MACROECONOMICS OF PARTIALLY IRREVERSIBLE INVESTMENT WITH FINANCING CONSTRAINTS

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Abstract

This thesis investigates firm-level partially irreversible investment (i.e., assets are not fully resaleable) with financing constraints. The goal is to study asset liquidity, the impacts on financing these assets, firm-level investment and liquidation, and the effects on optimal fiscal and monetary policy designs.

Chapter 1 studies why aggregate firm-level capital stock liquidation (i.e., selling of used assets plus mergers and acquisitions) slows down in recessions, in contrast to the traditional creative-destruction theory. In an environment of liquidation costs, I investigate when unproductive firms liquidate capital stock and how financing constraints alter the liquidation decisions. Importantly, financing constraints force productive firms to hire fewer workers and demand less borrowing. Therefore, the equilibrium wage rate and interest rate will tend to be low. A lower wage rate and a lower interest rate attract more unproductive firms to hold capital stock, such that liquidating capital stock will be prolonged. A recession generated by tightened financing constraints thus features less capital liquidation than in boom times.

Chapter 2 shows how asset liquidity can be endogenous. We construct a tractable model with endogenous asset liquidity through search frictions between buyers and sellers, whose participations are equilibrium outcomes. The model shows that variations of transaction ease of financial assets can generate procyclical asset liquidity and the flight to liquid government bonds in recessions, while standard productivity shocks cannot. The model sheds light on why financial system can have large feed-back impacts on investment and production. This work is co-authored with Sören Radde.

Chapter 3 continues with liquidity frictions where there are partially resaleable privately issued financial assets and fully resaleable government bonds. It designs the optimal policy and shows that government holding of some privately issued assets (i.e., unconventional monetary policy) can fully avoid the liquidity frictions. Without the policy, private agents over-save in government bonds which make the bond rate unnecessarily low. An interest
rate that is too low hurts investment by firms who use bonds as funding sources. The unconventional policy can raise the interest rate by using returns from privately issued financial assets. Aggregate investment and consumption can thus increase to the first-best level.
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Chapter 1

Delayed Capital Reallocation

1.1 Introduction

How do firms adjust their balance sheets of productive assets and liabilities in response to recurrent shocks to productivity or profitability? How does the existing capital stock reallocate across firms and interact with new investment? Two empirical observations provide some guidance of the balance sheet adjustment of U.S. corporate firms.

The first is aggregate capital stock reallocation. Figure 1.1 plots cyclical components of aggregate capital reallocation (solid line) and GDP (dashed line), which suggests that reallocation of existing productive capital is highly procyclical. Following Eisfeldt and Rampini (2006), capital reallocation includes sales of property, plants, and equipment and acquisitions from the COMPUSTAT database.\(^1\) This observation is in contrast with creative destruction theory in which more capital stock should be liquidated in recessions. Moreover, existing literature shows that firm-level total factor productivity (TFP) become more

\(^1\)Jovanovic and Rousseau (2002) also use this measure for studying the purchase of used assets. To give a sense of reallocation market size, in 2011, the reallocation from COMPUSTAT is about $0.65 trillion whereas the total U.S. fixed investment is about $1.6 trillion. Non-listed firms probably buy more used assets according to Eisfeldt and Rampini (2007). In sum, capital reallocation is comparable to new investment.
dispersed in recessions.\textsuperscript{2} Therefore, in recessions, there are highest potential benefits\textsuperscript{3} to reallocate which should imply the most capital reallocation.

The second looks at firms that liquidate assets. Figure 1.2 plots debt-to-asset ratios of firms over time during which they do not sell assets until time 0.\textsuperscript{4} We learn that most of the firms are reluctant to sell assets quickly. In addition, they shrink their debt burdens before selling: their liabilities are reduced relative to their assets.

Figure 1.1 is puzzling as Eisfeldt and Rampini (2006) point out: why is there less reallocation in recessions (especially when there are larger potential benefits to reallocate)? This paper asks what reason(s) can delay reallocation and generate larger TFP dispersion in recessions endogenously. Figure 1.2 suggests that the outside financing condition should be important in firms’ liquidation. The changes of the condition may affect the timing of reallocation and may explain why there is less reallocation but larger TFP dispersion in recessions.

To examine outside financing’s impact on capital reallocation, I construct a tractable dynamic general equilibrium model in which firms face idiosyncratic and aggregate shocks while being restricted by two frictions: asset illiquidity and financing constraints. The two frictions interact and generate capital reallocation delays from unproductive firms. In response to credit crunches, the delays are prolonged and the TFP dispersion thus expands.

\textsuperscript{2}For example, Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) shows that the dispersion of plant level total factor productivity increases in recessions, replicated in Figure 1.10 in the Appendix. Since the plant-level productivity dispersion measures potential gains from reallocation, the figure suggests that the gains are countercyclical. Other measures of TFP dispersion are also larger in recessions as in Table 1.8 in the Appendix.

\textsuperscript{3}Mergers and acquisitions (M&A) sometimes occur for market power motives; but in firm-level data, M&A generally increase efficiency as shown in Maksimovic and Phillips (2001). (A transaction that does not increase efficiency is “a minority of transactions”.)

\textsuperscript{4}Except for firms with very high leverage ratios, see Figure 1.11 in the Appendix. Covered firms are with asset sells in the years 2000-2012 in the SDC Platinum database and have corresponding information in the COMPUSTAT database. Those who sell multiple times are excluded. See the data description in the Appendix.
Figure 1.1: **Capital reallocation over cycles**
The series plotted are cyclical components of HP-filtered log data normalized by standard deviations. Solid line represents seasonally adjusted reallocation, i.e., the sum of sales of property, plant, and equipment (SPPE) and acquisition (AQC) in 2005 dollars. Dashed line represents real GDP in 2005 dollars. Shaded regions denote NBER recessions. For the separate cyclical patterns of SPPE and AQC, see Figure 1.12 in the Appendix. See also Table 1.6 and Table 1.7 in the Appendix for summary statistics and more statistics of cyclical patterns.

To be more specific, the key features are: (1) collateralized borrowing constraints, (2) capital resale discount\(^5,6\) (assets will be sold at discount in liquidation), and (3) fixed costs in running firms. In this economy, idiosyncratic productivity shocks create the benefits to reallocate capital stock. Productive firms expand by borrowing, but collateral constraints

\(^5\)Shleifer and Vishny (1992) summarize two usual reasons for resale costs. First, when firms are liquidating, the potential buyers with the highest valuation are often those in the same industry who generally also have financial troubles. Assets may not go to the highest valuation users. Second, because of antitrust reasons, assets may need to be sold to industry outsiders, causing lower values for assets.

\(^6\)Ramey and Shapiro (2001) provide empirical evidence of investment specificity and selling costs. They estimate the wedge between purchase price and resale price for different types of capital. Machine tools are sold at about a 69% discount off the purchase value, and structural equipment is sold at a 95% discount. These estimates suggest a large degree of specificity. Other evidence includes Holland (1990), in which a 50% to 70% discount is associated with the liquidation of the assets of a machine-tool manufacturer.
Figure 1.2: Debt-to-asset ratio before liquidation
Debt-to-asset ratios before selling assets of all firms who sold at least 50% of the assets in 2000-2012 (2071 such firms in total). Time 0 denotes the time when firms sell assets and time $t$ denotes $t$ quarters before selling assets. By construction, there is no assets selling at time $t$. For a more cross-sectional detail, see Figure 1.11 in the Appendix.

restrict the expansion so that not every capital stock can be reallocated. For example, not every production line of electric cars can be transferred to productive car companies.

In contrast, firms whose productivity falls are hesitant to sell assets because of the resale discount, gambling on the hopes that they might regain productivity soon. Meanwhile, these firms have accumulated a large amount of debt. The interest rate on the debt is higher than the rate of return on capital stock. They let the capital depreciate while pay down existing debt by shrinking dividends (modeled as consumption). If they persist in this unproductive way, profitability stays low and they gradually shrink. But they will eventually give up their capital when the option value of maintaining the depreciated capital is not enough to compensate for the fixed costs of operation. Thus, the model is able to generate balance sheet dynamics as in Figure 1.2 (see Figure 1.5 later).

The main result is that aggregate adverse shocks to borrowing constraints prolong the selling delay through the general equilibrium. Consider a credit crunch that further limits
efficient firms from expanding. These firms’ purchase of existing capital stock decreases. More importantly, economy-wide hiring drops and wage rates decrease such that the labor costs to run firms decrease. In response to lower input costs, the more inefficient firms postpone liquidation and less capital is sold. At the same time, these inefficient firms slowly pay down debt to reduce interest payments and to increase future borrowing capacity. In summary, the interaction is a result of the general equilibrium effect: when a financing problem restrict productive firms to expand, reduce demand for labor, and therefore lowers input costs, keeping assets and slowly deleveraging are more attractive to inefficient firms.

Because capital reallocation slows down during recessions, the idiosyncratic TFP dispersion across firms expands and the aggregate TFP declines with the tightened financing constraints, leading to a deepening recession. Thus, aggregate shocks to financing constraints interact with asset illiquidity, which helps explain why capital reallocation slows down in spite of larger potential benefits to reallocate during recessions. A major credit crunch after a banking crisis, such as the one in the U.S. in 2008, exemplifies these interactions.

Aggregate TFP shocks, however, generate different dynamics. When adverse aggregate TFP shocks hit, the profit rate is lower because of a lower productivity. Keeping capital is less profitable and inefficient firms have higher incentives to liquidate. Therefore, more reallocation and smaller TFP dispersion should be seen during recessions. Meanwhile, deleveraging is much smaller and more short-lived compared to responses after a credit crunch in which inefficient firms slowly pay down debt.

Finally, I estimate the two aggregate shocks (aggregate shocks to financing constraints and aggregate TFP shocks) using Bayesian estimation methods and simulate the economy

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8 Note that this is the standard creative destruction theory, but the opposite phenomena occur in data.
with only aggregate TFP shocks and only financial shocks. I confirm that aggregate TFP shocks alone cannot generate both observed procyclical capital reallocation and counter-cyclical TFP dispersion. Financial shocks are necessary to capture both dynamics. The joint dynamics thus offer some natural identification of the source(s) of business cycles.

The contribution of this paper is to consider the interaction of the two frictions. Without asset illiquidity, there will not be selling delay. Without financing constraints, productive firms can borrow as much as they want, pushing up the wage rate and interest rate. Thus, unproductive firms have small incentives of keeping assets, leaving a very short delays of selling assets.

The technical innovation of this paper is to propose a tractable method for firm dynamics with asset illiquidity and for the distribution of firms. Solutions to such model are usually complex and sometimes infeasible with aggregate shocks (not to mention estimations of the shocks). To maintain tractability, I simplify the problem by solving portfolio choices between bonds and capital stock with (real) “option values”, using finance portfolio choice theory, e.g., in Campbell and Viceira (2002). Therefore, the option value of capital depends on the portfolio weight (or leverage ratio) which is a new endogenous state variable.

Using the closed-form portfolio choice, individuals’ decision rules are easily aggregated. Note that finite moments are not enough to characterize the firm distribution. But the tractability of the distribution still leads to exact aggregation and avoids the approximation method as in Krusell and Smith (1998). Therefore, system dynamics can be analyzed by solving simple simultaneous non-linear difference equations.

**Literature Review.** Real option is the salient feature of this paper. Dixit and Pindyck (1994) and Caballero and Engel (1999) focus on the timing of irreversible investment. This

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9See, for example, Bloom (2009), and Khan and Thomas (2011), who use piece-wise functions to approximate individual value functions.

10I follow and extend previous works by Angeletos (2007), Kiyotaki and Moore (2012), and Buera and Moll (2012). Under the class of CRRA preferences, if individual production functions feature constant returns to scale, the wealth spent on capital and bonds is simplified to a portfolio choice between the two.
paper focuses on asset selling. Since assets may turn to be productive, running unproductive firms has an option value which may exceed the resale value. I show how to directly quantify the option value which is history dependent and summarized in firms’ leverage ratios.

The real option is linked to the delayed capital reallocation which generates larger dispersion during recessions. Implication of shocks to the dispersion of firm-specific conditions can be found, for example, in Bloom (2009), Arellano, Bai, and Kehoe (2012), and Gilchrist, Sim, and Zakrajsek (2010). But Bachmann and Bayer (2012a,b) show that large dispersion shocks are difficult to reconcile with other observations such as the investment rate dispersion. This paper shows how standard credit crunches can increase the dispersion endogenously through general equilibrium.

Further literature of macro implications of asset illiquidity and implications of financing constraints can be found in surveys by Caballero (1999) for capital illiquidity, and Bernanke, Gertler, and Gilchrist (1999) and more recently Brunnermeier, Eisenbach, and Yuliy (2012) for financing constraints. Whether asset illiquidity or financing constraints can quantitatively amplify TFP and output losses is a matter of some debate.

Innovation of this paper is to consider the interactions between asset illiquidity and financing constraints. The calibration shows that the aggregate TFP gap between the model

11 Partial irreversibility is important for the interaction in the model. Previous work on investment irreversibility focuses on zero resale value, or completely irreversible investment, such as in Abel and Eberly (1996, 1999) and Thomas (2002). With zero resale value, firms only consider when to buy instead of when to sell.

12 Thomas (2002) and Veracierto (2002) argue that irreversibility is not important in general equilibrium since idiosyncratic adjustments will be smoothed out. However, Kashyap and Gourio (2007) show that whether lumpy investment is important depends on production function of firms and the distribution of fixed costs. Recently, Kiyotaki and Moore (2012) study the illiquidity shocks and the amplification. Eisfeldt (2004) and Kurlat (2012) model the illiquidity through asymmetric information.

13 See financial constraints’ impact on long-run output and TFP losses in Buera, Kaboski, and Shin (2011), Moll (2010), and Midrigan and Xu (2012). For example, financing frictions in Midrigan and Xu (2012) cannot generate the misallocation observed in Hsieh and Klenow (2009). Moll (2010) shows that as firms have persistent idiosyncratic productivity shocks, they save enough to undo the frictions. See also financing constraints’ effect on short-run output and TFP fluctuations in Kocherlakota (2000), Cordoba and Ripoll (2004) and more recently Chen and Song (2012).
economy and an economy without illiquidity of capital or without financing constraints is significant in the steady state and expands during recessions caused by credit crunches. In this sense, the closest papers are perhaps Kurlat (2012) and Khan and Thomas (2011). Kurlat (2012) shows analytically why the secondary market for existing capital may shut down and its macroeconomic implications through adverse selection. He focuses on the resale prices by simplifying outside financing: entrepreneurs are not allowed to borrow. Instead, I focus on different degrees of borrowing constraints and the impact on the portfolio choices among capital and bonds. Khan and Thomas (2011) quantitatively examine reallocation efficiency for given degrees of resale costs and financing frictions, focusing mainly on numerical aspects. I extensively use analytical methods (by focusing on more specific process of idiosyncratic shocks) to better explain the interaction of the two frictions on the capital reallocation delays through general equilibrium, before calibration and estimation. More importantly, in contrast to both Kurlat (2012) and Khan and Thomas (2011), I look at the deleveraging behaviors of firms before liquidations. The deleveraging occurs because of risk-averse agents who try to smooth consumption. Thus, there are firms that borrow but are not constrained.

14The interactions in the model occur through general equilibrium. Credit crunches reduce wage rates because of a frictionless labor market. Empirically, despite wage rigidities, real wage rates decline during recessions, as found by Solon, Barsky, and Parker (1994) and Haefke, Sonntag, and Van Rens (2012). The decline of real wages is a consequence of lower wages of newly hired workers, in spite of moderate wage rigidity for longer term employees. Caggese and Cunat (2008) show firms can substitute flexible employment contracts for permanent employment contracts to reduce efficiency wages. Berger (2012) takes this a step further: firms hire more unproductive workers in expansions, but quickly fire them during recessions to reduce costs.

15Recently, Guerrieri and Lorenzoni (2011) look at deleveraging after credit crunches in households who face durable consumption goods illiquidity and financing constraints.
1.2 The Model

1.2.1 Preferences, Technology, and Information

Time is discrete and the horizon is infinite. There are two types of agents: households (with measure $L$) and entrepreneurs (with measure 1). Households are hand-to-mouth and supply labor inelastically. Entrepreneurs own production technology and some of them run firms.

Preferences. At time $t$, a typical entrepreneur $j$ has preferences over the consumption stream $c_{jt}, c_{jt+1}, c_{jt+2}, \ldots$ and leisure stream $(1-h_{jt}), (1-h_{jt+1}), (1-h_{jt+2}), \ldots$, given by

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} [u(c_{js}) + \eta(1-h_{js})]$$

(1.1)

where $\beta \in (0, 1)$ is the discount factor, $E_t$ is the conditional expectation operator, and $u(c) = c^{1-\sigma}/(1-\sigma)$. $\sigma$ is the relative risk-aversion parameter. To simplify, I use $\sigma = 1$, i.e., $u(c) = \log(c)$, leaving the general case in the Appendix. If $j$ runs the firm, $h_{jt} = 1$; if $j$ does not run the firm, $h_{jt} = 0$, and there is $\eta$ extra leisure utility. $\eta$ represent the fixed costs in running the business and will be important in the exit decisions later.\(^{16}\)

Production. In the beginning of time $t$, $j$’s firm uses capital $k_{jt}$ (installed in $t-1$) and hire labor $l_{jt}$ at a competitive wage rate $w_t$, to produce output:

$$y_{jt} = A_t z_{jt} k_{jt}^{\alpha} l_{jt}^{1-\alpha} = A_t (z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha}$$

where $\alpha \in (0, 1)$, $z_{jt}$ is the idiosyncratic productivity, and $A_t$ is aggregate productivity. Aggregate productivities $A_t$ are realized at the beginning of $t$, while idiosyncratic productivities $z_{jt}$ are known at time $t-1$. Similarly, entrepreneur $j$ learns $z_{jt+1}$ at time $t$. Let $a_t = (z_{jt}, z_{jt+1})$ denote the productivity pair at time $t$. Some entrepreneurs are productive at time $t$ ($z_{jt} = z^h$) while others are unproductive ($z_{jt} = z^l$), with $z^h > z^l > 0$. For conve-

\(^{16}\)Alternatively, an entrepreneur’s engagement in running the firm produce output that are the fixed costs required for production. Modeling fixed costs as $\eta$ utility will give rise to closed form solution later.
niecture, \( z^h = (z^h)^\alpha \) and \( \tilde{z}^h = (z^h)^\alpha \) denote the “measured” idiosyncratic productivity levels. The idiosyncratic productivity follows a two state Markov process\(^{17}\) where the transition probabilities are

\[
\begin{align*}
\text{Prob}(z_{jt+2} = z^l | z_{jt+1} = z^h) &= p^{hl} \\
\text{Prob}(z_{jt+2} = z^h | z_{jt+1} = z^l) &= p^{lh}
\end{align*}
\]

**Capital Accumulation.** Capital depreciates at a rate \( \delta \). Firms can invest in new capital stock, buy existing assets from the secondary market, or sell existing assets to the market. Inactive investment decisions are also allowed, i.e., \( j \) can choose to neither buy nor sell capital. One unit of efficient used assets, after being installed, is the same as one unit of new assets. Thus, the entrepreneur \( j \)'s capital stock evolves according to

\[ k_{jt+1} = (1 - \delta)k_{jt} + i_{jt} \]

where \( i_{jt} > 0, i_{jt} < 0 \) and \( i_{jt} = 0 \) denote buying, selling, and inaction in investment, respectively.

As in neo-classical growth model, a buyer pays one unit of consumption goods for investment goods. Thus, amplification from asset price channel is switched off. For each unit of used assets sold, only \( (1 - d) \) fraction is useful for other buyers which implies that sellers receive a payment of \( (1 - d) \) for each unit of asset sold from them.

In sum, it costs 1 to invest (new or old capital) and \( (1 - d) \) to retire a unit of old capital. If the firm changes its quantity of capital from \( k \) to \( k' \), the cost of doing so is

\[ \psi(k', k) = \begin{cases} 
(k' - (1 - \delta)k, & \text{if } k' > (1 - \delta)k \\
0, & \text{if } k' = (1 - \delta)k \\
-(1 - d)[(1 - \delta)k - k'], & \text{if } k' < (1 - \delta)k 
\end{cases} \]

\(^{17}\)Note that, \( 0 < p^{hl} < 1, \ 0 < p^{lh} < 1, \) and \( p^{hl} + p^{lh} < 1. \)
**Budget and Collateral Constraints.** Entrepreneur $j$ has access to the credit market. Denote the bond position as $b_{jt}$ at the beginning of $t$ and the interest rate from $t - 1$ to $t$ as $R_t$. The budget constraint of $j$ can be written as

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = y_{jt} - w_t l_{jt} + R_t b_{jt}.$$  

$j$ earns profits and interests, which are spent on consumption, new bonds, and paying the capital adjustment costs. Note that one can simplify profits further. Because firm $j$ has a constant return to scale (CRS) production technology, the instantaneous profits of $j$ are linear in $k_{jt}$.\(^{18,19}\)

$$
\Pi(z_{jt}, k_{jt}; w_t) = \max_{l_{jt}} \{ (A_t z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha} - w_t l_{jt} \} = (z_{jt} \pi_t) k_{jt}
$$

where $\pi_t = \alpha A_t \left( \frac{1-\alpha}{w_t} \right)^{(1-\alpha)/\alpha}$. Thus, the budget constraint can be simplified to

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = z_{jt} \pi_t k_{jt} + R_t b_{jt}. \quad (1.2)$$

Entrepreneur $j$ can short bonds (borrow), but not capital stock. Borrowing is bounded because $j$ faces collateral constraints similar to those in Kiyotaki and Moore (1997) and

\(^{18}\)To see this, the first-order condition for labor is $(A_t z_{jt} k_{jt})^{\alpha} (1-\alpha) l_{jt}^{1-\alpha} = w_t$, so that the optimal labor demand is $l_{jt}^* = A_t^{1/\alpha} z_{jt} k_{jt} \left[ \frac{1-\alpha}{w_t} \right]^{1/\alpha}$, from which profits are

$$
\Pi(z_{jt}, k_{jt}; w_t) = A_t (z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha} - w_t l_{jt} = A_t^{1/\alpha} z_{jt} k_{jt} \left[ \left( \frac{1-\alpha}{w_t} \right)^{(1-\alpha)/\alpha} - w_t \left( \frac{1-\alpha}{w_t} \right)^{1/\alpha} \right] = A_t^{1/\alpha} z_{jt} k_{jt} \left( \frac{1-\alpha}{w_t} \right)^{1/\alpha} \left[ \frac{w_t}{1-\alpha} - w_t \right] = z_{jt} \pi_t k_{jt}.
$$

\(^{19}\)After substitution, labor demand is $l_{jt}^* = (\frac{\pi_t}{\alpha k_{jt}})^{1/(1-\alpha)} z_{jt} k_{jt}$. Thus, total output produced by entrepreneur $j$ can be written as $y_{jt} = \frac{z_{jt} \pi_t k_{jt}}{\alpha}$. To interpret this result, $\alpha$ fraction of the output becomes $j$’s profits while the $1-\alpha$ fraction is paid through wages.
Hart and Moore (1994). The collateral constraint here includes the resale friction and an extra degree of financing friction $\theta_t$:

$$R_{t+1}b_{jt+1} \geq -\theta_t(1-d)(1-\delta)k_{jt+1}$$

(1.3)

where $1 - \theta_t$ is the “haircut”. Collateral constraint (1.3) says that debt value cannot exceed $\theta_t$ fraction of the resale value of the residual capital at $t + 1$. Also, for one unit of capital stock, the investing entrepreneur only needs to pay $1 - \theta(1-d)(1-\delta)/R_{t+1}$ as down payment. $\theta_t$ fluctuates and measures the financial market development, reflecting the external financing difficulties. For example, a permanently higher $\theta_t$ represents a better financial development, whereas a temporary decline in $\theta_t$ represents a sudden banking problem.

$\theta_t$ of (1.3) constrains capital stock allocation efficiency. Without (1.3), $z^h$ owners can obtain any funds needed to invest in capital stock. The economy would reach the efficient production frontier, and as many entrepreneurs as possible can enjoy leisure.

A Summary. Each entrepreneur $j$ maximizes (1.1) subject to (1.2) and (1.3), by choosing consumption $c_{jt}$, leisure $h_{jt}$, labor input $l_{jt}$, capital $k_{jt+1}$, and bonds $b_{jt+1}$, while taking the wage rate $w_t$ and the interest rate $R_{t+1}$ as given.

1.2.2 Recursive Equilibrium

I rewrite the entrepreneur’s problem recursively and then define recursive equilibrium. Denote aggregate state as $X = (\Gamma(k,b,a), \theta,A)$ where $\Gamma(k,b,a)$ is the distribution of individuals’ capital stock, bonds, and productivity pair at the beginning of each period. To emphasize, $\theta$ and $A$ are the primitive shocks, i.e., financial disturbances and aggregate productivity fluctuations are exogenous shocks. Let $V$ be the optimal value of an entrepreneur with $k$, $b$, and $a$, given the aggregate state variable $X$. The value function $V(k,b,a;X)$

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20This is a consequence of the fact that the human capital of the agent who is raising outside funds is inalienable. To ensure no “run away” default, the lender should be able to seize the tangible assets.
satisfies the Bellman equation:

\[ V(k, b, a; X) = \max \{ W^1(k, b, a; X), W^0(k, b, a; X) \} \]  

\[ W^1(k, b, a; X) = \max_{k' > 0, b' \geq -\theta(1-d)(1-\delta)k'} \{ u(z\pi k + Rb - \psi(k', k) - b') + \beta E[V(k', b', a'; X') | a, X] \} \]

\[ W^0(k, b, a; X) = \max_{b'} \{ u(z\pi k + Rb + (1-\delta)(1-d)k - b') + \eta + \beta E[V(0, b', a'; X') | a, X] \} \]

The first step maximization is over the two actions: (1) to run the firm and get \( W^1 \) and (2) not to run the firm and get \( W^0 \). The second step is to choose the optimal consumption and savings (in capital stock and in bonds). Note that \( W^0 \) has the leisure utility \( \eta \) today, as an entrepreneur who gets \( W^0 \) does not run the firm today and there is no output tomorrow. The existence and uniqueness of the value function are standard by contraction mapping, as in Chapter 9 of Stokey, Lucas, and Prescott (1989).

Finally, I define the recursive equilibrium to close the model:

**Definition 1 (The First Recursive Equilibrium Definition)** The equilibrium is a law of motion \( H \), policy functions \( l = g^l(k, b, a; X) \), \( k' = g^k(k, b, a; X) \), \( b' = g^b(k, b, a; X) \), and pricing functions \( \pi(X) \) and \( R'(X) \) such that:

(i) \( l, k' \) and \( b' \) solve the entrepreneur’s problem in (1.4) given the wage and the interest rate.

(ii) Markets for labor and bonds clear

\[ \int l_{jt} \, d\, j = L, \int b_{jt+1} \, d\, j = 0. \]

(iii) The distribution evolution \( H \) is consistent with policy functions.
1.3 Decision Rules

The challenge of equilibrium characterization is to track the distribution of firms. Fortunately, the economy turns out to be highly tractable. I begin by describing the general solution under the economy with an active secondary market, leaving the mathematical details for later. Doing so will give readers an idea of where the argument flow is and allow them to skip the details.

In the details, I first show some general properties of entrepreneurs’ recursive problems, regardless of the parameters. Then I shift the focus to certain parameters under which the equilibrium has both an active credit market and an active secondary asset market, since my focus is on the imperfect secondary market and delayed capital reallocation. In the next section, I show how the distribution can be easily handled.

1.3.1 A Quick Preview

It turns out that an individual entrepreneur’s policy depends only on the leverage ratio, i.e., capital stock over equity $k/(k+b)$. Under certain parameters, $z^h$ owners buy capital while $z^l$ owners hold on to it before liquidation, in the steady state and the neighborhood around the steady state. I focus on equilibrium of such because it has imperfect capital reallocation and possible binding financing constraints for productive firms. In numerical analysis, I confirm such equilibrium.

In steady state, the optimal policy functions can be shown in two ways. One is to examine tomorrow’s leverage given the leverage today (Figure 1.3a). When drawing $z^h$, entrepreneurs always lever up to some leverage ratio $\bar{\lambda}$, denoted as $z' = z^h$ line. When drawing $z^l$, entrepreneurs let the capital depreciate and pay back existing debt by consuming less. To see this, leverage tomorrow can be found through $z' = z^l$ and 45-degree lines. Leverage today can be mapped into leverage tomorrow by the following procedure. First, cut horizontally the $z^l$ line in which intersection point $G_1$ has $k/(k+b)$ as today’s lever-
Figure 1.3: Policy function illustration
(a) Policy function mapping leverage today to leverage tomorrow. The \( z' = z^h \) line denotes the target leverage when entrepreneurs draw \( z^h \). They target at \( \lambda \) independent of their leverage today. The \( z' = z^l \) line (which is below the 45-degree line) denotes the target leverage when drawing \( z^l \). The target leverage is lower than today’s leverage. When today’s leverage reaches \( \lambda \) and the entrepreneur still draws \( z' \), the entire firm will be liquidated and leverage will be 0. (b) Dynamics of \( k \) and \( b \). When entrepreneurs draw \( z^h \), their firms expand (increase \( k \) while decrease \( b \)) along the solid line. Whenever entrepreneurs draw \( z^l \), they step on the dashed line (one specific path): let \( k \) depreciate while paying back existing debt (increase \( b \)) until \( k/b = \frac{\lambda_1 - \lambda}{\lambda} \) when they liquidate the firm.

(a) Policy function 1
(b) Policy function 2

age. Then, cut vertically the \( z^l \) line in which the intersection \( G_2 \) has the same \( k/(k+b) \) as \( k'/(k'+b') \) of \( G_1 \), which is tomorrow’s leverage. Tomorrow’s leverage keeps decreasing if an entrepreneur keeps drawing \( z^l \) until leverage reaches some threshold \( \lambda_1 \). Then, the firm is liquidated since the capital stock will be very small and the fixed costs (the loss of leisure utility) will force the entrepreneur to do so.

Alternatively, one can examine the dynamics of \( k \) and \( b \) (Figure 1.3b). \( z^h \) owners always expand through the \( z' = z^h \) line so that the leverage remains as \( \lambda \) and \( k/b \) is kept as \( \frac{\lambda}{1-\lambda} \). For example, when \( \lambda \) is the leverage under the borrowing constraint, \( z^h \) owners are constrained by the credit limit. \( z^l \) owners, on the other hand, shrink their debt while letting the capital depreciate until they reach leverage \( \lambda_1 \) (i.e., \( k/b \) ratio is \( \frac{\lambda}{1-\lambda} \)) when their firms are liquidated. The region characterized by the two lines with slope \( \frac{\lambda}{1-\lambda} \) and \( \frac{\lambda}{1-\lambda} \) denotes the inaction region. Inside the region, the reward for changing capital stock is insufficient. From outside
the region (to the right of the $\frac{\lambda}{1-\lambda}$ slope line), the optimal policies are such as to proceed instantly to the $k = 0$ line, that is, to liquidate. Finally, $\bar{\lambda}$ and $\tilde{\lambda}$ will change in response to aggregate shocks.

### 1.3.2 General Properties

#### Properties of the Value Function

I establish some useful properties of the value function which will be used later. The value function behaves normally and has the “scale-invariant” property:

**Lemma 1 (Properties of the Value Function)** The value function $V$ has the following properties

1. $V(k, b, a; X)$ is increasing in $k$, $b$, and $a$, and concave in $(k, b)$.

2. $V$ satisfies

$$V(\gamma k, \gamma b, a; X) = V(k, b, a; X) + \log \frac{\gamma}{1-\beta}.$$  \hspace{1cm} (1.5)

**Proof.** See the Appendix. ■

One can prove Lemma 1 by contraction mapping, which maps the space of functions with properties (i) and (ii) to itself. Let leverage of a firm defined as $k/(k+b)$. (ii) of Lemma 1 says that value functions of entrepreneurs with the same leverage ratio and $a$ are affine transformations of each other. More importantly, target leverage of these entrepreneurs will be the same.\(^{21}\)

Lemma 1 also suggests that fixed costs do not affect the difference of values of two entrepreneurs with the same leverage and $a$. Later, this property is important in deriving the liquidation strategy (i.e., when should an entrepreneur liquidates the firm). Intuitively from Lemma 1, the liquidation strategy depends only on the leverage ratio $k/(k+b)$. In the appendix, I prove that this property still holds under general CRRA utility.

\(^{21}\)Their policies are $(k', b')$ and $(\gamma k', \gamma b')$ so the target leverages are $k'/(k' + b')$ and $\gamma k'/(\gamma k' + \gamma b')$. 

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To derive policy functions, one needs derivatives of the value function. A potential problem is that $\psi(k', k)$ has no derivative when $k' = (1 - \delta)k$. The left derivative is strictly smaller than the right one. Such functions are called sub-differentiable (at $k' = (1 - \delta)k$). If $\psi$ has a kink, so will $V$. Therefore, $V$ is also sub-differentiable at $k' = (1 - \delta)k$. Fortunately, the value function is an upper envelope and it will thus be super-differentiable (the opposite of sub-differentiable). The function that is both sub-differentiable and super-differentiable is differentiable.\footnote{As shown in Mordukhovich, Nam, and Yen (2006).}

**Lemma 2 (Differentiability)** $V(k, b, a; X)$ is differentiable for $k > 0$ and satisfies the envelope condition.

**Proof.** See the Appendix. ■

**Closed-form Policy Functions for $k' > 0$**

Let $z(a)$ and $z'(a)$ denote today’s and tomorrow’s productivity. Because of potential inaction investment decisions, it is useful to work with “shadow value” of capital, i.e., $q(k, b, a; X)$ that satisfies the envelope condition:

$$V_k(k, b, a; X) = u'(c(k, b, a; X))[z(a)\pi + q(k, b, a; X)(1 - \delta)],$$

(1.6)

for $k > 0$. $q$ measures the value of capital in consumption goods unit. It shows how much entrepreneurs value their capital internally, particularly when the investment decision is inaction. Later, it turns out to be useful in solving policy functions.

$q$ is equivalent to the marginal reward to adjust capital. When the marginal reward to increase capital reaches 1, a firm buys capital. When the marginal reward to decrease capital reaches $1 - d$, the firm sells it. When there are no active purchases or sales, the marginal reward to increase capital is $q$, which should be less than 1; the marginal reward

\footnote{The proof closely follows recent work by Clausen and Strub (2012).}
to decrease capital is \( q \), which should be greater than \( 1 - d \). Therefore, it is not optimal to adjust capital stock when

\[
1 - d < q(k, b, a; X) = \frac{V_k/u'(c) - z\pi}{1 - \delta} < 1.
\]

Inside the inaction region, \( q \) is the option value of staying. Such characterization is similar to that in Dixit (1997). Moreover, \( q \) depends only on leverage, keeping everything else fixed:

**Lemma 3 (Scale Invariance and Shadow Prices)** The value function \( V \) and the shadow value \( q \) have the following properties

i. \( V_k \) is homogeneous with degree \(-1\).

ii. For given \( a \) and \( X \), \( V_k/u'(c) \) depends only on \( k/(k+b) \), but not on \( k \) or \( b \) level.

iii. \( q(k, b, a; X) \) can be simplified to \( q(\frac{k}{k+b}, a; X) \).

**Proof.** See the Appendix.

One may also interpret \( q(\frac{k}{k+b}, a; X) \) as the “stock price” of each share of a firm with leverage \( k/(k+b) \) if the firm can be traded among entrepreneurs. When the firm is investing, each share of the stock is priced at 1. When sold, each share of the stock is priced at \( 1 - d \). When firms are inactive in investment, each share of the stock is \( q \in (1 - d, 1) \).

Having established the “competitiveness” of the \( q \), we can express the first-order conditions as:

**Proposition 1 (First-order Conditions)** Define \( \mu(k, b, a; X) \) as the Lagrangian multiplier to the borrowing constraint. The first-order condition for \( k' > 0 \) is

\[
u'(c)q\left(\frac{k}{k+b}, a; X\right) = B\mathbb{E}[V_k(k', b', a'; X') | a, X] + \mu(k, b, a; X)\theta(1 - \delta)(1 - d),\]

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where \( q(\frac{k}{k+b},a;X) \) is defined in equation (1.6). The first-order condition for \( b' \) is

\[
u'(c)R = \beta E[V_b(k',b',a';X')|a,X] + \mu(k,b,a;X)R,
\]

where \( V_b \) is \( V_b(k,b,a;X) = u'(c)R. \) Finally, \( \mu(k,b,a;X) > 0 \) when the borrowing constraint binds, and \( \mu(k,b,a;X) = 0 \) otherwise.

**Proof.** See the Appendix.

When \( \mu(k,b,a;X) = 0 \), the first-order condition and the envelop condition yield:

\[
E \left[ \frac{u'(c')z'(a)\pi' + (1 - \delta)q(\frac{k'}{k+b},a';X)}{u'(c)q(\frac{k}{k+b},a;X)} \right] = 1
\]

which exemplifies classic asset pricing formula “\( E_t[\Lambda_t(1 + r_t)] = 1 \)” or “\( E_t[\Lambda_t(1 + r_t - R_t)] = 0 \)”, where “\( \Lambda \)” is the stochastic discount factor and \( r \) is the return from an asset. Here, the return on capital is 

\[
\frac{z'(a)\pi' + (1 - \delta)q(\frac{k'}{k+b},a';X)}{q(\frac{k}{k+b},a;X)},
\]

where \( q(\frac{k}{k+b},a;X) \) takes different values depending on the decision of buying, selling, or being inactive.

The asset pricing formula also sheds some light on solving portfolio choices between capital stock and bonds. To see this, first define the rate of return of having capital \((k' > 0)\) as

\[
r'(k',b',a';X'|k,b,a;X) = \frac{z'(a)\pi' + (1 - \delta)q(\frac{k'}{k+b},a';X)}{q(\frac{k}{k+b},a;X)},
\]

define the net worth of an entrepreneur using the shadow value of capital as

\[
n(k,b,a;X) = z(a)\pi k + q(\frac{k}{k+b},a;X)(1 - \delta)k + Rb,
\]

and let \( \phi \) denote the fraction of net worth spent on capital. We have the closed-form solution as:
Proposition 2 (Closed-form Policy Functions) The policy function on consumption $c = c(k, b, a; X)$, capital $k' = k'(k, b, a; X) > 0$, and bonds $b' = b'(k, b, a; X)$ can be expressed as

$$c = (1 - \beta)n(k, b, a; X), \quad k' = \frac{\phi}{q(k, b, a; X)} Bn(k, b, a; X), \quad b' = (1 - \phi)\beta n(k, b, a; X).$$

where $\phi$ satisfies

$$\begin{cases} 
E \left[ \frac{r' - R'}{\phi r' + (1 - \phi)R'} \middle| a, X \right] = 0, & \text{if } E \left[ \frac{r'}{\phi r' + (1 - \phi)R'} \middle| a, X \right] = 1 \\
\phi = \frac{1}{1 - \sigma(1 - \delta)(1 - \delta)/qR'}, & \text{if } E \left[ \frac{r'}{\phi r' + (1 - \phi)R'} \middle| a, X \right] < 1 
\end{cases}$$

Finally, $k'$ is consistent with $q(k, b, a; X)$, so that $k' > (1 - \delta)k$ for $q(k, b, a; X) = 1$ and $k' < (1 - \delta)k$ for $q(k, b, a; X) = 1 - d$. Otherwise, $\phi$ should be such that $k' = (1 - \delta)k$.

Proof. See the case $\sigma = 1$ of the proof under general CRRA utility in the Appendix. ■

Notice that the stochastic discount factor here is $\Lambda' = \frac{1}{\phi r' + (1 - \phi)R'}$ such that asset pricing formula $E[\Lambda'(r' - R')] = 0$ holds. A typical entrepreneur consumes $(1 - \beta)$ fraction and saves the other $\beta$ fraction of the net worth. She uses the savings to invest in a portfolio. The portfolio consists of risky assets (capital stock) and risk-free assets (bonds), allowing shorting on risk-free assets but not on risky ones. If she invests $\phi$ fraction of a dollar in risky assets and the other $1 - \phi$ fraction in risk-free assets, the next period’s rate of return is $\phi r' + (1 - \phi)R'$. The goal of portfolio choice is to maximize the expected log rate of return (i.e., the solution of $\phi$).\textsuperscript{24}

Even though the saving rate is a constant ($\beta$) under log utility, different entrepreneurs save different fractions of the “accounting” net worth which is either $z\pi k + (1 - \delta)k + Rb$

\textsuperscript{24}Policy functions have closed-form expressions for any $\sigma$ (see the Appendix). But under general CRRA utility, the saving rate (not necessarily $\beta$) and portfolio weight $\phi$ intertwine with each other. The reason is that with general CRRA utility the income and substitution effect do not offset each other, for example illustrated in Campbell and Viceira (2002). The combination of the two effects are so-called “hedging demand” in the asset pricing literature. Depending on the investment opportunities in the long time frame, agents put different weights on capital and consume differently.
or \( z\pi k + (1 - \delta)(1 - d)k + Rb \). Unlike the accounting net worth, the “economic” net worth evaluates capital at shadow prices, which varies across entrepreneurs when the investment decisions opt for inaction.

### 1.3.3 The Inaction Regions and Liquidation Choices

I confine my attention to the equilibrium with an active credit market and an active secondary market. There may or may not be inaction in investment. When there is, there exists at least a \( q \) that is between \( 1 - d \) and 1. To characterize the inaction region, one only needs to check how the shadow price \( q(k/(k+b), a; X) \) varies as \( k/(k+b) \) and \( a \) change (for a given \( X \)). The inaction region is the set of \( k/(k+b) \) and \( a \) such that the shadow price is between \( 1 - d \) and 1.

In such equilibrium, \( z^h \) owners should always invest and \( z^l \) owners should not because:

\[
z^h \pi' + (1 - \delta) > R' \geq z^l \pi' + (1 - \delta)
\]

The first inequality should hold; otherwise no entrepreneurs will invest. The second inequality should hold. If \( R' < z^l \pi' + (1 - \delta) \), \( z^l \) owners always find a higher return from investing than the return from holding bonds regardless of drawing \( z^h \) or \( z^l \) tomorrow. They strictly prefer to invest and borrow to the credit limit. In that economy, everyone is a borrower, which is inconsistent with equilibrium definition since the bond market cannot clear.

Therefore, some or all \( z^h \) owners invest and borrow. Because of the linear rate of return in individual level, they have the same target leverage \( k'/(k' + b') \) tomorrow regardless of their leverage today (Proposition 3). \( k'/(k' + b') \) may or may not reach the leverage under credit limits.

For \( z^l \) owners, profits from capital stock are low. Their investment decision is either to hold or to sell. It turns out that an entrepreneur \( j \) who persistently draws \( z^l \) hold capital for
finite periods. The shadow price during the process of holding capital is monotonically decreasing until it reaches $1 - d$ when $j$ liquidates assets. Additionally, the leverage decreases before liquidation.

**Proposition 3 (Leverage and Deleverage)** In equilibrium with an active secondary market

i. $z^h$ owners borrow and invest. Moreover, they have the same target leverage $\frac{k'}{k'+b'} = \bar{\lambda}$.

ii. Denote today’s shadow price as $q$ and tomorrow’s shadow price as $q'$. Then,

$$q' \begin{cases} 
1 & \text{if } z' = z^h \\
< q & \text{if } z' = z^l
\end{cases} \quad \text{and} \quad \frac{k'}{k'+b'} \begin{cases} 
\bar{\lambda} & \text{if } z' = z^h \\
< \frac{k}{k+b} & \text{if } z' = z^l
\end{cases}$$

**Proof.** See the Appendix. ■

The deleveraging behavior during the inaction process are intuitive. For $z^l$ entrepreneurs, running business is not profitable compared to risk-free rate. Without resale costs, they will liquidate and repay all the debt. But with resale costs, those who just turn from $z^h$ to $z^l$ hold capital initially. They can still shrink interest payment in order to smooth consumption. Not surprisingly, capital is less and less valued.

After some periods of inaction, capital stock gradually shrinks to a very small amount. The fixed costs of running a business eventually force the $z^l$ owners to liquidate. To see this, $z^l$ owners compare the value of liquidating and holding. Once the value after liquidation is the same as the value of holding strategy, $z^l$ owners start to liquidate, that is, there exists a stopping time:

**Proposition 4 (Optimal Stopping Time)** For $z^l$ owners, there exists an optimal capital liquidation rule (stopping-time rule or exit rule). Let $n = z^l \pi + (1 - \delta)(1 - d) + R \frac{k - \lambda}{\lambda}$ and
suppose a finite $\lambda \in [0, \bar{\lambda}]$ is a root of

$$
\eta = \frac{\beta}{1-\beta} p^h E \left[ \log \left( 1 + (1-\delta) \frac{z^l \pi' + (1-\delta) - (1-d)R'}{\beta nR'} \right) \right] \bigg| X 
+ \frac{\beta}{1-\beta} p^l E \left[ \log \left( 1 + (1-\delta) \frac{z^l \pi' + (1-\delta)(1-d) - (1-d)R'}{\beta nR'} \right) \right] \bigg| X
$$

(1.7)

i When $\frac{k}{k+b} > \lambda$, $z^l$ owners are inactive in adjusting capital. When $\frac{k}{k+b} < \lambda$, they liquidate the whole firm. When $\frac{k}{k+b} = \lambda$, they are indifferent between holding or liquidating capital.

ii If no $\lambda$ satisfies equation (1.7), then no $z^l$ entrepreneur sells capital.

**Proof.** See the Appendix.

The indifference condition (1.7) is intuitive. Entrepreneurs are indifferent between liquidation and holding when the gains of liquidation (extra $\eta$ utility) equals the expected discounted costs of not doing so (the right hand side, extra value of holding capital stock one more period). In calculating the costs of not liquidating, for each unit of net worth saved in capital stock and bonds, the excess return is $\left( 1 + (1-\delta) \frac{z^l \pi' + (1-\delta) - (1-d)R'}{\beta nR'} \right)$ when drawing $z^h$ tomorrow and $\left( 1 + (1-\delta) \frac{z^l \pi' + (1-\delta)(1-d) - (1-d)R'}{\beta nR'} \right)$ when drawing $z^l$ tomorrow.

So far, I have shown the steps to establish the decision rules in Figures 1.3a and 1.3b. Now, the inaction region can be easily expressed by the set of leverage ratios and productivities

$$\{ \left( \frac{k}{k+b}, a \right) : \lambda \leq \frac{k}{k+b} \leq \bar{\lambda} \text{ and } z'(a) = z^l \}$$

where $\lambda$ is the lower bound while $\bar{\lambda}$ is the upper bound. To understand the changes of the inaction region, I show what leads to $(\bar{\lambda} - \lambda)$ changes. If $(\bar{\lambda} - \lambda)$ is larger, the inaction region expands. I focus on the borrowing constrained economy, i.e, $\bar{\lambda} = 1/(1 - \theta (1 - \delta)(1-d)/R')$. Intuitively, $z^l$ owners have more incentive to hold capital if (1) they are more patient (a larger $\beta$), (2) the fixed costs are smaller (a smaller leisure utility $\eta$), and
(3) the selling discount $d$ is higher. All of these increase the net benefits of holding capital and expand the inaction region.

**Corollary 1 (Changes of Inaction Region: Partial Equilibrium Effect)** If borrowing is constrained, the inaction region expands when

- $i$ \( \beta \) is higher, that is, \( \partial(\bar{\lambda} - \lambda) / \partial \beta > 0 \)
- $ii$ \( \eta \) is smaller, that is, \( \partial(\bar{\lambda} - \lambda) / \partial \eta < 0 \)
- $iii$ $d$ is higher, that is, \( \partial(\bar{\lambda} - \lambda) / \partial d < 0 \)

**Proof.** Define $m$ to be the right hand side of equation (1.7). From the proof of Proposition 4, \( \partial m / \partial \bar{\lambda} < 0 \). Notice that \( \partial m / \partial \beta < 0 \) and \( \bar{\lambda} \) does not depend on $\beta$. Then using the implicit function theorem, we know that \( \partial(\bar{\lambda} - \lambda) / \partial \beta > 0 \) which proves (i). (ii) can be proved by similar steps. (iii) can be proved by similar steps and by taking into account

\[
\frac{\partial \bar{\lambda}}{\partial d} = -\left(\bar{\lambda}\right)^2 \theta (1 - \delta) / R' .
\]

A larger degree of asset illiquidity directly expands the inaction region. In contrast, a larger degree of financing frictions (a lower $\theta$) does not have a direct effect, from equation (1.7). Moreover, when $\theta$ goes down, the highest leverage become smaller ($\bar{\lambda}$ is smaller) and ($\bar{\lambda} - \lambda$) decreases. But a lower $\theta$ has a general equilibrium effect. If financing frictions limit the expansion of productive firms so that aggregate demand shrinks and labor input costs are lower, $\pi'$ will be higher and $z'$ owners will wait until a even lower leverage before liquidation ($\bar{\lambda}$ is much smaller). In that case, ($\bar{\lambda} - \lambda$) increases in response to a lower $\theta$.

**Corollary 2 (Changes of Inaction Region: General Equilibrium Effect)** In borrowing constrained equilibrium, $\pi'(X')$ depends on $\theta$. The inaction region expands when

- $i$ profits rate is higher, that is, \( \partial(\bar{\lambda} - \lambda) / \partial \pi'(X') > 0 \)
- $ii$ financing constraints are tighter and \( \partial \pi'(X') / \partial \theta \) is negative and sufficiently small, that is, \( \partial(\bar{\lambda} - \lambda) / \partial \theta < 0 \)

24
Proof. By similar steps in Corollary 1.

The changes of profits rate and subsequent impacts on liquidation choices are essential for understanding the system dynamics in response to aggregate shocks. The Corollary provides us some intuition. For example, a credit crunch (a lower $\theta$) will reduce investment and employment, which leads to lower input costs and a higher profits rate. Therefore, after a credit crunch, inefficient firms have higher incentives to hold assets. Later, the intuition from the Corollary will be confirmed in the numerical analysis.

1.4 Recursive Equilibrium Revisit

So far, we know that entrepreneurs with the same leverage $k/(k + b)$ and productivity put the same portfolio weights on $k$ and $b$. Thus, I can define aggregate capital stock and aggregate bonds for a specific $k/(k + b)$ ratio, given a productivity pair $a$, i.e.,

$$K(x, a) = \int \{k, b: k/(k + b) = x\} k\Gamma(dk, db, a), \quad B(x, a) = \int \{k, b: k/(k + b) = x\} b\Gamma(dk, db, a)$$

Equilibrium can be redefined as a mapping $(K(x, a), B(x, a), A, \theta) \rightarrow (K'(x, a), B'(x, a), A', \theta')$. I apply this idea to characterize the evolution of the firm distribution. Subsection 1.4.1 shows the details and Subsection 1.4.2 redefines the equilibrium using the distribution in Subsection 1.4.1.

1.4.1 The Distribution of Firms

Since drawing $z^h$ always means investing, keeping track of the firm distribution is equivalent to keeping track of firms with the time length of having been drawing $z^l$. Thanks to Proposition 4, there is a stopping rule for entrepreneurs who run the firms but always draw $z^l$. 25
At the beginning of time $t$, let $s = 1, 2, ...$ denote the vintage of entrepreneurs, who have been drawing $s$ times of $z^l$. These firms did not invest in $t - 1$. Clearly, $s = 0$ denotes the state in which the entrepreneur just finished investing (or drew $z^h$ as time $t$ productivity in time $t - 1$). Drawing $z^h$ means entrepreneurs will go to vintage $s = 0$, whereas drawing $z^l$ means going to the next vintage, i.e., the vintage whose number equal current vintage number plus 1. Inside each vintage, the $k/(k + b)$ ratio is the same, which allows me to replace $q\left(\frac{k}{k+b},a;X\right)$ by vintage-specific price. When entrepreneurs with $k/(k+b)$ decide to go from vintage $s$ to $s'$, the shadow price of capital will be $q(s';X)$, which is vintage-specific and corresponds to a specific $k'(k' + b')$.

When the secondary market is active, there exists an integer $N_t < +\infty$ at time $t$, such that entrepreneurs who are from vintage $N_t + 1, N_t + 2, ...$ and draw $z^l$ hold no capital stock; while those who are from vintage $0, 1, ..., N_t$ and draw $z^l$ will be inactive in capital.

For simplicity, I focus on small exogenous shocks around the steady state such that the equilibrium vintages do not change, i.e., $N_t = N$ where $N$ is an endogenous constant integer. (Note that $N$ itself varies in different steady states). In numerical exercises, I verify that the shocks do not change vintage numbers $N$. When $N$ stays the same, entrepreneurs from vintage $N$ who draw $z^l$ again are indifferent between liquidating and keeping capital. They play a mixed strategy between staying and liquidating.

**Corollary 3** *In the equilibrium with capital reallocation, there exists an integer $N$ such that:*

i. *Entrepreneurs go to vintage 0 once they draw $z^h$.*

ii. *For those entrepreneurs who draw $z^l$, they go to the next vintage.*

- *Those in vintage 0 to $N-1$ hold on to capital.*

- *Those in vintage $N$ are indifferent between being inactive in capital or liquidating.*

- *Those in vintage $N+1$ liquidate the firm. Those in $N+2$, ... do not run the firm.*
Now, I describe the features of the vintages. One can group vintages after $N+2$ together to be one vintage, since entrepreneurs in vintages $N+2$, $N+3$,... only hold bonds, thanks to Corollary 3. To simplify, the probability of drawing $z^h$ and $z^l$ in each vintage is

$$\tilde{P} = \begin{pmatrix} p^{hh} & p^{hl} & \ldots & p^{lh} \\ p^{lh} & p^{ll} & \ldots & p^{ll} \end{pmatrix}_{(N+2) \times 2}$$

where $\tilde{P}_{11}$ and $\tilde{P}_{22}$ are the probability of drawing $z^h$ and $z^l$ in vintage $i$. The associated vintage specific productivity vector is

$$Z = \begin{bmatrix} z^h & z^l & \ldots & z^l \end{bmatrix}'_{(N+2) \times 1}$$

With slight abuse of notation, let $p^{jh} = \tilde{P}_{11}$ and $p^{jl} = \tilde{P}_{22}$ be the probability of drawing $z^h$ and $z^l$ as the new productivity respectively; let $z^i$ be the $i$th element of $Z$ which is the current productivity of an entrepreneur in vintage $i$.

A fraction of the entrepreneurs from vintage $N$ who draw $z^l$ goes to vintage $N+1$ and the other fraction liquidates capital and goes to vintage $N+2$. They are indifferent between holding or selling capital. As in Proposition 4, the optimal indifference leverage $\lambda$ solves \textsuperscript{25}

$$\eta = \frac{\beta p^{(N+1)h}}{1 - \beta} E \left[ \log \left( \frac{1 + (1 - \delta)}{\beta R_{i+1}(z^{N+1} \pi_{i+1} + (1 - \delta) - (1 - d)R_{t+1})} \right) \right] X_i$$

$$+ \frac{\beta p^{(N+1)l}}{1 - \beta} E \left[ \log \left( \frac{1 + (1 - \delta)}{\beta R_{i+1}(z^{N+1} \pi_{i+1} + (1 - \delta) - (1 - d)R_{t+1})} \right) \right] X_i$$

Finally, let $f^i_t$ ($i = 1, 2, \ldots, N+2$) be the fraction of entrepreneurs who go to vintage $i$ out of all entrepreneurs who draw vintage $i$ productivity $z^i$, and $1 - f^i_t$ be the other fraction of entrepreneurs who liquidate capital. Notice that $f^i_t = 1$ for $i = 0, 1, 2, \ldots, N$: those who

\textsuperscript{25}In equilibrium, $(1 - \lambda_t)/\lambda_t$ is equal to the ratio of $b^N_t/k^N_t$ in vintage $N$. Those from vintages $N+1$ and $N+2$ who also draw $z^l$ go to vintage $N+2$ by holding only bonds.
draw $z^h$ will always invest and go to vintage 0, and those who draw $z^l$ in vintage $i - 1$ will be inactive in investment and go to vintage $i$. Also, $f_i^{N+2} = 1$ because entrepreneurs who are from vintage $N + 1$ or $N + 2$ and draw $z^l$ will always hold only bonds and go to vintage $N + 2$. Finally, $f_i^{N+1} \in [0, 1)$ because entrepreneurs who are from vintage $N$ and who draw $z^l$ play mixed strategy: $f_i^{N+1}$ fraction of them go to vintage $N + 1$ and $1 - f_i^{N+1}$ fraction of them go to vintage $N + 2$.

Now, we can fully characterize the firm distribution evolution from $t$ to $t + 1$ in Figure 1.4.\(^{26}\)

### 1.4.2 Recursive Equilibrium Revisit

Thanks to the vintage distribution, I can leave aggregate state $X$ out and denote variables with vintage superscript and time subscript $t, t + 1, \ldots$. For example at time $t$, the capital shadow price of entrepreneurs who are going to vintage $i$ is $q_i^t$. For consistency, let $q_0^t = 1$ denote the buying price.\(^{27}\) Define the (risky) rate of return on capital from time $t$ to time $t + 1$ of entrepreneurs who are going to vintage $i$ as $r_i^{ij} t + 1$, where $i = 0, 1, \ldots, N + 1$ and $j \in h, l$ indicates drawing (time $t + 2$ productivity) $z^h$ or $z^l$ at time $t + 1$.\(^{28}\) Specifically, the vintage

\(^{26}\) If we expand the state space of idiosyncratic producibility and reclassify each vintage as a state, the matrix $P$ below is the transition probability matrix

$$
P = \begin{bmatrix}
    p^{hh} & p^{hl} \\
    p^{lh} & p^{ll} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    p^{lh} & \cdots & p^{ll} f_i^{N+1} & p^{ll} f_i^{N+1} (1 - f_i^{N+1}) & \cdots \\
    p^{lh} & \cdots & 0 & p^{ll} f_i^{N+1} & 0 \\
    p^{lh} & \cdots & 0 & 0 & p^{ll}
\end{bmatrix}
$$

where $P_{ij}$ denotes the probability from vintage $i$ to vintage $j$. The right eigenvector of $P^T$ associated with eigenvalue one is the population of entrepreneurs in each vintage in the steady state. See Chapter 2 of Ljungqvist and Sargent (2004) for details. In calibration, I use this property to calculate the cross-section standard deviation of TFP of the existing firms.

\(^{27}\) “Shadow price” of capital of entrepreneurs who are going to invest and go to vintage 0.

\(^{28}\) For example, at time $t + 1$, an entrepreneur in vintage 3 draws (time $t + 2$ productivity) $z^h$, her rate of return on capital from $t$ to $t + 1$ is $r_i^{3h} t + 1.$
Figure 1.4: Evolution of the distribution
Each box represents a vintage in which firms have the same $\lambda = \frac{k}{k+b}$ leverage ratio. The vintage number is identical to how many periods an entrepreneur has been drawing $z^l$. Entrepreneurs who draw $z^h$ invest and move to vintage 0. Entrepreneurs who are from vintage 0 to $N-1$ and draw $z^l$ are inactive. Entrepreneurs who are from vintage $N$ and draw $z^l$ are indifferent between liquidating or continuing production. Entrepreneurs in vintage $N+1$ or the last vintage $N+2$ hold only bonds if drawing $z^l$ (liquidate the firm or continuing holding only bonds). $f_i$ denotes the fraction of entrepreneurs who go to vintage $i$ out of all entrepreneurs who draw vintage $i$ productivity $z^l$. $1 - f_i$ then denotes the other fraction of entrepreneurs who do not go to vintage $i$ but liquidate their firms.

\[ r_{t+1}^{hl} = \frac{z^l \pi_{t+1} + (1 - \delta) q_{t+1}^0}{q_t^l} , r_{t+1}^{il} = \frac{z^l \pi_{t+1} + (1 - \delta) q_{t+1}^i}{q_t^l} , \text{ for } i = 1, 2, ..., N+1. \]
For convenience, denote \( \bar{r}_{i+1} \) as the average return, i.e., for \( i = 0, 1, 2, \ldots, N+1 \),
\[
\bar{r}_{i+1} = p_i^{ih} E[\bar{r}_{i+1} | X_t] + p_i^{il} E[R_{i+1} | X_t].
\]

Then, according to Proposition 2, the portfolio weight \( \phi \) on capital can be simplified as:

**Corollary 4 (Vintage-specific Portfolio Choices)** The capital weight \( \phi_i^j \) (\( i = 0, 1, 2, \ldots, N \)) for entrepreneurs who are going to vintage \( i \) solves
\[
\min \left\{ \frac{1}{1 - \theta_t (1 - \delta) (1 - d) / R_{t+1}} \phi_i^j = - \frac{R_{t+1} (\bar{r}_{i+1} - R_{t+1})}{(r_i^{ih} - R_{t+1}) (r_i^{il} - R_{t+1})} \right\}
\]

Now, we are ready to redefine the equilibrium. Denote \( K_i^j \) and \( B_i^j \) as the aggregate capital stock and bonds in vintage \( i \). Thanks to the closed-form decision rules in Proposition 2, the transition dynamics is highly tractable as in the following non-linear equations. Capital transition can be characterized by aggregate capital in vintage 0
\[
q_i^0 K_{i+1}^0 = f_i^0 \phi_i^0 \sum_{i=0}^{N+2} p_i^{ih} \beta [z_i^i \pi_i K_i^i + (1 - \delta) q_i^0 K_i^i + R_i B_i^{i-1}],
\]
(1.8)

by aggregate capital in vintage \( i = 1, 2, \ldots, N + 1 \)
\[
q_i^j K_{i+1}^j = f_i^j \phi_i^j p^{(i-1)l} \beta [z_i^{j-1} \pi_i K_i^{j-1} + (1 - \delta) q_i^{j-1} K_i^{j-1} + R_i B_i^{j-1}],
\]
(1.9)

and by aggregate capital in vintage \( N + 2 \)
\[
K_{N+2}^{N+2} = 0.
\]
(1.10)

The transition of bonds can be characterized by aggregate bonds in vintage 0
\[
B_{i+1}^0 = f_i^0 (1 - \phi_i^0) \sum_{i=0}^{N+2} p_i^{ih} \beta [z_i^i \pi_i K_i^i + (1 - \delta) q_i^0 K_i^i + R_i B_i^{i-1}],
\]
(1.11)
by aggregate bonds in vintage $i = 1, 2, ..., N + 1$,

$$B_{i+1}^i = f_i^i(1 - \phi_i^i)p^{(i-1)l}B_{t+1}^{i-1} + (1 - \delta)q_i^iK_{i+1}^i + R_iB_{i+1}^i, \quad (1.12)$$

and finally by aggregate bonds in vintage $N + 2$

$$B_{i+1}^{N+2} = \sum_{i=1}^{N}(1 - f_i^{i+1})p^{il}B_i^{i-1} + \sum_{i=N+1}^{N+2}f_i^{N+2}p^{il}B_i^{i-1} + \beta f_i^{i+1}K_i^{i-1} + (1 - \delta)q_i^{N+2}K_i^i + R_iB_{i+1}^i. \quad (1.13)$$

The aggregate capital in vintages $i = 1, 2, ..., N + 1$ satisfies

$$K_{i+1}^i = p^{(i-1)l}f_i^i(1 - \delta)K_{i+1}^{i-1}, \quad (1.14)$$

together with consistent $f_i^i$

$$f_i^i \left\{ \begin{array}{l}
= 1, \quad \text{if } i = 0, 1, ..., N \\
\in [0, 1), \quad \text{if } i = N + 1 \\
= 1, \quad \text{if } i = N + 2.
\end{array} \right. \quad (1.15)$$

The labor market and bond market clearing conditions are

$$\left( \frac{\pi_t}{\alpha A_t} \right)^{\frac{1}{1-a}} \left( \sum_{i=0}^{N+2} z_i^i K_i^i \right) = L, \quad \sum_{i=0}^{N+2} B_{i+1}^i = 0. \quad (1.16)$$

Finally, the stopping condition of an entrepreneur from vintage $N$ who draws $z_i^i$ again is

$$\eta = \beta p^{(N+1)h} \left[ \log \left( 1 + (1 - \delta) \frac{z_i^{N+1} \pi_{i+1} + (1 - \delta)q_{i+1}^0(1 - d)R_{i+1}}{\beta R_{i+3}(z_i^N \pi_i + (1 - \delta)(1 - d)q_i^0 + R_iN/K_i^N)} \right) \right]_{X_t}$$

$$+ \beta p^{(N+1)l} \left[ \log \left( 1 + (1 - \delta) \frac{z_i^{N+1} \pi_{i+1} + (1 - \delta)q_{i+1}^0(1 - d)q_i^0(1 - d)R_{i+1}}{\beta R_{i+3}(z_i^N \pi_i + (1 - \delta)(1 - d)q_i^0 + R_iN/K_i^N)} \right) \right]_{X_i^7}.$$
Definition 2 (The Second Recursive Equilibrium Definition) The recursive competitive equilibrium is functions \((\{\phi^i_t\}_{i=0}^{N+2}, \{f^i_t\}_{i=1}^{N+2}, \{q^i_t\}_{i=0}^{N+2}, \{K^i_{t+1}\}_{i=0}^{N+2}, \{B^i_{t+1}\}_{i=0}^{N+2}, \pi_t, R_{t+1}\) of state variables \((\{K^i_t\}_{i=0}^{N+2}, \{B^i_t\}_{i=0}^{N+2}, \theta_t, A_t)\) and a given initial condition \((\{K^i_0\}_{i=0}^{N+2}, \{B^i_0\}_{i=0}^{N+2}, \theta_0, A_0)\), such that:

i equations (1.8) to (1.17) are satisfied

ii \(\{\phi^i_t\}_{i=0}^{N}\) solve the portfolio problems in Corollary 4

iii \(q^i_0 = 1\) and \(q^i_{N+1} = q^i_{N+2} = 1 - d\)

iv together with the law of motion of \((\theta_t, A_t)\)

The capital market clearing is embedded in the capital transition dynamics, and one can easily verify that the goods market clearing condition is satisfied (i.e., the Walras’ Law holds).

1.4.3 Efficiency and Delayed Reallocation in the Steady State

The longer the waiting periods, the more capital reallocation is delayed and the less efficient is the economy (i.e., the lower is the aggregate TFP). In steady state, aggregate productivity \(A_t = 1\) and the aggregate TFP is defined as

\[
TFP = \frac{Y}{K^\alpha L^{1-\alpha}}
\]

where \(Y\) is the total output and \(K\) is the total capital stock. Note that \(\alpha\) fraction of the output is entrepreneurs’ profits. Output can be written as \(Y = \frac{\pi}{\alpha} (z^H K^H + z^L K^L)\), where \(K^H = K^0\) and \(K^L = \sum_{i=1}^{N+1} K^i\) denote the capital stock under \(z^H\) and \(z^L\) technology respectively. Together with the labor market clearing condition \((\frac{\pi}{\alpha})^{1-\alpha} (z^H K^H + z^L K^L) = L\), TFP can be simplified
to

\[
TFP = \frac{(z^h K^h + z^l K^l)\alpha}{(K^h + K^l)^\alpha} = \frac{(z^h K^h / K^l + z^l)\alpha}{(K^h / K^l + 1)^\alpha}.
\] (1.18)

When \(K^l \to 0\), all capital is installed under \(z^h\) technology, and the TFP reaches the upper bound \(\tilde{z}^h = (z^h)^\alpha\). When \(K^l > 0\), we know that the relative capital stock ratio \(K^h / K^l\) determines the economy efficiency. Intuitively, the longer the waiting period, the smaller \(K^h / K^l\) ratio and thus a lower TFP in the economy. The quantitative effects of delayed reallocation and aggregate TFP losses are the main targets in the next section.

What determines the delayed reallocation and thus the aggregate TFP of the economy? Intuitively, the essential parameters that affect the trade-off between liquidation and continued production are the relative productivity gap, the persistence of the transition matrix, the outside option utility \(\eta\), the resale costs \(d\), and the degree of financing frictions \(\theta\). For example, if the relative productivity gap is larger, holding capital has a higher benefit so waiting periods tend to be longer. But also the interest rate in the steady state is higher because \(z^h\) entrepreneurs can accumulate more capital and collateralized borrowing is easier. In that case, liquidation is more preferred. The net effects are unclear and further numerical examinations are needed.

However, the next proposition shows that labor supply and capital share do not have any impact on the trade-offs in the steady state, so is the absolute levels of \(z^h\) and \(z^l\) (as long as the relative gap remains the same, the vintage number does not change).

**Proposition 5 (Waiting Time)** Changing the following parameters does not change the steady state waiting periods \(N\):

1. Inelastic labor supply unit \(L\)
2. Capital share \(\alpha\) in the production function
3. \(z^h\) and \(z^l\) as long as the ratio of \(z^h / z^l\) stays the same.
Proof. (i) Suppose we have the solution for a given $L$. Consider changing $L$ to $(1 + \Delta)L$. The steady state equations are still satisfied by varying only $K^i$ and $B^i$ to be $(1 + \Delta)K^i$ and $(1 + \Delta)B^i$, while keeping other variables the same. Similar results hold for (ii) and (iii).

From now on, I will turn to quantitative exercises of the model. The above proposition shows that we should give more consideration to parameters other than labor supply, capital share, and the level of $\varepsilon^h$ or $\varepsilon^l$ (but $\varepsilon^h/\varepsilon^l$ is important).

## 1.5 Numerical Examples

### 1.5.1 Calibration and Estimation

While the model is stylized, I bring it to data as close as possible. I match the steady state result to several U.S. long-run economy characteristics. Further, I estimate the shocks to financing constraints and aggregate productivity for short-run analysis. Each period represents a quarter and the full calibrated parameters are in Table 1.1.

Following Veracierto (2002), the capital abstracts from components such as land, residential structure, and consumer durable goods. Thus, the capital corresponds to non-residential structures, plant, and equipment while the investment corresponds to the non-residential investment in the National Income and Product Accounts (NIPA). Meanwhile, the empirical counterpart for consumption should be non-durable goods and services consumption. Output is then defined as the sum of the consumption and the investment. The investment-to-output ratio is found to be 0.165, which translates into the capital share in the production function as $\alpha = 0.258$. The capital to annual output ratio is 1.5 which translates into a depreciation rate $\delta = 2.58\%$. $\beta = 0.9847$ targets at the risk-free interest rate. The interest rate is low in equity premium puzzle literature (e.g. Mehra and Prescott (1985)) and is commonly chosen to be 3% to 4% annually. Here, I chose 3.5%.
Table 1.1: **Calibrated Parameters**
Parameters calibrated to the long-run U.S. economy. $\alpha$ is the capital share, $\beta$ is the discount factor, $\delta$ is the capital stock depreciation rate, $\tilde{z}^h$ and $\tilde{z}^l$ are the high and low idiosyncratic productivities, $p^{hl}$ and $p^{lh}$ are the transition probabilities in the transition matrix, $\theta$ measures the tightness of financing constraints, $d$ is the proportional resale costs, and finally $\eta$ is the leisure utility that captures the fixed costs in running firms.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target</th>
<th>Source/Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.2580</td>
<td>Investment/Output Ratio: 0.165</td>
<td>NIPA data</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.9847</td>
<td>Quarterly Discount Rate</td>
<td>Common discount rate</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0258</td>
<td>Capital to Output Ratio: 1.5</td>
<td>NIPA data</td>
</tr>
<tr>
<td>$\tilde{z}^h$</td>
<td>1.1307</td>
<td>Standard Deviation of TFP: 5.7%</td>
<td>Basu, Fernald, and Kimball (2006)</td>
</tr>
<tr>
<td>$\tilde{z}^l$</td>
<td>1.0000</td>
<td>Normalization</td>
<td>Does not change $N$</td>
</tr>
<tr>
<td>$L$</td>
<td>4.0000</td>
<td>80% of the working-age population employed</td>
<td>Does not change $N$</td>
</tr>
<tr>
<td>$p^{hl}$</td>
<td>0.0665</td>
<td>Constrained firms: 64%</td>
<td>Almeida, Campello, and Weisbach (2004)</td>
</tr>
<tr>
<td>$p^{lh}$</td>
<td>0.0400</td>
<td>Reallocation/capital stock: 1.44%</td>
<td>COMPUSTAT and SDC data</td>
</tr>
<tr>
<td>$\tilde{\theta}$</td>
<td>0.4000</td>
<td>Average debt/asset: 0.325</td>
<td>Flow of funds data</td>
</tr>
<tr>
<td>$d$</td>
<td>0.1000</td>
<td>Reallocation/capital expenditure: 0.40</td>
<td>COMPUSTAT data</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.3000</td>
<td>Annual real interest rate 3.5%</td>
<td>Mehra and Prescott (1985)</td>
</tr>
</tbody>
</table>

The employment (labor measure) is set to be $L = 4$, so that roughly 80% of the working age population is employed. I normalize the low productivity $\tilde{z}^l = 1$. As shown by Proposition 5, $L$ and the level $z^l$ do not affect the waiting periods in the steady state.

For the productivity transition matrix, one only needs $p^{hl}$ and $p^{lh}$. The primary targets are (1) the fraction of firms that are constrained, and (2) the turn-over of capital reallocation over empirical relevant capital stock. The target of (1) is from the studies of Almeida, Campello, and Weisbach (2004), who identify the number of constrained firms to be 64% from COMPUSTAT data (which I average across from all alternative ways of measurement in their studies). Also, the turn-over of capital reallocation over total property, plant, and equipment is 5.7% annually and 1.4% quarterly in the COMPUSTAT data.

Once we have the transition matrix, we can determine $\tilde{z}^h$. I follow the cross-sectional standard deviation of productivity (5.7%) in Basu, Fernald, and Kimball (2006). Note that the measure is the standard deviation of TFP of existing firms in the model, excluding the TFP of entrepreneurs who exited before. Then, $\tilde{z}^h$ turns out to be 1.1307.
The parameters left are $\theta$, $d$, and $\eta$. These three affect decisions on leverage, investment, and liquidation. The haircut $\theta$ targets at leverage where empirically, the debt-to-asset ratio is averaged to be 0.325 from flow of funds data. The degree of asset irreversibility $d$ targets at reallocation. The fraction of capital reallocation over total capital purchase (roughly 35% quarterly) is stable. However, COMPUSTAT data only include publicly traded firms that are relatively large. Smaller firms, according to Eisfeldt and Rampini (2007), use more used capital. Therefore, 35% is naturally the lower bound and I chose 40% for benchmark calibration. Finally, the leisure utility $\eta$ measures “fixed costs” and controls how long a persistently unproductive firm will hold the assets and deleveraging. I chose $\eta = 0.30$ such that there will be 12 quarters of waiting periods, roughly the same as the number of quarters of deleveraging before selling found in the introduction.

The calibration is to capture the long-run steady state. For estimating the shocks and their persistence, I use output and capital reallocation (both after HP-filtered) as the observations. The unobservable shocks are financial shocks and aggregate productivity shocks. Therefore, the model linked the observations to the shocks. I use Bayesian methods to back out the information of the shocks, conditional on observations. Specifically, I assume $\theta_t = \theta \hat{\theta}_t$ and $A_t = \hat{A}_t$, where $\theta_t$ and $A_t$ follow AR(1) processes:

$$\hat{\theta}_t = \rho \hat{\theta}_{t-1} + \epsilon^\theta_t,$$

$$\hat{A}_t = \rho A_{t-1} + \epsilon^A_t.$$

Innovation process $\epsilon_t = [\epsilon^\theta_t, \epsilon^A_t]^T$ is Gaussian with $E[\epsilon_t] = 0$, $E[\epsilon_t \epsilon'_s] = 0$, $E[\epsilon_t \epsilon'_t] = \Sigma_\epsilon$ and

$$\Sigma_\epsilon = \begin{bmatrix} \sigma^2_{\theta} & 0 \\ 0 & \sigma^2_A \end{bmatrix}.$$

The estimation exercise is to back out $\sigma_A$, $\sigma_\theta$, $\rho_A$ and $\rho_\theta$, using Bayesian methods. The detail on estimation will become clear in the business cycle analysis.
Table 1.2: Steady State: Calibrated Benchmark
Each row represents a particular vintage. \( q \): shadow prices. \( \frac{k}{k+b} \): leverage. \( K \): total capital stock. \( B \): total bond assets. \( f \): the number of entrepreneurs who go to vintage \( i \) over the total number of entrepreneurs who draw \( z_i \) (vintage \( i \) productivity). “Binding” indicates whether the borrowing constraint is binding for entrepreneurs who are going to a specific vintage.

<table>
<thead>
<tr>
<th>Vintage</th>
<th>( q )</th>
<th>( \frac{k}{k+b} )</th>
<th>Binding?</th>
<th>( K )</th>
<th>( B )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.5330</td>
<td>Yes</td>
<td>33.419</td>
<td>-11.620</td>
<td>100%</td>
</tr>
<tr>
<td>1</td>
<td>0.9228</td>
<td>1.4680</td>
<td>No</td>
<td>2.165</td>
<td>-0.690</td>
<td>100%</td>
</tr>
<tr>
<td>2</td>
<td>0.9211</td>
<td>1.4454</td>
<td>No</td>
<td>2.025</td>
<td>-0.624</td>
<td>100%</td>
</tr>
<tr>
<td>3</td>
<td>0.9194</td>
<td>1.4230</td>
<td>No</td>
<td>1.894</td>
<td>-0.563</td>
<td>100%</td>
</tr>
<tr>
<td>4</td>
<td>0.9175</td>
<td>1.4009</td>
<td>No</td>
<td>1.771</td>
<td>-0.507</td>
<td>100%</td>
</tr>
<tr>
<td>5</td>
<td>0.9155</td>
<td>1.3789</td>
<td>No</td>
<td>1.656</td>
<td>-0.455</td>
<td>100%</td>
</tr>
<tr>
<td>6</td>
<td>0.9133</td>
<td>1.3572</td>
<td>No</td>
<td>1.549</td>
<td>-0.408</td>
<td>100%</td>
</tr>
<tr>
<td>7</td>
<td>0.9110</td>
<td>1.3357</td>
<td>No</td>
<td>1.449</td>
<td>-0.364</td>
<td>100%</td>
</tr>
<tr>
<td>8</td>
<td>0.9085</td>
<td>1.3143</td>
<td>No</td>
<td>1.355</td>
<td>-0.324</td>
<td>100%</td>
</tr>
<tr>
<td>9</td>
<td>0.9059</td>
<td>1.2932</td>
<td>No</td>
<td>1.267</td>
<td>-0.287</td>
<td>100%</td>
</tr>
<tr>
<td>10</td>
<td>0.9030</td>
<td>1.2723</td>
<td>No</td>
<td>1.185</td>
<td>-0.254</td>
<td>100%</td>
</tr>
<tr>
<td>11</td>
<td>0.9000</td>
<td>1.2516</td>
<td>No</td>
<td>0.607</td>
<td>-0.122</td>
<td>54.79%</td>
</tr>
<tr>
<td>12</td>
<td>0.9000</td>
<td>0.0000</td>
<td>No</td>
<td>0.000</td>
<td>16.218</td>
<td>0%</td>
</tr>
</tbody>
</table>

1.5.2 Interactions in the Steady State

The Calibrated Steady State

Under the calibrated parameters, there are 10 to 11 inactive quarters in the steady state. That is, entrepreneurs who turn from \( z^h \) to \( z^l \) and draw \( z^l \) for 10 quarters in a row neither buy nor sell capital during those 10 quarters (Table 1.2). When they unfortunately draws the 11th \( z^l \), one fraction of them sells the firm and saves in bonds while the other fraction decides to be inactive for another quarter. For those who still run firms but draw a 12th \( z^l \), they liquidate the entire firm and save the revenue in bonds until they become productive again.

As predicted, the real option value of capital decreases as the vintage number increases, which shows directly the reduced incentives to maintain the capital as a firm keeps drawing \( z^l \) and waiting. Meanwhile, the borrowing constraint only binds when firms invest. Once a firm draws \( z^l \), the financial constraint is slack since the firm pays down existing debt.
To illustrate, suppose entrepreneur $j$ has one unit of capital and was investing and borrowing before. Then her bond position is $-\theta(1 - \delta)(1 - d)/R$. Unfortunately, $j$ draws 11 quarters of $z^l$ in a row from time $t = 1$ on. In the 12th quarter ($t = 12$), $j$ draws $z^l$ again and decides to liquidate the entire firm. After that, $j$ keeps drawing two $z^l$ for quarters 13 and 14 but draws $z^h$ afterwards.

$j$ lets the capital depreciate in the first 11 quarters and liquidates it in the 12th quarter (firm dynamics in Figure 1.5), i.e., capital at the beginning of the 13th quarter is 0. During the inactive investment process, debt is being paid and leverage decreases. After liquidation, $j$ saves only in bonds and consume $(1 - \beta)$ of the bond value. Importantly, the leverage evolution before selling is similar to Figure 1.2.
\textit{j} continues to hold bonds until drawing \textit{z}^h again in the 15th quarter. Then, she uses her net worth as a down payment to borrow and invest. Though she borrows to the limit, capital stock after investing is less than one, the amount \textit{j} started with. The firm size is not as large as before because \textit{j} does not have enough resources to expand. Her business was not profitable under \textit{z}^l technology and capital was sold at a discount before. If \textit{j} keeps drawing \textit{z}^h, she can continue investing and capital stock can gradually go back to one.

\textbf{Delayed Reallocation and Aggregate TFP}

To examine the interactions of asset illiquidity and financing frictions, I vary $\theta$ in the $d > 0$ economy to see the changes of aggregate total factor productivity (TFP). The exercises can be thought of as comparing aggregate TFP across countries with different financing frictions but with the same degree of asset illiquidity. Then, I redo the exercise for the $d = 0$ economy. After that, I can compare how much the asset illiquidity can contribute to aggregate TFP losses. As $\theta$ becomes smaller, one can see how a tightening funding liquidity (a smaller $\theta$) has different impacts on the two economies. Such a comparison reveals the interaction of the two frictions in the steady state.

\textit{The $d > 0$ Economy.} Let $\theta$ decrease from $+\infty$ to 0. The $d$ economy features no borrowing constraint when $\theta = \theta > \theta_1^d = 0.6540$. $z^h$ firms have enough credit to reallocate all available capital from $z^l$ firms. For the calibrated benchmark $d$, every $z^l$ owners liquidate their firms when $\theta$ is above $\theta_1^d$. Capital stock is fully under $z^h$ technology and thus aggregate TFP equals $z^h$. Notice that if $d$ is large enough, $z^l$ owners may not sell their capital, even if $\theta$ is very large.

When $\theta$ reaches $\theta_2^d = 0.6214$, some previous $z^h$ entrepreneurs who just drew $z^l$ start to hold capital for one period (Figure 1.6). When $\theta_2^d = 0.6214$, the inaction region is the line that is the same as the borrowing constraint line in the $z^l$ plain (recall Figure 1.3b). $z^h$ owners invest and borrow to the limit; when turned into $z^l$ owners, their leverage ratio is the one under the borrowing constraint. As $\theta$ becomes even smaller, the inaction region
starts from a line to a fan as in Figure 1.3b. Persistently unlucky \( z^l \) owners wait longer and longer before selling capital. Capital reallocation is thus less and aggregate TFP is smaller (Figure 1.6).

When \( \theta = \theta_3^d = 0.3689 \), the secondary market shuts down so that no single \( z^l \) owner sells capital (Figure 1.7). In addition, all entrepreneurs save through running firms regardless of their productivity. The reason is that a larger degree of financial frictions further limit reallocation and borrowing. Therefore, both wage rate and risk-free interest rate will be low, i.e., \( \pi \) will tend to be large and \( R \) will tend to be small. When \( \theta < \theta_3^d \), the condition \( R' \geq z^l \pi' + (1 - \delta) \) under which \( z^l \) owners do not invest is no longer satisfied. Therefore, \( z^l \) owners always find investing in capital stock better than saving in bonds. The economy thus is characterized by autarky allocation (Figure 1.7) and no productivity risk-sharing exists through the financial market.

\[ \theta \in [0, \theta_3^d] \] is an extreme interaction between asset illiquidity and financial constraints. Asset illiquidity delays liquidation. Tighter borrowing constraints prolong the delay. Once the profit rate is high enough and the interest rate is low enough due to large financial

Figure 1.6: Aggregate TFP Losses and Waiting Periods
Steady state TFP and waiting periods as only \( \theta \) changes and when the steady state has capital reallocation. The red solid line denotes the waiting periods \( N + 1 \). The blue dashed line denotes aggregate TFP. Aggregate TFP is the percentage of \( \theta^h \).
frictions, no liquidation takes place and the credit market effectively shuts down. Therefore, the important message is that both markets can shut down together if the two frictions interact. Then the economy is the same as the $d = 0$ economy with $\theta = 0$, even though $d > 0$ and $\theta > 0$ ($\theta^d_3$ is still far from zero, which exemplifies the interaction).

**Figure 1.7: Aggregate TFP Losses in $d > 0$ and $d = 0$ economy**

Aggregate TFP as percentage of $\tilde{z}$ in the steady state, when only $\theta$ changes. The red solid line: $d > 0$ economy. The blue dashed line: $d = 0$ economy.

To summarize, when $\theta \in [\theta^d_2, \theta^d_1)$, the economy is inefficient only because investment from $z^h$ firms is constrained by financial frictions. When $\theta \in [\theta^d_3, \theta^d_2)$, $z^l$ owners delay selling and the economy is inefficient because of insufficient investment from $z^h$ firms and insufficient reallocation from $z^l$ firms. Finally, when $\theta \in [0, \theta^d_1)$, both secondary market and credit market shut down.

*The $d = 0$ Economy.* As a comparison, there is no inactive investment decisions in the $d = 0$ economy. The stationary economy can be characterized by two cut-offs, $0.5575 = \theta^0_2 < \theta^0_1 = 0.7121$. The financial constraint is slack when $\theta \geq \theta^0_1$. Because of constant return to scale technology, only a zero measure of firms operate. The rest of entrepreneurs
enjoy returns on bonds and leisure utility. When \( \theta \) decreases in the region \([\theta_2^0, \theta_1^0]\), more and more \( z^h \) firms produce.

When \( \theta \in [0, \theta_2^0] \), a fraction of \( z^l \) owners produces and TFP is less than \( z^h \). TFP is lower when \( \theta \) decreases in this region because more and more \( z^l \) firms produce. Notice that there is no asset illiquidity \((d = 0)\) so that the return on capital stock is risk-free. The return must be higher than interest rate, otherwise \( z^l \) owners will not operate and enjoy extra leisure. Therefore, these existing \( z^l \) firms will borrow to the credit limit.

To summarize, there is no delay of selling in \( d = 0 \) economy. When \( \theta \in [\theta_2^0, \theta_1^0] \), the economy is inefficient only because investment from \( z^h \) firms is constrained by financial frictions. When \( \theta \in [0, \theta_2^0] \), the economy is inefficient in two ways: not enough investment from \( z^h \) firms and not enough reallocation from \( z^l \) firms.

How much are the TFP losses from steady state when \( d = 0 \) changes to \( d = 0.1 \)? The answer obviously depends on what \( \theta \) the economy has (Figure 1.7). In the calibrated \( d = 0.1 \) economy, the TFP losses increase by almost 25%. In percentage terms of \( z^h \), the largest TFP losses are the following two cases. First, about 1.5% of \( z^h \) more losses when \( \theta = \theta_2^0 \). \( z^l \) owners produce in the \( d > 0 \) economy, but not in the \( d = 0 \) economy. Second, about 2.5% of \( z^h \) more losses when \( \theta = \theta_2^d \). The secondary market shuts down in the \( d > 0 \) economy but not in the \( d = 0 \) economy. TFP losses in other regions are typically from 0.5% to 1.5% of \( z^h \).

Such TFP losses are large and significant compared to the literature on financial frictions’ impact on capital misallocation.\(^{29}\) Given a degree of financial frictions, asset illiquidity can add losses of 0.5% to 1.5% of the efficient economy aggregate TFP (\( z^h \)). In the extreme case, there is about 2.5% more losses when borrowing is allowed but no lending is available (when both credit market and secondary market are effectively shut down). The

\(^{29}\)For example, Midrigan and Xu (2012) found that misallocation results in TFP losses of only about 0.3% in the benchmark calibrated economy and at most 5% when the credit market completely shuts down. Similarly, in Moll (2010) the magnitude of TFP losses depends on the persistence of idiosyncratic productivity shocks.
Table 1.3: Priors and Posteriors

“Prior s.d.” denotes the standard deviation of the prior. “Post mean” denotes the posterior mean. “5%” and “95%” denote the 5 and 95 percentile. Posteriors are drawn using Markov Chain Monte Carlo (MCMC) methods such as in An and Schorfheide (2007).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior Distribution</th>
<th>Prior mean</th>
<th>Prior s.d</th>
<th>Post mean</th>
<th>5%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_A )</td>
<td>Inverse Gamma</td>
<td>0.01</td>
<td>1</td>
<td>0.0045</td>
<td>0.0041</td>
<td>0.0049</td>
</tr>
<tr>
<td>( \sigma_\theta )</td>
<td>Inverse Gamma</td>
<td>0.01</td>
<td>0.05</td>
<td>0.0115</td>
<td>0.0103</td>
<td>0.0129</td>
</tr>
<tr>
<td>( \rho_A )</td>
<td>Beta</td>
<td>0.9</td>
<td>0.05</td>
<td>0.8721</td>
<td>0.8297</td>
<td>0.9227</td>
</tr>
<tr>
<td>( \rho_\theta )</td>
<td>Beta</td>
<td>0.9</td>
<td>0.05</td>
<td>0.9701</td>
<td>0.9472</td>
<td>0.9873</td>
</tr>
</tbody>
</table>

studies in the literature are thus sensitive to the introduction of asset illiquidity, a common phenomenon in the secondary market.

1.5.3 Interactions During Business Cycles

Returning to the cycle properties of capital reallocation, I experiment with standard aggregate TFP shocks and credit crunch shocks. With large aggregate shocks, the model becomes intractable because the number of vintages changes after large shocks, leaving complex dynamics to solve. Instead, I focus on small aggregate shocks such that the equilibrium vintages do not change. I solve the dynamics around the steady state using first-order perturbation methods. Then I verify that the shocks are small enough through the response of the fraction \( f^{N+1}_t \) of entrepreneurs that stay in vintage \( N = 10 \). If \( f^{N+1}_t \) is still less than 1, the vintages do not change.

Estimation Results

I use the HP-filtered cyclical components of real reallocation and real GDP data from 1984Q1 to 2011Q4 to estimate the standard deviation and the persistence parameters \( \rho_\theta \) and \( \rho_A \). I apply Bayesian methods to estimate the standard deviation and the persistence of the shocks, as standard in the DSGE model estimation.\(^{30}\) Prior and posterior information is in Table 1.3 and Figure 1.13.

\(^{30}\)See, for example Smets and Wouters (2003) and An and Schorfheide (2007).
I use the mean estimator for cycle analysis. Using the mode estimator will not change the result much since the mean and the mode are close to each other (Figure 1.13). There are several features of the mean estimators. The standard deviation of aggregate TFP shocks (shocks to \( A \)) is 0.45%, which is close to the estimation results found in the literature such as in Thomas (2002) (with 0.53%). Second, the size of the credit shocks (about 1.15%) is even larger than aggregate TFP shocks (0.45%). Finally, credit shocks (\( \rho_\theta = 0.9701 \)) are more persistent than TFP shocks (\( \rho_A = 0.8721 \)).

Even though I only use the two observed series (output and reallocation) for estimation (to avoid stochastic singularity issues because I focus on two shocks), the estimated aggregate TFP shocks and financing constraints shocks generate key business cycle statistics that are close to the data (Table 1.9).

**Financial Shocks and Aggregate Productivity Shocks**

Figure 1.8 show the impulses to a one standard deviation (1.15%) credit shocks and a one standard deviation (0.45%) aggregate productivity shocks.

In response to credit shocks, tightened financing constraints largely reduce the investment from \( z^h \) firms. Demand for labor shrinks and real wage rate decreases in equilibrium. Running firms now has lower labor input costs (therefore \( \pi \) increases). In response to lower input costs, more \( z^l \) firms delay selling assets. More selling delays lead to less reallocation and thus a larger TFP dispersion across firms. The direct consequence is that aggregate TFP is smaller and total output drops. As for the debt, there is persistent and sizable deleveraging. Though \( z^h \) firms can no longer raise as much debt as before, inactive \( z^l \) owners pay back more debt by shrinking consumption. After financial shocks, the reduced reallocation and the increased dispersion of TFP across firms are in line with the data.

Output and TFP responses are sizable given the small credit crunch that does not change the number of equilibrium vintages. This result is under the assumption that vintage number \( N \) does not change. Since the correlation between reallocation and output is lower in the
Figure 1.8: Experiment: Responses to two types of shocks
Responses to one standard deviation of negative financial shocks (shocks to $\theta$) and negative aggregate productivity shocks (shocks to $A$). Reallocation: capital reallocation. TFP Std: standard deviation of firm-level TFP employed. Aggregate TFP: the Solow residuals after adjusted by $A$ changes. The solid line denotes the response to financial shocks while the dashed line denotes the response to aggregate productivity shocks.

In response to aggregate productivity shocks, debt level changes little (i.e., a magnitude of 0.1%) compared to credit shocks. More importantly, responses have at least two aspects model than in the data (Table 1.9), if we can estimate under an endogenous $N$, the standard deviation of credit shocks should be larger than the mean estimator because it will increase $N$ (also because adverse $A$ shocks will reduce reallocation as will be clear soon). Therefore, the credit crunch in reality should be larger. In a larger credit crunch, the number of vintages can suddenly increase (capital reallocation suddenly disappears) and production efficiency is suddenly reduced.
that are not observed. First, capital reallocation is more initially. Since aggregate productivity drops, the profit rate of investing in capital is down (π_t responses). The z^l owners thus have less incentive to hold capital, and more capital is liquidated. Second, compared to the economy before the shocks, fewer z^l owners stay to operate firms such that the measured TFP dispersion is slightly smaller in recessions.

I have shown responses to a one-time financial shock and aggregate TFP shock. Financial shocks increases the return from running firms, which induces less capital reallocation from inefficient firms. However, aggregate TFP shocks generate the opposite dynamics. Though these exercises are impulses, they shed light on why aggregate TFP shocks might not be able to capture capital reallocation dynamics. In what follows, I confirm the intuition learned from impulse responses.

**Simulations**

The key for less reallocation in recessions is whether shocks can delay capital selling from z^l firms. To examine more thoroughly the reallocation-output co-movement and TFP dispersion-output co-movement, I simulate the model (i.e., financial shocks or aggregate productivity shocks repeatedly hit the economy), using parameters from the estimation. Table 1.4 shows the correlation of the key variables and output, using one type of shocks each time.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Co-movement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>Standard deviation to that of output</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Output</th>
<th>Reallocation</th>
<th>Reallocation</th>
<th>TFP dispersion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data:</td>
<td>1.42%</td>
<td>10.91</td>
<td>0.85</td>
</tr>
<tr>
<td>Model:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Only financial shocks</td>
<td>1.38%</td>
<td>11.03</td>
<td>0.83</td>
</tr>
<tr>
<td>Only aggregate TFP shocks</td>
<td>1.31%</td>
<td>9.11</td>
<td>0.18</td>
</tr>
</tbody>
</table>

46
Table 1.5: **Variance Decomposition**

<table>
<thead>
<tr>
<th></th>
<th>Output</th>
<th>Reallocation</th>
<th>Investment</th>
<th>TFP dispersion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Financial shocks</td>
<td>23.88%</td>
<td>99.29%</td>
<td>74.38%</td>
<td>99.23%</td>
</tr>
<tr>
<td>Aggregate TFP shocks</td>
<td>76.12%</td>
<td>0.71%</td>
<td>25.62%</td>
<td>0.77%</td>
</tr>
</tbody>
</table>

First, reallocation is more volatile in the economy with only financial shocks. From the impulse responses, aggregate TFP shocks have the opposite effects on reallocation. That is why we should observe a more volatile reallocation in responses to only financial shocks.

Second, aggregate TFP shocks generate a positive correlation between reallocation and output. After one-time aggregate TFP shock, eventually capital available for reallocation will be less, as in the impulse responses in Figure 1.8. Nevertheless, TFP dispersion shrinks in recessions from aggregate TFP shocks since more firms are liquidating, as in Figure 1.8.

To further decompose the effects from financial shocks and aggregate productivity shocks, I decompose the variance of reallocation and output explained by each type of shocks, using the mean estimators from the Bayesian exercise. As in Table 1.5, almost all the reallocation and TFP dispersions fluctuations are caused by financial shocks. In addition, financial shocks can also explain a large portion of the variation in investment and output. This result is because: (1) financial shocks lead to changes of “measured” aggregate TFP; (2) aggregate TFP shocks, similar as a neoclassical growth model, explain most of the output and a large portion of the investment. Therefore, it is not surprising that financial shocks can also explain a large portion of the variation in investment and output.

Finally, I apply Kalman smoother to reconstruct the implied financial shocks and aggregate productivity shocks conditional on the whole sample (Figure 1.9). The adverse financial shocks are particularly important during the 2008 recession. In addition, the shocks are relatively large during the 1990 recessions but quite mild during the 2000 recessions. If we relate the financial shocks to the capital reallocation time series in Figure 1.1, the drops of reallocation are large during 1990 and 2008 recessions but small during the 2000 one.
In summary, one needs both aggregate TFP shocks and credit crunch shocks to generate consumption, investment, and output dynamics as in Table 1.9; however, to capture both procyclical reallocation and countercyclical TFP dispersion, financial shocks are necessary. Therefore, dynamics of capital reallocation and the TFP dispersion in the data provide us some useful identification of the source(s) of business cycles.

1.6 Discussion

The interactions between asset illiquidity and financial frictions can be directly seen from the waiting periods in the steady state. Without asset illiquidity, there is no inactive investment decisions so that there is no waiting periods. Without borrowing constraints, $\zeta^h$ firms can borrow as much as possible to reallocate assets. The number of waiting periods is small and equal zero in our calibrated $d_0$ economy. Thus, in order to generate prolonged capital
reallocate delay during recessions, the interactions between the two frictions are the key ingredients.

In reality, recessions might originate from both aggregate TFP shocks and financial shocks. The relative importance, however, changes from recession to recession. The recent credit crunch since 2008 exemplifies a huge drop in $\theta$. Less capital reallocation and slow deleveraging\(^{31}\) are more significant than in past recessions. It is therefore reasonable to believe that financial shocks are essential in 2008 recessions and also important in previous recessions. Policy targeted at secondary market illiquidity should be able to help reverse the adverse shocks.

Importantly, the exercise does not imply that financial shocks are the only primitive shocks for recessions. Both aggregate TFP shocks and financial shocks are needed to generate business cycle statistics as in Table 1.9. Instead, this paper shows that if the economy features asset illiquidity, financial shocks are necessary to generate less capital reallocation and larger TFP dispersion during recessions.

Finally, this paper does not model changes of illiquidity. The first reason is that if illiquidity comes from asymmetric information, some good quality assets might be forced to be liquidated in recessions and mitigate the information problem as in Eisfeldt (2004). The second reason is that, if the increases of illiquidity are all because of fire-sale of real assets as in Shleifer and Vishny (1992), the larger TFP dispersion during recessions is hard to be justified. Fire-sale theories suggest the most efficient firms of using the assets are also in financial troubles, which should lead to a smaller TFP dispersion. The last and probably the most important one is that if the illiquidity can be amplified, then this paper proposes one cause for the initial drop of asset liquidity: a credit crunch can reduce the number of buyers and sellers simultaneously.\(^{31}\)

\(^{31}\)See Shirakawa (2012) and Koo (2011) for the evidence.
1.7 Final Remark

This paper begins with two empirical facts: (1) capital reallocation is procyclical, but the benefits to reallocate are countercyclical; (2) firms without surviving problems shrink liabilities relative to assets before selling assets. These two observations can be generated in the model with asset illiquidity and financing constraints in response to shocks to financing constraints, instead of aggregate TFP shocks. When negative financial shocks hit, inefficient firms are more willing to hold assets because of a lower input costs (a lower wage rate) and because of a lower interest rate. Therefore, financial shocks are important not only during the 2008 recession but also during previous ones.

The challenge to link individual firm’s asset liquidation and aggregate capital reallocation is the complex distribution of firms. I model the selling decision as a stopping-time problem that turns out to simplify the aggregate distribution dramatically. Meanwhile, the real option value of capital stock before liquidation shed some light on how firms price their assets internally.

One future prospect is how the resale costs endogenously interact with the depth of asset markets. The asset specificity costs, in that case, come from matching between buyers and sellers. Sellers may find it costly to search potential buyers, especially during downturns. In contrast, asset markets are generally deeper in economic booms. The resale discounts are smaller in boom times and delayed selling by inefficient firms is reduced. A better allocation of assets will deepen asset markets further, and labor market conditions will improve too. Therefore, policy targeted at the resale market depth may have a large effect by improving the efficiency of asset allocation and labor market. This channel may also shed light on unemployment issues and labor input costs for firms.
1.8 Appendix

1.8.1 Data Description

For capital reallocation, the quarterly COMPUSTAT contains useful information for ownership changes of productive assets from 1984Q1. Following Eisfeldt and Rampini (2006), who use annual COMPUSTAT data from 1971, I measure capital reallocation by sales of property, plant and equipment (SPPE, data item 107 with combined data code entries excluded), plus acquisitions (AQC, data item 129 with combined data code entries excluded). The measure captures transactions after which the capital is used by a new firm and new productivity thus applied. The advantage of using quarterly data compared to annual data is more observations. However, quarterly data is shown in the “cash flow statement” and there is a substantial seasonal pattern. Therefore, I apply seasonal adjustment to the data.

For debt-to-asset ratio of companies before selling assets, I merge quarterly COMPUSTAT and SDC file. SDC file contains merger and acquisition of all U.S. firms. I get information of all the companies who sold at least 50% of their assets in the SDC file after 2000 until the most recent available date (currently April 2012), then keep firms with information in COMPUSTAT and delete those who sell multiple times in the sample periods. The merging of COMPUSTAT and SDC allows me to trace back the leverage of companies before they sell assets.

For aggregate consumption, investment, and GDP, I obtain the data from FRED, a macroeconomic dataset managed by Federal Reserve Bank at St. Louis. Note that I exclude residential investment, consumer durable goods, government expenditure, and net export because the model abstract from these components.
1.8.2 Lemma 1

I prove a general result under general CRRA utility. First, define the Bellman operator $T$:

$$T V(k, b, a; X) = \max \{ W^1(k, b, a; X), W^0(k, b, a; X) \}$$

$$W^1(k, b, a; X) = \max_{k' > 0, R' b' \geq -\theta (1 - d) (1 - \delta) k'} u(z \pi k + R b - \psi(k', k) - b') + \beta E[V(k', b', a'; X') | a, X]$$

$$W^0(k, b, a; X) = \max_{b'} \{ u(z \pi k + R b + (1 - \delta) (1 - d) k - b') + \eta + \beta E[V(0, b', a'; X') | a, X] \}$$

The value function is the fixed point of the contraction mapping in the space $\mathcal{V}_1$ of well defined functions as in Stokey, Lucas, and Prescott (1989). Further, the Bellman operator $T$ is closed on the class of functions $\mathcal{V}_1$ satisfying the properties in the Lemma. I simplify notation by

$$w^1(k, b, k', b', a; X) = u(k, b, k', b', a; X) + \beta E[V(k', b', a'; X') | a, X]$$

$$w^0(k, b, k', b', a; X) = u(k, b, 0, b', a; X) + \eta + \beta E[V(0, b', a'; X') | a, X]$$

with slight abuse of notation of utility function $u(\cdot)$.

(i) Increasing in $a$, $k$ and $b$ and concavity


(ii) $V(\gamma k, \gamma b, a; X) = \gamma^{1 - \sigma} V(k, b, a; X) + \frac{\gamma^{1 - \sigma - 1}}{1 - \beta} + (1 - \gamma^{1 - \sigma}) \eta (1 - h)$

I will prove $T \mathcal{V}$ has the same property. Consider an agent with state $(k, b, a)$ and $(k', b')$ is the optimal policy. For any $\gamma > 0$, when the state is $(\gamma k, \gamma b, a)$, the policy $(\gamma k', \gamma b')$ are feasible, i.e., it satisfies budget and borrowing constraints. Therefore, given an consistent
Combining the two gives

\[ \mathcal{T}V(\gamma k, \gamma b, a; X) \geq w^h(\gamma k, \gamma b, \gamma k', \gamma b'; a; X) \]

\[ = (z \pi k + Rb - \psi(k', k) - b')^{1-\sigma} \frac{\gamma^{1-\sigma} - 1}{1-\sigma} + \eta(1-h) \]

\[ + \beta \gamma^{1-\sigma} E[V(k', b', a'; X')|a, X] + \beta \frac{\gamma^{1-\sigma} - 1}{1-\beta} \]

\[ = \gamma^{1-\sigma} [u(k, b, k', b'; X) + \eta (1-h)] + \gamma^{1-\sigma} \frac{1-\sigma}{1-\beta} + (1 - \gamma^{1-\sigma}) \eta (1-h) \]

\[ + \beta \gamma^{1-\sigma} E[V(k', b', z'; X')|a, X] + \beta \frac{\gamma^{1-\sigma} - 1}{1-\beta}, \]

and thus

\[ \mathcal{T}V(\gamma k, \gamma b, z; X) \geq \gamma^{1-\sigma} \mathcal{T}V(k, b, z; X) + \frac{\gamma^{1-\sigma} - 1}{1-\beta} + (1 - \gamma^{1-\sigma}) \eta (1-h). \]

Conversely, starting at \((\gamma k, \gamma b, a)\), scaling by \(1/\gamma\), and following similar procedure above, one has

\[ \mathcal{T}V(k, b, a; X) \geq (1/\gamma)^{1-\sigma} \mathcal{T}V(\gamma k, \gamma b, a; X) + \frac{(1/\gamma)^{1-\sigma} - 1}{1-\beta} + (1 - (1/\gamma)^{1-\sigma}) \eta (1-h). \]

Combining the two gives

\[ \mathcal{T}V(\gamma k, \gamma b, a; X) = \gamma^{1-\sigma} \mathcal{T}V(k, b, a; X) + \frac{\gamma^{1-\sigma} - 1}{1-\beta} + (1 - \gamma^{1-\sigma}) \eta (1-h). \]

Note that when \(\sigma = 1\), \(\mathcal{T}V(\gamma k, \gamma b, a; X) = \mathcal{T}V(k, b, a; X) + \frac{\log \gamma}{1-\beta}\). Finally, the difference between \(V(\gamma k, \gamma b, a; X)\) and \(V(k, b, a; X)\) does not depend on the fixed costs because

\[ V(\gamma k, \gamma b, a; X) - V(k, b, a; X) = (\gamma^{1-\sigma} - 1)(V - \eta (1-h)) + \frac{\gamma^{1-\sigma} - 1}{1-\beta}. \]

(Noticing that \(V - \eta (1-h)\) does not depend on \(\eta\)).
1.8.3 Lemma 2 and Proposition 1

The differentiability of $V(k, b, a; X)$ when $k' \geq (1 - \delta)k$ is trivial, which relies on the differentiability of standard dynamic programming problem as proved by Benveniste and Scheinkman (1979) or Stokey, Lucas, and Prescott (1989). Next, I prove the differentiability of $V(k, b, a; X)$ when $k' = (1 - \delta)k$.

I follow methods from Clausen and Strub (2012) in Banach space (the space of $k$ and $b$) and adjust to the dynamic programming problem in this paper. The general idea is that the value function is the upper envelop of value function of buying, inactive and selling. It is therefore super-differentiable. At the same time, it has potential downward kink (sub-differentiable) because of $\psi(k', k)$ function. Therefore, the value function will be both super-differentiable and sub-differentiable, and therefore differentiable. I omit the rest of the proof because it is standard in the convex analysis. But the details can be found in the same titled paper on my website.

1.8.4 Lemma 3

(i) To save notation, I abstract from aggregate state variable $X$. From the proof in Lemma 1,

$$V(\gamma(k + e), \gamma b, a) = \gamma^{1 - \sigma}V(k + e, b, a) + \frac{\gamma^{1 - \sigma} - 1}{1 - \beta} + (1 - \gamma^{1 - \sigma})\eta(1 - h).$$

Take a derivative with respect to $e$ and evaluate it at $e = 0$; one has $\gamma V_k(\gamma k, \gamma b) = \gamma^{1 - \sigma}V_k(k, b)$. Divide $\gamma$ on both sides and one can prove that $V_k$ is homogeneous with degree $-\sigma$. When $\sigma = 1$, $V$ is homogenous with degree -1 as in the main text.

(ii) Consider two entrepreneurs with $(k_0, b_0, a)$ and $(\gamma k_0, \gamma b_0, a)$. Using equation (1.5) of Lemma 1, the targeted capital stock and bonds are scaled up by $\gamma$ and thus the optimal consumption choices are $c_0$ and $\gamma c_0$ from the budget constraints. Therefore, using property (1) of this Lemma, $V_k/u'(c)$ is the same for the two entrepreneurs. More generally, $V_k/u'(c)$ depends only on $k/(k + b)$. 
(iii) By definition, \( q(k, b, a; X) = \left( \frac{V_k}{u'(c)} - z \pi \right) (1 - \delta)^{-1} \). Using (2), we know that \( q(k, b, a; X) \) can be written as \( q(k, b, a; X) \).

### 1.8.5 Proposition 2

I prove the policy functions for general CRRA utility. Using the net worth definition, I propose the following solution:

\[
V(k, b, a; X) = J(a; X) + \frac{(g(a; X)n(k, b, a; X))^{1-\sigma} - 1}{1 - \sigma} \frac{1}{1 - \beta},
\]  

(1.19)

and the associated policy functions

\[
c(k, b, a; X) = (1 - s(k, b, a; X))n(k, b, a; X)
\]

\[
qk'(k, b, a; X) = \phi(k, b, a; X)s(k, b, a; X)n(k, b, a; X)
\]

\[
b'(k, b, a; X) = (1 - \phi(k, b, a; X))s(k, b, a; X)n(k, b, a; X)
\]

(1.20)

where \( J, g, \phi \) and \( s \) are to be determined. Note that, \( s \) is the saving rate, \( \phi \) is the portfolio weight on capital, and

\[
\frac{V_k(k, b, a; X)}{u'(c)} = z \pi + q(1 - \delta), \quad \frac{V_b(k, b, a; X)}{u'(c)} = R.
\]

The first-order conditions with respect to \( k' \) and \( b' \) give:

\[
q c^{-\sigma} = \beta E \left[ \frac{(g')^{1-\sigma}(n')^{-\sigma}}{1 - \beta} \left( z' \pi' + (1 - \delta) q' \right) |a, X \right] + \mu \theta (1 - \delta)(1 - d)
\]

(1.21)

\[
c^{-\sigma} = \beta E \left[ \frac{(g')^{1-\sigma}(n')^{-\sigma}}{1 - \beta} R' |a, X \right] + \mu R',
\]

(1.22)
where $\mu$ is the Lagrangian multipliers attached to the borrowing constraint. When $\mu = 0$, multiply (1.21) by $\frac{\phi}{q}$ and (1.22) by $(1 - \phi)$, and then sum them up, we have

$$c^{-\sigma} = \beta E \left[ \frac{(g')^{1-\sigma}(n')^{-\sigma}}{1 - \beta} (\phi r' + (1 - \phi)R') | a, X \right], \quad (1.23)$$

where $r' = \frac{z'\pi' + (1 - \delta)q'}{q}$. When $\mu > 0$, we know that $\phi = \frac{1}{1 - \theta(1 - d)(1 - \delta)/qR}$ from the borrowing constraint $R'b' = -\theta(1 - \delta)(1 - d)k'$. Again multiply (1.21) by $\frac{\phi}{q}$ and (1.22) by $(1 - \phi)$, and then sum them up, we still have equation (1.23) because the part that has $\mu$ is cancelled out.

Next, notice that the envelope condition under the proposed value function is

$$V_k = \frac{g^{1-\sigma}n^{-\sigma}}{1 - \beta} (z\pi + (1 - \delta)q),$$

from which one has $\frac{g^{1-\sigma}n^{-\sigma}}{1 - \beta} = c^{-\sigma}$. Together with

$$n' = z'\pi'k' + q' (1 - \delta)k' + R'b' = [\phi r' + (1 - \phi)R'] sn \equiv \rho'sn, \quad (1.24)$$

equation (1.23) can be rewritten as

$$(1 - s)^{-\sigma} = \beta E[(1 - s')^{-\sigma}s^{-\sigma}(\rho')^{1-\sigma} | a, X], \quad (1.25)$$

- $\sigma = 1$, i.e., log utility

Equation (1.25) is simplified to be

$$\frac{s}{1 - s} = E \left[ \frac{\beta}{1 - s'} | a, X \right].$$
For convenience, let me temporarily get the time subscript back. After recursive substitution,

\[
\frac{s_t}{1-s_t} = \beta + \beta^2 + \ldots + \beta^j E_t \left[ \frac{s_{t+j}}{1-s_{t+j}} \right].
\]

Notice that \( s \in (0, 1) \), i.e., it is not optimal to save everything \((s = 1)\) or consume everything \((s = 0)\). Otherwise, the marginal utility of today or tomorrow will go to infinity because of CRRA utility assumptions. Then \( E_t \left[ \frac{s_{t+j}}{1-s_{t+j}} \right] \) is bounded by some positive numbers. Let \( j \to \infty \) and the solution is \( s_t = \beta \). Once the consumption choice is fixed, i.e., \( s = \beta \), \( \phi \) should be picked accordingly to solve equation (1.21) and (1.22), i.e.,

\[
\begin{align*}
E \left[ \frac{r' - R'}{\phi + (1-\phi)R'} \right] | a, X] &= 0, \quad \text{if } E \left[ \frac{r' - R'}{(1-\phi)R'} \right] | a, X] = 1 \\
\phi &= \frac{1}{1-\theta(1-\delta)(1-d)/qR'}, \quad \text{if } E \left[ \frac{r}{(1-\phi)R'} \right] | a, X] < 1.
\end{align*}
\]

\( \bullet \) \( \sigma \neq 1 \)

From equation (1.25), and the difference of equation (1.21) and equation (1.22), \( \phi \) and \( s \) jointly solve the recursive simultaneous equations:

\[
E \left[ \beta \left( \frac{1-s}{1-s'} \right) \rho' \right] | a, X] = 1
\]

\[
E \left[ \beta \left( \frac{1-s}{1-s'} \right) (r' - R') \right] | a, X] = 0, \quad \text{if } E \left[ \beta \left( \frac{1-s}{1-s'} \right) r \right] | a, X] = 1
\]

\[
\phi = \frac{1}{1-\theta(1-\delta)(1-d)/qR'}, \quad \text{if } E \left[ \beta \left( \frac{1-s}{1-s'} \right) r \right] | a, X] < 1.
\]

Notice that, \( \beta \left( \frac{1-s}{1-s'} \right) \) is the stochastic discount factor.

\( \bullet \) \( k' = 0 \)

When the optimal choice is \( k' = 0 \), there is only one first order condition for \( b' \),

\[
c^{-\sigma} = \beta E \left[ (c')^{-\sigma} R' \right], \quad (1.26)
\]
where I use the fact that the leisure utility is a constant term and will not be shown in the first order condition for $b'$, once (1.19) is plugged into the Bellman equation. Notice that

$$n' = z'\pi'k' + q'(1 - \delta)k' + R'b' = R'sn,$$

and consumption choice in (1.20), one has

$$(1 - s)^{-\sigma} - \sigma = \beta E[(1 - s')^{-\sigma} - \sigma (R')^{1-\sigma}],$$

which is the same as that in $k' = 0$. Everything else goes through the same way by replacing $\rho' = R'$.

- Verification

Finally, I verify the proposed value function (1.20) and policy functions (1.20) solve the Bellman equation. When $k' \neq 0$, substitute (1.19) back into the Bellman equation (1.4).

$$J_t + \frac{(g_t n_t)^{1-\sigma} - 1}{1 - \beta} = \frac{(1 - s_t) n_t^{1-\sigma} - 1}{1 - \sigma} + \beta E_{t+1} \left[ J_{t+1} + \frac{(g_{t+1} n_{t+1})^{1-\sigma} - 1}{1 - \beta} \right].$$

Plug in the envelop conditions $\frac{(s')^{1-\sigma}}{1 - \beta} = (1 - s')^{-\sigma}$, one has

$$J_t + \frac{(1 - s_t)^{-\sigma} n_t^{1-\sigma}}{1 - \sigma} = \frac{(1 - s_t) n_t^{1-\sigma}}{1 - \sigma} + \beta E_{t+1} \left[ J_{t+1} + \frac{(1 - s_{t+1})^{-\sigma} n_{t+1}^{1-\sigma}}{1 - \sigma} \right].$$

Using (1.24),

$$J_t + \frac{(1 - s_t)^{-\sigma} n_t^{1-\sigma}}{1 - \sigma} = \frac{(1 - s_t) n_t^{1-\sigma}}{1 - \sigma} + \beta E_{t+1} \left[ J_{t+1} + \frac{(1 - s_{t+1})^{-\sigma} \rho_{t+1} n_{t+1}^{1-\sigma}}{1 - \sigma} \right].$$

Then use (1.25), one can simplify the above equation to be

$$J_t = \beta E_t [J_{t+1}].$$
Therefore, $J_t$ does not depend on the net-worth $n_t$. When $k' = 0$, substitute (1.19) back into the Bellman equation (1.4) by noticing that an extra leisure utility

$$J_t + \frac{(g_t n_t)^{1-\sigma}-1}{1-\sigma} = \frac{((1-s_t)n_t)^{1-\sigma}-1}{1-\sigma} + \eta + \beta E_t [J_{t+1} + \frac{(g_{t+1} n_{t+1})^{1-\sigma}-1}{1-\beta}].$$

Then following the similar steps and one has

$$J_t = \beta E_t [J_{t+1}] + \eta.$$

Again, $J_t$ does not depend on the net-worth $n_t$. Then, I verify that the guessed value function is correct and the policy functions proposed solve the Bellman equation.

1.8.6 Proposition 3: Leverage and Deleverage

(i) I prove that in equilibrium, entrepreneurs who draw $z_{t+1} = z^h$ at time $t$ invest and borrow to a common target leverage $\tilde{\lambda}_t$. First, because idiosyncratic productivity follows the two state Markov process, it is straightforward to show that these entrepreneurs will invest. Second, I show that entrepreneurs will borrow to a common leverage if they invest. For notation simplicity, I use vintage specific shadow prices and rate of return. Moreover, instead of using leverage, I use the vintage specific portfolio weight on capital $\phi^0_t$. Suppose an entrepreneur have net-worth $n_t = (z_t \pi_t + (1-\delta) q_t^0) k_t + R_t b_t$, where $q_t^0 = 1$ denotes the buying price. When the entrepreneur decides to invest, the rate of return on capital is
\[
\frac{\partial^2 \pi_{t+1} + (1-\delta)q^0_{t+1}}{q^0_t} \text{ and } \frac{\partial^2 \pi_{t+1} + (1-\delta)q^1_{t+1}}{q^1_t}. \text{ The value from investing is }
\]

\[
V^{\text{buy}} = \log((1-\beta)n_t) + \beta p^{bh} E_t \left[ \phi_t^0 \beta n_t + R_{t+1}(1-\phi_t^0)\beta n_t \right] \]

\[
+ \beta p^{hl} E_t \left[ J_{t+1}^1 + \log\left( \frac{\partial^2 \pi_{t+1} + (1-\delta)q^1_{t+1}}{q^1_t} \phi_t^0 \beta n_t + R_{t+1}(1-\phi_t^0)\beta n_t \right) \right].
\]

Now consider one-shot deviation this period by taking a different portfolio weight on capital as \(\phi_t^{0'}\). The value of such one-shot deviation is

\[
V^{\text{in}} = \log((1-\beta)n_t) + \beta p^{hh} E_t \left[ \phi_t^{0'} \beta n_t + R_{t+1}(1-\phi_t^{0'})\beta n_t \right] \]

\[
+ \beta p^{hl} E_t \left[ J_{t+1}^1 + \log\left( \frac{\partial^2 \pi_{t+1} + (1-\delta)q^1_{t+1}}{q^1_t} \phi_t^{0'} \beta n_t + R_{t+1}(1-\phi_t^{0'})\beta n_t \right) \right],
\]

for some time-varying constant \(J_{t+1}^1\) and shadow value \(q^m_{t+1}\). Therefore, the difference between these two values is

\[
V^{\text{buy}} - V^{\text{in}} = \frac{\beta}{1-\beta} p^{bh} E_t \left[ \log\left( \frac{\phi_t^0 (r_{t+1}^{0h} - R_{t+1}) + R_{t+1}}{\phi_t^{0'} (\frac{\partial^2 \pi_{t+1} + (1-\delta)q^m_{t+1}}{q^m_t} - R_{t+1}) + R_{t+1}} \right) \right] X_t
\]

\[
+ \frac{\beta}{1-\beta} p^{hl} E_t \left[ \log\left( \frac{\phi_t^0 (r_{t+1}^{0l} - R_{t+1}) + R_{t+1}}{\phi_t^{0'} (\frac{\partial^2 \pi_{t+1} + (1-\delta)q^m_{t+1}}{q^m_t} - R_{t+1}) + R_{t+1}} \right) \right] X_t
\]

which does not depend on \(n_t\). So if there exist one shot deviation for some entrepreneurs who draw \(z^h\), then similar one-shot deviation always exist for any entrepreneurs who draw \(z^h\) so that no one will invest. Therefore, entrepreneurs who draw \(z^h\) will borrow to the same target leverage.
(ii) First, the option value decreases when drawing $z_{t+1} = z^l$. Suppose not, then the rate of return on capital from $t$ to $t+1$ are

$$\frac{z^l \pi_{t+1} + (1 - \delta)}{q_t}, \quad \frac{z^l \pi_{t+1} + (1 - \delta) q_{t+1}}{q_t}$$

with $q_{t+1} > q_t$. Notice that, the rate of return for an investing entrepreneur is

$$\frac{z^l \pi_{t+1} + (1 - \delta)}{1}, \quad \frac{z^l \pi_{t+1} + (1 - \delta) q^l_{t+1}}{1}$$

where $q^l_{t+1} < 1$. Therefore, the rate of return of capital for entrepreneurs who draw $z^l$ is higher than that of an investing entrepreneur, state by state because $q_t < q_{t+1} < 1$. This result suggest that $z^l$ entrepreneurs should invest rather than holding capital stock, a contradiction.

Second, I prove that entrepreneurs who draw $z^l$ and who hold capital will deleverage. Without loss of generality, consider an entrepreneur with $(k_t, b_t)$ and $k_t = 1$ who draws $z_{t+1} = z^l$ and lets the capital depreciate to $k_{t+1} = 1 - \delta$. It is straightforward to show that borrowing to the credit constraint limit is not optimal because productivity is low.

Suppose $b_{t+1} \leq (1 - \delta)b_t$. Then, the Euler equation (or the asset pricing formula) can be written as

$$\beta p^{lh} E \left[ \frac{z^l \pi_t + (1 - \delta) q_t + R_t b_t}{(z^l \pi_{t+1} + (1 - \delta))(1 - \delta) + R_{t+1} b_{t+1}} \right] \frac{z^l \pi_{t+1} + (1 - \delta)}{q_t} X_t \bigg| + \beta p^{hl} E \left[ \frac{z^l \pi_t + (1 - \delta) q_t + R_t b_t}{(z^l \pi_{t+1} + (1 - \delta) q_{t+1})(1 - \delta) + R_{t+1} b_{t+1}} \right] \frac{z^l \pi_{t+1} + (1 - \delta) q_{t+1}}{q_t} X_t \bigg| = 1.$$  

Notice that $b_{t+1} \leq (1 - \delta)b_t < 0$, the left-hand-side

$$LHS \leq \frac{\beta(z^l \pi_t + (1 - \delta) q_t + R_t b_t)}{(1 - \delta) q_t} \left[ p^{lh} E \left[ \frac{1}{1 + \frac{R_{t+1} b_t}{z^l \pi_{t+1} + (1 - \delta) q_{t+1}}} \right] X_t \right] + p^{hl} E \left[ \frac{1}{1 + \frac{R_{t+1} b_t}{z^l \pi_{t+1} + (1 - \delta) q_{t+1}}} \right] X_t \bigg|$$

$$< \frac{\beta(z^l \pi_t + (1 - \delta) q_t + R_t b_t)}{(1 - \delta) q_t} \frac{1}{1 + b_t},$$
where the last inequality uses the condition in equilibrium $z^l \pi_{t+1} + (1 - \delta) \leq R_{t+1}, b_t < 0,$ and $q_{t+1} < 1.$ Further,

$$q_t (1 - \delta)(1 + b_t) = q_t (1 - \delta) + q_t (1 - \delta) b_t$$

$$> q_t (1 - \delta) + b_{t+1} = \beta (z^l \pi_t + (1 - \delta) q_t + R_t b_t),$$

so that $LHS < 1,$ which contradict the Euler equation. Therefore, $b' > (1 - \delta) b$ and because $k_{t+1} = (1 - \delta) k_t,$ we know that

$$\frac{k_{t+1}}{k_{t+1} + b_{t+1}} < \frac{k_t}{k_t + b_t}.$$

### 1.8.7 Proposition 4: Existence of Stopping Time

Using Proposition 2, there is vintage specific (time varying) constant $J$ in the value function of entrepreneurs in that specific vintage. Denote $J^i (i = 0, 1, ..., N + 2)$ as the constant in the value function.

(1) I prove that it is never optimal to sell part of the capital stock using principle of unimprovability (to check one-shot deviation). Consider an agent with state $(1, \tilde{b}, a),$ where $z'(a) = z^l,$ i.e., she draws $z^l.$ The net worth is $n_t = z_t \pi_t + (1 - \delta)(1 - d) + R_t \tilde{b}.$ Suppose the partial selling strategy is optimal, and $0 < \tilde{k} < 1 - \delta$ is left and bonds are $\beta n_t - (1 - d) \tilde{k}.$ Such partial selling strategy gives value

$$V_{part}(\tilde{k}) = \log((1 - \beta)n_t)$$

$$\beta p^{hb} E\left[ j_{t+1}^b \frac{\log((z^l \pi_{t+1} + (1 - \delta)) \tilde{k} + R_{t+1} (\beta n_t - (1 - d) \tilde{k}))}{1 - \beta} | X_t \right]$$

$$\beta p^{hl} E\left[ j_{t+1}^k \frac{\log((z^l \pi_{t+1} + (1 - \delta) q_{t+1}) \tilde{k} + R_{t+1} (\beta n_t - (1 - d) \tilde{k}))}{1 - \beta} | X_t \right],$$

where $j_{t+1}^K$ is some time varying constant and $q_{t+1}$ is some consistent shadow price for the action tomorrow. However, there always exists an one shot deviation by inaction today in
which the shadow value of capital is \( q_t \geq 1 - d \). To see this, the one shot deviation gives value

\[
V^{in}(1 - \delta) = \log((1 - \beta)\bar{n}_t) + \beta p^{lh}E \left[f^0_{t+1} + \frac{\log((z^l_t \pi_{t+1} + (1 - \delta))(1 - \delta) + R_{t+1}(\beta \bar{n}_t - (1 - d)(1 - \delta)))}{1 - \beta} \right]_{X_t} + \beta p^{ll}E \left[f^{N+2}_{t+1} + \frac{\log((z^l_t \pi_{t+1} + (1 - \delta)q_{t+1})(1 - \delta) + R_{t+1}(\beta \bar{n}_t - (1 - d)(1 - \delta)))}{1 - \beta} \right]_{X_t},
\]

where \( \bar{n}_t = z_t \pi_t + (1 - \delta)q_t + R_t \tilde{b}_t \geq n_t \). Notice that \( V^{part}(\tilde{k}) \leq V^{part}(1 - \delta) \leq V^{in}(1 - \delta) \), where the first inequality uses the monotonicity of \( V^{part} \) and the second uses both monotonicity and \( \bar{n}_t \geq n_t \). Therefore, partial selling strategy is never optimal.

(2) We are left to prove when inaction strategy dominates full liquidation strategy, and vice versa. It is sufficient to look at the region where liquidation strategy dominates. Since I can always normalize capital stock by 1, the region is the set of \( b/k \) (or leverage \( k/(k + b) \)). Inside the region, I need to prove no one-shot deviation exists. Again, consider the same agent with state \((1, \tilde{b}, a)\). The net-worth is \( n_t = z_t \pi_t + (1 - \delta)(1 - d) + R_t \tilde{b} \). Liquidation strategy gives value

\[
V^{out} = \log((1 - \beta)n_t) + \eta + \beta p^{lh}E \left[f^0_{t+1} + \frac{\log(\beta n_t R_{t+1})}{1 - \beta} \right]_{X_t} + \beta p^{ll}E \left[f^{N+2}_{t+1} + \frac{\log(\beta n_t R_{t+1})}{1 - \beta} \right]_{X_t}.
\]

One shot inaction deviation strategy gives value

\[
V^{in} = \log((1 - \beta)n_t)
\]
where terms in the bracket are continuation values for drawing $z^h$ and $z^l$ tomorrow, respectively. The value difference between the two strategies is

\[
V^{in} - V^{out} = \frac{\beta}{1 - \beta} p^{lh} E \left[ \log \left( 1 + \frac{(1 - \delta) z^l \pi_{+1} + (1 - \delta) - (1 - d) R_{t+1}}{\beta n_t R_{t+1}} \right) | X_t \right] \\
+ \frac{\beta}{1 - \beta} p^{ll} E \left[ \log \left( 1 + \frac{(1 - \delta) z^l \pi_{+1} + (1 - \delta)(1 - d) - (1 - d) R_{t+1}}{\beta n_t R_{t+1}} \right) | X_t \right] - \eta.
\]

Notice that $\frac{\partial (V^{in} - V^{out})}{\partial n_t}$ is equal to

\[
- \frac{\beta}{1 - \beta} \left[ p^{lh} E \left[ \frac{z^l \pi_{+1}(1 - \delta) - (1 - d) R_{t+1}}{\beta n_t R_{t+1}} | X_t \right] + p^{ll} E \left[ \frac{z^l \pi_{+1}(1 - \delta)(1 - d) - (1 - d) R_{t+1}}{\beta n_t R_{t+1}} | X_t \right] \right]
\]

which must be less than 0 in equilibrium. To see this, I only need to prove that

\[
p^{lh} E \left[ \frac{z^l \pi_{+1} + (1 - \delta) - (1 - d) R_{t+1}}{1 - d} | X_t \right] + p^{ll} E \left[ \frac{z^l \pi_{+1} + (1 - \delta)(1 - d) - (1 - d) R_{t+1}}{1 - d} | X_t \right] > 0,
\]

or

\[
p^{lh} E \left[ \frac{z^l \pi_{+1} + (1 - \delta)}{1 - d} | X_t \right] + p^{ll} E \left[ \frac{z^l \pi_{+1} + (1 - \delta)(1 - d)}{1 - d} | X_t \right] > R_{t+1},
\]

so that the expected rate of return of capital stock (with price as the liquidation price) must be greater than the risk-free rate. Suppose not, then the rate of return of inactive entrepreneurs will be less than interest rate because

\[
p^{lh} E_t \left[ \frac{z^l \pi_{+1} + (1 - \delta)}{q_t} | X_t \right] + p^{ll} E_t \left[ \frac{z^l \pi_{+1} + (1 - \delta) q_{t+1}}{q_t} \right] < p^{lh} E_t \left[ \frac{z^l \pi_{+1} + (1 - \delta)(1 - d)}{1 - d} | X_t \right] \leq R_{t+1},
\]

by noticing that $1 - d < q_t < 1$ and $q_t > q_{t+1}$ (shadow price of capital decreases as it is held longer). This inequality says that inactive entrepreneurs’ strategy is not consistent. They earn a lower expected rate of return on capital than risk-free rate which implies that they should liquidate.
Therefore, $\frac{\partial (V_{in} - V_{out})}{\partial n_t} < 0$ and $\frac{\partial (V_{in} - V_{out})}{\partial \tilde{b}} < 0$. Notice that, $V_{in} - V_{out} \to -\eta$ as $\tilde{b} \to +\infty$ (so that $n_t \to +\infty$). If $V_{in} - V_{out}$ will ever cross 0 at some $\tilde{b} = 1 - \frac{\lambda_t}{\lambda'}$, then $V_{in} - V_{out} > 0$ when $\tilde{b} < 1 - \frac{\lambda_t}{\lambda'}$, and $V_{in} - V_{out} \leq 0$ when $\tilde{b} \geq 1 - \frac{\lambda_t}{\lambda'}$. Equivalently, entrepreneurs liquidate the capital stock when $\frac{k}{k + \tilde{b}} < \frac{\lambda_t}{\lambda'}$ and there is no one-shot deviation. In sum, if there is liquidation, the cut-off leverage $\frac{\lambda_t}{\lambda'}$ solves

$$\eta = \frac{\beta}{1 - \beta} p^{th} E \left[ \log \left( 1 + (1 - \delta) \frac{\beta n_t R_{t+1}}{\beta n_t R_{t+1}} \right) \right] |x_t] + \frac{\beta}{1 - \beta} p^{th} E \left[ \log \left( 1 + (1 - \delta) \frac{\beta n_t R_{t+1}}{\beta n_t R_{t+1}} \right) \right] |x_t],$$

where $n_t = z^l \pi_t + (1 - \delta)(1 - d) + R_t \frac{1 - \lambda_t}{\lambda'}$.

### 1.8.8 Extra Tables

Table 1.6: **Summary Statistics for COMPUSTAT Capital Reallocation**

Level variables are in millions of 2005 dollars for a given calendar quarter. “PP&E” stands for property, plant and equipment, “CapEx” for capital expenditures, "Reallocation" is the sum of acquisitions plus sales of PP&E, and “Investment” is defined as the capital expenditure plus acquisition. Total Reallocation/Total Previous PP&E ratio is computed as the sample mean of the numerator over the sample mean of the denominator to avoid the problem of firms with extremely large assets.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Median</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assets</td>
<td>2435.11</td>
<td>129.94</td>
<td>15712.73</td>
</tr>
<tr>
<td>PP&amp;E</td>
<td>602.24</td>
<td>17.16</td>
<td>3851.315</td>
</tr>
<tr>
<td>CapEx</td>
<td>20.12</td>
<td>1.23</td>
<td>101.23</td>
</tr>
<tr>
<td>Acquisitions</td>
<td>6.12</td>
<td>0.00</td>
<td>45.67</td>
</tr>
<tr>
<td>Sales of PP&amp;E</td>
<td>3.51</td>
<td>0.00</td>
<td>18.50</td>
</tr>
<tr>
<td>Total Sales of PP&amp;E/Total Reallocation</td>
<td>30.71%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Reallocation/Total Investment</td>
<td>32.1%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Reallocation/Total Previous PP&amp;E</td>
<td>1.44%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 1.7: **Capital reallocation**
Correlation of real GDP and the various definitions of capital reallocation, after taking natural log and then HP filtered. Numbers in the bracket are the standard deviation after correcting heteroscedasticity and autocorrelation. Acquisition: COMPUSTAT data items 129. SPPE: sales of property, plant and equipment, COMPUSTAT data item 107. AQC turnover: acquisition divided by total asset (item 6) last period. SPPE turnover: SPPE divided by total property, plant and equipment (item 8) last period. Total Reallocation is the sum of acquisition and SPPE. GDP is real GDP in 2005 dollars. All series are seasonal adjusted and "***" denotes 1% significance level.

<table>
<thead>
<tr>
<th>Correlation with GDP</th>
<th>Acquisition</th>
<th>SPPE</th>
<th>Reallocation</th>
<th>SPPE turnover</th>
<th>AQC turnover</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.840***</td>
<td>0.430***</td>
<td>0.854***</td>
<td>0.411***</td>
<td>0.786***</td>
</tr>
<tr>
<td></td>
<td>(0.064)</td>
<td>(0.148)</td>
<td>(0.057)</td>
<td>(0.128)</td>
<td>(0.071)</td>
</tr>
</tbody>
</table>

Table 1.8: **Benefits to reallocation**

<table>
<thead>
<tr>
<th>Standard Deviation of</th>
<th>TFP growth (2 SIC digit)</th>
<th>TFP growth (4 SIC digit)</th>
<th>Productivity Changes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corr with GDP</td>
<td>-0.465***</td>
<td>-0.384***</td>
<td>-0.437***</td>
</tr>
</tbody>
</table>
Table 1.9: **Key statistics in the data and in the model**
Data are cyclical components of HP filtered series from 1984Q1 to 2011Q4. Standard deviations denote the standard deviations of percentage deviations from trends.

<table>
<thead>
<tr>
<th></th>
<th>Volatility</th>
<th>Co-movement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Standard deviation</td>
<td>Standard deviation to that of output</td>
</tr>
<tr>
<td>Output</td>
<td>Consumption</td>
<td>Investment</td>
</tr>
<tr>
<td>Data: 1.42%</td>
<td>0.55</td>
<td>3.86</td>
</tr>
<tr>
<td>Model: 1.35%</td>
<td>0.61</td>
<td>4.01</td>
</tr>
</tbody>
</table>

### 1.8.9 Extra Graphs

Figure 1.10: **The potential benefits to capital reallocation**

Solid line (left scale) is the interquartile range (the gap between the 75% level and 25% level) of establishment level idiosyncratic TFP shocks (annual frequency), constructed by Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) through Annual Manufacturing Survey and Census of Manufacturing. Dashed line (right scale) is the cyclical component of HP-filtered log of real GDP (annual frequency) normalized by its standard deviation. Shaded regions denote NBER recessions.
Figure 1.11: **Debt-to-Asset Ratio before liquidation in different groups.**

Plotted series are debt-to-asset ratios before selling assets in each quantile group. Time 0 denotes the time when firms sell assets. Each firm is classified by their positions of debt-to-asset ratios quantile at time 0. Each plot traces back average debt-to-asset ratios in each quarter before time 0, in each quantile group. For example, the debt/asset ratio at time -10 in “50% - 75% quantile” plot, means the average debt/asset ratio of companies 10 quarters before selling assets in the 50% to 75% quantile group. This figure generally shows that firms that sell assets deleverage before they sell, in addition to firms that probably have surviving problems (the 75-100% quantile group).

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Figure 1.12: **Capital reallocation over cycles**

Figure 1.13: **Priors and Posteriors**
Priors and posteriors in graphs. Blue dashed lines denote priors. Red solid lines denote posteriors.
Chapter 2

Search-Based Endogenous Illiquidity

2.1 Introduction

Illiquidity of financial assets arises from impediments to transactions in these assets. Empirical evidence points to procyclical variation in the market liquidity of a wide range of financial assets.\(^1\) The view that asset liquidity dries up during recessions has been reinforced by the 2007-2009 financial crisis, when illiquidity problems were most pronounced in markets for corporate bonds, commercial paper, asset-backed securities and repurchase agreements.\(^2\)

\(^1\)Studies by Huberman and Halka (2001), Chordia, Roll, and Subrahmanyam (2001) and Chordia, Sarkar, and Subrahmanyam (2005) assert that market liquidity is procyclical and highly correlated across asset classes such as bonds and stocks in the US. This observation implies that common factors drive liquidity.

\(^2\)Dick-Nielsen, Feldhütter, and Lando (2012) identify a break in the market liquidity of corporate bonds at the onset of the subprime crisis. The liquidity component of spreads of all but AAA rated bonds increased and turnover rates declined, making refinancing on the corporate bond market more difficult. Similarly, Anderson and Gascon (2009) report that commercial paper (CP), which is largely traded on a search market with dealers as matchmakers, experienced illiquidity. At the same time, money market mutual funds, the main investors in the CP market, shifted away from commercial paper to government securities reflecting both a flight-to-safety and a flight-to-liquidity motive. Finally, as emphasized by Gorton and Metrick (2012), the repo market has registered strongly increasing haircuts during the crisis. These are attributed largely to concerns about the market liquidity of securities, i.e. claims on private sector cash flows, which are used as underlying collateral in repo agreements.
Macroeconomic models often ignore the impact of such variations in asset transaction on aggregate fluctuations. However, the cost and time involved in transforming illiquid financial assets into cash constitute important financial frictions, which strongly affect the funding conditions of both financial and nonfinancial firms. U.S. nonfinancial firms, for instance, fund 35% of fixed investment through primary (debt and equity issuance) and secondary markets (portfolio liquidations). Higher liquidation costs or longer trading delays in primary and secondary capital markets impair these firms’ funding capacities and thus feed directly into their investment and labor hiring decisions. Liquidity fluctuations may, hence, significantly affect business cycles.

The illiquidity of privately issued financial assets creates a role for liquid assets, as they provide an insurance against future funding constraints. These assets can be readily used to settle future financing obligations. During recessions when funding constraints are tough, firms are more willing to accumulate liquid assets (i.e. mainly government bonds, see Figure 2.1), a phenomenon usually called “flight to liquidity”. Accordingly, the correlation between GDP and the liquidity share, defined as the ratio of liquid assets to GDP, is highly negative (−0.72 post 1984). The portfolio rebalance towards liquid government bonds gives scope to unconventional monetary policy which can control their supply and change the portfolio composition held by firms.

This paper analyzes the macroeconomic effects of variations in asset liquidity. We seek to jointly explain the procyclicality of asset liquidity (and prices) and the countercyclical flight to liquidity in the corporate sector. We develop a dynamic stochastic general equilibrium model in which aggregate shocks generate endogenous fluctuations in asset liquidity. In this framework, asset liquidity is measured by the fraction of privately issued financial assets that can be sold or resold in a given period. The model shows how the variation in liquidity interacts with the funding conditions of firms over the business cycle. Our

\[^{3}\text{Debt and equity issuance cover 75.67\% of this financing gap, sales of liquid reserves make up a sizeable 20.74\% and illiquid reserves 3.69\%. For further details see Ajello (2012).}\]
framework makes three contributions: It (i) provides a tractable search-based approach to endogenous asset market liquidity in a general equilibrium model, (ii) generates procyclical fluctuations in both asset liquidity and asset prices, (iii) suggests that endogenous aggregate demand fluctuations, driven by pure liquidity shocks in financial asset markets, can account for the countercyclical liquidity share and cyclical properties of major macroeconomic variables, rather than driven by productivity (supply) shocks.

Our model extends a standard New Keynesian model with search frictions in the asset market. The economy is populated by four types of agents: a household sector whose members are either entrepreneurs (sellers of financial assets) or workers (investors), intermediate goods producers, final goods producers, and the government. Households make choices on consumption, labor, and investment. Intermediate goods producers rent capital from households and hire labor to produce. Final goods producers assemble intermediate

![Figure 2.1: Cyclical component of liquidity share (nominal liquid assets / nominal GDP) and real GDP as percent deviations from the trend. Liquid assets consist of all treasury bills. Shaded areas denote NBER recessions dates. Source: U.S. Flow of funds](image)

Student Version of MATLAB
inputs to final goods. Finally, the government affect the supply of fully liquid government bonds (or money) and may purchase claims to private assets.

Household members hold claims to government issued bonds and claims to privately issued financial assets, which we interpret as a catch-all for privately issued assets, such as corporate bonds and equity. Each member could be an entrepreneur or a worker each period, but only entrepreneurs have investment opportunities. To take full advantage of the opportunities, entrepreneurs issue financial claims to the investment, together with funding from liquidating financial asset. However, the private claims may not be fully liquid, because they have to be offered on a search market. Therefore, entrepreneurs cannot fully finance first-best investment level.

Moreover, search for appropriate counterparties is costly for a household and the asset price is determined in a bargaining process between buyers and sellers.\(^4\) This market structure intends to capture the features of OTC markets, in which a large fraction of corporate bonds, asset-backed securities and private equity is traded.\(^5\) Alternatively, we consider our framework as a reduced-form approach towards modelling financial intermediation. The search market framework captures the matching process between savers (investors) and the corporate sector through intermediaries (see Haan, Ramey, and Watson (2003)). We believe that the matching process captures an essential feature of financial intermediation. Therefore, our results may also shed light on the effects of modelling a financial sector.

We consider two types of exogenous disturbances: aggregate productivity shocks and shocks to the matching efficiency between buyers and sellers. In both cases, the participation of buyers in the search market drives endogenous procyclical asset liquidity and prices. Negative aggregate productivity shocks, for instance, decrease the return to capital, make investment into capital goods less attractive, and crowd out investors from the search market. Negative matching efficiency shocks (combined with nominal frictions), on

\(^4\)This setup borrows heavily from the labour search literature such as Ebell (2011).

the other hand, exert a negative income effect while making investment into liquid assets more attractive. This effect also reduces the incentive for investors to engage in costly search efforts. In either case, there are fewer matches, which endogenously decreases the volume of successful transactions. Hence, both the liquidity and prices of financial claims drop because of the reduced willingness of buyers to participate in the market. Lower asset liquidity and prices restrict funding available to entrepreneurs and, thereby, slow down aggregate investment.

While both productivity shocks and matching efficiency shocks generate procyclical asset liquidity and prices, only the latter induce a flight to liquidity and thus a countercyclical liquidity share. Persistent adverse TFP shocks lower both today’s and futures’ return to capital. Hence, investors have no incentive to hedge against future illiquidity of privately issued financial assets. Adverse matching efficiency shocks, on the other hand, do not deteriorate the quality of investment either today or tomorrow. Therefore, investors demand more hedging service provided by liquid assets. Accordingly, the liquidity share in their portfolios increases.

To our knowledge, we are the first to incorporate a search-based financial asset market with endogenous liquidity in a macro model and to analyze the feedback effects between liquidity and the real economy. The tractability of our approach derives from the convenient setup of the bargaining process, which takes advantage of households’ portfolio choice solution.

2.1.1 Related Literature

Our work is closely related to Kiyotaki and Moore (2012) (henceforth, KM), who motivate financial assets’ market liquidity by an exogenous constraint on the resalability of private paper. Their basic idea is that disinvestment takes time (due to frictions that are not modelled), such that only some fraction of asset holdings can be sold in a given period to finance new investment. Money or government bonds, on the other hand, can be readily sold when
needed and thus provide a liquidity service. A “spectrum of returns” emerges as a result of differences in asset liquidity. As a consequence, the irrelevance result of Wallace (1981) on central bank’s portfolios no longer holds. In fact, open market operations that change the composition of liquid and illiquid assets have real effects. Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011) analyse the stabilizing potential of such unconventional policy after an exogenous fall in liquidity. With standard monetary policy constrained by the “Zero Lower Bound”, they show that liquidity injections effectively dampen the liquidity shortfall.

However, Shi (2012) demonstrates that in these models exogenous adverse liquidity shocks induce asset price booms (in real consumption goods terms), because the shocks essentially amounts to reduce the supply of financial assets. Demand for financial assets, on the other hand, does not decrease substantially as investment projects’ quality does not change. This finding highlights the need for a theory of endogenous asset liquidity.

Treatment of illiquidity in these papers abstracts from feedback effects between macroeconomic conditions and asset market. In particular, it cannot account for endogenous cyclical variations in asset illiquidity. The search literature, on the other hand, provides a natural theory of endogenous liquidity. It has been applied to a wide range of markets such as housing (Wheaton, 1990; Ungerer, 2012), bank loans (Haan, Ramey, and Watson, 2003; Wasmer and Weil, 2004), as well as OTC markets for asset-backed securities, corporate bonds, US federal funds, private equity, and real estate (Duffie, Gârleanu, and Pedersen, 2005; Duffie, Garleanu, and Pedersen, 2007; Ashcraft and Duffie, 2007; Lagos, 2011; Feldhutter, 2011). This line of research shows that search frictions can explain substantial variation in a range of measures of liquidity, such as bid-ask spreads, trade volume (market depth), and trading delays. However, the majority of them does not consider the adverse feedback of illiquid asset markets on the real economy as in our paper.6

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6Asset illiquidity may further interact with financing constraints to induce delays in asset sales as in Cui (2013). This interaction prolongs shocks to the financing conditions of the private sectors and results in countercyclical productivity dispersion.
In a number of related studies, endogenous asset market liquidity has been motivated with information frictions. Eisfeldt (2004) develops a partial equilibrium model with adverse selection in asset markets. The model shows how investment and trading volume are amplified if asset liquidity endogenously varies with productivity. More recently, Guerrieri and Shimer (2012) provide a dynamic adverse selection model in an environment of exogenous fundamental asset returns. In this framework information frictions give rise to trading delays due to a shortage of buyers, similar to the search theoretic models of illiquidity.

However, these studies do not consider feedback effects on production and employment. To account for such effects, Kurlat (2012) extends KM with endogenous resaleability through adverse selection. Also in Bigio (2011), dispersion shocks to capital quality endogenously decrease the liquidity of private assets due to information asymmetries. Such shocks translate into substantial fluctuations in hours, investment and output when private assets are used as collateral in working capital loan contracts. However, these models do not consider alterative assets with different information properties, such as government bonds. In Eisfeldt and Rampini (2009), there is no secondary market for private assets and entrepreneurs accumulate liquid assets to fund investment opportunities. Because liquidity is accumulated out of retained earnings, the supply of liquidity correlates positively with productivity. Our approach differs from all of them, in that we motivate endogenous liquidity differentials across assets and preserve the role of a liquid asset as a lubricant of financial flows as in KM.

The rest of the paper is organized as follows: Section 2.2 presents the model and characterizes the equilibrium of our economy. Section 2.3 discusses impulse responses and policy experiments. In section 2.4, we conclude and outline avenues for further research.
2.2 The Model

Environment. Time is discrete and with infinite horizon. The economy comprises four sectors: households, intermediate goods producers, final goods producers, and the government. The members of each representative household are either entrepreneurs or workers. The key deviation from the New Keynesian DSGE model consists in search frictions afflicting the sale of financial assets from these household members. Government bonds, on the other hand, are fully liquid assets which can be traded freely on a spot market.

Timing. Each period $t$ is split into four phases: households’ decisions, production, investment and consumption. At the beginning of $t$, aggregate exogenous states are realized and the government policy rules are set. Then a representative household specifies policy rules for each household member, taking into account production, investment, and consumption at later stages. After the decision is made, production takes place, entrepreneurs and workers meet in the asset market to trade assets, invest according to the rules specified by the household, and finally consume when they are back to the household.

Government conducts conventional monetary policy via the control of the nominal interest rate; fiscal policy via taxes on households; and unconventional policy through purchasing or selling assets issued by private agents.

2.2.1 Households

Representative household structure. The economy comprises a continuum of representative households with a unit measure of members each. Each period, household members receive an idiosyncratic shock that determines their type in the middle of the period. With probability $\chi$, household members become entrepreneurs (we call them type $s$ because they “sell” financial assets). With probability $(1 - \chi)$, they become workers (we call them type $b$ because they “buy” financial assets). Workers earn wages by supplying their labour, while entrepreneurs have productive investment opportunities but do not earn wages. Type
shocks are \textit{i.i.d.} across members and through time. By the law of large numbers, each household thus consists of a fraction $\chi$ of entrepreneurs and a fraction $(1 - \chi)$ of workers. Both groups are temporarily separated such that resources cannot be re-allocated among household members during the period. Only at the very end of $t$, both types come together again to share their consumption goods and accumulated assets. Therefore, all members enter the next period with an equal share of the household’s assets.\footnote{The representative household structure with temporarily separated agents has been introduced in Lucas (1990) and applied to the KM framework in Shi (2012) and Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011).}

\textbf{A Representative Household}

\textit{Preferences.} The household determines entrepreneurs’ and workers’ choices in order to maximize

$$
E_t \sum_{h=0}^{\infty} \beta^{t+h} \left[ u(C_{t+h}) - \frac{\mu}{1 + \nu} L_{t+h}^{1 + \nu} \right]
$$

where $C_{t+h}$ is consumption and $L_{t+h}$ is labor supply in period $t + h$. Note that since both types of agents lump their consumption goods together at the end of the period, the household optimizes over household-wide consumption $C_{t+h}$. The full decision problem is developed in Section 2.2.1.

\textit{Portfolio.} Physical capital is held by households and lent to intermediate goods producers. Thus, capital earns a return. There is a claim to the return of every unit of capital, which is either retained by households or sold to outside investors. These claims can be sold at unit price $q_t$ (which will be determined in the search market structure in Section 2.2.2). In addition, households invest into risk-less (nominal) government bonds, with nominal price level $P_t$. Hence, at the onset of $t$, households have a portfolio of government bonds, equity claims on other households’ cash-flow on capital, and own physical capital. These assets are financed by equity claims issued on the return to own physical capital and net worth. This financing structure gives rise to the beginning-of-period balance sheet in Table 2.1.
Table 2.1: Household’s Balance Sheet

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>liquid bonds</td>
<td>$B_t/P_t$</td>
<td>equity issued</td>
</tr>
<tr>
<td>other’s equity</td>
<td>$q_tS_t^O$</td>
<td></td>
</tr>
<tr>
<td>capital stock</td>
<td>$q_tK_t$</td>
<td>net worth</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q_tS_t + B_t$</td>
</tr>
</tbody>
</table>

Portfolio adjustments are affected by search frictions: claims to capital cash-flow are sold on an over the counter (OTC) market. Here, only some fraction of offered assets is matched to appropriate buyers, such that some claims remain unmatched.\(^8\) We assume that an identical fraction of previously uncommitted returns to own physical capital, i.e. \((K_t - S_t^I)\), can be mortgaged. This simplification ensures that both types of assets not only yield the same return, but are equally liquid. They will thus yield the same price on the search market and can be treated as perfect substitutes, such that we only need to keep track of net equity, defined as

\[
S_t = S_t^O + (K_t - S_t^I)
\]

### Household Members

A typical household member \(j\) is endowed with equity \(s_{j,t}\) and bonds \(b_{j,t}\). There exists a search market for purchasing and selling equity. Buyers exert costly search effort \(e^{b}_{j,t}\) to acquire new or old equity. On the search market, each unit of buyers’ effort results in \(\phi_{b,t} \in [0, 1]\) matched purchases at unit cost \(\kappa_b\). Accordingly, the individual buyer expects to purchase an amount

\[
m_{j,t} = -\phi_{b,t} e^{b}_{j,t}
\]

of matched assets on the search market (note that a positive \(m_{j,t}\) corresponds to asset sales and a negative \(m_{j,t}\) corresponds to purchases). Sellers, on the other hand, decide which fraction \(e^{s}_{j,t} \in [0, 1]\) of their total assets to put up for sales. These assets consist of existing

\(^8\)The exact structure of the search market is detailed in Section 2.2.1.
equity claims on other households’ capital stock and their own capital stock, $s_{jt}$, plus claims on new investment to be issued, $i_t$. Each unit of sellers’ assets that are offered on the search market is matched with a buyer with probability $\phi_s, t \in [0, 1]$. Entrepreneurs, thus expect to sell

$$m_{jt} = \phi_s, t e^s_{jt} [ (1 - \delta) s_{jt} + i_{jt} ]$$

matched claims on the search market. The associated selling costs are $\kappa_s$ per unit of listed assets. Note that the respective matching probabilities $\phi_b, t$ and $\phi_s, t$ are taken as given. These probabilities will be determined on the search market. Importantly, a representative household takes search market structure $\{ q_t, \phi_b, t, \phi_s, t \}$ as given when choosing the search effort $e^b_{jt}$ and the fraction of assets to post on the market $e^s_{jt}$.

Now, we can write the budget constraint of our typical household member $j$ along with laws of motion for the equity position. Denote the nominal bond return as $R_t$ (which is pre-determined), and let $c_{jt}$, $l_{jt}$, and $i_{jt}$ be consumption, working hours, and investment respectively. Finally, let $\tau_t$ be the lump-sum taxes on household members. Accordingly, $j$’s budget constraint and equity evolution read

$$c_{jt} + \kappa_b e^b_{jt} + \kappa_s e^s_{jt} ( (1 - \delta) s_{jt} + i_{jt} ) + \frac{b_{jt+1}}{P_t} + \tau_t = w_t l_{jt} + r_t s_{jt} + q_t m_{jt} + R_t \frac{b_{jt}}{P_t}$$

$$s_{jt+1} = (1 - \delta) s_{jt} + i_{jt} - m_{jt}$$

Next, taking into account the specific functions of workers and entrepreneurs, we separate their budget constraints and equity evolutions.

A worker. If $j$ is a worker without investment opportunity, $j$ is a buyer of others’ equity claims because for the household-wide consumption smoothing. Then,

$$i_{jt} = 0, \quad e^b_{jt} \geq 0, \quad e^s_{jt} = 0$$
Substituting out expected purchases on the search market, $m_{j,t}$, the evolution of equity becomes

$$s_{j,t+1} = (1 - \delta)s_{j,t} + \phi_{b,t}e_{j,t}^b$$

while the flow-of-funds constraint for a worker simplifies to

$$c_{j,t} + \kappa s_{j,t} + \frac{b_{j,t+1}}{P_t} + \tau_t = w_t l_{j,t} + r_t s_{j,t} - q_t \phi_{b,t} e_{j,t}^b + R_t \frac{b_{j,t}}{P_t}.$$  \hfill (2.2)

Using the evolution of equity, the flow-of-funds constraint can be rewritten as

$$c_{j,t} + q_{b,t} s_{j,t+1} + \frac{b_{j,t+1}}{P_t} + \tau_t = w_t l_{j,t} + r_t s_{j,t} + (1 - \delta)q_{b,t} s_{j,t} + R_t \frac{b_{j,t}}{P_t}.$$  \hfill (2.3)

where $q_{b,t}$ is the effective buying price for one unit of assets and

$$q_{b,t} \equiv q_t + \frac{\kappa}{\phi_{b,t}}.$$  \hfill (2.4)

*An entrepreneur.* If $j$ is an entrepreneur with investment opportunity, $j$ is a seller of financial assets because the need to finance new investment. Then

$$i_{j,t} > 0, \quad e_{j,t}^s \geq 0, \quad e_{j,t}^b = 0$$

Similarly to the worker, we substitute out sales $m_{j,t}$, to retrieve the evolution of equity as

$$s_{j,t+1} = (1 - \phi_{s,t} e_{j,t}^s) \left[ (1 - \delta) s_{j,t} + i_{j,t} \right]$$

and the flow-of-funds constraint as

$$c_{j,t} + \kappa e_{j,t}^s \left[ (1 - \delta) s_{j,t} + i_{j,t} \right] + (1 - \phi_{s,t} e_{j,t}^s q_t) i_{j,t} + \frac{b_{j,t+1}}{P_t} + \tau_t = r_t s_{j,t} + \phi_{s,t} e_{j,t}^s q_t (1 - \delta) s_{j,t} + R_t \frac{b_{j,t}}{P_t}.$$
For convenience, one can substitute out $i_{j,t}$ from the evolution of equity to express the flow-of-funds constraint of an entrepreneur in terms of end of period equity $s_{j,t+1}$:

$$
c_{j,t} + \frac{1 - \phi_{s,t} e^s_{j,t} \left(q_t - \frac{\kappa_s}{\phi_{s,t}}\right)}{1 - \phi_{s,t} e^s_{j,t}} s_{j,t+1} + \frac{b_{j,t+1}}{P_t} + \tau_t = r_t s_{j,t} + (1 - \delta) s_{j,t} + R_t \frac{b_{j,t}}{P_t}. \tag{2.5}
$$

To simplify, let $q_{s,t}$ be the effective selling price for one unit of assets and $q_{r,t}$ as the down-payment price, where

$$
q_{r,t} = \frac{1 - \phi_{s,t} e^s_{j,t} q_{s,t}}{1 - \phi_{s,t} e^s_{j,t}}, \quad q_{s,t} = q_t - \frac{\kappa_s}{\phi_{s,t}}
$$

we can rewrite (2.5) as

$$
c_{j,t} + q_{r,t} s_{j,t+1} + \frac{b_{j,t+1}}{P_t} + \tau_t = r_t s_{j,t} + (1 - \delta) s_{j,t} + R_t \frac{b_{j,t}}{P_t}. \tag{2.6}
$$

Prices $q_{s,t}$ and $q_{r,t}$ have an intuitive economic interpretation. In an RBC world, the effective selling price of equity claims would be one. In the presence of search costs, however, equity claims are not fully resaleable. In this case, both the bargaining price $q_t$ and the effective selling price $q_{s,t}$ deviate from 1 and entrepreneurs are financing constrained.

If $q_t > q_{s,t} > 1$, as in our calibration, then $q_{r,t} < 1$ and entrepreneurs will effectively be able to leverage up. To understand this leverage effect, note that entrepreneurs can finance a fraction $\phi_{s,t} e^s_{j,t}$ of future equity claims $s_{j,t+1}$ with outside funding at price $q_{s,t}$. Hence, entrepreneurs only need to commit $1 - \phi_{s,t} e^s_{s,t} q_{s,t}$ of internal funds to finance future equity. At the same time, a fraction $1 - \phi_{s,t} e^s_{s,t}$ of previous equity and claims to new investment remain in the hands of entrepreneurs. By normalising the share of inside funding with the retained fraction we get $q_{r,t}$. This down-payment price captures the effect of search costs on equity accumulation: higher search costs decrease the effective sales price, which increases the down-payment that in turn depresses accumulated equity. Therefore, the entrepreneurs’ ability to leverage will be lower if search costs are higher.
Households’ Problem

Now, consider the aggregation of household members to all workers and all entrepreneurs. Let $j \in \{b,s\}$ indicate the household member being a worker (buying financial assets) or an entrepreneur (selling financial assets), and define aggregate variables as $X_{b,t} \equiv (1 - \chi)x_{b,t}$ and $X_{s,t} \equiv \chi x_{s,t}$. To simplify notation, we also switch to the recursive formulation, i.e., let $x$ and $x'$ denote $x_t$ and $x_{t+1}$. We aggregate the budget constraints of individual workers and entrepreneurs, (2.3) and (2.6), to

$$
C_b + q_b S'_b + \frac{B'_b}{P} + \mathcal{T}_b = wL + rS_b + (1 - \delta)q_b S_b + \frac{R B_b}{P} \tag{2.7}
$$

$$
C_s + q_r S'_s + \frac{B'_s}{P} + \mathcal{T}_s = rS_s + (1 - \delta)S_s + \frac{R B_s}{P} \tag{2.8}
$$

Let $\Gamma$ be the vector of aggregate state variables, the evolution of which is taken given by the household. Once we proceed to the equilibrium definition, $\Gamma \equiv (K, z_a, z_\phi)$ where $K$ is the total capital stock, $z_a$ is the productivity, and $z_\phi$ is the matching efficiency in the search market.\(^9\) Let $V(S, B; \Gamma)$ be the value of a household with equity claims $S$ and bonds $B$, given the collection of aggregate state variables $\Gamma$. Then the value satisfies the Bellman equation:

**Problem 1**

$$
V(S, B; \Gamma) = \max_{E_b, E_s, S'_b, S'_s, B'_b, B'_s, L} \left[ u(C_b + C_s) - \frac{\mu}{1 + \nu} L^{1 + \nu} + \beta \mathbb{E} \left[ V(S', B'; \Gamma') | \Gamma \right] \right], \quad \text{s.t.}
$$

$$
S_b = (1 - \chi) S, \quad S_s = \chi S
$$

$$
B_b = (1 - \chi) B, \quad B_s = \chi B
$$

$$
S' = S'_b + S'_s \quad B' = B'_b + B'_s
$$

\(^9\)The stochastic processes for the exogenous state variables are defined in the numerical examples.
As in the motivation, we are interested in an economy that financial assets are partially liquid. When entrepreneurs facing investment opportunities, they will liquidate these assets. Therefore, \( q_s > 1 \). Otherwise, they will not sell any assets. Then, it is easy to show that \( q_r < 1 < q_s < q < q_b \). We focus on the equilibrium with this condition satisfied because it implies that entrepreneurs are financing constrained and liquidity frictions matter. Therefore, households should have entrepreneurs spend their entire net worth on creating new equity because the effective price of equity is \( q_r \) which is lower than that \( (q_b) \) for the workers. In this case, entrepreneurs do not bring back consumption goods and liquid bonds to the household, such that consumption smoothing and precautionary bond holding are entirely delegated to workers. At the same time, the entrepreneurs will put every assets on sales. \(^{10}\)

**Lemma 4**  For \( q_s > 1 \), we have \( e_s = 1, E_s = \chi e_s = \chi, C_s = 0, \) and \( B_s' = 0 \). Moreover, the following condition satisfied

\[
q_r < 1 < q_s < q < q_b
\]

and

\[
q_r \equiv \frac{1 - \phi_s q_s}{1 - \phi_s}, \quad q_s \equiv q - \frac{\kappa_s}{\phi_s} \tag{2.9}
\]

It turns out that we can have a household-wide budget constraint. To see this, entrepreneurs’ budget constraint can be simplified to

\[
q_r S'_s + \chi \tau = (r + (1 - \delta)) S_s + R \frac{B_s}{P} \tag{2.10}
\]

We can back out entrepreneurs’ future equity as

\[
S'_s = \frac{[r + (1 - \delta)] S_s + R \frac{B_s}{P} - \chi \tau}{q_r} = \chi \frac{[r + (1 - \delta)] S_s + R \frac{B_s}{P} - \tau}{q_r} \tag{2.11}
\]

\(^{10}\)Alternatively, one can see all these claims from first order conditions on \( C_s, e_s, \) and \( B_s' \).
Hence, entrepreneurs’ end-of-period equity equals their entire networth over the down-payment $q_r$. For book keeping, we back out investment $I = \frac{S'_s}{(1-\phi_s)} - (1-\delta)\chi S$ as

$$I = \chi \left[ r + (1-\delta)\phi_s q_s \right] S + \frac{R B}{P} - \tau$$  \hspace{1cm} (2.12)

Now, the household-wide budget constraint is immediately available. Using $S'_s$ in (2.11), together with budget constraints of workers and entrepreneurs (2.7) and (2.8), $C = C_s + C_b$, $S = S_s + S_b$, and $B = B_s + B_b$, delivers the aggregate budget constraint for the whole household

$$C + q_b S' + \frac{B'}{P} + \left[ (1-\chi) + \chi \frac{q_b}{q_r} \right] \tau$$

$$= wL + \left[ (1-\chi)(r + (1-\delta)q_b) + \chi (r + (1-\delta)) \frac{q_b}{q_r} \right] S + \left[ (1-\chi) + \chi \frac{q_b}{q_r} \right] \frac{R B}{P}$$  \hspace{1cm} (2.13)

The aggregate budget constraint (2.13) takes all constraints of individual household members into account. It expresses the household portfolio choice problem in a convenient form: the household’s resources consist of wage payments, equity returns and the resale value of equity, as well as the bond value, which takes into account the liquidity service provided by bonds to entrepreneurs; these resources are spent on consumption, taxes and a saving portfolio that consists of new equity and new bonds. When there are no search frictions, $\kappa_b = \kappa_s = 0$, we will prove that $q = q_s = q = 1$, and the budget constraint collapses to the standard household budget constraint in a real business cycle (RBC) model.

The deviation from the underlying RBC structure is, thus, entirely due to search frictions. The household’s problem, Problem 1, can now be simplified to

**Problem 2**

$$V(S, B; \Gamma) = \max_{S', B', L, e_t} u(C) - \frac{\mu}{1 + v} L^{1+v} + \beta \mathbb{E} \left[ V(S', B'; \Gamma') \right] \quad s.t. \ (2.13)$$
We derive the first order necessary conditions to establish household’s behavior. As standard in the portfolio choice problem, these conditions are necessary and sufficient. The FOC for labor is

$$u'(C)w = \mu L^\nu$$  \hspace{1cm} (2.14)

The FOC for next periods’ equity $S'$ is

$$u'(C)q_b = \beta \mathbb{E}[V_s(S',B';\Gamma')]$$

Using the envelope condition, the above can be expressed in the form of a standard asset pricing formula:

$$\mathbb{E}\left[ \frac{\beta u'(C')}{u'(C)} r'_S \right] = 1$$  \hspace{1cm} (2.15)

where $r_S$ is the equity return in the perspective of the household

$$r'_S = \left[ (1 - \chi) (r' + (1 - \delta)q'_b) + \chi (r' + (1 - \delta)) \frac{q'_b}{q'_r} \right] / q_b.$$

The FOC for next period’s bond $B'$ is

$$\frac{u'(C)}{P} = \beta \mathbb{E}[V_B(S',B';\Gamma')]$$

which yields another asset pricing formula

$$\mathbb{E}\left[ \frac{\beta u'(C')}{u'(C)} r'_B \right] = 1$$  \hspace{1cm} (2.16)

where $r_B$ is the bond return in the perspective of the household

$$r'_B = \left[ (1 - \chi) + \chi \frac{q'_b}{q'_r} \right] \frac{PR'}{P'}. $$
When $\kappa_b = \kappa_s = 0$ and $q_b = q_s = q = 1$, equity can be fully sold and the bond loses its liquidity value. In this case, the FOC collapses to the standard consumption Euler equation $u'(C) = \beta E \left[ u'(C') \frac{P_R}{P} \right]$. Intuitively, as long as $\kappa_b, \kappa_s > 0$, bonds will provide a liquidity service so that their rate of return will generally be lower. This finding is one of our main analytical results:

**Proposition 6** Suppose $\kappa_b, \kappa_s > 0$ then $q_b > q_r$, so that nominal bonds have value and provide liquidity services. When, on the other hand, $\kappa_b, \kappa_s = 0$, nominal bonds can be entirely replaced by equity.

### 2.2.2 Search and Matching in the Equity Market

Matching between buyers and sellers takes place in a decentralized market. Buyers engage in aggregate search effort denoted $E_b$ where

$$E_b = \frac{S'_b - (1 - \delta)S_b}{\phi_b} = \frac{S' - S'_s - (1 - \delta)S_b}{\phi_b}$$

and sellers put all their new and old assets on sale

$$A_s = (1 - \delta) \chi S + I$$

Without government intervention, aggregate effort is $E_B = E_b$ and aggregate assets on sale are $A_S = A_s$. The number of aggregate matches $M$ is determined by the matching function

$$M = \xi e^{\phi} A_S^\gamma E_B^{1-\gamma}$$

where $\gamma \in (0, 1)$ is the elasticity of matches w.r.t posted assets, and $\xi e^{\phi}$ measures matching efficiency. Matching efficiency evolves according to the stochastic process
\[ z'_\phi = \rho \phi z_\phi + \varepsilon_\phi \]  

(2.18)

where \(0 < \rho_\phi < 1\) and \(\varepsilon_\phi\) is normal with mean 0 and variance \(\sigma^2_\phi\). The endogenous rate at which buyers encounter matching sellers (purchase rate) is

\[ \phi_b \equiv \frac{M}{E_B} = \xi e^{z_\phi} \left( \frac{E_B}{A_S} \right)^{-\gamma} \]  

(2.19)

while the endogenous sales rate is

\[ \phi_s \equiv \frac{M}{A_S} = \xi e^{z_\phi} \left( \frac{E_B}{A_S} \right)^{1-\gamma} \]  

(2.20)

Next, we show how the search market asset price \(q\) is determined in the bargaining process. Household members come to bargain on behalf of the household’s interest. We first denote \(v^b\) and \(v^s\) as the value of each individual buyer and seller. Notice that the household has already decided on search effort \(e^b\) and the fraction of assets to post in the search market \(e^s\), and all the contingent plans on consumption, labor supply, investment and bond accumulations when individuals come to bargain. Therefore, we treat the search costs as sunk costs when individuals come to bargaining. Then using individuals budget constraint (2.2) and (2.5) and noticing that the two individuals are bargaining over the margin \(m_{j,t}\), one can write \(v^b\) and \(v^s\) as

**Problem 3**

\[ v^b(m_j, e^b_j, s_j, b_j, l_j; \Gamma) = u(C_b + C_s) - \frac{\mu}{1 + \nu} L^{1+\nu} + \mathbb{E} [V(S', B'; \Gamma')] \text{ s.t.} \]

\[ c_j + k_b e^b_j + \frac{b'_j}{P} + \tau_j = w l_j + r s_j + q m_j + R \frac{b_j}{P} \]

\[ s'_j = (1 - \delta) s_j - m_j \]
\[ C_b + C_s = \int c_i di, \quad L = \int l_i di \]
\[ S' = \int s' \, di, \quad B' = \int b' \, di \]

**Problem 4**

\[ v^s(m_j, e_j^s, s_j, b_j, l_j; \Gamma) = u(C_b + C_s) - \frac{\mu}{1 + \nu} L^{1+\nu} + \mathbb{E} \left[ V(S', B'; \Gamma') \right] \]

s.t.

\[ c_j + \kappa_s e_s [(1 - \delta) s + i] + \frac{b_j'}{P} + \tau_j = r s_j + (1 - \delta) s_j + (q - \frac{1}{\phi_s}) m_j + R b_j \]

\[ s_j' = \frac{1 - \phi_s}{\phi_s} m_j \]

\[ C_b + C_s = \int c_i di, \quad L = \int l_i di \]
\[ S' = \int s' \, di, \quad B' = \int b' \, di \]

The key of our bargaining framework is that any buyers and sellers interact at the margin \( m_{j,t} \). By this we mean that buyers’ outside option is buying one unit less and sellers’ outside option is selling one unit less. Therefore, the surplus for both buyers and sellers is the respective marginal value of an assets. To be more specific, the buyers’ surplus is the marginal value to the household of an additional unit of matches, i.e.,

\[ -v^b_m = -u'(C)q + \beta \mathbb{E} \left[ V_s(S', B'; \Gamma') \right] \frac{\partial S'}{\partial s_j} \frac{\partial s_j'}{\partial (-m_j)} \]

Similarly, the sellers’ surplus is the marginal value to the household of an additional match for entrepreneurs

\[ v^s_m = u'(C) \left( q_s - \frac{1}{\phi_s} \right) + \beta \mathbb{E} \left[ V_s(S', B'; \Gamma') \right] \frac{\partial S'}{\partial s_j} \frac{\partial s_j'}{\partial m_j} \]
The price of a unit of assets in the search market $q$ is determined via Nash bargaining between a buyer and a seller, i.e. agents bargain over $q$ to maximize

$$\omega \ln (v_m^s) + (1 - \omega) \ln (-v_m^b)$$

where $\omega$ is the bargaining weight of sellers. The sufficient and necessary FOC yields

$$\frac{\omega}{u'(C) \left( q - \frac{1}{\phi_s} \right) + \frac{1 - \phi_s}{\phi_s} \beta \mathbb{E}[V_s(S', B'; \Gamma') | \Gamma]} = \frac{1 - \omega}{-u'(C)q + \beta \mathbb{E}[V_s(S', B'; \Gamma') | \Gamma]}$$

To simplify, the solution can be expressed by noticing that $u'(C) q_b = \beta \mathbb{E}[V_s(S', B'; \Gamma') | \Gamma]$ from the first order condition of the household

**Lemma 5** *The search market bargaining price is*

$$q = 1 + \kappa_b \left( \frac{\phi_s}{1 - \omega} - 1 \right). \quad (2.21)$$

The bargaining solution links the asset price $q$ to the degree of asset resaleability $\phi_s$ and the probability of meeting a matching seller $\phi_b$. Moreover, the bargaining solution (2.21) establishes an intuitive link between asset illiquidity and asset prices: When assets become harder to sell, prices also fall. This relation can be summarized as our main theoretical result:

**Proposition 7** *The search market price $q$ correlates positively with asset resaleability $\phi_s$ (i.e. $\frac{\partial q}{\partial \phi_s} > 0$) and negatively with search market “tightness” from the buyer perspective $\phi_b$ (i.e. $\frac{\partial q}{\partial \phi_b} < 0$).*

Intuitively, with less buyers’ search efforts, $\phi_b$ increases because fewer buyers are competing for the assets that are up for sale; at the same time, the resaleable fraction of the assets, $\phi_s$, decreases because fewer buyers imply fewer matches. The withdrawal of buyers from the market and the drop in market depth (i.e. the number of matches) reduce the over
all surplus from matching. Accordingly, the increase of $\phi_b$ and the decrease of $\phi_s$ in (2.21)
depress the bargained asset price. Our simple framework is, thus, suitable to deliver both
decreasing liquidity and falling asset prices.

When there are neither search costs nor posting costs, i.e. $\kappa_s = 0$ and $\kappa_b = 0$, the search
market price will go to $q = 1$, because there is no asset supply shortage. In this case (and
in the absence of price stickiness we introduce later) the economy collapses to the RBC
framework.

**Corollary 5** When $\kappa_s = 0$ and $\kappa_b = 0$

\[
q_s = q_b = q = 1.
\]

### 2.2.3 Intermediate Goods and Final Goods Producers

A continuum of intermediate goods producers indexed by $i \in [0, 1]$ assembles intermediate
goods $Y_i$ using capital and labor as inputs. Each intermediate producer $i$ operates in an
environment of monopolistic competition. When intermediate goods producers finish pro-
duction, consumption goods are produced by perfectly competitive final goods producers
through aggregating intermediate goods. Both intermediate and final goods producers are
owned by the households.

**Final goods producers.** Consumption goods producers combine a continuum of inter-
mediate products $Y_i, i \in [0, 1]$ according to

\[
Y = \left[ \int_0^1 Y_i^{\frac{\theta-1}{\sigma}} \, di \right]^{\frac{\sigma}{\theta - 1}}
\]

where $\theta \in (0, \infty)$ is the elasticity of substitution of inputs in production.

Let $P_i$ be the nominal price for the intermediate goods. Recall that the nominal price
level for final goods is $P$, then profit maximization of the final goods producer implies a
(downward-sloping) demand function for each intermediate good $i$

$$Y_i = \left(\frac{P_i}{P}\right)^{-\theta} Y$$

From the zero profits condition for final goods producers, the aggregate price level for final goods can be expressed as

$$P = \left[ \int_0^1 p_i^{1-\theta} di \right]^{\frac{1}{1-\theta}}$$

Intermediate goods producers. Each intermediate firm has access to a constant-returns-to-scale (CRS) technology for producing output from capital and labor. Firm $i$ rents capital $k_i$ (at rental rate $r$) and employs labor $l_i$ in a competitive labor market (at real wage $w$) to produce

$$Y_i = e^{z_a} k_i^\alpha l_i^{1-\alpha} = e^{z_a} F(k_i, l_i).$$

where $\alpha \in (0, 1)$ and $z_a$ follows

$$z_a' = \rho_a z_a + \epsilon_a$$ (2.22)

where $\epsilon_a$ is a normally distributed random variable with mean zero and standard deviation $\sigma_a$. Intermediate goods producers face menu costs in adjusting their relative prices.\footnote{Conceptually, we follow Rotemberg (1982) in introducing quadratic price adjustment costs to make price-setting a dynamic problem. Our particular specification of adjustment cost is adopted from Ireland (2004).}

Accordingly, $i$’s real current-period profit is

$$\frac{\Pi_i}{P} = \frac{P_i}{P} Y_i - m_{c,i} Y_i - \frac{\zeta}{2} \left( \frac{P_i}{\bar{P}_{t-1}} - 1 \right)^2 Y$$

$$= \left( \frac{P_i}{P} \right)^{1-\theta} Y - \left( \frac{P_i}{P} \right)^{-\theta} m_{c,i} Y - \frac{\zeta}{2} \left( \frac{P_i}{\bar{P}_{t-1}} - 1 \right)^2 Y$$

where we have substituted the individual demand function facing producer $i$. $m_{c,i} = \frac{r_k + w_l}{Y_i}$ is the marginal costs of producing $Y_i$, $\zeta$ measures the magnitude of price adjustment costs, and $\pi$ is the steady state gross inflation rate. Taking the cost-minimizing factor inputs $k_i$
and $l_i$ as given, intermediate goods producer $i$ sets his price $P_{i,t}$ in order to solve

$$\max_{P_{i,t}} E_t \sum_{s=0}^{\infty} \Lambda_{t+s} \frac{\Pi_{i,t+s}}{P_t}$$

where $\Lambda_{t+s} = \frac{\beta u'(C_{t+s})}{u(C_t)}$ is the stochastic discount factor of the households. The first-order conditions for this problem associated with each producer are, in recursive notation,

$$(\theta - 1) \left[ \frac{P_t}{P} \right]^{-\theta} \left( \frac{Y}{P} \right) = \theta \left[ \frac{P_t}{P} \right]^{-\theta - 1} m_{c,i} \left( \frac{Y}{P} \right) - \zeta \left( \frac{P_t}{\bar{\pi}P_{i-1}} - 1 \right) \left( \frac{Y}{\bar{\pi}P_{i-1}} \right)$$

$$+ \zeta E \left[ \left( \frac{\beta u'(C')}{u'(C)} \right) \left( \frac{P_t'}{\bar{\pi}P_t} - 1 \right) \left( \frac{Y'}{P_t} \right) \left( \frac{P_t'}{\bar{\pi}P_t} \right) \right]$$

If $\zeta = 0$, i.e. without price adjustments costs, the model collapses to the case of monopolistic competition. Then, relative prices are set at a constant mark-up $\frac{\theta}{\theta - 1}$ over nominal marginal costs

$$P_t = \frac{\theta}{\theta - 1} P m_{c,i}$$

We look for a symmetric equilibrium in which all intermediate goods producers set the same price, such that we can drop type subscript $i$ and $P_t = P$, $\forall i \in [0,1]$. The first-order conditions then collapse to

$$(\theta - 1) Y = \theta m_{c,Y} - \zeta \left( \frac{\pi}{\bar{\pi}} - 1 \right) \left( \frac{\pi}{\bar{\pi}} \right) Y + \zeta E \left[ \left( \frac{\beta u'(C')}{u'(C)} \right) \left( \frac{\pi'}{\bar{\pi}} - 1 \right) \left( \frac{\pi'}{\bar{\pi}} \right) Y' \right] \tag{2.23}$$

This expression is the New Keynesian Phillips Curve.

Finally, the rate of return on capital and the wage rate can be solved from the cost-minimization that are necessary to produce a given amount of intermediate goods $Y_t$.

$$\min_{k_i,l_i} r k_i + w l_i - m_{c,i} \left( Y_t - e^{\delta_a} k_i^{\alpha} l_i^{1-\alpha} \right)$$

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where the Lagrange multiplier $m_{c,i}$ represents the marginal costs of producing $Y_i$. Again imposing symmetry and aggregating over individual choices yields

$$w = m_c (1 - \alpha) e^{\hat{\sigma}z} \left( \frac{K}{L} \right)^{\alpha} = m_c F_L(K, L)$$  \hspace{1cm} (2.24)

$$r = m_c \alpha e^{\hat{\sigma}z} \left( \frac{K}{L} \right)^{\alpha} = m_c F_K(K, L)$$  \hspace{1cm} (2.25)

### 2.2.4 The Government

Following Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011) the government conducts fiscal policy, conventional monetary policy, and unconventional credit policy. We discuss the policies one by one in the following.

Fiscal policy can impose lump-sum tax. To focus on monetary policy, we do not allow the tax to vary, i.e.,

$$\tau = \bar{\tau}$$ \hspace{1cm} (2.26)

Conventional monetary policy consists of the central bank setting the nominal interest rate or money stock growth rate rule. The two types of conventional policy we consider are

1. Constant supply of liquid assets

$$B' = B_g \quad \text{or} \quad R' = 1$$ \hspace{1cm} (2.27)

$R' = 1$ means that liquid assets bear no nominal interest so that we can view them as fiat money.

2. Feedback interest rate rule

$$R' = \max \{R \pi^\psi, 0\}$$ \hspace{1cm} (2.28)
where $\pi = \frac{P}{P_{-1}}$ and $\psi > 1$.\footnote{We proceed with the first specification of conventional monetary policy, i.e. with the case of fiat money. Details for the case of interest-bearing bonds are available from the authors upon request. The qualitative results are robust to either specification.}

Unconventional policy corresponds to government purchases or sales of private paper (denoted by $S'_g > 0$ as the asset position) in the search market as a function of the liquidity of private paper

$$S'_g = \min \left\{ K\psi_k \left( \frac{\phi_s}{\bar{\phi}_s} - 1 \right), 0 \right\}$$

(2.29)

where $\bar{\phi}_s$ is the steady state of $\phi_s$.

Our approach to modelling such policy is intended to ensure that unconventional interventions affect the economy exclusively through their impact on portfolio compositions and asset market prices. In particular, unconventional policy does not directly relax any agents’ resaleability constraint. Also, when the government intervenes, it has to respect the search market structure.

The key question regarding our modelling strategy for unconventional policy is at what price the government can intervene. The full solution would require modelling the valuation of asset purchases or sales from the government perspective to obtain a bargaining solution. We do not opt for this, since optimal policy design is beyond the scope of the present paper. Instead, we let the government solely use the bargaining price from the market.\footnote{We make this assumption on two grounds: One, to mimic the actual purchases of public and private assets by central banks in response to the Great Recession; and two, to maintain tractability of the search market structure in the absence of optimal policy considerations. Our results should thus be interpreted as pointing to a lower bound of the effectiveness of government interventions.} This choice is equivalent to assuming that the government does not pay search costs to intervene on the market. Therefore, the aggregate inputs to the search market, effort and assets for sale, are modified to be

$$E_B = E_b + \frac{S'_g - (1 - \delta) S_g}{\phi_b}$$
Although we allow the government sector to have a better technology for entering the search market than private agents, we still view this approach as helpful for examining the mechanism of unconventional policy. In particular, our modelling strategy

(1) maintains the matching and bargaining framework even with government interventions,

(2) avoids the problem of different prices for asset sales and purchases from the government perspective (see equation (2.30)),

(3) still allows the market price, \( q \), to react to policy.

As per the government’s budget constraint (2.30), asset purchases are financed to a large extent through the issuance of liquid assets. Therefore, even given technological constraints, the government can potentially correct externalities in the economy with liquidity frictions by supplying liquid assets. Such policy feeds into private portfolio choices between (liquid) risk-free assets and private paper claims. Via these, unconventional interventions ultimately affect asset prices, search effort and asset liquidity (\( \phi_s \)) in the search market.

Fiscal policy interacts with monetary policy and both policies interact with the real economy. These interactions determine the price level \( P \) and the supply of nominal liquid assets \( B' \). Let the real bond be \( B_r \equiv B/P \) and let the supply of nominal liquid assets be \( B' = \eta B \) where \( \eta \) is the growth rate of liquid assets. Now, we can write the government budget constraint as

\[
qS'_g + RB_r + G = \tau + (r + (1 - \delta) q)S_g + \eta B_r
\]

(2.30)

where \( G \) corresponds to government spending. To have a tractable equilibrium definition, we work with real bonds as the liquidity measure. The inflation rate can thus be written as

\[
\pi = \frac{P}{P_{-1}} = \frac{\eta_{-1}B_{r,-1}}{B_r}
\]

(2.31)
which is used in the household maximization problem, i.e., (2.16).

### 2.2.5 Equilibrium Definition

The recursive competitive equilibrium is a mapping $(S, S_g, R, z_a, z_\phi) \to (S', S'_g, R', z'_a, z'_\phi)$, with associated consumption and investment choices $\{C, I\}$, capital and labor inputs $\{K, L\}$, marginal costs $\{m_c\}$, sales and purchase rates on the asset market $\{\phi_s, \phi_b\}$, government policy rules $\{\tau, R', S'_g\}$, real liquid assets and their growth rate $\{B_r, \eta\}$, asset, goods and factor prices $\{q, q_b, q_s, q_r, w, r, \pi\}$, and laws governing the evolution of $(z_a, z_\phi)$. In particular, the mapping satisfies

1. Individual optimality: given prices $\{q, q_b, q_s, q_r, w, r, \pi\}$ and search market characteristics $\{\phi_b, \phi_s\}$, the policy functions solve the representative household’s and intermediate good producers’ optimization problems; i.e., (2.14)-(2.16) are satisfied (replacing $\pi = \frac{P}{P'}$ by using (3.17)) with investment $I$ defined in (2.12); similarly, intermediate good producers’ optimality conditions (2.23)-(2.25) are satisfied.

2. Government policy rules: fiscal policy obeys (2.26), conventional monetary policy follows either (2.27) or (2.28), and unconventional monetary policy is conducted according to (2.29).

3. Market clearing\(^{14}\):
   
   (a) Households’ budget constraint (2.13) is satisfied.
   
   (b) Government’s budget constraint (2.30) is satisfied.
   
   (c) Search market clears:
   
   $\phi_s = (\phi_b)^{\frac{\tau - 1}{\tau}} (\xi e^{\omega})^{\frac{1}{\tau}}$

\(^{14}\)Labor market clearing $L = (1 - \chi)l$ is implicitly assumed.
The search market price $q$ is determined by Nash bargaining according to (2.21). While effective equity prices and the down-payment \( \{q_b, q_s, q_r\} \) are given by definitions (2.4) and (2.9).

(d) Equity market clears:

\[
S + S_g = K
\]

(e) Capital market clears:

\[
K' = (1 - \delta)K + I
\]

4. Exogenous matching efficiency and productivity evolve according to (2.18) and (2.22), respectively.

Note that the goods market clearing condition is implied by combining household members’ budget constraints and the government budget constraint. This yields the aggregate resource constraint

\[
C + I + G + \kappa_b E_b + \kappa_s ((1 - \delta) \chi S + I) = \left[1 - \frac{\xi}{2} \left(\frac{\pi}{\pi} - 1\right)^2\right] Y
\]

### 2.3 Numerical Results

We choose those parameters that are not related to the search market by following the literature (Table 2.2). To pin down the search-market related parameters, we choose four targets in the long run steady state: Tobin’s $q$, the liquidity share, the resaleable fraction of assets, and finally the purchase and selling price difference.\(^{15}\) The average Tobin’s $q$ in the U.S. ranges from 1.1 to 1.21 according to COMPUSTAT data (see Belo, Xue, and Zhang (2012)). From the flow of funds data, the liquidity share, defined as the total real liquid assets over real GDP is about 40%. Finally, since our illiquid assets represent all assets that are not government bonds, we choose the price spread to be relatively high as 6%.

\(^{15}\)See Table 2.3.
We have only four targets, although there are five parameters that relate to the search market, i.e. $\xi$, $\kappa_b$, $\kappa_s$, $\gamma$, and $\omega$. However, given our targets the matching efficiency parameter $\xi$ is always linked to $\gamma$ (This property is because of constant return to scale matching technology.\footnote{See the derivation of the steady state in the appendix.}) Therefore, it is sufficient to determine four parameters. Without loss of generality, we set $\gamma = 0.5$. The baseline calibration can be found in Table 2.2 in the appendix.

After determining the steady state, we log-linearize the system around the steady state to approximate the solution to the nonlinear model. Then we solve the linear rational expectation model with a persistent productivity and matching efficiency process and analyze unconventional government interventions in asset markets.

### 2.3.1 Technology Shock

Responses to a negative technology shock are displayed in Figure 2.2. In order to keep the analysis simple, we shut off unconventional policy and focus on the case of fiat money, i.e. conventional monetary policy follows (2.27).

A negative technology shock depresses the productivity of capital and its value to the household. Hence, capital accumulation is valued less which reduces the total surplus of an additional match on the search market. Accordingly, households have their workers engage less in costly search activity for investment. Less buy-side activity makes it harder for entrepreneurs to find counterparties, which translates into a sharp drop in the resaleable fraction $\phi_s$. A negative TFP shock thus triggers an endogenous decline in the liquidity of financial assets issued by private agents. At the same time (as demonstrated in our analytical result (2.21)), the fall in demand on the asset market also induces a lower asset price. The lower resale value of financial assets depresses entrepreneurs’ net worth and tightens financing constraint (2.12) for investment. Aggregate investment thus falls strongly.

Importantly, liquidity - the ease of initial issuance and resaleability - of financial assets is endogenously generated through the features of the search market. In the absence of
search frictions these liquidity effects would not occur. In the RBC world, a negative TFP shock would also affect the demand for capital goods. However, the optimal amount of investment would still be financed without frictions so that the asset price would stay constant at unity. In contrast, entrepreneurs in our framework are financing constrained and asset prices fall in response to negative TFP shocks. Therefore, investment contracts more strongly than in the RBC benchmark.

However, technology shocks fail to generate the empirically observed countercyclical response of the liquidity share in the economy. This result is somewhat surprising as one would expect the liquidity service provided by money (bonds) to become more valuable to the household when the liquidity of private assets declines. However, this is not the case for the following three reasons: first, the technology shock decreases available resources, such that households are exposed to a negative income effect that lowers their demand for any assets. Second, in view of the persistence of productivity shocks, even future investment will be unattractive. Hence, the incentive to hedge against persistent illiquidity is weak.
And third, as is standard in the New Keynesian literature (e.g., Ireland (2004)), nominal prices (and inflation) increase in response to the negative TFP shock. As the nominal price level reacts sluggishly, real factor prices cannot adjust flexibly and the marginal costs of production increase. This translates into inflationary pressures, which contribute to the strong decline in the real value of money (bonds) and, hence, the liquidity share.\footnote{The liquidity share even drops in the absence of the price effect due to nominal rigidities. See Figure 2.5 in the appendix.}

We glean two key observations from this exercise: (1) endogenizing asset resaleability can reconcile declining liquidity with falling asset prices; (2) even in the presence of endogenous procyclical liquidity, technology shocks cannot account for the observed counter-cyclical liquidity share.

### 2.3.2 Matching Efficiency Shock

In Kiyotaki and Moore (2012) and Shi (2012) asset resaleability, i.e. $\phi$, is exogenous. Their analysis focuses on a negative shock to the exogenous resaleability. The problematic feature of such a liquidity shock is that it acts like a supply shock: When $\phi$ drops, the supply of assets up for sale shrinks. However, productivity of capital is not affected by the shock such that the demand for private assets does not fall. Therefore, asset prices boom - a counter-factual phenomenon in recessions. In contrast, our framework is capable of generating an initial pro-cyclical response of asset prices even with exogenous liquidity shocks.

To investigate a liquidity shock in our endogenous liquidity framework, we present the dynamics after a pure liquidity problem, i.e. a negative shock to the matching efficiency process $z_{\phi}$. This shock is similar to a demand shock in the New Keynesian model: initially, it directly reduces the number of matches on the search market due to technological reasons. This induces less investment, a lower production level, and fewer resources in the future without any changes in TFP. Anticipating this effect, price-setting intermediate goods firms...
expect lower marginal costs. They react by reducing their prices to increase relative demand for their products. But they can do so only sluggishly due to adjustment costs. Hence, marginal costs fall today, the mark-up increases and factor rents decrease. This effect triggers a strong fall in employment and production today, leaving households with less resources today. As a consequence, the return to capital is lower today and tomorrow and financial assets become very unattractive and harder to sell.

Figure 2.3: Impulse responses after a negative matching efficiency shock (one percent).

At this point our search-based endogenous liquidity framework is critical to translate the negative income effect into a sufficient decline in demand on the asset market. Households prefer workers to substitute into liquid government bonds (flight to liquidity), rather than spending more resources in costly search on the asset market. Consequently, search effort (tantamount to asset demand) shrinks and asset resaleability \( \phi_s \) drops unequivocally both due to the exogenous shock and the endogenous withdrawal of buyers. The decline in the purchase rate, on the other hand, exhibits a hump-shaped response. The initially less pronounced drop precisely reflects the aforementioned endogenous decline in search effort.
In sum, the negative demand effect dominates the negative supply effect associated with the exogenous decrease in matching efficiency, at least initially. Accordingly, the asset price falls.\textsuperscript{18} Our model demonstrates that both endogenous liquidity and nominal price rigidities are necessary to generate this pro-cyclical response of asset prices after pure liquidity shocks.\textsuperscript{19}

Note that the matching efficiency shock also increases the hedging value of money (bonds). As mentioned, TFP is unaffected by the matching efficiency shock. Therefore, future investment remains attractive. To take advantage of future investment opportunities, households hedge against the persistent illiquidity of privately issued assets by expanding their government bond holdings. Demand for liquid government bonds increases today, which drives up the liquidity share in line with the data.

\subsection*{2.3.3 Unconventional Policy}

Finally, we investigate whether government purchases of claims to private assets (unconventional monetary policy) may have real effects. To illustrate, we set $\psi_k < 0$ so that the government starts to purchase private equity when the resaleability of assets is below the steady state level. We focus on aggregate matching efficiency shocks as the source of the disturbance, because, as shown before, the corresponding model dynamics replicate a series of stylized facts. Importantly, the government respects all technological constraints in the economy. The policy can only change the composition of liquid and illiquid assets in the economy.

Figure 2.4 displays the sensitivity of the model dynamics to asset purchases by the public sector on the search market. The steady state of the economy is the same as that

\textsuperscript{18} Figure 2.6 displays the transition to the new steady state after a permanent negative matching efficiency shock. This exercise reveals that the positive over-shooting of the asset price is due to the temporary nature of the shock. Hence, the greater the persistence of the shock, the more protracted the decline in the asset price.

\textsuperscript{19} In contrast to our result, nominal frictions in the KM model still cannot avoid the equity price boom as discussed in Shi (2012). This further supports our claim that endogenous liquidity frictions are key to generating lower asset prices in recessions.
of the previous economy without unconventional interventions (i.e. where the government holds no private assets). Once the matching efficiency shock hits, unconventional policy is activated according to rule (2.29).

The policy succeeds in reducing the need for liquidity in the private sector, which can be inferred from the price of liquid assets that decreases more strongly. This effect is due to the increased supply of government bonds (money), which we force the government to use in order to purchase private financial assets. On the other hand, the intervention has one major drawback: When the government is constrained to purchase assets on the search market it crowds out private buyers. This effect dominates the liquidity provision by the government, such that the net impact on investment and output is negative. These results
suggest that asset purchases could be more effective if conducted on a centrally cleared market or targeted at the balance sheets of intermediating institutions.\footnote{Our results seem to caution against the appropriateness of unconventional policy measures, such as asset purchases or the loosening of collateral frameworks, adopted by treasury departments and central banks in the U.S. and Europe. However, it should be noted that these measures were at least partly targeted at markets that had entirely collapsed (e.g. asset-backed securities). Public asset demand was intended to resuscitate these markets, in which case crowding out of private demand was hardly a concern. In this sense, our model provides an appropriate framework to think of interventions in depression, but still active markets.}

\section*{2.4 Conclusion}

We illustrate how asset liquidity - the ease of initial issuance and resaleability - can positively co-move with output through asset markets with search frictions, and how liquidity can have feedback effects on consumption, investment, and employment. Investor participation in the search market drives the liquidity and price of financial assets. Asset liquidity is important for financing new investment. Therefore, aggregate shocks that affect the incentive to purchase financial assets also affect the financing conditions of firms through endogenous liquidity fluctuations.

Our model matches the procyclicality of asset liquidity (and prices) observed in the data. Moreover, it shows that shocks that affect liquidity of financial assets rather than TFP can explain a countercyclical share of liquid assets relative to GDP, as well as cyclical properties of major macro variables. The endogenous asset liquidity is key to match this set of stylized facts. Implicitly, we also corroborate the finding in the previous literature that purely exogenous liquidity shocks lead to higher asset prices in recessions.

Importantly, we consider the matching framework as a shortcut for modelling financial intermediation. Financial intermediaries help channel funds from investors to firms which need outside funding. Although we do not explicitly model intermediaries’ balance sheets and borrowing/lending decision in our framework, the structure of financial intermediation...
is similar to the process of matching investors and entrepreneurs with investment opportunities.

Finally, open market operations such as asset purchase programs can potentially have real effects by easing liquidity frictions. However, such policies need to be carefully designed in order to avoid crowding-out of private market participants. Building on this result, future research could focus on the Ramsey problem of the optimal mix of conventional, unconventional and fiscal policy measures in the presence of endogenous illiquid asset markets. The interactions among secondary asset markets, policy responses, and nominal price levels can shed light on important policy debates.

2.5 Appendix

2.5.1 Equilibrium Conditions

Given the aggregate state variable \( \Gamma = (S, S_g, R; z_a, z_\phi) \), we are solving for

\[
\left( S', S'_g, R', \eta, B_r, C, I, L, K, mc, \Pi, \phi_b, \phi_s, q, q_r, q_b, q_s, r, w, \tau, \pi \right)
\]

together with the exogenous law of motion of \((z_a, z_\phi)\). Since there are 21 variables in total, one needs 21 equations.

1. Given the aggregate state and the price functions, the policy functions solve the representative household’s optimisation problem

\[
u'(C)w = \mu L^\nu
\]

\[
u'(C) = \beta E \left[ u'(C') \left( (1 - \chi) + \frac{q'_b}{q'_r} \right) \right] R'B'_r
\]

\[
u'(C) q_b = \beta E \left[ u'(C') \left( (1 - \chi) (r' + (1 - \delta) q'_b) + \chi (r' + (1 - \delta)) \frac{q'_b}{q'_r} \right) \right]
\]
\[ I = \chi \frac{[r + (1 - \delta) \phi s q_s] S + R B_p - \tau}{1 - \phi s q_s} \]

(a) Intermediate Goods Producer

\[(\pi - 1) \pi = \theta \frac{m_c - \theta - 1}{\theta} + \frac{\beta u'(C')}{u'(C)} (\pi' - 1) \pi' Y' Y \]

\[ r = m_c e^z F_K(K, L) \]

\[ w = m_c e^z F_L(K, L) \]

\[ \Pi = Y - (rK + wL) - \frac{\zeta}{2} (\pi - 1)^2 \]

2. Policy: (a). Fiscal Policy:

\[ \tau = \bar{\tau} \]

(b). Unconventional Monetary Policy: Purchasing rule:

\[ S'_s = \max \left\{ K \psi_k \left( \frac{\phi s}{\phi s} - 1 \right), 0 \right\} \]

(c). Conventional Monetary Policy: (nominal interest rate)

\[ R' = R(\pi)^\psi \pi \]

3. Price definition: The replacement cost of equity is defined as

\[ q_r \equiv \frac{1 - \phi s q_s}{1 - \phi s} \]

The effective purchasing price is defined as

\[ q_b \equiv q + \frac{\kappa_b}{\phi_b} \]
The effective selling price is defined as

\[ q_s = q - \frac{\kappa_s}{\phi_s} \]

4. Household budget constraint

\[ C + q_b S' + \eta B_r + \left( (1 - \chi) + \chi \frac{q_b}{q_r} \right) \tau = wL + \left( (1 - \chi)(r + (1 - \delta)q_b) + \chi(r + (1 - \delta)) \frac{q_b}{q_r} \right) S + \left( (1 - \chi) + \chi \frac{q_b}{q_r} \right) R B_r + \Pi \]

5. Government budget constraint: (Note: used to pin-down \( \eta \))

\[ q S_g' + R B_r + G = \tau + (r + (1 - \delta) q) S_g + \eta B_r \]

where \( \eta \equiv \frac{B_g'}{B_r}, B_r \equiv \frac{B}{p} \) and

\[ \pi = \frac{p}{p_{-1}} = \eta_{-1} \frac{B_{r-1}}{B_r} \]

6. Search Market Clearing

\[ \phi_s = \phi_b \frac{\gamma_{-1}}{\gamma} (\xi e^{\phi})^{\frac{1}{\gamma}} \]

\[ q = 1 + \kappa_s + \frac{\kappa_b}{\phi_b} \left( \frac{\phi_s}{1 - \omega} - 1 \right) \]

7. Equity Market Clearing:

\[ S + S_g = K \]

8. Capital Market Clearing:

\[ K' = (1 - \delta) K + I \]
9. Exogenous shocks

\[ z'_a = \rho_a z + \epsilon'_a \]

\[ z'_\phi = \rho_\phi z + \epsilon'_\phi \]

2.5.2 Steady State

In steady state, any variable \( X = X' \). The solution strategy is to guess \((K, L, B_r, \phi_b)\) and to express all other variables as functions of these. Given \((K, L, B_r, \phi_b)\), we have

\[ z'_a = \rho_a z + \epsilon'_a \rightarrow z_a = 0 \]

\[ z'_\phi = \rho_\phi z + \epsilon'_\phi \rightarrow z_\phi = 0 \]

\[ \eta = 1 \]

\[ \pi = 1 \]

\[ S'_g = \max \left\{ K\psi_k \left( \frac{\phi_r}{\phi_s} - 1 \right), 0 \right\} \rightarrow S_g = 0 \]

\[ \tau = \bar{\tau} \]

\[ qS'_g + RB_r + G = \tau + (r + (1 - \delta) q) S_g + \eta B_r \rightarrow R = \eta + \frac{\tau - G}{B_r} \]

\[ S + S_g = K \rightarrow S = K \]

\[ m_c = \theta - 1 \]

\[ r = m_c e^F K(L) \]

\[ w = m_c e^F L(K, L) \]

\[ \Pi = K^\alpha L^{1-\alpha} - (rK + wL) \]

\[ u'(C)w = \mu L^\gamma \rightarrow C = \left[ \frac{w}{\mu L^\gamma} \right]^{1/\sigma} \]
\[ u'(C) = \beta E \left[ u'(C') \left[ (1 - \chi) + \frac{q_b}{q_r} \right] \frac{R' B'_r}{\eta B_r} \right] \rightarrow q_b = \frac{\eta}{\beta R} - \frac{(1 - \chi)}{\chi} \]

\[ u'(C) q_b = \beta E \left[ u'(C') \left[ (1 - \chi) (r' + (1 - \delta) q_b) + \chi (r' + (1 - \delta)) \frac{q_b}{q_r} \right] \right] \]
\[ \rightarrow q_b = \frac{\beta \left\{ \frac{(1 - \chi) r + (r + (1 - \delta)) \left[ \frac{\eta}{\beta R} - (1 - \chi) \right]}{1 - \beta (1 - \chi) (1 - \delta)} \right\}} {1 - \beta (1 - \chi) (1 - \delta)} \]

\[ q_r = \frac{q_b}{q_r} = \frac{\beta \chi \left\{ (1 - \chi) r + (r + (1 - \delta)) \left[ \frac{\eta}{\beta R} - (1 - \chi) \right] \right\}} {1 - \beta (1 - \chi) (1 - \delta)} \frac{\left[ \frac{\eta}{\beta R} - (1 - \chi) \right]} {1 - \beta (1 - \chi) (1 - \delta)} \]

\[ K' = (1 - \delta) K + I \rightarrow I = \delta K \]

\[ \phi_s = \phi_b \frac{\xi}{\xi} \]

\[ q_b = q + \frac{\kappa_b}{\phi_b} \rightarrow q = q_b - \frac{\kappa_b}{\phi_b} \]

\[ q_s \equiv q - \frac{\kappa_s}{\phi_s} \rightarrow q_s = q - \frac{\kappa_s}{\phi_s} \]

To check for consistency of the initial guess, we solve the following five equations:

\[ C + q_b S + \eta B_r + \left[ (1 - \chi) + \frac{q_b}{q_r} \right] \tau = w L + \left[ (1 - \chi) (r + (1 - \delta) q_b) + \chi (r + (1 - \delta)) \frac{q_b}{q_r} \right] S \]
\[ + \left[ (1 - \chi) + \frac{q_b}{q_r} \right] R B_r + \Pi \]

\[ I = \chi \left[ \frac{r + (1 - \delta) (\phi_s q_s + \lambda (1 - \phi_s q_s))}{1 - \phi_s q_s} \right] S + R_p^B - \tau \]

\[ q_r \equiv \frac{1 - \phi_s q_s}{1 - \phi_s} \]

\[ q = 1 + \kappa_s + \frac{\kappa_b}{\phi_b} \left( \frac{\phi_s}{1 - \omega} - 1 \right) \]

2.5.3 Tables
### Table 2.2: Baseline calibration

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<th>Parameter</th>
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#### Intermediate goods production

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#### Capital goods production

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#### Search and Matching

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#### Government

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#### Shock processes

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<tr>
<td>Persistence, liquidity shock</td>
<td>$\rho_\phi$</td>
<td>0.9225</td>
<td>exogenous</td>
</tr>
<tr>
<td>Std. dev., liquidity shock</td>
<td>$\sigma_\phi$</td>
<td>0.01</td>
<td>1%</td>
</tr>
</tbody>
</table>

Notes: The model is calibrated for quarterly data.

### Table 2.3: Selected Moments: Data vs. Model

<table>
<thead>
<tr>
<th>Moment</th>
<th>Concept</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Search Market Sales Rate</td>
<td>$\phi_s$</td>
<td>24.7%</td>
<td>24.7%</td>
</tr>
<tr>
<td>Search Market Bid-ask Spread</td>
<td>$q_b - q_s$</td>
<td>1.56%</td>
<td>5.4%</td>
</tr>
<tr>
<td>Tobin’s q</td>
<td>$q$</td>
<td>1.05 - 1.21</td>
<td>1.1</td>
</tr>
<tr>
<td>Liquidity Share</td>
<td>$\beta$</td>
<td>52.07%</td>
<td>52.07%</td>
</tr>
</tbody>
</table>

Notes: to be completed
Figure 2.5: Impulse responses after a negative productivity shock (one percent) with monopolistic competition only.

Figure 2.6: Transition paths after a permanent negative matching efficiency shock (one percent).
Chapter 3

Optimal Policy with Liquidity Frictions:
A Primal Approach

3.1 Introduction

The recent financial crisis has shown that asset market liquidity fluctuations can have huge impacts on the real economy. However, if one driving force of the crisis was the liquidity problem of financial assets, it is surprising that there is little understanding on how the policy should react, or even whether it should react at all.

The risk-free rate and the degree of asset market liquidity affect the portfolio allocation of low-return-liquid and high-return-illiquid assets. The portfolio allocation further affects real investment, and then the real economy. One way of viewing this is by analyzing the ratio of the liquid government bonds value over the total asset value in firms’ balance sheet. As documented by many studies, the liquidity ratio has increased dramatically during the last recession of 2008-2009 in all sectors (even though the equity price was already bouncing back in 2009). The changes suggest a large portfolio re-balance towards liquid assets in all sectors during the past recession and afterwards. The question we aim to tackle, ¹

¹This empirical documentation at least includes Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011) and Cui and Radde (2013).
therefore, is how the optimal monetary policy should behave in a liquidity constrained economy.

We develop a model with liquidity frictions based on Kiyotaki and Moore (2012) and a Ramsey plan with instruments of taxation, nominal interest rates, and unconventional monetary policy, i.e., the purchase of financial assets issued by the private sector. The asset liquidity is measured by its resaleability. Government bonds can be fully resold but only a part of privately issued assets can be resold. Our findings show that with a proper policy design, the economy can attain the first-best allocation even though there are severe liquidity frictions.

Consider firms which face stochastic investment opportunities to increase capital stock. When a firm has an opportunity, it invests by using resources from issuing financial claims to the new capital stock and/or liquidating existing financial claims. Suppose financial claims issuance is not perfect, and suppose the firm’s existing financial claims are not fully resaleable. The financial frictions reduce the amount of resources that should be transferred from firms without investment opportunities to those which have. Not surprisingly, the frictions limit investment risk-sharing and reduce total output.

Therefore, firms will save in intrinsically valueless but fully liquid assets as extra internal resources for investment. In our model, the liquid assets are government bonds which can lubricate the transfer of resources and whose rate of return is risk-free. However, by holding onto government bonds, agents do not internalize their own effects on lowering the risk-free rate because they compete for limited liquid assets. If the risk-free rate is low, firms need even more liquid assets for future investment. In equilibrium, the agents over-save in government bonds and an extra provision of liquid assets are necessary.

Given the liquidity frictions in a competitive economy, we ask whether a constrained planner (i.e., who also respects the liquidity frictions) can improve the social welfare, and how. We embed policy instruments in a way that we are able to reach a closed-form Ramsey plan. The Ramsey plan is designed to influence the composition of liquid and partially
liquid assets. Importantly, we are not dealing with policies that can eliminate liquidity frictions.

For discussion simplicity in the following, we call the economy under optimal policy the Ramsey plan economy, while the economy with no tax and fixed supply of liquid assets the constant bond economy.

In the steady state, we show that the policy should pay a higher risk-free rate in the Ramsey plan economy than the constant bond economy, since agents have a propensity to over-save in government bonds. Intuitively, if the interest rate in equilibrium is raised to the time preference rate, the economy reaches the first-best. Those who have investment opportunities will have better internal financing. More wealth is then transferred from agents with funds but have no investment opportunities, to those with investment opportunities but not enough funds. The welfare gains, in our preferred calibration, amount to a 0.41% increase on permanent total consumption. At the same time, the Ramsey plan can have about a 5.19% increase on permanent total investment, and a 1.68% increase on permanent total output.

How to achieve the optimal allocation? To avoid the liquidity “shortage”, there should be a growing supply of government bonds. But the growth rate needs to be lower than the nominal interest rate, such that the inflation rate plus the time preference rate equals the nominal interest rate.\(^2\) We further show that the financing of interest payments should be from holding some privately issued financial assets, i.e., unconventional monetary policy. Notice that, the financing is not through taxing capital, which confirms the result of zero capital taxation in the optimal taxation literature.

Finally, we examine system dynamics after persistent liquidity problems in financial assets. The distinguished feature is that the asset price and other macro variables are stabilized very quickly. After liquidity shocks to existing privately issued financial claims, the

\(^2\)Even though we have a somewhat similar result to the “Friedman Rule”, running deflation is not necessary. Also, the reason behind the policy is very different from the “Friedman Rule”. The key reason in our model is the propensity to over-save, instead of the usual opportunity costs of holding money due to transaction needs.
private sector will re-balance towards government bonds. Thus, the demand for privately issued assets falls. The Ramsey plan purchases these assets and adds an extra demand. The extra demand maintains asset price and help firms’ investment since they benefit from liquidating existing assets.

We depart from early monetary policy literature such as Woodford (2003) by focusing on the Ramsey problem and using the primal approach, which can be solved analytically. The key for this tractability is to derive an implementability condition which summarizes all the decentralized market conditions, following the optimal taxation literature such as Golosov, Kocherlakota, and Tsyvinski (2003). In a representative agent model, a closed-form implementability condition is often achievable. A framework with heterogenous agents and incomplete markets setup usually does not have analytical implementability conditions and requires numerical computations. However, we still maintain the tractability. We find that the “implementability” condition equates the net-worth difference of different types of firms to the total gains if non-resaleable capital becomes resaleable. Intuitively, the larger the financial frictions are, the worse the investment opportunity risk-sharing is, and the larger the net-worth difference becomes afterwards.

To our knowledge, our contribution is to first analyze the optimal policy under liquidity frictions, particularly in a fully tractable approach. A standard New-Keynesian optimal policy uses the second order approximation to the objective function, usually finding a balance between the output gap and the inflation gap. Such strategy is not particularly attractive in the context of liquidity frictions, where heterogeneous agents and uninsured risks become the central theme. With the help of the closed-from implementability condition, heterogeneity can be summarized in one constraint for the policy maker.

---

3 A recent notable exception is Golosov and Sargent (2012) in which the extended model consists of labor income risks and borrowing constraints.

4 As we do not follow the strategy of approximating the objective function, we also do not have welfare ranking problems, since in principle one can consider all higher order terms.
Related Literature: The literature on financing frictions is very fast. Early contributions include at least Bernanke, Gertler, and Gilchrist (1999) and Kiyotaki and Moore (1997). The common theme is some form of borrowing constraint, which is related to imperfections in issuing financial claims in our model. Recently, there is a growing literature focusing on the interaction between imperfections in outside issuance and resaleability constraints of existing assets, such as in Kiyotaki and Moore (2012), Shi (2012), and Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011). The presence of outside issuance constraints and liquidity constraints open up the possibilities for government bonds or fiat money to circulate. Therefore, there will be both fully liquid government issued assets and partially liquid privately issued assets. By changing the two assets in the portfolio compositions, the policy can potentially affect the real economy. We contribute to this literature by developing a tractable optimal policy plan to highlight the externalities caused by the frictions and to show how the policy should redistribute wealth.

To deliver the tractability, we borrow the concept of implementability condition from the optimal taxation literature such as Chari and Kehoe (1999) and Werning (2007). By doing so, we avoid welfare ranking problem discussed in Clarida, Galí, and MarkGertler (1999) and Woodford (2003). Our contribution to this literature is embedding liquidity frictions to the Ramsey problem. We show that capital taxation should be zero in the long run, a similar result found before; the financing of interest rate payment should be from the return of holding privately issued financial assets. Therefore, it is not necessary to run deflation, i.e., the “Friedman rule” as in many previous work such as Chari and Kehoe (1999).

Finally, there is a recent literature of studying pecuniary externality in the presence of financing constraints, in which the competitive market is not efficient. The agent does not take into account her own effect on prices and imposes an externality on others. Examples at least include Bianchi (2009), Bianchi (2010), Korinek (2009), and Lorenzoni (2008)

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5 See a recent extensive survey by Brunnermeier, Eisenbach, and Yuli (2012).
6 See, for example, Gertler and Karadi (2011) and Gertler and Kiyotaki (2010).
who look at how far an economy with financial frictions is from the first-best. We depart from this strand by restricting attention to the constrained efficient allocation. The planner respects the resaleability constraint and only affect the composition of liquid and partially liquid assets. Surprisingly, the constrained efficient policy is also efficient.

3.2 A Macro Model of Liquidity Frictions

We construct a simple version of the Real Business Cycle (RBC) model that incorporate liquidity frictions similar to that of Kiyotaki and Moore (2012) (KM hereafter). Compared to KM, the key new things are that we take into consideration optimal policy, which can tax and set monetary policies (both conventional and unconventional ones). The government can collect tax from capital returns and wage bills. Conventional monetary policies are exemplified by setting nominal interest rates. For unconventional monetary policies, we consider open market operations on purchasing financial assets issued by private agents.7

Time is discrete and infinite. The economy has a continuum of entrepreneurs (measure 1), a continuum of households (measure $L$), representative firms, and a benevolent government. For simplicity, each household member is assumed to supply labor endowment (1 unit) inelastically and consume the wage bill $w_t$ each period. Doing so highlights asset resaleability problem and the feedback on investment, which is modelled in the following entrepreneurs problem.

7A more detailed explanation of such interpretations is given in Section 4.
3.2.1 Entrepreneurs

Preferences. At time $t$, each agent has expected utility\(^8\)

$$
\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log (c_{t+s}),
$$

(3.1)

where $\mathbb{E}_t$ denotes the conditional expectation and $c_{t+s}$ is consumption at time $t+s$.

Technology. A typical entrepreneur has capital stock $k_t$ at the beginning of each period $t$. The entrepreneur rents the capital to firms and receives the after-tax rental rate $r_t$. The details of the rental rate and production process will be clear soon.

Financing Investment. During period $t$, capital stock $k_t$ depreciates to be $(1 - \delta)k_t$, where $0 < \delta < 1$. Entrepreneurs can invest to increase capital stock if they have an investment opportunity. The arrival of an investment opportunity, i.e., the chance to produce new capital from general output, is independently distributed across entrepreneurs (Note: not across households) and through time. We assume that each entrepreneurs has probability $\chi$ to receive an investment opportunity. Investment completed in period $t$ will be available as capital in period $t+1$:

$$
k_{t+1} = (1 - \delta)k_t + i_t.
$$

Financing new investment is not frictionless. We have two key assumptions. First, we assume no insurance market against having an investment opportunity, so that the market is incomplete. In order to finance the investment, an entrepreneur can issue an equity claim to the future output from the investment, but only $0 < \theta < 1$ fraction of the investment can be issued. One micro-foundation is that entrepreneurs have to participate in the production to produce the full amount of future output, and outsiders may just be able to get $1 - \theta$ of the future output. Entrepreneurs thus can only issue $0 < \theta < 1$ fraction of the new investment as equity claims. If $\theta = 1$, then the market is complete, i.e., any investment opportunities

---

\(^8\)More generally, a CRRA utility framework has similar but more complicated solution and will not change the main results. I choose log utility because it allows simple aggregation later. The derivation is available upon request.
can be shared across entrepreneurs. To focus on the more interesting economy, we restrict the attention to $\theta < 1$.

The second assumption is that there are resale frictions for existing equity claims. Entrepreneurs can sell only up to a $0 < \phi < 1$ fraction of their own equity or other’s equity. For simplicity, both own equity that is backed by un-mortgaged capital stock and equity issued by others can be sold at most $\phi$ fraction. Therefore, having one’s own equity and as well as others’ equity are perfect substitutes. That is, entrepreneurs can re-mortgage their previously un-mortgaged capital stock up to a $\phi$ fraction; or they can sell their previously held financial assets up to a $\phi$ fraction.

For simplicity, we do not model where the $\phi$ is from. $\phi$ can be motivated by information asymmetry or search problems between buyers and sellers. For instance, the capital goods from the sellers might be of little use; or it takes time for the buyers to meet the right sellers. In both cases, each unit of assets has a probability $\phi$ to be successfully sold. Therefore, $\phi$ fraction of total assets are resaleable. Since we abstract from different asset categories while putting all assets (except fully liquid government bonds) together, $\phi$ measures the average degree of resale frictions. We allow $\phi$ to change stochastically over time.

Besides partially liquid assets, there are fully liquid government bonds circulated. In reality, bonds issued by private sectors are still less liquid than government bonds, which are backed up by taxes and government enforcement power. Therefore, any debt that is issued by firms and commercial banks should be thought of as the non-fully resaleable assets, i.e., the equity claims in the model. The return on the equity claims in the model thus should be regarded as average return from privately issued financial assets in reality.

Entrepreneurs can save in both liquid and illiquid assets. Thus, on the left hand side of an entrepreneur’s balance sheet, there could be government bonds, equity claims on others’ capital stock, and their own capital stock as in Table 3.1. To finance these assets, on the

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9Some recent studies, such as Kurlat (2012) and Cui and Radde (2013), further model how to have an endogenous $\phi$. 

right hand side of the balance sheet, there are equity claims on part of the capital stock, and the rest is the entrepreneurs’ net-worth.

Budget and Financing Constraints. At the beginning of $t$, let $s_t$ be the net equity shares and $b_t$ be bonds held by an individual entrepreneur. Thus, $s_t$ is defined as

$$s_t = \text{capital stock} + \text{holdings of others' equity} - \text{equity issued}$$

Because of equity resale frictions, we have a resaleability constraint

$$s_{t+1} \geq (1 - \theta) i_t + (1 - \phi_t) (1 - \delta) s_t, \quad (3.2)$$

which says the minimum equity held in the end of $t$ would be the sum of un-mortgaged investment plus equity that cannot be resold. Note that the private agents cannot issue government bonds

$$b_{t+1} \geq 0, \quad (3.3)$$

which is a non-negativity constraint on fully liquid assets.

Let $q_t$ be the (real) price of equity, and let $P_t$ be the nominal price level of government bonds, $R_t$ be the nominal interest rate of government bonds from time $t - 1$ to $t$. The flow-of-funds constraint for an entrepreneur is

$$c_t + i_t + q_t (s_{t+1} - i_t - (1 - \delta) s_t) + \frac{b_{t+1}}{P_t} = r_t s_t + \frac{R_t b_t}{P_t}, \quad (3.4)$$

<table>
<thead>
<tr>
<th>Bonds</th>
<th>Own Equity Issued</th>
</tr>
</thead>
<tbody>
<tr>
<td>Others’ Equity</td>
<td>Capital Stock</td>
</tr>
<tr>
<td>Net Worth</td>
<td></td>
</tr>
</tbody>
</table>
which says the entrepreneur uses return from equity, government bonds, to finance consumption, investment in capital stock, and net purchase of equity, and new government bonds.

A summary: a typical entrepreneur maximizes (3.1), subject to constraints (3.2), (3.3), and (3.4), by choosing consumption, investment, and a portfolio of equity claims and bonds next period.

**Entrepreneurs’ Recursive Problem**

For the ease of representation, we turn to recursive formulation of the model, i.e., let $x_t$ denote $x_t$ and $x_{t+1}$ denote $x_{t+1}$, for any variable $x$.

Let $v^j(s, b; \Gamma)$ be the optimal value for an entrepreneur $j$ with equity $s$ and bonds $b$, given aggregate state variable $\Gamma$, which includes exogenous shocks, endogenous state variables $K$ and aggregate bonds $B$, and government policy explained later. With a slight abuse of notation, $j \in \{i, n\}$ denotes the type of entrepreneurs. If $j = i$, the entrepreneur has investment opportunities; otherwise, the entrepreneur does not. The value satisfies the following Bellman equation:

**Problem 5**

$$
v^j(s, b; \Gamma) = \log(c) + \beta \chi \mathbb{E}[v^j(s', b', \Gamma') | \Gamma] + \beta (1 - \chi) \mathbb{E}[v^n(s', b', \Gamma') | \Gamma]$$

$$c + i + q(s' - i - (1 - \delta)s) + \frac{b'}{P} = rs + \frac{Rb}{P}$$

$$s' \geq (1 - \theta)i + (1 - \phi)(1 - \delta)s, \quad b' \geq 0$$

$$i = 0, \text{ if } j = n$$

where prices $q$, $P$, and $R'$ are taken as given.

---

10. The distribution of entrepreneurs over equity and bond holdings is not a state variable because policy functions are linear in equity and bonds. The solution detail will be clear soon.
Entrepreneurs’ Optimal Solution

We focus on the neighborhood around the steady state in which government bonds are valued. Once liquid assets are valued, they are used as alternative sources for savings since existing equity claims are not fully resaleable. Therefore, productive entrepreneurs will sell up to $\phi$ fraction of equity and their constraints (3.2) and (3.3) are all binding.\textsuperscript{11}

To reach this interesting equilibrium, we assume $\theta$ and $\bar{\phi}$ (the steady state value for $\phi$) are small enough such that investment cannot be undertaken fully by issuance or by selling existing equity, as in Kiyotaki and Moore (2012)\textsuperscript{12}

\begin{equation*}
Assumption: \delta \theta + \chi (1 - \delta) \bar{\phi} < (\beta - 1 + \delta) (1 - \chi).
\end{equation*}

Now, we solve for individual decisions. The detail derivation can be found in the appendix. Again, we use subscript $i$ or $n$ for the variables of entrepreneurs with investment opportunities (type $i$) or without investment opportunities (type $n$).

Type $i$ Entrepreneurs, under the above assumption, will borrow to the limit so that constraint (3.2) will bind and the flow-of-funds constraint becomes

\begin{equation*}
c_i + (1 - \theta q)_i = [r + q\phi (1 - \delta)] s_i + \frac{Rb_i}{P}.
\end{equation*}

\textsuperscript{11}Later, the optimal policy also respects the binding constraint, since we are looking only into policies that can be decentralized into a competitive equilibrium market as such.

\textsuperscript{12}To understand the assumption, suppose the assumption hold and the steady state capital is $K$. Note that the following is impossible,

\begin{equation*}
[\delta \theta + \chi (1 - \delta) \bar{\phi}] K > \delta (1 - \chi) K.
\end{equation*}

To see this, the right hand side is the saving of non-investing entrepreneurs (with populations $1 - \chi$) in steady state; the left hand side is the sum of new equity issued ($\delta \theta K$, which is the investment to compensate depreciation) and existing equity sold ($\chi (1 - \delta) \phi K$). Then the inequality says that investing entrepreneurs can transfer all the savings from non-investing entrepreneurs, which is not possible by the assumption. Thus, the first-best outcome cannot be achieved by individual savings.
The entrepreneur consume $1 - \beta$ fraction of the net-worth, as we have log-utility:

$$c_i = (1 - \beta) \left[ rs_i + [\phi q + (1 - \phi) q_r] (1 - \delta) s_i + \frac{b_i}{P} \right],$$  \hspace{1cm} (3.5)$$

where $q_r = \frac{1 - \theta q}{1 - \delta} < 1$ as $q > 1$. Investment can be backed out from the flow-of-funds constraint,

$$i = \frac{[r + q \phi (1 - \delta)] s_i + \frac{Rb_i}{P} - c_i}{1 - \theta q}. \hspace{1cm} (3.6)$$

Type $n$ entrepreneurs are with the flow-of-funds constraint

$$c_n + q s_n + \frac{b'_n}{P} = r s_n + q (1 - \delta) s_n + \frac{Rb_n}{P},$$

where the consumption can be solved as

$$c_n = (1 - \beta) \left[ rs_n + q (1 - \delta) s_n + \frac{Rb_n}{P} \right]. \hspace{1cm} (3.7)$$

Meanwhile, each of these entrepreneurs chooses a portfolio of bonds and equity. The portfolio choice yields a variant of asset pricing formula

$$1 = \mathbb{E} \left[ \Lambda'_n \frac{p R}{p'} | \Gamma \right] \hspace{1cm} 1 = \mathbb{E} \left[ \Lambda'_n r'_n | \Gamma \right], \hspace{1cm} (3.8)$$

where $\Lambda'_n$ is the stochastic discount factor and $r'_n$ is the (risky) return from equity. Both $\Lambda'_n$ and $r'_n$ can take two values, depending on the type realization next period:

$$\Lambda'_n = \frac{\beta}{\delta} c_n, \hspace{1cm} (3.9)$$

$$r'_n = \frac{r' + (1 - \delta) q'}{q} \hspace{1cm} r'_n = \frac{r' + (1 - \delta) \phi' q' + (1 - \delta) (1 - \phi') q_r}{q}. \hspace{1cm} (3.10)$$
where $c'_{nj}$ denotes the consumption of type $n$ entrepreneur, if their type is $j$ next period, and similarly $r'_{nj}$ denotes the return on equity of type $n$ entrepreneur, if their type is $j$ next period.

### 3.2.2 The Firms

There is a continuum of identical firms. A representative firm rents capital from entrepreneurs at a rental rate $\tilde{r}$, and hires labor at a wage rate $\tilde{w}$. The associated technology is to use capital stock $K$ and labor input $N$ to produce output $Y$ according to

$$Y = AF(K, N) = AK^\alpha N^{1-\alpha},$$

where labor input $N = L$ by labor market clearing. Since firms operate on a competitive market, the rental rate and the wage rate satisfy

$$\tilde{r} = AF_K(K, N) = \alpha A \left( \frac{K}{N} \right)^{\alpha-1}, \quad (3.11)$$

$$\tilde{w} = AF_N(K, N) = (1 - \alpha) A \left( \frac{K}{N} \right)^\alpha. \quad (3.12)$$

For simplicity, we consider simple flat-rate taxes on capital returns and wages, which can be mapped to dividend taxes and income taxes. Denote the flat tax rates on capital and labor as $\tau_k$ and $\tau_w$, so that

$$r = (1 - \tau_k) \tilde{r}, \quad (3.13)$$

$$w = (1 - \tau_w) \tilde{w}. \quad (3.14)$$

### 3.2.3 The Government

Our goal is optimal policy. In implementing the policy, the government acts like a constrained social planner.
As described before, the government tax the capital return $\tilde{r}$ and wage rate $\tilde{w}$ at flat rates $\tau_k$ and $\tau_w$. The flat rate taxation on capital has two advantages. First, imposing tax on returns from capital will not change the tractable solution for optimal policy on capital and bonds. Second, we view that the tax is usually a percentage on corporate profits, dividends, and wages; a constant fraction of the profits rate and wage rate is taxable should be a reasonable short-cut. Government expenditure is set to be 0 so that the main concern is not to finance government expenditure.

The conventional monetary policy consists of setting predetermined nominal interest rate $R'$. Unlike tax policy and purchase of privately issued assets, the interest rate is predetermined each period. Thus, we need an exogenously given rules to ensure a non-exploding equilibrium. In particularly, we specify the nominal interest rate according to a conventional Taylor rule

$$R' = \bar{R} \left( \frac{\pi}{\bar{\pi}} \right)^\psi,$$

where $\pi$ is the inflation rate defined as

$$\pi = \frac{P}{P_{-1}}.$$

Note that $\bar{R}$ and $\bar{\pi}$ are the steady state nominal interest rate and inflation.

The unconventional policy corresponds to the government purchase of financial assets issued by private agents. Since late 2007, central banks worldwide have implemented a new set of policies in which they buy financial assets with partial liquidity, such as mortgage-backed securities, using fully liquid assets. We consider, therefore, how open market operations should be used in such context.

Let $S_g$ denote the equity purchased from the private sector, i.e., the government hold $S_g$ unit of claim on capital stock $K$ at the beginning of $t$ (and the private sector thus hold $K - S_g$ unit of equity claims). Then $S'_g$ is a “quantity” choice variable of the government. Importantly, the government intervention does not relax any private agent’s resaleability
constraint. The unconventional policy changes the composition of portfolios (with liquid and illiquid assets) held by private sectors. The intervention will potentially have real effects because it affects expectation and prices.

The government budget constraint is

$$G + \frac{RB}{p} + qS'_g = \tau + \frac{B'}{p} + [r + (1 - \delta)q]S_g,$$

where $G$ is the government expenditure and $\tau = \tau_k\bar{K} + \tau_w\bar{w}N$ is the total tax revenue. For simplicity, we have assumed labor supply to be inelastic. In order to avoid equilibrium in which the government use labor (lump-sum) tax to achieve any allocation, we restrict the labor tax to be zero, i.e., $\tau_w = 0$.\(^{13}\) Moreover, we set $G = 0$ to avoid effects of government spending on the real economy.

We choose to work with the growth rate of liquid assets and inflation to simplify the government budget constraint. Let

$$B' = \eta B, B_r = \frac{B}{p},$$

where $\eta$ is the gross growth rate of liquid assets and $B_r$ is the real liquidity before interest rate. Therefore, inflation can be re-written as

$$\pi = \frac{p}{p_{-1}} = \frac{\eta_{-1}B_{r,-1}}{B_r}. \quad (3.17)$$

Now, the government budget constraint can be re-written as

$$RB_r + qS'_g = \tau + \eta B_r + [r + (1 - \delta)q]S_g. \quad (3.18)$$

\(^{13}\)Alternatively, we could model the households to make optimal choices in supplying labor. Due to the fact that in the economy the interest rate is lower than the time preference rate, we can show that the households will not save. Meanwhile, the optimal taxation on labor should be zero as well, because this policy will not distort labor supply and thus can increase the rate of return to capital to the largest possible. The detailed proof and argument are available upon requests.
The government uses taxes, new debt issuances, and payoff and resale value from equity holding, to finance government expenditures, debt payments, and purchase of new equity.

### 3.2.4 Recursive Equilibrium

Having described the economy, we now proceed to the competitive equilibrium. Denote $X_i = \chi x_i$, $X_n = (1 - \chi)x_n$, and $X = X_i + X_n$ as the aggregation of variable $x$ for type $i$ entrepreneurs, type $n$ entrepreneurs, and the sum of the two types of entrepreneurs. Total consumptions of investing entrepreneurs and saving entrepreneurs are

$$C_i = (1 - \beta) \chi [rS + \phi q + (1 - \phi) q_R] (1 - \delta) S + RB_r.$$  

(3.19)

$$C_n = (1 - \beta) (1 - \chi) [rS + q (1 - \delta) S + RB_r].$$  

(3.20)

Aggregate investment $I$ can be derived from (3.6):

$$(1 - \theta q) I = [r + q \phi (1 - \delta)] \chi S + \chi RB_r - C_i.$$  

(3.21)

Equity market clearing needs a consistent portfolio choice equation. Using asset pricing formula (3.8), we have

$$\mathbb{E} \left[ \Lambda_n' \left( r_n' - \frac{R'}{\pi'} \right) | \Gamma \right] = 0.$$  

(3.22)

Define the equity held by entrepreneurs without investment opportunities at the end of period $t$ as $S'_s$, where

$$S'_s = \theta I - (S'_g - (1 - \delta) S_g) + [\phi \chi + (1 - \chi)] (1 - \delta) S.$$  

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Then the net-worth for a type $n$ entrepreneur (this period) who become type $n$ and type $i$ entrepreneur next period are

\[ n_{nn}' = (r' + (1 - \delta) q') S_s' + R'B_r' \]  
(3.23)

\[ n_{ni}' = (r' + \phi' (1 - \delta) q' + (1 - \phi') (1 - \delta) q'_r) S_s' + R'B_r'. \]  
(3.24)

respectively. Then consumption next period can be backed out by

\[ c_{nn}' = (1 - \beta) n_{nn}', \quad c_{ni}' = (1 - \beta) n_{ni}'. \]

Now we obtain a portfolio equation for the equilibrium, by rewriting (3.22) and using (3.10), (3.23), and (3.24).

\[ (1 - \chi) E \left[ \frac{r_{nn}' - R}{n_{nn}'} | \Gamma \right] + \chi E \left[ \frac{r_{ni}' - R}{n_{ni}'} | \Gamma \right] = 0. \]  
(3.25)

Goods market clearing condition gives

\[ C_i + C_n + I + G = \alpha AK^\alpha N(1 - \alpha). \]  
(3.26)

Equity market clearing gives

\[ K' = S' + S_s'. \]  
(3.27)

\[ 14] Since $(1 - \chi)$ fraction of the savers will still be savers next period and with net-worth $[(r' + (1 - \delta) q') S_s' + R'B_r']$ while $\chi$ fraction of the savers will have investment opportunities next period and with net-worth $[(r' + \phi' (1 - \delta) q' + (1 - \phi') (1 - \delta) q'_r) S_s' + R'B_r']$. Therefore,

\[ n_{nn} = \frac{(r' + (1 - \delta) q') (1 - \chi) S_s' + (1 - \chi) R'B_r'}{1 - \chi}, \]

\[ n_{ni} = \frac{(r' + \phi' (1 - \delta) q' + (1 - \phi') (1 - \delta) q'_r) \chi S_s' + \chi R'B_r'}{\chi}. \]
Finally, the capital evolution is

\[ K' = (1 - \delta) K + I. \tag{3.28} \]

We now have the following recursive equilibrium definition:

**Definition 3** A recursive competitive equilibrium is a function \((C_i, C_m, I, K', S', B_r, r, \tilde{r}, w, \tilde{w}, \pi, q, \eta)\) of the aggregate state \((K, B_r, S, S_g, R, A, \phi)\), that satisfies

- aggregate consumption (3.19), (3.20), and aggregate investment (3.21),
- portfolio choice condition (3.25),
- rental rate (3.11) and (3.13),
- definition for inflation (3.17) and Taylor rule (3.15),
- the government budget constraint (3.18),
- aggregate capital evolution (3.28) and equity market clearing condition (3.27),
- goods market clearing condition (3.26),

given specific policy rules \(\{S'_g, \tau_k\}\) and exogenous stochastic processes \((A, \phi) \rightarrow (A', \phi')\).

Notice that, the bonds market clearing condition is satisfied according to the Walras law.\(^{15}\)

### 3.3 The Optimal Fiscal and Monetary Policy

After defining the competitive equilibrium, we proceed to the constrained optimal policy. If \(q > 1\), the competitive equilibrium is not Pareto efficient nor constrained efficient. In

\(^{15}\)To check this, simply by adding the social resources constraints and the government budget constraint. Finally, we have that the demand of bonds from non-investing entrepreneurs equals the supply of bonds.
the presence of illiquidity, private agents might over-save in liquid assets compared to what a social planner wants them to do. In sum, the social planner could redistribute wealth through the portfolio composition between $s_t$ and $b_t$ in the hand of private agents. Importantly, when conducting policy, the government respects the resale frictions in the economy. The private sector can issue at most $\theta$ fraction of new investment and can sell up to $\phi$ fraction of existing equity. Therefore, the government is not able to purchase every unit of equity from the private sector.

### 3.3.1 The Primal Approach

The approach to reach the optimal policy is in the same spirit of the public finance literature on obtaining an “implementability condition” with only quantity variables, e.g., Chari and Kehoe (1999) for a survey and more recently Golosov and Sargent (2012)). This is the so-called primal approach.

After constructing the equilibrium conditions of a decentralized market, we express prices to depend only on quantity allocations. The planner’s problem then collapse to choosing allocations under two constraints: one that defines a competitive equilibrium, and the other that defines social resources constraint. To do so, we first describe how one can obtain an implementability condition.

#### The Implementability Condition

First, using aggregate consumption of investing and non-investing entrepreneurs (3.5), together with (3.7), yields

\[
(1 - \beta)^{-1} (1 - \chi) C_n - (1 - \beta)^{-1} \chi C_i = (1 - \phi) (q - q_r) (1 - \delta) S
\]

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Second, we need to express $q$ and $q_r$ in terms of quantity allocation. This can be done by using (3.19) and (3.21),

$$\frac{\beta C_i}{1 - \beta} = (1 - \theta q) I + q_r (1 - \phi) (1 - \delta) \chi S,$$

from which we have

$$q = \frac{1 - d}{\theta} \text{ and } q_r = \frac{d}{1 - \theta},$$

where

$$d = 1 - \theta q = \frac{\beta C_i}{I + \frac{(1 - \delta)(1 - \phi)}{1 - \theta} \chi S}.$$

To understand (3.29), the left hand side $\frac{\beta C_i}{1 - \beta}$ is the savings of the investing entrepreneurs, since $\frac{C_i}{1 - \beta}$ is the net-worth of the investing entrepreneurs. All savings are in the form of equity (including own and others’ equity) because investing entrepreneurs hold no government bonds. The right hand side is the composition of the savings: for investment $I$, the investing entrepreneurs put $(1 - \theta q) I$ as down-payment; besides new investment, $(1 - \phi) (1 - \delta) \chi S$ of previous holding of equity is left and is valued at price $q_r$. Naturally, $d$ can be thought of as the down-payment rate: for each unit of investment the entrepreneur puts $1 - \theta q$ from internal funds as down-payment. Hence, the price of capital is one minus down-payment rate over the fraction of the investment ($\theta$) that can be initially issued.

In the last step, using $q$ and $q_r$, we obtain the implementability condition:

$$(1 - \beta)^{-1} (1 - \chi) C_n - (1 - \beta)^{-1} \chi C_i = (1 - \phi) \frac{(1 - \theta) - d}{\theta (1 - \theta)} (1 - \delta) S.$$

As in a standard implementability condition, there are no prices or taxes. To interpret the condition, recall that $\frac{C_n}{1 - \beta}$ is the net worth of the saving agents and $\frac{C_i}{1 - \beta}$ is the net worth of the investing agents. The left hand side is the net-worth difference of saving agents and investing agents, normalized by the populations. Such difference is determined by the
resaleability frictions of the equity held after depreciation. This difference is non-zero if the (shadow) price for saving agents \( q = \frac{1-d}{\theta} \) differs from the price for investing agents \( q_r = \frac{d}{1-\theta} \). That is, if \( q \neq 1 \). In summary, it states that the net-worth difference of two types of agents comes exactly from the price differences on non-resaleable assets due to financial frictions.

Importantly, one can see the limiting case in which there are no frictions, and perfect consumption risk-sharing. If investment can be fully financed by outside equity \( \theta = 1 \), then \( q = q_r \) and there will be no net-worth difference. If we relax the resaleability \( \phi = 1 \), the net worth difference will also be zero: equity is fully liquid and is the same as government bonds; there is no need for fully liquid assets, as one would expect in a usual real business cycle model.

Finally, on the technical side, our implementability condition follows the tradition that focuses on allocation of real assets and consumption. The real value of liquidity, \( RB_r \), is not in presence because the liquid assets are nominal and used to achieve the allocation of real goods and assets.

**The Ramsey Problem**

The government acts as a constrained social planner and maximizes social welfare, subject to the implementability condition (3.32) and the social resources constraint (3.26). In constructing the Ramsey plan, we work with equal welfare weight to each entrepreneur. Other welfare weights should work in a similar way as in the following. The constrained planner’s problem is then given by:

**Problem 6** Let \( u(C_i, C_n) \) denote the utility of entrepreneurs in the current period,

\[
u(C_i, C_n) = \chi \log \left( \frac{C_i}{\chi} \right) + (1 - \chi) \log \left( \frac{C_n}{1 - \chi} \right)\]

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Then a welfare maximizing social planner solves the following problem

\[ V(S, S_g; A, \phi) = \max_{C_i, C_n, S', S_g'} u(C_i, C_n) + \beta E \left[ V(S', S_g'; A', \phi') \mid A, \phi \right] \text{ s.t.} \]

implementability condition (3.32) and social resources constraint (3.26) are satisfied, given the law of motion of \((A, \phi)\) and the nominal interest rate policy \(R'\).

Note that the nominal interest rate We substitute out \(C_i\) and \(C_n\), so the constrained planner chooses \(S'\), and \(S_g'\). After some algebra, one can express

\[ C_i = \frac{\chi}{1 - D} \left[ AK^{\alpha}N^{1-\alpha} - K' + (1 - \delta) K - G - C_l - c_1 S \right] \]  
(3.33)

\[ C_n = AK^{\alpha}N^{1-\alpha} - K' + (1 - \delta) K - G - C_l - C_i, \]  
(3.34)

where

\[ D = \frac{c_2 S}{c_3 \left[ S' + S_g' - (1 - \delta)(S + S_g) \right] + c_4 S'}, \]

\[ c_1 = (1 - \chi)(1 - \beta)(1 - \phi)(1 - \delta)/\theta, \]

\[ c_2 = \beta \chi (1 - \chi)(1 - \phi)(1 - \delta), \]

\[ c_3 = \theta (1 - \theta), \quad c_4 = \theta (1 - \delta)(1 - \phi) \chi. \]

\(D\) measures the degree of frictions in the market. A larger \(D\) means a larger difference between individual entrepreneurs consumption \(C_i/\chi\) and \(C_n/(1 - \chi)\). In a perfect consumption risk-sharing world, for example \(\phi = 1\), \(C_i/\chi\) and \(C_n/(1 - \chi)\) are the same because \(c_1 = 0, c_2 = 0\) and \(D = 0\).

Let \(u_i\) and \(u_n\) denote the marginal utility on \(C_i\) and \(C_n\),

\[ u_i = \frac{\chi}{C_i}, \quad u_n = \frac{1 - \chi}{C_n}. \]
The first-order conditions for the planner read

\[
[S'] : \quad \frac{\partial C_i}{\partial S'} + \beta \mathbb{E} u_i' \frac{\partial C_i'}{\partial S'} + u_n \frac{\partial C_n}{\partial S'} + \beta \mathbb{E} u_i' \frac{\partial C_n'}{\partial S'} = 0
\]

\[
[S_g'] : \quad \frac{\partial C_i}{\partial S_g'} + \beta \mathbb{E} u_i' \frac{\partial C_i'}{\partial S_g'} + u_n \frac{\partial C_n}{\partial S_g'} + \beta \mathbb{E} u_i' \frac{\partial C_n'}{\partial S_g'} \geq 0.
\]

Note that because \( S_g \geq 0 \), the second FOC involves inequality. The second-order condition is checked numerically to ensure a maximum.

A full characterization of each term is relegated to the appendix and here only the final results are shown:

\[
u_n - \frac{\chi (u_n - u_i)}{1 - D} z_0 = \beta \mathbb{E} \left[ u_n' (\bar{r}' + 1 - \delta) \right] - \beta \mathbb{E} \frac{\chi (u_n' - u_i')}{1 - D'} z_1' \tag{3.35}
\]

\[
u_n - \frac{\chi (u_n - u_i)}{1 - D} z_0 \geq \beta \mathbb{E} \left[ u_n' (\bar{r}' + 1 - \delta) \right] - \beta \mathbb{E} \frac{\chi (u_n' - u_i')}{1 - D'} z_2', \tag{3.36}
\]

where

\[
z_0 = 1 + \frac{c_3 D^2}{c_2 S u_i},
\]

\[
z_1' = \bar{r}' + 1 - \delta - c_1' + \frac{D'}{S u_i} \left[ 1 + \frac{(1 - \delta) c_3' - c_4'}{c_2'} D' \right],
\]

\[
z_2' = \bar{r}' + 1 - \delta + \frac{D' (1 - \delta) c_3' D'}{c_2'}. \]

Again, in a perfect consumption risk-sharing world, \( u_i = u_n \). Using the first-order conditions above, we immediately have

\[
u_n = \beta \mathbb{E} \left[ u_n' (\bar{r}' + 1 - \delta) \right],
\]

which resembles the first-best solution. Marginal utility on consumption are equalized across entrepreneurs; the capital accumulation is such that the social benefits equal the
social costs. Our main task is to check whether policy can achieve this first-best solution. If the answer is positive, how can the policy achieve this allocation?

3.3.2 Policy Implementation

Once an allocation \( \{C_i, C_{n,t}, S_{t+1,1}, S_{t+1,1}\}^\infty_0 \) has been achieved, one can back out prices and taxes. Without the loss of generality, a planner has the initial starting knowledge of \( \eta_{-1}, B_{r,t-1} \) and \( R_0 \). Then, we take the following steps.

1. Back out asset price \( \{q_t\}^\infty_0 \) from (3.30) with \( \{d_t\}^\infty_0 \) calculated from (3.31).

2. Pre-tax rental rate is from (3.11). After-tax rental rate needs a loop to calculate. We first guess a \( \{r_t\}^\infty_0 \) and then tax on capital is \( \tau_k, t = 1 - \frac{r_t}{\tilde{r}_t} \).

3. Then, we back out the real value of liquidity \( \{R_tB_{r,t}\}^\infty_0 \) from the type \( i \) entrepreneurs’ consumption equation (3.19). Further, \( \{\eta_tB_{r,t}\}^\infty_0 \) can be backed out by the government budget constraint (3.18).

4. From the portfolio equation, one has

\[
(1 - \chi) \mathbb{E}_t \left[ \frac{(r_{t+1} + (1 - \delta) q_{t+1}) / q_t - \frac{R_{t+1}B_{r,t+1}}{\eta_t B_{r,t}}}{(r_{t+1} + (1 - \delta) q_{t+1}) S_{s,t+1} + R_{t+1}B_{r,t+1}} \right] = \chi \mathbb{E}_t \left[ \frac{R_{t+1}B_{r,t+1}}{\eta_t B_{r,t}} - [r_{t+1} + \phi_{t+1} (1 - \delta) q_{t+1} + (1 - \phi_{t+1})(1 - \delta) q_{r,t+1}] / q_t \right] \left( r_{t+1} + \phi_{t+1} (1 - \delta) q_{t+1} + (1 - \phi_{t+1})(1 - \delta) q_{r,t+1} S_{s,t+1} + R_{t+1}B_{r,t+1} \right].
\]

The guessed value of \( \{r_t\}^\infty_0 \) should satisfy the above portfolio equation.

5. Then \( B_{r,t} = \frac{R_t}{\eta_t B_{r,t}} \) and \( \eta_t = \frac{B_{r,t}}{\eta B_{r,t}} \). Knowing \( B_{r,t} \) and \( \eta_t \), inflation \( \pi_t \) can be backed out by (3.17). Then nominal interest rate \( R_{t+1} \) is set according to the Taylor rule (3.15).

While the full system dynamics are complicated, we present some analytical features of the Ramsey plan in the steady state.
Proposition 8 Suppose the economy is in the steady state and \( u_i = u_n \) is possible, then capital tax \( \tau_k \) should be set to zero.

Proof. See the Appendix. ■

The zero capital taxation result is similar to the literature of optimal taxation with representative households. When entrepreneurs are able to share their risks (note: we will illustrate this point later in which government can achieve this by directly holding some privately issued financial assets), the planner chooses not to distort capital accumulations. The intuition is that it is always better not to distort inter-temporal decisions. Notice that, even when the labor is modelled differently, the zero capital tax result still holds. The observation is from the planner’s first-order condition for \( S' \). Different modelling of labor will not affect that first-order condition.

Given that there are liquidity frictions \( (q > 1) \), the government will use policy to redistribute wealth. Since the funding of the policy does not come from taxation, it must come from the payoffs from holding privately issued financial assets.

Proposition 9 Suppose the economy is in the steady state. If perfect consumption risk-sharing is possible, i.e., \( u_i = u_n \), then \( S_g > 0 \).

Proof. See the Appendix. ■

The intuition behind this result is straightforward. Holding financial assets are necessary for future investment opportunities because it gives payoff and some resale value. However, the government does not have investment opportunities and it is better to allow the private sector to hold all financial assets, if there are no liquidity frictions or or the frictions are very small. When liquidity frictions become large, resources cannot transferred to the best use and thus the policy can step in to redistribute wealth. The policy uses the payoffs from holding \( S_g \), to reward the bond holders by paying a higher real interest rate. The higher interest rate in the steady state will be clear in the next corollary.
This result sheds some light on why the unconventional policy is “unconventional”. The policy targets at time of large disturbances, such as the 2008 U.S. financial crisis with large liquidity disturbances. Unconventional policy redistribute wealth in that period when privately issued financial assets are with large difficulties in transaction. However, in normal times, the resaleability of financial assets is not a major concern. When resaleability is high in normal times, financial assets should be kept only in the private sector to facilitate future investments.

Corollary 6 Suppose the economy is in the steady state and \( u_i = u_n \) is possible, then the (real) interest rate is \( R/\pi = \beta^{-1} \), which is larger than the interest rate \( R/\pi = 1 \) in the economy with constant money supply.

Proof. See the Appendix.

As long as there is incomplete market, type \( i \) and type \( n \) entrepreneurs cannot perfectly implement risk-sharing. Imperfect risk-sharing directs entrepreneurs to save more than socially optimal levels in risk-free government bonds, similar to Aiyagari (1994) model. Without active policy, the rate of return on risk-free assets will be too low. A low interest rate hurts entrepreneurs’ ability (those who have liquid assets on their balance sheet) to finance new investment and hinders the creation of new financial assets in the future. The policy, thus, is a redistribution device to achieve better risk-sharing, by taxing resources and rewarding those who hold liquid assets.

Similar to Aiyagari (1994), the agents in the economy tend to over-save in liquid assets because of the incomplete market set-up. However, the policy implication in our model is not to tax savings, but to reward savings. The key insight is that we have different rates of return. The interest rate rate is different from the rate of return from the capital stock. In Aiyagari (1994) model, a low interest rate implies that capital stock is over-accumulated because the interest rate equals the marginal product of capital. In our model, a low interest rate implies under-accumulation. To see this, note that agents will use liquid assets to finance new investment. If the real interest rate on liquid assets is low, the ability
to finance new investment will be low. Therefore, investment is insufficient and so is capital stock. This result is in sharp contrast to Aiyagari (1994) with uninsured labor income risks and individual savings. We will illustrate this result further in the following numerical exercises.

3.4 Numerical Examples

We calibrate the model in quarterly frequency. Some parameters (see Table 3.2) are standard in the literature such as depreciation rate, capital share in production and discount factor, while more elaboration should be given to $\chi$, $L$, $\phi$, $\theta$.

In the calibration, 6% of the entrepreneurs have investment opportunities every quarter, which is the number to match investment spikes observed from U.S. manufacturing plants in (Doms and Dunne (1998), Cooper, Haltiwanger, and Power (1999), and Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011)). $L$ should show the ratio of workers to entrepreneurs in the economy. The main difference between workers and entrepreneurs is the access to equity markets to fund the investment opportunity. Therefore, we calibrate this value to be in line with the participation rate of households from SCF in 2009 that we see in the equities market (15%), in line with previous studies from Mankiw and Zeldes (1991) and Heaton and Lucas (1999). Such number translates into $L = 6$. However, because of constant return to scale technology, the choice of inelastic labor supply only scale up and down the economy but not the composition of the economy. Therefore, the asset price and interest rate in the steady state are not affected by $L$ and the sensitivity of $L$ is not a concern.

For the financial frictions, Del Negro, Eggertsson, Ferrero, and Kiyotaki (2011) use the mean values for $\theta$ and $\phi$ to be 19%, matching total treasury bills outstanding to total assets. We take their value such that the total treasury bills over total assets in the flow of funds data
(from 1991Q1 to 2007Q4) are roughly the same in the model (11.25%) with constant bond. Later, we will vary $\phi$ to check robustness, which directly measures the resale frictions.

Finally, we target annual nominal interest rate at 4% and the Taylor coefficients\textsuperscript{16} is set to 1.05 in order to have uniquely determined equilibrium.

Table 3.2: Parameters
\textsuperscript{a}$\alpha$: capital share in the production function. $\beta$: discount factor. $\delta$: depreciation rate. $\theta$: initial equity issuance constraint for new investment. $\phi$: the fraction of existing equity claims that can be sold in the steady state. $L$: total labor supply unit. $\chi$: the probability of receiving investment opportunities. $\bar{R}$: nominal interest rate in the steady state. $\psi$: Taylor coefficient.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\delta$</th>
<th>$\theta$</th>
<th>$\phi$</th>
<th>$L$</th>
<th>$\chi$</th>
<th>$\bar{R}$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.33</td>
<td>0.995</td>
<td>0.025</td>
<td>0.19</td>
<td>0.19</td>
<td>6</td>
<td>0.06</td>
<td>1.01</td>
<td>1.05</td>
</tr>
</tbody>
</table>

3.4.1 Steady State

In order to show the effects of active policy, we compare the two economies, one with a constant bond and the other with a Ramsey plan. We label the former one as constant bond economy, and the latter as the Ramsey plan economy.

A distinguished feature of the Ramsey plan economy is that it can reach the first-best allocation (Table 3.3). There is full consumption risk-sharing because the implementability condition features zero on both sides. The asset price is one compared to the constant bond economy, which again reflects economy efficiency.

To implement the first-best allocation, the government uses liquid assets in exchange of illiquid assets. The holding of illiquid assets generate returns each period which can finance the payments to bond holders. To see this, the real interest rate is 1 in the constant bond economy, while the rate is $R/\pi = 1/\beta$ in the Ramsey plan economy. Therefore, the resources are transferred to those who hold liquid assets. Finally, as in Corollary 6, agents over-accumulate liquid assets and the interest rate is lower than the first-best interest rate.

The quantitative effects from the Ramsey plan is significant. Steady state output can increase by 1.68%, in which investment increases by 5.19% and consumption increases by

\textsuperscript{16}Any Taylor coefficients that is bigger than 1 is sufficient to give a well-defined equilibrium.
Table 3.3: Steady State Comparison
The comparison between the economy with constant bond supply and the economy with Ramsey plan. The consumption, investment, and output are normalized to be 100% in the constant bond economy. Liquidity ratio is total value of liquid assets over total value of liquid and illiquid assets, i.e., $B_r/(B_r + qK)$.

<table>
<thead>
<tr>
<th></th>
<th>Consumption</th>
<th>Investment</th>
<th>Output</th>
<th>Liquidity Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant bond</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>11.25%</td>
</tr>
<tr>
<td>Ramsey plan</td>
<td>+0.41%</td>
<td>+5.19%</td>
<td>+1.68%</td>
<td>16.60%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Nominal Rate</th>
<th>Inflation</th>
<th>$S_g/K$</th>
<th>Capital Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant bond</td>
<td>0%</td>
<td>0%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ramsey plan</td>
<td>4%</td>
<td>2%</td>
<td>19.99%</td>
<td>0</td>
</tr>
</tbody>
</table>

0.41%. In a growing economy, the growth rate of these macro variables should roughly increase by the same amount. To achieve such allocation, the inflation rate is increased from annual 0% to 2% so that the net real interest rate is increased from annual 0% to 2%. Importantly, as in Proposition 8, the financing of interest rate payment is not through taxing capital returns but through the holding of financial assets that are issued by private agents. Taxing capital returns will distort inter-temporal decisions and is generally unfavored.

How does the Ramsey plan changes when $\phi$ varies? When $\phi = 0.352$ (Table 3.4), the partial resaleable financial assets together with constant liquid bonds are enough to channel resources. The allocation reaches the first-best. Such allocation is identical to that in an economy with fully resaleable financial assets (in which there is no need for the existence of fully liquid government bonds). Government interventions are not necessary, even though the financial assets are partially liquid. As resaleability gets tighter, i.e., $\phi$ drops, the constant bond economy becomes more and more inefficient. The privately issued financial assets channel less and less fund to entrepreneurs with investment opportunities. On the other hand, the Ramsey plan economy can always implement the first-best, by holding more and more privately issued financial assets.

Finally, the holding of privately issued financial assets ($S_g$) and the real value of liquidity increase monotonically, as $\phi$ drops. But the $S_g$ and liquidity value increase much faster when $\phi$ drops initially from 0.352. This results potentially show that the unconventional
Table 3.4: Steady State with Different Liquidity Frictions

Steady state results when $\phi$ changes. Liquidity ratio is total value of liquid assets over total value of liquid and illiquid assets, i.e., $B_r/(B_r + qK)$. $S_g/K$ measures how large is the government purchase.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>0.352</th>
<th>0.35</th>
<th>0.30</th>
<th>0.25</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquidity Ratio</td>
<td>0.00%</td>
<td>0.30%</td>
<td>6.88%</td>
<td>11.92%</td>
<td>15.90%</td>
</tr>
<tr>
<td>$S_g/K$</td>
<td>0.00%</td>
<td>0.31%</td>
<td>7.43%</td>
<td>13.60%</td>
<td>18.99%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>0.19</th>
<th>0.18</th>
<th>0.17</th>
<th>0.16</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquidity Ratio</td>
<td>16.60%</td>
<td>17.27%</td>
<td>17.91%</td>
<td>18.53%</td>
<td>19.12%</td>
</tr>
<tr>
<td>$S_g/K$</td>
<td>19.99%</td>
<td>20.97%</td>
<td>21.93%</td>
<td>22.85%</td>
<td>23.76%</td>
</tr>
</tbody>
</table>

Policy is much more effective when an economy shifts from the first-best to one with some liquidity frictions.

3.4.2 Transition Dynamics

In this section, we show the system dynamics after aggregate shocks. We consider a drop in liquidity $\phi$ and a drop of productivity $A$ at the same time. In real practice, $A$ drop can be potentially motivated by misallocation of capital among firms with a different efficiency in using the capital. (For simplicity, we do not model the difference in productivity.) All of these attempts are to highlight the Ramsey plans in the presence of liquidity frictions. Next, we briefly explain why we consider both types of shocks.

As in Shi (2012), liquidity shocks generate some abnormal dynamics. Because of no deep modelling of the $\phi$, an exogenous decrease of $\phi$ does not change the output produced at the time when $\phi$ drops. Since agents fly to liquidity, the demand for investment decreases. Less investment means that agents consume more because output produced is kept
the same. Therefore, a recession that is generated by liquidity shocks features a higher consumption (at least initially).

Recent works, such as Ajello (2012) and Cui and Radde (2013), show that mechanisms that produce endogenous liquidity and sticky prices can generate less consumption after liquidity shocks. However, we abstract from modelling $\phi$ further, in order to highlight the Ramsey plans in response to liquidity frictions. To avoid a consumption dynamics anomaly, we consider shocks that combine changes of both $\phi$ and $A$. In specifying the evolution of $A_t$ and $\phi_t$, we follow the literature on assuming AR(1) processes. In particular,

$$\ln \phi_t = (1 - \rho_\phi) \ln \bar{\phi} + \rho_\phi \ln \phi + \epsilon^\phi_t$$

$$\ln A_t = \rho_A \ln A_{t-1} + \epsilon^A_t,$$

where $(\epsilon^\phi_t, \epsilon^A_t)^T$ are i.i.d, normal with mean zero and variance-covariance matrix:

$$
\begin{bmatrix}
\sigma^2_\phi & \sigma_{\phi A} \\
\sigma_{A \phi} & \sigma^2_A
\end{bmatrix}
$$

We follow Thomas (2002) on aggregate productivity shocks process where $\rho_A = 0.9225$ and $\sigma_A = 0.0052$. Since we target drops in both $\phi$ and $A$, there should be correlation between $\epsilon^\phi_t$ and $\epsilon^A_t$. We specify $\sigma_\phi = 0.05$ (5% standard deviation), $\rho_\phi = 0.8$, and $\sigma_{\phi A} =

17Meanwhile, in the neighborhood of the steady state in which asset price $q$ is strictly greater than 1, asset price increases after a decrease of $\phi$ since a smaller $\phi$ leads to less supply of financial assets and thus the asset price booms. We do not have such problems in the Ramsey plan economy because $q = 1$ in the steady state due to the Ramsey policy. However, we do not show the dynamics after only $\phi$ shocks (available upon request), because consumption still increases.

18The key idea is to add endogenous demand effects. First, after adverse liquidity shocks, entrepreneurs fly to liquidity, lend less capital, and invest less. The monopolistic competing firms find it harder to get capital, and in response hire less labor in a sticky price environment. Aggregate output is thus produced less, after liquidity shocks. As a result, liquidity shocks can generate both less consumption and investment. Second, endogenous liquidity creates a gap between the buying and selling price. Though liquidity shocks lead to less supply, they mainly affect the buying price. The selling price, which mainly affects the resale value, drops after liquidity shocks.
For robustness checks, we compare the system dynamics for $\sigma_{A\phi} = 0.3, 0.4, 0.5, \rho_{\phi} = 0.7, 0.8, 0.9, \text{ and } \sigma_{\phi} = 0.04, 0.05, 0.06$.

Liquidity shocks direct agents to demand more liquid assets (Figure 3.1). Inflation drops as a consequence. The demand for privately issued assets shrinks more than the tightened asset supply. The stronger demand reactions give a lower asset price. Because of a lower productivity $A$, less resources can be produced and agents consume less.

A notable feature of the policy is that although the liquidity shocks are persistent, the economy can be greatly stabilized. Asset price drops, but come back to the steady state very quickly because of the extra demand boosted from the unconventional policy. Therefore, investment will not be affected much.
All of these dynamics are not sensitive to a different persistence of liquidity shocks ($\rho_\phi$) and a different size of the liquidity shocks ($\sigma_\phi$). Figures 3.2 and 3.3 show the dynamics if we change $\rho_\phi$ and $\sigma_\phi$.

Notice that, the policy response only corrects liquidity frictions. This result can be seen from the system dynamics in Figure 3.1 with different $\sigma_\phi A$, i.e., the correlation between the aggregate productivity and the resaleability. Asset price movements and unconventional purchases are the same in these three cases, while dynamics of major macro variables (such as consumption and output) change under different productivity shocks. Therefore, the policy does not change in response to productivity shocks. Further, Figures 3.2 and 3.3 confirm this result. By holding productivity dynamics the same while changing the liquidity shocks, only unconventional purchases are different.

To achieve this allocation, the government mainly does unconventional purchases of financial assets, besides the usual interest rate policy. The purchase is mainly financed by a large supply of liquid bonds initially while enhanced by taxation one period later. Notice that, in reality, it is hard to adjust the tax in such a rapid speed and therefore that is probably why the monetary authority mainly uses bonds to implement the unconventional purchase.

Finally, we consider small enough shocks for the illustration and avoid the discussion of zero lower bound (see, for example, Adam and Billi (2007) and Eggertsson and Woodford (2003)) issues for the nominal interest rate. The nominal interest rate is always positive in our dynamics. However, if pure liquidity shocks are at least one of the business cycle sources, the above exercise already shows that unconventional purchases are necessary.

### 3.5 Conclusion

The closed-form implementability condition shows a clear Ramsey plan: with liquidity frictions the optimal policy can undo it by holding privately issued financial assets, supply-
ing more liquid assets, and pays a higher real interest rate. Even if liquidity fluctuations are persistent, the policy can greatly stabilize the economy.

While many studies rationalize the unconventional monetary policy because of reaching zero-lower bound of nominal interest rate, this paper shows that large enough liquidity frictions are already enough to call for an unconventional monetary policy. Note that all the analyses are based on the restriction in which the policy cannot change the private liquidity constraint directly. The policy can only change the portfolio allocation of liquid government bonds and other partially liquid financial assets.

One important future work is to study what is the Ramsey plan in the context of endogenous liquidity (how \( \phi \) changes endogenously by monetary policies). For simplicity, we set \( \phi \) changes as exogenous. Nevertheless, it could certainly depend on market expectations, asset qualities, and the tightness of the asset market. For example, if the illiquidity issues are caused by the search problems between buyers and sellers, will an unconventional policy improve the market efficiency or simply crowd out potential buyers? Does the unconventional policy have the same effects when there are large liquidity drops, compared to the policy when liquidity drops a little (i.e., normal recessions)? These issues are left for future research.

### 3.6 Entrepreneurs’ Optimal Solution

Here, we show the detailed steps in obtaining the optimal solution for an individual entrepreneur. It is helpful to write the entrepreneurs’ problem as

**Problem 7**

\[
\begin{align*}
v^i (s, b; \Gamma) &= \log (c) + \beta \chi \mathbb{E} \left[ v^i (s', b'; \Gamma') | \Gamma \right] + \beta (1 - \chi) \mathbb{E} \left[ v^n (s', b'; \Gamma') | \Gamma \right] \\
s.t. \quad c + q_r s' &= rs + (\varphi q + (1 - \varphi) q_r) (1 - \delta) s + \frac{r b}{P}
\end{align*}
\]
\( \varphi \leq \phi \)

\[
v^i(s, b; \Gamma) = \log(c) + \beta \chi \mathbb{E} [v^i(s', b'; \Gamma') | \Gamma] + \beta (1 - \chi) \mathbb{E} [v^u(s', b'; \Gamma') | \Gamma]
\]

s.t. \( c + qs' + \frac{b'}{p} = rs + q(1 - \delta)s + R \frac{b}{p} \)

In the equilibrium, \( q_r < 1 < q \), it is always better to choose \( \varphi = \phi \) so that \( (\varphi q + (1 - \varphi) q_r) \) can be maximized (to have maximal possible resources). The entrepreneurs’ problem can be re-written as

**Problem 8**

\[
v^i(s, b; \Gamma) = \log(c) + \beta \chi \mathbb{E} [v^i(s', b'; \Gamma') | \Gamma] + \beta (1 - \chi) \mathbb{E} [v^u(s', b'; \Gamma') | \Gamma]
\]

s.t. \( c + qs' + \frac{b'}{p} = rs + (\varphi q + (1 - \varphi) q_r)(1 - \delta)s + R \frac{b}{p} \)

\[
v^u(s, b; \Gamma) = \log(c) + \beta \chi \mathbb{E} [v^i(s', b'; \Gamma') | \Gamma] + \beta (1 - \chi) \mathbb{E} [v^u(s', b'; \Gamma') | \Gamma]
\]

s.t. \( c + qs' + \frac{b'}{p} = rs + q(1 - \delta)s + R \frac{b}{p}. \)

We propose the following solution:

\[
v^i(s, b; \Gamma) = J_i(\Gamma) + \frac{\log n_i(s, b; \Gamma)}{1 - \beta}, \quad v^u(s, b; \Gamma) = J_n(\Gamma) + \frac{\log n_n(s, b; \Gamma)}{1 - \beta}.
\]

where \( J_i \)s are constants and \( n \) is the net-worth of an entrepreneur

\[
n_i = rs + (\varphi q + (1 - \varphi) q_r)(1 - \delta)s + R \frac{b}{p}, \quad n_n = rs + q(1 - \delta)s + R \frac{b}{p},
\]

The associated policy functions are

\[
c_j = (1 - s_j)n_j, \quad q_j s_j' = \omega_j s_j n_j, \quad b'_j = (1 - \omega_j) s_j n_j
\]
where $s_j$ is the saving rate and $\omega_j$ is the portfolio weight on capital stock (both are to be determined), and $q_j$ is the implicit price (either $q$ or $q_r$).

(1) For a type $i$ entrepreneur, the first order condition with respect to $s'$ and $b'$ are

$$
q_r c^{-1} = \beta \chi \mathbb{E} \left[ (1-s_i)^{-1} (n_i')^{-1} (r' + (\phi' q' + (1-\phi') q_r') (1-\delta)) \mid \Gamma \right] + \beta (1-\chi) \mathbb{E} \left[ (1-s_i)^{-1} (n_n')^{-1} (r' + q' (1-\delta)) \mid \Gamma \right]
$$

$$
c^{-1} = \beta \chi \mathbb{E} \left[ (1-s_i)^{-1} (n_i')^{-1} \frac{pR'}{p'} \mid \Gamma \right] + \beta (1-\chi) \mathbb{E} \left[ (1-s_i)^{-1} (n_n')^{-1} \frac{pR'}{p'} \mid \Gamma \right] + \mu,
$$

where $\mu$ is the Lagrangian multiplier on the non-negativity constraint of $b$. When $\mu = 0$, multiply the first equation by $\frac{\omega_i q_r}{q_r}$ and the second equation by $(1-\omega_i)$, and add them up,

$$
c^{-1} = \chi \beta \mathbb{E} \left[ \frac{1}{(1-s_j) n_i'} \left( \omega_i r_{ii}' + (1-\omega_i) \frac{pR'}{p'} \right) \mid \Gamma \right] + (1-\chi) \beta \mathbb{E} \left[ \frac{1}{(1-s_j) n_n'} \left( \omega_i r_{in}' + (1-\omega_i) \frac{pR'}{p'} \right) \mid \Gamma \right],
$$

where

$$
r_{ii}' = \frac{r' + (\phi' q' + (1-\phi') q_r') (1-\delta)}{q_r}, \quad r_{in}' = \frac{r' + q' (1-\delta)}{q_r}.
$$

When $\mu > 0$, then $\omega_i = 1$ and we still get the same equation. Next, noticing that

$$
n_i' = r's' + (\phi' q' + (1-\phi') q_r') (1-\delta) s'
$$

$$
= \frac{r' + (\phi' q' + (1-\phi') q_r') (1-\delta)}{q_r} s_i n_i,
$$

$$
n_n' = r's' + q' (1-\delta) s' = \frac{r' + q (1-\delta)}{q_r} s_i n_i,
$$

we obtain

$$(1-s_i)^{-1} = \beta \mathbb{E} \left[ (1-s')^{-1} s_i^{-1} \mid \Gamma \right] \rightarrow \frac{s_i}{1-s_i} = \mathbb{E} \left[ \frac{\beta}{1-s'} \mid \Gamma \right].$$
Let us temporarily get time subscript back, then

\[
\frac{s_t}{1 - s_t} = \beta + \beta^2 + \ldots + \beta^{h_{t+h}} \left[ \frac{s_{t+h}}{1 - s_{t+h}} \right].
\]

Notice that due to log utility, it is not optimal to set \(s_t = 1\) or \(s_t = 0\) for every \(t\) and we have \(s_t = \beta\), if \(j \to \infty\).

(2) For a type \(n\) entrepreneur, the first-order conditions are

\[
qc^{-1} = \beta \chi E \left[ (1 - s_n)^{-1} (n_i')^{-1} \left( r' + (\phi'q' + (1 - \phi')q'_R) (1 - \delta) \right) \right] | \Gamma
\]

\[
+ \beta (1 - \chi) E \left[ (1 - s_n)^{-1} (n_n')^{-1} \left( r' + q' (1 - \delta) \right) \right] | \Gamma
\]

\[
c^{-1} = \beta \chi E \left[ (1 - s_n)^{-1} (n_i')^{-1} \frac{pR'}{p} | \Gamma \right]
\]

\[
+ \beta (1 - \chi) E \left[ (1 - s_n)^{-1} (n_n')^{-1} \frac{pR'}{p} | \Gamma \right] + \mu
\]

where \(\mu\) is the Lagrangian multiplier on the non-negativity constraint of \(b\). When \(\mu = 0\), multiply the first equation by \(\frac{\omega_i}{q_r}\) and the second equation by \((1 - \omega_i)\), and add them up

\[
c^{-1} = \chi \beta E \left[ \frac{1}{(1 - s_n)n_i'} \left( \omega_i r_{ni}' + (1 - \omega_i) \frac{pR'}{p} \right) \right] | \Gamma
\]

\[
+ (1 - \chi) \beta E \left[ \frac{1}{(1 - s_n)n_n'} \left( \omega_n r_{nn}' + (1 - \omega_n) \frac{pR'}{p} \right) \right] | \Gamma
\]

where

\[
r_{ni}' = \frac{r' + (\phi'q' + (1 - \phi')q'_R) (1 - \delta)}{q}, \quad r_{nn}' = \frac{r' + q' (1 - \delta)}{q}.
\]

When, \(\mu > 0\) then, \(\omega_i = 1\) and we still get the same equation. By following similar steps, we have \(s_n = \beta\) as well.

(3) Next we solve for the portfolio weights \(\omega_i\) and \(\omega_n\). Notice that, in the equilibrium with \(q > 1\), we have \(q_r < 1\). Therefore,

\[
r_{ni}' < r_{ii}', \quad r_{nn}' < r_{ni}'.
\]
Further notice that, in the neighborhood of the steady state

\[ r'_{nn} = \frac{r' + (1 - \delta) q'}{q} < r'_{ii} = \frac{r' + (1 - \delta) \phi' q' + (1 - \delta)(1 - \phi') q'}{q_r}. \]

Therefore

\[ r'_{ni} < r'_{nn} < r'_{ii} < r'_{in}. \]

The rate of return from government bond cannot be smaller than \( r'_{ni} \) (otherwise, type \( n \) entrepreneurs never acquire liquid assets) and cannot be larger than \( r'_{nn} \) (otherwise, type \( n \) entrepreneurs never acquire equity), so that

\[ r'_{ni} < \frac{PR'}{P'} < r'_{nn} < r'_{ii} < r'_{in}, \]

in the neighborhood of the steady state. Thus, we know that type \( i \) entrepreneurs put \( \omega_i = 1 \) on capital (never save in bonds) and type \( n \) entrepreneurs put \( 0 < \omega_n < 1 \) weight on equity where the portfolio weight \( \omega_n \) satisfy

\[
\chi \mathbb{E} \left[ \frac{\beta c r'_{ni}}{(1 - \beta) \left[ \omega_n r'_{ni} + (1 - \omega_n) \frac{pR'}{p'} \right]} \right] + (1 - \chi) \mathbb{E} \left[ \frac{\beta c r'_{nn}}{(1 - \beta) \left[ \omega_n r'_{nn} + (1 - \omega_n) \frac{pR'}{p'} \right]} \right] = 1, \\
\chi \mathbb{E} \left[ \frac{\beta c \frac{pR'}{p'}}{(1 - \beta) \left[ \omega_n r'_{ni} + (1 - \omega_n) \frac{pR'}{p'} \right]} \right] + (1 - \chi) \mathbb{E} \left[ \frac{\beta c \frac{pR'}{p'}}{(1 - \beta) \left[ \omega_n r'_{nn} + (1 - \omega_n) \frac{pR'}{p'} \right]} \right] = 1.
\]

which are the two equations in (3.8). To simplify, \( \omega_n \) satisfies

\[
\chi \mathbb{E} \left[ \frac{r'_{ni} - \frac{pR'}{p'}}{\omega_n r'_{ni} + (1 - \omega_n) \frac{pR'}{p'}} \right] + (1 - \chi) \mathbb{E} \left[ \frac{r'_{nn} - \frac{pR'}{p'}}{\omega_n r'_{nn} + (1 - \omega_n) \frac{pR'}{p'}} \right] = 0.
\]
3.7 Ramsey Problem

We first express $C_i$ and $C_n$ in terms of $S, S_g, S', \text{and } S_g'$. To do so, notice that the implementability condition and social resources constraint are

\[
(1 - \chi)^{-1} C_n - \chi^{-1} C_i = (1 - \beta) (1 - \phi) \frac{(1 - \theta) - d}{\theta (1 - \theta)} (1 - \delta) S
\]

\[
C_i + C_n + K' = \alpha AF(K, N) + (1 - \delta) K
\]

where $d = \frac{\beta}{1 + (1 - \sigma)(1 - \phi) \chi S}$, $K = S + S_g$, and $I = K' - (1 - \delta) K$. One can plug in $d$ and express $C_i, C_n$ only in terms of $S, S', S_g, \text{and } S_g'$:

\[
C_i = \frac{\chi}{1 - D} \left[ AF(K, N) - K' + (1 - \delta) K - G - C_i - c_1 S \right] \tag{3.37}
\]

\[
C_n = AF(K, N) - K' + (1 - \delta) K - G - C_i - C_i \tag{3.38}
\]

where

\[
D = \frac{c_2 S}{c_3 \left[ S' + S_g' - (1 - \delta) (S + S_g) \right] + c_4 S}, \quad K = S + S_g
\]

\[
c_1 = (1 - \chi) (1 - \beta) (1 - \phi) (1 - \delta) / \theta
\]

\[
c_2 = \beta \chi (1 - \chi) (1 - \phi) (1 - \delta)
\]

\[
c_3 = \theta (1 - \theta), \quad c_4 = \theta (1 - \delta) (1 - \phi) \chi.
\]

Now, one can re-write the Ramsey problem as

**Problem 9**

\[
V(S, S_g; \phi, A) = \max_{S', S_g'} u(C_i, C_n) + \beta \mathbb{E} [V(S', S_g'; \phi', A') | \phi, A]
\]

subject to (3.37) and (3.38).
Let \( u_i \) and \( u_n \) denote the marginal utility of \( C_i \) and \( C_n \):

\[
    u_i = \frac{\chi}{C_i}, \quad u_n = \frac{1 - \chi}{C_n}.
\]

The first-order condition for the planner is the following:

\[
\begin{align*}
    [S'] : & \quad u_i \frac{\partial C_i}{\partial S'} + \beta E u_i' \frac{\partial C_i'}{\partial S'} + u_n \frac{\partial C_n}{\partial S'} + \beta E u_n' \frac{\partial C_n'}{\partial S'} = 0 \\
    [S_g'] : & \quad u_i \frac{\partial C_i}{\partial S_g'} + \beta E u_i' \frac{\partial C_i'}{\partial S_g'} + u_n \frac{\partial C_n}{\partial S_g'} + \beta E u_n' \frac{\partial C_n'}{\partial S_g'} \geq 0,
\end{align*}
\]

where the “\( \geq \)” takes into account of \( S_g' = 0 \).

To fully characterize the problem requires some algebra. First,

\[
\frac{\partial D}{\partial S'} = -\frac{c_3 D^2}{c_2 S}; \quad \frac{\partial D}{\partial S_g'} = -\frac{c_3 D^2}{c_2 S_g}.
\]

\[
\frac{\partial D}{\partial S} = \frac{D}{S} + D \frac{c_3 (1 - \delta) - c_4}{c_3 [S' + S_g' - (1 - \delta) (S + S_g)] + c_4 S}
\]

\[
= \frac{D}{S} \left[ 1 + \frac{(1 - \delta) c_3 - c_4}{c_2 D} \right]
\]

\[
\frac{\partial D}{\partial S_g} = \frac{c_2 c_3 (1 - \delta) S}{c_3 [S' + S_g' - (1 - \delta) (S + S_g)] + c_4 S}^2 = \frac{(1 - \delta) c_3 D^2}{c_2 S}.
\]

Now, we use \( \frac{\partial D}{\partial S} \) and \( \frac{\partial D}{\partial S_g} \) to derive\(^\text{19}\)

\[
\frac{\partial C_i}{\partial S'} = u_i \chi \left[ \frac{(-1) (1 - D) + c_i (1 - D) \frac{\partial D}{\partial S'}}{(1 - D)^2} \right] = -\frac{\chi u_i}{1 - D} - \frac{\chi c_i D^2}{c_2 (1 - D) S}
\]

\(^\text{19}\)In order to save notation, we use \( \tilde{r} = AF_K(K,N) \)
\[
\beta \mathbb{E} u'_i \frac{\partial C'_i}{\partial S'} = \beta \mathbb{E} \chi u'_i \left[ \frac{(\bar{\varphi}' + 1 - \delta - c'_1) (1 - D') + \frac{c'_1(1 - D') \partial D'}{\chi}}{(1 - D')^2} \right]
\]

\[
= \beta \mathbb{E} \chi u'_i \frac{(\bar{\varphi}' + 1 - \delta - c'_1)}{1 - D'} + \beta \mathbb{E} \frac{\chi D' \left[ 1 + \frac{(1 - \delta)c'_3 - c'_4}{c'_2} \right]}{(1 - D') S'}
\]

\[
u_n \frac{\partial C_n}{\partial S'} = \nu_n \left[ -1 - \frac{\partial C_i}{\partial S'} \right] = -\nu_n + \frac{\chi u_n}{1 - D} + \frac{\chi c_3 D^2 u_n}{c_2 (1 - D) S u_i}
\]

\[
\beta \mathbb{E} u'_n \frac{\partial C'_n}{\partial S'} = \beta \mathbb{E} u'_n \left[ \bar{\varphi}' + 1 - \delta - \frac{\partial C'_i}{\partial S'} \right]
\]

\[
= \beta \mathbb{E} u'_n \left[ \bar{\varphi}' + 1 - \delta - \frac{\partial C'_i}{\partial S'} \right] - \beta \mathbb{E} \frac{\chi u'_n \left[ \bar{\varphi}' + 1 - \delta - c'_1 \right]}{1 - D'} - \beta \mathbb{E} \frac{\chi u'_n D' \left[ 1 + \frac{(1 - \delta)c'_3 - c'_4}{c'_2} \right]}{(1 - D') S'}.
\]

Similarly,

\[
u_i \frac{\partial C_i}{\partial S'_g} = \nu_i \chi \left[ \frac{(-1 - G' \left( S'_g \right)) (1 - D) + \frac{c'_1(1 - D) \partial D'_g}{\chi}}{(1 - D)^2} \right] = -\frac{\chi u_i}{1 - D} - \frac{\chi c_3 D^2}{c_2 (1 - D) S}
\]

\[
\beta \mathbb{E} u'_i \frac{\partial C'_i}{\partial S'_g} = \beta \mathbb{E} u'_i \chi \left[ \frac{\bar{\varphi}' + 1 - \delta - D' + \frac{c'_1(1 - D') \partial D'_g}{\chi}}{(1 - D')^2} \right]
\]

\[
= \beta \mathbb{E} \frac{\chi u'_i \left( \bar{\varphi}' + 1 - \delta \right)}{1 - D'} - \beta \mathbb{E} \frac{\chi (1 - \delta) c'_3 \left( D' \right)^2}{c'_2 (1 - D') S'}
\]

\[
u_n \frac{\partial C_n}{\partial S'_g} = \nu_n \left[ -1 - G' \left( S'_g \right) \right] = -\nu_n \left[ 1 + G' \left( S'_g \right) \right] + \frac{\chi u_n \left[ 1 + G' \left( S'_g \right) \right]}{1 - D} + \frac{\chi c_3 D^2 u_n}{c_2 (1 - D) S u_i}
\]
\[ \beta \mathbb{E} u_n' \frac{\partial C'_n}{\partial S'_g} = \beta \mathbb{E} u_n' \left( \tilde{r}' + 1 - \delta - \frac{\partial C'_i}{\partial S'_g} \right) \]

\[ = \beta \mathbb{E} u_n' (\tilde{r}' + 1 - \delta) - \beta \mathbb{E} \frac{\chi u_n' (\tilde{r}' + 1 - \delta)}{1 - D'} - \beta \mathbb{E} \frac{\chi u_n' (1 - \delta) c'_3 (D')^2}{c'_2 (1 - D') S'}. \]

Finally, we can simplify the first-order conditions for \( S' \) and \( S'_g \) as

\[ u_n - \frac{\chi (u_n - u_i)}{1 - D} z_0 = \beta \mathbb{E} \left[ u_n' (\tilde{r}' + 1 - \delta) \right] - \beta \mathbb{E} \frac{\chi (u_n - u'_i)}{1 - D'} z'_1 \]

\[ \geq \beta \mathbb{E} \left[ u_n' (\tilde{r}' + 1 - \delta) \right] - \beta \mathbb{E} \frac{\chi (u_n' - u'_i)}{1 - D'} z'_2, \]

where

\[ z_0 = 1 + \frac{c_3 D^2}{c_2 S u_i} \]

\[ z'_1 = \tilde{r}' + 1 - \delta - c'_1 + \frac{D'}{S' u'_i} \left[ 1 + \frac{(1 - \delta) c'_3 - c'_4}{c'_2} D' \right] \]

\[ z'_2 = \tilde{r}' + 1 - \delta + \frac{D'}{S' u'_i} \frac{(1 - \delta) c'_3 D'}{c'_2}. \]

We do not derive the second order conditions for the Ramsey problems, since the algebra becomes too tedious. Instead, we check all our calculations numerically by ensure that the first-order conditions give the welfare maximized solution. If perfect risk-sharing is possible, then \( u_n = u_i \) and we have

\[ u_n = \beta \mathbb{E} \left[ u_n' (\tilde{r}' + 1 - \delta) \right] \]

which says that investment is the amount such that marginal utility on consumption today and tomorrow are equalized. Then the first-order condition (3.40) has to be with equality so that \( S'_g > 0 \). This step proves Proposition 9.

At the same time, individual maximization has the same first-order condition but replace \( \tilde{r} \) as \( r \). Therefore, we go back to the standard optimal taxation result that if there is a long-run steady state, the capital return should not be taxed. This step proves Proposition 8.
Notice that with perfect risk-sharing, there should not be any difference in the rate of return from bonds and equity. Therefore, \( \frac{R}{\pi} = r + 1 - \delta = \bar{r} + 1 - \delta = 1/\beta \) and this step proves Corollary 6.
Figure 3.2: Impulse responses after liquidity shocks: different $\rho_{\phi}$

Figure 3.3: Impulse responses after liquidity shocks: different $\sigma_{\phi}$
Dynamics after liquidity shocks and productivity shocks. Blue dash lines: $\sigma_{\phi} = 0.04$, black dash dotted lines: $\sigma_{\phi} = 0.06$, and red solid lines: $\sigma_{\phi} = 0.05$. Output: $Y$. Investment: $I$. Consumption: $C$. Real liquidity: $B_r$. Asset price: $q$. Inflation: $\pi$. Nominal rate: $R$. Total tax: $\tau$. Bond supply growth: $\eta$. Government holding: $S_g$. Resaleability: $\phi$. Productivity: $A$. 

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