FINANCIAL MODELS FOR COMMODITY, ENERGY AND EQUITY MARKETS

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE PROGRAM IN
APPLIED AND COMPUTATIONAL MATHEMATICS
ADVISER: PROFESSOR RONNIE SIRCAR

NOVEMBER 2015
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Abstract

In this thesis, we propose several financial models to better understand the dramatic price behavior observed in the commodity and energy markets over the past decade. In the first part, we propose a feedback model to account for the “financialization” of commodities, referring to the increased correlations between the commodity and equity markets since the early 2000s. This is conjectured to be due to the influx of external (portfolio optimizing) traders through commodity index funds, for instance. We build a feedback model to capture some of these effects, in which traditional economic demand for a commodity, oil say, is perturbed by the influence of portfolio optimizers. We approach the full utility maximization problem with price impacts through a sequence of problems that can be reduced to linear PDEs, and we find correlation effects proportional to the long or short positions of the investors, along with a lowering of volatility.

The second part of this thesis is motivated by the recent free-fall in oil prices, from around $110 per barrel in June 2014 to less than $50 in January 2015. We apply the theory of mean field games to study the competitions between different energy sources. Indeed, the sustained price drop has been primarily attributed to OPEC’s strategic decision not to curb its oil production despite increased supply of shale oil in the US. In this context, we study how Cournot competitions can be analyzed as dynamic mean field games and illustrate how the traditional oil producers may react in counter-intuitive ways in face of competition from alternative energy sources.

In the third part, we apply techniques of optimal control to analyze a class of dynamic portfolio optimization problems that allow for models of return predictability, transaction costs, and stochastic volatility. We propose a multiscale asymptotic expansion when the volatility process is characterized by its time scales of fluctuation. The analyses of the nonlinear Hamilton-Jacobi-Bellman PDEs under fast mean-reverting and slowly fluctuating volatilities can be effectively combined for
multifactor multiscale stochastic volatility models. We present formal derivations of asymptotic approximations and demonstrate how the proposed algorithms improve our Profit&Loss using Monte Carlo simulations.
Acknowledgements

First of all, I would like to express my deepest gratitude to my adviser, Professor Ronnie Sircar, for his excellent guidance and encouragement. Thank you for guiding me through my doctoral research, and for being an invaluable mentor. I also appreciate the fellowship support from Princeton University the CV Starr Fellowship program, as well as the research and teaching assistantships I received while at Princeton.

I would also like to thank Professor René Carmona, Professor Patrick Cheridito, and Professor Benjamin Moll for helpful and stimulating discussion. Lastly I have benefited from enlightening mathematical discussions with my great officemates and friends, who provided an intellectually stimulating and fun environment at Princeton. Qianxiao Li, Yidong Dong, Jiequn Han, Borui Liu, Yuehaw Khoo and Cheng Tai, I would like to let you know how grateful I am to have you.

Thank you mom, dad and my two dear sisters for supporting me in every step of my journey. Thank you for always encouraging me to pursue my interests and for showering me with an abundance of love.

A special thanks to you, for all your love and support. Thank you for always being by my side, through laughter and tears. Thank you for letting me wake up every morning with a smile on my face. Jag älskar dig.
To my parents.
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Chapter 1

Introduction

This dissertation is devoted to better understand the dramatic price behavior of the commodity and energy markets (see Figure 1.1). The last decade has witnessed a quadruple price hike of crude oil from below $30 per barrel in 2003 to over $140 in 2008, followed by some 75% drop within three months in the aftermath of the Lehman Brothers bankruptcy, and more recently a staggering free-fall in oil price by more than half within six months in 2014. Applying tools from financial mathematics, in particular the theory of stochastic control and dynamic games, we illustrate that many key challenges one faces in the energy and commodity markets can be confronted from a quantitative point of view, using modern mathematical, statistical and numerical methods.

1.1 World energy market

2000s energy crisis For much of the two decades prior to 2003, the WTI crude oil price was under $30 per barrel. A series of events during 2003 had led the price to rise above $30. After briefly retreating for several months in late 2004 and early 2005, the oil price exceeded $65 by August 2005, reached $75 by the middle of 2006, and peaked at $145 in July 2008. The spectacular oil price hike has been attributed
to many factors, including the weakening of the US dollar, worries over peak oil due to a decline in fossil fuel reserves, political instability within various oil producing nations, and oil speculation.

Soon afterward, the unprecedented rise in oil price was followed by an even more spectacular crash in 2008. In a period of merely three months, a steady decline turned into a collapse after the onset of the global financial crisis in mid-September 2008. By October 2008, the price of oil had dropped some 60% down to below $65, and then bottomed out at $32 by the end of December 2008. While the collapse in oil prices was likely caused by the product of excessive financial speculation, a stronger US dollar, and a decline in European demand due to the global economy; many referred to this boom-and-bust as a bubble as futures prices far exceeded fundamental values as determined by the balance of demand and supply. The large scale speculative buying by commodity index funds was held as culprit.

“Financial markets tend to produce economic bubbles, and those bubbles tend to burst,” Rosneft CEO Igor Sechin wrote in an article for the Financial Times\footnote{Source: http://on.ft.com/1L0o4sV} In
late June and early July 2008, suspecting that a top was near, speculators in the oil futures market battled one another to unwind their position and exit the market. On July 15, 2008, a bubble-bursting sell-off began after remarks by President George W. Bush the previous day that the ban on oil drilling would be lifted. In the ensuing weeks, oil would come crashing down to earth as hedge funds, banks and pension funds unwound their positions. By September 2008, oil price fell below $100 for the first time in over six months, dropping below $92 in the aftermath of the Lehman Brothers bankruptcy. Not only did the financial crisis cause a global economic recession that decreased demand for oil, but the crisis also hurt the risk appetite of speculative investors for risky commodities in their portfolios, consequently pushing oil prices down even further.

The increasing role of institutional investors in the commodity markets has been termed the financialization of commodity markets. It is generally accepted that the commodity price dynamics has been depending increasingly heavily to the aggregate risk appetite and the investment behavior of commodity index investors. While the balance of supply and demand and traditional economic factors may have played a role in the rising, and subsequently crashing, of commodity prices, we seek to provide a possible explanation for the dramatic oil price swings by means of a mathematical model which incorporates the price impacts of commodity index investing by institutional investors.

2014-2015 global oversupply In 2009, the oil price recovered along with the economy, exceeding $100 in 2011 and most of 2012; and had enjoyed a relatively long period of stability for most of the first half of 2010s. Indeed, for much of the past decade, oil prices have been high – bouncing around $100 a barrel due to strong demand and frequent supply disruptions. Oil production by conventional means could not keep up with demand, so prices spiked. It was in fact the stated goal of OPEC
to “stabilise this oil price and keep it at a level around $100 [a barrel]”.\textsuperscript{2} This policy had worked until oil price went into free-fall in the second half of 2014, dropping by more than half over the course of six months.

The recent decline of oil price has indeed become one of the biggest energy story in the world. Back in June 2014, the price of crude oil was up around $115 per barrel. As of January 2015, it had fallen by more than half, down to $49 per barrel. Many OPEC’s member countries, like Saudi Arabia and Iran, need higher prices to balance their budgets. As prices plummeted, many market analysts expected to see OPEC reducing production to push prices back up. However, it surprised most market watchers that OPEC decided not to cut supply and to maintain their oil production quota at its annual meeting held in Vienna, Austria, on 27th November 2014.

The widely reported reason for the price drop, as well as OPEC’s surprising decision to do nothing, is fracking. Back in the mid-2000s, surging energy demand from developing countries like China and political instability in key oil nations like Iraq led to escalating oil prices. Between 2011 and 2014, oil prices have been hovering around $100 per barrel. Given the high oil prices, many energy companies found it profitable to begin extracting oil from difficult-to-drill places. In the United States, companies began using techniques like hydraulic fracturing and horizontal drilling to extract oil from shale formations in North Dakota and Texas. On the one hand, this new technology has added over 4 million extra barrels of crude oil per day to the global market since 2008 (compared to global production of about 75 million barrels per day).\textsuperscript{3} This surge in supply, together with a lack of demand due to sluggish global economic growth, led to a fall in oil price of nearly 50% over the second half of 2014.

On the other hand, fracking is relatively expensive to setup and to produce oil from. Producing shale oil from techniques like fracking costs more per barrel to break-even compared to conventional sources in the Middle East. The speculation is

\textsuperscript{2}Source: edition.cnn.com/2012/01/16/world/meast/saudi-oil-production/index.html\textsuperscript{3}Source: \url{http://www.eia.gov/forecasts/aeo/}
that OPEC is now engaged a “price war” with the fracking industries in the United States; by keeping the oil prices low, they seek to drive the US frackers out of business and regain market share. Broadly speaking, we investigate whether there is a game theoretic explanation for OPEC’s decision not to reduce supply in face of increased competition from the fracking industry. In other words, can we frame OPEC’s surprising decision as an optimal or equilibrium strategy in a game theoretic model?

1.2 Contributions of this thesis

The importance of better understanding the oil price behavior, and more broadly speaking the global energy production market, cannot be overstated. While energy markets are complex, with lots of pressing issues including financial speculation, exhaustibility of fossil fuels, concerns over climate change, etc., the crux of this thesis is that tremendous progress can be made in the understanding of the commodity and energy markets with the help of modern financial mathematics. What we accomplish in this thesis is to isolate pieces of the commodity and energy markets and build financial models to capture, explain and rationalize some of the most interesting issues and interactions among its participants, using the tools of stochastic analysis, simulation, differential equations, and stochastic control and optimization. Below we briefly describe our contributions to the literature in Chapters 2 to 5.

1.2.1 Financialization of Commodity Markets

The spectacular rise and fall of oil price during 2007-2009 serves to illustrate the financialization of commodity markets. The essential observation is that commodity prices are moving more in sync with financial markets throughout the 2000s, and in contrast to previously. This increase in correlation is illustrated in Figure 1.2 which shows the evolution of the time-dependent “beta” of the least squares linear
Financialization of the Commodities Markets varies with time. See for example section 7.5.2 entitled Linear Models with Time Varying Coefficients of the textbook [6] for details. The standard commodity indexes are reviewed in section 3, and a new generation of roll yield optimizing indexes is introduced in section 5.

![Time Series Plot of BETA.ts](image)

Figure 1.2: *Instantaneous Dependence (β) of the daily GSCI-TR returns upon the corresponding S&P 500 returns.* Source: Financialization of the Commodities Markets: A Non-technical Introduction of the volume [1].

regression of the Goldman Sachs Commodity Index (GSCI) Total Return against the returns of the S&P 500 index. We refer to the chapter entitled *Financialization of the Commodities Markets: A Non-technical Introduction* of the volume [1] for implementation details. It is generally accepted that correlations between equity returns and commodity index returns have increased significantly due to the increased leverage and the exponential growth of speculations in the oil futures market dwarfing the real balance of supply and demand.

This increased correlation is conjectured to be due to the influx of external traders through commodity index funds, for instance. The increase in speculative behavior in the commodity market has caught much attention of the press, and many market analysts attribute the wild swings in oil prices during 2007-2009 to Wall Street speculators who are gambling in the loosely regulated commodity markets for gas and oil. As Senator Bernie Sanders puts it:

> A decade ago, speculators controlled only about 30% of the oil futures market. Today, Wall Street speculators control nearly 80% of this market.

Oil and gas prices have almost nothing to do with economic fundamentals. 

...speculation was driving up the price of a barrel of oil by as much as 40%

In Chapter 2 we build a feedback model to try and capture some of these effects, in which traditional economic demand for a commodity, oil say, is perturbed by the influence of portfolio optimizers. Modelling of feedback effects due to program trading typically begins with an economy of two types of investors, the first whose behavior upholds a reference model\footnote{The reference model is typically chosen to be the Black-Scholes model in the case of equity market; however in our case of commodity markets, the Schwartz one-factor model is more appropriate.} and the second who trade to maximize utility or to insure other portfolios. In other words, the problem can be considered as a portfolio optimization problem in the presence of price impact; and as we shall see, this price impact will precisely induce a cross correlation between the equity and commodity markets.

We approach the full problem of utility maximizing with a risky asset whose dynamics are impacted by trading through a sequence of problems that can be reduced to linear PDEs. We find the explicit form of the perturbed commodity price in the presence of investor feedback effects, and we develop observations on the changes in volatility, and correlation with stocks, that the demand of the portfolio optimizers creates. Consistent with empirical observations, we find correlation effects proportional to the long or short positions of the investors, along with a lowering of volatility. Chapter 2 is an adapted version of Chan et al. \cite{31}.

1.2.2 OPEC’s War on Fracking

The spectacular drop in oil price, from above $110 in June 2014 to below $50 in January 2015, demonstrates the results of strategic interactions between energy producers of different sources. Indeed, the price drop has been sustained by OPECs
strategic decision not to curb its oil production in the face of increased supply of shale gas and oil in the US.

Oil price is, in some sense, determined by a small number of large players, in particular the Organization of Petroleum Exporting Countries (OPEC) – a cartel of oil producing countries in middle east controlling about 40% of global energy supply. Given that many OPEC member countries depend on oil price at around $80 per barrel to balance their budgets, it was widely expected at the end of 2014 that OPEC would control the supply and restore oil price to a higher level. However, when the OPEC met in November 2014, it surprises most market watchers that they decided not to cut supply and maintain their oil production quota. Broadly speaking, the question we ask is whether there is there a game theoretic explanation for OPEC’s decision not to cut oil supply, and whether one can frame OPEC’s decision as an optimal strategy or equilibrium strategy in a game theory model.

The surprising decision of OPEC not to cut production in face of excessive oil supply due to fracking has caught much attention of the press. Underlying this “price war” in the crude market between OPEC and the US frackers is the issue of heterogeneity in production costs – fracking is a relatively expensive new technology, producing shale oil from fracking costs more per barrel than producing traditional sources from the middle east (See Figure 1.3). In other words, by keeping oil price at $50-60 per barrel, at which level fracking is not profitable, OPEC aims to force the fracking industry out of business and regain its role as the world’s dominant energy supplier.

The important issue underlying their interactions is heterogeneity, in terms of exhaustibility, production costs, and emissions. For example, fossil fuels such as oil and coal are cheap, exhaustible, and dirty in terms of emissions; while solar, wind, and hydroelectric power are relatively expensive to set up and produce from, inexhaustible to all intents and purposes, and clean in terms of emissions. The purpose
At the broadest level, we model energy production market as an oligopoly, in which the resolution of the competition is resolved by Nash equilibrium. In this context, it is natural to think of the energy production market as Cournot competition, in which firms set production quantity and market price reacts to the supply. The complementary framework, or Bertrand competition, assumes that firms set prices and receive a quantity demanded. For the energy market, the Cournot framework seems more reasonable, as in the expected scenario of OPEC reducing production quotas to increase oil price.

In Chapter 3 we focus on a single dimension of the issue of heterogeneity – we study the competitive behavior of energy producers facing exhaustibility (e.g. fossil
fuels). We study how continuous time Bertrand and Cournot competitions, in which firms producing similar goods compete with one another by setting prices or quantities respectively, can be analyzed as continuum dynamic mean field games. Interactions are of mean field type in the sense that the demand faced by a producer is affected by the others through their average price or quantity. Motivated by energy or consumer goods markets, we consider the setting of a dynamic game with uncertain market demand, and under the constraint of finite supplies (or exhaustible resources).

The continuum game is characterized by a coupled system of partial differential equations: a backward HJB PDE for the value function, and a forward Kolmogorov PDE for the density of players. Asymptotic approximation enables us to deduce certain qualitative features of the game in the limit of small competition. The equilibrium of the game is further studied using numerical solutions, which become very tractable by considering the tail distribution function instead of the density itself. This also allows us to consider Dirac delta distributions to use the continuum game to mimic finite \( N \)-player nonzero-sum differential games, the advantage being having to deal with two coupled PDEs instead of \( N \). We find that, in accordance with the two-player game, a large degree of competitive interaction causes firms to slow down production. The continuum system can therefore be used qualitative as an approximation to even small player dynamic games.

In Chapter 4, we extend the baseline model of Chapter 3 and demonstrate how the introduction of an alternative energy producer to a dynamic oligopoly of traditional energy producers can induce “game changers” in the global energy market. Within the framework of continuum mean field game models, we illustrate how the traditional oil producers may react in counterintuitive ways in face of competition from alternative energy sources. We investigate the competitive interactions between a single alternative energy producer to a continuum of traditional oil producers from
the perspective of three distinct time scales (see Table 1.1). Chapters 3 and 4 are based on the results of Chan and Sircar [28, 29].

Table 1.1: *Three distinct time scales representing different idealizations of the global energy market.*

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Global Energy Market</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Long-term</strong></td>
<td>Over a longer horizon, alternative energy sources become more competitive as their production costs decrease further due to technological advances. While traditional fossil fuels are not necessarily depleted, renewable energy gains considerable market share due to increasing (scarcity) costs of fossil fuel extraction. We study how the global economy transitions from the traditional/exhaustible energy production to its renewable counterpart.</td>
</tr>
<tr>
<td><strong>Intermediate</strong></td>
<td>Over an intermediate horizon, the production of traditional energy (fossil fuels e.g. oil) is still the cheapest. However, alternative energy sources (shale oil or solar power) are gaining market shares due to decreasing production costs. Traditional energy producers may strategically increase their production rate to compete for market share with the alternative energy producers. This may be the logic behind OPEC’s decision not to cut crude oil output.</td>
</tr>
<tr>
<td><strong>Short-term</strong></td>
<td>In short time frame, traditional energy sources are not going to run out. The major determinant of the energy price level is supply shocks due to exploration successes, itself arising from investment in research and development. We model the joint strategic decision of (costly) exploration effort and production rate in this context.</td>
</tr>
</tbody>
</table>
1.2.3 Optimal Trading with Return Predictability

In Chapter 5 we ask the more mundane question of how we can profit from the insights gained in Chapters 2 to 4. In other words, given a predictive factor (a.k.a. \textit{alpha}) for the future return of a financial asset, how do we trade optimally to maximize the risk-adjusted return? We consider a class of dynamic portfolio optimization problems that allow for models of return predictability, transaction costs, and stochastic volatility. Our model extends that of Gárleanu and Pedersen and captures a number of common features of practical interest, including return predictability, transaction costs, and stochastic volatility.

Determining the dynamic optimal portfolio in this general setting is usually intractable, we provide asymptotic approximation to the optimal trading strategies which are explicitly computable under many stochastic volatility models of practical interest (e.g. Heston, exponential Ornstein-Uhlenbeck, and the 3/2-model). Tractability is maintained by viewing the more flexible model as a perturbation around the well-understood constant volatility problem considered by Gárleanu and Pedersen \cite{garleanu2008dynamic}.

Specifically, we propose a multiscale asymptotic expansion when the volatility process is characterized by its time scales of fluctuation. The analysis of the nonlinear HJB PDE is a singular perturbation problem when volatility is fast mean-reverting; and it is a regular perturbation when the volatility is slowly varying. These analyses can be combined for multifactor multiscale stochastic volatility model. We present formal derivations of asymptotic approximations and demonstrate how the proposed algorithms improve our Profit&Loss using Monte Carlo simulations.

Moreover, the correction terms give rise to economically sensible trading strategies. We find that under fast-scale stochastic volatility, the investor should optimally deleverage his portfolio when the current volatility level is higher than the long-term average, regardless of the return-volatility correlation. On the other hand, the
return-volatility correlation plays a more important role under the slow-scale stochastic volatility. When the correlation between the volatility and return factors is positive, the investor optimally decreases his trading rate as he anticipates a higher return estimate is accompanied by a higher volatility. This chapter is an adapted version of Chan and Sircar [30].
Chapter 2

A Feedback Model for the
Financialization of Commodity Markets

Recent empirical studies find evidence that commodity prices have become more correlated with financial markets since the early 2000s. This increased correlation is called the financialization of commodity markets and is conjectured to be due to the influx of external (portfolio optimizing) traders through commodity index funds, for instance. We build a feedback model to try and capture some of these effects, in which traditional economic demand for a commodity, oil say, is perturbed by the influence of portfolio optimizers. We approach the full problem of utility maximizing with a risky asset whose dynamics are impacted by trading through a sequence of problems that can be reduced to linear PDEs, and we find correlation effects proportional to the long or short positions of the investors, along with a lowering of volatility. This chapter is adapted from the article [31].
2.1 Introduction

2.1.1 Background and motivation

Recent empirical studies, for example [114, 19], have documented the “financialization” of commodity and energy futures markets due to an influx of external traders through investment vehicles such as commodity index funds or ETFs. They report that price movements of goods such as oil which, prior to the last decade, were mainly governed by supply and demand of users of the commodity, now exhibit much greater correlation with the movements of equity markets.

In the wake of the recent financial crisis, increased attention is being paid by politicians and regulators to the consequences of securitizing and trading derivatives on nontraditional investment assets. One result of the active trading of a securitized asset may be a fundamental shift in the economics that drives the price of the underlying. This change could be explained by professional investors beginning to trade actively in commodities for the purposes of portfolio diversification and speculation. At the beginning of the last decade, such trading activity in commodities increased dramatically, for example by hedge funds, and coincided with an increase in their correlation with the stock market. Causation between the change in trading activity and the change in the nature of commodity prices is still a subject of debate, as there are various global economic factors that may have played a role. While a purely mathematical model will not be able to separate out these possible effects, a possible explanation can be offered by analyzing the price impact of portfolio optimizers in a simple and idealized framework.

We build a model in which the demand of utility maximizing traders is introduced into an environment of a fundamentals-driven commodity price. We find the explicit form of the perturbed commodity price in the absence of investor feedback effects,
and we develop observations on the changes in volatility, and correlation with stocks, that the demand of the portfolio optimizers creates.

2.1.2 Related literature

Our model is related to two different strands of literature: the literature of commodity modeling and in particular the financialization of commodity market, and the literature on the price impact and feedback effect of large traders.

Commodity modeling and financialization  The modeling of commodity prices is discussed extensively in various sources. The books [42, 55, 101] give a primarily economic background on factors that drive prices for commodities and their derivatives. Benth et al. [12] details different models driven by time-inhomogeneous jump processes for electricity spot price dynamics, and the recent book of Swindle [113] provides a comprehensive overview of energy, commodities, and their derivatives. In a well known paper, Schwartz [108] fits parameters to a geometric Ornstein-Uhlenbeck model for the price of various commodities based on empirical data. The classic portfolio optimization analysis of Merton [92] has been extended by various authors to deal with mean-reverting processes. Jurek and Yang [70] derive explicit solutions to an optimal portfolio allocation and consumption problem for a portfolio optimizer seeking to profit from a mean-reverting pairs trade which follows an arithmetic Ornstein-Uhlenbeck process. Benth and Karlsen [13] solve the two-asset Merton problem for a risk-free asset and a risky asset with a geometric Ornstein-Uhlenbeck price process. These works provide a basis of comparison for the optimal stock/commodity allocation in our more complicated model.

Study of the financialization of commodity markets is somewhat new. In a recent paper, Tang and Xiong [114] discuss the history and development of the financialization of commodities as a result of increased index investing activity in the past decade.
They find evidence of increased exposure of commodities prices to shocks in other asset classes via regression analysis on empirical data. A report published by FTI UK Holdings Limited [79] on the impact of speculation in commodity markets weighs the merits of speculative trading as providing liquidity for parties that need to hedge against the potential instability that speculation can create in a market. Brunetti and Büyüksahin [18] find, in a forecasting sense, that speculators are not causing any price movement, and moreover, speculative trading activity reduces volatility levels. In a subsequent paper, Büyüksahin and Robe [20] show that increased participation by hedge funds that trade in both equity and commodity markets could strengthen the correlation between the rates of return on commodities and equities rises. More recently, Silvennoinen and Thorp [110] and Henderson, Pearson, and Wang [61] provide further evidence for the financialization of commodity prices. There is a fast-growing literature of financialization of commodity markets, and we refer to Gilbert [56], Irwin and Sanders [65], Kaufmann [71], Mayer [91] and Singleton [111].

Price impact and feedback The problem of large agents having price impact in small or illiquid markets is one that will be at play in our model, as the introduction of speculators will have an impact on the commodity price. Çetin, Jarrow and Protter [26] build a model of liquidity risk in which a stochastic supply curve affects participants in a market that have large trade sizes. Bank and Kramkov [8] develop a game-theoretic large trader liquidity model. Jonsson and Keppo [67] treat a model where the portfolio position of a large agent has a particular impact on the price of call options. Bank and Baum [17] derive a general framework for dynamic liquidity effects of large traders, where assumptions about imperfect liquidity of an asset cause large purchases of the asset to affect its price.

The feedback effect of option hedging strategies in continuous time models was considered in a number of papers in the 1990s. Specifically, Frey and Stremme [49],
Schönbucher and Wilmott [117], Sircar and Papanicolaou [104], and Platen and Schweizer [103] all studied the impact of option hedging strategies on stock prices. These models typically begin with reference traders who produce reference price dynamics such as geometric Brownian motion, but the equilibrium prices are then perturbed by the presence of noise traders who are hedging derivatives. These have the effect of increasing market volatility because, for example, hedging a short call option position entails selling stock when the stock price goes down, and therefore the presence of a significant number of hedgers has destabilizing price impact. This analysis explains to some extent the finding of the Brady Report into the 1987 crash, which attributed some cause to the presence of program trading for hedging option positions.

The feedback effect of portfolio optimizers was analyzed by Nayak and Papanicolaou [97]. Their reference setting consists of a stock following the geometric Brownian Motion, and a portfolio optimizer having the power utility. They explicitly analyze the feedback effects on the price process when the relative influence of portfolio optimizers is small, and they analyze the system numerically under more general assumptions. Their main conclusion is that rational trading from solving a Merton portfolio optimization problem is stabilizing, and therefore lowers volatility, in contrast to what was found for option hedging strategies.

In a different but related problem, Christensen et al. [34] employ an equilibrium approach where the asset price is determined endogenously using a market clearing condition. With exponential utility, they solve for each investor’s optimal investment strategy and find the equilibrium dynamics.

Models for financialization This chapter builds a simple feedback model to capture some of the empirical effects documented as the financialization of commodity market. We review several alternative avenues explored in the literature.
Sockin and Xiong [112] build a one-period feedback model for the financialization of commodities emphasizing the consequence of information frictions and production complementarity. They find that an increase in commodity futures prices may drive up producers’ commodity demand and thus the spot price. In their model, the futures price is used as a proxy for global economic strength and other producers’ production decisions, and in certain circumstances this information effect can dominate the cost effect and result in a positive demand elasticity. Basak and Pavlova [9] explore the effects of financialization in a model that features institutional investors alongside traditional futures markets participants, but they focus on the case where the institutional investors are evaluated relative to a benchmark index.

In a different context, Cont and Wagalath [35] illustrated how feedback effects due to distressed selling of mutual funds lead to endogenous correlations between asset classes. However, price impacts due to distressed selling are exogenously given by a block-shaped order book model and the funds follows a passive buy-and-hold strategy unless the fund value drops below certain threshold. In a static setting, Leclercq and Praz [75] consider an equilibrium based model that emphasizes the role of information aggregation of the commodity futures market. They demonstrate that speculation in futures markets facilitates hedging by suppliers, and hence decreases expected spot prices and increases the correlation between the financial and commodity markets.

2.1.3 Organization and Results

In Section 2.2 we introduce a feedback model and derive an iterative sequence of problems that increasingly incorporate the feedback effect and capture the impact of financialization. In Section 2.3 we derive the HJB equation from dynamic programming principle for the stage-\( k \) problem and show that the HJB equation can be linearized with a power transformation. In Section 2.4 we present the numerical solutions to the first couple of stages and quantify the induced correlation between the
equity and commodity markets. In Appendix A.2 we return to the feedback model of Nayak and Papanicolaou [97] and present an explicit solution to the first stage in the feedback sequence which greatly facilitates the model analysis. We conclude in Section 2.6.

2.2 Feedback Model with Market Users and Portfolio Optimizers

We build a model leading to the price of a commodity in which there are two main distinct groups creating demand: market users and portfolio optimizers. The price is determined from a supply-demand market clearing condition.

2.2.1 Reference model with market users

Market users trade in the commodity market for direct industrial use or for hedging their operational exposures. We assume their demand for commodity is driven by a stochastic incomes process $I_t$ which, roughly speaking, captures economic growth, and can be thought of as determining the amount of capital available to the market users for purchasing the commodity.

For simplicity of exposition and explicit calculations, we will take $(I_t)$ to be a geometric Ornstein-Uhlenbeck process described by

$$ \frac{dI_t}{I_t} = a(m - \log I_t) dt + b dW^c_t, $$

where $a, b > 0$ and $W^c$ is a standard Brownian motion. This captures in a simple way periods of geometric growth along with mean-reversion or stochastic cyclicality.

Given a commodity price $Y_t$, the demand $D(Y_t, I_t)$ from market users is increasing in $I_t$ and decreasing in $Y_t$. Again for simplicity and explicitness, we use the isoelastic
demand function

\[ D(Y_t, I_t) = \frac{I_t^\lambda}{Y_t}, \]  

(2.1)

where \( \lambda > 0 \). This kind of demand structure in a continuous-time model is used for instance in [49, 104, 97]. We also assume a fixed constant supply \( A \) of the commodity available for trading at each time period. That is, we ignore growth or decline of supply over the short-run.

In the reference model in which there are only market users (or reference traders), we label the price process \( Y_t = Y_t^{(0)} \). The market clearing condition \( D(Y_t, I_t) = A \) gives

\[ Y_t^{(0)} = \frac{I_t^\lambda}{A}, \]

showing that the reference commodity price \( Y^{(0)} \) dynamics is also a geometric Ornstein-Uhlenbeck process, which is the commonly-employed Schwartz [108] one-factor model of mean-reverting commodities prices:

\[ \frac{dY_t^{(0)}}{Y_t^{(0)}} = a \left( \bar{m} - \log Y_t^{(0)} \right) dt + \lambda b dW_t^c, \]  

(2.2)

where

\[ \bar{m} = \lambda m - \log A + \frac{1}{2a} \lambda (\lambda - 1)b^2. \]  

(2.3)

### 2.2.2 Incorporating portfolio optimizers

The portfolio optimizers, on the other hand, have no direct operational or hedging interest in the commodity, but seek to invest in the commodity market so as to maximize their expected utility at a fixed terminal horizon \( T \). We assume for tractability the constant relative risk-aversion (CRRA) utility function with risk aversion \( \gamma > 0 \):

\[ U(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1. \]  

(2.4)
In addition to the commodity market, the portfolio optimizers can invest in the risk-free money-market account with constant interest rate $r$, and a single representative stock index which follows the geometric Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^s,$$  \hspace{1cm} (2.5)

where $W_t^s$ is a standard Brownian motion independent of $W_t^c$. By this choice, we have assumed that the pre-financialized commodity price $Y^{(0)}$ in (2.2) is independent of the equity market, so later feedback correlation is in comparison to this case.

Denoting by $\theta_t$ the investment in the commodity market by the portfolio optimizers, the aggregate demand for commodity is given by $D(Y_t, I_t) + \tilde{\varepsilon} \theta_t / Y_t$, where $\tilde{\varepsilon} > 0$ parametrizes the relative size of the portfolio optimizers compared to the market users. The market-clearing condition is

$$D(Y_t, I_t) + \tilde{\varepsilon} \frac{\theta_t}{Y_t} = A,$$

which leads to

$$Y_t = \frac{I_t^\lambda}{A} + \varepsilon \theta_t,$$  \hspace{1cm} (2.6)

where $\varepsilon = \tilde{\varepsilon} / A$. This causes the equilibrium price process for $Y_t$ to deviate from the geometric Ornstein-Uhlenbeck process $Y^{(0)}$.

The portfolio optimizers’ position $\theta_t$ is modeled as coming from an expected utility maximizing criterion. However, unlike in the classical Merton problem, their actions impact the commodity they trade through (2.6), and we describe the extent of this feedback as related to the degree to which they are aware of their own price influence. Because they impact the commodity price $Y$ and their trades are governed by portfolio diversification concerns, $\theta_t$ is affected by the stock index price $S_t$ and this induces
correlation between commodity and equity returns, that is, financialization of the commodity price. The goal is to try and quantify this effect.

In our model we do not get into the details of how commodities are traded, for instance through commodity index funds or ETFs, but instead think of all these investments as linked to futures contracts which are themselves linked to the physical delivery of the commodity which is necessarily finite. The grouping of the traders into two large groups means that each group has a significant price impact due to the finiteness at each time of the supply $A$ available for trading.

### 2.2.3 Fixed point characterization of problem

We say that a pair $(\hat{\pi}^*, \hat{\theta}^*)$, respectively denoting the fractions of wealth invested in the stock and commodity markets, is an equilibrium solution to our utility maximization problem with feedback if the following are satisfied:

1. The stock price $S_t$ is given by (2.5) and the commodity price $Y_t$ is determined by the market clearing condition

$$D(Y_t, I_t) + \varepsilon \frac{\hat{\theta}_t X_t}{Y_t} = A,$$

where $X_t$ is the controlled wealth process with strategies $(\hat{\pi}^*, \hat{\theta}^*)$ such that

$$\frac{dX_t}{X_t} = \frac{\hat{\pi}_t^*}{S_t} dS_t + \frac{\hat{\theta}_t^*}{Y_t} dY_t + r \left(1 - \hat{\pi}_t^* - \hat{\theta}_t^*\right) dt.$$

2. The pair $(\hat{\pi}^*, \hat{\theta}^*)$ maximizes the expected utility of terminal wealth $Z_T$

$$\sup_{(\hat{\pi}, \hat{\theta}) \in A} \mathbb{E} \left[U(Z_T) | \mathcal{F}_t \right],$$
under the budget constraint

\[
\frac{dZ_t}{Z_t} = \frac{\hat{\pi}_t}{S_t} dS_t + \frac{\hat{\theta}_t}{Y_t} dY_t + r \left(1 - \hat{\pi}_t - \hat{\theta}_t\right) dt.
\]

The set of admissible strategies \( \mathcal{A} \) will be given explicitly in Section 2.3.1 when we define the value function and derive the associated HJB equation.

### 2.2.4 Feedback iteration

We propose a feedback iteration to capture the successive improvement of trading strategies due to the increasing awareness of self-impact by the commodity traders. This approach follows an iterative chain of reasoning which is best illustrated in the simple setting of the famous “guessing 2/3 of the average” game.

**Interlude - guessing 2/3 of the average**

In this game, a number of players are asked to pick a number between 0 and 100, with the winner of the game being the one that is closest to 2/3 times the average number picked by all players.

We can solve this game iteratively as follows:

**Stage-0** A typical player ignores the other players and choose a random number between 0 and 100.

**Stage-1** He realizes that if the other players are following the stage-0 strategy, the average is about 50; he can take advantage of this and update his guess to be 100/3.

**Stage-2** He refines on the stage-1 strategy and notice that if the other players are following the stage-1 strategy, then the average is 100/3 and he should update his guess to be 200/9.
This chain of reasoning goes on: at stage-$k$, any individual player anticipates that the other players are following the stage-$(k - 1)$ strategy, and take into account their aggregate effect (the sample average in this setting) when determining the stage-$k$ strategy. As $k \to \infty$, the only rational guess is zero and this is called the Nash equilibrium.

Empirical studies show that, however, people do not behave as this simple model predicts. For instance, Nagel [96] conducted an experiment in which students were asked to guess what 2/3 of the average of their guesses will be, within limits of 0 and 100. She found that the average guess for the groups was around 35 and very few students chose 0. Moreover, it was observed that many students chose 33 and 22, which are respectively 2/3 of the midpoint 50 and 2/3 of 2/3 of the midpoint. The number of steps of iterated reasoning most students seemed to be doing were between 0 and 3 rounds.

2.2.5 Stage-$k$ portfolio optimization problem

In light of the guessing game, we model the aggregate portfolio optimizers as comprising a large number of commodity traders. Each individual trader is too small to affect the market price, but their aggregate demand does have an impact, which is enforced by the market-clearing constraint. We first analyze the stage-$k$ problem and return to the explicit solution for stage-0 in Section 2.3.3 [1].

In general, the stage-$k$ commodity price process $Y^{(k)}$ for $k \geq 1$ no longer follows a geometric Ornstein-Uhlenbeck process, as was the case in the reference model $k = 0$ in (2.2). We suppose the dynamics of $Y^{(k)}$ can be written as

$$\frac{dY^{(k)}}{Y^{(k)}} = P^{(k)} \, dt + Q^{(k)} \, dW^c_t + R^{(k)} \, dW^s_t,$$

(2.7)

---

1We emphasize that, as in the 2/3 averaging game, the sequence of stages the portfolio optimizers run through are reasoning steps that are not temporally implemented. The choice of how many stages to go is a particular proposal related to the awareness of feedback effect of the portfolio optimizers (or the students in the 2/3 averaging game.)
for some coefficients $P^{(k)}, Q^{(k)},$ and $R^{(k)}$. The aggregate wealth process for the bulk of the portfolio optimizers is denoted $X_t$, and they employ the stage-$(k-1)$ strategy $(\pi^{(k-1)}, \theta^{(k-1)})$, where $\pi_t^{(k-1)}$ is the dollar amount held in the stock index $S$ at time $t$, and $\theta_t^{(k-1)}$ is the dollar amount held in the commodity $Y^{(k)}$ at time $t$. Their self-financing aggregate wealth process $X$ follows

$$dX_t = \frac{\pi_t^{(k-1)}}{S_t} dS_t + \frac{\theta_t^{(k-1)}}{Y_t^{(k)}} dY_t^{(k)} + r(X_t - \pi_t^{(k-1)} - \theta_t^{(k-1)}) dt$$

$$= \left( rX_t + \pi_t^{(k-1)}(\mu - r) + \theta_t^{(k-1)}(P^{(k)} - r) \right) dt$$

$$+ \theta_t^{(k-1)} Q_t^{(k)} dW^c_t + \left( \pi_t^{(k-1)} \sigma + \theta_t^{(k-1)} R_t^{(k)} \right) dW^s_t.$$ (2.8)

For our inductive hypothesis, we suppose that $P^{(k)} = P^{(k)}(t, X_t, Y_t^{(k)})$ and similarly for $Q^{(k)}$ and $R^{(k)}$, which come from the solution of the stage-$(k-1)$ problem; and that $\pi_t^{(k-1)}$ and $\theta_t^{(k-1)}$ are Markovian controls of the form $\pi_t^{(k-1)} = \pi^{(k-1)}(t, X_t, Y_t^{(k)})$ and $\theta_t^{(k-1)} = \theta^{(k-1)}(t, X_t, Y_t^{(k)})$. We have suppressed the argument $(t, X_t, Y_t^{(k)})$ in (2.8). Under these hypotheses, $(X_t, Y_t^{(k)})$ is a Markov process with respect to the filtration generated by $(W^c, W^s)$.

In stage-0 the portfolio optimizers do not trade the commodity so we have that $\theta^{(-1)} = 0$ and $Y^{(0)}$ is given by the reference model (2.2) from which we see that

$$P^{(0)}(t, x, y) = a(\tilde{m} - \log y), \quad Q^{(0)}(t, x, y) = \lambda b, \quad R^{(0)} = 0,$$

and in particular they do not depend on $X_t$.

At stage-$k$, all but one of the commodity traders follow the stage-$(k-1)$ strategy. We imagine that a single “smart” trader seeks to outperform the others by taking into consideration the price impact of their stage-$(k-1)$ strategy. We denote his stage-$k$ portfolio by $(\pi^{(k)}, \theta^{(k)})$ and the self-financing condition determines the following
wealth process $Z_t$ for the “smart” trader:

$$dZ_t = \frac{\hat{\pi}_t^{(k)}}{S_t} dS_t + \frac{\hat{\theta}_t^{(k)}}{Y_t^{(k)}} dY_t^{(k)} + r(Z_t - \pi_t^{(k)} - \theta_t^{(k)}) \, dt$$

$$= \left( rZ_t + \pi_t^{(k)}(\mu - r) + \theta_t^{(k)} \left( P^{(k)}(t, X_t, Y_t^{(k)}) - r \right) \right) \, dt$$

$$+ \hat{\theta}_t^{(k)} Q^{(k)}(t, X_t, Y_t^{(k)}) \, dW^c_t + \left( \pi_t^{(k)} \sigma + \hat{\theta}_t^{(k)} R^{(k)}(t, X_t, Y_t^{(k)}) \right) \, dW^s_t.$$

His goal is to maximize expected utility at the terminal time $T$. This leads to a Merton problem we must solve to determine the stage-$k$ optimal portfolio.

### 2.2.6 Deriving the stage-$(k+1)$ dynamics from stage-$k$ strategies

Given the stage-$k$ optimal portfolio $\pi^{(k)}$ and $\theta^{(k)}$ of the “smart” trader, we determine its effect on the stage-$(k+1)$ commodity price process. As is well known and we will confirm in the next section, because of power utility, the optimal Merton strategies are of the form $\hat{\pi}_t^{(k)} = \tilde{\pi}^{(k)}(t, X_t, Y_t^{(k)}) Z_t$ and $\hat{\theta}_t^{(k)} = \tilde{\theta}^{(k)}(t, X_t, Y_t^{(k)}) Z_t$. That is to say, they are given as fractions of the current wealth $Z_t$ where the fractions are determined by the current levels of $X_t$ and $Y_t^{(k)}$.

After having solved for the stage-$k$ optimal portfolio, the “smart” trader realizes that the other traders will follow the same reasoning and trade according to the stage-$k$ strategy. The aggregate position on the commodity is then $\hat{\theta}^{(k)}(t, X_t, Y_t^{(k+1)}) X_t$, and the stage-$(k+1)$ market clearing constraint reads

$$Y_t^{(k+1)} = \frac{I^\lambda}{A} + \varepsilon \hat{\theta}^{(k)}(t, X_t, Y_t^{(k+1)}) X_t. \quad (2.10)$$

We can determine the dynamics of the stage-$(k+1)$ commodity price process $Y_t^{(k+1)}$ by applying Itô’s formula to (2.10) and matching coefficients of the $dt$, $dW^c_t$, $dW^s_t$. 

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and \(dW_t^s\) terms. We can then solve for \(P^{(k+1)}, Q^{(k+1)},\) and \(R^{(k+1)}\) in terms of \(\pi^{(k)}\) and \(\theta^{(k)}\).

**Proposition 1.** The dynamics of the stage-\((k+1)\) commodity price process \(Y^{(k+1)}\) is

\[
\frac{dY_t^{(k+1)}}{Y_t^{(k+1)}} = P^{(k+1)}(t, X_t, Y^{(k+1)}) \, dt + Q^{(k+1)}(t, X_t, Y^{(k+1)}) \, dW_t^c + R^{(k+1)}(t, X_t, Y^{(k+1)}) \, dW_t^s,
\]

where

\[
Q^{(k+1)}(t, x, y) = \frac{\lambda b(y - \varepsilon x \hat{\theta}^{(k)})}{y - \varepsilon x (y \partial_y \hat{\theta}^{(k)} + \hat{\theta}^{(k)} x \partial_x \hat{\theta}^{(k)} + (\hat{\theta}^{(k)})^2)},
\]

\[
R^{(k+1)}(t, x, y) = \frac{\varepsilon x \sigma \hat{\pi}^{(k)}(x \partial_x \hat{\theta}^{(k)} + \hat{\theta}^{(k)})}{y - \varepsilon x (y \partial_y \hat{\theta}^{(k)} + \hat{\theta}^{(k)} x \partial_x \hat{\theta}^{(k)} + (\hat{\theta}^{(k)})^2)},
\]

\[
P^{(k+1)}(t, x, y) = \frac{a(y - \varepsilon x \hat{\theta}^{(k)}) \left( \bar{m} - \log(y - \varepsilon x \hat{\theta}^{(k)}) \right) + \varepsilon x \left( P_1^{(k+1)}(t, x, y) + \frac{P_2^{(k+1)}(t, x, y)}{2} \right)}{y - \varepsilon x (y \partial_y \hat{\theta}^{(k)} + \hat{\theta}^{(k)} x \partial_x \hat{\theta}^{(k)} + (\hat{\theta}^{(k)})^2)},
\]

(2.11)

with

\[
P_1^{(k+1)}(t, x, y) = \partial_t \hat{\theta}^{(k)} + \frac{1}{2} \left( \left( Q^{(k+1)} \right)^2 + \left( R^{(k+1)} \right)^2 \right) y^2 \partial_y \hat{\theta}^{(k)} + \mu \hat{\theta}^{(k)} \hat{\pi}^{(k)} + r \hat{\pi}^{(k)} \left( 1 - \hat{\pi}^{(k)} - \hat{\theta}^{(k)} \right)
\]

\[
+ \left( \left( Q^{(k+1)} \right)^2 + \left( R^{(k+1)} \right)^2 \right) \hat{\theta}^{(k)} \sigma R^{(k+1)} \hat{\pi}^{(k)} \right) y \partial_y \hat{\theta}^{(k)}
\]

\[
P_2^{(k+1)}(t, x, y) = \left( \hat{\theta}^{(k)} Q^{(k+1)} \right) \frac{1}{2} \left( \left( \hat{\pi}^{(k)} \right)^2 + \left( \hat{\theta}^{(k)} \right)^2 \right) x^2 \partial_x \hat{\theta}^{(k)}
\]

\[
+ \frac{1}{2} \left( \left( \hat{\theta}^{(k)} Q^{(k+1)} \right)^2 + \left( \hat{\pi}^{(k)} \sigma + \hat{\theta}^{(k)} R^{(k+1)} \right)^2 \right) x y \partial_{xy} \hat{\theta}^{(k)},
\]

and \(\bar{m}\) is given by \(2.3\).

**Proof.** Apply Itô formula to the market clearing constraint \((2.10)\) and substitute the dynamics of \(X_t\) using \((2.8)\) with \(k\) replaced by \(k + 1\). Finally substitute \(I_t\) in terms of \(X_t\) and \(Y_t^{(k+1)}\) using \((2.10)\). \(\square\)
Therefore, given $\theta^{(k)}$ and $\pi^{(k)}$, we can determine the stage-$(k + 1)$ commodity price dynamics $P^{(k+1)}$, $Q^{(k+1)}$, and $R^{(k+1)}$. Roughly speaking, the coefficient $R^{(k+1)}$ determines the stock-commodity correlation. Notice that the expressions for $P, Q,$ and $R$ are valid only for small $\varepsilon$ as their denominators may become zero. However, we will see in Section 2.4 that the solution to the stage-$k$ problem is well-behaved since the underlying price processes never get to the problematic region due to a repulsive potential. We focus here on the isoelastic demand function to illustrate the main features of financialization in a specific setting, but remark that the same techniques can be applied to more general demand functions, provided that they are invertible.

2.3 HJB analysis

In this section, we use dynamic programming to derive an HJB PDE that determines the optimal strategies the “smart” trader follows in each stage of the feedback iteration. We show that it can be reduced to a linear PDE and give the explicit solution in stage-0.

2.3.1 Value function and HJB equation

The value function for the “smart” trader in stage-$k$ described in Section 2.2.5 is defined by

$$V(t, x, y, z) = \sup_{(\pi^{(k)}, \theta^{(k)}) \in \mathcal{A}} \mathbb{E} \left[ U(Z_T) | X_t = x, Y_t^{(k)} = y, Z_t = z \right],$$

where $X_t$ follows (2.8), $Y_t$ follows (2.7), and $Z_t$ follows (2.9), and we have defined the set of admissible strategies $\mathcal{A}$ to contain adapted processes $(\pi_t, \theta_t)$ such that $\mathbb{E} \int_0^T |\pi_t|^2 + |\theta_t|^2 \, dt < \infty$. See for instance [100, Chapter 3]. Following the usual
Bellman’s principle, we obtain the stage-$k$ HJB equation

$$V_t + \mathcal{L}_x V + rzV_z + \sup_{\nu \in \mathbb{R}^2} \left[ \frac{1}{2} \nu^T C_1 \nu V_{zz} + \nu^T (\mu_1 - r)V_z + \nu^T \sigma_1 \sigma_2^T \nabla_x V_z \right] = 0,$$

for $t < T$ and $x, y, z > 0$. Here we have denoted the trading strategy by $\nu = (\pi^{(k)}, \theta^{(k)})^T$, $x = (x, y)^T$, $\nabla_x = (\partial_x, \partial_y)^T$, and the drift vector and volatility matrix of the pair of tradeable assets $(S, Y^{(k)})$ by

$$\mu_1 = \begin{pmatrix} \mu \\ P^{(k)} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & \sigma \\ Q^{(k)} & R^{(k)} \end{pmatrix}, \quad C_1 = \sigma_1 \sigma_1^T.$$

Also defined are

$$\sigma_2 = \begin{pmatrix} \theta^{(k-1)} Q^{(k)} & \theta^{(k-1)} R^{(k)} + \pi^{(k-1)} \sigma \\ Q^{(k)} y & R^{(k)} y \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} \theta^{(k-1)} P^{(k)} + \pi^{(k-1)} \mu + r(x - \theta^{(k-1)} - \pi^{(k-1)}) \\ P^{(k)} y \end{pmatrix},$$

as well as

$$\mathcal{L}_x = \frac{1}{2} \sum_{i,j=1}^2 (C_2)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \mu_2^T \nabla_x,$$

$$C_2 = \sigma_2 \sigma_2^T,$$

where we identify $(x_1, x_2) = (x, y)$. The terminal condition is $V(T, x, y, z) = U(z)$.

### 2.3.2 Analysis of the HJB equation

From the point of view of the “smart” trader, he is facing a complete market Merton problem, being able to trade two assets, the commodity and the stock, driven by two Brownian motions $W^c$ and $W^s$. Therefore we expect that the HJB equation can be reduced to a linear equation via a Cole-Hopf type transformation. We show that it is indeed the case in the following.
Proposition 2. The value function is given by

\[ V(t, x, z) = \frac{z^{1-\gamma}}{1-\gamma} (G(t, x))^\gamma, \] (2.12)

where \( G(t, x) \) solves the linear PDE problem

\[ G_t + \mathcal{L}_x G + \left( \frac{1-\gamma}{\gamma} \right) (\sigma_2 \sigma_1^{-1}(\mu_1 - r))^T \nabla_x G + \frac{\zeta}{\gamma} G = 0, \] (2.13)

with terminal condition \( G(T, x) = 1 \), where

\[ \zeta = \left( r(1-\gamma) + \frac{1-\gamma}{2\gamma} M \right), \quad M = (\mu_1 - r)^T C_1^{-1}(\mu_1 - r). \] (2.14)

The optimal portfolio \( \nu^*_t = (\pi^{(k)}, \theta^{(k)})^T \) is given by

\[ \nu^*_t = \left( \frac{1}{\gamma} C_1^{-1}(\mu_1 - r) + (\sigma_2 \sigma_1^{-1})^T \nabla_x G \right) Z_t. \] (2.15)

Proof. Optimization Assuming for now that \( V_{zz} \) is negative (that is, it inherits the concavity from the terminal condition), the supremum in the HJB equation is given by

\[ \nu^* = -\frac{1}{V_{zz}} C_1^{-1} \left( (\mu_1 - r) + \sigma_1 \sigma_2^T \nabla_x G \right) V_z. \] (2.16)

The HJB equation can then be written as

\[ V_t + \mathcal{L}_x V + rzV_z - \frac{1}{2V_{zz}} \left( MV_z^2 + 2(\mu_1 - r)^T C_1^{-1} \sigma_2^T \nabla_x V_z \right) V_z + \left( \sigma_1 \sigma_2^T \nabla_x V_z \right)^T C_1^{-1} \sigma_1 \sigma_2^T \nabla_x V_z = 0 \]

with terminal condition \( V(T, x, z) = U(z) \), where \( M \) is defined in (2.14).

Separation of variables Making the transformation

\[ V(t, x, z) = \frac{z^{1-\gamma}}{1-\gamma} g(t, x) \]
results in

\[ g_t + \mathcal{L}_x g + \left( \frac{1 - \gamma}{\gamma} \right) (\mu_1 - r)^T C_1^{-1} \sigma_1 \sigma_2 \nabla_x g + \zeta g + \frac{1 - \gamma}{2 \gamma} (\sigma_1 \sigma_2^T \nabla_x g)^T C_1^{-1} (\sigma_1 \sigma_2^T \nabla_x g) = 0, \]

(2.17)

with terminal condition \( g(T, x) = 1 \), where \( \zeta \) is defined in (2.14).

**Cole-Hopf transformation** We make the power transformation \( g = G^\delta \), previously introduced by Zariphopoulou [118], and observe that the nonlinear term in (2.17) becomes

\[ \delta G^{\delta - 1} \left[ \frac{\delta}{2 \gamma G} (\sigma_1 \sigma_2^T \nabla_x G)^T C_1^{-1} (\sigma_1 \sigma_2^T \nabla_x G) \right]. \]

Using the definition of \( C_1 \) and \( C_2 \), we note that

\[ (\sigma_1 \sigma_2^T \nabla_x G)^T C_1^{-1} (\sigma_1 \sigma_2^T \nabla_x G) = (\nabla_x G)^T C_2 (\nabla_x G) \]

so the nonlinear term is

\[ \delta G^{\delta - 1} \left[ \frac{\delta}{2 \gamma G} (\nabla_x G)^T C_2 (\nabla_x G) \right]. \]

Meanwhile,

\[ \mathcal{L}_x G^\delta = \delta G^{\delta - 1} \left[ \frac{1}{2} \sum_{i,j=1}^2 (C_2)_{ij} \frac{\partial^2 G}{\partial x_i \partial x_j} + \frac{1}{2} (\delta - 1) \frac{(\nabla_x G)^T C_2 (\nabla_x G)}{G} \right], \]

so the nonlinear terms cancel provided we choose

\[ \frac{1}{2} (\delta - 1) + \delta \frac{1 - \gamma}{2 \gamma} = 0 \quad \implies \quad \delta = \gamma. \]

With this choice, we obtain the linear PDE for \( G(t, x) \) given in (2.13). Inserting transformations (2.12) for \( V \) into (2.16) gives the optimal portfolio (2.15) in terms of \( G \).
Notice in particular that the optimal holdings in commodity and stock index are proportional to wealth, as we expect for power utility, and we write:

$$\pi^{(k)}_t = \hat{\pi}^{(k)}(t, X_t, Y_t)Z_t, \quad \theta^{(k)}_t = \hat{\theta}^{(k)}(t, X_t, Y_t)Z_t,$$

where the functions \( \hat{\pi}^{(k)}(t, x, y) \) and \( \hat{\theta}^{(k)}(t, x, y) \) can be read from the components of \( \nu^* \) in (2.15).

**Remark 1.** In Appendix A.2 we study an application of our feedback model to the equity market where the stage-1 HJB equation can be solved analytically.

### 2.3.3 Stage-0 PDE and explicit solution

As it turns out, the stage-0 PDE problem has an explicit solution. This is the Merton problem with geometric OU dynamics as studied in [13] and [70], for instance. From the market-clearing constraint (without any influence of portfolio optimizers), we see that the stage-0 commodity price dynamics (2.2) is simply a geometric Ornstein-Uhlenbeck process. That is, we have \( P^{(0)} = a(\tilde{m} - \log y), \quad Q^{(0)} = \lambda b, \) and \( R^{(0)} = 0. \)

**Proposition 3.** The stage-0 value function is given by

$$V(t, y, z) = \frac{z^{1-\gamma}}{1-\gamma} \exp \left( f_0(t) + f_1(t) \log y + f_2(t)(\log y)^2 \right), \quad (2.18)$$
where

\[
f_2(t) = \frac{a(1 - \gamma)}{2 \lambda^2 b^2} \frac{\sinh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right)}{\sinh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right) + \sqrt{\gamma} \cosh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right)},
\]

\[
f_1(t) = (1 - \gamma) \frac{r - \tilde{a} \tilde{m}}{\lambda^2 b^2 \gamma} \frac{\sinh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right) + \sqrt{\gamma} \left( \frac{r}{\lambda^2 b^2} - \frac{\gamma}{2} \right) \left( \cosh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right) - 1 \right)}{\sinh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right) + \sqrt{\gamma} \cosh \left( \frac{a}{\sqrt{\gamma}} (T - t) \right)}
\]

\[
f_0(t) = k(T - t) + \int_t^T \left\{ \left( \frac{a \tilde{m} - (1 - \gamma)r}{\gamma} - \frac{\lambda^2 b^2}{2} \right) f_1(s) + \lambda^2 b^2 f_2(s) + \frac{\lambda^2 b^2}{2 \gamma} f_1^2(s) \right\} ds,
\]

(2.19)

and

\[
k = \frac{1 - \gamma}{2} \left( \frac{(a \tilde{m} - r)^2}{\lambda^2 b^2 \gamma} + 2r + \frac{(\mu - r)^2}{\sigma^2 \gamma} \right).
\]

**Proof.** After separating out the wealth variable \( z \) and making the linearizing transformation, the stage-0 equation (2.13) for \( G \) is

\[
G_t + \frac{1}{2} \lambda^2 b^2 y^2 G_{yy} + rx G_x + \frac{1}{\gamma} \left( a(\tilde{m} - \log y) - (1 - \gamma)r \right) y G_y + \frac{\zeta}{\gamma} G = 0
\]

with terminal condition \( G(T, x, y) = 1 \). Observe that the terminal condition does not depend on \( x \), and that the term \( rx G_x \) drops out from the PDE if we look for a solution \( G(t, y) \) as a function of \( t \) and \( y \) only. Intuitively it is clear that the stage-0 value function should not depend on the aggregate wealth of the speculative traders as they have no influence on the commodity price.

Next, we make the transformation \( G(t, y) = H(t, u) \) where \( u = \log y \). This results in

\[
H_t + \frac{1}{2} \lambda^2 b^2 (H_{uu} - H_u) + \frac{1}{\gamma} \left( a(\tilde{m} - u) - (1 - \gamma)r \right) H_u + \frac{\zeta}{\gamma} H = 0.
\]

Recall that \( M \) is defined by

\[
M = (\mu_1 - r)^T C_1^{-1} (\mu_1 - r) = \frac{(a(\tilde{m} - u) - r)^2}{\lambda^2 b^2} + \frac{(\mu - r)^2}{\sigma^2}.
\]
We have a PDE of the form

\[ H_t + (c_0 + c_1 u) H_u + \frac{1}{2} \lambda^2 b^2 H_{uu} + \frac{1}{\gamma} (c_2 + c_3 u + c_4 u^2) H = 0, \quad (2.20) \]

with terminal condition \( H(T, u) = 1 \), where \( c_0 \) through \( c_4 \) are constants given in Appendix A.1. As shown there, the solution is of the form

\[ H(t, u) = \exp \left( \frac{1}{\gamma} \left( f_0(t) + f_1(t) u + f_2(t) u^2 \right) \right), \quad (2.21) \]

where \( f_0, f_1, \) and \( f_2 \) satisfies a system of ordinary differential equations, whose solutions lead to (2.18).

The optimal portfolio can be recovered from the value function using (2.15):

\[ \hat{\pi}^{(0)}(t, y) = \frac{\mu - r}{\sigma^2 \gamma} \quad \text{and} \quad \hat{\theta}^{(0)}(t, y) = \left( F_1(t) + F_2(t) \log y \right) \quad (2.22) \]

where

\[ F_1(t) = \frac{1}{\gamma} \left( f_1(t) + \frac{1}{\lambda^2 b^2} (a \tilde{m} - r) \right) \quad \text{and} \quad F_2(t) = \frac{1}{\gamma} \left( 2 f_2(t) - \frac{a}{\lambda^2 b^2} \right). \]

We observe that the investment in the stock \( \hat{\pi}^{(0)} \) follows the fixed-mix strategy as in the Merton problem [92]. The fraction of wealth \( \hat{\theta}^{(0)} \) invested in the commodity, however, depends on both the remaining time to the investment horizon \( T - t \) and the commodity price \( Y_t \).

**Remark 2.** The verification argument for candidate solutions of the HJB equation is classical and a rigorous treatment follows closely Benth and Karlsen [113, Section 4] and Zariphopoulou [118, Theorem 3.2].
2.4 Numerical Results

For stage-1 and onwards, we have to resort to numerical methods to solve the PDE problems. As an illustration, we consider the following set of parameter values detailed in Table 2.1.

Table 2.1: Parameter values for numerical illustration.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Significance</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>demand from market users</td>
<td>1.0</td>
</tr>
<tr>
<td>$a$</td>
<td>mean-reversion rate</td>
<td>0.3</td>
</tr>
<tr>
<td>$r$</td>
<td>risk-free rate</td>
<td>0.02</td>
</tr>
<tr>
<td>$m$</td>
<td>mean of commodity log-price</td>
<td>3.0</td>
</tr>
<tr>
<td>$b$</td>
<td>volatility of commodity price</td>
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</tr>
<tr>
<td>$\mu$</td>
<td>drift of stock price</td>
<td>0.08</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>volatility of stock price</td>
<td>0.2</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>coefficient of risk aversion</td>
<td>1.5</td>
</tr>
<tr>
<td>$A$</td>
<td>market supply</td>
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</tr>
<tr>
<td>$\varepsilon$</td>
<td>relative size of portfolio optimizers</td>
<td>0.5</td>
</tr>
<tr>
<td>$T$</td>
<td>Investment horizon</td>
<td>2.0</td>
</tr>
</tbody>
</table>

2.4.1 Solution via finite difference

The stage-$k$ PDE (2.13) can be written in expanded form (where we suppress the $k$ superscripts)

\[
0 = G_t + \frac{1}{2}(Q^2 + R^2)y^2G_{yy} + \frac{1}{2}\left((\hat{\theta}R + \hat{\pi}\sigma)^2 + (\hat{\theta}Q)^2\right)x^2G_{xx} + \left((Q^2 + R^2)\hat{\theta} + \sigma R\hat{\pi}\right)xyG_{xy}
\]

\[
+ \frac{1}{\gamma}\left(P - (1 - \gamma)r\right)yG_y + \frac{1}{\gamma}\left(\hat{\theta}(P - r) + \hat{\pi}(\mu - r) + \gamma r\right)xG_x
\]

\[
+ \frac{1}{\gamma}\left(r(1 - \gamma) + \frac{1 - \gamma}{2\gamma} \frac{1}{\sigma^2 Q^2}\left(\sigma^2(P - r)^2 - 2\sigma R(\mu - r)(P - r) + (Q^2 + R^2)(\mu - r)^2\right)\right)G.
\]
After using the log transformation \( u = \log x \) and \( v = \log y \), we may write the discretized equation as

\[
0 = \frac{G^{n+1}_{i,j} - G^n_{i,j}}{\Delta t} + \frac{1}{2} \left( (Q^n_{i,j})^2 + (R^n_{i,j})^2 \right) \left( \frac{G^{n+1}_{i,j+1} - 2G^n_{i,j+1} + G^{n+1}_{i,j-1}}{(\Delta v)^2} - \frac{G^{n+1}_{i,j} - G^{n+1}_{i,j-1}}{2\Delta u} \right) \\
+ \frac{1}{2} \left( (\tilde{P}^n_{i,j} + \tilde{R}^n_{i,j})^2 \right) \left( \frac{G^{n+1}_{i+1,j} - 2G^n_{i+1,j} + G^{n+1}_{i-1,j}}{(\Delta u)^2} - \frac{G^{n+1}_{i+1,j} - G^{n+1}_{i,j-1}}{2\Delta u} \right) \\
+ \left( (Q^n_{i,j})^2 + (R^n_{i,j})^2 \right) \left( \frac{G^{n+1}_{i+1,j+1} - 2G^n_{i+1,j+1} + G^{n+1}_{i,j+1}}{(\Delta u)^2} - \frac{G^{n+1}_{i+1,j} + G^{n+1}_{i,j-1}}{4\Delta u} \right) \\
+ \frac{1}{\gamma} \left( P^n_{i,j} - (1 - \gamma) \right) \left( \frac{G^n_{i+1,j+1} - G^n_{i,j+1}}{2\Delta v} \right) + \frac{1}{\gamma} \left( \tilde{P}^n_{i,j} (P^n_{i,j} - r) + \tilde{R}^n_{i,j} (\mu - r) + \gamma r \right) \left( \frac{G^n_{i+1,j+1} - G^n_{i,j+1}}{2\Delta u} \right) \\
+ \frac{1}{\gamma} \left( r(1 - \gamma) + \frac{1}{2\gamma} \frac{1}{(\sigma Q^n_{i,j})^2} \left( \sigma^2 (P^n_{i,j} - r)^2 - 2\sigma R^n_{i,j} (\mu - r)(P^n_{i,j} - r) + (Q^n_{i,j})^2 + (R^n_{i,j})^2 (\mu - r)^2 \right) \right) G^{n+1}_{i,j}.
\]

(2.23)

In the scheme, the first subscript denotes the \( u \) coordinate in the uniform \((u,v)\)-grid, while the second corresponds to the \( v \) coordinate. The superscript represents the time step. We note that we are solving a terminal value problem: we start with \( G^N_{i,j} = 1 \) for all \( i, j \) and step backward in time using (2.23) which is the explicit Euler scheme.

**Truncation of domain** We approximate the Cauchy problem (2.13) with one on a bounded domain \([0,T] \times [u_{\text{min}}, u_{\text{max}}] \times [v_{\text{min}}, v_{\text{max}}] \). We choose a uniform mesh

\[
G = \{ (t^n, u_i, v_j) : n = 0, 1, \ldots, N, \ i = 0, 1, \ldots, I, \ j = 0, 1, \ldots, J \}
\]

where \( t^n = n\Delta t, u_i = u_{\text{min}} + i\Delta u, \) and \( v_j = v_{\text{min}} + j\Delta v. \)

**Boundary conditions** In the limit \( x \to 0 \), the price impact of portfolio optimizers vanishes because they have no capital to invest, and we should recover the stage-0 solution. We therefore use the analytic solution (2.18) for the stage-0 problem to impose a Dirichlet boundary condition at \( x = 0 \) (or equivalently \( u \to -\infty \)).

As for the other three boundaries, we derive approximate boundary conditions by the PDE itself, which are discretized by one-sided finite differences without requiring
Figure 2.1: Comparison between Monte-Carlo and numerical PDE solutions to (2.13). The error bars represent ±2 standard error of the Monte-Carlo estimates. Parameter values are as in Table 2.1.

any additional information concerning the behavior of the solution for large $x$ and for large/small $y$.

Notice that in (2.11), the denominators in the definitions of $P$, $Q$, and $R$ can be zero for $y$ sufficiently small or $x$ sufficiently large. Therefore our model can breakdown when the commodity price is sufficiently low and the portfolio optimizers take over a sufficiently large share of market. However, we check that the truncation of domain does not result in significant error by solving the linear PDE (2.13) using Monte-Carlo simulations. In all instances we have tested, the difference between the PDE solutions and the Monte-Carlo estimates are within the Monte-Carlo standard error. See Figure 2.1 for instance where we compare the Monte-Carlo estimates with the numerical PDE solutions to (2.13).

Stage-0 benchmark

Since we have an explicit solution to the stage-0 problem, it serves as a benchmark for testing our numerical scheme.

In Figure 2.2b we plot the optimal fraction of wealth invested in the commodity market, computed using (2.15) by differentiating the numerical solution to the linear
Stage-0 value function.  
(b) Stage-0 optimal commodity holding.

Figure 2.2: Analytic and numerical solutions to the stage-0 HJB equation.

equation (2.13). Also shown is the optimal portfolio \( \tilde{\theta}(0) \) from formula (2.22). Notice that the two curves are almost indistinguishable, and we check that the numerical solution has relative error less than 1.5% throughout the domain of interest. This validates the use of the numerical scheme in the following sections.

Stage-1 numerical solution

We now solve the stage-1 HJB equation numerically. The commodity price dynamics \( P^{(1)}, Q^{(1)}, \) and \( R^{(1)} \) can be computed from the analytic solution to the stage-0 problem. Indeed, using that the stage-0 optimal portfolio \( \tilde{\theta}(0) \) does not depend on \( x \), equation (2.11) can be simplified considerably:

\[
Q^{(1)}(t, x, y) = \frac{\lambda b (y - \varepsilon x \tilde{\theta}(0))}{y - \varepsilon x \left(y \partial_x \tilde{\theta}(0) + (\tilde{\theta}(0))^2\right)}, \\
R^{(1)}(t, x, y) = \frac{\varepsilon x \hat{\pi}^{(0)} \tilde{\theta}(0)}{y - \varepsilon x \left(y \partial_x \tilde{\theta}(0) + (\tilde{\theta}(0))^2\right)}, \\
P^{(1)}(t, x, y) = \frac{a \left(y - \varepsilon x \tilde{\theta}(0)\right) \left(\tilde{\theta}(0) - \log(y - \varepsilon x \tilde{\theta}(0))\right) + \varepsilon x F(t, x, y)}{y - \varepsilon x \left(y \partial_x \tilde{\theta}(0) + (\tilde{\theta}(0))^2\right)},
\]

(2.24)
Figure 2.3: Stage-0 and 1 optimal portfolio as functions of the current commodity level $y$.

where

$$
F(t, x, y) = \partial_t \hat{\theta}^{(0)} + \frac{1}{2} \left( (Q^{(1)})^2 + (R^{(1)})^2 \right) y^2 \partial_{yy} \hat{\theta}^{(0)} + \mu \hat{\theta}^{(0)} \hat{\pi}^{(0)} + r \hat{\theta}^{(0)} \left( 1 - \hat{\pi}^{(0)} - \hat{\theta}^{(0)} \right) + \left( (Q^{(1)})^2 + (R^{(1)})^2 \right) \hat{\theta}^{(0)} + \sigma R^{(1)} \hat{\pi}^{(0)} y \partial_y \hat{\theta}^{(0)}.
$$

Using the explicit finite difference scheme (2.23), we can solve for the value function $G$. The optimal portfolio $(\pi^{(1)}, \theta^{(1)})$ is then recovered from (2.15), see Figure 2.3 for the result. We observe that the stock holding $\pi^{(1)}$ now varies with the level of the commodity price $y$, where in the pre-financialization stage-0, it was independent.

**Volatility and correlation**

Ultimately we are interested in studying how price impact effects the volatility of the commodity price and its correlation with the stock price. To this end, we note that the stage-$k$ volatility $\eta^{(k)}$ of the commodity price and the stage-$k$ correlation $\rho^{(k)}$ with the stock price are given by

$$
\eta^{(k)} = \sqrt{(Q^{(k)})^2 + (R^{(k)})^2}, \quad \rho^{(k)} = \frac{R^{(k)}}{\sqrt{(Q^{(k)})^2 + (R^{(k)})^2}}.
$$

In Figures 2.4a and 2.4b we plot the stage-0, stage-1, and stage-2 volatilities and correlations as functions of $Y_0 = y$ with $X_0 = 1, T = 2$. 
Stage-0 We simply have
\[ \eta^{(0)} = \lambda b \quad \text{and} \quad \rho^{(0)} = 0. \]

Stage-1 The stage-1 volatility and correlation can be computed explicitly using the explicit solution to the stage-0 problem. From (2.24), we see that
\[ \eta^{(1)} = \lambda b \left( 1 + \frac{\varepsilon X_t}{Y_t} \left( Y_t \partial_y \hat{\theta} + \hat{\theta}^2 - \hat{\theta} \right) + \mathcal{O}(\varepsilon^2) \right) \]
\[ \rho^{(1)} = \frac{\varepsilon \sigma X_t \hat{\pi} \hat{\theta}}{\sqrt{\lambda^2 b^2 \left( Y_t - \varepsilon X_t \hat{\theta} \right)^2 + \left( \varepsilon \sigma X_t \hat{\pi} \hat{\theta} \right)^2}} = \frac{\varepsilon \sigma X_t \hat{\pi} \hat{\theta}}{\lambda b Y_t \hat{\pi} \hat{\theta} + \mathcal{O}(\varepsilon^2)} \]
(2.25)
to leading order in \( \varepsilon \), where \( \hat{\pi} \) and \( \hat{\theta} \) are the stage-0 optimal portfolio. Notice in particular that the sign of \( \rho_t \) is given by the sign of \( \hat{\pi}^{(0)}(t,Y_t^{(1)}) \), as long as the Merton ratio \( \hat{\pi}^{(0)} \) is positive, or equivalently \( \mu > r \). Therefore, when the commodity price is low (resp. high), portfolio optimizers are long (resp. short) the commodity and feedback correlation is positive (resp. negative).

Stage-2 From the (numerical) solution to the stage-1 portfolio optimization problem, we can determine the stage-2 commodity price dynamics using (2.11). This gives the stage-2 volatility of the commodity price as well as it correlation with the stock. We notice that the volatility of the commodity price reduces further in stage-2 (compared to stage-1) when the commodity is near its long-term mean; while the stage-2 correlation is greater when the commodity price is below its mean level, and approximately the same as in stage-1 when above.

2.4.2 Comparative Statics
We observe how \( \rho^{(1)}(t,X_t,Y_t^{(1)}) \) varies in the other model parameters, by modifying each of \( \gamma, \varepsilon, a, \) and \( b \) independently while holding the other parameters constant, see Figure 2.5. Since the instantaneous correlation depends on the commodity price \( Y_t^{(1)} \),
(a) Volatility of the commodity price.  
(b) Correlation between the commodity and stock.

Figure 2.4: Volatility and correlation at stage-0, stage-1, and stage-2.

(a) How \( \rho \) varies in \( \gamma \)  
(b) How \( \rho \) varies in \( \varepsilon \)  
(c) How \( \rho \) varies in \( \alpha \)

Figure 2.5: Comparative statics.

we consider a fixed low commodity price (50\% of its mean, in blue) and a fixed high commodity price (150\% of its mean, in red).

As we observed analytically earlier in this section, we see that when the commodity price is below (resp. above) its mean and the portfolio optimizers long (resp. short) the commodity, the correlation is positive (resp. negative). As the risk aversion
\( \gamma \) increases, the correlation induced by the perturbation of the risk-averse portfolio optimizers tends to zero. A sufficiently risk-averse trader will invest in neither the stock nor the commodity and thus will have no market impact. We also see that increasing \( \varepsilon \) increases the magnitude of the induced correlation. If the aggregate wealth of the portfolio optimizers is large enough, they overtake the market and the model breaks down. Interestingly, increasing the speed of mean-reversion \( a \) also increases the magnitude of the induced correlation. In the extreme case with a very large \( a \), the portfolio optimizers will be able to achieve large and reliable gains in either a short or long position in the commodity whenever it deviates from its mean, and they choose a highly leveraged position in the commodity and it caused a high induced correlation.

### 2.4.3 Simulation

We can compute the empirical volatility and correlation arising from our feedback model. See Figures 2.6 and 2.7 for the daily sampled volatility and correlation over a 2-year horizon using 5,000 paths in two scenarios: low commodity volatility \((b = 0.3)\) and high commodity volatility \((b = 1)\). We see that financialization causes a decrease in commodity volatility in both scenarios. This is expected because the portfolio optimizers are buying low and selling high, and their trading has a stabilizing effect on the commodity price. Moreover, we see that the stage-0 correlation is sharply peaked at zero; while at stage-1, a relatively low (resp. high) commodity volatility \( b \) can induce a mostly negative (resp. positive) correlation between the commodity and stock.
Figure 2.6: Empirical volatility between commodity and stock at stage-0 and stage-1. Monte-Carlo simulation uses 5,000 sample paths and starting points $S_0 = 1$, $X_0 = 50$, and $Y_0 = e^3$.

Figure 2.7: Empirical correlations between commodity and stock at stage-0 and stage-1. Monte-Carlo simulation uses 5,000 sample paths and starting points $S_0 = 1$, $X_0 = 50$, and $Y_0 = e^3$.
2.5 Empirical Analysis

2.5.1 Cross-correlations Between Commodities

We gathered daily prices of ten of the most heavily-traded commodity futures prices (see Table 2.2) over the period from 1990-2011.

In line with the hypotheses of Tang and Xiong [114], we choose 2004 as the dividing time-point between the unperturbed commodity price movement and the perturbed price movement due to the influx of index investors.

We find that the correlation between each commodity and the S&P 500 in the “non-indexed period” (1990-2004) is, averaging over the ten commodities, -0.0078435. In the “indexed period” (2004-2011), the average commodity-to-stock correlation is 0.1883. This dramatic increase supports the idea that commodities became more correlated with stocks after 2004.

In comparing all possible pairs of cross-correlations between these commodities, we find that the average value of the correlation over all commodity pairs is 0.05988 in the non-indexed period and 0.1860 in the indexed period, which supports the idea of increased commodity cross-correlation as discussed in [114].

2.5.2 Parameter Estimation

For each of the ten commodities, using the historical data, we can estimate the parameters of the stage-0 price process through the use of maximum likelihood estimation. The continuous form of the price process is, by (2.2),

\[ dY_t^{(0)} = a \left( \bar{m} - \log(Y_t^{(0)}) \right) Y_t^{(0)} dt + \lambda b Y_t^{(0)} dW_t. \]  

(2.26)

By the non-closed-form approach detailed by Franco [47], we show, in Table 2.2, the maximum likelihood estimators for the parameters of this geometric Ornstein-
Uhlenbeck process for each commodity, where \( a_{1990} \) denotes the best-fit value for the parameter \( a \) over the period beginning at 1990, and so on.

Table 2.2: \textit{Parameter estimates: the ten most heavily-traded commodity futures fitted to the geometric Ornstein-Uhlenbeck process. We note that the parameters are normalized daily.}

<table>
<thead>
<tr>
<th>Commodity</th>
<th>( a_{1990} )</th>
<th>( a_{2004} )</th>
<th>( m_{1990} )</th>
<th>( m_{2004} )</th>
<th>( b_{1990} )</th>
<th>( b_{2004} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBOT Corn Futures Prices</td>
<td>0.0023</td>
<td>0.0008</td>
<td>5.5204</td>
<td>6.5317</td>
<td>0.0116</td>
<td>0.0160</td>
</tr>
<tr>
<td>CSCE Cocoa Futures Prices</td>
<td>0.0024</td>
<td>0.0020</td>
<td>7.1868</td>
<td>7.8537</td>
<td>0.0169</td>
<td>0.0187</td>
</tr>
<tr>
<td>NYMEX Crude Oil Futures</td>
<td>0.0026</td>
<td>0.0028</td>
<td>3.1673</td>
<td>4.3921</td>
<td>0.0181</td>
<td>0.0190</td>
</tr>
<tr>
<td>NYCE Cotton Futures Prices</td>
<td>0.0018</td>
<td>0.0014</td>
<td>4.2041</td>
<td>4.3829</td>
<td>0.0141</td>
<td>0.0185</td>
</tr>
<tr>
<td>CSCE Coffee Futures Prices</td>
<td>0.0015</td>
<td>0.0013</td>
<td>4.6238</td>
<td>5.3168</td>
<td>0.0217</td>
<td>0.0158</td>
</tr>
<tr>
<td>NYMEX Natural Gas Futures Contract</td>
<td>0.0019</td>
<td>0.0031</td>
<td>1.2436</td>
<td>1.8634</td>
<td>0.0274</td>
<td>0.0265</td>
</tr>
<tr>
<td>CSCE Sugar No. 11 Futures Prices</td>
<td>0.0023</td>
<td>0.0013</td>
<td>2.1763</td>
<td>3.2023</td>
<td>0.0170</td>
<td>0.0190</td>
</tr>
<tr>
<td>CBOT Soybean Futures Prices</td>
<td>0.0018</td>
<td>0.0011</td>
<td>6.4326</td>
<td>7.0582</td>
<td>0.0102</td>
<td>0.0147</td>
</tr>
<tr>
<td>COMEX Silver Futures</td>
<td>0.0042</td>
<td>0.0005</td>
<td>6.1989</td>
<td>8.9836</td>
<td>0.0111</td>
<td>0.0179</td>
</tr>
<tr>
<td>CBOT Wheat Futures Price</td>
<td>0.0023</td>
<td>0.0015</td>
<td>5.8315</td>
<td>6.4878</td>
<td>0.0131</td>
<td>0.0180</td>
</tr>
</tbody>
</table>

We notice that most commodities experience a reduction in their best-fit value of \( a \) from the non-indexed period to the indexed period, though it increases for the two energy commodities. For each commodity, the best-fit value of the long-term mean log-price \( m \) increases, though not inconsistent with what would be expected due to inflation between the two periods. There is no clear trend in the best-fit values of \( b \) for the commodities over the two time periods.

2.5.3 Correlations In Extreme Price Cases

Table 2.3 shows the correlations between the commodities and the S&P 500. This illustrates the significant increase in correlation between commodities and stocks that we have tried to model.
Table 2.3: Correlations between the commodities and the S&P 500.

<table>
<thead>
<tr>
<th>Commodity, correlated to S&amp;P</th>
<th>non-indexed period</th>
<th>indexed period</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBOT Corn Futures Prices</td>
<td>0.0099</td>
<td>0.2231</td>
</tr>
<tr>
<td>CSCE Cocoa Futures Prices</td>
<td>-0.0182</td>
<td>0.1473</td>
</tr>
<tr>
<td>NYMEX Crude Oil Futures</td>
<td>-0.0546</td>
<td>0.3287</td>
</tr>
<tr>
<td>NYCE Cotton Futures Prices</td>
<td>0.0116</td>
<td>0.1511</td>
</tr>
<tr>
<td>CSCE Coffee Futures Prices</td>
<td>0.0290</td>
<td>0.1928</td>
</tr>
<tr>
<td>NYMEX Natural Gas Futures Contract</td>
<td>-0.0092</td>
<td>0.1035</td>
</tr>
<tr>
<td>CSCE Sugar No. 11 Futures Prices</td>
<td>-0.0082</td>
<td>0.1499</td>
</tr>
<tr>
<td>CBOT Soybean Futures Prices</td>
<td>0.0124</td>
<td>0.2074</td>
</tr>
<tr>
<td>COMEX Silver Futures</td>
<td>-0.0516</td>
<td>0.1848</td>
</tr>
<tr>
<td>CBOT Wheat Futures Price</td>
<td>0.0005</td>
<td>0.1943</td>
</tr>
</tbody>
</table>

2.5.4 Application: model calibration

As an application of the proposed model, we perform a calibration exercise to demonstrate that our model can generate the size of empirical correlation typically observed in the commodity market as shown in Table 2.3. The pre-financialized commodity dynamics is taken to be the Schwartz one-factor model (2.2); using oil as an example, we take the market calibrated parameters found by Schwartz [108, Table 4] and estimate the size of portfolio optimizers in our model (using εX as a proxy) to generate realistic correlation observed in the commodity market.

Figure 2.8a shows the empirical correlation between the stage-1 commodity price and the stock price, for different values of εX. Using 1000 Monte-Carlo simulations, we show the average empirical correlation as well as the 25th and 75th percentiles. We see that with realistic market parameters, our model can generate the typical
Levels of empirical correlation.

Figure 2.8: Model calibration using Monte-Carlo simulations. Parameters for commodity dynamics are taken from Schwartz [108, Table 4]: $a = 0.301, b = 0.334, m = 3.093$. Other parameters are $A = 1, \gamma = 1, \lambda = 1, \mu = 0.1, r = 0, \sigma = 0.15, T = 2$.

correlation level with the size of portfolio optimizers $\varepsilon X$ or order one. For instance, for $\varepsilon X = 5$, a time series analysis of two-year daily data could reveal a correlation between 0.2% at the 25-percentile and 30% at the 75-percentile. Figure 2.8b shows the corresponding distribution of empirical correlation for $\varepsilon X = 5$.

2.6 Conclusion

Despite the speculative nature of the portfolio optimizers in this model, they will frequently act to stabilize commodity prices through their trading. As in the simple economic argument of Friedman [50], the portfolio optimizers generally buy the commodity when the commodity price is below its mean and sell the commodity when the commodity price is above its mean, creating a demand effect which keeps the price nearer to its mean. We have shown that this volatility reduction occurs when the amount invested is somewhat near an unleveraged long position.

Correlation between commodities and the stock market is also of significant practical interest, and we have shown that the sign of the stage-1 induced correlation in our model will be the same as the sign of the fraction of wealth invested. This leads
to a high positive correlation when the commodity price is unusually low, which is undesirable, but it also leads to a high negative correlation when the price is unusually high, which is desirable for investors as it will, in some sense, cause the commodity price to move in the opposite direction as the overall economy during the times when the commodity price is high.

Overall, through numerical simulations, for a few different batches of reasonable market parameters, the net effect of the introduction of the portfolio optimizers seems to be a significant reduction in commodity price volatility and induced correlation with the stock market.
Chapter 3

Bertrand & Cournot Mean Field Games

In this chapter and the next, we study how dynamic game theory, in particular the framework of mean field games, can be applied to study the interactions between energy producers of traditional and alternative sources. In particular, we are interested in the issue of blockading: how low must conventional oil reserves go before it becomes profitable to start producing from more expensive but sustainable sources? However, to introduce ideas and notations in the simplest setting, this chapter will focus on the case of linear demand functions and a market without the alternative energy producers. The strategic actions of traditional energy producers in face of potential entry of an alternative energy producer will be studied thoroughly in Chapter 4. To illustrate the wide applicability of the mean field game approach to economic problems, the exposition in this chapter will focus on the Bertrand model, which is applicable for consumer goods markets; in Chapter 4 we return to the Cournot market model which is more suitable to the energy production market. We show in Appendix B that the two models are in fact equivalent in the continuum limit.
Specifically, we study in this chapter how continuous time Bertrand and Cournot competitions, in which firms producing similar goods compete with one another by setting prices or quantities respectively, can be analyzed as continuum dynamic mean field games. Interactions are of mean field type in the sense that the demand faced by a producer is affected by the others through their average price or quantity. Motivated by energy or consumer goods markets, we consider the setting of a dynamic game with uncertain market demand, and under the constraint of finite supplies (or exhaustible resources). The continuum game is characterized by a coupled system of partial differential equations: a backward Hamilton-Jacobi-Bellman PDE for the value function, and a forward Kolmogorov PDE for the density of players. Asymptotic approximation enables us to deduce certain qualitative features of the game in the limit of small competition. The equilibrium of the game is further studied using numerical solutions, which become very tractable by considering the tail distribution function instead of the density itself. This also allows us to consider Dirac delta distributions to use the continuum game to mimic finite $N$-player nonzero-sum differential games, the advantage being having to deal with a couple system of two PDEs, instead of $N$. We find that, in accordance with the two-player game, a large degree of competitive interaction causes firms to slow down production. The continuum system can therefore be used qualitative as an approximation to even small player dynamic games. This chapter is adapted from the article [28].

3.1 Introduction

Oligopoly models of markets with a small number of competitive players go back to the classical works of Cournot [36] and Bertrand [14] in the 1800s. These have typically been static (or one-period) models, where the existence and construction of a Nash equilibrium have been extensively studied. We refer to Vives [115] for a
survey. In the context of nonzero-sum dynamic games between $N$ players, each with their own resources, the computation of a solution is a challenging problem, typically involving coupled systems of $N$ nonlinear partial differential equations (PDEs), with one value function per player. This is further complicated when the players’ resources are exhaustible and the market structure changes over time as players deplete their capacities and drop out of competition. See, for instance, [60] in a Cournot framework, and [76] in a Bertrand model.

On the other hand, mean field games proposed by Lasry and Lions [74] and independently by Huang et al. [63, 64] allow one to handle certain types of competition in the continuum limit of an infinity of small players by solving a coupled system of two PDEs. The interaction here is such that each player only sees and reacts to the statistical distribution of the states or actions of other players. Optimization against the distribution of other players leads to a backward (in time) Hamilton-Jacobi-Bellman (HJB) equation; and in turn their actions determine the evolution of the state distribution, encoded by a forward Kolmogorov equation. We refer to the survey article [59] and the recent monograph [10] for further background.

Our goal in this chapter is to understand the relationship between oligopoly games in the traditional Nash equilibrium sense and their appropriately-defined mean field counterpart, especially the approximation of one by the other. We look at oligopoly models for markets with differentiated but substitutable goods, and in continuous time with potentially random fluctuations in demands. The firms have different production capacities representing their different sizes, and the fraction of firms remaining decreases over time as smaller firms exhaust their capacities and disappear from the market.

In Cournot competition where firms choose quantities of production, an example might be oil, coal and natural gas in the energy market, while in a Bertrand model, where firms set prices, an example might be competition between food producers
where consumers have preference for one type of food, but reduce their demand for it depending on the average price of substitutes. We shall see that, in the \textit{continuum} mean field versions of these games, where there is an infinite number of firms, the Cournot and Bertrand models are equivalent, in the sense that they result in the same equilibrium prices and quantities. However, for concreteness, we will focus our exposition in this chapter on Bertrand competition, and return to Cournot competition in the next chapter. Throughout, we work with \textit{linear} demand systems.

In dynamic oligopoly problems with a finite number of players, the HJB system of PDEs does not admit an explicit solution, except possibly in the monopoly case, and one needs numerical means for computing the value functions as well as the equilibrium strategies, which of course quickly becomes infeasible as the number of players goes beyond three. Moreover, even in the two-player case, these equations are hard to handle when the competition is strong. To overcome this problem, we study the market dynamics when the number of firms tends to infinity and the resulting interactions are modeled as a mean field game (MFG).

\subsection*{3.1.1 Related Literature}

Our model is related to two different strands of literature: the literature of dynamic oligopoly with exhaustibility, and the literature on the use of mean field games in economic applications.

\textbf{Dynamic Oligopoly} The study of static oligopoly models of markets with a small number of competitive players goes back to the classical works of Cournot \cite{cournot1838} and Bertrand \cite{bertrand1883} in the 1800s. More recently, energy markets have been modeled through dynamic games. Harris \textit{et al.} \cite{harris2010} characterize a dynamic Cournot game in an oligopoly market by systems of nonlinear Hamilton-Jacobi PDEs. Ledvina and Sir-car \cite{ledvina2012} study the corresponding Bertrand game in which firms compete with one
another by setting prices. We refer to Dockner [40] for an introduction to the applications of dynamic games in economics and management science.

Hotelling [62] introduced one of the first models for the management of an exhaustible resource. In a monopoly setting, Hotelling solved a calculus-of-variations problem and showed that the marginal value of reserves grows at the discount rate along the optimal extraction path, which is now referred to as Hotelling’s rule. The competition between a single exhaustible producer and \( N - 1 \) renewable producers has been considered by Ledvina and Sircar [78]. This simplified setup provides insights into the effect of blockading: how low must oil reserves go before it becomes profitable for the renewable producers to enter the market. It also leads to a modified piecewise version of Hotelling’s rule.

Other aspects of the exhaustibility issue are renewability and exploration. For example, while fossil fuels are ultimately exhaustible, they are also replenishable by (costly) exploration efforts. The optimal planning of exploration effort has been considered by Pindyck [102] and many others in the monopoly context, and Ludkovski and Sircar [82] in a dynamic duopoly. Dynamic Cournot games when the demand function is stochastic are studied in Ludkovski and Yang [84]. For a recent survey of game theoretic models for energy production, we refer to Ludkovski and Sircar [83].

**Mean Field Games** Second, there is also a literature on the use of mean field games to economic applications. Since the seminal papers by Lasry and Lions [74] and Huang et al. [63] 64, this approximation technique has attracted considerable interest recently as the corresponding \( N \)-player dynamic games are almost always intractable using PDE methods. In the context of energy production, Guéant et al. [58] 59 have considered a mean field version of a Cournot game with a quadratic cost function; while in this chapter and the next, we apply asymptotic and numerical
methods to study how substitutability affects the market equilibrium in Bertrand and Cournot mean field games.

There are many other applications of mean field games in economics, and we list only a few. Lucas and Moll [81] study knowledge growth in an economy with many agents of different productivity levels. In particular, what they call a “balanced growth path” resembles our sustainable economy in Section 4.5. Carmona et al. [24] present a mean field game model for analyzing systemic risk. Mean field games analysis has been adopted to study the optimal execution problem in algorithmic trading by Jaimungal and Nourian [66].

Recently, several authors propose to analyze the mean field games using probabilistic techniques. Carmona and Delarue [23] provide a probabilistic analysis of a class of stochastic differential games for which interaction between players is of mean field type and show that solutions of the mean field game do indeed provide approximate Nash equilibria for games with a large number of players. In a subsequent paper, Carmona et al. [22] discuss the similarities and the difference between the MFG approach and control of McKean-Vlasov dynamics via analysis of forward-backward stochastic differential equations. Carmona and Lacker [25] provide a weak formulation of stochastic optimal control allowing more general mean field interactions including rank and nearest-neighbor effects. For a comprehensive study of the uniqueness and existence of equilibrium strategies of a general class of mean field games, including the linear-quadratic framework, we refer to Bensoussan et al. [10].

3.1.2 Organization and Results

• In Section 3.2, we present the model for Bertrand competition with differentiated goods and discuss the finite player and continuum limit MFG for the one-period problem. We see that the MFG equilibrium is formally the limit of finite player Nash equilibrium.
• In Section 3.3, we introduce the dynamic Bertrand MFG problem with exhaustible capacities and set up the resulting forward/backward PDE system. We give explicit calculations for the monopoly problem.

• In Section 3.4, we obtain an asymptotic expansion in powers of a parameter that represents the extent of competition between the firms in a deterministic game. This captures the principle effects and quantifies the effect of product substitutability.

• In Section 3.5, we present the numerical solution of the forward/backward system of PDEs that allows us to characterize the price strategies and resulting demands of firms in the stochastic game.

• This allows us to compare and contrast the pricing strategies in the MFG approximation to the N-player game in Section 3.6. Here we use the Dirac delta functions to mimic the finite player case, and by considering the tail distribution function instead of the density itself, numerical solutions become very tractable, especially in the deterministic setting.

We conclude in Section 3.7.

### 3.2 Linear Demand, Continuum Limit and Static MFG

To motivate the form of demand functions we are going to use in the continuum MFG, we first study a finite market with N firms who produce substitutable goods which compete for market share in a one-period game. Associated to each firm \(i \in \{1, \ldots, N\}\) are variables \(p_i \in \mathbb{R}, q_i \in \mathbb{R}_+\) representing the price and quantity at which firm \(i\) offers its good for sale to the market. We denote by \(p\) the vector of
prices whose $i$th element is $p_i$, similarly the $i$th element of the vector $q$ is $q_i$. In the Cournot model, players choose quantities as a strategic variable in non-cooperative competition with the other firms, and the market determines the price of each good. In a Bertrand competition, which will be the main focus of the present chapter, firms set prices, and the market determines its demand for each type of good.

### 3.2.1 Linear Demand System

The market model is specified by linear inverse demand functions, which give prices as a function of quantity produced, and are the basis of Cournot competition. Our firms are suppliers, and so quantities are nonnegative. For $q \in \mathbb{R}_+^N$, the price received by player $i$ is $p_i = P_i(q)$ where

$$P_i(q) = 1 - (q_i + \varepsilon \bar{q}_i), \quad \text{where } \bar{q}_i = \frac{1}{N-1} \sum_{j \neq i} q_j, \quad i = 1, \cdots, N, \quad \text{and } 0 \leq \varepsilon < N-1. \quad (3.1)$$

The inverse demand functions are decreasing in all of the quantities. In the linear model (3.1), some of the prices $p_i = P_i(q)$ may be negative, meaning player $i$ produces so much that he has to pay to have his goods taken away, but we will see negative prices do not arise in competitive equilibrium.

The parameter $\varepsilon$ measures the degree of interaction or product substitutability, in the sense that the price received by an individual firm decreases as the other firms increase production of their goods. In particular, the case $\varepsilon = 0$ corresponds to independent goods. In this chapter, we will consider the two cases of fixed and finite $N$ as well as the continuum limit where $N = \infty$, and study how varying the interaction parameter $\varepsilon$ affects the competitive equilibrium.

The dependence of the price player $i$ receives depends on the quantities produced by his competitors through their mean $\bar{q}_i$. That is, the interaction is of mean field type. In particular, the inverse demand (3.1) takes the form $1 - p = Aq$, where $A$
can be written as a rank-one update to the identity matrix

\[ A = \left( 1 - \frac{\varepsilon}{N - 1} \right) I + \frac{\varepsilon}{N - 1} \mathbf{1} \mathbf{1}^T, \quad \text{where} \quad \mathbf{1} = (1, \ldots, 1)^T. \]

As a consequence, the demand function \( \mathbf{q} = A^{-1}(1 - \mathbf{p}) \) can be computed explicitly using the Sherman-Morrison formula [109]. However, the demands must be nonnegative and so the inversion process is as follows. First order the price vector \( \mathbf{p} \in \mathbb{R}^N \) such that \( p_1 \leq \cdots \leq p_N \), so that resulting demands will be decreasing in the player number. We need to find the largest \( n \leq N \) such that player \( n \) receives nonnegative demand, while the players above him setting higher prices will have their demands set to zero.

More specifically, for given \( n \) we invert the first \( n \) equations in (3.1) to give

\[ D_i^{(n)}(\mathbf{p}) = a_n - b_n p_i + c_n \bar{p}_n^i, \quad \bar{p}_n^i = \frac{1}{n-1} \sum_{j \neq i}^n p_j, \]

(3.2)

where the positive coefficients \((a_n, b_n, c_n)\) are given by

\[ a_n = \frac{1}{1 + \varepsilon \frac{n-1}{N-1}}, \quad b_n = \frac{1 + \varepsilon \frac{n-2}{N-1}}{(1 + \varepsilon \frac{n-1}{N-1}) (1 - \frac{\varepsilon}{N-1})}, \quad c_n = \frac{\varepsilon \frac{n-1}{N-1}}{(1 + \varepsilon \frac{n-1}{N-1}) (1 - \frac{\varepsilon}{N-1})}. \]

(3.3)

Note that our assumption \( \varepsilon < N - 1 \) in (3.1) implies the system is invertible and that \( b_n > c_n > 0 \), and we also observe that \( a_n + c_n = b_n \).

Then we find the largest \( n \) such that \( D_n^{(n)}(\mathbf{p}) \geq 0 \), and the actual demands are given by

\[ q_i = D_i(\mathbf{p}) = \begin{cases} a_n - b_n p_i + c_n \bar{p}_n^i, & i = 1, 2, \ldots, n \\ 0 & i = n + 1, \ldots, N. \end{cases} \]

(3.4)

These demand functions are the basis of Bertrand competition. The demand for player \( i \)’s good, \( q_i = D_i(\mathbf{p}) \), is decreasing in his own price and increasing in the prices of his competitors, again through their mean \( \bar{p}_n^i \).
Remark 3. Such a linear demand system can be derived from a quadratic utility function of the form

\[ U(q) = \sum_{i=1}^{N} q_i - \frac{1}{2} \left( \sum_{i=1}^{N} q_i^2 + \frac{\varepsilon}{N-1} \sum_{j \neq i} q_i q_j \right), \]

and solving the utility maximization problem \( \max_{q \in \mathbb{R}_+^N} U(q) - q \cdot p \) for a representative consumer; the first-order condition gives equation (3.1). We refer to Vives [115, Chapter 6] for more details.

Continuum Analog In the continuum limit \( N \to \infty \), the continuous variable corresponding to \( n/N \) in the finite player game is denoted by \( \eta \in [0, 1] \), which is the proportion of players who receive positive demand. The demand received by a representative producer decreases with his own price \( p \) and increases with the average price \( \bar{p} \) charged by the other producers. We define the demand function, by analogy with (3.2) and (3.3), to be

\[ D^{(\eta)}(p, \bar{p}) = a(\eta) - b(\eta)p + c(\eta)\bar{p}, \quad \eta > 0, \] (3.5)

where the continuum limits of (3.3) are

\[ a(\eta) = \frac{1}{(1 + \varepsilon\eta)}, \quad b(\eta) = 1, \quad c(\eta) = \frac{\varepsilon\eta}{(1 + \varepsilon\eta)}, \] (3.6)

and the average price \( \bar{p} \) will be defined for the static game in the next section, and for the dynamic game in Section 3.3. We note also that \( c(\eta) < b(\eta) = 1 \), and \( a(\eta) + c(\eta) = b(\eta) = 1 \).

Remark 4. In the traditional Cournot literature (see, for example, [115]), the inverse demand functions, \( P_i \) in (3.1), are written as depending on the other players’ total quantity instead of the average \( \bar{q}_i \), which amounts to taking \( \varepsilon = N - 1 \). In this case,
the goods are termed homogeneous in the sense that the prices are the same for each players’ goods and the one price is \( p_i(q) \equiv p(q) = 1 - \sum^N q_j \). This is the case considered in the static and dynamic games in [60]. For a Bertrand game, when \( \varepsilon = N - 1 \), the system (3.1) is not invertible, and this corresponds to the classical Bertrand “winner takes all” model in which all demand goes to the player offering the lowest price. This type of competition typically results in a Nash equilibrium in which players set prices equal to cost and so make zero profit. This “tough” competitive effect is not usually seen in markets, and a differentiated (or substitutable) goods Bertrand model is more reasonable. This is the basis of the static and dynamic games in [76]. For a discussion and comparison between the four static games (Cournot and Bertrand with homogeneous or differentiated goods), we refer to [78].

**Remark 5.** In this section, we are going to associate each player with a constant marginal cost of production, even though there is no explicit production cost in the dynamic MFG. But as we will see in Section 3.3, the exhaustibility of production capacity induces a shadow cost that will play the role of marginal cost in this section.

### 3.2.2 Static Bertrand Games

Here we consider the static Bertrand game of \( N \) players and the continuum MFG version of it, and show that the limit of the former’s Nash equilibrium price vector as \( N \to \infty \) gives the solution of the latter.

#### N-player Games

There are \( N \) players who have constant marginal costs of production \( 0 \leq s_1 \leq \cdots \leq s_N \), which are “small enough” (made precise in (3.11) below) such that all the players are active, in the sense of receiving positive demand in the Nash equilibrium computed below. In other words, when using the demand functions (3.4), we only have to consider \( n = N \). (The case where some costs are higher and so some firms are
“blockaded” from competition and receive zero demand in equilibrium is considered in detail in Chapter 4 and in [78].

The optimization problem faced by each firm $i$ is

$$
\max_{p_i} (a_N - b_N p_i + c_N \bar{p}_i) (p_i - s_i), \quad \bar{p}_i = \frac{1}{N-1} \sum_{j \neq i} p_j. \quad (3.7)
$$

The first order condition for each player gives

$$
p_i^* = \frac{1}{2b_N} (a_N + b_N s_i + c_N \bar{p}_i), \quad i = 1, \ldots, N. \quad (3.8)
$$

This is the best response of player $i$ if the other players play prices with average $\bar{p}_i$. We say $(p_1^*, \ldots, p_N^*)$ is a Nash equilibrium when the best response equations (3.8) intersect, that is

$$
\bar{p}_i = \frac{1}{N-1} \sum_{j \neq i} p_j^* \quad \text{for all } i = 1, \ldots, N.
$$

Throughout, we will not attach a * to $\bar{p}$.

The system of equations (3.8) is scalarized by averaging over the players. Summing over $i$ and dividing by $N$, we find that the average price $\bar{p} = \frac{1}{N} \sum_{i=1}^{N} p_i^*$ is given by

$$
\bar{p} = \frac{1}{2b_N - c_N} (a_N + b_N \bar{s}^{(N)}), \quad \text{where} \quad \bar{s}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} s_i. \quad (3.9)
$$

Note that $\bar{p}$ is well-defined as $b_N > c_N$. In particular it depends on the costs only through their average $\bar{s}^{(N)}$.

The individual Nash equilibrium prices $p_i^*$ are given by (3.8), where $\bar{p}_i$, the average without player $i$, can be written in terms of the full average $\bar{p}$ as

$$
\bar{p}_i = \frac{N}{N-1} \bar{p} - \frac{1}{N-1} p_i^*.
$$
the difference $|\bar{p}_i - \bar{p}|$ being small as $N \to \infty$. Substituting back in to (3.8), we obtain

$$p^*_i = \frac{1}{2b_N + \frac{c_N}{N-1}} \left( b_N s_i + a_N + \frac{Nc_N}{N-1} \bar{p} \right).$$  

(3.10)

The resulting demands or quantities sold are given by $q^*_i = (a_N - b_N p^*_i + c_N \bar{p}_i)$. A sufficient and necessary condition that all players receive positive demand, that is $q^*_i > 0$, is that the highest cost $s_N$ is small enough. In [78, Theorem 4.1], this is shown to be equivalent to

$$s_N < s^\text{max}_N := \frac{\theta_N + \varepsilon \bar{s}^{(N-1)}}{\theta_N + \varepsilon}, \text{ where } \theta_N = \left( 1 - \frac{\varepsilon}{N-1} \right) \left( 2 + \frac{(2N-3)}{N-1} \varepsilon \right).$$  

(3.11)

which restricts how far the highest cost $s_N$ can be from the average $\bar{s}^{(N-1)} = \frac{1}{N-1} \sum_{j=1}^{N-1} s_j$ of the lower cost players. See [78] for further details, as well as the Nash equilibrium in other cases where (3.11) does not hold.

If we assume that as more players are added the limit $\bar{s} = \lim_{N \to \infty} \bar{s}^{(N-1)}$ is finite, then we have

$$s^\text{max}_N \to \frac{2 + \varepsilon \bar{s}}{2 + \varepsilon} \text{ as } N \to \infty.$$  

(3.12)

We shall see shortly that this limit coincides with the maximum marginal cost in the continuum mean field game.

Moreover, using the limits of $a_N, b_N, c_N$ as $N \to \infty$ given in equation (3.6), with $\eta = 1$ in the case that all firms are active in equilibrium, we have

$$\bar{p} \to \frac{1}{2 + \varepsilon} + \frac{1 + \varepsilon}{2 + \varepsilon} \bar{s},$$

and so

$$p^*_i \to \frac{1}{2} \left( s_i + \frac{2}{2 + \varepsilon} + \frac{\varepsilon}{2 + \varepsilon} \bar{s} \right), \text{ and } q^*_i \to \frac{1}{2} \left( \frac{2}{2 + \varepsilon} + \frac{\varepsilon}{2 + \varepsilon} \bar{s} - s_i \right).$$  

(3.13)
We shall again recover the same result from solving the continuum mean field game.

**Continuum MFG**

In the static continuum mean field game, there is an infinite number of players labeled by \( x > 0 \), with associated density \( M(x) \) and marginal cost of production \( s(x) \). (In the static model, the label \( x \) is really not necessary: one could label the players by their costs \( s \) and put a density \( M(s) \). However, in the dynamic game of Section 3.3, \( x \) will denote the remaining capacities of the players, so we retain it here). As in the finite-player example above, we suppose that the marginal costs \( s(x) \) are “small enough” (made precise in (3.19) below) for all \( x > 0 \) such that all the players are active, in the sense of receiving positive demand in the MFG solution we now compute. In other words, we take \( \eta = 1 \) in the MFG demand function (3.5)-(3.6).

A player at location \( x \) optimizes his profit as though he is unable to affect the mean price \( \bar{p} \):

\[
\max_{\bar{p}} \left( a(1) - \bar{p} + c(1)\bar{p} \right) \left( p(x) - s(x) \right),
\]

where we have used the continuum demand function defined in (3.5) and (3.6). Given \( \bar{p} \), the first order condition gives

\[
p^*(x) = \frac{1}{2} \left( s(x) + a(1) + c(1)\bar{p} \right).
\]

The mean price \( \bar{p} \) is

\[
\bar{p} = \int_0^{\infty} p^*(x)M(x) \, dx,
\]

and now the continuum system (3.15) is scalarized by multiplying by \( M \) and integrating over \( x \), which leads to

\[
\bar{p} = \frac{1}{2 - c(1)} \left( \bar{s} + a(1) \right), \quad \text{where} \quad \bar{s} = \int_0^{\infty} s(x)M(x) \, dx.
\]
Note that \( \bar{p} \) is well-defined as \( c < 1 \). Substituting back into (3.15), we find the optimal price \( p^*(x) \):

\[
p^*(x) = \frac{1}{2} \left( s(x) + \frac{2}{2 + \varepsilon} + \frac{\varepsilon}{2 + \varepsilon} \bar{s} \right),
\]

which is the continuum analog of the limit of \( p^*_i \) in (3.13). The demands are given by

\[
q^*(x) = D^{(1)}(p^*(x), \bar{p}) = a(1) - p^*(x) + c(1)\bar{p} = \frac{1}{2} \left( \frac{2}{2 + \varepsilon} + \frac{\varepsilon}{2 + \varepsilon} \bar{s} - s(x) \right),
\]

which is the continuum analog of the limit of \( q^*_i \) in (3.13). The \( q^*(x) \) in (3.18) are positive for all \( x > 0 \) if and only if

\[
s(x) < \frac{2 + \varepsilon \bar{s}}{2 + \varepsilon}, \quad \forall x > 0,
\]

which is the continuum analog of (3.11) and identical to the \( N \to \infty \) limit (3.12). Note also that it turns out the profit function is simply the square of the quantity:

\[
\Pi(x) = q^*(x) (p^*(x) - s(x)) = (q^*(x))^2.
\]

Eventually we would like to use the MFG machinery to approximate finite player games, and one way to do this is by taking the density \( M \) to be a series of delta functions centered at the points \( \{x_i \mid s(x_i) = s_i\} \): \( M = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \). Then, as in the discrete case, we have \( \bar{p} = \frac{1}{N} \sum_{i=1}^{N} p^*(x_i) \) and \( \bar{s} = \frac{1}{N} \sum_{i=1}^{N} s(x_i) \). The optimal price set by the player at \( x_i \) is

\[
p^*(x_i) = \frac{1}{2} (a(1) + s_i + c(1)\bar{p}), \quad s_i = s(x_i),
\]

where \( \bar{p} \) is given in (3.16). Comparing this with the discrete case in equations (3.10) and (3.9), we see that the only differences are \( (a(1), b(1), c(1)) \approx (a_N, b_N, c_N) \) and
\( \frac{N}{N-1} \approx 1 \), which quantifies the \( O(N^{-1}) \) approximation of the finite player static game by the continuum MFG.

### 3.2.3 Cournot-Bertrand Equivalence as \( N \to \infty \) and in the Continuum

Next, we contrast the difference between the static \( N \)-player Cournot and Bertrand competitions, and how the difference vanishes in the continuum limit. We focus on interior equilibrium in which all firms participate.

**Comparison of Cournot and Bertrand \( N \)-Player Games**

For the \( N \)-player games, let \( q \) and \( p \) be the vector of quantities and prices respectively. We denote by \( P \) the \( N \times N \) matrix with 1’s on the diagonal and \( \varepsilon/(N-1) \) everywhere else, so that the inverse demand system (3.1) for the Cournot problem can be written

\[
p = 1 - Pq, \quad \text{where} \quad 1 = (1, \cdots, 1)^T.
\]

From this, we have \( q = P^{-1}(1 - p) \), which gives the demand system (3.4) (with \( n = N \)) for the Bertrand problem:

\[
q = a_N 1 - Qp, \quad \text{where} \quad Q = P^{-1}.
\]

We note that \( Q \) is the matrix with \( b_N \) on the diagonal and \( -c_N/(N-1) \) everywhere else, and \( a_N 1 = P^{-1}1 \) (which is just the observation \( a_N + c_N = b_N \)).

**Cournot competition** Here firms choose quantities, and so player \( i \) solves

\[
\max_{q_i} q_i(1 - (Pq)_i - s_i),
\]
which leads to the Nash equilibrium intersection of the first-order conditions \((I + P)q^* = 1 - s\), where \(s\) is the vector of costs. Therefore, the Cournot equilibrium quantities are given by \(q^* = (I + P)^{-1}(1 - s)\), and the corresponding prices \(p_c^* = 1 - Pq^*\) are given by

\[
p_c^* = (I + P)^{-1}1 + P(I + P)^{-1}s.
\] (3.20)

Note that the average Cournot quantity \(\bar{q}\) and price \(\bar{p}_c\) satisfy \((1 + \epsilon)\bar{q} = (1 - \bar{p}_c)\), which follows from \(1^TP = (1 + \epsilon)1^T\); and it is straightforward to compute

\[
\bar{q} = \frac{1 - \bar{s}(N)}{2 + \epsilon}, \quad \bar{p}_c = \frac{1}{2 + \epsilon} + \frac{1 + \epsilon}{2 + \epsilon} \bar{s}(N),
\] (3.21)

where \(\bar{s}(N)\) is the average of the costs. One can also compute the profit of each player:

\[
\Pi_i = q_i^*(p_{c,i}^* - s_i) = (q_i^*)^2.
\]

**Bertrand competition** Here, as in [3.7], firm \(i\) solves \(\max_{p_i}(p_i - s_i)(a_N - (Qp)_i)\), which leads to the first order conditions, intersected to find the Nash equilibrium \(p^*\) that solves the linear system:

\[
(Q + b_N I)p^* = a_N 1 + b_N s.
\]

Multiplying by \(P\) and using that \(a_N 1 = P^{-1}1\), we have \((I + b_N P)p^* = 1 + b_N Ps\), which gives

\[
p^* = (I + b_N P)^{-1}1 + b_N P(I + b_N P)^{-1}s,
\]

where we have used an obvious commutation between \(P\) and \((I + b_N P)^{-1}\) (both being rank one updates of a diagonal matrix). This coincides with the prices (3.20) from the Cournot competition only if \(b_N = 1\), which is true only when the number of players goes to infinity. Indeed, since \(b_N = 1 + O(N^{-1})\), the difference between the
Cournot and Bertrand solutions is $O(N^{-1})$. The Bertrand equilibrium demands are $q_b^* = a_N 1 - Q p^*$ and we can compute that the profit is given by:

$$\Pi = q_{b,i}^* (p_i^* - s_i) = \frac{1}{b_N} (q_{b,i}^*)^2.$$

Furthermore, we also have that the average Bertrand demand $\bar{q}_b$ is given in terms of the average price $\bar{p}$ by $(1 + \varepsilon) \bar{q}_b = (1 - \bar{p})$, which follows from $1^T Q = (b_N - c_N) 1^T$ and $b_N - c_N = a_N = \frac{1}{1 + \varepsilon}$. From the formula (3.9), we have

$$\bar{p} = \frac{1}{2 + \varepsilon - \frac{2 \varepsilon}{N-1}} + \frac{1 + \varepsilon}{2 + \varepsilon} \left( \frac{N-2}{N-1} \bar{s}(N) \right), \quad \bar{q}_b = \frac{1}{1 + \varepsilon} \left( \frac{1 + \varepsilon}{2 + \varepsilon - \frac{2 \varepsilon}{N-1}} - \frac{1 + \varepsilon}{2 + \varepsilon} \left( \frac{N-2}{N-1} \bar{s}(N) \right) \right),$$

and so the average Bertrand price and quantity do not equal their Cournot counterparts given in (3.21), but $|\bar{p} - \bar{p}_c|, |\bar{q}_b - \bar{q}| \to 0$ as $N \to \infty$ at rate $N^{-1}$.

**Cournot Continuum MFG**

This observation anticipates that the Bertrand and Cournot continuum MFGs lead to the same equilibrium prices and quantities. To see this, consider the continuum Cournot game where, as in Section 3.2.2, there is an infinite number of players labeled by $x > 0$, with associated density $M(x)$ and marginal cost of production $s(x)$. The optimization problem faced by a player at position $x > 0$ is

$$\max_q q \left( 1 - q - \varepsilon \bar{q} - s(x) \right), \quad \bar{q} = \int q M,$$

where the continuum inverse demand function $P(q, \bar{q}) = 1 - q - \varepsilon \bar{q}$ is the analog of (3.1).

The first order condition gives

$$q^*(x) = \frac{1}{2} \left( 1 - s(x) - \varepsilon \bar{q} \right),$$

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and integrating with respect to $M$ leads to

$$\bar{q} = \frac{1}{2 + \varepsilon} (1 - \bar{s}).$$

Therefore we obtain

$$q^*(x) = \frac{1}{2} \left( \frac{2}{2 + \varepsilon} + \frac{\varepsilon}{2 + \varepsilon} \bar{s} - s(x) \right),$$

which is the same as found from the Bertrand continuum MFG, equation (3.18).

We shall see in Appendix B.2 that Cournot-Bertrand equivalence also hold in the dynamic mean field game with exhaustible resources. We introduce the Bertrand version of this problem in the next section.

### 3.3 Dynamic Mean Field Game with Exhaustible Capacities

We look now at a dynamic problem in which firms sell goods over time, but have different capacities or inventories, modeling that they are of different size. As they exhaust their supplies or reserves, they no longer participate and the market shrinks, but the demand functions adjust consistently according to (3.5). In the static game of the previous section, players were differentiated by their costs of production. In the dynamic $N$-player Bertrand games considered in [76], firms have different finite production capacities and, for simplicity, zero production costs. However, the firms will be faced with nonzero shadow costs due to the exhaustibility of production capacities. Here, we analyze the continuum mean field version of this problem.

There is an infinity of producers setting prices for their goods. At time $t = 0$, the density of players with remaining capacity $x > 0$ is given by $M(x)$, where $\int M = 1$. Initial capacity $x$ allows us to distinguish between bigger and smaller players. As time
evolves, some players exhaust their capacity by selling all their goods and drop out of competition, and we denote by $m(t,x)$ the “density” of firms with positive capacity at time $t > 0$. Let $\eta(t)$ be the fraction of active firms remaining at time $t$:

$$\eta(t) = \int_{\mathbb{R}_+} m(t,x) \, dx, \quad \mathbb{R}_+ = (0, \infty). \quad (3.22)$$

In general, we expect $\eta(t) < 1$ for large enough $t > 0$, and it plays the role of $n/N$ in the discrete setting. We define the exhaustion time $T$ of the game to be the first time $\eta$ hits zero, and all the quantities introduced in the following are defined for $t < T$.

Given the price $p(t,x)$ set by one of these players at time $t$, the expected demand for his good is

$$q(t,x) = D(\eta(t)) (p(t,x), \bar{p}(t)) = a(\eta(t)) - p(t,x) + c(\eta(t))\bar{p}(t), \quad (3.23)$$

where the functions $a$ and $c$ were defined in $(3.6)$, and $\bar{p}$ is the average price at time $t$ given by

$$\bar{p}(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}_+} p(t,x)m(t,x) \, dx. \quad (3.24)$$

The average price $\bar{p}$ is the continuum counterpart of $\bar{p}_i^n$ in $(3.2)$, which denotes the average price charged by the remaining firms except the $i$th one. In the continuum limit, a single firm no longer affects the average price, and hence $\bar{p}$ depends on $t$ but not on $x$.

The actual demands are subject to random fluctuations and we model this with an additive Gaussian white noise $\tilde{W}_t$ so that the actual demand is given by $q(t,x) - \sigma \tilde{W}_t$.

The remaining capacity (or reserves) $(X_t)_{t \geq 0}$ of any producer depletes according to the actual demand and follows the dynamics

$$dX_t = -q(t,X_t) \, dt + \sigma \, dW_t, \quad (3.25)$$
as long as \( X_t > 0 \), and \( X_t \) is absorbed at zero. As in [59], the Brownian motion \( W \) is specific to the agent considered, and \( \sigma \geq 0 \) is a constant.

A firm that starts with capacity \( x > 0 \) at time \( t \geq 0 \) sets prices over the horizon \([t, T)\) to maximize the lifetime profit discounted at constant rate \( r > 0 \) over Markov controls \( p_t = p(t, X_t) \), with the corresponding demand \( q_t = q(t, X_t) \) given by (3.23). The value function of the firm is defined by

\[
v(t, x) = \sup_p \mathbb{E} \left\{ \int_t^\infty e^{-(s-t)} p_s q_s 1_{\{X_s > 0\}} ds \mid X_t = x \right\}, \quad x > 0.
\]

The indicator function describes that the player is exhausted when \( X_t \) hits zero and he no longer can produce and earn revenue.

**Dynamic Programming and the HJB Equation**

The associated HJB equation is

\[
\partial_t v + \frac{1}{2} \sigma^2 \partial_{xx}^2 v - rv + \max_{p \geq 0} \left[ (a(\eta(t)) - p + c(\eta(t))\bar{p}(t)) (p - \partial_x v) \right] = 0. \tag{3.26}
\]

We observe that the internal optimization is the static MFG equilibrium problem (3.14), but with effective shadow cost (or scarcity) \( s(x) \mapsto \partial_x v(t, x) \). The first-order condition in (3.26) gives

\[
p^*(t, x) = \frac{1}{2} \left( a(\eta(t)) + \partial_x v(t, x) + c(\eta(t))\bar{p}(t) \right).
\tag{3.27}
\]

Substituting (3.27) into (3.24), we have

\[
\bar{p}(t) = \frac{1}{\eta(t)} \int_{\mathbb{R}_+} \frac{1}{2} \left( a(\eta(t)) + \partial_x v(t, x) + c(\eta(t))\bar{p}(t) \right) m(t, x) \, dx.
\]
from which we obtain

$$\bar{p}(t) = \frac{1}{2} - c(\eta(t)) \left( a(\eta(t)) + \frac{1}{\eta(t)} \int_{\mathbb{R}_+} \partial_x v(t, x) m(t, x) \, dx \right). \quad (3.28)$$

From (3.23), the optimal (equilibrium) demand is given by

$$q^*(t, x) = \frac{1}{2} \left( a(\eta(t)) - \partial_x v(t, x) + c(\eta(t)) \bar{p}(t) \right). \quad (3.29)$$

Therefore, the HJB equation (3.26) is

$$\partial_t v + \frac{1}{2} \sigma^2 \partial_x^2 v - rv + \frac{1}{4} \left( a(\eta(t)) - \partial_x v + c(\eta(t)) \bar{p}(t) \right)^2 = 0, \quad x > 0. \quad (3.30)$$

When a player reaches $x = 0$, his reserves are exhausted and he no longer earns revenue, so we have $v(t, 0) = 0$. Moreover, at time $T$ all capacities are exhausted and $v(T, x) = 0$. The time $T$ is not known a priori and has to be determined endogenously.

The average price $\bar{p}$ is computed from the density $m(t, x)$ of the distribution of reserves $X_t$ which evolve by dynamics (3.25), with $q = q^*$ given by (3.29). It is the solution of the forward Kolmogorov equation

$$\partial_t m - \frac{1}{2} \sigma^2 \partial_x^2 m + \partial_x \left( -\frac{1}{2} \left( a(\eta(t)) - \partial_x v + c(\eta(t)) \bar{p}(t) \right) m \right) = 0 \quad (3.31)$$

$$m(0, x) = M(x),$$

where $\bar{p}$ depends on $m$ through (3.28) and $M$ is the given initial density of reserves.

We will distinguish between stochastic and deterministic cases:

- In the first case $\sigma > 0$, we shall assume that $m$ is a $C^{1,2}$ function and therefore a classical solution to (3.31). When $x$ is close to zero, the short-term behavior is dominated by the Brownian motion. Once a player is driven to zero he cannot be revived by the Brownian motion, meaning that the effect of noise is
predominately one-sided. Hence we have the boundary condition \( m(t, 0) = 0 \) because players close to zero will “rapidly” be absorbed into zero.

- In the deterministic case \( \sigma = 0 \), there is no boundary condition at \( x = 0 \). In this case we also want to consider the case where the initial distribution is a sum of delta functions to mimic the situation of finite player games. Therefore we assume the initial density \( M \) and the later density \( m \) exist in the sense of distributions, and that the inner products with test functions are \( C^1 \) in time.

The system (3.30) and (3.31) is an instance of what Lasry and Lions [74] have called a mean field game. The forward/backward system of PDEs is coupled through \( \bar{p} \) in (3.28) and \( \eta \) in (3.22). Existence, uniqueness and regularity of solutions to the MFG system is an ongoing challenge and a subject of active research, and we do not attempt to prove these properties here. In the following we shall assume sufficient regularity of \( v \) and \( m \) for our asymptotic calculations to hold, but remark that the perturbation is around the monopoly case \( \varepsilon = 0 \) which is explicitly solvable, and regularity of solution can be seen directly. The validity of our assumptions is backed up by numerical experiments in Sections 3.5 and 3.6. We also note that our model does not fit into the linear-quadratic framework studied by Bensoussan et al. [11], for which there are explicit solutions, because of the nonlinear dependence of \( \bar{p} \) on the state variable and mean field term.

### 3.3.1 Lifetime Production and Total Profit

We define here two useful objects of study for analyzing the effects of competition. The output rate \( Q \) at time \( t \) can be defined as

\[
Q(t) = \int_{\mathbb{R}_+} q^\ast(t, x)m(t, x) \, dx.
\] (3.32)
From (3.23), \( q^*(t, x) = a(\eta(t)) - p^*(t, x) + c(\eta(t))\bar{p}(t) \), and so we can also write the output rate \( Q \) in terms of the average equilibrium price as

\[
Q(t) = [a(\eta(t)) - \bar{p}(t) + c(\eta(t))\bar{p}(t)]\eta(t).
\]

In the deterministic setting, we expect that the integral of the output rate over time to be simply the total initial capacity. We define the lifetime production to be

\[
\text{Lifetime production} = \int_0^T Q(t) \, dt,
\]

and this quantity is invariant under change in the level of competition. This invariance can serve as a useful check for the numerical quality of our code when an explicit solution is not available. The following proposition makes precise the above observation.

**Proposition 4.** *In the deterministic setting, the lifetime production depends only on the initial capacity distribution via*

\[
\int_0^T Q(t) \, dt = \int_{\mathbb{R}^+} xM(x) \, dx.
\]

*Proof.* Notice equation (3.31) is \( \partial_t m = \partial_x(q^*m) \). Hence we have

\[
-\frac{d}{dt} \int_{\mathbb{R}^+} x m(t, x) \, dx = - \int_{\mathbb{R}^+} x \partial_t m \, dx = - \int_{\mathbb{R}^+} x \frac{d}{dx} [q^*m] \, dx = \int_{\mathbb{R}^+} q^*(t, x)m(t, x) \, dx = Q(t),
\]

where the boundary terms in the integration-by-parts clearly vanish. Then integrating this expression in time gives

\[
\int_0^T Q(t) \, dt = \int_0^T \left( -\frac{d}{dt} \int_{\mathbb{R}^+} x m(t, x) \, dx \right) \, dt = \int_{\mathbb{R}^+} xM(x) \, dx.
\]
We define the total profit rate \( \Pi(t) \) by

\[
\Pi(t) = \int_{\mathbb{R}^+} p^*(t, x)q^*(t, x)m(t, x) \, dx
\]

\[
= \frac{1}{4} \left( (a(\eta(t)) + c(\eta(t))\bar{p}(t))^2 \eta(t) - \int_{\mathbb{R}^+} (\partial_x v(t, x))^2 m(t, x) \, dx \right).
\]  

(3.34)

We will use this to demonstrate the effect of competition in Section 3.5.

### 3.3.2 Monopoly with Deterministic Demand

Ultimately, our goal is to quantify the effects of market competition in the continuum mean field setting. We do so by studying the first-order corrections to the value function and density in the presence of small but nonzero degree of product substitutability. In preparation for the asymptotic expansion, we first look at the case when \( \varepsilon = 0 \), which implies that \( c \) in (3.6) is identically zero, so the goods are independent and the players are monopolists in their own markets. In this subsection, we also suppose that demand is deterministic, so that \( \sigma = 0 \). Later on in Section 3.4 we will see that the monopoly solution corresponds precisely to the zeroth order expansion in our asymptotic approximation.

From equation (3.6), when \( \varepsilon = 0 \), \( a \) is constant and equal to 1. Let \( (v_0, m_0) \) be the value function and density in this case. Then equation (3.30) becomes

\[
\partial_t v_0 - rv_0 + \frac{1}{4}(1 - \partial_x v_0)^2 = 0, \quad x > 0,
\]

with \( v_0(t, 0) = 0 \). The solution is time-independent and given by \( v_0(t, x) = v_0(x) \) solving

\[
\frac{1}{4}(1 - v'_0)^2 = rv_0, \quad v_0(0) = 0.
\]

(3.35)
Proposition 5. The solution \(v_0\) to the ODE with boundary condition given in (3.35) is

\[
v_0(x) = \frac{1}{4r} \left(1 + \mathbb{W}(\theta(x))\right)^2, \quad \theta(x) = -e^{-2rx-1}, \tag{3.36}
\]

where \(\mathbb{W}\) is the Lambert \(W\)-function defined by the relation \(x = \mathbb{W}(x)e^{\mathbb{W}(x)}\) with domain \(x \geq -e^{-1}\) and for \(x \in (-1/e, 0)\), we take the principal branch \(\mathbb{W}(x) > -1\).

The economically sensible solution is given by the principal branch of \(\mathbb{W}\), as this ensures that the shadow (or scarcity) cost \(v_0'(x) \to 0\) as \(x \to \infty\).

Proof. One can check that \(-1 \leq \mathbb{W}(z) < 0\) for \(z \in [-e^{-1}, 0)\), with \(\mathbb{W}(-e^{-1}) = -1\), and for \(z > -e^{-1}\)

\[
\mathbb{W}'(z) = \frac{\mathbb{W}(z)}{z(1+\mathbb{W}(z))}.
\]

It is now clear that equation (3.36) indeed satisfies equation (3.35) and the zero boundary condition.

From the explicit solution of \(v_0\), we can rewrite equation (3.25) as the ordinary differential equation (ODE)

\[
X'(t) = -\frac{1}{2r} \left(1 + \mathbb{W}(\theta(X(t)))\right), \quad X(0) = x_0. \tag{3.37}
\]

Proposition 6. Equation (3.37) can be solved in closed form and the solution is given by

\[
X(t; x_0) = x_0 - \frac{1}{2} t + \frac{1}{2r} (1 - e^{rt}) \mathbb{W}(\theta(x_0)). \tag{3.38}
\]

Proof. Consider the function defined by \(f(t) = \mathbb{W}(\theta(X(t)))\). A short calculation shows that \(f'(t) = rf(t)\), and hence

\[
\mathbb{W}(\theta(X(t))) = \mathbb{W}(\theta(x_0)) e^{rt}. \tag{3.39}
\]

Using this one can integrate the ODE (3.37) to get (3.38).
We observe that equation (3.39) is the famous Hotelling’s rule [62]. Recall that the shadow cost in the monopoly setting is simply

\[ v'_0(x) = -\mathbb{W}(\theta(x)). \] (3.40)

Then equation (3.39) says that \( v'_0(X(t)) = v'_0(x_0)e^{rt} \). In other words, along the optimal extraction path, the shadow cost grows at the discounting rate \( r \). In the case of linear demand, Hotelling’s rule can be equivalently written as

\[ \frac{d}{dt} \left( p^*(t, X(t)) - \frac{1}{2} \right) = r \left( p^*(t, X(t)) - \frac{1}{2} \right), \quad \Rightarrow \quad p^*(t, X(t)) = \frac{1}{2} \left( 1 - e^{rt} \right) + p^*(0, x_0)e^{rt}. \]

That is, the (shifted) optimal price \( p^* \) grows at the discount rate \( r \) along the optimal extraction path.

We define the hitting time \( \tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) to be the time to exhaustion in the deterministic monopoly market starting at initial capacity \( x_0 \):

\[ \tau(x_0) = \inf\{t \geq 0 \mid X(t; x_0) \leq 0\}. \] (3.41)

**Proposition 7.** The hitting time is given explicitly by:

\[ \tau(x_0) = 2x_0 + \frac{1}{r} \left( 1 + \mathbb{W}(\theta(x_0)) \right). \] (3.42)

Formula (3.42) follows directly from (3.38). The hitting time \( \tau \) is monotonic increasing as expected; for large \( x_0 \), \( \tau \) grows linearly in \( x_0 \). In fact, in the deterministic setting, when the initial density \( M \) has compact support \([0, x_{\text{max}}]\), it follows that the exhaustion time \( T \) is given by \( T = \tau(x_{\text{max}}) \).
For the density, recalling from equation (3.6) that \( a \equiv 1 \) and \( c \equiv 0 \) when \( \varepsilon = \sigma = 0 \), the forward Kolmogorov equation (3.31) becomes
\[
\partial_t m_0 - \frac{1}{2} \partial_x ((1 - v_0')m_0) = 0, \tag{3.43}
\]
with \( m_0(0, x) = M(x) \).

**Proposition 8.** The solution to equation (3.43) is given by
\[
m_0(t, x) = \frac{1 + \mathcal{W}(\theta(x))e^{-rt}}{1 + \mathcal{W}(\theta(x))} M(X(-t; x)). \tag{3.44}
\]

**Proof.** The solution follows from the method of characteristics. For fixed \((t, x)\), we define the characteristic curve by the ODE
\[
z'(s) = -\frac{1}{2} \left( 1 + \mathcal{W}(\theta(z(s))) \right), \quad z(t) = x, \tag{3.45}
\]
whose solution is precisely the monopoly capacity trajectory started at \((t, x)\): \( z(s) = X(s - t; x) \). Along the curve \( z(s) \), the function \( \tilde{m}_0(s) = (1 + \mathcal{W}(\theta(z(s))))m_0(s, z(s)) \) is constant, and in particular we have
\[
m_0(t, x) = \frac{1}{1 + \mathcal{W}(\theta(x))} \tilde{m}_0(t) = \frac{1 + \mathcal{W}(\theta(z(0))))}{1 + \mathcal{W}(\theta(x))} M(z(0)) = \frac{1 + \mathcal{W}(\theta(x))e^{-rt}}{1 + \mathcal{W}(\theta(x))} M(X(-t; x)),
\]
using Hotelling’s rule (3.39) and the definition of \( X \).

It turns out that the integral of \( m_0 \), or the proportion of remaining active firms \( \eta_0 \), can be computed in closed-form.

**Proposition 9.** Given the distribution \( M : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) of the initial capacity \( x_0 \), the proportion \( \eta_0 : \mathbb{R}_+ \rightarrow [0, 1] \) of remaining players, as defined by equation (3.22), is given by
\[
\eta_0(t) = 1 - F \left( \tau^{-1}(t) \right) \tag{3.46}
\]
where $F$ denotes the cumulative distribution function (CDF) of the initial distribution $M$, and $\tau^{-1}$ is the inverse function of $\tau$ in (3.42), explicitly given by

$$
\tau^{-1}(t) = \frac{t}{2} - \frac{1}{2r} \left(1 - e^{-rt}\right).
$$

**Proof.** First we note both sides of equation (3.46) evaluate to 1 at $t = 0$. It suffices to show that the time derivatives on both sides are equal for all $t > 0$. Using equation (3.43), we have

$$
\eta'_0(t) = \int_{\mathbb{R}^+} \partial_t m_0(t, x) \, dx = \frac{1}{2} \int_{\mathbb{R}^+} \partial_x ((1 - v'_0)m_0) \, dx = \frac{1}{2} \left(1 + \mathbb{W}(\theta(x)) \right) m_0(t, x) \bigg|_{0^+}^\infty
$$

$$
= -\frac{1}{2} \left(1 - e^{-rt}\right) M(X(-t; 0)),
$$

where we have used that $m_0(t, x) \to 0$ as $x \to \infty$. Now, since that $X(-t; 0) = \tau^{-1}(t)$, we recognize the last line is precisely the time derivative of $1 - F(\tau^{-1}(t))$, the right-hand side of equation (3.46). \qed

**Proposition 10.** In the case of monopoly, the average equilibrium price $\bar{p}_0$ is given by

$$
\bar{p}_0(t) = \frac{1}{2} \left(1 - \frac{I(t)}{\eta_0(t)}\right),
$$

(3.47)

where $I$ satisfies the first-order linear ODE

$$
I'(t) - rI(t) = -\eta'_0(t), \quad I(0) = \int_{\mathbb{R}^+} \mathbb{W}(\theta(x)) M(x) \, dx.
$$

(3.48)

**Proof.** We see from equation (3.28) with $\varepsilon = 0$ that the average equilibrium price is given by (3.47), where, using the formula (3.36) for $v_0$, we define
\[ I(t) = \int_{0}^{\infty} \mathbb{W}(\theta(x)) m_0(t, x) \, dx. \] Then we compute

\[ I'(t) = \int_{\mathbb{R}_+^+} \mathbb{W}(\theta(x)) \cdot \partial_t m_0(t, x) \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}_+^+} \mathbb{W}(\theta(x)) \cdot \partial_x ((1 - v'_0)m_0) \, dx \]
\[ = \frac{1}{2} \mathbb{W}(\theta(x)) \left( 1 + \mathbb{W}(\theta(x)) e^{-rt} \right) M(X(t; x)) \bigg|_{0}^{\infty} + r \int_{\mathbb{R}_+^+} \mathbb{W}(\theta(x)) m_0(t, x) \, dx \]
\[ = -\eta'_0(t) + rI(t). \]

which gives \[ (3.48). \]

This representation is convenient for numerical purposes because all we need is to solve a first-order linear ODE to obtain the time evolution of the average equilibrium price \( \bar{p}_0 \). Moreover, we will see in Section \[ 3.4 \] that this splitting is used to prove certain qualitative features of first-order corrections to the game in the case of small substitutability.

Finally, recall the output rate \( Q(t) \) defined in \[ (3.32) \]. In the monopoly setting, this simplifies to

\[ Q_0(t) = \eta_0(t)(1 - \bar{p}_0(t)). \]

### 3.4 Small Competition Asymptotics under Deterministic Demand

Having studied the monopoly problem where many quantities, including the value function and density, are explicitly solvable, we are now ready to investigate the effect of competition. We first note that \( \varepsilon = 0 \) (or equivalently \( a \equiv 1 \) and \( c \equiv 0 \) from equation \[ (3.6) \]) is equivalent to stating that firms have independent goods in the sense that they operate in markets without competing with one another. When \( \varepsilon > 0 \), firms produce goods that are actually in competition with one another. Our approach
is to formally construct a perturbation expansion around the non-competitive case for small $\varepsilon > 0$ to view the effects of a small amount competition. Throughout this section, we work with deterministic demand where $\sigma = 0$.

We will look for an approximation to the PDE system of the form

$$
\begin{align*}
v(t, x) &= v_0(t, x) + \varepsilon v_1(t, x) + \varepsilon^2 v_2(t, x) + \cdots, \\
m(t, x) &= m_0(t, x) + \varepsilon m_1(t, x) + \varepsilon^2 m_2(t, x) + \cdots.
\end{align*}
$$

(3.49)

To leading order in $\varepsilon$, the demand coefficients $a$ and $c$ in (3.6) are given by

$$
\begin{align*}
a(\eta(t)) &= 1 - \varepsilon \eta_0(t) + \cdots, \\
c(\eta(t)) &= \varepsilon \eta_0(t) + \cdots,
\end{align*}
$$

where $\eta$ denotes the proportion of remaining players, and hence in the expansion

$$
\eta = \eta_0 + \varepsilon \eta_1 + \cdots, \quad \text{we have} \quad \eta_i(t) = \int_{\mathbb{R}^+} m_i(t, x) \, dx, \quad i = 0, 1, \cdots. \quad (3.50)
$$

We also expand the average equilibrium price $\bar{p}(t) = \bar{p}_0(t) + \varepsilon \bar{p}_1(t) + \cdots$, where, from (3.28), we find

$$
\bar{p}_0(t) = \frac{1}{2} \left( 1 + \frac{1}{\eta_0(t)} \int_{\mathbb{R}^+} \partial_x v_0(t, x) \cdot m_0(t, x) \, dx \right).
$$

### 3.4.1 First-order Correction to Value Function

Inserting the expansion (3.49) for $v$ into equation (3.30) and collecting terms independent of $\varepsilon$, we recover the monopoly equation (3.35) for $v_0$ whose solution is given by equation (3.36). We obtain the following equation for $v_1$ by equating terms of order $\varepsilon$:

$$
\partial_t v_1 - \frac{1}{2} (1 - v_0') \partial_x v_1 - rv_1 = \frac{1}{2} (1 - v_0') (1 - \bar{p}_0) \eta_0, \quad v_1(t, 0) = 0. \quad (3.51)
$$
Observe that equation (3.51) does not involve $m_1$, and this greatly simplifies the solution process for otherwise the forward-backward structure of mean field games typically requires an iterative solver, with each iteration involving a finite-difference solution to some PDEs, as we do in Section 3.5.

**Proposition 11.** The correction to the value function $v_1$ is given by

$$v_1(t, x) = -\frac{1}{2} \int_0^{\tau(x)} \left( e^{-rs} + \mathbb{W}(\theta(x)) \right) \cdot (1 - \bar{p}_0(t + s)) \cdot \eta_0(t + s) \, ds, \quad (3.52)$$

where $\tau(x)$ is given by equation (3.42). In particular, the first-order correction $v_1(t, x)$ is negative for all $(t, x)$.

**Proof.** Using (3.40), equation (3.51) can be rewritten as

$$\partial_t v_1 - \frac{1}{2} \left( 1 + \mathbb{W}(\theta(x)) \right) \partial_x v_1 - rv_1 = \frac{1}{2} \left( 1 + \mathbb{W}(\theta(x)) \right) (1 - \bar{p}_0) \eta_0.$$

This is a first-order transport equation and can be solved by the method of characteristics. Given fixed $(t, x)$, we define the characteristic curve $z$ as in (3.45). Then the discounted first-order correction $\tilde{v}_1(s) = v_1(s, z(s))e^{-r(s-t)}$ satisfies the ODE along the characteristic curve

$$\tilde{v}_1'(s) = \frac{1}{2} \left( 1 + \mathbb{W}(\theta(z(s))) \right) (1 - \bar{p}_0(s)) \eta_0(s)e^{-r(s-t)}, \quad \tilde{v}_1(t + \tau(x)) = 0.$$

Integrating from $t$ to $t + \tau(x)$, we obtain

$$v_1(t, x) = \tilde{v}_1(t) = -\frac{1}{2} \int_t^{t+\tau(x)} \left( 1 + \mathbb{W}(\theta(z(s))) \right) e^{-r(s-t)} (1 - \bar{p}_0(s)) \eta_0(s) \, ds.$$
using Hotelling’s rule (3.39). This yields (3.52) after a change of variable \( s \mapsto s + t \).

It follows readily that \( v_1 \) is negative for all \((t, x)\) since

\[
(1 - \bar{p}_0)\eta_0 = \frac{1}{2}(\eta_0 + I) = \frac{1}{2} \int_{\mathbb{R}^+} \left(1 + \mathbb{W}(\theta(x))\right)m_0(t, x) \, dx \geq 0.
\]

\[\square\]

### 3.4.2 First-order Correction to Density

Inserting the expansion (3.49) for \( m \) into equation (3.31) and collecting terms independent of \( \varepsilon \) we recover equation (3.43) whose solution is given by equation (3.44). We obtain the following equation for \( m_1 \) by equating terms of order \( \varepsilon \):

\[
\partial_t m_1 - \frac{1}{2} \partial_x \left( (1 - v_0')m_1 + (-\eta_0 + \eta_0\bar{p}_0 - \partial_x v_1)m_0 \right) = 0, \quad m_1(0, x) = 0. \tag{3.53}
\]

**Proposition 12.** The solution to equation (3.53) is given by

\[
m_1(t, x) = \int_0^t \frac{1 + \mathbb{W}(\theta(x))e^{-r(t-s)}}{1 + \mathbb{W}(\theta(x))} g(s, X(s-t; x)) \, ds, \tag{3.54}
\]

where

\[
g(t, x) = \frac{1}{2} \partial_x \left( (-\eta_0 + \eta_0\bar{p}_0 - \partial_x v_1)m_0 \right),
\]

and \( X(t; x) \) was given in equation (3.38).

**Proof.** This follows from the method of characteristics. For fixed \((t, x)\), we define the characteristic curve by equation (3.45). Along the characteristic curve, the function

\[
\tilde{m}_1(s) = \left(1 + \mathbb{W}(\theta(z(s)))\right)m_1(s, z(s))
\]

\[82\]
satisfies the ODE

\[ \tilde{m}_1'(s) = g(s, z(s)) \left( 1 + \mathbb{W}(\theta(z(s))) \right), \quad \tilde{m}_1(0) = 0. \]

Integrating from 0 to \( t \), and using the definition of \( \tilde{m}_1 \) we obtain

\[ m_1(t, x) = \int_0^t g(s, z(s)) \frac{1 + \mathbb{W}(\theta(z(s)))}{1 + \mathbb{W}(\theta(x))} \, ds = \int_0^t g(s, X(s-t; x)) \frac{1 + \mathbb{W}(\theta(x)) e^{-r(t-s)}}{1 + \mathbb{W}(\theta(x))} \, ds. \]

We plot the first-order corrections to the value function and density in Figures 3.1a and 3.1b respectively. Notice that the density correction is positive for large values of \( x \), then changes sign and becomes negative for smaller values of \( x \). This suggests that firms are exhausting their production capacity more slowly in the presence of competition and hence the capacity distribution is shifted away from zero. We will see that this is indeed the case in a later Proposition 13.

Remark 6. We observe that expansion of the form (3.49) may lead to negative density. This leads us to consider a multiplicative asymptotic perturbation for \( m \) given by

\[ m(t, x) \approx m_0(t, x) \left( 1 + \tanh \left( \frac{\varepsilon m_1(t, x)}{m_0(t, x)} \right) \right). \] (3.55)

Using that \( \tanh x = x + O(x^3) \) for small \( x \), one can show that (3.55) agrees with (3.49) up to first order in \( \varepsilon \). Moreover, since \( \tanh x \in (-1, 1) \) for all \( x \), we obtain a positivity-preserving asymptotic perturbation for \( m \).
3.4.3 First-order Correction to Demands and Capacity Trajectories

Expanding the equilibrium demand $q^*$ given by (3.29) as $q^* = q_0 + \varepsilon q_1 + \mathcal{O}(\varepsilon^2)$, we have

$$q_0 = \frac{1}{2} (1 - v'_0), \quad q_1 = \frac{1}{2} (-\eta_0 + \eta_0 \bar{p}_0 - \partial_x v_1).$$

Solving equation (3.25) yields the first-order correction to the capacity trajectories, which we plot in Figure 3.1d for various values of $\varepsilon$. In Figure 3.1c we plot the first order correction to the equilibrium demand $q_1$ and observe that it is negative. We show that this is true in general in the following proposition.

Proposition 13. The first order correction to the equilibrium demand $q$ is negative, that is $q_1 \leq 0$ for $0 \leq t \leq T$ and $x \geq 0$. 

Figure 3.1: Small substitutability expansion under deterministic demand, where $r = 0.2$ and the initial capacity is chosen to following the beta distribution with shape parameters $\alpha = 2, \beta = 4$. 
Proof. In light of equation (3.51), it suffices to show that $\partial_t v_1 - rv_1 \geq 0$ since $1 + \mathbb{W}(\theta(x)) \geq 0$ for $x \geq 0$. To this end, we first notice that $e^{-rs} + \mathbb{W}(\theta(x)) \geq 0$ for $0 \leq s \leq \tau(x)$ and $x \geq 0$. Now using the explicit form of $v_1$ in equation (3.52), it suffices to show that

$$(-\eta_0 + \eta_0 \bar{p}_0)' - r(-\eta_0 + \eta_0 \bar{p}_0) \geq 0.$$ 

First we rewrite

$$\eta_0 \bar{p}_0 = \frac{1}{2} \left( \eta_0 + \int_{\mathbb{R}_+} \partial_x v_0 \cdot m_0 \, dx \right) = \frac{1}{2} (\eta_0 - I)$$

where $I$ satisfies the differential equation (3.48).

It follows that $(-\eta_0 + \eta_0 \bar{p}_0) = -\frac{1}{2} (\eta_0 + I)$. The claim follows immediately since

$$(-\eta_0 + \eta_0 \bar{p}_0)' - r(-\eta_0 + \eta_0 \bar{p}_0) = \frac{1}{2} r \eta_0 \geq 0.$$ 

The asymptotic approximation to first order demonstrates the principal effect of competition in the continuum mean field game compared with the monopoly case: demand for the goods drop, the firms sell at a slower rate and so take longer to exhaust their capacities, and so their density is shifted away from zero. We investigate these effects further by numerical methods in the next two sections. Expressions for the higher order terms in the expansions are given in Appendix B.1.

3.5 Numerical Analysis

We have been able to capture many of the qualitative features of the model analytically in the case of deterministic demand using the asymptotic expansions of the
previous section. However, in order to fully analyze the case where $\varepsilon$ is not so small, or in the stochastic model $\sigma > 0$, we have to solve the PDE system numerically.

In the deterministic monopoly game, the density $m(t, x)$ becomes singular as $x \rightarrow 0$ when players begin to exhaust their capacities. Similar problematic behavior is expected in the case of substitutable goods, and this renders accurate numerical solution to the forward equation difficult. It turns out, however, that the tail distribution function $\bar{\eta}(t, x)$, defined by

$$\bar{\eta}(t, x) = \int_x^\infty m(t, y) \, dy,$$

is more amenable to numerical treatment. Substituting equation (3.56) into the forward Kolmogorov equation (3.31), we get for $t \geq 0$ and $x \geq 0$

$$\partial_t \bar{\eta}(t, x) - \frac{1}{2} \sigma^2 \partial_{xx} \bar{\eta}(t, x) - \frac{1}{2} \left( a(\eta(t)) - \partial_x v(t, x) + c(\eta(t)) \bar{p}(t) \right) \partial_x \bar{\eta}(t, x) = 0,$$

with initial condition

$$\bar{\eta}(0, x) = \int_x^\infty M(y) \, dy.$$

Note that while $M$ may be singular at certain values of $x$ (for instance certain cases of the Beta distribution or, later when we take $M$ to be a sum of delta functions), $\bar{\eta}(0, x)$ is bounded and, even in the extreme example of delta functions, still piecewise continuous.

3.5.1 Solution Strategy

We employ an iterative algorithm to calculate the MFG solution. Starting with initial guesses $\left( \eta^0, \bar{p}^0 \right)$ for $\left( \eta, \bar{p} \right)$, we follow for $n = 0, 1, 2, \ldots$:
Step 1. Given \((\eta^n, \bar{p}^n)\), solve the HJB equation (3.30) to calculate \(v^n\):

\[
\partial_tv^n + \frac{1}{2}\sigma^2\partial^2_{xx}v^n - rv^n + \frac{1}{4}\left(a(\eta^n(t)) + \partial_xv^n(t, x) + c(\eta^n(t))\bar{p}^n(t)\right)^2 = 0, \quad v^n(t, 0) = 0.
\]

We have the terminal condition \(v^n(T, x) = 0\). In practice we do not know \(T\) and choose \(T_{\text{max}}\) to be significantly larger than (for instance double) its monopoly counterpart \(\tau(x_{\text{max}})\). Then the strategy \(p^{n,*}\) and the corresponding demand \(q^{n,*}\) are given by

\[
p^{n,*}(t, x) = \frac{1}{2}\left(a(\eta^n(t)) + \partial_xv^n(t, x) + c(\eta^n(t))\bar{p}^n(t)\right),
\]

\[
q^{n,*}(t, x) = \frac{1}{2}\left(a(\eta^n(t)) - \partial_xv^n(t, x) + c(\eta^n(t))\bar{p}^n(t)\right).
\]

Step 2. Given the price \(p^{n,*}\) and demand \(q^{n,*}\), solve equation (3.57) for \(\bar{\eta}^{n+1}\):

\[
\partial_t\bar{\eta}^{n+1} - \frac{1}{2}\sigma^2\partial^2_{xx}\bar{\eta}^{n+1} - q^{n,*}(t, x)\partial_x\bar{\eta}^{n+1}(t, x) = 0, \quad \bar{\eta}^{n+1}(0, x) = \bar{\eta}(0, x).
\]

Then generate new \((\eta^{n+1}, \bar{p}^{n+1})\) from

\[
\eta^{n+1}(t) = \bar{\eta}^{n+1}(t, 0), \quad m^{n+1}(t, x) = \partial_x\bar{\eta}^{n+1}(t, x),
\]

\[
\bar{p}^{n+1}(t) = \frac{1}{\eta^{n+1}(t)} \int_{\mathbb{R}_+} p^{n,*}(t, x)m^{n+1}(t, x) \, dx.
\]

When \((\eta^{n+1}, \bar{p}^{n+1})\) is close enough to \((\eta^n, \bar{p}^n)\), we call \((v^n, m^n)\) a solution to the MFG. Steps 1 and 2 themselves involve PDE solvers using finite difference which we describe in more detail.

Step 1 Using the method of lines, we discretize the HJB equation in the space dimension but not in time and solving the resulting system of ODEs using the fourth-order Runge-Kutta method.
Step 2  In the stochastic case, we apply the standard finite difference method. In the deterministic case, we can write the forward Kolmogorov equation in terms of the tail distribution \( \bar{\eta} \) as

\[
\partial_t \bar{\eta}^{n+1}(t, x) - q^{n, \ast}(t, x) \partial_x \bar{\eta}^{n+1}(t, x) = 0, \quad \bar{\eta}^{n+1}(0, x) = \int_x^\infty M(y) \, dy.
\]

In this form, it is clear that \( \bar{\eta}^{n+1} \) just gets transported along the characteristics

\[
\frac{d}{dt} x^{n+1}(t) = -q^{n, \ast}(t, x^{n+1}(t)), \quad x^{n+1}(0) = x_0.
\]

It therefore suffices to solve a family of ODEs with different initial \( x_0 \).

3.5.2 Results and Observations

Deterministic Bertrand Games (\( \sigma = 0 \))

We illustrate the numerical results with one instance of the model, where we choose \( \varepsilon = 0.3, r = 0.2 \) and assume a beta distribution with shape parameters \( \alpha = 2, \beta = 4 \) for the initial capacity. See Figure 3.2 for the convergence graph in this case, we note that the iterative solver converges very rapidly, often in less than 10 iterations.

Figure 3.2: Convergence of \( L_1 \) error in the iterative solver, for deterministic Bertrand competition, with model parameters as in Figure 3.3. Here we take the final iteration of our algorithm as a proxy as the true solution, and measure the \( L_1 \) error by \( \| \eta - \eta^{\text{true}} \|_{L_1} \).
Figure 3.3d compares the average equilibrium price $\bar{p}$ in Bertrand competition against the monopoly case. Figure 3.3c shows the proportion of remaining players in Bertrand competition as well as the monopoly case. Figure 3.3e compares the output rate $Q$ in Bertrand competition against the monopoly case. Figure 3.3a shows the capacity trajectories for various initial values in Bertrand competition as well as the monopoly case.

In the presence of competition, firms are more cautious and exhaust their production capacity more slowly, as shown in Figure 3.3c and 3.3a. This is in accordance with Proposition 13 where an asymptotic expansion is used to show that production is slowed down in the presence of competition. Although each firm reduces their production level, and hence the decrease in the output rate $Q$ as shown in Figure 3.3e, as the game proceeds further, the production level in the presence of competition is actually above the monopoly level. This is because firms are more cautious in the competitive market and find themselves with higher production capacities as the game unfolds, even though they each choose a smaller production level than they would in monopoly, overall we still see an increase in the production level. This is in accordance with Proposition 4 since the time integral of $Q$ has to be invariant to the degree of competition $\varepsilon$. Figure 3.3f shows the total profit rate $\Pi(t)$ defined in (3.34): it is initially higher in the case $\varepsilon = 0$ of many firms producing independent goods, but declines more quickly as resources become scarce and firms drop out than when $\varepsilon > 0$ and competition enforces greater discipline in price setting close to exhaustion.

**Stochastic Bertrand Games ($\sigma > 0$)**

For $\sigma > 0$, the forward equation can no longer be solved using the method of characteristics, and we need to specify the boundary conditions for $\bar{\eta}(t, x)$ and solve the
Figure 3.3: Various descriptive statistics of deterministic Bertrand competition, where $r = 0.2$ and the initial capacity is assumed to follow a beta distribution with shape parameters $\alpha = 2, \beta = 4$. 
PDE using finite differences. We choose

\[
\lim_{x \to \infty} \bar{\eta}(t, x) = 0, \quad \lim_{x \to 0} \partial_x \bar{\eta}(t, x) = 0.
\]

This is appropriate because we have an absorbing boundary at \( x = 0 \), that is to say, once a firm hits zero, it is out of the game and cannot return.

Solving the full PDE using our iterative solver, we compare the average equilibrium price in the presence of noise, with and without competition, see Figure 3.4. We notice that a high level of noise pushes down the average equilibrium price \( \bar{p} \) as well as shortens the duration of game. This is expected since the effect of our stochastic term is predominantly one-sided, once a firm exhausts their production capacity, they cannot be revived by the Brownian motion. Moreover, a high level of noise washes out the effects of competition, since the Brownian motion dominates the interactions between competitive firms.

### 3.6 MFG Approximations to Deterministic Finite Player Games

Since real-world situations involve games with only a finite number of players, we consider using the MFG framework to study the \( N \)-player game. The goal is to apply the MFG methodology to provide an efficient way to model situations with a moderate to large number of players in the deterministic dynamic game where \( \sigma = 0 \).

The idea is to approximate an \( N \)-player game by an initial density of the form

\[
M(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_0^i),
\]
where $\delta$ is the Dirac-delta function, and $x_0^i$ corresponds to the initial capacity of the $i$th player. In the deterministic setting, we denote the hitting time of player $i$ to reach 0 by $\tau_i$, where $i = 1, 2, \ldots, N$. The remaining proportion of active players $\eta(t)$ is then a pure jump function, with jump times given by $(\tau_i)_{1 \leq i \leq N}$. The average equilibrium price $\bar{p}(t)$ will also jump across $\tau_i$. We describe the algorithm we use to solve the MFG in this setting.

### 3.6.1 Discretization Algorithm

Again the solution to the discrete MFG depends on an iterative algorithm. We modify our scheme slightly to take advantage of the discrete nature of the problem at hand. Starting with the initial guess $v^0 = v_0$ in (3.36) for the value function $v$, we follow...
Step 1. Given \( v^n \), solve \( N \) ODEs

\[
\frac{d}{dt} x^{n,i}(t) = -q^{n,i}(t, x^{n,i}(t)), \quad x^{n,i}(0) = x^i_0.
\]

We obtain, in particular, \( \tau^{n,i} \) for \( i = 1, 2, \ldots \), which is the hitting time of the \( i \)th player, in the \( n \)th iteration. Then for \( \tau^{n,k-1} < t \leq \tau^{n,k} \), we have

\[
\eta^n(t) = \eta^n_k = \frac{N - k + 1}{N},
\]

\[
\bar{p}^n(t) = \frac{1}{2 - c(\eta^n(t))} \left( a(\eta^n(t)) + \frac{1}{\eta^n(t)} \frac{1}{N} \sum_{i=k}^N \partial_x v^n(t, x^{n,i}(t)) \right).
\]

Step 2. Given \( (\eta^n, \bar{p}^n) \), we solve the HJB PDE (3.30) to obtain a new guess for the value function \( v^{n+1} \).

We iterate until the updated approximation for \( (\eta, \bar{p}) \) is close enough to the previous iterate. See Figure 3.5d and 3.5e for the average equilibrium price \( \bar{p} \) and output rate \( Q \), respectively, in a numerical example, with initial capacity distribution of the \( N \)-players is specified in the caption. As \( N \) increases, we can use this discrete algorithm to approximate the continuous MFG, thus providing another way to solve the MFG PDE system.

3.6.2 Comparison with two-player Bertrand Competition

In the two-player case, each firm has a fixed lifetime capacity of production at time \( t = 0 \) denoted by \( x_i(0) \), and where \( x_i(t) \) denotes the remaining capacity at time \( t \). Each firm \( i \) chooses a dynamic pricing strategy, \( p_i = p_i(x(t)) \) where \( x(t) = (x_1(t), x_2(t)) \).

Given these prices, each firm \( i \) receives market demand at a rate \( D_i(p_1, p_2) \), where \( D_i \) are given in (3.4) with \( n = N = 2 \). Their capacities deplete as:

\[
\frac{dx_i}{dt}(t) = -D_i(p_1(x(t)), p_2(x(t))).
\]
Figure 3.5: MFG approximation to the $N$-player game, in the case of 10 players with initial sizes given by $(0.08, 0.14, 0.19, 0.24, 0.29, 0.34, 0.39, 0.45, 0.53, 0.66)$, and $\varepsilon = 0.3$. 
The value functions of the two firms are

\[ V_i(x_1, x_2) = \sup_{p_i \geq 0} \left\{ \int_0^\infty e^{-rt} p_i(x(t)) D_i(p_1(x(t)), p_2(x(t))) 1_{x_i(t) > 0} \, dt \right\} . \]

As long as both players have resources, this is a duopoly. After the first player has exhausted his capacity, the other player has a monopoly until he also runs out of reserves. As detailed in [77], a dynamic programming argument for nonzero-sum differential games yields that these value functions, if they have sufficient regularity, satisfy the following system of PDEs:

\[
\sup_{p_i \geq 0} \left\{ -D_1(p_1, p_2) \frac{\partial V_i}{\partial x_1} - D_2(p_1, p_2) \frac{\partial V_i}{\partial x_2} + p_i D_i(p_1, p_2) \right\} - r V_i = 0, \quad (x_1, x_2) \in \mathbb{R}^2_+,
\]

\[ V_i|_{x_i=0} = 0, \quad V_i|_{x_j=0} = v_0(x_i), \quad i = 1, 2, \quad j \neq i, \]

where \( v_0 \) given in (3.36) is the monopoly value function.

In [77], the following asymptotic approximation is constructed in the small \( \varepsilon \) limit. The value function of player \( i \) is expanded as

\[ V_i(x_1, x_2) = v_0(x_i) + \varepsilon v_1^{(1)}(x_1, x_2) + \mathcal{O}(\varepsilon^2). \]

The correction \( v_1^{(1)} \) is given, for \( x_1 > x_2 \), by

\[
v_1^{(1)}(x_1, x_2) = \frac{1}{4r} \left( e^{-r \Lambda(x_2)} \left( 1 + r \Lambda(x_2) \right) - e^{-r \Lambda(x_1)} \left( 1 - r \Lambda(x_2) \right) + e^{-r(\Lambda(x_1)+\Lambda(x_2))} - 1 \right),
\]

(3.58)

where \( \Lambda(x) = -\frac{1}{r} \log (-\mathcal{W}(\theta(x))) \), and, for \( x_2 \geq x_1 \), by reversing the roles of \( x_1 \) and \( x_2 \) in (3.58). The solution for \( v_2^{(1)} \) is the same: \( v_2^{(1)} = v_1^{(1)} \). From these, approximations to the Nash equilibrium prices and demands can be computed, and hence approximate game trajectories \( x_i(t) \).
**Numerical Example** In the two-player game, the two firms have sizes 0.25 and 0.75 respectively, while the continuum MFG approximation uses two (equally weighted) delta functions centered at 0.25 and 0.75. See Figure 3.6 for the capacity trajectories in the monopoly case (dotted), computed using the two-player asymptotic expansion (dashed), and the MFG approximation (solid).

We notice that the MFG approximation predicts a more cautious behavior since the firms are producing at a slower pace than the two-player solution. However, while the continuum MFG overstates the extent of competition when used to approximate the 2-player game, the trajectories are remarkably close.

![Figure 3.6: MFG approximation for two-player game (solid), compared with the asymptotic expansion (dashed) and the monopoly solution (dotted). Here we take $\varepsilon = 0.3$.](image)

**3.7 Conclusion**

We have studied nonzero-sum stochastic differential games arising from Bertrand competitions of mean field type with linear demand functions in the limit of an infinite number of players. By considering the case where there is a small degree of interaction between the firms, we are able to construct an asymptotic approximation that captures many of the qualitative features of the ordinary differential games. Numerical solutions provide further insight into the stochastic case and when there
is a higher degree of competition. By considering the tail distribution instead of the density itself, numerical solution becomes very tractable.

We find that, in the presence of competition, firms tend to be more cautious and slow down their production, and hence the duration of the game increases. Moreover, firms find themselves left with a higher production capacity as the game proceeds, and therefore total production $Q$ goes up even though each individual firm reduces their production level compared to the monopoly case. This leads to a more stable output level throughout the lifetime production profile.

Moreover, we consider the case where the initial distribution of the production capacity is a sum of delta functions. This setting mimics the case of finite player games and allows us to compare the MFG solution with the two-player asymptotic solution. Surprisingly, as seen in Figure 3.6, the game trajectories are quite close even when approximating a two-player game. Therefore the continuum MFG technology has excellent promise in approximating very difficult nonzero-sum differential game problems with a small number of players.
Chapter 4

Fracking, Renewables & Mean Field Games

The dramatic decline in oil prices, from around $110 per barrel in June 2014 to less than $50 in January 2015, highlights the importance of competition between different energy sources. Indeed, the sustained price drop has been primarily attributed to OPEC’s strategic decision not to curb its oil production in the face of increased supply of shale oil in the US, spurred by the technological innovation of “fracking”.

In this chapter, we study how continuous time Cournot competitions, in which firms producing similar goods compete with one another by setting quantities, can be analyzed as continuum dynamic mean field games. In this context, we illustrate how the traditional oil producers may react in counter-intuitive ways in face of competition from alternative energy sources. This chapter is adapted from the article [29].

4.1 Introduction

The recent rapid fall in the price of oil is arguably the biggest energy story of the past year. Back in June 2014, the price of Brent crude was up around $115 per barrel. As of January 23, 2015, it had fallen by more than half, down to $49 per
barrel (see Figure 4.1). The dramatic decline in oil prices illustrates the evolution of the global energy market as competition between different energy sources expands. Indeed, the sustained price drop was prompted in large part by OPEC’s decision not to curb its oil production in the face of increased supply of shale gas and oil in the US, itself arising from technological advances such as hydraulic fracturing and horizontal drilling, collectively referred to as fracking.

The goal of the present chapter is to explain how dynamic game theory, in particular mean field games proposed by Lasry and Lions [74] and Huang et al. [63, 64], can be used to explain some of the strategic interactions between various energy producers.

![CBM10 - Crude Oil Brent (ICE)](image)

Figure 4.1: End of day Commodity Futures Price Quotes for Crude Oil Brent. Source: [www.nasdaq.com](http://www.nasdaq.com)

**How OPEC sets production** The Organization of Petroleum Exporting Countries (OPEC) is a cartel of oil-producing nations that accounts for about 40% of the world’s oil production. Comprising of twelve member countries (see Table 4.1 for its member countries), OPEC mandates to “coordinate and unify the petroleum policies” of its members and to “ensure the stabilization of oil markets in order to secure an efficient, economic and regular supply of petroleum to consumers, a steady income to producers, and a fair return on capital for those investing in the petroleum
industry.\footnote[1]{Source: \url{http://www.opec.org/opec_web/en/publications/345.htm}} OPEC typically meets twice a year to set production quotas. As with most commodities, the price of oil is mainly dictated by supply and demand. Since the supply of oil was determined in large part by OPEC, the higher they set their quotas, the lower the oil price.

As shown in Figure 4.1 oil prices have been high since 2010 – bouncing around $110 per barrel because of escalating oil consumption in countries like China and political instability in key oil nations like Iraq. Given the high oil prices, many energy companies (most notably Chevron Corporation, Exxon Mobil Corp and ConocoPhillips Co) found it profitable to begin extracting oil from difficult-to-drill places. In the United States, companies began using techniques like hydraulic fracturing and horizontal drilling to extract oil from shale formations in North Dakota and Texas.

Table 4.1: \textit{OPEC has twelve member countries: six in the Middle East, four in Africa, and two in South America.} Source: \url{http://en.wikipedia.org}

<table>
<thead>
<tr>
<th>Country</th>
<th>Production (bbl/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algeria</td>
<td>2,125,000 (16th)</td>
</tr>
<tr>
<td>Angola</td>
<td>1,948,000 (17th)</td>
</tr>
<tr>
<td>Ecuador</td>
<td>485,700 (30th)</td>
</tr>
<tr>
<td>Iran</td>
<td>4,172,000 (4th)</td>
</tr>
<tr>
<td>Iraq</td>
<td>3,200,000 (7th)</td>
</tr>
<tr>
<td>Kuwait</td>
<td>2,494,000 (10th)</td>
</tr>
<tr>
<td>Libya</td>
<td>2,210,000 (15th)</td>
</tr>
<tr>
<td>Nigeria</td>
<td>2,211,000 (14th)</td>
</tr>
<tr>
<td>Qatar</td>
<td>1,213,000 (21st)</td>
</tr>
<tr>
<td>Saudi Arabia</td>
<td>8,800,000 (2nd)</td>
</tr>
<tr>
<td>United Arab Emirates</td>
<td>2,798,000 (8th)</td>
</tr>
<tr>
<td>Venezuela</td>
<td>2,472,000 (11th)</td>
</tr>
</tbody>
</table>

\textbf{Plummeting oil price} Hydraulic fracturing, or “fracking”, is the process through which oil and gas are released from shale deposits deep underground by means of drilling and injecting pressurized liquid made of water, sand, and chemicals. According to the US Energy Information Administration, there are over 500,000 active
natural gas wells in the US as of 2011, adding significantly to the world oil supply. To put this in context, the US fracking industry has added nearly 4 million extra barrels of crude oil per day to the global market since 2008 (compared to global production of about 75 million barrels per day). This surge in supply, together with a lack of demand due to sluggish global economic growth, led to a fall in oil price of nearly 50% over the second half of 2014. As oil prices tumbled, most observers expected to see OPEC, the world’s largest oil cartel, cut back on production to push prices back up.

OPEC’s war on fracking This brings us to the OPEC Conference in Vienna on 27 November 2014. Some countries, like Venezuela and Iran, wanted the cartel to cut back on production in order to boost the price. On the other side of the debate, Saudi Arabia didn’t want to give up market share and refused to reduce production – in the hopes that lower oil prices would help impede expansion of the fracking industry. In the end, despite the oversupply on the world market, OPEC failed to agree on a response and ended up keeping production unchanged. So the price of oil began declining even further.

The price of oil has hovered in the $40-55/barrel range for most of the first half of 2015, but it is estimated that many fracking companies need prices above $60-80 to break even. There is now speculation that many fracking operations may be forced into closure. The theory is that OPEC is now engaged in a “price war” with the US frackers. Led by powerful oil nations such as Saudi Arabia, OPEC is seeking to drive the fracking industry out of business, once again regain its place as the world’s pre-eminent source of oil, and stabilize oil prices well above the present level. This is the central issue of blockading we want to model using dynamic game theory.
4.1.1 Competitive oligopolistic view

We take a competitive oligopolistic view of an idealized global energy market, in which game theory describes the outcome of competition. The classical works of Cournot [36] and Bertrand [14] study oligopoly markets based on the assumptions about the strategic variables a firm chooses to compete with its rivals. The Bertrand model assumes that firms compete on price while the Cournot model assumes that the competition is on output quantity. The Cournot framework of oligopoly is appropriate for energy production in which major players determine their output relative to their production costs, as in the expected scenario that OPEC will cut production in order to increase the market price of oil.

The computation of Nash equilibria in nonzero-sum dynamic games between \( N \) players, each with their own resources, is a challenging problem, typically involving coupled systems of \( N \) nonlinear Hamilton-Jacobi-Bellman (HJB) PDEs. In the continuum limit of an infinity of small players, however, the mean field game approach allows one to handle certain types of competition by solving a coupled system of two PDEs. The interaction here is such that each player only sees and reacts to the statistical distribution of the states or actions of other players. Optimization against the distribution of other players leads to a backward HJB equation; and in turn their actions determine the evolution of the state distribution, encoded by a forward Kolmogorov equation. This continuum approximation allows for analytical and computational results which are hard to obtain from the \( N \)-player system.

The goal of the present chapter is to extend the basic mean field game model in Chapter 3 and study the competition between the traditional energy producers and alternative sources. In this setting, the economy is framed as a Cournot competition where the market model is specified by inverse demand functions, which give prices as a function of quantities produced. In the context of a global energy market, we model OPEC by a continuum of oil producers with low costs of production, but each
member nation has a finite reserve. The other side of the economy is represented by an alternative energy producer (e.g., the fracking industry in the US or renewable production such as from solar technology) with relatively costly production. However, the alternative energy producer is distinguished from the traditional oil producers by its relative abundance of production capacity. Throughout this chapter, we make the simplifying assumption that the alternative energy source is inexhaustible relative to the traditional source.

“The Stone Age did not end for lack of stone, and the Oil Age will end long before the world runs out of oil.” With recent technological advances such as renewables and fracking, this intriguing quote of former Saudi oil minister Sheikh Zaki Yamani may not be far-fetched and the end of traditional oil age may be upon us. In this chapter, our central innovation is to consider the interaction and competition between traditional and alternative energy producers. We do so by considering three distinct time scales representing different idealizations of the global energy market. Table 4.2 describes the three time scales under consideration and the main features of each horizon.
Table 4.2: *Three distinct time scales representing different idealizations of the global energy market.*

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Global Energy Market</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Long-term</strong></td>
<td>Over a longer horizon, alternative energy sources become more competitive as their production costs decrease further due to technological advances. While traditional fossil fuels are not necessarily depleted, renewable energy gains considerable market share due to increasing (scarcity) costs of fossil fuel extraction. We study how the global economy transitions from the traditional/exhaustible energy production to its renewable counterpart (Section 4.3).</td>
</tr>
<tr>
<td><strong>Intermediate</strong></td>
<td>Over an intermediate horizon, the production of traditional energy (fossil fuels e.g. oil) is still the cheapest. However, alternative energy sources (shale oil or solar power) are gaining market share due to decreasing production costs. Traditional energy producers may strategically increase their production rate to compete for market share with the alternative energy producers. This may be the logic behind OPEC’s decision not to cut crude oil output. Section 4.1.2 below illustrates the issue of blockading in the simplest setting of static games. Competition between exhaustible fossil fuels and renewable alternatives is studied in a dynamic model in Section 4.4.</td>
</tr>
<tr>
<td><strong>Short-term</strong></td>
<td>Over a short time frame, traditional energy sources are not going to run out. The major determinant of the energy price level is supply shocks due to exploration successes, itself arising from investment in research and development. We model the joint strategic decision of (costly) exploration effort and production rate in this context (Section 4.5).</td>
</tr>
</tbody>
</table>
4.1.2 Example: static MFG and blockading

To illustrate the effect of blockading in the simplest setting, we consider a static (one-period) competition between traditional oil producers and an alternative energy producer. In this setting there is a continuum of traditional oil producers labelled by its “position” $x$ and density $m(x)$. The producer at position $x$ has cost of production $s(x)$. In addition, there is an alternative energy producer with cost $s_0$.

We denote the quantity of the traditional (resp. alternative) producers by $q(x)$ (resp. $\hat{q}$), and the average production of the traditional producers by $Q$:

$$Q = \int q(x) m(x) \, dx.$$  

Here we use linear inverse demand functions:

$$p(q, Q, \hat{q}) = 1 - q - Q - \hat{q}, \quad \hat{p}(\hat{q}, Q) = 1 - \hat{q} - Q. \quad (4.1)$$

The price $p$ received by an oil producer is decreasing in his own production quantity $q$, the average quantity $Q$ produced by the other players, and the quantity $\hat{q}$ of the alternative energy producer. Similarly, the price $\hat{p}$ received by the alternative energy producer is decreasing in his own production quantity $\hat{q}$ as well as the average production quantity $Q$ of the traditional oil producers.

In a Nash equilibrium $(q^*(x), \hat{q}^*)$ for the “$\infty + 1$” players, each one maximizes profit as a best response to the other players’ equilibrium strategies:

$$\sup_{q \geq 0} q(1 - q - Q - q^* - s(x)), \quad \sup_{\hat{q} \geq 0} \hat{q}(1 - \hat{q} - Q - s_0),$$
where now \( Q = \int q^* m \). If there is an interior maximum (i.e. each player having positive equilibrium production), then we have

\[
q^*(x) = \frac{1}{2} (1 - Q - \tilde{q}^* - s(x)), \quad \tilde{q}^* = \frac{1}{2} (1 - Q - s_0). \tag{4.2}
\]

Integrating \( q^* \) against \( m \) and solving for \( Q \) using the above expression for \( \tilde{q}^* \) yields

\[
Q = \frac{1}{3} (1 - \tilde{q}^* - \langle s \rangle) = \frac{1}{5} (1 + s_0 - 2\langle s \rangle), \quad \text{where} \quad \langle s \rangle = \int_{\mathbb{R}_+} s(x)m(x) \, dx.
\]

Consequently, from (4.2) we derive

\[
\tilde{q}^* = \frac{1}{5} (2 - 3s_0 + \langle s \rangle), \quad q^*(x) = \frac{1}{5} \left( 1 - \frac{5}{2} s(x) + s_0 + \frac{1}{2} \langle s \rangle \right).
\]

Blockading of the alternative producer occurs when \( \tilde{q}^* \leq 0 \), or equivalently when \( s_0 \geq (2 + \langle s \rangle) / 3 \) in terms of production costs. The interpretation is that the alternative energy producer is blockaded when his cost \( s_0 \) is too high compared to the average production cost \( \langle s \rangle \) of the traditional producers. In this case the alternative producer produces nothing \( \tilde{q}^* = 0 \), and the traditional producers take over the market

\[
Q = \frac{1}{3} (1 - \langle s \rangle), \quad \text{which leads to} \quad q^*(x) = \frac{1}{3} \left( 1 - \frac{3}{2} s(x) + \frac{1}{2} \langle s \rangle \right).
\]

In this case, we say that the alternative energy producer is \textit{blockaded} from production. Figure 4.2 shows that as \( s_0 \) decreases (representing increased competitiveness of the costly alternative energy source), the traditional low-cost producer may strategically choose not the reduce production in an attempt to keep the alternative producer blockaded. In our context, this may be OPEC holding back on cuts in production to drive shale oil producers out of the market and into bankruptcy.
Figure 4.2: Static Cournot duopoly with linear demand (4.1). When $s_0$ is large relative to fixed $\langle s \rangle$, the alternative energy producer is blockaded from production, and the traditional oil producers take over the entire market.

We study in Section 4.4 a dynamic version of this game incorporating exhaustibility of the traditional fuel.

4.1.3 Organization and Results

We study the interaction between the traditional and alternative energy producers from three perspectives: competition, transition, and exploration.

- In Section 4.2 we revisit the basic framework for dynamic Cournot mean field games with exhaustible resources. We generalize and extend the results of Chapter 3 to include nonlinear demand functions.

- Section 4.3 considers an economy transitioning from exhaustible energy sources to renewable ones. The exhaustible producers can switch to an alternative energy source (e.g. solar or hydroelectric power) when they run out of reserves. As we shall see, this essentially introduces a Neumann boundary condition to the PDE problem. In the regime of small exhaustibility, the first order correction to the value function satisfies a partial integro-differential equation which is
explicitly solvable. In Section 4.3.4, we consider a smooth transition to an alternative energy source. The interpretation is that as the exhaustible resource begins to run out, the cost of production increases for instance due the need for deeper drilling or more costly extraction procedures such as fracking.

- Section 4.4 investigates the competitive interaction between an alternative energy producer with traditional oil producers who have exhaustible resources. Here the alternative energy producer has marginal cost of production $s > 0$, but inexhaustible supplies. We shall see the blockading of the renewable producer when his production cost $s$ is high enough and when the exhaustible resource is still abundant.

- Section 4.5 deals with exploration and exhaustibility. Here we incorporate the stochastic effect of resource exploration into the dynamic Cournot mean field games. We study the equilibrium production rate and exploration effort level in a sustainable economy. This corresponds to a steady-state solution to the MFG PDE problem.

We conclude in Section 4.6.

### 4.2 Dynamic Cournot model

The basic mean field game model in Chapter 3 will serve as a baseline for our analysis of competition between the traditional energy producers and the alternative sources. While the exposition there focuses on a Bertrand (price-setting) model, it is shown in Appendix B that the dynamic Cournot and Bertrand games are identical in the continuum limit. Our focus in this chapter is the global energy market in which the Cournot framework is more appropriate. We elaborate on and extend the dynamic Cournot mean field game model to nonlinear demand functions in this section. To
introduce notation and ideas, we concentrate in this section only on the exhaustible oil producers without competition from the alternative producers, which are introduced in Section 4.3.

4.2.1 Dynamic continuum mean field games

In the dynamic problem, firms produce energy by depleting their reserves of a fossil fuel, and different producers have different levels of initial reserves. When they exhaust their reserves, they no longer participate and the market shrinks. There is an infinity of players labelled by their reserves \( x > 0 \), with initial density of reserves \( M(x) \). They choose production rates \( q_t \) which deplete the remaining reserve \( X_t \); moreover the reserve level may be subject to random fluctuations (e.g. due to noisy seismic estimation of the oil or gas well). The reserve level \( X_t \) follows the dynamics

\[
dX_t = -q_t \, dt + \sigma \, dW_t,
\]

as long as \( X_t > 0 \), and \( X_t \) is absorbed at zero. Here \( W \) is a standard Brownian motion, and \( \sigma \geq 0 \) is a constant.

A firm that starts with reserve \( x > 0 \) at time \( t \geq 0 \) sets quantities to maximize the lifetime profit discounted at constant rate \( r > 0 \) over Markov controls \( q_t = q(t, X_t) \), with the corresponding price \( p_t = p(t, X_t) \) given by the inverse demand to be specified below. Hence the value function of the firm is defined by

\[
v(t, x) = \sup_{q} \mathbb{E} \left\{ \int_t^\infty e^{-r(u-t)} p_u q_u 1_{\{X_u>0\}} \, du \middle| X_t = x \right\}, \quad x > 0. \tag{4.3}
\]

The game runs till some exhaustion time \( T \) (which may be infinite) when all producers have exhausted their reserves, and \( T \) has to be determined endogenously as part of the problem.
The price received by the player depends on his own production quantity \( q_t \) as well as the mean production rate \( Q(t) \), according to

\[
p_t = P(q + \varepsilon Q), \quad Q(t) = \int_{\mathbb{R}_+} q(t, x)m(t, x) \, dx, \tag{4.4}
\]

where \( m(t, x) \) denotes the density of producers’ reserves at time \( t > 0 \), and \( P \) is a decreasing inverse demand function. We note that the price received by a representative producer depends on the other players through their mean production rate \( Q \), and so the interaction is of mean field type. The parameter \( \varepsilon \) measures the degree of interaction or product substitutability, in the sense that the price received by an individual firm decreases as the other firms increase production of their goods.

In Chapter 3, the inverse demand function is taken to be linear \( P(Q) = 1 - Q \). In this section, we extend this to nonlinear inverse demand function \( P \) of power type:

\[
P(Q) = \begin{cases} 
\frac{\eta}{1-\rho} \left( 1 - \left( \frac{Q}{\eta} \right)^{1-\rho} \right), & \rho \neq 1, \\
\eta (\log \eta - \log Q), & \rho = 1.
\end{cases} \tag{4.5}
\]

The parameter \( \rho \) is known as the relative prudence. Notice that we recover the special case of linear demand by setting \( \rho = 0 \) and \( \eta = 1 \). Following [76], we focus on the case where \( \rho < 1 \) in which the choke price \( P(0^+) = \eta/(1 - \rho) \) is finite. This family of pricing function is shown in [60] to be particularly tractable for the computation of Nash equilibrium in the static Cournot game. It is further analyzed in [85].

### 4.2.2 Dynamic programming and the HJB equation

The HJB equation associated to (4.3) is

\[
\partial_t v + \frac{1}{2} \sigma^2 \partial_{xx} v - rv + \max_{q \geq 0} q (P(q + \varepsilon Q(t)) - \partial_x v) = 0, \quad x > 0, \tag{4.6}
\]
where the inverse demand $P$ is given by (4.5). Notice that the embedded optimization can be interpreted as a static profit maximization problem for a producer with (shadow) cost $\partial_x v$ facing competitors’ production $Q(t)$. The optimal production quantity $q^*$ is given (implicitly) by the first order condition

$$P'(q^* + \varepsilon Q(t))q^* + P(q^* + \varepsilon Q(t)) - \partial_x v(t, x) = 0, \quad Q(t) = \int_{\mathbb{R}_+} q^*(t, x)m(t, x)\, dx. \quad (4.7)$$

When a player runs out of reserves, he no longer produces or makes income, and so we have the boundary condition $v(t, 0) = 0$. At the exhaustion time $T$, $v(T, x) = 0$, but as mentioned before, the terminal time $T$ when all oil runs out has to be determined endogenously.

Given the rate of depletion $q^*(t, x)$, we can determine the ‘population dynamics’ of the producers by the forward Kolmogorov equation

$$\partial_t m - \frac{1}{2} \sigma^2 \partial_{xx}^2 m - \partial_x (q^* m) = 0, \quad (4.8)$$

with $m(0, x) = M(x)$. The system (4.6) and (4.8) is an example what Lasry and Lions [74] have called a mean field game. The backward evolution equation (4.6) represents the firms’ decisions based on anticipating how the game unfolds in the future; and the forward evolution equation (4.8) represents where they actually end up, based on their strategic decision and initial distribution. The forward/backward system of PDEs is coupled through the dependence on $Q$ in the HJB equation (4.6) and the dependence on $\partial_x v$ in the forward Kolmogorov equation (4.8).

### 4.2.3 Small competition asymptotics

Solving this coupled system of equations is highly non-trivial. There does not exist anything like general existence and uniqueness theorems for PDE systems of this kind, and we do not attempt to prove these properties here. The stochastic case has been
studied numerically in Section 3.5, and here we focus on the deterministic setting \( \sigma = 0 \).

The main goal is to determine the principal effect of competitiveness on the market equilibrium. From the inverse demand function \( f(x) \), we see that \( \varepsilon \) parameterizes the degree of interaction among firms; the limit \( \varepsilon = 0 \) corresponds to independent markets, where each firm has a monopoly in his own market. In the small competition regime, one can formally look for an approximation to the PDE system of the form

\[
v(t, x) = v_0(t, x) + \varepsilon v_1(t, x) + \varepsilon^2 v_2(t, x) + \cdots, \\
m(t, x) = m_0(t, x) + \varepsilon m_1(t, x) + \varepsilon^2 m_2(t, x) + \cdots.
\]

Solving for the value function \( v \) and the density \( m \) perturbatively leads to approximations to \( q^* \) and \( Q \) in (4.7):

\[
q^*(t, x) = q_0(t, x) + \varepsilon q_1(t, x) + \cdots, \\
Q(t) = Q_0(t) + \varepsilon Q_1(t) + \cdots.
\]

**Monopoly value function and production rate**

The monopoly value function \( v_0 \) is determined by setting \( \varepsilon = 0 \) in the HJB equation (4.6), and after performing the maximization, we have:

\[
\partial_t v_0 - rv_0 + C \left( \frac{\eta}{1 - \rho} - \partial_x v_0 \right)^\beta = 0, \\
v_0(0) = 0,
\]

(4.9)

where \( C = \left( \frac{\beta - \beta \eta^{-1 - \rho}}{1 - \rho} \right) \) and \( \beta = \frac{2 - \rho}{1 - \rho} \). The solution to (4.9) is given in the following proposition, which can be verified by direct substitution.

**Proposition 14.** The leading order (monopoly) value function is time-independent \( v_0(t, x) = v_0(x) \), and it is implicitly given by

\[
\frac{\beta C}{r} \left( \frac{1 - \rho}{\eta} \right)^{1 - \beta} B \left( \frac{1 - \rho}{\eta} \left( \frac{r}{C} \right)^{1/\beta} v_0^{1/\beta}; \beta, 0 \right) = x,
\]

(4.10)
where the incomplete beta function $B(z; a, b)$ is defined by

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1} \, dt.$$  

In particular, we recover the case of linear inverse demand studied in Section 3.3 if we set $\rho = 0$ since in this case $\beta = 2$, $B(z; 2, 0) = -z - \log(1 - z)$, and so $v_0$ can be written in terms of the Lambert-W function.

**Production trajectory and Hotelling’s rule**  In the monopoly case, the optimal production quantity $q_0$ for each individual firm is given by setting $\varepsilon = 0$ in (4.7):

$$P'(q_0)q_0 + P(q_0) - v'_0 = 0. \quad (4.11)$$

It follows from (4.11) that

$$q_0(x) = \left( \frac{P(0) - v'_0(x)}{\beta \eta} \right)^{\frac{1}{1-\rho}}. \quad (4.12)$$

It is easy to show that $v_0(x)$ is an increasing concave function, with $v_0(0) = 0$ and $v_0(\infty) = \eta^2(2-\rho)^{-\beta}/r$. Therefore, $v'_0(x)$ is a decreasing function, with $v'_0(0) = P(0) = \frac{\eta}{1-\rho}$ and $v'_0(\infty) = 0$. Consequently, $q_0(x)$ is an increasing function of $x$, with $q_0(0) = 0$ and $q_0(\infty) = \eta(2 - \rho)^{-\frac{1}{1-\rho}}$. Therefore, players produce at a finite rate, and as they run out of reserves, they decrease their production rate to zero at $x = 0$.

Let $x_0(t; x)$ denote the optimal monopoly production trajectory starting from $x$:

$$x'_0(t) = -q_0(x_0(t)), \quad x_0(0) = x, \quad \text{and define} \quad S(t) = v'_0(x_0(t)),$$

so $S(t)$ is the shadow cost along the optimal production trajectory $x_0(t)$.

**Proposition 15.** The classical Hotelling’s rule holds also for the continuum mean field monopoly, i.e. the shadow cost grows at the discount rate $r$ along the optimal pro-
duction trajectory: \( S'(t) = rS(t) \). It follows that the market price \( \mathcal{P}(t) = P(q_0(x_0(t))) \) satisfies the following linear ODE:

\[
\mathcal{P}'(t) = r \left( \mathcal{P}(t) - \frac{\eta}{2 - \rho} \right). \tag{4.13}
\]

**Proof.** First we write the monopoly ODE (4.9) as \( rv_0 = q_0(P(q_0) - v_0') \), where \( q_0 \) satisfies the first order condition (4.11). Then differentiating the ODE with respect to \( x \), and using (4.11), we obtain \( rv_0' = -q_0v_0'' \). Now we compute the growth rate of the shadow cost \( S(t) \) along the optimal production trajectory:

\[
S'(t) = \frac{d}{dt} v'_0(x_0(t)) = -v''_0(x_0(t)) q_0(x_0(t)) = rS(t).
\]

By direct calculation, the market price is a linear function of \( v_0' \): \( P(q_0(x)) = \frac{\eta + v_0'(x)}{2 - \rho} \). It follows easily that \( \mathcal{P} \) also satisfies a linear ODE, which is given by (4.13). \( \square \)

**Monopoly exhaustion times**

We define the hitting time \( \tau : \mathbb{R}_+ \to \mathbb{R}_+ \) to be the time to exhaustion in the deterministic monopoly market starting at initial reserve \( x \):

\[
\tau(x) = \inf\{t \geq 0 \mid x_0(t; x) = 0\}. \tag{4.14}
\]

Even though there does not seem to be an explicit expression for the reserve trajectory, the exhaustion time \( \tau(x) \) can be given explicitly in the following proposition.

**Proposition 16.** The exhaustion time \( \tau(x) \) is given explicitly by

\[
\tau(x) = \frac{1}{r} \log \left( \frac{P(0)}{v_0'(x)} \right). \tag{4.15}
\]
Moreover, the exhaustion time $\tau$ can be inverted in closed-form to give

$$
\tau^{-1}(t) = \frac{\beta C}{r} P(0)^{\beta-1} K \left( 1 - e^{-rt}; \beta, 0 \right).
$$

(4.16)

Proof. From the definition (4.14) of $\tau(x)$, and using that $v'_0$ is a monotonic function with $v'_0(0) = P(0)$, we write

$$
\tau(x) = \inf \{ t \geq 0 \mid v'_0(x_0(t)) = P(0) \} = \inf \{ t \geq 0 \mid v'_0(x)e^{rt} = P(0) \},
$$

where we have used Hotelling’s rule given in Proposition 15. The expression (4.15) follows. From the HJB equation (4.9), we derive

$$
v_0(x) = \frac{C}{r} P(0)^{\beta} (1 - e^{-r\tau(x)})^{\beta}.
$$

Plugging this into the expression (4.10), and identifying $x = \tau^{-1}(t)$ lead to (4.16). 

In the special case of linear inverse demand (where $\rho = 0$ and $\eta = 1$), we have $\tau(x) = -r^{-1} \log \left(-W(\theta(x))\right)$, where $W$ is the Lambert-W function, and $\theta(x) = -e^{-2rx-1}$, which recovers the one found in Section 3.3.

When the initial density $M(x)$ has compact support $[0,x_{\max}]$, all oil reserves are exhausted at finite time $T = \tau(x_{\max})$; otherwise $T = \infty$.

Monopoly density function

In the deterministic monopoly setting, the forward Kolmogorov equation (4.8) reads

$$
\partial_t m_0 - \partial_x [q_0 m_0] = 0, \quad m_0(t,x) = M(x).
$$

(4.17)
Proposition 17. The monopoly density is given by

\[ m_0(t, x) = \frac{q_0(\tau^{-1}(t + \tau(x)))}{q_0(x)} M(\tau^{-1}(t + \tau(x))) = \frac{d}{dx}F(\tau^{-1}(t + \tau(x))), \]

(4.18)

where \( F \) denotes the cumulative distribution function (CDF) of the initial density \( M \).

Moreover, the proportion \( \eta_0 : \mathbb{R}_+ \rightarrow [0, 1] \) of remaining firms is given by

\[ \eta_0(t) = \int_{\mathbb{R}_+} m_0(t, x) \, dx = 1 - F(\tau^{-1}(t)). \]

Proof. The explicit solution to (4.17) follows from the method of characteristics; while the second expression for \( m_0 \) in (4.18) follows from straightforward manipulation. The computation of \( \eta_0 \) follows similarly to Proposition 9.

First order asymptotics

Let \( G \) be the supremum in the PDE (4.6): \( G(\varepsilon) = \max_{q \geq 0} q (P(q + \varepsilon Q) - \partial_x v) \).

The first order condition is given in (4.7), and so, to first order in \( \varepsilon \), we have \( G \approx G(0) + \varepsilon G'(0) \), where

\[ G'(0) = (P(q_0) - v_0)q_1 + q_0(q_1 + Q_0)P'(q_0) - q_0 \partial_x v_1 = q_0(Q_0P'(q_0) - \partial_x v_1). \]

This leads to the equation satisfied by the first order correction \( v_1 \):

\[ \partial_t v_1 - rv_1 - q_0 \partial_x v_1 = -q_0 P'(q_0)Q_0, \quad v_1(t, 0) = 0. \]

Proposition 18. The first-order correction to the value function \( v_1 \) is given by

\[ v_1(t, x) = -\frac{\eta}{2 - \rho} \int_0^{\tau(x)} (e^{-ru} - e^{-r\tau(x)}) Q_0(t + u) \, du. \]

Proof. The proof follows similarly to Proposition 11.

\[ \square \]

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Clearly $v_1 < 0$, so that competition reduces the value function from the monopoly limit $v_0$, as is expected. The first order correction $q_1$ to the optimal production quantity is

$$q_1 = \frac{\partial_x v_1 - Q_0 (P'(q_0) + q_0 P''(q_0))}{2 P'(q_0) + q_0 P''(q_0)} = \frac{\partial_x v_1 - (1 - \rho) Q_0 P'(q_0)}{(2 - \rho) P'(q_0)}. \quad (4.19)$$

The first-order correction to density $m_1$ satisfies the following equation

$$\partial_t m_1 - \partial_x [q_0 m_1 + q_1 m_0] = 0, \quad m_1(0, x) = 0.$$

**Proposition 19.** The first-order correction to density $m_1$ is given by

$$m_1(t, x) = \int_0^t q_0 \left( x_0(u - t; x) \right) g(u, x_0(u - t; x)) du,$$

where the inhomogeneous term is given by $g = \partial_x q_1 m_0 + q_1 \partial_x m_0$.

**Proof.** The proof follows similarly to Proposition 12.

The following proposition demonstrates that the principal effect of competitive interaction is that firms to slow down production and increase the exhaustion time.

**Proposition 20.** For concave pricing functions $P$ (i.e. $\rho < 0$), the first order correction $q_1$ to the equilibrium production rate is negative.

**Proof.** From the expression (4.19) for $q_1$, it suffices to show that

$$\partial_x v_1 - (1 - \rho) Q_0 P'(q_0) \geq 0.$$

Indeed, defining $F = \beta q_0 [\partial_x v_1 - (1 - \rho) Q_0 P'(q_0)]$, a straightforward calculation leads to

$$F(t, x) = -r v_0'(x) \int_0^{\tau(x)} Q_0(t + u) du + (1 - \rho) Q_0(t) (P(0) - v_0'(x)).$$
One can readily check that $F(t,0) = 0$ and $\partial_x F(t,x) \geq 0$ for concave $P$, and so from (4.19), $q_1 \leq 0$.

4.3 Transition to renewable resources

In this section, we consider a model in which the exhaustible producers can switch to a more expensive alternative energy source (e.g. solar or hydroelectric power) when they run out of reserves. This model corresponds to the long horizon in Table 4.2. This yields a continuum mean field version of the continuous-time Cournot model of Harris et al. [60]. In the context of energy production, resources, such as oil or natural gas, have finite supply, and exhaustibility enters as boundary conditions for the PDEs. As we shall see, this essentially introduces a Neumann boundary condition to the PDE problem. We suppose there is an alternative, but costly, technology (for example solar power), and study the system using asymptotic approximation. In the regime of small exhaustibility, the first order correction to the value function satisfies a partial integro-differential equation which is explicitly solvable.

4.3.1 Deterministic model setup

The energy market is modeled by a Cournot game which is specified by the inverse demand: $P(q,Q) = 1 - q - Q$, where $Q$ is the mean energy production (from both traditional and alternative sources). A firm with reserves $x > 0$ at time $t \geq 0$ sets quantities to maximize lifetime discounted profits. Its value function is given by

$$v(t,x) = \sup_q \int_t^\infty e^{-r(u-t)} p_u q_u 1_{\{X_u > 0\}} du, \quad X_t = x,$$

subject to the deterministic dynamics $dX_t = -q_t \, dt$. 

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The HJB equation associated to (4.20) reads

\[ \partial_t v - rv + \max_{q \geq 0} q (1 - q - Q - \partial_x v) = 0. \]

Plugging in the feedback strategy

\[ q^*(t, x) = \frac{1}{2} (1 - Q(t) - \partial_x v(t, x)), \] (4.21)

the HJB equation becomes

\[ \partial_t v - rv + \frac{1}{4} (1 - Q(t) - \partial_x v(t, x))^2 = 0. \] (4.22)

On hitting the boundary \( x = 0 \), the player switches to an alternative inexhaustible source at marginal cost of production \( s \). Playing against the mean production \( Q \), his equilibrium strategy is \( q^*(t, 0) = \frac{1}{2} (1 - Q - s) \). Assuming continuity of the equilibrium production rates \( q_t \) up to the boundary \( x = 0 \) leads to the Neumann boundary condition \( \partial_x v(t, 0) = s \). The interpretation is that, on running out of the exhaustible resource, the shadow cost \( \partial_x v(t, x) \) of the player turns into the real cost \( s \).

The mean production \( Q(t) \) comes from two sources: the exhaustible and inexhaustible parts. We assume that no player produces from the more expensive source as long as the cheaper one is available. Thus we can write

\[ Q(t) = \frac{1}{2} (1 - Q(t) - s) (1 - \eta(t)) + \int_{\mathbb{R}^+} q^*(t, x)m(t, x) \, dx, \] (4.23)

where \( \eta \) is the proportion of players using the cheap energy source:

\[ \eta(t) = \int_{\mathbb{R}^+} m(t, x) \, dx, \] (4.24)
and so \((1 - \eta)\) is the fraction of firms who have exhausted their traditional fuel reserves and are now producing from the alternative inexhaustible source at marginal cost \(s\).

Solving for \(Q\) in \((4.23)\), we obtain

\[
Q = \frac{1}{3} \left( 1 - s(1 - \eta) - \int_{\mathbb{R}^+} m \partial_x v \, dx \right). \tag{4.25}
\]

The forward Kolmogorov equation for \(m\) is

\[
\partial_t m - \frac{1}{2} \partial_x ((1 - Q - \partial_x v) m) = 0, \tag{4.26}
\]

with \(m(0, x) = M(x)\).

### 4.3.2 Small exhaustibility expansion

**Inexhaustible limit** In the inexhaustible limit where \(s = 0\), the firms are indifferent to using oil or alternative energy sources. Consequently, the value function \(v_0\) is a constant. From \((4.25)\) with \(s = 0\) and \(\partial_x v_0 = 0\), we derive a constant mean production \(Q_0 = 1/3\), which from the HJB equation \((4.22)\) gives \(v_0(t, x) = (9r)^{-1}\), also satisfying the boundary condition \(\partial_x v_0(t, 0) = 0\). From \((4.26)\), the density \(m\) is transported at constant speed \(m_0(t, x) = M(x + t/3)\).

We are interested in the case when \(s\) is small but non-zero. To this end we formally look for an expansion in the small \(s\) regime:

\[
v(t, x) = v_0(t, x) + sv_1(t, x) + \mathcal{O}(s^2), \tag{4.27}
\]

\[
m(t, x) = m_0(t, x) + sm_1(t, x) + \mathcal{O}(s^2).
\]

Of course, we have just found that \(v_0\) is independent of \(t\) and \(x\) and is given by \((9r)^{-1}\).
**First order correction: value function**  Plugging the formal expansion (4.27) into the HJB equation (4.22) we get

\[
\partial_t v_1 - rv_1 - \frac{1}{3} (Q_1 + \partial_x v_1) = 0,
\]

with \(\partial_x v_1(t, 0) = 1\). Here \(Q_1\) is the first order correction to the mean production, given by

\[
Q_1 = -\frac{1}{3} \left( 1 - \eta_0 + \int_{\mathbb{R}^+} m_0 \partial_x v_1 \, dx \right).
\]

Therefore, we obtain the partial integro-differential equation (PIDE) for \(v_1\)

\[
\partial_t v_1 - rv_1 - \frac{1}{3} \partial_x v_1 + \frac{1}{9} \left( 1 - \int_{\mathbb{R}^+} m_0 (1 - \partial_x v_1) \, dx \right) = 0, \quad \partial_x v_1(t, 0) = 1. \tag{4.28}
\]

By considering an additively separable solution of the form \(v_1(t, x) = f(x) + g(t)\), the solution to the above PIDE can be readily computed:

\[
v_1(t, x) = -\frac{1}{3r} e^{-3rx} - \int_0^t I(u) e^{r(t-u)} \, du, \tag{4.29}
\]

where \(I(t) = \frac{1}{9} \left( 1 - \int_0^\infty M(x + t/3) \left( 1 - e^{-3rx} \right) \, dx \right)\). That \(v_1\) is additively separable simplifies the expression for \(Q_1\) considerably:

\[
Q_1(t) = -\frac{1}{3} \left( 1 - \eta_0 + \int_{\mathbb{R}^+} m_0 \partial_x v_1 \, dx \right) = -3I(t). \tag{4.30}
\]

In particular, we see that \(Q_1\) is always negative. The economic interpretation is that increasing the costs of inexhaustible energy source slows down production of the exhaustible one. From (4.21), we can compute the equilibrium production rate:

\[
q^*(t, x) = \frac{1}{3} - \frac{s}{2} \left( Q_1(t) + e^{-3rx} \right) + O(s^2).
\]
First order correction: density  The first order correction to density \( m_1 \) satisfies

\[
\partial_t m_1 - \frac{1}{3} \partial_x m_1 + \frac{1}{2} \partial_x ((Q_1 + \partial_x v_1) m_0) = 0, \quad m_1(0, x) = 0. \tag{4.31}
\]

Having already solved for \( v_1 \), we can write down the solution analytically

\[
m_1(t, x) = \int_0^t h \left( u, x + \frac{t - u}{3} \right) du, \quad h(t, x) = -\frac{1}{2} \partial_x ((Q_1 + \partial_x v_1) m_0). \tag{4.32}
\]

4.3.3 Numerical illustration

We illustrate the effect of exhaustibility using a numerical example. Suppose the initial distribution of reserves is given by an exponential distribution with parameter \( \lambda \) (i.e. \( M(x) = \lambda e^{-\lambda x} \)). Then we can compute the value function correction

\[
v_1(t, x) = -\frac{1}{9} \left( e^{rt} - 1 \right) \left( \frac{1}{r} - \frac{3e^{-\lambda t/3}}{\lambda + 3r} \right) - \frac{e^{-3rx}}{3r},
\]

and the density correction

\[
m_1(t, x) = \frac{1}{2} \lambda e^{-\lambda(x+t/3)} \left( \frac{r \left( 3 - 3e^{-\lambda t/3} \right)}{\lambda + 3r} + \frac{(\lambda + 3r) (1 - e^{-rt}) e^{-3rx} - \frac{3t}{3}}{r} \right).
\]

The leading order correction to the mean production rate is

\[
Q_1(t) = -\frac{1}{3} \left( 1 - \frac{3re^{-\lambda t/3}}{\lambda + 3r} \right).
\]

Observe that the correction to the mean quantity \( Q_1 \) is always negative, in other words, exhaustibility shows down production, as shown in Figure 4.3.
4.3.4 Variable production costs

We illustrate that the technique of asymptotic expansion can be applied to the variable production cost model in [38]. In their setting, instead of an abrupt transition to the alternative technology, a firm’s marginal production cost $\tilde{s}(x)$ gradually increases up to $s$ as the traditional energy reserve runs out $x \to 0$. For illustrative purpose, we will take the variable production costs to be $\tilde{s}(x) = se^{-\gamma x}$. The interpretation is that the exhaustible producer has non-zero cost of extraction; in particular, as reserves begin to run out, costs for exhaustible producers often increase (deeper drilling, more expensive extraction technology required), and the exhaustible producer may choose to invest in R&D (research and development, including exploration) which adds to the marginal production costs.
Model setup  With non-zero production cost \( \tilde{s}(x) \), the value function of a representative firm is

\[
v(t, x) = \sup_q \int_t^\infty e^{-r(u-t)} (p_u - \tilde{s}(X_u)) q_u 1_{\{X_u > 0\}} du, \quad X_t = x,
\]

subject to the dynamics \( dX_t = -q_t \, dt \). The associated HJB equation is

\[
\partial_t v - rv + \max_{q \geq 0} q (1 - q - Q - \partial_x v - \tilde{s}(x)) = 0,
\]

with \( \partial_x v(t, 0) = 0 \). The forward Kolmogorov equation for \( m \) is

\[
\partial_t m - \frac{1}{2} \partial_x ((1 - Q - \partial_x v - \tilde{s}(x)) m) = 0,
\]

with \( m(0, x) = M(x) \), where the mean production \( Q(t) \) comes from both the exhaustible and inexhaustible producers:

\[
Q = \frac{1}{3} \left( 1 - s(1 - \eta) - \int_{\mathbb{R}^+} m(\partial_x v + \tilde{s}(x)) \, dx \right).
\]

Small exhaustibility expansion  We formally look for an asymptotic expansion in the small \( s \) regime. The inexhaustible limit \( s = 0 \) in the present model admits a solution of a constant mean production \( Q_0 = 1/3 \) with \( v_0(t, x) = (9r)^{-1} \). The density \( m_0 \) is transported at constant speed \( m_0(t, x) = M(x + t/3) \). The leading-order corrections to the value function and density in the expansion \((4.27)\) are given by the following proposition.
Proposition 21. The leading-order correction to the value function $v_1$ is additively separable:

$$v_1(t, x) = f(x) + g(t), \quad \text{where}$$

$$f(x) = \frac{3re^{-\gamma x} + \gamma e^{3rx}}{9r^2 + 3\gamma r}, \quad g(t) = \frac{1}{3} \int_0^t Q_1(u)e^{r(t-u)} \, du,$$  

$$Q_1(t) = -\frac{1}{3} \left(1 - \int_{\mathbb{R}^+} M(x + t/3) \left(1 - e^{-\gamma x} - f'(x)\right) \, dx\right).$$

Moreover, the leading-order correction to the density $m_1$ is given explicitly by

$$m_1(t, x) = \int_0^t h\left(u, x + \frac{t-u}{3}\right) \, du, \quad h(t, x) = -\frac{1}{2} \partial_x \left((Q_1(t) + f'(x) + e^{-\gamma x}) m_0\right).$$

Proof. Plugging the formal expansion (4.27) into the HJB equation (4.34) we get

$$\partial_t v_1 - rv_1 - \frac{1}{3} (Q_1 + \partial_x v_1 + e^{-\gamma x}) = 0,$$

with $\partial_x v_1(t, 0) = 0$. Here $Q_1 = -\frac{1}{3} \left(1 - \eta_0 + \int_{\mathbb{R}^+} m_0(\partial_x v_1 + e^{-\gamma x}) \, dx\right)$ is the first order correction to the mean production. Therefore, we obtain the PIDE for $v_1$:

$$\partial_t v_1 - rv_1 - \frac{1}{3} \partial_x v_1 + \frac{1}{9} \left(1 - \int_{\mathbb{R}^+} m_0(1 - \partial_x v_1 - e^{-\gamma x}) \, dx\right) = \frac{1}{3} e^{-\gamma x}, \quad \partial_x v_1(t, 0) = 0.$$

By considering the additively separable solution of the form (4.37), the solution to the above PIDE can be readily computed to give (4.38) and (4.39).

As for the density, the first order correction $m_1$ satisfies

$$\partial_t m_1 - \frac{1}{3} \partial_x m_1 + \frac{1}{2} \partial_x \left((Q_1 + \partial_x v_1 + e^{-\gamma x}) m_0\right) = 0, \quad m_1(0, x) = 0.$$

Using the method of characteristics, we obtain the solution (4.40) analytically. \qed
Notice in particular that as the transition rate $\gamma$ goes to infinity, the total production rate $Q_1$ in this case reduces to $\left(4.30\right)$ as $1 - e^{-\gamma x} - f'(x) \to 1 - e^{-3rx}.$

**Numerical illustration** We illustrate the effect of exhaustibility using a numerical example, where we assume that the initial reserves have an exponential density with parameter $\lambda$. Figure 4.4 shows the effect of the transition rate $\gamma$ on the mean production rate $Q \approx Q_0 + sQ_1$. Notice that a smoother transition (i.e. decreasing $\gamma$) leads to higher production rate.

![Figure 4.4: Effect of the transition rate $\gamma$ on the mean production rate $Q \approx Q_0 + sQ_1$. Parameters used are $r = 0.2, \lambda = 1, s = 0.2$.](image)

### 4.4 Competition with a renewable Producer

In this section we consider the competition between the traditional oil producers with an alternative energy producer. This model corresponds to the intermediate time horizon in Table 4.2. The economy consists of a continuum of firms depleting a non-renewable energy source with zero marginal cost, and an alternative producer with inexhaustible reserves, but higher cost of production $s > 0$. This corresponds to sustainable production from “green” sources (e.g. solar power, or to leading order approximation, the fracking industry in the US). The two classes of producers compete against each other through the Cournot game equilibrium.
For the exhaustible producers, the remaining reserves ($X_t$) follow the dynamics

$$dX_t = -q_t \, dt + \sigma \, dW_t,$$

as long as $X_t > 0$, where $W$ is a standard Brownian motion and $q_t = q(t, X_t)$ is his rate of production at time $t$. Each exhaustible producer has oil resources which he extracts at zero costs, and which is subject to random fluctuation (e.g. due to noisy seismic estimates of oil reserves). When he runs out, he cannot produce anymore and we have $q(t, 0) = 0$. The renewable player produces from an alternative source which is expensive but abundant: his marginal cost of production is $s > 0$. His rate of production is denoted by $\hat{q}(t)$.

The Cournot market is specified by linear inverse demand functions. The inverse demand faced by an exhaustible producer producing at rate $q$ unit is given by

$$P(q, \hat{q}, Q) = 1 - q - \delta \hat{q} - \varepsilon Q,$$

where $Q$ is the mean production rate of the exhaustible producers, and $\hat{q}$ is the production rate of the renewable producer. The inverse demand faced by the renewable producer is similarly given by

$$\hat{P}(\hat{q}, Q) = 1 - \hat{q} - \delta Q.$$

Here $\varepsilon$ is the interaction parameter between exhaustible producers, and $\delta$ is the interaction parameter between exhaustible producers and the renewable producer.
The value functions for the traditional and alternative producers are their discounted lifetime profit, respectively given by:

\[
v(t, x) = \sup_{q \geq 0} \mathbb{E} \left\{ \int_t^\infty e^{-r(u-t)} q u p_u 1_{\{X_u > 0\}} \, dt \ \bigg| \ X_t = x \right\},
\]

\[
g(t) = \sup_{\hat{q} \geq 0} \int_t^\infty e^{-r(u-t)} \hat{q}_u (\hat{p}_u - s) 1_{\{\eta(u) > 0\}} \, du + \int_t^\infty e^{-r(u-t)} \frac{1}{4} (1 - s)^2 1_{\{\eta(u) = 0\}} \, du,
\]

(4.41)

where \( \eta \) is the fraction of exhaustible producers with reserves remaining, given by

\[
\eta(t) = \int_{\mathbb{R}_+} m(t, x) \, dx.
\]

The second term in the definition of \( g \) expresses that the renewable producer has a monopoly when all the exhaustible producers are out of reserves.

We also stress that \( \hat{q} \) must be non-negative: for large enough \( s \), we will see that the renewable player is blockaded in that his cost of producer is so high and his competitors’ reserves of the cheaper resource are so plentiful, that his equilibrium strategy is not to produce anything until the exhaustible producers have run down their reserves some more. When \( s = 1 \), the renewable player never participates in the game, and the above model reduces to the standard Cournot mean field game studied in Section 4.2.

**Dynamic programming and HJB equations**  The HJB equations associated to (4.41) read

\[
\partial_t v + \frac{1}{2} \sigma^2 \partial_{xx} v + \sup_q \left[ q \left( 1 - q - \varepsilon Q(t) - \delta \hat{q}(t) - \partial_x v \right) \right] = rv,
\]

\[
g'(t) + \sup_{\hat{q}} \left[ \hat{q} \left( 1 - \hat{q} - \delta Q(t) - s \right) \right] = rg.
\]

(4.42)
From the optimal feedback control $q^*(t, X_t)$ of the exhaustible producer, the density $m$ of reserves $X_t$ follows the forward Kolmogorov equation

$$\partial_t m - \frac{1}{2} \sigma^2 \partial^2_{xx} m - \partial_x (q^* m) = 0,$$

with $m(0, x) = M(x)$. The total production by exhaustible producer is given by

$$Q(t) = \int_{\mathbb{R}_+} q^*(t, x) m(t, x) \, dx. \quad (4.43)$$

(a) If the renewable producer is not blockaded, the feedback production rates are

$$q^*_{nb}(t, x) = \frac{1}{4} \left( 2 - \delta - (2 \varepsilon - \delta^2) Q(t) + \delta s - 2 \partial_x v \right), \quad \tilde{q}^*(t) = \frac{1}{2} \left( 1 - \delta Q(t) - s \right),$$

and the HJB equations become

$$\partial_t v + \frac{1}{2} \sigma^2 \partial^2_{xx} v + \frac{1}{16} \left( 2 - \delta - (2 \varepsilon - \delta^2) Q(t) + \delta s - 2 \partial_x v \right)^2 = rv,$$

$$g'(t) + \frac{1}{4} \left( 1 - \delta Q(t) - s \right)^2 = rg.$$

(b) If the renewable producer is blockaded, we have $\tilde{q}^* = 0$ and

$$q^*_b(t, x) = \frac{1}{2} \left( 1 - \varepsilon Q(t) - \partial_x v \right).$$

In this case the HJB equation becomes

$$\partial_t v + \frac{1}{2} \sigma^2 \partial^2_{xx} v + \frac{1}{4} (1 - \varepsilon Q(t) - \partial_x v)^2 = rv, \quad g'(t) = rg.$$
**Forward Kolmogorov equation** Combining the two cases, we can write the forward Kolmogorov equation of the reserves density as

$$0 = \partial_t m - \frac{1}{2} \sigma^2 \partial_{xx}^2 m - \partial_x \left[m \left(1_B q_b^*(t, x) + 1_c q_{nb}^*(t, x)\right)\right],$$

(4.44)

where $1_B$ is the blockading indicator function.

### 4.4.1 Numerical solutions

To study the full MFG equation system (4.42) and (4.44), we need to solve a coupled system of forward/backward PDE system with a free boundary. We propose an iterative algorithm to calculate the MFG solution and optimal production rate. Starting with an initial guess $Q^0$ for the total production, we follow for $n = 1, 2, \ldots$

**Step 1.** Given the mean production $Q^{n-1}$ from the previous iteration, solve the optimal control problem by numerically solving the HJB equations:

(a) The optimal strategy of the renewable producer is simply

$$\hat{q}^n(t) = \frac{1}{2} \left(1 - \delta Q^{n-1}(t) - s\right)^+. $$

The renewable producer is “blockaded” whenever $1 - \delta Q(t) - s < 0$.

(b) The exhaustible producer solves the optimal control problem

$$\partial_t v^n + \frac{1}{2} \sigma^2 \partial_{xx}^2 v^n + \frac{1}{4} \left(1 - \varepsilon Q^{n-1}(t) - \partial_x v^n\right)^2 1_B$$

$$+ \frac{1}{16} \left(2 - \delta - (2\varepsilon - \delta^2)Q^n(t) + \delta s - 2\partial_x v^n\right)^2 1_c = rv^n.$$

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The feedback production strategy of the exhaustible producer is

\[ q^n(t, x) = \frac{1}{2} \left( 1 - \varepsilon Q^{n-1}(t) - \partial_x v^n \right) 1_B + \frac{1}{4} \left( 2 - \delta - (2\varepsilon - \delta^2)Q^{n-1}(t) + \delta s - 2\partial_x v^n \right) 1^c_B. \]

**Step 2.** Given the optimal production strategy \( q^n \), we can solve the forward Kolmogorov equation

\[ \partial_t m^n - \frac{1}{2} \sigma^2 \partial_{xx} m^n - \partial_x [m^n q^n] = 0. \]

This gives the mean production \( Q^n \) for the next iteration

\[ Q^n(t) = \int_{\mathbb{R}^+} q^n(t, x) m^n(t, x) \, dx. \]

As observed in Section 3.5 it is computationally convenient to consider the tail distribution function

\[ \eta(t, x) = \int_x^\infty m(t, y) \, dy. \]

The forward Kolmogorov equation in terms of the tail distribution function \( \eta \) reads

\[ \partial_t \eta(t, x) - \frac{1}{2} \sigma^2 \partial_{xx} \eta(t, x) - q(t, x) \partial_x \eta(t, x) = 0, \]

with initial condition

\[ \eta(0, x) = \int_x^\infty M(y) \, dy, \]

which allows us to handle point masses in the initial distribution \( M \).

### 4.4.2 Results and discussion

We take the deterministic setting \( \sigma = 0 \) and choose \( \varepsilon = \delta = 1 \) for the following numerical illustrations.
Initialization and convergence of algorithm  The initial guess of the mean production $Q^0$ is taken to be the explicit result (4.10) derived in the context of monopoly Cournot competition (without the renewable producer). The left panel of Figure 4.5 illustrates the initialization $Q^0$ for the numerical algorithm. From the right panel, we observe that our iterative algorithm converges rapidly, typically within 10 iterations. We also notice that the exhaustible producers slow down production in the presence of a renewable competitor.

![Graph](image)

Figure 4.5: Left panel: initialization of iterative algorithm. Right panel: convergence of iterative algorithm. The parameter are $\varepsilon = \delta = 1, r = 0.2, M \sim Beta(2, 4)$ and $s = 0.9$.

Blockading of renewable producer  If the constant marginal cost of production $s$ is high enough, the renewable producer can be blockaded and produces nothing. The idea is that when the marginal cost of production is high, and when the exhaustible resource is plentiful, it may be advantageous for the renewable producer to hold back production and wait until the exhaustible players diminish their reserves. We see numerical evidence that blockading occurs when the marginal cost of production $s = 0.9$.

4.4.3 Strategic blockading entry of renewable resources  

We now return to OPEC’s strategic decision not to curb its oil production in face of increased supply of shale gas and oil in the US. In Figure 4.7, we consider the
mean production rate $Q$ of the exhaustible producers when they are rivalled with an alternative source of different marginal costs $s$. The left panel shows the production profile $Q(t)$ over time; while the right panel plots the short term production $Q(0)$ as a function of the renewable energy cost $s$. When $s = 1$, we know that the alternative energy producer does not participate and the traditional energy producers have the entire market to themselves. However, we see that as $s$ decreases from 1, the exhaustible producers may strategically increase their mean production in the short run (and hence driving energy price down) to keep the renewable energy out of the market. Therefore, our model is capable of providing a dynamical explanation to OPEC’s decision to maintain oil production in order to compete for market share with the fracking industry in the US.

4.5 Resource Discovery

In this section, we study the stochastic effect of resource exploration in dynamic Cournot mean field game models of exhaustible resources. This model corresponds to the short horizon in Table 4.2. The exhaustible producers may invest in exploration, with effort level indicated by $a_t \geq 0$ and cost $\mathcal{C}(a_t)$. The production capacity $X_t$
Figure 4.7: Left panel: the mean production rate $Q$ for 5 different values of production costs $s = 0, 0.2, 0.4, 0.6, 0.8$. Right panel: initial production rate of the exhaustible producers. Notice the strategic blockading of entry for large $s$. Other parameters are as in Figure 4.5.

decreases at a (controlled) production rate $q_t \geq 0$, and increases through jumps thanks to discrete new discoveries. Exploration successes are represented by a point process $N_t$ with (controlled) intensity $\lambda a_t$, where $\lambda$ is given model parameter. Suppose that each discovery leads to an increase in reserves by a fixed amount $\delta > 0$, then we have the following dynamics:

$$dX_t = -q_t dt + \delta dN_t.$$ 

Similar models with resource exploration have been considered by Deshmukh and Pliska [39] for monopolies and Ludkovski and Sircar [82] for duopolies.

The cost of exploration is captured by a positive, non-decreasing function $C(\cdot)$. We will further assume that $C$ is strictly convex to guarantee that optimal effort levels are finite. The economic interpretation is based on a spatial model of the deposits of non-renewable resources (e.g. fossil fuels in different geographical regions). In the simplest case of a Poisson random measure with constant rate $\lambda$, exploration of a region $A$ yields amount $\nu(A) \sim \text{Poisson}(\lambda |A|)$. In this model, the exploration effort $a$ mimics the speed at which one sweeps through areas searching for deposits. The convex costs $C$ comes from diseconomies of scale at higher sweeping speeds.
The objective function is now the discounted lifetime revenue minus exploration cost:

\[
v(t, x) = \sup_{q,a} \mathbb{E} \left\{ \int_t^\infty e^{-r(u-t)} \left\{ q_u p_u 1_{\{X_u > 0\}} - \mathcal{C}(a_u) \right\} \, du \bigg| X_t = x \right\}. \tag{4.46}
\]

With inverse demand \( 1 - (q + \varepsilon Q) \), the HJB equation corresponding to the value function is

\[
\partial_t v + \sup_{q \geq 0} \{ q (1 - q - \varepsilon Q - \partial_x v) \} + \sup_{a \geq 0} \{ a \lambda \Delta v - \mathcal{C}(a) \} - rv = 0, \tag{4.47}
\]

where the delay term \( \Delta v(t, x) = v(t, x + \delta) - v(t, x) \) and the mean production is given by

\[
Q(t) = \int_{\mathbb{R}_+} q^*(t, x)m(t, x) \, dx.
\]

The optimal production rate and effort level are given by

\[
a^*(t, x) = (\mathcal{C}')^{-1} (\lambda \Delta v(t, x)), \quad q^*(t, x) = \frac{1}{2} (1 - \varepsilon Q(t) - \partial_x v(t, x)).
\]

Given the optimal controls, the population dynamics \( m(t, x) \) is governed by the forward Kolmogorov equation:

\[
\partial_t m(t, x) - \partial_x (q^*(t, x)m(t, x)) - \lambda \{ a^*(t, x - \delta)m(t, x - \delta) - a^*(t, x)m(t, x) \} = 0.
\tag{4.48}
\]

Note that the introduction of random jumps leads to a system of non-local PDEs.

### 4.5.1 Sustainable economy

Motivated by what Lucas and Moll [81] call a “balanced growth path”, we look for stationary solution to the above mean field game equation system. The interpretation is a sustainable energy market in which resource extraction is balanced by the
exploration successes. The stationary equations are

\[ rv(x) = \sup_{q \geq 0} \{ q (1 - q - \varepsilon Q - v'(x)) \} + \sup_{a \geq 0} \{ a \lambda \Delta v - C(a) \}, \]

\[ 0 = -\frac{d}{dx} (q^*(x)m(x)) - \lambda \{ a^*(x - \delta) m(x - \delta) - a^*(x) m(x) \}, \]

\[ a^*(x) = (C')^{-1} (\lambda \Delta v(x)), \quad q^*(x) = \frac{1}{2} (1 - \varepsilon Q - v'(x)), \]

\[ Q = \int_{\mathbb{R}_+} q^*(x)m(x) \, dx. \] (4.49)

\[ 4.5.2 \text{ Computational algorithm} \]

Following \[82\], we take power costs

\[ C(a) = \frac{1}{\beta} a^\beta + \kappa a, \quad \beta > 1, \kappa > 0. \]

Since \( C'(0) = \kappa \), a strictly positive \( \kappa \) guarantees a finite saturation point \( x_{sat} < \infty \) such that \( a^*(x) = 0 \) for \( x > x_{sat} \), and \( (X_t) \) does become arbitrarily large infinitely often. In this case, the optimal effort is given by

\[ a^*(x) = (\lambda \Delta v(x) - \kappa)^{\gamma - 1}_+, \quad \gamma = \frac{\beta}{\beta - 1}. \]

The HJB equation can be written as

\[ rv(x) = \frac{1}{4} (1 - \varepsilon Q - v'(x))^2 + \frac{1}{\gamma} (\lambda \Delta v(x) - \kappa)^\gamma_+. \]

The boundary condition \( v(0) \) is determined by optimizing the level of exploration effort \( a \) while the producer is stuck at \( x = 0 \) waiting for his first exploration success, which his waiting time exponentially distributed with mean \( (\lambda a)^{-1} \). This leads to

\[ v(0) = \sup_{a \geq 0} \mathbb{E} \left[ e^{-rt} v(\delta) - \int_0^t e^{-rt} C(a) \, dt \right] = \sup_{a \geq 0} \frac{a \lambda v(\delta) - C(a)}{\lambda a + r}. \] (4.50)

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Numerically solving for the value function \( v \) is challenging due to the implicit boundary condition and the presence of a “forward” delay term on the semi-infinite domain \( \mathbb{R}_+ \). We resolve this difficulty by using an iterative scheme. Starting with an initial guess of the value function \( v_0 \) and mean production rate \( Q_0 \), for \( n \geq 1 \) we numerically solve the following inductively.

**Value function**  We replace the forward delay term by its counterpart from the previous iteration:

\[
rv_n(x) = \frac{1}{4} (1 - \varepsilon Q_n - v'_n(x))^2 + \frac{1}{\gamma} (\lambda (v_{n-1}(x + \delta) - v_n(x)) - \kappa)_+, \tag{4.51}
\]

with boundary condition

\[
v_n(0) = \sup_a \frac{a \lambda v_{n-1}(\delta) - C(a)}{\lambda a + r}.
\]

Observe that (4.51) is a standard first-order nonlinear ordinary differential equation with “source” term \( v_{n-1}(\cdot + \delta) \) and can be solved using standard tools, such as Runge-Kutta methods.

**Density**  Given the value function \( v_n \) we can determine the optimal production rate \( q^*_n(x) \) and optimal exploration level \( a^*_n(x) \):

\[
a^*_n(x) = (\lambda \Delta v_n(x) - \kappa)_+^{-1}, \quad q^*_n(x) = \frac{1}{2} (1 - \varepsilon Q_{n-1} - v'_n(x)).
\]

The stationary solution to the forward Kolmogorov equation (4.48) is determined by

\[
0 = -\frac{d}{dx} (q^*_n(x)m_n(x)) - \lambda \{ a^*_n(x - \delta)m_{n-1}(x - \delta) - a^*_n(x)m_n(x) \}.
\]
We obtain the stationary solution by solving the time-dependent problem (4.48) and take the large time limit. By using the finite volume method we ensure that the density integrates to one. Now we integrate \( m_n \) over the optimal feedback production rate \( q_n^* \) to update the mean production \( Q_n \):

\[
Q_n = \int_{\mathbb{R}^+} q_n^*(x)m_n(x) \, dx.
\]

### 4.5.3 Numerical illustration

Figure 4.8 illustrates the numerical solution for the sustainable economy (4.49). We observe in Figure 4.8a that while the production rate \( q^* \) is monotone increasing in \( x \), the exploration level \( a^* \) is monotone decreasing. Figures 4.8b, 4.8c and 4.8d show the sample path for the evolution of the game solution over time. The system state is described by \( (X_t) \) in the top right panel which drives the feedback controls \( q^*(X_t) \) and \( a^*(X_t) \) in the bottom panels. One can readily observe that higher reserves lower exploration rates and increase production. The recurrent behavior of \( (X_t) \) is apparent, as the resource is repeatedly exhausted until a new discovery replenishes reserves and allows to restart production.

### 4.6 Conclusion

In this chapter, we apply the Cournot mean field game model to the global energy market. We focus on the interaction between traditional oil producers and alternative sources (e.g. solar, hydroelectric power, or fracking). Specifically, we investigate the issue from three perspectives: competition, transition, and exploration. This leads to three extensions of the basic Cournot MFG model.

**Transition** As the traditional oil producers run out of reserves, they can transition to energy production with alternative sources. This essentially introduce a
Neumann boundary condition to our PDE problem. In the regime of small exhaustibility, we find explicit correction to the value function and optimal production rate by solving a partial integro-differential equation.

**Competition** We find that if the alternative energy source has a high enough cost of production, the traditional energy producers may strategically increase production rate (and hence lowering the energy price ever further) to keep the alternative energy producer blockaded. This explains OPEC’s strategic decision not to reduce production quotas in the face of falling oil prices due in large part to the US fracking boom.

**Exploration** We have studied the impact of exploration and discovery in Cournot models of exhaustible resources. We characterize a sustainable energy market
as a stationary solution to the forward Kolmogorov equation. We find that higher reserves lower exploration rates and increase production.
Chapter 5

Optimal Trading with Predictable Return and Stochastic Volatility

We consider a class of dynamic portfolio optimization problems that allow for models of return predictability, transaction costs, and stochastic volatility. Determining the dynamic optimal portfolio in this general setting is almost always intractable. We propose a multiscale asymptotic expansion when the volatility process is characterized by its time scales of fluctuation. The analysis of the nonlinear Hamilton-Jacobi-Bellman PDE is a singular perturbation problem when volatility is fast mean-reverting; and it is a regular perturbation when the volatility is slowly varying. These analyses can be combined for multifactor multiscale stochastic volatility model. We present formal derivations of asymptotic approximations and demonstrate how the proposed algorithms improve our Profit&Loss using Monte Carlo simulations. This chapter is adapted from the article [30].

5.1 Introduction

Dynamic portfolio optimization provides institutional investors in active asset management a framework for determining optimal investment strategies. This central
and essential problem has a long history dating back to Mossin [95], Samuelson [105], and Merton [92, 93]. In his seminal paper, Merton [93] derives explicit solutions for the continuous-time portfolio optimization problem. In this classical setting, the stocks are modeled as geometric Brownian motions (with constant volatilities), and the objective is to maximize the expected utility of terminal wealth by allocating investment capital between risky stocks and a riskless money-market account. Under the constant relative risk aversion (CRRA) utility, Merton shows that the optimal control is a fixed mix strategy.

While this work has brought forth important structural insights, its restrictive assumptions about investor objectives and market dynamics (necessary for exact analytical solutions) have prevented more widespread applications to practical trading algorithms. Following this seminal paper, there has been a significant literature aiming to relax its assumptions and to incorporate the impact of various frictions, such as transaction costs and stochastic volatility, on the optimal portfolio choice.

A tractable alternative is to formulate the dynamic portfolio optimization problem as a linear-quadratic control. Gàrleanu and Pedersen [51] derive a closed-form optimal dynamic portfolio policy for a model with linear dynamics for return predictors, quadratic transaction costs, and quadratic penalty terms for risk. However, the explicit solution depends sensitively on the quadratic cost structure with linear dynamics. The goal of this chapter is to study the dynamic portfolio optimization problem allowing for more realistic market dynamics without sacrificing model tractability. Specifically, our model captures a number of common features of practical interest, while maintaining tractability by viewing the more flexible model as a perturbation around the well-understood constant volatility problem.

Return predictability. The usual goals of hedge fund managers and proprietary traders are to predict future security returns and trade to profit from their predictions. Such predictions are not limited to simple unconditional bullish or
bearish forecasts of future returns, but often involve predictions on short and long-term expected returns using a factor-based approach such as momentum and mean-reversion. Different factors often have different predicting strengths and mean-reversion speeds.

**Transaction costs.** Dynamic portfolio optimization often involves frequent turnover and hence significant transaction costs. Such trading costs can arise from sources ranging from the bid-offer spread or execution commissions to price impact, where the manager’s own trading affects the subsequent evolution of prices. Intuitively, the investor would like to keep his portfolio close to the “optimal” portfolio in the absence of transaction costs; however, due to transaction costs, it may not be optimal to trade all the way to the target all the time.

**Stochastic volatility.** Stochastic volatility has been recognized as an important factor of asset price modeling because it is seen as an explanation of a number of well-known empirical findings such as volatility smile and volatility clustering. The need for multifactor modeling of the volatility process is alluded to by Chacko and Viceira [27]; they observe that vastly different estimates of the mean-reversion speed of volatility can be obtained by using high and low frequency data.

In this chapter, our central innovation is to propose a framework for the dynamic trading problems allowing for many features relevant for practical trading algorithms described above. Our formulation maintains the tractability of the Gârleanu and Pedersen problem by analyzing the dynamic trading problem under stochastic volatility under the lens of multiscale asymptotics. We further demonstrate that our formulation provides explicit correction terms to the constant volatility strategy which can be efficiently computed for a large class of volatility models of practical interest;
moreover, through Monte-Carlo simulations, we show how the proposed algorithms improve the trading profit&loss.

Specifically, our dynamic portfolio optimization problem is analytically tractable. In many stochastic volatility models of practical interest (e.g. Heston, exponential Ornstein-Uhlenbeck, and the 3/2-model), the correction terms to the constant volatility strategies can be explicitly given. Moreover, the correction terms give rise to economically sensible trading strategies. We find that under fast-scale stochastic volatility, the investor should optimally deleverage his portfolio when the current volatility level is higher than the long-term average, regardless of the return-volatility correlation. On the other hand, the return-volatility correlation plays a more important role under the slow-scale stochastic volatility. When the correlation between the volatility and return factors is positive, the investor optimally decreases his trading rate as he anticipates a higher return estimate is accompanied by a higher volatility. Moreover, we demonstrate that the effect of slow-scale stochastic volatility is more significant than the fast-scale volatility in our infinite-horizon optimal trading problem. In fact, the leading order correction in the fast-scale volatility expansion vanishes identically and one has compute the second order expansion to consider the principal effect of fast-scale volatility.

5.1.1 Related literature

Our model is related to three different strands of literature: the literature of dynamic portfolio choice with return predictability and transaction costs, the modeling of price impact in algorithmic trading, and the use of asymptotic approximation in the presence of multiscale stochastic volatility.

First, we consider the literature on dynamic portfolio choice. The vast body of work begins with the seminal paper of Merton [93]. Following this paper, there has been a significant literature aiming to incorporate various frictions, such as trans-
action costs, stochastic volatility, and partial information, on the dynamic portfolio optimization problem. Transaction costs were first introduced into the Merton portfolio problem by Magill and Constantinides [87] and later further investigated by Dumas and Luciano [41]. Liu and Loewenstein [80] study the optimal trading strategy for a CRRA investor in the presence of transaction costs and obtain closed-form solutions when the finite horizon is uncertain. Bichuch and Sircar [15] analyze the problem using asymptotic approximations and find approximations to the optimal policy and the optimal long-term growth rate.

There is also significant literature on portfolio optimization that incorporates return predictability (see, e.g., Campbell and Viceira [21]). Balduzzi and Lynch [6, 86] illustrate the impact of return predictability and transaction costs on the utility costs and the optimal rebalancing rule by discretizing the state space of the dynamic program. Wachter [116] solves, in closed form, the optimal portfolio choice problem for an investor with utility over consumption under mean-reverting returns without transaction costs.

Several authors have also considered the portfolio problems under more realistic market dynamics such as stochastic interest rates or stochastic volatility. For the case of stochastic interest rates the reader is referred to Korn and Kraft [72]. Kraft [73], Boguslavskaya and Muravey [17] consider a variation of the Merton problem within the framework of the Heston model and finite time horizon. Chacko and Viceira [27] consider a similar problem with a different specification of the market price of risk and a slightly different stochastic volatility model; they also note the need for multifactor volatility model to adequately capture the persistence and variability characteristics of the volatility process that are most relevant to long-term investors. Fouque \textit{et al.} [46] build on this empirical observation and study the Merton portfolio optimization problem in the presence of multiscale stochastic volatility using asymptotic approximations.
Moreover, there is also an emerging body of literature on partial information and expert opinions. Fouque et al. [43] analyze the Merton problem when the growth rate is an unobserved Gaussian process. By applying the Kalman filter on observations of the stock price, they track the level of the growth rate and determine the optimal portfolio maximizing expected terminal utility. Frey et al. [48] investigate optimal portfolio strategies in a market where the drift is driven by an unobserved Markov chain. Information on the state of this chain is obtained from stock prices and expert opinions in the form of signals at random discrete time points. Using hidden Markov model filtering results and Malliavin calculus, Sass and Haussmann [106] numerically determine the optimal strategy under a multi-stock market model where prices satisfy a stochastic differential equation with instantaneous rates of return modeled as a continuous time Markov chain with finitely many states.

Gărleanu and Pedersen [51] achieve a closed-form solution for a model with linear dynamics for return predictors, quadratic functions for transaction costs, and quadratic penalty terms for risk. Glasserman and Xu [57] develop a linear-quadratic formulation for portfolio optimization that offers robustness to modeling errors or mis-specifications. Moallemi and Saglam [94] allow for more flexible models with trading constraints and risk considerations, but at the expense of restricting to the class of linear rebalancing policies. In similar spirit, Passerini and Vazquez [99] extend the model of Gărleanu and Pedersen [51] to include linear trading costs and using both limit and market orders. They find that the presence of linear costs induces a “no-trading” zone when using market orders, and a corresponding “market-making” zone when using limit orders. The more complex models are not analytically tractable, and Passerini and Vazquez propose a heuristic “recipe” that approximates the value function by dropping certain terms in the Hamilton-Jacobi-Bellman (HJB) equation.

Second, there is a large body of work on the modeling of price impact in algorithmic trading. The typical problem studied in this literature is the so-called “optimal
execution problem.” This arises when an investor holding a large number of shares wants to liquidate his position over a given horizon. Rapid selling of the stock may depress the stock price, while order slicing may add to the uncertainty in the sale price. This tradeoff between expected execution cost and risk is first formulated by Almgren and Chriss [4, 5] in a couple of seminal papers. Under the assumptions that execution costs are linear in the trading rate and the choice of risk criterion is the quadratic variation, Almgren and Chriss derive a closed-form analytical solution to the optimal execution problem.

The Almgren and Chriss model has then been generalized in various directions. A number of authors have investigated the optimal execution problem with respect to different risk criteria. For example, Schied and Schöneborn [107] consider the maximization of expected utility of the proceeds of an asset sale; while Gatheral and Schied [52] quantify the risk associated with a liquidation strategy as the time-averaged value-at-risk (VaR) and provide a closed-form solution to the optimal execution problem. More recently, limit order book dynamics has been incorporated into the optimal execution problem. This leads to the concept of transient price impact, that is, price impact that decays over time. Obizhaeva and Wang [98] proposed one of the first models for linear transient price impact. This model has been generalized by Gatheral et al. [54] and Alfonsi et al. [2]. For a recent survey of the market impact models used in algorithmic order execution, we refer to Gatheral and Schied [53].

Third, there is also a literature on the use of asymptotic approximation in the presence of multiscale stochastic volatility. This approximation technique has attracted considerable interest recently in derivative pricing and optimal investment problems. As detailed in the recent book of Fouque et al. [45], singular and regular perturbation methods have been developed over a number of years to provide effective approximations for the linear option pricing problems.
More recently, asymptotic analysis has been extended to simplify a number of non-linear problems. Jonsson and Sircar [68, 69] apply singular perturbation to the partial hedging problem and optimal investment problem, both for fast mean-reverting stochastic volatility. Fouque et al. [46] extend the results for the nonlinear Merton problem for general utility functions using multiscale stochastic volatility asymptotics. While the basic solution approach is similar, we stress that our work differs from these papers in several critical ways. First and foremost, transaction costs are not taken into account in the aforementioned literature; and we believe that the explicit modeling of transaction costs is crucial for a practical trading algorithm to keep turnover under control. Second, we consider a mean-variance optimization problem with an infinite trading horizon, which is more popular among industry practitioners. Finally, as opposed to the asymptotic expansion in Fouque et al. [46], we compute explicitly up to the second order correction in the fast-scale stochastic volatility.

Summary In the table below we summarize the models for dynamic trading in the literature. Type refers to continuous or discrete-time model; (g)BM stands for (geometric) Brownian motion; SV stands for stochastic volatility; pred. stands for predictability; proportional refers to proportional transaction costs.

<table>
<thead>
<tr>
<th>Model</th>
<th>Type</th>
<th>Price dynamics</th>
<th>Trading friction</th>
<th>Objective</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merton [93]</td>
<td>Continuous</td>
<td>gBM</td>
<td>None</td>
<td>Utility</td>
<td>Analytic</td>
</tr>
<tr>
<td>Liu and Loewenstein [60]</td>
<td>Continuous</td>
<td>gBM</td>
<td>proportional</td>
<td>Utility</td>
<td>Analytic</td>
</tr>
<tr>
<td>Bichuch and Sircar [12]</td>
<td>Continuous</td>
<td>gBM+SV</td>
<td>proportional</td>
<td>Utility</td>
<td>Asymptotic</td>
</tr>
<tr>
<td>Kraft [73]</td>
<td>Continuous</td>
<td>gBM+Heston</td>
<td>None</td>
<td>Utility</td>
<td>Analytic</td>
</tr>
<tr>
<td>Chacko and Viceira [24]</td>
<td>Continuous</td>
<td>gBM+3/2</td>
<td>None</td>
<td>Utility</td>
<td>Analytic</td>
</tr>
<tr>
<td>Fouque et al. [46]</td>
<td>Continuous</td>
<td>gBM+SV</td>
<td>None</td>
<td>Utility</td>
<td>Asymptotic</td>
</tr>
<tr>
<td>Găreanu and Pedersen [22]</td>
<td>Continuous/discrete</td>
<td>BM+pred.</td>
<td>Quadratic</td>
<td>Mean-variance</td>
<td>Analytic</td>
</tr>
<tr>
<td>Moallemi and Saglam [21]</td>
<td>Discrete</td>
<td>BM+pred.</td>
<td>Quadratic</td>
<td>Mean-variance</td>
<td>Linear rebalancing rules</td>
</tr>
<tr>
<td>Passerini and Vazquez [57]</td>
<td>Continuous</td>
<td>BM+pred.</td>
<td>Linear-quadratic</td>
<td>Mean-variance</td>
<td>Approximate</td>
</tr>
<tr>
<td>Our model</td>
<td>Continuous</td>
<td>BM+pred.+SV</td>
<td>Quadratic</td>
<td>Mean-variance</td>
<td>Asymptotic</td>
</tr>
</tbody>
</table>

Table 5.1: Dynamic trading models: problems, models and solution approaches.
5.1.2 Organization and Results

In Section 5.2 we introduce the Gărleanu and Pedersen [51] model in discrete time. This section serves to provide some structural intuitions of the optimal trading problem. Section 5.3 introduces the continuous-time model with multiscale stochastic volatility. We derive the HJB equation for the optimal portfolio problem and give the analytical solution in the special case of constant volatility. To keep the presentation manageable, we focus on the analysis of the two factors separately. We begin in Section 5.4 with the case of fast mean-reverting stochastic volatility, which leads to a singular perturbation problem for the associated HJB PDE. In Section 5.5 we analyze the case of slowly fluctuating volatility, which leads to a regular perturbation problem. Section 5.6 discusses how the fast and slow results can be combined for approximations under multiscale stochastic volatility. In Section 5.7 we illustrate our results with numerical examples. Section 5.8 concludes and suggests directions of extension.

5.2 Introduction in discrete time

The goal of this section is to use discrete-time dynamic programming to illustrate how transaction costs influence investment decisions. In their seminal paper, Gărleanu and Pedersen [51] examine a dynamic, transaction-cost-sensitive version of the Markowitz portfolio optimization problem [89, 88] with multiple stocks and multiple return predictors, examining how the portfolio should dynamically be adjusted as new information arrives. For expositional purposes, we will focus on the special case when there is just a single stock and a single return predictor.

Denote by $q_t$ the investor’s position in this stock at time $t$. The excess return is given by $r_{t+1} = P_{t+1} - (1 + r^f)P_t$, where $P_t$ is stock price at time $t$ and $r^f$ is the risk-free rate. We suppose that at each time $t$, the investor has an estimate of the
stock’s anticipated return \( x_t \), so that
\[
rt+1 = xt + \hat{\varepsilon}_{t+1},
\]
where \( \hat{\varepsilon} \) is white noise with mean zero and variance \( \sigma^2 \). We assume mean-reverting dynamics for \( x_t \):
\[
x_{t+1} - xt = -\varphi x_t + \varepsilon_t,
\]
where \( \varepsilon \) is an independent white noise with mean zero and variance \( \Omega \).

We assume quadratic transaction costs: a trade of size \( \Delta q \) incurs transaction costs
\[
\frac{1}{2} K (\Delta q)^2
\]
for some constant \( K > 0 \). The interpretation is that trades move the market transiently by an amount linear in the trade size \( \Delta q \).

### 5.2.1 Investor’s problem and dynamic programming

At time 0, starting with position \( q_{-1} \) of stock and return estimate \( x_0 \) for the coming period’s return, the investor chooses \( q_0 \) to maximize the discounted lifetime risk-adjusted return less transaction cost, i.e.
\[
V(q_{-1}, x_0) = \max_{q_0, q_1, \ldots} \mathbb{E} \left[ \sum_{t=0}^{\infty} (1 - \rho)^{t+1} \left( rt_{t+1} q_{t+1} - \frac{\gamma}{2} \sigma^2 q_t^2 \right) - \frac{(1 - \rho)^t}{2} (\Delta q_t)^2 K \right],
\]
where the constant \( \gamma \) is a risk aversion parameter and \( \rho \) is the discount rate. The principle of dynamic programming states that
\[
V(q_{-1}, x_0) = \max_{q_0} \left\{ -\frac{1}{2} (\Delta q_0)^2 K + (1 - \rho) \left( q_0 x_0 - \frac{\gamma}{2} \sigma^2 q_0^2 \right) + (1 - \rho) \mathbb{E} [V(q_0, x_1)] \right\}.
\]
\[(5.1)\]
This is a linear-quadratic stochastic control problem, so it is natural to use the linear-quadratic ansatz

\[ V(q, x) = -\frac{1}{2}A_{qq}q^2 + A_{qx}qx + \frac{1}{2}A_{xx}x^2 + A_0 \]  

(5.2)

for some constants \( A_{qq}, A_{qx}, A_{xx}, A_0 \). To find these constants and the optimal investment policy, we substitute the ansatz (5.2) into the dynamic programming equation (5.1). The left hand side reads

\[ -\frac{1}{2}A_{qq}q^2 - 1 + A_{qx}x_0 + \frac{1}{2}A_{xx}x_0^2 + A_0; \]

while the right hand side is a quadratic polynomial in \( q_0 \):

\[
\max_{q_0} \left\{ -\frac{1}{2}q_0^2 (K + (1 - \rho)\gamma\sigma^2 + (1 - \rho)A_{qq}) 
+ q_0 (x_0(1 - \rho)(1 - \varphi)A_{qx} + x_0(1 - \rho) + Kq_{-1}) 
+ \left( \frac{1}{2}(1 - \rho)A_{xx} (x_0^2(1 - \varphi)^2 + \Omega^2) + A_0(1 - \rho) - \frac{1}{2}Kq_{-1}^2 \right) \right\}. 
\]

Writing this as \(-\frac{1}{2}q_0^2 P + q_0 Q + R\) we see that \( q_0 = P/Q \) and the maximum is

\[ \frac{1}{2} \frac{Q^2}{P} + R, \]

and matching coefficients determines the values of \( A_{qq}, A_{qx}, A_{xx}, \) and \( A_0 \).

5.2.2 Interpretation of the optimal policy

Differentiating the dynamic programming equation (5.1) with respect to \( q_{-1} \) gives

\[ -A_{qq}q_{-1} + A_{qx}x_0 = -(q_{-1} - q_0)K. \]  

(5.3)
To interpret this relation, let $q_*$ maximize the value function for given $x_0$, i.e.

$$ q_* = \arg \max_q V(q, x_0) = \frac{A_{qq} x_0}{A_{qq}}. $$

At first one might expect $q_0 = q_*$; but this is not true due to market frictions: if $q_{-1}$ is far from $q_*$, the investor would incur large transaction costs to do that trade. Instead, by rearranging (5.3) we obtain

$$ q_0 = q_{-1} \left( 1 - \frac{A_{qq}}{K} \right) + \frac{A_{qq}}{K} q_*.$$

Although the “target amount” is $q_*$, due to transaction costs it is not optimal to trade all the way there – instead the investor go to a choice just part way between $q_{-1}$ and $q_*$. Note that one can see, using the explicit formula for $A_{qq}$, that $0 < A_{qq}/K < 1$.

### 5.3 Continuous-time model

We now return to the continuous-time model of dynamic portfolio optimization problems with return predictability, transaction costs, and stochastic volatility. For expository purposes, we will consider a single asset with price $P_t$ and a single return predictor $x_t$. The dynamics of the price is given by

$$ dP_t = \alpha_t dt + \sigma(Y_t, Z_t) dB_t, $$

where $B_t$ is a standard Brownian motion. Without loss of generality, we will decompose the drift $\alpha_t$ into a constant $\bar{\alpha}$ and intraday component $x_t$ with zero long-term mean: $\alpha_t := \bar{\alpha} + x_t$. We will model the signal $x_t$ with an Ornstein-Uhlenbeck process,

$$ dx_t = -\kappa x_t dt + \sqrt{\eta} dW_t^{(0)}. \quad (5.4) $$

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5.3.1 Multiscale stochastic volatility

We work under the multiscale stochastic volatility framework used in Fouque et al. [44, 46] for option pricing and portfolio optimization, where there is one fast volatility factor, and one slow. Here, the volatility is a function $\sigma$ of a fast factor $Y$ and a slow factor $Z$: $\sigma(Y_t, Z_t)$. The volatility-driving factors $(Y, Z)$ are described by:

$$
\begin{align*}
    dY_t &= \frac{1}{\varepsilon} b(Y_t) \, dt + \frac{1}{\sqrt{\varepsilon}} a(Y_t) \, dW_t^{(1)}, \\
    dZ_t &= \delta c(Y_t) \, dt + \sqrt{\delta} g(Y_t) \, dW_t^{(2)},
\end{align*}
$$

(5.5)

where the standard Brownian motions $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are correlated as follows:

$$
    d\langle W^{(i)}, W^{(j)} \rangle_t = \rho_{ij} \, dt, \quad i, j = 1, 2,
$$

where $|\rho_1| < 1$, $|\rho_2| < 1$, $|\rho_{12}| < 1$, and $1 + \rho_1 \rho_2 \rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, in order to ensure positive definiteness of the covariance matrix of the three Brownian motions. The model is described by the coefficients $\bar{\alpha}, \kappa, \eta, \sigma, a, b, c, g$. The parameters $\varepsilon$ and $\delta$, when small, characterize the fast and slow variation of $Y$ and $Z$ volatility factors respectively.

We assume that the process $Y_t = Y_t^{(1)}_{t/\varepsilon}$ in distribution, where $Y^{(1)}$ is an ergodic process with unique invariant distribution $\Phi$, independent of $\varepsilon$. Moreover, we assume that $Z_t = Z_t^{(1)}$ in distribution, where $Z^{(1)}$ is a diffusion process with drift and diffusion coefficients $c$ and $g$ respectively. We do not need any ergodicity assumptions on $Z^{(1)}$ for the slow scale asymptotics in the limit $\delta \downarrow 0$.  

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5.3.2 Market friction

Trading is costly in our model, and the transaction cost (TC) associated with trading 
$\text{dq}_t$ shares within a time interval $dt$ is

$$TC(u_t) = Ku_t^2, \quad dq_t = u_t \, dt.$$

where $q_t$ is his position at time $t$, so that $u_t$ is the rate of trading. The level of trans-
action cost is parameterized by some positive constant $K > 0$. The interpretation
is that is transaction price of the asset is above the unaffected price process when
$u_t > 0$; and the difference is proportional to the rate of trading.

5.3.3 Hamilton-Jacobi-Bellman equation

The investor’s objective is to choose the dynamic trading strategy $(u_t)_t$ to maximize
the present value of all future expected excess returns, penalized for risks and trading
costs,

$$\max_{u} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \left( q_s \alpha_s - \frac{\gamma}{2} \sigma(Y_s, Z_s)^2 q_s^2 - \frac{K}{2} u_s^2 \right) ds,$$

where the constant $\gamma$ is a risk aversion parameter. We define the value function

$$v(q, x, y, z) = \sup_u \mathbb{E}_{q, x, y, z} \int_0^{\infty} e^{-\rho t} \left( q_t (\bar{\alpha} + x_t) - \frac{\gamma}{2} \sigma(Y_t, Z_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) dt,$$

where we have adopted the notation

$$\mathbb{E}_{q, x, y, z}[\cdot] = \mathbb{E}[\cdot | q_0 = q, x_0 = x, Y_0 = y, Z_0 = z],$$

and the supremum is taken over admissible strategies that are $\mathcal{F}_t$-progressively mea-
surable, square-integrable (i.e., $\int_0^T u_t^2 \, dt < \infty$ a.s. for all $T > 0$), and such that (5.4)
and (5.5) has a unique strong solution on $[0, \infty)$. The usual dynamic programming
principle leads to the HJB equation

\[
0 = \sup_u \left\{ q(\bar{\alpha} + x) - \frac{\gamma}{2} \sigma^2(y, z) q^2 - \frac{1}{2} K u^2 - \rho v + w v_q - \kappa x v_x + \frac{1}{2} \eta v_{xx} \right.
\]

\[
+ \frac{1}{\varepsilon} L_0 v + \delta M_2 v + \frac{1}{\sqrt{\varepsilon}} \rho_1 \sqrt{\eta} a(y) v_{xy} + \sqrt{\delta} \rho_2 \sqrt{\eta} g(z) v_{xz} + \frac{1}{\varepsilon} \rho_{12} a(y) g(z) v_{yz} \right\},
\]

(5.6)

where \( L_0 \) and \( M_2 \) are, respectively, the infinitesimal generators of the process \( Y^{(1)} \) and \( Z^{(1)} \):

\[
L_0 = \frac{1}{2} a(y)^2 \frac{\partial^2}{\partial y^2} + b(y) \frac{\partial}{\partial y}, \quad M_2 = \frac{1}{2} g(z)^2 \frac{\partial^2}{\partial z^2} + c(z) \frac{\partial}{\partial z}.
\]

Plugging in the optimal trading rate

\[
u^* = \frac{1}{K} v_q,
\]

(5.7)

we obtain

\[
0 = q(\bar{\alpha} + x) - \frac{\gamma}{2} \sigma^2(y, z) q^2 + \frac{1}{2} K v^2 - \rho v - \kappa x v_x + \frac{1}{2} \eta v_{xx} + \frac{1}{\varepsilon} L_0 v + \delta M_2 v
\]

\[
+ \frac{1}{\sqrt{\varepsilon}} \rho_1 \sqrt{\eta} a(y) v_{xy} + \sqrt{\delta} \rho_2 \sqrt{\eta} g(z) v_{xz} + \frac{1}{\varepsilon} \rho_{12} a(y) g(z) v_{yz}.
\]

(5.8)

We note that (5.8) is a nonlinear PDE which is not easily solved either analytically or numerically. Our approach is to view this problem as a perturbation around the special case of constant volatility problem studied by Gârleanu and Pedersen [51].
5.3.4 Constant volatility solution

In the case of constant volatility $\sigma$, the value function $v$ does not depend on the volatility factors $y$ and $z$. The HJB equation simplifies to

$$0 = q(\bar{\alpha} + x) - \frac{\kappa}{2} \sigma^2 q^2 + \frac{1}{2K} v^2 - \rho v - \kappa x v_x + \frac{1}{2} \eta v_{xx}. \quad (5.9)$$

Gârleanu and Pedersen [51] provide a closed-form solution (with $\bar{\alpha} = 0$)

$$v(q, x) = -\frac{1}{2} A_{qq} q^2 + A_{qx} q x + \frac{1}{2} A_{xx} x^2 + A_0,$$

where

$$A_{qq} = \frac{K}{2} \left( \sqrt{\rho^2 + 4\gamma \sigma^2 K} - \rho \right), \quad A_{qx} = \left( \kappa + \rho + \frac{A_{qq}}{K} \right)^{-1},$$

$$A_{xx} = \frac{A_{qx}^2}{K(2\kappa + \rho)}, \quad A_0 = \frac{\eta}{2\rho} A_{xx}.$$

We will denote by $\mathcal{GP}(q, x; \sigma^2)$ the constant volatility solution. The optimal trading rate is given by

$$u^*(q, x) = \frac{1}{K} (-A_{qq} q + A_{qx} x).$$

Figure 5.1 shows the optimal trading rate $u^*$ as a function of the position $q$ and signal $x$. Figure 5.2 shows the dependance of the parameters $A_{qq}$ and $A_{qx}$ on the volatility level $\sigma$.

5.4 Fast Mean-Reverting Stochastic Volatility

We first analyze the optimal trading problem under fast mean-reverting stochastic volatility. For simplicity of exposition and without loss of generality, we will take
We have the following dynamics for a stock or index price process $P$:

$$
    dP_t = x_t \, dt + \sigma(Y_t) \, dB_t,
$$

$$
    dx_t = -\kappa x_t \, dt + \sqrt{\eta} \, dW_t^{(0)},
$$

$$
    dY_t = \frac{1}{\varepsilon} b(Y_t) \, dt + \frac{1}{\sqrt{\varepsilon}} a(Y_t) \, dW_t^{(1)},
$$

where $\bar{\alpha} = 0$ throughout.$^1$

We note that the case of nonzero $\bar{\alpha}$ can be analyzed analogously, though with more cumbersome formulas which do not shed light on the structure of the optimal trading problem.
where \( W^{(0)} \) and \( W^{(1)} \) are Brownian motions on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})\) with instantaneous correlation coefficient between volatility and stock return shocks \( \rho_1 \in (-1, 1) \).

The investor chooses his optimal trading strategy to maximize the present value of the future stream of expected excess returns, penalized for risk and trading costs:

\[
\max_{(u_t)_{t \geq 0}} \mathbb{E} \int_0^\infty e^{-\rho t} \left( q_t x_t - \frac{\gamma}{2} \sigma(Y_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) dt.
\]

We define the value function

\[
v^\varepsilon(q, x, y) = \sup_u \mathbb{E}_{q, x, y} \left\{ \int_0^\infty e^{-\rho t} \left( q_t x_t - \frac{\gamma}{2} \sigma(Y_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) dt \right\},
\]

where the supremum is taken over admissible strategies that are \( \mathcal{F}_t \)-progressively measurable. The associated HJB equation PDE for \( v^\varepsilon \) is

\[
0 = \sup_u \left\{ qx - \frac{\gamma}{2} \sigma^2(y) q^2 - \frac{1}{2} K u^2 - \rho v^\varepsilon + uv^\varepsilon - Kx v^\varepsilon_x + \frac{1}{2} \eta v^\varepsilon_{xx} + \frac{1}{\varepsilon} \mathcal{L}_0 v^\varepsilon + \rho_1 \sqrt{\eta} a(y) v^\varepsilon_{xy} \right\}.
\]

Maximizing the quadratic expression in \( u \), the optimal trading rate is given in feedback form by

\[
u^* = \frac{1}{K} v^\varepsilon_q.
\]

Inserting the maximizer \( u^* \) into (5.11) leads to the HJB equation

\[
0 = qx - \frac{\gamma}{2} \sigma^2(y) q^2 + \frac{1}{2K} \left( v^\varepsilon_q \right)^2 - \rho v^\varepsilon - Kx v^\varepsilon_x + \frac{1}{2} \eta v^\varepsilon_{xx} + \frac{1}{\varepsilon} \mathcal{L}_0 v^\varepsilon + \rho_1 \sqrt{\eta} a(y) v^\varepsilon_{xy}.
\]

Analytically or numerically solving the nonlinear PDE (5.12) is a difficult problem. In the limit \( \varepsilon \downarrow 0 \), it is a singular perturbation problem, and our approach is to construct an asymptotic approximation of the solution.
5.4.1 Expansion of the value function

We look for an asymptotic expansion of the value function of the form

\[ v^\varepsilon(q, x, y) = v^{(0)}(q, x, y) + \sqrt{\varepsilon}v^{(1)}(q, x, y) + \varepsilon v^{(2)}(q, x, y) + \varepsilon^{3/2}v^{(3)}(q, x, y) \cdots . \]

Inserting this expansion into (5.8), and collecting terms in successive powers of \( \varepsilon \), we obtain at the highest order \( \varepsilon^{-1} \):

\[ \mathcal{L}_0 v^{(0)} = 0. \]

Since \( \mathcal{L}_0 \) takes derivatives in \( y \), this equation is satisfied with \( v^{(0)}(q, x) \) independent of \( y \). With this choice, we have \( v^{(0)}_y = 0 \), so expanding the nonlinear term in (5.12) up to order \( \varepsilon \) gives:

\[ \left( v^\varepsilon_q \right)^2 = \left( v^{(0)}_q \right)^2 + 2\sqrt{\varepsilon}v^{(0)}_q v^{(1)}_q + \varepsilon \left( (v^{(1)}_q)^2 + 2v^{(0)}_q v^{(2)}_q \right) + \cdots . \]

Therefore, at the next order \( \varepsilon^{-1/2} \) in the expansion of the PDE, there is no contribution from the nonlinear term, and we have

\[ \mathcal{L}_0 v^{(1)} + \rho_1 \sqrt{\eta a(y)} v^{(0)}_{xy} = 0. \]

With our choice \( v^{(0)}(q, x) \), we obtain simply \( \mathcal{L}_0 v^{(1)} = 0 \). Again, we satisfy this equation with \( v^{(1)} = v^{(1)}(q, x) \), independent of \( y \).

Then, collecting the order one terms leads to:

\[ \mathcal{L}_0 v^{(2)} + qx - \gamma^2 (y) q^2 + \frac{1}{2K} \left( v^{(0)}_q \right)^2 - \rho v^{(0)} - \kappa x v^{(0)} + \frac{1}{2} \eta v^{(0)}_{xx} + \rho_1 \sqrt{\eta a(y)} v^{(1)}_{xy} = 0. \] (5.13)
5.4.2 Zeroth order term $v^{(0)}$

Equation (5.13) is a Poisson equation for $v^{(2)}$ whose solvability condition (Fredholm Alternative) requires that

$$
\left\langle qx - \frac{\gamma}{2} \sigma^2(y) q^2 + \frac{1}{2K} (v^{(0)}_q)^2 - \rho v^{(0)} - \kappa x v^{(0)}_x + \frac{1}{2} \eta v^{(0)}_{xx} \right\rangle = 0,
$$

where $\langle \cdot \rangle$ is defined by the unique invariant distribution $\Phi$ of the ergodic process $Y^{(1)}$:

$$
\langle g \rangle = \int g(y) \Phi(dy),
$$

for any smooth function $g$. Using that $v^{(0)}(q, x)$ is independent of $y$, the solvability condition simplifies to

$$
qx - \frac{\gamma}{2} (\sigma^2(y) - \langle \sigma^2 \rangle) q^2 + \frac{1}{2K} (v^{(0)}_q)^2 - \rho v^{(0)} - \kappa x v^{(0)}_x + \frac{1}{2} \eta v^{(0)}_{xx} = 0. \quad (5.14)
$$

Notice that (5.14) is the nonlinear PDE (5.9) for the optimal portfolio problem with constant volatility $\sqrt{\langle \sigma^2 \rangle}$, and so,

$$
v^{(0)}(q, x) = \mathcal{G} \mathcal{P}(q, x; \langle \sigma^2 \rangle).
$$

5.4.3 First order term $v^{(1)}$

Combining Equations (5.13) and (5.14), we can write

$$
\mathcal{L}_0 v^{(2)} = g,
$$

where $g(y) = \frac{\gamma}{2} (\sigma^2(y) - \langle \sigma^2 \rangle) q^2$. The solution of the Poisson equation (5.16) can be expressed as

$$
v^{(2)} = -\int_0^\infty \mathbf{P}_t g(y) \, dt + C(q, x),
$$

(5.17)
where \( C(q, x) \) is some “constant” of integration in \( y \), and the transition semigroup \( P_t \) is defined by its action on bounded measurable functions \( g \):

\[
P_t g(y) = \mathbb{E} \left\{ g(Y_t^{(1)}) \middle| Y_0^{(1)} = y \right\}.
\]

See, for instance, Fouque et al. [45, Section 3.2].

Continuing, at order \( \sqrt{\epsilon} \) in the expansion of the PDE, we have

\[
\frac{1}{K} v^{(0)}_q v^{(1)}_q - \rho v^{(1)} - \kappa x v^{(1)}_x + \frac{1}{2} \eta v^{(1)}_{xx} + \mathcal{L}_0 v^{(3)} + \rho_1 \sqrt{\eta} a(y) v^{(2)}_{xy} = 0.
\] (5.18)

Equation (5.18) is a Poisson equation for \( v^{(3)} \) whose solvability condition is

\[
\frac{1}{K} v^{(0)}_q v^{(1)}_q - \rho v^{(1)} - \kappa x v^{(1)}_x + \frac{1}{2} \eta v^{(1)}_{xx} = 0.
\] (5.19)

Observe that (5.19) is a linear homogeneous PDE for \( v^{(1)} \), we can choose \( v^{(1)} = 0 \) identically. With this choice, we have \( \mathcal{L}_0 v^{(3)} = 0 \). Again, we satisfy this equation with \( v^{(3)} = v^{(3)}(q, x) \), independent of \( y \).

### 5.4.4 Second order term \( v^{(2)} \)

At order \( \epsilon \) in the expansion of the PDE, we have

\[
\frac{1}{2K} \left( v^{(1)}_q \right)^2 + \frac{1}{K} v^{(0)}_q v^{(2)} - \rho v^{(2)} - \kappa x v^{(2)}_x + \frac{1}{2} \eta v^{(2)}_{xx} + \mathcal{L}_0 v^{(4)} + \rho_1 \sqrt{\eta} a(y) v^{(3)}_{xy} = 0.
\]

The solvability condition gives

\[
0 = \frac{1}{K} v^{(0)}_q C_q - \rho C - \kappa x C_x + \frac{1}{2} \eta C_{xx}.
\]

This is a linear equation without source term, we can choose \( C \) to be identically zero. Therefore, from (5.17) we see that the leading order correction to the value function
is given by

\[
v^{(2)}(q, y) = -\frac{\gamma}{2} q^2 \int_0^\infty \mathbf{P}_t \left( \sigma^2(y) - \langle \sigma^2 \rangle \right) dt =: -\frac{1}{2} q^2 \varphi(y).
\]

(5.20)

**Example 1.** Suppose that \( \sigma^2(y) = y \) and the volatility factor \( Y_t^{(1)} \) is the Cox-Ingersoll-Ross \([37]\) process, that is

\[
b(y) = \theta (\mu - y), \quad a(y) = \tilde{\sigma} \sqrt{y}.
\]

Applying the transition semigroup \( \mathbf{P}_t \) on the function \( g \) gives

\[
\mathbf{P}_t g(y) = \mathbb{E} \left[ g(Y_t^{(1)}) \bigg| Y_0^{(1)} = y \right]
= \frac{\gamma}{2} \mathbb{E} \left[ Y_t^{(1)} - \mu \bigg| Y_0^{(1)} = y \right] q^2
= \frac{\gamma}{2} e^{-\theta t} (y - \mu) q^2.
\]

Then Equation (5.17) immediately gives

\[
v^{(2)} = -\frac{\gamma}{2\theta} (y - \mu) q^2.
\]

One can readily check that the above does indeed solve the HJB equation (5.16).

**Example 2.** As an alternative example, let us consider the exponential Ornstein-Uhlenbeck stochastic volatility model \([90]\). In this case we have \( \sigma(y) = me^y \) and the volatility factor \( Y_t^{(1)} \) is the Ornstein-Uhlenbeck process, that is

\[
b(y) = -\theta y, \quad a(y) = \tilde{\sigma}.
\]
Applying the transition semigroup $P_t$ on the function $g$ gives

$$P_t g(y) = \mathbb{E} \left[ g(Y_t^{(1)}) \mid Y_0^{(1)} = y \right] = \frac{\gamma}{2} q^2 m^2 e^{k^2/\alpha} \left( e^{2ye^{-at} - \frac{k^2}{\alpha} e^{-2at}} - 1 \right).$$

There does not appear to be a closed-form expression for the time-integral of the function $P_t g(y)$, but we can compute explicitly the leading order correction when the volatility factor fluctuates around its mean level

$$v^{(2)} = -\frac{\gamma}{2} q^2 m^2 e^{k^2/\alpha} \int_0^\infty \left( e^{2ye^{-at} - \frac{k^2}{\alpha} e^{-2at}} - 1 \right) dt$$

$$\approx -\frac{\gamma}{4\alpha} q^2 m^2 e^{k^2/\alpha} \left( 4y\Gamma \left( \frac{1}{2}, \frac{k}{\alpha^2} \right) - \left[ \hat{\gamma} + \Gamma \left( 0, \frac{k^2}{\alpha} \right) + \log \left( \frac{k^2}{\alpha} \right) \right] \right),$$

where $\hat{\gamma} \approx 0.5772$ is the Euler-Mascheroni constant and $\Gamma$ is the incomplete gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt.$$  

### 5.4.5 Optimal Portfolio

We now analyze and interpret how the principal expansion terms $v^{(0)}$ and $v^{(2)}$ for the value function can be used in the expression for the optimal portfolio $u^*$ in (5.7), which leads to an approximate feedback policy of the form

$$u^*(q, x, y) = u^{(0)}(q, x, y) + \varepsilon u^{(1)}(q, x, y) + \cdots.$$

The zeroth order trading rate is independent of $y$:

$$u^{(0)}(q, x) = \frac{1}{K} (A_{qx}x - A_{qq}q).$$
This is simply the Gârleanu and Pedersen [51] constant volatility strategy evaluated at the long-term variance \( \langle \sigma^2 \rangle \).

Differentiating (5.20) gives the principal correction to optimal trading rate:

\[
    u^{(2)}(q, y) = -\frac{q}{K} \varphi(y).
\]

In the case where the volatility factor is driven by a Cox-Ingersoll-Ross process (see Example 1), the expression simplifies to

\[
    u^{(2)}(q, y) = -\frac{\gamma}{\theta K} q (y - \mu). \tag{5.21}
\]

Notice that \( u^{(2)} \) and \( q \) have opposite signs when \( \sigma^2(y) > \langle \sigma^2 \rangle \). The economic interpretation is that the investor should optimally deleverage his portfolio when the current volatility level is higher than the long-term average.

### 5.5 Slow scale volatility asymptotics

We now perform an asymptotic analysis under the assumption that stochastic volatility is slowly fluctuating. The model reads

\[
    dP_t = x_t \, dt + \sigma(Z_t) \, dB_t,
\]

\[
    dx_t = -\kappa x_t \, dt + \sqrt{\eta} \, dW^{(0)}_t,
\]

\[
    dZ_t = \delta c(Z_t) \, dt + \sqrt{\delta g(Z_t)} \, dW^{(2)}_t,
\]

where \( W^{(0)} \) and \( W^{(2)} \) are Brownian motions with instantaneous correlation coefficient between volatility and stock return shocks \( \rho_2 \in (-1, 1) \), and \( \delta \) is the small time-scale parameter for expansion. The HJB equation for the value function

\[
    v^\delta(q, x, z) = \sup_u \mathbb{E}_{q, x, z} \left\{ \int_0^\infty e^{-\rho t} \left( q_t x_t - \frac{\gamma}{2} \sigma(Z_t)^2 q_t^2 - \frac{K}{2} u_t^2 \right) \, dt \right\},
\]

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is

\[
0 = \sup_u \left\{ qx - \frac{\gamma}{2} \sigma^2(z) q^2 - \frac{1}{2} Ku^2 - \rho v^\delta + uv^\delta - \kappa xv^\delta + \frac{1}{2} \eta v^\delta_{xx} + \delta M_2 v^\delta + \sqrt{\delta \rho_2} \sqrt{\eta g(z)} v^\delta_{xz} \right\}.
\]

(5.22)

The optimal trading rate is

\[ u^* = \frac{1}{K} v_q. \]

Plugging this into the HJB equation (5.22) yields

\[
0 = qx - \frac{\gamma}{2} \sigma^2(z) q^2 + \frac{1}{2K} \left( v^\delta_q \right)^2 - \rho v^\delta - \kappa xv^\delta + \frac{1}{2} \eta v^\delta_{xx} + \delta M_2 v^\delta + \sqrt{\delta \rho_2} \sqrt{\eta g(z)} v^\delta_{xz},
\]

5.5.1 Slow scale expansion

We look for expansion of the value function of the form

\[
v^\delta(q, x, y) = v^{(0)}(q, x, z) + \sqrt{\delta} v^{(1)}(q, x, z) + \delta v^{(2)}(q, x, z) + \delta^{3/2} v^{(3)}(q, x, z) + \cdots. \quad (5.23)
\]

Then it follows by setting \( \delta = 0 \) that \( v^{(0)} \) solves

\[
0 = qx - \frac{\gamma}{2} \sigma^2(z) q^2 + \frac{1}{2K} \left( v^{(0)}_q \right)^2 - \rho v^{(0)} - \kappa xv^{(0)} + \frac{1}{2} \eta v^{(0)}_{xx}.
\]

Therefore, the principal term is the Gârleanu and Pedersen value function explicitly given by \( G\mathcal{P}(q, x, \sigma^2(z)) \). As with the fast factor zeroth order approximation to the value function given in (5.15), the zeroth order approximation in the slow factor model is the constant volatility value function, but with \( \sigma^2(z) \), the current volatility, instead of the averaged quantity \( \langle \sigma^2 \rangle \).
5.5.2 Slow scale value function correction

Taking the order $\sqrt{\delta}$ term after inserting the expansion (5.23) into the PDE (5.22) leads to

$$0 = \frac{1}{K} v_q^{(0)} v_q^{(1)} - \rho v^{(1)} - \kappa x v_x^{(1)} + \frac{1}{2} \eta v_{xx}^{(1)} + \rho_2 \sqrt{\eta g(z)} v_x^{(0)}.$$  (5.24)

With a linear ansatz

$$v^{(1)}(q, x, z) = B_q(z) q + B_x(z) x,$$

we can write down the solution to the HJB equation

$$B_q(z) = \frac{\rho_2 \sqrt{\eta g(z)} A_{qx}(z)}{\rho + A_{qq}(z)/K}, \quad B_x(z) = \frac{\rho_2 \sqrt{\eta g(z)} A_{xx}(z) + A_{qx}(z) B_q(z)/K}{\rho + \kappa},$$  (5.25)

where $A_{qx}(z)$ and $A_{xx}(z)$ are the corresponding coefficients in $\mathcal{GP}(q, x, \sigma^2(z))$.

5.5.3 Optimal trading strategy

The optimal trading strategy in feedback form is given by

$$u^*(q, x, z) = \frac{1}{K} (A_{qx} x - A_{qq} q) + \frac{1}{K} \sqrt{\delta} B_q(z).$$

It follows from a straightforward calculation that $B_q$ and $\rho_2$ has opposite signs, provided that the function $\sigma$ is monotonically increasing in the volatility factor $Z$.

When the correlation $\rho_2$ between the volatility and return factors is positive, the investor optimally decrease his trading rate as he anticipates a higher return estimate is accompanied by a higher volatility. Conversely, if the return-volatility correlation $\rho_2$ is negative, the investor optimally increases his trading rate $u$. 

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5.6 Multiscale stochastic volatility

We return to the two-factor multiscale volatility model (5.5), introduced in Section 5.3, where there is one fast volatility, and one slow. We show that the separate fast and slow expansions to first order essentially combine, but with slight modification of the averaged parameters involved.

Under our simplifying assumption of zero expected stock return $\bar{\alpha}$, the stock price process follows

$$dP_t = x_t \, dt + \sigma(Y_t, Z_t) \, dB_t,$$

$$dx_t = -\kappa x_t \, dt + \sqrt{\eta} \, dW_t^{(0)},$$

$$dY_t = \frac{1}{\varepsilon} b(Y_t) \, dt + \frac{1}{\sqrt{\varepsilon}} a(Y_t) \, dW_t^{(1)},$$

$$dZ_t = \delta c(Z_t) \, dt + \sqrt{\delta g} \, dW_t^{(2)}.$$

The value function

$$v^{\varepsilon,\delta}(q, x, y, z) = \sup_u \mathbb{E}_{q,x,y,z} \left\{ \int_0^\infty e^{-\rho t} \left( q x_t - \frac{\gamma}{2} \sigma(Y_t, Z_t)^2 q_t^2 \right) \, dt \right\},$$

has the associated HJB equation

$$0 = qx - \frac{\gamma}{2} \sigma^2(y, z) q^2 + \frac{1}{2K} (v_q^{\varepsilon,\delta})^2 - \rho v^{\varepsilon,\delta} - \kappa x v_x^{\varepsilon,\delta} + \frac{1}{2} \eta v_{xx}^{\varepsilon,\delta} + \frac{1}{\varepsilon} \mathcal{L}_0 v^{\varepsilon,\delta}$$

$$+ \delta \mathcal{M}_2 v^{\varepsilon,\delta} + \rho_1 \sqrt{\frac{\eta}{\varepsilon}} a(y) v_{xy}^{\varepsilon,\delta} + \sqrt{\delta} \rho_2 \sqrt{\eta g(z)} v_{xz}^{\varepsilon,\delta} + \rho_{12} \sqrt{\frac{\delta}{\varepsilon}} a(y) g(z) v_{yz}^{\varepsilon,\delta}. \quad (5.26)$$

The optimal strategy in feedback form is given by

$$u^*(q, x, y, z) = \frac{1}{K} v_q^{\varepsilon,\delta}. \quad (5.27)$$
5.6.1 Combined expansion in slow and fast scales

For expositional purposes, we focus on the leading order corrections to the value function from the fast and slow scale volatilities. Appendix C.1 presents the full second order asymptotic expansion to the multiscale stochastic volatility problem. First we construct an expansion in powers of $\sqrt{\delta}$:

$$v_{\varepsilon, \delta} = v_{\varepsilon, 0} + \sqrt{\delta} v_{\varepsilon, 1} + \delta v_{\varepsilon, 2} + \cdots,$$

(5.28)

so that $v_{\varepsilon, 0}$ is obtained by setting $\delta = 0$ in the equation for $v_{\varepsilon, \delta}$:

$$0 = qx - \frac{\gamma}{2} \sigma^2(y, z) q^2 - \frac{1}{2K} \left( v_{\varepsilon, 0}^q \right)^2 - \rho v_{\varepsilon, 0} + uv_{\varepsilon, 0} - \kappa x v_{\varepsilon, 0}^x + \frac{1}{2} \eta v_{\varepsilon, 0}^{xx} + \frac{1}{\varepsilon} \mathcal{L}_0 v_{\varepsilon, 0} + \rho \sqrt{\frac{\eta}{\varepsilon}} a(y) v_{\varepsilon, 0}^{xy}.$$

(5.29)

This is the same HJB problem (5.12) as for the value function $v^\varepsilon$ except that the volatility depends on the current level $z$ of the slow volatility factor, which enters as a parameter in the PDE (5.29). It is clear that when we construct an expansion for $v_{\varepsilon, 0}$ in powers of $\sqrt{\varepsilon}$:

$$v_{\varepsilon, 0} = v^{(0)} + \sqrt{\varepsilon} v^{(1, 0)} + \varepsilon v^{(2, 0)} + \cdots,$$

we will obtain, as in Section 5.4, that $v^{(0)}$ is the Gârleanu and Pedersen value function with constant volatility $\bar{\sigma}^2(z)$:

$$v^{(0)}(q, x, z) = \mathcal{GP}(q, x; \bar{\sigma}^2(z)),$$

(5.30)

where $\bar{\sigma}^2(z) = \langle \sigma^2(\cdot, z) \rangle$. That is, the variance is squared-averaged over the fast factor with respect to its invariant distribution, and evaluated at the current level of slow factor.
Following Section 5.4, the correction term \( v^{(1,0)} \) is identically zero and \( v^{(2,0)} \) is given by

\[
v^{(2,0)} = -\frac{\gamma}{2} q^2 \int_0^\infty P_t (\sigma^2(y, z) - \bar{\sigma}^2(z)) \, dt =: -\frac{1}{2} q^2 \varphi(y, z).
\] (5.31)

Next we return to the slow scale expansion (5.28) and extract the order \( \sqrt{\delta} \) terms in (5.26) to obtain the following equation for \( v_{\varepsilon,1} \):

\[
0 = \frac{1}{K} v^{(1,0)} + \eta v^{(2,1)} + \frac{1}{2} \varepsilon v^{(2,1)} + \rho_1 \sqrt{\eta} \varphi(x, z) v^{(2,1)} + \rho_2 \sqrt{\eta} g(z) v^{(2,0)} + \rho_{12} a(y) g(z) v^{(2,0)}.
\] (5.32)

We look for an expansion

\[
v^{(1)} = v^{(0,1)} + \sqrt{\varepsilon} v^{(1,1)} + \varepsilon v^{(2,1)} + \cdots,
\] (5.33)

where we are only interested here in the first term which will give the principal slow scale correction to the value function.

The order \( \varepsilon^{-1} \) terms in (5.32) give \( \mathcal{L}_0 v^{(0,1)} = 0 \) and we take \( v^{(0,1)} = v^{(0,1)}(q, x, z) \) independent of \( y \). At order \( \varepsilon^{-1/2} \), we have \( \mathcal{L}_0 v^{(1,1)} = 0 \) and so again \( v^{(1,1)} = v^{(1,1)}(q, x, z) \).

At order one:

\[
0 = \frac{1}{K} v^{(0)} v^{(1,0)} + \rho v^{(0,1)} - \kappa x v^{(1,0)} + \frac{1}{2} \eta v^{(2,1)} + \mathcal{L}_0 v^{(2,1)}
\]

\[
+ \rho_1 \sqrt{\eta} a(y) v^{(1,1)} + \rho_2 \sqrt{\eta} g(z) v^{(0)} + \rho_{12} a(y) g(z) v^{(1,0)}.
\] (5.34)

Viewed as a Poisson equation for \( v^{(2,1)} \), this yields the following solvability condition for \( v^{(0,1)} \):

\[
0 = \frac{1}{K} v^{(0)} v^{(1,0)} + \rho v^{(0,1)} - \kappa x v^{(0,1)} + \frac{1}{2} \eta v^{(0,1)} + \rho_2 \sqrt{\eta} g(z) v^{(0)}.
\]

This is the same PDE problem (5.24) as for the slow scale correction in Section 5.5, except with \( \sigma(z) \) replaced by \( \bar{\sigma}(z) \). We conclude that \( v^{(0,1)}(q, x, z) = B_q(z) q + B_x(z) x, \)
with

\[ B_q(z) = \frac{\rho_2\sqrt{\eta g(z)}A'_{qx}(z)}{\rho + A_{qq}(z)/K}, \quad B_x(z) = \frac{\rho_2\sqrt{\eta g(z)}A'_{xx}(z) + A_{qx}(z)B_q(z)/K}{\rho + \kappa}, \tag{5.35} \]

where \( A_{qx}(z) \) and \( A_{xx}(z) \) are the corresponding coefficients in \( \mathcal{GP}(q, x, \bar{\sigma}^2(z)) \).

In summary, the leading-order multiscale correction is given by

\[ v^{\varepsilon,\delta}(q, x, y, z) = \mathcal{GP}(q, x; \bar{\sigma}^2(z)) - \frac{\varepsilon}{2} q^2 \varphi(y, z) + \sqrt{\delta} (B_q(z)q + B_x(z)x) + \cdots. \tag{5.36} \]

In Appendix C.1 we present the full second order asymptotic expansion; the results are conveniently summarized in Table 5.2.

**Table 5.2: Summary of the full second order asymptotic expansion to the multiscale stochastic volatility problem** \( (5.26) \).

<table>
<thead>
<tr>
<th></th>
<th>( \mathcal{O}(1) )</th>
<th>( \mathcal{O}(\sqrt{\varepsilon}) )</th>
<th>( \mathcal{O}(\varepsilon) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{O}(1) )</td>
<td>( \mathcal{GP}(q, x; \bar{\sigma}^2(z)) )</td>
<td>0</td>
<td>( v^{(2,0)} = (5.31) )</td>
</tr>
<tr>
<td>( \mathcal{O}(\sqrt{\delta}) )</td>
<td>( v^{(0,1)} = (5.35) )</td>
<td>( v^{(1,1)} = (C.2) )</td>
<td></td>
</tr>
<tr>
<td>( \mathcal{O}(\delta) )</td>
<td>( v^{(0,2)} = (C.5) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 5.6.2 Multiscale optimal portfolio

The optimal portfolio up to orders \( \varepsilon \) and \( \sqrt{\delta} \) for the multiscale model is obtained by inserting the value function approximation \( (5.36) \) into the optimal strategy feedback function \( (5.27) \), which leads to

\[ u^{\varepsilon,\delta} = \frac{1}{K} (A_{qx}(z)x - A_{qq}(z)q) - \frac{\varepsilon}{K} q\varphi(y, z) + \frac{\sqrt{\delta}}{K} B_q(z) + \cdots, \tag{5.37} \]
where

\[ A_{qq}(z) = \frac{K}{2} \left( \sqrt{\rho^2 + 4\gamma(z)^2} - \rho \right), \quad A_{qx}(z) = \left( \kappa + \rho + \frac{A_{qq}(z)}{K} \right)^{-1}, \]

and

\[ B_q(z) = \frac{\rho^2 \sqrt{\eta g(z)} A'_{qx}(z)}{\rho + A_{qq}(z)/K}. \]

The formula (5.37) for the approximate optimal trading rate up to order \( \epsilon \) and \( \sqrt{\delta} \) highlights the contribution from the volatility factor-returns correlations. The principal (zero order) strategy

\[ u^{(0)}(q, x, z) = \frac{1}{K} (A_{qx}(z)x - A_{qq}(z)q) \]

is a moving Gârleanu and Pedersen strategy with respect to the slow factor \( Z \), as in the slow-only case (Section 5.5).

## 5.7 Examples & numerical solutions

We present numerical examples to demonstrate that the asymptotic approximation can be computed efficiently under a wide variety of models of practical interest. Then we demonstrate how the proposed algorithms improve our Profit&Loss using Monte-Carlo simulations.

### 5.7.1 Heston stochastic volatility model

Kraft [73] considered the one-factor stochastic volatility model in which the volatility factor \( Z_t \) is a CIR process:

\[ \sigma(z) = z^{1/2}, \quad c(z) = m - z, \quad g(z) = \beta \sqrt{z}, \]
Figure 5.3: Value functions in the slow scale volatility model \((5.38)\) for a range of \(\delta\), for three different values of the volatility factor. Parameters used are \(\rho = 0.2, \gamma = 1, m = 0.5, \beta = 0.25, K = 1, \rho_2 = 0.5, \eta = 0.5, \kappa = 1\).

that is

\[
\begin{align*}
\frac{dP_t}{P_t} &= x_t\,dt + \sqrt{Z_t}\,dB_t, \\
\frac{dx_t}{x_t} &= -\kappa x_t\,dt + \sqrt{\eta}\,dW_t^{(0)}, \\
\frac{dZ_t}{Z_t} &= \delta(m - Z_t)\,dt + \sqrt{\delta\beta}\sqrt{Z_t}\,dW_t^{(2)}.
\end{align*}
\]

\[(5.38)\]

We assume the standard Feller condition \(\beta^2 < 2m\), which we note does not involve the time scale parameter \(\delta\).

In Figure 5.3, we show the value function over a range of the time scale parameter \(\delta\), for three different values of the volatility factor. The leading-order correction \(\nu^{(1)}\) to the value function is proportional to \(\sqrt{\delta}\) and \(\rho_2\). When the correlation between the slow volatility and stock return shocks is positive, the principal impact of stochastic volatility is lowering of the value function. Intuitively, when the stock return \(x_t\) goes up, the optimal stock holding \(q_t\) also goes up; but in the case of positive correlation \(\rho_2\), this is also followed by an increase in uncertainty, causing the investor to be more conservative and reduce leverage. Figure 5.4 shows the principal effect of a slow-scale stochastic volatility on the optimal trading rate \(u\).
Figure 5.4: Optimal trading rate $u$ in the slow scale volatility model (5.38). Parameters used are as in Figure 5.3 and $\delta = 0.5$.

5.7.2 Chacko and Viceira model

As another example, we illustrate our approximation with a model considered in Chacko and Viceira [27]:

$$
\sigma(z) = z^{-1/2}, \quad c(z) = m - z, \quad g(z) = \beta \sqrt{z},
$$

that is

$$
dP_t = x_t \, dt + \sqrt{\frac{1}{Z_t}} \, dB_t, \\
dx_t = -\kappa x_t \, dt + \sqrt{\eta} \, dW_t^{(0)}, \\
dZ_t = \frac{1}{\varepsilon} (m - Z_t) \, dt + \frac{1}{\sqrt{\varepsilon}} \beta \sqrt{Z_t} \, dW_t^{(1)}. 
$$

Equation (5.20) applied to the current setting gives the leading order correction to the optimal trading rate in the presence of fast-scale stochastic volatility. Chou and Lin [33] show that the probability transition density of the CIR process is

$$
p_t(x, y) = \frac{2}{\beta^2 (1 - e^{-\ell t})} \exp \left[ \frac{2 \left( x + ye^{\ell t} \right)}{\beta^2 (1 - e^{\ell t})} \right] \left( \frac{ye^{\ell t}}{x} \right)^{\nu/2} I_{\nu} \left( -\frac{4\sqrt{xye^{\ell t}}}{\beta^2 (1 - e^{\ell t})} \right), \quad \nu = \frac{2m}{\beta^2} - 1,
$$
where $I_\nu$ is the modified Bessel function of the first kind of index $\nu$. With this we can compute the expected variance at time $t$

\[
\mathbb{P}_t \sigma^2(y) = \frac{\zeta_t e^{-y \mu_t}}{q} \, _1F_1(q, 1 + q, y \mu_t), \quad \langle \sigma^2 \rangle = \frac{2}{2m - \beta^2},
\]

where

\[
\zeta_t = \frac{2}{\beta^2 (1 - e^{-t})}, \quad \mu_t = \zeta_t e^{-t}, \quad q = \frac{2m}{\beta^2} - 1,
\]

and $_1F_1(\cdot, \cdot, \cdot)$ is the Kummer confluent hypergeometric function. The correction to the optimal control under the fast-scale Chacko and Viceira volatility process can be computed with a single numerical integral:

\[
u^{(2)}(q, y) = -\frac{q}{K} \varphi(y) = -\frac{\gamma}{K} q \int_0^\infty \mathbb{P}_t \left( \sigma^2(y) - \langle \sigma^2 \rangle \right) dt.
\]

As shown in Figure 5.5, the correction term $u^{(2)}$ is nonlinear in $y$, in contrast to the Heston model in Example 1.
5.7.3 Monte Carlo simulation

We have tested our trading strategy using a Monte Carlo simulation under the fast and slow-scale optimal trading algorithm described in Section 5.4. We simulate the Heston stochastic volatility model of Example 1 using the 3-stage Rossler Stochastic Runge-Kutta scheme. Figure 5.6 demonstrates the gain in P&L using the optimal trading strategy (5.21) over the constant volatility Gârleanu and Pedersen strategy.

For the fast-scale stochastic volatility, we compare our proposed strategy (5.21) with the zeroth order Gârleanu and Pedersen constant volatility strategy. With parameters \( \rho = 0.2, \gamma = 5, m = 0.5, \beta = \sqrt{0.5}, K = 1, \rho_1 = 0.5, \eta = 0.5, \kappa = 1, \epsilon = 0.25, \) we record a gain in P&L of 23.91 bps. See Figure 5.6 for the distribution of the difference in P&L between the proposed strategy (5.21) and the Gârleanu and Pedersen strategy.

In the case of slow-scale stochastic volatility, we compare the proposed leading-order correction to the optimal trading rate with the Gârleanu and Pedersen [51] strategy with the current volatility \( \sigma(Z_t) \). The gain in P&L depends on the initial value of volatility factor \( Z_0 \). Table 5.7.3 demonstrates the gain in P&L of the proposed trading strategies under the slow-scale Heston stochastic volatility model for different starting value of the volatility factor \( Z_0 \). We note that the proposed algorithm provides an improvement on the trading P&L for all starting values \( Z_0 \); while the gain in P&L is most significant when the initial volatility factor \( Z_0 \) is below its long-term level \( m \).

5.8 Conclusion

The impact of stochastic volatility on the problem of dynamic trading can be studied and quantified through asymptotic approximation, which are tractable to compute. We have derived the first two terms of the approximations for the Gârleanu and
Figure 5.6: Gain in P&L in the fast and slow-scale stochastic volatility model. Left panel: distribution of the difference in P&L between the proposed strategy and the Garleanu and Pedersen strategy; parameters chosen are $\gamma = 5, \beta = \sqrt{0.5}, \rho_1 = 0.5, \epsilon = 0.25$. Right panel: gain in P&L in the slow-scale Heston stochastic volatility model for different starting value of the volatility factor $Z_0$; parameters chosen are $\gamma = 1, \beta = 0.25, \rho_2 = 0.5, \sqrt{\delta} = 0.25$. Other parameters are as in 5.3.

Table 5.3: Gain in P&L in the slow-scale Heston stochastic volatility model; parameters chosen are $\gamma = 1, \beta = 0.25, \rho_2 = 0.5, \sqrt{\delta} = 0.25$. Other parameters are as in 5.3.

<table>
<thead>
<tr>
<th>$Z_0$</th>
<th>mean (bps)</th>
<th>std error (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>40.157</td>
<td>15.497</td>
</tr>
<tr>
<td>0.2</td>
<td>29.617</td>
<td>9.750</td>
</tr>
<tr>
<td>0.3</td>
<td>10.560</td>
<td>6.826</td>
</tr>
<tr>
<td>0.4</td>
<td>10.928</td>
<td>5.242</td>
</tr>
<tr>
<td>0.5</td>
<td>7.005</td>
<td>4.061</td>
</tr>
<tr>
<td>0.6</td>
<td>7.686</td>
<td>3.298</td>
</tr>
<tr>
<td>0.7</td>
<td>5.183</td>
<td>2.686</td>
</tr>
<tr>
<td>0.8</td>
<td>0.637</td>
<td>2.339</td>
</tr>
<tr>
<td>0.9</td>
<td>3.705</td>
<td>2.041</td>
</tr>
</tbody>
</table>

Pedersen [51] value function, when volatility is driven by a single fast or slow factor, and Section 5.6 shows how these can be combined to incorporate both long and short time scales of volatility fluctuations. Using numerical examples and Monte-Carlo simulations, we have demonstrated that our proposed strategies is efficient to compute and they offer improvement in the Profit&Loss when the volatility process is characterized by its time scales of fluctuation.

There are a number of directions where similar techniques may play an effective role and we mention a few.
1. The impact of stochastic liquidity on the optimal portfolio, first formulated by Almgren [3] in continuous time and later extended by Cheridito and Sepin [32] in discrete time, is clearly of interest and a challenge. We refer to Gatheral and Schied [53] for modern developments and background. The joint asymptotics to study the impact on portfolio choice of friction from both stochastic liquidity and stochastic volatility will be considered in a future paper.

2. In the present chapter, the trading signals are considered observable. In practice, they are often observed with high signal-to-noise ratio. It is therefore important to quantify the impact of partial observations on the optimal trading behavior. In similar spirit to the Black and Litterman [16] model, one can incorporate investors’ views on upcoming performance are incorporated along with any degree of uncertainty that the investor may have in these views. The treatment of optimal trading with partial observations and intermittent insertion of expert opinions is the subject of an upcoming paper.
Appendix A

Appendices to Chapter 2

A.1 Solution to stage-0 PDE

The stage-0 equation (2.20) is

\[ H_t + (c_0 + c_1 u) H_u + \frac{1}{2} \lambda^2 b^2 H_{uu} + (c_2 + c_3 u + c_4 u^2) H = 0 \]

with terminal condition \( H(T, u) = 1 \), where \( c_0 \) through \( c_4 \) are given by

\[
\begin{align*}
  c_0 &= \frac{1}{\gamma} \left( a \bar{m} - (1 - \gamma) r \right) - \frac{1}{2} \lambda^2 b^2 \\
  c_1 &= -\frac{a}{\gamma} \\
  c_2 &= k \\
  c_3 &= -\frac{1 - \gamma}{\gamma} \frac{1}{\lambda^2 b^2} a (a \bar{m} - r) \\
  c_4 &= \frac{1 - \gamma}{\gamma} \frac{a^2}{2 \lambda^2 b^2}.
\end{align*}
\]

Substituting the ansatz (2.21) into the above yields

\[
f_0' + f_1' u + f_2' u^2 + (c_0 + c_1 u)(f_1 + 2 f_2 u) + \frac{1}{2} \lambda^2 b^2 \left( 2 f_2 + \frac{1}{\gamma} (f_1 + 2 f_2 u)^2 \right) + (c_2 + c_3 u + c_4 u^2) = 0.
\]
This yields three ordinary differential equations we have to solve to determine $H$:

$$f_0' + c_0 f_1 + \frac{1}{2} \lambda^2 b^2 (2f_2 + f_1) + c_2 = 0 \quad (A.1a)$$

$$f_1' + c_1 f_1 + 2c_0 f_2 + \frac{2}{\gamma} \lambda^2 b^2 f_1 f_2 + c_3 = 0 \quad (A.1b)$$

$$f_2' + \frac{2}{\gamma} \lambda^2 b^2 f_2^2 + 2c_1 f_2 + c_4 = 0 \quad (A.1c)$$

with terminal conditions $g_i(T) = 0$ for $i = 0, 1, 2$. These equations can be solved in closed form leading to (2.19); by inspection of the explicit formula, the solution does not explode in finite time provided that $\gamma > 0$.

A.2 Application to Equity Market

In a related but simpler setting, Nayak and Papanicolaou [97] studied a feedback model for the equity market in which a single stock $Y$ is traded by reference traders (who play the role of the commodity users in Section 2.2 of our analysis). The reference traders have a stochastic income process $I_t$. However, in the equity model, this is taken to be a geometric Brownian motion. Similar to the demand-supply analysis of Section 2.2.1 with an isoelastic demand function (2.1), when there are only reference traders, the stage-0 dynamics of the stock price $Y^{(0)}$ is also a geometric Brownian motion

$$\frac{dY_t^{(0)}}{Y_t^{(0)}} = \alpha_0 \, dt + \sigma_0 \, dW_t, \quad (A.2)$$

for some parameters $\alpha_0$ and $\sigma_0$ and Brownian motion $W$, in contrast to (2.2) where this is an expOU in the commodity model. In addition, there are portfolio optimizers, who seek to maximize their expected CRRA utility (2.4) in a fixed horizon $T$. They will trade in the equity market and their demand will cause the stock price dynamics to deviate from (A.2).
A.2.1 Stage-0 portfolio optimization

Since the unperturbed stock price process (A.2) is the geometric Brownian motion, the stage-0 portfolio optimization is simply the Merton problem [92]. The optimal portfolio is given by a fixed-mix strategy

\[ \theta_t^{(0)} = \hat{\theta}_0 Z_t := \frac{\alpha_0 - r}{\sigma_0^2} Z_t, \]

where \( Z_t \) is the wealth process of the portfolio optimizers.

A.2.2 Stage-1 portfolio optimization

As in Section 2.2.2, the stage-1 stock price dynamics is derived from the market clearing constraint, where we measure the relative size of the portfolio optimizers and reference traders by \( \epsilon \). The stage-1 drift \( \alpha_1 \) and volatility \( \sigma_1 \) (analogs of \( P^{(1)} \) and \( Q^{(1)} \) in Section 2.2.5) are given by

\[
\alpha_1(X_t, Y_t) = \frac{\alpha_0 \left( Y_t - \epsilon X_t \hat{\theta}_0 \right) + \epsilon X_t r \hat{\theta}_0 (1 - \hat{\theta}_0)}{Y_t - \epsilon X_t \hat{\theta}_0^2}, \quad \sigma_1(X_t, Y_t) = \frac{\sigma_0 (Y_t - \epsilon X_t \hat{\theta}_0)}{Y_t - \epsilon X_t \hat{\theta}_0^2}.
\]

(A.3)

where \( X_t \) is the aggregate wealth of the portfolio optimizers who follows the stage-0 strategy.

We denote the stage-1 strategy by \( \theta^{(1)} = \hat{\theta}^{(1)} Z_t \), where \( Z_t \) is the wealth process of the “smart” traders. Analogous to Section 2.3.1, his value function \( V \) is determined by the HJB PDE problem

\[
0 = V_t + rzV_z + \left( \hat{\theta}_0 (\alpha_1 - r) + r \right) xV_x + \frac{1}{2} \hat{\theta}_0^2 \sigma_1^2 x^2V_{xx} + \alpha_1 yV_y + \frac{1}{2} \sigma_1^2 y^2V_{yy} + \hat{\theta}_0 \sigma_1^2 xyV_{xy} + \sup_{\hat{\theta}} \left( \hat{\theta} \left( (\alpha_1 - r) zV_z + \sigma_1^2 (\hat{\theta}_0 xzV_{xz} + yzV_{yz}) \right) + \frac{1}{2} \hat{\theta}_0^2 \sigma_1^2 z^2V_{zz} \right)
\]

(A.4)
with terminal condition \( V(T, x, y, z) = U(z) \).

We observe that \( \alpha_1 = \alpha_1(x, y) \) and \( \sigma_1 = \sigma_1(x, y) \) are in fact functions only of the ratio \( \xi = x/y \). We look for a similarity solution in the variable \( \xi = x/y \). After separation of variables \( V(t, x, y, z) = \frac{z^{1-\gamma}}{1-\gamma} G(t, \xi) \), \( G \) solves

\[
0 = G_t + \frac{1}{2} \sigma_1^2 (1-\tilde{\theta}_0) \xi^2 G_{\xi \xi} + \frac{1}{\gamma} (\alpha_1 - r - \gamma \sigma_1^2) (1-\tilde{\theta}_0) \xi G_\xi + \frac{1-\gamma}{\gamma} \left( \frac{(\alpha_1 - r)^2}{2\gamma \sigma_1^2} + r \right) G
\]

with terminal condition \( G(T, \xi) = 1 \). The key observation to solving (A.5) is that the stage-1 Sharpe ratio \( (\alpha_1 - r)/\sigma_1 \) is equal to its stage-0 counterpart \( (\alpha_0 - r)/\sigma_0 \) and, in particular, is independent of \( \xi \). This allows us to deduce the analytic solution to (A.4) and give the following proposition.

**Proposition 22.** The stage-1 value function \( V \) does not depend on the current stock price \( Y_t \) or the aggregate wealth process \( X_t \):

\[
V(t, x, y, z) = \frac{z^{1-\gamma}}{1-\gamma} \exp \left( (1-\gamma) \left( \frac{(\alpha_0 - r)^2}{2\gamma \sigma_0^2} + r \right) (T-t) \right).
\]

The stage-1 optimal portfolio is given by the Merton ratio, evaluated at the stage-1 drift \( \alpha_1 \) and volatility \( \sigma_1 \):

\[
\tilde{\theta}^{(1)}(x, y) = \frac{(\alpha_1 - r)}{\gamma \sigma_1^2} = \frac{(\alpha_0 - r)}{\gamma \sigma_0} \frac{1}{\sigma_1} = \tilde{\theta}_0 \left( \frac{y - \epsilon x \tilde{\theta}_0}{y - \epsilon x \tilde{\theta}_0} \right).
\]

(A.6)

We have checked that (A.6) has excellent agreement with the order \( \epsilon \) asymptotic expansion derived by Nayak and Papanicolaou [97] in the small feedback regime.

**Remark 7.** It is also possible to show, by similar calculations and similarity solutions, that the stage-\( k \) optimal strategy \( \tilde{\theta}^{(k)} \) depends only on the ratio \( \xi_t = X_t/Y_t \).
A.3 Exponential Utility

In a tractable instance of our feedback model, with Ornstein-Uhlenbeck income process, linear demand function, and exponential utility for the portfolio optimizer, we are able to characterize the full fixed point problem by a coupled system of integral-differential equations.

Specifically, we will take the income process \((I_t)\) to be an Ornstein-Uhlenbeck process

\[ dI_t = a(m - I_t) \, dt + b \, dW^c_t, \]

and use the linear demand function \(D(Y_t, I_t) = I_t - Y_t\). In this simplified setting, the market clearing condition \(D(Y_t, I_t) = A\) leads to

\[ dY_t = a(\tilde{m} - Y_t) \, dt + b \, dW^c_t, \]

where we have defined \(\tilde{m} = m - A\). We see that the pre-financialized commodity price \(Y_t\) is the Ornstein-Uhlenbeck process.

A.3.1 Stage-0 optimal investment problem

We motivate the fixed-point characterization of the full feedback problem with the stage-0 optimal investment problem. As in Section 2.2 there is a single representative stock index with dynamics (2.5). The portfolio optimizer allocates his wealth into investments in the stock and the commodity. The wealth process \(X_t\) follows

\[
\begin{align*}
    dX_t &= \frac{\pi_t}{S_t} \, dS_t + \frac{\theta_t}{Y_t} \, dY_t + r(X_t - \pi_t - \theta_t) \, dt \\
    &= \left[ \mu \pi_t + a(\tilde{m} - Y_t) \frac{\theta_t}{Y_t} + r(X_t - \pi_t - \theta_t) \right] \, dt + \sigma \pi_t \, dW^s_t + b \frac{\theta_t}{Y_t} \, dW^c_t.
\end{align*}
\]
The value function for the exponential investor is the maximum expected terminal utility
\[ v(t, x, y) = \sup_{\pi, \theta} \mathbb{E} [U(X_T) \mid X_t = x, Y_t = y], \quad U(x) = -e^{-\gamma x}. \]

The usual dynamic programming principle leads to the HJB equation
\[
0 = v_t + rxv_x + a(\bar{m} - y)v_y + \frac{1}{2}b^2v_{yy} \\
+ \sup_{\pi, \theta} \left\{ (\mu - r)v_x \pi + \frac{1}{2}\sigma^2v_{xx}\pi^2 + \left[ \left( \frac{a(\bar{m} - y)}{y} - r \right)v_x + \frac{b^2v_{xy}}{y} \right] \theta + \frac{b^2}{2y^2}v_{xx}\theta^2 \right\}.
\]

The optimization problem nested within the PDE gives
\[
\pi^* = -\left( \frac{\mu - r}{\sigma^2} \right)\frac{v_x}{v_{xx}}, \quad \theta^* = -y \left[ \frac{(a(\bar{m} - y) - ry)v_x}{b^2v_{xx}} + \frac{v_{xy}}{v_{xx}} \right].
\]

Plugging these in we obtain
\[
0 = v_t + rxv_x + a(\bar{m} - y)v_y + \frac{1}{2}b^2v_{yy} - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{v_x^2}{v_{xx}} - \frac{1}{2b^2v_{xx}} \left[ (a\bar{m} - (a + r)y)v_x + b^2v_{xy} \right]^2.
\]

Proposition 23. The solution to the HJB equation (A.7) is given by
\[
v(t, x, y) = -\exp \left( A(t)x + B(t)y^2 + C(t)y + D(t) \right), \quad (A.8)
\]

where
\[
A(t) = -\gamma e^r(T-t), \quad B(t) = \frac{(a + r)^2}{4b^2r} \left( 1 - e^{2r(T-t)} \right), \quad C(t) = \frac{am(a + r)}{b^2r} \left( e^{r(T-t)} - 1 \right).
\]

The optimal investment in commodity can be written in feedback form
\[
\theta(t, x, y) = y \gamma e^{-r(T-t)} \left[ \frac{a\bar{m}}{b^2} + C(t) - \left( \frac{a + r}{b^2} - 2B(t) \right)y \right].
\]
In particular, the number of shares demanded by the portfolio optimizer is linear in the stage-1 commodity price $Y_t^{(1)}$. The modified market clearing condition reads

$$A = D(I_t, Y_t^{(1)}) + \epsilon \frac{\theta_t}{Y_t^{(1)}} = I_t - Y_t^{(1)} + \epsilon \gamma e^{-r(T-t)} \left[ \frac{am}{b^2} + C(t) - \left( \frac{a + r}{b^2} - 2B(t) \right) Y_t^{(1)} \right].$$

We therefore obtain

$$Y_t^{(1)} = I_t - A + \epsilon \gamma e^{-r(T-t)} \left( \frac{am}{b^2} + C(t) \right) \frac{1}{1 + \epsilon \gamma e^{-r(T-t)} \left( \frac{a + r}{b^2} - 2B(t) \right)} =: F_0(t) + F_1(t)I_t.$$  

It follows that the stage-1 commodity dynamics is given by

$$dY_t^{(1)} = \left[ amF_1(t) + F_0'(t) + F_0(t) \left( a - \frac{F_1'(t)}{F_1(t)} \right) Y_t^{(1)} \right] dt + bF_1(t) dW_t^c. \quad (A.9)$$

In particular, the commodity price $Y_t$ is still independent to the stock price $S_t$. Intuitively, portfolio optimizers with exponential utility do not induce financialization since their trading in the commodity markets is not affected by the wealth generated in the financial markets.

### A.3.2 Full problem

We now consider the fixed-point characterization of full problem, where the price impact of the portfolio optimizers is fully incorporated in their trading decision. Motivated by the fact that the stage-1 commodity dynamics (A.9) is again an Ornstein-Uhlenbeck process, we postulate that the stage-$\infty$ commodity price will be a time-inhomogeneous Ornstein-Uhlenbeck process

$$dY_t = a(t)(m(t) - Y_t)dt + b(t) dW_t^c. \quad (A.10)$$
The HJB equation now reads

\[ 0 = v_t + r x v_x + a(t)(m(t) - y) v_y + \frac{1}{2} b(t)^2 v_{yy} - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{v_x^2}{v_{xx}} \]

\[ - \frac{(a(t)m(t) - (a(t) + r)y) v_x + b(t)^2 v_{xy}}{2b(t)^2 v_{xx}}. \]

The same ansatz (A.8) leads to the following system of ODEs:

\[
\begin{cases}
0 = A'(t) + r A(t), & A(T) = -\gamma; \\
0 = B'(t) + 2r B(t) - \frac{1}{2} \left( \frac{a(t) + r}{b(t)} \right)^2, & B(T) = 0; \\
0 = C'(t) + r C(t) + \frac{a(t)m(t)}{b(t)^2} (a(t) + r), & C(T) = 0; \\
0 = D'(t) + b(t)^2 B(t) + \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{a(t)m(t)}{b(t)} \right)^2, & D(T) = 0.
\end{cases}
\]

The market clearing condition reads

\[
A = I_t - Y_t + \epsilon \gamma e^{-r(T-t)} \left[ \frac{a(t)m(t)}{b(t)^2} + C(t) - \left( \frac{a(t) + r}{b(t)^2} - 2B(t) \right) Y_t \right].
\]

**Proposition 24.** The full problem is characterized by the dynamics (A.10) where \(a, m, b\) satisfy the following integro-differential equations

\[
0 = -\epsilon \gamma b(t) (a'(t) + (-a(t) + a + r)(a(t) + r)) + 2\gamma \epsilon (a(t) + r)b'(t)
\]

\[ + b(t)^3 (2\gamma \epsilon (B(t)(-a(t) + a + r) + B'(t)) + (a(t) - a)e^{r(T-t)}) \]

\[ 0 = \left( 1 + \epsilon \gamma e^{-r(T-t)} \left( \frac{a(t) + r}{b(t)^2} - 2B(t) \right) \right) b(t) - b, \]

\[ 0 = b(t)^3 \epsilon e^{r(T-t)} (a(A - m) + m(t)) - \epsilon \gamma (b(t) (m(t) (a'(t) + (a + r - 1)a(t) - r) + a(t)m'(t))) \]

\[ + \epsilon \gamma (2a(t)m(t)b'(t) - b(t)^3 ((a + r)C(t) + 2B(t)m(t) + C'(t))). \]

(A.11)
where

\[
B(t) = -\frac{1}{2} \int_t^T e^{-2r(t-s)} \left( \frac{a(s) + r}{b(s)} \right)^2 ds, \quad C(t) = \int_t^T e^{-r(t-s)} \frac{a(s)m(s)(a(s) + r)}{b(s)^2} ds.
\]

In particular, the stage-∞ commodity price \( Y_t \) is independent to the stock price \( S_t \); there is no financialization under the feedback model with exponential utility.

**Proof.** An application of the Itô’s lemma leads to the following differential algebraic equation:

\[
0 = B'(t) + 2rB(t) - \frac{1}{2} \left( \frac{a(t) + r}{b(t)} \right)^2, \quad B(T) = 0,
\]

\[
0 = C'(t) + rC(t) + \frac{a(t)m(t)}{b(t)^2} (a(t) + r), \quad C(T) = 0,
\]

\[
a(t) = a - \frac{F_1'(t)}{F_1(t)}, \quad b(t) = bF_1(t), \quad m(t) = \frac{amF_1(t) + F_0'(t) + F_0(t) \left( a - \frac{F_1'(t)}{F_1(t)} \right)}{a - \frac{F_1'(t)}{F_1(t)}},
\]

\[
F_0(t) = \frac{-A + \epsilon \gamma e^{-r(T-t)} \left( \frac{a(t)m(t)}{b(t)^2} + C(t) \right)}{1 + \epsilon \gamma e^{-r(T-t)} \left( \frac{a(t)+r}{b(t)^2} - 2B(t) \right)}, \quad F_1(t) = \frac{1}{1 + \epsilon \gamma e^{-r(T-t)} \left( \frac{a(t)+r}{b(t)^2} - 2B(t) \right)}.
\]

Solving the ODEs for \( B \) and \( C \) leads to proposition.

**Remark 8.** Alternatively, one can view the above proposition as a fixed-point iteration of the stage-\( k \) problem under exponential utility. Following the steps outlined in Section A.3.1, one can derive the stage-(\( k + 1 \)) dynamics

\[
dY_t^{(k+1)} = a_{k+1}(t)(m_{k+1}(t) - Y_t^{(k+1)})dt + b_{k+1}(t) dW_t^c. \tag{A.12}
\]

where

\[
b_{k+1}(t) = bF_1^{(k)}(t), \quad a_{k+1}(t) = a - \frac{d}{dt} F_1^{(k)}(t),
\]

\[
m_{k+1}(t) = F_0^{(k)}(t) + \frac{amF_1^{(k)}(t) + \frac{d}{dt} F_0^{(k)}(t)}{a_{k+1}(t)}. \tag{A.13}
\]

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with \( b_0(t) \equiv b, a_0(t) \equiv a, m_0(t) \equiv \bar{m} \), and

\[
F^{(k)}_0(t) = \frac{-A + e^{-r(T-t)} \left( \frac{a_k(t)m_k(t)}{b_k(t)^2} + \int_t^T e^{-r(t-s)} \frac{a_k(s)m_k(s)(a_k(s)+r)}{b_k(s)^2} \, ds \right)}{1 + e^{-r(T-t)} \left( \frac{a_k(t)+r}{b_k(t)^2} + \int_t^T e^{-2r(t-s)} \left( \frac{a_k(s)+r}{b_k(s)} \right)^2 \, ds \right)},
\]

\[
F^{(k)}_1(t) = \frac{1}{1 + e^{-r(T-t)} \left( \frac{a_k(t)+r}{b_k(t)^2} + \int_t^T e^{-2r(t-s)} \left( \frac{a_k(s)+r}{b_k(s)} \right)^2 \, ds \right)}.
\]

While the convergence properties of the family \((a_k, b_k, m_k)\) to \((a, b, m)\) in Proposition 24 seems to be rather involved, we note that (by removing the \( k \) in (A.13) and (A.14)) the limit necessarily satisfies (A.11).
Appendix B

Appendices to Chapter 3

B.1 Asymptotic correction of arbitrary order

In this section we give expressions for successive terms in the small $\varepsilon$ expansions for $v$ and $m$ in the deterministic mean field game where $\sigma = 0$. It is straightforward to compute the expansions for the coefficients $\alpha(t) = a(\eta(t))$ and $\gamma(t) = c(\eta(t))$ using the expansion for $\eta(t)$ in (3.50). We also expand $\bar{p}(t)$. The terms in the three series are labeled $\alpha_k(t)$, $\gamma_k(t)$ and $\bar{p}_k(t)$ respectively, but we do not give their cumbersome formulas here.

Value function

Inserting the expansion of $v$ into the PDE (3.30) with $\sigma = 0$, we find that the $k$th-order value function correction $v_k$ satisfies the equation

$$\partial_t v_k - rv_k - \frac{1}{2} \left( 1 + \mathbb{W}(\theta(x)) \right) \partial_x v_k = f_k(t, x), \quad v_k(t, 0) = 0,$$

(B.1)
where the inhomogeneous term $f_k$ is given by

$$
f_k(t, x) = -\frac{1}{4} \left\{ \sum_{i=1}^{k-1} \xi_i \xi_{k-i} + 2(1 - v'_0) \left( \alpha_k + \sum_{j=0}^{k} \gamma_j \bar{p}_{k-j} \right) \right\}, \quad \xi_n = \alpha_n - \partial_x v_n + \sum_{i=0}^{n} \gamma_i \bar{p}_{n-i}.
$$

Observe that $f_k$ depends only on $v_j$ and $m_j$ for $0 \leq j < k$. Having solved for these $v_j$ and $m_j$, the $v_k$ equations can be solved in closed-form:

$$
v_k(t, x) = -\int_0^{\tau(x)} e^{-rs} f_k(t + s, X(s; x)) ds, \quad \text{(B.2)}
$$

where $X(t; x)$ is the capacity trajectory starting from $x$, given by (3.38).

Density

The $k$th-order density correction $m_k$ satisfies the equation

$$
\partial_t m_k - \frac{1}{2} \partial_x \left( (1 - v'_0) m_k \right) = g_k(t, x), \quad m_k(0, x) = 0, \quad \text{(B.3)}
$$

where the inhomogeneous term $g_k$ is given by

$$
g_k(t, x) = \frac{1}{2} \sum_{i=1}^{k} \partial_x (\xi_i m_{k-i}).
$$

Again we see that $g_k$ depends only on $v_j$ and $m_j$ for $0 \leq j < k$. Then $m_k$ can be written as

$$
m_k(t, x) = \int_0^t \frac{1 + \mathbb{W}(\theta(x)) e^{-r(t-s)}}{1 + \mathbb{W}(\theta(x))} g_k(s, X(s - t; x)) ds. \quad \text{(B.4)}
$$
B.2 Cournot-Bertrand Equivalence in the Stochastic Dynamic CMFG

In this section we show that in the continuum mean field setting, the dynamic Cournot game and Bertrand games are identical. We first derive the Cournot MFG PDEs.

B.2.1 Dynamic Cournot Mean Field Game

As in the Bertrand game described in Section 3.3, there is an infinity of players on $x > 0$ with initial density $M(x)$. Here they choose quantities of production $q_t = q(t, X_t)$ which deplete the remaining capacity of the producers $(X_t)$ following the dynamics

$$dX_t = -q(t, X_t) \, dt + \sigma \, dW_t,$$

as long as $X_t > 0$, and $X_t$ is absorbed at zero. Here $W$ is a Brownian motion, and $\sigma > 0$ is a constant. The Cournot market model is specified by the inverse demand function $p_t = 1 - (q_t + \bar{q}(t))$, where $\bar{q}$ is the mean production. We will denote by $m^c(t, x)$ the “density” of firms with positive capacity at time $t > 0$, and by $\eta^c(t) = \int_{\mathbb{R}^+} m^c(t, x) \, dx$ the fraction of active firms remaining. Then the average quantity is

$$\bar{q}(t) = \int_{\mathbb{R}^+} q(t, x) m^c(t, x) \, dx.$$

The value function $v^c$ of the producers is

$$v^c(t, x) = \sup_q \mathbb{E} \left\{ \int_t^\infty e^{-r(s-t)} p_s q_s 1_{\{X_s > 0\}} \, ds \bigg| X_t = x \right\}, \quad x > 0. \quad (B.5)$$

In analogy to the Bertrand game, we define the Cournot exhaustion time $T^c$ to be the first time $\eta^c$ reaches zero. The following quantities are defined for $t < T^c$. The
associated HJB equation is

\[ \partial_t v^c + \frac{1}{2} \sigma^2 \partial_{xx}^2 v^c - rv^c + \max_{q \geq 0} \left( 1 - (q + \varepsilon \bar{q}(t)) - \partial_x v^c \right) q = 0, \quad x > 0. \tag{B.6} \]

The internal optimization is the static continuum mean field Cournot game (Section 3.2.3) with cost function \( s(x) \mapsto \partial_x v^c \). The first-order condition gives

\[ q^*(t, x) = \frac{1}{2} (1 - \varepsilon \bar{q}(t) - \partial_x v^c(t, x)), \tag{B.7} \]

with the optimal (equilibrium) price given by \( p^*(t, x) = \frac{1}{2} (1 - \varepsilon \bar{q}(t) + \partial_x v^c(t, x)) \).

Therefore, the HJB equation becomes

\[ \partial_t v^c + \frac{1}{2} \sigma^2 \partial_{xx}^2 v^c - rv^c + \frac{1}{4} \left( 1 - \varepsilon \bar{q}(t) - \partial_x v^c \right)^2 = 0, \quad x > 0. \tag{B.8} \]

When all the reserves are exhausted, the game is over and \( v^c(t, 0) = 0 \).

The density \( m^c(t, x) \) of the distribution of reserves is the solution of the forward Kolmogorov equation

\[ \partial_t m^c - \frac{1}{2} \sigma^2 \partial_{xx}^2 m^c - \frac{1}{2} \partial_x \left( (1 - \varepsilon \bar{q}(t) - \partial_x v^c) m^c \right) = 0, \tag{B.9} \]

with \( m^c(0, x) = M(x) \). The average demand is computed by averaging \( \text{[B.7]} \) with respect to \( m^c \), which leads to

\[ \bar{q}(t) = \frac{1}{2 + \varepsilon \eta(t)} \left( \eta(t) - \int_{\mathbb{R}^+} \partial_x v^c(t, x) m^c(t, x) \, dx \right). \tag{B.10} \]
B.2.2 Equivalence of Bertrand and Cournot Problems

We start by recalling the Bertrand MFG equations:

\[
0 = \partial_x v + \frac{1}{2} \sigma^2 \partial_{xx} v - rv + \frac{1}{4} (a(\eta(t)) - \partial_x v + c(\eta(t)) \bar{p}(t))^2
\]

\[
0 = \partial_t m - \frac{1}{2} \sigma^2 \partial_{xx} m - \frac{1}{2} \partial_x ((a(\eta(t)) - \partial_x v + c(\eta(t)) \bar{p}(t)) m)
\]

\[
\bar{p}(t) = \frac{1}{2 - c(\eta(t))} \left( a(\eta(t)) + \frac{1}{\eta(t)} \int \partial_x v(t,x) m(t,x) dx \right),
\]

where \( \eta(t) = \int_{\mathbb{R}_+} m(t,x) dx. \)

If we define

\[
\bar{q}_b(t) = \frac{1}{2 + \varepsilon \eta(t)} \left( \eta(t) + \int \partial_x v(t,x) m(t,x) dx \right),
\]

then it follows that \((1 + \varepsilon \eta) \bar{q}_b(t) = \eta(1 - \bar{p}(t))\) and hence

\[
a(\eta(t)) + c(\eta(t)) \bar{p}(t) = 1 - \varepsilon \bar{q}_b(t).
\]

Then equations \([\text{B.11}]\) can be written

\[
0 = \partial_x v + \frac{1}{2} \sigma^2 \partial_{xx} v - rv + \frac{1}{4} (1 - \varepsilon \bar{q}_b(t) - \partial_x v)^2
\]

\[
0 = \partial_t m - \frac{1}{2} \sigma^2 \partial_{xx} m - \frac{1}{2} \partial_x ((1 - \varepsilon \bar{q}_b(t) - \partial_x v) m),
\]

and these are exactly the Cournot CMFG equations \([\text{B.8}], [\text{B.9}]\) and \([\text{B.10}]\). As the boundary conditions are the same, we have \( v \equiv v^c, m \equiv m^c \) and \( \bar{q} \equiv \bar{q}_b \), and the Bertrand and Cournot dynamic MFG problems are equivalent.
Appendix C

Appendices to Chapter 5

C.1 Full second order asymptotics

In this appendix we provide the full second order asymptotic expansion to the multiscale stochastic volatility model in Section 5.6. At order $\varepsilon^{1/2}$ of equation (5.32), we have

$$0 = \frac{1}{K} v_q^{(0)} v^{(1,1)}_q - \rho v^{(1,1)} - \kappa x v^{(1,1)}_x + \frac{1}{2} \eta v^{(1,1)}_{xx} + L_0 v^{(3,1)} + \rho_{12} a(y) g(z) v^{(2,0)}_y.$$  

When viewed as a Poisson equation for $v^{(3,1)}$, this gives the solvability condition for $v^{(1,1)}$:

$$0 = \frac{1}{K} v_q^{(0)} v^{(1,1)}_q - \rho v^{(1,1)} - \kappa x v^{(1,1)}_x + \frac{1}{2} \eta v^{(1,1)}_{xx} + G(z) q^2, \quad (C.1)$$

where the source term $G$ is given by

$$G(z) = -\frac{1}{2} \rho_{12} g(z) \left\langle a(y) \frac{\partial^2}{\partial y \partial z} \phi(y, z) \right\rangle.$$

The equation (C.1) can be solved using a quadratic ansatz

$$v^{(1,1)}(q, x, z) = \frac{1}{2} C_{qq}(z) q^2 + C_{qx}(z) q x + \frac{1}{2} C_{xx}(z) x^2 + C_0(z), \quad (C.2)$$
where

\[ C_{qq}(z) = \frac{G(z)}{\frac{A_{qq}(z)}{K} + \rho \frac{1}{2}}, \quad C_{qx}(z) = \frac{A_{qx}(z)C_{qq}(z)}{\frac{A_{qq}(z)}{K} + \rho + \kappa}, \quad C_{xx}(z) = \frac{A_{xx}(z)C_{qx}(z)}{\frac{A_{qq}(z)}{K} + \rho \frac{1}{2}}, \quad C_0(z) = \frac{\eta}{2\rho} C_{xx}(z). \]

Returning to the slow scale expansion (5.28), we extract the order \( \delta \) term in (5.26) to obtain the following equation for \( v^{0,2} \):

\[
0 = \frac{1}{2K} \left( (v_q^{0,1})^2 + 2v_q^{0,1}v_q^{0,2} - \rho v^{0,2} - \kappa x v_x^{0,2} + \frac{1}{2} \eta v_{xx}^{0,2} + \frac{1}{\varepsilon} L_0 v^{0,2} \right) + M_2 v \varepsilon, 0 + \rho \frac{1}{\sqrt{\eta}} a(y) v_{xy}^{0,2} + \rho_2 \frac{1}{\sqrt{\eta}} g(z) v_{xz}^{0,2}.
\]

(C.3)

The order \( \varepsilon^{-1} \) terms in (C.3) give \( L_0 v^{0,2} = 0 \) and we take \( v^{0,2} = v^{0,2}(q, x, z) \) independent of \( y \). At order \( \varepsilon^{-1/2} \), we have \( L_0 v^{1,2} = 0 \) and so again \( v^{1,2} = v^{1,2}(q, x, z) \).

At order one:

\[
0 = \frac{1}{2K} \left( (v_q^{0,1})^2 + \frac{1}{K} v_q^{0} v_q^{0,2} - \rho v^{0,2} - \kappa x v_x^{0,2} + \frac{1}{2} \eta v_{xx}^{0,2} + L_0 v^{0,2} + M_2 v^{0,0} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{0,1} \right).
\]

(C.4)

When viewed as a Poisson equation for \( v^{2,2} \), this yields the following solvability condition for \( v^{0,2} \):

\[
0 = \frac{1}{2K} \left( (v_q^{0,1})^2 + \frac{1}{K} v_q^{0} v_q^{0,2} - \rho v^{0,2} - \kappa x v_x^{0,2} + \frac{1}{2} \eta v_{xx}^{0,2} + M_2 v^{0,0} + \rho_2 \sqrt{\eta} g(z) v_{xz}^{0,1} \right).
\]

This can again be solved using a linear-quadratic ansatz

\[
v^{0,2}(q, x, z) = \frac{1}{2} D_{qq}(z)q^2 + D_{qx}(z)qx + \frac{1}{2} D_{xx}(z)x^2 + D_0(z),
\]

(C.5)
where

\[ D_{qq}(z) = -\frac{\mathcal{M}_2 A_{qq}(z)}{\rho + \frac{2}{\kappa} A_{qq}(z)}, \quad D_{qx}(z) = \frac{\mathcal{M}_2 A_{qx}(z) + \frac{1}{\kappa} A_{qx}(z) D_{qq}(z)}{\kappa A_{qq}(z) + \rho + \kappa}, \]

\[ D_{xx}(z) = \frac{\frac{1}{\kappa} A_{qx}(z) D_{qx}(z) + \frac{1}{2} \kappa A_{xx}(z)}{\kappa + \rho/2}, \quad D_0(z) = \frac{1}{\rho} \left( H(z) + \frac{1}{2} \eta D_{xx}(z) \right), \]

and

\[ H(z) = \frac{1}{2\kappa} B_q(z)^2 + \mathcal{M}_2 A_0(z) + \rho_2 \sqrt{\eta} g(z) B'_x(z). \]
Bibliography


