expect the multiplicative version of a hypertoric variety to play an important role.

Finally, Nick Proudfoot has pointed out that the quantum cohomology of hypertoric varieties appears to specialize (in an appropriate sense) to the intersection cohomology of their affinization for a specific value of $q$. Explaining this phenomenon and generalizing it to other symplectic resolutions would be highly desirable.
Bibliography


