SYNTHETIC DIAGNOSTICS PLATFORM FOR FUSION PLASMA AND A TWO-DIMENSIONAL SYNTHETIC ELECTRON CYCLOTRON EMISSION IMAGING CODE

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Abstract

Magnetic confinement fusion is one of the most promising approaches to achieve fusion energy. With the rapid increase of the computational power over the past decades, numerical simulation have become an important tool to study the fusion plasmas. Eventually, the numerical models will be used to predict the performance of future devices, such as the International Thermonuclear Experiment Reactor (ITER) or DEMO. However, the reliability of these models needs to be carefully validated against experiments before the results can be trusted. The validation between simulations and measurements is hard particularly because the quantities directly available from both sides are different. While the simulations have the information of the plasma quantities calculated explicitly, the measurements are usually in forms of diagnostic signals. The traditional way of making the comparison relies on the diagnosticians to interpret the measured signals as plasma quantities. The interpretation is in general very complicated and sometimes not even unique. In contrast, given the plasma quantities from the plasma simulations, we can unambiguously calculate the generation and propagation of the diagnostic signals. These calculations are called synthetic diagnostics, and they enable an alternate way to compare the simulation results with the measurements.

In this dissertation, we present a platform for developing and applying synthetic diagnostic codes. Three diagnostics on the platform are introduced. The reflectometry and beam emission spectroscopy diagnostics measure the electron density, and the electron cyclotron emission diagnostic measures the electron temperature. The theoretical derivation and numerical implementation of a new two dimensional Electron cyclotron Emission Imaging code is discussed in detail. This new code has shown the potential to address many challenging aspects of the present ECE measurements, such as runaway electron effects, and detection of the cross phase between the electron temperature and density fluctuations.
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Chapter 1

Introduction

1.1 Fusion Energy

1.1.1 Fossil fuels and the demand of new energy resources

Energy is one of the most fundamental resources all lives depend on. It is the nature of life to keep the internal structure, and according to the second law of thermodynamics, external energy is required to overcome the natural trend towards equilibrium, or as that means to life, destruction.

In this sense, all kinds of resources are valuable only because they play a role in the process of generating energy, and keeping us alive. For example, air is a resource solely because the oxygen is required to extract the chemical energy we have stored in our cells.

It is fair to say that the history of humanity, is indeed a history of inventing new ways to collect energy. Farming is a reliable and controllable way to harvest solar energy. The use of fire opens a door of utilizing chemical energy stored by plants. The invention of sail enables people to use wind power, and travel across the ocean. Steam engines make it possible to efficiently use coal, and lead to the Industrial Revolution. Since then, the intensive use of the fossil fuels, i.e. coal, oil, natural gas, etc., has
enabled an accelerating improvement in people’s life. The total demand of energy has
grown exponentially as shown in Fig 1.1. It is clear from Fig 1.1 that the main energy
resources we rely on today are still the fossil fuels, which is essentially solar energy
stored by plants over millions of years. The US Energy Information Administration
(EIA) estimated that almost half of the world’s total conventional oil reserves will
have been exhausted by 2030[2]. These estimates include existing oil reserves and
anticipated reserves resulting from new technologies and discoveries. It is a striking
fact that we have exhausted almost half of the energy reserve stored by the earth over
millions of years within a century, and the demand is still growing. If we can not
discover a new kind of energy resource before reaching the limit of fossil fuels, which
may be in less than a hundred of years if the current energy structure does not change,
it is hard to imagine how we can maintain modern life. It is even scarier if we consider
that no country would willingly slow down its economical development because of
reaching the limit of the domestic reserve of energy. Conflicts over the remaining energy resources will very likely happen before the resources are actually depleted. It is our most urgent task to find a new rich, sustainable, and safe energy resource. Nuclear energy, especially fusion energy, is one of the most promising candidates.

1.1.2 Advantages of fusion energy

Nuclear energy takes two forms. It is the energy that is released when 1) very heavy atoms split into lighter atoms, called fission; or 2) very light atoms fuse to form heavier atoms, called fusion.

In contrast to other new energy resources, such as wind, hydraulic, and photovoltaic energy which are all utilizing a part of the solar energy, nuclear energy is using the energy stored from the very beginning of the formation of matter. This makes nuclear energy a fundamentally different source from all the other main energy resources.

Nuclear fission was first used for generating electricity in the 1950s, and became commercially available in the 1960s. In 2016, about 20% of electricity, or 8.5% of the total power, consumed in the US is provided by nuclear fission plants. However, no new fission plants have started operation in the US since 1996[3].

The main concerns regarding nuclear fission plants include operational safety and disposal of the radioactive waste. The former has been a serious concern of the public since the nuclear accident at the Fukushima Daiichi plant in Japan resulting from an earthquake and tsunami in March 2011, which led to a worldwide review of the current nuclear energy regulations. In history, there were two other infamous fission plant accidents, the Three Mile Island accident in 1979 in the US and the Chernobyl accident in 1986 in the USSR. Considerable resources have also been devoted to searching to a solution to the waste disposal problem since 1982, but the plans proposed are still mainly to deposit the waste in lands with little population[3].
The sustainability of fission energy is another issue. It is estimated that the known economically accessible uranium resources can support the current working fission plants to run for roughly 200 years\cite{4}. With the currently non-economical technologies, such as extracting uranium from sea water and/or fuel-cycling fast-breeder reactors, the fission fuel may be available significantly longer, up to hundreds of thousand years. But this is still much shorter compared to the potential of fusion energy.

Nuclear fusion plants, if successfully built, will have advantages over the fission plants in all of the three above-mentioned aspects.

The fusion reactors are safer than the fission ones mainly because there is no runaway situations for fusion reaction. In fact, the biggest challenge in current research of fusion energy is to maintain the fusion reaction for a long period of time. Failure in any part of the reactor may at most lead to the destruction of the reactor. For magnetically confined fusion, the amount of fuel loaded in the machine is limited due to the low density of the plasma and the size of the vessel. So, accidents like the one in Chernobyl which were due to loss of control and lead to massive release of radioactive material can hardly happen on fusion plants.

Fusion produces $^4$He, which is totally clean, and neutrons. No radioactive waste is directly produced by the fusion reactions. Although the energetic neutrons produced by fusion may introduce excited and radioactive materials in the machine wall, these pollutants can be minimized by the careful design of the plant.

Finally, the fuels of fusion plants are deuterium and tritium. Deuterium can be extracted from sea water, and is virtually unlimited. Tritium does not exist in nature because of its short half life, but can be produced from Lithium-6 ($^6$Li$^\alpha$). An estimate shows that the total known Lithium resource on earth can support fusion plants running for around 3000 years at a power output equal to roughly the current level of fission plants, provided that all the $^6$Li is reserved for fusion. If Lithium extraction
from sea water proves economically profitable, the Lithium supply will be sufficient for 23 million years [4]. Furthermore, if deuterium-deuterium fusion is eventually achieved, Lithium is no longer a requirement, the fusion energy will be practically unlimited.

Because of these advantages, fusion energy has been considered a major energy source for the future.

1.2 Challenges in Realizing Fusion Energy

1.2.1 Lawson criterion and challenges in the magnetic confinement fusion

The criterion for a thermonuclear fusion reactor to have a gain greater than one, i.e. the ratio between fusion generated power and the input power larger than one, was first discussed by J.D. Lawson in 1955[5]. For D-T fusion, the required temperature $T \sim 25 \text{keV} \approx 250$ million degrees, with a density $n$ and energy confinement time $\tau_E$ product equal to $10^{20} \text{m}^{-3}\text{s}$. The energy confinement time $\tau_E$ is defined as the total plasma energy divided by the rate of energy loss. If the constraint is on the achievable total pressure $P = nT$, the criterion is that the triple product $nT\tau_E \geq 3 \times 10^{21} \text{keV m}^{-3}\text{s}$[6]. This is a very severe criterion to meet.

The required temperature is too high for normal material to sustain. So, two approaches have been developed. Inertial Confinement Fusion (ICF) completely avoids the use of any container for the hot fusion plasma, and relies on compressing and heating the fuel so rapidly that the fusion reaction exhausts the fuel before it spreads and cools down. The other approach uses a strong magnetic field to “contain” the hot plasma, and is therefore called Magnetic Confinement Fusion (MCF). The key idea is that, when the fuel is heated to a temperature of the tens of keV, the atoms are fully ionized. The charged particles are confined perpendicularly to the magnetic field due
to the Lorentz force, so a pressure gradient may be sustained across the field. While
the fusion criterion is met in the core of the plasma, the temperature and density can
be significantly lower at the edge so the machine wall can sustain for a long period –
ideally years – of operation. The rest of this dissertation is mainly applicable to MCF,
but some general concepts, such as synthetic diagnostics – numerical simulations of
the diagnostic signals used for interpretation and prediction of the measurements, –
can be applied to ICF as well.

Recent research on MCF, in terms of plasma physics, can be mainly divided into
two categories: stability and transport. A successful MCF reactor should first be
able to sustain a fusion plasma in a steady state for a certain period of time, over
which fusion reactions take place and the energy is collected. This leads to the study
of the stability of magnetized plasmas. Furthermore, in order to achieve the Lawson
criterion, the core plasma pressure needs to be high, while the whole plasma power loss
needs to be low. This requirement leads to the study of energy and particle transport
inside the plasma. In both of these areas, very difficult problems have to be solved.
This is mainly because when large pressure gradients exist, they provide huge sources
of free energy and make the plasma more unstable both globally and locally. The
global instabilities may undermine the stability, while the local instabilities degrade
the energy confinement.

1.2.2 Challenges in the validation of theories and numerical
simulations

Over the past six decades, enormous effort has been made to tackle the difficult prob-
lems in MCF. From linear theory of stability, to non-linear treatment of turbulent
transport, analytic studies have shown to be fundamental and groundbreaking. On
the other hand, it is experiments which not only demonstrate the theoretical pre-
dictions, but also discover new phenomena and raise new questions to the theorists.
Along with the rapid growth of the power of computers, a third approach, numerical simulation, has become a more and more powerful method to study plasma physics. With the unparalleled computing power, numerical simulations can apply the theories of plasma to very complicated situations which are virtually intractable using analytic approaches.

Although much more powerful in dealing with realistic conditions than pure analytic calculations, numerical simulations are, by nature, still theoretical models. The validity of these models, just as for analytic ones, needs to be demonstrated by experiments.

The main difficulty of comparing simulation results with measurements is that the physical quantities available from simulations are very different from the measured ones. In simulations, the data is given in plasma quantities, e.g. density and temperature, while experimentally, a set of diagnostics is used to observe the plasma, which do not immediately measure plasma quantities. For instance, the electron temperature can be inferred from the measured electron cyclotron emission power, but the relation is sometimes complicated (see Sec 2.4.1).

Traditionally, it is the diagnosticians’ job to interpret the measured signals and provide knowledge about the underlying plasma quantities. But in most cases, the measurements do not contain full information about the original plasma state, and therefore the exact plasma quantities can not be reconstructed from the measurements.

This difficulty of validation is the motivation for the development of synthetic diagnostics.
1.3 Synthetic Diagnostics

Synthetic diagnostics provide an alternative approach to theory-experiment comparisons. While interpreting measurements is difficult or even impossible in many cases, \textit{forward modeling} of the diagnostic process can be carried out with the full plasma information given by simulation and/or theory. In contrast to the interpretative approach, no \textit{a priori} assumptions about the plasma and diagnostic processes are needed since the plasma is known and the diagnostic response is simulated. One limitation comes from the models used to simulate the diagnostic response. The validity of these models can be independently examined through either theoretical assessments or numerical studies. The synthetic results can then be compared directly to the measured ones, and can provide valuable information about both the simulations and experiments.

Figure \ref{fig:1.2} shows the typical work flow of a complete comparison between simulations and experiments using synthetic diagnostics. Synthetic diagnostics play a central role in this comparison. They transform the output from simulations or theory into the synthetic signals that are then processed in the same way as the measured signals. The synthetic and experimental results can then be compared in an unambiguous way.

Another important use of synthetic diagnostics is to quantify uncertainties and sensitivities of the underlying measurements. They are thus important tools for validation, and fit in the proposed guidelines for best validation practices\cite{7}. In addition, once the diagnostic response is known, the synthetic diagnostics can also be used to optimize diagnostic hardware\cite{8}.

In this dissertation, a platform for developing and applying synthetic diagnostics is presented, as well as the theoretical derivation and the numerical implementation of a two-dimensional synthetic Electron Cyclotron Emission diagnostic code that includes refraction, diffraction, emission, and absorption all in one calculation.
Figure 1.2: Synthetic diagnostic work flow. The synthetic diagnostics generate the diagnosed signals based on the theoretically predicted or simulated plasmas. These synthetic signals can then be processed and analyzed the same way as the real signals obtained from experiments. The comparison of the results is then unambiguous.
Chapter 2

Synthetic Diagnostics Platform

In this chapter, we introduce the Synthetic Diagnostics Platform (SDP), which is a platform for both development and application of synthetic diagnostic codes\cite{9}. Its modular structure makes adding new diagnostics straightforward. Programmed in Python, it can benefit from the strong support provided by the Python scientific computing community \cite{10,11}. It is also possible to include C/Fortran code into SDP, enabling further optimizations when high computational efficiency is vital. SDP can read output from various simulation codes, and generate test plasmas based on theoretical models. After the synthetic diagnostic runs, the synthetic signals can be further processed in various analysis and post-processing modules. A group of supporting packages provides basic utilities and standard interfaces, including geometry, I/O, and unit conversion.

A brief introduction to the data interfaces available on SDP is given in Sec \ref{sec:interfaces}. Three available synthetic diagnostics are then introduced: a reflectometry diagnostic based on the FWR codes\cite{12} (Sec \ref{sec:reflectometry}), a Beam Emission Spectroscopy diagnostic (Sec \ref{sec:beamspectroscopy}), and a traditional one-dimensional synthetic Electron Cyclotron Emission Imaging (ECEI) diagnostic (Sec \ref{sec:ecei}). The theoretical derivation and numerical im-
plementation of a more advanced two-dimensional synthetic ECEI code are the main content of this dissertation, and are presented in Chapters 3 and 4.

2.1 Interfaces to Simulations

The key task of data interfaces between the plasma simulations and the synthetic diagnostics is to load the calculated plasma quantities on the simulation grids, and interpolate them onto the grids used by the synthetic diagnostics.

In SDP, the synthetic diagnostic modules are developed in Cartesian coordinates. For 2D calculations, plasma data is needed on a rectangular mesh in radial ($R$) and vertical ($Z$) directions. For 3D calculations, the radial direction is called $x$, and vertical direction $y$, the third dimension is perpendicular to the poloidal plane, i.e. $\hat{z} \equiv \hat{x} \times \hat{y}$, and is approximately the toroidal direction.

Currently, SDP provides data interfaces for several advanced plasma simulation codes.

The M3D-C code\cite{i3} is a continuum simulation solving the Magneto-Hydro-Dynamics (MHD) equations using first order continuous finite element method. It provides some Python functions for obtaining the plasma quantities at specified locations, which is directly used to generate data on SDP mesh. No further conversion is needed.

The GTC \cite{i4} and GTS\cite{i5} codes are Particle-In-Cell (PIC) simulations for core turbulence studies. They both use flux coordinates, ($\psi_p, \theta, \zeta$), where the grids for field quantities are aligned with the magnetic flux surfaces. Data available on the GTC and GTS meshes in general have the form $f_i = f(\psi_{p,i}, \theta_i, \zeta_i)$, where $f$ is any calculated quantity, e.g. the perturbed electric potential $\tilde{\phi}$, and $i$ labels the grid points. The details about the flux coordinates used by GTS and GTC, and the specific arrangements of the grids in memory can be found in \cite{i5} and \cite{i6}.
The XGC-1 code is another major PIC simulation designed to overcome the difficulties related to the complex magnetic field topology near the plasma edge, both inside and outside the magnetic separatrix\cite{17}. The equations XGC-1 solves are written in cylindrical coordinates \((R, \Phi, Z)\). However, the calculation mesh is set up in such a way that the grids are still on a series of constant magnetic flux “surfaces,” although outside the separatrix, the magnetic field lines do not form closed surfaces. For interpolation purposes, the XGC-1 mesh can be treated in a very similar way as that in GTC or GTS.

Special care is needed for the interpolation in the toroidal direction. The fluctuations studied in the simulations usually have very small wave vectors along the field line, while the toroidal wave vectors can be much larger. Because of this feature, the simulation grids conform to the field lines. As a result, only a small number of poloidal cross-sections are required to resolve the parallel mode structure. This is much coarser than the resolution needed to resolve the toroidal mode structure if the grids are placed along the toroidal direction. Therefore, the interpolation must also be done along the field line instead of along the toroidal direction.

A more detailed discussion on the interpolation schemes used in SDP is provided in Appendix A.

2.2 Synthetic Reflectometry

In this section, we briefly introduce the synthetic reflectometry capability in SDP. We are using a stand-alone synthetic reflectometry code FWR2D\cite{12}, and provide input and output data interfaces, and some post-processing modules. An introduction to the reflectometry measurement is given in Sec 2.2.1. The basic features of the FWR2D code and its interface on SDP are discussed in Sec 2.2.2 and 2.2.3. Sec 2.2.4 shows a
comparison between the Millimeter-wave Imaging Reflectometry (MIR) measurement and the synthetic MIR calculation using FWR2D.

2.2.1 Introduction to Reflectometry

Reflectometry is used to measure electron density profiles and fluctuations\cite{18, 19}. It uses the fact that electromagnetic waves with a certain frequency and polarization can propagate in part of the plasma and get reflected back. The relative phase between the reflected wave and a reference wave contains information about the location of reflection, which is in turn related to the local electron density. Millimeter-wave Imaging Reflectometry\cite{20} (MIR) is an extension of traditional reflectometry, which adds lenses and an array of receivers to provide resolution in the vertical direction.

2.2.2 FWR2D/3D Code

FWR2D is a two dimensional combined WKB/Finite-Difference Time-Domain (FDTD) code which solves the electromagnetic wave equation within a two-dimensional plasma, in which a reflection layer exists\cite{12}. It exploits the fact that the probing waves used for reflectometry measurement usually propagate nearly perpendicularly to the flux surfaces. Away from the reflection layer, an efficient paraxial/WKB approximation is used to solve for the slowly varying wave amplitude, while the rapid phase variation is governed by the local dispersion relation. Near the reflection layer, where the WKB approximation breaks down, the time-dependent full wave equation is solved. The code has been verified and validated against experiments\cite{21}.

The 3D counterpart FWR3D\cite{22} has a very similar structure as FWR2D, but it allows three dimensional plasma and fields. This enables the study of physics, such as polarization mixing and magnetic shear effects.
2.2.3 Interface to FWR2D input and output

SDP contains interfaces for FWR2D/3D to the GTS[15] and XGC-1[17] simulation codes. Both codes provide plasma data on poloidal planes, and use flux coordinates on each plane. On each poloidal cross-section, SDP uses triangulation and 2D interpolation provided by Scipy[23] and matplotlib[24] to obtain data on a regular Cartesian mesh. Since the fluctuations under study have long parallel wavelengths, we interpolate the data linearly along field lines between planes. Details of the data interface are discussed in Sec 2.1 and Appendix A. Post-processing modules are available for converting FWR2D/3D output into complex amplitudes comparable to the signals measured by reflectometers. FWR2D has been widely used as a design and optimization tool for reflectometry systems[25, 26].

2.2.4 Synthetic MIR calculation on DIII-D Edge Harmonic Oscillations

EHOs are coherent oscillations which are usually observed near the plasma edge in Quiescent H-mode (QH-mode) discharges[27]. They are related to the enhanced transport near the plasma edge, and are believed to be crucial for the suppression of Edge Localized Modes (ELMs)[28]. In a recent study of Edge Harmonic Oscillations (EHOs)[8], a careful comparison between MIR measurements and simulation results of M3D-C[13] code[13] was performed by forward modeling using FWR2D.

The MIR diagnostic on DIII-D[20] was used to measure edge electron density perturbation during EHOs. It has 12 vertically separated sight-lines, and four frequencies, corresponding to 4 different radial cutoff locations.

The three dimensional resistive MHD simulation code[13] M3D-C[1] was used to study the nature of the EHOs. M3D-C[1] reads in plasma profiles from a DIII-D discharge (shot #157102) and calculates the linear mode structure for a range of
toroidal mode numbers. The peak amplitude is set to be 2% of the equilibrium density based on the measured magnetic fluctuation level. Synthetic MIR is then applied to M3D-C$^1$ data, and a poloidal wave number spectrum is generated for each probing frequency.

Figure 2.1 shows the comparison between MIR measurement and synthetic MIR result at two frequencies, 57 GHz and 58 GHz, corresponding to two radial locations. Poloidal wave number spectra for the n=1 component of the EHO are calculated based on the phase difference and the vertical distance of all possible pairs of vertical receivers. While a relatively peaked spectrum is measured at 57 GHz, as shown in Fig 2.1(a), no significant features are observed at 58 GHz, Fig 2.1(b). This behavior has been successfully reproduced by the synthetic MIR applied to the M3D-C$^1$ simulation. After passing through the same post-processing routine, the synthetic MIR gives a spectrum for the 57GHz channel with the location and height of the peak which is in fair agreement with measurement. A more careful comparison requires a quantitative assessment of the uncertainties in the measurement, as well as in the simulation. The latter again relies on the synthetic diagnostic tools. Nonetheless, the comparison shown here has demonstrated the potential of synthetic MIR in validation.

2.3 Synthetic Beam Emission Spectroscopy

In this section, we’ll first briefly introduce the Beam Emission Spectroscopy (BES) measurement (Sec 2.3.1). Then an overview of the synthetic BES module is given in Sec 2.3.2. This module was developed by Loic Haussammann, and detailed information can be found in his master thesis[29]. Finally, Sec 2.3.3 provides a demonstration of the capability of the synthetic BES module.
2.3.1 Introduction to BES

BES is a diagnostic method to measure local electron density fluctuations. Line emission resulting from the interaction between the plasma and an injected neutral beam is measured. In many machines, the $D_\alpha$ emission is used. This emission results from the spontaneous relaxation of the electron in the excited beam atom from energy level $n = 3$ down to $n = 2$. The intensity of the observed light is proportional to the local electron density in the plasma\[30\]. A complicating factor in the interpretation of BES signals comes from the fact that excitation and relaxation don’t happen instantaneously. An excited atom travels a finite distance before emitting a photon. This effect sets a resolution limit along the beam direction. The size and alignment of receiving optic fibers and lenses can also affect the received intensity and the resolution.
2.3.2 Synthetic BES

Synthetic BES codes has been used in several studies to validate numerical simulations against measurements \[31, 32\]. In a validation study of the GYRO simulation code \[33\], synthetic BES is applied to the calculated electron density perturbations via the “Point Spread Function” (PSF) \[34, 35\], which is very similar to the Instrumental Function (IF) in the case of electron cyclotron emission diagnostic that will be discussed in Sec 2.4.

Instead of applying a PSF calculated on the equilibrium to the simulated perturbations, the synthetic BES module on SDP can calculate the collected emission intensity with all the available simulation data self-consistently. The finite life time effect is included, as well as the realistic geometry of the optics and fibers \[29, 36\]. It contains routines for calculating neutral beam densities, excitation rates, and optical reception volumes.

A. Population of excited atoms

In order to model the emission intensity, we need to calculate the density of the atoms in given excited state, \(n_{ex}\). Because on most machines the \(n = 3 \rightarrow 2\) transition is measured, the \(n = 3\) energy level atom density is required. This density is closely related to the density of atoms in the ground level, which is defined as the “neutral beam density”, \(n_b\), in the rest of this section. In our synthetic BES, we use the effective stopping coefficient \(S_{cr}\) and the effective emission coefficient \(\langle \sigma v \rangle\) for the \(n = 3 \rightarrow 2\) transition that is provided by Atomic Data and Analysis Structure (ADAS) database \[37, 38, 39\] to calculate \(n_b\) and \(n_{ex}\).

ADAS uses a collisional-radiative model to solve the quasi-equilibrium distribution of a “N-shell” system, which allows an Hydrogen-like atom to be excited up to energy level \(N\) before fully ionized. Given the ground level density \(n_b\) and the ion density \(n_{\infty}\), the model calculates the equilibrium density of each energy level
by balancing the number of atoms leaving that level with the number of atoms entering the level. These energy level transitions include spontaneous, radiative, and collisional processes. Collisions from plasma electrons, ions, and impurities are all included. This dynamical equilibrium is achieved in a fast time scale compared to the ionization process.

On a slower time scale, the ground level density \( n_b \) decreases due to ionizations. This process can be described by an effective stopping coefficient \( S_{cr} \) which is defined as the effective rate of atoms leaving the ground state divided by the plasma electron density [37, 40]. Thus, \( n_b \) obeys the decay equation, and along the central line of beam it takes the form:

\[
v_b \frac{dn_b(z)}{dz} = -S_{cr}(z)n_e(z)n_b(z),
\]

(2.1)

where \( n_e \) is the electron density in plasma, \( v_b \) the speed of beam particles, and \( z \) the distance along central line of beam. The beam profile in the plane perpendicular to the central beam line is approximated as a fixed Gaussian function, i.e. \( n_b(\vec{x}) = n_b(z) \exp(-r^2/\Delta^2) \), \( \Delta \) is a constant controlling the width of the beam. All of the beam particles are assumed to be streaming along the \( z \) direction with the same speed.

The density of neutrals at excited level \( n = 3 \), \( n_{ex}(\vec{x}) \), is then obtained by solving

\[
\vec{v}_b \cdot \nabla n_{ex}(\vec{x}) + \frac{n_{ex}}{\tau} = R_{ex}n_b,
\]

(2.2)

where \( 1/\tau \) is the effective rate of transitions out of \( n = 3 \) level, \( R_{ex}n_b \) is the total number of atoms transitioned into \( n = 3 \) level per unit time per unit volume at the given ground level density.

Eq\.[2.2] was derived from the steady state continuity equation for the neutral particles with excitation level 3. The first term is the divergence of the flux, since we
have assumed the incompressibility of the flow. The second term is the sink of excited particles, and the right-hand-side term is the source.

Since the collisional-radiative model assumes a local dynamical equilibrium between the states, the total number of atoms that transition into \( n = 3 \) state per unit time, should equal the total number of atoms that transition out of \( n = 3 \) state at the same time. In fairly low density plasmas \( (n_e \leq 10^{13}\text{cm}^{-3}) \), the collisional transitions can be neglected compared to the spontaneous transitions from \( n = 3 \) to \( n = 2 \) and \( n = 1 \). Thus, we can approximate the effective emission rate to be the source, i.e.

\[
R_{ex} n_b = \langle \sigma v \rangle_{\text{all}} n_e n_b, \tag{2.3}
\]

where \( \langle \sigma v \rangle_{\text{all}} = \langle \sigma v \rangle_{31} + \langle \sigma v \rangle_{32} \) is the sum of the effective emission coefficients from energy level 3 to both level 1 and level 2. These coefficients are defined as the spontaneous transitions between the energy levels divided by \( n_e n_b \), so the right-hand-side in the above equation is exactly the total spontaneous emission from the \( n = 3 \) excited state. In practice, we only use \( \langle \sigma v \rangle_{32} \) in our code. The reason is that \( \langle \sigma v \rangle_{\text{all}} / \langle \sigma v \rangle_{32} = (A_{32} + A_{31})/A_{32} \) is a constant, where \( A_{ij} \) are the Einstein coefficients of spontaneous transition from \( n = i \) to \( n = j \). So the use of only \( \langle \sigma v \rangle_{32} \) introduces just a constant multiplied to the final measured intensity \( I \). Since we are interested in the relative level of the electron density fluctuations, \( \tilde{n}_e/n_{e0} \sim \tilde{I}/I_0 \), this constant does not change the result.

The advantage of this approximation is that we can directly use the emission coefficient available in ADAS. However, this approximation underestimates the total source, and the error gets larger when electron density gets higher because of the stronger collisional transitions.

We arrive at the equation for \( n_{ex} \) that we solve in the synthetic BES code,

\[
\vec{v}_b \cdot \nabla n_{ex}(\vec{x}) + \frac{n_{ex}}{\tau} = \langle \sigma v \rangle_{32} n_e n_b. \tag{2.4}
\]
Since we have assumed $\vec{v}_b$ is purely in the $z$ direction, Eq 2.4 can be written as an ordinary differential equation in $z$, and the solution is easily found to be

$$n_{ex}(x, y, z) = \frac{1}{v_b} \int_0^\infty n_{col}(x, y, z - z') \exp\left(-\frac{z'}{v_b \tau}\right) dz',$$

(2.5)

where $n_{col}(\vec{x}) \equiv \langle \sigma v \rangle_{32} (\vec{x}) n_e(\vec{x}) n_b(\vec{x})$ is the local source of the $n = 3$ excited atoms as discussed above.

Currently, a constant $\tau$ is used in Eq 2.5 because its dependency on electron density is weak when $n_e < 10^{13} cm^{-3}$, which is usually the case at plasma edge. Inclusion of the density dependency of $\tau$ is needed for calculation of emission from the high density regions.

Note that the coefficients $S_{cr}$ and $\langle \sigma v \rangle_{32}$ are both functions of the neutral particle energy, the electron density, and the electron temperature. This is why they have spatial dependencies in Eq 2.1 and 2.5.

Finally, the number of photons emitted per unit volume per unit time, is then $\varepsilon(\vec{x}) = n_{ex}(\vec{x})/\tau_A$, where $\tau_A = 1/A_{32}$ is the effective lifetime of the excited state due to purely spontaneous emissions from $n = 3$ to $n = 2$, which is collected by a set of optic fibers.

**B. Integration along the optical path**

The observed intensity, $I$, is obtained by integration over detection volume of the optic fiber:

$$I = \frac{1}{4\pi} \int ds \int_{\Sigma(s)} \varepsilon(\vec{x}) \Omega(s, d\sigma) d\sigma \int_{\Sigma(s)} \Omega(s, d\sigma) d\sigma,$$

(2.6)

where $s$ denotes the distance along the optical fiber central line of sight, $\Sigma(s)$ the optic fiber’s view area at $s$, and $\Omega$ the effective receiving solid angle of the area element $d\sigma$ given the size of the lens and the optic fiber. Fig 2.2 shows the 3 kinds of relations between the emission source location and the optic system. Special care
is taken for the case shown in Fig 2.2(c) where the receiving solid angle $\Omega$ is the overlapped region between the lens receiving angle and the fiber receiving angle. Detailed discussion of the numerical technique used to calculate $\Omega$ can be found in [29].

Figure 2.2: Three different kinds of emission sources in optic integrations. (a) $\Omega$ constrained by the lens; (b) $\Omega$ constrained by the fiber; (c) $\Omega$ constrained by both the lens and the fiber.

### 2.3.3 Finite life-time broadening

A comparison between XGC-1 simulated edge electron density fluctuations and synthetic BES signals is shown in Fig 2.3. The non-instantaneous emission model gives rise to an inward shift, broadening, and a reduction of the signal. These effects can significantly change the observed radial and vertical correlations [29], which are generally used to characterize underlying density fluctuations. The synthetic BES module can predict and quantify the potential disagreement between the simulations and observations, and provide information about the spatial resolution limits of the BES measurements.
Figure 2.3: Comparison between (a) simulated electron density fluctuations (color map chosen to emphasize edge fluctuations), (b) derived $n_e$ fluctuations at the focuses of the BES fibers, and (c) the synthetic BES signal. Note the degradation in spatial resolution and the decrease (different color scale in figs. b and c) in the BES signal. Figure courtesy to Loic Hausammann.

2.4 Synthetic Electron Cyclotron Emission Imaging

2.4.1 Introduction to ECE and ECE Imaging

Electron Cyclotron Emission is a very important diagnostic for electron temperature\cite{42, 43}. Magnetized electrons emit electromagnetic waves with frequencies around harmonics of their cyclotron frequencies, $\Omega_c$. When the electron density is sufficiently large, most of the emission is reabsorbed. So, measured from outside of the plasma, the emission spectrum near cyclotron harmonics will be blackbody-
like. Since the blackbody spectrum is proportional to the temperature in the long wave-length limit, the observed ECE power is then closely related to local electron temperature where the emission originates. In magnetically confined fusion devices, the magnetic field strength changes mainly along major radius. So the origin of ECE with frequency $\omega$ is localized in the radial direction where $\omega \approx n\Omega_c(R)$. In the perpendicular propagation and cold plasma limit, neglecting the relativistic and Doppler shift effects, the radial resolution is mainly constrained by the bandwidth of the receiver. For DIII-D, it is typically 1–2 centimeters. Vertically, the resolution is determined by the aperture of the antenna, and for DIII-D, is typically about 5cm.

Traditionally, ECE is used to measure the equilibrium electron temperature profile which is essentially constant on a flux surface. As indicated in Fig 2.4a, the antenna is usually placed at the mid-plane, where the flux surfaces are nearly perpendicular to the radial direction. Thus the vertical resolution does not limit measurement accuracy.

In contrast, the ECEI system is designed to measure electron temperature fluctuations. As illustrated in Fig 2.4b, it adds a vertical array of receivers and a set of optical lenses to achieve enhanced vertical resolution, $\sim 1$cm, and provides a 2D image of the electron temperature [44, 45]. This unique capability enables the direct visualization of the radial and poloidal mode structure, and provides a great opportunity for validation of numerical calculations. A comparison between the ECEI measured Alfven Eigenmodes (AEs) structure on DIII-D and the simulation results from GTC has shown very good agreement, and demonstrates the potential of the ECEI technique [46].

Another technique commonly used to increase the signal to noise ratio of the ECE measurements is the Correlation ECE (CECE) [47]. It correlates the signals from two radially close ECE channels, thus significantly reduces the uncorrelated noise. The synthetic ECE codes introduced in this dissertation can be easily deployed to compare
with CECE measurements by applying the same correlation process to the synthetic ECE signals. Therefore, our discussion in the following sections will focus on the traditional ECE and ECEI systems.

In the ECE case, the receiver is located on the mid-plane, and oriented radially towards the plasma. In the ECEI case, the receivers are slightly tilted in the vertical direction, but the radiation collected is still mainly propagating radially. This feature allows us to use the paraxial approximation to calculate the propagation as discussed in detail in Sec 3.1.3. The importance of refraction and diffraction effects depends on the electron density gradient and the frequency difference between the ECE frequency and the local cutoff frequency. When the ECE resonance location is away from the cutoff location by several wavelengths, the refraction and diffraction effects are fairly weak, so we treat them as small corrections to strictly radial propagation.

We introduce the Instrumental Function (IF), \( f(R, Z) \), of a ECE channel as

\[
T_{rad} = \int f(R, Z) T_e(R, Z) \, dR \, dZ.
\]  

(2.7)

to account for the fact that the measured temperature is a weighted sum of the real electron temperature over one or more plasma volumes. The IF describes where the emission sources are, and how their contributions are weighted in the measured temperature.

Radially, the IF is typically peaked near a cold resonance layer. Its spatial width has contributions from three effects: the bandwidth of the receiver, the relativistic down-shift of the electron cyclotron frequency (Sec 3.1.2C), and the Doppler shift of the resonant frequency (See, for example, Eq 3.71). In the DIII-D ECEI system, the bandwidth is typically 1 GHz. The electron temperature is \( \sim 2 \) keV. For second harmonic ECE, \( \omega \sim 100 \)GHz. Under these parameters, we have roughly \( \Delta \omega_{rel} \sim 0.2 \)GHz. The Doppler shift is important only when \( k_{||} \) is significant. In experiments,
Figure 2.4: (a) Traditional ECE system. Radial resolution is set by the monotonically changing magnetic field strength and the frequency resolution of the receiver. The vertical resolution is determined by the aperture of the antenna. In DIII-D, the radial resolution is around 1–2 cm. The vertical resolution is typically around 5 cm. (b) ECEI system. With the imaging lenses, the vertical receiving beam width is significantly reduced, and an array of receivers is used to collect emission from different vertical locations. The typical vertical resolution is about 1 cm, which is roughly 4 \times \text{vacuum wavelength}. (Figure obtained from [44])
the receiver is oriented nearly perpendicular to the magnetic field, so it mainly collects radiation for which $k_\parallel \approx 0$. So, in the DIII-D ECEI system, the radial resolution is mainly determined by the bandwidth. Although relativistic broadening doesn’t determine the resolution, it is nonetheless very important when optical thickness is low, as shown in Sec 2.4.3A. Note that depending on the plasma conditions, the instrumental function $f(R,Z)$ may not be normalized to 1. In fact, we’ll see that $\int f(R,Z)dRdZ \leq 1$ in Sec 2.4.2C from the one-dimensional calculation.

Although our synthetic ECE code supports arbitrary receiver bandwidth functions specified by users, in the demonstrations shown in this section and in Sec 4.2, 4.3 and 4.4, in order to make the relativistic and Doppler effects more visible, we have ignored the receiver bandwidth, and used only the central frequency of each ECEI channel to calculate the ECE power.

In an earlier validation study of the GYRO code [33], synthetic ECE was used to compare the simulated core temperature fluctuations with the measurements on DIII-D[32, 31, 35]. The IF used in this work is a Gaussian function in both radial and vertical directions, with the radial half-width determined by an estimation of the radial spread, and the vertical half-width determined by the laboratory measured beam width of the receiving optic system [35].

The synthetic ECE modules introduced in this dissertation (Sec 2.4.2, Chap 3 and 4) accurately calculate the ECE IFs by including the most important physics, and can be used as advanced tools for both the validation of simulation codes and the design of new ECE diagnostics.

### 2.4.2 One-dimensional Synthetic ECEI

The one-dimensional synthetic ECEI code (ECEI1D) calculates the received spectral power $P_s(\omega)$ by treating the emission as a collection of ray bundles, and solving the
Equation of Radiative Transfer for each bundle. (See Appendix B for details on the Equation of Radiative Transfer.)

For one bundle of rays, the Equation of Radiative Transfer is

$$N_r(s)^2 \frac{d}{ds} \frac{I(s)}{N_r(s)^2} = \eta(s) - \alpha(s)I(s)$$  \hspace{1cm} (2.8)

where $s$ is the spatial coordinate along the central ray path, $I$ the specific intensity of radiation, i.e. the radiation power flux per unit area, solid angle and frequency, and $N_r$ the ray refractive index. $\eta = \eta(s, \omega)$ is the local emissivity of the chosen mode of radiation. $\alpha = \alpha(s, \omega)$ is the corresponding absorption coefficient.

A formal solution to Eq. 2.8 is:

$$\frac{I(s)}{N_r^2(s)} = \frac{I(s_0)}{N_r^2(s_0)} \exp \left( - \int_{s_0}^{s} \alpha(s')ds' \right) + \int_{s_0}^{s} \frac{ds'}{N_r^2(s')} \exp \left( - \int_{s'}^{s} \alpha(s'')ds'' \right) \eta(s') \frac{N_r^2(s')}{N_r^2(s)}. \hspace{1cm} (2.9)$$

Here, $s_0$ is the start point of the integration path, which is sketched in Fig 2.5. The incident intensity $I(s_0)$ typically results from reflection from walls and/or internal cutoffs. When these are small and/or the optical thickness

$$\tau(s) \equiv \int_{s_0}^{s} \alpha(s')ds'$$  \hspace{1cm} (2.10)

is large, $\tau \gg 1$, then it is often possible to neglect the first term on the right hand side of Eq 2.9.

We can obtain the specific intensity, $I(\theta, \phi)$, for a collection of ray bundles, where $\theta$ and $\phi$ are the inclination angle and the azimuth angle of the central rays respect to the normal direction of the antenna. The total received spectral power $P(\omega)$ is then

$$P_s(\omega) = \int I(\theta, \phi)A_\omega(\theta, \phi)d\Omega,$$ \hspace{1cm} (2.11)
Figure 2.5: Sketch of the integration path in ECEI1D. Path starts at the inner edge of the plasma, where the outgoing ECE intensity is assumed to be zero, and ends at the receiving antenna. The light path is determined by the ray tracing technique, and labeled by the spherical angles $\theta$ and $\phi$

where $A_\omega(\theta, \phi)$ is the effective receiving area of the antenna for the specific wave frequency, polarization, and incident angles $(\theta, \phi)$. It is shown in [48] Sec 10.2 that $A_\omega(\theta, \phi)$ is proportional to the antenna’s Gain $G_\omega(\theta, \phi)$,

$$G_\omega(\theta, \phi) = \frac{4\pi A_\omega(\theta, \phi)}{\lambda^2},$$  \hspace{1cm} (2.12)$$

where $\lambda = \frac{2\pi c}{\omega}$ is the wave length in vacuum. $G_\omega$ is normalized as $\int G_\omega d\Omega = 4\pi$, so $\int A_\omega d\Omega = \lambda^2$. In particular, if the antenna’s receiving angle is very narrow, $A_\omega(\theta, \phi)$ is effectively a properly normalized $\delta$-function at $\theta = 0$, and Eq [2.11] becomes

$$P_s(\omega) = I\lambda^2 = \frac{4\pi^2 c^2}{\omega^2} I,$$  \hspace{1cm} (2.13)$$

where $I$ is the intensity integrated along the central ray. This is the formula we use in ECEI1D.
In order to calculate $I$, using Eq 2.9, we need $\alpha(s)$, $\eta(s)$, and $N_r(s)$ evaluated along the ray trajectory $\vec{r}(s)$. We’ll introduce the formula for $\alpha(s)$ in Sec 2.4.2A. The relationship between $\eta(s)$ and $\alpha(s)$ for a plasma in thermal equilibrium is described in Sec 2.4.2B. Finally, the ray-tracing method for calculation of $\vec{r}(s)$ is described in Sec 2.4.2C.

A. Absorption coefficient

We assume that the plasma responds linearly to the wave electric field with amplitude $\vec{E}$. The absorption coefficient $\alpha$ is formally defined as (See Appendix B and Eq B.25)

$$\alpha \equiv \frac{P_{\text{loss}}}{\left| \vec{S}_{\text{EM}} + \vec{S}_P \right|},$$

where

Power absorption: $P_{\text{loss}} = \frac{\omega r \vec{E}^* \cdot \vec{\varepsilon} \cdot \vec{E}}{8\pi},$ \hspace{1cm} (2.15)

Poynting flux: $\vec{S}_{\text{EM}} = \frac{c}{8\pi} \text{Re}(\vec{E} \times \vec{B}^*),$ \hspace{1cm} (2.16)

Coherent particle power flux: $\vec{S}_P = -\frac{\omega}{16\pi} \vec{E}^* \cdot \frac{\partial \vec{\varepsilon}^H}{\partial \vec{k}} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E}^*,$ \hspace{1cm} (2.17)

and $\vec{\varepsilon} = \vec{\varepsilon}(\omega, \vec{k})$ is the plasma dielectric tensor, with $\vec{\varepsilon}^H$ and $\vec{\varepsilon}^A$ its Hermitian and anti-Hermitian parts.

Since both the numerator and the denominator in Eq 2.14 are proportional to $|\vec{E}|^2$, and $\alpha$ doesn’t depend on the magnitude of $\vec{E}$, but only on its polarization. The polarization of the wave field for a specific wave vector $\vec{k}$ is in turn determined by the wave equations involving the dielectric tensor.

In the ECEI1D code, we only consider the radiation propagating purely perpendicularly to the magnetic field, with no wave electric field along the magnetic field direction. This is usually called an Extraordinary(X)-mode. In experiments, the 2nd
harmonic X-mode ECE is usually chosen because it has a larger optical thickness than the 2nd harmonic O-mode, and is less vulnerable to the cutoffs compared to the fundamental O-mode. We present the key equations needed in ECEI1D for evaluating $\alpha$ and refer the readers to Bornatici’s review paper on ECE\cite{43}, Section 3.1.1 for details.

The absorption coefficient due to the $n^{th}$ harmonic resonance, $\alpha_n$, is expressible as

$$\alpha_n = A_n \alpha_{n,o},$$

where

$$\alpha_{n,o} \equiv \frac{n^{(2n-1)}}{2^n n!} \left( \frac{\omega_{pe}}{\omega_{ce}} \right)^2 \left( \frac{v_{th}}{c} \right)^{2(n-2)} \frac{|\omega_{ce}|}{c} \left[ -\text{Im} F_{n+3/2}(z_n) \right]$$

is the absorption coefficient in the tenuous-plasma limit, $(\omega_{pe}/\omega_{ce})^2 \ll n^2$, where

$$\omega_{pe} = \sqrt{\frac{4\pi n_e e^2}{m}},$$

$$\omega_{ce} = -\frac{eB}{mc},$$

$$z_n = \left( \frac{c}{v_{th}} \right)^2 \frac{\omega - n|\omega_{ce}|}{\omega},$$

$$v_{th} = \sqrt{\frac{T_e}{m}},$$

and the function $F_q(z)$ is the $k_\parallel = 0$ limit of the weakly relativistic dispersion function $F_q(\phi, \psi)$ discussed in Sec. 4.1.1. Detailed discussion can also be found in \cite{43} section 2.3.7.

The coefficient $A_n$ in Eq 2.18 is

$$A_n \equiv (\text{Re} N_\perp)^{(2n-3)} |1 + a_n|^2 b_n.$$  

It describes the corrections to the absorption coefficient due to finite plasma density.
The \((\text{Re}N_\perp)^{(2n-3)}\) term describes refractive effect.

\[
N_\perp^2 = \begin{cases} 
-(1 + b) + [(1 + b)^2 + 4aN_\perp^2c]\sqrt{2a} & (n = 2) \\
N_\perp^2_\perp, c & (n \geq 3)
\end{cases}
\tag{2.25}
\]

is the refractive index, where

\[
N_\perp^2_\perp, c \equiv 1 - \frac{(\omega_{pe})^2}{(\omega)^2} \frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{ce}^2 - \omega_{pe}^2},
\tag{2.26}
\]

is the cold plasma X-mode refractive index, and \(a\) and \(b\) are coefficients which appear in the quadratic dispersion equation for \(n = 2\) case,

\[
a \equiv -\frac{1}{2} \left( \frac{\omega_{pe}}{\omega_{ce}} \right)^2 \frac{\omega^2 - \omega_{ce}^2}{\omega^2 - \omega_{ce}^2 - \omega_{pe}^2} F_{7/2}(z_2),
\tag{2.27}
\]

\[
b \equiv -2 \left( 1 - \frac{\omega_{pe}^2}{\omega(\omega + |\omega_{ce}|)} \right) a.
\tag{2.28}
\]

The \(a_n\) term in Eq 2.24, defined as \(iE_x/E_y\), describes polarization effects, and can be evaluated as

\[
a_n \equiv \begin{cases} 
\frac{1}{2} \left( \frac{\omega_{pe}}{\omega_{ce}} \right)^2 \frac{1 + 3N_\perp^2cF_{7/2}(z_2)}{3 - (\omega_{pe}/\omega_{ce})^2 [1 + \frac{3}{2}N_\perp^2cF_{7/2}(z_2)]} & (n = 2) \\
\left( \frac{\omega_{pe}/\omega_{ce}}{\omega_{ce}} \right)^2 & (n \geq 3)
\end{cases}
\tag{2.29}
\]

The \(b_n\) term, related to the energy flux, deviates from 1 when the coherent particle power flux \(\bar{S}_p\) is non-negligible, i.e. for \(n = 2\).

\[
b_n \equiv \begin{cases} 
\left| 1 + \frac{1}{2} \left( \frac{\omega_{pe}}{\omega_{ce}} \right)^2 (1 + a_2)^2 \text{Re}F_{7/2}(z_2) \right|^{-1} & (n = 2) \\
1 & (n \geq 3)
\end{cases}
\tag{2.30}
\]
B. Kirchhoff’s Law of Radiation: relation between emissivity and absorption coefficient

The emissivity $\eta$ is related to the absorption coefficient $\alpha$ via the Kirchhoff’s Law of Radiation [48].

The Kirchhoff’s Law of Radiation originally concerns the spectral radiation power emitted and absorbed by a body as a whole, stated as follows:

For a body of any arbitrary material in thermodynamic equilibrium, emitting and absorbing electromagnetic radiation, the ratio of its emitted power to the absorbed power at a specific frequency is equal to a universal function only of the radiative frequency and the temperature. This universal function describes the perfect black-body emitted power.

The proof given by Gustav Kirchhoff was based on the argument of energy balance and the second law of thermodynamics[49].

The form of the Kirchhoff’s Law we will use here, is given by Kirchhoff’s student, Max Planck. In his proof of the Kirchhoff’s Law of Radiation, Planck considered the emission and absorption within a spatially varying medium, and provided the following expression of the law:

$$\frac{\eta}{N_r^2 \alpha} = B(\omega, T) \tag{2.31}$$

where $\eta$ and $\alpha$ are the plasma emissivity and absorption coefficient specific to the frequency and polarization. $N_r$ is the ray refractive index, Eq B.57.

The function $B(\omega, T)$ is known to be the Planck’s Black-Body Radiation:

$$B(\omega, T) = \frac{\hbar \omega^3}{8\pi^3 c^2} \frac{1}{e^{\frac{\hbar \omega}{k T}} - 1} \tag{2.32}$$
where $\bar{h} \equiv \hbar/2\pi$ is the Planck constant, and, as usual, we use energy unit for temperature and drop the Boltzmann constant $k_B$ in front of $T$. For electron cyclotron waves in fusion plasmas, $\hbar \omega \ll T$, so we can expand the exponential in the denominator to obtain the Rayleigh-Jeans Law

$$B(\omega, T) = \frac{\omega^2 T}{8\pi^3 c^2}.$$  

(2.33)

Using Kirchhoff’s Law, Eq. 2.31, Rayleigh-Jeans Law, Eq. 2.33, and choosing the initial point $s_0$ where $I(s_0) = 0$, we have

$$\frac{I(s)}{N_r(s)} = \int_{s_0}^{s} ds' \exp \left( - \int_{s'}^{s} \alpha(s'') ds'' \right) \alpha(s') \frac{\omega^2 T(s')}{8\pi^3 c^2}. \quad (2.34)$$

It is worth noting that the Kirchhoff’s Law, Eq. 2.31, is only valid for material in thermodynamic equilibrium. For fusion plasmas, strictly speaking, this is not the case. However, for our purposes, it is usually a good approximation to consider the plasma in local thermal equilibrium described by a local Maxwellian distribution in velocity space. In this case, every plasma element can actually be considered as in thermal equilibrium at its own temperature. Since the emission is a local phenomenon that depends only on the source plasma element’s status, it suggests that the Kirchhoff’s Law should be applicable to this plasma element.

To explicitly show the Kirchhoff’s Law applies, let’s consider a thought experiment in which we remove this locally equilibrium plasma element and put it into a container that’s at the same temperature $T$. Now the plasma element is in exact thermal equilibrium with the environment, thus Kirchhoff’s Law applies. The uniqueness of thermal equilibrium distribution at a certain temperature guarantees that the distribution of the electrons in the plasma element won’t change. Since absorption and emission only depend on the distribution function of the plasma, they should not change either.
C. Light path: ray-tracing method

The last piece we need to evaluate is the light path \( \vec{r}(s) \), along which the integration is carried out. The absorption coefficient \( \alpha \), and plasma temperature \( T \) are both functions of the spatial coordinate \( \vec{r} \). They are eventually evaluated along \( \vec{r}(s) \).

To calculate the light path, we solve ray-tracing equations\(^{[50]}\):

\[
\frac{d\vec{r}}{dt} = \frac{\partial \omega(\vec{r}, \vec{k})}{\partial \vec{k}}, \\
\frac{d\vec{k}}{dt} = -\frac{\partial \omega(\vec{r}, \vec{k})}{\partial \vec{r}},
\]

where the frequency \( \omega(\vec{r}, \vec{k}) \) obeys the dispersion relation. In ECEI1D, we simply use the cold X-mode dispersion relation\(^{[51]}\):

\[
\left( \frac{ck}{\omega} \right)^2 = \frac{RL}{S},
\]

where \( R \equiv 1 - \omega_{pe}^2/\omega(\omega + \omega_{ce}) \), \( L \equiv 1 - \omega_{pe}^2/\omega(\omega - \omega_{ce}) \), and \( S \equiv (L + R)/2 \).

Note that the spatial dependency is in the plasma frequency \( \omega_{pe}^2 = 4\pi e^2 n_e(\vec{r})/m_e \), and the electron cyclotron frequency \( \omega_{ce} = -eB(\vec{r})/m_ec \).

We define

\[
D(\omega, \vec{k}, \vec{r}) \equiv \left( \frac{ck}{\omega} \right)^2 - \frac{RL}{S}.
\]

Then the dispersion relation is simply \( D(\omega, \vec{k}, \vec{r}) = 0 \), and we have

\[
\frac{d\vec{r}}{dt} = -\frac{\partial D/\partial \vec{k}}{\partial D/\partial \omega}, \\
\frac{d\vec{k}}{dt} = \frac{\partial D/\partial \vec{r}}{\partial D/\partial \omega},
\]

\[34\]
The partial derivatives of $D$ are analytically written out in terms of $dn_e/d\vec{r}$ and $dB/d\vec{r}$. The density and magnetic field strength gradients are then evaluated numerically.

Integrating Eq 2.38 with the initial condition $(\vec{r}_0, \vec{k}_0)$ determined by the antenna location and orientation, we obtain the light path $\vec{r}(t)$. Since $s$ is the length along the path, we can also integrate it along the path and get $\vec{r}(s)$.

Finally, the ECE spectral power $P_s(\omega)$ is obtained from Eq 2.13 and 2.34,

$$P_s(\omega) = \frac{4\pi^2 c^2}{\omega^2} I = \frac{4\pi^2 c^2}{\omega^2} \int_{s_0}^{s} ds' \exp \left(-\int_{s'}^{s} \alpha(s'')ds''\right) \alpha(s') \frac{\omega^2 T(s')}{8\pi^3 c^2}$$

$$(2.39)$$

where $\tau(s)$ is the optical thickness defined in Eq 2.10.

It is now clear that the usual definition of the ECE temperature $T_{rad} \equiv 2\pi P_s(\omega)$ is essentially a weighted average of the real temperature along the ray. The Instrumental Function introduced in Sec 2.4.1 becomes $f(\tau') = \exp(\tau' - \tau)$. The integration of $f$ is simply

$$\int_{0}^{\tau} f(\tau')d\tau' = 1 - e^{-\tau} \leq 1,$$

$$(2.40)$$

and it equals 1 only when $\tau \to \infty$, which is the optically thick limit.

### 2.4.3 Shine Through Effect Near Plasma Edge

In this section, we show an example application of the 1D synthetic ECE code in a situation where the optical thickness varies from small to large across channels. The ideal ECE assumption that the received intensity measures temperature at the cold resonance layer is no longer valid for the optically thin channels, and their instrument functions need to be calculated carefully.
When ECE is used to measure the electron temperature profile near the plasma edge, the “shine-through” phenomenon often occurs due to the low plasma density. At low densities, the plasma is no longer optically thick, and it becomes transparent for the ECE radiations\cite{45}. This is particularly important for measurements at the plasma pedestal in H-mode.

At the bottom of the pedestal, and immediately outside of it, the ECE system observes higher effective electron temperatures than obtained by other diagnostics, sometimes even a bump outside the separatrix\cite{45}.

Figure 2.6 shows a typical shine through example for a DIII-D-like plasma. The vertically integrated Instrument Function of three different channels are shown in color. The optically thick channel, as shown in magenta, is ideal for local temperature measurements. The emission comes from a radially localized region right inside the cold resonance. The optically gray channel shown in cyan has a cold resonance at the bottom of the pedestal, and receives the emission power mainly from the lower half of the pedestal. Due to the decrease in optical thickness, this channel receives emission from a source that extends to the mid-pedestal region. The optically thin channel, shown in red, has its cold resonance well outside the pedestal. The emission from the hotter top pedestal “shines through” the low absorption edge and is collected by the receiver. As a result, the measured power is even stronger than that from the edge channel (depicted in cyan).

When the density and temperature are perturbed near the edge, the three channels respond differently. The optically thick channel receives mainly blackbody radiation, so it responds only to the temperature fluctuations, as expected. The optically gray channel behaves in a complex way. While the local ECE emission is changing due to the temperature perturbation, the density perturbation affects the optical thickness, and thus controls the width of the Instrumental Function. An increase in density will make it optically thicker, and more localized to the pedestal bottom. This then
tends to lower the measured ECE power. The final result depends on the relative strength and phase of the temperature and density fluctuations. We will present a density dominant example in Section 4.5 where the ECE signal is out of phase with the temperature fluctuations. The shine-through channel is also affected by both temperature and density perturbations. However, because it is far from the optically thick regime, an increase in density will always give a larger received power. So it will respond to both temperature and density perturbations positively. We will also discuss this effect in detail in Sec 4.5.

Figure 2.6: Synthetic ECE electron temperature profile at the edge of a DIII-D like plasma (solid), the input electron temperature (dashed) and density (dotted). Instrumental functions are shown as colored lines for optically thick (magenta), gray (cyan) and thin (red) channels (absolute values not shown). The corresponding cold resonances are indicated with the arrows in the corresponding colors.
Chapter 3

Theoretical Derivations of ECEI2D

The one-dimensional ECEI code introduced in Sec 2.4 is adequate when the received ECE power can be treated as an incoherent sum of the power of a collection of non-interacting ray bundles. This is usually the case for profile ECE diagnostic, and when ECEI is applied to fluctuations with vertical wave length significantly larger than the viewing beam width. However, when the typical wave length of the fluctuations is comparable to the waist width of the viewing beam, the 1D approximation is no longer valid. We need a 2D model to take into account the diffraction and refraction effects due to the variation of the plasma within the range of a single ray bundle.

In this chapter, we first present the derivation of the two-dimensional ECE calculation using the Reciprocity Theorem (Sec 3.1). Then in Sec 3.2, we compare the 2D ECE formalism with the 1D model, and show that they agree when the approximations in the 1D model are valid. The numerical implementation and some results are given in Chapter 4.
3.1 Reciprocal Calculation of the ECE Power

The main difficulty in developing a two dimensional synthetic ECE model is that the emission originated from different plasma elements are incoherent. In order to obtain the total power received at the antenna, we need to calculate the propagation of each emitted wave separately. This calculation involves solving the wave equations hundreds of times, which is computationally expensive.

The key to efficiently solve the two dimensional ECEI problem is the reciprocity theorem [52]. It relates the original problem with its reciprocal problem, where a wave is launched from the antenna, propagates into the reciprocal plasma, and is absorbed by the electron cyclotron resonance. It is shown that the received power in the original problem can be expressed in terms of a source current correlation tensor and the reciprocal wave field. Using this method, only one wave propagation calculation is required, thus the computational efficiency is greatly enhanced.

In this section, we’ll first present a formal proof of the reciprocity theorem (Sec 3.1.1), then apply it to the ECE case. The methods to obtain the two key components for evaluation of the ECE power – the source current correlation tensor and the reciprocal wave fields – are derived and discussed in detail in Sec 3.1.2 and 3.1.3 respectively.

3.1.1 Reciprocity Theorem

This discussion follows that in [52].

A. The original radiation problem and its Green’s function solution

Consider electromagnetic radiation emerging from the spatially inhomogeneous plasma, being received by an antenna and transmitted through a wave-guide, as shown in Fig 3.1(a). The radiated wave field \((\vec{E}, \vec{H})\) in the plasma satisfies the
Figure 3.1: (a) Original emission problem, (b) reciprocal problem. The small circles illustrate the $\delta$-like emission sources. The arcs stand for the propagating wave fronts.

Maxwell’s Equations,

\begin{align}
\nabla \times \vec{E}(\omega, \vec{r}) &= \frac{i\omega}{c} \vec{H}(\omega, \vec{r}) \tag{3.1} \\
\nabla \times \vec{H}(\omega, \vec{r}) &= -\frac{i\omega}{c} \vec{D}(\omega, \vec{r}) + \frac{4\pi}{c} \vec{j}_s(\omega, \vec{r}) \tag{3.2}
\end{align}

where $\vec{D}(\omega, \vec{r}) \equiv \int \vec{\epsilon}(\omega, \vec{r}, \vec{r}') \cdot \vec{E}(\omega, \vec{r}') d\vec{r}'$ is the electric displacement vector. $\vec{j}_s$ is the source current of the radiation. In the ECE case, this is the current due to electron’s gyro motion. Accordingly, $\vec{\epsilon}(\omega, \vec{r}, \vec{r}')$ is the dielectric tensor of the “smooth” plasma, without considering the radiation source current. A detailed discussion on the source current and the dielectric tensor is given in Appendix C.
In this dissertation, we’ll follow the convention of the frequency domain quantities $A(\omega)$ used in Piliya’s paper[52]. The frequency $\omega$ is defined to be positive. Any real quantity in the time domain, e.g. $E(t)$, is related to its frequency domain counterpart $E(\omega)$ by
\[
E(t) = \text{Re} \left( \int_{0}^{+\infty} E(\omega) e^{-i\omega t} d\omega \right). \tag{3.3}
\]
Similar to the normal Fourier Transformation, it is straightforward to show that $E(\omega)$ can then be written as,
\[
E(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} E(t) e^{i\omega t} dt. \tag{3.4}
\]
Note that $E(t)$ is real, so $E(-\omega) = E^*(\omega)$. We can show these definitions are self-consistent by substituting Eq.3.4 into Eq.3.3 and obtain an identity. These definitions will also be used to calculate the frequency domain source current in Sec. C.10.

To solve the Maxwell’s Equations, we need proper boundary conditions. Instead of putting the boundary at the wall, and use rather complicated boundary conditions, we consider the machine completely inside of our calculation region. The machine walls are then described by some proper dielectric tensors for metals and dielectrics. The boundary conditions are then at $r \to \infty$ where the solution takes an asymptotic form of outgoing plane waves. This treatment simplifies our calculation on the boundary. In principle, the complications now reside in the dielectric tensor $\epsilon$ because it includes the machine wall. However, in the code, we have ignored the machine wall and used only the plasma dielectric tensor. Further development to include the wall is required for calculations that involves reflections from the wall.

In particular, we require our boundary to go through the waveguide, and within the waveguide, the solution is also purely outgoing. Without loss of generality, we consider only a single fundamental mode. The discussion can be easily applied to multiple-mode case due to the mutual orthogonality of the fundamental modes.
As shown in Fig. 3.1(a), we use the notation

\[
\begin{pmatrix}
\vec{E} \\ \vec{H}
\end{pmatrix} = A(\omega) \begin{pmatrix}
\vec{E}^{(\text{out})} \\ \vec{H}^{(\text{out})}
\end{pmatrix},
\]

(3.5)

where \(A(\omega)\) is the complex amplitude and \((\vec{E}^{(\text{out})}, \vec{H}^{(\text{out})})\) are the electric and magnetic fields of the outgoing fundamental waveguide mode, whose integrated Poynting flux is normalized to unity,

\[
\frac{c}{8\pi} \int_{S_{wg}} \vec{E}^{(\text{out})} \times \vec{H}^{(\text{out})\ast} \cdot d\vec{S} = 1,
\]

(3.6)

where \(S_{wg}\) is the area of the waveguide cross-section, and \(d\vec{S}\) is oriented outwards. Note that the integrand is real for fundamental waveguide modes.

In the ECE case, \(A(\omega)\) is a randomly fluctuating quantity due to the random nature of the source current \(j_s\). The correlation function of \(A(\omega)\) has the form:

\[
\langle A(\omega)A^\ast(\omega') \rangle = |A\omega|^2 \delta(\omega - \omega'),
\]

(3.7)

where \(\langle \cdots \rangle\) is the ensemble average over a test particle distribution function. A more detailed discussion about the test particle approach is given in Appendix C.

Then the total received ECE power \(P_s\) can be written as

\[
P_s = \int_0^\infty |A\omega|^2 d\omega = \int_0^\infty P_s(\omega) d\omega,
\]

(3.8)

where \(P_s(\omega)\) is the spectral power density. Because the ECE diagnostic measures the emitted power, \(P_s(\omega)\), the problem of calculating the ECE signal reduces to the calculation of \(A(\omega)\) and its correlation function.
In general, when the radiated field is weak, only the linear response of the plasma needs to be considered. The complex amplitude $A(\omega)$ can be written as a linear functional of the source currents,

$$A(\omega) = \int \vec{j}_s(\omega; \vec{r}') \cdot \vec{g}(\omega; \vec{r}') d\vec{r}',$$  \hspace{1cm} (3.9)$$

where $\vec{g}(\omega; \vec{r}')$ is a function depending only on the “smooth” plasma and its environment, e.g. metals and dielectrics around the plasma, but not on the source current itself. It is clear that $\vec{g}$ is closely related to the Green’s function solution to the Maxwell’s Eq 3.1 and 3.2. The integral in Eq 3.9 is taken over the whole plasma volume since there are no source currents outside the plasma.

In principle, we can obtain the Green’s function solution for Eq 3.1 and 3.2 by using a $\delta$-function as the source current at each plasma location, and propagating its radiated wave into the antenna and calculating the generated waveguide mode amplitude. The total amplitude is then obtained using Eq 3.9 with the source current being generated by a collection of randomly initialized test electrons. This situation is shown in Fig 3.1(a). The small circles represent each $\delta$-like source current, and the curves indicate the propagation of the radiated waves. Computationally, this method requires a very large number of wave propagation calculations which is computationally expensive even with approximations and simplifications.

B. The reciprocal problem and its correspondence to the original problem

The Reciprocity Theorem provides an alternative way to calculate the $\vec{g}$ function defined in Eq 3.9. This is achieved by considering the reciprocal problem.

Consider the reciprocal problem as shown in Fig 3.1(b). Now, an ingoing electromagnetic wave is propagated through the waveguide and towards the plasma. The plasma is replaced by its “transposed” counterpart, whose dielectric tensor $\epsilon^T$ is
related to the origin dielectric tensor by

\[ \tilde{\epsilon}^T(\omega, \vec{r}, \vec{r}') = \tilde{\epsilon}^T(\omega, \vec{r}', \vec{r}), \]  

(3.10)

where superscript \( T \) denotes the matrix transpose. Note that the positions of \( \vec{r} \) and \( \vec{r}' \) are switched due to reciprocity.

The incident wave, denoted by \((\vec{E}^{(in)}, \vec{H}^{(in)})\), is taken to have unit energy flow, as defined in Eq. 3.6 and is in the same wave-guide fundamental mode as considered in the original problem.

The total wave field generated by this incident wave is denoted as \((\vec{E}^{(+)}, \vec{H}^{(+)})\). As in the original problem, we require \((\vec{E}^{(+)}, \vec{H}^{(+)})\) represents asymptotically the out-going wave far away from the plasma, except in the waveguide, where

\[
\begin{pmatrix}
\vec{E}^{(+)} \\
\vec{H}^{(+)}
\end{pmatrix} = \begin{pmatrix}
\vec{E}^{(in)} \\
\vec{H}^{(in)}
\end{pmatrix} + R \begin{pmatrix}
\vec{E}^{(out)} \\
\vec{H}^{(out)}
\end{pmatrix}. 
\]  

(3.11)

The wave field includes both incoming and outgoing waves, and the coefficient of reflection is a complex constant denoted by \( R \). The value of \( R \) is not relevant to the calculations below, so we just formally write it here.

The Maxwell’s equations for \((\vec{E}^{(+)}, \vec{H}^{(+)})\) within the “transposed” plasma are very similar to the original ones, (Eq 3.1 and 3.2,) but don’t have the source current term,

\[
\nabla \times \vec{E}^{(+)}(\omega, \vec{r}) = \frac{i\omega}{c} \vec{H}^{(+)}(\omega, \vec{r}), 
\]  

(3.12)

\[
\nabla \times \vec{H}^{(+)}(\omega, \vec{r}) = -\frac{i\omega}{c} \vec{D}^{(+)}(\omega, \vec{r}), 
\]  

(3.13)

where \( \vec{D}^{(+)}(\omega, \vec{r}) = \int \tilde{\epsilon}^T(\omega, \vec{r}, \vec{r}') \cdot \vec{E}^{(+)}(\omega, \vec{r}')d\vec{r}' \).
Taking dot products between $\vec{H}^{(+)}$ and Eq 3.1, $\vec{E}^{(+)}$ and Eq 3.1, $-\vec{H}$ and Eq 3.12 $-\vec{E}$ and Eq 3.12 and adding all 4 equations, we have

$$\nabla \cdot (\vec{E} \times \vec{H}^{(+)} + \vec{H} \times \vec{E}^{(+}) = \frac{i\omega}{c}(\vec{E} \cdot \vec{D}^{(+)} - \vec{D} \cdot \vec{E}^{(+}) + \frac{4\pi}{c} \vec{j}_s \cdot \vec{E}^{(+)}. \quad (3.14)$$

Integrate this equation over a large volume $V \to \infty$ bounded by a surface $S_0$, and assume formally the waveguide goes through $S_0$. After the volume integration, the left hand side becomes a surface integral because it was a divergence,

$$\int_V \nabla \cdot (\vec{E} \times \vec{H}^{(+)} + \vec{H} \times \vec{E}^{(+})dV = \int_{S_0} (\vec{E} \times \vec{H}^{(+)} + \vec{H} \times \vec{E}^{(+}) \cdot d\vec{S}, \quad (3.15)$$

where $d\vec{S}$ is the surface element on $S_0$ and points outwards from the plasma. Since the surface is far away from the plasma, we can use the asymptotic form of the solution as discussed before. Out of the waveguide, the solutions to both the original and the reciprocal problems represent outgoing waves from a point-like plasma. Locally, on a surface element, they can be taken as plane waves propagating away from the plasma. Since they are propagating in the same direction, it can be shown that the $\vec{E} \times \vec{H}^{(+)}$ and $\vec{H} \times \vec{E}^{(+})$ terms cancel each other. The same cancellation happens in the waveguide for the outgoing components of the two solutions. However, the ingoing component of the reciprocal problem gives a non-zero contribution to the integration. So, using Eq 3.5 and 3.6, we finally have

$$\int_{S_{wg}} (\vec{E} \times \vec{H}^{(+)} + \vec{H} \times \vec{E}^{(+}) \cdot d\vec{S} = \frac{16\pi}{c} A(\omega). \quad (3.16)$$

The first term on the right hand side of Eq 3.14 vanishes after integration over $V$, because of the relation between $\vec{e}$ and $\vec{\epsilon}^T$, Eq 3.10, and the definition of $\vec{D}$ and $\vec{D}^{(+)}. \quad 45$
That is
\[
\int \vec{E} \cdot \vec{D}^{(+)} - \vec{D} \cdot \vec{E}^{(+)} \, d\vec{r} = \int d\vec{r} \int d\vec{r}' \vec{E}(\vec{r}) \cdot \varepsilon^T(\omega, \vec{r}, \vec{r}') \cdot \vec{E}^{(+)}(\vec{r}') - \vec{E}^{(+)}(\vec{r}) \cdot \varepsilon(\omega, \vec{r}, \vec{r}') \cdot \vec{E}(\vec{r}') \tag{3.17}
\]
\[
= \int \int d\vec{r} \int d\vec{r}' \vec{E}(\vec{r}) \cdot \varepsilon^T(\omega, \vec{r}', \vec{r}) \cdot \vec{E}^{(+)}(\vec{r}') - \vec{E}^{(+)}(\vec{r}) \cdot \varepsilon(\omega, \vec{r}, \vec{r}') \cdot \vec{E}(\vec{r}') \nonumber
\]
\[
= \int \int d\vec{r} \int d\vec{r}' \vec{E}^{(+)}(\vec{r}') \cdot \varepsilon(\omega, \vec{r}', \vec{r}) \cdot \vec{E}(\vec{r}) - \vec{E}^{(+)}(\vec{r}) \cdot \varepsilon(\omega, \vec{r}, \vec{r}') \cdot \vec{E}(\vec{r}') \nonumber
\]
\[
= 0. 
\]
Equating the remaining terms, we have
\[
A(\omega) = \frac{1}{4} \int \vec{j}_s(\omega, \vec{r}) \cdot \vec{E}^{(+)}(\omega, \vec{r}) \, d\vec{r} \tag{3.18}
\]
Comparing with Eq 3.9, since the source current \( \vec{j}_s \) is arbitrary, we immediately get
\[
g(\omega, \vec{r}) = \frac{1}{4} \vec{E}^{(+)}(\omega, \vec{r}). \tag{3.19}
\]
Since the reciprocal wave field \( \vec{E}^{(+)} \) is a coherent wave in the transposed plasma, we can obtain it by solving the wave propagation equations only once. This method is much cheaper computationally, than solution of Eq 3.1 and 3.2 for multiple instances of \( \vec{j}_s(\omega, \vec{r}) \).

C. Received radiation power

Using Eq 3.7, 3.8 and 3.18, we have the following formula for the spectral power received,
\[
P_s(\omega) = \frac{1}{16} \int_{-\infty}^{\infty} \vec{E}^{(+)}(\omega; \vec{r}) \vec{K}(\omega; \vec{r}, \vec{r}') \vec{E}^{(+)*}(\omega; \vec{r}') \, d\vec{r} \, d\vec{r}' \tag{3.20}
\]
where \( \vec{K} \) is the source current correlation tensor defined by
\[ \hat{K}_{ik}(\omega; \vec{r}, \vec{r}') \delta(\omega - \omega') \equiv \langle j_{s,i}(\omega; \vec{r}) j_{s,k}^*(\omega'; \vec{r}') \rangle \]  

(3.21)

In electron cyclotron emission case, this source current is the current induced by the cyclotron motion of statistically independent bare test electrons. The ensemble average is taken over the probability distribution of the initial states. A detailed discussion on how the source current of ECE can be derived from test particle theory is given in Appendix C.

The total power registered by a specific receiver can then be obtained by integrating the spectral radiation power, Eq. 3.20 over the proper band width \( f(\omega) \) in frequency domain,

\[ P_s = \int_0^\infty f(\omega) P_s(\omega) \, d\omega, \]  

(3.22)

Numerically, \( f(\omega) \) is taken to be a weighted summation of some \( \delta \)-functions, \( f(\omega) = \sum_i f_i \delta(\omega - \omega_i) \), \( \sum f_i = 1 \). Thus, \( P_s = \sum_i f_i P_s(\omega_i) \). In the work shown in Sec. 4.2 - 4.5, \( f(\omega) = \delta(\omega - \omega_0) \) is used for simplicity, with \( \omega_0 \) the central frequency of the band width.

### 3.1.2 Source Current Correlation Tensor for ECE

In the previous section, we’ve introduced the reciprocal approach to calculate the received emission power \( P_s \), Eq. 3.22. Evaluating \( P_s \) requires the spectral power \( P_s(\omega) \), given by Eq. 3.20 which in turn needs the reciprocal field \( \vec{E}^{(+)} \) and the source current correlation tensor \( \hat{K} \).

In this section, we’ll discuss the calculation of \( \hat{K} \). The method we use to obtain \( \vec{E}^{(+)} \) is the topic of Sec. 3.1.3.

In general, depending on the nature of the emission, the correlation length of the source currents can be comparable to the plasma inhomogeneity length scale. The calculation of \( \hat{K} \) is thus complicated. However, in the ECE radiation case, the source
current is produced by the electron gyro motion. For typical fusion plasmas, the electron’s gyro-radius is much smaller than the plasma inhomogeneity scale. This allows us to use a “local uniformity” approximation to significantly simplify the calculation of $\hat{K}$, and relate the result to those derived from uniform plasmas.

Sec. [3.1.2A] provides a summary of the geometry and approximations we use in the derivation. Sec [3.1.2B] introduces the $k$-space current correlation tensor, $\hat{K}_k$, and gives an expression for non-relativistic electrons. Sec [3.1.2C] generalizes the calculation to include relativistic effects, and Sec [3.1.2D] shows that for relativistic isotropic Maxwellian electron distribution, $\hat{K}_k$ is proportional to the anti-Hermitian part of the plasma dielectric tensor, which is directly related to the Kirchhoff’s Law of Radiation.

**A. General Notions on Geometry and Approximations**

First, we’ll consider some general features regarding the geometry of the plasma and the common ECEI systems, and the corresponding approximations we can use to simplify our calculation.
The geometry of the plasma and the coordinate system we use are shown in Fig 3.2. The $x$ coordinate is along the major radius, with 0 at the machine center. The $y$ coordinate is vertical, with 0 at the mid plane, and increasing upwards. The $z$ coordinate is along $\hat{x} \times \hat{y}$ direction, which is locally aligned with the toroidal direction. We use a two-dimensional plasma model, where all the physical quantities are uniform along $\hat{z}$ direction. Thus, we have ignored the toroidal curvature because the toroidal extent of the ECEI measurements, $D_z$, is negligibly small ($\sim 1$ cm in DIII-D) compared to the major radius of the machine, $R_0$ ($\sim 2$ m for DIII-D, 5 m for ITER).

Since the poloidal magnetic field is much weaker than the toroidal field in most tokamaks, we assume that the magnetic field consists of only the toroidal component along $\hat{z}$ direction. The magnetic field strength, while being uniform in $z$, is changing slowly in $x$ and $y$,

$$\left| \frac{\rho_e}{B} \frac{\partial B}{\partial x} \right| \ll 1,$$

$$\left| \frac{\rho_e}{B} \frac{\partial B}{\partial y} \right| \ll 1,$$

where $\rho_e \equiv v_{th}/|\omega_{ce}|$ is the typical gyro-radius of thermal electrons, and $v_{th} \equiv \sqrt{T_e/m}$ is the electron thermal velocity, $\omega_{ce} = -eB/mc$ the electron gyro-frequency. For a typical DIII-D plasma, $B \sim 2 \times 10^4$ G, $|\omega_{ce}| \sim 3.5 \times 10^{11}$ rad/s, $T_e \sim 10$ keV, the corresponding $v_{th} \sim 4.2 \times 10^9$ cm/s $\sim c/7$, so $\rho_e \sim 0.01$ cm. For ITER, $B \sim 5 \times 10^4$ G, $T_e \sim 30$ keV, $\rho_e \approx 0.007$ cm. Since $B$ is changing in the scale of $R_0$, it is clear that Eq 3.23 is well satisfied.

Now we state some approximations about the electron distribution function $f$. In most realistic cases, we are interested in phenomena with typical time scale $\sim 10^{-4}$ s, which is much longer than the time of the ECE radiation propagating out of the plasma, $\sim 10^{-8}$ s, which in turn is longer than the typical electron cyclotron period $\sim 10^{-11}$ s. This allows us to ignore the time and gyro-angle dependencies in $f$. $f$ is also assumed uniform in $z$, so it is only a function of the electron guiding
center coordinates \((X,Y)\) and the parallel and perpendicular velocity \((v_\parallel, v_\perp)\), \(f = f(X,Y, v_\parallel, v_\perp)\). We assume \(f\) satisfy the “local uniformity” condition,

\[
\left| \frac{\rho_e \partial f}{f \partial X} \right| \ll 1, \quad \left| \frac{\rho_e \partial f}{f \partial Y} \right| \ll 1,
\]

(3.24)

This assumption is valid for most MHD modes and ion scale phenomena, but may not be fully satisfied when electron scale phenomena are under consideration.

Note that for plasmas in equilibrium, \(f\) is changing over the scale of the minor radius \(a\), while \(B\) is changing over the scale of the major radius \(R\). Since \(R > a\), Eq 3.24 is more restrictive than Eq 3.23.

The antenna is assumed being located near the mid-plane, and oriented mainly towards \(-\hat{x}\) direction. The effective size of the antenna receiving area is characterized by two parameters, the vertical size \(D_y\), and the toroidal size \(D_z\). We assume they are much longer than the typical wave length \(\lambda\) of the ECE radiation, so the convergence or divergence of the received emission beam is weak, and we may use the paraxial approximation,

\[
\left| \frac{1}{E^{(+)}} \frac{\partial E^{(+)}}{\partial y} \right| \sim \frac{1}{D_y} \ll \frac{1}{\lambda}, \quad \left| \frac{1}{E^{(+)}} \frac{\partial E^{(+)}}{\partial z} \right| \sim \frac{1}{D_z} \ll \frac{1}{\lambda},
\]

(3.25)

where \(E^{(+)} = |\vec{E}^{(+)}|\) is the amplitude of the reciprocal field.

In addition, we’ll assume that the emitted wave does not go through a region close to the cutoff, so the wave length never gets very large, and the following relation holds through out the propagation,

\[
\left| \frac{\lambda \partial f}{f \partial X} \right| \ll 1.
\]

(3.26)

This relation will lead to the normal WKB approximation when we solve for the wave field \(\vec{E}^{(+)}\).
Note that $\rho_e/\lambda = n v_{th}/2\pi v_{ph} \approx n/2\pi (v_{th}/c) \ll 1$ for most common ECE measurements where the harmonic number $n \leq 3$, and electron temperature $T_e \leq 30\text{keV}$.

Eq 3.26 is therefore more restrictive than both Eq 3.24 and 3.23.

Now, we are ready to calculate the source current correlation tensor. We’ll start with the non-relativistic case, and generalize the calculation to relativistic electrons.

**B. Non-relativistic Case**

The starting point of the calculation is to determine the origin of the source current $\vec{j}_s(\omega, \vec{r})$ in Eq.3.2. One critical difference between the ECE radiation and the radiation caused by an external source (discussed in the previous section) is that the ECE “source currents” are generated by the discrete motion of the electrons in the plasma, while the collective behavior of the electrons has already been described in the dielectric tensor. A detailed discussion of constructing this current based on the 1-particle distribution function from a test particle point of view is given in Appendix C. The key insight in this calculation is that the electromagnetic field we are solving now should be understood as a field generated by an ensemble of statistically independent “test” electrons. The source current is the current produced by the “bare” test electrons, and the dielectric tensor describes the proper “shielding” of the field by the plasma. The solution to this test particle problem agrees with the solution to the original problem in the sense that, to the first order in the plasma parameter, $\epsilon_p \equiv n_e \lambda_D^3$, both give the same ensemble average of the Poynting flux $\langle \vec{E} \times \vec{B} \rangle$, which is the key quantity we are calculating in the ECE radiation problem. Here, we’ll use the source current given in Appendix C directly.

From Appendix C, Eq.C.71-C.73, we can write the total source current as the inco-
herent sum over all electrons:

\[
\begin{align*}
\mathbf{j}_\mathbf{s}_x(\mathbf{r}, \omega) &= \sum_i j^i_{sx}(\mathbf{r}, \omega) = \sum_i \frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\perp i0} \sin \phi_i(t) \delta(\mathbf{r} - \mathbf{r}_i(t)) \, dt, \\
\mathbf{j}_\mathbf{s}_y(\mathbf{r}, \omega) &= \sum_i j^i_{sy}(\mathbf{r}, \omega) = \sum_i -\frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\perp i0} \cos \phi_i(t) \delta(\mathbf{r} - \mathbf{r}_i(t)) \, dt, \\
\mathbf{j}_\mathbf{s}_z(\mathbf{r}, \omega) &= \sum_i j^i_{sz}(\mathbf{r}, \omega) = \sum_i -\frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\parallel i0} \delta(\mathbf{r} - \mathbf{r}_i(t)) \, dt,
\end{align*}
\]  

(3.27)

(3.28)

(3.29)

where \(\mathbf{r}_i(t)\) is the position of the \(i\)th electron at time \(t\), given by

\[
\begin{align*}
\mathbf{r}_i(t) &= (x_i(t), y_i(t), z_i(t)), \\
x_i(t) &= X_{i0} + \rho_{i0} \cos \phi_i(t), \\
y_i(t) &= Y_{i0} + \rho_{i0} \sin \phi_i(t), \\
z_i(t) &= Z_{i0} + v_{\parallel i0} t,
\end{align*}
\]  

(3.30)

and \(X_{i0}, Y_{i0}\) are coordinates of the guiding-center. In our ordering, to the lowest order, we ignore all the drift motions, so \(X_{i0}\) and \(Y_{i0}\) are constants. The subscript \(i\) denotes the \(i\)th electron, and \(0\) denotes the initial values. \(v_{\perp i0}\) and \(v_{\parallel i0}\) are the electron’s velocity perpendicular and parallel to the magnetic field. \(\rho_{i0} \equiv v_{\perp i0}/|\omega_{ce}|\) is the gyro-radius, \(\phi_i(t) = \phi_{i0} - \omega_{ce} t\) is the gyro-angle at time \(t\). A detailed discussion about the electron’s gyro-motion in uniform straight magnetic field can be found in Appendix [C.9].

Note that for simplicity, we have not explicitly written out the dependency of \(\mathbf{j}_e(\mathbf{r}, \omega)\) on each electron’s initial coordinates \(\mathbf{R}_{i0} = (X_{i0}, Y_{i0}, Z_{i0})\) and \(\mathbf{v}_{i0} = (v_{\perp i0}, v_{\parallel i0}, \phi_{i0})\), but this dependency will be used to calculate the ensemble average.
According to the correlation formula derived in Appendix C.7, Eq C.47, we have for the source current correlation,

\[
\langle \vec{j}_s(\vec{r},t)\vec{j}_s^*(\vec{r}',t') \rangle = \frac{N}{V} \int \vec{j}_s^1(\vec{r},t)\vec{j}_s^{1*}(\vec{r}',t') \ f_0(\vec{R}_{10},\vec{v}_{10})d\vec{R}_{10}d\vec{v}_{10},
\]

(3.31)

where \( \vec{j}_s^1 \) is the source current due to electron number 1, \( \vec{R}_{10} \) and \( \vec{v}_{10} \) are its initial guiding center and velocity coordinates. For simplicity, we’ll drop the subscript 1 in the rest of this chapter. \( N \) is the total number of electrons, \( V \) the total plasma volume. \( f_0 \) is the initial 1-particle distribution, which we assume is uniform in both \( Z_0 \) and \( \phi_0 \), i.e. \( f_0(X_0,Y_0,v_{\parallel 0},v_{\perp 0}) \). The normalization convention is the same as in Appendix C,

\[
\frac{1}{V} \int f_0(\vec{R}_0,\vec{v}_0)d\vec{R}_0d\vec{v}_0 = 1,
\]

(3.32)

The electron number density \( n(\vec{R}_0) \) is then

\[
n(\vec{R}_0) = \frac{N}{V} \int f_0(\vec{R}_0,\vec{v}_0)d\vec{v}_0
\]

(3.33)

\( f_0 \) also satisfies the local uniformity assumption discussed in Sec 3.1.2A, Eq 3.24.

Using Eq 3.27-3.29, we have

\[
\langle \vec{j}_s(\vec{r},\omega)\vec{j}_s^*(\vec{r}',\omega') \rangle = \frac{N e^2}{V \pi^2} \int_{-\infty}^{\infty} e^{i\omega t-i\omega' t'} \hat{V}(\delta(\vec{r}-\vec{r}(t))\delta(\vec{r}'-\vec{r}(t'))) \ f_0 d\vec{R}_0d\vec{v}_0dt\,dt'
\]

(3.34)

where the “cross-velocity tensor” \( \hat{V} \) is defined as

\[
\hat{V}(\vec{v}_0,\phi_0,t,t') \equiv \vec{v}(t)\vec{v}(t')
\]

\[
= \begin{pmatrix}
    v_{\perp 0}^2 \sin \phi(t) \sin \phi(t') & -v_{\perp 0}^2 \sin \phi(t) \cos \phi(t') & -v_{\perp 0} v_{\parallel 0} \sin \phi(t) \\
    -v_{\perp 0}^2 \cos \phi(t) \sin \phi(t') & v_{\perp 0}^2 \cos \phi(t) \cos \phi(t') & v_{\perp 0} v_{\parallel 0} \cos \phi(t) \\
    -v_{\perp 0} v_{\parallel 0} \sin \phi(t') & v_{\perp 0} v_{\parallel 0} \cos \phi(t') & v_{\parallel 0}^2
\end{pmatrix}
\]

(3.35)
Substituting Eq.3.34 into Eq.3.21, we have

$$\hat{K}(\vec{r},\vec{r}',\omega)\delta(\omega-\omega') = \frac{N e^2}{V \pi^2} \int_{-\infty}^{\infty} e^{i\omega t - i\omega' t'} \tilde{V}(\vec{r}-\vec{r}(t))\delta(\vec{r}' - \vec{r}(t')) f_0 \, d\vec{R}_0 d\vec{v}_0 dt dt', \quad (3.36)$$

a. Locality of $\hat{K}$ in locally uniform plasma.

We'll first carry out the integration of $\vec{R}_0$, and show that for locally uniform plasmas, $\hat{K}$ only explicitly depends on $\vec{r}' - \vec{r}$, and is only non-zero when $|\vec{r}' - \vec{r}| \sim \rho_e$. Notice that $\vec{R}_0$ only appears in the distribution function $f_0$, and the delta-functions

$$\delta(x - X_0 - \rho_0 \cos \phi(t)) \delta(x' - X_0 - \rho_0 \cos \phi(t'))$$
$$\delta(y - Y_0 - \rho_0 \sin \phi(t)) \delta(y' - Y_0 - \rho_0 \sin \phi(t'))$$
$$\delta(z - Z_0 - v_{\|0} t) \delta(z' - Z_0 - v_{\|0} t'). \quad (3.37)$$

After integration over $X_0$, $Y_0$ and $Z_0$, three of the delta-functions disappear. The $X_0$, $Y_0$, and $Z_0$ in the other three delta-functions and $f_0$ are substituted by $x - \rho_0 \cos \phi(t)$, $y - \rho_0 \sin \phi(t)$, and $z - v_{\|0} t$ respectively. Note that in principle, $\rho_0$ and $\phi(t)$ are also functions of $X_0$ and $Y_0$ because they depend on magnetic field strength. However, since the magnetic field strength hardly changes over a electron gyro-radius, (Eq 3.23) we can take the magnetic field at $(x,y)$ instead of $(X_0, Y_0)$. Similarly, using local uniformity of $f_0$,

$$f_0(x - \rho_0 \cos \phi(t), y - \rho_0 \sin \phi(t), v_{\|0}, v_{\perp0}) \approx f_0(x, y, v_{\|0}, v_{\perp0}). \quad (3.38)$$
So, we have

\[
\int \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \ f_0(X_0, Y_0, v_{\|0}, v_{\perp0}) \ d\vec{R}_0 \\
= \delta(d_x + \rho_0[\cos \phi(t) - \cos \phi(t')]) \ \delta(d_y + \rho_0[\sin \phi(t) - \sin \phi(t')]) \\
\times \delta(d_z + v_{\|0}(t - t')) \ f_0(x, y, v_{\|0}, v_{\perp0}),
\]

(3.39)

where \( \vec{d} \equiv \vec{r}' - \vec{r} \).

We have shown that after integration over \( \vec{R}_0 \), the delta-functions depend explicitly on \( \vec{d} \equiv (\vec{r}' - \vec{r}) \), and weakly on \( \vec{r} \) through magnetic field in \( \rho_0 \), \( \phi(t) \) and \( \phi(t') \), while \( f_0 \) is assumed to change slowly in \( \vec{r} \). So the original current correlation tensor can be written in the form

\[
\hat{K}(\vec{r}, \vec{r}', \omega) = \hat{K}(\vec{d}, \omega; x, y),
\]

(3.40)

which emphasizes the dependency on \( \vec{d} \), and weak dependency on \( x \) and \( y \).

Particularly, since we have assumed uniformity in \( z \)-direction, \( \hat{K} \) does not depend on \( z \), i.e. \( \hat{K} \) is only a function of \( d_z \) but not \( z \).

The integration over velocity space variables \( (v_{\perp0}, \phi_0, v_{\|0}) \) is non-trivial because of the complicated dependencies of \( \vec{v}_0 \) over \( \rho_0 \), \( \phi(t) \) and cross-velocity tensor \( \hat{V} \). So, instead of compute the integration of \( \vec{v}_0 \) in the expression of \( \hat{K} \), Eq 3.36 we’ll use the locality of \( \hat{K} \) we just shown, and derive a \( k \)-space current correlation tensor which is easier to calculate.

**b. Source current correlation tensor in \( k \)-space**

Now we’ll go back to Eq 3.20 and derive a new expression of \( P_s \) using the locality of \( \hat{K} \) in local uniform plasma obtained in the previous section. We are essentially taking Fourier Transform of \( \hat{K} \) with respect to \( \vec{d} = \vec{r}' - \vec{r} \). The result is a new current correlation tensor \( \hat{K}_k \) which has a very similar form as that in uniform plasma.
We start with the integration over $z, z'$. Since $\hat{K}$ is strictly independent of $z$, we have

\begin{align*}
\int dz \int dz' \, \vec{E}^{(+)}(z; \omega, x, y) \cdot \hat{K}(z' - z) \cdot \vec{E}^{(+)*}(z'; \omega, x', y') &= \int dz' \left( \int dz \, \vec{E}^{(+)}(z) \cdot \hat{K}(z' - z) \right) \cdot \vec{E}^{(+)*}(z') \\
&= \frac{1}{2\pi} \int dk_z \, \mathcal{F} \left( \int dz \, \vec{E}^{(+)}(z) \cdot \hat{K}(z' - z) \right) \cdot \mathcal{F}^* \left( \vec{E}^{(+)*}(z') \right) \\
&= \frac{1}{2\pi} \int dk_z \, \vec{E}_{kz} \cdot \hat{K}_{kz} \cdot \vec{E}_{kz}^* \tag{3.41}
\end{align*}

where $\mathcal{F}$ denotes Fourier transform against $z'$, and superscript $*$ means taking complex conjugate. The second equal sign uses Parseval’s formula \cite{54} for Fourier transform, and the third uses convolution theorem. The quantities in the last line are defined as

\begin{align*}
\vec{E}_{kz} &\equiv \int_{-\infty}^{\infty} \vec{E}^{(+)}(z; x, y, \omega) \exp(ik_z z) dz, \tag{3.42} \\
\hat{K}_{kz} &\equiv \int_{-\infty}^{\infty} \hat{K}(d_z; d_x, d_y, x, y, \omega) \exp(ik_z d_z) dd_z. \tag{3.43}
\end{align*}

Note that the convention of the sign of $k_z$ in Fourier transform has been changed here since $E^{(+)}$ is launched from the antenna, which is opposite to the real emitted wave. The electric field $\vec{E}_{kz}^{(+)}$ can be further approximated using WKB solution in $x$ direction.

\begin{align*}
\vec{E}_{kz} = \vec{E}_0(x, y; k_z, \omega) \exp \left( -i \int_{x_0}^{x} k_x(x', y) dx' - i\varphi_0 \right), \tag{3.44}
\end{align*}

where $\vec{E}_0$ is an envelope that varies in the large spatial scale, and $k_x$ is the wave vector which also changes in the large spatial scale. Note that we choose the ingoing wave by setting the sign in front of the phase integral to be $-$, since $\vec{E}^{+}$ is launched from the antenna on the low field side. The initial phase, $\varphi_0$, is related to the start point of the phase integration $x_0$. 

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The WKB approximation is valid when

\[
\frac{1}{k_x^2} \left| \frac{\partial k_x}{\partial x} \right| \ll 1.
\]

It is satisfied in ECEI measurements where normally the 2nd or higher harmonics of electron cyclotron frequency are used, and the wave is sufficiently far away from the cutoff.

Substitute Eq.3.41 and 3.44 into Eq.3.20, we have

\[
P_s(\omega) = \frac{1}{32\pi} \int dk_z \int dy \int dx' \int dx \bar{E}_0(x, y; k_z, \omega) \exp \left(-i \int_{x_0}^{x} k_x(x')dx' - i\varphi_0 \right) \cdot
\]
\[
\hat{K}_{k_z} \cdot \bar{E}_0^*(x',y';k_z,\omega) \exp \left(i \int_{x_0}^{x'} k_x(x'')dx'' + i\varphi_0 \right)
\]
\[
= \frac{1}{32\pi} \int dk_z \int dy \int dx \bar{E}_0(x, y; k_z, \omega) \cdot
\]
\[
\int \hat{K}_{k_z}(d_x, d_y; \omega, x, y) \exp \left(i \int_{x}^{x+d_x} k_x(x'')dx'' \right) \cdot \bar{E}_0^*(x + d_x, y + d_y, k_z, \omega) dd_x dd_y
\]
\[
\approx \frac{1}{32\pi} \int dk_z \int dx dy \ \bar{E}_0(x, y; k_z, \omega)
\]
\[
\cdot \left( \int \hat{K}_{k_z}(d_x, d_y; x, y, \omega) \exp (ik(x)d_x) dd_x dd_y \right) \cdot \bar{E}_0^*(x, y; k_z, \omega).
\]

(3.45)

Here we have used the fact that $\hat{K}_{k_z}$ is nonzero only when $x' - x$ and $y' - y$ are of order $\rho_e$. So for functions that change slowly in space, e.g. $k_x$ and $\bar{E}_0$, the value at $x', y'$ is almost the same as at $x, y$. And the phase integration can be approximated with the multiplication. The integrations over $x'$ and $y'$ also turn into local integrations over $d_x$ and $d_y$.

We define

\[
\hat{K}_k(x, y, k_z, \omega) \equiv \int \hat{K}_{k_z}(d_x, d_y; k_z, \omega, x, y) \exp (ik_x(x)d_x) \ d d_x d d_y
\]
\[
= \int \hat{K}(d_x, d_y, d_z; \omega, x, y) \exp(ik_x d_z) \exp(ik_z(x)d_x) \ d d_x d d_y d d_z
\]

(3.46)
The new formula for $P_s(\omega)$ is finally

$$P_s(\omega) = \frac{1}{32\pi} \int \! dk_z dx dy \bar{E}_0(x, y, k_z, \omega) \cdot \bar{K}_k(x, y, k_z; \omega) \cdot \bar{E}_0^*(x, y, k_z, \omega). \quad (3.47)$$

c. Integration over $\bar{d}$

Now, with Eq. 3.36, 3.39 and 3.46, we can continue to calculate $\bar{K}_k$ by interchanging the order of integration in $\bar{d}$ and in velocity space variables $(v_{\perp i0}, \phi_{i0}, v_{||i0})$. We have first the integration over $\bar{d}$ space, which involves the three $\delta$-functions and the phase exponential,

$$\int \delta(d_x + \rho_0[\cos \phi(t) - \cos \phi(t')]) \delta(d_y + \rho_0[\sin \phi(t) - \sin \phi(t')])$$

$$\delta(d_z + v_{||0}(t - t')) e^{ik_x dx} e^{ik_x dz} \frac{dd_x dd_y dd_z}{2\pi^3} \exp(-ik_x \rho_0[\cos \phi(t) - \cos \phi(t')]) \exp(-ik_{||0}(t - t')). \quad (3.48)$$

Conventionally, we denote $k_{||} = k_z$ and $k_{\perp} = \sqrt{k_{||}^2 + k_{y}^2} \approx k_z(1 + k_{y}^2/2k_{||}^2) \approx k_z$.

Because $k_y \ll k_x$ where the correction is second order in $k_y/k_x$.

After using the Jacobi-Anger identity

$$e^{iz \cos \phi} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{-in\phi}, \quad (3.49)$$

the exponential of $k_{\perp}$ can then be written as

$$\exp(-ik_{\perp} \rho_0[\cos \phi(t) - \cos \phi(t')])$$

$$= \sum_{n} (-i)^n J_n(k_{\perp} \rho_0) e^{i n \phi(t)} \sum_{n'} i^{n'} J_{n'}(k_{\perp} \rho_0) e^{-i n' \phi(t')}, \quad (3.50)$$

where $J_n(z)$ is the $n^{th}$ order Bessel Function of the first kind.
Now, $\hat{K}_k$ is in the form

$$\hat{K}_k\delta(\omega - \omega') = \frac{e^2 n_e(r)}{\pi^2} \int dv_{\perp 0} \int dv_{\parallel 0} \int d\phi_0 f^1_{\nu}(v_{\parallel 0}, v_{\perp 0}) \int dt dt' e^{i\omega t - i\omega' t'} \times \hat{V} \sum_n (-i)^n J_n(k_{\perp \rho_0}) e^{in\phi(t)} \sum_{n'} i^{n'} J_{n'}(k_{\perp \rho_0}) e^{-in'\phi(t')} \exp(-ik_{\parallel v_{\parallel 0}}(t - t')),$$

(3.51)

where $\hat{V}$ is defined in Eq. [3.35] and has $\phi_0$, $t$ and $t'$ dependences.

From Eq. [3.48] and [3.50] we directly see that higher harmonic components can be generated by the gyro-motion of a single electron when $k_{\perp \rho_0}$ is finite. Intuitively, the resonance is between the electron’s x-direction motion and the electric field with finite $k_x$. Since the x-direction motion is $\propto \cos(\omega_{ce} t)$, the phase variation in the field due to this motion contains all the harmonics, and thus may resonate with all the harmonic frequencies. The strength depends on how well the electron can stay in constant phase. When $k_{\perp \rho_0} \to 0$, gyro-motion doesn’t introduce a phase variation, so the electron can’t resonant with any frequencies. In the opposite limit, $k_{\perp \rho_0} \to \infty$, the phase variation due to gyro-motion is so large that no frequency can match with it, the electron will not resonant with the field either. Only when $k_{\perp \rho_0} \sim n$ a strong resonance with frequency $\omega = n|\omega_{ce}|$ is present. This behavior is described by the Bessel functions.

d. Integration over $\phi_0$, $t$, and $t'$

Our next step is to carry out the integration over $\phi_0$, $t$ and $t'$. These integrations result in several $\delta$-functions that cancel the summation in $n'$, and the $\delta(\omega - \omega')$ term on the left-hand-side of Eq [3.51] The detailed calculation is given in Appendix D.

After integration over $\phi_0$, $t$ and $t'$, we have

$$\hat{K}_k = \frac{4e^2 N}{V} \int_{-\infty}^{\infty} dv_{\perp 0} \int_{-\infty}^{\infty} dv_{\parallel 0} 2\pi f_0(x, y, v_{\parallel 0}, v_{\perp 0}) \sum_{n=-\infty}^{\infty} \hat{T}_n \delta(\omega + n|\omega_{ce}| - k_{\parallel v_{\parallel 0}}),$$

(3.52)
where

$$
\hat{T}_n \equiv \begin{pmatrix}
(n^2v_{\perp 0}^2/k_{\perp 0}^2)J_n^2 & i(nv_{\perp 0}^2/k_{\perp 0})J_nJ_n' & -(nv_{\perp 0}v_{\parallel 0}/k_{\perp 0})J_n^2 \\
-i(nv_{\perp 0}^2/k_{\perp 0})J_nJ_n' & v_{\perp 0}^2J_n' & iv_{\perp 0}v_{\parallel 0}J_nJ_n' \\
-(nv_{\perp 0}v_{\parallel 0}/k_{\perp 0})J_n^2 & -iv_{\perp 0}v_{\parallel 0}J_nJ_n' & v_{\parallel 0}^2J_n^2
\end{pmatrix}
$$

(3.53)

The resonance condition is determined by $\delta(\omega + n|\omega_{ce}| - k_{\parallel}v_{\parallel 0})$, which includes the harmonic number $n$, and the Doppler shift term $k_{\parallel}v_{\parallel 0}$. The strength of the resonance is given by the tensor $\hat{T}_n$. If we compare the xx-component and the zz-component, and assume $v_{\perp 0}^2 \sim v_{\parallel 0}^2$, we can see that the xx-component is $(n^2/k_{\perp 0}^2)$ times larger than the zz-component. The xy, yx, and yy components are of the same order as xx-component. So, in fairly low temperature, when $k_{\perp 0}^2 \rho_e^2 \ll 1$, the X-mode wave resonance is much stronger than the O-mode one. This is the main reason that X-mode ECE is usually chosen in experiments.

**e. Integration over $v_{\perp 0}$ and $v_{\parallel 0}$ for Maxwellian distributions**

A close-form expression can be obtained by assuming specific analytic forms of the distribution functions. Here, we will calculate $\hat{K}_k$ for $f_0$ being the multiplication of a density in space, and two Maxwellian distributions in perpendicular and parallel velocity space, i.e.

$$
N/Vf_0(x, y, v_{\parallel 0}, v_{\perp 0}) = n_e(x, y) f_\perp(v_{\perp 0}; x, y) f_\parallel(v_{\parallel 0}; x, y),
$$

(3.54)

$$
f_\perp(v_{\perp 0}; x, y) = \left(\frac{m}{2\pi T_\perp(x, y)}\right)^{1/2} v_{\perp 0} \exp\left(-\frac{mv_{\perp 0}^2}{2T_\perp(x, y)}\right),
$$

(3.55)

$$
f_\parallel(v_{\parallel 0}; x, y) = \left(\frac{m}{2\pi T_\parallel(x, y)}\right)^{1/2} \exp\left(-\frac{m(v_{\parallel 0} - V_\parallel(x, y))^2}{2T_\parallel(x, y)}\right).
$$

(3.56)

$n_e(x, y)$ is the total electron density. The density, temperature, and mean flow are all function of space coordinates, so we drop the $(x, y)$ in the rest of this section for clarity.
Now, the integration over $v_\parallel 0$ can be carried out by integrating $f_\parallel$ against the $\delta$-function. When $f_\parallel$ is a Maxwellian, these terms can be written as the imaginary part of the Plasma Dispersion Function \[55\].

\[
\int f_\parallel \delta(\omega + n|\omega_{ce}| - k_\parallel v_\parallel 0) \begin{pmatrix} 1 \\ v_\parallel 0 \\ v_\parallel 0^2 \end{pmatrix} dv_\parallel 0 = \begin{pmatrix} 1 \\ \omega + n|\omega_{ce}| \\ (\omega + n|\omega_{ce}|)^2 \end{pmatrix} \frac{\text{Im}(Z_0(\zeta_n))}{\pi k_\parallel v_{\parallel 0}^{\text{th}}} \tag{3.57}
\]

Where $v_{\parallel 0}^{\text{th}} \equiv \sqrt{2T_\parallel/m}$, and $\zeta_n \equiv (\omega + n|\omega_{ce}| - k_\parallel V_\parallel)/k_\parallel v_{\parallel 0}^{\text{th}}$ measures the distance from the resonant frequency, which is defined at $\zeta_n = 0$.

The integration over $v_\perp 0$ is harder to carry out. For Maxwellian $f_\perp$, Eq.3.55 we have integrations in the form

\[
\int dv_\perp 0 \begin{pmatrix} v_\perp 0 J_n^2 \\ v_\perp 0 J_n J'_n \\ v_\perp 0^3 J_n^2 \end{pmatrix} \exp\left(-\frac{m v_\perp 0^2}{2T_\perp}\right),
\]

where $J_n = J_n(k_\perp v_\perp 0/|\omega_{ce}|)$.

To carry out these integrations, we need the identity \[56\]

\[
\int dt\ t J_\nu(at) J_\nu(bt) \exp(-p^2 t^2) = \frac{1}{2p^2} \exp\left(-\frac{a^2 + b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right), \tag{3.58}
\]

where $I_\nu(z)$ is the $\nu$th order modified Bessel function of the first kind.

Taking the derivative with respect to $a$, we have

\[
\int dt\ t^2 J_\nu'(at) J_\nu(bt) \exp(-p^2 t^2) = -\frac{1}{4p^4} \exp\left(-\frac{a^2 + b^2}{4p^2}\right) (a I_\nu - b I'_\nu), \tag{3.59}
\]

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and take derivatives with respect to both $a$ and $b$,

$$
\int dt \ t^3 J'_\nu(at)J'_\nu(bt) \exp(-p^2 t^2) = \frac{1}{8p^6} \exp\left(-\frac{a^2 + b^2}{4p^2}\right) [abI_\nu - (a^2 + b^2 - 2p^2)I'_\nu + abI''_\nu].
$$

(3.60)

In our case, $a = b = k_\perp/|\omega_{ce}|$, $p = 1/v_{\perp th}$, and $v_{\perp th} \equiv \sqrt{2T_\perp/m}$.

So,

$$
\int dv_{\perp 0} \begin{pmatrix}
v_{\perp 0} J^2_n \\
v^2_{\perp 0} J_n J'_n \\
v^3_{\perp 0} J^2_n
\end{pmatrix} \exp\left(-\frac{mv^2_{\perp 0}}{2T_\perp}\right) = \frac{T_\perp}{m} e^{-\lambda} \begin{pmatrix}
I_n(\lambda) \\
-\sqrt{\frac{\lambda T_\perp}{m}} [I_n(\lambda) - I'_n(\lambda)] \\
\sqrt{\frac{\lambda T_\perp}{m}} [I_n(\lambda) - (2 - 1/\lambda)I'_n(\lambda) + I''_n(\lambda)]
\end{pmatrix}
$$

(3.61)

where $\lambda \equiv k_\perp^2 v_{\perp th}^2/2\omega_{ce}^2 = k_\perp^2 T_\perp/m\omega_{ce}^2$.

Since $I_n$ satisfies the Bessel’s equation, we can directly obtain

$$
I''_n(\lambda) = \frac{n^2}{\lambda^2} I_n(\lambda) - \frac{1}{\lambda} I'_n(\lambda) + I_n(\lambda),
$$

(3.62)

where the terms are sorted in order of $\lambda$.

Substituting Eq 3.62 into Eq 3.61, we have

$$
\int dv_{\perp 0} \begin{pmatrix}
v_{\perp 0} J^2_n \\
v^2_{\perp 0} J_n J'_n \\
v^3_{\perp 0} J^2_n
\end{pmatrix} \exp\left(-\frac{mv^2_{\perp 0}}{2T_\perp}\right) = \frac{T_\perp}{m} e^{-\lambda} \begin{pmatrix}
I_n(\lambda) \\
-\sqrt{\frac{\lambda T_\perp}{m}} [I_n(\lambda) - I'_n(\lambda)] \\
\sqrt{\frac{\lambda T_\perp}{m}} \left[\left(\frac{n^2}{\lambda^2} + 2\right) I_n(\lambda) - 2I'_n(\lambda)\right]
\end{pmatrix}
$$

(3.63)

Finally, substituting Eq 3.57 and 3.63 into Eq 3.52, we have

$$
\hat{K}_k = \frac{\omega_{pe} T_\perp}{\pi^2} \sum_{n=-\infty}^{+\infty} e^{-\lambda} a_n \hat{Y}_n(\lambda),
$$

(3.64)
where

\[ \hat{Y}_n(\lambda) \equiv \begin{pmatrix} \frac{n^2}{\lambda} I_n & -in (I_n - I'_n) & -n \sqrt{\frac{2}{\lambda}} I_n b_n \\ in (I_n - I'_n) & \left( \frac{n^2 I_n(\lambda)}{\lambda} + 2\lambda I_n - 2\lambda I'_n \right) & -i\sqrt{2\lambda} (I_n - I'_n) b_n \\ -n \sqrt{\frac{2}{\lambda}} I_n b_n & i\sqrt{2\lambda} (I_n - I'_n) b_n & 2I_n b_n^2 \end{pmatrix}, \tag{3.65} \]

and

\[ a_n \equiv \frac{Im(Z_0(\zeta_n))}{k_{||} v_{th}}, \]

\[ b_n \equiv \frac{\omega + n|\omega_{ce}|}{k_{||} v_{perp th}}. \tag{3.66} \]

Note that in low temperature plasma, \( \lambda \ll 1 \), the higher harmonic resonances are small due to the Bessel function \( I_n(\lambda) \sim \lambda^n \). For \( n = 2 \) X-mode ECE, the resonance is of order \( I_2/\lambda \sim \lambda \), while the \( n = 2 \) O-mode resonance is of order \( I_2 \sim \lambda^2 \). This confirms the assertion that X-mode resonance is much stronger to the O-mode of the same harmonic number.

Another key feature of Eq 3.64 is that it allows an anisotropic plasma with different perpendicular and parallel electron temperatures, and a non-zero parallel mean flow. A relativistic generalization for such an anisotropic plasma has not been derived, and is a important future upgrade to the ECEI2D code.

In addition, the expression of \( \hat{K}_k \) we have here is very similar to the expression of the dielectric tensor obtained for a non-relativistic Maxwellian electron distribution (See for example [51] Chapter 10). A more general discussion on this similarity is presented in Sec 3.1.2D.

C. Relativistic generalization

In section 3.1.2B, we have derived a new expression of the measured power \( P_s(\omega) \) in a 2D locally uniform plasma (Eq 3.47), and calculated the new current correlation tensor
\( \hat{K}_k(\vec{k}, \vec{r}, \omega) \) in a non-relativistic limit, and formulas of \( \hat{K}_k \) are given for both a general velocity distribution function (Eq 3.52) and an anisotropic Maxwellian distribution (Eq 3.64).

As discussed in Sec 2.4, the relativistic down-shift of the ECE resonance is important in most fusion plasma devices. In this section, we discuss the extension of our calculation to the relativistic cases.

For relativistic electrons, two places need to be modified:

- Electron’s equation of motion,
- Electron’s distribution function.

a. Relativistic electron’s equation of motion

The relativistic equations of motion for an electron in a uniform static magnetic field are known to be \[57\]:

\[
\frac{d\vec{p}}{dt} = -\frac{e}{c} \vec{v} \times \vec{B}, \quad (3.67)
\]

\[
\frac{dE}{dt} = 0, \quad (3.68)
\]

where \( \vec{p} = \gamma m_e \vec{v} \) is the relativistic momentum, \( E = \sqrt{m_e^2 c^4 + p^2 c^2} \) the total energy, and \( \gamma = \sqrt{1 + \frac{p^2}{m_e^2 c^2}} \) the relativistic factor.

The solution is easily obtained by observing that since \( E \) is conserved, \( \gamma \) is constant in time, so

\[
\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B, \quad (3.69)
\]

where \( \vec{\omega}_B = -\frac{e}{mc} \vec{B} = \frac{\vec{\omega}_{ce}}{\gamma} \).

When comparing Eq 3.69 with Eq C.64, we can see that the only difference between the relativistic and the non-relativistic case is the cyclotron frequency \( \omega_B = \frac{\omega_{ce}}{\gamma} \).
instead of \( \omega_{ce} \). So, the solution is formally the same as the non-relativistic case, Eq C.65 and C.66 whereby all \( \omega_{ce} \) are changed to \( \omega_{ce}/\gamma \) in the definition of \( \rho_0 \) and \( \phi(t) \), i.e. \( \rho_0 = \gamma v_{\perp 0}/|\omega_{ce}|, \phi(t) = \phi_0 - \omega_{ce}t/\gamma \).

Similar to the non-relativistic case, we can construct the source current using the electron trajectory, and eventually obtain the integral expression of the current correlation tensor, which is formally the same as Eq 3.34 while the gyro-frequency in \( \vec{r}(t) \) and \( \phi(t) \) is modified by the relativistic effect.

The following steps are exactly the same as the non-relativistic case, after integration over the initial guiding center coordinates \( \vec{R}_0 \), and introducing the \( k \)-space current correlation tensor \( \hat{K}_k \), we get

\[
\hat{K}_k \delta(\omega - \omega') = \frac{e^2 N}{\pi^2 V} \int d\vec{v}_0 f_0(x, y, v_{\perp 0}, v_{\parallel 0}) \int dt dt' e^{i\omega t - i\omega' t'} \\
\times \hat{V} \exp(-ik_{\perp}v_{\parallel 0}(t - t')) \delta(\omega_{ce}/\gamma - k_{\parallel}v_{\parallel 0}).
\]

We can continue by using the Jacobi-Anger identity, Eq 3.49 and integrating over \( \phi_0, t \) and \( t' \), resulting in an expression similar to Eq 3.52.

\[
\hat{K}_k = \frac{4e^2 N}{V} \int dv_{\perp 0} \int dv_{\parallel 0} f_0(x, y, v_{\perp 0}, v_{\parallel 0}) \sum_{n=-\infty}^{\infty} \hat{T}_n \delta(\omega + \frac{n|\omega_{ce}|}{\gamma} - k_{\parallel}v_{\parallel 0}),
\]

where

\[
\hat{T}_n \equiv \begin{pmatrix}
(n^2 \omega_{ce}^2/\gamma^2 k_\perp^2)J_n^2 & i(nv_{\perp 0}|\omega_{ce}|/\gamma k_{\perp})J_n J_n' & -(nv_{\parallel 0}|\omega_{ce}|/\gamma k_{\perp})J_n^2 \\
-i(nv_{\parallel 0}|\omega_{ce}|/\gamma k_{\perp})J_n J_n' & v_{\perp 0}^2 J_n^2 & iv_{\perp 0}v_{\parallel 0} J_n J_n' \\
-(nv_{\parallel 0}|\omega_{ce}|/\gamma k_{\perp})J_n^2 & -iv_{\perp 0}v_{\parallel 0} J_n J_n' & v_{\parallel 0}^2 J_n^2
\end{pmatrix}.
\]

The evaluation of Eq 3.71 requires a relativistic distribution function, which is the topic of the second part of this section.

b. Relativistic distribution function
Since the electron speed can not exceed the speed of light in vacuum, \( c \), it’s more convenient to write the relativistic electron distribution function in terms of momentum \( \vec{p} \) instead of velocity \( \vec{v} \). Therefore, it will be convenient to rewrite Eq \ref{3.70} in terms of \( \vec{p} \). It is straightforward to substitute \( \vec{v} \) with \( \vec{p}/\gamma m \), so that \( \hat{V} \equiv \vec{v}(t)\vec{v}(t') = \vec{p}(t)\vec{p}(t')/\gamma^2 m^2 \), and \( \rho_0 = \gamma v_\perp/|\omega_{ce}| = p_\perp/m|\omega_{ce}| \).

Again, after taking out the electron density part from the distribution,

\[
\frac{N}{V} f_0(x, y, p_{\perp 0}, p_{\parallel 0}) = n_e(x, y) f_p(p_{\perp 0}, p_{\parallel 0}).
\]

(3.73)

We have

\[
\hat{K}_k \delta(\omega - \omega') = \frac{e^2 n_e}{\pi^2} \int d\vec{p}_0 f_p(p_{\perp 0}, p_{\parallel 0}) \int dt dt' e^{i\omega t - i\omega' t'} \times \frac{\vec{p}\vec{p}'}{\gamma^2 m^2} \exp\left(-i k_\perp p_{\perp 0} \cos \phi(t) - \cos \phi(t')\right) \exp\left(-i k_\parallel p_{\parallel 0} (t - t')/\gamma m\right).
\]

(3.74)

In Sec \ref{sec:3.1.2C}, we start with this expression, and use an isotropic relativistic Maxwellian distribution \[58\] as \( f_p \),

\[
f_p^1(p) = \frac{\mu}{4\pi m^3 c^3 K_2(\mu)} e^{-\mu \gamma(p)}
\]

(3.75)

where \( \mu \equiv m_e c^2/T_e \), and \( K_\nu(z) \) is the \( \nu \)th order modified Bessel function of the second kind.

We can also proceed with the summation of Bessel’s functions formalism, and rewrite Eq \ref{3.71} in terms of \( \vec{p} \),

\[
\hat{K}_k = 4e^2 n_e(\vec{r}) \int dp_0 d\theta f_p(p_0, \theta) \sum_{n=-\infty}^\infty \hat{T}_n \delta \left( \omega + \frac{n|\omega_{ce}|}{\gamma} - \frac{k_\parallel p_0 \cos \theta}{\gamma m_e} \right),
\]

(3.76)
with \( \hat{T}_n \) still defined as in Eq 3.72 but \( J_n = J_n(k_\perp p_0 \sin \theta/m_e|\omega_{ce}|) \) is a function of \( p_0 \) and \( \theta \). Here \( p_0 \) is the modulus of \( \vec{p}_0 \), \( \theta \equiv \arctan(p_\perp/p_\parallel) \) is the pitch angle of the electron.

This integration can be simplified further when we are considering strongly relativistic electrons, e.g. \( \gamma \geq 2 \), and assuming a nearly perpendicularly propagating wave, i.e. \( k_\parallel \ll k_\perp/2 \). Therefore \( \omega \approx c k_\perp \). Since \( |v_\parallel| \ll n|\omega_{ce}|(\gamma - 1)/\gamma \) will hold for electrons with arbitrarily large momentum. So, we may neglect the \( k_\parallel v_\parallel \) term in the resonance condition and get \( \delta(\omega + n|\omega_{ce}|/\gamma) \), where \( \gamma \) is only a function of \( p_0 \), so the integration of \( p_0 \) can be carried out easily using the \( \delta \)-function property\(^{[59]} \),

\[
\delta(\omega + n|\omega_{ce}|/\gamma) = \sum_i \delta(x - x_i) \frac{|f'(x_i)|}{|f'(x)|} \tag{3.77}
\]

where \( x_i \) are the roots of \( f(x) \) and \( f'(x) \neq 0 \) anywhere.

We finally get

\[
\hat{K}_k = 4e^2n_e(\vec{r})mc \sum_{n=-\infty}^{\infty} \frac{n^2\omega_{ce}^2}{\omega^3 z_n} \int d\theta f_p(z_n mc, \theta) \hat{T}_n^\theta(\theta), \tag{3.78}
\]

where

\[
\hat{T}_n^\theta(\theta) = \begin{pmatrix}
(\omega^2/k_\perp^2)J_n^2 & i(\omega/k_\perp \beta_n c \sin \theta)J_n J'_n & -(\omega/k_\perp \beta_n c \cos \theta)J_n^2 \\
-i(\omega/k_\perp \beta_n c \sin \theta)J_n J'_n & \beta_n^2 c^2 \sin^2 \theta J_n^2 & i(\beta_n^2 c^2 \sin \theta \cos \theta)J_n J'_n \\
-(\omega/k_\perp \beta_n c \cos \theta)J_n^2 & -i(\beta_n^2 c^2 \sin \theta \cos \theta)J_n J'_n & \beta_n^2 c^2 \cos^2 \theta J_n^2
\end{pmatrix},
\tag{3.79}
\]

\( z_n \equiv \sqrt{n^2\omega_{ce}^2/\omega^2 - 1} \), \( \beta_n \equiv z_n \omega / n|\omega_{ce}| \), and \( J_n = J_n(z_n k_\perp c / |\omega_{ce}| \cdot \sin \theta) \).

Eq 3.79 now involves only an integral over the pitch angle \( \theta \), which can be evaluated numerically for an arbitrary \( f_p \). Note that the integral is non-trivial because the Bessel functions have \( \theta \) dependence, and need to be evaluated efficiently. In Sec 4.6 when we
calculate the synthetic ECE power due to runaway electrons, we use this expression for the source current correlation tensor.

D. Relativistic Isotropic Maxwellian

In this section, we derive an important relation between the source current correlation tensor $\hat{K}_k$ and the plasma dielectric tensor $\hat{\epsilon}$, when the electron’s distribution function $f_p$ is a relativistic isotropic Maxwellian. This relation corresponds to the Kirchhoff’s Law of Radiation introduced in Sec 2.4.2B. In Sec 3.2, we’ll use this relation and show that the 2D formalism we have derived is consistent with the 1D calculation given in Sec 2.4.2.

Starting from Eq 3.74, we have:

$$
\hat{K}_k \delta(\omega - \omega') = \frac{e^2 n_e(\vec{r})}{\pi^2} \int_{-\infty}^{+\infty} e^{i\omega t} dt \int_{-\infty}^{+\infty} e^{-i\omega' t'} dt' \int d^3\vec{p} \ f_p(p) \times \frac{\vec{p}\vec{p}'}{\gamma^2 m^2} \exp \left(-i \frac{k_\perp (p_y - p_y')}{|\omega_{ce}| m} \right) \exp \left(-i \frac{k_\parallel p_z (t - t')}{\gamma m} \right), \quad (3.80)
$$

where we have written $\vec{p}(t)$ and $\vec{p}'(t')$ as $\vec{p}$ and $\vec{p}'$ for short. We also used the solution $p_y(t) = p_{\perp 0} \cos(\phi(t))$, and $p_z(t) = p_{z0}$ in the exponential part. Since the plasma distribution is isotropic, and $p$ is conserved along the trajectory, $f_p(p_0) = f_p(p)$. In addition, using the solution of $\vec{p}(t)$, it’s easy to show that $d^3\vec{p} = d^3\vec{p}_0$. We can then change the integration over initial condition $\vec{p}_0$ into integration over $\vec{p}$ which is the momentum at time $t$. 

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Now, upon using the dimensionless quantities defined in (Sec 2.3.1),

\[ \bar{\omega} \equiv \omega / |\omega_{ce}| , \]

\[ \tau \equiv \frac{|\omega_{ce}|}{\gamma} (t - t') , \]

\[ \vec{N} \equiv \frac{\vec{k}}{\omega} = (N_\perp, 0, N_\parallel) , \]

\[ \bar{\omega}_{pe} \equiv \omega_{pe} / |\omega_{ce}| , \]

\[ \bar{p} \equiv \bar{p}/mc , \]

\[ \bar{f}_p \equiv (mc)^3 f_p , \]

and noticing that \( \bar{p}' \) is a function of \( \bar{p} \) and \( \tau \),

\[ \bar{p}' = \begin{pmatrix} \cos \tau & \sin \tau & 0 \\ -\sin \tau & \cos \tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \bar{p} , \]

we can write Eq.3.80 as

\[ K_{k,ij} = \frac{\omega T_e}{\pi^2} \epsilon^a_{ij} , \]

where

\[ \epsilon^a_{ij} \equiv -\frac{1}{2i} \frac{i\bar{\omega}^2_{pe}}{\bar{\omega}} \int_{-\infty}^{+\infty} d\tau \int d^3\vec{p} \frac{1}{\vec{p}} \left( -\frac{mc^2\bar{p} \bar{f}_p(\bar{p})}{T_e \gamma} \right) \]

\[ \times \bar{p}_i \bar{p}'_j \exp \left( -i\bar{\omega} \left[ N_\perp (\bar{p}_y - \bar{p}'_y) + N_\parallel \bar{p}_z \tau - \gamma \tau \right] \right) \]

Using the expression \( \bar{f}_p(\bar{p}) \) for an isotropic relativistic Maxwellian, Eq.3.75, we can easily show that

\[ \frac{d\bar{f}_p(\bar{p})}{d\bar{p}} = -\frac{mc^2\bar{p} \bar{f}_p(\bar{p})}{T_e \gamma(\bar{p})} , \]
So, we can see that the tensor $\vec{\epsilon}^a$ we defined in Eq 3.89 is closely related to the plasma dielectric tensor $\vec{\epsilon}$ defined in [43] (Eq 2.3.16),

$$
\epsilon_{ij}(\vec{k}, \omega) = \delta_{ij} - i \frac{\tilde{\omega} \tilde{p}_e}{\tilde{\omega}} \int d^3 \tilde{p} \frac{\tilde{p}_i}{\tilde{p}} \frac{d\bar{f}_1^i}{d\bar{p}}
\times \int_0^\infty d\tau \bar{p}_j' (\tau) \exp (-i \tilde{\omega} \left[ N_{\perp} (\bar{p}_y - \bar{p}'_y) + N_{\parallel} \bar{p}_z \tau - \gamma \tau \right]) .
$$

(3.91)

The only differences between Eq 3.89 and Eq 3.91 are a factor of $1/2i$, an additional $\delta_{ij}$, and the lower limit in the integration of $\tau$.

Now we’ll show that

$$
\vec{\epsilon}^a = \vec{\epsilon}^A \equiv \frac{1}{2i} (\vec{\epsilon} - \vec{\epsilon}^\dagger),
$$

(3.92)

where $\vec{\epsilon}^A$ is the anti-Hermitian part of the dielectric tensor. The $\dagger$ denotes the Hermitian transpose which is defined as the combination of taking complex conjugate and the normal matrix transpose.

First, we notice that the $\delta_{ij}$ term belongs to the Hermitian part of the $\vec{\epsilon}$ tensor, thus doesn’t affect Eq 3.92. We will not consider this term in the following discussion.

Then, we’ll deal with the integration limit in $\tau$.

We define

$$
I_{ij}(\tau) \equiv -i \frac{\tilde{\omega}^2 \tilde{p}_e}{\tilde{\omega}} \int d^3 \tilde{p} \frac{1}{\tilde{p}} \left( -\frac{mc^2 \bar{p}_i \bar{F}_1^i(\bar{p})}{T_e \gamma} \right)
\times \bar{p}_i \bar{p}'_j \exp (-i \tilde{\omega} \left[ N_{\perp} (\bar{p}_y - \bar{p}'_y) + N_{\parallel} \bar{p}_z \tau - \gamma \tau \right]) ,
$$

(3.93)
then we have

\[ \epsilon^{(a)} = \frac{1}{2i} \int_{-\infty}^{\infty} d\tau I(\tau) \]

\[ = \frac{1}{2i} \left( \int_{0}^{\infty} d\tau I(\tau) + \int_{-\infty}^{0} d\tau I(\tau) \right) \]

\[ = \frac{1}{2i} \left( \int_{0}^{\infty} d\tau I(\tau) + \int_{0}^{\infty} d\tau I(-\tau) \right) \]  \hspace{1cm} (3.94)

The first half is already in the form of \( \epsilon^{(a)} \), we need to know how \( I(-\tau) \) is related to \( I(\tau) \).

Let’s consider the transformation of \( \vec{p}'(\tau) \) under \( \tau \to -\tau \).

From Eq \( 3.87 \) we see that \( \vec{p}' \) is the momentum vector rotated by an angle of \( -\tau \) from \( \vec{p} \). When \( \tau \to -\tau \), \( \vec{p}' \) becomes a vector rotated by \( \tau \) from \( \vec{p} \), in other words, \( \vec{p} \) and \( \vec{p}' \) exchange places. Fig \( 3.3 \) illustrates this interchange of \( \vec{p} \) and \( \vec{p}' \). Note that the transformation defined by Eq \( 3.87 \) conserves the modulus, i.e. \( \vec{p} = \vec{p}' \), and the phase space element, \( d^3\vec{p} = d^3\vec{p}' \), so if we exchange \( \vec{p} \) and \( \vec{p}' \) in Eq \( 3.93 \), the Maxwellian part does not change.
So, we have

\[
I_{ij}(-\tau) = -\frac{\tilde{\omega}_e^2}{\tilde{\omega}} \int d^3\bar{p} \frac{1}{\bar{p}} \left( -\frac{mc^2\bar{p}\vec{p}_1^T(\bar{p})}{T_e\gamma} \right) \\
\times \bar{p}_i\bar{p}_j'(-\tau) \exp \left( -i\tilde{\omega} \left[ N_\perp(\bar{p}_y - \bar{p}_y'(-\tau)) - N_\parallel\bar{p}_z\tau + \gamma\tau \right] \right) \\
= -\frac{\tilde{\omega}_e^2}{\tilde{\omega}} \int d^3\bar{p} \frac{1}{\bar{p}} \left( -\frac{mc^2\bar{p}\vec{p}_1^T(\bar{p})}{T_e\gamma} \right) \\
\times \bar{p}_i'\tau(\tau)\bar{p}_j \exp \left( -i\tilde{\omega} \left[ N_\perp(\bar{p}_y'(\tau) - \bar{p}_y) - N_\parallel\bar{p}_z\tau + \gamma\tau \right] \right) \\
= -I_{ji}^*(\tau) \\
= -I_{ij}^\dagger(\tau)
\]

(3.95)

Note that the exchanged indexes \(i\) and \(j\) indicates a matrix transpose, which is a direct result from exchanging \(\vec{p}\) and \(\vec{p}'\).

Substituting Eq 3.95 into Eq 3.94, while using Eq 3.90, 3.91, and 3.93, we finally obtain Eq 3.92.

Eq 3.88 is then a very important relation between the source current correlation tensor \(\hat{K}_k\) and the anti-Hermitian part of the dielectric tensor \(\epsilon^{\leftrightarrow}_A\). We’ll write it again in tensor form:

\[
\hat{K}_k(\vec{k},\omega;\vec{r}) = \frac{\omega T_e(\vec{r})}{\pi^2} \epsilon^{\leftrightarrow}_A(\vec{k},\omega;\vec{r})
\]

(3.96)

This relation is derived in a very general way, so it does not depend on the specific method and approximations used to carry out the integration. In fact, in the non-relativistic limit, comparing with the dielectric tensor given in [51], it can be shown that our expression of \(\hat{K}_k\), Eq 3.64, satisfies this relation when the distribution function is isotropic, i.e. \(T_\perp = T_\parallel\), and \(V_\parallel = 0\).

In Sec 3.2, we’ll show that this relation is very closely related to the Kirchhoff’s Law of Radiation, and guarantees the equivalence of the reciprocal approach and the radiative transfer approach introduced in Sec 2.4.2.
3.1.3 Paraxial and WKB Approximations in Wave Propagation

In Sec 3.1.2 we have introduced the $k$-space source current correlation tensor $\hat{K}_k$ and the corresponding formula to calculate the received emission power $P_s$, i.e. Eq 3.47. The way to calculate $\hat{K}_k$ from a given electron distribution function has been discussed in detail. In this section, we’ll discuss the method to calculate the wave field amplitude $\vec{E}_0$, which is the other quantity required to evaluate $P_s$.

We have shown that $\vec{E}_0$ is related to the full reciprocal wave field $\vec{E}^{(+)}$ by the Fourier Transform in $z$ direction (Eq 3.42) and the WKB treatment in $x$ direction (Eq 3.44). We’ll start with the equation for $\vec{E}^{(+)}$, and derive the equation for $\vec{E}_0$ step by step.

A. Full Wave Equation for the reciprocal field in Locally Uniform Plasma

As introduced in Sec 3.1.1 $\vec{E}^{(+)}(\omega, \vec{r})$ is the wave field generated in the “transposed” plasma by sending in a specifically polarized wave with unit power from the receiving antenna. The evolution of the wave is in general governed by the Maxwell’s Equations similar to Eq 3.1-3.2, without the source current term. Using the definition of $\vec{D}(\omega, \vec{r})$, we obtain the equation for $\vec{E}^{(+)}$,

$$\nabla \times \nabla \times \vec{E}^{(+)} - \frac{\omega^2}{c^2} \int \epsilon_\omega(\omega, \vec{r}, \vec{r}') \cdot \vec{E}^{(+)}(\omega, \vec{r}') d^3\vec{r}' = 0, \quad (3.97)$$

where $\epsilon_\omega(\omega, \vec{r}, \vec{r}')$ is the dielectric tensor in $\vec{r}$ space in the transposed plasma, and $\nabla$ denotes the gradient operator in $\vec{r}$. Since the propagation is always in the transposed plasma, in this section, we no longer write the dielectric tensor as $\epsilon^T$, but just $\epsilon$ for simplicity.

Since the electron cyclotron frequency is much higher than the ion cyclotron frequency $\omega_{ci}$, and normally much higher than the ion plasma frequency $\omega_{pi}$, we can
safely neglect the ion dynamics in the dielectric tensor. Similar to the current correlation tensor, in a locally uniform plasma, \( \epsilon(\omega, \vec{r}, \vec{r'}) \) only explicitly depends on \( \vec{r} - \vec{r}' \), and parametrically depends on \( \vec{r} \) through plasma quantities \( \vec{B}(\vec{r}), n_e(\vec{r}), \) and \( T_e(\vec{r}) \), i.e. \( \epsilon = \epsilon(\omega, \vec{r} - \vec{r'}; \vec{r}) \).

B. Paraxial approximation and WKB approximation

In most ECEI systems, e.g. the one installed on DIII-D [tobias’ commissioning 2010] (see Sec 2.4.1), the receiving antennas are designed to be oriented mainly in the radial direction, i.e. \( x \) direction. So, the fastest variation in \( E(+) \) is the phase variation along \( x \). Further more, we’ll assume the waist width of the microwave beam is much larger than the wave length, so we can use the paraxial approximation which is valid when,

\[
\frac{\left| \frac{\partial E^{(+)} / \partial y}{k_x E^{(+)}} \right|}{k_x E^{(+)}} \approx \frac{\left| \frac{\partial E^{(+)} / \partial z}{k_x E^{(+)}} \right|}{k_x E^{(+)}} \approx \delta \ll 1. \tag{3.98}
\]

The antennas are also located at the Low Field Side (left-hand-side) near the mid plane, so along the light paths, the equilibrium plasma parameters, especially the electron density, are changing mainly in \( x \) direction. In most relevant scenarios, away from the cutoff, the typical scale length of the plasma quantities is much longer than the wave length. This eventually leads to the criterion for using the WKB approximation [60],

\[
\frac{d k_x / dx}{k_x^2} \ll 1, \tag{3.99}
\]

and the solution in the form,

\[
E^{(+)} = E_0^{(+)} \exp \left( -i \int_{x_0}^{x} k_x(x')dx' \right), \tag{3.100}
\]
where \( \vec{E}_0^{(+)} \) is the slowly varying amplitude. We'll assume the following ordering,

\[
\left| \frac{\partial \vec{E}_0^{(+)} / \partial x}{|k_x \vec{E}_0^{(+)}|} \right| \sim \delta^2.
\] (3.101)

Note that in Eq (3.100) we have chosen the in-going wave by taking the “-” sign in front of the phase integral, and requiring \( k_x > 0 \).

The uniformity in \( z \) direction can lead to a further simplification of the wave equation by introducing the Fourier Transformed amplitude \( \vec{E}_0 \),

\[
\vec{E}_0(x, y, k_z, \omega) \equiv \int_{-\infty}^{+\infty} \vec{E}_0^{(+)}(x, y, z, \omega) e^{i k_z z} dz.
\] (3.102)

It’s straightforward to show that the \( \vec{E}_0(x, y, k_z, \omega) \) defined here is the same as defined in Eq (3.42) and (3.44).

Substituting Eq (3.102) and (3.100) into Eq (3.97), we obtain the wave equation for \( \vec{E}_0 \),

\[
\hat{\mathbf{L}} \cdot \vec{E}_0 - \frac{\omega^2}{c^2} \epsilon(\omega, -k_x, -k_z; x, y) \cdot \vec{E}_0 = 0,
\] (3.103)

where

\[
\hat{\mathbf{L}} \equiv \begin{pmatrix}
   k_z^2 - \frac{\partial^2}{\partial y^2} & \left( -i k_x + \frac{\partial}{\partial x} \right) \frac{\partial}{\partial y} & -i k_z \left( -i k_x + \frac{\partial}{\partial x} \right) \\
   \frac{\partial}{\partial y} \left( -i k_x + \frac{\partial}{\partial x} \right) & -\left( -i k_x + \frac{\partial}{\partial x} \right)^2 + k_z^2 & -i k_z \frac{\partial}{\partial y} \\
   -i k_z \left( -i k_x + \frac{\partial}{\partial x} \right) & \left( -i k_x + \frac{\partial}{\partial x} \right) \frac{\partial}{\partial y} & -\left( -i k_x + \frac{\partial}{\partial x} \right)^2 - \frac{\partial^2}{\partial y^2}
\end{pmatrix},
\] (3.104)

and \( \epsilon(\omega, k_x, k_z; x, y) \equiv \int \epsilon(\omega, \vec{r}' - \vec{r}; x, y) \exp(ik_x(x' - x) + ik_z(z' - z)) d^3\vec{r}' \) is the dielectric tensor in \( k \)-space. Note that we have ignored the \( k_y \) contribution, because for the polarization and harmonics we are interested in, e.g. 2nd harmonic X-mode radiation, the correction due to \( k_y \) will be smaller than \( (k \perp \rho_e)^2 \delta \).
C. Solution via a Perturbation Method

Eq 3.103 can be solved using the perturbation method [61], by assuming the solution is a series of perturbations:

\[ \vec{E}_0 = \sum_n \vec{E}_n^n, \]  

(3.105)

with the ordering \( |E_0^{n+1}|/|E_0^n| \sim \delta \). Using the ordering given by Eq 3.98 and 3.101, we can write operator \( \hat{L} \approx \hat{L}^0 + \hat{L}^1 + \hat{L}^2 \), where

\[
\hat{L}^0 \equiv \begin{pmatrix}
0 & 0 & 0 \\
0 & k_x^2 & 0 \\
0 & 0 & k_x^2
\end{pmatrix},
\]

(3.106)

\[
\hat{L}^1 \equiv \begin{pmatrix}
0 & -ik_x \frac{\partial}{\partial y} & -k_x k_z \\
-ik \frac{\partial}{\partial y} k_x & 0 & 0 \\
-k_x k_z & 0 & 0
\end{pmatrix},
\]

(3.107)

\[
\hat{L}^2 \equiv \begin{pmatrix}
k_z^2 - \frac{\partial^2}{\partial y^2} & 0 & 0 \\
0 & i(k_x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} k_x) + k_z^2 & -ik_z \frac{\partial}{\partial y} \\
0 & -ik_z \frac{\partial}{\partial y} & i(k_x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} k_x) - \frac{\partial^2}{\partial y^2}
\end{pmatrix}.
\]

(3.108)

We also write \( \hat{\epsilon}^c(\omega, k_x, k_z; x, y) = \hat{\epsilon}^c(\omega; x, y_0) + \hat{\delta}\epsilon(\omega, k_x, k_z; x, y) \), where

\[
\hat{\epsilon}^c(\omega; x, y_0) \equiv \begin{pmatrix}
S & -iD & 0 \\
iD & S & 0 \\
0 & 0 & P
\end{pmatrix}.
\]

(3.109)
is the cold plasma dielectric tensor \[51\] on radial line with a fixed vertical location \(y = y_0\) along which the wave mainly propagates, with

\[
S \equiv 1 - \frac{\omega^2_{pe}}{\omega^2 - \omega^2_{ce}},
\]

\[
D \equiv \frac{|\omega_{ce}|}{\omega} \frac{\omega^2_{pe}}{\omega^2 - \omega^2_{ce}},
\]

\[
P \equiv 1 - \frac{\omega^2_{pe}}{\omega^2},
\]

Note that \(S\), \(D\) and \(P\) are functions of \(\omega\) and \(x\), since we have fixed \(y = y_0\).

Then \(\dddot{\delta \epsilon}\) consists of 2 parts. The first part comes from the cold dielectric tensor elements evaluated at \(y\) that deviates from \(y_0\). The second part includes the thermal effects which give rise to the dependency of \(k_x\) and \(k_z\), as well as the anti-Hermitian part which corresponds to the absorption.

We’ll assume that

\[
\left| \vec{E}_0 \cdot \dddot{\delta \epsilon} \cdot \vec{E}_0 \right| \sim \delta^2
\]  

Now we can solve Eq 3.103 order by order.

**a. 0th order equation**

For the lowest order in \(\delta\), we have

\[
\hat{L}^0 \cdot \vec{E}_0^0 - \frac{\omega^2_{ce}}{c^2} \epsilon^c \cdot \vec{E}_0^0 = 0.
\]  

Using Eq 3.106 and 3.109, we have

\[
\dddot{M} \cdot \vec{E}_0^0 = 0,
\]  

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where

\[ \tilde{M} \equiv \begin{pmatrix} S & -iD & 0 \\ iD & S - N_x^2 & 0 \\ 0 & 0 & P - N_x^2 \end{pmatrix}, \tag{3.116} \]

and \( N_x^2 = c^2 k_x^2 / \omega^2 \) is the refractive index.

This equation is exactly the same as that in the uniform cold plasma, and the solution is well known\[51\]. However, here, we will solve it using some new notations which will help us in solving the 1st and 2nd order equations.

First we obtain the eigenvalues of \( \tilde{M} \) and the corresponding eigenvectors,

\[ \tilde{M} \cdot \vec{e}_i = \lambda_i \vec{e}_i, \tag{3.117} \]

where

\[ \lambda_1 = P - N_x^2, \]
\[ \lambda_2 = S - \frac{N_x^2 + \sqrt{N_x^4 + 4D^2}}{2}, \tag{3.118} \]
\[ \lambda_3 = S - \frac{N_x^2 - \sqrt{N_x^4 + 4D^2}}{2}, \]

and

\[ \begin{align*}
\vec{e}_1 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
\vec{e}_2 &= \frac{1}{\sqrt{(S - \lambda_2)^2 + D^2}} \begin{pmatrix} iD \\ S - \lambda_2 \\ 0 \end{pmatrix}, \\
\vec{e}_3 &= -\frac{1}{\sqrt{(S - \lambda_2)^2 + D^2}} \begin{pmatrix} S - \lambda_2 \\ iD \\ 0 \end{pmatrix}.
\end{align*} \tag{3.119} \]
Since $\hat{M}$ is Hermitian, it’s easy to show that the eigenvectors belong to different eigenvalues are orthogonal to each other, i.e.

$$\vec{e}_i^* \cdot \vec{e}_j = \delta_{ij}. \quad (3.120)$$

The existence condition of a non-trivial solution to Eq 3.115 requires one of the eigenvalues being 0, thus gives us the normal cold dispersion relations,

$$N_x^2 = P \quad \text{for } (\lambda_1 = 0), \quad (3.121)$$
$$N_x^2 = \frac{S^2 - D^2}{S} \quad \text{for } (\lambda_2 = 0), \quad (3.122)$$

note that $\lambda_3 = 0$ doesn’t have a algebraic solution for $N_x^2$.

Conventionally, the first solution is called “Ordinary mode” or “O-mode”, the second called “Extraordinary mode” or “X-mode”. The corresponding polarization vectors are, of course, $\vec{e}_1$ and $\vec{e}_2$ with $N_x^2$ substituted by Eq 3.121 and 3.122.

$$e_0^x = e_0^y = 0, \quad e_0^z = 1 \quad \text{(O-mode)}, \quad (3.123)$$
$$\frac{e_0^x}{e_0^y} = \frac{iD}{S}, \quad e_0^z = 0 \quad \text{(X-mode)}, \quad (3.124)$$

b. 1st order equation

Taking the first order terms in Eq 3.103, we have

$$\frac{c^2}{\omega^2} \hat{L}^1 \cdot \vec{E}_0^0 - \hat{M} \cdot \vec{E}_0^1 = 0, \quad (3.125)$$

with $\vec{E}_0^0 = E_0^0 \vec{e}^0$ is the 0th order solution that we obtained in the Sec 3.1.3B, where $\vec{e}^0$ is either the O-mode or X-mode polarization eigenvector given in Eq 3.123 and
In addition, \( \hat{M} \) will also be different for O-mode and X-mode waves since \( N_x^2 \) takes different values.

We have shown that \( \vec{e}^0 \) is the eigenvector of \( \hat{M} \) with the 0 eigenvalue, i.e.

\[
\hat{M} \cdot \vec{e}^0 = 0. \tag{3.126}
\]

Since \( \hat{M} \) is Hermitian, we also have

\[
\vec{e}^{0*} \cdot \hat{M} = 0. \tag{3.127}
\]

So, \( \vec{e}^{0*} \). Eq \( 3.125 \) gives us the solubility condition:

\[
\vec{e}^{0*} \cdot \hat{L}^1 \cdot \vec{E}_0^0 = 0. \tag{3.128}
\]

It is straightforward to show that the following identity

\[
\vec{e}^{0*} \cdot \hat{L}^1 \cdot \vec{e}^0 = 0, \tag{3.129}
\]

is exact for both O-mode and X-mode. Note that \( S, D \) and \( P \) are functions of \( x \) but not \( y \), so \( \partial \vec{e}^0 / \partial y = 0 \) for both O-mode and X-mode.

Eq \( 3.128 \) is satisfied for arbitrary \( E_0^0 \). We need to go to the next order for an equation that determines \( E_0^0 \).

c. 2nd order equation

The 2nd order equation reads,

\[
\left( \frac{c^2}{\omega^2} \hat{L}^2 - \hat{\delta} \right) \cdot \vec{E}_0^0 + \frac{c^2}{\omega^2} \hat{L}^1 \cdot \vec{E}_0^1 - \hat{M} \cdot \vec{E}_0^2 = 0. \tag{3.130}
\]
We again take the dot product between $\mathbf{e}^0 \cdot \mathbf{E}_0$ and Eq 3.130, use the Hermitian property of $\hat{M}$ and the fact that $\mathbf{e}^0$ is the eigenvector of $\hat{M}$ with the eigenvalue of zero. Therefore, we have

$$
\mathbf{e}^0 \cdot \left( \frac{c^2}{\omega^2} \mathbf{E}^2 - \delta \mathbf{e} \right) \cdot \mathbf{E}_0^0 + \mathbf{e}^0 \cdot \frac{c^2}{\omega^2} \mathbf{L}^1 \cdot \mathbf{E}_0^1 = 0. 
$$

(3.131)

Now we need to use Eq 3.125 to solve $\mathbf{E}_0^1$ in terms of $\mathbf{E}_0^0$. We do this for O-mode and X-mode polarizations separately since the expressions are slightly different. From now on, we drop the subscript 0 in all $\mathbf{E}_0^n$, and the superscript in $\mathbf{e}^0$ for clarity, and adopt the notation $\mathbf{e}^{O/X}_i$ for the eigenvectors defined by Eq 3.119 specifically for O/X-mode dispersions, Eq 3.121 and 3.122.

**O-mode**

In O-mode, $\lambda_1 = 0$, and $\mathbf{e} = \mathbf{e}^O_1$.

Substituting Eq 3.121 into Eq 3.118, we have

$$
\lambda_2 = S - \frac{P + \sqrt{P^2 + 4D^2}}{2},
$$

$$
\lambda_3 = S - \frac{P - \sqrt{P^2 + 4D^2}}{2},
$$

(3.132)

Since the magnitude of $\mathbf{E}^0$ is arbitrary, we can always assume $\mathbf{E}_1^1$ is orthogonal to $\mathbf{E}_0^0$ by absorbing the parallel part into $\mathbf{E}_0^0$. So, not losing generality, we can write $\mathbf{E}_1^1 = c_2 \mathbf{e}^O_2 + c_3 \mathbf{e}^O_3$ where $c_2$ and $c_3$ are determined from Eq 3.125, as shown below.

Taking dot product between $\mathbf{e}^O_2^*$ and Eq 3.125, we have

$$
\frac{c^2}{\omega^2} \mathbf{e}^O_2^* \cdot \mathbf{L}^1 \cdot \mathbf{e}_1^O \mathbf{E}^0_1 - \lambda_2 c_2 = 0.
$$

(3.133)

So

$$
c_2 = \frac{c^2}{\omega^2} \frac{\mathbf{L}^1_2 \mathbf{E}^0}{\lambda_2},
$$

(3.134)
where 
\[ \mathcal{L}^n_{ij} \equiv \tilde{e}_i^O \cdot \hat{\mathcal{L}}^n \cdot \tilde{e}_j^O \] (3.135)
is the \(ij\)-th tensor element of \(\hat{\mathcal{L}}^n\) in the representation with basis \((\tilde{e}_1^O, \tilde{e}_2^O, \tilde{e}_3^O)\).

Similarly, \(\tilde{e}_3^O \cdot \text{Eq 3.125}\) gives us
\[ c_3 = \frac{c_3^2}{\omega^2} \frac{L_{31}^1}{\lambda_3} E^0. \] (3.136)

Finally, we have
\[ E^1 = \left( \frac{c_3^2}{\omega^2} \sum_{i=2,3} \frac{L_{i1}^1}{\lambda_i} \tilde{e}_i^O \right) E^0. \] (3.137)

Substituting Eq 3.137 into Eq 3.131 and noting that \(\tilde{e}^O\) is replaced by \(\tilde{e}_1^O\), we obtain
\[ \left( \frac{c_3^2}{\omega^2} L_{11}^2 - \delta \epsilon_{11} + \frac{c_3^2}{\omega^4} \sum_{i=2,3} \frac{L_{1i}^1 L_{i1}^1}{\lambda_i} \right) E^0 = 0, \] (3.138)
where \(L_{11}^2, L_{1i}^1, \text{ and } L_{i1}^1\) are the tensor elements as defined in Eq 3.135 and \(\delta \epsilon_{11} \equiv \tilde{e}_1^O \cdot \tilde{\epsilon} \cdot \tilde{e}_1^O = \delta \epsilon_{zz} = \epsilon_{zz} - P\).

Using the expression of \(\tilde{e}_1^O\) given by Eq 3.119 and \(L^1\) in Eq 3.107, we obtain
\[ L_{12}^1 = -L_{21}^1 = \frac{-i D k_x k_z}{\sqrt{(P + \sqrt{P^2 + 4D^2})^2/4 + D^2}}, \]
\[ L_{13}^1 = L_{31}^1 = \frac{-(P + \sqrt{P^2 + 4D^2}) k_x k_z}{2 \sqrt{(P + \sqrt{P^2 + 4D^2})^2/4 + D^2}}, \] (3.139)
and after some algebra, we get
\[ \sum_{i=2,3} \frac{L_{i1}^1 L_{i1}^1}{\lambda_i} = \frac{S - P}{S(S - P) - D^2 k_x k_z^2}. \] (3.140)

Using the expressions for \(S, D, \text{ and } P\), (Eq 3.110-3.112) we can further prove that
\[ S - P = S(S - P) - D^2, \] (3.141)
Finally, we have
\[
\sum_{i=2,3} \frac{L_i^1 L_{i1}^1}{\lambda_i} = \frac{\omega^2}{c^2} P k_z^2, \tag{3.142}
\]
where we have used the O-mode dispersion relation, Eq 3.121, to write \(k_z^2\) as \(P \omega^2/c^2\).

The \(L_{11}^2\) term in Eq 3.138 can be easily obtained thanks to the simple expression of the O-mode polarization vector,
\[
L_{11}^2 = i (k_x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} k_x) - \frac{\partial^2}{\partial y^2}, \tag{3.143}
\]
which is just the \(zz\) component of \(L^2\) given in Eq 3.108.

So, substituting Eq 3.143, 3.142 into Eq 3.138 we finally get the equation for \(E^0\) in O-mode polarization
\[
\left[ 2i k_x \frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2} + P k_z^2 - \frac{\omega^2}{c^2} \delta \epsilon_{OO} \right] F_O^0 = 0, \tag{3.144}
\]
where
\[
F_O^0 = k_x^{1/2} E^0. \tag{3.145}
\]
We have written \(\delta \epsilon_{11}\) as \(\delta \epsilon_{OO}\) to emphasize this is the tensor element projected by O-mode polarization vector.

We’ll leave the discussion of the properties of this equation, and how it is numerically solved in Sec 4.1.2. The notation of \(F_O^0\) is introduced to have a unified notation with the X-mode equation, as derived below.

**X-mode**

The X-mode calculation is very similar to that of the O-mode calculation, with \(\lambda_2 = 0\), and \(\vec{e} = \vec{e}_2^X\).
Substituting the X-mode dispersion relation (Eq 3.122) into Eq 3.118, we have

$$\lambda_1 = P - \frac{S^2 - D^2}{S},$$
$$\lambda_3 = \frac{S^2 + D^2}{S}. \tag{3.146}$$

After taking similar steps as in the O-mode case, by taking the dot products $\vec{e}_X^* \cdot \vec{e}_1^* \cdot \vec{e}_3^*$ with Eq 3.125, we can express $\vec{E}^1$ in terms of $E^0$,

$$E^1 = \left( \frac{c^2}{\omega^2} \sum_{i=1,3} \frac{L^1_i}{\lambda_i} \vec{e}_i^X \right) E^0, \tag{3.147}$$

whereby all the tensor elements are now taken in the representation with the X-mode basis ($\vec{e}_1^X, \vec{e}_2^X, \vec{e}_3^X$).

The equation of $E_0$ reads

$$\left( \frac{c^2}{\omega^2} L^2_{22} - \delta\epsilon_{22} + \frac{c^4}{\omega^4} \sum_{i=1,3} \frac{L^1_i L^1_{i2}}{\lambda_i} \right) E^0 = 0. \tag{3.148}$$

Now the $L^2_{22}$ and $\delta\epsilon_{22}$ terms are slightly more complicated than those in the O-mode case, because of the different polarization vector.

Using the eigenvectors, Eq 3.119 with X-mode dispersion relation, Eq 3.122 and the expression for $\hat{L}^2$, Eq 3.108, we have

$$L^2_{22} = \frac{i S}{\sqrt{S^2 + D^2}} \left( \frac{k_x}{\partial_x} + \frac{\partial}{\partial x} k_x \right) S \frac{1}{\sqrt{S^2 + D^2}} - \frac{D^2}{S^2 + D^2} \frac{\partial^2}{\partial y^2} + k^2, \tag{3.149}$$

whereby $S$ and $D$ are functions of $x$, so we have to pay attention to the order of the operator $\partial/\partial x$ relative to the $S$ and $D$ terms.

We also have

$$\delta\epsilon_{22} = \frac{1}{S^2 + D^2} \left( D^2 \delta\epsilon_{xx} - i DS (\delta\epsilon_{xy} - \delta\epsilon_{yx}) + S^2 \delta\epsilon_{yy} \right). \tag{3.150}$$
Using $\mathcal{L}^1$ given in Eq 3.107, we have

$$\begin{align*}
\mathcal{L}_{12}^1 &= -\mathcal{L}_{21}^1 = \frac{-iD_k k_z}{\sqrt{S^2 + D^2}}, \\
\mathcal{L}_{23}^1 &= \mathcal{L}_{32}^1 = ik_x \frac{\partial}{\partial y}.
\end{align*}$$

So,

$$\sum_{i=1,3} \frac{\mathcal{L}_{2i}^1 \mathcal{L}_{i2}^1}{\lambda_i} = \frac{\omega^2}{c^2} \left[ \frac{(S^2 - D^2)D^2}{(S^2 + D^2)(SP - S^2 + D^2)} k_z^2 - \frac{S^2 - D^2}{S^2 + D^2} \frac{\partial^2}{\partial y^2} \right],$$

where we have used the X-mode dispersion relation $k_x^2 = (\omega^2/c^2)(S^2 - D^2)/S$.

Substituting Eq 3.146, 3.149, and 3.152 into Eq 3.148, we get

$$\left[ 2i k_x \frac{\partial}{\partial x} - \frac{\partial^2}{\partial y^2} + \left( \frac{S^2 + D^2}{S^2} - \frac{D^2(S^2 - D^2)}{S^2(S^2 - SP - D^2)} \right) k_z^2 - \frac{\omega^2 S^2 + D^2}{S^2} \delta_{xx} \right] F_X^0 = 0,$$

where

$$F_X^0 = k_{x}^{1/2} \frac{S}{\sqrt{S^2 + D^2}} E^0.$$

Eq 3.144 and 3.153 are the equations we solve for O-mode and X-mode wave field amplitudes. They are both first order differential equations in $x$, and can be solved in similar ways. The numerical scheme we use to solve these equations is discussed in detail in Sec 4.1.2.

With the reciprocal wave amplitude $\vec{E}_0$ and the $k$-space source current correlation tensor $\hat{K}_k$, we can evaluate the ECE power $P_s$ using Eq 3.47. This is the complete 2D formalism of our synthetic ECE code.

### 3.2 Comparison between 1D and 2D formalisms

So far, we have introduced two ways to calculate the received ECE radiation power.
The 1D formalism, given in Sec 2.4.2, solves the equation of radiative transfer which describes the intensity of radiation along the light path. The received spectral power $P_s(\omega)$ is given by Eq 2.39 for a unidirectional antenna.

The 2D calculation is based on the reciprocity theorem, and expresses the received spectral power as an integration involving both the reciprocal electric field amplitude and the source current correlation tensor, Eq 3.47.

In this section, we’ll show these two approaches are equivalent when the 1D model is valid.

### 3.2.1 Expressions of the spectral power in 1D and 2D calculations

In the 1D model, the spectral power formula is (See Eq 2.11)

$$P_s(\omega) = \int I(\theta, \phi)A_\omega(\theta, \phi)d\Omega. \quad (3.155)$$

For a unidirectional antenna, and using the solution to the equation of radiative transfer, we obtain Eq 2.39

$$P_s^{1D}(\omega) = \frac{1}{2\pi} \int_{s_0}^{s} ds' \exp \left(-\int_{s'}^{s} \alpha(s'')ds'' \right) \alpha(s')T_e(s'), \quad (3.156)$$

where $\alpha$ is defined by Eq 2.14

$$\alpha \equiv \frac{\omega \mathbf{E}^*(\omega) \cdot \varepsilon^A \cdot \mathbf{E}(\omega)}{8\pi |\mathbf{S}|}, \quad (3.157)$$

where $\varepsilon^A$ is the anti-Hermitian dielectric tensor, and $\mathbf{S} \equiv \mathbf{S}_{EM} + \mathbf{S}_P$ is the spectral power flux of the wave. The spectral wave field $\mathbf{E}(\omega)$ is related to the emission field
in space and time by

$$\vec{E}(\vec{r}, t) = \text{Re} \int_0^\infty \vec{E}(\omega) \exp(-i\omega t + i\vec{k} \cdot \vec{r}) \, d\omega$$  \hspace{1cm} (3.158)$$

On the other hand, our 2D approach arrives at the expression for the received spectral power given by Eq 3.47,

$$P_{2D}^s(\omega) = \frac{1}{32\pi} \int dk_z dx dy \vec{E}_0(\omega, k_z; x, y) \cdot \vec{K}_k(\omega, k_x(x), k_z; x, y) \cdot \vec{E}_0^*(\omega, k_z; x, y).$$  \hspace{1cm} (3.159)

The complex amplitude $\vec{E}_0(\omega, k_z; x, y)$ is related to the reciprocal field $\vec{E}^{(+)}(\vec{r}, t)$ as

$$\vec{E}^{(+)}(\vec{r}, t) = \text{Re} \int_0^\infty d\omega \vec{E}_{\vec{k}}(\vec{r}, \omega) \exp(-i\omega t)$$
$$= \text{Re} \int_0^\infty d\omega \int_{-\infty}^{\infty} dk_z \vec{E}_0 \exp \left(-i\omega t - ik_z z - i \int_{x_0}^{x} k_x(x') dx' \right),$$  \hspace{1cm} (3.160)

with $\vec{E}^{(+)}(\vec{r}, \omega)$ chosen to have the same polarization as the spectral field $\vec{E}(\omega)$ at the antenna, and is normalized according to Eq 3.6,

$$\int_{S_{wg}} \vec{S}^{(+)} \cdot d\vec{A} = 1,$$  \hspace{1cm} (3.161)

where $\vec{S}^{(+)}(\vec{r}, \omega) = (c/8\pi)\text{Re}[\vec{E}^{(+)}(\vec{r}, \omega) \times \vec{B}^{(+)*}(\vec{r}, \omega)]$ is the spectral power flux coming out of the waveguide.

In addition, in Sec 3.1.2D, we have shown that when the electron distribution function is isotropic Maxwellian, the source current dielectric tensor is proportional to the anti-Hermitian part of the plasma dielectric tensor, Eq 3.96.

$$\hat{K}_k(\vec{k}, \omega; \vec{r}) = \frac{\omega T_e(\vec{r})}{\pi^2} \epsilon^{+/A}(\vec{k}, \omega; \vec{r})$$  \hspace{1cm} (3.162)
3.2.2 1D Plasma configuration

To show that the two expressions for $P_s(\omega)$, Eq 3.156 and 3.159, are equivalent in the limit that the 1D description is valid, let’s consider a plasma that is uniform in both $y$ and $z$ directions, and the magnetic field $\vec{B} = B(x)\hat{z}$ and plasma density $n_e(x)$ change slowly in $x$, so that the WKB solution in $x$ is valid. An unidirectional antenna is placed outside the plasma at $(x_a, 0, 0)$, facing the negative $x$ direction. The electron cyclotron radiation of frequency $\omega$ is generated within a range of $x$ inside the plasma. So at the left edge of the plasma, $x = 0$, the ECE radiation intensity propagating towards the antenna is zero, $I(x = 0) = 0$.

![Figure 3.4: Plasma and antenna layout for the comparison between 1D and 2D ECEI calculations.](image)

Figure 3.4 shows the layout the plasma and the antenna. Eq 3.156 now becomes

$$P_s^{1D}(\omega) = \frac{1}{2\pi} \int_0^{x_a} dx \exp \left( - \int_x^{x_a} \alpha(x') dx' \right) \alpha(x) T_e(x),$$

(3.163)
where $\alpha(x)$ is calculated using Eq 3.157 with a plane wave propagating purely along $x$ direction, i.e. $k_x = k_x(x)$, $k_y = k_z = 0$.

For the 2D calculation, Eq 3.159 is also simplified, and becomes

$$P_s^{2D}(\omega) = \frac{1}{32\pi} \int dk_z dx dy \, E_0(\omega, k_z; x) \cdot \hat{K}_k(\omega, k_x(x), k_z; x) \cdot \bar{E}_0^*(\omega, k_z; x). \quad (3.164)$$

The unidirectional antenna approximation tells us that the complex amplitude $\vec{E}_0$ is non-zero only at $k_z = 0$, and the proper normalization for $\vec{E}_0$ can be obtained by applying the Parseval’s Theorem to Eq 3.161 along $z$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dy dk_z \left| \bar{S}_0(\omega, k_z, y, x) \right| = \int_{-\infty}^{\infty} dy dz \left| \bar{S}^{(+)}(\omega, x, y, z) \right| = 1, \quad (3.165)$$

where $\bar{S}_0 = \bar{S}_{0,EM} + \bar{S}_{0,P}$ is formally the “power flux” related to $\vec{E}_0$, with the terms $\bar{S}_{0,EM}$ and $\bar{S}_{0,P}$ defined the same as those in Eq 2.16 and 2.17, but with the dielectric tensor of the transposed plasma.

Substituting Eq 3.162 into Eq 3.164, we have

$$P_s^{2D}(\omega) = \frac{1}{32\pi} \int dk_z dx dy \, \frac{\omega T_e(x)}{\pi^2} \bar{E}_0(\omega, k_z; x) \cdot \hat{\varepsilon}^A(\vec{k}, \omega; x) \cdot \bar{E}_0^*(\omega, k_z; x)$$

$$= \frac{1}{4\pi^2} \int_0^{x_a} dx \, T_e(x) \int dy dk_z \left| \bar{S}_0(\omega, k_z, x) \right| \frac{\omega \bar{E}_0(\omega, k_z; x) \cdot \hat{\varepsilon}^A(\vec{k}, \omega; x) \cdot \bar{E}_0^*(\omega, k_z; x)}{8\pi \left| \bar{S}_0(\omega, k_z, x) \right|}$$

$$= \frac{1}{2\pi} \int_0^{x_a} dx \exp \left( -\int_{x_a}^{x} \alpha^T(x')|dx'| \right) \alpha^T(x) T_e(x). \quad (3.166)$$

In the last step, we have used the exponentially decaying solution of the time averaged energy conservation equation, Eq B.27, in the transposed plasma,

$$\frac{d}{ds} |\bar{S}_0| = \alpha^T |\bar{S}_0|, \quad (3.167)$$
where $\alpha^T$ is the absorption coefficient in the transposed plasma,

$$\alpha^T = \frac{\omega \vec{E}_0^* \cdot \epsilon^{TA} \cdot \vec{E}_0^*}{8\pi|\vec{S}_0|}. \tag{3.168}$$

The initial value $|\vec{S}_0(x_a)|$ is normalized according to Eq 3.165, and becomes one after integration over $y$ and $k_z$.

The dielectric tensor of the transposed plasma is defined as the transpose of the original one with $\vec{r}$ and $\vec{r}'$ interchanged, i.e. $\epsilon^T(\omega, \vec{r}, \vec{r}') = \epsilon^T(\omega, \vec{r}', \vec{r})$. It is straightforward to show that, upon using normal Fourier transform, the interchange of spatial coordinates corresponds to changing $\vec{k}$ to $-\vec{k}$ in $k$-space. Note that this change of sign has already been taken into account when we define the reciprocal field, Eq 3.160. So, in our notation, the dielectric tensor associated with an in-going wave vector $k_x$ in the reciprocal problem is simply the transpose of the dielectric tensor of the original plasma associated with the outgoing wave vector $k_x$. So we have

$$\alpha^T = \frac{\omega \vec{E}_0^* \cdot \left(\epsilon^{TA}\right)^T \cdot \vec{E}_0^*}{8\pi|\vec{S}_0|} = \frac{\omega \vec{E}_0^* \cdot \epsilon^{A} \cdot \vec{E}_0^*}{8\pi|\vec{S}_0|}. \tag{3.169}$$

This equation is also used during the final step of Eq 3.166.

Comparing Eq 3.163 and 3.166, we can see that the only difference is in the two absorption coefficients $\alpha$ and $\alpha^T$, given by Eq 3.157 and 3.169, respectively. We’ll show that $\alpha = \alpha^T$, and the 1D and 2D approaches give us the same spectral power $P_s(\omega)$.

### 3.2.3 Equality of the absorption coefficients

The absorption coefficients only depend on the polarization of the wave, and do not depend on the amplitude. Denote the polarization of $\vec{E}$ and $\vec{E}_0$ wave as $\vec{e}$ and $\vec{e}^{(+)}$, respectively. Comparing Eq 3.157 and 3.169 and noting that $\vec{S}_0$ is evaluated with
the transposed dielectric tensor, we can see that if $\bar{c}^{(+)} = \bar{c}^*$, we’ll have identical expressions for both $\alpha$ and $\alpha^T$.

We can calculate the polarization using the wave equation and the definition of the transposed dielectric tensor. We have, for X-mode

$$\frac{e_y}{e_x} = \frac{\epsilon_{xx}}{\epsilon_{xy}}, \quad (3.170)$$

$$\frac{e_y^{(+)}}{e_x^{(+)}} = \frac{\epsilon_{xx}}{\epsilon_{yx}}, \quad (3.171)$$

$$e_z = e_z^{(+)} = 0, \quad (3.172)$$

and for O-mode,

$$e_x = e_x^{(+)} = 0, \quad (3.173)$$

$$e_y = e_y^{(+)} = 0, \quad (3.174)$$

$$e_z = e_z^{(+)} = 1. \quad (3.175)$$

It is clear that for O-mode, $\bar{c}^{(+)} = \bar{c}^*$. In our calculation, the polarization is determined by the Hermitian part of the dielectric tensor, so $\epsilon_{yx} = \epsilon_{xy}^*$ and $\epsilon_{xx} = \epsilon_{xx}^*$.

So $\bar{c}^{(+)} = \bar{c}^*$ also holds for X-mode.

So we conclude that the two absorption coefficients are indeed identical. Thus $P_{s1D}^1$ and $P_{s2D}^2$ given by Eq 3.163 and 3.164 are the same.

### 3.2.4 Discussion on general plasma cases

The discussion regarding the simplified 1D plasma and unidirectional antenna case can be generalized to plasmas slowly changing in both $x$ and $y$, and to antennas with finite spread of receiving angles. As long as the received power can be considered as a simple summation of a collection of ray bundles, we can treat the reciprocal field
as the superposition of a collection of wave fields in which each field corresponds to a bundle of ray. Then we can use a similar argument to show that the powers calculated for this ray bundle from the two approaches are the same.

The 1D approach fails when the plasma can not be treated as uniform across a single bundle of rays. This is the case when the vertical wave length of the plasma perturbations is comparable with the waist of the receiving beam. The 2D calculation is required to correctly treat the diffraction and refraction in this situation. A demonstration of these effects is given in Sec 4.3.

It is also important to realize that the relation between the source current correlation tensor $\hat{K}_k$ and the anti-Hermitian part of the dielectric tensor $\tilde{\epsilon}^A$, Eq 3.162, is only true for Maxwellian electron distribution. In the ECE case, the Kirchhoff’s Law of Radiation, Eq 2.31, is directly related to this relation.

When plasma distribution function is not a Maxwellian, the 1D formula Eq 3.156 is no longer valid. The correct calculation involves the evaluation of both the emissivity $\eta$ and absorption coefficient $\alpha$ independently. However, the 2D formula Eq 3.159 is still correct, while the dielectric tensor and source current correlation tensor need to be calculated independently as well. A demonstration of applying the reciprocal formalism to Runaway Electron induced ECE is given in Sec 4.6.
Chapter 4

Numerical Implementation of ECEI2D

4.1 Numerical Implementation

In this section, we will discuss the numerical implementations of two key components in the current ECEI2D module. In Sec 4.1.1, we introduce the method we use to numerically evaluate the weakly relativistic plasma dispersion function to an arbitrary order, as well as an interpolation method used to accelerate the evaluation. In Sec 4.1.2, the numerical scheme is introduced that we use to solve the second order wave equation, Eq 3.144 and 3.153.

4.1.1 Weakly Relativistic Plasma Dispersion Function

The weakly relativistic plasma dispersion functions $F_q(\phi, \psi)$ are introduced by Shkarofsky\cite{Shkarofsky62} to evaluate the plasma dielectric tensor with weakly relativistic isotropic electrons.

In the weakly relativistic limit, the electron temperature and the perpendicular wave
vector under consideration are assumed small in the sense that

\[ \mu \equiv \frac{c^2}{v_{th}^2} \gg 1, \]  
\[ \lambda \equiv k_\perp^2 \rho_e^2 \ll 1, \]  

(4.1) \hspace{1cm} (4.2)

where \( v_{th} \equiv \sqrt{T_e/m} \) is the thermal velocity of electrons, \( \rho_e \equiv v_{th}/|\omega_{ce}| \) the typical electron gyro-radius. In most fusion plasmas, this is the most relevant regime for ECEI measurements.

A. Weakly Relativistic Dielectric Tensor

The dielectric tensor can be written in terms of the \( F_q \) functions\[63],

\[ \epsilon_{xx} = 1 - \frac{\mu \omega_{pe}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} n^2 \lambda^{p+n-1} F_{p+n+3/2}, \]  
\[ \epsilon_{yy} = 1 - \frac{\mu \omega_{pe}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} [ (p+n)^2 - \frac{p(p+2n)}{2n+2p-1} ] \lambda^{p+n-1} F_{p+n+3/2}, \]  
\[ \epsilon_{xy} = -\epsilon_{yx} = i \frac{\mu \omega_{pe}^2}{\omega^2} \sum_{N=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} N(p+n) \lambda^{p+n-1} F_{p+n+3/2}, \]  
\[ \epsilon_{zz} = \epsilon_{zz} = 1 - \frac{\mu \omega_{pe}^2 k_\perp k_\parallel c^2}{\omega^3 |\omega_{ce}|} \sum_{N=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} N \lambda^{p+n-1} F'_{p+n+5/2}, \]  
\[ \epsilon_{yz} = -\epsilon_{zy} = -i \frac{\mu \omega_{pe}^2 k_\perp k_\parallel c^2}{\omega^3 |\omega_{ce}|} \sum_{N=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} (p+n) \lambda^{p+n-1} F'_{p+n+5/2}, \]  
\[ \epsilon_{zz} = 1 - \frac{\mu \omega_{pe}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{pn} \lambda^{p+n} \left( F_{p+n+5/2} + 2\psi^2 F'_{p+n+7/2} \right), \]  

(4.3) \hspace{1cm} (4.4) \hspace{1cm} (4.5) \hspace{1cm} (4.6) \hspace{1cm} (4.7) \hspace{1cm} (4.8)

where \( n \equiv |N| \), and \( a_{pn} = (-1)^p(n+p - \frac{1}{2})!/[p!(n+\frac{1}{2}p)!(n+\frac{1}{2}p - \frac{1}{2})!2^n] \) is the coefficients in the expansion of \( e^{-\frac{z}{2}} I_N(z) = \sum_{p=0}^{\infty} a_{pn} z^{n+p} \).
The weakly relativistic dispersion function $F_q$ is defined as

$$F_q(\phi, \psi) = -ie^{\psi^2} \int_0^\infty dt (1 - it)^{-q} \exp \left(-i\phi^2 t + \psi^2/(1 - it)\right), \quad (4.9)$$

where $\psi = k_{\parallel}c^2/\omega v_t \sqrt{2}$, $\phi^2 = \psi^2 - \mu \delta$, $\delta = (\omega - N|\omega_c|)/\omega$, and The signs of the real and imaginary part of $\phi$ is defined to be Re$\phi > 0$, Im$\phi < 0$. $F_q'$ denotes the derivative respect to $\phi^2$.

**B. Recursive Evaluation of $F_q$**

We use the recursion relation given in [63] to evaluate the $F_q$ function.

$$F_{1/2}(\phi, \psi) = -\frac{1}{2\phi} \left[Z(\psi - \phi) + Z(-\psi - \phi)\right], \quad (4.10)$$

$$F_{3/2}(\phi, \psi) = -\frac{1}{2\psi} \left[Z(\psi - \phi) - Z(-\psi - \phi)\right], \quad (4.11)$$

$$F_{q+2}(\phi, \psi) = (1 + \phi^2 F_q - qF_{q+1})/\psi^2. \quad (4.12)$$

where

$$Z(\zeta) = \sqrt{\pi} \int_{-\infty}^\infty dt e^{-t^2}/(t - \zeta) \quad (4.13)$$

is the well-known plasma dispersion function [55], and can be related to the Faddeeva function $w(z)$, or complementary error function $\text{erfc}(z)$,

$$Z(\zeta) = i\pi w(\zeta) = i\pi e^{-\zeta^2} \text{erfc}(-i\zeta) = i\pi e^{-\zeta^2}(1 + \frac{2i}{\sqrt{\pi}}) \int_0^\zeta e^{t^2} \text{d}t. \quad (4.14)$$

In our work, the Faddeeva function is evaluated using the Scipy library, which uses the continued fraction formula given in [65] for sufficiently large $|\zeta|$, and switches to a faster algorithm described by [66] for smaller $|\zeta|$.

It is straightforward to show that the derivative of plasma dispersion function $Z'(\zeta) \equiv$
\[ \frac{dZ(\zeta)}{d\zeta} \text{ can be related to } Z(\zeta) \text{ through} \]

\[ Z'(\zeta) = -2(1 + \zeta Z(\zeta)). \quad (4.15) \]

This provides us a recursive way to evaluate higher order derivatives of \( Z \).

The \( m \)-th derivatives of \( F_q \) respect to \( \phi^2 \), \( F^m_q \) can be evaluated recursively in \( q \) as:

\[ F^m_{q+2} = (\phi^2 F^m_q - qF^m_{q+1} + mF^{m-1}_q)/\psi^2. \quad (4.16) \]

So, for \( \psi \neq 0 \) and \( \phi \neq 0 \), we have the following scheme to calculate all the \( F_q \) and \( F^m_q \):

1. Evaluate \( F_{1/2} \) and \( F_{3/2} \) using Eq 4.10 and 4.11.
2. Calculate \( F_q \) using the recursion relation Eq 4.12 starting from \( F_{5/2} \);
3. Calculate \( F^m_{1/2} \) and \( F^m_{3/2} \) based on the derivatives of \( Z(\zeta) \) and Eq 4.15;
4. Calculate \( F^m_q \) using the recursion relation Eq 4.16.

When \( \psi = 0 \), which corresponds to \( k_\parallel = 0 \) case, the recursive relations Eq 4.12 and 4.16 become singular. The right-hand-side expression remains finite only when the numerator is also zero, which gives us the recursion relations determining \( F_{q+1} \) and \( F^m_{q+1} \),

\[ F_{q+1}(\phi, 0) = (1 + \phi^2 F_q)/q, \quad (4.17) \]
\[ F^m_{q+1}(\phi, 0) = (\phi^2 F^m_q + mF^{m-1}_q)/q. \quad (4.18) \]

The formula for \( F_{3/2} \), Eq 4.11 also becomes zero divided by zero when \( \psi = 0 \), and L’Hospital’s rule can be used to show that in this case,

\[ \lim_{\psi \to 0} F_{3/2}(\phi, \psi) = -Z'(-\phi). \quad (4.19) \]
Then calculation can be carried out recursively as before.

Finally, special care is needed when $\phi = 0$, since now $F_{1/2} \rightarrow \infty$. However, this won’t break any of the higher order functions, because this singularity is only first order in $\phi$, and is removed in the recursion relation Eq 4.12 by the $\phi^2$ coefficient in front. The same thing happens for $F_m^q(0, \psi)$. In general, $F_m^q(0, \psi) \rightarrow \infty$ for $q \leq m + 1/2$, but is finite for larger $q$. We can easily check that in our expression for the dielectric tensor, Eq 4.3-4.8, there is no singularity when $\phi \rightarrow 0$.

### C. Interpolation Method for Fast Evaluation of $F_q$

In inhomogeneous plasmas, the evaluation of the dielectric tensor for high harmonics ($n > 3$) and/or higher order Finite Larmor Radius (FLR) correction ($p > 3$) requires calculation of $F_q$ and $F_m^q$ with large $q$ at many $(\psi, \phi)$ values. The recursive method introduced in Sec 4.1.1B can be too slow for this purpose. In order to accelerate the evaluation of these functions, we pre-calculate these functions on a mesh in $(\psi, \phi)$, and linearly interpolate the values at points in between.

Fig 4.1 shows the imaginary part of $F_{5/2}$ on a $(\psi, \phi^2)$ mesh. It is clear that as $|\psi| \rightarrow \infty$, the function decays slowly along the $\psi^2 = \phi^2$ path. This is bad for
interpolation. Because in our cases, $\psi$ can vary over a large range because of the varying temperature from outside to inside of the plasma, as well as the a series of possible $k_\parallel$ values. For a fixed $\psi$, $\phi$ can also change significantly because of $\mu\delta$ term in the definition of $\phi^2$. So, if we want to get small interpolation errors, e.g. 1% relative error, for all the relevant $\psi$ and $\phi$ values, we need to use a very large mesh and have good resolution where $|\psi| \approx \phi$. This requires a lot of memory and a special mesh.

A better choice of variables can be obtained by noticing that the function is non-zero mainly along $\psi^2 - \phi^2 = \mu\delta \sim 0$. We can then use a mesh in ($\psi$, $\mu\delta$) space. In this case, the function decays to zero in both $|\psi| \to \infty$ and $|\mu\delta| \to \infty$, as shown in Fig 4.2(a). We can then choose proper cutoffs in $\psi$ and $\mu\delta$ according to the desired error level. In practice, we choose $-1000 \leq \psi \leq 1000$, and $-1500 \leq \mu\delta \leq 1500$. So outside the mesh, the absolute value of the function is less than $10^{-3}$. In addition, since the function is large mainly near $\psi = 0, \mu\delta = 0$, we use a “cubic” mesh to increase the density of grid points near 0. The cubic mesh is uniform after taking cubic root. By default, we have 1001 grid points in each dimension. An odd number of grid points is essential so that we have the peak of the function at (0, 0) evaluated exactly.

Fig 4.2(b) shows the value of $-\text{Im}(\mathcal{F}_{5/2})$ evaluated by the interpolation method. Fig 4.2(c) and (d) demonstrates that the error due to interpolation is fairly small where $-\text{Im}\mathcal{F}_{5/2}$ is significant. The largest error is along $\psi^2 \approx \mu\delta$, i.e. $\phi \approx 0$, and when $\psi$ is also small. This feature may be used to further optimize the mesh for the interpolator, but is not included in our work.

4.1.2 Finite Difference Scheme for Wave Propagation Solver

In this section, we’ll discuss the numerical method we use to solve the wave equations for $E_0$ derived in Sec 3.1.3.
The equations for both O-mode (Eq 3.144) and X-mode (Eq 3.153) can be written in a general form,

\[ 2ik_x \frac{\partial F}{\partial x} - \frac{\partial^2 F}{\partial y^2} + \hat{C}(x, y) F = 0, \]  

(4.20)

where

\[ F = \begin{cases} 
    k_x^{1/2} E_0, & \text{(O-mode)} \\
    k_x^{1/2} \frac{s}{\sqrt{s^2 + D^2}} E_0, & \text{(X-mode)} 
\end{cases} \]  

(4.21)

\[ \hat{C}(x, y) = \begin{cases} 
    Pk_z^2 - \frac{\omega}{c \tau} \delta \epsilon_{OO}, & \text{(O-mode)} \\
    \left( \frac{s^2 + D^2}{s^2} - \frac{D^2(s^2 - D^2)}{s^2(s^2 - sP - D^2)} \right) k_z^2 - \frac{\omega}{c \tau} \frac{s^2 + D^2}{s^2} \delta \epsilon_{XX}, & \text{(X-mode)} 
\end{cases} \]  

(4.22)
We define operator $\hat{B} \equiv -\partial^2/\partial y^2$, and $\hat{A} \equiv -i(\hat{B} + \hat{C})/2k_x$. Then, Eq 4.20 can be written as

$$\frac{\partial F}{\partial x} = \hat{A}F,$$  \hspace{1cm} (4.23)

which is a first order differential equation in $x$ with a linear operator $\hat{A}$.

Using the Magnus expansion theorem\textsuperscript{67}, the solution of Eq 4.23 can be formally written as

$$F(x) = \exp(\hat{\Omega}(x, x_0))F(x_0),$$  \hspace{1cm} (4.24)

where

$$\hat{\Omega}(x, x_0) = \sum_{n=1}^{\infty} \hat{\Omega}_n(x, x_0),$$  \hspace{1cm} (4.25)

is the Magnus exponential series. The first two terms in the Magnus series are

$$\hat{\Omega}_1(x, x_0) = \int_{x_0}^{x} \hat{A}(x_1)dx_1,$$  \hspace{1cm} (4.26)

$$\hat{\Omega}_2(x, x_0) = \int_{x_0}^{x} \int_{x_0}^{x_1} [\hat{A}(x_1), \hat{A}(x_2)]dx_1dx_2,$$  \hspace{1cm} (4.27)

where $[\hat{A}(x_1), \hat{A}(x_2)] \equiv \hat{A}(x_1)\hat{A}(x_2) - \hat{A}(x_2)\hat{A}(x_1)$ is the commutator between operator $\hat{A}$ evaluated at $x_1$ and $x_2$.

Note that the exponential of an operator, or in the discrete case a matrix, is defined in a way analogous to the Taylor expansion of the exponential function, i.e.

$$\exp(\hat{A}) \equiv 1 + \hat{A} + \frac{\hat{A} \cdot \hat{A}}{2} + \cdots + \frac{\hat{A}^n}{n!} + \cdots,$$  \hspace{1cm} (4.28)

where the numerator of each term is a tensor product of $n \hat{A}$, and the denominator is factorial of $n$. Special care is needed to differentiate this “symmetric” definition with the “asymmetric” one given in \textsuperscript{68}.
If we propagate $F(x)$ using a small step size in $x$, $x - x_0 = \Delta x$, satisfying

$$\Delta x \| \hat{A} \|_2 \sim \delta \ll 1,$$  \hspace{1cm} (4.29)

where $\| \hat{A} \|_2 \equiv \sqrt{\lambda_{\text{max}}(\hat{A}^\dagger \hat{A})}$ is the Euclidean norm of operator $\hat{A}$, $\lambda_{\text{max}}(\hat{A}^\dagger \hat{A})$ is the largest eigenvalue of matrix $\hat{A}^\dagger \hat{A}$, we may show that $\Omega_n \sim o(\delta^n)$.

Here, we’ll focus on finding a numerical scheme that integrates $F$ from $x_0$ to $x_f$ with an accuracy in the exponent up to the first order in $\delta$. Note that the total steps $N = (x_f - x_0)/\Delta x \sim 1/\delta$, and the total error in the phase is the summation of the error in each step, in order to have first order accuracy, we need to keep up to $\delta^2$ terms in each step, i.e. keep up to $\Omega_2$ terms in the Magnus series.

The commutator in $\Omega_2$ can be further evaluated as

$$\left[ \hat{A}(x_1), \hat{A}(x_2) \right] = \hat{A}(x_1)\hat{A}(x_2) - \hat{A}(x_2)\hat{A}(x_1)$$

$$= -\frac{1}{4k^2}([\hat{B}, \hat{C}(x_2)] - [\hat{B}, \hat{C}(x_1)]) \sim o(\delta),$$

(4.30)

We can actually see that $\Omega_2 \sim o(\delta^3)$ in our specific case, so we can drop the $\Omega_2$ term as well, i.e.

$$F(x + \Delta x) = \exp \left( \int_x^{x+\Delta x} \hat{A}(x')dx' + o(\delta^3) \right) F(x)$$

$$= \exp \left( -\int_x^{x+\Delta x} \frac{i(\tilde{B}(x') + \tilde{C}(x'))}{2k_x} dx' + o(\delta^3) \right) F(x)$$

$$= \exp \left( \frac{1}{k^2} \int_x^{x+\Delta x} \frac{i(\tilde{B}(x') + \tilde{C}(x'))}{2k_x} 2dx' + o(\delta^3) \right) F(x)$$

(4.31)

The next step is to evaluate the phase integration numerically. We can, in principle, express operator $\hat{B} = -\partial^2/\partial y^2$ as a matrix in discrete $y$ space, and carry out the phase integration purely in $y$ space. However, when the number of grid points in $y$, $N_y$, is large, this method involves $N_y \times N_y$ matrix multiplying with size $N_y$ vector, which is of efficiency $N_y^2$ without further optimization. It is worth noting that, since
\( \hat{B} \) is the square of partial derivative of \( y \), which is a local operator in \( y \), the matrix representation will be essentially band diagonal. Optimizations with this feature in mind may eventually achieve the efficiency of \( O(N_y) \).

We have, however, used a numerical scheme that has the efficiency \( O(N_y \log N_y) \), which is better than a naive implementation of the matrix representation method. And for practically used \( N_y \sim 100 \), it is of the same order of the most optimal \((O(N_y))\) method.

We observe that the eigenstate of operator \( \hat{B} \) is simply the Fourier transformed function \( \tilde{F}(k_y) = \mathcal{F}(F(y)) \). If we can propagate \( F \) under operator \( \hat{B} \) and \( \hat{C} \) separately, we may take advantage of the Fast Fourier Transform (FFT) technique, and evaluate \( \hat{B} \) with efficiency \( O(N \log N) \). However, this is not simple since operators \( \hat{B} \) and \( \hat{C} \) do not commute.

For short, we define

\[
\hat{b}(x, \Delta x) \equiv -\frac{i}{2k_x} \int_x^{x+\Delta x} \hat{B}(x')dx',
\]

\[
\hat{c}(x, \Delta x) \equiv -\frac{i}{2k_x} \int_x^{x+\Delta x} \hat{C}(x')dx',
\]

then, it’s clear that \( \hat{b} \sim \hat{c} \sim \Delta x \sim \delta \), and \( F(x + \Delta x) = \exp(\hat{b}(x, \Delta x) + \hat{c}(x, \Delta x))F(x) \).

A particularly useful identity is the Baker-Campbell-Hausdorff formula\[70\],

\[
e^\hat{X} e^\hat{Y} = \exp(\hat{X} + \hat{Y} + \frac{[\hat{X}, \hat{Y}]}{2} + o(\delta^3)),
\]

where we have truncated the series up to terms of order \( o(\delta^2) \).

Using this formula twice, we can show that

\[
e^{\hat{c}(x+\Delta x/2, \Delta x/2)} e^{\hat{b} \hat{c}(x, \Delta x/2)} = \exp \left( \hat{b} + \hat{c}(x, \Delta x) + o(\delta^3) \right),
\]

\[102\]
where the commutator of $\hat{b}$ and $\hat{c}$ cancels to the order of $\delta^2$.

Now we have separated $\hat{b}$ and $\hat{c}$, the next step is to evaluate them to the required accuracy. Using a local Taylor expansion in $x$ for operators $\hat{B}$ and $\hat{C}$, we have

$$\hat{b}(x, \Delta x) = -\frac{i}{2k_x} \hat{B}(x + \Delta x/2)\Delta x + o(\delta^3),$$

$$\hat{c}(x, \Delta x/2) = -\frac{i}{2k_x} \left( \frac{\hat{C}(x)\Delta x}{2} + \frac{\hat{C}'(x)\Delta x^2}{8} \right) + o(\delta^3), \quad (4.36)$$

$$\hat{c}(x + \Delta x/2, \Delta x/2) = -\frac{i}{2k_x} \left( \frac{\hat{C}(x + \Delta x)\Delta x}{2} - \frac{\hat{C}'(x + \Delta x)\Delta x^2}{8} \right) + o(\delta^3),$$

Substituting Eq. 4.36 into Eq. 4.35, and note that the $\hat{C}'$ terms cancels up to the second order in $\delta$, we have the numerical scheme

$$\exp\left(\hat{b} + \hat{c}(x, \Delta x)\right) \approx \exp\left(-\frac{i}{2k_x} \frac{\hat{C}(x + \Delta x)\Delta x}{2}\right) \exp\left(-\frac{i}{2k_x} (\hat{B}(x + \Delta x/2)\Delta x)\right) \times \exp\left(-\frac{i}{2k_x} \frac{\hat{C}(x)\Delta x}{2}\right), \quad (4.37)$$

which evolves the phase accurately up to order $\delta^2$.

The right-hand-side propagator is now the composition of three propagators, each contains either $\hat{B}$ or $\hat{C}$. We can switch between eigenstates of $\hat{B}$ and $\hat{C}$ using FFT and the inverse transformation, then evaluate the propagator on its eigenstates.

### 4.2 Benchmark with ECEI1D

#### 4.2.1 Wave propagation

The first benchmark is to show the consistency between the 2D WKB paraxial wave solver (Sec 3.1.3) and the ray-tracing method (Sec. 2.4.2C).

A modeled cylindrical plasma is used. The characteristic frequencies are shown in Fig. 4.3. The O-mode and X-mode wave frequencies are chosen to be high enough.
compared to the corresponding cutoff frequencies throughout the plasma, so the
WKB approximation is valid. The incident wave is slightly tilted upwards in y di-
rection \( k_y / k_x = \tan 0.1 \), and the paraxial approximation is still good. A Gaussian
beam with waist width \( w_0 \sim 20\lambda \) is used in the 2D propagation. As a result, the
divergence of the beam is weak, and the beam can be effectively described by one
central ray.

Fig 4.4 shows the benchmark results. The light path calculated by the ray-tracing
method matches well with the center of the 2D wave field solution for both O-mode
and X-mode. The bending of the ray due to refraction is correctly calculated by the
2D wave solver.
4.2.2 Shine Through Effect

As the second benchmark, we run the ECEI2D code to study the shine through case discussed in Sec 2.4.3.

Fig 4.5 shows similar curves as those in Fig 2.6. The radial resolution of the Instrument Function curves is slightly lower due to a larger step size used in ECEI2D.

Fig 4.6 shows a direct comparison of the effective temperature of ECE calculated by ECEI1D and ECEI2D codes. The two codes agree very well through the whole calculation area, from the optically thick core to the optically thin edge.
Figure 4.5: Synthetic ECE electron temperature profile calculated by ECEI2D at the edge of a DIII-D like plasma (solid), the input electron temperature (dashed) and density (dotted). Sources of emission power are shown for optically thick (magenta), gray (cyan) and thin (red) channels. The corresponding cold resonances are indicated with the arrows in the corresponding colors. C.F. Fig 2.6.

Figure 4.6: Benchmark between ECEI1D and ECEI2D codes on the Shine through case. 32 frequencies are chosen, corresponding to the 2nd ECE harmonic frequencies at equally spaced radial locations along mid-plane across the pedestal region. The synthetic ECE temperature is calculated using ECEI1D and ECEI2D codes. The red triangles indicate ECEI1D results, and the blue squares are ECEI2D results. The modeled plasma temperature and density profiles are shown in dashed and dotted lines, respectively.
The successful benchmarks have shown that the ECEI2D code is consistent with ECEI1D when the 1D model is valid. In the next two sections, we will show two studies that are enabled by the new two-dimensional synthetic ECEI capability.

4.3 Refraction Near Cutoff

When the ECE frequency is close to the local cutoff frequency, we expect that refraction effects become important. We use a simple analytic plasma model to show this effect, in which large electron temperature and density fluctuations ($\tilde{T}_e/T_{e0} = 5\%$, $\tilde{n}_e/n_{e0} = 10\%$) propagate upwards vertically. The radial wave length is set to be much larger than the vertical wave length.

Figure 4.7 shows how the plasma area where the observed emission originates is perturbed by electron density fluctuations that are in phase with the electron temperature fluctuations. In Fig 4.7, a time trace of ECE signal gets squeezed near the peaks of density and temperature fluctuations, and flattened near the troughs. The reason for this can be seen in Figs. 4.7b, 4.7c, and 4.7d. From Fig 4.7c and 4.7d, we see that the source of ECE tends to stay in the trough of the density perturbation. Since the temperature and density perturbations are in phase, the ECE measured temperature over this period is roughly the same, and leads to the flattened part of the time trace. When the peak of density perturbation is aligned with the unperturbed source location (Fig 4.7b), the ECE source gets elongated, and moves from the upper trough to the lower trough.

Figure 4.8 demonstrates the possibility of using ECE measurement for detecting the cross-phase between electron density and temperature fluctuations of a single coherent mode. The strong refraction effect near the cutoff is the key to this application. Fig 4.8a shows the ECE $T_e$ fluctuation without $n_e$ perturbations. A perfect single frequency mode is seen, indicated by a delta function at the fundamental frequency.
Figure 4.7: (a) Synthetic ECE effective temperature (solid), and actual electron temperature (dashed) and density (dotted) at the cold resonance as a function of time. Three time snapshots (vertical solid lines) are chosen to show the distinct ECE emission source pattern at different stages. The ECE emission sources (black dot) on top of iso-density contour (black curve) are shown at (b) 25 µs, (c) 60 µs, and (d) 90 µs. The “+” mark indicates the location of cold resonance in equilibrium. The marker size is set by the vertical width of the beam.

(10kHz) in the power spectrum, shown in Fig 4.8b. When an in-phase density perturbation is added, as shown in Fig 4.8c, the ECE signal has a flattened trough, and the higher harmonic components show up in the frequency domain, as shown in Fig 4.8d. When the cross-phase \( \phi \) between the density and temperature perturbations is changed to 90°, the flattening region moves to the middle of a period, and a strong second harmonic component is found in the simulation, with almost no fundamental component (Figs. 4.8e and f). In the \( \phi = 180^\circ \) case, the power spectrum looks similar to the \( \phi = 0^\circ \) case (Figs. 4.8g and h).

### 4.4 Higher Harmonics Overlap

The second demonstration is the non-local measurement of ECE due to higher harmonics overlap.

Fig 4.9 shows the characteristic frequencies in the modeled plasma. The 8 X-mode
Figure 4.8: The time traces and power spectra of synthetic temperature fluctuations under different density-temperature cross phases. The solid lines in plots (a), (c), (e), (g) show synthetic ECE temperature deviations from the time averaged values normalized to the time averaged values. The dashed lines show the electron temperature fluctuations, and the dash-dotted lines show density fluctuations with arbitrary unit. (b), (d), (f), and (h) are the corresponding power spectra of the ECE signals, normalized to the maximum value. (a, b) no density perturbation. 10% density perturbation with (c, d) 0°, (e, f) 90°, and (g, h) 180° cross-phase between \( \tilde{n}_e \) and \( \tilde{T}_e \).

Figure 4.9: Characteristic frequencies for the modeled plasma. Eight horizontal black lines indicate the X-mode ECE frequencies.
Figure 4.10: Third harmonic resonance effect on a profile ECE measurement. Dash-dot line is the real electron temperature profile. The synthetic Te calculated with only 2nd harmonic effects included and with 3rd harmonic effects are depicted by the blue dots and the red triangles, respectively. The Blue and red dashed lines show the radial Instrument Functions (IFs) without and with 3rd harmonic effect for the inner most channel indicated by blue and red arrows, respectively.

ECE frequencies are shown with horizontal black lines. They are chosen in the 2nd harmonic electron cyclotron frequency range and their radial resonance locations are evenly spaced. All of the cold 3rd harmonic resonances are outside the plasma.

Fig 4.10 shows the synthetic ECE results. We can see that the synthetic ECE temperatures calculated with only 2nd harmonic resonances included (blue circles) match the synthetic ECE calculations with 3rd harmonic contributions included very well, except for the inner most three points. This is because the inner most three channels have frequencies that are close to the 3rd harmonic frequency at the edge, and resonances with hot electrons happen due to the relativistic down-shift of the cyclotron frequency. Consequently, the Instrumental Functions (IFs), defined by Eq 2.7, are finite at both the edge and the core. Therefore, the measured ECE power is a combination of the core 2nd harmonic emission and the edge 3rd harmonic emission. The effective temperature is lowered because of the cooler edge contribution.
This higher harmonic overlap effect sets a constraint on the frequency range the ECE profile measurement can operate in. The range gets narrower when the electron temperature is higher because of a stronger relativistic down-shift at the edge. In the DIII-D ECEI system, the finite band-width and the relativistic down-shift dominate over the Doppler shift (Sec 2.4).

Our Synthetic ECE code can be used to predict the operational frequency range, and help to design new diagnostics. It also helps to quantitatively confirm that the existing ECE diagnostics are not affected by higher harmonic overlaps, and/or to understand the data when the overlaps happen.

4.5 ECEI on DIII-D Edge Harmonic Oscillations

In this section, we show an application of ECEI2D on the DIII-D EHO case discussed in Sec. 2.2. Figure 4.11 shows a comparison between the simulated M3D-C results, the synthetic ECEI signals, and the measured ECEI signals. The significant difference between simulation and synthetic results is due to the complex response of the ECEI diagnostic in the edge of the plasma. A direct comparison between the simulated $\delta T_e$ and the measurement interpreted $\delta T_e$ without using a synthetic diagnostic gives the inaccurate impression that they completely disagree. When the synthetic ECEI diagnostic is applied to the simulated $\delta T_e$, the response is in qualitative agreement with the measurements, as shown in Figure 4.11(b) and 4.11(c). The strong shine through features are reproduced as well as the out of phase intensity fluctuations at the pedestal bottom.

Although the synthetic and measured signals are in qualitative agreement, they still don’t agree quantitatively. The most significant disagreement is the radial location of the shine through layer and the negative response channel. The whole structure of the synthetic signals is shifted about 2cm radially outwards compared to the measured
Figure 4.11: Comparison between M3D-C simulation (a), synthetic ECEI response (b), and ECEI measurement (c) of the electron temperature fluctuations of an EHO (DIII-D, shot #157102 at 2420 ms).

signals. This shift may be due to the uncertainties in the equilibrium reconstruction.

4.6 ECE spectrum with runaway electrons

In this section, we show a qualitative comparison between the measured ECE spectrum during the Runaway Electron (RE) growth and the synthetic ECE spectrum with a simulated RE generation.

Fig 4.12 shows the ECE spectrum observed on DIII-D with a Michelson ECE system [71] when the RE population is growing. The experiment is designed to have a low plasma density to study RE generation [72]. Fig 4.12 shows the change of the ECE power spectrum over time. The slope of the spectrum decreases as the RE population grows. Matching the exact growth rate of the REs in the simulation is very hard because of the uncertainty in the measured parallel electric field. We try to understand qualitatively how REs may contribute to the observed ECE radiation, and demonstrate that our formalism based on the reciprocity theorem (Sec 3.1) can be applied to strongly non-Maxwellian electrons, such as REs.
Figure 4.12: Measured ECE spectrum in a low density, runaway electron generation experiment on DIII-D for shot #157209. Different colored lines stand for the spectra measured at different time, over roughly 2s. The arrow indicates the time evolution. Runaway electron population is growing over time, and resulting a significant increase of ECE power in the high frequency part. Figure courtesy to Carlos Paz-Soldan.

In order to investigate the RE induced ECE radiations, a stand-alone synthetic ECE code is developed by Chang Liu at PPPL. It uses the formalism based on reciprocity theorem (Sec 3.1), and calculates the source current correlation tensor and plasma dielectric tensor of a plasma with uniform $n_e$ and $T_e$ in a radially decreasing magnetic field ($B \sim 1/R$). The RE distribution function is simulated by CODE. The calculated electron distribution slowly evolves as shown in Fig 4.13. The RE distribution is very low in density($\sim 10^{-7}$ of the thermal electron density), but extremely relativistic ($p > mc$). With this strongly non-Maxwellian, extremely relativistic distribution, the weakly relativistic dielectric tensor formula introduced in Sec 4.1.1 is no longer valid. We need to numerically evaluate the velocity space integration in both the source current dielectric tensor, Eq 3.71, and the dielectric tensor. The reciprocal wave field is then solved with WKB approximation, and the ECE power is obtained through Eq 3.20.
Fig 4.14 shows the result for the effective radiation temperature as a function of frequency. The step-like structure is due to the uniform plasma temperature in the model. It can be understood as follows. When the frequency increases, the resonance for a given harmonic moves inwards to the higher magnetic field region. As long as the resonance is still within the plasma, due to the uniformity of the modeled plasma, the measured ECE power is roughly constant. A rapid decrease in the ECE power occurs when a certain harmonic resonance moves outside of the plasma, and the total optical thickness drops to the level determined by the higher harmonic resonances. The drops at $\omega \sim 2.5\omega_{ce0}$ and $\omega \sim 4\omega_{ce0}$ correspond to the 2nd and 3rd harmonic resonances moving out of the plasma, respectively. Since the REs are extremely relativistic, and contribute to the ECE power through a combination of many harmonics, the loss of a lower harmonic thermal resonance doesn’t strongly affect the ECE power due to REs. This is shown by the blue line in Fig 4.14, where the drops are much more insignificant compared to those in the black line which corresponds to no REs. In fact, the existence of the thermal resonances may absorb part of the RE induced ECE power, and lower the total received power. This can be seen also in the blue line, where a drop in ECE power appears around $\omega \approx 2.4\omega_{ce0}$. This is where the 3rd harmonic resonance enters the plasma, and absorbs part of the ECE power at the edge. The synthetic ECE spectra qualitatively agree with the observed spectra, and show the flattening due to the increased density of REs.

These results lead to two questions. How does a negligibly small amount of REs ($\sim 10^{-7}$ of the thermal population, see Fig 4.13) provide such a significant ECE power increase? Why the ECE spectrum gets flatter as RE density grows?

For a non-Maxwellian RE population, the emissivity, represented by the current correlation tensor $\hat{K}_k$ Eq 3.80 and the absorption, represented by the anti-Hermitian
Figure 4.13: Runaway electron distribution at zero pitch angle simulated by CODE. The distribution function is shown at three times. The thermal bulk is shown as the narrow peak near $p = 0$. The Dreicer effect is shown as the extension of the $p$ range over time, which is due to the parallel acceleration. The subplot at upper right corner shows the exponential growth of $f$ at $p = 0$. Figure courtesy to Chang Liu.

In fact, if we define the effective temperature for the REs,

$$T_e^{\text{eff}} = \frac{\pi^2}{\omega} \frac{\hat{K}_{k,xx}}{\epsilon_{xx}^A},$$

it is straightforward to show that $T_e^{\text{eff}} \gg T_e$ for a typical RE distribution function, shown in Fig. 4.13. This is due to the fact that, according to Eq. 3.80 and Eq. 3.91, $\hat{K}_k \propto f(\vec{p})$ while $\epsilon_{xx}^A \propto df/dp$. The flattened RE distribution tail significantly enhances the
emission to absorption ratio. This results in a very strong ECE radiation power from a small number of REs.

Since the RE’s perpendicular energy is much larger than that of the thermal electrons, the emission from higher harmonic resonances is stronger. This causes a flatter ECE spectrum due to REs as compared to that from thermal electrons. The measured spectrum is the sum of the thermal and RE contributions. As the RE density grows, the measured spectrum becomes more and more dominated by the RE contribution, thus flattens.

The sensitivity of the ECE spectrum to the REs shows the potential of using ECE as a measurement for the RE population and/or distribution function. Comprehensive synthetic ECE codes are essential for designing these new diagnostics, and understanding the measurements.
Chapter 5

Summary and Future Plans

5.1 Summary

In this dissertation, we have presented a software platform for development and applications of synthetic diagnostics (Chapter 2). A derivation of a new formalism for calculating the Electron Cyclotron Emission (ECE) signal in a two-dimensional plasma has been given in Chapter 3, and the numerical implementation and results were shown in chapter 4.

5.1.1 Synthetic Diagnostic Platform

We have developed a Synthetic Diagnostics Platform (SDP) for fusion plasmas. The SDP is useful for both developing and applying synthetic diagnostics. Data interfaces to several plasma simulation codes are provided, so the synthetic diagnostic codes can be easily applied. The diagnostics on SDP can be used to validate theory and simulation results against measurements, analyze the diagnostic responses, and design and demonstrate new diagnostic devices.

With the data interfaces and post-processing modules provided by SDP, the synthetic reflectometry codes FWR2D and FWR3D are available for the GTS, XGC-1,
GTC, and M3D-C\textsuperscript{1} simulations. Application of FWR2D to an Edge Harmonic Oscillations (EHO) calculation of M3D-C\textsuperscript{1} has shown fair agreement with the Millimeter-wave Imaging Reflectometer (MIR) measurements on DIII-D tokamak (Sec 2.2).

A synthetic Beam Emission Spectroscopy (BES) code was developed on SDP by Loic Hausammann. The broadening of the observed BES signal due to the finite life-time effect has been demonstrated (Sec 2.3).

Two synthetic Electron Cyclotron Emission (ECE) codes were developed on SDP. The one-dimensional ECE code implements the traditional calculation using the equation of radiative transfer, and is applied to a DIII-D-like plasma to demonstrate the shine through effect near the plasma edge (Sec 2.4).

The development of the new two-dimensional ECE code is a main part of this dissertation and is summarized in the following sections.

### 5.1.2 Reciprocal calculation of the ECE

The development of a new two-dimensional synthetic ECE code based on the Reciprocity Theorem is a main part of this dissertation, and was derived in Chapter 3.

The new formalism provides an efficient way to calculate the refraction and diffraction effects in ECE. It also enables the calculation of ECE using an arbitrary electron distribution, which is needed to study the ECE sensitivity to non-Maxwellian electrons, e.g. runaway electrons, and the design of the next generation ECE diagnostics.

The specific form of the reciprocity theorem for ECE is provided in Sec 3.1 and the two key components of the calculation – the source current correlation tensor (Sec 3.1.2) and the reciprocal wave amplitude (Sec 3.1.3) – are discussed in detail. A general relation between the source current correlation tensor and the anti-Hermitian part of the plasma dielectric tensor for Maxwellian electron distributions is derived (Sec 3.1.2D). It is shown to be closely related to the Kirchhoff’s Law of Radiation in the demonstration of the equivalence of the 1D and 2D ECE formalisms (Sec 3.2).
5.1.3 Numerical implementation and results

The reciprocal formalism for the synthetic ECE has been implemented on SDP with the weakly relativistic dielectric tensor using isotropic Maxwellian electron distributions. The numerical implementation of the dielectric tensor, and the wave propagation solver is discussed in detail in Sec 4.1.

The 2D code has been benchmarked against the 1D results, and has shown good agreement (Sec 4.2).

The application to the M3D-C simulation has shown that the synthetic ECE is crucial to compare the simulation results with the ECEI measurement from the edge of the DIII-D plasma (Sec 4.5).

The potential use of the reciprocal formalism of synthetic ECE to the study of the runaway electrons (REs) induced ECE signals has been explored and demonstrated on a model plasma. The synthetic ECE simulation has found an enhanced emission particularly notable at high frequencies with the increase of RE density, which qualitatively agrees with the measurements (Sec 4.6). With the upgrade of allowing non-Maxwellian electron distributions, the new synthetic ECE code can be an indispensable tool for designing proposed ECE diagnostics for REs and for understanding the measured signals.

The new synthetic ECE code can also be conveniently applied to GTC, GTS, XGC-1, and M3D-C for validating these codes.

5.2 Future Plans

1. Upgrade in ECEI2D to fully support Runaway Electron calculations

Experimentalists from both DIII-D and COMPASS tokamaks are planning to install new ECE diagnostics to measure the runaway electron density and distributions. To simulate the ECE response requires the calculation of the dielectric
tensor and the source current correlation tensor for an arbitrary electron distribution function. Thus, accurate and efficient numerical integration schemes are needed to perform the required velocity space integrations. The generation of the runaway electron distribution functions will depend on other codes or theoretical models.

2. Integration with OMFIT

OMFIT is a platform which provides data interfaces for major fusion experimental databases and many analysis/simulation codes\textsuperscript{[75]}. SDP can be incorporated into OMFIT easily.

3. Development of new synthetic diagnostics

A lot of other important diagnostics are not yet included in SDP. Thomson Scattering can provide both electron temperature and density measurements, CHarge Exchange Recombination Spectroscopy (CHERS) is another key diagnostic that is widely used to measure localized ion temperature, plasma rotation, and the impurity densities. Synthetic diagnostics for these measurements need to be developed on SDP to have a more comprehensive suite of diagnostics.
Appendix A

Interpolation schemes in SDP data interface

In this appendix, we introduce the interpolation schemes used in the data interfaces for the GTS, GTC, and XGC-1 simulations.

A.1 Interpolation on poloidal cross-sections

The 2D interpolation problem in SDP can be stated as follows:

Given the function values \( f_i = f(\psi_i, \theta_i) \) on the points in the flux coordinates \((\psi_i, \theta_i)\), find the values of \( f \) at some Cartesian coordinate \((R_j, Z_j)\).

It is clear that we need first a coordinate conversion either from \((\psi, \theta)\) to \((R, Z)\) or backwards, then an interpolation on the converted coordinates.

The choice of the coordinate transformation gives us two schemes.

In GTS, since the coordinate transformations \( \psi(R, Z) \) and \( \theta(R, Z) \) are provided by the equilibrium solver, we can easily convert \((R_j, Z_j)\) into \((\psi_j, \theta_j)\). Then, we can locate the two flux surfaces with grids that contains \( \psi_j \), \( \psi_{\text{inner}} < \psi_j < \psi_{\text{outer}} \). First, we linearly interpolate the function in \( \theta \) on each flux surface, we obtain \( f(\psi_{\text{inner}}, \theta_j) \) and...
\( f(\psi_{\text{outer}}, \theta_j) \). Then, linearly interpolate \( f \) in \( \psi \), we get \( f(\psi_j, \theta_j) \), which is the value at \((R_j, Z_j)\).

In GTC and XGC-1, the \( R, Z \) coordinates of the grid points are available. We use the Delaunay triangulation algorithm provided by the python package matplotlib\(^{24}\) to generate the Delaunay Triangulation of the grid points, and linearly interpolate the values on each triangle.

### A.2 Interpolation between poloidal cross-sections

The interpolation in the toroidal direction is more involved than the 2D interpolation on each poloidal plane.

The equilibrium profiles are assumed symmetric in the toroidal direction, \( \Phi \), so no interpolation in \( \Phi \) is needed for them. However, the perturbed quantities depend on \( \Phi \). In the simulations, the grid points are placed along magnetic field lines\(^{15}\). Since the perturbed quantities changes slowly along the field lines, only a few poloidal cross-sections are required to resolve the mode structure, typically 32 or 64. However, this resolution is usually not enough to resolve the mode structure along the toroidal direction, because the toroidal mode number can be of order 100 for some fluctuations. In order to obtain the values accurately, our interpolation must also be along the field line. This means we need to know the values at the two points where the field line intersects with the two closest poloidal planes. When we have the coordinates of these points, We can obtain the values using the 2D interpolation on each plane.

For GTS, since we interpolate in the \((\psi, \theta)\) space, we need the \( \psi \) and \( \theta \) value of the two points. By definition, \( \psi \) is constant along the field line. We only need to calculate \( \theta \). GTS uses a “straight field-line” coordinate system\(^{15}\), so that on each flux surface, the safety factor \( q \equiv d\zeta/d\theta \) along a field line is a constant. Given the Cartesian coordinates \((x_j, y_j, z_j)\) of the required interpolation point, we first convert
them into the Cylindrical coordinates \((R_j, \Phi_j, Z_j)\), then into GTS flux coordinates \((\psi_j, \theta_j, \zeta_j)\), where \(\psi(R_j, Z_j)\) and \(\theta(R_j, Z_j)\) are provided by GTS, and \(\zeta_j = -\Phi\). Then we locate the toroidal section that contains this point, calculate the two end points’ coordinates \((\psi_j, \theta_{\text{lower}})\) and \((\psi_j, \theta_{\text{upper}})\) using the safety factor \(q_j\) that is obtained by linearly interpolating \(q\) on \(\psi\).

For GTC and XGC-1, 2D interpolation is done in \((R, Z)\) space. In order to obtain the coordinates of the two end points, we integrate \(R\) and \(Z\) according to the equations of the field line,

\[
\frac{dR}{d\Phi} = \frac{B_R}{B_\Phi}, \\
\frac{dZ}{d\Phi} = \frac{B_Z}{B_\Phi},
\]

where \(B_R\), \(B_Z\) and \(B_\Phi\) are the components of the equilibrium magnetic field which can be evaluated for any \((R, Z)\) using the 2D interpolation. The final interpolation along the field line depends on the arc lengths between the evaluating point and the two end points.

We use linear interpolation to minimize the effects of the unwanted noise in the original simulations. Smoothing splines or other numerical fitting methods may be used in the future when needed to provide smoother functions and their derivative for synthetic diagnostic calculations.
Appendix B

Equation of Radiative Transfer

In this Appendix, we discuss the derivation of the Equation of Radiative Transfer used by the ECEI1D code (Sec 2.4), and the calculation of the ray refraction index appeared in the Equation.

The Equation of Radiative Transfer is a simplified description of the electromagnetic wave energy propagation in a refractive medium. It contains the refraction, emission, and absorption effects in the medium, but does not consider wave-wave interactions.

B.1 Energy Conservation in Uniform, Weakly Lossy Media

Before we derive the Equation of Radiative Transfer in a general medium, we’ll first introduce some notations in a uniform medium.

Consider electromagnetic radiation propagating in a uniform medium. Ignoring the discreteness of the charges, and using a smooth dielectric tensor $\epsilon(\omega, \vec{k})$, the
Maxwell’s Equations in such a medium can be written as

\[
\vec{k} \cdot \vec{D} = 0, \tag{B.1}
\]
\[
\vec{k} \cdot \vec{B} = 0, \tag{B.2}
\]
\[
\vec{k} \times \vec{E} = \frac{\omega}{c} \vec{B}, \tag{B.3}
\]
\[
\vec{k} \times \vec{B} = -\frac{\omega}{c} \vec{D}, \tag{B.4}
\]

where

\[
\vec{D} \equiv \vec{\varepsilon}(\omega, \vec{k}) \cdot \vec{E} = \vec{E} + \frac{4\pi i}{\omega} \vec{j}, \tag{B.5}
\]

and \(\vec{E}, \vec{B}\) are the electric and magnetic fields of a plane wave, defined as

\[
\vec{E}(\vec{r}, t) = \text{Re} (\vec{E} e^{-(i\omega t - i\vec{k} \cdot \vec{r})}),
\]
\[
\vec{B}(\vec{r}, t) = \text{Re} (\vec{B} e^{-(i\omega t - i\vec{k} \cdot \vec{r})}). \tag{B.6}
\]

Since the Maxwell equations are homogeneous, the existence of a non-zero solution for \(\vec{E}\) and \(\vec{B}\) gives us an equation relating \(\omega\) and \(\vec{k}\), called the dispersion relation, and is normally written as

\[
\omega = \omega(\vec{k}). \tag{B.7}
\]

The dispersion relation depends on the dielectric tensor of the medium.

In a slightly lossy medium, \(\vec{\varepsilon} = \vec{\varepsilon}^H + i\vec{\varepsilon}^A\), where

\[
\frac{\omega^H}{\varepsilon} \equiv \frac{1}{2}(\varepsilon^\ast + \varepsilon^\ast \dagger), \tag{B.8}
\]
\[
\frac{\omega^A}{\varepsilon} \equiv \frac{1}{2i}(\varepsilon - \varepsilon^\ast \dagger) \tag{B.9}
\]

are the Hermitian and anti-Hermitian parts of the dielectric tensor, respectively. The \(\dagger\) denotes the Hermitian transpose which is obtained by taking the transpose of the complex conjugate tensor.
The wave vector $\vec{k}$ and frequency $\omega$ that satisfy the dispersion relation will, in general, be complex.

$$\vec{k} = \vec{k}_r + i\vec{k}_i,$$

$$\omega = \omega_r + i\omega_i.$$  \hspace{1cm} (B.10)

We’ll assume in the following that $|k_i| \ll |k_r|$ and $|\omega_i| \ll \omega_r$ which corresponds to a weakly absorbing medium.

Now, the dot product of $B^*$ and Eq B.3 minus the dot product of $E^*$ and Eq B.4 gives us

$$\vec{k} \cdot (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) = \frac{\omega}{c} (|\vec{B}|^2 + \vec{E}^* \cdot \epsilon^\dagger \cdot \vec{E}).$$ \hspace{1cm} (B.11)

Similarly, taking complex conjugate of Eq B.3 and B.4 dotting with $\vec{B}$ and $\vec{E}$, respectively, then subtracting the latter from the former, and multiplying $c/16\pi$, we obtain the complex conjugated form of Eq B.11

$$\vec{k}^* \cdot (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) = \frac{\omega^*}{c} (|\vec{B}|^2 + \vec{E}^* \cdot \epsilon^\dagger \cdot \vec{E}).$$ \hspace{1cm} (B.12)

Subtracting Eq B.12 from Eq B.11, we have

$$2k_i \cdot \frac{c}{16\pi} (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) = \frac{1}{8\pi} \left[ \omega_i |\vec{B}|^2 + \frac{1}{2i} \vec{E}^* \cdot (\omega e^* - \omega^* e^\dagger) \cdot \vec{E} \right].$$ \hspace{1cm} (B.13)

Note that we have multiplied $c/8\pi$ to have the coefficients consistent with the original Poynting’s Theorem,

$$\nabla \cdot \left( \frac{c}{4\pi} \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right) = -\frac{1}{4\pi} \left( \vec{B}(\vec{r},t) \cdot \frac{\partial \vec{B}(\vec{r},t)}{\partial t} + \vec{E}(\vec{r},t) \cdot \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \right),$$ \hspace{1cm} (B.14)

and the relation between the complex amplitude $\vec{E}, \vec{B}$ and the real $\vec{E}(\vec{r},t), \vec{B}(\vec{r},t)$, Eq B.6.
Eq B.13 is the zero frequency part of B.14, which is essentially the *time averaged* Poynting’s Theorem.

When we expand $\omega \leftrightarrow \epsilon(\omega, \vec{k})$ around $\omega_r \leftrightarrow \epsilon(\omega_r, \vec{k}_r)$, and keep up to the first order terms in $\vec{k}_i$ and $\omega_i$, we obtain

$$\omega \leftrightarrow \epsilon(\omega + i\omega_i, \vec{k} + i\vec{k}_i) \approx \omega_r \leftrightarrow \epsilon(\omega_r, \vec{k}_r) + i\vec{k}_i \cdot \frac{\partial \omega \leftrightarrow \epsilon}{\partial \vec{k}} \bigg|_{\omega_r, \vec{k}_r} + i\omega_i \frac{\partial \omega \leftrightarrow \epsilon}{\partial \omega} \bigg|_{\omega_r, \vec{k}_r}. \quad (B.15)$$

Substituting Eq B.15 into Eq B.13 and using Eq B.8 and B.9, we finally obtain

$$2\vec{k}_i \cdot \left( \frac{c}{16\pi} (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) - \frac{1}{16\pi} \vec{E}^* \cdot \frac{\partial \omega \leftrightarrow H}{\partial \vec{k}} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E} \right) = 2\omega_i \left( \frac{|\vec{B}|^2}{16\pi} + \frac{1}{16\pi} \vec{E}^* \cdot \frac{\partial \omega \leftrightarrow H}{\partial \omega} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E} \right) + \frac{\omega_i \vec{E}^* \cdot \epsilon \leftrightarrow A \cdot \vec{E}}{8\pi}. \quad (B.16)$$

Note that we have intentionally multiplied the equation by a factor of 2, so that the $2\vec{k}_i$ and $2\omega_i$ coefficients reflect the power decay in space and time, instead of the amplitude decay.

The terms in Eq B.16 have the following physical interpretations.

The first term on the left-hand side

$$\vec{S}_{EM} = \frac{c}{16\pi} (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) \quad (B.17)$$

is the time averaged Poynting flux, and represents the flow of the electromagnetic energy of the radiation.

The second term,

$$\vec{S}_P = -\frac{1}{16\pi} \vec{E}^* \cdot \frac{\partial \omega \leftrightarrow H}{\partial \vec{k}} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E} \quad (B.18)$$

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represents the non-electromagnetic energy flow associated with the wave, and is interpreted as being the energy associated with the particles moving coherently with the wave fields [51].

The first term on the right-hand side of Eq. (B.16)

\[ U_M = \frac{1}{16\pi} |\vec{B}|^2, \]  

(B.19)

represents the time averaged magnetic energy density. The second term

\[ U_{E+P} = \frac{1}{16\pi} \vec{E}^* \cdot \left. \frac{\partial \omega^H}{\partial \omega} \right|_{\omega_r, \vec{k}_r} \cdot \vec{E}, \]  

(B.20)

is the sum of the electrical energy density and the energy density related to the coherent particles. The last term on the right-hand side,

\[ P_{\text{loss}} = \frac{\omega_r \vec{E}^* \cdot \vec{A} \cdot \vec{E}}{8\pi}, \]  

(B.21)

represents the time-averaged power density absorbed by the media.

Equation (B.16) represents the time-averaged energy conservation law in a weakly lossy media. In fact, we can write it in the form

\[ - \nabla \cdot \langle \vec{S} \rangle = \frac{\partial}{\partial t} \langle U \rangle + \alpha \left| \langle \vec{S} \rangle \right|, \]  

(B.22)

where

\[ \langle \vec{S} \rangle \equiv (\vec{S}_{\text{EM}} + \vec{S}_P) \exp(2\omega t - 2\vec{k}_r \cdot \vec{x}) \]  

(B.23)

is the total energy flux averaged over a time scale that is long compared to the wave period, but short compared to the decay of the wave amplitude. The exponential
coefficient represents the decay in time and space. Similarly,

\[ \langle U \rangle \equiv (U_M + U_{E+P}) \exp(2\omega_i t - 2\vec{k}_i \cdot \vec{x}) \]  

(B.24)

is the total time-averaged energy density, and

\[ \alpha \equiv \frac{P_{\text{loss}}}{|\vec{S}_{\text{EM}} + \vec{S}_P|} \]  

(B.25)

is the effective absorption coefficient.

So far, the discussion is restricted to a plane wave with single \( \omega \) and \( \vec{k} \). It is straightforward to generalize the derivation to a frequency spectrum. We can relate the spectral power flux \( \vec{S}_\omega \), spectral energy density \( U_\omega \), and spectral lossy power \( P_{\text{loss,}\omega} \) to the quantities for the plane wave by

\[ \langle \vec{S} \rangle = \vec{S}_\omega d\omega, \]
\[ \langle U \rangle = U_\omega d\omega, \]
\[ P_{\text{loss}} = P_{\text{loss,}\omega} d\omega. \]  

(B.26)

Then we immediately see that the equation for the spectral quantities has the same form as Eq B.22,

\[ - \nabla \cdot \vec{S}_\omega = \frac{\partial}{\partial t} U_\omega + \alpha |\vec{S}_\omega|, \]  

(B.27)

**B.2 Group Velocity of the Radiation**

The group velocity of the radiation is defined as

\[ \vec{v}_g \equiv \langle \vec{S} \rangle / \langle U \rangle, \]  

(B.28)

where \( \langle \vec{S} \rangle \) and \( \langle U \rangle \) are defined in Eq B.23 and B.24.
Starting with the real part of Eq $\text{B.11}$, we allow $\vec{k}_r$ to take a small variation $\delta \vec{k}_r$. The real frequency $\omega_r$ will therefore have a variation $\delta \omega_r = \left( \partial \omega_r(\vec{k}_r)/\partial \vec{k}_r \right) \cdot \delta \vec{k}_r$ according to the dispersion relation. We also expend $(\omega_r + \delta \omega_r) e^{\alpha H} (\omega_r + \delta \omega_r, \vec{k}_r + \delta \vec{k}_r)$ in a similar manner as Eq $\text{B.15}$ up to the first order in $\delta \vec{k}_r$ and $\delta \omega_r$, we obtain

$$
\delta \vec{k}_r \cdot \left( \frac{c}{16\pi} (\vec{E} \times \vec{B}^* + \vec{E}^* \times \vec{B}) - \frac{1}{16\pi} \vec{E}^* \cdot \frac{\partial \omega_r e^{\alpha H}}{\partial \vec{k}_r} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E} \right) 
= \frac{\delta \omega_r}{16\pi} \left( |\vec{B}|^2 + \vec{E}^* \cdot \frac{\partial \omega_r e^{\alpha H}}{\partial \omega_r} \bigg|_{\omega_r, \vec{k}_r} \cdot \vec{E} \right).$

(B.29)

Note that this is exactly

$$
\delta \vec{k}_r \cdot \langle \vec{S} \rangle = \delta \vec{k}_r \cdot \frac{\partial \omega_r}{\partial \vec{k}_r} \langle U \rangle.
$$

(B.30)

Since $\delta \vec{k}_r$ is arbitrary, using the definition of $\vec{v}_g$, Eq $\text{B.28}$, we then have

$$
\vec{v}_g = \frac{\partial \omega_r}{\partial \vec{k}_r}.
$$

(B.31)

This equation is read as the energy for a specific mode with real frequency $\omega_r$ and wave vector $\vec{k}_r$ propagates along the vector $\partial \omega_r / \partial \vec{k}_r$, where $\omega_r = \omega_r(\vec{k}_r)$ satisfies the dispersion relation for this mode.

### B.3 Equation of Radiative Transfer

#### B.3.1 Uniform Media

The Equation of Radiative Transfer is an equation describing the propagation of the electromagnetic radiation power in a stationary medium. In the case that the medium is also uniform, it can be derived from Eq $\text{B.22}$ by letting the time derivative of the energy density to be zero (the stationary condition), and taking $\langle \vec{S} \rangle$ as the energy flux of a single mode with frequency $\omega$ and a small cone of $\vec{k}$s around a central $\vec{k}$.
such that the energy flux is propagating in the direction within a small solid angle $d\Omega$ around the central direction of $\vec{v}_g(\omega, \vec{k})$. The spectral flux is in general proportional to $d\Omega$, i.e.

$$\vec{S}_\omega = I_\omega \vec{a} \, d\Omega,$$

(B.32)

where $\vec{a} \equiv \vec{v}_g / |\vec{v}_g|$ is the unit vector along the central group velocity, and $I_\omega$ the proportionality coefficient. $I_\omega$ has the unit of power per unit area per unit frequency per unit solid angle, and is usually called the “intensity” of the radiation.

In a uniform medium, the group velocity is constant, so

$$\nabla \cdot \vec{S}_\omega = \vec{a} \cdot \nabla I_\omega \, d\Omega = \frac{dI_\omega}{ds} \, d\Omega,$$

(B.33)

where $s$ denotes the length along the propagation path. Substituting Eq B.33 into Eq B.22, we have

$$\frac{dI_\omega}{ds} = -\alpha I_\omega.$$

(B.34)

This is the Equation of Radiative Transfer in a purely dissipative, uniform medium.

In general, if we include the emissivity of the media due to the motion of discrete charges, the equation reads

$$\frac{dI_\omega}{ds} = \eta_\omega - \alpha I_\omega,$$

(B.35)

where the left-hand side term describes the change of intensity per unit length along the propagation path. The emissivity term $\eta_\omega$ on the right describes the emitted power per unit volume and unit solid angle, the second term on right-hand side is the absorption. We’ll drop the subscript $\omega$ in the future for simplicity, all the intensity and emissivity should be considered as the spectral quantities.
B.3.2 Weakly Non-uniform, Isotropic Medium

When the medium is slightly non-uniform, the propagation of the radiation will change due to refraction. In the limit that the Geometric Optics is valid, i.e. the WKB criteria Eq. 3.26 holds, we can still describe the propagation of the radiation as a bundle of rays with group velocities defined in Eq. B.31. In addition, when the medium is isotropic, the change in group velocity due to the refraction across an interface between regions with two different dielectric constants can be described by the Snell’s Law (See, e.g. [76]),

\[ n_1 \sin \xi_1 = n_2 \sin \xi_2 , \]  

where \( n_1 \) is the refractive index of the medium where the light is coming from, and \( n_2 \) the refractive index of the medium the light is going into, \( \xi_1 \) and \( \xi_2 \) are the angles of incident and transmission, respectively, as shown in Fig. B.1.

A pencil of incident radiation centered at incident angle \( \xi_1 \) within the solid angle \( d\Omega_1 \), will go out of the interface as a pencil centered at \( \xi_2 \) with solid angle \( d\Omega_2 \). In a
lossless medium, the total power flux going through an unit area \(da\) aligned with the interface is conserved, i.e.

\[
I_1 \cos \xi_1 \, da \, d\Omega_1 \, d\omega = I_2 \cos \xi_2 \, da \, d\Omega_2 \, d\omega.
\]  
(B.37)

where \(d\Omega_i = \sin \xi_i \, d\xi_i \, d\phi\), with \(\phi\) defined as the azimuthal angle on the interface.

A relation between \(d\xi_1\) and \(d\xi_2\) can be obtained by taking the derivative on both sides of Eq B.36 with respect to \(\xi\),

\[
n_1 \cos \xi_1 \, d\xi_1 = n_2 \cos \xi_2 \, d\xi_2.
\]  
(B.38)

Multiply with Eq B.36, we have

\[
n_1^2 \sin \xi_1 \, \cos \xi_1 \, d\xi_1 = n_2^2 \sin \xi_2 \, \cos \xi_2 \, d\xi_2.
\]  
(B.39)

Substitute it into Eq B.37, we can finally get

\[
\frac{I_1}{n_1^2} = \frac{I_2}{n_2^2},
\]  
(B.40)

or equivalently, the quantity \(I/n^2\) is conserved after refraction.

To include the emission and absorption of the medium, similar to the uniform case, we need to introduce the emissivity \(\eta\) and the absorption coefficient \(\alpha\). Two terms are added to the left-hand side of Eq B.37,

\[
[I_1 + (\eta - \alpha I_1) \, ds] \cos \xi_1 \, da \, d\Omega_1 \, d\omega = I_2 \cos \xi_2 \, da \, d\Omega_2 \, d\omega.
\]  
(B.41)
Again, after dividing Eq B.39 on both sides, we have the Equation of Radiative Transfer in a non-uniform isotropic medium,

$$\frac{d}{ds} \frac{I}{n^2} = \eta - \alpha I.$$  \hspace{1cm} (B.42)

### B.3.3 Weakly Non-uniform, Anisotropic Medium

When the medium is anisotropic, the discussion above needs to be modified because the phase velocity and the group velocity of the radiation no longer coincide, and Snell’s Law does not hold in general. Here, we will derive the relation between the radiation intensity before and after passing the interface of media with different refractive indexes, by considering a group of photons going through the interface.

**Conservation of photons**

As in the previous section, we consider a pencil of radiation going through a unit area $da$, centered at the incident angle $\xi_1$ before refraction, and at angle $\xi_2$ after refraction. The solid angle elements are $d\Omega_1$ and $d\Omega_2$ respectively. Note that now, $\xi_1$, $\xi_2$, $d\Omega_1$, $d\Omega_2$ are all defined in the group velocity space, which is not the same as those in the wave vector space.

Here, let’s suppose the photons within the above mentioned pencil also comes from an element $d^3\vec{x}d^3\vec{k}$ in the phase space $(\vec{x},\vec{k})$. The equations of motion for a photon in the phase space is given by the ray tracing equations

$$\frac{d\vec{x}}{dt} = \frac{\partial \omega}{\partial \vec{k}},$$

$$\frac{d\vec{k}}{dt} = -\frac{\partial \omega}{\partial \vec{x}}.$$ \hspace{1cm} (B.43)
These equations are in the form of a Hamiltonian system. In fact, if we define the photon’s Hamiltonian and canonical momentum as

\[ \varepsilon = \hbar \omega, \]
\[ \mathbf{p} = \hbar \mathbf{k}, \]

we immediately recover the same equations of motion.

Therefore, because the system is Hamiltonian, the phase space volume is conserved along the trajectory of the photon. In our case, it means before and after the refraction, the photons will occupy the same volume in \((\mathbf{x}, \mathbf{k})\) space, i.e.

\[ d^3 \mathbf{x}_1 d^3 \mathbf{k}_1 = d^3 \mathbf{x}_2 d^3 \mathbf{k}_2, \]

or, if written in a spherical coordinate system,

\[ w_1 \cos \xi_1 d a \, dt \, k_1^2 \, d k_1 \Omega = w_2 \cos \xi_2 d a \, dt \, k_2^2 \, d k_2 \Omega, \]

where \( \mathbf{w} \equiv \partial \omega / \partial \mathbf{k} \) is the group velocity, \( \omega \equiv |\mathbf{w}| \), \( d \Omega_k \) is the solid angle element in \( \mathbf{k} \) space. Noticing that \( n \equiv c k / \omega \), and \( d k = |\partial k / \partial \omega| d \omega \), we obtain

\[ w_1 \cos \xi_1 d a \, n_1^2 \left| \frac{\partial k_1}{\partial \omega} \right| d \omega d \Omega_{k_1} = w_2 \cos \xi_2 d a \, n_2^2 \left| \frac{\partial k_2}{\partial \omega} \right| d \omega d \Omega_{k_2}. \]

The energy conservation in the lossless case is still in the form of Eq B.37 divided by Eq B.47, we have

\[ \frac{I_1}{n_1^2} = \frac{I_2}{n_2^2}, \]

where

\[ n_\mathbf{r} \equiv n^2 w \left| \frac{\partial k}{\partial \omega} \right| \frac{d \Omega_k}{d \Omega} \]
is the square of the effective refractive index along the ray. It is clear that $n_r$ equals $n$ in the isotropic case, where $w = |\partial \omega / \partial k|$, and $d\Omega_k$ is the same as $d\Omega$. And Eq B.48 becomes Eq B.40 as expected.

As in the isotropic case, after taking into account the emission and absorption from the medium, we can write the Equation of Radiative Transfer in a weakly non-uniform, anisotropic medium as

$$\frac{d}{ds} \frac{I}{n_r^2} = \eta - \alpha I n_r^2,$$  \hspace{1cm} (B.50)

We’ll calculate the $d\Omega_k/d\Omega$ term in the next section for a medium that is symmetric under rotation with respect to an axis, which is usually the case for a magnetized plasma, where the symmetry axis is along the magnetic field.

**Relation between $d\Omega$ and $d\Omega_k$**

In this section, we calculate the relation between the solid angle element in wave vector space, $d\Omega_k$, and that in group velocity space, $d\Omega$, in order to obtain an explicit expression for the ray refractive index $n_r$ defined in Eq B.49.

Consider radiation propagating in a magnetized plasma satisfying the WKB approximation criteria, Eq 3.26. Locally, we can obtain the dispersion relation between the wave frequency $\omega$ and the wave vector $\vec{k}$. Because of the existence of the magnetic field, the dispersion relation is in general anisotropic, i.e. $\omega$ is a function of both the magnitude of $\vec{k}$ and the direction of $\vec{k}$. Figure B.2 shows a typical dispersion relation in magnetized plasmas. The iso-frequency surface in $\vec{k}$ space is symmetric in the azimuthal angle $\varphi$ around the magnetic field, but not along the inclination angle $\theta$. The result of the anisotropy is that the group velocity $\vec{w} \equiv \partial \omega / \partial \vec{k}$ is no longer along the direction of $\vec{k}$. As shown in Fig B.2, the group velocity can be expressed in terms
Figure B.2: The group velocity in an anisotropic medium. The medium is assumed to be symmetric in the azimuthal direction, i.e. $\omega = \omega(k, \theta)$ does not depend on $\varphi$. The group velocity is perpendicular to the iso-frequency surface, which may not be in the same direction as the wave vector $\vec{k}$, but will be in the same $\varphi$ plane. $\beta$ is defined as the angle from $\vec{k}$ to $\vec{w}$, with positive $\beta$ defined to be in the same direction as $\theta$ and $\xi$.

of the dispersion relation as

$$\vec{w} = \frac{\partial \omega}{\partial k} \hat{k} + \frac{1}{k} \frac{\partial \omega}{\partial \theta} \hat{\theta}$$  \hspace{1cm} (B.51)

Note that there is no $\varphi$ component because we have assumed symmetry in $\varphi$. We define $\xi$ to be the inclination angle of $\vec{w}$, and $\beta \equiv \xi - \theta$ the angle between $\vec{w}$ and $\vec{k}$.

Then, the two solid angle elements can be formally written as

$$d\Omega_k = \sin \theta d\theta d\varphi,$$ \hspace{1cm} (B.52)

$$d\Omega = \sin \xi d\xi d\varphi.$$ \hspace{1cm} (B.53)
Their ratio is
\[
\frac{d\Omega_k}{d\Omega} = \frac{\sin \theta \, d\theta}{\sin \xi \, d\xi} = -\sin \theta \frac{\partial \cos \xi}{\partial \theta},
\]
where \(\xi = \theta + \beta\), and from Fig B.2 we can see that
\[
\tan \beta = \frac{1}{k} \frac{\partial \omega}{\partial k} \bigg|_{\omega},
\]
\[
\tan \beta = \frac{\partial \omega}{\partial \theta} \quad \text{(B.55)}
\]

**Ray refractive index in magnetized plasma**

We now can obtain an explicit formula for the ray refractive index \(n_r\) in a magnetized plasma.

From the definition of \(\beta\), and the relation between \(\vec{w}\) and \(\partial \omega/\partial k\) shown in Fig B.2 we have
\[
\vec{w} = \frac{\partial \omega}{\partial k} \cos \beta \quad \text{(B.56)}
\]

Substitute Eq B.54 and B.56 into Eq B.49 we finally get
\[
n_r^2 = n^2 \left| \frac{\sin \theta}{\cos \beta \left\{ \frac{\partial \cos(\theta + \beta)}{\partial \theta} \right\}} \right|, \quad \text{(B.57)}
\]

where, from Eq B.55
\[
\beta = \arctan \left( -\frac{1}{k} \frac{\partial k}{\partial \theta} \right) \quad \text{(B.58)}
\]

This result can also be found in Bekefi’s book [48] and Bornatici’s review paper on ECE [43].
Appendix C

Source Current of ECE

In this chapter, we discuss the formulation of the ECE source current from a test particle point of view. We start with the Klimontovich description of the plasma\cite{77}, and the distribution function in 6N-dimensional phase space, then derive the BBGKY-chain of equations\cite{78}. The time-averaged ECE radiation power is then calculated by taking the ensemble average of the Poynting flux. We then show that a test particle problem can be constructed and related to the original problem in such a way that the test particles are statistically independent, and the ensemble average of the Poynting flux is the same as the original one up to first order in the plasma parameter, $\epsilon_p \equiv 1/n_e L_{De}^3$, where $L_{De} \equiv v_{th}/\omega_{pe}$ is the electron Debye length (See e.g. \cite{79}). Finally we show that the Maxwell’s Equations used in the reciprocity theorem, Eq 3.1 and 3.2 are exactly the equations for the field generated by the test particles, where the source current term is due to the bare charges, and the dielectric tensor represents the “shielding” by the plasma. A formula for the source current used in Sec 3.1.23 is given in the end of the chapter.
C.1 Klimontovich description and the distribution function in 6N-dimensional phase space

For simplicity, we consider only electrons here, since the ions are too heavy to be responsive to the fields in the electron cyclotron frequency range. Ions are assumed to be a stationary background with a uniformly distributed charge density.

For an N-particle plasma system, the Klimontovich approach\[77\] considers the distribution function in the 6-dimensional phase space

\[
\mathcal{N}(\vec{r}, \vec{v}, t) = \sum_{i} \delta(\vec{r} - \vec{r}_i(t)) \delta(\vec{v} - \vec{v}_i(t)),
\]  

(C.1)

where \(i\) labels the electrons from 1 to \(N\). \(\vec{r}_i(t)\), \(\vec{v}_i(t)\) are the \(i^{th}\) particle position and velocity at time \(t\), respectively, which obey Newton's law

\[
\frac{d}{dt} \vec{r}_i = \vec{v}_i,
\]

\[
\frac{d}{dt} \vec{v}_i = \frac{\vec{F}_i}{m},
\]

(C.2)

where

\[
\vec{F}_i = \vec{F}_{i,ext} + \sum_{j \neq i} \vec{F}_{i,j},
\]

\[
\vec{F}_{i,ext} \equiv -e \left[ \vec{E}_{ext}(\vec{r}_i) + \frac{1}{c} \vec{v}_i \times \vec{B}_{ext}(\vec{r}_i) \right],
\]

\[
\vec{F}_{i,j} \equiv -e \left[ \vec{E}_j(\vec{r}_i) + \frac{1}{c} \vec{v}_i \times \vec{B}_j(\vec{r}_i) \right],
\]

(C.3)

describes the electromagnetic force applied to particle \(i\) by external sources and other particles, and \(\vec{E}_j(\vec{r}_i)\) and \(\vec{B}_j(\vec{r}_i)\) mean the field generated by particle \(j\) at location of particle \(i\).
The electromagnetic field generated by the \( j \)th electron is determined by Maxwell’s Equations

\[
\nabla \cdot \vec{E}_j = 4\pi \rho_j, \quad (C.4)
\]
\[
\nabla \cdot \vec{B}_j = 0, \quad (C.5)
\]
\[
\nabla \times \vec{E}_j = -\frac{1}{c} \frac{\partial \vec{B}_j}{\partial t}, \quad (C.6)
\]
\[
\nabla \times \vec{B}_j = \frac{1}{c} \frac{\partial \vec{E}_j}{\partial t} + \frac{4\pi}{c} \vec{j}_j. \quad (C.7)
\]

\( \vec{E}_j, \vec{B}_j, \vec{j}_j \) and \( \rho_j \) are all functions of \( \vec{r} \) and \( t \). \( \vec{j}_j \) and \( \rho_j \) are the current and charge density produced by the \( j \)th particle,

\[
\vec{j}_j(\vec{r}, t) = -e\vec{v}_j(t) \delta(\vec{r} - \vec{r}_j(t)), \quad (C.8)
\]
\[
\rho_j(\vec{r}, t) = -e \delta(\vec{r} - \vec{r}_j(t)). \quad (C.9)
\]

It is clear that as soon as an initial state of all the individual particles is given, the whole system is determined and may evolve deterministically in time. The state of all the particle positions and velocities stays in a \( 6N \)-dimensional phase space. In general, we do not know the exact initial state of all the particles. Instead, the information we have about the system can only provide us with a probability distribution of the initial state, denoted as \( \mathcal{D}(X, t = 0) = \mathcal{D}_0 \), where \( X \) includes all \( N \) particles’ position and velocity coordinates. The initial distribution \( \mathcal{D}_0(X) \) needs to be chosen carefully based on the global parameters of the system, e.g. volume, local temperature, etc., as well as all orders of correlations between the particles.

In principle, we can evolve this probability distribution based on the Liouville equation

\[
\left[ \frac{\partial}{\partial t} + \sum_{i=1}^{N} \vec{v}_i \cdot \nabla_{\vec{r}_i} + \sum_{i=1}^{N} \frac{\vec{F}_i}{m} \cdot \nabla_{\vec{v}_i} \right] \mathcal{D}(X, t) = 0, \quad (C.10)
\]
which is essentially the continuity equation for $D$ under the 6N-dimensional flow determined by the particles equations of motion.

The ensemble average of an observable $A(\vec{r}, t)$, is then defined as the integration of $A(\vec{r}, t)$ given the system at $(\vec{X}, t)$ times the probability of the system being at $(\vec{X}, t)$, over the whole 6N state space,

$$\langle A \rangle(\vec{r}, t) \equiv \int A(\vec{r}, t|\vec{X}, t) D(\vec{X}, t) \, d\vec{X} \quad \text{(C.11)}$$

Note that this definition doesn’t imply an instantaneous response of $A$ to the particles described by $\vec{X}$. In fact, since the system is deterministic, a state at time $t$, $(\vec{X}, t)$, contains the full information of the whole history of the system. Thus the dependence of $A$ on the state $(\vec{X}, t)$ is actually a dependence on its whole history. We therefore may as well write $A = A(\vec{r}, t|\vec{X}_0)$ so that the dependence is shown explicitly on the initial state.

### C.2 BBGKY-hierarchy

By taking moments of Eq\textsuperscript{[C.10]} we can obtain the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations\textsuperscript{[78]},

$$\left\{ \frac{\partial}{\partial t} + \sum_{i=1}^{s} \vec{v}_i \cdot \nabla \vec{v}_i + \frac{1}{m} \sum_{i=1}^{s} \left[ \vec{F}_{i,ext} + \sum_{j \neq i}^{s} \vec{F}_{i,j} \right] \cdot \nabla \vec{v}_i \right\} f_s$$

$$+ \frac{N - s}{V} \sum_{i=1}^{s} \int \frac{\vec{F}_{i,s+1}}{m} \cdot \nabla \vec{v}_i f_s \, dX_{s+1} \quad = 0,$$

where

$$f_s(\vec{X}_1, \ldots, \vec{X}_s, t) \equiv V^s \int D(\vec{X}, t) \, dX_{s+1} \ldots dX_N, \quad \text{(C.13)}$$

is called the “$s$-particle” distribution function. $V \equiv \int d\vec{r}_1$ is the total volume of the plasma.
Eq C.12 is a direct result of integrating Eq C.10 over $X_{s+1}, \ldots, X_N$, and using the properties of $D(X, t)$ which are vanishing at plasma boundary and symmetric under interchange of two particles.

It is worth noting that solving the complete BBGKY equations, Eq C.12, is not easier than solving the Liouville’s Equation, Eq C.10. In fact, the $N$-particle distribution function is just $V^N D(X, t)$, so the last BBGKY equation is exactly the Liouville’s Equation for $D(X, t)$.

However, in the regime where the plasma parameter $\epsilon_p$ is small, it has been shown that the BBGKY hierarchy can be solved approximately by expanding in $\epsilon_p$ [80]. In this case, instead of solving the full $N$-particle distribution function, we can use a closure for a much lower order distribution function, and solve for only a few of the lowest order distribution functions. For example, to the first order in $\epsilon_p$, we have the solution

$$f_s(X_1, \ldots, X_s, t) = \prod_{i=1}^{s} f(X_i, t) + \frac{1}{2} \sum_{k=1}^{s} \sum_{j \neq k}^{s} \left[ \prod_{i \neq j,k}^{s} f(X_i, t) \right] G(X_j, X_k, t), \quad (C.14)$$

where $f(X_1, t)$ and $G(X_1, X_2, t)$ satisfy the following equations

$$\left( \frac{\partial}{\partial t} + \bar{v}_1 \cdot \nabla \bar{r}_1 + \frac{\vec{F}_M}{m} \cdot \nabla \bar{v}_1 \right) f(X_1, t) = -\frac{n}{m} \int \vec{F}_{1,2} \cdot \nabla \bar{v}_1 G(X_1, X_2, t) dX_2, \quad (C.15)$$

$$\left( \frac{\partial}{\partial t} + O(X_1, t) + O(X_2, t) \right) G(X_1, X_2, t) \quad (C.16)$$

$$= -\frac{\vec{F}_{1,2}}{m} \cdot [f(X_2, t) \nabla \bar{v}_1 f(X_1, t) - f(X_1, t) \nabla \bar{v}_2 f(X_2, t)],$$

with

$$\vec{F}_M(X_1, t) \equiv \vec{F}_{1,ext} + n \int \vec{F}_{1,2} f(X_2, t) dX_2, \quad (C.17)$$

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is the sum of forces due to the external field and the mean field of the 1-particle distribution function. The operator $O$ is defined as

$$O(X_{1(2)}, t)G(X_1, X_2, t) \equiv \left[ \vec{v}_{1(2)} \cdot \nabla r_{1(2)} + \frac{\vec{F}_M(X_{1(2)}, t)}{m} \cdot \nabla \vec{v}_{1(2)} \right] G(X_1, X_2, t)$$

\[ + \frac{n}{m} \nabla \vec{v}_{1(2)} f(X_{1(2)}, t) \cdot \int F_{1,2(2,1)} G(X_1, X_2, t) dX_{2(1)}, \] (C.18)

In addition to the convection of $G$ under mean field force $F_M$, the $O$ operator also contains the convection of $f$ under the force due to $G$, which is of the same order as $G$.

### C.3 Correlation of fields

In ECE calculation, we need to calculate the ensemble averaged Poynting flux of the emission field at the antenna. This is essentially calculating the correlation between the emission electric field and magnetic field, $\langle E_i(\vec{r}, t)B_j(\vec{r}, t) \rangle$, where $\vec{r}$ is the location of the antenna, and the ensemble average is over the electron distribution function, $\mathcal{D}(X, t)$.

In this section, we discuss how the correlation of two general fields can be formally expressed in terms of $f$ and $G$.

Suppose we have two field quantities,

$$A(\vec{r}, t) = \sum_{i=1}^{N} a(X_i(t)|\vec{r}),$$

$$B(\vec{r}, t) = \sum_{i=1}^{N} b(X_i(t)|\vec{r}).$$

where $a(X_i|\vec{r}, t)$ and $b(X_i|\vec{r}, t)$ are the corresponding fields at $\vec{r}$ generated by a single particle at $X_i$ at time $t$. The key assumption about this representation is that $a(X_i(t)|\vec{r})$ depends only on the state of the $i^{th}$ electron. In general, the field $a(X_i|\vec{r}, t)$
also depends on the history of the $i^{th}$ particle, which in turn depends on both the external fields and the fields generated by other particles. To be consistent with our ordering in plasma parameter, here we use the lowest order trajectory, which includes only the external fields, and ignore the higher order plasma generated fields. This means the $i^{th}$ particle trajectory is totally determined by its state $X_i(t)$ at time $t$. So, the single particle field $a(X_i|\vec{r},t)$ indeed only depends on the $i^{th}$ particle state.

The correlation of $A$ and $B$ at different spatial locations, but at the same time, is then defined as

$$\langle A(\vec{r},t)B(\vec{r}',t) \rangle = \int dX \mathcal{D}(X,t) \sum_{i=1}^{N} \sum_{j=1}^{N} a(X_i|\vec{r}) b(X_j|\vec{r}') \quad \text{(C.20)}$$

For each $\{i,j\}$ pair, when $i = j$, integration of all but one coordinate can be carried out. When $i \neq j$, two of the coordinates will be left. So, using the definition of $f_1$ and $f_2$, and the symmetry of $\mathcal{D}$ under switching two particles, we have

$$\langle A(\vec{r},t)B(\vec{r}',t) \rangle = n \int f_1(X_1,t) a(X_1|\vec{r}) b(X_1|\vec{r}') dX_1 + n^2 \int f_2(X_1,X_2,t) a(X_1|\vec{r}) b(X_2|\vec{r}') dX_1 dX_2 \quad \text{(C.21)}$$

Substitute $f_1$ and $f_2$ with $f$ and $G$ using Eq [C.14], we have the formal expression for the correlation

$$\langle A(\vec{r},t)B(\vec{r}',t) \rangle = \langle A \rangle \langle B \rangle + n \int f(X_1,t) a(X_1|\vec{r}) b(X_1|\vec{r}') dX_1 + n^2 \int G(X_1,X_2,t) a(X_1|\vec{r}) b(X_2|\vec{r}') dX_1 dX_2 \quad \text{(C.22)}$$

where $\langle A(\vec{r},t) \rangle = n \int f(X_1,t)a(X_1|\vec{r})$ is the ensemble average of $A$.

In order to calculate the correlation, we need to solve the coupled equations Eq [C.15] and [C.16] for both $f$ and $G$, which is in general very hard.
C.4 Test particle problem

In this section, we introduce a test particle problem, and derive the governing equation for the perturbed 1-particle distribution function $\delta f$ by the test particle. A very useful relation between $\delta f$ and $G$ is given, and will lead to another way of calculating the correlation.

To start, let’s consider we put one “test” electron into the plasma we have described in the previous section with the distribution functions $D(X,t)$, and $f_s(X_1,\ldots,X_s,t)$. We denote the phase space coordinates of this test particle as $(X_0,t)$. The Liouville’s Equation now have an additional force term due to the test particle,

$$\left[ \frac{\partial}{\partial t} + N \sum_{i=1}^{N} \vec{v}_i \cdot \nabla \vec{r}_i + \sum_{i=1}^{N} \frac{\vec{F}_i}{m} \cdot \nabla \vec{v}_i \right] \hat{D}(X,t) = - \frac{1}{m} \sum_{i=1}^{N} \vec{F}_{i,0} \cdot \nabla \vec{v}_i \hat{D}, \quad (C.23)$$

and consequently, the equations for the first and second distribution functions $\hat{f}$, and $\hat{G}$ become

$$\left( \frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla \vec{r}_1 + \frac{\hat{F}_M + \hat{F}_{1,0}}{m} \cdot \nabla \vec{v}_1 \right) \hat{f}(X_1,t) = - \frac{n}{m} \int \vec{F}_{1,2} \cdot \nabla \vec{v}_1 \hat{G}(X_1,X_2,t)dX_2, \quad (C.24)$$

$$\left( \frac{\partial}{\partial t} + \hat{O}(X_1,t) + \hat{O}(X_2,t) \right) \hat{G}(X_1,X_2,t) = - \frac{\vec{F}_{1,2}}{m} \cdot \left[ \hat{f}(X_2,t)\nabla \vec{v}_1 \hat{f}(X_1,t) - \hat{f}(X_1,t)\nabla \vec{v}_2 \hat{f}(X_2,t) \right], \quad (C.25)$$

where the new mean field force $\hat{F}_M$ is defined by the new 1-particle distribution function $\hat{f}$ (See Eq[C.17]), and the new $\hat{O}$ operator differs from the old one defined in
Eq \[C.18\] in both the force term and the 1-particle distribution function,

\[
\hat{O}(X_i, t) \hat{G}(X_1, X_2, t) \equiv \left[ \vec{v}_i \cdot \nabla_{\vec{r}_i} + \frac{\vec{F}_M + \vec{F}_{i,0}}{m} \cdot \nabla_{\vec{v}_i} \right] \hat{G}(X_1, X_2, t) + \frac{n}{m} \nabla_{\vec{v}_i} \hat{f}(X_i, t) \cdot \int \vec{F}_{i,i'} \hat{G}(X_1, X_2, t) dX_{i'}.
\]  

(C.26)

Define \( \delta f \equiv \hat{f} - f \), \( \delta G \equiv \hat{G} - G \). In our plasma ordering, this perturbation due to the test particle is a higher order term compared to the original quantity. Since \( G \) is already a first order quantity, the \( \delta G \) term is second order and can be neglected in the first order corrections. Subtract Eq \[C.15\] from Eq \[C.24\] and keep only the leading order terms, we have

\[
\left( \frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla_{\vec{r}_1} + \frac{\vec{F}_M}{m} \cdot \nabla_{\vec{v}_1} \right) \delta f(X_1, t) + \frac{\delta \vec{F}_M + \vec{F}_{i,0}}{m} \cdot \nabla_{\vec{v}_1} f(X_1, t) = 0,
\]  

(C.27)

where

\[
\delta \vec{F}_M(X_1, t) \equiv n \int \vec{F}_{1,2} \delta f(X_2, t) dX_2,
\]  

(C.28)

is the electromagnetic force applied to particle 1 due to the perturbed distribution function \( \delta f \).

Since the perturbed distribution function \( \delta f \) is a result of the test particle at \( X_0 \), we may explicitly write

\[
P(X_0 | X_1, t) \equiv \delta f(X_1, t).
\]  

(C.29)

Note that the test particle phase space coordinate \( X_0 = X_0(t) \) is a function of time, and obeys Newton’s law

\[
\frac{d}{dt} \vec{r}_0 = \vec{v}_0,
\]  

(C.30)

\[
\frac{d}{dt} \vec{v}_0 = \frac{\vec{F}_M(X_0, t)}{m}.
\]  

(C.31)
So, substitute Eq C.29 into Eq C.27 and note that the $\partial/\partial t$ is calculated with only $X_1$ fixed, we have the equation for $P(X_0|X_1,t)$,

\[
\left( \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla \tilde{r}_0 + \frac{\vec{F}_M(X_0,t)}{m} \cdot \nabla \tilde{v}_0 \right) P(X_0|X_1,t)
\]

\[+ O(X_1,t)P(X_0|X_1,t) = -\frac{\vec{F}_{1,0}}{m} \cdot \nabla \vec{v}_1 f(X_1,t). \]

(C.32)

This is the equation governing the test particle induced distribution function perturbation. The first term is the former $\partial \delta f/\partial t$ term, with the contribution of the test particle movement explicitly written out.

We will claim without proof here that the solution to Eq C.32 is related to the solution of Eq C.16, $G(X_1, X_2, t)$, as follows,

\[
G(X_1, X_2, t) = f(X_1, t)P(X_1|X_2, t) + f(X_2, t)P(X_2|X_1, t)
\]

\[+ n \int dX_3 f(X_3, t)P(X_3|X_1, t)P(X_3|X_2, t). \]

(C.33)

The terms in Eq C.33 have very straightforward physical interpretations. The $f(X_1, t)P(X_1|X_2, t)$ term stands for the contribution from the perturbed distribution at $X_2$ due to a particle being at $X_1$. The second term is opposite, stands for the contribution of the perturbed distribution at $X_1$ due to a particle at $X_2$. The last term describes the correlation between $X_1$ and $X_2$ coming from both of the locations being affected by a particle at $X_3$.

It is shown in [53] that, up to the first order in the plasma parameter, Eq C.33 is indeed the solution to Eq C.16 given this relation being initially satisfied.
C.5 Test particle expression of the correlation

Now, substitute Eq C.33 into the expression of the correlation, Eq C.22, we have

\[
\langle A(\vec{r}, t) B(\vec{r}', t) \rangle = \langle A \rangle \langle B \rangle + n \int f(X_1, t) a(X_1|\vec{r}) b(X_1|\vec{r}') dX_1
\]

\[+ n^2 \int f(X_1, t) P(X_1|X_2, t) a(X_1|\vec{r}) b(X_2|\vec{r}') dX_1 dX_2 \]

\[+ n^2 \int f(X_2, t) P(X_2|X_1, t) a(X_1|\vec{r}) b(X_2|\vec{r}') dX_1 dX_2 \]

\[+ n^3 \int f(X_3, t) P(X_3|X_1, t) P(X_3|X_2, t) a(X_1|\vec{r}) b(X_2|\vec{r}') dX_1 dX_2 dX_3. \]

(C.34)

Define the “dressed” field,

\[
\hat{a}(X_1|\vec{r}, t) \equiv a(X_1|\vec{r}) + n \int a(X_2|\vec{r}) P(X_1|X_2, t) dX_2, \quad (C.35)
\]

we have

\[
\langle A(\vec{r}, t) B(\vec{r}', t) \rangle = \langle A \rangle \langle B \rangle + n \int f(X_1, t) a(X_1|\vec{r}) b(X_1|\vec{r}') dX_1
\]

\[+ n \int f(X_1, t) a(X_1|\vec{r}) (\hat{b}(X_1|\vec{r}', t) - b(X_1|\vec{r}')) dX_1 \]

\[+ n \int f(X_2, t) (\hat{a}(X_2|\vec{r}, t) - a(X_2|\vec{r})) b(X_2|\vec{r}') dX_2 \quad (C.36)\]

\[+ n \int f(X_3, t) (\hat{a}(X_3|\vec{r}, t) - a(X_3|\vec{r})) (\hat{b}(X_3|\vec{r}', t) - b(X_3|\vec{r}')) dX_3 \]

\[= \langle A \rangle \langle B \rangle + n \int f(X_1, t) \hat{a}(X_1|\vec{r}, t) \hat{b}(X_1|\vec{r}', t) dX_1. \]

(C.36)

Using Eq C.36, without solving for the two particle correlation function G, we can calculate the correlation using only the 1-particle distribution function f(X, t), and the perturbed distribution function P(X_0|X, t).
Now, let’s consider a system consists of \( N \) statistically independent test particles, with the total distribution function

\[
\hat{D}(\mathbf{X}, t) = \frac{1}{V N} \prod_{i=1}^{N} f(X_i, t),
\]

(C.37)

where \( f(X_i, t) \) is the 1-particle distribution of our original plasma.

We define the total dressed fields

\[
\hat{A} \equiv \sum_{i=1}^{N} \hat{a}(X_i, t),
\]

\[
\hat{B} \equiv \sum_{i=1}^{N} \hat{b}(X_i, t).
\]

(C.38)

Then, the correlation between \( \hat{A} \) and \( \hat{B} \) is by definition,

\[
\langle \hat{A} \hat{B} \rangle = \int \hat{A} \hat{B} \hat{D}(\mathbf{X}, t) d\mathbf{X}
\]

\[
= \langle \hat{A} \rangle \langle \hat{B} \rangle + n \int f(X_1, t) \hat{a}(X_1|\vec{r}, t) \hat{b}(X_1|\vec{r}', t) dX_1.
\]

(C.39)

We may immediately notice that Eq C.39 and Eq C.36 are very similar.

In the next section, we show that if \( f \) changes much slower compared to the electron cyclotron period, the first term on the right-hand-side of the two equations doesn’t contribute to the ECE calculation. Thus instead of solving for the fields \( A \) and \( B \) in the original plasma, we can solve for the dressed fields \( \hat{A} \) and \( \hat{B} \) in the test particle system, and obtain the same correlation result. The derivation of the dressed fields equations is given in Sec. C.8. We’ll see that these equations are identical to the equations we used at the beginning of Sec 3.1.1 which discusses the reciprocity theorem, Eq 3.1 and 3.2.
C.6  Unperturbed solution for 1-particle distribution function

In the case where a strong external field exists, for example, a strong toroidal magnetic field in a magnetic fusion device, the plasma-generated self-consistent field can be considered as a small quantity compared to the external field. Eq C.14 can then be linearized around a “equilibrium” solution $\bar{f}(X_1, t)$ with only the external field,

$$\left( \frac{\partial}{\partial t} + \bar{v}_1 \cdot \nabla_{\bar{r}_1} + \frac{\vec{F}_{1,\text{ext}}}{m} \cdot \nabla_{\bar{v}_1} \right) \bar{f}(X_1, t) = 0, \quad (C.40)$$

which is exactly the Liouville’s equation with particle trajectories in only the external field,

$$\frac{d}{dt} \bar{f}(X_1, t) = 0. \quad (C.41)$$

With a given initial condition, $\bar{f}(X_1, 0) = \bar{f}_0(X_{10})$, we can directly obtain the solution by integrating along the particle trajectory,

$$\bar{f}(X_1, t) = \int \bar{f}_0(X_{10}) \delta(X_1 - X_1(X_{10}, t)) dX_{10}, \quad (C.42)$$

where $X_1(X_{10}, t)$ is the particle position at time $t$ starting from the initial position $X_{10}$.

In principle, the initial state $\bar{f}_0(X_{10})$ is arbitrary. In the ECE case, a key assumption about $\bar{f}_0$ is that $\bar{f}_0$ is uniform along particle trajectory in external field, i.e. $\bar{f}_0$ is uniform along magnetic field, and along the gyro-motion. This immediately guarantees that $\bar{f}$ does not explicitly depend on $t$. 
Substituting Eq C.42 into Eq C.36 and C.39, we have

\[
\langle A(\vec{r},t)B(\vec{r}',t) \rangle = \langle A \rangle \langle B \rangle + n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10},t)|\vec{r},t)\hat{b}(X_1(X_{10},t)|\vec{r}',t)dX_{10},
\]

\[
\langle \hat{A}(\vec{r},t)\hat{B}(\vec{r}',t) \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10},t)|\vec{r},t)\hat{b}(X_1(X_{10},t)|\vec{r}',t)dX_{10},
\]

(C.43)

whereby the integration over the particle location \(X_1\) at time \(t\) becomes an integration over the initial particle location \(X_{10}\), and the particle trajectory information is included in \(X_1(X_{10},t)\).

When \(\bar{f}\) is independent of \(t\), both \(\langle A \rangle \langle B \rangle\) and \(\langle \hat{A} \rangle \langle \hat{B} \rangle\) are constants, so they don’t contribute to the results in the ECE frequency range.

So, in the ECE relevant frequency range,

\[
\langle A(\vec{r},t)B(\vec{r}',t) \rangle = \langle \hat{A}(\vec{r},t)\hat{B}(\vec{r}',t) \rangle = n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10},t)|\vec{r},t)\hat{b}(X_1(X_{10},t)|\vec{r}',t)dX_{10},
\]

(C.44)

where \(\hat{A}\) and \(\hat{B}\) are the total dressed fields generated by statistically independent test electrons.

C.7 Generalization to a two-time correlation

A generalization to a two-time correlation \(\langle A(\vec{r},t)B(\vec{r}',t') \rangle\) can be obtained by essentially the same argument starting from the two-time distribution function \(D(Xt, X't')\)
The result can be written in almost the same form as (C.43)

\[ \langle A(\vec{r}, t) B(\vec{r}', t') \rangle = \langle A \rangle \langle B \rangle + n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10}, t) | \vec{r}, t) \hat{b}(X_1(X_{10}, t) | \vec{r}', t') dX_{10}. \quad \text{(C.45)} \]

\[ \langle \hat{A}(\vec{r}, t) \hat{B}(\vec{r}', t') \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10}, t) | \vec{r}, t) \hat{b}(X_1(X_{10}, t) | \vec{r}', t') dX_{10}. \quad \text{(C.46)} \]

Again, with our choice of \( \bar{f}_0 \), within ECE frequency range, the product of averaged fields is not relevant. We have

\[ \langle A(\vec{r}, t) B(\vec{r}', t') \rangle = \langle \hat{A}(\vec{r}, t) \hat{B}(\vec{r}', t') \rangle = n \int \bar{f}_0(X_{10}) \hat{a}(X_1(X_{10}, t) | \vec{r}, t) \hat{b}(X_1(X_{10}, t) | \vec{r}', t') dX_{10}. \quad \text{(C.47)} \]

### C.8 Dressed field equations

In the ECE calculation, the key quantity is the ensemble averaged Poynting flux in frequency domain, \( \langle \vec{S}(\omega) \rangle = (c/8\pi) \langle \vec{E}(\omega) \times \vec{B}^*(\omega) \rangle \), at the antenna. The frequency domain fields are related to the time domain ones through the Fourier Transform defined by Eq (3.4), so we have

\[ \langle \vec{S}(\vec{r}, \omega) \rangle \delta(\omega - \omega') = \frac{c}{8\pi^3} \int_{-\infty}^{\infty} \langle \vec{E}(\vec{r}, t) \times \vec{B}^*(\vec{r}, t') \rangle e^{i(\omega t - \omega't')} dt dt'. \quad \text{(C.48)} \]

As discussed in the Sec C.5, we can construct a system of statistically independent test particles, and calculate the ensemble average of the Poynting flux of the “dressed” electromagnetic field.

Now, we’ll show that the equations for the “dressed” electromagnetic field are actually Eq (3.1) and (3.2), where the source current term is due to the bare test particles, and the dielectric tensor comes from the dressing term.
Using the definition of the dressed field, Eq [C.35], we have the dressed electric and magnetic fields,

\[
\vec{E}(X_1|\vec{r},t) = \vec{E}(X_1|\vec{r},t) + n \int \vec{E}(X_2|\vec{r},t) P(X_1|X_2,t) dX_2,
\]
\[
\vec{B}(X_1|\vec{r},t) = \vec{B}(X_1|\vec{r},t) + n \int \vec{B}(X_2|\vec{r},t) P(X_1|X_2,t) dX_2,
\]  
(C.49)

where \(\vec{E}(X_1)\) and \(\vec{B}(X_1)\) denotes the electric and magnetic fields at location \(\vec{r}\) and time \(t\) produced by a bare test electron that is at \(X_1(t)\). \(\vec{E}\) and \(\vec{B}\) does not only depend on the particle’s location and velocity at time \(t\), but also depend on its whole history. Since the test particles equation of motion is determined only by the mean field, given the history of the 1-particle distribution function \(f(t)\), one point on the particle trajectory is sufficient to represent the whole history.

Formally, we can write the Maxwell Equations for the \(\vec{E}\) and \(\vec{B}\) as

\[
\nabla \times \vec{E}(X_1|\vec{r},t) = -\frac{1}{c} \frac{\partial \vec{B}(X_1|\vec{r},t)}{\partial t},
\]
\[
\nabla \times \vec{B}(X_1|\vec{r},t) = \frac{1}{c} \frac{\partial \vec{E}(X_1|\vec{r},t)}{\partial t} + \frac{4\pi}{c} \vec{j}(X_1|\vec{r},t),
\]  
(C.50)

with

\[
\vec{j}(X_1|\vec{r},t) = -e\vec{v}_1 \delta(\vec{r}_1(t) - \vec{r})
\]  
(C.51)

being the current produced by a bare test particle.

Define the “dressing” field \(\vec{E}_d\) and \(\vec{B}_d\) as

\[
\vec{E}_d(X_1|\vec{r},t) \equiv n \int E(X_2|\vec{r}) P(X_1|X_2,t) dX_2,
\]
\[
\vec{B}_d(X_1|\vec{r},t) \equiv n \int B(X_2|\vec{r}) P(X_1|X_2,t) dX_2,
\]  
(C.52)
we may obtain the equations for $\vec{E}_d$ and $\vec{B}_d$ by changing subscript 1 to 2 in Eq. C.50, multiplying with $nP(X_1|X_2, t)$ and integrating over $X_2$,

$$\nabla \times \vec{E}_d(X_1|\vec{r}, t) = -\frac{1}{c} \frac{\partial \vec{B}_d(X_1|\vec{r}, t)}{\partial t},$$

$$\nabla \times \vec{B}_d(X_1|\vec{r}, t) = \frac{1}{c} \frac{\partial \vec{E}_d(X_1|\vec{r}, t)}{\partial t} + \frac{4\pi}{c} n \int \vec{j}(X_2) P(X_1|X_2, t) dX_2.$$  \hspace{1cm} (C.53)

The current term is physically due to the perturbed distribution function introduced by the test particle at $X_1$.

Add Eq. C.50 and Eq. C.53 and sum over all particles, we have the equations for the total dressed fields, $\vec{E}_T(\vec{r}, t) \equiv \sum \vec{E}(X_i|\vec{r}, t)$ and $\vec{B}_T(\vec{r}, t) \equiv \sum \vec{B}(X_i|\vec{r}, t)$

$$\nabla \times \vec{E}_T(\vec{r}, t) = -\frac{1}{c} \frac{\partial \vec{B}_T(\vec{r}, t)}{\partial t},$$

$$\nabla \times \vec{B}_T(\vec{r}, t) = \frac{1}{c} \frac{\partial \vec{E}_T(\vec{r}, t)}{\partial t} + \frac{4\pi}{c} ( n \int \vec{j}(X_2) \sum_{i=1}^{N} P(X_i|X_2, t) dX_2 + \sum_{i=1}^{N} \vec{j}(X_i) ).$$  \hspace{1cm} (C.54)

Now, we define the perturbed distribution function by all the test particles,

$$\delta f(\vec{r}, \vec{v}, t) \equiv \sum_{i=1}^{N} P(X_i|\vec{r}, \vec{v}, t).$$  \hspace{1cm} (C.55)

Then using the definition of $\vec{j}(X_2)$, Eq C.51, the dressing current term in Eq C.54 can be readily written as

$$\vec{j}_d(\vec{r}) = -ne \int \vec{v} \delta f(\vec{r}, \vec{v}) d\vec{v}.$$  \hspace{1cm} (C.56)
From the equation for $P(X_0|X_t)$, Eq. [C.32] and sum over all the particles, we obtain the equation for $\delta f$,

$$\left[ \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \vec{r} + \frac{\vec{F}_M(\vec{r}, t)}{m} \cdot \nabla \vec{v} \right] \delta f = \frac{e}{m} \left( \vec{E}_T + \frac{1}{c} \vec{v} \times \vec{B}_T \right) \cdot \nabla_v f, \quad (C.57)$$

which has exactly the same form as the linearized Vlasov equation, only that now on the right-hand-side the fields are generated by both $\delta f$ and the bare test particles.

Exactly analogous to solving the linearized Vlasov system, we can integrate along the unperturbed trajectory [51], and eventually express the dressing current in terms of the total dressed field,

$$\vec{j}_d(\vec{r}, t) = \int \hat{\chi}(\vec{r}, \vec{r}', t - t') \cdot \vec{E}_T(\vec{r}', t') d\vec{r}' dt', \quad (C.58)$$

where $\hat{\chi}$ is the susceptibility tensor calculated by the unperturbed distribution function $f$.

Finally, after taking Fourier transform in time, Eq. [C.54] turns into the same form as Eq. 3.1 and 3.2,

$$\nabla \times \vec{E}_T(\omega, \vec{r}) = \frac{i \omega}{c} \vec{B}_T(\omega, \vec{r}) \quad (C.59)$$

$$\nabla \times \vec{B}_T(\omega, \vec{r}) = -\frac{i \omega}{c} \int \hat{\epsilon}(\omega, \vec{r}, \vec{r}') \vec{E}_T(\omega, \vec{r}') d\vec{r}' + \frac{4 \pi}{c} \hat{j}_s(\omega, \vec{r}) \quad (C.60)$$

with $\hat{\epsilon}(\omega, \vec{r}, \vec{r}') \equiv \hat{\chi} + \hat{\chi}(\omega, \vec{r}, \vec{r}')$ the Fourier transform of $\chi(\vec{r}, \vec{r}', t - t')$. The source current $\hat{j}_s(\omega, \vec{r})$ is the Fourier transform of the source current in time,

$$\vec{j}_s(\vec{r}, t) = \sum_{i=1}^{N} \vec{j}(X_i|\vec{r}, t) \quad (C.61)$$

$$= -e \sum_{i=1}^{N} \vec{v}_i(t) \delta(\vec{r} - \vec{r}_i(t)), \quad (C.61)$$
which is the total current produced by all the bare test particles.

### C.9 Electron trajectories in straight external magnetic field

To calculate the source current given by Eq C.61, we need to specify the test electron trajectories in phase space, \((\vec{r}_i(t), \vec{v}_i(t))\), as governed by Newton’s Law, Eq C.30. With the presence of a strong external magnetic field \(\vec{B}_0\), as in Sec C.2, we ignore terms due to the plasma self-generated electromagnetic fields in Eq C.30 and keep only the external field term. The equations of motion become

\[
\frac{d}{dt} \vec{r}_i(t) = \vec{v}_i(t), \tag{C.62}
\]

\[
\frac{d}{dt} \vec{v}_i(t) = -\frac{e}{mc} \vec{v}_i \times \vec{B}_0. \tag{C.63}
\]

For a straight uniform magnetic field \(\vec{B}_0\), the velocity equation can be written as

\[
\frac{d}{dt} \vec{v}_i(t) = \vec{v}_i \times \vec{\omega}_{ce}, \tag{C.64}
\]

where \(\vec{\omega}_{ce} \equiv -e\vec{B}_0/mc\) is the electron gyro-frequency, which is a constant.

The solution is simply

\[
x_i(t) = X_{i0} + \rho_{i0} \cos \phi_i(t), \]

\[
y_i(t) = Y_{i0} + \rho_{i0} \sin \phi_i(t), \tag{C.65}
\]

\[
z_i(t) = Z_{i0} + v_{i0} t,
\]
and
\[
\begin{align*}
v_{xi} &= -v_{\perp i0} \sin \phi_i(t), \\
v_{yi} &= v_{\perp i0} \cos \phi_i(t), \\
v_{zi} &= v_{||i0},
\end{align*}
\]

where \(X_{i0}, Y_{i0}\) are coordinates of the guiding-center, in straight field line geometry, they are constants of motion. The subscription 0 denotes the initial values. \(\rho_{i0} \equiv v_{\perp i0}/|\omega_{ce}|\) is the gyro-radius, \(\phi_i(t) = \phi_{i0} - \omega_{ce}t\) is the gyro-angle at time \(t\). An illustration of electron motion is shown in Fig.\ref{fig:gyro-motion} with the background \(B\) field in the \(z\) direction, i.e. pointing outward from the paper, and the electron is moving in the counter clockwise direction, the gyro-angle is increasing in time which agrees with our convention of signs in previous definitions.

Inclusion of relativistic effect in this simple magnetic field is also straightforward since the electron energy is conserved. The only change is due to the relativistic mass increase, \(m \to \gamma m\), where \(\gamma = 1/\sqrt{1 - v_i^2/c^2}\) is the relativistic factor.
\( \gamma \geq 1 \), this effect lowers the cyclotron frequency, so is usually called a “relativistic down-shift” effect.

## C.10 Final formula for source current of electron cyclotron emission

To end this chapter, let’s write the source current term \( \vec{j}(\omega, \vec{r}) \) in Eq.[C.60] in the form we will be using in Sec.3.1.2.

First, substitute the electron trajectory, Eq.[C.65] and [C.66] into the expression of the source current, Eq.[C.61] we have

\[
\vec{j}_s(\vec{r},t) = -e \sum_i \delta(x - x_i(t)) \delta(y - y_i(t)) \delta(z - z_i(t)) \\
\times \int \vec{v} \delta(v_x + v_{\perp i0} \sin \phi_i(t)) \delta(v_y - v_{\perp i0} \cos \phi_i(t)) \delta(v_z - v_{\parallel i0}) \, d\vec{v} 
\]

(C.67)

Write out in 3 components, we have:

\[
j_{sx}(\vec{r},t) = e \sum_i v_{\perp i0} \sin \phi_i(t) \delta(x - x_i(t)) \delta(y - y_i(t)) \delta(z - z_i(t)) \quad \text{(C.68)}
\]

\[
j_{sy}(\vec{r},t) = -e \sum_i v_{\perp i0} \cos \phi_i(t) \delta(x - x_i(t)) \delta(y - y_i(t)) \delta(z - z_i(t)) \quad \text{(C.69)}
\]

\[
j_{sz}(\vec{r},t) = -e \sum_i v_{\parallel i0} \delta(x - x_i(t)) \delta(y - y_i(t)) \delta(z - z_i(t)) \quad \text{(C.70)}
\]
and in frequency domain, we have the Fourier transformed ones

\[ j_{sx}(\vec{r}, \omega) = \sum_i j_{ex}^i(\vec{r}, \omega) \]
\[ = \sum_i \frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\perp i0} \sin \phi_i(t) \delta(x - x_i(t))\delta(y - y_i(t))\delta(z - z_i(t)) \, dt \]  
\[ (C.71) \]

\[ j_{sy}(\vec{r}, \omega) = \sum_i j_{ey}^i(\vec{r}, \omega) \]
\[ = \sum_i -\frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\perp i0} \cos \phi_i(t) \delta(x - x_i(t))\delta(y - y_i(t))\delta(z - z_i(t)) \, dt \]  
\[ (C.72) \]

\[ j_{sz}(\vec{r}, \omega) = \sum_i j_{ez}^i(\vec{r}, \omega) \]
\[ = \sum_i -\frac{e}{\pi} \int_{-\infty}^{\infty} e^{i\omega t} v_{\parallel i0} \delta(x - x_i(t))\delta(y - y_i(t))\delta(z - z_i(t)) \, dt \]  
\[ (C.73) \]
Appendix D

Integration over $\phi_0$, $t$ and $t'$ in $\hat{K}_k$ calculation

During the calculation of $\hat{K}_k$ in Sec 3.1.2 Eq 3.51 we have encountered an integration over $\phi_0$, $t$ and $t'$ of the following form:

$$I(\omega, \omega') = \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} (-i)^n J_n(k_\perp \rho_0) i^{n'} J_{n'}(k_\perp \rho_0) \times \int dt dt' \exp(i(\omega - k_\parallel v_\parallel 0)t - i(\omega' - k_\parallel v_\parallel 0)t') \int d\phi_0 f_\phi^1(\phi_0) \hat{V}_\phi e^{i\phi(t)} e^{-i\phi(t')},$$

(D.1)

where $\hat{V}_\phi$ is the part of $\hat{V}$ (defined in Eq 3.35) that depends on $\phi(t)$ and $\phi(t')$,

$$\hat{V}_\phi \equiv \begin{pmatrix} \sin \phi(t) \sin \phi(t') & -\sin \phi(t) \cos \phi(t') & -\sin \phi(t) \\ -\cos \phi(t) \sin \phi(t') & \cos \phi(t) \cos \phi(t') & \cos \phi(t) \\ -\sin \phi(t') & \cos \phi(t') & 1 \end{pmatrix}.$$

(D.2)

$f_\phi^1$ is assumed to be uniform, i.e. $f_\phi^1 = 1/2\pi$.

We will calculate the xx-component as an example, the rest of the tensor components can be calculated using the same technique.
D.1 Calculation of xx-component

The xx-component of \( \hat{V}_\phi \) is \( \sin \phi(t) \sin \phi(t') \).

Noting that \( \phi(t) = \phi_0 + |\omega_{ce}|t \), and \( \sin \phi(t) = 1/(2i)(\exp(i\phi(t)) - \exp(-i\phi(t))) \), we have

\[
\sin \phi(t) \sin \phi(t') e^{in\phi(t)} e^{-in'\phi(t')}
\]

\[
= -\frac{1}{4} [e^{i(n+1)\phi(t)} e^{-i(n'-1)\phi(t')} + e^{i(n-1)\phi(t)} e^{-i(n'+1)\phi(t')}
\]

\[
- e^{i(n+1)\phi(t)} e^{-i(n'+1)\phi(t')} - e^{i(n-1)\phi(t)} e^{-i(n'-1)\phi(t')}
\]

\[
= -\frac{1}{4} [\exp(i(n - n' + 2)\phi_0 + (n + 1)|\omega_{ce}|t - (n' - 1)|\omega_{ce}|t')
\]

\[
+ \exp(i(n - n' - 2)\phi_0 + (n - 1)|\omega_{ce}|t - (n' + 1)|\omega_{ce}|t')
\]

\[
- \exp(i(n - n')\phi_0 + (n + 1)|\omega_{ce}|t - (n' + 1)|\omega_{ce}|t')
\]

\[
- \exp(i(n - n')\phi_0 + (n - 1)|\omega_{ce}|t - (n' - 1)|\omega_{ce}|t')
\]

Integrating over \( \phi_0 \), we obtain terms like:

\[
\int_0^{2\pi} \exp(i(n - n' + 2)\phi_0) d\phi_0 = 2\pi \delta(n - n' + 2).
\]

whereby the \( \delta \)-function is defined on integers,

\[
\delta(n) \equiv \begin{cases} 
0 & (n \neq 0) \\
1 & (n = 0) 
\end{cases}.
\]

Then, after integrating over \( t \) and \( t' \), we obtain terms like:

\[
\int_{-\infty}^{\infty} \exp(i(\omega - k||v||_0 + (n + 1)|\omega_{ce}|)t) = 2\pi \delta(\omega + (n + 1)|\omega_{ce}| - k||v||_0).
\]

This \( \delta \)-function is the usual one defined on real numbers.
Collecting all the terms, we have

\[
\int dt \int dt' \exp(i(\omega - k\|v\|_i(t) - t - i(\omega' - k\|v\|_i(t) t')
\times \int d\phi_0 f_\phi^\phi(\phi_0) \sin \phi_1(t) \sin \phi_1(t') e^{i\phi(t) - i\phi(t')}
\]

\[
= -\pi^2[\delta(n - n' + 2) \cdot \delta(\omega + (n + 1)\|\omega_{ce}\| - k\|v\|_i) \cdot \delta(\omega + (n' - 1)\|\omega_{ce}\| - k\|v\|_i) + \delta(n - n' - 2) \cdot \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i) \cdot \delta(\omega + (n' + 1)\|\omega_{ce}\| - k\|v\|_i) - \delta(n - n') \cdot \delta(\omega + (n + 1)\|\omega_{ce}\| - k\|v\|_i) \cdot \delta(\omega + (n' + 1)\|\omega_{ce}\| - k\|v\|_i) - \delta(n - n') \cdot \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i) \cdot \delta(\omega + (n' - 1)\|\omega_{ce}\| - k\|v\|_i)]]
\]

The double summation over \(n\) and \(n'\) can then be carried out easily because of the terms like \(\delta(n + n')\),

\[
I_{xx} = \pi^2 \delta(\omega - \omega') \sum_n \sum_{n'} (-i)^n i^{n'} J_n(k_{\perp \rho_{i0}}) J_{n'}(k_{\perp \rho_{i0}})
\times [- \delta(\omega + (n + 1)\|\omega_{ce}\| - k\|v\|_i) \delta(n - n' + 2) - \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i) \delta(n - n' - 2) + (\delta(\omega + (n + 1)\|\omega_{ce}\| - k\|v\|_i) + \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i) \delta(n - n')]]
\]

\[
= \pi^2 \delta(\omega - \omega') \sum_n [(J_n^2 + J_n J_{n+2}) \delta(\omega + (n + 1)\|\omega_{ce}\| - k\|v\|_i) + \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i) \delta(n - n') + J_n^2 + J_n J_{n-2}) \delta(\omega + (n - 1)\|\omega_{ce}\| - k\|v\|_i)]]
\]

\[
= \pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} (J_n^2 + 2J_n J_{n+1} + J_{n-1}^2) \delta(\omega + n\|\omega_{ce}\| - k\|v\|_i)
\]

\[
= \pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} \frac{4n^2}{(k_{\perp \rho_{i0}})^2} J_n^2(k_{\perp \rho_{i0}}) \delta(\omega + n\|\omega_{ce}\| - k\|v\|_i)
\]

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The last line we used identity $J_n(z) = (z/2n)(J_{n+1}(z) + J_{n-1}(z))$.

## D.2 Other components and final result

Similar to xx-component, the other components can be calculated using exactly the same technique.

After some algebra, we get:

\[
I_{xy} = -I_{yx} = i\pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} \frac{4n}{k_\perp \rho_0} J_n J'_n \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

\[
I_{yy} = \pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} 4J'_n \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

\[
I_{xz} = I_{zx} = -\pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} \frac{4n}{k_\perp \rho_0} J_n^2 \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

\[
I_{yz} = -I_{zy} = i\pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} 4J_n J'_n \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

\[
I_{zz} = \pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} 4J'_n \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

where $J'_n(z) \equiv d/dz J_n(z)$ is the derivative of $J_n$ respect to its argument.

Finally, we can write the result in tensor form,

\[
I = 4\pi^2 \delta(\omega - \omega') \sum_{n=-\infty}^{+\infty} \delta(\omega + n|\omega_{ce}| - k_\parallel v_{||i0})
\]

\[
\begin{pmatrix}
(n^2/k_\perp^2 \rho_0^2) J_n^2 & i(n/k_\perp \rho_0) J_n J'_n & -(n/k_\perp \rho_0) J_n^2 \\
-i(n/k_\perp \rho_0) J_n J'_n & J'_n^2 & iJ_n J'_n \\
-(n/k_\perp \rho_0) J_n^2 & -iJ_n J'_n & J_n^2
\end{pmatrix}
\]

(D.3)

This is the formula we have used in Sec 3.1.2Bd.
## Appendix E

### Physical constants

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# Appendix F

## Tables of Symbols

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