THREE ESSAYS IN FINANCIAL ECONOMICS

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Abstract

The dissertation consists of three chapters. In the first chapter I study heterogeneous beliefs speculative bubbles in a setup in which investment is endogenous and affects information flow. I show that during periods of major technological innovations it is socially optimal to have a bubble in asset prices, because it mitigates learning externalities. However, the speculative bubble that arises in decentralized economy is not perfectly aligned with the optimal bubble level and typically is too small at the beginning of the bubble episode and too large close to the end.

The second chapter (joint with Martin Schmalz) presents a model of Bayesian learning about multiple parameters of firms’ fundamentals, which explains asymmetries between upturns and downturns both in asset pricing and in corporate finance. Good performance in good times can be due to either desirable good idiosyncratic performance or to undesirable positive correlation with a market-wide factor. In contrast, good performance in bad times can either come from desirable good idiosyncratic performance or desirable negative correlation with a market-wide factor. As a result, in downturns prices react more strongly to news about fundamentals, stock picking earns higher returns and boards tend to fire CEOs more frequently.

The last chapter is an extension of the classical market microstructure model of Kyle (1985). Information about a security’s fundamentals arrives much less frequently than trading occurs. As a result, the market maker, trying to infer from the trading volume whether the insiders have received new information, will react much stronger to larger orders than to smaller ones and the relationship between price concession and the size of the order becomes non-linear. Thus, the market can be very liquid for small orders and very illiquid for large ones.
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# Contents

Abstract ................................................................. iii
Acknowledgments ....................................................... iv

1 Speculative Bubbles and Real Investment 1
   1.1 Introduction .................................................... 1
   1.2 Real Investment with Endogenous Learning ................. 7
   1.3 Speculative Bubble with Local Disagreement ............. 13
   1.4 Speculative Bubble with Endogenous Investment ........ 21
   1.5 Conclusion ..................................................... 27
   1.6 Appendix ....................................................... 29

2 Revealing Downturns 34
   2.1 Introduction .................................................... 34
   2.2 Model ............................................................ 39
   2.3 Applications .................................................... 48
   2.4 Discussion ..................................................... 52

3 Uncertain Insider Trading 59
   3.1 Introduction .................................................... 59
   3.2 Model and Equilibrium ....................................... 62
CONTENTS

3.3 Numerical solution ................................................. 67
3.4 Results and Interpretation ....................................... 68
3.5 Problems, Limitations and Further Work ..................... 71
3.6 Appendix .......................................................... 73

Bibliography .......................................................... 81
Chapter 1

Speculative Bubbles and Real Investment

1.1 Introduction

Bubbles usually arise with the advent of major technological innovations, when the uncertainty about the new technology’s productivity is still high. The only way to resolve this uncertainty is to invest in the technology and to observe the outcome. Thus, investments in the new technology sector generate additional information about it, which affects both prices of existing stocks and future investment decisions. While most theoretic models of bubbles (see Brunnermeier (2009)) take information structure and asset supply as given, I study implications for bubbles episodes of endogenous real investment that affects information flow.

The main contribution of this paper is a welfare analysis of bubbles. If we look at a particular bubble episode, for example the Internet bubble, a lot of wasteful investments were made in obscure projects that never paid off, and investors lost large fractions of their wealth. However, it is not obvious whether the bubble was
bad for the society as a whole. Investments in the new technology (even those that lose money) provide additional information about this technology and thus allow to make better investment decisions in the future. Decentralized investors will not fully internalize this learning externality. As a result, if prices are equal to fundamental values the overall level of investment will be less than socially optimal. A bubble in asset prices during periods of major technological innovations increases the level of investment and can mitigate the externality. Thus, it is possible that it is because of the bubble that internet technologies developed so fast. Probably, if not for the bubble, it would have taken much longer for “Google” and “Facebook” to appear and maybe we would not be able to enjoy using them right now.

A popular way to explain the existence of bubbles, such as the Internet bubble, is investors’ disagreement (Harrison and Kreps (1978)). If investors disagree, then in the presence of short sale constraints, the price of an asset can be even higher than the valuation of the most optimistic agents. This happens because beliefs might change in the future and current optimists will have an option to resell the asset to new future optimists. Disagreement can arise from investors’ overconfidence, because of which investors overreact to the information they receive (Scheinkman and Xiong (2003)). However, it is not straightforward how to study welfare implications of bubbles in a model with heterogeneous beliefs, because it is not clear beliefs of which group of investors we should take to calculate social welfare.

In the paper, I develop a new approach for modeling heterogeneous beliefs bubbles which is suitable for doing welfare analysis. I consider a model with local disagreement in which investors disagree about things that are resolved in the nearest

\[ \text{\footnotesize\(^1\)One possible approach is suggested by Brunnermeier and Xiong (2011): one alternative is better than the other if it is preferable in the beliefs of all groups of agents.} \]
future. In this case all disagreements are small and short-lived\(^2\). Then, I consider a limiting case in which the horizon of the disagreement converges to zero. In the limit, beliefs converge to rational (which allows to do welfare analysis), however the bubble still has a positive size. The intuition is as follows. To have a heterogeneous beliefs speculative bubble investors do not need to disagree right away, it is sufficient to know that there might be disagreements in the future. As a result, in a model with a large number of small short-lived disagreements, all future disagreements contribute to the current level of the bubble, and the bubble can be large. On the other hand, for welfare analysis, that is to say whether the current level of investment is socially optimal or not, only the current level of disagreement matters and it is negligible. In addition, the approach allows to get tractable results. Since beliefs converge to rational we do not need to keep track of the current differences in beliefs between groups of investors. As a result, the number of state variables which the bubble depends on is reduced.

Now we can compare the speculative bubble that arises in the markets with the optimal bubble level that we need to mitigate the learning externality. I show that while both the optimal bubble and speculative bubble are large in the presence of high uncertainty, they are not perfectly aligned. In particular, the optimal bubble is decreasing in the size of the new technology sector, while the speculative bubble is increasing. The intuition is that when the new technology sector is large, investors are already learning fast enough and there is little benefit from additional overinvestment. On the other hand, a large new technology sector creates a lot of information about the new technology, which implies a lot of speculation and large

\(^2\)Short horizon of investors’ disagreements seems to be a reasonable assumption when studying bubble episodes that last several years. Also, even if long-term disagreements exist, short-term disagreements are responsible for most of the speculative trading and therefore the bubble.
speculative bubble. As a result, for a particular bubble episode, the speculative bubble is too small at the beginning, when new technology sector is still small, and it is too large at the peak, when the new technology sector is already large.

The model is dynamic and in continuous time. After a new technology becomes available, at any point of time there is a pool of potential new technology projects. They differ in quality and can be financed by issuing equity. Thus, the number of projects that get financing depends on the current new technology valuations. Once financed a project becomes a new technology stock which, while it exists, pays dividends depending on the unknown productivity of the technology. The more new technology stocks there are in the economy, the more information investors receive and the faster they learn about the new technology’s productivity. A bubble in asset prices increases investment in the new technology sector and allows investors to learn faster, which leads to more information and thus better investment decisions in the future.

The model differs from the standard learning models in that the number of signals that investors receive changes over time and is endogenously determined. Also, investment in the new technology is modeled via projects getting financing, not via firms choosing the optimal scale of production. I choose this structure since usually during bubble episodes (think of the Internet bubble), the growth of the new technology sector is driven largely by new startups getting financing. When prices are high, even bad projects that are related to the new technology sector are financed (it seems that this is precisely what was happening during the Internet bubble). The continuous-time model allows to derive differential equations describing the optimal level of the bubble and the speculative bubble that arises in
the market. Using these equations it is possible to characterize how these bubble levels depend on the state of the economy and compare them.

Another insight of the model is that bubble episodes arise stochastically and the same technology can be adopted with and without a major bubble episode. If investors initially receive very good news about a given technology, this will lead to a lot of investment in the new technology sector, which will then constitute a sizable fraction of the economy. A large new technology sector in the presence of high uncertainty implies a lot of speculative trading and a large speculative bubble. Moreover, a large new technology sector will lead to faster price correction and to large wealth losses when this correction occurs. On the other hand, if initial news are not so good then investment in the new technology will be moderate and the speculative bubble will be small. Investors will learn only gradually about the potentially good technology. Also, since the new technology sector is small, there will not be any noticeable wealth gains. Looking ex post, bubble episodes that we observe in the history should correspond to the technologies for which investors initially receive very good news.

Finally, the model offers another explanation of increases in volatility and betas in the new technology stocks during bubble episodes (different from the one in Pastor and Veronesi (2009b)). Stock price volatilities of the new technology stocks are small at the beginning, because the new technology sector is small and not much new information is released. They are largest around the peak, when the new technology sector and information flow are large, but there is still a lot of unresolved uncertainty. They are small again in the end, when there is no uncertainty left. Betas are high at the peak of the bubble, because at this time volatility of the new
technology stocks is large and the new technology sector constitutes a big share of the economy.

There are several strands of related literature. Harrison and Kreps (1978) developed the original idea of heterogeneous beliefs speculative bubbles. Scheinkman and Xiong (2003) consider a continuous time model in which disagreement is driven by overconfidence and show how speculative bubbles are related to trading volume. Hong, Scheinkman, and Xiong (2006) show that increase in asset supply decreases the speculative bubble if investors are risk-averse. Morris (1996) studies bubbles in a learning environment and shows that bubbles can arise even if agents are completely rational but hold different priors. Panageas (2005) considers heterogeneous beliefs bubbles in a setup in which investment is endogenous, but the informational shocks are exogenously given.

The ideas that bubble episodes are related to technological innovations (which usually involve a lot of uncertainty) and that investors need to learn about productivity of new technologies I borrow from Pastor and Veronesi (2009b).

Welfare of rational bubbles with learning externalities was studied in overlapping-generations macro models. Grossman and Yanagawa (1993) showed that bubbles can be inefficient because they crowd savings away from physical capital and thus reduce growth rates. Olivier (2000) showed that if the bubble is in equity of the productive sector, then it can be welfare improving.

There are several other papers in which investment is endogenous and affects learning. Johnson (2007) explains overinvestment during major technological innovations with a model in which the new technology is characterized by unknown returns to scale, and investment increases the speed of learning. In his paper a representative agent makes all investment decisions. Thus, there are no external-
CHAPTER 1. SPECULATIVE BUBBLES AND REAL INVESTMENT

ities, and there is no role for bubbles. Veldkamp (2005) shows that busts should happen faster than booms, because in the boom more investments are made and people learn faster that economy switched to a bad state. Chamley and Gale (1994) show that there could be an inefficient delay in the investment decisions, because asymmetrically informed agents wait for others to reveal their private information.

The paper is structured as follows. Section 1.2 describes the model of investment with endogenous learning and shows that it can be socially optimal to have bubbles in asset prices. Section 1.3 develops the model of heterogeneous beliefs bubbles with local disagreement. In section 1.4 the speculative heterogeneous beliefs bubble is compared with the optimal bubble level.

1.2 Real Investment with Endogenous Learning

1.2.1 The Model

This section describes the main model of the paper, in which there is a technology with uncertain productivity and investments in this technology generate additional information about it. In such a setup, the socially optimal price level is always higher than the fundamental value of assets. In other words, it is socially optimal to have a bubble in asset prices. We will also derive a differential equation that describes how the optimal bubble level depends on the state of the economy.

There is a large group of risk neutral investors and there are two technologies in the economy. The old technology generates a risk free return $r$ and is available in unlimited supply. We will assume that at any moment of time some fraction of wealth is invested in the old technology.
At \( t = 0 \), the new technology with unknown productivity \( \theta \) becomes available. The prior distribution of \( \theta \) is normal \( \theta \sim \mathcal{N}\left(\theta_0, \frac{1}{\tau_0}\right) \). A new-technology stock \( i \), while it exists, pays dividend \( dD_t^i \) depending on the productivity of the new technology

\[
dD_t^i = \theta dt + \sigma_D dB_t^{D,i}
\]

and generates an additional signal \( Y_t^i \)

\[
dY_t^i = \theta dt + \sigma_Y dB_t^{Y,i}
\]

\( dB_t^{D,i} \) and \( dB_t^{Y,i} \) are independent Brownian motions (also independent across different stocks).

To allow depreciation we will assume that new technology stocks are liquidated at rate \( \lambda \). If \( K_t \) is the current number of new technology stocks, the number of stocks liquidated at time \( t \) is

\[
dL_t = \lambda K_t dt
\]

At each moment of time, \( Ndt \) randomly chosen investors become entrepreneurs and get a project. The project available at time \( t \) requires investment \( H \), which is a random variable with known cumulative distribution function \( F \). The project must be either financed or discarded right away and once financed becomes a standard new technology stock, described above. Suppose \( p_t \) is the current new technology stock price (it is the same for all new technology stocks). The entrepreneur can issue equity to finance the project and, therefore the project is financed if \( H < p_t \). The probability of becoming entrepreneur is small, thus learning externalities will
not be internalized. Therefore, the total number of projects financed is\(^3\)

\[ dM_t = Ndt \cdot \text{Prob} \{ H < p_t \} = NF(p_t) \cdot dt \]

Combining liquidation and financing we get the following equation for the evolution of the number of new technology stocks.

\[ dK_t = dM_t - dL_t = (NF(p_t) - \lambda K_t)dt \]

### 1.2.2 Updating Beliefs

By observing dividends and additional signals, investors gradually learn the actual productivity of the new technology. Since all noise signals are independent, the average dividend and average additional signal are sufficient statistics for learning about the productivity of the new technology.

\[
\begin{align*}
    dD_t &= \frac{1}{K_t} \sum dD^i_t = \theta dt + \frac{\sigma_D}{\sqrt{K_t}} dB^D_t \\
    dY_t &= \frac{1}{K_t} \sum dY^i_t = \theta dt + \frac{\sigma_Y}{\sqrt{K_t}} dB^Y_t
\end{align*}
\]

where \( dB^D_t = \frac{1}{\sqrt{K_t}} \sum dB^{D,i}_t \) and \( dB^Y_t = \frac{1}{\sqrt{K_t}} \sum dB^{Y,i}_t \) are independent Brownian motions (sums are over all stocks \( i \) available at time \( t \)). Since shocks from different stocks are averaged out, the information provided by averages is more precise and the variance of the noise term is decreasing in \( K_t \).

\(^3\)I assume here that the numbers of new technology stocks and projects are sufficiently large and we can use averages for investment and liquidation. The assumption does not affect the results, but greatly simplifies analysis and notation. On the other hand, I still assume that the number of stocks is not so large (or information received is sufficiently noisy) that investors immediately learn the productivity of the new technology.
Given the structure of the signals, conditional beliefs about the productivity of the new technology always remain normal. We will use the following notation.

\[
\theta |\{\text{Information at } t\} \sim \mathcal{N}\left(\hat{\theta}_t, \frac{1}{\tau_t}\right)
\]

Here instead of conditional variance, we use the conditional precision of beliefs \(\tau_t\).

With this notation we can write equations for the evolution of beliefs in a simple and intuitive form\(^4\)

\[
d\hat{\theta}_t = \frac{\tau_Y K_t}{\tau_t} (dY_t - \hat{\theta}_t dt) + \frac{\tau_D K_t}{\tau_t} (dD_t - \hat{\theta}_t dt)
\]

\[
d\tau_t = (\tau_Y + \tau_D) K_t dt
\]

where \(\tau_Y = \frac{1}{\sigma_Y^2}\) and \(\tau_D = \frac{1}{\sigma_D^2}\). At each moment of time, the precisions of signals \(\tau_Y K_t dt\) and \(\tau_D K_t dt\) (for \(dY_t\) and \(dD_t\) respectively) are added to the current precision of beliefs. The impact of each signal on the conditional mean is proportional to the ratio of the precision of the signal to the current precision of beliefs.

The state of the economy can be completely described by current believes \(\hat{\theta}_t, \tau_t\) and the size of the new technology sector \(K_t\). The following lemma summarizes equations for evolution of state variables.

\(^4\)Using theorem 12.7 from Liptser and Shiryayev (1978)
Lemma 1.1. State variables $\hat{\theta}_t, \tau_t$ and $K_t$ evolve according to

$$d\hat{\theta}_t = \sqrt{\frac{(\tau_Y + \tau_D)K_t}{\tau}} d\tilde{B}_t$$

$$d\tau_t = (\tau_Y + \tau_D)K_t dt$$

$$dK_t = (N \cdot F(p_t) - \lambda K_t) dt$$

where $d\tilde{B}_t = \sqrt{\frac{K_t}{\tau_Y + \tau_D}} \left( \tau_Y (dY_t - \hat{\theta}_t dt) + \tau_D (dD_t - \hat{\theta}_t dt) \right)$ is a standard Brownian motion for investors.

1.2.3 Optimal Bubble Level

The fundamental value of a new technology stock is the present value of its dividends

$$q_t = \int_0^\infty e^{-(r+\lambda)u} E_t [dD_{t+u}] = \frac{1}{r + \lambda} \hat{\theta}_t = \gamma \hat{\theta}_t$$

where we denote $\gamma = \frac{1}{r+\lambda}$ (the term $\lambda$ appears because of the possibility of liquidation).

Since investors are risk neutral we can think of social welfare as a present value of all future aggregate consumption. If we take for the benchmark the case in which investors do not invest at all in the new technology, then relative to the benchmark social welfare is going to be

$$\{\text{Social Welfare}\} = \{\text{PV of All Dividends from the New Technology Stocks}\}$$
Consider a new technology project available at time $t$. If the project is financed, the present value of all future dividends is $\gamma \hat{\theta}_t$, and the required investment is $H$. Thus, if the project is financed, social welfare gains from this transaction are

$$\gamma \hat{\theta}_t - H$$

Instantaneous social welfare gains from all projects financed at time $t$ is the average of welfare gains over all the projects that are financed ($H < p_t$)

$$U(\hat{\theta}_t, p_t) = N \cdot E[\gamma \hat{\theta}_t - H | H < p_t] = N \cdot \int_{-\infty}^{p_t} (\gamma \hat{\theta}_t - x) dF(x)$$

The above expression is maximized at $p_t = \gamma \hat{\theta}_t$. Thus, if investment did not affect learning, rational prices would be socially optimal.

The social value function at time $t$ is the highest possible value of the discounted sum of the future instantaneous welfare gains.

$$V(\hat{\theta}_t, \tau_t, K_t) = \max_{\{p_{t+u}\}} \int_0^\infty e^{-ru} U(\hat{\theta}_{t+u}, p_{t+u}) du$$

To find the optimal price level we can write the Hamilton-Jacobi-Bellman equation

$$\max_{p_t} \left( U(\hat{\theta}_t, p_t) - rV_t + E[dV_t] \right) = 0$$

Using the equations for the evolution of the state variables from lemma 1.1 we can get a partial differential equation for $V(\hat{\theta}_t, \tau_t, K_t)$ and calculate the optimal price.
level. The following proposition summarizes the results ($V_K$ is a derivative of the value function with respect to $K$)

**Proposition 1.2.** The optimal price level is

$$p_t = \gamma \hat{\theta}_t + V_K$$

The optimal bubble level is always positive

$$b_t^\theta = V_K > 0$$

The value function $V(\hat{\theta}_t, \tau_t, K_t)$ solves the following differential equation

$$U(\hat{\theta}_t, p_t) - rV + \left[ V_{\theta\theta} \frac{\tau_t K_t}{2\tau_t^2} + V_\tau \tau_t K_t + V_K (NF(p_t) - \lambda K_t) \right] = 0$$

Intuitively, financing negative NPV projects today will lead to more information and thus better investment decisions tomorrow. Since prices drive investment it is always optimal to have a bubble.

### 1.3 Speculative Bubble with Local Disagreement

#### 1.3.1 Overview of the Local Disagreement Approach

This section presents the model of the heterogeneous beliefs speculative bubble with local disagreement. The setup considered is quite general and can potentially be applied in various other settings. We will consider an asset whose value depends on unknown fundamental variable and assume that investors are overconfident about some of the information that they receive. We will be able to derive a simple differential equation that determines the level of the speculative bubble.
The speculative bubble arises in the presence of disagreement because current optimists will have an option to resell the asset to new future optimists if beliefs change. For the bubble to be large, it is not necessary for differences in opinions to be large. In fact, if at some point of time one group of investors is significantly more optimistic than everyone else, it will take a lot of time for beliefs to change and speculative bubble will be small. On the other hand, if disagreements are small, but relatively short-lived, then investors will trade very frequently and the resulting bubble can be very large.

We will consider a model in which investors disagree only locally, i.e., they disagree about things that are resolved in the nearest future. This naturally creates a setup in which disagreements are small and short-lived. Then, we will look at the limiting case in which the horizon of the disagreement converges to zero. In the limit, beliefs of investors become rational, but the bubble still persists. Since the level of the bubble will not depend on the current difference in beliefs, it is much easier to characterize it.

1.3.2 The Model

Information in reality is received in discrete pieces and a continuous time learning process is just a useful approximation. If investors are overconfident, then each piece of information could potentially lead to a speculative trade. We will consider a discrete-time model, that approximates a continuous time learning process, in which information is received in pieces of fixed size (in terms of precision) and then analyze what will happen in the limit as the precision of pieces of information converges to zero.
CHAPTER 1. SPECULATIVE BUBBLES AND REAL INVESTMENT

Assume there is an asset whose rational value depends on the unknown fundamental variable $\theta$. Each period, investors receive a signal about $\theta$.

$$\Delta Y_t = \theta \Delta t + \sigma_Y \varepsilon_t \sqrt{\Delta t} = \Delta t \left( \theta + \frac{1}{\sqrt{\tau_Y \Delta t}} \varepsilon_t \right)$$

where $\varepsilon_t \sim N(0, 1)$ is a random shock (and $\tau_Y = \frac{1}{\sigma_Y^2}$). As $\Delta t$ tends to 0, the process converges to the standard continuous time learning process

$$dY_t = \theta dt + \sigma_Y dB_t$$

The interval length $\Delta t$ is chosen in a way that precision of each signal received is equal to $\Delta \tau$, that is $\tau_Y \Delta t = \Delta \tau$. Faster learning (larger $\tau_Y$) means information is received more frequently, and thus there are more opportunities for investors to speculate.

There are two groups of investors $j = A, B$ who believe they can predict the realization of signal $\Delta Y_t$ before it was actually observed: at period $t$ before $\Delta Y_t$ is revealed each group receives an additional speculative signal $S_t^j \sim N(0, 1)$ which they believe is positively correlated with $\Delta Y_t$, but in reality is just noise.\(^5\)

$$corr(\Delta Y_t, S_t^j) = \rho \cdot \sqrt{\pi \Delta \tau}$$

Here, $\rho$ measures the level of overconfidence, the additional term $\Delta \tau$ prevents the bubble from exploding in the limit, (and $\pi \approx 3.14$ is to simplify results). It turns out that a small amount of overconfidence is sufficient to have a positive bubble.

\(^5\)Overconfidence is modeled here as investors reacting to irrelevant information. The results are the same as in a model in which investors receive relevant information, but overreact to it.
in the limit. The speculative signals are publicly observable and each group holds correct beliefs about the signal of the other group.

All disagreement is resolved after the realization of $\Delta Y_t$ is observed, thus it is indeed short lived. After receiving the speculative signals investors may trade on their information (there are no costs of trading). Agents are risk neutral, an the risk free rate is $1 + r\Delta t$. The timing of the model within period $t$ is presented in figure 1.1.

1.3.3 Analysis

Assuming $\theta$ has a normal prior, the conditional distribution of $\theta$ is going to be always normal. Again, we will use the following notation

$$\theta|\{\text{Information at } t\} \sim \mathcal{N}\left(\hat{\theta}_t, \frac{1}{\tau_t}\right)$$

Denote by $q_t$ the fundamental value of the asset and by $b_t$ the value of the resale option (or the speculative bubble component) at the end of period $t$ after signal $\Delta Y_t$ is received. Note that there is no disagreement at this point. We will take
$q_t$ as exogenously given, and then calculate the value of the speculative bubble component for a given process of $q_t$.

The value of the bubble component $b_{t-\Delta t}$ in period $t - \Delta t$ is the discounted sum of the expected bubble at period $t$ and of the expected gains from trading at period $t$

$$b_{t-\Delta t} = \frac{1}{1 + r\Delta t} (E_{t-\Delta t}[\Delta G_t] + E_{t-\Delta t}[b_t])$$

$\Delta G_t$ are gains from trading (or difference in valuations of buyer and seller if trade occurs) at time $t$.

**Lemma 1.3.** Expected gains from trading are equal to (for relatively small $\Delta \tau$)

$$E_{t-\Delta t}[\Delta G_t] = \frac{\rho(q_{t,\theta} + b_{t,\theta})\tau_Y}{\tau_t} \Delta t$$

(here $q_{t,\theta}$ and $b_{t,\theta}$ are derivates of $q_t$ and $b_t$ and with respect to $\hat{\theta}_t$)

The expression is quite intuitive. Instantaneous gains from trading over interval $\Delta t$ are proportional to the level of disagreement $\rho$, sensitivity of the price to the news about the fundamental variable $q_{t,\theta} + b_{t,\theta}$ and to how informative the signal that investors receive $\frac{\tau_Y}{\tau_t}$ is.

We can rewrite the equation determining $b_{t-\Delta t}$ in the following way

$$E_{t-\Delta t}[\Delta b_t] = \frac{\Delta t}{1 + r\Delta t} \left( r E_{t-\Delta t}[b_t] - \frac{\rho(q_{t,\theta} + b_{t,\theta})\tau_Y}{\tau_t} \right)$$

If $\Delta t$ is small, the equation approximates a differential equation that can be used to describe the bubble. The next proposition shows this formally.
Proposition 1.4. Suppose $b_t(\Delta \tau)$ is the speculative bubble in the discrete time approximating model for a given $\Delta \tau$. If $b_t(\Delta \tau) \to b_t$ as $\Delta \tau \to 0$, then $b_t$ solves the following differential equation

$$E_t[db_t] = \left( rb_t - \frac{\rho(q_t, \theta + b_t, \theta)\tau_Y}{\tau_t} \right) dt$$

which is equivalent to

$$b_t = E_t \left[ \int_t^\infty e^{-r(s-t)} \frac{\rho(q_s, \theta + b_s, \theta)\tau_Y}{\tau_s} ds \right]$$

The model can be generalized to the case in which precision of signal $\tau_{Y,t}$ changes over time. In this case, in the discrete-time model the length of the time interval between two signals will vary with $\tau_{Y,t}$. In addition, the unknown fundamental variable $\theta$ can change over time and signal $Y_t$ does not have to be the only signal from which investor learn about $\theta$ (just the only one about which they disagree).

Proposition 1.5. Assume there is an asset whose fundamental value $q_t$ depends on the beliefs about unknown fundamental variable $\theta_t$ and investors observe signal $Y_t$

$$dY_t = \theta_t dt + \frac{1}{\sqrt{\tau_{Y,t}}} dB_t$$

about which they disagree. If $b_t(\Delta \tau)$ is the speculative bubble in the corresponding discrete time approximating model and $b_t(\Delta \tau) \to b_t$ as $\Delta \tau \to 0$, then $b_t$ solves the following differential equation

$$E_t[db_t] = \left( rb_t - \frac{\rho(q_t, \theta + b_t, \theta)\tau_{Y,t}}{\tau_t} \right) dt$$
which is equivalent to

\[ b_t = E_t \left[ \int_t^\infty e^{-r(s-t)} \left( \frac{\rho (q_{s,\theta} + b_{s,\theta}) \tau_{Y,s}}{\tau_s} \right) ds \right] \]

The speculative bubble component \( b_t \) can depend on beliefs about \( \theta \), because future values of \( q_{t,\theta} \) and \( \tau_{Y,t} \) might depend on them (for example, it could be that information flow \( \tau_{Y,t} \) increases when prices go up). This dependence can also create an additional amplification channel for the bubble. If \( b_{t,\theta} \) is positive (as well as \( q_{t,\theta} \)), then prices will react more strongly to news about fundamentals, compared to the fully rational case (\( \rho = 0 \)). Investors who are overconfident about their ability to predict changes in fundamentals will be willing to pay even more for this asset. On the other hand, if \( q_{t,\theta} \) and \( \tau_{Y,t} \) do not depend on beliefs then \( b_{t,\theta} = 0 \) and the expression for the bubble is simplified.

**Corollary 1.6.** If \( q_{t,\theta} \) and \( \tau_{Y,t} \) do not depend on beliefs about the fundamental variable \( \hat{\theta}_t \), then \( b_{t,\theta} = 0 \) and the speculative bubble is equal to

\[ b_t = \rho \int_t^\infty e^{-r(s-t)} \left( \frac{q_{s,\theta} \tau_{Y,s}}{\tau_s} \right) ds \]

**1.3.4 Example**

To illustrate how to apply the model we can consider a simple example motivated by Scheinkman and Xiong (2003). Assume there is an asset that pays dividend \( dD_t \) depending on unknown fundamental variable \( \theta_t \) which changes over time according to

\[ d\theta_t = -\lambda (\theta_t - \bar{\theta}) + \sigma_\theta dB_t^\theta \]
In addition to dividends, agents observe a signal $dY_t$ which also depends on $\theta_t$. The processes for $dD_t$ and $dY_t$ are given by

$$dD_t = \theta_t dt + \sigma_D dB_t^D$$

$$dY_t = \theta_t dt + \sigma_Y dB_t^Y$$

The fundamental value of the asset is the present value of its dividends

$$q_t = \int_t^\infty e^{-rs} E_t[dD_s] = \frac{\bar{\theta}}{r} + \frac{1}{\lambda + r}(\hat{\theta}_t - \bar{\theta})$$

If beliefs about $\theta$ are at its stationary level, the precision of beliefs is constant and equal to

$$\tau_t = \tau = \tau_\theta \left( \lambda + \sqrt{\lambda^2 + \frac{\tau_D + \tau_Y}{\tau_\theta}} \right)$$

(here $\tau_\theta = \sigma_\theta^{-2}$, $\tau_D = \sigma_D^{-2}$ and $\tau_Y = \sigma_Y^{-2}$).

Suppose investors are overconfident and believe they can predict changes in $dY_t$ with level of overconfidence $\rho$ (in the way that is described above). We can write a differential equation for the bubble, which in this case is simplifies to

$$b_t = \int_t^\infty e^{-r(s-t)} \rho q_\theta \frac{\tau_Y}{\tau_s} ds = \frac{\rho q_\theta \tau_Y}{r \tau} = \frac{\rho}{r(\lambda + r)} \left( \lambda + \sqrt{\lambda^2 + \frac{\tau_D + \tau_Y}{\tau_\theta}} \right)$$

The difference from Scheinkman and Xiong (2003) is that disagreements are short-lived. As a result, the bubble does not depend on current differences in beliefs and we can get a simple closed form expression for the bubble.
1.4 Speculative Bubble with Endogenous Investment

1.4.1 Bubble Differential Equation

Using the techniques developed in the previous section we can now study heterogeneous beliefs bubbles in the setup in which investment is endogenous and affects information flow. The average dividend and signal are changing according to

\[ dD_t = \theta dt + \frac{\sigma_D}{\sqrt{K_t}} dB^D_t \]

\[ dY_t = \theta dt + \frac{\sigma_Y}{\sqrt{K_t}} dB^Y_t \]

Assume investors are overconfident about signal \( dY_t \) (equivalently, they could be overconfident about signals of individual stocks) with degree of overconfidence is \( \rho \).

The precision of signal \( dY_t \) is \( \tau_{Y,t} = \tau_Y K_t \), thus we can write the following differential equation for the bubble (additional term \( \lambda \) comes from the fact that stocks might be liquidated, \( q_{t,\theta} = \gamma \)).

\[ E_t[db_t] = (r + \lambda)b_t - \rho(\gamma + b_{t,\theta}) \frac{\tau_Y K_t}{\tau_t} \] \( dt \)

The speculative bubble is a function of the state variables \( \hat{\theta}_t, \tau_t \) and \( K_t \), the evolution of which is described by lemma 1.1 with \( p_t = \gamma \hat{\theta}_t + b_t \). Since (here \( \tau_\sigma = \tau_Y + \tau_D \))

\[ E_t[db_t] = \left( b_t \tau_\sigma K + b_K \left( NF(\gamma \hat{\theta} + b) - \lambda K \right) + b_{\theta\theta} \frac{\tau_\sigma K}{2\tau^2} \right) dt \]
we can write a partial differential equation for \( b(\hat{\theta}_t, \tau_t, K_t) \), which can be solved numerically

\[
b_\tau \tau \sigma K + b_K \left( N F(\gamma \hat{\theta} + b) - \lambda K \right) + b_\theta \tau \sigma K \frac{\tau \sigma K}{2 \tau^2} = (r + \lambda) b - \rho (\gamma + b_\theta) \frac{\tau Y K_t}{\tau_t}
\]

### 1.4.2 Bubble for Fixed Industry Size

To understand the intuition of how the speculative bubble should depend on the state variables we can consider a special case of the model in which the size of the new technology industry is exogenously given, \( K_t = K \). In this case beliefs do not affect the level of the bubble (that is \( b_{t,\theta} = 0 \)) and instead of differential equations we get the following integral

\[
b_t = \rho \gamma \int_t^\infty e^{-(r+\lambda)(u-t)} \frac{\tau Y K_t}{\tau u} du
\]

Given that the precision of beliefs now grows at a constant rate (\( \tau_u - \tau_t = (\tau Y + \tau_D) K \cdot (u - t) \) for \( s > t \), we get a closed form expression for the bubble.

**Lemma 1.7.** *If the new technology industry size is fixed, the speculative bubble is equal to*

\[
b(\tau, K) = \frac{\rho}{r + \lambda} \frac{\tau Y}{\tau Y + \tau_D} \Psi \left( \frac{(\tau Y + \tau_D) K}{(r + \lambda) \tau} \right)
\]

*where \( \Psi(u) = \int_0^\infty \frac{we^{-z}}{(1+uz)}dz \) is an increasing function of \( u \) (\( \Psi(u) \approx u \) for small \( u \)).*

The speculative bubble is decreasing in the precision of beliefs \( \tau \) and increasing in the new technology industry size \( K \). The more investors know about the productivity of the new technology, the less disagreement new information brings, and as a result the smaller is the bubble. Larger \( K \) makes signals arrive faster, which means
that the same speculative trades will happen sooner. As a result, since investors discount the future, the speculative bubble is also going to be larger.

The expression above also allows us to understand how the bubble should depend on beliefs about productivity $\hat{\theta}_t$ in general. If beliefs are high then the new technology industry will be growing, and if beliefs are low, it will be shrinking. Thus, the bubble should be an increasing function of $\hat{\theta}_t$.

### 1.4.3 Optimal Bubble vs Speculative Bubble

In this subsection, we will compare numerically the optimal bubble level and the speculative bubble. The parameters used in the numerical solution are in table 1.1. The are not carefully calibrated, but the results do not depend on particular values chosen.

The optimal bubble and the speculative bubble are the functions of state variables $\hat{\theta}_t, \tau_t$ and $K_t$. Figures 1.2 shows how they depend on belief about productivity $\hat{\theta}$ and the new technology industry size $K$ for two different values of the precision of beliefs, $\tau = 1$ and $\tau = 5$. We can see that there are significant differences. As expected, the speculative bubble is increasing in the size of the new technology sector and in the beliefs about the productivity of the new technology. The optimal bubble level, on the other hand, is decreasing with the size of the new technology sector.
Figure 1.2: Optimal bubble and speculative bubble. The figure shows how the optimal bubble level and the speculative bubble depend on the the state variables (τ – precision of beliefs, θ – current level of beliefs, K – the size of the new technology sector).
CHAPTER 1. SPECULATIVE BUBBLES AND REAL INVESTMENT

sector. If the new technology sector is large, then investors are already learning fast about the new technology’s productivity and there is little additional benefit from investing even more. The peak in the relationship between the optimal bubble and the beliefs about the productivity arises because the distribution of the required investment for projects is normal. If beliefs are very high, then investors finance all the projects that are available, and there is little benefit in knowing precisely the productivity of the new technology. We can also see that as \( \tau \) increases, both the optimal and the speculative bubbles gradually converge to zero. Table 1.2 summarizes all comparative statics results.

1.4.4 Dynamic Analysis

So far we looked at how the speculative bubble and the optimal bubble level depend on the state variables. We can also analyze how they are going to change over time for a particular bubble episode.

We can rewrite the equation from lemma 1.1 for the evolution of beliefs about productivity in the following way

\[
    d\hat{\theta}_t = \left(\frac{\tau_Y}{\tau_t} + \frac{\tau_D}{\tau_t}\right)K_t \left(\theta - \hat{\theta}_t\right)dt + \frac{\sqrt{(\tau_Y + \tau_D)K_t}}{\tau_t} dB_t
\]

where \( dB_t = \sqrt{\frac{\tau_Y}{\tau_Y + \tau_D}} dB_t^Y + \sqrt{\frac{\tau_D}{\tau_Y + \tau_D}} dB_t^D \) is also a Brownian motion. Thus, for a technology with a given productivity \( \theta \), beliefs will gradually converge to this productivity, but how exactly this will happen will depend on the realization of the process of shocks \( B_t \). Figure 1.3 shows how the state variables, the speculative bubble and the optimal bubble level will evolve for two such realizations (solid and dashed lines). For the “solid” realization, investors initially receive very good news, which in the presence of high uncertainty is amplified by the additional
Figure 1.3: Dynamic analysis. The figure shows how the state variables, the speculative bubble and the optimal bubble level evolve over time for two different realizations of shocks (which are brownian motions). The actual productivity of the technology is the same in both case and it equal to $\theta = \theta_0 + 0.5$. Initial precision of beliefs $\tau_0 = 2$. 
CHAPTER 1. SPECULATIVE BUBBLES AND REAL INVESTMENT

Table 1.2: Comparative Statics for Optimal and Speculative Bubbles

<table>
<thead>
<tr>
<th></th>
<th>Optimal Bubble</th>
<th>Speculative Bubble</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size of New Technology Sector, $K_t$</td>
<td>decreasing</td>
<td>increasing</td>
</tr>
<tr>
<td>Beliefs about Productivity, $\hat{\theta}_t$</td>
<td>depends on distribution of projects</td>
<td>increasing</td>
</tr>
<tr>
<td>Precision of Beliefs, $\tau_t$</td>
<td>decreasing</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

speculative bubble and leads to overinvestment in the new technology sector. For the “dashed” realization the initial news are neutral. Investors gradually learn the actual productivity of the new technology, the speculative component in asset prices is never large and there are no spikes in the investment. Bubbles arise stochastically, that is, the same technology could be adopted with and without a major bubble episode. Looking ex post bubble episodes happen for those technologies, for which investors initially receive very good news.

Also, we can compare the optimal bubble level with the speculative bubble in the dynamics. We see that the optimal bubble is large at first and then declines, as the size of the new technology sector increases, while the speculative bubble is the largest when the new technology sector is large, but there is still a lot of uncertainty remaining.

1.5 Conclusion

Information flow is affected by investment decisions. In the paper I want to argue that this has several implications for understanding bubble episodes.

The main result of the paper is a welfare analysis of bubbles. It is shown that during periods of major technological innovations, which involve a lot of uncertainty about the productivity of the new technology, it can be socially optimal to have bubbles in asset prices, because they can mitigate learning externalities. It is also
shown that bubbles that arise in financial markets are not perfectly aligned with the optimal bubble level. In particular, they are likely to be too small in the beginning of the bubble episode and too large at the peak. The methodological contribution of the paper is the local disagreement approach to modeling heterogeneous beliefs speculative bubbles. The approach is well-suited for a welfare analysis of speculative bubbles, as at any point time beliefs of investors are almost rational. It also allows to get tractable results. Thus, bubbles can be studied in more complicated settings, such as the one presented in this paper with endogenous investment that affects learning. In addition, the model is consistent with various empirical facts about bubble episodes such as investment spikes, increases in volatilities and betas of the new technology stocks and increases in trading volume during the episode.

One straightforward extension of the model is to incorporate trading costs, which will involve only a minor adjustment to the differential equation for the bubble. In the present setup, investors trade whenever they receive new information, no matter how significant it could be. Trading costs will limit the benefits from speculation and thus reduce the size of the speculative bubble. A possible application is to calculate the optimal trading tax and describe how it should depend on the state of the economy. It is likely that benefits from the trading tax are the largest when there is an already large new technology sector with yet very uncertain prospects.

The paper offers a qualitative welfare analysis for bubble episodes. The next step would be to evaluate empirically the optimal bubble level and the size of the speculative bubble. Possibly, the size of the speculative bubble can be determined from the trading volume.
1.6 Appendix

Proof of Lemma 1.1

The only thing we need to show is that $d\tilde{B}_t$ defined in the lemma is a standard Brownian motion. According to theorem 12.7 from Liptser and Shiryayev (1978)

$$d\tilde{B}^Y_t = \frac{1}{\sigma_Y/\sqrt{K_t}}(dY_t - \hat{\theta}_t dt) = \sqrt{\tau_Y K_t}(dY_t - \hat{\theta}_t dt)$$

$$d\tilde{B}^D_t = \frac{1}{\sigma_D/\sqrt{K_t}}(dD_t - \hat{\theta}_t dt) = \sqrt{\tau_D K_t}(dD_t - \hat{\theta}_t dt)$$

are independent standard Brownian motions for investors. Therefore

$$d\tilde{B}_t = \sqrt{\frac{\tau_Y}{\tau_Y + \tau_D}}d\tilde{B}^Y_t + \sqrt{\frac{\tau_D}{\tau_Y + \tau_D}}d\tilde{B}^D_t$$

is also a standard Brownian motion.

Proof of Proposition 1.2

Using the equations from lemma 1.1 (here $\tau_\sigma = \tau_Y + \tau_D$)

$$E[dV_t] = V_\theta \frac{\tau_\sigma K_t}{2\tau_t^2} + V_\tau \tau_\sigma K_t + V_K (NF(p_t) - \lambda K_t)$$

We can rewrite HJB Equation as

$$\max_{p_t} \left\{ U(\hat{\theta}_t, \gamma \hat{\theta}_t + V_K) - rV + \left[ V_\theta \frac{\tau_\sigma K_t}{2\tau_t^2} + V_\tau \tau_\sigma K_t + V_K (NF(p_t) - \lambda K_t) \right] \right\} = 0$$
The derivative of welfare gains with respect to price is

$$\frac{\partial U}{\partial p_t} = N(\gamma \hat{\theta}_t - p_t) f(p_t)$$

where $f(x) = \frac{dF(x)}{dx}$ is pdf of the distribution of projects. Thus, F.O.C. for $p_t$

$$N(\gamma \hat{\theta}_t - p_t) f(p_t) + NV_K f(p_t) = 0$$

and optimal price

$$p_t = \gamma \hat{\theta}_t + V_K$$

The value function is increasing in the new technology industry size $V(K + \Delta K) \geq V(K)$ since information from additional $\Delta K$ new technology stocks can be ignored for choosing the optimal price. Therefore $V_K \geq 0$.

**Proof of Lemma 1.3**

The signal $\frac{\Delta Y_t}{\Delta t}$ has precision $\Delta \tau$, thus beliefs are updated according to

$$\Delta \tau_t = \tau_t - \tau_{t-\Delta t} = \Delta \tau$$

$$\Delta \hat{\theta}_t = \hat{\theta}_t - \hat{\theta}_{t-\Delta t} = \frac{\Delta \tau}{\tau_t} \left( \frac{\Delta Y_t}{\Delta t} - \hat{\theta}_{t-\Delta t} \right)$$

Value of the asset at the end of period $t$

$$p_t = q_t + b_t \approx (q_t + b_t)|_{\Delta \hat{\theta}_t=0} + (q_{t,\theta} + b_{t,\theta}) \Delta \hat{\theta}_t$$
CHAPTER 1. SPECULATIVE BUBBLES AND REAL INVESTMENT

Since correlation between $\Delta Y_t$ is equal to $\rho \sqrt{\pi \Delta \tau}$, then expected value of the signal by each group of investors is

$$E^j_t[\Delta Y_t] - \hat{\theta}_{t-1} \Delta t = \rho \sqrt{\pi \Delta \tau} \cdot \sqrt{\text{Var}_{t-\Delta t}[\Delta Y_t]} S^j_t =$$

$$= \rho \sqrt{\pi \Delta \tau} \cdot \sqrt{\Delta t^2 \left( \frac{1}{\tau_t-\Delta t} + \frac{1}{\Delta \tau} \right)} S^j_t \approx \rho \sqrt{\pi \Delta t} \cdot S^j_t$$

where $E^j[\ldots]$ conditional expectation in the information set of group $j$. As a result

$$E^j_t[\Delta \hat{\theta}_t] = \frac{\Delta \tau}{\tau_t} \cdot \rho \sqrt{\pi} S^j_t = \frac{\rho \sqrt{\pi} \tau Y}{\tau_t} \cdot S^j_t \Delta t$$

Denote by $p^A_t$ and $p^B_t$ the price that investors of each group are willing to pay for the asset after receiving their signals.

$$p^j_t = E_t^j[p_t] = (q_t + b_t)|_{\Delta \theta_t = 0} + (q_{t,\theta} + b_{t,\theta})E_t^j[\Delta \hat{\theta}_t] =$$

Difference in valuation

$$p^B_t - p^A_t = (q_{t,\theta} + b_{t,\theta}) \left( E_t^B[\Delta \hat{\theta}_t] - E_t^A[\Delta \hat{\theta}_t] \right) = \frac{\rho \sqrt{\pi} (q_{t,\theta} + b_{t,\theta}) \tau Y \Delta t}{\tau_t} (S^B_t - S^A_t)$$

Assume group A holds the asset at the beginning of period $t$ then gains from resale and expected gains from resale are

$$G_t = \max (0, p^B_t - p^A_t) = \max (0, S^B_t - S^A_t) \cdot \frac{\rho \sqrt{\pi} (q_{t,\theta} + b_{t,\theta}) \tau Y \Delta t}{\tau_t}$$

$$E_{t-\Delta t}[\Delta G_t] = \frac{\rho (q_{t,\theta} + b_{t,\theta}) \tau Y}{\tau_t} \Delta t$$
Proof of Proposition 1.4

The equation for the bubble is

\[ b_t = \frac{1}{1 + r\Delta t} (E_t[\Delta G_{t+\Delta t}] + E_t[b_{t+\Delta t}]) \]

Solving this forward

\[ b_t = \sum_{k=1}^{\infty} \frac{E_t[\Delta G_{t+k\Delta t}]}{(1 + r\Delta t)^k} \]

Given that

\[ E_t[\Delta G_{t+k}] = E_t \left[ \frac{\rho(q_{t+k\Delta t,\theta} + b_{t+k\Delta t,\theta})\tau Y_{t+k\Delta t}}{\tau_{t+k\Delta t}} \right] \]

we have

\[ b_t = E_t \left[ \sum_{k=1}^{\infty} \frac{1}{(1 + r\Delta t)^k} \frac{\rho(q_{t+k\Delta t,\theta} + b_{t+k\Delta t,\theta})\tau Y_{t+k\Delta t}}{\tau_{t+k\Delta t}} \right] \]

As \( \Delta t \to 0 \) expression under the expectation converges to the integral

\[ \int_t^\infty e^{-r(s-t)} \frac{\rho(q_{s,\theta} + b_{s,\theta})\tau Y_{s}}{\tau_s} ds \]

Therefore

\[ b_t = E_t \left[ \int_t^\infty e^{-r(s-t)} \frac{\rho(q_{s,\theta} + b_{s,\theta})\tau Y_{s}}{\tau_s} ds \right] \]

and we proved the second part of the proposition. We can rewrite this expression as

\[ b_t e^{-rt} + \int_0^t e^{-r(s-t)} \frac{\rho(q_{s,\theta} + b_{s,\theta})\tau Y_{s}}{\tau_s} ds = E_t \left[ \int_0^\infty e^{-r(s-t)} \frac{\rho(q_{s,\theta} + b_{s,\theta})\tau Y_{s}}{\tau_s} ds \right] \]

The process defined by expectation on the right is a martingale, thus the process on the left is a martingale as well. Therefore

\[ E_t[db_t]e^{-rt} - rb_t e^{-rt} dt + e^{-rt} \frac{\rho(q_{t,\theta} + b_{t,\theta})\tau Y_t}{\tau_t} dt = 0 \]
which is equivalent to

\[ E_t[db_t] = \left( rb_t - \frac{\rho(q_t, \theta + b_t, \theta) \gamma_t}{\tau_t} \right) dt \]

**Proof of Proposition 1.5**

The proof is analogous to the proof of proposition 1.4.

**Proof of Lemma 1.7**

The speculative bubble is equal to

\[ b_t(\tau, K) = \rho \gamma \int_t^\infty e^{-(r + \lambda)(s - t)} \frac{\tau Y K}{\tau_s} ds = \]

\[ = \frac{\rho}{r + \lambda} \cdot \frac{\tau Y}{\tau Y + \tau D} \int_t^\infty e^{-(r + \lambda)(s - t)} \frac{(\tau Y + \tau D)K}{\tau + (\tau Y + \tau D)K \cdot (s - t)} ds \]

If we denote \( z = (r + \lambda)(s - t) \) and \( u = \frac{(\tau Y + \tau D)K}{(r + \lambda)r} \) we can rewrite the above integral as

\[ b_t(\tau, K) = \frac{\rho}{r + \lambda} \cdot \frac{\tau Y}{(\tau Y + \tau D)} \int_t^\infty \frac{ue^{-z}}{1 + uz} dz \]
Chapter 2

Revealing Downturns

2.1 Introduction

Asymmetries between upturns and downturns puzzle researchers both in asset pricing and corporate finance. For example, there is evidence that fund managers are more effective at picking stocks in downturns than in upturns. In corporate finance, boards tend to fire CEOs more often in downturns than in upturns, which contradicts relative performance evaluation.\(^1\)

The asymmetries in asset pricing have been addressed with rather intricate partial equilibrium models of effort choice and attention allocation over the business cycle (Glode (2011), Kacperczyk, Nieuwerburgh, and Veldkamp (2011)), and the combination of negatively skewed consumption and the convexity of the flow-performance relationship of funds (Kaniel and Kondor (2011)). A possible reason for higher forced CEO turnover in downturns is that boards, for some reason, fail to fulfill their monitoring duties in upturns. Jenter and Kanaan (2006) discuss and

provide support for an alternative hypothesis: performance in recessions may be more informative about the quality of the CEO, or the CEO-firm match.\footnote{Other papers addressing asymmetries between upturns and downturns with learning models include Veldkamp (2005) and Van Nieuwerburgh and Veldkamp (2006), who are interested in the speed of booms and busts. Their agents learn about real economic activity, which fluctuates over the business cycle.}

This paper proposes a simple theory according to which news in downturns indeed carry more relevant information about the utility that risk-averse investors derive from holding the asset than news in upturns. It thereby reduces the number and strength of the “behavioral” assumptions needed to explain the data. Our model proposes that risk-averse investors like good idiosyncratic performance (“cash-flow alpha” – henceforth, $a$), but dislike high correlation with a market-wide factor (“cash-flow beta” – henceforth, $b$). They are uncertain about both parameters. High cash-flows in good times are both a signal for high $a$ and high $b$. One is a good signal, one is a bad signal for the value of holding the asset to the investor. Exceptionally high cash flows in good times are therefore a somewhat ambiguous signal about firm value. Investors may sense that the reported performance is likely to have been achieved with exceptionally high risk exposure. Therefore, investors will not attach high confidence to good news in good times. Similarly, low cash flows in an upturn can be due to low $a$, or due to a negative $b$. This, again, is a somewhat mixed signal: the sub-par performance might be due to negative market exposure, which will turn out to be a valuable hedge for bad times. Therefore, prices will not adjust strongly in response to bad news in good times either. The model thus predicts that the reaction of market prices to news and the associated volatility is less strong in upturns.

In contrast, relatively good performance in bad times can be either due to high cash-flow alpha or due to low cash-flow beta, both of which are a positive signal
about firm value. Similarly, bad performance in bad times is clearly a bad signal about firm value: it can either be due to bad idiosyncratic performance or high market correlation – both of which are undesirable attributes. In sum, cash flow news in downturns are unambiguous signals about firm value, in one direction or another. Therefore, investors place higher “weights” on information pertaining to firm performance in downturns than to performance in upturns. Table 2.1 illustrates the conclusions investors draw from observing good and bad news in good and bad times.

<table>
<thead>
<tr>
<th></th>
<th>high payoff</th>
<th>low payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>market upturn</td>
<td>high $a$, high $b$</td>
<td>low $a$, low $b$</td>
</tr>
<tr>
<td>market downturn</td>
<td>high $a$, low $b$</td>
<td>low $a$, high $b$</td>
</tr>
</tbody>
</table>

Table 2.1: The table shows inferences about $a$ and $b$ from observing stocks’ cash flows in market upturns and downturns.

A more technical way of describing the mechanics is as follows. Estimates of $a$ and $b$ are positively correlated in upturns, and negatively in downturns. As a result, investors are better able to distinguish in bad times between “good firms” that have high $a$ and low $b$ and “bad firms” that have low $a$ and high $b$.

Yet another way of understanding the idea is to realize that to value the stock, investors put high weights on observations made when the stochastic discount factor is high, i.e., in downturns. The reason is that risk-averse investors care more about the performance of an asset in bad times, which is when they operate at a very steep part of their utility function. No effort choice or rational inattention (as used in the above-discussed extant theories) is needed to derive this result.

The theory explains the asymmetries in corporate finance and asset pricing as follows. Since (ia) prices react more strongly to news received in downturns, (ib) the cross-sectional dispersion of returns is higher at that time. As news will thus
change valuations more significantly in downturns, the return to analyzing them correctly and trading on superior information is higher then. As a result of the larger spread between before-news and after-news asset prices, (ii) active funds will perform better in market downturns. Lastly, if replacing a CEO comes at a cost (see Taylor (2010)), boards will only move when they are reasonably sure that the manager in place performs worse than what they can expect to get by randomly drawing a new CEO from the pool of candidates. As the left tails of market values becomes fatter in downturns, more firms’ valuations fall below the threshold at which boards rationally sense that replacing the CEO is thus sufficiently likely to add shareholder value. The model therefore predicts that (iii) boards are more likely to fire CEOs after observing the performance of the CEO-firm match in downturns.

Predictions (ia) and (ib) are in this form new to the literature. Predictions generating (ii) and (iii) have been made with more complicated models with stronger assumptions before, as described above. Providing empirical evidence for the claims (i) is subject of current research. We do not replicate the existing empirical evidence for (ii) and (iii), as ample evidence exists, which serves as the motivation of this paper.\(^3\)

The paper contributes to a growing literature on Bayesian learning in financial markets.\(^4\) In contrast to much of the literature which focuses on learning about the mean of a single asset’s payoff, we investigate the implications of learning about the covariance parameters of the assets’ fundamentals as well. The trick to keep the

\(^3\)Maybe because the cross-sectional dispersion is irrelevant for relative pricing in a neo-classical model, it has not received much attention in the literature (one exception is Duffee (2001)). However, it is relevant if stock pickers can exist, which is the presumption in our framework.

\(^4\)Pastor and Stambaugh (2012) study the seeming unconditional underperformance of active mutual funds. See Pastor and Veronesi (2009a) for an overview over less recent research.
model tractable is to consider an economy with many assets, whose payoffs depend on several unknown parameters. For a large number of assets, there is no leaning about the aggregate payoff from cross-sectional observations, and as a result no learning about the stochastic discount factor.

The closest papers may be Veronesi (1999) and Veronesi and Ribeiro (2002). The respective purpose of their and our paper is different, however. Veronesi (1999) explains the asymmetric reaction to good and bad news at any time. We explain the asymmetric reaction to any type of news in good and bad times. Veronesi and Ribeiro (2002) explain higher co-movement of stock returns in recessions. We explain higher cross-sectional dispersion in recessions. The two are not mutually exclusive, see Figure 2.1 for an illustration.

In Veronesi and Ribeiro (2002), investors update their beliefs about the state of the world by observing potentially multiple securities’ cash flows, whose covariance matrix is known. In our model, investors can perfectly infer the state of the world at all times, but the covariance matrix is uncertain.

We do not discuss in detail our model’s implications for asymmetric patterns of time-series volatility between upturns and downturns, as in Mele (2007), and of investment decisions as in Dangl and Wu (2011).

The paper is organized as follows. Section 2.2 has the basic model. Section 2.3 discusses applications. Section 2.4 concludes.
Figure 2.1: Two stocks can be uncorrelated or negatively correlated in upturns and perfectly correlated in downturns, while relative valuations do not diverge in upturns, but diverge strongly in downturns.

2.2 Model

2.2.1 General Case

The Setup

There is a large number of assets $i = 1, 2, \ldots, N$ in the economy. Each asset $i$ pays dividends $Y^i_t$ at time $t$. The random vector $Y_t = (Y_t^1, \ldots, Y_t^N)$ is iid. The distribution of $Y^i_t$ depends on parameters $\psi^i$, which are drawn from a known distribution. There is an overlapping generations representative agent with vNM utility $u$. 
As $N$ is large, there is no learning about the aggregate dividend $Y_t = \sum Y^i_t = N \cdot E^\psi [Y^i_t]$. As a result, there is no learning about the stochastic discount factor (sdf) $m_t$, which prices the uncertain dividend stream of each asset

$$p^i_t = \frac{1}{R} E_t[m_{t+1}(p^i_{t+1} + Y^i_{t+1})] \quad E_t[m_{t+1}] = 1$$

**Lemma 2.1.** The sdf in an OLG model is iid and given by

$$m_{t+1} = \frac{u'(Y_{t+1})}{E_t[u'(Y_{t+1})]}$$

As there is no learning about $m_t$, we can solve recursively for the price of asset $i$:

$$p^i_t = \sum_{k=1}^{\infty} \frac{1}{R^k} E_t[(m_{t+1} \cdots m_{t+k}) \cdot Y^i_{t+k}]$$

$$= \sum_{k=1}^{\infty} \frac{1}{R^k} E_t(m_{t+1}) \cdots E_t(m_{t+k-1}) \cdot E_t(m_{t+k} \cdot Y^i_{t+k})$$

$$= \sum_{k=1}^{\infty} \frac{1}{R^k} E_t[m_{t+k} \cdot Y^i_{t+k}] = \sum_{k=1}^{\infty} \frac{1}{R^k} E_t[m_{t+1} \cdot Y^i_{t+1}]$$

$$= \frac{1}{R - 1} E_t[m_{t+1} \cdot Y^i_{t+1}]$$

**Intuition**

The intuition for the main result is as follows. As per lemma 2.1, since agents are risk averse, the sdf is higher in downturns. Thus, for valuing the asset it is more important to know how the asset performs in downturns. As a result, the agent will
put more weight on observations in downturns, compared to observations in the upturns.

To illustrate the intuition, we can approximate

$$E_t[m_{t+1} \cdot Y_{t+1}^i] \approx \frac{1}{t} \sum_{\tau=1}^{t} m_{\tau} Y_{\tau}^i$$

a consistent estimate of $p_i^t \cdot (R-1)$. (Note that there is no asymmetric information in this model. The price therefore reflects the best estimate of value.)

As a result,

$$p_i^t \approx \frac{1}{R-1} \frac{1}{t} \sum_{\tau=1}^{t} m_{\tau} Y_{\tau}^i$$

Since the $sdf$ is higher in the downturns, the price will react more to the observations received in downturns. There is not more information in prices in downturns, but the information there is is more important to gauge the utility the investor derives from holding the asset. We show this result more formally in a special case that assumes normal parameter distributions.

2.2.2 The Normal Case

The Setup

To obtain analytical solutions, we specialize the model. Let firm-level dividends depend on two parameters $a^i$ and $b^i$, $Y_t^i = a^i + b^i \xi_t + \epsilon_t^i$. $a^i$ is firm $i$’s cash-flow alpha, or idiosyncratic performance parameter. $b^i$ is the firm’s cash-flow beta, i.e., the correlation of the firm’s cash flows with a market-wide shock $\xi_t$, which is distributed normally $\xi_t \sim \mathcal{N}(0, \sigma_{\xi}^2)$. For each security, $a^i$ and $b^i$ are drawn from
\[
\begin{pmatrix}
a^i \\
b^i
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
\bar{a} \\
\bar{b}
\end{pmatrix},
\begin{pmatrix}
\sigma_a^2 & \sigma_{ab} \\
\sigma_{ab} & \sigma_b^2
\end{pmatrix}
\]

Moreover, assume the OLG agents have utilities over cash flows \( u(Y) = -\exp(-\gamma Y) \).

**Lemma 2.2.** The price of asset \( i \) is

\[
p_t = \frac{1}{R - 1} E_t[m_{t+1} Y^i_{t+1}] = \frac{1}{R - 1} E_t[a^i - \phi b^i]
\]

where \( \phi = \gamma \tilde{N} \tilde{b} \cdot \sigma_\xi^2 \).

The parameter \( \phi \) captures the degree of risk aversion and the riskiness of the economy. Intuitively, risk-averse investors’ willingness to pay for the asset increases in the asset’s cash-flow alpha, and decreases in its cash-flow beta.

**Intuition**

Our main result is best illustrated by comparing in which state \( \xi_t \) of the economy the alignment is greatest between what the investor cares about and what she observes. The investor is interested in learning about \( \pi^i_t = (R - 1) \cdot p^i_t = E_t[a^i - \phi b^i] \), and observes cash flows \( Y^i_t = a^i + b^i \cdot \xi_t + \varepsilon^i_t \), from which she can also infer the state of the economy \( \xi_t \). That is, she calculates

\[
\pi^i_t = E_t \left[ a^i - \phi b^i \mid a^i + b^i \cdot \xi_t + \varepsilon^i_t \right]
\]

\[
= E_t \left[ a^i - \phi b^i \mid a^i - \phi b^i + b^i(\xi_t + \phi) + \varepsilon^i_t \right]
\]

What the investor observes and what she is interested in learning is most aligned when \( \xi_t + \phi \) is close to zero, or when \( \xi_t \) is close to \(-\phi\), which is smaller than 0. In
other words, one component of the noise is “shut off” when the market-wide shock is (moderately) negative.

Formal Results

We now show the results more formally. Conditional on the realization of $\xi_t$, beliefs about the distributions of $Y^i_t$ are mutually normally distributed. Thus, the conditional belief $\Omega_t$ at time $t$ about parameters $\psi^i = \begin{pmatrix} a^i \\ b^i \end{pmatrix}$ will always remain normal

$$\Omega^i_t = \psi^i | I_t \sim N(\mu_t, \Sigma_t)$$

The standard equations for Bayesian updating of beliefs apply

$$\mu_t = \mu_{t-1} + \frac{cov[\psi^i, Y^i_t]}{var[Y^i_t]} (Y^i_t - E[Y^i_t])$$

$$\Sigma_t = \Sigma_{t-1} - \frac{cov[\psi^i, Y^i_t] cov[\psi^i, Y^i_T]}{var[Y^i_t]}$$

Here all expectations are conditional on information available at $t - 1$ and $\xi_t$. For example, $cov[\psi^j, Y^i_t] = cov[\psi^j, Y^i_t | I_{t-1}, \xi_t]$.

If we denote

$$\Sigma_{t-1} = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix}$$

Then

$$var[Y^i_t] = \sigma_a^2 + 2\sigma_{ab}\xi_t + \sigma_b^2\xi_t^2 + \sigma_e^2$$

$$cov[\psi^j, Y^i_t] = \begin{pmatrix} \sigma_a^2 + \sigma_{ab}\xi_t \\ \sigma_{ab} + \sigma_b^2\xi_t \end{pmatrix}$$
We can rewrite the asset price in vector notation

\[ p_i^t = \frac{1}{R-1} E_t [a^i - \phi b^i] = \frac{1}{R-1} (1 - \phi) \cdot \mu_t \]

With that notation in place, we can derive the price change in response to a piece of news.

**Lemma 2.3.** The price change of asset i from time \( t - 1 \) to time \( t \), when the realization of the macro-shock is \( \xi_t \), is

\[ p_i^t - p_i^{t-1} = \frac{\lambda(\xi_t)}{R-1} \cdot (Y_i^t - E[Y_i^t]) \]

where

\[ \lambda(\xi_t) = \frac{\sigma_a^2 - \phi \sigma_{ab} - (\phi \sigma_b^2 - \sigma_{ab}) \xi_t}{\sigma_a^2 + 2 \sigma_{ab} \xi_t + \sigma_b^2 \xi_t^2 + \sigma_\varepsilon^2} \]

We can characterize how strongly prices react to news by the variance of the price changes.

\[ Var_p = Var[p_t - p_{t-1}] = \frac{(\lambda(\xi_t))^2}{(R-1)^2} \cdot Var[Y_i^t] \]

\[ = \frac{1}{(R-1)^2} \cdot \frac{(\sigma_a^2 - \phi \sigma_{ab} - (\phi \sigma_b^2 - \sigma_{ab}) \xi_t)^2}{\sigma_a^2 + 2 \sigma_{ab} \xi_t + \sigma_b^2 \xi_t^2 + \sigma_\varepsilon^2} \]

For a moment, think of the prior belief of correlations of \( a \) and \( b \), \( \sigma_{ab} \) as close to zero. For \( \sigma_{ab} \), the expression is simplified to

\[ Var_p = \frac{1}{(R-1)^2} \cdot \frac{(\sigma_a^2 - \phi \sigma_b^2 \xi_t)^2}{\sigma_a^2 + \sigma_b^2 \xi_t^2 + \sigma_\varepsilon^2} \]
Clearly, prices react more strongly to news received when $\xi_t$ is smaller, i.e., in downturns. A proposition that makes this claim formally follows.

**Proposition 2.4.** There exist $0 < \sigma_{ab}^p < \sigma_a \sigma_b$ such that

- $\text{Var}_p(\xi_t = -x) > \text{Var}_p(\xi_t = +x)$ for $-\sigma_a \sigma_b \leq \sigma_{ab} < \bar{\sigma}_{ab}
- \text{Var}_p(\xi_t = -x) < \text{Var}_p(\xi_t = +x)$ for $\bar{\sigma}_{ab} < \sigma_{ab} \leq \sigma_a \sigma_b

The proposition says that there is an asymmetry in price responses between market upturns and downturns (see figure 2.2).

![Figure 2.2](image)

Figure 2.2: The figure illustrates that when $\sigma_{ab}$ is less than some positive cutoff $\bar{\sigma}_{ab}$, the variance of prices changes in market downturns is higher than in market upturns.

Unless the correlation between $a^i$ and $b^i$ is high (which means that we already know a lot about how the asset behaves in the downturns) the prices will react
more strongly to news received in market downturns. Figure 2.3 shows an example of how the variance of price changes depends on the realization of the market-wide factor.

![Figure 2.3: Plot of the variance of the price changes over the realization of the market-wide shock $\xi$, for different values of $\rho = \frac{\sigma_{ab}}{\sigma_a \sigma_b}$. Unless $\rho_{ab}$ is strongly positive, the part of the graph left of $\xi = 0$ tends to be higher than the part to the right of $\xi = 0$, i.e., variance is higher in downturns than in upturns. That means, unless $\sigma_{ab}$ is very high, the cross-sectional dispersion in response to an observation is higher in downturns then in upturns. Simulation results for $\phi = 0.1$, $\sigma_a^2 = 1$, $\sigma_b^2 = 1$, $\sigma_{ab} = 0$, $\sigma_\epsilon = 1$, and $R = 1$.]

With empirical applications in mind, we also derive implications for returns (rather than prices, as above). Define returns as

$$r_t = (p_t + Y_t) - p_{t-1}$$
An innovation in returns is then given by

\[ \eta_t = r_t - E[r_t] = \]

\[ = \left( \frac{\lambda(\xi_t)}{R - 1} + 1 \right) (Y_t^i - E[Y_t^i]) \]

The variance of return innovations is then

\[ \text{Var}_r = \text{Var}[\eta_t] = \left( \frac{\lambda(\xi_t)}{R - 1} + 1 \right)^2 \cdot \text{Var}[Y_t^i] \]

Analogous to proposition 2.4, we can then derive

**Proposition 2.5.** There exist \( \bar{\sigma}_{ab}^r(\sigma_a^2, \sigma_b^2, x) > 0 \) such that for any \( \sigma_{ab} < \bar{\sigma}_{ab}^r \)

\[ \text{Var}_r(\xi_t = -x) > \text{Var}_r(\xi_t = +x) \]

It is useful to note that the cross-sectional variance of price responses and returns are equal to the time series variances that we calculated in propositions 2.4 and 2.5.

Conditional on the realization of \( \xi_t \), dividends \( Y_t^i \) are independent across different stocks. As a result, the price changes and returns are independent across different stocks. Moreover, they have the same distributions for different stocks. Therefore, for a large number of stocks, the cross-sectional variance of price changes across stocks is equal to the conditional time series variances for a given stock.

### 2.2.3 Model Limitations and Extensions

As is common in the learning literature, we explicitly solve our model only for the normal case. It should be clear from the general setup, however, that normality is
CHAPTER 2. REVEALING DOWNTURNS

not driving the direction of our results. However, relaxing the normality assumption would certainly be useful for numerical evaluations and calibrations of the model. Similar things apply for the independence assumption for $\xi_t$. This assumption should be relaxed in more applied follow-up papers.

In the model, the parameters $a^i$ and $b^i$ are fixed and do not change over time. As a result, investors would eventually learn the true values. In the real world, the parameters change over time in response to changes of leadership, the competitive landscape, and innovations. As a result, investors never perfectly learn the true values. A stationary model that reflects these features of reality with time-changing parameters leads to similar results. To simplify the exposition, we therefore restrict the model in this paper to the “static case” above.

2.3 Applications

2.3.1 CEO Turnover

We continue with the normal case: firm $i$ pays dividend $Y^i = a^i + b^i\xi_t + \varepsilon^i_t$ at time $t$; $a_i$ and $b_i$ represent a match between the CEO and the firm. $a_i$ can be interpreted as firm $i$ CEOs ability to generate cash flows in firm $i$, e.g., by efficiently running the organization. $b_i$ can be interpreted as the CEOs strategy in firm $i$. Some strategies generate high cash flows when the macro-economy is doing well (e.g., developing big luxury cars), while others are less profitable in upturns, but fare better in economic downturns (e.g., developing small fuel-efficient cars). For the moment, let there be no strategic choice of the strategy by the CEO. When a new CEO is hired, $a_i$ and $b_i$ are drawn from
A CEO can be in place for at most two periods. After observing the performance of the CEO after one term in office, shareholders have an option to fire and replace the CEO with a new one that is chosen from a pool of available CEOs. The timing is illustrated in Figure 2.4.

There is no general equilibrium effect of firing the CEO. Replacing the CEO, however, comes at a cost $C$, which includes severance pay to the outgoing CEO and search and hiring frictions for the new CEO. We assume this cost is relatively
large. As a result, the sdf is not affected by the option to fire, the option is far out of the money, and the ex ante value of option is small.

The value of the firm with a new CEO (who is unlikely to get fired in the next period) is

\[
p^{new} = \frac{1}{R} E_t^{new} [a^i - \phi b^i] + \frac{1}{R^2} \left( E_t^{new} [a^i - \phi b^i] + p^{new} \right)
\]

\[
p^{new} = \frac{1}{R - 1} (\bar{a} - \phi \bar{b})
\]

The value of the firm with the old CEO (who will leave the firm after the next period for sure due to mandatory retirement) is

\[
p^{old} = \frac{1}{R} E_t^{old} [a^i - \phi b^i] + \frac{1}{R} p^{new} = \frac{1}{R} E_t^{old} [a^i - \phi b^i] + \frac{1}{R(R - 1)} (\bar{a} - \phi \bar{b})
\]

Shareholders decide to replace the CEO if and only if replacing is a NPV-positive project:

\[
p^{new} > p^{old} + C \iff E_t^{old} [a^i - \phi b^i] < \bar{a} - \phi \bar{b} - R \cdot C
\]

Given that shareholders observe \( Y_i^i \),

\[
E_t^{old} [a^i - \phi b^i] - (\bar{a} - \phi \bar{b}) = (Y_i^i - \bar{Y}_i) \cdot \frac{cov[Y_i^i, a^i - \phi b^i]}{var[Y_i^i]}
\]
\[ Y^i_t = (\bar{a} + \bar{b}\xi_t) \cdot \frac{\sigma_a^2 - \phi \sigma_{ab} - (\phi \sigma_b^2 - \sigma_{ab})\xi_t}{\sigma_a^2 + 2\sigma_{ab}\xi_t + \sigma_b^2\xi_t^2 + \sigma_{\varepsilon}^2} \]

The dispersion of market prices of firms with “old” CEOs is higher when \( \xi \) is negative.

\[ V(\xi_t) = \text{Var}[E_t^{\text{old}}[a^i - \phi b^j]] = \frac{(\sigma_a^2 - \phi \sigma_{ab} - (\phi \sigma_b^2 - \sigma_{ab})\xi_t)^2}{\sigma_a^2 + 2\sigma_{ab}\xi_t + \sigma_b^2\xi_t^2 + \sigma_{\varepsilon}^2} \]

Analogous to the proof of proposition 2.4, there exists 0 < \( \bar{\sigma}_{ab} \) < \( \sigma_a \sigma_b \), such that

- \( V(\xi_t = -x) > V(\xi_t = +x) \) when \( \sigma_a \sigma_b < \sigma_{ab} < \bar{\sigma}_{ab} \)
- \( V(\xi_t = -x) < V(\xi_t = +x) \) when \( -\sigma_a \sigma_b < \sigma_{ab} < \bar{\sigma}_{ab} \)

As a result, more firms’ market value falls below the cutoff \( C \) when the market-wide shock is negative. Therefore, more boards – rationally – decide to replace their CEO. See Figure 2.5 for an illustration.

In words: unless \( \sigma_{ab} \) is very large, firms tend to fire their CEOs in downturns more often than in upturns.

### 2.3.2 Active Funds Puzzle

Our model also provides a simple explanation why active funds perform better in market downturns relative to upturns. We only provide an explanation for this asymmetry. We do not provide a rationale why active funds underperform the market unconditionally, which is a more controversial claim in the literature.

In a world with only active investors and noise/liquidity traders as counter parties, prices reflect fundamental information due to active funds’ trading of mispriced
assets. Fundamental news lets funds detect such mispriced assets. Active investors buy or sell the mispriced asset until it adjusts to fair value. During the process of the price adjustment they can buy or sell the asset at better than a fair price. The counter-party that loses money is noise and liquidity traders. During market downturns, funds detect more severe mispricing, and make more money as a result.

We do not microfound the process of price adjustment as existing models (e.g., Kyle (1985)) have shown that active funds make is proportional to the variance in price changes – and that variance is higher in downturns.

2.4 Discussion

This paper provided a rationale for asymmetries between upturns and downturns both in asset pricing and in corporate finance. The theory assumes that (i) investors
try to infer firm value from cash flow news, (ii) both firm value and cash flows depend on multiple parameters, one of which is a factor affecting all stocks. As a result, the extent to which the parameters governing cash flows and value are aligned is different in upturns and downturns. Therefore, investors learn more about firm value from any given observation sampled in a downturn rather than one sampled in an upturn. As market prices thus react more strongly to news in downturns, active investing based on superior information (“stock picking”) tends to produce greater abnormal returns, and boards tend to fire CEOs in downturns more frequently than in upturns.

The focus of future research should be on a thorough empirical validation of the empirical predictions of the model. One is that a greater cross-sectional dispersion after downturns is caused by a stronger stock market reaction to news in downturns than in upturns. An additional and more direct consistency test is to track the confidence intervals of stock analysts over the business cycle. These tests are subject to ongoing research.

Appendix

Proof of Lemma 2.1. (Stochastic Discount Factor for the OLG model)

Consider an agent that buys at time $t$ an asset at price $p_z$ that pays $Z_{t+1}$ at $t+1$. The agent consumes $Y_{t+1}$ at $t+1$. Then, the expected utility of the agent

$$U(x) = E_t[u(Y_{t+1} + x(Z_{t+1} - Rp_z))]$$
The representative agent’s utility is maximized when \( x = 0 \)

\[
0 = U'(x)|_{x=0} = E_t[u'(Y_{t+1})(Z_{t+1} - Rp_x)]
\]

Therefore,

\[
p_z = \frac{1}{R} \frac{E_t[u'(Y_{t+1})]}{E_t[u'(Y_{t+1})]} Z_{t+1}
\]

and the stochastic discount factor is equal to

\[
m_{t+1} = \frac{u'(Y_{t+1})}{E_t[u'(Y_{t+1})]}
\]

**Proof of Lemma 2.2. (Valuation for the Case of Normally Distributed Parameters)**

For normally distributed \( Y \)

\[
E[e^{\gamma Y}] = e^{\gamma E[Y] + \frac{\gamma^2}{2} V[Y]}
\]

\[
E[Y \cdot e^{\gamma Y}] = (E[Y] + \gamma V[Y]) \cdot e^{\gamma E[Y] + \frac{\gamma^2}{2} V[Y]} = (E[Y] + \gamma V[Y]) \cdot E[e^{\gamma Y}]
\]

Aggregate consumption is equal to \( Y_{t+1} = N(\bar{a} + \bar{b} \cdot \xi_{t+1}) \) and the stochastic discount factor is

\[
m_{t+1} = \frac{u'(Y_{t+1})}{E_t[u'(Y_{t+1})]}
\]

Therefore, for exponential utility \( (u'(Y_{t+1}) = \gamma e^{-\gamma Y_{t+1}}) \)

\[
G(\psi^i) = E_t[m_{t+1} Y^i_{t+1} | \psi^i] = \frac{1}{E_t[e^{-\gamma Y_{t+1}}]} E_t \left[ e^{-\gamma Y_{t+1}} (a^i + b^i \xi_{t+1} + \varepsilon^i_{t+1}) | \psi^i \right]
\]
Since (using above formulas)

\[ E_t \left[ e^{-\gamma Y_{t+1}} \xi_{t+1} \right] = E_t \left[ e^{-\gamma N(\bar{a} + \bar{\bar{b}} \cdot \xi_{t+1})} \xi_{t+1} \right] = -\gamma \bar{N} N \sigma^2_{\xi} \cdot E_t \left[ e^{-\gamma Y_{t+1}} \right] \]

then \( G(\psi^i) = a^i - \phi \cdot b^i \) where \( \phi = \gamma \bar{N} \bar{b} \cdot \sigma^2_{\xi} \).

Thus, the price of the asset follows

\[ p_t^i = \frac{1}{R - 1} E_t[G(\psi^i)] = \frac{1}{R - 1} E_t[a^i - \phi b^i] \]

**Proof of Lemma 2.3**

In the defined notation

\[ \text{var}[Y_t^i] = \sigma^2_a + 2\sigma_{ab} \xi_t + \sigma^2_b \xi_t + \sigma^2_\varepsilon \]

\[ \text{cov} \left[ \begin{pmatrix} a^i \\ b^i \end{pmatrix}, Y_t^i \right] = \begin{pmatrix} \sigma^2_a + \sigma_{ab} \xi_t \\ \sigma_{ab} + \sigma^2_b \xi_t \end{pmatrix} \]

\[ \mu_t = \mu_{t-1} + \frac{Y_t^i - E[Y_t^i]}{\text{var}[Y_t^i]} \begin{pmatrix} \sigma^2_a + \sigma_{ab} \xi_t \\ \sigma_{ab} + \sigma^2_b \xi_t \end{pmatrix} \]

Therefore

\[ p_t - p_{t-1} = (1, -\phi)(\mu_t - \mu_{t-1}) = \]

\[ \frac{Y_t^i - E_{t-1}[Y_t^i]}{\text{var}[Y_t^i]}(\sigma^2_a + \sigma_{ab} \xi_t - \phi(\sigma_{ab} + \sigma^2_b \xi_t)) \]

and we get the following expression for \( \lambda(\xi_t) \)

\[ \lambda(\xi_t) = \frac{\sigma^2_a - \sigma_{ab} \phi - \xi_t(\phi \sigma^2_b - \sigma_{ab})}{\sigma^2_a + \sigma^2_b \xi_t + \sigma^2_\varepsilon + 2\sigma_{ab} \xi_t} \]
CHAPTER 2. REVEALING DOWNTURNS

Proof of Proposition 2.5

Although we know that

\[ \text{Var}_P = \frac{(\sigma_a^2 - \sigma_{ab}\phi - \xi_t(\phi\sigma_b^2 - \sigma_{ab}))^2}{\sigma_a^2 + \sigma_b^2\xi_t^2 + \sigma_\varepsilon^2 + 2\sigma_{ab}\xi_t} \]

it is easier to get to the result indirectly. The updated conditional variance is

\[ \Sigma_t = \begin{pmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ab} & \sigma_b^2 \end{pmatrix} - \frac{1}{\text{var}[Y_t]} \begin{pmatrix} \sigma_a^2 + \sigma_{ab}\xi_t \\ \sigma_{ab} + \sigma_b^2\xi_t \end{pmatrix}^T \begin{pmatrix} \sigma_a^2 + \sigma_{ab}\xi_t \\ \sigma_{ab} + \sigma_b^2\xi_t \end{pmatrix} = \]

\[ = \frac{1}{\text{var}[Y_t]} \begin{pmatrix} \sigma_a^2\sigma_a + (\sigma_a^2\sigma_b - \sigma_{ab})\xi_t^2 & \sigma_{ab}\sigma_a^2 - (\sigma_a^2\sigma_b - \sigma_{ab})\xi_t \\ \sigma_{ab}\sigma_a^2 - (\sigma_a^2\sigma_b - \sigma_{ab})\xi_t & \sigma_b^2\sigma_a + (\sigma_a^2\sigma_b - \sigma_{ab}) \end{pmatrix} \]

Then (here \( \Phi = (1, -\phi) \))

\[ \text{Var}[p_t - p_{t-1}] = \text{Var}[\Phi(\mu_t - \mu_{t-1})] = \Phi \text{Var}[\mu_t - \mu_{t-1}]\Phi^T = \]

\[ = \Phi(\Sigma_{t-1} - \Sigma_t)\Phi^T \]

Since \( \Sigma_{t-1} \) does not depend on \( \xi_t \) we can consider only the second term.

\[ H = \Phi\Sigma_t\Phi^T = \frac{\sigma_\varepsilon^2(\sigma_a^2 - 2\phi\sigma_{ab} + \phi^2\sigma_b^2) + (\sigma_a^2\sigma_b^2 - \sigma_{ab})(\phi^2 + 2\phi\xi_t + \xi_t^2)}{\sigma_a^2 + 2\sigma_{ab}\xi_t + \sigma_b^2\xi_t^2 + \sigma_\varepsilon^2} \]

If we denote

\[ A = \sigma_\varepsilon^2(\sigma_a^2 - 2\phi\sigma_{ab} + \phi^2\sigma_b^2) + (\sigma_a^2\sigma_b^2 - \sigma_{ab})(\phi^2 + x) \]

\[ B = 2\phi(\sigma_a^2\sigma_b^2 - \sigma_{ab}) \]
CHAPTER 2. REVEALING DOWNTURNS

\[ C = \sigma_a^2 + \sigma_b^2 x^2 + \sigma_{\xi}^2 \]

\[ D = 2\sigma_{ab} \]

then \( H(\xi_t) = \frac{A + B\xi_t}{C + D\xi_t} \) and

\[ H_{\xi=+x} > H_{\xi=-x} \iff A \cdot D < B \cdot C \]

The last equation is equivalent to

\[ (\sigma_{\xi}^2 (\sigma_a^2 - 2\phi\sigma_{ab} + \phi^2 \sigma_b^2) + \sigma_a^2 \sigma_b^2 - \sigma_{ab}^2)(\phi^2 + x^2)) \cdot 2\sigma_{ab} < \]

\[ 2\phi(\sigma_a^2 \sigma_b^2 - \sigma_{ab}^2) \cdot (\sigma_a^2 + \sigma_b^2 x^2 + \sigma_{\xi}^2) \]

\[ \sigma_{ab} \left( \sigma_{\xi}^2 \frac{\sigma_a^2 - 2\phi\sigma_{ab} + \phi^2 \sigma_b^2}{\sigma_a^2 \sigma_b^2 - \sigma_{ab}^2} + (\phi^2 + x^2) \right) < 2\phi(\sigma_a^2 + \sigma_b^2 x^2 + \sigma_{\xi}^2) \]

We can rewrite the left hand side as

\[ \sigma_{ab} \left( \sigma_{\xi}^2 \frac{\sigma_a^2 - 2\phi\sigma_{ab} + \phi^2 \sigma_b^2}{\sigma_a^2 \sigma_b^2 - \sigma_{ab}^2} + (\phi^2 + x^2) \right) = \]

\[ \sigma_{ab} \left( \sigma_{\xi}^2 \frac{(\sigma_a - \phi\sigma_b)^2}{\sigma_a^2 \sigma_b^2 - \sigma_{ab}^2} + (\phi^2 + x^2) \right) + \frac{2\phi\sigma_{ab}}{\sigma_a\sigma_b + \sigma_{ab}} \]

The expression is negative for negative \( \sigma_{ab} \), is equal to zero when \( \sigma_{ab} = 0 \), and is an increasing function of \( \sigma_{ab} \) for positive \( \sigma_{ab} \). Moreover, \( H(\xi_t = -x) > H(\xi_t = x) \) for \( \sigma_{ab} = \sigma_a\sigma_b \). Thus, there exists a positive cutoff \( 0 < \sigma^p_{ab} < \sigma_a\sigma_b \) such that

- \( \text{Var}_p[\xi_t = -x] > \text{Var}_p[\xi_t = +x] \) for \( -\sigma_a\sigma_b \leq \sigma_{ab} < \sigma^p_{ab} \)

- \( \text{Var}_p[\xi_t = -x] < \text{Var}_p[\xi_t = +x] \) for \( \sigma^p_{ab} < \sigma_{ab} \leq \sigma_a\sigma_b \)
Proof of Proposition 2.5

We know that

\[ \text{Var}_r = \left( \frac{\lambda}{R-1} + 1 \right)^2 \cdot \text{Var}[Y_i^r] = \]

\[ = \frac{\lambda^2 \text{Var}[Y_i^r]}{(R-1)^2} + \frac{2\lambda \text{Var}[Y_i^r]}{R-1} + \text{Var}[Y_i^r] = \]

\[ = \text{Var}_p + \frac{1}{R-1} \left( 2\lambda \text{Var}[Y_i^r] + \text{Var}[Y_i^r] \right) \]

From proposition 2.4 there exists \( \bar{\sigma}_{pa} \) such that for \( \sigma_{ab} < \bar{\sigma}_{pa} \)

\[ \text{Var}_p[\xi_t = -x] > \text{Var}_p[\xi_t = +x] \]

Now

\[ Q = 2\lambda \text{Var}[Y_i^r] + (R-1)\text{Var}[Y_i^r] = \]

\[ 2(\sigma_a^2 - \phi \sigma_{ab} - \xi_t(\phi \sigma_b^2 - \sigma_{ab})) + (R - 1)(\sigma_a^2 + \sigma_b^2 \xi_t^2 + \sigma_e^2 + 2\sigma_{ab} \xi_t) = \]

\[(R + 1)\sigma_a^2 - 2\phi \sigma_{ab} + (R - 1)(\sigma_b^2 \xi_t^2 + \sigma_e^2) - 2\xi_t(\phi \sigma_b^2 - \sigma_{ab} R) \]

Again there exists \( \bar{\sigma}_{qb} \) such that for \( \sigma_{ab} < \bar{\sigma}_{qb} \)

\[ Q(\xi_t = -x) > Q(\xi_t = +x) \]

Taking \( \bar{\sigma}_{ab} = \min(\bar{\sigma}_{pa}, \bar{\sigma}_{qb}) \) we get the result of the proposition.
Chapter 3

Uncertain Insider Trading

3.1 Introduction

Market liquidity is a subject of active research in financial economics. It is not easy to give a precise definition of liquidity, but broadly speaking an asset is liquid if it can be easily traded. There are different reasons why an asset can be illiquid, and one of them is asymmetric information\(^1\). In his seminal paper, Kyle (1985) proposed a model explaining how asymmetric information can affect trading of a given security. His approach turned out to be very influential and inspired a lot of further work.

In the simplest version of Kyle’s model an informed trader and uninformed liquidity trader trade a security with a competitive market maker. The informed trader has some private information about the value of the security, while the uninformed trades for exogenous reasons and is insensitive to the price. The market maker chooses the price of the trade, but importantly, she does not know as much

\(^1\)Others include transaction costs, inventory risk, and search costs. See Amihud, Mendelson, and Pedersen (2005) or Biais, Glosten, and Spatt (2005) for a broader discussion of this issue.
as the informed trader about the value of the security and she cannot distinguish
the order flows from the informed and uninformed traders. As a result, the market
maker has to infer the information about the value of the security from the order
flow. The informed trader behaves strategically, that is, she takes into account the
fact that her trading affects prices.\(^2\)

The model assumes for tractability that the value of the security and the order
flow from uninformed traders are normally distributed. It is shown that in equilib-
rium the price \( p \) at which the security is traded linearly depends on the aggregate
order flow \( y \)

\[
p(y) = p_0 + \lambda \cdot y
\]

The parameter \( \lambda \) increases when the informational advantage of the informed trader
becomes more severe and decreases when the uninformed trader trades more. Thus,
Kyle’s model presents a very simple and tractable approach to modeling the price
impact of trading. The coefficient \( \lambda \) can be used as a measure of illiquidity (the
higher is \( \lambda \) the more costly is to trade this security). It can be easily estimated
from the observed financial data. In fact, several studies addressing the issue of
whether illiquidity is priced (Brennan and Subrahmanyam (1996); Amihud (2002))
used Kyle’s \( \lambda \) as a measure of illiquidity.

However, there is an important limitation of Kyle’s model and of most of the
subsequent literature. In Kyle’s model (even in the dynamic versions of one) the
information structure is essentially static. The market maker always knows that
there is an informed trader, who at any given moment of time has certain infor-
mation and is trading on it. In reality news affecting valuation of the security is
released much less frequently than the trading occurs, and importantly, the market

\(^2\)The other approach that deals with private information in trading suggested by Glosten and Milgrom (1985).
maker might not know when exactly it is released. Thus, she needs to infer from
the trading volume, not only what information insiders have, but also whether they
actually know something.

In the paper I consider a simple extension of Kyle’s model which addresses
this issue. The setup is similar to Admati and Pfleiderer (1988). The model is
dynamic, the security is traded in several rounds, however the informational of
advantage of the insider over the market is only one period. In this case the analysis
is similar to the original static Kyle’s model. However, we assume that each period
the insider receives information only with certain small probability. Thus, the insider
knows more about the security only during a small number of the trading rounds,
but the market maker does not know when exactly this happens. Because, of this
assumption the model becomes non-linear, and it has to be solved numerically.

I get the following results. I show that the price impact of trading volume
becomes essentially non-linear. For small volumes of trade, the price remains almost
unchanged, while for large volumes there can be significant deviations. That means
that the market can be very liquid for relatively small orders and can suddenly
become very illiquid when a large order arrives. The market maker infers whether
the informed trader has some additional information from observing trading volume.
If the volume is relatively small she may believe that most likely no information
was released. On the other hand, if the volume is high, then mostly likely something
has happened, and the price should be adjusted accordingly.

One implication of the non-linear price response function is that it becomes
impossible to unambiguously compare the illiquidity of the two given stocks: one
stock can be more liquid for small trades, the other can be more liquid for larger
trades. Thus, Kyle’s $\lambda$ can be an imprecise measure of illiquidity. Also, I show that
in such model even if the underlying informational shocks are normally distributed, the distribution of returns is going to have fat tails.

This paper contributes to the market microstructure literature. Some of the recent surveys include Biais, Glosten, and Spatt (2005) and Hasbrouck (2007). More general information about markets and trading institutions can be found in Harris (2002). O’Hara (1998) and Brunnermeier (2001) discuss theoretical models of market microstructure. Also, recently there was a lot of work on how illiquidity is priced. Amihud, Mendelson, and Pedersen (2005) survey both the theoretical and empirical literature on the subject.

The paper is structured as follows. Section 3.2 discusses the model and defines the equilibrium. Section 3.3 describes the numerical procedure used to find the equilibrium. Section 3.4 presents results and their interpretation. Section 3.5 discusses potential problems and limitations and outlines further work. The appendix contains proofs and figures.

### 3.2 Model and Equilibrium

**The Model**

There is a security which is traded in $T$ trading rounds $t = 1, \ldots, T$. Its value evolves according to $V_t = V_{t-1} + v_t$, where $v_t$ is the shock realized in the period $t$ ($V_0$ is given). After all trading rounds everyone gets the liquidation value $V_T$ for every share he or she holds.

Every period with probability $\alpha$ some information is released about the security. In this case $v_t$ is distributed according to given known distribution with continuous density $f_v$ (it can be normal distribution as in the original Kyle’s model). With
probability 1 − α nothing happens and \( v_t = 0 \). All \( v_t \) are independent across trading rounds.

There are three agents in the market: an informed trader (insider), an uniformed liquidity trader and a market maker. The informed trader trades for speculative reasons and tries to maximize her profit. She observers the shock \( v_t \) before trading round \( t \). The uninformed trader trades for exogenous reasons. In each trading round her demand \( u_t \) is random and is drawn from some known distribution with density \( f_u \). This demand is insensitive to the price. All \( u_t \) are independent across trading rounds. The market maker observes shock \( v_t \) after trading round \( t \). Thus, the informed trader has one period informational advantage over the market maker. Market making is assumed to be competitive, that is, the market maker earns zero profit. Both the insider and the market maker are risk neutral.

Trading proceeds as follows. The insider learns \( v_t \) before each trading round and then she submits her demand \( x_t \), which can depend on all previous history of signals and prices (the insider does not observe the past trading volumes of the uninformed trader). Then, the demand \( u_t \) of the uniformed trader is realized. The market maker observers the total demand \( y_t = x_t + u_t \) and based on this demand and all information from the past periods available to her, she chooses the price \( p_t \) at which all trades occur. The market maker does not know which part of the demand comes from the informed insider and which part comes from the uninformed traders.
CHAPTER 3. UNCERTAIN INSIDER TRADING

The Equilibrium

The strategy of the informed trader is $X = (x_1, \ldots, x_T)$, where $x_t = x_t(v_1, \ldots, v_t, p_1, \ldots, p_{t-1})$.\(^3\)

The strategy of market marker is $P = (p_1, \ldots, p_T)$, where $p_t = p_t(y_1, \ldots, y_t, v_1, \ldots, v_{t-1})$.

In equilibrium

- The informed trader maximizes her expected profit

$$\pi(X, P) = \sum_{t=1}^{T} E[(V_T - p_t)x_t] \rightarrow \max_X$$  \hspace{1cm} (3.1)

- The market maker earns zero expected profit. Thus, price reflects all information available to her

$$p_t(y_1, \ldots, y_t, v_1, \ldots, v_{t-1}) = E[V_T|y_1, \ldots, y_t, v_1, \ldots v_{t-1}] \hspace{1cm} (3.2)$$

Since the informed trader has only one period informational advantage over the market maker she has to trade on her information immediately. We can rewrite the expected profit of informed trader as

$$\pi(X, P) = \sum_{t=1}^{T} E[(V_T - p_t)x_t] = \sum_{t=1}^{T} E[(v_t - (p_t - V_{t-1}))x_t]$$

Thus, if the strategy of the market maker is uniform across trading rounds (that is, if $p_t - V_{t-1}$ depends only on $y_t$) then the optimal demand of the informed trader will depend only on the realization of the current shock $v_t$

$$x_t(v_1, \ldots, v_t, p_1, \ldots, p_{t-1}) = x(v_t)$$

\(^3\)Here and further we will assume that things like $p_1, \ldots, p_{t-1}$ for $t = 1$ denote just an empty list. That is, $x_1 = x_1(v_1)$. 
CHAPTER 3. UNCERTAIN INSIDER TRADING

Now consider the market maker. In trading round \( t \) past shocks \( v_1, \ldots, v_{t-1} \) are known to her and the current trading volume \( y_t = x(v_t) + u_t \) depends only on realization of current shocks \( v_t \) and \( u_t \). Thus, \( y_1, \ldots, y_{t-1} \) do not provide any additional relevant information. As a result

\[
p_t(y_1, \ldots, y_t, v_1, \ldots, v_{t-1}) = E[V_T | y_1, \ldots, y_t, v_1, \ldots, v_{t-1}] =
\]

\[
= E \left[ V_0 + \sum_{t=1}^{T} v_t | y_1, \ldots, y_t, v_1, \ldots, v_{t-1} \right] =
\]

\[
= V_0 + \sum_{t=1}^{t-1} v_t + E[v_t | y_1, \ldots, y_t, v_1, \ldots, v_{t-1}] =
\]

\[
= V_{t-1} + E[v_t | y_t] = V_{t-1} + p(y_t)
\]

where by \( p(y_t) \) we denote \( E[v_t | y_t] \). Thus, it is true that \( p_t - V_{t-1} \) depends only on \( y_t \). We are going to restrict ourselves only to such uniform equilibria.

Since in the considered model the shocks \( u_t \) are \( v_t \) are uniform and independent across trading rounds and the strategies of the agents depend only on current realizations of shocks, we can omit time subscripts and consider a generic trading round. The model becomes essentially static, and we can describe it as follows. With probability \( \alpha \) the value of the security \( v \) is drawn from a distribution with density \( f_v \) and with probability \( 1 - \alpha \) it is equal to 0. The demand from the uniformed trader \( u \) is a random variable with density \( f_u \). The informed trader knows \( v \) and submits her demand \( x(v) \). The market maker chooses the price of trading \( p(y) \) based on the total volume \( y = x(v) + u \). In equilibrium, the informed trader maximizes her profit and the price contains all information is available to the market maker.

\[
\pi(x(v), p) = E[(v - p(x(v) + u))x(v)] \rightarrow \max_{x(v)} \tag{3.3}
\]
CHAPTER 3. UNCERTAIN INSIDER TRADING

\[ p(y) = E[v|y = x(v) + u] \] (3.4)

The only difference from the original Kyle’s model is that the value of the security is no longer normally distributed. However, I would like to argue that this distinction is important and that this special distribution is likely to appear in the markets since trading happens much more frequently than new information is released.

We are particularly interested in \( p(y) \), which we call a price impact or a price response function. This function describes the behavior of the market. From the point of view of an outside investor who wants to trade this security, this function contains all relevant information about the costs of trading.

In the considered model trading occurs in rounds, while in reality it usually proceeds continuously. However, if it usually takes around \( \tau \) units of time for new information to be incorporated into prices, then we can consider an interval of length \( \tau \) as a trading round. The parameter \( \alpha \) here is a probability that information is released during an interval of length \( \tau \). Thus, If new information is released on average once in an interval of length \( \bar{T} \), then \( \alpha \) is approximately equal to the ratio of these time intervals \( \alpha \approx \frac{\tau}{\bar{T}} \). Based on this intuition I would like to argue that \( \alpha \) can be relatively small and that the values of \( \alpha \) for which results are presented in the paper are reasonable.

As shown in the appendix, equilibrium conditions (3.3) and (3.4) are equivalent to the following expressions.

\[
x(v) = \arg \max_x \int f_u(y - x) \cdot (v - p(y)) x \, dv
\]

\[
p(y) = \frac{\alpha \int v \cdot f_u(y - x(v)) \cdot f_v(v) \, dv}{\alpha \int f_u(y - x(v)) \cdot f_v(v) \, dv + (1 - \alpha) f_u(y)}
\]
3.3 Numerical solution

The considered model is very simple and quite close to original Kyle’s version, but since we lose the normality of the underlying distributions, it is hard to expect that we can find an analytical solution. Therefore, the model is solved numerically.\(^4\)

For numerical solution, we assume that \(f_u\) and \(f_v\) are densities of the standard normal distribution \(N(0, 1)\). The unitary variance is not restrictive since we can always scale the obtained results. If \(x(v), p(y)\) are equilibrium strategies for \(f_u \sim N(0, 1), f_v \sim N(0, 1)\) and \(\bar{x}(v), \bar{p}(y)\) are equilibrium strategies for \(f_u \sim N(0, \sigma_u^2), f_v \sim N(0, \sigma_v^2)\) then

\[
\bar{x}(v) = \sigma_u \cdot x\left(\frac{v}{\sigma_v}\right)
\]

\[
\bar{p}(y) = \sigma_v \cdot p\left(\frac{y}{\sigma_u}\right)
\]

To avoid dealing with infinite integrals we are making some further simplifications. We assume that \(f_v\) is a density of a cutoff of normal distribution on the interval \([-v_{\text{max}}, v_{\text{max}}]\) and also assume that the market exists only as long as total demand is not too big (\(y\) it lies in the interval \([-y_{\text{max}}, y_{\text{max}}]\)). Although, it might be arguable whether these assumptions are in fact reasonable, we would like to claim that as long as \(y_{\text{max}}\) and \(v_{\text{max}}\) are large enough, they should not affect the equilibrium strategies in the relevant for us regions of variables. In most calculations we used the values \(y_{\text{max}} = 11, v_{\text{max}} = 20\). For this values (and for \(\alpha = 0.05\)) the probabilities that \(y\) and \(v\) will get out of bounds in equilibrium were only \(3.6769 \cdot 10^{-16}\) and \(5.5072 \cdot 10^{-89}\) correspondingly, which are fairly small numbers.

\(^4\)MATLAB was used to perform the calculations. The code is available on request.
The numerical calculations are organized in the following way. We start with some initial price response function \( p_0(y) \). (In the calculations I used the original Kyle’s solution \( p_0(y) = \frac{y}{2} \)). Then, assuming that we have \( k \)-th iteration \( p_k(y) \) we calculate the optimal response of the informed trader to this function according to the formula

\[
x(v) = \arg \max_x \int_{-y_{\text{max}}}^{y_{\text{max}}} f_u(y - x) \cdot (v - p_k(y)) \cdot x \, dy.
\] (3.5)

After that, we calculate the next iteration of the price response function \( p_{k+1}(y) \) according to the equilibrium pricing rule of the market maker

\[
p_{k+1}(y) = \frac{\alpha \int_{-y_{\text{max}}}^{y_{\text{max}}} v \cdot f_u(y - x(v)) \cdot f_v(v) \, dv}{\alpha \int_{-y_{\text{max}}}^{y_{\text{max}}} f_u(y - x(v)) \cdot f_v(v) \, dv + (1 - \alpha) f_u(y)}
\] (3.6)

If the iterative procedure converges to some function \( p(y) \) we found an equilibrium (we assume that convergence is achieved when difference between two successive iterations is smaller than chosen \( \varepsilon_p \)).

### 3.4 Results and Interpretation

#### Price Response Function

The insider’s equilibrium strategy and the price response function for \( \alpha = 0.05 \) are presented in figures 3.1 and 3.2. We see the price depends on the trading volume non-linearly. When the volume is small, the price almost does not change. But in rare events when the volume is high the price reacts much stronger. Thus, we might say that most of the time the stock is very liquid, but in the event of high demand, it becomes suddenly very illiquid.
CHAPTER 3. UNCERTAIN INSIDER TRADING

The intuition of why this happens is as follows. The market maker has to infer not only what information the informed trader has, but also whether she actually received some additional information. If the trading volume is relatively small the market maker may tend to believe that no information was released. On the other hand, if the market maker observers a high volume she might think that it is likely that something happened, and she might adjust prices more strongly. Figure 3.3 shows how the market maker’s beliefs about whether the new information was released depends on the trading volume.

We can be also interested in how the parameter \( \alpha \) affects the equilibrium price response function of and the strategy of the informed trader. The comparative statics results are shown in figures 3.4 and 3.5. As \( \alpha \) gets smaller the flat region around zero in the price response function becomes larger and more pronounced. If \( \alpha = 1 \) we get the original Kyle’s result.

**Distribution of Price Changes**

One way to describe the behavior of the market is to look at the distribution of returns between trading rounds.

\[
\Delta p_t = p_{t+1} - p_t = v_t + p(y_{t+1}) - p(y_t)
\]

Empirically the distributions of returns usually have fat tails. For the equilibrium price response function and the strategy of the informed trader we can calculate the kurtosis of the the distribution of the price changes (which is a measure of how fat are the tails of the distribution). The table below shows how the relative kurtosis \( \frac{E(\Delta p_t^4)}{E(\Delta p_t^2)^2} - 3 \) depends on \( \alpha \).
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>kurtosis of distribution of $\Delta p_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.90</td>
<td>0.1087</td>
</tr>
<tr>
<td>0.70</td>
<td>0.4560</td>
</tr>
<tr>
<td>0.50</td>
<td>1.1602</td>
</tr>
<tr>
<td>0.30</td>
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</tr>
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<td>29.8533</td>
</tr>
<tr>
<td>0.02</td>
<td>82.6337</td>
</tr>
<tr>
<td>0.01</td>
<td>174.5732</td>
</tr>
</tbody>
</table>

We see that as $\alpha$ gets smaller, the kurtosis increases. Thus, even though all informational shocks are normally distributed, because these shocks are spread across time, the distribution of returns is going to have fat tails.

**Measuring Liquidity**

In the original Kyle’s model, when the price response function was linear, it was straightforward that we can use the slope of this function $\lambda$ as a measure of illiquidity of the stock. The higher is $\lambda$ the more trading costs a potential investor will incur. Unfortunately, things are complicated in the considered setup. Figure 3.6 shows the price response function for two securities, which have different values of parameters. The difference between them is that for the first security (the solid line) new information is released more frequently, but every piece of information is less valuable. We see that for small trades the second security is more liquid (price is less affected by trading), but if someone wants to trade a large number of shares...
(in a given period of time) it is cheaper to trade the first one. Thus, it is impossible to unambiguously state that one security is more liquid than the other.

The question arises how to compare the illiquidity of two securities in such situation. A possible approach is to start with some exogenous distribution \( f_e \) of possible trades for outside investors and calculate their average trading costs

\[
Costs = \int f_e(y) \cdot yp(y) \, dy = E^e[y \cdot p(y)]
\]

If we normalize this expression by variance of trading volume we will get

\[
Costs \sim \frac{E^e[y \cdot p(y)]}{E^e[y^2]}
\]

Now let us consider empirically estimated Kyle’s \( \hat{\lambda} \). The average value of such estimate is going to be

\[
E[\hat{\lambda}] = \frac{E[y \cdot p(y)]}{E[y^2]}
\]

which looks very similar to the expression above. However, the distribution of trades here is specific to the given market. If, for example, in some market large trades are very likely than the estimated Kyle’s \( \hat{\lambda} \) is going to overstate the importance of such trades. Thus, even though the empirically estimated Kyle’s \( \hat{\lambda} \) is still a relevant measure of average illiquidity, it can be somewhat imprecise.

### 3.5 Problems, Limitations and Further Work

In the considered model (as well as in the original Kyle’s model) it is assumed that the informed trader does not observe the volume of uninformed trading. This creates
several potential problems. First of all, in such setup the informed trader cannot be sure at which price his trade is going to be executed. He can expect to earn money on average, but it might happen that he will lose money on a particular trade (which is a little bit unrealistic, because usually a trader can control the price of execution). Also, it seems that for such setup it is hard to prove any results about the existence and uniqueness of equilibrium. Finally, because of the unknown trading volume, the price response function in the obtained results is not steadily increasing but has some small oscillations (see figure 3.4 for small values of $\alpha$).

We can consider an alternative specification in which the uninformed trading volume is observed by the informed trader. In this case we can prove that for bounded distributions the equilibrium exists and unique\footnote{{see Rochet and Vila (1994).}}. It is not included in the paper, but for this specification I was able to obtain similar results for the price response function (although they are somewhat less pronounced). However, in such setup the informed trader is always going to trade against the uninformed, even if no new information is released. I would like to argue that in reality the informed traders are more likely to trade when they have some information rather than when they do not have one. The reason is that any trader cannot be perfectly sure that no other traders in the market have superior information. In that respect the original model might a better approximation of reality.

Another possible extension is to consider a more realistic model in which the informed trader does not have to trade immediately on her information, but in which the more she waits the more likely is that the other market participants learn the same information.
3.6 Appendix

Expressions for Equilibrium Conditions

**Probabilities.** Denote by $A$ the event that information is released in the current trading round (by $\bar{A}$ that it is not released). Probabilities of this events are

$$Pr(A) = \alpha, \quad Pr(\bar{A}) = 1 - \alpha$$

Mutual density of $v$ and $y = x(v) + u$ given that the event $A$ happened is

$$\rho_A(y, v) = f_u(y - x(v)) \cdot f_v(v)$$

Density of $y$ given that the event $A$ happened is equal to

$$f_A(y) = \int \rho_A(y, v) dv = \int f_u(y - x(v)) \cdot f_v(v) dv$$

Density of $y$ given that the event $A$ did not happen (in this case $y = u$) is

$$f_{\bar{A}}(y) = f_u(y)$$

**Market Maker.** Since market making is competitive than

$$p(y) = E[v|y]$$

which means that $E[p(y)h(y)] = E[v \cdot h(y)]$ for any measurable $h(y)$, which means that
\[ E[p(y)h(y)] = \]
\[ = Pr(A) \int \int dy \, dp(y)h(y)f_A(y, v) + (1 - Pr(A)) \int dy \, p(y)h(y)f_{\bar{A}}(y) = \]
\[ = \int dy \, h(y) \cdot p(y) \left( \alpha \int dv \, f_A(y, v) + (1 - \alpha) \int dy \, f_{\bar{A}}(y) \right) \quad (3.7) \]

Analogously (if \( \bar{A} \) happened then \( v = 0 \)),
\[ E[v \cdot h(y)] = \int dy \, h(y) \cdot \alpha \int dv \cdot f_A(y, v) \quad (3.8) \]

Since \( h(y) \) is an arbitrary measurable function, then the expressions under the integrals in (3.7) and (3.8) should be equal. As a result,
\[ p(y) = \frac{\alpha \int dv \cdot f_A(y, v)}{\alpha \int dv \, f_A(y, v) + (1 - \alpha) f_u(y)} = \]
\[ = \frac{\alpha \int dv \cdot f_u(y - x(v)) \cdot f_v(v)}{\alpha \int dv \, f_u(y - x(v)) \cdot f_v(v) + (1 - \alpha) f_u(y)} \]

**Informed trader.** If no information is released than the insider is not going to trade at all. The expected profit of the insider given \( v \) and given that the event \( A \)
\[ \tilde{\pi}(v) = E_{A,v}[(v - p(y)) \cdot x] = E_{A,v}[(v - p(x + u)) \cdot x] = \]
\[ = \int du \, f_u(u) \cdot (v - p(x + u))x \, du = \]
\[ = \int dy \, f_u(y - x) \cdot (v - p(y))x \, dy \]

Insider maximizes her profit. Thus,
\[ x(v) = \arg \max_x \int dy \, f_u(y - x) \cdot (v - p(y))x \]
Figure 3.1: Price response function ($\alpha = 0.05$). The figure shows how in equilibrium the price that the market maker sets responds to the combined trading volume of the informed and uniformed traders.
Figure 3.2: Strategy of informed trader ($\alpha = 0.05$). The figure shows the size of order that the informed trader submits depends on the information she receives.
Figure 3.3: Conditional probability of informed trading ($\alpha = 0.05$). The figures shows how the conditional beliefs of the market maker about whether the insider received some information in a given period depends on the size of the aggregate trading volume.
Figure 3.4: Price response function, comparative statics. The figure shows how the equilibrium price response function depends on the probability of informed trading $\alpha$. 
Figure 3.5: Strategy of the informed trader, comparative statics. The figure shows how the equilibrium strategy of the insider depends on the probability of informed trading $\alpha$. 

\[ x(v) \]

- $\alpha = 1$
- $\alpha = 0.5$
- $\alpha = 0.05$
- $\alpha = 0.01$
Figure 3.6: Comparison of the price response functions. The figure contains the price response function of two securities that have different values of parameters. We see that it is impossible to unambiguously say which security is more liquid.
Bibliography


BIBLIOGRAPHY


