Set-Valued Risk Measures

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Operations Research and Financial Engineering
Adviser: Birgit Rudloff

June 2014
Abstract

Set-valued risk measures are defined on $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ space with $0 \leq p \leq \infty$. The results presented are in the dynamic framework, with the image space of the dynamic risk measures in the power set of $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ for all times $t$. Primal and dual representations are deduced for closed (conditionally) convex and (conditionally) coherent risk measures. Definitions of different time consistency properties in the set-valued framework are given. It is shown that the recursive form for multivariate risk measures is equivalent to a strong time consistency property called multi-portfolio time consistency. Further, multi-portfolio time consistency is shown to be equivalent to an additive property for the acceptance sets. When considering closed convex risk measures, it is possible to prove that multi-portfolio time consistency is equivalent to a cocycle condition on the sum of minimal penalty functions. In the closed coherent case, multi-portfolio time consistency is equivalent to a generalized version of stability of the dual variables. Additionally, utilizing these equivalent properties, it is possible to generate the multi-portfolio time consistent version of any set-valued risk measure.

Under a finite probability space, we propose an algorithm for calculating multi-portfolio time consistent set-valued risk measures in discrete time. Market models for $d$ assets with transaction costs or illiquidity and possible trading constraints are considered on a finite probability space. The set of capital requirements at each time and state is calculated recursively backwards in time along the event tree. Additionally, we motivate why the proposed procedure can be viewed as a set-valued Bellman’s principle. We give conditions under which the backwards calculation of the sets reduces to solving a sequence of linear, respectively convex vector optimization problems.

As examples of dynamic set-valued risk measures, we consider and provide numerical examples of superhedging under proportional and convex transaction costs, the
relaxed worst case risk measure, average value at risk (as well as a description of the multi-portfolio time consistent version), and the set-valued entropic risk measure.

Finally, we give an overview of three other methods for defining (dynamic) set-valued risk measures. In particular, we prove under which assumptions results within these approaches coincide, and how properties like the primal and dual representations and time consistency in the different approaches compare to each other.
Acknowledgements

During my doctoral studies, I have had the privilege to work with some extraordinary people who have helped me get to this point.

First, and foremost, I would like to thank my advisor, Professor Birgit Rudloff, for all of her support. Her patience over the years to help make me the meticulous researcher I now am cannot be overstated. I am extremely grateful to her for always having the time to discuss the many projects I was working on, and helping me to prioritize my time. Without that guidance this thesis would not have been able to be completed.

I would like to thank Professor Stefan Weber at Leibniz University of Hannover for his time and collaborations. His ideas for applications of set-valued risk measures, though not appearing in this thesis, have benefited me immensely.

I would like to thank Professor Ronnie Sircar for his support.

I would also like to thank my fellow graduate students for keeping me focused and inspired to finish this thesis.

Finally, I would like to thank my family and friends for keeping me sane throughout my academic studies.
To my parents.
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Chapter 1

Introduction

The concept of coherent risk measures was introduced in an axiomatic way in [4, 5] to find the minimal capital required to cover the risk of a portfolio. The notion was relaxed by introducing convex risk measures in [41, 42]. The risk that a risk measure captures is a type of downside risk, as risk measures assume monotonicity, i.e. if one portfolio or contingent claim is almost surely more profitable than another then that initial portfolio would have a lower risk. In these papers the risk was measured only at time zero, in a frictionless market, for univariate claims, and with only a single eligible asset that can be used for the capital requirements and serves as the numéraire. We call this the static scalar framework.

The static assumptions were relaxed by considering dynamic risk measures, where the risk evaluation of a portfolio is updated as time progresses and new information become available. Time consistency is a useful property for dynamic risk measures; it gives a relation between risks at different times. Conceptually, a risk measure is time consistent if, a priori, it is known that a future time one portfolio is more risky than another then at any prior time the same relation holds as well. It is a well known result in the scalar framework that time consistency is equivalent to a recursive form which allows for calculations via Bellman’s principle. For primal and dual representations,
as well as time consistency properties, we refer to [73, 14, 28, 76, 22, 40, 6, 24, 23, 1, 42] for a discrete time setting and [43, 26, 27] for dynamic risk measures in continuous time. An equivalent property for time consistency in the conditionally coherent case is given by the stability of the dual probability measures as seen in [1, 22, 42, 40, 6]. In the conditionally convex case a property on the sum of penalty functions was deduced in [40, 22, 15, 16, 1]. This property is referred to as the cocycle property in [15, 16].

Eliminating the assumption that the financial markets are frictionless required a new framework. Since the ‘value’ of a portfolio is not uniquely determined anymore when bid and ask prices or market illiquidity exist, it is natural to consider portfolios as vectors in physical units instead, i.e. a portfolio is specified by the number of each of the asset which is held as opposed to their value. But even in the absence of transaction costs multivariate claims might be of interest, e.g. when assets are denoted in different currencies with fluctuating exchange rates, or different business lines with no direct exchange or different regularity rules are considered, see [21]. In contrast to frictionless univariate models also the choice of the numéraire assets matters, which lead to different approaches: pick a numéraire and allow capital requirements to be in this numéraire, which allows a risk manager to work with scalar risk measures again (see e.g. [33, 7, 44, 63, 77]); or use the more general numéraire-free approach and allow risk compensation to be made in a basket of assets which leads to risk measures that are set-valued. In the numéraire free model, initiated by Kabanov in [59], the value of the risk measure is a collection of portfolio vectors describing, for example, the amount of money in different currencies that compensate for the risk of\(X\). Then, ‘minimal capital requirements’ correspond to the set of efficient points (the efficient frontier) of the set of risk compensating portfolio vectors. Mathematically it is easier to work with the set of all risk compensating portfolio vectors than with the set of its minimal elements (nevertheless, in both cases the value of the risk measure would be a set!). For example, the whole set is convex for convex risk measures, whereas
the set of minimal elements is rarely a convex set. Thus, considering multivariate risks in a market with transaction costs leads naturally to set-valued risk measures. This approach was first studied in Jouini, Meddeb, Touzi [58] in the coherent case. Several extensions have been made. We will introduce four approaches to deal with dynamic multivariate risk measures, and compare and relate them (in chapter 6) by giving conditions under which the results obtained in each approach coincide. The four approaches we discuss are

1. a set-optimization approach;
2. a measurable selector approach;
3. an approach utilizing set-valued portfolios; and
4. a family of multiple asset scalar risk measures.

The first three approaches correspond to the numéraire-free framework, whereas the last approach includes scalar risk measures where a numéraire asset is chosen. In this text we primarily focus on the set-optimization approach.

In [49, 52, 48, 50] the results of [58] were extended to the convex case and a stochastic market model. The extension of the dual representation results were made possible by an application of convex analysis for set-valued functions (set-optimization), see the work by Hamel [47]. The values of risk measures and its minimal elements in the static framework have been studied and computed in [67, 53, 51, 68] via Benson’s algorithm for linear vector optimization (see e.g. [66]) in the coherent and polyhedral convex case, respectively via an approximation algorithm in the convex case, see [68]. The dynamic case and time consistency was studied in [34, 36]. We call this approach the set-optimization approach.

Chapter 2 studies the axiomatic definition of the dynamic set-valued risk measures in the set-optimization approach as discussed in [34, 36]. The primal representation is
deduced, and the dual representation of convex and coherent risk measures using the set-valued duality approach by [47] is given. Surprisingly, the conditional expectations we would expect in the dual representation of dynamic risk measures (having the scalar case in mind) appear only after a transformation of dual variables from the classical dual pairs in the set-valued theory of [47] to dual pairs involving vector probability measures. This leads to dual representation in the spirit of the well known scalar case (theorem 2.2.8 and corollary 2.2.9). Additionally, dual representations for conditionally convex and coherent risk measures are given.

In chapter 3 we discuss generalizing the scalar time consistency property to the set-valued framework as was done in [34, 36]. The difficulty lies in the fact that the time consistency property for scalar risk measures can be generalized to set-valued risk measures in different ways. The most intuitive generalization we call time consistency. We will show that the equivalence between a recursive form of the risk measure and time consistency, which is a central result in the scalar case, does not hold in the set-valued framework. Instead, we propose an alternative generalization, which we call multi-portfolio time consistency and show that this property is indeed equivalent to the recursive form as well as to an additive property for the acceptance sets. Multi-portfolio time consistency is a stronger property than time consistency, however in the scalar framework both notions coincide. Section 3.3 deduces an equivalent characterization of multi-portfolio time consistency for set-valued normalized closed (conditionally) convex risk measures. This is given by a property on the sum of minimal penalty functions, called the cocycle property in the scalar case in [15, 16], and is the extension of the scalar result of [40, 22, 15, 16, 1]. The proof of this results is entirely different from the proof in the scalar case as the scalar method leads to difficulties in the set-valued case, due to the union in the set-valued recursive form. Section 3.4 discusses two equivalent characterizations of multi-portfolio time consistency for set-valued normalized closed (conditionally) coherent risk measures. The
first is the result for convex risk measures applied to the coherent case. This characterization has not been explicitly stated in the scalar case, but is useful for generating multi-portfolio time consistent risk measures (see e.g. [23]). The second property is the set-valued generalization of stability of the dual variables, and generalizes the work in [1, 22, 42, 40, 6]. In section 3.5, the idea described in the scalar case in [24] to generate a new time-consistent risk measure from any dynamic risk measure by composing them backwards in time is studied in the set-valued case. In particular, we give a method for composing a risk measure backwards in time to create a multi-portfolio time consistent version of the risk measure. Special attention is given to the composed form for (conditionally) convex and coherent risk measures.

In chapter 4 we use the results from [37] to show that this recursive form can be seen as a set-valued version of Bellman’s principle. On one hand, it enables us to calculate the value of a risk measure, that is, the set of all risk compensating initial portfolio holdings, backwards in time. This is in the spirit of dynamic programming. On the other hand, one can show that the principle of optimality holds true: the truncated optimal strategy calculated at time \( t = 0 \) is still optimal for the optimization problems appearing at any later time point \( t > 0 \). We provide conditions under which the recursive form is equivalent to a sequence of linear vector optimization problems, which can be solved with Benson’s algorithm, see e.g. [13, 66, 51, 29]. This is the case for most coherent, but also for convex polyhedral risk measures. More generally, we give conditions under which the recursive form is equivalent to a sequence of convex vector optimization problems, that can be approximately solved by the algorithms proposed in [68]. We will give a financial interpretation of the optimal strategies, which provides an intuition for the nested formulation of the risk measures.

As examples of such dynamic risk measures in the set-optimization approach, we study (in chapter 5) the superhedging risk measure (under proportional and convex transaction costs), the relaxed worst case risk measure, average value at risk, and the
entropic risk measure in chapter 5. Of particular interest we consider whether these risk measures are multi-portfolio time consistent and discuss the computation of the multi-portfolio time consistent version of each.

Finally, in chapter 6 we consider the other three approaches for studying risk measures in illiquid markets as was done in [35]. In doing so we first consider the work by Ben Tahar and Lépinette [11]; they extended the results of [58] for coherent risk measures to the dynamic case. We call this the measurable selector approach as it considers the value of a risk measures as a random set, and then provides a primal and dual representation for the measurable selectors in that set. In [11], time consistency properties were also introduced and some equivalent characterizations discussed.

Most recently, in [21], set-valued coherent risk measures were considered as functions from random sets into the upper sets. The transaction costs model, and other financial considerations like trading constraints, or illiquidity, are then embedded into the construction of “set-valued portfolios.” A subclass of risk measures in this framework can be constructed using a vector of scalar risk measures and [21] gives upper and lower bounds as well as dual representations for this subclass. We present here the dynamic extension of this approach. Time consistency properties have not yet been studied within this framework. However, by comparing and relating the different approaches we will see that a larger subclass can be obtained by using the set-valued risk measures of the set-optimization approach, which provides already a link to dual representations and time consistency properties for this larger subclass.

The fourth approach is to consider a family of dynamic scalar risk measures to evaluate the risk of a multivariate claim. This approach has not been studied so far in the dynamic case outside of [35]. In the special case of frictionless markets, the family of scalar risk measures coincides with scalar risk measures using multiple eligible assets as discussed in [33, 7, 44, 63, 77]. Also the scalar static risk measure of multivariate claims with a single eligible asset studied in [19]; the scalar liquidity adjusted risk
measures in market with frictions as studied in [80]; and the scalar superhedging price in markets with transaction costs, see [12, 18, 71, 57, 74, 75, 67], are special cases of this approach. Thus, the family of dynamic scalar risk measures for portfolio vectors generalizes these special cases in a unified way to allow for frictions, multiple eligible assets, and multivariate portfolios in a dynamic framework. The connection to the set-optimization approach allows one to utilize the dual representation and time consistency results deduced there.

Other papers in the context of set-valued risk measures are [10], where an extension of the tail conditional expectation to the set-valued framework of [58] was presented and a numerical approximation for calculation was given; and [20], where set-valued risk measures in a more abstract setting were studied and a consistent structure for scalar-valued, vector-valued, and set-valued risk measures (but for constant solvency cones) was created. Furthermore, in [20] distribution based risk measures were extended to the set-valued framework via depth-trimmed regions. More recently, vector-valued risk measures were studied in [9].
In this chapter, we introduce some notations and review basic definitions and main results about duality of set-valued dynamic risk measures from [34, 36]. The static set-valued risk measures (discussed in e.g. [50, 48]) are implicitly described by the conditional risk measures at time 0.

As we are working in a dynamic framework, we consider a general space of times $\mathbb{T}$ over which risk should be measured. We consider the case where there is a terminal time $T$. Thus this time space $\mathbb{T}$ includes the continuous ($\mathbb{T} = [0,T]$) or discrete ($\mathbb{T} = \{0,1,\ldots,T\}$) cases.

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ satisfying the usual conditions where $\mathcal{F}_T = \mathcal{F}$. Let $d \geq 1$ be the number of assets under consideration. Let $\| \cdot \|$ denote an arbitrary norm in $\mathbb{R}^d$ and let $\mathcal{L}_p^\mathcal{F} := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ for any $p \in [0, +\infty]$ (with $\mathcal{L}^p := \mathcal{L}_p^\mathcal{F}$). If $p = 0$ then $\mathcal{L}_0^\mathcal{F}$ is the linear space of the equivalence classes of $\mathcal{F}_t$-measurable functions $X : \Omega \rightarrow \mathbb{R}^d$. For $p > 0$, $\mathcal{L}_p^\mathcal{F}$ denotes the linear space of $\mathcal{F}_t$-measurable functions $X : \Omega \rightarrow \mathbb{R}^d$ such that $\|X\|_p = (\int_{\Omega} |X(\omega)|^p d\mathbb{P})^{1/p} < +\infty$ for $p \in (0, \infty)$, and $\|X\|_\infty = \text{ess sup}_{\omega \in \Omega} |X(\omega)| < +\infty$ for $p = +\infty$. We consider the dual pair $(\mathcal{L}_p^\mathcal{F}, \mathcal{L}_q^\mathcal{F})$ with $p \in [1, +\infty]$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, and endow it with the norm topology, respectively the $\sigma(\mathcal{L}_\infty^\mathcal{F}, \mathcal{L}_1^\mathcal{F})$-topology on $\mathcal{L}_\infty^\mathcal{F}$ in the case $p = +\infty$.  


We denote by \( \mathcal{L}_t^p(D_t) := \{ Z \in \mathcal{L}_t^p : Z \in D_t \ \mathbb{P}\text{-a.s.}\} \) those random vectors in \( \mathcal{L}_t^p \) that take \( \mathbb{P}\text{-a.s.} \) values in \( D_t \). Note that an element \( X \in \mathcal{L}_t^p \) has components \( X_1, \ldots, X_d \) in \( L_t^p := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \). (In-)equalities between random vectors are always understood componentwise in the \( \mathbb{P}\text{-a.s.} \) sense. The multiplication between a random variable \( \lambda \in L^\infty \) and a set of random vectors \( D \subseteq \mathcal{L}^p \) is understood elementwise, i.e. \( \lambda D = \{ \lambda Y : Y \in D \} \subseteq \mathcal{L}^p \) with \( (\lambda Y)(\omega) = \lambda(\omega)Y(\omega) \). The multiplication and division between (random) vectors is understood in the elementwise sense, i.e. \( x \cdot y := (x_1y_1, \ldots, x_dy_d)^T \) and \( x/y := (x_1/y_1, \ldots, x_d/y_d)^T \) for \( x, y \in \mathbb{R}^d \) \( (x, y \in \mathcal{L}_t^p) \) with \( y_i \neq 0 \) (almost surely) for every index \( i \in \{1, \ldots, d\} \).

Let \( \mathcal{L}_{t,+}^p := \{ X \in \mathcal{L}_t^p : X \in \mathbb{R}^d_+ \ \mathbb{P}\text{-a.s.}\} \) denote the closed convex cone of \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-measurable random vectors with \( \mathbb{P}\text{-a.s.} \) non-negative components. Additionally let \( \mathcal{L}_{t,++}^p := \{ X \in \mathcal{L}_t^p : X \in \mathbb{R}^{d+}_+ \ \mathbb{P}\text{-a.s.}\} \) be the \( \mathcal{F}_t \)-measurable random vectors which are \( \mathbb{P}\text{-a.s.} \) positive. Similarly define \( \mathcal{L}_{+}^p := \mathcal{L}_{T,+}^p \) and \( \mathcal{L}_{++}^p := \mathcal{L}_{T,++}^p \). Let \( Y \succeq X \) for \( X, Y \in \mathcal{L}^p \) denote \( Y - X \in \mathcal{L}^p_+ \).

As in [59] and discussed in [78, 60], the portfolios in this text are in “physical units” of an asset rather than the value in a fixed numéraire via some price. That is, for a portfolio \( X \in \mathcal{L}_t^p \), the values of \( X_i \) (for \( 1 \leq i \leq d \)) are the number of units of asset \( i \) in the portfolio at time \( t \).

### 2.1 Multivariate risk measures

We define set-valued risk measures as a function mapping a contingent claim into a set of portfolios which can be used to cover the risk of that claim as in [34, 36]. In this way we must introduce the set of potential risk covering portfolios. Let us assume \( m \) of the \( d \) assets are eligible (\( 1 \leq m \leq d \)), that is, they can be used to compensate for the risk of a portfolio. Without loss of generality we can assume these are the first \( m \) assets, then \( M = \mathbb{R}^m \times \{0\}^{d-m} \) denotes the subspace of eligible assets. By
section 5.4 and proposition 5.5.1 in [60], \( M_t := \mathcal{L}^p_t(M) \) is a closed (weak\(^*\) closed if \( p = +\infty \)) linear subspace of \( \mathcal{L}^p_t \). Let us denote \( M_{t,+} := M_t \cap \mathcal{L}^p_{t,+} = (\mathcal{L}^p_{t,+})^m \times \{0\}^{d-m} \) and \( M_{t,-} := -M_{t,+} \). Similarly, we define \( M_{t,++} := (\mathcal{L}^p_{t,++})^m \times \{0\}^{d-m} \) and \( M_{t,--} := -M_{t,++} \).

**Example 2.1.1.** In the scalar framework (see e.g. [6, 73, 28, 22, 76, 14, 40, 24, 23]) the risk is covered by the numéraire only. This corresponds to \( M = \mathbb{R} \times \{0\}^{d-1} \); thus the set of eligible assets is \( M_t = L^p_t \times \{0\}^{d-1} \) when the numéraire is taken to be the first asset.

### 2.1.1 Set-valued dynamic risk measures

A conditional risk measure \( R_t \) is a function which maps a \( d \)-dimensional random vector \( X \) into a subset of the power set of the space \( M_t \) of eligible portfolios at time \( t \). The set \( R_t(X) \) contains those portfolio vectors that can be used to cover the risk of the input portfolio. Consider \( \mathcal{L}^p \) with \( p \in [0, +\infty] \) as the pre-image space and recall that \( M_t \subseteq \mathcal{L}^p_t \). In this text we study risk measures

\[
R_t : \mathcal{L}^p \to \mathcal{P} (M_t; M_{t,+}) := \{D \subseteq M_t : D = D + M_{t,+}\}.
\]

Conceptually, the image space \( \mathcal{P} (M_t; M_{t,+}) \) contains the proper sets to consider. This is since if some portfolio \( u \in M_t \) covers the risk of a portfolio then so should any eligible portfolio that is (almost surely) greater than \( u \), i.e. an element of \( u + M_{t,+} \). The choice of the image space is further discussed in remark 2.1.10.

**Definition 2.1.2.** A function \( R_t : \mathcal{L}^p \to \mathcal{P} (M_t; M_{t,+}) \) is a **conditional risk measure** at time \( t \) if it is

1. \( M_t \)-translative: \( R_t (X + u) = R_t(X) - u \) for every \( u \in M_t \);
2. \( \mathcal{L}^p_{+} \)-monotone: \( Y \succeq X \Rightarrow R_t(Y) \supseteq R_t(X) \);
3. finite at zero: $\emptyset \neq R_t(0) \neq M_t$.

Conceptually, $R_t(X)$ is the set of eligible portfolios which compensate for the risk of the portfolio $X$ at time $t$. In this sense, $M_t$-translativity implies that if part of the risk is covered, only the remaining risk needs to be considered. This property in the scalar framework is sometimes called cash-invariance. $\mathcal{L}_+^p$-monotonicity also has a clear interpretation: if a random vector $Y \in \mathcal{L}_+^p$ dominates another random vector $X \in \mathcal{L}_+^p$, then there should be more possibilities to compensate the risk of $Y$ (in particular cheaper ones) than for $X$. Finiteness at zero means that there is an eligible portfolio at time $t$ which covers the risk of the zero payoff, but not all portfolios compensate for it.

Beyond these bare minimum of properties given in definition 2.1.2, there are additional properties that are desirable and are described in definition 2.1.5. Notable among these is $K_t$-compatibility (defined below). This property allows for a portfolio manager to take advantage of trading opportunities when assessing the risk of a position. For such a concept we need a market model. Let us consider a market with proportional and convex transaction costs as in [70], which is modeled by a sequence of convex solvency regions $(K_t)_{t \in \mathbb{T}}$ describing a convex market model. $K_t$ is a convex solvency region at time $t$ if it is an $\mathcal{F}_t$-measurable set such that for (almost) every $\omega \in \Omega$, $K_t(\omega)$ is closed and convex with $\mathbb{R}_+^d \subseteq K_t(\omega) \subseteq \mathbb{R}^d$. When the market model is additionally conical, the solvency cone $K_t$ can be generated by the time $t$ bid-ask exchange rates between any two assets, for details see [59, 78, 60] and examples 2.1.3 and 2.1.4 below. The financial interpretation of the solvency region at time $t$ is the set of positions which can be exchanged into a nonnegative portfolio at time $t$ by trading according to the market prices (including the market impact costs).
Example 2.1.3. In frictionless markets, the solvency cone $K_t(\omega)$ at time $t$ and in state $\omega \in \Omega$ is the positive half-space defined by

$$K_t(\omega) = \{ k \in \mathbb{R}^d : S_t(\omega)^T k \geq 0 \}$$

where $S_t$ is the (random) vector of prices in some numéraire.

Example 2.1.4. Assume that we have a sequence of (random) bid-ask matrices $(\Pi_t)_{t \in \mathbb{T}}$, as in [78], where $\pi_{t}^{ij}$ is the (random) number of units of asset $i$ needed to purchase one unit of asset $j$ at time $t$ (where $i, j \in \{1, 2, ..., d\}$). Then the corresponding market model is given by the sequence of solvency cones $(K_t)_{t \in \mathbb{T}}$ defined such that $K_t(\omega) \subseteq \mathbb{R}^d$ is the convex cone spanned by the unit vectors $e^i$ for every $i \in \{1, 2, ..., d\}$ and the vectors $\pi_{t}^{ij}(\omega)e^i - e^j$ for every $i, j \in \{1, 2, ..., d\}$.

Definition 2.1.5. A conditional risk measure $R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ is called

- (conditionally) convex if for all $X, Y \in \mathcal{L}^p$, for all $\lambda \in [0, 1]$ ($\lambda \in L^\infty_t([0, 1])$)

$$R_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y);$$

- (conditionally) positive homogeneous if for all $X \in \mathcal{L}^p$, for all $\lambda \in \mathbb{R}_{++}$ ($\lambda \in L^\infty_t(\mathbb{R}_{++})$)

$$R_t(\lambda X) = \lambda R_t(X);$$

- subadditive if for all $X, Y \in \mathcal{L}^p$

$$R_t(X + Y) \supseteq R_t(X) + R_t(Y);$$

- (conditionally) coherent if it is (conditionally) convex and (conditionally) positive homogeneous, or subadditive and (conditionally) positive homogeneous;
• normalized if \( R_t(X) = R_t(X) + R_t(0) \) for every \( X \in \mathcal{L}^p \);

• K-compatible (i.e., allowing trading at time \( t \)) if \( R_t(X) = \bigcup_{k \in K} \hat{R}_t(X - k) \) for some set \( K \subseteq \mathcal{L}^p \) and some risk measure \( \hat{R}_t \);

• local if \( 1_D R_t(1_D X) = 1_D R_t(X) \) for all \( X \in \mathcal{L}^p \) and \( D \in \mathcal{F}_t \), where the stochastic indicator function is denoted by \( 1_A(\omega) \) being 1 if \( \omega \in A \) and 0 if \( \omega \notin A \);

• decomposable if \( R_t(1_D X + 1_D Y) = 1_D R_t(X) + 1_D R_t(Y) \) for every \( X, Y \in \mathcal{L}^p \) and \( D \in \mathcal{F}_t \);

• closed if the graph of \( R_t \), defined by

\[
\text{graph } R_t = \{(X, u) \in \mathcal{L}^p \times M_t : u \in R_t(X)\} ,
\]

is closed (\( \sigma(\mathcal{L}^\infty, \mathcal{L}^1) \)-closed if \( p = +\infty \));

• A conditional risk measure at time \( t \) is convex upper continuous (c.u.c.) if

\[
R_t^{-1}(D) := \{X \in \mathcal{L}^p : R_t(X) \cap D \neq \emptyset\}
\]

is closed for any closed set \( D \in \mathcal{G}(M_t; M_{t,-}) := \{D \subseteq M_t : D = \cl\co(D + M_{t,-})\} \).

It is called conditionally c.u.c. if \( R_t^{-1}(D) \) is closed for any conditionally convex closed set \( D \in \mathcal{G}(M_t; M_{t,-}) \).

Before discussing the properties mentioned above, we will define dynamic risk measures. A dynamic risk measure is a series of conditional risk measures. In this way we can construct the process of risk compensating portfolios.

Definition 2.1.6. \((R_t)_{t \in T} \) is a dynamic risk measure if \( R_t \) is a conditional risk measure for every \( t \in T \).
Definition 2.1.7. A dynamic risk measure \((R_t)_{t \in T}\) is said to have one of the properties given in definition 2.1.5 if \(R_t\) has this property for every \(t \in T\).

A dynamic risk measure \((R_t)_{t \in T}\) is called market-compatible with respect to the market model \((K_t)_{t \in T}\) with respect to the market model \((K_t)_{t \in T}\) (i.e. trading is allowed at any time point \(t \in T\)) if \(R_t\) is \(\sum_{s=t}^{T} L^p_s(K_s)\)-compatible for every \(t \in T\), typically assumed to be discrete in time.

We now discuss the properties from definition 2.1.5 in the order given.

(Conditional) convexity is regarded as a useful property for dynamic risk measures because it defines a regulatory scheme which promotes diversification as discussed in [41] in the scalar static case and [50] in the set-valued static case. The values of a (conditionally) convex set-valued conditional risk measure \(R_t\) are (conditionally) convex. The distinction between convexity and conditional convexity is whether the combination of portfolios can depend on the state of the market. Trivially it can be seen that conditional convexity is a stronger property than convexity. Typically, in the scalar framework (see e.g. [28, 40, 22]), only conditional convexity is considered; though [61, 14, 15, 16] consider convex local risk measures instead.

If \(R_t\) is positive homogeneous then \(R_t(0)\) is a cone and if \(R_t\) is sublinear (positive homogeneous and subadditive), then \(R_t(0)\) is a convex cone which is included in the recession cone of \(R_t(X)\) for all \(X \in L^p\). Therefore, as in the scalar framework, any closed coherent risk measure is normalized.

Let us give an interpretation of the normalization property. A normalized closed conditional risk measure \(R_t\) satisfies \(M_{t,+} \subseteq R_t(0)\) and \(R_t(0) \cap M_{t,-} = \emptyset\). Thus, nonnegative portfolios cover the risk of the zero payoff, but strictly negative portfolios cannot cover that risk. This clearly is the set-valued analog of the scalar normalization property given by \(\rho(0) = 0\). As it is typical in the set-valued framework, there is not a unique generalization of the normalization property for scalar risk measures to the set-valued case. Note that for set-valued risk measures normalization could be defined
also in a ‘weaker’ sense, as it was done in [48, 52], by directly imposing $M_{t,+} \subseteq R_t(0)$ and $R_t(0) \cap M_{t,-} = \emptyset$. In the closed coherent set-valued case both notions coincide, see property 3.1 in [58].

As presented in [34], any conditional risk measure (that is finite at zero) can be normalized in the following way. The normalized version $\tilde{R}_t$ of a conditional risk measure $R_t$ is defined by

$$
\tilde{R}_t(X) = R_t(X) - R_t(0) := \{ Z \in M_t : Z + R_t(0) = R_t(-Z) \subseteq R_t(X) \}
$$

for every $X \in \mathcal{L}^p$. The operation $A - B$ for sets $A, B$ is sometimes called the Minkowski difference (see e.g. [46]) or geometric difference (see e.g. [72]). This difference notation trivially shows how the normalization procedure introduced above relates to the normalized version for scalar risk measures given by $\rho_t(X) - \rho_t(0)$. The normalized version $\tilde{R}_t(X)$ will also satisfy $M_{t,+} \subseteq \tilde{R}_t(0)$ and $\tilde{R}_t(0) \cap M_{t,-} = \emptyset$. Further note that a risk measure is equivalent to its normalized version, that is $R_t(X) = R_t(X) - R_t(0)$, if and only if $R_t$ is normalized and $0 \in R_t(0)$.

For notational simplicity, we use the shorthand $K_t$-compatibility for $\mathcal{L}_t^p(K_t)$-compatibility when $K_t$ is a random set (rather than a set of random variables). As previously mentioned, $K_t$-compatibility means that trading at time $t$ is allowed according to a market defined by the solvency region defined by $K_t$. If $u \in R_t(X)$ for $K_t$-compatible risk measure (with respect to $\hat{R}_t$), then there exists some $k \in \mathcal{L}_t^p(K_t)$ (i.e. $-k$ can be purchased in the market without additional capital) so that $u$ compensates for the risk of the modified portfolio $X - k$ (with respect to the risk measure $\hat{R}_t$). Further when $K_t$ is a convex cone, $R_t$ is $K_t$-compatible if and only if $R_t(X) = \bigcup_{k \in \mathcal{L}_t^p(K_t)} R_t(X - k)$. Market-compatibility is the corresponding property for dynamic risk measures, when trading at each future time point $t \in \mathbb{T}$ is allowed. If $(K_t)_{t \in \mathbb{T}}$ is a sequence of solvency cones then market-compatibility is equivalent to
$R_t$ being $K_s$-compatible for every $s \geq t$ and for every time $t \in \mathbb{T}$. Note that the definition of $K$-compatibility (and market-compatibility) are different than in [34], the definitions given within this text generalize those in [34] by allowing for a larger class of market models and not restricting trades to the eligible portfolios.

The local property is a desirable property for dynamic set-valued risk measures. It has been studied for scalar dynamic risk measures for example in [28, 23, 39]. The local property ensures that the output of a risk measure at time $t$ at a specific state in $\mathcal{F}_t$ only depends on the payoff in scenarios that can still be reached from this state, as we would expect of a risk compensating portfolio.

**Proposition 2.1.8.** Any conditionally convex risk measure $R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+})$ is local.

The proof is an adaption from the proof of lemma 3.4 in [39].

*Proof.* Let $X \in \mathcal{L}^p$ and $D \in \mathcal{F}_t$. By convexity it is obvious that

$$R_t(1_D X) \supseteq 1_D R_t(X) + 1_D \cdot R_t(0)$$
$$= 1_D R_t(1_D (1_D X) + 1_D \cdot X) + 1_D \cdot R_t(0) \supseteq 1_D R_t(1_D X) + 1_D \cdot R_t(0).$$

Multiplying through by $1_D$, then the left and right sides are equal, therefore $1_D R_t(1_D X) = 1_D R_t(X)$.

Decomposability is a stronger property than locality, though in the scalar case both notions coincide (see e.g. [28]). While the typical examples of risk measures will be decomposable, we give one example below of a risk measure that is not decomposable (but is local).

**Example 2.1.9.** Let $A := \text{cl}(K + \mathcal{L}_+^{\infty}) \neq \mathcal{L}_+^{\infty}$ for some cone $K \subseteq \mathcal{L}_0^{\infty}$ such that $A$ is an acceptance set (see the following section). Let $R_t(X) = \{ u \in M_t : X + u \in A \}$ for all times $t$. It can be shown that $(R_t)_{t \in \mathbb{T}}$ is closed, convex, and multi-portfolio.
time consistent (see chapter 3 for details), but is not decomposable. This might be important if one is interested in the static risk measure $R_0(X)$, sticks to the decision made at time $t = 0$ and just reevaluates the risk at time $t > 0$ based on the same acceptability criterion used at $t = 0$ to determine e.g. if the initial deposit $u \in R_0(X)$ can be reduced at time $t > 0$ while keeping acceptability. Furthermore, if $M = \mathbb{R}^d$, then $R_t$ is local, but not decomposable.

For convex duality results a closedness property is necessary. Any set-valued function $R_t$ with a closed graph has closed values, i.e. $R_t(X)$ is a closed set for every $X \in \mathcal{L}^p$.

Note that any c.u.c. risk measure is closed. This follows from $(X, u) \in \text{graph } R_t$ if and only if $X + u \in R_t^{-1}(M_{t,-})$. In the literature, upper continuity is defined analogously to c.u.c., but with respect to all closed sets $D \subseteq M_t$ rather than the subset $\mathcal{G}(M_t; M_{t,-})$, see [54, 45, 8, 55] (in the latter two references upper continuity is called upper semicontinuity, we follow the naming practiced by [54, 45] because upper semicontinuity can also refer to a different property for set-valued functions). As we do not need the upper continuity property for all closed sets $D \subseteq M_t$, but only for $D \in \mathcal{G}(M_t; M_{t,-})$, we labeled the corresponding property convex upper continuity.

**Remark 2.1.10.** From $M_t$-translativity and monotonicity it follows that $R_t(X) = R_t(X) + M_{t,+}$, which mathematically justifies the choice of the image space of a conditional risk measure to be $\mathcal{P}(M_t; M_{t,+})$ instead of the full powerset $2^{M_t}$. The image space of a closed convex conditional risk measure is

$$\mathcal{G}(M_t; M_{t,+}) = \{ D \subseteq M_t : D = \text{cl co} (D + M_{t,+}) \}.$$ 

**2.1.2 Dynamic acceptance sets**

Acceptance sets are intrinsically linked to risk measures. A set-valued risk measure provides the portfolios which compensate for the risk of a contingent claim, whereas
a portfolio is an element of the acceptance set if its risk does not need to be covered. Typically, a conditional acceptance set at time $t$ is given by a regulator or risk manager. We show below in remark 2.1.12 that there exists a bijective relation between risk measures and acceptance sets. Expanding upon the definition given in [50] for the acceptance set of a static set-valued risk measure, we now define acceptance sets for dynamic set-valued risk measures.

**Definition 2.1.11.** A $A_t \subseteq \mathcal{L}^p$ is a conditional acceptance set at time $t$ if it satisfies the following:

1. $M_t \cap A_t \neq \emptyset,$

2. $M_t \cap (\mathcal{L}^p \setminus A_t) \neq \emptyset,$ and

3. $A_t + \mathcal{L}^p_+ \subseteq A_t.$

The properties given in definition 2.1.11 are the minimal requirements for a conditional acceptance set. As noted in [50] these are conditions that any rational regulator would agree upon. They imply that there is an eligible portfolio at time $t$ that is accepted by the regulator, but there exists an eligible portfolio which would itself require compensation to make it acceptable. The last condition means that if a portfolio is acceptable to the regulator then any portfolio with at least as much of each asset will also be acceptable.

**Remark 2.1.12.** It can be seen that there is a one-to-one relationship between acceptance sets and risk measures in the dynamic setting. In particular, if we define the set

$$A_{R_t} := \{X \in \mathcal{L}^p : 0 \in R_t(X)\}$$
for a given conditional risk measure \( R_t \), then \( A_{R_t} \) satisfies the properties of definition 2.1.11 and thus is a conditional acceptance set. On the other hand, defining

\[
R_t^{A_t}(X) := \{ u \in M_t : X + u \in A_t \}
\]  

(2.1.2)

for a given conditional acceptance set \( A_t \) ensures that \( R_t^{A_t} \) is a conditional risk measure. Further, it can trivially be shown that \( A_t = A_{R_t^{A_t}} \) and \( R_t(\cdot) = R_t^{A_{R_t}}(\cdot) \). Equation (2.1.2) is often referred to as the primal representation for the risk measure and provides the capital requirement interpretation of a risk measure.

**Example 2.1.13.** The worst case risk measure is defined by the acceptance set \( \mathcal{L}_+^p \), that is \( R_{WC}^t(X) = \{ u \in M_t : X + u \in \mathcal{L}_+^p \} \).

The following proposition is a list of corresponding properties between classes of conditional risk measures and classes of conditional acceptance sets. If a risk measure \( R_t \) has the (risk measure) property then \( A_{R_t} \) has the corresponding (acceptance set) property, and vice versa: if an acceptance set \( A_t \) has the (acceptance set) property then \( R_t^{A_t} \) has the corresponding (risk measure) property.

**Proposition 2.1.14.** The following properties are in a one-to-one relationship for a conditional risk measure \( R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+}) \) and an acceptance set \( A_t \subseteq \mathcal{L}^p \) at time \( t \):

1. \( R_t \) is \( C \)-monotone, and \( A_t + C \subseteq A_t \), where \( C \subseteq \mathcal{L}^p \);

2. \( R_t \) maps into the set

\[
\mathcal{P}(M_t; C) = \{ D \subseteq M_t : D = D + C \},
\]

and \( A_t + C = A_t \), where \( C \subseteq M_t \);

3. \( R_t \) is (conditionally) convex, and \( A_t \) is (conditionally) convex;
4. \( R_t \) is (conditionally) positive homogeneous, and \( A_t \) is a (conditional) cone;

5. \( R_t \) is subadditive, and \( A_t + A_t \subseteq A_t \);

6. \( R_t \) is sublinear, and \( A_t \) is a convex cone;

7. \( R_t \) has a closed graph, and \( A_t \) is closed;

8. \( R_t(X) \neq \emptyset \) for all \( X \in \mathcal{L}^p \), and \( \mathcal{L}^p = A_t + M_t \);

9. \( R_t(X) \neq M_t \) for all \( X \in \mathcal{L}^p \), and \( \mathcal{L}^p = (\mathcal{L}^p \setminus A_t) + M_t \).

**Proof.** All of these properties follow from adaptations from proposition 6.5 in [50].

1. Let \( C \subseteq \mathcal{L}^p \). If \( R_t \) is \( C \)-monotone and \( Y - X \in C \), then if \( X \in A_t \) it immediately follows that \( Y = X + (Y - X) \in A_t \), i.e. \( A_t + C \subseteq A_t \). Conversely, if \( A_t + C \subseteq A_t \) and \( Y - X \in C \), then

\[
R_t(X) = \{ u \in M_t : X + u \in A_t \} = \{ u \in M_t : X + (Y - X) + u \in A_t + C \}
\]

\[
\subseteq \{ u \in M_t : Y + u \in A_t \} = R_t(Y).
\]

2. Let \( C \subseteq M_t \). If \( R_t(X) \in \mathcal{P}(M_t; C) \) for every \( X \in \mathcal{L}^p \), then for any \( X \in A_t \) and \( c \in C \) it follows that \( 0 \in R_t(X) \subseteq R_t(X) + C - c = R_t(X + c) \), and thus \( A_t + C \subseteq A_t \); to show that \( A_t + C \supseteq A_t \) we will assume this is false, let \( Y \in A_t \) and assume for every \( c \in C \) it follows that \( Y - c \not\in A_t \), but

\[
0 \in R_t(Y) = R_t(Y) + C = \bigcup_{c \in C} R_t(Y - c)
\]

which is a contradiction. Conversely, if \( A_t + C = A_t \) and \( X \in \mathcal{L}^p \), then \( R_t(X) + C = \{ u + c : u \in M_t, c \in C, X + u \in A_t \} = \{ u \in M_t : \exists c \in C : X + u - c \in A_t \} = \{ u \in M_t : X + u \in A_t + C = A_t \} = R_t(X) \).

3.-6. Trivially.

7. Let \( R_t \) have a closed graph and let \( (X_n)_{n \in \mathbb{N}} \subseteq A_t \) be a sequence (respectively a net if \( p = +\infty \)) converging to \( X \in \mathcal{L}^p \). By the closed graph we know that
\((X_n, 0) \rightarrow (X, 0) \in \text{graph } R_t\) and therefore \(X \in A_t\). Conversely, let \(A_t\) be closed and let \((X_n, u_n)_{n \in \mathbb{N}} \subseteq \text{graph } R_t\) be a sequence (respectively a net if \(p = +\infty\)) converging to \((X, u) \in \mathcal{L}^p \times M_t\). By the primal representation we have that \(X_n + u_n \rightarrow X + u \in A_t\) and therefore \((X, u) \in \text{graph } R_t\).

8. If \(R_t(X) \neq \emptyset\) for every \(X \in \mathcal{L}^p\), then for any \(X \in \mathcal{L}^p\) there exists some \(u \in M_t\) such that \(X - u \in A_t\). Thus it immediately follows that \(\mathcal{L}^p \subseteq A_t + M_t\), and \(\supseteq\) follows trivially. Conversely, if \(\mathcal{L}^p = A_t + M_t\), then for any \(X \in \mathcal{L}^p\) there exists some \(u \in M_t\) such that \(X - u \in A_t\), i.e. \(-u \in R_t(X)\).

9. If \(R_t(X) \neq M_t\) for every \(X \in \mathcal{L}^p\), then for any \(X \in \mathcal{L}^p\) there exists some \(u \in M_t\) such that \(X - u \not\in A_t\). Thus it immediately follows that \(\mathcal{L}^p \subseteq (\mathcal{L}^p \setminus A_t) + M_t\), and \(\supseteq\) follows trivially. Conversely, if \(\mathcal{L}^p = (\mathcal{L}^p \setminus A_t) + M_t\), then for any \(X \in \mathcal{L}^p\) there exists some \(u \in M_t\) such that \(X - u \in \mathcal{L}^p \setminus A_t\), i.e. \(-u \not\in R_t(X)\).

\(\Box\)

It can be seen that property 2 in proposition 2.1.14 corresponds to normalization \((C = R_t(0) = A_t \cap M_t)\).

**Example 2.1.15.** (Example 2.1.13 continued) The worst case risk measure is a closed coherent risk measure. In fact, the worst case acceptance set is the smallest closed normalized acceptance set.

Due to the one-to-one relation between acceptance sets and risk measures, we will henceforth use the convention that \(A_t = A_{R_t}\) for a conditional risk measure \(R_t\).

### 2.2 Dual representation

In section 2.1, we discussed the primal representation for conditional risk measures. In this section, we develop a dual representation by a direct application of the set-valued duality developed by [47] as was presented in [34]. This representation provides
a probability based representation for finding the set of risk compensating portfolios. In particular, we will demonstrate that, as in the scalar framework, closed convex and coherent risk measures have a representation as the supremum of penalized conditional expectations.

There are multiple approaches that have been used to obtain duality results for scalar dynamic risk measures. In [26, 6] a dual representation is given as the logical extension of the static case and shown to have the desired properties without directly involving a duality theory for the conditional risk measure. In [76] an omega-wise approach is used, as it reduces to an omega-wise application of biconjugation of (extended) real-valued functions. A popular approach, used in [28, 40, 23, 22, 61, 16], proves the dual representation for dynamic risk measures through a mathematical trick using the static dual representation. Another approach is given by a direct application of vector-valued duality to the conditional risk measure as in [65, 64] and the first part of [39]. But this approach needs additional strong assumptions (non-emptiness of the subdifferential), and as in the setting of [65, 64], does not allow for local properties. A further possibility is to use the module approach introduced and applied in [38, 39].

For set-valued dynamic risk measures, Ben Tahar and Lépinette, in [11], define the dual form by an intersection of supporting hyperplanes. But this allows the treatment of coherent risk measures only. In the present text, we apply the set-valued approach based on [47], which was applied to dynamic risk measures in [34] (and could also be used in the scalar case) and works for both the coherent and the convex case. This is the most intuitive approach for us as the risk measures under consideration are by nature set-valued functions.

In the scalar framework, most of the papers consider conditionally convex dynamic risk measures (see [28, 40, 22]). In [61, 14, 16] the dual representation is deduced for convex and local dynamic risk measures, and in [22] it was shown that
any risk measure on $L^\infty$ satisfies the local property. We provide an analogous result for conditionally convex dynamic risk measures (reproduced from [36]). Additionally, using the set valued approach we are able to provide a dual representation for any convex dynamic risk measure for any $p \in [1, +\infty]$, and thus extend, as a byproduct, also the scalar case.

Since we will be considering conjugate duality, we need to assume for this section that $p \in [1, +\infty]$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. That means, for any time $t$, we consider the dual pair $(\mathcal{L}^p_t, \mathcal{L}^q_t)$ and endow it with the norm topology, respectively the $\sigma(\mathcal{L}^\infty_t, \mathcal{L}^1_t)$-topology on $\mathcal{L}^\infty_t$ in the case $p = +\infty$. As before let the set of eligible portfolios $M_t$ be a closed subspace of $\mathcal{L}^p_t$ for all times $t$.

For all $t \in T$, we denote the positive dual cone of $C$ by $C^+$. We should note that $\mathcal{L}^p_t(K_t)^+ = \mathcal{L}^q_t(K_t^+)$ for random cones $K_t$, see section 6.3 in [50]. Denote

$$M_t^+ = \{v \in \mathcal{L}^q_t : \mathbb{E}[v^Tu] = 0 \forall u \in M_t\}$$

and

$$M_{t,+}^+ = \{v \in \mathcal{L}^q_t : \mathbb{E}[v^Tu] \geq 0 \forall u \in M_{t,+}\}.$$  

Recall from remark 2.1.10 that $R_t(X) = R_t(X) + M_{t,+}$ by translativity and monotonicity and that a closed convex conditional risk measure $R_t$ maps into the set $\mathcal{G}(M_t; M_{t,+}) = \{D \subseteq M_t : D = \text{cl co} (D + M_{t,+})\}$. Let us denote the positive half-space with respect to $v \in \mathcal{L}^q_t$ by

$$G_t(v) = \{x \in \mathcal{L}^p_t : 0 \leq \mathbb{E}[v^Tx]\}.$$  

### 2.2.1 Dual variables

We will construct the biconjugate for closed convex and coherent risk measures in the same way as in [34], that is by applying the results from [47] given in appendix A.1
with the dual pairs \((\mathcal{L}^p, \mathcal{L}^q)\) and \((\mathcal{L}^p_t, \mathcal{L}^q_t)\) taking the place of \((\mathcal{X}, \mathcal{X}^*)\) and \((\mathcal{Z}, \mathcal{Z}^*)\) respectively. As noted in fact 3 of section 6.3 in [50], any closed convex risk measure is a proper function by finiteness at zero. Therefore by theorem A.1.3, any closed convex or coherent risk measure is equivalent to its biconjugate, which we refer to as the dual representation.

The set-valued duality, given in appendix A.1, greatly reduces the work for finding the dual representation for dynamic risk measures as compared to the scalar framework. This is because the set-valued duality theory works with the same type of image space that (dynamic) set-valued risk measures map into (see remark 2.1.10), and not necessarily just the extended reals.

In contrast to the static risk measure case discussed in [48, 50], we need to consider functions mapping into the power set of \(\mathcal{L}^p_t\) and thus generalize definition 3.1 and proposition 3.2 in [50] to this more general case.

Then, the set-valued functionals of definition A.1.1 (and as discussed in appendix A.1) are given as follows

**Definition 2.2.1.** Given \(Y \in \mathcal{L}^q\), \(v \in \mathcal{L}^q_t\), then the function \(\tilde{F}_{(Y,v)}^t : \mathcal{L}^p \to 2^{M_t}\) is defined by

\[
\tilde{F}_{(Y,v)}^t[X] = \{ u \in M_t : \mathbb{E}[X^TY] \leq \mathbb{E}[v^Tu] \}.
\]

In the following proposition we consider the relation between properties of these functionals and conditions on the sets of dual variables.

**Proposition 2.2.2.** Let \(R_t(X) = \tilde{F}_{(Y,v)}^t[-X]\) for some \(Y \in \mathcal{L}^q\), \(v \in \mathcal{L}^q_t\), then \(R_t\)

1. is additive and positive homogeneous with

\[
\tilde{F}_{(Y,v)}^t[0] = G_t(v) \cap M_t = \{ u \in M_t : 0 \leq \mathbb{E}[v^Tu] \};
\]

2. has a closed graph, and hence closed values, namely closed half-spaces;
3. is finite at 0 if and only if it is finite everywhere if and only if \( v \in L^q \setminus M^t \), moreover \( R_t(X) \in \{ M^t, \emptyset \} \) for all \( X \in \mathcal{L}^p \) if and only if \( v \in M^t \);

4. satisfies \( R_t(X) = R_t(X) + C \) for some cone \( C \subseteq M^t \) with \( 0 \in C \) for all \( X \in \mathcal{L}^p \) if and only if \( v \in C^+ \);

5. is \( C \)-monotone for some cone \( C \subseteq \mathcal{L}^p \) if and only if \( Y \in C^+ \);

6. is \( M^t \)-translative if and only if

\[
R_t(X) = R_t(X) + \hat{u} \in \mathcal{E} \left[ Y \mid \mathcal{F}_t \right] + M^t
\]

if and only if \( v \in C^+ \);

7. has the corresponding acceptance set given by

\[
A_t = \{ X \in \mathcal{L}^p : 0 \leq \mathbb{E} [ Y^T X ] \}
\]

Proof. This is an adaption of proposition 3.2 in [50] by using example 2 and proposition 6 in [47] with the linear space \( Z \) chosen to be \( \mathcal{L}^p_t \).

1. \( \tilde{F}^t_{(Y,v)}[0] = G_t(v) \cap M^t \) trivially. The additivity and positive homogeneity follow much more clearly using the following remark taken from [48].

Remark 2.2.3. For every choice of \( Y \in \mathcal{L}^q \) and \( v \in \mathcal{L}_t^q \), it follows that

\[
\tilde{F}^t_{(Y,v)}[X] = \mathbb{E} [ Y^T X ] \hat{u} + \tilde{F}^t_{(Y,v)}[0]
\]

for any \( \hat{u} \in M^t \) such that \( \mathbb{E} [ v^T \hat{u} ] = 1 \) by

\[
\tilde{F}^t_{(Y,v)}[X] = \{ u \in M^t : 0 \leq \mathbb{E} [ v^T (u - \mathbb{E} [ Y^T X ] \hat{u}) ] \}
\]

To show that \( \tilde{F}^t_{(Y,v)} \) is additive, let \( X_1, X_2 \in \mathcal{L}^p \), noting that \( \tilde{F}^t_{(Y,v)}[0] = \tilde{F}^t_{(Y,v)}[0] + \tilde{F}^t_{(Y,v)}[0] \)

\[
R_t(X_1 + X_2) = \tilde{F}^t_{(Y,v)}[-X_1 - X_2] = -\mathbb{E} [ Y^T (X_1 + X_2) ] + \tilde{F}^t_{(Y,v)}[0]
\]

\[
= -\mathbb{E} [ Y^T X_1 ] + \tilde{F}^t_{(Y,v)}[0] - \mathbb{E} [ Y^T X_2 ] + \tilde{F}^t_{(Y,v)}[0]
\]

\[
= \tilde{F}^t_{(Y,v)}[-X_1] + \tilde{F}^t_{(Y,v)}[-X_2] = R_t(X_1) + R_t(X_2).
\]
To show that $\bar{F}_t^{(Y,v)}$ is positive homogeneous, let $X \in \mathcal{L}^p$ and $\lambda > 0$

$$R_t(\lambda X) = \bar{F}_t^{(Y,v)}[-\lambda X] = \{ u \in M_t : 0 \leq \mathbb{E}[v^T u] + \lambda \mathbb{E}[Y^T X] \}$$

$$= \{ \lambda u : u \in M_t, 0 \leq \lambda \mathbb{E}[v^T u + Y^T X] \}$$

$$= \lambda \{ u \in M_t : 0 \leq \mathbb{E}[v^T u + Y^T X] \} = \lambda R_t(X).$$

2. The graph of $R_t$ is given by $\{(X,u) \in \mathcal{L}^p \times M_t : -\mathbb{E}[Y^T X] \leq \mathbb{E}[v^T u] \}$ is a closed half-space in $\mathcal{L}^p \times M_t$ since $X \mapsto \mathbb{E}[Y^T X]$ is a continuous linear functional on $\mathcal{L}^p$.

3. (a) Given $v \in M_t^\perp$ then $R_t(X) = \{ u \in M_t : 0 \leq \mathbb{E}[Y^T X] \} \in \{ \emptyset, M_t \}$.

(b) Given $R_t(X) \in \{ \emptyset, M_t \}$ then either

i. $0 \leq \mathbb{E}[v^T u + Y^T X]$ for every $u \in M_t$, and therefore $v \in M_t^\perp$, or

ii. $0 > \mathbb{E}[v^T u + Y^T X]$ for every $u \in M_t$, and therefore $v \in M_t^\perp$.

4. It is trivial to see that $\mathbb{E}[v^T u] \leq \mathbb{E}[v^T (u + c)]$ for every $c \in C$ if and only if $v \in C^+$.

5. First we will show that $Y \in C^+$ implies $C$-monotonicity, and then we will show the converse statement.

(a) Let $X_1, X_2 \in \mathcal{L}^p$ such that $X_2 - X_1 \in C$, and let $u \in R_t(X_1)$. If $Y \in C^+$, then

$$-\mathbb{E}[Y^T X_2] + \mathbb{E}[Y^T (X_2 - X_1)] \leq \mathbb{E}[v^T u]$$

and $\mathbb{E}[Y^T (X_2 - X_1)] \geq 0$. Therefore $u \in R_t(X_2)$ and thus $R_t$ is $C$-monotone.
(b) Let $Y \not\in C^+$ and assume $R_t$ is $C$-monotone. Let $X \in C$ such that $E[Y^TX] < 0$ then

$$R_t(-X) = \{ u \in M_t : E[Y^TX] \leq E[v^Tu] \} \supseteq \{ u \in M_t : 0 \leq E[v^Tu] \} = R_t(0).$$

But by $C$-monotonicity it follows that $R_t(-X) \subseteq R_t(0)$, which is a contradiction.

6. Let $\bar{u} \in M_t$,

$$R_t(X + \bar{u}) = \tilde{F}^t_{(Y,v)}[-X - \bar{u}] = \{ u \in M_t : 0 \leq E[Y^T(X + \bar{u}) + v^Tu] \}$$

$$= \{ u \in M_t : 0 \leq E[Y^TX + (E[Y|\mathcal{F}_t] - v)^T\bar{u} + v^T(u + \bar{u})] \}$$

$$= \{ u + \bar{u} : u \in M_t, 0 \leq E[Y^TX + (E[Y|\mathcal{F}_t] - v)^T\bar{u} + v^T(u + \bar{u})] \} - \bar{u}.$$

This is equal to $R_t(X) - \bar{u}$ if and only if $E[(E[Y|\mathcal{F}_t] - v)^T\bar{u}] = 0$. Since this must be true for every $\bar{u} \in M_t$ it follows that $E[Y|\mathcal{F}_t] - v \in M^+_t$, which is true if and only if $v \in E[Y|\mathcal{F}_t] + M^+_t$.

7. Trivially. \qed

Remark 2.2.4. Just as translativity and monotonicity imply that $R_t(\cdot) = R_t(\cdot) + M_{t,+}$, it can be seen that $M^+_{t,+} \supseteq E[Y|\mathcal{F}_t] + M^+_t$ if $Y \in \mathcal{L}^q_t$.

Remark 2.2.5. The functions $X \mapsto \tilde{F}^t_{(Y,v)}[X], \tilde{F}^t_{(Y,v)}[-X]$ map into the collection $G(M_t; M^+_{t,+})$ if and only if $v \in M^+_{t,+}$. By remark 2.2.4, the image space is $G(M_t; M^+_{t,+})$ and the functions are finite at 0 if the dual variables $(Y, v) \in \mathcal{L}^q_t \times [(E[Y|\mathcal{F}_t] + M^+_t) \setminus M^+_t]$. 

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Using set-valued biconjugation as discussed in appendix A.1 it is possible to give a dual representation for closed convex risk measures already. However, with the dual representation for scalar dynamic risk measures in mind, we would expect the conditional expectations to appear in the dual representation (i.e. also in definition 2.2.1) for set-valued dynamic risk measures. We accomplish this by transforming the classical dual variables \((Y, v)\), appearing above, into dual pairs involving vector probability measures \(Q \in \mathcal{M}\). We denote by \(\mathcal{M} := \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P})\) the set of all vector probability measures with components being absolutely continuous with respect to \(\mathbb{P}\). That is, \(Q_i : \mathcal{F} \to [0, 1]\) is a probability measure on \((\Omega, \mathcal{F})\) such that for all \(i \in \{1, 2, ..., d\}\) we have \(\frac{dQ_i}{d\mathbb{P}} \in L^1\). Using this transformation, defined below in lemma 2.2.6, we demonstrate a clear comparison to the dual representation of conditional risk measures in the scalar framework, as given in [1, 42] for example. Further, let \(\mathcal{M}^e\) denote the set of \(d\)-dimensional probability measures equivalent to \(\mathbb{P}\). We say \(Q = \mathbb{P}|_{\mathcal{F}_t}\) for vector probability measures \(Q\) and some time \(t \in \mathbb{T}\) if for every \(D \in \mathcal{F}_t\) it follows that \(Q_i(D) = \mathbb{P}(D)\) for all \(i = 1, ..., d\).

We will use a \(\mathbb{P}\)-almost sure version of the \(Q\)-conditional expectation of \(X \in \mathcal{L}^p\) (for \(Q := (Q_1, ..., Q_d)^T \in \mathcal{M}\)) given by

\[
E^Q[X | \mathcal{F}_t] := E\left[\xi_{t,T}(Q) \cdot X | \mathcal{F}_t\right],
\]

where \(\xi_{t,s}(Q) = (\bar{\xi}_{t,s}(Q_1), ..., \bar{\xi}_{t,s}(Q_d))^T\) for any times \(t, s \in \mathbb{T}\) where \(t \leq s\) with

\[
\bar{\xi}_{t,s}(Q_i)[\omega] := \begin{cases} 
\frac{E\left[\frac{dQ_i}{d\mathbb{P}} | \mathcal{F}_s\right](\omega)}{E\left[\frac{dQ_i}{d\mathbb{P}} | \mathcal{F}_t\right](\omega)} & \text{on } E\left[\frac{dQ_i}{d\mathbb{P}} | \mathcal{F}_t\right](\omega) > 0 \\
1 & \text{else}
\end{cases}
\]

for every \(\omega \in \Omega\), see e.g. [23, 34]. For any probability measure \(Q_i \ll \mathbb{P}\) and any times \(t, r, s \in \mathbb{T}\) where \(t \leq r \leq s\), it follows that \(\frac{dQ_i}{d\mathbb{P}} = \bar{\xi}_{0,T}(Q_i), \bar{\xi}_{t,s}(Q_i) = \bar{\xi}_{t,r}(Q_i)\bar{\xi}_{r,s}(Q_i)\), and \(E[\bar{\xi}_{t,s}(Q_i) | \mathcal{F}_t] = 1\) almost surely.
Throughout this text (particularly in chapter 3) we use the notation \(Q^s\) to denote the modification of \(Q \in \mathcal{M}\) defined by \(\frac{dQ^s}{dP} = \xi_{s,T}(Q)\) so that \(Q^s = P|_{\mathcal{F}_s}\).

We define the function \(w^s_t: \mathcal{M} \times \mathcal{L}^q_t \rightarrow \mathcal{L}^0_s\) for any \(t, s \in \mathbb{T}\) with \(t \leq s\) by

\[
w^s_t(Q, w) := w \cdot \xi_{t,s}(Q) = (w_1 \xi_{t,s}(Q_1), ..., w_d \xi_{t,s}(Q_d))^T
\]

for any \(Q \in \mathcal{M}\) and \(w \in \mathcal{L}^q_t\). Note that \(w^T \mathbb{E}^Q[X|\mathcal{F}_t] = \mathbb{E}^Q[w^T(Q, w)^T X|\mathcal{F}_t]\) for any \(Q \in \mathcal{M}\), \(w \in \mathcal{L}^q_t\), and \(X \in \mathcal{L}^p\). We define the set of dual variables to be

\[
W_t = \{(Q, w) \in \mathcal{M} \times (M_t^+ \setminus M_t^⊥) : w^T(Q, w) \in \mathcal{L}^q_t, \ Q = P|_{\mathcal{F}_t}\}
\]

Recall that the half-space in \(\mathcal{L}^p_t\) with normal direction \(w \in \mathcal{L}^q_t\setminus\{0\}\) is denoted by

\[
G_t(w) := \{u \in \mathcal{L}^p_t : 0 \leq \mathbb{E}[w^T u]\}.
\]

Further recall the definitions

\[
C^+ = \{v \in \mathcal{L}^q_t : \mathbb{E}[v^T u] \geq 0 \ \forall u \in C\}
\]

is the positive dual cone of a cone \(C \subseteq \mathcal{L}^p_t\) for any time \(t\), and

\[
M_t^+ = \{v \in \mathcal{L}^q_t : \mathbb{E}[v^T u] = 0 \ \forall u \in M_t\}.
\]

In lemma 2.2.6 below, a one-to-one correspondence between the dual variables \((Y, v)\) from set-valued duality theory and dual variables based on probability measures \((Q, w)\) is established. Then with the probability measure based dual variables, we see that the values of the set-valued functionals \(F^t_{(Q, w)}\) are half-spaces shifted by the \(Q\)-conditional expectation.
Lemma 2.2.6. 1. Let \( Y \in L^q_+ \) and \( v \in (E[Y|F_i] + M_t^+) \setminus M_t^+ \), thus we assume \( X \mapsto \tilde{F}_{(Y,v)}[-X] \) of definition 2.2.1 to be \( M_t \)-translative, \( L^p_+ \)-monotone and to be finite at 0. Then there exists a \( Q \in \mathcal{M} \) and a \( w \in M_t^+ \setminus M_t^+ \) such that \( w_t^T(Q,w) \in L^q_+ \), \( Q = \mathbb{P}|_{F_i} \), and \( \tilde{F}_{(Y,v)}[X] = F_t^l(Q,w)[X] \) for any \( X \in L^p \), where

\[
F_t^l(Q,w)[X] = \{ u \in M_t : E[wTQ[X|F_i]] \leq E[wT u] \}
\]

\[
= (E^Q[X|F_i] + G_t(w)) \cap M_t. \tag{2.2.1}
\]

2. Vice versa, let \( Q \in \mathcal{M} \) and \( w \in M_t^+ \setminus M_t^+ \) such that the relationship \( w_t^T(Q,w) \in L^q_+ \) holds and \( Q = \mathbb{P}|_{F_i} \). Then there exists a \( Y \in L^q_+ \) and \( v \in (E[Y|F_i] + M_t^+) \setminus M_t^+ \) such that \( \tilde{F}_{(Y,v)}[X] = F_t^l(Q,w)[X] \) for any \( X \in L^p \).

Proof. 1. Let \( w = E[Y|F_i] \) then \( w \in L^q_{t,+} \). Then since \( v \in (E[Y|F_i] + M_t^+) \setminus M_t^+ \) we have \( v \in w + M_t^+ \) or equivalently \( w \in v + M_t^+ \). Additionally we have, by remark 2.2.4, that \( v \in M_t^+ \), therefore \( w \in M_t^+ + M_t^+ \). And because \( v \notin M_t^+ \) we have \( w \notin M_t^+ \), therefore \( w \in M_t^+ + M_t^+ \). It can easily be seen that the set \( M_t^+ + M_t^+ \) is equal to \( M_t^+ \).

Let \( \xi_{r,s}(\omega) : \begin{cases} E[Y_i|F_r](\omega) & \text{if } E[Y_i|F_r](\omega) > 0 \\ 1 & \text{else} \end{cases} \) for all times \( r < s \) and every state \( \omega \in \Omega \). Then we can define a vector probability measure \( Q \in \mathcal{M} \) such that \( \frac{dQ}{dP} : = \xi_{t,T}^i \). Therefore, since \( Y_i = w_i \xi_{t,T}^i \) for every \( i = 1, \ldots, d \), it trivially follows that \( Y = w_t^T(Q,w) \in L^q_+ \). Further, \( Q = \mathbb{P}|_{F_i} \) since \( E[\frac{dQ}{dP}|F_i] = 1 \) almost surely for every \( i = 1, \ldots, d \). From the above, we can conclude that \( \tilde{F}_{(Y,v)} = F_t^l(Q,w) \) by

\[
E[Y^T X] = E[w_t^T(Q,w)^T X] = E[wT E^Q[X|F_i]]
\]

for \( X \in L^p \) and \( E[vT u] = E[wT u] \) for \( u \in M_t \) since \( w \in v + M_t^+ \).
2. Let $Y = w^T_t(Q, w) \in \mathcal{L}_+^q$ then trivially we have

$$\mathbb{E}[Y | \mathcal{F}_t] = \mathbb{E}[w \cdot \xi_{t,T}(Q) | \mathcal{F}_t] = w$$

and $\mathbb{E}[Y^T X] = \mathbb{E}[w^T_t(Q, w)^T X]$ for $X \in \mathcal{L}^p$. From the assumption and $M_{t,+}^+ \setminus M_t^+ = M_{t,+}^+ \setminus M_t^+ + M_t^+$ it holds $w \in M_{t,+}^+ + M_t^+$. Thus, $w = w_{M_{t,+}^+} + w_{M_t^+}$ where $w_{M_{t,+}^+} \in M_{t,+}^+$ and $w_{M_t^+} \in M_t^+$. In particular, $w_{M_{t,+}^+} = w - w_{M_t^+} \in \mathbb{E}[Y | \mathcal{F}_t] + M_t^+ \subseteq \mathcal{L}_+^q$. Set $v = w_{M_{t,+}^+}$. Furthermore, $w \notin M_t^+$ implies $v \notin M_t^+$. Thus $v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^+) \setminus M_t^+$. We have $\mathbb{E}[w^T u] = \mathbb{E}[v^T u]$ for every $u \in M_t$ since $w \in v + M_t^+$.

\[ \Box \]

### 2.2.2 Convex and coherent risk measures

Utilizing set-valued duality, proposed in [47], and the transformation of the dual variables as described by lemma 2.2.6, we can now give the dual representation for set-valued closed convex and coherent dynamic risk measures, as was done in [34, 36]. As we demonstrated in section 2.2.1, the set of dual variables can be defined by

$$\mathcal{W}_t = \{(Q, w) \in \mathcal{M} \times (M_{t,+}^+ \setminus M_t^+) : w^T_t(Q, w) \in \mathcal{L}_+^q\}.$$ 

In this way we have two dual variables, the first is a (vector) probability measure and the second contains the ordering of the eligible portfolios. The additional coupling condition of a pair of dual variables $(Q, w)$ guarantees the $\mathbb{Q}$-conditional expectation of a ($\mathbb{P}$-a.s.) greater portfolio is dominant in the ordering defined by $w$ as well.

**Definition 2.2.7.** A function $-\gamma_t : \mathcal{W}_t \to \mathcal{G}(M_t; M_{t,+})$ is a penalty function at time $t$ if it satisfies

1. $\cap_{(Q, w) \in \mathcal{W}_t} -\gamma_t(Q, w) \neq \emptyset$ and $-\gamma_t(Q, w) \neq M_t$ for at least one $(Q, w) \in \mathcal{W}_t$, and
2. \(-\gamma_t(Q, w) = \text{cl}(\gamma_t(Q, w) + G_t(w)) \cap M_t\) for all \((Q, w) \in W_t\).

This result is an extension of the scalar dual representation given in [28, 76]. Then, the duality theorem 4.2 from [50] extends to the dynamic case in the following way.

**Theorem 2.2.8.** A function \(R_t: L^p \to G(M^t; M^t, +)\) is a closed convex conditional risk measure if and only if

\[
R_t(X) = \bigcap_{(Q, w) \in W_t} \left[ -\beta_t(Q, w) + (E^Q[-X|\mathcal{F}_t] + G_t(w)) \cap M_t \right] \quad (2.2.2)
\]

for every \(X \in L^p\), where \(-\beta_t\) is the minimal penalty function defined by

\[
-\beta_t(Q, w) = \text{cl} \bigcup_{Z \in A_t} \left( E^Q[Z|\mathcal{F}_t] + G_t(w) \right) \cap M_t. \quad (2.2.3)
\]

**Proof.** This proof follows from theorem 2 in [47]. For our proof we follow the same logic as theorem 4.2 in [50], the proof of which is given in section 6.3 of that same paper.

First note that \(R_t\) is a proper closed convex set-valued function by finiteness at 0. In particular, \(R_t(X) \neq M_t\) for every \(X \in L^p\) by proposition 5 in [47].

The set-valued Fenchel convex conjugate (as given in [47]) for the risk measure \(R_t\) is given by

\[-R_t^*(Y, v) := \text{cl} \bigcup_{X \in L^p} \left( R_t(X) + \tilde{F}^t_{(Y, v)}[-X] \right)\]

for any \(Y \in L^q\) and \(v \in L^q_t\). Note that the image of \(-R_t^*\) is of the form \(u + G_t(v)\) for some \(u \in M_t\) or an element of \(\{0, M_t\}\). The set-valued biconjugate is then given by

\[R_t^{**}(X) := \bigcap_{Y \in L^q, v \in M^t, + \setminus \{0\}} \left( -R_t^*(Y, v) + \tilde{F}^t_{(Y, v)}[X] \right)\]

for any \(X \in L^p\). By the set-valued Fenchel-Moreau theorem, as given in theorem 2 of [47], we have that \(R_t = R_t^{**}\). Since \(\tilde{F}^t_{(Y, v)}[-X] = \tilde{F}^t_{(-Y, v)}[X]\), we can define \(-\tilde{R}_t^*\).
and use it to define the biconjugate by

\[-\hat{R}_t^*(Y, v) := \text{cl} \bigcup_{X \in L^p} \left( R_t(X) + \tilde{F}_{(Y,v)}^t[X] \right) = -R_t^*(-Y, v)\]

\[R_t^{**}(X) = \bigcap_{Y \in L^q, v \in M_t^\perp, \{0\}} \left( -\hat{R}_t^*(Y, v) + \tilde{F}_{(Y,v)}^t[-X] \right).\]

We now show that

\[-\hat{R}_t^*(Y, v) = \begin{cases} 
\text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}^t[Z] & \text{if } Y \in L^q_+, v \in (\mathbb{E}[Y|\mathcal{F}_t] + M_t^\perp) \setminus M_t^\perp \\
M_t & \text{else}
\end{cases}.
\]

It is immediately clear that

\[-\hat{R}_t^*(Y, v) \supseteq \text{cl} \bigcup_{Z \in A_t} \left( R_t(Z) + \tilde{F}_{(Y,v)}^t[Z] \right) \supseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}^t[Z]
\]

for any \( Y \in L^q \) and \( v \in L^q_t \).

If \( Y \notin L^q_+ \) then there exists some \( X_0 \in L^p_+ \) such that \( \mathbb{E}[Y^TX_0] < 0 \). By the definition of the acceptance set, there exists some \( u_0 \in A_t \cap M_t \), and thus by monotonicity it follows that \( u_0 + L^p_+ \subseteq A_t \). Therefore, we can immediately conclude that

\( \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}^t[Z] \supseteq \text{cl} \bigcup_{Z \in u_0 + L^p_+} \tilde{F}_{(Y,v)}^t[Z] \supseteq \text{cl} \bigcup_{\lambda > 0} \tilde{F}_{(Y,v)}^t[u_0 + \lambda X_0] = M_t \), and thus

\[-\hat{R}_t^*(Y, v) = M_t \text{ whenever } Y \notin L^q_+.
\]

If \( v \in M_t^\perp \) then it immediately follows that \( \tilde{F}_{(Y,v)}^t[X] = \begin{cases} M_t & \text{if } \mathbb{E}[Y^TX] \leq 0 \\
\emptyset & \text{else}
\end{cases}.
\]

This implies that \( \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}^t[Z] \in \{\emptyset, M_t\} \), and the same holds for \(-\hat{R}_t^*(Y, v)\). Since \( R_t(0) \neq \emptyset \), it follows that \(-\hat{R}_t^*(Y, v) \neq \emptyset \), and thus must be equal to \( M_t \).
If \( v \neq \mathbb{E} [Y|\mathcal{F}_t] + M_t^1 \) then for any \( u_0 \in M_t \)

\[
\tilde{F}_{(Y,v)}[u_0] = \left\{ u \in M_t : 0 \leq \mathbb{E} \left[ v^T u + \mathbb{E} [-Y|\mathcal{F}_t]^T u_0 \right] \right\}
\]

\[
= \left\{ u - u_0 : u \in M_t, 0 \leq \mathbb{E} [v^T (u - u_0) + (v + \mathbb{E} [-Y|\mathcal{F}_t])^T u_0] \right\} + u_0
\]

\[
= \left\{ u \in M_t : 0 \leq \mathbb{E} [v^T u + (v + \mathbb{E} [-Y|\mathcal{F}_t])^T u_0] \right\} + u_0.
\]

Since \( v + \mathbb{E} [-Y|\mathcal{F}_t] \not\in M_t^1 \), for any \( u \in M_t \) there exists some \( u_0 \in M_t \) such that

\[
0 \leq \mathbb{E} [v^T u + (v + \mathbb{E} [-Y|\mathcal{F}_t])^T u_0].
\]

Therefore, it holds that \( \text{cl} \bigcup_{u_0 \in M_t} \left( \tilde{F}_{(Y,v)}[u_0] - u_0 \right) = M_t \). By finiteness at 0, in particular \( R_t(0) \neq \emptyset \), it follows that

\[
-\hat{R}_t^*(Y, v) \supseteq \text{cl} \bigcup_{u_0 \in M_t} \left( R_t(u_0) + \tilde{F}_{(Y,v)}[u_0] \right) = \text{cl} \bigcup_{u_0 \in M_t} \left( R_t(0) + \tilde{F}_{(Y,v)}[u_0] - u_0 \right) = M_t.
\]

It remains to show that \( -\hat{R}_t^*(Y, v) \subseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}[Z] \) for any \( Y \in \mathcal{L}_+^q \) and \( v \in (\mathbb{E} [Y|\mathcal{F}_t] + M_t^1) \setminus M_t^1 \). Let \( u \in R_t(X) \), then by proposition 2.2.2

\[
\text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}[Z] \supseteq \tilde{F}_{(Y,v)}[X + u] = \tilde{F}_{(Y,v)}[X] + u.
\]

Therefore \( R_t(X) + \tilde{F}_{(Y,v)}[X] \subseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}[Z] \) for every \( X \in \mathcal{L}_+^p \), and hence \( -\hat{R}_t^*(Y, v) \subseteq \text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}[Z] \).

By the above, the biconjugate for risk measures need only be defined over the set of dual variables \( \{(Y, v) \in \mathcal{L}_+^q \times \mathcal{L}_+^l : v \in (\mathbb{E} [Y|\mathcal{F}_t] + M_t^1) \setminus M_t^1 \} \). By lemma 2.2.6 and the definition of the penalty function \( -\beta_t(\mathbb{Q}, w) \), the biconjugate \( R_t^{**} \) is equivalent to
the dual representation

$$R_t^*(X) = \bigcap_{(Q,w) \in W_t} \left[ -\beta_t(Q,w) + \left( \mathbb{E}^Q[-X|\mathcal{F}_t] + G_t(w) \right) \cap M_t \right].$$

Finally it can immediately be shown that $-\beta_t$ is a penalty function. $\square$

**Corollary 2.2.9.** A function $R_t : \mathcal{L}^p \to \mathcal{G}(M_t;M_{t,+})$ is a closed coherent conditional risk measure if and only if

$$R_t(X) = \bigcap_{(Q,w) \in W_t^{\text{max}}} \left( \mathbb{E}^Q[-X|\mathcal{F}_t] + G_t(w) \right) \cap M_t$$

for every $X \in \mathcal{L}^p$, where $W_t^{\text{max}}$ is the maximal set of dual variables defined by

$$W_t^{\text{max}} = \left\{ (Q,w) \in \mathcal{W}_t : w^T_t(Q,w) \in A_t^+ \right\}$$

**Proof.** This corollary follows from an adaption of equation (6.4) in proposition 6.7 in [50] to the dynamic case.

Using the logic of the proof of theorem 2.2.8 above, we will show that

$$\text{cl} \bigcup_{Z \in A_t} \tilde{F}_{(Y,v)}^t[Z] = \begin{cases} G_t(v) \cap M_t & \text{if } Y \in A_t^+ \\ M_t & \text{else} \end{cases}.$$ 

Then using the results of lemma 2.2.6 it follows that the minimal penalty function

$$-\beta_t(Q,w) = \begin{cases} G_t(w) \cap M_t & \text{if } w^T_t(Q,w) \in A_t^+ \\ M_t & \text{else} \end{cases}$$

and thus $W_t^{\text{max}} = \{(Q,w) \in \mathcal{W}_t : -\beta_t(Q,w) \neq M_t\}$.

If $Y \in A_t^+$ then $\mathbb{E}[Y^TZ] \geq 0$, and thus $\tilde{F}_{(Y,v)}^t[Z] \subseteq \tilde{F}_{(Y,v)}^t[0] = G_t(v) \cap M_t$, for every $Z \in A_t$. Therefore it trivially follows that $\text{cl} \bigcup_{Z \notin A_t} \tilde{F}_{(Y,v)}^t[Z] = G_t(v) \cap M_t.$
If $Y \not\in A_t^+$ then there exists some $\hat{Z} \in A_t$ such that $\mathbb{E}[Y^T\hat{Z}] < 0$. In particular, by positive homogeneity, for any $\lambda > 0$ it follows that $\lambda \hat{Z} \in A_t$, and thus $\text{cl} \bigcup_{Z \in A_t} F_{(Y,v)}[\lambda Z] \supseteq \text{cl} \bigcup_{\lambda > 0} F_{(Y,v)}[\lambda \hat{Z}] = M_t$. 

If a dynamic risk measure is additionally $K$-compatible with respect to some cone $K \subseteq \mathcal{L}^p$, then the dual representation can be given with respect to $W_{t,K} \subseteq W_t$. We define the set of $K$-compatible dual variables by

$$W_{t,K} = \{ (Q, w) \in W_t : w^T(Q, w) \in K^+ \} \subseteq W_t.$$ 

In fact, the dual representation for a market-compatible risk measure requires a simple switch in the monotonicity cone from $\mathcal{L}^p_+$ to $K$. Similarly, if $R_t$ is additionally normalized then it is possible to consider only those dual variables such that $w \in R_t(0)^+$ by switching the ordering cone from $M_{t,+}$ to $R_t(0)$ (with corresponding image space $G(M_t, R_t(0))$).

### 2.2.3 Conditionally convex and coherent risk measures

Let us turn to the special case of conditional convexity and coherence, which is the usual property imposed on dynamic risk measures in the scalar case. In this section we present two representations for conditionally convex and coherent risk measures. The first, proposed in [34] and provided in corollary 2.2.10, utilizes the dual representation given in theorem 2.2.8 (corollary 2.2.9) by assuming additional properties on the penalty functions (the maximal set of dual variables). The second, proposed in [36] and provided in corollary 2.2.12, uses conditional “half-spaces” (instead of the half-spaces $G_t$) to define a new dual representation.
Consider a penalty function $-\gamma_t : \mathcal{W}_t \rightarrow \mathcal{G}(M_t; M_{t,+})$ at time $t$ such that for any $(Q, w) \in \mathcal{W}_t$ and any $\lambda \in L^\infty_t((0, 1))$ it holds

$$-\gamma_t(Q, w) \supseteq \lambda(-\gamma_t(Q, \lambda w)) + (1 - \lambda)(-\gamma_t(Q, (1 - \lambda)w)).$$

(2.2.6)

**Corollary 2.2.10.** A function $R_t : L^p \rightarrow \mathcal{G}(M_t; M_{t,+})$ is a closed conditionally convex conditional risk measure if and only if the minimal penalty function $-\beta_t$ at time $t$, defined in equation (2.2.3), satisfies (2.2.6).

Further, a function $R_t : L^p \rightarrow \mathcal{G}(M_t; M_{t,+})$ is a closed conditionally coherent conditional risk measure if and only if the maximal dual set $\mathcal{W}_t^{\text{max}}$, defined in equation (2.2.5), is conditionally conical in the second variable, i.e. for any $(Q, w) \in \mathcal{W}_t^{\text{max}}$ and $\lambda \in L^\infty_{t,+}$ then $(Q, \lambda w) \in \mathcal{W}_t^{\text{max}}$.

**Proof.** Using theorem 2.2.8, only two things remain to show: First, if $-\beta_t$ is a penalty function satisfying (2.2.6), then the risk measure defined by (2.2.2) is conditionally convex. Second, the minimal penalty function $-\beta_t$ of a conditionally convex risk measure satisfies (2.2.6).

First, if $-\beta_t$ is a penalty function satisfying (2.2.6), then for any $\lambda \in L^\infty_t((0, 1))$

$$R_t(\lambda X + (1 - \lambda)Y)$$

$$\subseteq \bigcap_{(Q, w) \in \mathcal{W}_t} [-\beta_t(Q, w) + (\mathbb{E}^Q[-(\lambda X + (1 - \lambda)Y)|\mathcal{F}_t] + G_t(w)) \cap M_t]$$

$$\subseteq \bigcap_{(Q, w) \in \mathcal{W}_t} [-\beta_t(Q, w) + (\lambda \mathbb{E}^Q[-X|\mathcal{F}_t] + (1 - \lambda)\mathbb{E}^Q[-Y|\mathcal{F}_t] + G_t(w)) \cap M_t]$$

$$\subseteq \bigcap_{(Q, w) \in \mathcal{W}_t} [-\beta_t(Q, w) + (\lambda \mathbb{E}^Q[-X|\mathcal{F}_t] + G_t(w)) \cap M_t$$

$$+ ((1 - \lambda)\mathbb{E}^Q[-Y|\mathcal{F}_t] + G_t(w)) \cap M_t]$$
= \bigcap_{(Q,w) \in W_t} \left[ -\beta_t(Q,w) + \lambda \left( \mathbb{E}^Q [-X | \mathcal{F}_t] + G_t(\lambda w) \right) \cap M_t \right]
+ (1 - \lambda) \left( \mathbb{E}^Q [-Y | \mathcal{F}_t] + G_t((1 - \lambda)w) \right) \cap M_t
\supseteq \bigcap_{(Q,w) \in W_t} \left[ \lambda(-\beta_t(Q,\lambda w)) + (1 - \lambda)(-\beta_t(Q,(1 - \lambda)w)) \right]
+ \lambda \left( \mathbb{E}^Q [-X | \mathcal{F}_t] + G_t(\lambda w) \right) \cap M_t
+ (1 - \lambda) \left( \mathbb{E}^Q [-Y | \mathcal{F}_t] + G_t((1 - \lambda)w) \right) \cap M_t
\supseteq \bigcap_{(Q,w) \in W_t} \left[ \lambda(-\beta_t(Q,\lambda w)) + \lambda \left( \mathbb{E}^Q [-X | \mathcal{F}_t] + G_t(\lambda w) \right) \cap M_t \right]
+ (1 - \lambda) \left( -\beta_t(Q,(1 - \lambda)w) \right)
+ (1 - \lambda) \left( \mathbb{E}^Q [-Y | \mathcal{F}_t] + G_t((1 - \lambda)w) \right) \cap M_t
= \lambda \bigcap_{(Q,w) \in W_t} \left[ -\beta_t(Q,\lambda w) + \mathbb{E}^Q [-X | \mathcal{F}_t] + G_t(\lambda w) \right) \cap M_t \right]
+ (1 - \lambda) \bigcap_{(Q,w) \in W_t} \left[ -\beta_t(Q,(1 - \lambda)w)) + \mathbb{E}^Q [-Y | \mathcal{F}_t] + G_t((1 - \lambda)w) \right) \cap M_t \right]
\supseteq \lambda R_t(X) + (1 - \lambda) R_t(Y).

The last line above follows since if \((Q,w) \in W_t\) then it follows that \((Q,\lambda w) \in W_t\).

The conditional convexity of \(R_t\) can be extended to any \(\lambda \in L_t^\infty([0,1])\) by taking a sequence \((\lambda_n)_{n \in \mathbb{N}} \subseteq L_t^\infty((0,1))\) which converges almost surely to \(\lambda\). Then by dominated convergence \(\lambda_n X\) converges to \(\lambda X\) in the norm topology (\(\sigma(L^\infty, L^1)\) topology if \(p = +\infty\)). Therefore, for any \(X,Y \in L^p\)

\[
R_t(\lambda X + (1 - \lambda)Y) = R_t(\lim_{n \to \infty} (\lambda_n X + (1 - \lambda_n)Y))
\supseteq \liminf_{n \to \infty} R_t(\lambda_n X + (1 - \lambda_n)Y)
\supseteq \liminf_{n \to \infty} [\lambda_n R_t(X) + (1 - \lambda_n)R_t(Y)]
\supseteq \lambda R_t(X) + (1 - \lambda) R_t(Y)
\]
by $R_t$ closed (see proposition 2.34 in [66]) and conditionally convex on the interval $0 < \lambda_n < 1$. Note that we use the convention from [66] that the limit inferior of a net of sets $(B_i)_{i \in I}$ is given by $\liminf_{i \in I} B_i = \bigcap_{i \in I} \text{cl} \bigcup_{j \geq i} B_j$.

Conversely, let $R_t$ be a conditionally convex risk measure, then its acceptance set $A_t$ is conditionally convex as well. Therefore for any pair of dual variables $(Q, w) \in W_t$ and any $\lambda \in L^\infty((0, 1))$

\[-\beta_t(Q, w) = \text{cl} \bigcup_{Z \in A_t} (E^Q[Z|\mathcal{F}_t] + G_t(w)) \cap M_t\]

\[\sup_{Z \in \lambda A_t + (1-\lambda)A_t} (E^Q[Z|\mathcal{F}_t] + G_t(w)) \cap M_t\]

\[= \text{cl} \bigcup_{Z_1, Z_2 \in A_t} (E^Q[Z_1 + (1-\lambda)Z_2|\mathcal{F}_t] + G_t(w)) \cap M_t\]

\[\sup_{Z \in A_t} \lambda (E^Q[Z|\mathcal{F}_t] + G_t(\lambda w)) \cap M_t\]

\[+ \text{cl} \bigcup_{Z \in A_t} (1 - \lambda) (E^Q[Z|\mathcal{F}_t] + G_t((1-\lambda)w)) \cap M_t\]

\[= \lambda(-\beta_t(Q, \lambda w)) + (1 - \lambda)(-\beta_t(Q, (1-\lambda)w)).\]

The proof for the conditionally coherent case follows analogously. \qed

**Example 2.2.11.** (Example 2.1.13 continued) As the worst cost risk measure is a closed conditionally coherent risk measure with acceptance set $A_t = L^p_+$ then the maximal set of dual variables is given by

$$W^\text{WC}_t := \{(Q, w) \in W_t : w^T_t(Q, w) \in L^q_+ \} = W_t$$

which trivially is conditionally conical in the 2nd variable.

The next corollary sharpens the above duality results for the conditionally convex case by using the conditional "half-space" in $L^p_t$ with normal direction $w \in L^p_t \setminus \{0\}$
denoted by
\[ \Gamma_t(w) := \{ u \in \mathcal{L}_t^p : 0 \leq w^T u \ \text{P-a.s.} \} . \]

This provides a stronger dual representation result in the spirit of decomposability, see remark 2.2.13 below. Note that the set of dual variables \( \mathcal{W}_t \) stays the same in the conditional framework since \( M_{t,+} = \{ v \in \mathcal{L}_t^q : v^T u \geq 0 \ \text{P-a.s.} \ \forall u \in M_{t,+} \} \) and \( M_t^\perp = \{ v \in \mathcal{L}_t^q : v^T u = 0 \ \text{P-a.s.} \ \forall u \in M_t \} \).

**Corollary 2.2.12.** A function \( R_t : \mathcal{L}^p \to \mathcal{G}(M_t; M_{t,+}) \) is a closed conditionally convex risk measure if and only if
\[
R_t(X) = \bigcap_{(Q,w) \in \mathcal{W}_t} \left[ -\alpha_t(Q, w) + (\mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w)) \cap M_t \right], 
\tag{2.2.7}
\]
where \(-\alpha_t\) is the minimal conditional penalty function given by
\[
-\alpha_t(Q, w) = \text{cl} \bigcup_{Z \in A_t} \left( \mathbb{E}^Q[Z|\mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t. 
\tag{2.2.8}
\]

\( R_t \) is additionally conditionally coherent if and only if
\[
R_t(X) = \bigcap_{(Q,w) \in \mathcal{W}_t^\text{max}} \left( \mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t. 
\tag{2.2.9}
\]

**Proof.** First, we can reformulate the conditional penalty function as
\[
-\alpha_t(Q, w) = \left\{ u \in M_t : \text{ess inf}_{Z \in A_t} w^T \mathbb{E}^Q[Z|\mathcal{F}_t] \leq w^T u \right\} .
\]

So, for any \((Q, w) \in \mathcal{W}_t\)
\[
-\alpha_t(Q, w) + (\mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w)) \cap M_t
\]
\[
= \left\{ u \in M_t : \text{ess inf}_{Z \in A_t} w^T \mathbb{E}^Q[Z - X|\mathcal{F}_t] \leq w^T u \right\},
\]

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using $w \notin M_t^\perp$. We will prove that a conditionally convex risk measure has the representation above by showing that

$$\tilde{R}_t(X) := \bigcap_{(Q, w) \in W_t} [-\alpha_t(Q, w) + (E^Q[-X | F_t] + \Gamma_t(w)) \cap M_t] = R_t(X)$$

where $R_t(X)$ is given by the dual representation (2.2.2) in theorem 2.2.8.

1. Let $u \in \tilde{R}_t(X)$, i.e. $w^T u \geq \text{ess inf}_{Z \in A_t} w^T E^Q[Z - X | F_t]$ for any $(Q, w) \in W_t$. It immediately follows that $E[w^T u] \geq E[\text{ess inf}_{Z \in A_t} w^T E^Q[Z - X | F_t]]$ for any $(Q, w) \in W_t$. By $\mathcal{F}_t$-decomposability of the acceptance set $A_t$ (which follows from conditional convexity), i.e. $1_D A_t + 1_{D^c} A_t \subseteq A_t$ for any $D \in \mathcal{F}_t$, we can apply [81, theorem 1] to interchange the expectation and the essential infimum. Thus, by the representation in theorem 2.2.8, $u \in R_t(X)$.

2. Now let $u \in R_t(X)$. Assume $u \notin \tilde{R}_t(X)$, i.e. there exists some $(Q, w) \in W_t$ such that

$$P(w^T u < \text{ess inf}_{Z \in A_t} w^T E^Q[Z - X | F_t]) > 0.$$ 

Define $D := \{w^T u < \text{ess inf}_{Z \in A_t} w^T E^Q[Z - X | F_t]\} \in \mathcal{F}_t$, then one has

$$1_D w^T u < \text{ess inf}_{Z \in A_t} 1_D w^T E^Q[Z - X | F_t]$$

on $D$, and the same holds for the expectation. As above, by conditional convexity we can interchange the expectation and the infimum, thus we recover that

$$E[1_D w^T u] < \inf_{Z \in A_t} E[1_D w^T E^Q[Z - X | F_t]].$$

From the equality $M_t^\perp = \{v \in L^q_t : v^T u = 0 \text{ P-a.s. } \forall u \in M_t\}$ it follows $(Q, 1_D w) \in W_t$, which is a contradiction to $u \in R_t(X)$. 

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It remains to show that \( R_t \) defined by (2.2.7) is conditionally convex. Let \( X, Y \in \mathcal{L}^p \) and let \( \lambda \in L_t^\infty([0,1]) \),

\[
R_t(\lambda X + (1 - \lambda)Y) = \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ u \in M_t : \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - (\lambda X + (1 - \lambda)Y) | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
= \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ u \in M_t : \essinf_{Z \in \mathcal{A}_t} w^T E^Q [\lambda(Z_X - X) + (1 - \lambda)(Z_Y - Y) | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
= \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ u \in M_t : \lambda \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - X | \mathcal{F}_t] + (1 - \lambda) \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - Y | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
\supseteq \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ \lambda u : u \in M_t, \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - X | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
+ \left\{ (1 - \lambda) u : u \in M_t, \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - Y | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
\supseteq \lambda \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ u \in M_t : \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - X | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
+ (1 - \lambda) \bigcap_{(Q,w) \in \mathcal{W}_t} \left\{ u \in M_t : \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z - Y | \mathcal{F}_t] \leq w^T u \right\}
\]

\[
= \lambda R_t(X) + (1 - \lambda) R_t(Y).
\]

For the conditionally coherent case, we first note that \(-\alpha_t(Q, w) \supseteq \Gamma_t(w) \cap M_t\) for every \((Q, w) \in \mathcal{W}_t\) since \(0 \in A_t\) by positive homogeneity. In fact,

\[
-\alpha_t(Q, w) = \begin{cases} G_0(w(\omega)) \cap M & \omega \in D := \{ \essinf_{Z \in \mathcal{A}_t} w^T E^Q [Z | \mathcal{F}_t] \geq 0 \} \\ M & \omega \in D^c \end{cases}
\]

\[
= 1_D \Gamma_t(w) \cap M_t + 1_{D^c} M_t.
\]
Define $-\hat{\alpha}_t$ by

$$
-\hat{\alpha}_t(Q, w) = \begin{cases} 
\Gamma_t(w) \cap M_t & \text{if } \text{ess inf}_{Z \in A_t} w^T \mathbb{E}^Q[Z | F_t] = 0 \ \mathbb{P}\text{-a.s.} \\
M_t & \text{else}
\end{cases}.
$$

By construction we have $-\alpha_t \subseteq -\hat{\alpha}_t$, and thus

$$
R_t(X) \subseteq \bigcap_{(Q, w) \in W_t^{\max}} \left( \mathbb{E}^Q[-X | F_t] + \Gamma_t(w) \right) \cap M_t.
$$

Conversely, by theorem 2.2.9, $u \in R_t(X)$ if and only if $u \in M_t$ and

$$
0 \leq \inf_{(Q, w) \in W_t^{\max}} \mathbb{E} \left[ w^T \mathbb{E}^Q [X + u | F_t] \right].
$$

But $u \in \bigcap_{(Q, w) \in W_t^{\max}} (\mathbb{E}^Q[-X | F_t] + \Gamma_t(w)) \cap M_t$ implies $u \in M_t$ as well as $0 \leq \inf_{(Q, w) \in W_t^{\max}} \mathbb{E} [w^T \mathbb{E}^Q[X + u | F_t]]$, i.e. $u \in R_t(X)$.

It remains to show that $R_t$ defined by (2.2.9) is conditionally coherent, but as this follows similarly to the conditionally convex case above we will omit it.

Remark 2.2.13. A risk measure is closed and conditionally convex if and only if it is closed, convex, and decomposable.

## 2.3 Stepped risk measures

In this section, we consider the stepped risk measures, and in particular the dual representation of closed (conditionally) convex and coherent stepped risk measures $R_{t,s} : M_s \to \mathcal{P}(M_t; M_{t,+})$. This is used in sections 3.3 and 3.4 as the stepped penalty functions and stepped sets of dual variables play a role when discussing equivalent characterizations of multi-portfolio time consistency. For the dual representation we
use set-valued duality defined in [47] analogously as for conditional risk measures in section 2.2. The results given here are reproduced from appendix C of [36].

Given a risk measure $R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+})$, a stepped risk measure is the restriction of $R_t$ to $M_s$, i.e. $R_{t,s} = R_t|_{M_s}$. It can be seen that

$$R_{t,s}(X) := \{ u \in M_t : X + u \in A_{t,s} \}$$

for $X \in M_s$ where $A_{t,s}$ is the $(s-t)$-stepped acceptance set defined by $A_{t,s} = A_t \cap M_s$.

Therefore, if $R_t$ is closed convex (coherent) then $R_{t,s}$ is closed convex (coherent). Furthermore, if $R_t$ is $\mathcal{L}^p$-monotone, then $R_{t,s}$ is $M_{s,+}$-monotone.

For the remainder of this section we assume that $p \in [1, +\infty]$ with $q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.3.1. Let $R_t$ be a closed convex risk measure. The set of dual variables for $R_{t,s} : M_s \to \mathcal{P}(M_t; M_{t,+})$ with $t \leq s$ is given by

$$\mathcal{W}_{t,s} = \{ (Q, w) \in \mathcal{M} \times (M_{t,+} \setminus M_t^+) : w_t^s(Q, w) \in M_{s,+}^+, Q = \mathbb{P}|_{\mathcal{F}_t} \}$$

Proof. By the logic of proposition 2.2.2, the set of (classical) stepped dual variables are given by $\{(Y, v) : Y \in M_{s,+}^+, v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^+) \setminus M_t^+)\}$. It remains to show that for any dual pair $(Y, v)$ there exists a $(Q, w) \in \mathcal{W}_{t,s}$ such that $\tilde{F}_{(Y, v)}^t[X] = F_{(Q, w)}^t[X]$ for any $X \in M_s$, and vice versa, where $\tilde{F}_{(Y, v)}^t[X] := \{ u \in M_t : \mathbb{E}[X^T Y] \leq \mathbb{E}[v^T u] \}$.

1. Let $(Q, w) \in \mathcal{W}_{t,s}$. Then, we will show that there exists a dual pair

$$((Y, v) : Y \in M_{s,+}^+, v \in (\mathbb{E}[Y | \mathcal{F}_t] + M_t^+) \setminus M_t^+)$$

such that $\tilde{F}_{(Y, v)}^t[X] = F_{(Q, w)}^t[X]$ for any $X \in M_s$. Let $Y = w_t^s(Q, w) \in M_{s,+}^+$ (by remark 2.3.2 and lemma A.2.1(1)), therefore we have $\mathbb{E}[X^T Y] = \mathbb{E}[w_t^s(Q, w)^T X] = \mathbb{E}[w^T \mathbb{E}[X | \mathcal{F}_t]]$ and $\mathbb{E}[Y | \mathcal{F}_t] = w$. From $w \in M_{t,+}^+ \setminus M_t^+$ we
can rewrite \( w = w_{M_1^+} + w_{M_2^+} \). Thus \( v = w_{M_2^+} = w - w_{M_1^+} \in \mathbb{E}[Y \mid \mathcal{F}_t] + M_1^+ \).

Finally, \( w \not\in M_1^+ \) implies \( v \not\in M_1^+ \), and \( \mathbb{E}[w^Tu] = \mathbb{E}[v^Tu] \) for every \( u \in M_1 \) since \( w \in v + M_1^+ \).

2. Let \( Y \in M_{s,+}^+ \) and \( v \in (\mathbb{E}[Y \mid \mathcal{F}_t] + M_1^+) \setminus M_1^+ \). We want to show there exists some pair \((Q_i, w) \in W_{t,s}\) such that \( \tilde{F}_{(Y,v)}^t[X] = F_{(Q_i,w)}^t[X] \) for any \( X \in M_s \). Let \( w \in \mathbb{E}[(Y + M_1^+) \cap \mathcal{L}_{s,+}^q \mid \mathcal{F}_t] \) (which is nonempty), i.e., \( w = \mathbb{E}[Y + m^+ \mid \mathcal{F}_t] \) for some \( m^+ \in M_s^+ \) and \( Y + m^+ \in \mathcal{L}_{s,+}^q \). Then it can easily be seen that \( w \in v + M_1^+ \) for \( v \in (\mathbb{E}[Y \mid \mathcal{F}_t] + M_1^+) \setminus M_1^+ \subseteq M_{s,+}^+ \). Thus \( w \in M_{s,+}^+ + M_1^+ \) and with \( v \not\in M_1^+ \) this implies \( w \in M_{s,+}^+ \setminus M_1^+ \). From \( w \in v + M_1^+ \) it follows that \( \mathbb{E}[w^Tu] = \mathbb{E}[v^Tu] \) for every \( u \in M_1 \).

Additionally, choose \( Q \in \mathcal{M} \) such that \( d_{Q_i}^\xi = \xi_{0,s}(Q_i) \) where

\[
\xi_{r,s}(Q_i)[\omega] = \begin{cases} 
\frac{\mathbb{E}[Y_i + m_i^+ \mid \mathcal{F}_r](\omega)}{\mathbb{E}[Y_i + m_i^+ \mid \mathcal{F}_r](\omega)} & \text{if } \mathbb{E}\left[Y_i + m_i^+ \mid \mathcal{F}_r\right](\omega) > 0 \\
1 & \text{else}
\end{cases}
\]

for any \( 0 \leq r \leq s \) and almost every \( \omega \in \Omega \). Define \( Q^t \in \mathcal{M} \) by its density \( d_{Q^t}^\xi = \xi_{t,s}(Q_i) \). Then \( w_t^i(Q^t, w) = w_t^i(Q, w) = Y + m^+ \in M_{s,+}^+ \subseteq M_{s,+}^+ \). Therefore, \( \mathbb{E}[w^T \mathbb{E}[X \mid \mathcal{F}_t]] = \mathbb{E}[w_t^i(Q^t, w)^TX] = \mathbb{E}[Y^TX] \) for every \( X \in M_s \).

\[\square\]

**Remark 2.3.2.** For any choice of eligible portfolios \( M_t \), it trivially follows that \( W_{t,s} \supseteq W_t \) for any \( t \leq s \).

**Remark 2.3.3.** If we consider the case when \( M_t = \mathcal{L}_t^p \) for all times \( t \), then an inspection of the proof of lemma 2.2.6 shows that \( W_{t,s} = W_t \).

The lemma below gives a dual representation for closed convex stepped risk measures.
Lemma 2.3.4. The dual representation for any closed convex stepped risk measure

\( R_{t,s} : M_s \to G(M_t; M_{t,+}) \) with \( t \leq s \) is given by

\[
R_{t,s}(X) = \bigcap_{(Q, w) \in \mathcal{W}_{t,s}} \left[ \beta_{t,s}(Q, w) + (E^Q [-X | \mathcal{F}_t] + G_t(w)) \cap M_t \right]
\]

for any \( X \in M_s \) where

\[
\beta_{t,s}(Q, w) = \text{cl} \bigcup_{X \in A_{t,s}} (E^Q [X | \mathcal{F}_t] + G_t(w)) \cap M_t.
\]

(2.3.1)

Proof. This is an adaption of theorem 2.2.8 to stepped risk measures using lemma 2.3.1.

We use the above results to give a dual representation for closed coherent stepped risk measures.

Corollary 2.3.5. The dual representation for any closed coherent stepped risk measure

\( R_{t,s} : M_s \to G(M_t; M_{t,+}) \) with \( t \leq s \) is given by

\[
R_{t,s}(X) = \bigcap_{(Q, w) \in \mathcal{W}_{t,s}^{\text{max}}} (E^Q [-X | \mathcal{F}_t] + G_t(w)) \cap M_t
\]

for any \( X \in M_s \) where

\[
\mathcal{W}_{t,s}^{\text{max}} = \{(Q, w) \in \mathcal{W}_{t,s} : w^*(Q, w) \in A_{t,s}^+ \}.
\]

Proof. Note that \( \beta_{t,s}(Q, w) = \text{cl} \bigcup_{X \in A_{t,s}} (E^Q [X | \mathcal{F}_t] + G_t(w)) \cap M_t = G_t(w) \cap M_t \) if and only if for every \( X \in A_{t,s} \) we have

\[
E \left[ w^T E^Q [X | \mathcal{F}_t] \right] = E \left[ w^*_t(Q, w)^T X \right] \geq 0,
\]
i.e. \( w_t^s(Q, w) \in A^+_{t,s} \). Thus, for a \( M_{s,+} \)-monotone closed coherent stepped risk measure \( R_{t,s} \) with \( t, s \in \mathbb{T} \) and \( t \leq s \) it holds that for any \((Q, w) \in W_{t,s}\)

\[
-\beta_{t,s}(Q, w) = G_t(w) \cap M_t \iff w_t^s(Q, w) \in A^+_{t,s}.
\]

An application of lemma 2.3.1 provides the desired result. \(\square\)

Finally, we use the above duality results to extend corollary 2.2.12 to the stepped risk measures.

**Corollary 2.3.6.** The dual representation for any closed conditionally convex stepped risk measure \( R_{t,s} : M_s \to \mathcal{G}(M_t; M_{t,+}) \) with \( t \leq s \) is given by

\[
R_{t,s}(X) = \bigcap_{(Q,w) \in W_{t,s}} \left[ -\alpha_{t,s}(Q, w) + \left( \mathbb{E}^Q[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t \right]
\]

for any \( X \in M_s \) where

\[
-\alpha_{t,s}(Q, w) = \text{cl} \bigcup_{X \in A_{t,s}} \left( \mathbb{E}^Q[X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t.
\]

If \( R_{t,s} \) is additionally conditionally coherent then

\[
R_{t,s}(X) = \bigcap_{(Q,w) \in W_{t,s}^{\text{max}}} \left( \mathbb{E}^Q[-X \mid \mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t.
\]

**Proof.** This is an adaption of corollary 2.2.12 to stepped risk measures using the results of lemma 2.3.4 and corollary 2.3.5. \( \square \)
Chapter 3

Time Consistency

In this chapter we study two different time consistency properties for set-valued dynamic risk measures proposed, and discussed, in [34, 36]. One of these properties - called multi-portfolio time consistency - we study in great detail in this chapter. Most generally, time consistency properties concern how risk relates through time. A risk manager would want the risk compensating portfolios to not conflict with each other as time progresses. In the scalar framework time consistency is defined by

\[ \rho_s(X) \geq \rho_s(Y) \Rightarrow \rho_t(X) \geq \rho_t(Y) \]

for all times \( t < s \) and portfolios \( X, Y \in L^p \). This means that if it is known, a priori, that one portfolio is less risky than another in (almost) every state of the world, this relation holds backwards in time. In particular, by compensating for the risks of \( X \) then the risks of \( Y \) would also be covered.

A key result in the scalar framework is the equivalence between time consistency and a recursive representation given by \( \rho_t(X) = \rho_t(-\rho_s(X)) \) for all portfolios \( X \) and times \( t < s \). This recursive form relates to Bellman’s principle which provides an efficient method for calculation. Time consistency, the recursive formulation, and
other equivalent properties have been studied in great detail in the scalar case in papers such as [6, 73, 28, 22, 76, 15, 40, 24, 23].

3.1 Time consistency for set-valued risk measures

When generalizing to the set-valued framework we present two different properties which are analogous to time consistency in the scalar framework. The first property we present, which we call ‘time consistency’ is the intuitive generalization of scalar time consistency. The other property, which we call ‘multi-portfolio time consistency,’ is shown to be equivalent to the recursive form for set-valued risk measures. We will demonstrate how both notions are related to each other and under which conditions they coincide.

Definition 3.1.1. A dynamic risk measure \((R_t)_{t \in \mathbb{T}}\) is called time consistent if for all times \(t, s \in \mathbb{T}\) where \(t < s\) and \(X, Y \in L^p\) it holds

\[
R_s(X) \subseteq R_s(Y) \Rightarrow R_t(X) \subseteq R_t(Y).
\]

As in the scalar case described previously, time consistency implies that if it is known at some future time that every eligible portfolio which covers the risk of a claim \(X\) also would cover the claim \(Y\), then any prior time if a portfolio covers the risk of \(X\) it would also do so for \(Y\).

The recursive form for set-valued risk measures is understood pointwise, that is

\[
R_t(-R_s(X)) := \bigcup_{Z \in R_s(X)} R_t(-Z).
\]

The generalization of the recursion from the scalar framework can then be given by \(R_t(X) = R_t(-R_s(X))\). This can be seen as a set-valued Bellman’s principle, which is discussed in more detail in chapter 4.


3.2 Multi-portfolio time consistency

We introduce multi-portfolio time consistency in definition 3.2.1 below, as was done in [34]. In theorem 3.2.2 we show that multi-portfolio time consistency is equivalent to the recursive form for normalized risk measures. In example 3.2.8, we show that time consistency is a weaker property than multi-portfolio time consistency. Furthermore, we show in lemma 3.2.9 and remark 3.2.10 below that both concepts coincide in the scalar case.

Definition 3.2.1. A dynamic risk measure \((R_t)_{t \in \mathbb{T}}\) is called multi-portfolio time consistent if for all times \(t, s \in \mathbb{T}\) with \(t < s\), portfolios \(X \in \mathcal{L}^p\), and sets \(Y \subseteq \mathcal{L}^p\) the implication

\[
R_s(X) \subseteq \bigcup_{Y \in \mathcal{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathcal{Y}} R_t(Y)
\]

is satisfied.

Note that multi-portfolio time consistency is trivially equivalent to the above definition with the single portfolio \(X\) replaced by a set of portfolios. That is, \((R_t)_{t \in \mathbb{T}}\) is multi-portfolio time consistent if and only if for all times \(t, s \in \mathbb{T}\) with \(t < s\), sets of portfolios \(X, Y \subseteq \mathcal{L}^p\)

\[
\bigcup_{X \in \mathcal{X}} R_s(X) \subseteq \bigcup_{Y \in \mathcal{Y}} R_s(Y) \Rightarrow \bigcup_{X \in \mathcal{X}} R_t(X) \subseteq \bigcup_{Y \in \mathcal{Y}} R_t(Y).
\]

Further, note that in a discrete time setting, \(\mathbb{T} = \{0, 1, \ldots, T\}\), it is sufficient to consider multi-portfolio time consistency on a single time-step, i.e. \(s = t + 1\).

The intuitive reasoning for multi-portfolio time consistency is that if at some time any risk compensation portfolio for \(X\) also compensates the risk of some portfolio \(Y\) in the set \(\mathcal{Y}\), then at any prior time the same relation should hold true.

We prove now that the recursive structure is equivalent to multi-portfolio time consistency for normalized risk measures. Additionally, we demonstrate the equiva-
lence between these properties and a property on the acceptance sets. We will later show, in section 3.5, how to use these properties to construct multi-portfolio time consistent risk measures.

**Theorem 3.2.2.** For a normalized dynamic risk measure \((R_t)_{t \in \mathbb{T}}\) the following are equivalent:

1. \((R_t)_{t \in \mathbb{T}}\) is multi-portfolio time consistent;

2. for all times \(t, s \in \mathbb{T}\) with \(t < s\), \(X \in \mathcal{L}^p\), and \(Y \subseteq \mathcal{L}^p\)

\[
R_s(X) = \bigcup_{Y \in Y} R_s(Y) \Rightarrow R_t(X) = \bigcup_{Y \in Y} R_t(Y);
\]

3. \(R_t\) is recursive, that is for all times \(t, s \in \mathbb{T}\) with \(t < s\)

\[
R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) =: R_t(-R_s(X)). \tag{3.2.2}
\]

4. for all times \(t, s \in \mathbb{T}\) with \(t < s\)

\[
A_t = A_s + A_{t,s}.
\]

**Proof.** It can trivially be seen that property 1 implies property 2. By \((R_t)_{t \in \mathbb{T}}\) normalized it follows that for every \(X \in \mathcal{L}^p\) and \(t, s \in \mathbb{T}\) with \(t < s\) it holds

\[
\bigcup_{Z \in R_s(X)} R_s(-Z) = \bigcup_{Z \in R_s(X)} (R_s(0) + Z)
\]

\[
= R_s(0) + R_s(X) = R_s(X).
\]

Thus by property 2 and setting \(Y = -R_s(X)\), the recursive form defined in equation (3.2.2) is derived and thus property 3. It remains to show that 3 implies multi-portfolio time consistency as defined in definition 3.2.1, i.e. property 1. If \(X \in \mathcal{L}^p\)
and \(Y \subseteq \mathcal{L}^p\) such that \(R_s(X) \subseteq \bigcup_{Y \in Y} R_s(Y)\) and let \((R_t)_{t \in T}\) be recursive (as defined in equation (3.2.2)) then

\[
R_t(X) = \bigcup_{Z \in R_s(X)} R_t(-Z) \subseteq \bigcup_{Z \in \bigcup_{Y \in Y} R_s(Y)} R_t(-Z) = \bigcup_{Y \in Y} \bigcup_{Z \in R_s(Y)} R_t(-Z) = \bigcup_{Y \in Y} R_t(Y).
\]

Finally by lemma 3.2.6 below property 4 is equivalent to the recursive form, i.e. is equivalent to \((R_t)_{t \in T}\) being multi-portfolio time consistent.

**Remark 3.2.3.** In [11], a property called “consistent in time” for set-valued coherent risk measures (in the measurable selector approach) is defined. A coherent risk measure with closed values satisfies consistency in time if and only if for all times \(t, s \in \mathbb{T}\) with \(t < s\) the recursive form

\[
R_t(X) = \text{cl}\text{env}_{\mathcal{F}_t} \bigcup_{Z \in R_s(X)} R_t(-Z)
\]

is satisfied. In the above equation, for any \(D \subseteq \mathcal{L}^p\), let \text{env}_{\mathcal{F}_t} D\ denote the smallest \(\mathcal{F}_t\)-decomposable (see page 148 in [69] or page 260 in [60]) subset of \(\mathcal{L}^p\) containing \(D\).

For more details see section 6.1.

**Remark 3.2.4.** As it is trivially noticed using the acceptance set definition (and also is implicitly understood in the other definitions) for multi-portfolio time consistency, the choice of eligible assets would impact whether a risk measure is multi-portfolio time consistent. Therefore, it is possible that under one choice of eligible portfolios the risk measure is recursive, but under another choice of eligible portfolios that same risk measure is not recursive.

**Example 3.2.5.** (Example 2.1.13 continued) For the worst case risk measure, the acceptance sets are given by \(A_t = \mathcal{L}^p_+\) for all times \(t\), therefore \(A_t = A_{t,s} + A_s\) for all times and any choice of eligible assets \(M\). Thus, the worst case risk measure is recursive and multi-portfolio time consistent.
The following lemma is used in the proof of theorem 3.2.2. It is the set-valued generalization of lemma 4.3 in [40] or lemma 11.14 in [42] and gives a relationship between the sum of conditional acceptance sets and properties of the corresponding risk measure.

**Lemma 3.2.6.** Let \((R_t)_{t \in \mathbb{T}}\) be a dynamic risk measure. Let \(t, s \in \mathbb{T}\) such that \(t < s\) and let \(X \in \mathcal{L}^p\) and \(D \subseteq \mathcal{L}^p\). It holds

1. \(X \in A_s + D \cap M_s \iff -R_s(X) \cap D \neq \emptyset;\)

2. \(R_t(X) \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z) \iff A_t \subseteq A_s + A_{t,s};\)

3. \(R_t(X) \supseteq \bigcup_{Z \in R_s(X)} R_t(-Z) \iff A_t \supseteq A_s + A_{t,s};\)

**Proof.**

1. \((\Rightarrow)\) Given that \(X \in A_s + D \cap M_s\) then \(X = X_s + X^D\) such that \(X_s \in A_s\) and \(X^D \in D \cap M_s\). Therefore it can be seen that \(R_s(X) = R_s(X_s + X^D);\) and by \(X^D \in M_s\) and \(M_s\)-translativity \(-X^D \in R_s(X_s) - X^D = R_s(X)\). Then \(X^D \in -R_s(X)\) and by assumption \(X^D \in D \cap M_s\).

\((\Leftarrow)\) Given that there exists a \(Y \in -R_s(X)\) such that \(Y \in D \cap M_s\) and trivially \(X = X - Y + Y\), then \(R_s(X - Y) = R_s(X) + Y \ni 0\) since \(-Y \in R_s(X)\). Therefore \(X - Y \in A_s\) and \(X \in A_s + D \cap M_s\).

2. \((\Rightarrow)\) Let \(X \in A_t\) then \(0 \in R_t(X) \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z)\). This implies that there exists a \(Y \in -R_s(X)\) such that \(0 \in R_t(Y)\), and thus \(Y \in A_{t,s}\). Therefore \(X \in A_s + A_{t,s}\) by condition 1.

\((\Leftarrow)\) Let \(X \in \mathcal{L}^p\) and \(Y \in R_t(X)\) then \(X + Y \in A_t \subseteq A_s + A_{t,s}\). By condition 1, there exists a \(Z \in R_s(X)\) such that \(\hat{Z} \in A_{t,s}\) and \(-Z \in R_t(X)\). But then there exists a \(\hat{Z} \in A_{t,s}\) such that \(\hat{Z} \in A_{t,s}\). Since \(\hat{Z} \in A_{t,s}\) it holds \(0 \in R_t(\hat{Z}) = R_t((-Z + Y)) = R_t(-Z) - Y\). Therefore \(Y \in R_t(-Z)\). Thus for all \(Y \in R_t(X)\) there
exists a \( Z \in R_s(X) \) such that \( Y \in R_t(-Z) \). This can be rewritten as
\[
R_t(X) \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z).
\]

3. (\( \Rightarrow \)) Let \( X \in A_s + A_{t,s} \), then there exists a \( Y \in -R_s(X) \) such that \( Y \in A_{t,s} \) by condition 1. Therefore, \( R_t(X) \supseteq \bigcup_{Z \in R_s(X)} R_t(-Z) \supseteq R_t(Y) \ni 0 \) and thus \( X \in A_t \).

(\( \Leftarrow \)) Let \( X \in \mathcal{L}^p \), any \( Z \in R_s(X) \), and any \( Y \in R_t(-Z) \), then \( Y + X = Y - Z + Z + X \). In particular, \( X + Z \in A_s \) since \( Z \in R_s(X) \) if and only if \( 0 \in R_s(X) - Z = R_s(X + Z) \), and \( Y - Z \in A_{t,s} \) since \( Y \in R_t(-Z) \) if and only if \( 0 \in R_t(-Z) - Y = R_t(Y - Z) \) and \( Y - Z \in M_s \) by \( Y \in M_t \) and \( -Z \in M_s \) with \( M_t + M_s \subseteq M_s \) (by \( M_t \subseteq M_s \)). Therefore, \( Y + X \in A_s + A_{t,s} \subseteq A_t \). This implies \( 0 \in R_t(Y + X) = R_t(X) - Y \). Therefore \( Y \in R_t(X) \) and for every \( Z \in R_s(X) \) it follows that \( R_t(-Z) \subseteq R_t(X) \). Therefore \( \bigcup_{Z \in R_s(X)} R_t(-Z) \subseteq R_t(X) \).

\[ \square \]

**Remark 3.2.7.** As can be seen from lemma 3.2.6, the normalization property is not used for the equivalences between properties 3 and 4 in theorem 3.2.2. Furthermore, if a dynamic risk measure \((R_t)_{t \in T}\) that is not normalized follows the recursive form defined in equation (3.2.2), then \((R_t)_{t \in T}\) is multi-portfolio time consistent (but not necessarily vice versa).

Now that we have given the main result on multi-portfolio time consistency for this section, we are interested in how multi-portfolio time consistency and time consistency relate to each other. First, we show that the two properties are not equivalent in general (see example 3.2.8). Second, we give sufficient conditions for these properties to coincide in lemma 3.2.9. Remark 3.2.10 demonstrates that these sufficient conditions are always satisfied in the scalar framework.
Example 3.2.8. Consider a one-period model with \( t \in \{0,T\} \). Define the sets \( A_0 = \{X \in \mathcal{L}^p : X_1 \in L^p_t\} \) and \( A_T = \{X \in \mathcal{L}^p : X_1 \in L^p_{T+}\} \). Let the space of eligible assets be given by \( M = \mathbb{R} \times \{0\}^{d-1} \). Clearly, \( A_0 \) and \( A_T \) satisfy the properties of a normalized acceptance set (see definition 2.1.11 and proposition 2.1.14 (2) on the set \( C = A_t \cap M_t, \ t \in \{0,T\} \)) and denote the corresponding dynamic risk measure \( R_t(X) := \{u \in M_t : X + u \in A_t\} \) for \( t \in \{0,T\} \). Then, it holds \( A_0 \supseteq A_T + A_{0,T} = A_T \) since \( A_{0,T} = \{X \in M_T : X_1 \geq 0\} \). Thus, the risk measure \( (R_t)_{t \in \mathbb{T}} \) is not multi-portfolio time consistent by theorem 3.2.2. However, \( (R_t)_{t \in \mathbb{T}} \) is time consistent since, by the definition of \( A_T \) and \( M_T \), \( R_T(X) \subseteq R_T(Y) \) if and only if \( Y_1 - X_1 \in L^p_t \). This implies \( R_0(X) = R_0((X_1,0,...,0)\mathbb{T}) \subseteq R_0((Y_1,0,...,0)\mathbb{T}) = R_0(Y) \) by monotonicity and the definition of \( A_0 \).

As we have shown with this example, the recursive form and time consistency are not equivalent in general. We now consider sufficient conditions for the two properties to be equivalent.

Lemma 3.2.9. Let \( (R_t)_{t \in \mathbb{T}} \) be a normalized time consistent dynamic risk measure such that for all times \( t,s \in \mathbb{T} \) with \( t < s \) and every \( X \in \mathcal{L}^p \) there exists a \( \hat{Z} \in R_s(X) \) such that \( R_s(-\hat{Z}) \supseteq R_s(X) \) (or, equivalently \( R_s(X) = \hat{Z} + R_s(0) \)). Then, \( (R_t)_{t \in \mathbb{T}} \) is multi-portfolio time consistent.

Proof. Let \( X \in \mathcal{L}^p, t,s \in \mathbb{T} \) with \( t < s \), and \( Z \in R_s(X) \) arbitrarily chosen. Then, \( M_s \)-translativity and normalization of \( R_s \) imply \( R_s(-Z) = R_s(0) + Z \subseteq R_s(X) \), hence \( R_t(-Z) \subseteq R_t(X) \) by time consistency. Thus, \( \bigcup_{Z \in R_s(X)} R_t(-Z) \subseteq R_t(X) \). On the other hand, by assumption there exists a \( \hat{Z} \in R_s(X) \) such that \( R_s(-\hat{Z}) \supseteq R_s(X) \). Hence, \( R_t(-\hat{Z}) \supseteq R_t(X) \) by time consistency and \( \bigcup_{Z \in R_s(X)} R_t(-Z) \supseteq R_t(X) \). 

Remark 3.2.10. For a scalar normalized time consistent dynamic risk measure \( (\rho_t)_{t \in \mathbb{T}} \) with \( \rho_t : L^p \to L^p_t \), the corresponding set-valued dynamic risk measure \( (R_t^\rho)_{t \in \mathbb{T}} \) defined on \( L^p \) is given by \( R_t^\rho(X) = \rho_t(X) + L^p_{t,+} \) and thus automatically satisfies the
assumptions of lemma 3.2.9. This shows that in the scalar case time consistency is equivalent to multi-portfolio time consistency.

Now we will take a more in depth look at market-compatibility for dynamic risk measures under a conical market model. For this reason we will only be considering the discrete time setting \( \mathbb{T} = \{0, 1, \ldots, T\} \) for the remainder of this section. It should be noticed that market-compatibility in definition 2.1.7 is different from that given in [34] given by \( R_t \) is \( K_t \)-compatible for all times \( t \). However, it turns out that for multi-portfolio time consistent risk measures both notions coincide under a conical market model, which justifies the use of the same name. We first give a general result related to compatibility and then give a corollary specifically on market-compatibility.

**Lemma 3.2.11.** For all \( t \in \mathbb{T} \) let \( A_t, C_t, D_t \subseteq \mathcal{L}^p \) with \( A_s = A_{s+1} + D_s \) for all \( s \in \mathbb{T} \). Then the following three statements are equivalent.

1. \( A_t = A_t + C_t \) for all times \( t \in \mathbb{T} \),

2. \( A_t = A_t + C_s \) for all times \( t, s \in \mathbb{T} \) with \( s \geq t \),

3. \( A_t = A_t + \sum_{s=t}^{T} C_s \) for all times \( t \in \mathbb{T} \).

**Proof.** Let \( A_t, C_t, D_t \subseteq \mathcal{L}^p \) for every time \( t \in \mathbb{T} \) with \( A_s = A_{s+1} + D_s \) for all \( s \in \mathbb{T} \). We will do this by proving 1 implies 2, 2 implies 3, and finally 3 implies 1.

1. Let \( A_t = A_t + C_t \) for all times \( t \in \mathbb{T} \). Let \( s, t \in \mathbb{T} \) with \( s > t \) (if \( s = t \) then the result follows by definition).

\[
A_t = A_{t+1} + D_t = A_s + \sum_{r=t}^{s-1} D_r = A_s + C_s + \sum_{r=t}^{s-1} D_r = A_t + C_s.
\]
2. Let \( A_t = A_t + C_s \) for all times \( t, s \in \mathbb{T} \) with \( s \geq t \). Let \( t \in \mathbb{T} \).

\[
A_t = A_t + C_t = A_{t+1} + D_t + C_t = A_{t+1} + D_t + \sum_{s=t}^{t+1} C_s
\]

\[
= A_T + \sum_{s=t}^{T-1} D_s + \sum_{s=t}^{T} C_s = A_t + \sum_{s=t}^{T} C_s.
\]

3. Let \( A_t = A_t + \sum_{s=t}^{T} C_s \) for all times \( t \in \mathbb{T} \). Let \( t \in \mathbb{T} \).

\[
A_t = A_t + \sum_{s=t}^{T} C_s = A_{t+1} + D_t + \sum_{s=t}^{T} C_s = A_{t+1} + D_t + C_t = A_t + C_t.
\]

Corollary 3.2.12. Let \((R_t)_{t \in \mathbb{T}}\) be a normalized multi-portfolio time consistent dynamic risk measure. \( R_t \) is \( K_t \)-compatible for a sequence of solvency cones \( K_t \) at each time \( t \) if and only if \((R_t)_{t \in \mathbb{T}}\) is market-compatible.

Proof. By proposition 2.1.14, \((R_t)_{t \in \mathbb{T}}\) is market-compatible if and only if \( A_t = A_t + \sum_{s=t}^{T} \mathcal{L}_s^p(K_s) \) for every \( t \in \mathbb{T} \). By \((R_t)_{t \in \mathbb{T}}\) normalized and multi-portfolio time consistency, we have \( A_t = A_{t,t+1} + A_{t+1} \) for every \( t \in \mathbb{T} \). Thus the result follows from lemma 3.2.11 where \( C_t := \mathcal{L}_t^p(K_t) \) for all times \( t \in \mathbb{T} \) and \( D_t := A_{t,t+1} \) for all times \( t \in \mathbb{T} \).

3.3 Convex risk measures and multi-portfolio time consistency

In this section, we want to study the impact of multi-portfolio time consistency on the penalty function of a (conditionally) convex risk measure, as was done in [36]. In the scalar case it could be shown that (multi-portfolio) time consistency is equivalent to an additive property of the penalty functions, see e.g. [40, 22, 15, 16, 1], which is called
the cocycle property in [15, 16]. We will show that a corresponding result is also true in the set-valued case. However, it is much harder to prove than in the scalar case. The reason is that, when following the proofs along the lines of [40, 16], an additional infimum (that is the union in the recursion) appears in the set-valued case, which is not present in the scalar case. One would need to apply a minimax theorem in order to exchange the infimum and the supremum, but it is hard to verify the constraint qualification. Thus, we follow a different route in proving the main theorem about the equivalence between multi-portfolio time consistency and an additive property of the penalty functions. In the heart of this new proof lies a Hahn-Banach separation argument, which we will provide before presenting the main theorem.

As we will be working exclusively with the dual representation in this section we assume that $p \in [1, +\infty]$ with dual space defined by $q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

The Hahn-Banach argument uses the functions $F_{(Q,w)}^t : \mathcal{L}^p \rightarrow 2^{M_t}$ defined by

$$F_{(Q,w)}^t[X] := \left\{ u \in M_t : \mathbb{E} \left[ w^T \mathbb{E}^Q [X|\mathcal{F}_t] \right] \leq \mathbb{E} \left[ w^T u \right] \right\}$$

$$= (\mathbb{E}^Q [X|\mathcal{F}_t] + G_t(w)) \cap M_t,$$

for $(Q, w) \in \mathcal{W}_t$. As shown in section 2.2, the functions $F_{(Q,w)}^t$ are the main ingredients in the duality theory for set-valued risk measures as they replace the continuous linear functions used in scalar duality theory. In particular, these functions appear in the dual representation (2.2.2) of risk measures and in the definition of the minimal penalty function (2.2.3).

We are now ready to formulate the Hahn-Banach argument, which characterizes when a portfolio is acceptable.
Lemma 3.3.1. Let $A_t \subseteq \mathcal{L}^p$ be a closed convex acceptance set and let $X \in \mathcal{L}^p$. Then, $X \notin A_t$ if and only if there exists a $(Q, w) \in \mathcal{W}_0$ such that

$$F^0_{(Q, w)}[X] \supseteq \text{cl} \bigcup_{Z \in A_t} F^0_{(Q, w)}[Z].$$

Proof. If $X \notin A_t$ then there exists a $Y \in \mathcal{L}^q_+$ with $E[Y^TX] < \inf_{Z \in A_t} E[Y^TZ]$ by the separating hyperplane theorem. (If we choose $Y \notin \mathcal{L}^q_+$ it follows that $\inf_{Z \in A_t} E[Y^TZ] = -\infty$ by $A_t + \mathcal{L}^p_+ \subseteq A_t$ which leads to a contradiction.) This implies that

$$\tilde{F}^0_{(Y, v)}[X] := \{u \in M : \mathbb{E}[Y^TX] \leq v^Tu\} \supseteq \{u \in M : \inf_{Z \in A_t} \mathbb{E}[Y^TZ] \leq v^Tu\} = \text{cl} \bigcup_{Z \in A_t} \tilde{F}^0_{(Y, v)}[Z]$$

for any $v \notin M^\perp$ since $f(u) = v^Tu$ is a continuous linear operator from $M$ to $\mathbb{R}$. In particular this is true for any $v \in (\mathbb{E}[Y] + M^\perp) \setminus M^\perp$. As given in lemma 2.2.6, there exists a pair $(Q, w) \in \mathcal{W}_0$ such that $F^0_{(Q, w)}[\cdot] = \tilde{F}^0_{(Y, v)}[\cdot]$, therefore $F^0_{(Q, w)}[X] \supseteq \text{cl} \bigcup_{Z \in A_t} F^0_{(Q, w)}[Z]$. If $F^0_{(Q, w)}[X] \supseteq \text{cl} \bigcup_{Z \in A_t} F^0_{(Q, w)}[Z]$ for some $(Q, w) \in \mathcal{W}_0$, then $\mathbb{E}^Q[X] \neq \mathbb{E}^Q[Z]$ for all $Z \in A_t$. Therefore $X \notin A_t$. \hfill \Box

We now state the main results of this section. Its proofs are based on the Hahn-Banach argument given above and several lemmas provided in the appendix, sections A.2 and A.3, that concern e.g. the relation of dual variables at different times. Recall that we use the notation $Q^s$ to denote the modification of $Q \in \mathcal{M}$ defined by $\frac{dQ^s}{dP} = \xi_{s,T}(Q)$.
Theorem 3.3.2. Let \((R_t)_{t \in \mathbb{T}}\) be a dynamic normalized closed convex risk measure. Then \((R_t)_{t \in \mathbb{T}}\) is multi-portfolio time consistent if and only if for every \((Q, w) \in \mathcal{W}_t\)

\[-\beta_t(Q, w) = \text{cl} \left( -\beta_{t,s}(Q, w) + \mathbb{E}^Q [-\beta_s(Q^s, w^s_t(Q, w))|\mathcal{F}_t] \right)\]

and \(A_{t,s} + A_s\) is closed, for all \(t, s \in \mathbb{T}\) where \(t < s\).

Proof. From theorem 3.2.2, a normalized dynamic risk measure is multi-portfolio time consistent if and only if \(A_t = A_{t,s} + A_s\) for every \(t, s \in \mathbb{T}\) where \(t < s\).

1. Assume \((R_t)_{t \in \mathbb{T}}\) is a normalized closed convex multi-portfolio time consistent risk measure, i.e. assume \(A_t = A_{t,s} + A_s\). It immediately follows that for any pair of dual variables \((Q, w) \in \mathcal{W}_t\)

\[
\text{cl} \left( -\beta_{t,s}(Q, w) + \mathbb{E}^Q [-\beta_s(Q^s, w^s_t(Q, w))|\mathcal{F}_t] + \mathbb{E}^Q \left[ X_{t,s} \mid \mathcal{F}_t \right] + G_t(w) \right) \cap M_t \]

By Theorem 3.2.2, it follows that

\[
\text{cl} \left( -\beta_{t,s}(Q, w) + \mathbb{E}^Q [-\beta_s(Q^s, w^s_t(Q, w))|\mathcal{F}_t] + \mathbb{E}^Q \left[ X_{t,s} \mid \mathcal{F}_t \right] + G_t(w) \right) \cap M_t = -\beta_t(Q, w).
\]
Equation (3.3.1) follows from lemma A.2.3, and equation (3.3.2) follows from proposition 1.23 in [66]. Note that if \((Q, w) \in \mathcal{W}_t\) then \((Q, w) \in \mathcal{W}_{t,s}\) (see remark 2.3.2) and \((Q^s, w^s_t(Q, w)) \in \mathcal{W}_s\) (see lemma A.2.1).

2. Conversely, assume \(A_{t,s} + A_s\) is closed and the cocycle condition is satisfied, i.e.

\[-\beta_t(Q, w) = \text{cl}(\beta_{t,s}(Q, w) + \mathbb{E}^Q[\beta_s(Q^s, w^s_t(Q, w)) | \mathcal{F}_t])\]

for every \((Q, w) \in \mathcal{W}_t\).

Note that for any \((Q, w) \in \mathcal{W}_0\) it holds \(w^s_0(Q) = w^s_t(Q^t, w^s_0(Q))\).

Let \(X \in A_{t,s} + A_s\), then by the tower property and corollary A.2.4, for every \((Q, w) \in \mathcal{W}_0\)

\[
\begin{align*}
F^0_{(Q,w)}[X] &\subseteq \mathbb{E}^Q \left[ \text{cl}(\beta_{t,s}(Q^t, w^s_t(Q, w)) + \mathbb{E}^Q[\beta_s(Q^s, w^s_0(Q, w)) | \mathcal{F}_t]) \right] \\
&= \mathbb{E}^Q \left[ \beta_t(Q^t, w^s_0(Q, w)) \right] = \text{cl} \bigcup_{Z \in A_t} F^0_{(Q,w)}[Z].
\end{align*}
\]

The last equality follows from lemma A.2.3. If \(X \not\in A_t\) then, by lemma 3.3.1, there exists a pair \((Q, w) \in \mathcal{W}_0\) such that \(F^0_{(Q,w)}[X] \supseteq \text{cl}\bigcup_{Z \in A_t} F^0_{(Q,w)}[Z]\). However, this is a contradiction to the above, therefore \(X \in A_t\) and thus

\[
A_{t,s} + A_s \subseteq A_t. \tag{3.3.3}
\]

Let \(X \in A_t\), then (using corollary A.2.4 and lemma A.2.3) for every \((Q, w) \in \mathcal{W}_0\)

\[
\begin{align*}
F^0_{(Q,w)}[X] &\subseteq \mathbb{E}^Q \left[ \beta_t(Q^t, w^s_0(Q, w)) \right] \\
&= \mathbb{E}^Q \left[ \text{cl}(\beta_{t,s}(Q^t, w^s_t(Q, w)) + \mathbb{E}^Q[\beta_s(Q^s, w^s_0(Q, w)) | \mathcal{F}_t]) \right] \\
&= \text{cl} \bigcup_{Z \in A_{t,s} + A_s} F^0_{(Q,w)}[Z].
\end{align*}
\]

If we assume that \(X \not\in A_{t,s} + A_s\) (which is closed by assumption and is a convex acceptance set by lemma A.3.4, where the assumption \(A_{t,s} + A_s \subseteq A_t\)
of lemma A.3.4 is satisfied by \((3.3.3)\) then, by lemma 3.3.1, there exists a pair \((Q, w) \in W_0\) such that

\[
F^0_{(Q, w)}[X] \supseteq \operatorname{cl} \bigcup_{Z \in A_{t,s} + A_s} F^0_{(Q, w)}[Z].
\]

This is a contradiction to the above, therefore \(X \in A_{t,s} + A_s\).

\[\square\]

**Corollary 3.3.3.** Let \((R_t)_{t \in \mathbb{T}}\) be a dynamic normalized c.u.c. convex risk measure. Then, \((R_t)_{t \in \mathbb{T}}\) is multi-portfolio time consistent if and only if

\[
-\beta_t(Q, w) = \operatorname{cl} \left( -\beta_{t,s}(Q, w) + E^Q \left[ -\beta_s(Q^s, w^s_t(Q, w)) \mid \mathcal{F}_t \right] \right)
\]

holds for every \((Q, w) \in W_t\) for all \(t, s \in \mathbb{T}\) where \(t < s\).

**Proof.** In light of theorem 3.3.2 it only remains to show that convex upper continuity of \((R_t)_{t \in \mathbb{T}}\) implies the closedness of \(A_{t,s} + A_s\). This follows from remark A.3.3 and lemma A.3.2.

\[\square\]

In the above theorem and corollary, we have demonstrated that the sum of penalty functions gives an equivalent characterization of multi-portfolio time consistency. This allows us to define risk measures by the penalty functions alone and verify whether the corresponding c.u.c. convex risk measure is multi-portfolio time consistent. See, for example, section 5.4 and theorem 5.4.2 for such an application of corollary 3.3.3.

We now consider the conditionally convex case when a dual representation w.r.t. equivalent probability measures, i.e. w.r.t. the dual set \(W^e_t\), holds.
Corollary 3.3.4. Let $(R_t)_{t \in \mathbb{T}}$ be a dynamic normalized closed conditionally convex risk measure with dual representation

$$R_t(X) = \bigcap_{(Q, w) \in W_t^c} [-\alpha_t(Q, w) + (E^Q [-X| F_t] + \Gamma_t(w)) \cap M_t]$$

for every $X \in \mathcal{L}_p$ where $W_t^c = \{(Q, w) \in W_t : Q \in \mathcal{M}^c\}$. Then $(R_t)_{t \in \mathbb{T}}$ is multi-portfolio time consistent if and only if for every $(Q, w) \in W_t$

$$-\alpha_t(Q, w) = \text{cl} \left( -\alpha_{t,s}(Q, w) + E^Q [-\alpha_s(Q^s, w_s(Q, w)| F_t) \right)$$

and $A_{t,s} + A_s$ is closed, for all $t, s \in \mathbb{T}$ where $t < s$.

Proof. This follows using the same logic as in the proof of theorem 3.3.2 (with the closure operator added where necessary) using the results on the conditional expectation of $\alpha_s$ and $\Gamma_s$ in lemma A.2.5 and corollary A.2.6.

If $(R_t)_{t \in \mathbb{T}}$ is, additionally to the assumptions of Corollary 3.3.4, conditionally c.u.c., then $(R_t)_{t \in \mathbb{T}}$ is multi-portfolio time consistent if and only if the cocycle condition on $\alpha_t$ holds. This corresponds to the results of corollary 3.3.3.

3.4 Coherent risk measures and multi-portfolio time consistency

In this section, we want to study multi-portfolio time consistency in the (conditionally) coherent case. In particular, we want to find equivalent characterizations of multi-portfolio time consistency with respect to the set of dual variables. In the scalar framework an equivalent property is given by stability of the dual variables, also called m-stability, which was studied for the case when the dual probability mea-
sures are absolutely continuous to the real world probability measure $\mathbb{P}$ in $[1, 22]$, and when the dual probability measures are equivalent to $\mathbb{P}$ in $[26, 40, 6]$.

As we will be working exclusively with the dual representation in this section we assume that $p \in [1, +\infty]$ with dual space defined by $q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

**Remark 3.4.1.** In this section and section 3.5, we will for simplicity only present the results for multi-portfolio time consistency assuming convex upper continuity, akin to corollary 3.3.3 above. The results can be given for closed risk measures as well, as it was done in theorem 3.3.2 and corollary 3.3.4.

For the results below we use the definition of the maximal set of stepped dual variables $\mathcal{W}^{\text{max}}_{t,s} \subseteq \mathcal{W}_{t,s}$ as defined in section 2.3. That is,

$$
\mathcal{W}^{\text{max}}_{t,s} = \{(Q, w) \in \mathcal{W}_{t,s} : -\beta_{t,s}(Q, w) = G_t(w) \cap M_t\}
= \{(Q, w) \in \mathcal{W}_{t,s} : w_T^*(Q, w) \in A_{t,s}^+\}.
$$

All the results for the conditionally coherent case stay the same as for the coherent case (except that the assumption c.u.c. can be weakened to conditionally c.u.c.) as the set of dual variables does not change (compare (2.2.7) and (2.2.2)). This is also true for the maximal set of stepped dual variables as

$$
\mathcal{W}^{\text{max}}_{t,s} = \{(Q, w) \in \mathcal{W}_{t,s} : -\alpha_{t,s}(Q, w) = \Gamma_t(w) \cap M_t\}
$$

since $A_{t,s}^+ = \{v \in \mathcal{L}^q_t : \forall u \in A_{t,s} : v^T u \geq 0 \ \mathbb{P}\text{-a.s.}\}$ if $R_t$ is conditionally coherent.

**Remark 3.4.2.** For any closed coherent risk measure $R_t$ (not necessarily multi-portfolio time consistent) it can trivially be seen that $\mathcal{W}^\text{max}_t \subseteq \mathcal{W}^\text{max}_{t,s}$ since $\mathcal{W}_t \subseteq \mathcal{W}_{t,s}$ (see remark 2.3.2) and $w_T^*(Q, w) \in A^+_t$ implies $w_T^*(Q, w) \in A^+_{t,s}$.
The first result we provide, which will be useful for generating a c.u.c. coherent multi-portfolio time consistent risk measure in section 3.5, is a corollary to theorem 3.3.2 (respectively corollary 3.3.3) above.

Let us define the set $H_t^s : 2^{W_s} \to 2^{W_t}$ by

$$H_t^s(D) := \{(Q, w) \in W_t : (Q^s, w^s(Q, w)) \in D\}$$

for any $t, s \in \mathbb{T}$ where $t < s$, and any $D \subseteq W_s$.

**Corollary 3.4.3.** Let $(R_t)_{t \in \mathbb{T}}$ be a dynamic normalized c.u.c. coherent risk measure. Then, $(R_t)_{t \in \mathbb{T}}$ is multi-portfolio time consistent if and only if for all $t, s \in \mathbb{T}$ where $t < s$ it holds

$$W_{t_{\text{max}}} = W_{t,s_{\text{max}}} \cap H_t^s(W_{s_{\text{max}}}).$$

**Proof.** This follows trivially from corollary 3.3.3 and corollary A.2.4 by noting that for any times $t$ and $s > t$

$$-\beta_t(Q, w) = \begin{cases} G_t(w) \cap M_t & \text{if } (Q, w) \in W_{t_{\text{max}}} \\ M_t & \text{else} \end{cases}$$

$$-\beta_{t,s}(Q, w) = \begin{cases} G_t(w) \cap M_t & \text{if } (Q, w) \in W_{t,s_{\text{max}}} \\ M_t & \text{else.} \end{cases}$$

And since $W_{t,s} \supseteq W_t$ (see remark 2.3.2) for any times $t < s$, the result follows.

We now want to study the pasting of dual variables and the generalization of stability to the set-valued case.
For $Q, R \in \mathcal{M}$ we denote by $Q \oplus^s R$ the pasting of $Q$ and $R$ at $s$, i.e. the vector probability measures $S \in \mathcal{M}$ defined via
\[ \frac{dS}{d\mathbb{P}} = \xi_{0,s}(Q) \cdot \xi_{s,T}(R). \]

Note that if $S = Q \oplus^s R$ then $\tilde{\xi}_{t,r}(S_i) = \tilde{\xi}_{t,r}(Q_i)$ for $t \leq r \leq s$, but $\tilde{\xi}_{t,r}(S_i)$ is not necessarily equal to $\tilde{\xi}_{t,r}(R_i)$ for $r \geq t > s$. If $Q = \mathbb{P}|_{\mathcal{F}_t}$ for some $t \leq s$ (i.e. $\tilde{\xi}_{0,t}(Q_i) = 1$ almost surely for every $i \in \{1, ..., d\}$), then it trivially follows that $w^*_i(S, w) = w^*_s(R, w^*_t(Q, w))$ for $t, s, r \in \mathbb{T}$ with $t \leq s \leq r$ and any $w \in \mathcal{L}^q_t$. In the set-valued framework we define stability as a property with respect to two other sets. This is due to the fact that our dual variables consists of pairs. Naturally, stability is a property that imposes conditions on both components of a pair $(Q, w)$.

**Definition 3.4.4.** A set $W_t \subseteq W_t$ is called **stable** at time $t$ with respect to $W_{t,s}$ and $W_s$ for $s > t$ if

1. $(Q, w) \in W_t$ implies $(Q^*, w^*_t(Q, w)) \in W_s$ and

2. $(Q, w) \in W_{t,s}$ and $R \in \mathcal{M}$ with $(R, w^*_t(Q, w)) \in W_s$ implies $(Q \oplus^s R, w) \in W_t$.

**Remark 3.4.5.** In the scalar framework, stability is defined with respect to stopping times, see e.g. [26]. We are able to weaken this assumption in the set-valued framework due to the total ordering given by the half space $G_t(w)$ generated by the second dual variable, see lemma A.2.3 for more details.

The main theorem of this section is given below. It provides an equivalence between the stability of the sets of dual variables $\mathcal{W}^{max}_t$ and multi-portfolio time consistency. We present an additional property which is equivalent to stability and therefore to multi-portfolio time consistency. This additional property, given in equation (3.4.1), is a generalization of property 2 of corollary 1.26 from [1].
Theorem 3.4.6. Let \((R_t)_{t \in T}\) be a normalized c.u.c. coherent risk measure, then the following are equivalent:

1. \((R_t)_{t \in T}\) is multi-portfolio time consistent;

2. \(W_{t,s}^{\text{max}}\) is stable at time \(t\) with respect to \(W_{t,s}^{\text{max}}\) and \(W_{s}^{\text{max}}\) for every time \(s \in T\) such that \(t < s\);

3. for all times \(t, s \in T\) where \(t < s\)

\[
W_{t}^{\text{max}} = \left\{ (Q \oplus^s R, w) : (Q, w) \in W_{t,s}^{\text{max}}, (R, w^s_t(Q, w)) \in W_{s}^{\text{max}} \right\}. \tag{3.4.1}
\]

Proof. We will show that multi-portfolio time consistency implies stability, stability implies equation (3.4.1), and finally, that equation (3.4.1) implies multi-portfolio time consistency.

First, assume \((R_t)_{t \in T}\) is multi-portfolio time consistent. We want to show that \(W_{t}^{\text{max}}\) is stable at time \(t\) with respect to \(W_{t,s}^{\text{max}}\) and \(W_{s}^{\text{max}}\), as given in definition 3.4.4.

1. By corollary 3.4.3 it follows that \(W_{t}^{\text{max}} \subseteq H^s_t(W_{s}^{\text{max}})\) and thus \((Q, w) \in W_{t}^{\text{max}}\) implies \((Q^s, w^s_t(Q, w)) \in W_{s}^{\text{max}}\).

2. Let \(t < s\), \((Q, w) \in W_{t,s}^{\text{max}}\) and \(R \in \mathcal{M}\) with \((R, w^s_t(Q, w)) \in W_{s}^{\text{max}}\). We need to show that \((S, w) \in W_{t}^{\text{max}}\) where \(S = Q \oplus^s R\) is the pasting of \(Q\) and \(R\) at \(s\).

   (a) For any index \(i = 1, \ldots, d\), \(S_i = \mathbb{P}|\mathcal{F}_i\) since \(E[dS_i/dR | \mathcal{F}_i] = E[dQ_i/dR | \mathcal{F}_i] = 1\) almost surely.

   (b) \((S, w) \in \mathcal{W}_t\) since \(S \in \mathcal{M}\), \(w \in M_{t,+}^+ \setminus M^+_t\), and

\[
w^T_t(S, w) = w^s_t(R, w^s_t(Q, w)) \in \mathcal{L}^q_+.
\]
(c) \((S, w) \in \mathcal{W}_t^{\text{max}}\) if \(-\beta_t(S, w) = G_t(w) \cap M_t\). By corollary 3.3.3 it follows that for any \((S, w) \in \mathcal{W}_t\)

\[-\beta_t(S, w) = \text{cl}(\beta_{t,s}(S, w)) + \mathbb{E}^S [\beta_{t,s}(S, w)](\mathcal{F}_t)\].

We can see that \(-\beta_{t,s}(S, w) = -\beta_{t,s}(Q, w)\) by the tower property. One can show that \(-\beta_s(S^*, w_1^*(S, w)) = -\beta_s(R, w_1^*(Q, w))\) by

\[-\beta_s(S^*, w_1^*(S, w)) = \text{cl} \bigcup_{X \in A_s} \{x \in M_s : E[w_1^*(S, w)^T X] \leq E[w_1^*(Q, w)^T x]\} = \text{cl} \bigcup_{X \in A_s} \{x \in M_s : E[w_1^*(Q, w)^T X] \leq E[w_1^*(S, w)^T x]\} = \text{cl} \bigcup_{X \in A_s} \{x \in M_s : E[w_1^*(Q, w)^T X] \leq E[w_1^*(Q, w)^T x]\} = -\beta_s(R, w_1^*(Q, w)).

Therefore,

\[-\beta_t(S, w) = \text{cl}(\beta_{t,s}(S, w)) + \mathbb{E}^S [\beta_{t,s}(S, w)](\mathcal{F}_t)
= \text{cl}(\beta_{t,s}(Q, w)) + \mathbb{E}^Q [\beta_{t,s}(Q, w)](\mathcal{F}_t)
= \text{cl}(G_t(w) \cap M_t) + \mathbb{E}^Q [G_s(w_1^*(Q, w)) \cap M_s](\mathcal{F}_t)
= G_t(w) \cap M_t,

using \((Q, w) \in \mathcal{W}_t^{\text{max}}\) and \((R, w_1^*(Q, w)) \in \mathcal{W}_s^{\text{max}}\). The last line follows from corollary A.2.4.
Therefore for every time \( t \), \( \mathcal{W}_t^{\text{max}} \) is stable at time \( t \) with respect to \( \mathcal{W}_{t,s}^{\text{max}} \) and \( \mathcal{W}_s^{\text{max}} \) for every \( s > t \).

Now we will demonstrate that stability implies equation (3.4.1). If for every time \( t \), \( \mathcal{W}_t^{\text{max}} \) is stable at time \( t \) with respect to \( \mathcal{W}_{t,s}^{\text{max}} \) and \( \mathcal{W}_s^{\text{max}} \) for every \( s > t \) then trivially “\( \supseteq \)" in equation (3.4.1) follows by the second property of stability. By the first property of stability and remark 3.4.2, for any \( (Q, w) \in \mathcal{W}_t^{\text{max}} \) it follows that \( (Q, w) \in \mathcal{W}_{t,s}^{\text{max}} \) and \( (Q^s, w^s_t(Q, w)) \in \mathcal{W}_s^{\text{max}} \). Since \( Q = Q \oplus^s Q^s \) for any time \( s \) and any probability measure \( Q \in \mathcal{M} \), then “\( \subseteq \)" in equation (3.4.1) trivially follows.

Finally, we will prove that equation (3.4.1) implies that for every \( (Q, w) \in \mathcal{W}_t \)

\[
-\beta_t(Q, w) = \text{cl} \left( -\beta_{t,s}(Q, w) + \mathbb{E}^Q [ -\beta_s(Q^s, w^s_t(Q, w)) | \mathcal{F}_t ] \right)
\]

which in turn implies multi-portfolio time consistency by corollary 3.3.3. We define the set \( \tilde{\mathcal{W}}_t := \{ (Q \oplus^s \mathbb{R}, w) : (Q, w) \in \mathcal{W}_{t,s}^{\text{max}}, (\mathbb{R}, w^s_t(Q, w)) \in \mathcal{W}_s^{\text{max}} \} \) for notational purposes.

1. We will show that the inclusion \( \mathcal{W}_t^{\text{max}} \subseteq \tilde{\mathcal{W}}_t \) implies the penalty function inclusion

\[
-\beta_t(Q, w) \supseteq \text{cl}(-\beta_{t,s}(Q, w) + \mathbb{E}^Q [-\beta_s(Q^s, w^s_t(Q, w))] | \mathcal{F}_t)
\]

for every \( (Q, w) \in \mathcal{W}_t \).

(a) Let \( (S, w) \in \mathcal{W}_t^{\text{max}} \). Then, \( -\beta_t(S, w) = G_t(w) \cap M_t \). Additionally, there exists a \( Q, \mathbb{R} \in \mathcal{M} \) such that \( (Q, w) \in \mathcal{W}_{t,s}^{\text{max}}, (\mathbb{R}, w^s_t(Q, w)) \in \mathcal{W}_s^{\text{max}} \), and \( S = Q \oplus^s \mathbb{R} \). This implies

\[
-\beta_{t,s}(S, w) = -\beta_{t,s}(Q, w) = G_t(w) \cap M_t, \quad \text{and} \quad -\beta_s(S^s, w^s_t(S, w)) = -\beta_s(\mathbb{R}, w^s_t(Q, w)) = G_s(w^s_t(Q, w)) \cap M_s = G_s(w^s_t(S, w)) \cap M_s.
\]

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Therefore, corollary A.2.4 yields

\[-\beta_t(S, w) = G_t(w) \cap M_t \]
\[= \text{cl} \left( G_t(w) \cap M_t + \mathbb{E}^S [G_s(w^*_t(S, w)) \cap M_s | \mathcal{F}_t] \right) \]
\[= \text{cl} \left( -\beta_{t,s}(S, w) + \mathbb{E}^S [-\beta_s(S^*, w^*_t(S, w)) | \mathcal{F}_t] \right). \]

(b) Let \((S, w) \in \mathcal{W}_t \setminus \mathcal{W}_t^{\max}\), then \(-\beta_t(S, w) = M_t\), and thus

\[-\beta_t(S, w) \supseteq \text{cl} \left( -\beta_{t,s}(S, w) + \mathbb{E}^S [-\beta_s(S^*, w^*_t(S, w)) | \mathcal{F}_t] \right). \]

2. We will show that the inclusion \(\mathcal{W}_t^{\max} \supseteq \widehat{\mathcal{W}}^*_t\) implies the penalty function inclusion \(-\beta_t(Q, w) \subseteq \text{cl}(-\beta_{t,s}(Q, w) + \mathbb{E}^{|F|}[-\beta_s(Q^*, w^*_t(Q, w)) | \mathcal{F}_t])\) for every \((Q, w) \in \mathcal{W}_t\).

(a) Let \((S, w) \in \widehat{\mathcal{W}}^*_t\). By the assumption we have \(-\beta_t(S, w) = G_t(w) \cap M_t\) and \((S, w) \in \mathcal{W}_t^{\max}\). Additionally there exists \(Q, R \in \mathcal{M}\) such that \((Q, w) \in \mathcal{W}_t^{\max}\), \((R, w^*_t(Q, w)) \in \mathcal{W}_s^{\max}\), and \(S = Q \oplus^s R\). As above, this implies that the penalty functions are half spaces, i.e. \(-\beta_{t,s}(S, w) = G_t(w) \cap M_t\) and \(-\beta_s(S^*, w^*_t(S, w)) = G_s(w^*_t(S, w)) \cap M_s\). Therefore we have the equality

\[-\beta_t(S, w) = \text{cl} \left( -\beta_{t,s}(S, w) + \mathbb{E}^S [-\beta_s(S^*, w^*_t(S, w)) | \mathcal{F}_t] \right). \]

(b) Let \((S, w) \in \mathcal{W}_t \setminus \mathcal{W}_t^{\max}\), i.e. for every \(Q, R \in \mathcal{M}\) such that \(S = Q \oplus^s R = Q \oplus^s R^s\) either \((Q, w) \not\in \mathcal{W}_t^{\max}\) or \((R^s, w^*_t(Q, w)) \not\in \mathcal{W}_s^{\max}\). This implies for any \(Q, R \in \mathcal{M}\) where \(S = Q \oplus^s R\) either \(-\beta_{t,s}(S, w) = -\beta_t(Q, w) = M_t\) or \(-\beta_s(S^*, w^*_t(S, w)) = -\beta_s(R^s, w^*_t(Q, w)) = M_s\). Thus, \(\text{cl}(-\beta_{t,s}(S, w) + \mathbb{E}^S[-\beta_s(S^*, w^*_t(S, w)) | \mathcal{F}_t]) = M_t\), and \(-\beta_t(S, w) \subseteq \text{cl}(-\beta_{t,s}(S, w) + \mathbb{E}^S[-\beta_s(S^*, w^*_t(S, w)) | \mathcal{F}_t])\). \(\square\)
The above theorem provides two equivalent representations for multi-portfolio time consistency for coherent risk measures. This generalizes the stability property for scalar risk measures, which is a well known result. Conceptually, stability means that pasting together dual variables creates another possible dual variable, which logically corresponds with time consistency concepts.

We conclude this section by noting that the superhedging risk measure (section 5.1) under a conical market model satisfies stability, stated explicitly in remark 5.1.4.

### 3.5 Composition of one-step risk measures

For this section we will restrict ourselves to the discrete time setting $T = \{0, 1, ..., T\}$. As in section 2.1 in [23] and section 4 in [24], a (multi-portfolio) time consistent version of any scalar dynamic risk measure can be created through backwards recursion. In the following we recall the corresponding results from proposition 3.11 and corollary 3.14 in [34] in the set-valued framework as well as the equivalent formulation for c.u.c. convex and coherent risk measures from corollary 5.3 in [36]. These results are useful for deducing the dual representations of composed (and thus multi-portfolio time consistent) risk measures.

Since multi-portfolio time consistency is a restrictive property for risk measures (in the scalar framework both value at risk and average value at risk are not time consistent, and in section 5.3 we show that the set-valued average value at risk is not multi-portfolio time consistent either), we would like to have a way to construct multi-portfolio time consistent versions of any risk measure. As in section 2.1 in [23] and section 4 in [24], a (multi-portfolio) time consistent version of any scalar dynamic risk measure can be created through backwards recursion. The same is true for set-valued risk measures as discussed in the following proposition.
Proposition 3.5.1. Let \((R_t)_{t \in T}\) be a dynamic risk measure on \(\mathcal{L}^p\), \((\tilde{R}_t)_{t \in T}\) defined for all \(X \in \mathcal{L}^p\) by

\[
\tilde{R}_T(X) = R_T(X), \tag{3.5.1}
\]

\[
\forall t \in T \setminus \{T\}: \tilde{R}_t(X) = \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) \tag{3.5.2}
\]

is multi-portfolio time consistent. Furthermore, \((\tilde{R}_t)_{t \in T}\) satisfies properties 1 and 2 in definition 2.1.2 of dynamic risk measures, but may fail to be finite at zero. Additionally, if \((R_t)_{t \in T}\) is (conditionally) convex ((conditionally) coherent, convex and c.u.c.) then \((\tilde{R}_t)_{t \in T}\) is (conditionally) convex ((conditionally) coherent, convex and c.u.c.).

Proof. Let \(t \in T \setminus \{T\}\), \(X \in \mathcal{L}^p\), and \(Y \subseteq \mathcal{L}^p\) such that \(\tilde{R}_{t+1}(X) \subseteq \bigcup_{Y \in Y} \tilde{R}_{t+1}(Y)\), then

\[
\tilde{R}_t(X) = \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) \subseteq \bigcup_{Z \in \bigcup_{Y \in Y} \tilde{R}_{t+1}(Y)} R_t(-Z) \]

\[
= \bigcup_{Y \in Y} \bigcup_{Z \in \tilde{R}_{t+1}(Y)} R_t(-Z) = \bigcup_{Y \in Y} \tilde{R}_t(Y).
\]

Thus, \((\tilde{R}_t)_{t \in T}\) is multi-portfolio time consistent. The property \(M_t \subseteq M_{t+1}\) ensures \(\tilde{R}_t\) to be \(M_t\)-translative for all \(t\). \(\mathcal{L}^p_+\)-monotonicity follows from the corresponding property for \(R_t\).

If \((R_t)_{t \in T}\) is (conditionally) convex (positive homogeneous) then by backwards induction \((\tilde{R}_t)_{t \in T}\) is (conditionally) convex (positive homogeneous).

By \(\tilde{R}_T = R_T\), then convex upper continuity trivially holds at time \(T\). Using backwards induction we will assume \(\tilde{R}_{t+1}\) is c.u.c., then \(\tilde{R}_t\) is the composition of convex and c.u.c. set-valued functions. Thus, by proposition A.3.1, \(\tilde{R}_t\) is c.u.c. \(\square\)
(\tilde{R}_t)_{t \in \mathbb{T}} defined as in equations (3.5.1), (3.5.2) is multi-portfolio time consistent, but not necessarily normalized or finite at zero. If (\tilde{R}_t)_{t \in \mathbb{T}} is normalized, then (\tilde{R}_t)_{t \in \mathbb{T}} is recursive itself, see also remark 3.2.7. Thus, we are interested to find conditions under which (\tilde{R}_t)_{t \in \mathbb{T}} is normalized, or finite at zero.

**Proposition 3.5.2.** Let (R_t)_{t \in \mathbb{T}} and (M_t)_{t \in \mathbb{T}} be as in proposition 3.5.1 and let (\tilde{R}_t)_{t \in \mathbb{T}} be defined as in (3.5.1), (3.5.2). Then, (\tilde{R}_t)_{t \in \mathbb{T}} is normalized and finite at zero if (R_t)_{t \in \mathbb{T}} is normalized and either of the following are true:

1. $R_t(0) = M_t$ for every time $t$, or

2. $\ (R_t)_{t \in \mathbb{T}}$ is a time consistent risk measure with $0 \in R_t(0)$ for all times $t$.

If $(R_t)_{t \in \mathbb{T}}$ is a normalized coherent risk measure with $0 \in R_t(0)$ for all times $t$ then $(\tilde{R}_t)_{t \in \mathbb{T}}$ is normalized, coherent and for $t \in \mathbb{T}$ either finite at zero or $\tilde{R}_t(X) \in \{0, M_t\}$ for every $X \in \mathcal{L}^p$.

**Proof.** If $R_T$ is normalized then it immediately follows that $\tilde{R}_T$ is normalized as well, indeed $\tilde{R}_T(0) = R_T(0)$. Thus, using backwards induction, we want to show that $\tilde{R}_t(X) = \tilde{R}_t(X) + \tilde{R}_t(0)$.

1. Let $\tilde{R}_{t+1}(0) = R_{t+1}(0) = M_{t+1,+}$, thus $Z \in \tilde{R}_{t+1}(0)$ implies $R_t(-Z) \subseteq R_t(0)$ by $\mathcal{L}^p_+$-monotonicity. Therefore

$$\tilde{R}_t(0) = \bigcup_{Z \in R_{t+1}(0)} R_t(-Z) = R_t(0) = M_{t,+}.$$ 

Trivially it then holds that $\tilde{R}_t(X) = \tilde{R}_t(X) + \tilde{R}_t(0)$.

2. Let $\tilde{R}_{t+1}(0) = R_{t+1}(0)$. By lemma 3.5.4 below $R_t$ is $R_{t+1}(0)$-monotone. Therefore it immediately follows that $R_t(-Z) \subseteq R_t(0)$ for every $Z \in \tilde{R}_{t+1}(0)$ (with $0 \in \tilde{R}_{t+1}(0)$), and

$$\tilde{R}_t(0) = \bigcup_{Z \in R_{t+1}(0)} R_t(-Z) = R_t(0).$$
\( \tilde{R}_t \) is normalized since \( \tilde{R}_t(X) + \tilde{R}_t(0) = \bigcup_{Z \in \tilde{R}_{t+1}(X)} [R_t(-Z) + R_t(0)] = \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) = \tilde{R}_t(X) \) by \( R_t \) normalized.

If \((R_t)_{t \in T}\) is a normalized closed coherent risk measure then \((\tilde{R}_t)_{t \in T}\) is coherent by proposition 3.5.1, and thus \( \tilde{R}_t(X) \supseteq \tilde{R}_t(X) + \tilde{R}_t(0) \) by subadditivity. To show the other direction assume that \( 0 \in \tilde{R}_{t+1}(0) \) (by \( 0 \in \tilde{R}_T(0) \) and backwards induction). It follows that \( \tilde{R}_t(0) = \bigcup_{Z \in \tilde{R}_{t+1}(0)} R_t(-Z) \supseteq R_t(0) \ni 0 \) since \( 0 \in R_t(0) \). This implies \( \tilde{R}_t(X) + \tilde{R}_t(0) \supseteq \tilde{R}_t(X) \). Therefore \( \tilde{R}_t \) is a normalized risk measure and \( \tilde{R}_t(0) \neq \emptyset \). However, it still may be the case that \( \tilde{R}_t(0) = M_t \). By normalization if \( \tilde{R}_t(0) = M_t \) then \( \tilde{R}_t(X) \in \{ \emptyset, M_t \} \) for any \( X \in \mathcal{L}^p \).

**Remark 3.5.3.** Trivially, for any normalized coherent conditional risk measure \( R_t \), it can be seen that \( R_t(0) \) is a convex cone, thus \( 0 \in R_t(0) \) if \( R_t \) is closed. Therefore the condition \( 0 \in R_t(0) \) in corollary 3.5.2 can be dropped when \( R_t \) is closed and coherent.

Property 2 in proposition 3.5.2 above shows that in the set-valued framework, even though time-consistency is not equivalent to multi-portfolio time consistency, it can be a useful property for the creation of multi-portfolio time consistent risk measures.

**Lemma 3.5.4.** If \((R_t)_{t \in T}\) is a time consistent risk measure and \( R_s \) is normalized for some time \( s \), then \( R_t \) is \( R_s(0) \)-monotone for any time \( t \leq s \).

**Proof.** Let \( X, Y \in \mathcal{L}^p \) such that \( Y - X \in R_s(0) \), then \( R_s(Y) = R_s(Y - X + X) = R_s(X) - (Y - X) = R_s(X) + R_s(0) - (Y - X) \supseteq R_s(X) \) by \( Y - X \in M_s \), \( R_s \) normalized, and \( 0 \in R_s(0) - (Y - X) \). Then by time consistency, \( R_s(Y) \supseteq R_s(X) \) implies \( R_t(Y) \supseteq R_t(X) \) for any \( t \leq s \), and therefore \( R_t \) is \( R_s(0) \)-monotone.

The following corollary provides a possibility to construct multi-portfolio time consistent risk measures by backward composition using the (one step) acceptance sets of any dynamic risk measure.

**Corollary 3.5.5.** Let \((R_t)_{t \in T}\) be a dynamic risk measure on \( \mathcal{L}^p \) and \((A_t)_{t \in T}\) its dynamic acceptance sets. The following are equivalent:
1. \((\tilde{R}_t)_{t\in\mathbb{T}}\) is defined as in (3.5.1) and (3.5.2);

2. \(\tilde{A}_T = A_T\) and \(\tilde{A}_t = \tilde{A}_{t+1} + A_{t,t+1}\) for each time \(t \in \mathbb{T}\setminus\{T\}\), where \(\tilde{A}_s\) is the acceptance set of \(\tilde{R}_s\) for all \(s\).

**Proof.** The proof is analogous to the proof of lemma 3.2.6, where \(R\) and \(A\) is replaced by \(\tilde{R}\) and \(\tilde{A}\) at the appropriate places. \(\square\)

**Corollary 3.5.6.** Let the assumptions of corollary 3.5.5 be satisfied. Additionally let \(p \in [1, +\infty]\) and

3. \((R_t)_{t\in\mathbb{T}}\) be c.u.c. and convex with minimal penalty function \((-\beta_t)_{t\in\mathbb{T}}\). Then, \((-\tilde{\beta}_t)_{t\in\mathbb{T}}\) defined recursively by

\[
-\tilde{\beta}_T(Q_T, w_T) = -\beta_T(Q_T, w_T), \\
-\tilde{\beta}_t(Q_t, w_t) = \text{cl} \left( -\beta_{t,t+1}(Q_t, w_t) + \mathbb{E}_Q \left[ -\tilde{\beta}_{t+1}(Q_{t+1}^t, w_{t+1}^t(Q_t, w_t)) \Big| \mathcal{F}_t \right] \right)
\]

where \(t \in \mathbb{T}\) and \((Q_t, w_t) \in \mathcal{W}_t\), is equivalently defined by

\[
-\tilde{\beta}_t(Q, w) := \text{cl} \bigcup_{Z \in \tilde{A}_t} (\mathbb{E}_Q [Z | \mathcal{F}_t] + G_t(w)) \cap M_t, \quad \text{(3.5.3)}
\]

where \((\tilde{A}_t)_{t\in\mathbb{T}}\) is obtained by the recursion in property 2 in corollary 3.5.5. The dynamic risk measure \((\tilde{R}_t)_{t\in\mathbb{T}}\) corresponding to \((\tilde{A}_t)_{t\in\mathbb{T}}\) is c.u.c. convex and multi-portfolio time consistent (but may fail to be finite at zero). Further, if \(\tilde{R}_t\) is finite at zero then \(\tilde{R}_t\) is equivalent to its dual form with penalty function \(-\tilde{\beta}_t\) and half-spaces \(G_t(w)\).

4. \((R_t)_{t\in\mathbb{T}}\) be conditionally c.u.c. and conditionally convex with dual representation (3.3.4) w.r.t. \(\mathcal{W}_t^w\) and minimal penalty function \((-\alpha_t)_{t\in\mathbb{T}}\). Then, \((-\tilde{\alpha}_t)_{t\in\mathbb{T}}\)
defined recursively by 

\[-\tilde{\alpha}_T(Q, w_T) = -\alpha_T(Q, w_T),\]

\[-\tilde{\alpha}_t(Q_t, w_t) = \text{cl} \left( -\alpha_{t+1}(Q_t, w_t) + \mathbb{E}^Q \left[ -\tilde{\alpha}_{t+1}(Q_{t+1}, w_{t+1}(Q_t, w_t)) \right] | \mathcal{F}_t \right) \]

where \( t \in \mathbb{T} \) and \((Q_t, w_t) \in \mathcal{W}_t^e \), is equivalently defined by

\[-\tilde{\alpha}_t(Q, w) := \text{cl} \bigcup_{Z \in \tilde{A}_t} \left( \mathbb{E}^Q [ Z | \mathcal{F}_t ] + \Gamma_t(w) \right) \cap M_t, \quad (3.5.4)\]

where \((\tilde{A}_t)_{t \in \mathbb{T}}\) is obtained by the recursion in property 2 in corollary 3.5.5. The dynamic risk measure \((\tilde{R}_t)_{t \in \mathbb{T}}\) corresponding to \((\tilde{A}_t)_{t \in \mathbb{T}}\) is conditionally c.u.c., conditionally convex and multi-portfolio time consistent (but may fail to be finite at zero). Further, if \(\tilde{R}_t\) is finite at zero then \(\tilde{R}_t\) is equivalent to its dual form with penalty function \(-\tilde{\alpha}_t\) and conditional half-spaces \(\Gamma_t(w)\).

5. \((R_t)_{t \in \mathbb{T}}\) be (conditionally) c.u.c. and (conditionally) coherent with maximal dual set \((\mathcal{W}_t^{\max})_{t \in \mathbb{T}}\). Then, \((\tilde{\mathcal{W}}_t^{\max})_{t \in \mathbb{T}}\) defined recursively by

\[-\tilde{\mathcal{W}}_T^\text{max} = \mathcal{W}_T^\text{max},\]

\[-\tilde{\mathcal{W}}_t^\text{max} = \mathcal{W}_t^{\max} \cap H_t^{t+1}(\tilde{\mathcal{W}}_{t+1}^{\max}),\]

where \( t \in \mathbb{T} \), is equivalently defined by

\[-\tilde{\mathcal{W}}_t^{\text{max}} := \left\{ (Q, w) \in \mathcal{W}_t : w_T^T(Q, w) \in \tilde{A}_t^+ \right\},\]

where \((\tilde{A}_t)_{t \in \mathbb{T}}\) is obtained by the recursion in property 2 in corollary 3.5.5. The dynamic risk measure \((\tilde{R}_t)_{t \in \mathbb{T}}\) corresponding to \((\tilde{A}_t)_{t \in \mathbb{T}}\) is (conditionally) c.u.c., (conditionally) coherent and multi-portfolio time consistent, and is finite at zero if and only if \(\tilde{\mathcal{W}}_t^{\text{max}} \neq \emptyset\) for all times \( t \).
Proof. 3. The proof of corollary 3.3.3 demonstrates the equivalence between the sum of penalty functions and the sum of acceptance sets, where $A$ and $-\beta$ have to be replaced by $\tilde{A}$ and $-\tilde{\beta}$ at the appropriate places. Regarding the assumptions of corollary 3.3.3: c.u.c. and convexity follow from proposition 3.5.1 and normalization is not needed for this equivalence as stated in remark 3.2.7. Notice that lemma 3.3.1 does not require the finite at zero properties for acceptance sets. Finally, if $\tilde{R}_t$ is finite at zero, then it is equivalent to its dual representation with minimal penalty function $-\tilde{\beta}_t$ by theorem 2.2.8.

4. The proof is analog to property 3, using corollary 3.3.4 instead of corollary 3.3.3 and corollary 2.2.12 instead of theorem 2.2.8. Adapting proposition A.3.1 to the conditional case yields $(\tilde{R}_t)_{t \in T}$ conditionally c.u.c.

5. Using the definition in (3.5.3), corollary 3.4.3, where $W, A$ and $-\beta$ is replaced by $\tilde{W}, \tilde{A}$ and $-\tilde{\beta}$ at the appropriate places, yields the equivalence between the two definitions of $(\tilde{W}_t^{\max})_{t \in T}$. Convex upper continuity and (conditionally) coherence follow from proposition 3.5.1. Conditional convex upper continuity follows if $(R_t)_{t \in T}$ is conditionally coherent by adapting proposition A.3.1 to the conditional case. Additionally, $\tilde{R}_t(0) \neq \emptyset$, see proof of proposition 3.5.2. Furthermore, $\tilde{R}_t(0) \neq M_t$ implies that $\tilde{R}_t$ is proper and thus the dual representation holds true. Then, there exists a $(\tilde{Q}, w) \in W_t$ such that $-\tilde{\beta}_t(\tilde{Q}, w) \neq M_t$, i.e. $\tilde{W}_t^{\max} \neq \emptyset$. And $\tilde{R}_t(0) = M_t$ implies, by proposition 13 (iv) in [47], $M_t = \tilde{R}_t(0) \subseteq \tilde{R}_t^{**}(0) = \bigcap_{(\tilde{Q}, w) \in \tilde{W}_t^{\max}} G_t(w) \cap M_t$ and thus $\tilde{W}_t^{\max} = \emptyset$.

The above results can be used to show that the convex superhedging portfolios are multi-portfolio time consistent (see section 5.1.1), and to deduce a multi-portfolio time consistent version of the average value at risk by backward recursion (see section 5.3) that is analogous to that given in the scalar framework provided in [23].
Chapter 4

Computation and a Set-Valued Bellman’s Principle

In this chapter, as in [37], we want to answer the question if it is possible to use the nested formulation (3.2.2), or, more generally, the backward composition (3.5.1), (3.5.2) to explicitly calculate the set $R_t(X)$, respectively the multi-portfolio time consistent version $\tilde{R}_t(X)$, backwards in time. If this is possible it would justify calling this procedure a set-valued Bellman’s principle, yielding a dynamic programming method for set-valued functions. This would be an interesting insight in itself within the field of set-optimization with applications beyond the one considered here.

For this chapter we will assume a finite sample space $\Omega$ with the power set sigma algebra, i.e. $\mathcal{F} = 2^{\Omega}$ over a discrete time space $T = \{0, 1, ..., T\}$. Since for a finite probability space we can choose any $p \in [0, +\infty]$, for this chapter we will work entirely with $p = 0$. We define $\Omega_t$ as the set of atoms in $\mathcal{F}_t$. For any $\omega_t \in \Omega_t$ we denote the successor nodes by

$$\text{succ}(\omega_t) = \{\omega_{t+1} \in \Omega_{t+1} : \omega_{t+1} \subseteq \omega_t\}.$$  

We use the convention that for an $\mathcal{F}_t$-measurable random variable $u$, we denote by $u(\omega_t)$ the value of $u$ at node $\omega_t$, that is $u(\omega_t) := u(\omega)$ for some $\omega \in \omega_t$. Further, we
denote by $R_t(X)[\omega_t] := \{u(\omega_t) : u \in R_t(X)\}$ the collection of projections of elements of $R_t(X)$ onto $\omega_t$. $R_t(X)[\cdot]$ is a random set not necessarily measurable.

4.1 An omega-wise representation for risk measures

In order to study a possible calculation of a random set $\tilde{R}_t(X)$ backwards in time on a finite event tree, one first needs to check if one can calculate the random set $\tilde{R}_t(X)$ $\omega_t$-wise at each node. This is not a concern when dealing with scalar risk measures, but in the set-valued case, certain conditions are needed to ensure that

$$u \in \tilde{R}_t(X) \iff u(\omega_t) \in \tilde{R}_t(X)[\omega_t] \quad \forall \omega_t \in \Omega_t.$$ 

Since we work on a finite probability space, for an $\omega_t$-wise approach to (3.5.1), (3.5.2) one will need $(\tilde{R}_t)_t \in T$ to have $\mathcal{F}_t$-decomposable images, which is satisfied if $(\tilde{R}_t)_t \in T$ has closed and conditionally convex images. Furthermore, for the multi-portfolio time consistent version to only depend on the possible future (successor) states, one will need $(R_t)_t \in T$ to be local.

**Remark 4.1.1.** Assuming $(R_t)_t \in T$ to be conditionally convex implies both, $(R_t)_t \in T$ being local (see proposition 2.1.8), as well as $(\tilde{R}_t)_t \in T$ being conditionally convex (see proposition 3.5.1) and thus having conditionally convex images. The more challenging property is to ensure that $(\tilde{R}_t)_t \in T$ has closed images. Let us give two examples, where $(\tilde{R}_t)_t \in T$ is closed (and thus has closed images): a) if the dynamic risk measure $(R_t)_t \in T$ is convex upper continuous and convex (see proposition 3.5.1), or b) if the dynamic risk measure $(R_t)_t \in T$ is polyhedral (that is if graph$(R_t)$ is a convex polyhedron, i.e. the intersection of finitely many closed half-spaces).
If \((\tilde{R}_t)_{t \in T}\) defined in (3.5.1), (3.5.2) is conditionally convex, but does not already have closed images, one needs to consider its closed-valued version, i.e.

\[
\bar{R}_T(X) := \text{cl}(R_T(X))
\]

\(
\forall t \in \{0, 1, ..., T - 1\} : \bar{R}_t(X) := \text{cl} \bigcup_{Z \in \bar{R}_{t+1}(X)} R_t(-Z)
\)

for all portfolios \(X \in \mathcal{L}_0\). Even though \((\tilde{R}_t)_{t \in T}\) itself might not have an \(\omega_t\)-wise representation in this case, we will show that \((\tilde{R}_t)_{t \in T}\) can be approximated for arbitrarily small \(\delta > 0\) by \((\bar{R}_t)_{t \in T}\) that admits an \(\omega_t\)-wise representation. The approximation is understood in the following sense. \((\bar{R}_t)_{t \in T}\) is called an approximation of \((\tilde{R}_t)_{t \in T}\) if

\[
\bar{R}_t(X) + \delta m \mathbf{1} \subseteq \tilde{R}_t(X) \subseteq \bar{R}_t(X)
\]

for any time \(t\), for every portfolio \(X \in \mathcal{L}_0\), for any approximation tolerance \(\delta > 0\) and any \(m \in \text{int}(M_+)\) in the subspace topology, i.e. with the topology given by \(\tau_M := \{A \cap M : A \in \tau\}\) where \(\tau\) is the topology on \(\mathbb{R}^d\).

**Theorem 4.1.2.** Let \((R_t)_{t \in T}\) be a conditionally convex dynamic risk measure. Let \((\tilde{R}_t)_{t \in T}\) denote its multi-portfolio time consistent version as defined in (3.5.1), (3.5.2). Then, we can calculate an approximation \((\bar{R}_t)_{t \in T}\) of \((\tilde{R}_t)_{t \in T}\) in an \(\omega_t\)-wise manner by

\[
\bar{R}_T(X)[\omega_T] = \text{cl}(R_T(X)[\omega_T]), \quad \forall \omega_T \in \Omega_T
\]

and for every state \(\omega_t \in \Omega_t\) and all times \(t \in T\setminus\{T\}\)

\[
\bar{R}_t(X)[\omega_t] = \text{cl}\bigcup \{R_{t,t+1}(-Z)[\omega_t] : Z(\omega_{t+1}) \in \bar{R}_{t+1}(X)[\omega_{t+1}] \forall \omega_{t+1} \in \text{succ}(\omega_t)\}.
\]

**Proof.** Define \((\bar{R}_t)_{t \in T}\) by equations (4.1.1), (4.1.2). Clearly, \((\bar{R}_t)_{t \in T}\) has closed images. One can show that \((\bar{R}_t)_{t \in T}\) is conditionally convex by backward induction. The
assertion is clearly true for $\bar{R}_T$. Now assume $\bar{R}_{t+1}$ is conditionally convex and let 
$X, Y \in \mathcal{L}^0$ and $\lambda \in L_1^\infty([0, 1])$. Then,

$$
\lambda \bar{R}_t(X) + (1 - \lambda) \bar{R}_t(Y) = \lambda \cl \bigcup_{Z_X \in R_{t+1}(X)} R_t(-Z_X) + (1 - \lambda) \cl \bigcup_{Z_Y \in R_{t+1}(Y)} R_t(-Z_Y)
$$

$$
\subseteq \cl \left( \cl \bigcup_{Z_X \in R_{t+1}(X)} \lambda R_t(-Z_X) + \cl \bigcup_{Z_Y \in R_{t+1}(Y)} (1 - \lambda) R_t(-Z_Y) \right)
$$

$$
= \cl \left( \bigcup_{Z_X \in R_{t+1}(X), Z_Y \in R_{t+1}(Y)} \lambda R_t(-Z_X) + \bigcup_{Z_Y \in R_{t+1}(Y)} (1 - \lambda) R_t(-Z_Y) \right)
$$

$$
= \cl \left( \bigcup_{Z_X \in R_{t+1}(X), Z_Y \in R_{t+1}(Y)} [\lambda R_t(-Z_X) + (1 - \lambda) R_t(-Z_Y)] \right)
$$

$$
\subseteq \cl \left( \bigcup_{Z_X \in R_{t+1}(X), Z_Y \in R_{t+1}(Y)} R_t(-(\lambda Z_X + (1 - \lambda) Z_Y)) \right)
$$

$$
= \cl \bigcup_{Z \in \lambda R_{t+1}(X) + (1 - \lambda) R_{t+1}(Y)} R_t(-Z)
$$

$$
\subseteq \cl \bigcup_{Z \in R_{t+1}(\lambda X + (1 - \lambda) Y)} R_t(-Z) = \bar{R}_t(\lambda X + (1 - \lambda) Y).
$$

Since the probability space is assumed to be finite, $\Omega_t$ is by definition the finest partition of $\Omega$ in $\mathcal{F}_t$ and $1_{\omega_t} u \in 1_{\omega_t} \bar{R}_t(X)$ if and only if $u(\omega_t) \in \bar{R}_t(X)[\omega_t]$. By lemma 6.1.17, which can be applied since $(\bar{R}_t)_{t \in T}$ has closed and conditionally convex images, it follows that $u \in \bar{R}_t(X)$ if and only if $u(\omega_t) \in \bar{R}_t(X)[\omega_t]$ for every $\omega_t \in \Omega_t$. Thus, one can calculate $\bar{R}_t(X)$ $\omega_t$-wise. Therefore the terminal condition (4.1.3) holds trivially. Let $t \in T \setminus \{T\}$ and $\omega_t \in \Omega_t$, using (4.1.2) and the local property for $R_t$ (see proposition 2.1.8) it follows that

$$
1_{\omega_t} R_t(X) = 1_{\omega_t} \cl \bigcup_{Z \in R_{t+1}(X)} R_t(-Z)
$$

$$
= \cl \bigcup_{Z \in R_{t+1}(X)} 1_{\omega_t} R_t(1_{\omega_t}(-Z)) = \cl \bigcup_{Z \in 1_{\omega_t} R_{t+1}(X)} 1_{\omega_t} R_t(-Z).
$$
Note that $Z \in 1_{\omega_t} \tilde{R}_{t+1}(X)$ if and only if $Z(\omega_{t+1}) \in \tilde{R}_{t+1}(X)[\omega_{t+1}]$ for every $\omega_{t+1} \in \text{succ}(\omega_t)$ and $Z(\omega_{t+1}) = 0$ otherwise. But as we only need to consider $1_{\omega_t}Z$ by $R_t$ local, the only constraints on $Z$ are imposed by $\omega_{t+1} \in \text{succ}(\omega_t)$. Thus, (4.1.4) follows.

Finally we will show that $\bar{R}_t$ is an approximation of $\tilde{R}_t$. By definition, $\bar{R}_T(X) := \text{cl}(\tilde{R}_T(X))$ for all $X \in \mathcal{L}^0$, which implies that $\bar{R}_T(X) + \delta_T m 1 \subseteq \tilde{R}_T(X) \subseteq \bar{R}_T(X)$ for any portfolio $X \in \mathcal{L}^0$, for any $\delta_T > 0$, and any $m \in \text{int}(M_+)$. For the proof by induction, assume that $\bar{R}_{t+1}$ is an approximation of $\tilde{R}_{t+1}$. Then for $t < T$ we have

$$\bar{R}_t(X) = \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) \subseteq \text{cl} \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) = \bar{R}_t(X)$$

$$\subseteq \text{cl} \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z + \delta_{t+1}m 1) = \text{cl} \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) - \delta_{t+1}m 1$$

$$\subseteq \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z) - (\epsilon + \delta_{t+1})m 1 = \tilde{R}_t(X) - (\epsilon + \delta_{t+1})m 1$$

for any $\epsilon, \delta_{t+1} > 0$, and any $m \in \text{int}(M_+)$. The first inclusion on the second line follows from the induction hypothesis. The first inclusion on the third line follows since $\text{cl}(\tilde{R}_t(X)) \subseteq \tilde{R}_t(X) - \epsilon m$ for any $\epsilon > 0$ and all $X \in \mathcal{L}^0$. Denote $\delta_t := \epsilon + \delta_{t+1} > 0$. Therefore for any time $t$ and any $\delta_t > 0$ we have that

$$\bar{R}_t(X) + \delta_t m 1 \subseteq \tilde{R}_t(X) \subseteq \bar{R}_t(X).$$

In the recursion (4.1.4), the calculation is dependent on $R_{t,t+1}(-Z)[\omega_t]$ which is, by locality of $R_t$, equal to $R_{t,t+1}(-1_{\omega_t}Z)[\omega_t]$, i.e. $R_{t,t+1}(-Z)[\omega_t]$ only depends on the part of $Z$ that can be attained from state $\omega_t$. For a local risk measure, we can therefore define $R_{t,t+1}(\cdot)[\omega_t]$ on $M_{t+1}[\omega_t]$ (where

$$M_{t+1}[\omega_t] = \{Z \in L^0 \cup \text{succ}(\omega_t), 2^{\text{succ}(\omega_t)}, \mathbb{P}(\cdot|\omega_t); \mathbb{R}^d) : \mathbb{P}(Z \in M) = 1\}$$
is the equivalence class of $\mathcal{F}_{t+1}$-measurable random variables in the eligible space $M$
starting from node $\omega_t$ by $R_{t,t+1}(Z)[\omega_t] := R_{t,t+1}(\hat{Z})[\omega_t]$ for $Z \in M_{t+1}[\omega_t]$ and some
$\hat{Z} \in M_{t+1}$ with $Z(\omega_{t+1}) = \hat{Z}(\omega_{t+1})$ for every $\omega_{t+1} \in \text{succ}(\omega_t)$.

**Remark 4.1.3.** If $(\tilde{R}_t)_{t \in T}$ defined in (3.5.1), (3.5.2) does have closed images already,
$(\tilde{R}_t)_{t \in T}$ coincides with $(\bar{R}_t)_{t \in T}$ and thus can be calculated in an $\omega_t$-wise manner by
(4.1.3), (4.1.4).

Observe that (4.1.2) is a set-valued optimization problem in the complete lattice
$\mathcal{G}(M_t; M_{t,+}) := \{D \subseteq M_t : D = \text{cl co}(D + M_{t,+})\}$, and (4.1.4) is a set-valued optimization problem in the complete lattice $\mathcal{G}(M; M_+)$. That is, the objective function
$R_{t,t+1}$ at node $\omega_t$ is a set-valued function that is minimized over the constraint set
$\tilde{Z}_t := \{Z \in M_{t+1}[\omega_t] : Z(\omega_{t+1}) \in \bar{R}_{t+1}(X)[\omega_{t+1}] \forall \omega_{t+1} \in \text{succ}(\omega_t)\}$:

$$\bar{R}_t(X)[\omega_t] = \text{cl} \bigcup_{Z \in \tilde{Z}_t} R_{t,t+1}(-Z)[\omega_t] = \inf_{Z \in \tilde{Z}_t} R_{t,t+1}(-Z)[\omega_t]. \quad (4.1.5)$$

Recall from [66] that the infimum over a function $f : M_{t+1}[^{\omega_t}] \to \mathcal{G}(M; M_+)$ is
given by $\inf_{X \in \mathcal{X}} f(X) = \text{cl co} \bigcup_{X \in \mathcal{X}} f(X)$ for $\mathcal{X} \subseteq M_{t+1}[\omega_t]$, where the convex hull in front of the union can be dropped here as $\bar{R}_t(X)[\omega_t]$ is convex by theorem 4.1.2.

Using the idea of [79], one can transform the set-valued problem (4.1.5) into a linear vector optimization problem that can be solved e.g. by Benson’s algorithm if the risk measure $(R_t)_{t \in T}$ is polyhedral. More generally, if $(R_t)_{t \in T}$ is the upper image of a convex vector optimization problem, one can transform the set-valued problem (4.1.5) into a convex vector optimization problem that can be approximately solved by the algorithms discussed in [68]. In both cases, one uses that the value of the set-optimization problem (4.1.5) can be written as the value of a vector optimization problem

$$\bar{R}_t(X)[\omega_t] = \inf_{(Z,Y) \in \tilde{Z}_t} \Phi_t(Z,Y). \quad (4.1.6)$$

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for the linear vector-valued function $\Phi_t(Z, Y) = \{Y\}$, feasible set

$$Z_t = \{(Z, Y) \in M_{t+1}[\omega_t] \times M : Y \in R_{t,t+1}(-Z)[\omega_t],
Z(\omega_{t+1}) \in R_{t+1}(X)[\omega_{t+1}] \forall \omega_{t+1} \in \text{succ}(\omega_t)\}$$

and ordering cone $M_+$. Let $\Phi_t(Z) = \{\Phi_t(Z) : Z \in Z_t\}$ denote the image of the feasible set. The set $\text{cl}(\Phi_t(Z) + M_+)$ is called the upper image of the vector optimization problem (4.1.6). We now discuss the constraints by looking at two cases: the polyhedral case and the convex case.

### 4.2 Linear vector optimization and polyhedral risk measures

Recall that a risk measure $R_t$ is polyhedral if $\text{graph}(R_t)$ is a convex polyhedron, i.e. the intersection of finitely many closed half-spaces. It is also equivalent to $R_t$ having a polyhedral acceptance set. For a polyhedral risk measure $(R_t)_{t \in T}$, problem (4.1.6) is a linear vector optimization problem.

**Proposition 4.2.1.** If the dynamic risk measure $(R_t)_{t \in T}$ is conditionally convex and polyhedral, its multi-portfolio time consistent version $(\tilde{R}_t)_{t \in T}$, defined in (3.5.1), (3.5.2), can be calculated $\omega_t$-wise, where in each node $\omega_t \in \Omega_t$, $t \in T \setminus \{T\}$, the linear vector optimization problem (4.1.6) has to be solved.

**Proof.** The $\omega_t$-wise representation follows from theorem 4.1.2. Now, let us show that problem (4.1.6) is a linear vector optimization problem. By $(R_t)_{t \in T}$ polyhedral and since $R_{t,t+1}[\omega_t]$ maps into $G(M_+; M_+)$, $R_{t,t+1}(-Z)[\omega_t]$ is the upper image of a linear vector optimization problem (see remark 5.1 in [51]), thus

$$R_{t,t+1}(-Z)[\omega_t] = \{P_t(x) + M_+ : B_t x \geq b_t\} \quad (4.2.1)$$
for a vector $x = (Z, z)^T$ that might include some auxiliary variable $z$, and for matrices $P_t$ and $B_t$ and vectors $b_t$ of appropriate dimensions. Then, the constraints $Y \in R_{t,t+1}(-Z)[\omega_t]$ can be equivalently written as


g \in \text{graph} R_{t,t+1}[\omega_t] \iff \tilde{M}^T(Y - P_t(x)) \geq 0, \quad B_t x \geq b_t,

depth 4

where the matrix $\tilde{M}$ contains the generating vectors of the positive dual $M^+_+ \cap M^+$. Thus, these constraints are linear. To obtain linearity of the other constraints, note that $(R_t)_{t \in \Omega}$ polyhedral implies $(R_t)_{t \in \Omega}$ closed (by definition) and $(\tilde{R}_t)_{t \in \Omega}$ closed. To see the last implication observe that $R_t$ is polyhedral if and only if $A_t$ is a polyhedron. The acceptance set of $\tilde{R}_t$ is given by $\tilde{A}_t = A_{t,t+1} + \tilde{A}_{t+1}$, see corollary 3.5.5. $A_{t,t+1}$ is a polyhedron since $R_{t,t+1}$ is polyhedral, by backwards recursion we assume $\tilde{A}_{t+1}$ is a polyhedron, and the sum of polyhedra is a polyhedron. Therefore $\tilde{A}_t$ is a polyhedron, which is equivalent to $\tilde{R}_t$ being polyhedral (and thus closed as well). Thus, in the polyhedral case, $(\tilde{R}_t)_{t \in \Omega}$ coincides with $(\tilde{R}_t)_{t \in \Omega}$. The linearity of the constraints $Z \in \tilde{R}_{t+1}(X)$ follow by induction. The constraints from the terminal condition $Z(\omega_T) \in \tilde{R}_T(X)[\omega_T] = R_T(X)[\omega_T]$ are linear for all $\omega_T \in \Omega_T$ by $R_T$ polyhedral. Thus, let us assume the constraints $Z(\omega_{t+1}) \in \tilde{R}_{t+1}(X)[\omega_{t+1}]$ are linear for all $\omega_{t+1} \in \text{succ}(\omega_t)$ for a given node $\omega_t \in \Omega_t$, then we need to show that $\tilde{R}_t(X)[\omega_t]$ is polyhedral. Since $\Omega$ is assumed to be finite, problem (4.1.6) is clearly a linear vector optimization problem. By (4.1.5), (4.1.6) and $R_t(t+1)(-Z)[\omega_t]$ mapping into $\mathcal{G}(M; M^+_+)$, we have $\tilde{R}_t(X)[\omega_t] = \Phi_t(Z_t) + M^+$, which is for finite $\Omega$ closed and polyhedral. Thus, $\tilde{R}_t(X)[\omega_t]$ is the upper image of the linear vector optimization problem (4.1.6).

Thus, if one assumes $(R_t)_{t \in \Omega}$ to be conditionally convex and polyhedral, $\tilde{R}_t(X)[\omega_t]$ is for every $t$ and $\omega_t$ the upper image of a linear vector optimization problem, which is polyhedral and can be calculated by Benson’s algorithm (see [51]). Then, the
set $\tilde{R}_t(X)$ can be calculated backwards in time, by solving at each node a linear vector optimization problem. Note that at time $t$, the problems for each $\omega_t \in \Omega_t$ can be calculated in parallel instead of sequentially which reduces computational time. Several examples will be discussed in chapter 5.

Benson’s algorithm is an appropriate tool to solve the vector optimization problem (4.1.6) as it takes advantage of the fact that the dimension $\dim(M)$ of the image space is usually significantly smaller than the dimension $d \times |\text{succ}(\omega_t)| + d + |z|$ of the pre-image space, where $|\text{succ}(\omega_t)|$ denotes the number of successor nodes of $\omega_t$ and $|z|$ denotes the dimension of the auxiliary variables in (4.2.1).

In practice (especially if $M$ is higher dimensional), when the number of vertices of the set $\tilde{R}_t(X)[\omega_t]$ is very high, one would calculate an $\epsilon$-approximation of $\tilde{R}_t(X)[\omega_t]$ having fewer vertices, see remark 4.10 in [51]. Then, for the backward recursion, one would need to know how the approximation errors accumulate over time. This will be discussed in propositions 4.3.2 and 4.3.5 below in a more general framework.

### 4.3 Convex vector optimization and conditionally convex risk measures

As the upper image of a convex vector optimization problem can only be calculated by a polyhedral approximation yielding an inner as well as an outer approximation with respect to some error level $\epsilon$ (see e.g. [30, 68]), we introduce $\epsilon$-approximations of sets, respectively functions. Throughout, we fix a parameter $m \in \text{int} M_+$.

**Definition 4.3.1.** Given a set $S \in \mathcal{P}(M; M_+)$ and an error level $\epsilon > 0$, we call a set $S^\epsilon \in \mathcal{P}(M; M_+)$ an $\epsilon$-approximation of $S$, if

\[ S^\epsilon + \epsilon m \subseteq S \subseteq S^\epsilon. \]
Given a set-valued function \( F : \mathcal{L}^0 \to \mathcal{P}(M_t; M_{t,+}) \) and an error level \( \epsilon > 0 \), we call the function \( F^\epsilon : \mathcal{L}^0 \to \mathcal{P}(M_t; M_{t,+}) \) an \( \epsilon \)-approximation of \( F \) if

\[
F^\epsilon(X) + \epsilon m 1 \subseteq F(X) \subseteq F^\epsilon(X) \text{ for every } X \in \mathcal{L}^0.
\]

In the convex case one can in general only approximately calculate the constraint set \( \bar{R}_{t+1}(X) \) in the backward recursion (4.1.4), respectively (4.1.5). Let us study the robustness of the set-optimization problem (4.1.5) to this perturbation of the constraints.

**Proposition 4.3.2.** Let \( \epsilon > 0 \). Let \( \bar{R}_t^\epsilon[X][\omega_{t+1}] \) be an \( \epsilon \)-approximation of the set \( \bar{R}_{t+1}(X)[\omega_{t+1}] \in \mathcal{P}(M; M_{+}) \) for each \( \omega_{t+1} \in \text{succ}(\omega_t) \), then \( \bar{R}_t^\epsilon[X][\omega_t] \) defined by

\[
\bar{R}_t^\epsilon(X)[\omega_t] := \text{cl} \bigcup_{Z \in \bar{Z}_t^\epsilon} R_{t,t+1}(-Z)[\omega_t],
\]

(4.3.1)

with

\[
\bar{Z}_t^\epsilon := \{ Z \in M_{t+1}[\omega_t] : Z(\omega_{t+1}) \in \bar{R}_t^\epsilon(X)[\omega_{t+1}] \forall \omega_{t+1} \in \text{succ}(\omega_t) \},
\]

is an \( \epsilon \)-approximation of \( \bar{R}_t(X)[\omega_t] \) defined in (4.1.4).

**Proof.** The assumption implies \( \bar{Z}_t \subseteq \bar{Z}_t^\epsilon \subseteq \bar{Z}_t - \epsilon m 1 \). This, together with (4.1.5) and transitivity of \( R_t \), yields

\[
\bar{R}_t(X)[\omega_t] = \text{cl} \bigcup_{Z \in \bar{Z}_t} R_{t,t+1}(-Z)[\omega_t] \subseteq \text{cl} \bigcup_{Z \in \bar{Z}_t^\epsilon} R_{t,t+1}(-Z)[\omega_t] = \bar{R}_t^\epsilon(X)[\omega_t]
\]

\[
\subseteq \text{cl} \bigcup_{Z + \epsilon m 1 \in \bar{Z}_t} R_{t,t+1}(-Z)[\omega_t] = \text{cl} \bigcup_{Z \in \bar{Z}_t} R_{t,t+1}(-Z)[\omega_t] - \epsilon m
\]

\[
= \bar{R}_t(X)[\omega_t] - \epsilon m.
\]

Thus, \( \bar{R}_t^\epsilon(X)[\omega_t] + \epsilon m \subseteq \bar{R}_t(X)[\omega_t] \subseteq \bar{R}_t^\epsilon(X)[\omega_t] \). i.e. \( \bar{R}_t^\epsilon(X)[\omega_t] \) is a \( \epsilon \)-approximation of \( \bar{R}_t(X)[\omega_t] \). \( \square \)
Next, we discuss under which conditions problem (4.3.1) is a convex vector optimization problem and which additional assumptions are necessary to apply the algorithm proposed in [68] to calculate a polyhedral $\epsilon$-approximation of the upper image of this convex vector optimization problem.

**Assumption 4.3.3.**

a) Let the objective function in (4.3.1) be of the form
\[
R_{t,t+1}(Z) = \{ \Psi_t(Z, z) + M_+ : g_t(Z, z) \leq 0 \}
\]
for an $M_+$-convex vector function $\Psi_t$, a component-wise convex vector function $g_t$ and a vector $z$, all of appropriate and finite dimensions.

b) Let the function $\Psi_t$ in a) be continuous, and let the feasible set have nonempty interior, i.e. let $X = \{(Z, z) : g_t(Z, z) \leq 0\}$ satisfy $\text{int} X \neq \emptyset$.

Assumption 4.3.3 a) means that the closure of $R_{t,t+1}(Z)_{\omega_t}$ is itself the upper image of a convex vector optimization problem.

**Proposition 4.3.4.** Let the objective function $R_{t,t+1}(Z)_{\omega_t}$ in (4.3.1) satisfy assumption 4.3.3 a) and let $\bar{R}_{t+1}(X)_{\omega_{t+1}}$ be a polyhedron for each $\omega_{t+1} \in \text{succ}(\omega_t)$. Then, problem (4.3.1) is a convex vector optimization problem.

**Proof.** Similar to (4.1.6), the set-valued problem (4.3.1) can be written as a vector optimization problem
\[
\inf_{(Z,Y) \in Z_t^*} \Phi_t(Z, Y),
\]
by setting $\Phi_t(Z, Y) = \{Y\}$ (which is a linear vector function), defining the feasible set as
\[
Z_t^* = \{(Z, Y) \in M_{t+1}[\omega_t] \times M : Y \in R_{t,t+1}(-Z)_{\omega_t} \}
Z(\omega_{t+1}) \in \bar{R}_{t+1}(X)_{\omega_{t+1}} \forall \omega_{t+1} \in \text{succ}(\omega_t)\}.
\]
and using $M_+$ as the ordering cone. The constraints $Z(\omega_{t+1}) \in \bar{R}_{t+1}^\varepsilon(X)[\omega_{t+1}]$ are by assumption linear. Under assumption 4.3.3 a), the constraints $Y \in R_{t,t+1}(-Z)[\omega_t]$ in (4.1.6) can be equivalently written as

$$(Y, -Z) \in \text{graph } R_{t,t+1}[\omega_t] \iff \hat{M}^T(\Psi_t(-Z, z) - Y) \leq 0, \quad g_t(-Z, z) \leq 0,$$

where the matrix $\hat{M}$ contains the generating vectors of $M_+^\varepsilon$. $\hat{M}^T(\Psi_t(-Z, z) - Y)$ is a component wise convex vector function since $\Psi_t$ is a $M_+$-convex vector function. Thus, these are convex constraints and (4.3.1) is a convex vector optimization problem.

The additional assumptions 4.3.3 b) are necessary to ensure, that problem (4.3.1) can be (approximately) solved by the algorithms presented in [68]. In detail, under assumptions 4.3.3 and if the feasible set $\mathcal{X} := \{(Z, z) : g_t(Z, z) \leq 0\}$ is compact, [68, theorems 4.9 and 4.14] state that the algorithms in [68] provide an $\varepsilon$-approximation of the upper image of (4.3.1), i.e. a polyhedral $\varepsilon$-approximation of $\bar{R}_t(X)$, if they terminate. However, the compactness assumption is typically not satisfied in the setting of risk measures. In that case [68, remark 3 in section 4.3] shows that the algorithms presented in [68] still return an $\varepsilon$-approximation of the upper image of (4.3.1) as long as all the scalar optimization problems within the algorithm can be solved and the algorithm terminates. In the example of the set-valued entropic risk measure considered in section 5.4, this will indeed be the case.

Since in general problem (4.3.1) can only be solved approximately (e.g. by the algorithms in [68]), one also needs to study how the approximation errors made at different time points accumulate over time.

**Proposition 4.3.5.** Let $\varepsilon, \gamma > 0$. If $\bar{R}_t^\varepsilon(\omega_t)$ is a $\gamma$-approximation of $\bar{R}_t^\varepsilon(\omega_t)$ defined in (4.3.1), then $\bar{R}_t^{\varepsilon,\gamma}(X)[\omega_t]$ is an $(\varepsilon + \gamma)$-approximation of $\bar{R}_t(X)[\omega_t]$ defined in (4.1.4).
Proof. \( \bar{R}^{\epsilon,\gamma}_t(X)\omega_t \) being a \( \gamma \)-approximation of \( \bar{R}^{\epsilon}_t(X)\omega_t \) means

\[
\bar{R}^{\epsilon,\gamma}_t(X)\omega_t + \gamma m \subseteq \bar{R}^{\epsilon}_t(X)\omega_t \subseteq \bar{R}^{\epsilon,\gamma}_t(X)\omega_t.
\]

Proposition 4.3.2 shows that \( \bar{R}^{\epsilon}_t(X)\omega_t \) is an \( \epsilon \)-approximation of \( \bar{R}^\epsilon_t(X)\omega_t \), i.e.

\[
\bar{R}^{\epsilon}_t(X)\omega_t + \epsilon m \subseteq \bar{R}^\epsilon_t(X)\omega_t \subseteq \bar{R}^{\epsilon}_t(X)\omega_t.
\]

Both chains of inclusions yield

\[
\bar{R}^{\epsilon,\gamma}_t(X)\omega_t + (\epsilon + \gamma)m \subseteq \bar{R}^\epsilon_t(X)\omega_t \subseteq \bar{R}^{\epsilon,\gamma}_t(X)\omega_t.
\]

We are now ready to prove the main result of this section. Recall that the aim was to (approximately) calculate the multi-portfolio time consistent risk measure \( \tilde{R}^\epsilon_t(X) \) backwards in time in the spirit of a set-valued Bellman’s principle. We will see that \( \tilde{R}^\epsilon_t(X) \) can be obtained by solving at each node backwards in time a convex vector optimization problem. In practice, these problems can only be approximately solved. But we are able to determine the overall approximation error, when the approximation error at each node is chosen to be \( \epsilon > 0 \). One could of course also vary this error level at different nodes or different time points and obtain corresponding results.

**Proposition 4.3.6.** Let \( (R_t)_{t \in T} \) be a conditionally convex dynamic risk measure satisfying assumption 4.3.3. Let \( \epsilon > 0 \). Then for any time \( t \) and given \( X \in \mathcal{L}^0 \), we can find a \( (T - t + 1)\epsilon + \delta \)-approximation of the multi-portfolio time consistent version \( (\tilde{R}_t(X))_{t \in T} \) defined in (3.5.1), (3.5.2), by calculating backwards in time at each node \( \omega_t \in \Omega_t \) an \( \epsilon \)-approximation of the upper image of the convex vector optimization problem (4.3.1). Here \( \delta > 0 \) can be chosen arbitrarily small.
Proof. Assumption 4.3.3 and the local property of $(\mathcal{R}_t)_{t \in \mathbb{T}}$ imply that all the assumptions of theorem 4.1.2 are satisfied, thus a $(\delta)$-approximation $(\bar{\mathcal{R}}_t)_{t \in \mathbb{T}}$ of $(\hat{\mathcal{R}}_t)_{t \in \mathbb{T}}$ can be calculated $\omega_t$-wise for arbitrarily small $\delta > 0$ by (4.1.4), (4.1.3).

For $t = T$ one obtains an $\epsilon$-approximation $\bar{\mathcal{R}}^*_T(X)$ of $\hat{\mathcal{R}}_T(X) = \text{cl}(\hat{\mathcal{R}}_T(X)) = \text{cl}(\mathcal{R}_T(X))$ by calculating an $\epsilon$-approximation of the upper image of the convex vector optimization problem $\mathcal{R}_T(X)[\omega_T] = \{ \Psi_T(X,z) + M_+ : g_T(X,z) \leq 0 \}$ (see assumption 4.3.3 a)) at each node $\omega_T \in \Omega_T$. $\bar{\mathcal{R}}^*_T(X)$ is by construction polyhedral (see the algorithms in [68]) and is the input for problem (4.3.1) at time $t = T - 1$, which is by proposition 4.3.4 then a convex vector optimization problem. Its solution would by proposition 4.3.2 yield an $\epsilon$-approximation $\bar{\mathcal{R}}^*_{T-1}(X)$ of $\hat{\mathcal{R}}_{T-1}(X)$, but one can in general only calculate an $\epsilon$-solution. This $\epsilon$-solution yields an $\epsilon$-approximation of $\bar{\mathcal{R}}^*_{T-1}(X)$, which is by proposition 4.3.5 a $2\epsilon$-approximation of $\hat{\mathcal{R}}_{T-1}(X)$.

Going backwards like this yields for any $t$ a $(T - t + 1)\epsilon$-approximation of $\hat{\mathcal{R}}_t(X)$, which is by theorem 4.1.2 and the logic of adding up approximation errors as in proposition 4.3.5 a $(T - t + 1)\epsilon + \delta$-approximation of the multi-portfolio time consistent version $\hat{\mathcal{R}}_t(X)$ for arbitrarily small $\delta > 0$. \qed

### 4.4 Interpretation and relation to Bellman’s principle

The risk measure $(\bar{\mathcal{R}}_t)_{t \in \mathbb{T}}$, while constructed backwards in time, has a nice financial interpretation involving portfolio injections made as time progresses, that is an interpretation forwards in time: For every choice of a risk compensating portfolio holding $Z_0 \in \bar{\mathcal{R}}_0(X)$ at time $t = 0$, there exists, by equation (3.5.2), a sequence of portfolio holdings $(Z_t)_{t \in \mathbb{T} \setminus \{0\}}$ such that

$$Z_t \in \bar{\mathcal{R}}_t(X) \quad (4.4.1)$$
and
\[ Z_{t-1} \in R_{t-1}(-Z_t). \tag{4.4.2} \]

Inclusion (4.4.1) means \( Z_0, Z_1, ..., Z_T \) are the risk compensating portfolio holdings at times 0, 1, ..., \( T \). An intuitive interpretation of (4.4.2) can be obtained by the following reformulation. Defining the portfolio injections (respectively withdrawals if negative) \((u_t)_{t \in \mathbb{T}}\) that are needed to update the risk compensating portfolio holdings by \( u_t = Z_t - Z_{t-1} \) (with \( u_0 = Z_0 \)), the two conditions on \((Z_t)_{t \in \mathbb{T}}\) can be rewritten in terms of \((u_t)_{t \in \mathbb{T}}\) as follows
\[ u_t \in \tilde{R}_t(X + \sum_{s=0}^{t-1} u_s) \]
for every time \( t \), and
\[ 0 \in R_t(-u_{t+1}), \tag{4.4.3} \]
for \( t \in \mathbb{T}\setminus\{T\} \). Inclusion (4.4.3) means the risk of the portfolio injection needed at time \( t + 1 \) is acceptable at time \( t \) with respect to the one-period risk measure \( R_t \).

This gives the main interpretation of the backward composition of \((R_t)_{t \in \mathbb{T}}\). At each one-period step the original measure \((R_t)_{t \in \mathbb{T}}\) is used, but it is used in a time consistent way in the sense of Bellman.

One can observe Bellman’s principle of optimality: The at \( t \) truncated optimal solution \((Z_s)_{s=t}^T\) obtained at time 0 from (3.5.2) and a given \( Z_0 \in \tilde{R}_0(X) \) is still optimal at any later time point \( t \in \mathbb{T} \). To see that, note that for the risk compensating portfolio holding \( Z_t \in \tilde{R}_t(X) \), \((Z_s)_{s=t}^T\) satisfies the conditions \( Z_s \in \tilde{R}_s(X) \) and \( Z_{s-1} \in R_{s-1}(-Z_s) \), \( s \in \{t, ..., T\} \) from (3.5.2).

Let us now explain on how to compute \((\tilde{R}_t(X))_{t \in \mathbb{T}}\) and how to obtain for a given \( Z_0 \in \tilde{R}_0(X) \) at time \( t = 0 \) a sequence \((Z_t)_{t \in \mathbb{T}\setminus\{0\}}\) of risk compensating portfolio holdings on the realizing path. \((\tilde{R}_t)_{t \in \mathbb{T}}\) can be calculated with the approach discussed in sections 4.2 and 4.3. Benson’s algorithm also calculates a solution of the linear vector optimization problems in the sense of definition 2.20 in [66],
respectively, an $\epsilon$-solution in the sense of definition 3.3 in [68] for a convex vector optimization problem. These finite solution sets are then used to calculate the sequence of risk compensating portfolio holdings $(Z_t)_{t \in T}$, respectively the injection/withdrawal strategy $(u_t)_{t \in T}$, forwards in time on the realizing path by solving an additional linear program, specified in the following, at each point in time. Let

$$\mathcal{X}_t[\omega_t] = \{(Z^{i}_{t+1}[\omega_t], Y^{i}_{t}[\omega_t]) : i = 1, \ldots, n, n \in \mathbb{N}\} \subseteq M_{t+1}[\omega_t] \times M$$

be the ($\epsilon$)-solution set to the vector optimization problem (4.1.6). Let us first explain the method in case of a linear vector optimization problem: For any $Z_0$ in the risk measure $\tilde{R}_0(X)$, there exists a convex combination of elements of the solution such that $Z_0 \geq \sum_{i=1}^{n} \lambda^{*}_{i} Y^{i}_{0}$ on the efficient frontier. This coefficient vector $\lambda^{*} \in \mathbb{R}^{n}_{+}$ can be found by solving any linear optimization problem of the form

$$\min_{\lambda \in \mathbb{R}^{d}_{+}} c^T (Y^{1}_{0}, \ldots, Y^{n}_{0}) \lambda \quad \text{subject to} \quad (Y^{1}_{0}, \ldots, Y^{n}_{0}) \lambda \leq Z_0, \quad \mathbb{1}^T \lambda = 1 \quad (4.4.4)$$

with $c \in \mathbb{R}^{d}_{+} \setminus \{0\}$. The coefficient vector $\lambda^{*} \in \mathbb{R}^{n}_{+}$ can then be used to define $Z^{*}_{0} := \sum_{i=1}^{n} \lambda^{*}_{i} Y^{i}_{0}$ on the efficient frontier of $\tilde{R}_0(X)$. Notice that $Z^{*}_{0} = Z_0$ if $Z_0$ is already on the efficient frontier. Additionally, the next time step full capital requirement is given by $Z_{1} := \sum_{i=1}^{n} \lambda^{*}_{i} Z^{i}_{1}$, which might not be on the efficient frontier of $\tilde{R}_1(X)$. This process is repeated through the event tree forwards in time. The choice of cost vector $c$ (or alternatively a nonlinear cost function) determines the possible liquidation/withdrawal strategy akin to that discussed for the superhedging risk measure in [67].

In the case of a convex vector optimization problem, one can calculate only a polyhedral $\tilde{\epsilon}$-approximation (e.g. $\tilde{\epsilon} = (T + 1)\epsilon + \delta$ as in proposition 4.3.6) $\tilde{R}^{\tilde{\epsilon}}_0(X)$ of $\tilde{R}_0(X)$. Thus, when choosing the initial capital, one would pick a minimal capital from the calculated inner approximation, and not the true set, i.e. $u_0 \in \tilde{R}^{\tilde{\epsilon}}_0(X) + \tilde{\epsilon}m$. Noting that an $\epsilon$-solution of problem (4.1.6) provides a solution to the linear
vector optimization problem whose upper image is the inner approximation, the same
procedure as in the linear case can be applied, just replacing $\tilde{R}_0(X)$ by its inner
polyhedral approximation. One obtains an $\tilde{\epsilon}_t$-optimal strategy of risk compensating
portfolios $(Z_t)_{t \in T}$, respectively portfolio injections $(u_t)_{t \in T}$, with $\tilde{\epsilon}_t = (T - t + 1)\epsilon + \delta$
when using the same error level $\epsilon > 0$ in each iteration step, see proposition 4.3.6.

4.5 Computation of market extensions

Market extensions are considered when one is not only interested in putting a ‘capital
requirement’ $u \in R_t(X)$ at time $t$ aside and holding it until time $T$ to make $X$ risk
neutral, but in exploiting the trading opportunities at the market to minimize the
amount of capital needed for risk compensation. For the definition of the market
extension below, we will set $M = \mathbb{R}^d$, i.e. we consider the full space of eligible assets.
A justification for that comes from a mathematical as well as an interpretational
aspect, which will be detailed in remark 4.5.4 below. But one can already understand
that choice by realizing that the role of $M$ comes mainly from a regulatory point
of view. A regulator might only allow capital requirements to be made in certain
currencies for example, and these capital requirements are held until time $T$. But the
market extension is more linked to internal risk measurement and management as
one is exploring trading opportunities in possibly all assets, and thus will hold at any
time $t$ a portfolio in possibly all assets, so there is no need in restricting the capital
requirements to be made in certain assets only. The *market extension* $(R_t^{mar})_{t \in T}$
of a dynamic risk measure $(R_t)_{t \in T}$ is given by

$$R_t^{mar}(X) := \bigcup_{k \in K_t} R_t(X - k)$$
for some $K_t \subseteq \mathcal{L}_t^0$ modeling the set of attainable claims. When $K_t \subseteq \mathcal{L}_t^0$ then it immediately follows that $R_t^{mar}(X) = R_t(X) + K_t$. Let us give a few examples, all are special cases of the set-valued portfolios introduced in [21] and used in section 6.2.

**Example 4.5.1.** In a market with proportional transaction costs, trading is modeled by a sequence of solvency cones $(K_t)_{t \in T}$, see [59, 78, 60]. $K_t$ is a **solvency cone** at time $t$ if it is an $\mathcal{F}_t$-measurable cone such that for every $\omega \in \Omega$, $K_t[\omega]$ is a closed convex cone with $\mathbb{R}_+^d \subseteq K_t[\omega] \subseteq \mathbb{R}^d$. $K_t$ is generated by the bid and ask prices between any two assets at time $t$. In a market with proportional transaction costs, one would set $K_t = \mathcal{L}_t^0(K_t)$.

**Example 4.5.2.** More generally, in markets with illiquidity (convex transaction costs) as in [70], trading is modeled by a sequence of convex solvency regions $(K_t)_{t \in T}$. $K_t$ is a **convex solvency region** at time $t$ if it is an $\mathcal{F}_t$-measurable set such that for every $\omega \in \Omega$, $K_t[\omega]$ is a closed convex set with $\mathbb{R}_+^d \subseteq K_t[\omega] \subseteq \mathbb{R}^d$. Then, one would set $K_t = \mathcal{L}_t^0(K_t)$.

**Example 4.5.3.** One could also incorporate **trading constraints** on the size of transactions by considering convex random sets $D_t$ (not necessarily mapping into $\mathcal{G}(\mathbb{R}_+^d, \mathbb{R}^d)$) as follows. Given $t \in T$, let $D_t : \Omega \to 2^{\mathbb{R}_+^d}$ (with $2^{\mathbb{R}_+^d}$ denoting the power set of $\mathbb{R}_+^d$) be an $\mathcal{F}_t$-measurable function such that $D_t[\omega]$ is a closed convex set and $K_t[\omega] \cap D_t[\omega] \neq \emptyset$ for every $\omega \in \Omega$. Then, one would set $K_t = \mathcal{L}_t^0(K_t \cap D_t)$.

Considering the market extension of the multi-portfolio time consistent version of $(R_t)_{t \in T}$ with respect to $K_t \subseteq \mathcal{L}_t^0$, one realizes that it coincides with the multi-portfolio time consistent version of the market extension of $(R_t)_{t \in T}$.

\[
\tilde{R}_t^{mar}(X) := \bigcup_{k \in K_t} \bigcup_{Z \in \tilde{R}_t^{mar}(X-k)} R_t(-Z) = \bigcup_{Z \in \tilde{R}_t^{mar}(X)} (R_t(-Z) + K_t). \tag{4.5.1}
\]

Thus, the two operations ‘market extension’ and ‘multi-portfolio time consistent version’ are interchangeable under the assumption $M = \mathbb{R}^d$. 

In analogy to section 4.4 for the regulator risk measure, one can obtain a nice financial interpretation of the market extended composed risk measure \((\tilde{R}^{mar}_{t})_{t \in T}\) involving portfolio injections and trades made forwards in time. In addition to the sequence of portfolios holdings \((Z_t)_{t \in T}\) one obtained for the regulator risk measure, there additionally exists by equation (4.5.1) a sequence of trades \((k_t)_{t \in T}\) such that \(Z_t \in \tilde{R}^{mar}_{t}(X)\), \(k_t \in K_t\), and \(Z_t - k_t \in R_t(-Z_{t+1})\) for every choice of a risk compensating portfolio holding \(Z_0 \in \tilde{R}^{mar}_{0}(X)\) at time \(t = 0\). That means \(Z_0, Z_1, ..., Z_T\) are the risk compensating portfolio holdings before trades at times \(0, 1, ..., T\) and \(Z_0 - k_0, Z_1 - k_1, ..., Z_T - k_T\) are the risk compensating portfolio holdings after the trades at times \(0, 1, ..., T\). Equivalently, the portfolio injections (respectively withdrawals if negative) \((u_t)_{t \in T}\), needed to update the risk compensating portfolio holdings and defined by \(u_t = Z_t - Z_{t-1} + k_{t-1}\) (with \(u_0 = Z_0\)), satisfy

\[
u_t \in \tilde{R}^{mar}_{t}(X + \sum_{s=0}^{t-1} (u_s - k_s) - k_t)\]

for every time \(t\), and

\[0 \in R_t(-u_{t+1}),\]

for \(t \in T \setminus \{T\}\). This gives the same interpretation as with the composed regulator risk measure discussed in section 4.4: The portfolio injections of the next time period \(t + 1\) are random, but acceptable with respect to the one-period risk measure \(R_{t,t+1}\).

Let us now discuss how to calculate \(\tilde{R}^{mar}_{t}(X)\). Assume \(K_t\) is closed and conditionally convex for all times \(t\), then \((\tilde{R}^{mar}_{t})_{t \in T}\) can be calculated with the approach discussed in sections 4.2 and 4.3 by adding \(K_t[\omega_t]\) to the vector optimization problem (4.1.6) at each time \(t\). In particular, if \(K_t = \mathcal{L}_t^0(K_t)\) for a solvency cone \(K_t\) for all times \(t\) as in example 4.5.1, Benson’s algorithm can be applied directly to problem (4.1.6), but replacing the ordering cone \(M_+ = \mathbb{R}_+^d\) with \(K_t[\omega_t]\).
As with the ‘regulator risk measure’ considered in the previous sections, \( \tilde{R}^{mar}_t(X) \) might not be closed, but an arbitrarily close approximation is given by its closed-valued variant

\[
\bar{R}^{mar}_t(X) := \text{cl} \bigcup_{Z \in R^{mar}_{t+1}(X)} (R_t(-Z) + K_t).
\]

Solving at each node backwards in time the vector optimization problem with objective \( \Phi_t(Z, Y + k) \) and feasible region \( (Z, Y, k) \in Z_t \times K_t[\omega] \) (with \( \tilde{R}_{t+1} \) replaced by \( \tilde{R}_t^{mar} \)) yields \( (\bar{R}_t^{mar})_{t \in \mathcal{T}} \). Benson’s algorithm yields the solution set \( \bar{X}^{mar,\omega}_t \subseteq L^0_t[\omega] \times \mathbb{R}^d \times K_t[\omega] \) (with \( L^0_{t+1}[\omega] = L^0_d(\cup \text{succ}(\omega), \mathbb{P}(\cdot | \omega)) \)), which can be used to calculate the sequence \((Z_t)_{t \in \mathcal{T} \setminus \{0\}}\) and now additionally the sequence of trades \((k_t)_{t \in \mathcal{T}}\) forwards in time on the realizing path for a given \( Z_0 \in \tilde{R}_0^{mar}(X) \) (or its inner approximation) at time \( t = 0 \). Utilizing (4.4.4), we can find a convex combination of the \( Y \) elements of the solution to describe any portfolio on the efficient frontier of the risk measure, respectively of the inner approximation of it in the convex, non-polyhedral case. The same convex coefficients are then used for both the trading strategy and the next time step full capital requirements, which are then used as the starting value in the next period.

**Remark 4.5.4.** Let us comment on the choice of \( M = \mathbb{R}^d \) in this subsection. After calculating the risk measure \( (\tilde{R}_t^{mar})_{t \in \mathcal{T}} \), it is of course possible to choose a subspace of eligible portfolios \( M \) and choose the capital requirements to be in that space (if \( \tilde{R}_t^{mar}(X) \cap M_t \) is non-empty).

However, if the subspace \( M \neq \mathbb{R}^d \) were to be chosen first and used for the recursive computation, then the market extension of the multi-portfolio time consistent version would in general not be equal to the multi-portfolio time consistent version of the market extension. This would cause several problems as the market extension of the multi-portfolio time consistent version, i.e. \( \tilde{R}_t^{mar,M}(X) := \bigcup_{k \in K_t} \bigcup_{Z \in \tilde{R}^{mar}_{t+1}(X-k)} R_t(-Z) \cap M_t \), while retaining the capital injection interpretation given above, is in general not multi-
portfolio time consistent. Furthermore, this approach is more restrictive than the one proposed above as it holds $\tilde{R}_t^{mar,M}(X) \subseteq \tilde{R}_t^{mar}(X) \cap M_t$.

On the other hand, the multi-portfolio time consistent version of the market extension, while being multi-portfolio time consistent, does not admit a good economic interpretation.

All of these problems disappear if $M = \mathbb{R}^d$. For this, and the motivation given at the beginning of this subsection, we suggest to use $M = \mathbb{R}^d$ when considering market extensions and, if needed, one can choose $u_t \in \tilde{R}_t^{mar}(X) \cap M_t$ afterwards.
Chapter 5

Examples

In this chapter we will present examples of dynamic risk measures in the set-optimization framework proposed in chapter 2. In particular, we will study the primal and dual representations for certain convex and coherent risk measures, and also discuss whether each is multi-portfolio time consistency. When not multi-portfolio time consistency we use the results from section 3.5 to find the multi-portfolio time consistent version. Additionally, in this chapter we will implement the algorithms for the recursive calculation of polyhedral and conditionally convex risk measures presented in chapter 4. Specifically we will consider the superhedging price, relaxed worst case risk measure, average value at risk, and entropic risk measure, which were considered in [34, 36, 37].

We consider a multi-dimensional tree that approximates the $d-1$ asset prices (denoted in the domestic currency) and assume stock price dynamics under the physical measure $\mathbb{P}$ are given by correlated geometric Brownian motions:

$$dS^i_t = S^i_t(\mu_i dt + \sigma_i dW^i_t), \quad i = 1, ..., d - 1$$

for Brownian motions $W^i$ and $W^j$ with correlation $\rho_{ij} \in [-1, 1]$. To create a tree for the correlated risky assets, we follow the approach in [62]. We expand the tree
structure produced in such a method by allowing for $n^{d-1}$ (recombining) branches for any natural number $n \geq 2$ and consider some maximum change from a parent to child node given by $\nu \in \mathbb{R}_{++}$ (instead of $2^{d-1}$ branches and $\nu = 1$ from the binomial model presented in [62]). That is, since every asset can rise or fall, we consider the set of possible up-down scenarios given by

$$
\mathcal{E} = \left\{ (w_1, \ldots, w_{d-1})^T : w_i \in \left\{ -\nu, -\nu + \frac{2\nu}{n-1}, \ldots, \nu - \frac{2\nu}{n-1}, \nu \right\} \forall i = 1, \ldots, d-1 \right\}.
$$

We note that in the situation with only a single risky asset, $n = 2$, and $\nu = 1$, this reduces to the Cox-Ross-Rubinstein binomial tree model. To calculate the (conditional) probabilities of reaching a successor node by partitioning the space $\mathbb{R}^{d-1}$ into $n^{d-1}$ boxes so that each element of $\mathcal{E}$ resides in a unique box. The probability of rising or falling by level $e \in \mathcal{E}$ is given by the probability of the surrounding box under a (multivariate) normal distribution.

For simplicity, we additionally assume that the proportional transaction costs are constant for each of the risky assets, given by $\gamma = (\gamma_1, \ldots, \gamma_{d-1})^T \in \mathbb{R}_{++}^{d-1}$ (possibly 0). Thus the bid and ask prices are given by $(S^b_t)^i = S^i_t(1 - \gamma_i)$ and $(S^a_t)^i = S^i_t(1 + \gamma_i)$ respectively for every $i = 1, \ldots, d-1$. In the case that $\gamma_i = 0$ then the bid-ask spread is 0 for the $i^{th}$ risky asset.

Assume the existence of a risk-free asset with dynamics $(B_t)_{t \in \mathbb{T}}$ and no bid-ask spread, i.e. $B^b_t = B^a_t$ at all times $t$. Further, we consider the case where cash (i.e. the risk-free asset) is an intermediary for all transactions. That is, the exchange between any two assets is done via cash and not directly. Under proportional transaction costs, the above simplifying assumptions ensure that the solvency cone $K_t$ at time $t$ is generated by the columns of the matrix

$$
\begin{pmatrix}
(S^a_t)^T & -(S^b_t)^T \\
-I_{d-1 \times d-1} & I_{d-1 \times d-1}
\end{pmatrix}
$$
where \( I_{d-1 \times d-1} \) denotes the identity matrix with \( d-1 \) rows and columns.

For the examples that contain convex transaction costs, we assume that there are quadratic market impact costs, i.e. when trading \( x \) units of a risky asset it adds a cost proportional to \( x^2 \). We incorporate this into our model by defining the actualized price of a transaction by \( \pi(S,x) = S - \theta \cdot S \cdot x \) for some parameter \( \theta \in \mathbb{R}^{d-1}_+ \) where \( S \) is the initial price (bid or ask if selling or buying respectively) and \( x \) is the number of units of the asset being traded. That is, the average actualized price per unit is given by \( \pi(S,x) \). For simplicity, we assume that the trading strategy chosen does not impact the future market. Thus, a trade at time \( t \) does not affect the market at time \( t+1 \).

### 5.1 Superhedging price

In this section, we define the dynamic extension of the set of superhedging portfolios in markets with proportional transaction costs as presented in [34, 36, 59, 78, 60, 50, 67]. In subsection 5.1.1 we extend the results herein to include convex transaction costs, as was shown in [36]. Under a conical market model, we show that the set of superhedging portfolios yields a set-valued market-compatible coherent dynamic risk measure that is multi-portfolio time consistent.

Consider a discrete time setting such that \( \mathbb{T} = \{0, 1, ..., T\} \).

Let the random variable \( V_t : \Omega \rightarrow \mathbb{R}^d \) be a portfolio vector at time \( t \) such that the values of \( V_t(\omega) \) are in physical units as described in [59, 78]. That is, the \( i^{th} \) element of \( V_t(\omega) \) is the number of asset \( i \) in the portfolio in state \( \omega \in \mathcal{F}_t \) at time \( t \). An \( \mathbb{R}^d \)-valued adapted process \((V_t)_{t \in \mathbb{T}}\) is called a self-financing portfolio process for the market given by the solvency cones \((K_t)_{t \in \mathbb{T}}\) if

\[
\forall t \in \mathbb{T} : V_t - V_{t-1} \in -K_t \ \mathbb{P}-\text{a.s.}
\]
where \( V_{-1} = 0 \).

Let \( C_{t,T} \subseteq \mathcal{L}^p \) be the set of \( \mathcal{L}^p \)-valued random vectors \( V_T : \Omega \to \mathbb{R}^d \) that are the values of a self-financing portfolio process at terminal time \( T \) with endowment 0 at time \( t \). From this definition it follows that \( C_{t,T} = \sum_{s=t}^{T} -L^p_s(K_s) \).

An \( \mathbb{R}^d_+ \)-valued adapted process \( Z = (Z_t)_{t \in T} \) is called a consistent pricing process for the market model \( (K_t)_{t \in T} \) if \( Z \) is a martingale under the physical measure \( \mathbb{P} \) and

\[
\forall t \in T : Z_t \in K_t^+ \setminus \{0\} \quad \mathbb{P}\text{-a.s.}
\]

The market is said to satisfy the robust no arbitrage property \( (\text{NA}^r) \) if there exists a market process \( (\tilde{K}_t)_{t \in T} \) satisfying

\[
K_t \subseteq \tilde{K}_t \quad \text{and} \quad K_t \setminus K_t \subseteq \text{int} \tilde{K}_t \quad \mathbb{P}\text{-a.s.}
\]

for all \( t \in T \) such that

\[
\tilde{C}_{0,T} \cap L^0_d(\mathcal{F}_T, \mathbb{R}^d_+ \setminus \{0\}) = \{0\},
\]

where \( \tilde{C}_{0,T} \) is generated by the self-financing portfolio processes with

\[
\forall t \in T : V_t - V_{t-1} \in -\tilde{K}_t \quad \mathbb{P}\text{-a.s.}
\]

The time \( t \) version of theorem 4.1 in [78], or theorem 5.2 in [50] reads as follows.

**Theorem 5.1.1.** Assume that the market process \( (K_t)_{t \in T} \) satisfies the robust no arbitrage condition \( (\text{NA}^r) \), then the following conditions are equivalent for \( X \in \mathcal{L}^p \) and \( u \in \mathcal{L}^p_t \):

1. \( X - u \in C_{t,T}, \ i.e. \ there exists a self-financing portfolio process \((V_s)_{s \in T}\) with \( V_s = 0 \) if \( s < t \), and \( V_s \in \mathcal{L}^p_s \) for each time \( s \geq t \) such that

\[
u + V_T = X.
\]
2. For every consistent pricing process \((Z_t)_{t \in T}\) with \(Z_t \in \mathcal{L}_t^q\) for each time \(t\), it holds that

\[
\mathbb{E} [X^T Z_T] \leq \mathbb{E} [u^T Z_t] .
\]

**Proof.** This is a trivial adaptation of theorem 4.1 in [78], or theorem 5.2 in [50]. \(\Box\)

Clearly any element \(u \in \mathcal{L}_t^p\) satisfying equation (5.1.2) is a superhedging portfolio of \(X\) at time \(t\). Thus, as an extension to the static case in [50], the set of superhedging portfolios defines a closed coherent market-compatible dynamic risk measure on \(\mathcal{L}^p\) as described in the corollary below.

**Corollary 5.1.2.** An element \(u \in \mathcal{L}_t^p\) is a superhedging portfolio at time \(t\) for the claim \(X \in \mathcal{L}^p\) if and only if \(u \in \text{SHP}_t(X)\) with

\[
\text{SHP}_t(X) := \{ u \in \mathcal{L}_t^p : -X + u \in -C_{t,T} \} .
\]

If the market process \((K_t)_{t \in T}\) satisfies the robust no arbitrage condition \((NA^r)\), then \((R_t)_{t \in T}\) defined by \(R_t(X) := \text{SHP}_t(-X)\) is a closed conditionally coherent market-compatible dynamic risk measure on \(\mathcal{L}^p\) and has the following dual representation

\[
\text{SHP}_t(X) = \bigcap_{(Q, \omega) \in \mathcal{W}_{t,K}} \left( \mathbb{E}^{Q} [X | \mathcal{F}_t] + \Gamma_t(\omega) \right) ,
\]

where \(t \in T\) and

\[
\mathcal{W}_{t,K} = \{ (Q, \omega) \in \mathcal{W}_{t,K} : \forall s \in \{ t, \ldots, T \} : w_s^s(Q, \omega) \in \mathcal{L}_s^q(K_s^+) \} .
\]

**Proof.** Theorem 5.1.1 condition 1 implies equation (5.1.3) immediately. Setting \(M_t = \mathcal{L}_t^p\) for all times \(t = 0, 1, \ldots, T\), and since \(-C_{t,T} = \sum_{s=t}^T \mathcal{L}_s^p(K_s)\) it follows from \(K_s(\omega)\) being a convex cone with \(\mathbb{R}_+^d \subseteq K_s(\omega)\) for all \(s \in \{ t, \ldots, T \}\) and for almost all \(\omega \in \Omega\) that the set \(-C_{t,T}\) is an acceptance set at time \(t\) as it satisfies definition
2.1.11. This also trivially implies that $-C_{t,T}$ is market-compatible. Furthermore, $-C_{t,T} \subseteq L^p$ is a conditionally convex cone and closed in $L^p$ (follows as in [78]). Thus, $R_t(X) = \text{SHP}_t(-X)$ as defined in equation (5.1.3) is by proposition 2.1.14 a closed, conditionally coherent, and market-compatible conditional risk measure.

By theorem 5.1.1 condition 2 the set $\text{SHP}_t(X)$ of superhedging portfolios of $X$ at time $t$ described in equation (5.1.3) can also be written in the form

$$\text{SHP}_t(X) = \bigcap_{Z \in \text{CPP}_t} \{ u \in L^p_t : \mathbb{E}[Z^T_t X] \leq \mathbb{E}[Z^T_t u] \}$$  \hspace{1cm} (5.1.5)

where $\text{CPP}_t$ is the set of consistent pricing processes starting at time $t$ such that $Z_s \in L^q_s(K^+_t) \setminus \{0\}$ for all $s \geq t$. Equation (5.1.5) is equivalent to (5.1.4) as there is a one-to-one relationship between the set $\text{CPP}_t$ and $\mathcal{W}_{\{t,...,T\}}$: Given a consistent pricing process $Z$, we can create a pair $(Q, w) \in \mathcal{W}_{\{t,...,T\}}$ by defining $w := \mathbb{E}[Z^T_t F_t] = Z_t \in L^q_t(K^+_t) \setminus \{0\}$ and

$$\frac{dQ_i}{dP} := \frac{(Z_T)_i}{\mathbb{E}[(Z_T)_i]}.$$

Conversely, a pair $(Q, w) \in \mathcal{W}_{\{t,...,T\}}$ yields a consistent pricing process $Z$ starting from time $t$ by letting $Z_T = w^T_t (Q, w)$ and $Z_s = \mathbb{E}[Z^T_T F_s]$ for all $s = t,...,T$.

Finally, by corollary 2.2.12 it follows that we can apply the same set of dual variables $\mathcal{W}_{\{t,...,T\}}$ for the conditional dual representation (i.e. the $\Gamma_t$ representation) as in the set-optimization version (i.e. the $G_t$ representation).

The probability measures $Q$ with $(Q, w) \in \mathcal{W}_{\{t,...,T\}}$ can be seen as equivalent martingale measures. Indeed, the component $Q_i$, $i = 1,...,d$ is a martingale measure if asset $i$ is chosen as numéraire.

It remains to show that the dynamic superhedging set (as a coherent risk measure) is multi-portfolio time consistent.
Lemma 5.1.3. Under the \( (NA^r) \) condition, the set-valued function \( R_t(X) := \text{SHP}_t(-X) \) defined in corollary 5.1.2 is a normalized multi-portfolio time consistent dynamic risk measure.

Proof. We have already shown that the acceptance set of \( R_t(X) = \text{SHP}_t(-X) \) is \( A_t = -C_{t,T} = \sum_{s=t}^T \mathcal{L}_s^p(K_s) \). The one step acceptance set is given by \( A_{t,t+1} = A_t \cap \mathcal{L}_{t+1}^p = \mathcal{L}_t^p(K_t) + \left( \sum_{s=t+1}^T \mathcal{L}_s^p(K_s) \right) \cap \mathcal{L}_{t+1}^p \). Since it holds

\[
\sum_{s=t+2}^T \mathcal{L}_s^p(K_s) + \left( \sum_{s=t+2}^T \mathcal{L}_s^p(K_s) \right) \cap \mathcal{L}_{t+1}^p = \sum_{s=t+2}^T \mathcal{L}_s^p(K_s),
\]

it can easily be seen that \( A_t = A_{t,t+1} + A_{t+1} \) is satisfied for any time \( t \). Theorem 3.2.2 implies that \( (R_t)_{t \in \mathbb{T}} \) is a multi-portfolio time consistent dynamic risk measure if it is normalized. Note that \( \text{SHP}_t(0) = A_t \cap \mathcal{L}_t^p \) and since the solvency cones contain \( \mathbb{R}_+^d \), it holds \( \text{SHP}_t(0) \supseteq \mathcal{L}_t^p \), and by \( (NA^r) \) we have \( \text{SHP}_t(0) \cap \mathcal{L}_{t,-}^p = \emptyset \) for every time \( t \), which implies for coherent risk measures that \( (R_t)_{t \in \mathbb{T}} \) is normalized (as mentioned in section 2.1 and shown in property 3.1 in [58]).

Remark 5.1.4. Because closure and multi-portfolio time consistency imply that the sum \( A_{t,s} + A_s \) is closed, it is not necessary to have convex upper continuity in theorem 3.4.6 (see also remark 3.4.1). Therefore, stability of the dual set \( \mathcal{W}_{\{t,...,T\}} \) follows from theorem 3.4.6 where, for any \( t, s \in \mathbb{T} \) with \( t < s \), the set of stepped dual variables \( \mathcal{W}_{\{t,...,s\}} \subseteq \mathcal{W}_{t,s} \) is given by

\[
\mathcal{W}_{\{t,...,s\}} := \left\{ (Q, w) \in \mathcal{W}_{t,s} : w_t^*(Q, w) \in \mathcal{L}_t^p(K_t^+) \forall r \in \{t, ..., s\}, \right. \\
\left. w_t^*(Q, w) \in (\mathcal{L}_s^p \cap \sum_{r=s+1}^T \mathcal{L}_r^p(K_r))^+ \right\}.
\]

Alternatively, one can directly prove that for any time \( t \) and any \( s > t \),

\[
\mathcal{W}_{\{t,...,T\}} = \left\{ (Q \oplus^s \mathbb{R}, w) : (Q, w) \in \mathcal{W}_{\{t,...,s\}}, (\mathbb{R}, w_t^*(Q, w)) \in \mathcal{W}_{\{s,...,T\}} \right\}
\]
holds, which is by theorem 3.4.6 equivalent to stability.

5.1.1 Convex superhedging price

We now consider the set of superhedging portfolios under convex transaction costs, for instance a market impact cost. We will use the results from section 3.5 to show that the convex superhedging portfolios are multi-portfolio time consistent by backward recursion.

Consider the setting with a full space of eligible assets, i.e. $M_t = \mathcal{L}_t^p$ for all times $t$. Also consider a market with convex transaction costs as in [70], which is modeled by a sequence of convex solvency regions $(K_t)_{t \in \mathbb{T}}$. $K_t$ is a solvency region at time $t$ if it is an $\mathcal{F}_t$-measurable set such that for (almost) every $\omega \in \Omega$, $K_t[\omega]$ is a closed set with $\mathbb{R}_+^d \subseteq K_t[\omega] \subsetneq \mathbb{R}_+^d$. Let the robust no scalable arbitrage condition be satisfied, i.e. the sequence of recession cones of $K_t$ satisfy robust no arbitrage (see e.g. [70]).

As in the conical case, denote the set of self-financing portfolios starting from zero capital at time $t$ by

$$C_{t,T} := - \sum_{s=t}^{T} \mathcal{L}_s^p(K_s).$$

Thus the set of superhedging portfolios is, again, given by

$$SHP_t(X) := \{ u \in \mathcal{L}_t^p : -X + u \in -C_{t,T} \}.$$

The convex superhedging portfolios can also be defined via the dual representation with penalty functions

$$-\alpha_t^{SHP}(Q, w) := \sum_{s=t}^{T} \left\{ u \in \mathcal{L}_t^p : \text{ess inf}_{k \in \mathcal{L}_s^p(K_t)} \mathbb{E}^Q \left[ k \mid \mathcal{F}_t \right] \leq w^T u \right\},$$

$$-\beta_t^{SHP}(Q, w) := \sum_{s=t}^{T} \left\{ u \in \mathcal{L}_t^p : \sigma_{K_s}(w^s(Q, w)) \leq \mathbb{E} \left[ w^T u \right] \right\}.$$
where \( \sigma_{K_s}(w^*_t(Q, w)) = \inf_{k \in \mathcal{L}^*_p(K_s)} \mathbb{E}[w^*_t(Q, w)^T k] \) is the support function for the selectors of \( K_s \). The proof of such a representation follows immediately from the dual representation of the worst-case risk measure modified by a market model (see e.g. [3]).

One can use corollary 3.5.5 and proposition 3.5.1 to show that the convex superhedging portfolios are multi-portfolio time consistent: Consider acceptance sets \((A_t)_{t \in T}\) given by \( A_T = \mathcal{L}_T^p(K_T) \) and \( A_t = \mathcal{L}_t^p(K_t) + \mathcal{L}^p_{t+1} \) for \( t < T \). Thus, \( A_{t,t+1} = \mathcal{L}_t^p(K_t) + \mathcal{L}^p_{t+1} \). Then, the acceptance set \(-C_{t,T}\) of the convex superhedging set can be recovered by backward recursion of \((A_t)_{t \in T}\), that is \(-C_{T,T} = A_T\) and \(-C_{t,T} = -C_{t+1,T} + A_{t,t+1}\) for \( t < T \). Thus, by corollary 3.5.5 and proposition 3.5.1 the convex superhedging portfolios are multi-portfolio time consistent.

Under the robust no scalable arbitrage condition \(-C_{t,T}\) is closed. Therefore, by theorem 3.3.2, convex upper continuity is not necessary in corollary 3.5.6, or for the cocycle condition to be satisfied. And indeed, we can recover the minimal penalty function \(-\beta^{SHP}_t\) by the backward recursion of penalty functions as given in corollary 3.5.6

\[
-\beta^{SHP}_T(Q, w) = -b_T(Q, w)
\]

\[
-\beta^{SHP}_t(Q, w) = \text{cl}(-b_{t,t+1}(Q, w) + \mathbb{E}^Q[ -\beta^{SHP}_{t+1}(Q^{t+1}, w^{t+1}(Q, w)) | \mathcal{F}_t])
\]

for any \((Q, w)\) \(\in\) \(W_t\), where

\[
-b_T(Q, w) := \text{cl} \bigcup_{X \in A_T} \left( \mathbb{E}^Q[X | \mathcal{F}_t] + G_t(w) \right)
\]

\[
= \left\{ u \in \mathcal{L}_T^p : \sigma_{K_T}(w) \leq \mathbb{E}[w^T u] \right\},
\]

\[
-b_{t,t+1}(Q, w) := \text{cl} \bigcup_{X \in A_{t,t+1}} \left( \mathbb{E}^Q[X | \mathcal{F}_t] + G_t(w) \right)
\]

\[
= \left\{ u \in \mathcal{L}_t^p : \sigma_{K_t}(w) \leq \mathbb{E}[w^T u] \right\}.
\]
5.1.2 Numerical computation

We can use the algorithm from chapter 4 to compute the set of superhedging prices under either a conical or convex market model. We do this by assuming a finite probability space, as in chapter 4, and computing the market-compatible version of the worst case risk measure. Let \((R_t)_{t \in T}\) be the worst case risk measure, that is \(R_t : \mathcal{L}_t^0 \to \mathcal{P}(\mathcal{L}_t^0; \mathcal{L}_{t+}^0)\) with

\[
R_t(X) = \{ u \in \mathcal{L}_t^0 : X + u \in \mathcal{L}_t^0 \}.
\]

The worst case risk measure \((R_t)_{t \in T}\) is conditionally convex and polyhedral with \(R_{t,t+1}(-Z)[\omega_t]\) in (4.2.1) given as the upper image of a linear vector optimization problem. By proposition 4.2.1 one can calculate \((R_t)_{t \in T}\) \(\omega_t\)-wise since the worst case risk measure is multi-portfolio time consistent.

Consider the market extension of the worst case risk measure, where trading is modeled by a sequence of solvency regions \((K_t)_{t \in T}\). The multi-portfolio time consistent market extension \((\tilde{R}_t^{mar})_{t \in T}\) with \(K_t = \mathcal{L}_t^0(K_t)\) is nothing else than the superhedging risk measure. In particular, for a given claim \(X \in \mathcal{L}_t^0\), the set \(SHP_t(X) := \tilde{R}_t^{mar}(-X)\) is the set of superhedging portfolios of \(X\).

Under proportional transaction costs, modeled by a sequence of solvency cones \((K_t)_{t \in T}\), by proposition 4.2.1 and the discussion in section 4.5, the set of superhedging portfolios \(SHP_t(X)\) of \(X\) can be calculated backwards in time by solving a sequence of linear vector optimization problems (4.1.6) with ordering cone \(K_t[\omega_t]\). This backwards recursive algorithm is exactly the one proposed in [67], see also [75], which could be obtained using the simple structure of the worst case risk measure \((R_t)_{t \in T}\), which yields a great simplification to the recursive structure (3.2.2), respectively (3.5.1), (3.5.2) in that case. Note that this is not possible for more complicated risk measures,
which means that the method in [67, 75] cannot be generalized to other risk measures, whereas the approach discussed in chapter 4 is widely applicable.

In fact, the algorithm presented in [67] relies on a successive calculation of superhedging prices backwards in time and leads to a sequence of linear vector optimization problems that can be solved by Benson’s algorithm. In the following remark, reproduced from [34], we show that the recursive form, which is equivalent to multi-portfolio time consistency, leads to and simplifies the proof of the recursive algorithm given in [67].

**Remark 5.1.5.** Since multi-portfolio time consistency is equivalent to recursiveness (theorem 3.2.2), the set of superhedging portfolios satisfies, under a conical market and (NA^r),

$$\text{SHP}_t(X) = \bigcup_{Z \in \text{SHP}_{t+1}(X)} \text{SHP}_t(Z) =: \text{SHP}_t(\text{SHP}_{t+1}(X)).$$ (5.1.6)

The recursiveness leads in a straight forward manner to the recursive algorithm given by theorem 3.1 in [67] and thus simplifies the proof of that theorem significantly. Direct observation or using lemma 3.2.11 for $(R_t)_{t \in T}$ defined by $R_t(X) := \text{SHP}_t(-X)$ being market-compatible (corollary 5.1.2), normalized and multi-portfolio time consistent (lemma 5.1.3), yields $-C_{t+1,T} = -C_{t+1,T} + \sum_{s=t+1}^{T} \mathcal{L}_s^0(K_s)$ for each $t$, and in particular

$$-C_{t+1,T} = -C_{t+1,T} + \left( \sum_{s=t+1}^{T} \mathcal{L}_s^0(K_s) \right) \cap \mathcal{L}_{t+1}^0.$$

Then, proposition 2.1.14 implies $\text{SHP}_{t+1}(X) = \text{SHP}_{t+1}(X) + \left( \sum_{s=t+1}^{T} \mathcal{L}_s^0(K_s) \right) \cap \mathcal{L}_{t+1}^0$ which leads to $\text{SHP}_{t+1}(X) + (-C_{t,T}) \cap \mathcal{L}_{t+1}^0 = \text{SHP}_{t+1}(X) + \mathcal{L}_{t}^0(K_t)$. Thus, the
recursive form (5.1.6) reads as

$$SHP_t(X) = \bigcup_{z \in SHP_{t+1}(X)} \{ u \in L_t^0 : -Z + u \in -C_{t:T} \}$$

$$= \bigcup_{z \in SHP_{t+1}(X)} \{ u \in L_t^0 : -Z + u \in -C_{t:T} \cap L_{t+1}^0 \}$$

$$= \{ u \in L_t^0 : u \in SHP_{t+1}(X) + (-C_{t:T} \cap L_{t+1}^0) \}$$

$$= \{ u \in L_t^0 : u \in SHP_{t+1}(X) + L_t^0(K_t) \} = SHP_{t+1}(X) \cap L_t^0 + L_t^0(K_t),$$

for \( t \in \{ T - 1, ..., 0 \} \). Together with \( SHP_T(X) = X + K_T \) one obtains a recursive algorithm, which is shown in [67] to be equivalent to a sequence of linear vector optimization problems that can be solved by Benson’s algorithm.

With the results from chapter 4, the sets of superhedging portfolios, over a sequence of convex solvency regions \((K_t)_{t \in T}\) with trading constraints given by \((D_t)_{t \in T}\), can be explicitly calculated as long as the set \( K_t \cap D_t \) is polyhedral. In this case \((SHP_t)_{t \in T}\) is a conditionally convex polyhedral, but in general not coherent risk measure (which may not be finite at zero). If the set \( K_t \cap D_t \) is not polyhedral, one can calculate an \( \epsilon \)-approximation of \( SHP_t(X) \) using the results in section 4.3.

**Example 5.1.6.** Consider a market with convex transaction costs and two assets (risk-free bond and a risky asset). We consider the 2 time step Cox-Ross-Rubinstein model. We consider a market with proportional transaction costs given by \( \gamma = 5\% \) and a temporary market impact cost given by \( \theta = 10^{-7} \).

Let the risk-free rate of return be 10%. Let the drift for the risky asset be \( \mu = 12.5\% \) and the volatility given by \( \sigma = 0.5 \). Consider the initial value of the risky asset to be \( S_0 = \$1 \) (measured in the risk-free asset).

Let \( X \) be the terminal payoff from selling an at-the-money European call option, i.e. with strike price \$1. Running the convex algorithm presented in this text, with the
single time step error given by $\epsilon = 0.1$, the efficient frontier of the time 0 superhedging set $SHP_0(X)$ is approximately given by figure 5.1.1.

Figure 5.1.1: Convex superhedging price under proportional and convex transaction costs

5.2 Relaxed worst case

The relaxed worst case risk measure was introduced in the static framework in example 5.2 of [51]. The idea beyond the relaxed worst case risk measure is to modify the worst case risk measure so that portfolios with “small” negative components can still be acceptable.

In this text we consider the dynamic extension of such a risk measure. By definition it is a polyhedral and conditionally convex, but not conditionally coherent, risk measure; the acceptance set at time $t$ is given by

$$A_t^{RWC} = (\epsilon + \mathcal{L}_+^\infty) \cap \mathcal{L}^\infty(G)$$
for some level $\varepsilon \in \mathbb{R}^d_+$ and some finitely generated convex cone $G \supseteq \mathbb{R}^d_+$ and $G \neq \mathbb{R}^d$.

Note that if $G = \mathbb{R}^d_+$ or $\varepsilon = 0$ then the relaxed worst case risk measure is equivalent to the worst case risk measure.

Now consider the finite probability space discussed in chapter 4 and consider the discrete time setting with $T = \{0, 1, \ldots, T\}$. Let $(R_t)_{t \in T}$ be the relaxed worst case risk measure with acceptance set $A_t^{RWC}$. Then, by proposition 4.2.1 one can calculate its multi-portfolio time consistent version $(\tilde{R}_t)_{t \in \mathbb{T}}$ $\omega_t$-wise, where in each node $\omega_t \in \Omega_t$, $t \in \mathbb{T} \setminus \{T\}$, the linear vector optimization problem (4.1.6) has to be solved. It should be noted that in general $R_t \neq \tilde{R}_t$, i.e. the relaxed worst case risk measure is not multi-portfolio time consistent. In example 5.2.1 below, we consider the market extension $(\tilde{R}_t^{mar})_{t \in \mathbb{T}}$ as discussed in section 4.5 under a conical market model.

**Example 5.2.1.** Consider a market with proportional transaction costs and three assets (risk-free bond and two correlated risky assets). We will estimate the market with a binomial tree model with $T = 20$ time steps over a one year time horizon. Consider a market with proportional transaction costs defined by $\gamma = 5\%$.

Let the risk-free rate of return be 10%. Let the drift for the risky assets be given by $\mu_1 = 15\%$ and $\mu_2 = 30\%$. Let the volatility for the risky assets be given by $\sigma_1 = 0.5$ and $\sigma_2 = 1$. Let the correlation be given by $\rho = 0.5$. Consider the initial value of the risky assets to be $S_0 = (\$1, \$1)^T$ (measured in the risk-free asset).

Let $X$ be the terminal payoff of an outperformance option with strike price $K = \$1.10$, i.e. $X = (-KI_{(\max(S_T^2) \geq K)}, I_{(S_T^2)^1 \geq (S_T^1)^2, (S_T^2)^{1,2} \geq K}), I_{(S_T^2)^{2,1}, (S_T^2)^{2} \geq K})^T$.

Consider the relaxed worst case risk measure with constant parameters $\varepsilon_i = .25$ for $i = 0, 1, 2$ and $G$ is the convex cone generated by the vectors $(1, -0.25, -0.25)^T$, $(-0.25, 1, -0.25)^T$, and $(-0.25, -0.25, 1)^T$. The market extended multi-portfolio time consistent version of the relaxed worst case risk measure can be calculated via the polyhedral algorithm presented in this paper; the efficient frontier of the time 0 risk measure $\tilde{R}_0^{mar}(X)$ is given by figure 5.2.1.
In this section we discuss the dynamic set-valued average value at risk with time $t$ parameters $\lambda^t \in \mathcal{L}_t^\infty$, $\epsilon \leq \lambda^t_i \leq 1$ for every index $i$ and for some $\epsilon \in \mathbb{R}_{++}$. As the underlying space we consider $\mathcal{L}_t^\infty$ with the weak* topology $\sigma(\mathcal{L}_t^\infty, \mathcal{L}_t^1)$. We define the average value at risk, as in [34, 36], by the following dual representation

$$AV@R^\lambda_t(X) := \bigcap_{(Q,w) \in \mathcal{W}^\lambda_t} (\mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w)) \cap M_t \quad (5.3.1)$$

for any $X \in \mathcal{L}^\infty$ and set of eligible portfolios given by a closed subspace $M_t \subseteq \mathcal{L}_t^\infty$.

We denote

$$\mathcal{W}^\lambda_t := \left\{ (Q, w) \in \mathcal{W}_t : \frac{w}{\lambda^t} - w^T_t (Q, w) \in \mathcal{L}_1^t \right\}$$

$$= \left\{ (Q, w) \in \mathcal{W}_t : 0 \preceq w \cdot \frac{dQ}{dP} \preceq w/\lambda^t \right\},$$
to be the set of dual variables for the average value at risk. A primal definition of
the static set-valued average value at risk can be found in [53], and for the dynamic
version below in proposition 5.3.3.

**Proposition 5.3.1.** \((AV@R^\lambda_t)_{t \in \mathbb{T}}\) is a normalized closed conditionally coherent dy-
namic risk measure.

**Proof.** \((AV@R^\lambda_t)_{t \in \mathbb{T}}\) is a closed conditionally coherent dynamic risk measure by defi-
nition, see corollary 2.2.12.

To prove normalization we use the set-valued duality representation (i.e. w.r.t.
\(G_t\) rather than \(\Gamma_t\)). Take \(u_0 \in AV@R^\lambda_t(0)\) and \(u_X \in AV@R^\lambda_t(X)\). Then, for every
\((Q, w) \in W^\lambda_t\) it holds that \(u_0 \in G_t(w) \cap M_t\) and \(u_X \in (E^Q[-X|F_t] + G_t(w)) \cap M_t\). It follows that \(u_X + u_0 \in G_t(w) \cap M_t + (E^Q[-X|F_t] + G_t(w)) \cap M_t \subseteq (E^Q[-X|F_t] + G_t(w)) \cap M_t\), and therefore \(u_X + u_0 \in AV@R^\lambda_t(X)\). The other
direction trivially follows from \(0 \in AV@R^\lambda_t(0)\).

**Remark 5.3.2.** If \(\lambda^i_t = 1\) for every index \(i\) with \(M_t = L^\infty_t\) then it can be seen that

\[ W^\lambda_t = \{\mathbb{P}\} \times L^1_{t,+} \setminus \{0\} \]

and thus \(AV@R^\lambda_t(X) = E[-X|F_t] + L^\infty_{t,+}\) for any \(X \in L^\infty\). Therefore, for any choice
of \(M_t\) we have \(AV@R^\lambda_t(X) = (E[-X|F_t] + L^\infty_{t,+}) \cap M_t\) for any \(X \in L^\infty\).

In the following proposition, we provide the acceptance set and thus the primal
representation for the dynamic average value at risk given in (5.3.1). This proves that
(5.3.1) is the dynamic version of the closure of the static average value at risk defined
via its acceptance set \(A_0^\lambda\) in [53]. We note that the proof is similar to [53].

**Proposition 5.3.3.** The acceptance set associated with the conditional average value
at risk at time \(t\) and parameter \(\lambda^t\) is given by \(\bar{A}^\lambda_t = cl(A^\lambda_t)\) where

\[ A^\lambda_t = \left\{ X \in L^\infty : \exists Z \in L^\infty_+ : X + Z \geq \frac{E[Z|F_t]}{\lambda^t} \right\} \]
and \( W^\lambda_t \) is the maximal dual set.

Proof. By corollary 2.2.9, in order to show that \( \bar{A}^\lambda_t \) is the acceptance set for \( AV@R^\lambda_t \) and \( W^\lambda_t \) is the maximal dual set, we need to verify that

\[
W^\lambda_t = \left\{ (Q, w) \in W_t : w_t^T(Q, w) \in (A^\lambda_t)^+ \right\},
\]

since \( (A^\lambda_t)^+ = (\bar{A}^\lambda_t)^+ \).

It can easily be seen that \( A^\lambda_t = \bigcup_{Z \in \mathcal{L}^\infty_+} (E[Z|\mathcal{F}_t]/\lambda^t - Z) + \mathcal{L}^\infty_+ \). Additionally, \( \bigcup_{Z \in \mathcal{L}^\infty_+} (E[Z|\mathcal{F}_t]/\lambda^t - Z) \) is a \( (\mathcal{F}_t\text{-conditional}) \) cone containing 0 (letting \( Z = 0 \)). Therefore \( w_t^T(Q, w) \in (A^\lambda_t)^+ \) if and only if

\[
w_t^T(Q, w) \in \left( \bigcup_{Z \in \mathcal{L}^\infty_+} (E[Z|\mathcal{F}_t]/\lambda^t - Z) \right)^+ \cap \mathcal{L}^1_+
\]

by property (v) on page 7 in [82]. By \( (Q, w) \in W_t \), it already follows that \( w_t^T(Q, w) \in \mathcal{L}^1_+ \). Thus the maximal dual set for \( \bar{A}^\lambda_t \) is given by

\[
\left\{ (Q, w) \in W_t : w_t^T(Q, w) \in \left( \bigcup_{Z \in \mathcal{L}^\infty_+} (E[Z|\mathcal{F}_t]/\lambda^t - Z) \right)^+ \right\}.
\]

The condition \( w_t^T(Q, w) \in \left( \bigcup_{Z \in \mathcal{L}^\infty_+} (E[Z|\mathcal{F}_t]/\lambda^t - Z) \right)^+ \) is true if and only if for every \( Z \in \mathcal{L}^\infty_+ \)

\[
0 \leq E[w_t^T(Q, w)^T(E[Z|\mathcal{F}_t]/\lambda^t - Z)]
= E[w_t^T(Q, w)^T(E[Z|\mathcal{F}_t]/\lambda^t) - E[w_t^T(Q, w)^TZ]]
= E[(w/\lambda^t)^TZ] - E[w_t^T(Q, w)^TZ]
= E[(w/\lambda^t - w_t^T(Q, w))^TZ].
\]
This means that this additional condition is equivalent to

\[ \frac{w}{\lambda^t} - w_t^T(Q, w) \in L^1_+ . \]

Therefore \( W_\lambda^t = \{ (Q, w) \in W_t : w_t^T(Q, w) \in (A_\lambda^t)^+ \} \).

If \( \mathbb{P}(\max_i \lambda_i^t < 1) > 0 \) then it can easily be seen that \( (AV@\mathbb{R})_{t \in \mathbb{T}} \) is not recursive since there is no relation between the set of dual variables through time (and similarly the scalar average value at risk is well known to not be time consistent). Therefore, there is no relation between the acceptance sets between time \( t \) and time \( t + 1 \). Thus by proposition 5.3.1 and theorem 3.2.2 the set-valued average value at risk is not a multi-portfolio time consistent risk measure.

By the procedure described in proposition 3.5.1 a multi-portfolio time consistent version of the set-valued average value at risk can be obtained, analogous to the scalar version defined in [23]. For the remainder of this section we assume a discrete time setting \( \mathbb{T} = \{ 0, 1, ..., T \} \) and \( \lambda_i^t < 1 \) for all times \( t \) and indices \( i \). We now prove the dual representation of the composed dynamic set-valued average value at risk given in theorem 5.3.4 by using corollary 3.5.6.

**Theorem 5.3.4.** Let \( M_t = L_\lambda^\infty \) for all times \( t \). The composed version of the average value at risk \( (AV@\mathbb{R})_{t \in \mathbb{T}} \) is, by corollary 3.5.6, a multi-portfolio time consistent c.u.c. conditionally coherent risk measure with dual representation

\[
\widehat{AV@\mathbb{R}}^\lambda_t(X) := \bigcap_{(Q,w) \in \widetilde{W}_\lambda^t} \left( \mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w) \right) \cap M_t,
\]
where

\[
\tilde{W}_t^\lambda = \left\{ (Q, w) \in W_t : \frac{w_t^s(Q, w)}{\lambda_t^s} \geq w_{t+1}^s(Q, w) \ \forall s \in \{t, ..., T - 1\} \right\} \\
= \left\{ (Q, w) \in W_t : \forall s \in \{t, ..., T - 1\} : \right.
\]

\[
\mathbb{P} \left( \xi_{s,s+1}(Q_i) \leq \frac{1}{\lambda_t^s} \text{ or } w_i = 0 \right) = 1 \ \forall i \in \{1, ..., d\}\right\}.
\]

Proof. First, by proposition 5.3.5 we know that \((AV@R_t^\lambda)_{t \in T}\) is a c.u.c. dynamic conditionally coherent risk measure.

By corollary 3.5.6, \((\tilde{AV}@R_t^\lambda)_{t \in T}\) is the multi-portfolio time consistent version of \((AV@R_t^\lambda)_{t \in T}\) if and only if

\[
\tilde{W}_T^\lambda = W_T^\lambda \\
\tilde{W}_t^\lambda = H_{t+1}^t \left( \tilde{W}_{t+1}^\lambda \right) \cap W_{t,t+1}^\lambda
\]

where \(\tilde{W}_t^\lambda \neq \emptyset\) for all times \(t\). Trivially it can be seen that \(\tilde{W}_T^\lambda = W_T\). Furthermore, \(W_T^\lambda = W_T\) since \(\frac{1}{\lambda_t^t} - 1 \geq 0\) for every \(i = 1, ..., d\) (by \(\epsilon \leq \lambda_t^T < 1\)) and \(w = w_T^T(Q, w) \in \mathcal{L}_t^1\) by \((Q, w) \in W_T\), and therefore the product is almost surely nonnegative. By remark 5.3.7 and lemma 5.3.6 it holds

\[
W_{t,s}^\lambda = \left\{ (Q, w) \in W_t : w/\lambda_t^s \geq w_t^s(Q, w) \right\}
\]

Furthermore, using lemma A.2.1 and \(w_{t+1}^s(Q, w) = w_t^s(Q, w)\) it follows

\[
H_{t+1}^t \left( \tilde{W}_{t+1}^\lambda \right) = \left\{ (Q, w) \in W_t : \frac{w_t^s(Q, w)}{\lambda_t^s} \geq w_{t+1}^s(Q, w) \ \forall s \in \{t + 1, ..., T - 1\} \right\}.
\]

Noting that \(w_t^s(Q, w) = w\) for any time \(s\), the recursive form of the dual variables \(\tilde{W}_t^\lambda = H_{t+1}^t(\tilde{W}_{t+1}^\lambda) \cap W_{t,t+1}^\lambda\) is proven. \(\tilde{W}_t^\lambda \neq \emptyset\) holds since \((\mathbb{P}, w) \in \tilde{W}_t^\lambda\) for any \(w \in \mathcal{L}_{t,+}^1 \setminus \{0\}\). This is because \((\mathbb{P}, w) \in W_t\) and for any \(s \geq t\) it follows \(w_t^s(\mathbb{P}, w) = w\)
and

$$\mathbb{E} \left[ (w/\lambda^s - w)^T Z \right] \geq 0$$

for every $Z \in \mathcal{L}_{s+1}^\infty$.

Finally, $w_t^s(Q, w) / \lambda^s - w_t^{s+1}(Q, w) \in \mathcal{L}_{s+1, +}^1$ if and only if it is componentwise nonnegative, i.e. $w_t^s(Q, w)_i \leq \xi_{s, s+1}(Q, w)_i \geq 0$ almost surely for every index $i = 1, ..., d$. Since $w_t^s(Q, w)_i \geq 0$ by $(Q, w) \in \mathcal{W}_t$, one has $(Q, w) \in \tilde{\mathcal{W}}_t^\lambda$ if and only if $P(\xi_{s, s+1}(Q, w)_i \leq \lambda_s^i \text{ or } w_t^s(Q, w)_i = 0) = 1$ for every $i \in \{1, ..., d\}$. Notice that for any $\omega \in \Omega$ we have $w_t^s(Q, w)_i[\omega] = 0$ if and only if $w_t[\omega] = 0$ or $\xi_t(Q, w)_i[\omega] = 0$ for some time $r \in (t, s]$, but $\xi_t(Q, w)_i[\omega] = 0$ implies $\xi_{s, s+1}(Q, w)_i[\omega] = 1 \leq \lambda_s^i[\omega]$. Thus we recover the final form

$$\tilde{\mathcal{W}}_t^\lambda = \left\{ (Q, w) \in \mathcal{W}_t : \forall s \in \{t, ..., T - 1\} : \right.$$

$$\left. P\left( \xi_{s, s+1}(Q, w)_i \leq \lambda_s^i \text{ or } w_t = 0 \right) = 1 \forall i \in \{1, ..., d\} \right\}.$$  

\(\square\)

In proposition 5.3.1 it was shown that $(AV@R_t^\lambda)_{t \in \mathbb{T}}$ is a normalized closed conditionally coherent dynamic risk measure. And in proposition 5.3.5 below, we show that the average value at risk with $M_t = \mathcal{L}^\infty_t$ is c.u.c. This result is used in the proof of theorem 5.3.4 above.

**Proposition 5.3.5.** Let $M_t = \mathcal{L}^\infty_t$ for all times $t$, then $(AV@R_t^\lambda)_{t \in \mathbb{T}}$ is a c.u.c. risk measure.

**Proof.** Let $X \in \mathcal{L}^\infty$, then

$$AV@R_t^\lambda(X) = \left\{ u \in \mathcal{L}^\infty_t : X + u \in \bar{A}_t^\lambda \right\}$$

$$= \left\{ u \in \mathcal{L}^\infty_t : X_i + u_i \in \bar{A}_t^{\lambda_i} \forall i = 1, ..., d \right\}$$

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where
\[ \bar{A}^\lambda_{t,i} = \text{cl}\left\{ X \in L^\infty : \exists Z \in L^\infty_+ : X + Z \geq \frac{1}{\lambda_i^t} \mathbb{E}[Z|\mathcal{F}_t] \right\} . \]

Therefore, \( u \in AV@R^\lambda_t(X) \) if and only if \( u \in \mathcal{L}^\infty_t \) with \( u_i \geq \rho^\lambda_t(X_i) \) \( \mathbb{P} \)-almost surely, where \( \rho^\lambda_t \) is the scalar dynamic average value at risk. The result then follows by proposition A.3.5.

We conclude the discussion of the average value at risk by considering the stepped version, which is necessary for the construction of the composed version.

**Lemma 5.3.6.** The stepped average value at risk from time \( t \) to \( s \) (for \( 0 \leq t < s \leq T \)) with time \( t \) parameters \( \lambda^t \in \mathcal{L}^\infty_t \) where \( \epsilon \leq \lambda^t_i < 1 \) for every index \( i \) and for some \( \epsilon \in \mathbb{R}_+ \) is given by

\[
AV@R^\lambda_{t,s}(X) := \bigcap_{(Q,w) \in W^\lambda_{t,s}} \left( \mathbb{E}^Q[-X|\mathcal{F}_t] + \Gamma_t(w) \right) \cap M_s
\]

for any \( X \in \mathcal{L}^\infty \) where

\[
W^\lambda_{t,s} = \left\{ (Q,w) \in W_{t,s} : \forall Z \in \mathcal{L}^\infty_{s,+}, \mathbb{E}\left[ (w/\lambda^t - w^s_t(Q,w))^TZ \right] \geq \sup \left\{ \mathbb{E}[w^s_t(Q,w)^TD] : D \in \mathcal{L}^\infty_{s,-} \cap \left[ M_s + (\mathbb{E}[Z|\mathcal{F}_t]/\lambda^t - Z) \right] \right\} \right\}
\]

is the associated maximal stepped dual set.

**Proof.** Using the definition of the acceptance set for \( AV@R^\lambda_t \) given in proposition 5.3.3, we find the stepped acceptance set is given by \( \bar{A}^\lambda_{t,s} = \text{cl}(A^\lambda_{t,s}) \) where

\[
A^\lambda_{t,s} = \left\{ X \in M_s : \exists Z \in \mathcal{L}^\infty_+ : X + Z \geq \mathbb{E}[Z|\mathcal{F}_t]/\lambda^t \right\} = \left\{ X \in M_s : \exists Z \in \mathcal{L}^\infty_{s,+} : X + Z \geq \mathbb{E}[Z|\mathcal{F}_t]/\lambda^t \right\} = \left( \bigcup_{Z \in \mathcal{L}^\infty_{s,+}} \left( \mathbb{E}[Z|\mathcal{F}_t]/\lambda^t - Z \right) + \mathcal{L}^\infty_{s,+} \right) \cap M_s.
\]
By corollary 2.3.5 and \((A_{t,s}^\lambda)^+ = (\bar{A}_{t,s}^\lambda)^+\), the maximal stepped dual set is given by

\[
\left\{ (Q, w) \in W_{t,s} : w^*_t(Q, w) \in (A_{t,s}^\lambda)^+ \right\}.
\]

It can trivially be seen that \(X \in A_{t,s}^\lambda\) if and only if \(X = \mathbb{E}[Z | \mathcal{F}_t]/\lambda^t - Z + D\) for some \(Z \in \mathcal{L}_{s,+}^\infty\) and \(D \in \mathcal{L}_{s,+}^\infty \cap [M_s + (Z - \mathbb{E}[Z | \mathcal{F}_t]/\lambda^t)]\). Therefore \(w^*_t(Q, w) \in (A^\lambda_{t,s})^+\) if and only if for any \(Z \in \mathcal{L}_{s,+}^\infty\) and \(D \in \mathcal{L}_{s,+}^\infty \cap [M_s + (Z - \mathbb{E}[Z | \mathcal{F}_t]/\lambda^t)]\)

\[
0 \leq \mathbb{E} \left[ w^*_t(Q, w)^T (\mathbb{E}[Z | \mathcal{F}_t]/\lambda^t - Z + D) \right] \\
= \mathbb{E} \left[ (w/\lambda^t - w^*_t(Q, w))^T Z \right] + \mathbb{E} \left[ w^*_t(Q, w)^T D \right].
\]

That is, for every \(Z \in \mathcal{L}_{s,+}^\infty\)

\[
\sup \left\{ \mathbb{E} [w^*_t(Q, w)^T D] : D \in \mathcal{L}_{s,-}^\infty \cap [M_s + (\mathbb{E}[Z | \mathcal{F}_t]/\lambda^t - Z)] \right\} \\
\leq \mathbb{E} \left[ (w/\lambda^t - w^*_t(Q, w))^T Z \right].
\]

\[\square\]

**Remark 5.3.7.** The dual representation in lemma 5.3.6 simplifies significantly if all assets are eligible, i.e., if \(M_t = \mathcal{L}_{t}^\infty\) for all times \(t\). Then, the maximal dual sets for the stepped average value at risk can be equivalently given by

\[
\mathcal{W}_{t,s}^\lambda = \left\{ (Q, w) \in \mathcal{W}_t : 0 \leq w \cdot \xi_{t,s}(Q) \leq w/\lambda^t \right\}
\]

for all times \(0 \leq t < s \leq T\), where \(\mathcal{W}_{t,s} = \mathcal{W}_t\) by remark 2.3.3. This dual representation can be interpreted as the extension of the stepped scalar representation given in [23].

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5.3.1 Numerical computation

Assume a discrete time $T = \{0, 1, ..., T\}$ setting over a finite probability space, as in chapter 4. As discussed above, $(AV@R^\lambda_t)_{t \in T}$ is a normalized closed conditionally coherent dynamic risk measure. Further, since we are only considering a finite probability space, we can immediately conclude (as in [53]) that the dynamic average value at risk is a polyhedral risk measure. Therefore, by proposition 4.2.1, we can calculate the multi-portfolio time consistent version for a general space of eligible portfolios $M_t$ via equations (4.1.3), (4.1.4). In theorem 5.3.4 the multi-portfolio time consistent version $(\tilde{AV}@R^\lambda_t)_{t \in T}$ was explicitly defined via its dual representation under the full space of eligible assets, i.e. $M = \mathbb{R}^d$. In example 5.3.8 we will compute the market extended composed average value at risk $(\tilde{AV}@R^{\lambda,mar}_t)_{t \in T}$ with proportional transaction costs.

Example 5.3.8. Consider a market with proportional transaction costs and two assets (a risk-free bond and a risky asset). We estimate the market with a recombining tree model with 25 branches and $T = 2$ time steps over one year time horizon. Further consider the maximal possible rise or fall in the Brownian motion to be given by $\nu = 2$. Consider the market with high proportional transaction costs, defined by $\gamma = 30\%$.

Let the risk-free rate of return be 10%. Let the drift for the risky asset be $\mu = 12.5\%$ and the volatility given by $\sigma = 0.5$. Consider the initial value of the risky asset to be $S_0 = $1 (measured in the risk-free asset).

Consider the average value at risk with constant parameter $\lambda = (30\%, 30\%)^T$ on the terminal payoff $X$ of an at the money European put option, i.e. with strike price $\$1$. Running the polyhedral algorithm presented in this paper, the efficient frontier of the time 0 composed market extended average value at risk $\tilde{AV}@R^{\lambda,mar}_0 (X)$ is given by figure 5.3.1.
5.4 Entropic risk measure

The set-valued entropic risk measure was studied in [3] in a single period static framework. In this text we present a dynamic version of the entropic risk measure as it was defined in [36].

For the purposes of this section let $p = +\infty$ and $q = 1$, and consider the weak* topology. Let $M_t = \mathcal{L}^\infty_t$ for all times $t$. Further, consider parameters $\lambda^t \in \mathcal{L}^\infty_{t,+}$ and $C_t \in \mathcal{G}(\mathcal{L}^\infty_t; \mathcal{L}^\infty_{t,+})$ with $\lambda^t_i \geq \epsilon$ for every index $i$ and for some $\epsilon \in \mathbb{R}_{++}$, $0 \in C_t$ and $C_t \cap \mathcal{L}^\infty_{t,-} = \emptyset$. The set $C_t$ models the set of acceptable expected utilities, thus $C_t = \mathcal{L}^\infty_{t,+}$ is the most restrictive (conservative) choice.

The dynamic entropic risk measure with parameters $\lambda^t$ and $C_t$ can then be defined by

$$R^\text{ent}_t(X; \lambda^t, C_t) := \{ u \in \mathcal{L}^\infty_t : \mathbb{E}[u_t(X + u)|\mathcal{F}_t] \in C_t \}$$

$$u_t(x) = (u_{t,1}(x_1), ..., u_{t,d}(x_d))^T$$ for any $x \in \mathbb{R}^d$ and $u_{t,i}(z) = \frac{1-e^{-\lambda^t_i z}}{\lambda^t_i}$ for $z \in \mathbb{R}$ and $i = 1, ..., d$. 

Figure 5.3.1: Composed average value at risk under high proportional transaction costs
It can be shown in analogy to propositions 4.4 and 5.1 of [3] that the entropic risk measure is conditionally convex, closed and equal to

\[ R_t^{ent}(X; \lambda^t, C_t) = \rho_t^{ent}(X; \lambda^t) + \tilde{C}_t(\lambda^t, C_t) \]  

(5.4.1)

for any \( X \in \mathcal{L}^\infty \), where

\[ \rho_t^{ent}(X; \lambda^t) = \frac{\log(\mathbb{E}[\exp(-\lambda^t \cdot X) | \mathcal{F}_t])}{\lambda^t} \]

and

\[ \tilde{C}_t(\lambda^t, C_t) = -\frac{\log \left[ \left( \mathbf{1} - \lambda^t \cdot C_t \right) \cap \mathcal{L}^\infty_{t,++} \right]}{\lambda^t} \]

with \( \mathbf{1} = (1, ..., 1)^T \in \mathbb{R}^d \) and the exponential and logarithm are taken componentwise for a vector and elementwise for a set, e.g. \( \exp(z) = (\exp(z_1), ..., \exp(z_d))^T \) for any \( z \in \mathbb{R}^d \).

The dual form of the dynamic entropic risk measure can be deduced as follows. This is a trivial extension from the work in [3], so we will omit the proof.

**Lemma 5.4.1.** Let \( 0 \leq t < s \leq T \). The minimal stepped penalty functions of the stepped entropic risk measure are given by

\[ -\alpha_{t,s}^{ent}(Q, w; \lambda^t, C_t) := -\frac{\hat{H}_{t,s}(Q|\mathcal{P})}{\lambda^t} + \tilde{C}_t(\lambda^t, C_t) + \Gamma_t(w), \]

\[ -\beta_{t,s}^{ent}(Q, w; \lambda^t, C_t) := -\frac{\hat{H}_{t,s}(Q|\mathcal{P})}{\lambda^t} + \tilde{C}_t(\lambda^t, C_t) + G_t(w) \]

for any \( (Q, w) \in \mathcal{W}_t \) where

\[ \hat{H}_{t,s}(Q|\mathcal{P}) := \mathbb{E}^Q \left[ \log(\xi_{t,s}(Q)) | \mathcal{F}_t \right] . \]
The dual representation of the entropic risk measure is given by (2.2.2) with minimal penalty function \( -\beta^\text{ent}_t := -\beta^\text{ent}_{t,T} \). Note that \( \hat{H}_{t,T}(Q|P) = \mathbb{E}^Q[\log(\frac{dQ}{dP}) | \mathcal{F}_t] \) is the conditional relative entropy.

In theorem 5.4.2 below we discuss a specific example of the entropic risk measure which satisfies the cocycle condition, and is therefore multi-portfolio time consistent.

**Theorem 5.4.2.** The restrictive entropic risk measure defined by

\[
R^\text{ent}_t(X; \lambda) := \{ u \in \mathcal{L}^\infty_t : \mathbb{E}[u_t(X + u) | \mathcal{F}_t] \succeq 0 \}
\]

for every \( X \in \mathcal{L}^\infty \) is multi-portfolio time consistent, where for any time \( t \in \mathbb{T} \) the vector utility function is given by \( u_t(x) = (u_{t,1}(x_1), \ldots, u_{t,d}(x_d))^T \) for any \( x \in \mathbb{R}^d \) and \( u_{t,i}(z) = \frac{1-e^{-\lambda z}}{\lambda} \) for \( z \in \mathbb{R} \) and \( i = 1, \ldots, d \).

**Proof.** First note that the restrictive entropic risk measure is nothing but the entropic risk measure with \( C_t = \mathcal{L}^\infty_{t,+} \) and constant risk aversion level \( \lambda \in \mathbb{R}^d_{++} \).

It can easily be shown that \( \check{C}_t(\lambda, \mathcal{L}^\infty_{t,+}) = \mathcal{L}^\infty_{t,+} \). Therefore the restrictive entropic risk measure is normalized, convex, and closed. By applying lemma 5.4.1 one obtains that its stepped penalty functions, defined in (2.3.1) (with \( 0 \leq t < s \leq T \)), are given by

\[
-\beta^\text{ent}_{t,s}(Q, w; \lambda) := -\frac{\hat{H}_{t,s}(Q|P)}{\lambda} + G_t(w)
\]

for any \((Q, w) \in \mathcal{W}_t\), where \( \hat{H}_{t,s}(Q|P) := \mathbb{E}^Q[\log(\xi_{t,s}(Q)) | \mathcal{F}_t] \). The dual representation of the entropic risk measure is given by (2.2.2) with minimal penalty function

\[
-\beta^\text{ent}_t := -\beta^\text{ent}_{t,T}.
\]

It can immediately be seen that

\[
-\beta^\text{ent}_t(Q, w; \lambda) = \operatorname{cl}(-\beta^\text{ent}_{t,s}(Q, w; \lambda)
+ \mathbb{E}^Q[-\beta^\text{ent}_s(Q^s, u^s_t(Q, w); \lambda) | \mathcal{F}_t])
\]

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for all times $0 \leq t < s \leq T$ and any dual variables $(Q, w) \in \mathcal{W}_t$. Thus, the restrictive entropic risk measure satisfies the cocycle property. Proposition A.3.5 yields (using representation (5.4.1) for the restrictive entropic risk measure) the convex upper continuity of $R_{t}^{\text{ent}}$. Therefore the set-valued entropic risk measure with constant risk aversion parameter $\lambda$ and restrictive thresholds $\mathcal{L}_{t}^{\infty}$ is multi-portfolio time consistent by corollary 3.3.3.

5.4.1 Numerical computation

As in chapter 4, for this section assume the discrete time setting $\mathbb{T} = \{0, 1, ..., T\}$ and an underlying finite probability space.

An approximate calculation of the static entropic risk measure was shown in [68] via solving a convex vector optimization problem. With the method presented in chapter 4 we are able to compute an approximation $(\bar{R}_{t}^{\text{ent}})_{t \in \mathbb{T}}$ of the multi-portfolio time consistent version $(\tilde{R}_{t}^{\text{ent}})_{t \in \mathbb{T}}$ by backward composition for a general space of eligible portfolios $M_t$, (stochastic) risk aversion parameters $\lambda^t$, and polyhedral parameters $C_t$. Recall from above that the entropic risk measure is c.u.c. and multi-portfolio time consistent in the case that $M = \mathbb{R}^d$, constant $\lambda \in \mathbb{R}^d_+$, and $C_t = \mathcal{L}_t^{\infty}$, i.e. $R_{t}^{\text{ent}} = \tilde{R}_{t}^{\text{ent}} = \bar{R}_{t}^{\text{ent}}$.

From the definition of the entropic risk measure with $C_t$ polyhedral, it is clear that $R_{t}^{\text{ent}}$ satisfies assumption 4.3.3. Thus, by proposition 4.3.6 one can calculate an approximation of the multi-portfolio time consistent version $(\tilde{R}_{t}^{\text{ent}}(X))_{t \in \mathbb{T}}$ by calculating backwards in time at each node $\omega_t \in \Omega_t$ an $\epsilon$-approximation of the upper image of the convex vector optimization problem (4.3.1) using the algorithms presented in [68]. In example 5.4.3 we consider the market extension of the entropic risk measure under a conical market model.

Example 5.4.3. Consider a market with proportional transaction costs and two assets (a risk-free bond and a risky asset). We estimate the market with a recombining
tree model with \( n = 3 \) child branches from each node branches, \( \nu = 1 \) size of possible rise and fall, and \( T = 5 \) time steps over a one year time horizon. Consider a market with large proportional transaction costs and let the bid-ask spread for the risky asset be given by \( \gamma = 30\% \).

Let the risk-free rate of return be 10\%. Let the drift for the risky asset be \( \mu = 15\% \) and the volatility given by \( \sigma = 0.5 \). Consider the initial value of the risky asset to be \( S_0 = \$1 \) (measured in the risk-free asset).

Let \( X \) be the terminal payoff from selling an out of the money European call option with strike price $1.10.

Consider the entropic risk measure with constant parameters \( \lambda_i = 10\% \) for \( i = 1, 2 \) and \( C_t = \mathcal{L}_t^0(C) \) where \( C = \text{cone}((1, -.25)^T, (-.25, 1)^T) \) be the convex cone generated by the vectors \((1, -.25)^T\) and \((-0.25, 1)^T\). Running the convex algorithm presented in this paper, with the approximation error at each step given by \( \epsilon = 10^{-3} \). The efficient frontier of the time 0 composed market extended entropic risk measure is approximately given by figure 5.4.1.

![Figure 5.4.1: Composed entropic risk measure under high proportional transaction costs](image-url)
Chapter 6

Comparison to Other Approaches

In this chapter we compare the properties for various techniques for dynamic multivariate risk measures, as was done in [35]. It will be shown that the set-valued portfolio approach to dynamic risk measures is the most general model into which every other approach can be embedded. It will be shown in section 6.2.1 that under weak assumptions on the construction of the set-valued portfolios, the set-optimization approach (presented in chapter 2) is equivalent to the set-valued portfolio approach. Because additional properties for dynamic risk measures have been studied previously for the set-optimization approach and due to the (often) one-to-one relation with the set-portfolio approach, we will present the relations in this section as comparisons with the set-optimization approach.

Note that, as with chapter 2, and as opposed to chapter 4, we consider a general filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ satisfying the usual conditions and a general space of times $\mathbb{T}$.

Throughout this chapter we will work with a stronger definition for finiteness at zero. We say a risk measure is finite at zero if $R_t(0) \neq \emptyset$ and $R_t(0)[\omega] := \{u(\omega) : u \in R_t(0)\} \neq M$ for almost every $\omega \in \Omega$. For an acceptance set the second finiteness property corresponds to $M \cap (\mathbb{R}^d \setminus A_t[\omega]) \neq \emptyset$ for almost every $\omega \in \Omega$. 

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6.1 Measurable selector approach

The measurable selector approach was proposed in [11] and is an extension of [58] to the dynamic framework. Only coherent risk measures are considered in this approach as the technique used to deduce the dual representation relies on coherency. The risk measures are assumed to be compatible to a conical market model at the final time $T$, i.e. portfolios are compared based on the final “values”. In so doing, a new pre-image space denoted by $B_{K_T,m}$ is introduced, which is defined below and is discuss in remark 6.1.13. Recall that the space of eligible assets is given by $M = \mathbb{R}^m \times \{0\}^{d-m}$, i.e. $m \leq d$ of the $d$ assets can be used to cover risk.

Let $\mathcal{S}_T^d$ be the set of $\mathcal{F}_T$-measurable random sets in $\mathbb{R}^d$. Recall that a mapping $\Gamma : \Omega \rightarrow 2^{\mathbb{R}^d}$ is an $\mathcal{F}_T$-measurable random set if

$$\text{graph } \Gamma = \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \Gamma[\omega]\}$$

is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (where $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-algebra). The random set $\Gamma$ is closed (convex, conical) if $\Gamma[\omega]$ is closed (convex, conical) for almost every $\omega \in \Omega$.

Let $K_T \in \mathcal{S}_T^d$ satisfy the following assumptions:

k1. for almost every $\omega \in \Omega$: $K_T[\omega]$ is a closed convex cone in $\mathbb{R}^d$;

k2. for almost every $\omega \in \Omega$: $\mathbb{R}^d_+ \subseteq K_T[\omega] \neq \mathbb{R}^d$;

k3. for almost every $\omega \in \Omega$: $K_T[\omega]$ is a proper cone, i.e. $K_T[\omega] \cap -K_T[\omega] = \{0\}$.

It is then possible to create a partial ordering in $\mathcal{L}_T^0$ defined by $K_T$ such that $X \geq_{K_T} Y$ if and only if $\mathbb{P}(X-Y \in K_T) = 1$. The solvency cones with friction, see e.g. [59, 78, 60], satisfy the conditions given above for $K_T$.

Define $B_{K_T,m} := \{X \in \mathcal{L}_T^0 : \exists c \in \mathbb{R}_+ : c1_{d,m} \geq_{K_T} X \geq_{K_T} -c1_{d,m}\}$ where the $i$-th component of $1_{d,m} \in \mathbb{R}^d$ is $1^i_{d,m} = \begin{cases} 1 & \text{if } i \in \{1, \ldots, m\} \\ 0 & \text{else} \end{cases}$. Then we can define a norm on
$B_{K_T,m}$ by $\|X\|_{K_T,m} := \inf \{ c \in \mathbb{R}_+ : c 1_{d,m} \geq_{K_T} X \geq_{K_T} -c 1_{d,m} \}$, and $(B_{K_T,m}, \|\cdot\|_{K_T,m})$ defines a Banach space.

Let $S_t^{d,m} \subseteq S^d_t$ be such that $\Gamma \in S_t^{d,m}$ if $\Gamma \in S^d_t$ and $\Gamma[\omega] \subseteq M$ for almost every $\omega \in \Omega$.

**Definition 6.1.1.** A risk process is a sequence $(\tilde{R}_t)_{t \in \mathbb{T}}$ of measurable set valued mappings $\tilde{R}_t : B_{K_T,m} \rightarrow S_t^{d,m}$ satisfying

1. $\tilde{R}_t(X)$ is a closed $\mathcal{F}_t$-measurable random set for any $X \in B_{K_T,m}$, $\tilde{R}_t(0) \neq \emptyset$, and $\tilde{R}_t(0)[\omega] \neq M$ for almost every $\omega \in \Omega$.

2. For any $X,Y \in B_{K_T,m}$ with $Y \geq_{K_T} X$ it holds $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$.

3. $\tilde{R}_t(X + u) = \tilde{R}_t(X) - u$ for any $X \in B_{K_T,m}$ and $u \in M_t$.

A risk process is **conditionally convex** at time $t$ if for all $X,Y \in B_{K_T,m}$ and $\lambda \in L^1_t([0,1])$ it holds $\lambda \tilde{R}_t(X) + (1 - \lambda) \tilde{R}_t(Y) \subseteq \tilde{R}_t(\lambda X + (1 - \lambda)Y)$.

A risk process is **conditionally positive homogeneous** at time $t$ if for all $X \in B_{K_T,m}$ and $\lambda \in L^0_{t,++}$ with $\lambda X \in B_{K_T,m}$ it holds $\tilde{R}_t(\lambda X) = \lambda \tilde{R}_t(X)$.

A risk process is **conditionally coherent** at time $t$ if it is both conditionally convex and conditionally positive homogeneous at time $t$.

A risk process is **normalized** at time $t$ if $\tilde{R}_t(X) + \tilde{R}_t(0) = \tilde{R}_t(X)$ for every $X \in B_{K_T,m}$.

Thus, the values $\tilde{R}_t(X)$ of a risk process are $\mathcal{F}_t$-measurable random sets in $\mathbb{R}^d$. Primal and dual representations can be provided for the measurable selectors of this set. Recall that $\gamma$ is a $\mathcal{F}_t$-measurable selector of a $\mathcal{F}_t$-random set $\Gamma$ if $\gamma(\omega) \in \Gamma[\omega]$ for almost every $\omega \in \Omega$. Then the measurable selectors in $L^p$ are given by $L^p_t(\Gamma) = \{ \gamma \in L^p_t : \mathbb{P}(\gamma \in \Gamma) = 1 \}$.

**Definition 6.1.2.** Given a risk process $(\tilde{R}_t)_{t \in \mathbb{T}}$, then $S_{\tilde{R}} : \mathbb{T} \times B_{K_T,m} \rightarrow 2^{M_t}$ is a selector risk measure if $S_{\tilde{R}}(t, X) := L^p_t(\tilde{R}_t(X))$ for every time $t$ and portfolio
The bounded selector risk measure is defined by \( S_R^\infty(t, X) := S_R(t, X) \cap B_{K_T, m} \).

**Definition 6.1.3.** A set \( A_t \subseteq B_{K_T, m} \) is a **conditional acceptance set** at time \( t \) if:

1. \( A_t \) is closed in the \( (B_{K_T, m}, \| \cdot \|_{K_T, m}) \) topology.
2. If \( X \in B_{K_T, m} \) such that \( X \geq_{K_T} 0 \) then \( X \in A_t \).
3. \( B_{K_T, m} \cap M_t \nsubseteq A_t \).
4. \( A_t \) is \( \mathcal{F}_t \)-decomposable, i.e. if for any finite partition \( (\Omega_n^t)_{n=1}^N \subseteq \mathcal{F}_t \) of \( \Omega \) and any family \( (X_n)_{n=1}^N \subseteq A_t \), then \( \sum_{n=1}^N 1_{\Omega_n^t} X_n \in A_t \).
5. \( A_t \) is a conditionally convex cone.

**Remark 6.1.4.** The definition for \( \mathcal{F}_t \)-decomposability given above can be found in [69, page 148] or [60, page 260]. Note that this definition for \( \mathcal{F}_t \)-decomposability differs from that in [11], as in that paper \( \mathcal{F}_t \)-decomposability is considered with respect to countable rather than finite partitions (in fact only the partition \( \{ D, D^c \} \subseteq \mathcal{F}_t \) is necessary). We weakened the condition by adapting the proof of theorem 1.6 of chapter 2 from [69] when \( p = +\infty \) to the space \( B_{K_T, m} \).

**Proposition 6.1.5** (Proposition 3.13 of [11]). Given a conditionally coherent risk process \( \tilde{R}_t \) at time \( t \), then \( A_t := \left\{ X \in B_{K_T, m} : 0 \in \tilde{R}_t(X) \right\} \) is a conditional acceptance set at time \( t \).

A primal representation of the selector risk measure is given as follows.

**Theorem 6.1.6** (Theorem 3.14 in [11]). Let \( A_t \) be a closed subset of \( (B_{K_T, m}, \| \cdot \|_{K_T, m}) \). Then \( A_t \) is a conditional acceptance set if and only if there exists some conditionally coherent risk process \( \tilde{R}_t \) at time \( t \) such that the associated bounded selector risk measure \( S_R^\infty \) satisfies \( S_R^\infty(t, X) = \left\{ u \in M_t : X + u \in A_t \right\} \) for all \( X \in B_{K_T, m} \).
Below, we give the dual representation for coherent selector risk measures as done in theorem 4.2 and theorem 4.8 of [11]. This dual representation can be viewed as the intersection of supporting half-spaces for the selector risk measure, which is the reason that coherence is needed in this approach.

From [11], it is known that \((B_{K_T,m}, \| \cdot \|_{K_T,m})\) is a Banach space, let \(ba_{K_T,m}\) be the topological dual of \(B_{K_T,m}\), and let \(ba_{K_T,m}^+\) denote the positive linear forms, that is

\[
ba_{K_T,m}^+ := \{ \phi \in ba_{K_T,m} : \phi(X) \geq 0 \forall X \geq_{K_T} 0 \}.
\]

**Definition 6.1.7** (Definition 4.1 of [11]). A set \( \Lambda \subseteq ba_{K_T,m} \) is called \( \mathcal{F}_t\)-stable if for all \( \lambda \in L_{t,+}^\infty \) and \( \phi \in \Lambda \), the linear form \( \phi^\lambda : X \ni B_{K_T,m} \mapsto \phi(\lambda X) \) is an element of \( \Lambda \).

**Theorem 6.1.8** (Theorem 4.2 of [11]). Let \((\tilde{R}_t)_{t \in \mathbb{T}}\) be a sequence of \((S_{d,m}^t)_{t \in \mathbb{T}}\)-valued mappings on \(B_{K_T,m}\). Then the following are equivalent:

1. \((\tilde{R}_t)_{t \in \mathbb{T}}\) is a conditionally coherent risk process.

2. There exists a nonempty \(\sigma(ba_{K_T,m}, B_{K_T,m})\)-closed subset \(Q_t \neq \{0\}\) of \(ba_{K_T,m}^+\) which is \(\mathcal{F}_t\)-stable and satisfies the equality

\[
S^\infty_R(t, X) = \{ u \in M_t \cap B_{K_T,m} : \phi(X + u) \geq 0 \forall \phi \in Q_t \}.
\]  

(6.1.1)

We finish the discussion of the dual representation by considering the case when the risk process additionally satisfies a “Fatou property” as defined below.

**Definition 6.1.9.** A sequence \((\tilde{R}_t)_{t \in \mathbb{T}}\) of \((S_{d,m}^t)_{t \in \mathbb{T}}\)-valued mappings on \(B_{K_T,m}\) is said to satisfy the Fatou property if for all \(X \in B_{K_T,m}\) and all times \(t\)

\[
\limsup_{n \to +\infty} S^\infty_R(t, X_n) \subseteq S^\infty_R(t, X)
\]

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for any bounded sequence \((X_n)_{n \in \mathbb{N}} \subseteq B_{K_T,m}\) which converges to \(X\) in probability.

Note that in the above definition the limit superior is defined to be

\[
\limsup_{n \to +\infty} B_n = \overline{\bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} B_m}
\]

for any sequence of sets \((B_n)_{n \in \mathbb{N}}\).

For the following theorem we assume two additional properties on the convex cone \(K_T\):

k4. for almost every \(\omega \in \Omega\): \(\mathbb{R}_+^d \setminus \{0\} \subseteq \text{int}(K_T[\omega])\) or equivalently \(K_T[\omega]^+ \setminus \{0\} \subseteq \text{int}(\mathbb{R}_+^d)\);

k5. \(K_T\) and \(K_T^+\) are both generated by a finite number of linearly independent and bounded generators denoted respectively by \((\xi_i)_{i=1}^N\) and \((\xi_i^+)_{i=1}^{N^+}\).

Let \(L^{1,m}_{\ast}(K_T^+) := \left\{ Z \in L^0(K_T^+) : 1_{d,m}^T Z \in L^1 \right\}\). In the following theorem we use \(L^{1,m}_{\ast}(K_T^+)\) as a dual space for \(B_{K_T,m}\). For \(Z \in L^{1,m}_{\ast}(K_T^+)\), the linear form \(\phi_Z(X) := \mathbb{E}[Z^T X]\) belongs to \(ba_{K_T,m}^+\). We use the dual norm

\[
\|Z\|_{d,m} := \sup \left\{ \|\mathbb{E}[Z^T X]\| : X \in B_{K_T,m}, \|X\|_{K_T,m} \leq 1 \right\}
\]

for any \(Z \in L^{1,m}_{\ast}(K_T^+)\).

**Theorem 6.1.10** (Theorem 4.8 of [11]). Let \((\tilde{R}_t)_{t \in \mathbb{T}}\) be a conditionally coherent risk process on \(B_{K_T,m}\) and let \(K_T\) satisfy property k1 – k5. The following are equivalent:

1. For every time \(t \in \mathbb{T}\), there exists a closed conditional cone \(\{0\} \neq Q_t^1 \subseteq L^{1,m}_{\ast}(K_T^+)\) (in the norm topology, with norm \(\|\cdot\|_{d,m}\)) such that for any \(X \in B_{K_T,m}\)

\[
S_R^\infty(t, X) = \left\{ u \in M_t \cap B_{K_T,m} : \forall Z \in Q_t^1 : \mathbb{E}[Z^T (X + u)] \geq 0 \right\}.
\]
2. \((\tilde{R}_t)_{t \in T}\) satisfies the Fatou property.

3. \(C_t := \left\{ X \in B_{K_T,m} : 0 \in \tilde{R}_t(X) \right\}\) is \(\sigma(B_{K_T,m}, \mathcal{L}^{1,m}(K_T^+))\)-closed.

We conclude this section by discussing time consistency properties as they were defined in the measurable selector approach in [11]. As in the set-optimization approach in chapter 3, one would like to define a property that is equivalent to a recursive form. For this reason we extend the risk process to be a function of a set. For a set \(X \subseteq B_{K_T,m}\), let us define \(\tilde{R}_t(X) \in S_{t,m}^d\) via its selectors, that is

\[
\mathcal{L}^0_t(\tilde{R}_t(X)) \cap B_{K_T,m} = \text{cl} \text{env}_{\mathcal{F}_t} \bigcup_{X \in X} S_{\tilde{R}}^\infty(t, X) := S_{\tilde{R}}^\infty(t, X),
\]

where, for any \(\Gamma \subseteq B_{K_T,m}\), \(\text{env}_{\mathcal{F}_t} \Gamma\) denotes the smallest \(\mathcal{F}_t\)-decomposable set (see definition 6.1.3) which contains \(\Gamma\). This means that the measurable selectors of the risk process of a set are defined by the closed and \(\mathcal{F}_t\)-decomposable version of the pointwise union. Note that if \(X = \{X\}\) then this reduces to the prior definition on portfolios. The risk process of a set is defined in this way because the selection risk measure must be closed and \(\mathcal{F}_t\)-decomposable-valued to guarantee the existence of an \(\mathcal{F}_t\)-measurable random set \(\tilde{R}_t(X)\) such that \(S_{\tilde{R}}^\infty(t, X) = \mathcal{L}^0_t(\tilde{R}_t(X)) \cap B_{K_T,m}\).

**Definition 6.1.11.** A risk process \((\tilde{R}_t)_{t \in T}\) is called consistent in time if for any \(t, s \in T\) with \(t < s\) and \(X \in B_{K_T,m}\), \(Y \subseteq B_{K_T,m}\)

\[
\tilde{R}_s(X) \subseteq \tilde{R}_s(Y) \Rightarrow \tilde{R}_t(X) \subseteq \tilde{R}_t(Y).
\]

The following theorem gives equivalent characterizations of consistency in time, the last one being a recursion in the spirit of Bellman’s principle.

**Theorem 6.1.12** (Theorem 5.9 of [11]). A normalized risk process \((\tilde{R}_t)_{t \in T}\) on \(B_{K_T,m}\) is consistent in time if any of the following equivalent conditions hold:
1. If $\tilde{R}_s(X) \subseteq \tilde{R}_s(Y)$ for $X \in B_{K_T,m}$ and $Y \subseteq B_{K_T,m}$, then $\tilde{R}_t(X) \subseteq \tilde{R}_t(Y)$ for $t \leq s \leq T$.

2. If $\tilde{R}_s(X) = \tilde{R}_s(Y)$ for $X \in B_{K_T,m}$ and $Y \subseteq B_{K_T,m}$, then $\tilde{R}_t(X) = \tilde{R}_t(Y)$ for $t \leq s \leq T$.

3. For all $X \in B_{K_T,m}$, $S^\infty_R(t,X) = S^\infty_R(t, -S^\infty_R(s,X))$ for $t \leq s \leq T$.

### 6.1.1 Set-optimization approach versus measurable selectors

In order to compare these two approaches, one first needs to agree on the same pre-image and image space. One possibility would be to define the risk measures of chapter 2 on the space $B_{K_T,m}$. This can be done as the theory involved (set-optimization) works for any locally convex space as the pre-image space. The other possibility is to consider the measurable selectors approach on $L^p$ spaces. This in not a problem for the definition of risk processes given in definition 6.1.1, but could pose a problem for primal and dual representations, see discussion in remark 6.1.13 for more details. However, since for the comparison results we just work with the definitions, we follow this path here. Thus, consider $L^p$ spaces for $p \in [0, +\infty]$ endowed with the metric topology (that is the norm topology for $p \geq 1$), even for $p = +\infty$ which is in contrast to chapters 2 and 3 (as well as [34, 36]) where the weak* topology is used for $p = +\infty$. We show that when the dynamic risk measure has closed and conditionally convex images, the set-optimization and the measurable selectors approach coincide.

**Remark 6.1.13.** While the space $B_{K_T,m}$ shares many properties with $L^\infty$, the two do not coincide in general. If $m = d$ or additional assumptions (e.g. substitutability from [58]) are satisfied, then $L^\infty \subseteq B_{K_T,m}$. If $m = d$ and $K_T = \mathbb{R}^d_+$ almost surely, then $B_{K_T,m} = L^\infty$. However, in general the two spaces are not comparable in the set-inclusion relation. Therefore, without additional assumptions, it is not trivial to use the representation results from [11] for the space $L^\infty$. Furthermore, the assumptions
for the Fatou duality (theorem 6.1.10) exclude the special case $K_T = \mathbb{R}^d_+$ and thus
exclude the case $B_{K_T,m} = L^\infty$ when $m = d$. However, the definition for risk process
can be given for $L^p$ spaces (and this is used in this section). But complications arise
in both the primal and dual definition, as e.g. boundedness is used in the proofs in

The following theorem and corollary 6.1.15 below state that there is a one-to-
one relation between conditional risk measures $R_t$ with closed and $\mathcal{F}_t$-decomposable
images and closed risk processes $\tilde{R}_t$. In corollary 6.1.20 we demonstrate that any
conditional risk measure with closed and conditionally convex images also has $\mathcal{F}_t$-
decomposable images.

**Theorem 6.1.14.** Let $\tilde{R}_t : L^p \to S^{d,m}_t$ be a risk process at time $t$ (see definition 6.1.1),
then $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$, defined by $R_t(X) := L^p_t(\tilde{R}_t(X))$ for any $X \in L^p$, is a
conditional risk measure at time $t$ (see definition 2.1.2) with $\mathcal{F}_t$-decomposable images.

Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time $t$ (see definition 2.1.2) with closed and $\mathcal{F}_t$-decomposable images, then there exists a risk process
$\tilde{R}_t : L^p \to S^{d,m}_t$ (see definition 6.1.1) such that $R_t(X) = L^p_t(\tilde{R}_t(X))$ for any $X \in L^p$.

**Proof.** 1. Let $\tilde{R}_t : L^p \to S^{d,m}_t$ be a risk process at time $t$. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$
be defined by $R_t(X) := L^p_t(\tilde{R}_t(X))$ for any $X \in L^p$. It remains
to show that $R_t$ is a conditional risk measure at time $t$. $L^p_+$-monotonicity: let
$X, Y \in L^p$ such that $Y \succeq X$, then $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$, and thus $R_t(X) \supseteq R_t(X)$. $M_t$-translativity: let $X \in L^p$ and $u \in M_t$, then $R_t(X + u) = L^p_t(\tilde{R}_t(X + u)) = L^p_t(\tilde{R}_t(X) - u) - u = R_t(X) - u$. Finiteness at zero: By $\tilde{R}_t(0) \neq \emptyset$
almost surely then trivially $R_t(0) = L^p_t(\tilde{R}_t(0)) \neq \emptyset$. By $\tilde{R}_t(0) \neq M$ almost
surely then if $u(\omega) \in M \setminus \tilde{R}_t(0)[\omega]$ for almost every $\omega \in \Omega$ such that $u \in M_t$, 
then $u(\omega) \not\in R_t(0)[\omega]$ for almost every $\omega \in \Omega$. $\mathcal{F}_t$-decomposable images: Let
$(\Omega^t_n)_{n=1}^N \subseteq \mathcal{F}_t$ for some $N \in \mathbb{N}$ be a finite partition of $\Omega$ and let $(u_n)_{n=1}^N \subseteq R_t(X)$
then $\sum_{n=1}^{N} 1_{\Omega_n} u_n \in M_t$, then since $R_t(X)$ are the measurable selectors of $\tilde{R}_t(X)$ it immediately follows that $\sum_{n=1}^{N} 1_{\Omega_n} u_n \in R_t(X)$.

2. Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time $t$ with closed and $\mathcal{F}_t$-decomposable images. By proposition 5.4.3 in [60] (for $p \in [0, +\infty)$) and theorem 1.6 of chapter 2 from [69] (for $p = +\infty$), it follows that $R_t(X) = \mathcal{L}_t^p(\tilde{R}_t(X))$ for some almost surely closed random set $\tilde{R}_t(X)$ for every $X \in L^p$.

Trivially, we can see that $\tilde{R}_t(X) \subseteq M$ almost surely. It remains to show that $\tilde{R}_t$ is a risk process at time $t$. Let $X \in L^p$, then $\tilde{R}_t(X)$ is a closed $\mathcal{F}_t$-measurable random set [60, proposition 5.4.3] and [69, chapter 2 theorem 1.6]. Finiteness at zero of $R_t$ implies finiteness at zero of $\tilde{R}_t$. Consider $X, Y \in L^p$ with $Y \succeq X$, then $R_t(Y) \supseteq R_t(X)$, which implies that $\tilde{R}_t(Y) \supseteq \tilde{R}_t(X)$. Let $X \in L^p$ and $u \in M_t$, then $R_t(X + u) = R_t(X) - u$. This implies $\mathcal{L}_t^p(\tilde{R}_t(X + u)) = \mathcal{L}_t^p(\tilde{R}_t(X)) - u = \mathcal{L}_t^p(\tilde{R}_t(X) - u)$, i.e. $\tilde{R}_t(X + u) = \tilde{R}_t(X) - u$ almost surely.

\[ \square \]

In the below corollaries the conditional risk measure associated with the risk process (and vice versa) is defined as in theorem 6.1.14 above.

**Corollary 6.1.15.** Let $\tilde{R}_t : L^p \to S_t^{d,m}$ be a conditionally convex (conditionally positive homogeneous, normalized) risk process at time $t$ then the associated conditional risk measure is conditionally convex (conditionally positive homogeneous, normalized).

Let $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditionally convex (conditionally positive homogeneous, normalized) conditional risk measure at time $t$ with closed and $\mathcal{F}_t$-decomposable images, then the associated risk process is conditionally convex (conditionally positive homogeneous, normalized).

**Proof.** 1. Let $\tilde{R}_t : L^p \to S_t^{d,m}$ be a risk process at time $t$ and $R_t$ be the associated conditional risk measure. Let $\tilde{R}_t$ be conditionally convex. Take $X, Y \in L^p$,
\( \lambda \in L_t^{\infty}([0,1]). \) Then,

\[
\lambda R_t(X) + (1 - \lambda) R_t(Y) = \lambda \mathcal{L}_t^p(\tilde{R}_t(X)) + (1 - \lambda) \mathcal{L}_t^p(\tilde{R}_t(Y)) \\
= \mathcal{L}_t^p(\lambda \tilde{R}_t(X) + (1 - \lambda) \tilde{R}_t(Y)) \\
\subseteq \mathcal{L}_t^p(\tilde{R}_t(\lambda X + (1 - \lambda) Y)) = R_t(\lambda X + (1 - \lambda) Y).
\]

Let \( \tilde{R}_t \) be conditionally positive homogeneous. Take \( X \in \mathcal{L}^p \) and \( \lambda \in L_t^{\infty}. \) Then, \( \lambda R_t(X) = \lambda \mathcal{L}_t^p(\tilde{R}_t(X)) = \mathcal{L}_t^p(\lambda \tilde{R}_t(X)) = R_t(\lambda X). \) Let \( \tilde{R}_t \) be normalized and let \( X \in \mathcal{L}^p. \) Then, \( R_t(X) + R_t(0) = \mathcal{L}_t^p(\tilde{R}_t(X)) + \mathcal{L}_t^p(\tilde{R}_t(0)) = \mathcal{L}_t^p(\tilde{R}_t(X) + \tilde{R}_t(0)) = \mathcal{L}_t^p(\tilde{R}_t(X)) = R_t(X). \)

2. Let \( R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+}) \) be a conditional risk measure at time \( t \) and let \( \tilde{R}_t \) be the associated risk process. Let \( R_t \) be conditionally convex. Take \( X, Y \in \mathcal{L}^p \) and \( \lambda \in L_t^{\infty}([0,1]). \) Then,

\[
\mathcal{L}_t^p(\lambda \tilde{R}_t(X) + (1 - \lambda) \tilde{R}_t(Y)) = \lambda R_t(X) + (1 - \lambda) R_t(Y) \\
\subseteq R_t(\lambda X + (1 - \lambda) Y) = \mathcal{L}_t^p(\tilde{R}_t(\lambda X + (1 - \lambda) Y)).
\]

By [69, chapter 2 proposition 1.2 (iii)] it holds

\[
\lambda \tilde{R}_t(X) + (1 - \lambda) \tilde{R}_t(Y) \subseteq \tilde{R}_t(\lambda X + (1 - \lambda) Y)
\]

almost surely. The proof for conditional positive homogeneity and normalization is analog.

\[\square\]

As discussed in chapter 3 and section 6.1, we have time consistency properties for both the set-optimization and measurable selector approach to risk measures. Therefore, we would like to be able to compare multi-portfolio time consistency (def-
inition 3.2.1) and consistency in time (definition 6.1.11). These properties coincide in their notation, however as we show below the two properties only coincide under additional assumptions.

**Corollary 6.1.16.** Let \((\tilde{R}_t)_{t \in \mathbb{T}}\) be a normalized conditionally convex consistent in time risk process, then the associated dynamic risk measure is multi-portfolio time consistent if it is convex upper continuous.

Let \((R_t)_{t \in \mathbb{T}}\) be a normalized multi-portfolio time consistent dynamic risk measure with closed and \(\mathcal{F}_t\)-decomposable images for all times \(t\), then the associated risk process is consistent in time.

**Proof.** 1. Let \((\tilde{R}_t)_{t \in \mathbb{T}}\) be a normalized conditionally convex risk process which is consistent in time such that the associated dynamic risk measure \((R_t)_{t \in \mathbb{T}}\) is convex upper continuous. By theorem 6.1.12, it immediately follows that \(R_t(X) = \text{cl env}_{\mathcal{F}_t} \bigcup_{Z \in R_s(X)} R_t(-Z)\) for any \(X \in \mathcal{L}^p\) and any times \(t, s \in \mathbb{T}\) such that \(t \leq s\). By corollary 6.1.15 above, \((R_t)_{t \in \mathbb{T}}\) is conditionally convex.

We will show that the recursive form \(\bigcup_{Z \in R_s(X)} R_t(-Z)\) is \(\mathcal{F}_t\)-decomposable. Let \(N \in \mathbb{N}\), \((u_n)_{n=1}^N \subseteq \bigcup_{Z \in R_s(X)} R_t(-Z)\) and \((\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t\) is a partition of \(\Omega\). Denote by \(Z_n \in R_s(X)\) the element such that \(u_n \in R_t(-Z_n)\) for every \(n \in \{1, ..., N\}\). By lemma 6.1.17, it follows that \(\sum_{n=1}^N 1_{\Omega_t^n} Z_n \in R_s(X)\). Then
we can see

\[ \sum_{n=1}^{N} 1_{\Omega^n_t} u_n \in \sum_{n=1}^{N} 1_{\Omega^n_t} R_t(Z_n) = \sum_{n=1}^{N} 1_{\Omega^n_t} R_t(1_{\Omega^n_t} Z_n) \]

\[ = \sum_{n=1}^{N} 1_{\Omega^n_t} R_t(1_{\Omega^n_t} \sum_{m=1}^{N} 1_{\Omega^m_t} Z_m) = \sum_{n=1}^{N} 1_{\Omega^n_t} R_t(\sum_{m=1}^{N} 1_{\Omega^m_t} Z_m) \]

\[ \subseteq \{ u \in M_t : \exists J \subseteq \{1, \ldots, N\} \text{ s.t. } P(\cup_{j \in J} A_j) = 1, \]

\[ 1_{\Omega^n_t} u \in 1_{\Omega^m_t} R_t(\sum_{m=1}^{N} 1_{\Omega^m_t} Z_m) \forall j \in J \} \]

\[ = R_t(\sum_{m=1}^{N} 1_{\Omega^m_t} Z_m) \subseteq \bigcup_{Z \in R_t(X)} R_t(-Z). \]

In the above we use the local property for conditionally convex risk measures (proposition 2.1.8) and lemma 6.1.17. Therefore, \( \bigcup_{Z \in R_t(X)} R_t(-Z) \) is \( \mathcal{F}_t \)-decomposable, and thus \( R_t(X) = \text{cl} \bigcup_{Z \in R_t(X)} R_t(-Z) \). And as seen in appendix A.3, if \( (R_t)_{t \in \mathbb{T}} \) is convex upper continuous then \( \bigcup_{Z \in R_t(X)} R_t(-Z) \) is closed for any \( X \in \mathcal{L}^p \). Therefore, \( R_t(X) = \bigcup_{Z \in R_t(X)} R_t(-Z) \), i.e. \( R_t(X) \) multi-portfolio time consistent.

2. Let \( (R_t)_{t \in \mathbb{T}} \) be a normalized multi-portfolio time consistent dynamic risk measure with closed and \( \mathcal{F}_t \)-decomposable images for all time \( t \). Let \( (\hat{R}_t)_{t \in \mathbb{T}} \) be the associated risk process. By theorem 3.2.2, it follows that \( R_t(X) = \bigcup_{Z \in R_t(X)} R_t(-Z) \) for any \( X \in \mathcal{L}^p \) and any times \( t, s \in \mathbb{T} \) such that \( t \leq s \). Since \( R_t \) has closed and \( \mathcal{F}_t \)-decomposable images then it additionally follows that \( \bigcup_{Z \in R_t(X)} R_t(-Z) = \text{clenv}_{\mathcal{F}_t} \bigcup_{Z \in R_t(X)} R_t(-Z) \) for any \( X \in \mathcal{L}^p \). Therefore, \( \mathcal{L}_t^p(\hat{R}_t(X)) = \mathcal{L}_t^p(\hat{R}_t(-R_s(X))) \) and thus, by theorem 6.1.12, it follows that \( (\hat{R}_t)_{t \in \mathbb{T}} \) is consistent in time.

\( \square \)
The convex upper continuity in the first part of the above theorem could we weakened as one only needs \( \bigcup_{Z \in R_t(\mathcal{X})} R_t(-Z) \) is closed for any \( X \in \mathcal{L}^p \) and \( t \leq s \).

Up to this point we have made the additional assumption for conditional risk measures of chapter 2 to be \( \mathcal{F}_t \)-decomposable. The following results (lemma 6.1.17 and corollary 6.1.20 below) demonstrate that a conditional risk measure with closed and conditionally convex images satisfies a property stronger than \( \mathcal{F}_t \)-decomposable images as the property remains true for any (possibly uncountable) partition as well.

**Lemma 6.1.17.** Let \( (R_t)_{t \in \mathbb{T}} \) be a dynamic risk measure with closed and conditionally convex images. Let \( (A_i)_{i \in I} \subseteq \mathcal{F}_t \) be a partition of \( \Omega \). Then

\[
R_t(X) = \left\{ u \in M_t : \exists J \subseteq I \text{ s.t. } \mathbb{P}(\bigcup_{j \in J} A_j) = 1, \ 1_{A_j} u \in 1_{A_j} R_t(X) \ \forall j \in J \right\}
\]

for any \( X \in \mathcal{L}^p \) and any time \( t \).

Before giving the proof we give a remark on the uncountable summation as it will be used in part 2 (b) of the proof.

**Remark 6.1.18.** As given in [17, Chapter 3 Section 5] and [25, Chapter 3 Section 3.9], the arbitrary summation on a Hausdorff commutative topological group is given by

\[
\sum_{j \in J} f_j = \lim_{K \in \mathcal{J}} \sum_{k \in K} f_k,
\]

for any \( \{f_j \in \mathcal{X} : j \in J\} \) where \( \mathcal{X} \) is a Hausdorff commutative topological group, such that \( \mathcal{J} = \{K \subseteq J : \#K < +\infty\} \), i.e. \( \mathcal{J} \) are the finite subsets of \( J \). Note that \( \mathcal{J} \) is a net with order given by set inclusion and join given by the union.

In particular, for our concerns, the metric topologies for \( \mathcal{L}^p_t \) for \( p \in [0, +\infty] \) are all Hausdorff commutative topological groups. (If \( p = 0 \) then we consider convergence in measure, which is equivalent to a metric space with metric \( d(f, g) = \int_{\Omega} \frac{|f-g|}{1+|f-g|} d\mathbb{P} \) (lemma 13.40 in [2])).
Proof of lemma 6.1.17. Note that $1_D R_t(X) = \{1_D u : u \in R_t(X)\}$ for any $D \in \mathcal{F}_t$.

For notational convenience let

$$\hat{R}_t(X) := \{ u \in M_t : \exists J \subseteq I \text{ s.t. } \mathbb{P}(\cup_{j \in J} A_j) = 1, \ 1_{A_j} u \in 1_{A_j} R_t(X) \ \forall j \in J \}.$$ 

1. The inclusion $R_t \subseteq \hat{R}_t$ follows straightforward: Let $u \in R_t(X)$, then by definition $1_D u \in 1_D R_t(X)$ for any $D \in \mathcal{F}_t$, and in particular this is true for $D = A_i$ for any $i \in I$. Therefore it follows that $u \in \hat{R}_t(X)$.

2. To prove $\hat{R}_t \subseteq R_t$ we will consider the two cases: finite and infinite partitions.

Let $u \in \hat{R}_t(X)$ and $J \subseteq I$ the underlying subindex. Then $u = \sum_{j \in J} 1_{A_j} u$ almost surely, therefore $u \in R_t(X)$ if and only if $\sum_{j \in J} 1_{A_j} u \in R_t(X)$ since they are in the same equivalence class. Let $\#J$ denote the cardinality of the set $J$. Note that by definition $1_{A_j} u \in 1_{A_j} R_t(X)$ for every $j \in J$.

(a) If $\#J < +\infty$, i.e. if $J$ is a finite set, then trivially

$$\sum_{j \in J} 1_{A_j} u \in \sum_{j \in J} 1_{A_j} R_t(X) \subseteq R_t(X)$$

by closedness and conditional convexity of $R_t(X)$ as shown in proposition 6.1.19 below. And thus $u \in R_t(X)$.

(b) Consider the case $\#J = +\infty$, i.e. if $J$ is not a finite set. Let $u \in \hat{R}_t(X)$, that is there exists $J \subseteq I$ with $\mathbb{P}(\cup_{j \in J} A_j) = 1$ such that $1_{A_j} u \in 1_{A_j} R_t(X)$ for all $j \in J$, or equivalently $1_{A_j} (u - \hat{u}) \in 1_{A_j} R_t(X + \hat{u})$ for all $j \in J$ for some $\hat{u} \in R_t(X)$ by using the translation property of $R_t$. We want to show $u \in R_t(X)$, respectively $u - \hat{u} \in R_t(X + \hat{u})$. Recall the summation as given
in remark 6.1.18, and the notation $\mathcal{J} = \{K \subseteq J : \#K < +\infty\}$.

$$u - \hat{u} = \sum_{j \in J} 1_{A_j}(u - \hat{u}) \in \sum_{j \in J} 1_{A_j} R_t(X + \hat{u})$$

$$= \left\{ \sum_{j \in J} 1_{A_j} Z_j : Z_j \in R_t(X + \hat{u}) \forall j \in J \right\}$$

$$= \left\{ \lim_{K \in \mathcal{J}} \sum_{k \in K} 1_{A_k} Z_k : Z_j \in R_t(X + \hat{u}) \forall j \in J \right\} \quad (6.1.3)$$

$$= \left\{ \lim_{K \in \mathcal{J}} \left( \sum_{k \in K} 1_{A_k} Z_k + 1_{(\cup_{j \in J \setminus K} A_j)0} \right) : Z_j \in R_t(X + \hat{u}) \forall j \in J \right\}$$

$$\subseteq \left\{ \lim \inf_{K \in \mathcal{J}} \left( \sum_{k \in K} 1_{A_k} Z_k + 1_{(\cup_{j \in J \setminus K} A_j)\check{Z}} \right) : Z_j, \check{Z} \in R_t(X + \hat{u}) \forall j \in J \right\} \quad (6.1.4)$$

$$\subseteq \lim \inf_{K \in \mathcal{J}} \left\{ \sum_{k \in K} 1_{A_k} Z_k + 1_{(\cup_{j \in J \setminus K} A_j)Z} : Z_k, Z \in R_t(X + \hat{u}) \forall k \in K \right\}$$

$$= \lim \inf_{K \in \mathcal{J}} \left( \sum_{k \in K} 1_{A_k} R_t(X + \hat{u}) + 1_{(\cup_{j \in J \setminus K} A_j)} R_t(X + \hat{u}) \right)$$

$$= \lim \inf_{K \in \mathcal{J}} R_t(X + \hat{u}) = R_t(X + \hat{u}). \quad (6.1.5)$$

Equation (6.1.3) follows from the definition of an arbitrary summation as given in [17, 25], see remark 6.1.18. Inclusion (6.1.4) follows from $0 \in R_t(X + \hat{u})$ since $\hat{u} \in R_t(X)$. Equation (6.1.5) follows from the finite case given above applied to the partition $((A_k)_{k \in K}, \cup_{j \in J \setminus K} A_j)$. Note that $\cup_{j \in J \setminus K} A_j \in \mathcal{F}_t$ by $(\mathcal{F}_t)_{t \in T}$ a filtration satisfying the usual conditions (and $\mathcal{F}_t$ is a sigma algebra). Furthermore, note that we define the limit inferior as in [66] to be $\lim \inf_{n \in N} B_n = \bigcap_{n \in N} \text{cl} \bigcup_{m \geq n} B_m$ for a net of sets $(B_n)_{n \in N}$.

The following proposition is used in the proof of lemma 6.1.17.
Proposition 6.1.19. A closed set $D \subseteq \mathcal{L}^p_t$ is conditionally convex if and only if for any $N \in \mathbb{N}$ where $N \geq 2$

$$\sum_{n=1}^{N} \lambda_n D \subseteq D \quad (6.1.6)$$

for every $(\lambda_n)_{n=1}^{N} \in \Lambda_N := \{(x_n)_{n=1}^{N} : \sum_{n=1}^{N} x_n = 1 \text{ a.s.}, x_n \in L^\infty_{t,+} \forall n \in \{1, \ldots, N\}\}.$

Proof. $\Leftarrow$ If $N = 2$ then this is the definition of conditional convexity. If $N > 2$ then choose $(\lambda_n)_{n=1}^{N}$ such that $\lambda_n = 0$ almost surely for every $n > 2$, this then reduces to the case when $N = 2$ and thus $D$ is conditionally convex.

$\Rightarrow$ We first define a set of multipliers for strict convex combinations

$$\Lambda_N^\succ = \left\{(x_n)_{n=1}^{N} : \sum_{n=1}^{N} x_n = 1 \text{ a.s.}, x_n \in L^\infty_{t,+} \forall n \in \{1, \ldots, N\}\right\}.$$

Then the result for $\Lambda_N^\succ$ for any $N \in \mathbb{N}$ follows as in the static case (i.e. when $x_n \in \mathbb{R}_{++}$) by induction.

Let $(\lambda_n)_{n=1}^{N} \in \Lambda_N$. Then there exists a sequence of $((\lambda^m_n)_{n=1}^{N})_{m=0}^{+\infty} \subseteq \Lambda_N^\succ$ which converges almost surely to $(\lambda_n)_{n=1}^{N}$ (i.e. for any $n \in \{1, \ldots, N\}$, $(\lambda^m_n)_{m=0}^{+\infty}$ converges almost surely to $\lambda_n$, and for every $m$ it holds $\sum_{n=1}^{N} \lambda^m_n = 1$ almost surely). By the dominated convergence theorem, it follows that $\lambda^m_n X$ converges to $\lambda_n X$ in the metric topology for any $X \in \mathcal{L}^p_t$. Therefore for any $(X_n)_{n=1}^{N} \subseteq D$ (and let $\bar{X}_m = \sum_{n=1}^{N} \lambda^m_n X_n \in D$ for any $m$)

$$\sum_{n=1}^{N} \lambda_n X_n = \sum_{n=1}^{N} \lim_{m \to +\infty} \lambda^m_n X_n = \lim_{m \to +\infty} \sum_{n=1}^{N} \lambda^m_n X_n = \lim_{m \to +\infty} \bar{X}_m \in D$$

by $\bar{X}_m$ convergent (since it is the finite sum of converging series) and $D$ closed.

Corollary 6.1.20. Any conditional risk measure $R_t$ with closed and conditionally convex images has $\mathcal{F}_t$-decomposable images.
Proof. Let $R_t$ be a conditional risk measure with closed and conditionally convex images, and let $X \in \mathcal{L}^p$. Let $(\Omega_t^n)_{n=1}^N \subseteq \mathcal{F}_t$, for some $N \in \mathbb{N}$, be a finite partition of $\Omega$. By lemma 6.1.17,

$$R_t(X) = \left\{ u \in M_t : \exists J \subseteq \{1, ..., N\} : \mathbb{P}(\cup_{j \in J} \Omega_t^j) = 1, \forall j \in J : 1_{\Omega_t^j} u \in 1_{\Omega_t^j} R_t(X) \right\}.$$ 

Therefore, if $(u_n)_{n=1}^N \subseteq R_t(X)$, then $1_{\Omega_t^m} \sum_{n=1}^N 1_{\Omega_t^n} u_n = 1_{\Omega_t^m} u_m \in 1_{\Omega_t^m} R_t(X)$ for every $m \in \{1, ..., N\}$, and thus $\sum_{n=1}^N 1_{\Omega_t^n} u_n \in R_t(X)$. \qed

We showed that when the dynamic risk measure has closed and conditionally convex images, the set-optimization approach of chapter 2 and the measurable selector approach of section 6.1 coincide. As a conclusion, the set-optimization approach which is using convex analysis results for set-valued functions, i.e. set-optimization, seems to be the richer approach as it allows the handling of primal and dual representations for $\mathcal{L}^p$ spaces ($p \in [1, +\infty]$) as well as for the space $B_{K_T,m}$ (or any other locally convex pre-image space). Furthermore, it allows the consideration of conditionally convex (and not necessarily conditionally coherent) risk measures as well as convex risk measures, whereas the measurable selectors approach relies heavily on the conditional coherency assumption.

6.2 Set-valued portfolio approach

The approach for considering sets of portfolios, so called set-valued portfolios, as the argument of a set-valued risk measure was proposed in [21]. The reasoning for considering set-valued portfolios is to take the risk, not only of a portfolio $X$, but of every possible portfolio that $X$ can be traded for in the market, into account. We denote by $X$ the random set of portfolios for which $X \in \mathcal{L}^p$ can be exchanged. The concept of set-valued portfolios appears naturally when trading opportunities in the
market are taken into account. Below we provide two examples, one in which no trading is allowed and another in which any possible trade can be used. There are other examples provided in [21] on how a set-valued portfolio can be obtained, and the definition of the risk measure is independent of the method used to construct set-valued portfolios.

**Example 6.2.1.** The random mapping \( X = X + \mathbb{R}_d \) for a random vector \( X \in \mathcal{L}^p \) describes the case when no exchanges are allowed.

**Example 6.2.2.** (Example 2.3 of [21]) The random mapping \( X = X + K \) for a random vector \( X \in \mathcal{L}^p \) and a lower convex (random) set \( K \), such that \( \mathcal{L}^p(K) \) is closed, defines the set-valued portfolios related to the exchanges defined by \( K \). If \( K \) is a solvency cone (see e.g. [59, 78, 60]) or the sum of solvency cones at different time points, then \( K = -K \) is an exchange cone, and the associated random mapping defines a set-valued portfolio. The setting of example 6.2.1 corresponds to the case where \( K = \mathbb{R}_d \).

We slightly adjust the definitions given in [21] to include the dynamic extension of such risk measures, to incorporate the set of eligible portfolios \( M_t \), and go beyond the coherent case.

Let \( S^d_f \) denote the set of \( \mathcal{F} \)-random sets in \( \mathbb{R}_d \) (as in section 6.1 above). Let \( \tilde{S}^d_f \subseteq S^d_f \) be those random sets that are nonempty, closed, convex and lower, that is for \( X \in X \) also \( Y \in X \) whenever \( X - Y \in \mathbb{R}_d \) \( \mathbb{P} \)-a.s. As in [21], we consider set-valued portfolios \( X \in \tilde{S}^d_f \). By proposition 2.1.5 and theorem 2.1.6 in [69], the collection of \( p \)-integrable selectors of \( X \), that is \( \mathcal{L}^p(X) \), is a nonempty, closed, \( \mathcal{F} \)-conditionally convex, lower and \( \mathcal{F} \)-decomposable set, which is an element of \( \mathcal{G}(\mathcal{L}^p; \mathcal{L}^p) \). In [21], \( \tilde{S}^d_f \) is used as the pre-image set, one could also use the family \( \{ \mathcal{L}^p(X) : X \in \tilde{S}^d_f \} \subseteq \mathcal{G}(\mathcal{L}^p; \mathcal{L}^p) \) of sets of selectors as the pre-image set, which is particular useful when dynamic risk measures are considered and recursions due to multi-portfolio time consistency
become important. Recall that \( \mathcal{P}(M_t; M_{t,+}) := \{ D \subseteq M_t : D = D + M_{t,+} \} \) denotes the set of upper sets, which are used as the image space for the risk measures. Closed (conditionally) convex risk measures map into \( \mathcal{G}(M_t; M_{t,+}) \).

In the following definition for convex risk measures we consider a modified version of set-addition used in [21] which is denoted by \( \boxplus \). For two random sets \( X, Y \in \mathcal{S}^d_t \), \( X \boxplus Y \in \mathcal{S}^d_t \) is the random set such that \( \mathcal{L}^p(X \boxplus Y) = \text{cl}[\mathcal{L}^p(X) + \mathcal{L}^p(Y)] \). Note that, by lemma 2.1 in [21], if the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is non-atomic then we can consider the typical Minkowski addition instead of \( \boxplus \).

Definition 6.2.3 (Definition 2.10 of [21]). A function \( R_t : \mathcal{S}^d_t \rightarrow \mathcal{P}(M_t; M_{t,+}) \) is called a set-valued conditional risk measure if it satisfies the following conditions.

1. Cash invariance: \( R_t(X + u) = R_t(X) - u \) for any \( X \) and \( u \in M_t \).

2. Monotonicity: Let \( X \subseteq Y \) almost surely, then \( R_t(Y) \supseteq R_t(X) \).

The risk measure \( R_t \) is said to be closed-valued if its values are closed sets.

The risk measure \( R_t \) is said to be (conditionally) convex if for every set-valued portfolio \( X, Y \) and \( \lambda \in [0, 1] \) (respectively \( \lambda \in L^\infty_t([0,1]) \))

\[
R_t(\lambda X \boxplus (1 - \lambda)Y) \supseteq \lambda R_t(X) + (1 - \lambda)R_t(Y).
\]

The risk measure \( R_t \) is said to be (conditionally) positive homogeneous if for every \( X \) and \( \lambda > 0 \) (respectively \( \lambda \in L^\infty_{t,++} \))

\[
R_t(\lambda X) = \lambda R_t(X).
\]

The risk measure \( R_t \) is said to be (conditionally) coherent if it is (condition-
ally) convex and (conditionally) positive homogeneous.

The closed-valued variant of \( R_t \) is denoted by \( \bar{R}_t(X) = \text{cl}(R_t(X)) \) for every set-valued portfolio \( X \in \mathcal{S}^d_t \).
A set-valued portfolio \( \mathbf{X} \) is acceptable if \( 0 \in R_t(\mathbf{X}) \), i.e. we can define the acceptance set \( \mathbf{A}_t \subseteq \mathcal{S}^d_T \) by \( \mathbf{A}_t := \{ \mathbf{X} : 0 \in R_t(\mathbf{X}) \} \). And a primal representation for the risk measures can be given by the usual definition \( R_t(\mathbf{X}) = \{ u \in M_t : \mathbf{X} + u \in \mathbf{A}_t \} \) due to cash invariance.

We now consider a subclass of set-valued conditional risk measures presented in [21, section 3] that are constructed using a scalar dynamic risk measure for each component. For the remainder of this section we consider the case when \( M_t = \mathcal{L}^p_t \).

In [21], only (scalar) law invariant coherent risk measures were considered for this approach, we will consider the more general case.

Let \( \rho^1_t, ..., \rho^d_t \) be dynamic risk measures defined on \( L^p \) with values in \( \mathcal{L}^p_t \cup \{ +\infty \} \).

For a random vector \( \mathbf{X} = (X_1, ..., X_d)^T \in \mathcal{L}^p \) we define

\[
\rho_t(\mathbf{X}) = (\rho^1_t(X_1), ..., \rho^d_t(X_d))^T.
\]

We say the vector \( \mathbf{X} \in \mathcal{L}^p \) is **acceptable** if \( \rho_t(\mathbf{X}) \preceq 0 \), i.e. \( \rho^i_t(X_i) \leq 0 \) for all \( i = 1, ..., d \). We say the set-valued portfolio \( \mathbf{X} \) is **acceptable** if there exists a \( \mathbf{Z} \in \mathcal{L}^p(\mathbf{X}) \) such that \( \rho_t(\mathbf{Z}) \preceq 0 \).

**Definition 6.2.4** (Definition 3.3 of [21]). *The constructive conditional risk measure* \( R_t : \mathcal{S}^d_T \to \mathcal{P}(\mathcal{L}^p_t; \mathcal{L}^p_{t,+}) \) *is defined for any set-valued portfolio* \( \mathbf{X} \) *by*

\[
R_t(\mathbf{X}) = \{ u \in \mathcal{L}^p_t : \mathbf{X} + u \text{ is acceptable} \},
\]

*which is equivalent to*

\[
R_t(\mathbf{X}) = \bigcup_{\mathbf{Z} \in \mathcal{L}^p(\mathbf{X})} (\rho_t(\mathbf{Z}) + \mathcal{L}^p_{t,+}). \tag{6.2.1}
\]

*The closed-valued variant is defined by* \( \bar{R}_t(\mathbf{X}) := \text{cl}(R_t(\mathbf{X})) \) *for every* \( \mathbf{X} \in \mathcal{S}^d_T \).
In [21], the constructive (static) risk measures have been called selection risk measures, we modified the name here in accordance to the title of the paper [21] to avoid confusion with the measurable selector approach from section 6.1.

**Example 6.2.5.** Consider the no-exchange set-valued portfolios from example 6.2.1. Then the constructive conditional risk measure associated with any vector of scalar conditional risk measures is given by

\[ R_t(X) = \rho_t(X) + \mathcal{L}_{t,+}^p. \]

**Theorem 6.2.6** (Theorem 3.4 of [21]). Let \( \rho_t \) be a vector of dynamic risk measures, then \( R_t \) and \( \bar{R}_t \) given in definition 6.2.4 are both set-valued conditional risk measures.

If \( \rho_t \) is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an atomless probability space), then \( R_t \) and \( \bar{R}_t \) are convex (conditionally convex, positive homogeneous, conditionally positive homogeneous, law invariant convex on an atomless probability space).

Furthermore, [21] gives conditions under which the constructive (static) risk measure \( R_0 \) defined in (6.2.1) in the coherent case is closed, or Lipschitz and deduces upper and lower bounds for it and dual representations in certain special cases. Numerical examples for the calculation of upper and lower bounds are given.

### 6.2.1 Set-optimization approach versus set-valued portfolios

As in the prior sections, consider \( \mathcal{L}^p \) spaces with \( p \in [0, +\infty] \).

**Theorem 6.2.7.** Given a conditional risk measure \( R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+}) \) (see definition 2.1.2), then the function \( R_t : S^d_t \to \mathcal{P}(M_t; M_{t,+}) \) defined by

\[
R_t(X) := \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z) \quad (6.2.2)
\]
for any set-valued portfolio $X$ is a set-valued conditional risk measure (see definition 6.2.3).

Given a set-valued conditional risk measure $R_t : \tilde{S}_T \rightarrow \mathcal{P}(M_t; M_{t,+})$ (see definition 6.2.3) and a mapping $X : \mathcal{L}^p \rightarrow \tilde{S}_T$ of the set-valued portfolio associated with a (random) portfolio vector such that $X$ is monotone and translative, i.e. $X(X) \subseteq X(Y)$ if $Y \succeq X$ and $X(X + u) = X(X) + u$ for any $X \in \mathcal{L}^p$ and $u \in M_t$, then the function $R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ defined by

$$R_t(X) := R_t(X(X)) \quad (6.2.3)$$

for any $X \in \mathcal{L}^p$ is a conditional risk measure (see definition 2.1.2) which might not be finite at zero.

**Proof.** 1. Let $R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure as in definition 2.1.2. Let $R_t(X) := \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z)$ for any set-valued portfolio $X$. We wish to show that $R_t$ satisfies definition 6.2.3.

   (a) Trivially $R_t(X) \in \mathcal{P}(M_t; M_{t,+})$ for any set-valued portfolio $X$.

   (b) Cash invariance: let $X$ be a set-valued portfolio and let $u \in M_t$, then

   $$R_t(X + u) = \bigcup_{Z \in \mathcal{L}^p(X + u)} R_t(Z) = \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z + u)$$

   $$= \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z) - u = R_t(X) - u.$$

   (c) Monotonicity: Let $X \subseteq Y$ almost surely, then

   $$R_t(X) = \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z) \subseteq \bigcup_{Z \in \mathcal{L}^p(Y)} R_t(Z) = R_t(Y).$$
2. Let $R_t : \tilde{S}_T^d \to \mathcal{P}(M_t; M_{t,+})$ be a set-valued conditional risk measure as in definition 6.2.3. Let $X : \mathcal{L}^p \to \tilde{S}_T^d$ be a mapping of portfolio vectors to set-valued portfolios that is monotone and translative. Let $R_t(X) := R_t(X(X))$ for any $X \in \mathcal{L}^p$. We wish to show that $R_t$ satisfies definition 2.1.2.

(a) $\mathcal{L}_t^p$-monotonicity: Let $X, Y \in \mathcal{L}^p$ such that $Y \succeq X$. Then $X(Y) \supseteq X(X)$, and thus $R_t(X) = R_t(X(X)) \subseteq R_t(X(Y)) = R_t(Y)$.

(b) $M_t$-translativity: Let $X \in \mathcal{L}^p$ and $u \in M_t$, then

$$R_t(X + u) = R_t(X(X + u)) = R_t(X(X) + u) = R_t(X(X)) - u = R_t(X) - u.$$

\[ \square \]

The above theorem states that conditional risk measures as in definition 2.1.2 can be used to construct set-valued conditional risk measure (see definition 6.2.3). This is in analogy to construction (6.2.1), but yields a larger class of risk measures. If one restricts oneself to set-valued portfolios $X : \mathcal{L}^p \to \tilde{S}_T^d$ which are monotonic and with $X(X + u) = X(X) + u$ for any $X \in \mathcal{L}^p$ and $u \in M_t$, then conditional risk measures as in definition 2.1.2 are one-to-one to set-valued conditional risk measure as in definition 6.2.3. This is the case whenever the set of portfolios $X$ represents the set of portfolios that can be obtained from $X \in \mathcal{L}^p$ following certain exchange rules (including transaction costs and illiquidity). The advantage of considering $R_t$ as a function of the set $X(X)$ as opposed to a function of $X$ as in (6.2.3) is that $R_t$ might be law invariant (see theorem 6.2.6), whereas $R_t$ is in general not law invariant.

**Example 6.2.8.** If $X(X) := X + K$ for some (almost surely) closed convex lower set $K$ such that $\mathcal{L}^p(K)$ is closed, then trivially $X(X)$ is a set-valued portfolio and satisfies monotonicity and translativity.
If $X(X)$ is as in example 6.2.8 and $K$ is additionally a convex cone, then for a given set-valued conditional risk measure $R_t$, the associated conditional risk measure $R_t$ defined by (6.2.3) is $K$-compatible.

Note, that constructions very similar to (6.2.2) appear a) in [53, 3] to define the market extension (that is a $C_{t,T}$-compatible version) of a risk measures $R_t$ by

$$R_{t}^{mar}(X) := \bigcup_{Z \in X + C_{t,T}} R_t(Z),$$

where $C_{t,T} = -\sum_{s=t}^{T} \mathcal{L}_{s}(K_s)$ and $(K_t)_{t \in \mathbb{T}}$ is a sequence of solvency cones modeling the bid-ask prices of the $d$ assets, and b) in section 3.5 to define a multi-portfolio time consistent risk measure $(\tilde{R}_t)_{t \in \mathbb{T}}$ by backward recursion of a discrete time dynamic risk measure $(R_t)_{t \in \mathbb{T}}$ via $\tilde{R}_T(X) = R_T(X)$ and

$$\tilde{R}_t(X) := \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z)$$

for $t \in \{T - 1, ..., 0\}$.

The following two corollaries provide additional relations between the conditional risk measures of the set-optimization approach and the set-valued portfolio conditional risk measures. Specifically, they provide sufficient conditions for (conditional) convexity and coherence of one type of risk measure to be associated with a (conditionally) convex and coherent risk measure of the other type.

**Corollary 6.2.9.** Let $R_t : \mathcal{L}_p \rightarrow \mathcal{P}(M_t; M_{t,+})$ be a convex (conditionally convex, positive homogeneous, conditionally positive homogeneous) conditional risk measure (see definition 2.1.2) at time $t$, then the associated set-valued conditional risk measure (see definition 6.2.3) $R_t$ defined by (6.2.2) is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous).
Proof. Let $R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure and let $R_t(X) := \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z)$ for any $X \in \mathcal{S}_T^d$. 

1. Let $R_t$ be convex. Consider $X, Y \in \mathcal{S}_T^d$ and $\lambda \in [0, 1]$. Then,

$$R_t(\lambda X \boxplus (1 - \lambda)Y) = \bigcup_{Z \in \mathcal{L}^p(\lambda X \boxplus (1 - \lambda)Y)} R_t(Z) = \bigcup_{Z \in \text{cl}(\lambda \mathcal{L}^p(X) + (1 - \lambda) \mathcal{L}^p(Y))} R_t(Z)$$

$$\supseteq \bigcup_{Z \in \mathcal{L}^p(X)} R_t(\lambda Z_X + (1 - \lambda)Z_Y)$$

$$\supseteq \bigcup_{Z \in \mathcal{L}^p(Y)} \left[ \lambda R_t(Z_X) + (1 - \lambda)R_t(Z_Y) \right]$$

$$= \lambda \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z_X) + (1 - \lambda) \bigcup_{Z \in \mathcal{L}^p(Y)} R_t(Z_Y)$$

$$= \lambda R_t(X) + (1 - \lambda)R_t(Y).$$

2. Let $R_t$ be conditionally convex. Then the proof is analogous to the convex case above.

3. Let $R_t$ be positive homogeneous. Consider $X \in \mathcal{S}_T^d$ and $\lambda > 0$. It holds

$$R_t(\lambda X) = \bigcup_{Z \in \mathcal{L}^p(\lambda X)} R_t(Z) = \bigcup_{Z \in \mathcal{L}^p(X)} R_t(\lambda Z) = \lambda \bigcup_{Z \in \mathcal{L}^p(X)} R_t(Z) = \lambda R_t(X).$$

4. Let $R_t$ be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.

\[ \square \]

**Corollary 6.2.10.** Let $R_t : \mathcal{S}_T^d \rightarrow \mathcal{P}(M_t; M_{t,+})$ be a set-valued conditional risk measure (see definition 6.2.3) at time $t$, and let $X : \mathcal{L}^p \rightarrow \mathcal{S}_T^d$ of the set-valued portfolio associated with a (random) portfolio vector be monotonic and translative. Let $R_t$ be the associated conditional risk measure (see definition 2.1.2).
1. If \( R_t \) is convex and
\[
X(\lambda X + (1 - \lambda) Y) \supseteq \lambda X(X) \oplus (1 - \lambda) X(Y)
\]
for every \( X, Y \in \mathcal{L}^p \) and \( \lambda \in [0,1] \) (\( X \) is closed-convex), then \( R_t \) is convex.

2. If \( R_t \) is conditionally convex and
\[
X(\lambda X + (1 - \lambda) Y) \supseteq \lambda X(X) \oplus (1 - \lambda) X(Y)
\]
for every \( X, Y \in \mathcal{L}^p \) and \( \lambda \in L^\infty([0,1]) \) (\( X \) is conditionally closed-convex), then \( R_t \) is conditionally convex.

3. If \( R_t \) is positive homogeneous and
\[
X(\lambda X) = \lambda X(X)
\]
for every \( X \in \mathcal{L}^p \) and \( \lambda > 0 \) (\( X \) is positive homogeneous), then \( R_t \) is positive homogeneous.

4. If \( R_t \) is conditionally positive homogeneous and
\[
X(\lambda X) = \lambda X(X)
\]
for every \( X \in \mathcal{L}^p \) and \( \lambda \in L^\infty_{t++} \) (\( X \) is conditionally positive homogeneous), then \( R_t \) is conditionally positive homogeneous.

Proof. Let \( R_t : \mathcal{S}_T^d \to \mathcal{P}(M_t; M^+_t) \) be a set-valued conditional risk measure, let \( X \) be as above and let \( R_t(X) := R_t(X(X)) \) for every portfolio vector \( X \in \mathcal{L}^p \).

1. Let \( R_t \) be convex and \( X \) be closed-convex. Let \( X, Y \in \mathcal{L}^p \) and \( \lambda \in [0,1] \).

\[
R_t(\lambda X + (1 - \lambda) Y) = R_t(\lambda X(\lambda X + (1 - \lambda) Y)) \supseteq\ R_t(\lambda X(\lambda X) \oplus (1 - \lambda) X(Y))
\]

\[
\supseteq \lambda R_t(X(X)) + (1 - \lambda) R_t(X(Y)) = \lambda R_t(X) + (1 - \lambda) R_t(Y).
\]

2. Let \( R_t \) be conditionally convex and \( X \) be conditionally closed-convex. Then the proof is analogous to the convex case above.

3. Let \( R_t \) and \( X \) be positive homogeneous. Let \( X \in \mathcal{L}^p \) and \( \lambda > 0 \).

\[
R_t(\lambda X) = R_t(X(\lambda X)) = R_t(\lambda X(X)) = \lambda R_t(X(X)) = \lambda R_t(X).
\]

4. Let \( R_t \) and \( X \) be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.
Example 6.2.11. (Example 6.2.8 continued) Let \( X(X) := X + K \) for every \( X \in \mathcal{L}^p \) for some random set \( K \). If \( K \) is (almost surely) convex and closed then \( X \) is \((\mathcal{F}_-)\)conditionally closed-convex (and thus closed-convex as well). If \( K \) is (almost surely) a cone then \( X \) is \((\mathcal{F}_-)\)conditionally positive homogeneous (and thus positive homogeneous as well).

In light of theorem 6.2.7, equation (6.2.3) and corollary 6.2.10 for set-valued portfolios of the form \( X(X) := X + K \) for all \( X \in \mathcal{L}^p \) and some random closed convex cone \( K \), one obtains the following. The dual representation of a constructive risk measure \( R_0 \) with coherent components \( \rho^1, \ldots, \rho^d \) given in equation (12) in [21] coincides with a special case of the dual representation of a \( K_T \)-compatible risk measure \( R_0 \) given in Theorem 4.2 in [50], by choosing \( A = \times_{i=1}^d A_i \) (\( A_i \) being the acceptance set of \( \rho_i \)), \( M = R^d \), \( K_I = R^d_+ \) and \( K_T = -K \):

\[
R_0(X) = R_0(X + K) = \bigcap_{w \in \mathbb{R}^d_+(\{0\}, Q, \mathcal{Q}, w \cdot \mathcal{Q} \in (-K)^+)} \{ u \in \mathbb{R}^d : w^T E^Q[X] \leq u^T \},
\]

where \( \mathcal{Q} = \times_{i=1}^d \mathcal{Q}_i \) and \( \mathcal{Q}_i \) denotes the set of probability measures in the dual representation of \( \rho_i \). This also follows from corollary 2.2.9, where the set of dual variables is

\[
\mathcal{W}_0^{\text{max}} = \{(Q, w) \in \mathcal{W}_{0, K_T} : w_i^T(Q, w) \in A_i^+ \} = \{(Q, w) \in \mathcal{W}_0 : Q \in \mathcal{Q} \},
\]

with

\[
\mathcal{W}_{0, K_T} := \{(Q, w) \in \mathcal{M} \times \mathbb{R}^d_+ \setminus \{0\} : w_0^T(Q, w) \in \mathcal{L}^q(K_T^+) \}
\]

due to \( K_T \)-compatibility of \( R_0 \).

Additional to dual representations for constructive risk measure, theorem 6.2.7 allows to deduce dual representations of a larger class of conditional risk measure.
for set-valued portfolios (definition 6.2.3) by using equation (6.2.2) and the duality results for set-valued risk measures of the set-optimization approach.

### 6.3 Family of scalar risk measures

Consider $L^p$, $p \in [1, +\infty]$ with the norm topology for $p \in [1, +\infty)$ and the weak* topology for $p = +\infty$. Recall that for this chapter a set $A_t \subseteq L^p$ is a **conditional acceptance set** at time $t$ if it satisfies $M_t \cap A_t \neq \emptyset$, $M \cap (\mathbb{R}^d \setminus A_t[\omega]) \neq \emptyset$ for almost every $\omega \in \Omega$, and $A_t + L^p_+ \subseteq A_t$.

We define a family of scalar conditional risk measures $\rho_w^t$ with parameter $w \in M_t^+ \setminus M_t^\perp$ via their primal representation. The scalar risk measures map into the random variables with values in the extended real line, that is, into the space $\bar{L}_t^0 := L_0^0(\mathbb{R} \cup \{\pm \infty\})$.

**Definition 6.3.1.** A function $\rho_w^t : L^p \rightarrow \bar{L}_t^0$ satisfying

$$
\rho_w^t(X) = \text{ess inf} \{ w^T u : u \in M_t, X + u \in A_t \}
$$

(6.3.1)

for a parameter $w \in M_t^+ \setminus M_t^\perp$ and a conditional acceptance set $A_t$ is called a multiple asset conditional risk measure at time $t$.

Clearly, the scalar risk measures defined above are scalarizations of a set-valued risk measure from the set-optimization approach (see chapter 2) defined by primal representation $R_t(X) := \{ u \in M_t : X + u \in A_t \}$, where the scalarizations are taken with respect to vectors $w \in M_t^+ \setminus M_t^\perp$, that is

$$
\rho_w^t(X) = \text{ess inf}_{u \in R_t(X)} w^T u = \text{ess inf} \{ w^T u : u \in M_t, X + u \in A_t \}.
$$

(6.3.2)

Note, that when $R_t$ is $K$-compatible (that is $A_t = A_t + L^p_t(K)$) for some $\mathcal{F}_t$-measurable random cone $K \subseteq M$, then $\rho_w^t(X)[\omega] = -\infty$ on $w(\omega) \not\in K[\omega]^+$ for any
X ∈ ℒ^p. Thus, one can restrict oneself in this case to parameters w in the basis of ℒ^q_t(K^+) \ M^\perp_t.

We will give some examples from the literature of scalar risk measures of form (6.3.1).

**Example 6.3.2.** In [33, 7, 44, 63, 77] risk measures of form (6.3.1) have been studied in the static case.

In a frictionless market let the time t prices be given by the (random) vector S_t. In this case the solvency cones (see [59, 78, 60]) (K_t[ω]) are given by

\[ K_t[ω] = \{ x \in \mathbb{R}^d : S_t(ω)^T x ≥ 0 \} \]

where the normal vector \( S_t(ω) \) is the unique vector in the basis of \( K_t[ω]^+ \). Let \( A_t = A_t + ℒ^p_t(K_t \cap \tilde{M}) + ℒ^p_t(K_T) \) (where \( \tilde{M} \) is \( \mathcal{F}_t \)-measurable random set equal to \( M \) almost surely),

\[
\hat{ρ}_t^{S_t}(X) = \text{ess inf} \left\{ S_t^T u : u ∈ M_t, X + u ∈ A_t \right\} = \hat{ρ}_t^{S_t}(S_T^TX)
\]

for any \( X ∈ ℒ^p \) (since \( ℒ^q_t((K_t \cap \tilde{M})^+) := ℒ^q_t(K_t^+ + M^\perp_t) \)). It can be seen that \( \hat{ρ}_t^{S_t}(Z) = \text{ess inf} \left\{ S_t^T u : u ∈ M_t, Z + S_t^T u ∈ \tilde{A}_t \right\} \) with \( \tilde{A}_t = \{ S_T^TX : X ∈ A_t \} \) is the dynamic version of the risk measures with multiple eligible assets defined in [33, 7, 44, 63, 77] (and with single eligible assets (which is not necessarily the original numéraire) defined in [32, 31]). \( \tilde{A}_t \) satisfies definition 2.1 of [33] for an acceptance set.

**Example 6.3.3.** [19] discusses scalar static risk measure of multivariate claims, when only a single eligible asset is considered, that is

\[
ρ(X) = \inf \{ u ∈ \mathbb{R} : X + u e_1 ∈ A \}
\]

for \( X ∈ ℒ^\infty \), where \( A ⊆ ℒ^\infty \) is an acceptance set. We can see that this has the form

\[
ρ(X) = \inf \{ e_1^T u : u ∈ \mathbb{R} × \{0\}^{d-1}, X + u ∈ A \},
\]

i.e. the scalarization of a set-valued risk measure with \( M = \mathbb{R} × \{0\}^{d-1} \) and \( w = e_1 \).
Example 6.3.4. In [80] so called liquidity-adjusted risk measure \( \rho^V : \mathcal{L}^\infty \to \mathbb{R} \), which are scalar static risk measure of multivariate claims in markets with frictions, are studied, when only a single eligible asset is considered. The primal representation

\[
\rho^V(X) = \inf \{ k \in \mathbb{R} : X + ke_1 \in A^V \}
\]

for \( A^V := \{ X \in \mathcal{L}^\infty : V(X) \in A \} \), where \( V \) is a real valued function providing the value of a portfolio \( X \) under liquidity and portfolio constraints and \( A \subseteq L^\infty \) is the acceptance set of a scalar convex risk measure in the sense of [41]. Clearly, \( \rho^V(X) \) is of form (6.3.1).

Example 6.3.5. In [12, 18, 71, 57, 74] (and many other papers) the scalar superhedging price in a market with two assets and transaction costs has been studied. The \( d \) asset case is treated in [75, 67]. Let \( (K_t)_{t \in \mathbb{T}} \) be the sequence of solvency cones modeling the market with proportional transaction costs.

The \( d \) dimensional version of the dual representation of the scalar superhedging price given in Jouini, Kallal [57] reads as follows. Let \( X \in \mathcal{L}^p \) be a payoff in physical units. Under the robust no arbitrage condition (see section 5.1), the scalar superhedging price \( \pi^a_i(X) \) in units of asset \( i \in \{1, \ldots, d\} \) at time \( t = 0 \) is given by

\[
\pi^a_i(X) = \sup_{(S_t, Q) \in Q^i} \mathbb{E}^Q[S_T^TX], \tag{6.3.3}
\]

where \( Q^i \) is the set of all processes \( (S_t)_{t \in \mathbb{T}} \) and their equivalent martingale measures \( Q \) with \( \frac{dQ}{dP} \in L^1(\mathcal{F}_T) \), \( S_t^i \equiv 1 \), \( \mathbb{E} \left[ \frac{dQ}{dP} \right] \mathcal{F}_t \) \( S_t \in \mathcal{L}^a_i(K_t^+) \) for all \( t \). Theorem 7.1 in [67] shows that (6.3.3) can be obtained by scalarizing the coherent set-valued risk measure with acceptance set \( A_0 = \sum_{s=0}^T \mathcal{L}^a_s(K_s) \) and single eligible asset (asset \( i \), which is also the numéraire asset, i.e. \( M = \{ u \in \mathbb{R}^d : u_j = 0 \ \forall j \neq i \} \) w.r.t. the unit vector \( w = e_i \in (K_0 \cap M)^+ \). Thus, \( \pi^a_i \) is a special case of (6.3.1).
Of course any standard scalar risk measure in a frictionless markets with single eligible asset as in [41, 5] is also special cases of (6.3.1), but in that case there is no advantage to explore the relationship with a set-valued risk measure via (6.3.2). In any other case, i.e. if one of the following is considered: multiple eligible assets, multivariate claims, transaction costs or other market frictions, it can be advantageous to explore (6.3.2) as the dual representation of the corresponding set-valued risk measure given in chapter 2 can lead to a dual representation of the scalarization as demonstrated in (6.3.3). Furthermore, even if one is interested in only one particular scalarization (as it is the case in all the examples above), the dual representation of the scalar risk measure might involve the whole family of scalarizations (as in example 6.3.5, where the constraints $S_t \in K_t^+$ a.s. for all $t$ enter the scalar problem in (6.3.3)). This is related to time consistency properties of the scalar risk measure and multi-portfolio time consistency of the corresponding set-valued risk measure (see definition 3.2.1). In this section we are only concerned with the connection between a family of scalar risk measures and a set-valued risk measure. Lemma 6.3.10 below gives very mild conditions under which a set-valued risk measures can be equivalently represented by a family of scalar risk measures. Results about dual representations and the study of time consistency properties of the family of scalar risk measures are left for further research.

The main motivation to study a family of scalar risk measures in this section is that it allows to generalize all of the examples given above in a unified way by allowing multiple eligible assets, multivariate claims and frictions in the form of transaction costs, as well as considering a dynamic setting. As example 6.3.5 suggests, viewing a scalar risk measure in a market with frictions as being a scalarization of a set-valued risk measure has the advantage of obtaining dual representations and conditions on time consistency by using the corresponding results of the set-valued risk measure.
A different approach concerning a family of scalar risk measures and multiple eligible assets in a frictionless market was taken in [56]. In that paper, given a set of eligible assets (with values $S^i_T$ for $i = 1, ..., n$), the risk of the portfolio $X$ is the set of values \( \{ \sum_{i=1}^n \rho^{S^i_T}_t (X_i) S^i_T : X = \sum_{i=1}^n X_i \} \) where $\rho^{S^i_T}_t$ is a risk measure in asset $i$ (with change of numéraire). However, we will not discuss this approach further since lemma 4.10 of that paper demonstrates that $\rho^{S^0_T}_t (X) \leq \rho^{S^0_T}_t (- \sum_{i=1}^n \rho^{S^i_T}_t (X_i) S^i_T)$ for any choice of numéraire 0 and any allocation of $X = \sum_{i=1}^n X_i$, i.e. the family of risks (as a portfolio) has risk bounded below by the risk of the initial portfolio no matter the numéraire chosen.

In the following proposition we show that the multiple asset conditional scalar risk measures satisfy monotonicity and a translative property. These properties are usually given as the definition of a risk measure in the literature given in the above examples. However, here we consider the primal representation 6.3.1 as the starting point.

**Proposition 6.3.6.** Let $\rho^w_t : L^p \to \bar{L}^0_t$ be a multiple asset conditional scalar risk measure at time $t$ for pricing vector $w \in M^+_t \setminus M^\perp_t$. Then $\rho^w_t$ satisfies the following conditions.

1. If $Y \succeq X$ for $X, Y \in L^p$, then $\rho^w_t(Y) \leq \rho^w_t(X)$.

2. $\rho^w_t(X + u) = \rho^w_t(X) - w^T u$ for all $X \in L^p$ and $u \in M_t$.

Further, if we consider the family of such risk measures over all pricing vectors $w \in M^+_t \setminus M^\perp_t$ then we have the following finiteness properties.

3. $\rho^w_t(0) < +\infty$ for every $w \in M^+_t \setminus M^\perp_t$.

4. $\rho^w_t(0) > -\infty$ for some $w \in M^+_t \setminus M^\perp_t$.

**Proof.** Let $\rho^w_t(X) := \essinf \left\{ w^T u : u \in M_t, X + u \in A_t \right\}$ for every $X \in L^p$, every $w \in M^+_t \setminus M^\perp_t$, and some conditional acceptance set $A_t$. 

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1. Let $X, Y \in \mathcal{L}^p$ such that $Y \succeq X$. Let $w \in \mathcal{M}_t^+ \setminus \mathcal{M}_t^\perp$.

$$\rho_t^w(Y) = \essinf \{ w^T u : u \in \mathcal{M}_t, Y + u \in \mathcal{A}_t \}$$
$$= \essinf \{ w^T u : u \in \mathcal{M}_t, X + (Y - X) + u \in \mathcal{A}_t \}$$
$$\leq \essinf \{ w^T u : u \in \mathcal{M}_t, X + u \in \mathcal{A}_t \} = \rho_t^w(X).$$

2. Let $X \in \mathcal{L}^p$ and $u \in \mathcal{M}_t$. Let $w \in \mathcal{M}_t^+ \setminus \mathcal{M}_t^\perp$.

$$\rho_t^w(X + u) = \essinf \{ w^T m : m \in \mathcal{M}_t : X + m + u \in \mathcal{A}_t \}$$
$$= \essinf \{ w^T (m - u) : m \in \mathcal{M}_t, X + m \in \mathcal{A}_t \} = \rho_t^w(X) - w^T u.$$

3. Fix some $\omega \in \Omega$. $\rho_t^w(0)[\omega] = +\infty$ for some $w \in \mathcal{M}_t^+ \setminus \mathcal{M}_t^\perp$ if and only if $\mathcal{A}_t[\omega] \cap M = \emptyset$, which by $\mathcal{A}_t \cap \mathcal{M}_t \neq \emptyset$ is false.

4. Fix some $\omega \in \Omega$. $\rho_t^w(0)[\omega] = -\infty$ for every $w \in \mathcal{M}_t^+ \setminus \mathcal{M}_t^\perp$ if and only if $(\mathbb{R}^d \setminus \mathcal{A}_t[\omega]) \cap M = \emptyset$, which by definition is false.

6.3.1 Set-optimization approach versus family of scalar risk measures

In the static setting, the relation between set-valued risk measures and multiple asset scalar risk measures has been studied in [48, 50, 33].

**Theorem 6.3.7.** Let $R_t : \mathcal{L}^p \to \mathcal{P}(\mathcal{M}_t; \mathcal{M}_t^+)$ be a conditional risk measure at time $t$ (see definition 2.1.2), then $\rho_t^w : \mathcal{L}^p \to \bar{\mathcal{L}}_t^0$, defined by

$$\rho_t^w(X) := \essinf_{u \in R_t(X)} w^T u$$
for any \( X \in \mathcal{L}^p \), is a family of multiple asset scalar risk measures indexed by \( w \in M_{t,+}^+ \setminus M_t^\perp \) at time \( t \) (see definition 6.3.1).

Let \( \{ \rho^w_t : \mathcal{L}^p \to \bar{L}_0^t : w \in M_{t,+}^+ \setminus M_t^\perp \} \) be a family of multiple asset scalar risk measures at time \( t \) indexed by \( w \in M_{t,+}^+ \setminus M_t^\perp \) (see definition 6.3.1), then \( R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+}) \), defined by

\[
R_t(X) := \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ u \in M_t : \rho^w_t(X) \leq w^T u \ \mathbb{P} \text{-a.s.} \}
\]

for any \( X \in \mathcal{L}^p \), is a conditional risk measure at time \( t \) (see definition 2.1.2).

**Proof.**

1. This follows from definition 6.3.1 and (6.3.2).

2. We will show that \( R_t(X) := \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ u \in M_t : \rho^w_t(X) \leq w^T u \ \mathbb{P} \text{-a.s.} \} \) is a conditional risk measure. We use the properties of \( \rho^w_t \) given in proposition 6.3.6.

   (a) \( \mathcal{L}^p_t \)-monotonicity: let \( X, Y \in \mathcal{L}^p \) such that \( Y \succeq X \), then \( \rho^w_t(Y) \leq \rho^w_t(X) \) almost surely for every \( w \in M_{t,+}^+ \setminus M_t^\perp \). Therefore \( R_t(Y) \supseteq R_t(X) \).

   (b) \( M_t \)-translativity: let \( X \in \mathcal{L}^p \) and \( u \in M_t \), then

\[
R_t(X + u) = \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ m \in M_t : \rho^w_t(X + u) \leq w^T m \ \mathbb{P} \text{-a.s.} \}
\]

\[
= \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ m \in M_t : \rho^w_t(X) - w^T u \leq w^T m \ \mathbb{P} \text{-a.s.} \}
\]

\[
= \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ m \in M_t : \rho^w_t(X) \leq w^T (u + m) \ \mathbb{P} \text{-a.s.} \}
\]

\[
= \bigcap_{w \in M_{t,+}^+ \setminus M_t^\perp} \{ m \in M_t : \rho^w_t(X) \leq w^T m \ \mathbb{P} \text{-a.s.} \} - u
\]

\[
= R_t(X) - u.
\]

(c) Finiteness at zero: \( R_t(0) \neq \emptyset \) since \( \rho^w_t(0) < +\infty \) for every \( w \in M_{t,+}^+ \setminus M_t^\perp \), and \( R_t(0)[\omega] \neq M \) since there exists a \( v \in M_{t,+}^+ \setminus M_t^\perp \) with \( \rho^v_t(0) > -\infty \).
Remark 6.3.8. If $R_t$ is normalized, with closed and conditionally convex images, and $w \in R_t(0)^+ \backslash M_t^\perp$ then $\rho_t^w(0) = 0$, i.e. $\rho_t^w$ normalized in the scalar framework.

Apart from closedness, many properties are one-to-one for conditional risk measures $R_t$ and the corresponding family of scalarizations. The corresponding results for the static case can be found in lemma 5.1 and lemma 6.1 of [48]. An example showing that closedness of $R_t$ does not necessarily imply closedness of all scalarizations can be found in the beginning of section 5 in [48] for the case $t = 0$.

Corollary 6.3.9. Let $R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+})$ be a convex (conditionally convex, positive homogeneous, conditionally positive homogeneous) conditional risk measure at time $t$ with closed and $\mathcal{F}_t$-decomposable images, then the associated family of multiple asset scalar risk measures is convex (conditionally convex, positive homogeneous, conditionally positive homogeneous).

Let $\{\rho_t^w : \mathcal{L}^p \to \bar{\mathcal{L}}_t^0 : w \in M_{t,+}^+ \backslash M_t^\perp\}$ be a family of convex (positive homogeneous, conditionally positive homogeneous, lower semicontinuous) multiple asset scalar risk measures at time $t$ indexed by $w \in M_{t,+}^+ \backslash M_t^\perp$ then the associated conditional risk measure is convex (positive homogeneous, conditionally positive homogeneous, closed).

Additionally, if $\{\rho_t^w : \mathcal{L}^p \to \bar{\mathcal{L}}_t^0 : w \in M_{t,+}^+ \backslash M_t^\perp\}$ is a family of lower semicontinuous conditionally convex risk measures then the associated conditional risk measure is conditionally convex.

Proof. Let $R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+})$ be a conditional risk measure at time $t$. Let $\rho_t^w : \mathcal{L}^p \to \bar{\mathcal{L}}_t^0$ be defined by $\rho_t^w(X) := \text{ess inf}_{u \in R_t(X)} w^T u$ for every $X \in \mathcal{L}^p$. 

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(a) Let $R_t$ be convex. Let $X, Y \in \mathcal{L}^p$, $\lambda \in [0, 1]$, and $w \in M_{t,+}^+ \backslash M_t^+$. 

\[
\rho_t^w(\lambda X + (1 - \lambda)Y) = \text{ess inf}_{u \in R_t(\lambda X + (1 - \lambda)Y)} w^T u \\
\leq \text{ess inf}_{u \in \lambda R_t(X) + (1 - \lambda)R_t(Y)} w^T u \\
= \lambda \text{ ess inf}_{u \in \lambda R_t(X)} w^T u_x + (1 - \lambda) \text{ ess inf}_{u \in \lambda R_t(Y)} w^T u_y \\
= \lambda \rho_t^w(X) + (1 - \lambda) \rho_t^w(Y).
\]

(b) Let $R_t$ be conditionally convex. Then the proof is analogous to the convex case above.

(c) Let $R_t$ be positive homogeneous. Let $X \in \mathcal{L}^p$, $\lambda > 0$, and $w \in M_{t,+}^+ \backslash M_t^+$.

\[
\rho_t^w(\lambda X) = \text{ess inf}_{u \in R_t(\lambda X)} w^T u = \text{ess inf}_{u \in \lambda R_t(X)} w^T u = \lambda \text{ ess inf}_{u \in R_t(X)} w^T u = \lambda \rho_t^w(X).
\]

(d) Let $R_t$ be conditionally positive homogeneous. Then the proof is analogous to the positive homogeneous case above.

2. Let $\{\rho_t^w : \mathcal{L}^p \rightarrow \bar{\mathcal{L}}^0 : w \in M_{t,+}^+ \backslash M_t^+\}$ be a family of multiple asset scalar risk measures at time $t$ indexed by $w \in M_{t,+}^+ \backslash M_t^+$. Let $R_t : \mathcal{L}^p \rightarrow \mathcal{P}(M_t; M_{t,+})$ be defined by $R_t(X) := \bigcap_{w \in M_{t,+}^+ \backslash M_t^+} \{u \in M_t : \rho_t^w(X) \leq w^T u \ \mathbb{P} \text{-a.s.}\}$ for every $X \in \mathcal{L}^p$. 

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(a) Let $\rho^w_t$ be convex for every $w \in M^+_t \backslash M^+_t$. Let $X, Y \in \mathcal{L}^p$ and $\lambda \in (0,1)$.

$$R_t(\lambda X + (1 - \lambda)Y) = \bigcap_{w \in M^+_t \backslash M^+_t} \{ u \in M_t : \rho^w_t(\lambda X + (1 - \lambda)Y) \leq w^T u \mathbb{P} \text{ a.s.} \} \supseteq \bigcap_{w \in M^+_t \backslash M^+_t} \{ u \in M_t : \lambda \rho^w_t(X) + (1 - \lambda) \rho^w_t(Y) \leq w^T u \mathbb{P} \text{ a.s.} \} \supseteq \bigcap_{w \in M^+_t \backslash M^+_t} \{ \lambda u_X : u_X \in M_t, \rho^w_t(X) \leq w^T u_X \mathbb{P} \text{ a.s.} \} + \{(1 - \lambda)u_Y : u_Y \in M_t, \rho^w_t(Y) \leq w^T u_Y \mathbb{P} \text{ a.s.} \} \supseteq \lambda \bigcap_{w \in M^+_t \backslash M^+_t} \{ u_X \in M_t : \rho^w_t(X) \leq w^T u_X \mathbb{P} \text{ a.s.} \} + (1 - \lambda) \bigcap_{w \in M^+_t \backslash M^+_t} \{ u_Y \in M_t : \rho^w_t(Y) \leq w^T u_Y \mathbb{P} \text{ a.s.} \} = \lambda R_t(X) + (1 - \lambda)R_t(Y) .$$

Let $\lambda = 0$ (the case for $\lambda = 1$ is analogous), then $R_t(\lambda X + (1 - \lambda)Y) = \lambda R_t(X) + (1 - \lambda)R_t(Y)$ for any conditional risk measure and the result follows.

(b) Let $\rho^w_t$ be positive homogeneous for every $w \in M^+_t \backslash M^+_t$. Let $X \in \mathcal{L}^p$ and $\lambda > 0$.

$$R_t(\lambda X) = \bigcap_{w \in M^+_t \backslash M^+_t} \{ u \in M_t : \rho^w_t(\lambda X) \leq w^T u \mathbb{P} \text{ a.s.} \} = \bigcap_{w \in M^+_t \backslash M^+_t} \{ u \in M_t : \lambda \rho^w_t(X) \leq w^T u \mathbb{P} \text{ a.s.} \} = \bigcap_{w \in M^+_t \backslash M^+_t} \{ \lambda u : u \in M_t, \rho^w_t(X) \leq w^T u \mathbb{P} \text{ a.s.} \} = \lambda R_t(X).$$

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(c) Let $\rho_t^w$ be conditionally positive homogeneous for every $w \in M_{t,+}^+ \setminus M_t^\perp$.

Then the proof is analogous to the positive homogeneous case above.

(d) Let $\rho_t^w$ be lower semicontinuous for every $w \in M_{t,+}^+ \setminus M_t^\perp$. Consider a sequence $(X_n, u_n)_{n \in \mathbb{N}} \subseteq \text{graph } R_t$ (respectively a net if $p = +\infty$) with $\lim_{n \to +\infty} (X_n, u_n) = (X, u)$. Note that $(X_n, u_n) \in \text{graph } R_t$ if and only if $\rho_t^w(X_n) \leq v^Tu_n$ for every $v \in M_{t,+}^+ \setminus M_t^\perp$.

$$\rho_t^w(X) \leq \liminf_{n \to +\infty} \rho_t^w(X_n) \leq \liminf_{n \to +\infty} w^Tu_n = w^Tu.$$ 

The last equality above follows from $u_n \to u$ in $L_t^p$ implies $w^Tu_n \to w^Tu$ in $L_t^1$ (by Hölder’s inequality). Thus, $(X, u) \in \text{graph } R_t$.

(e) Let $\rho_t^w$ be lower semicontinuous and conditionally convex for every $w \in M_{t,+}^+ \setminus M_t^\perp$. Let $X, Y \in L^p$ and $\lambda \in L_t^\infty([0,1])$ then the proof is analogous to the convex case above.

We now extend conditional convexity to the case for $\lambda \in L_t^\infty([0,1])$ in the same way as was accomplished in the proof of corollary 2.2.10, noting that $R_t$ is closed by $\rho_t^w$ lower semicontinuous. Take a sequence $(\lambda_n)_{n=0}^\infty \subseteq L_t^\infty$ such that $0 < \lambda_n < 1$ for every $n \in \mathbb{N}$ which converges almost surely to $\lambda$. Then by dominated convergence $\lambda_n X$ converges to $\lambda X$ in the norm topology (weak* topology if $p = +\infty$) for any $X \in L^p$. Therefore, for any $X, Y \in L^p$

$$R_t(\lambda X + (1 - \lambda)Y) = R_t(\lim_{n \to +\infty} (\lambda_n X + (1 - \lambda_n)Y))$$

$$\sup \liminf_{n \to +\infty} R_t(\lambda_n X + (1 - \lambda_n)Y)$$

$$\sup \liminf_{n \to +\infty} [\lambda_n R_t(X) + (1 - \lambda_n)R_t(Y)]$$

$$\geq \lambda R_t(X) + (1 - \lambda)R_t(Y)$$
by $R_t$ closed (see proposition 2.34 in [66]) and conditionally convex on the
interval $0 < \lambda_n < 1$. Note that we use the convention from [66] that the
limit inferior of a sequence of sets $(B_i)_{i \in \mathbb{N}}$ is given by
$\lim \inf_{i \to +\infty} B_i = \bigcap_{i \in \mathbb{N}} \text{cl} \bigcup_{j \geq i} B_j$.

In the following lemma we show that when the conditional risk measure has closed
and conditionally convex images, the family of scalarizations can be used to recover
the conditional risk measure.

**Lemma 6.3.10.** Let $R_t : \mathcal{L}^p \to \mathcal{P}(M_t; M_{t,+})$ be a dynamic risk measure with closed
and conditionally convex images. Then, for any $X \in \mathcal{L}^p$

$$R_t(X) = \bigcap_{w \in M_{t,+} \setminus M_t^\bot} \{ u \in M_t : \rho^w_t(X) \leq w^T u \ \mathbb{P} \text{-a.s.} \} \quad (6.3.4)$$

where $\rho^w_t(X) := \text{ess inf}_{u \in R_t(X)} w^T u$ is the multiple asset scalar risk measure associated
with $R_t$.

**Proof.** $\subseteq$: By definition it is easy to see that $u \in R_t(X)$ implies that $w^T u \geq \rho^w_t(X)$
for any $w \in M_{t,+} \setminus M_t^\bot$.

$\supseteq$: Let $u \in \bigcap_{w \in M_{t,+} \setminus M_t^\bot} \{ u \in M_t : \rho^w_t(X) \leq w^T u \ \mathbb{P} \text{-a.s.} \}$. Assume $u \notin R_t(X)$. Then since $R_t(X)$ is closed and convex we can apply the separating hyperplane
theorem. In particular, there exists some $v \in M_{t,+} \setminus M_t^\bot$ such that

$$\mathbb{E} \left[ v^T u \right] < \inf_{\hat{u} \in R_t(X)} \mathbb{E} \left[ v^T \hat{u} \right]$$
(if $v \not\in M_{t,+}^+ \setminus M_t^+$ then $\inf_{\hat{u} \in R_t(X)} \mathbb{E}[v^T \hat{u}] = -\infty$ by $R_t(X) = R_t(X) + M_{t,+}$). This implies that

$$
\mathbb{E}[\rho_t^v(X)] = \mathbb{E}\left[\operatorname{ess inf}_{\hat{u} \in R_t(X)} v^T \hat{u}\right] \leq \mathbb{E}[v^T u] < \inf_{\hat{u} \in R_t(X)} \mathbb{E}[v^T \hat{u}]
$$

By corollary 6.1.20, if $u_1, u_2 \in R_t(X)$ then $u_1 \wedge u_2 \in R_t(X)$ as well. Therefore by theorem 1 of [81] (and $\{v^T u : u \in R_t(X)\} \subseteq L_1^t$), it follows that

$$
\mathbb{E}\left[\operatorname{ess inf}_{\hat{u} \in R_t(X)} v^T \hat{u}\right] = \inf_{\hat{u} \in R_t(X)} \mathbb{E}[v^T \hat{u}].
$$

This is a contradiction and thus $u \in R_t(X)$.

$\square$
Appendix A

Appendix

A.1 Set-valued duality

In this section we will give a quick introduction and review to set-valued duality results developed by [47]. For the duration of this section we consider the topological dual pairs \((X, X^*)\) and \((Z, Z^*)\) with the (partial) ordering on \(Z\) defined by the cone \(C \subseteq Z\). Let us consider the set-valued function \(f : X \rightarrow G(Z; C)\).

Set-valued duality uses two dual variables, which we denote \(x^* \in X^*\) and \(z^* \in C^+ = \{z^* \in Z^* : \forall z \in C : \langle z^*, z \rangle \geq 0\}\). The first dual variable, \(x^*\), behaves the same as the dual element in the scalar case. The second dual variable, \(z^*\), reflects the order relation in the image space as it is an element in the positive dual cone of the ordering cone.

Using these dual variables one can define a class of set-valued functions that serve as a (set-valued) replacement for continuous linear functions used in the scalar duality theory, and continuous linear operators used in vector-valued theories, as follows. Let \(2^Z\) denote the power set of \(Z\), including the empty set.
Definition A.1.1 ([47] example 2 and proposition 6). Given \( x^* \in \mathcal{X}^* \) and \( z^* \in \mathcal{Z}^* \), the function \( F_{(x^*,z^*)}^Z: \mathcal{X} \to 2^\mathcal{Z} \) is defined for any \( x \in \mathcal{X} \) by

\[
F_{(x^*,z^*)}^Z(x) := \{ z \in \mathcal{Z} : \langle x^*,x \rangle \leq \langle z^*,z \rangle \} .
\]

Before we can define the convex conjugate, we must define the set-valued infimum and supremum. As in [47, 66], the set-valued infimum and supremum on \( \mathcal{G}(\mathcal{Z};\mathcal{C}) \) are given by

\[
\inf_{x \in A} f(x) := \text{cl co} \bigcup_{x \in A} f(x); \quad \sup_{x \in A} f(x) := \bigcap_{x \in A} f(x)
\]

for any \( A \subseteq \mathcal{X} \) and \( f: \mathcal{X} \to \mathcal{G}(\mathcal{Z};\mathcal{C}) \). In this way we can then define the (negative) set-valued convex conjugate as

\[
-f^*(x^*,z^*) = \inf_{x \in \mathcal{X}} \left[ F_{(x^*,z^*)}^Z(-x) + f(x) \right] = \text{cl co} \bigcup_{x \in \mathcal{X}} \left[ F_{(x^*,z^*)}^Z(-x) + f(x) \right] \quad (A.1.1)
\]

and the set-valued biconjugate as

\[
f^{**}(x) = \sup_{(x^*,z^*) \in \mathcal{X}^* \times C^+ \setminus \{0\}} \left[ -f^*(x^*,z^*) + F_{(x^*,z^*)}^Z(x) \right]
\]

\[
= \bigcap_{(x^*,z^*) \in \mathcal{X}^* \times C^+ \setminus \{0\}} \left[ -f^*(x^*,z^*) + F_{(x^*,z^*)}^Z(x) \right]. \quad (A.1.2)
\]

From this form it is clear to see how the set-valued functions \( F_{(x^*,z^*)}^Z \) replace the linear functionals in convex analysis in the scalar framework.

Remark A.1.2. If \( f: \mathcal{X} \to \mathcal{G}(\mathcal{Z};\mathcal{C}) \) is convex then the (negative) conjugate is equivalent to

\[
-f^*(x^*,z^*) = \text{cl} \bigcup_{x \in \mathcal{X}} \left[ F_{(x^*,z^*)}^Z(-x) + f(x) \right] .
\]

Finally, the Fenchel-Moreau theorem for set-valued functions reads as follows.
Theorem A.1.3 (Theorem 2 in [47]). A proper function \( f : \mathcal{X} \to \mathcal{G}(\mathcal{Z};C) \) (i.e. \( f(x) \neq \mathcal{Z} \) for every \( x \in \mathcal{X} \) and \( f(x) \neq \emptyset \) for some \( x \in \mathcal{X} \)) is closed and convex if and only if \( f(x) = f^{**}(x) \) for every \( x \in \mathcal{X} \).

A.2 On the relationship of dual variables at different times

In considering how closed convex (and coherent) risk measures relate through time we must consider how the sets of dual variables relate. This is especially important for the results of sections 3.3 and 3.4, and the results herein were first shown in [36]. In the following lemma we provide such a relationship between elements of \( \mathcal{W}_t \) and elements of \( \mathcal{W}_s \) for any times \( t, s \) with \( t \leq s \). In fact we define a mapping on \( \mathcal{W}_t \) which is equivalent (in the set-valued replacement for continuous linear functionals) in \( \mathcal{W}_s \). In the scalar framework this type of property is not needed since the set \( \mathcal{W}_t \) can be simplified to any \( Q \ll P \) for any time \( t \) (where \( Q = P|_{\mathcal{F}_t} \)).

Lemma A.2.1. For any choice of times \( t \) and \( s > t \) it follows that:

1. \( \{(Q^s, w^s(Q, w)) : (Q, w) \in \mathcal{W}_t \} \subseteq \mathcal{W}_s \),

2. for every \( (R, v) \in \mathcal{W}_s \) there exists \( (Q, w) \in \mathcal{W}_t \) such that \( F^s_{(R,v)}(Q,w) = F^s_{(Q^s, w^s(Q,w))} \).

Proof. 1. \( \{(Q^s, w^s(Q, w)) : (Q, w) \in \mathcal{W}_t \} \subseteq \mathcal{W}_s \) if and only if for every \( (Q, w) \in \mathcal{W}_t \) it follows that \( w^s(Q, w) \in M^+_s \setminus M^+_s \) and \( w^T(Q^s, w^s(Q, w)) \in \mathcal{L}_+^q \).

(a) Let \( (Q, w) \in \mathcal{W}_t \). Show \( w^s(Q, w) \in M^+_s \setminus M^+_s \):

i. Let \( m_s \in M^+_{s,+} \), then

\[
\mathbb{E} \left[ w^s(Q, w)^T m_s \right] = \mathbb{E} \left[ w^T \mathbb{E}^Q \left[ m_s | \mathcal{F}_t \right] \right] \geq 0
\]

since \( \mathbb{E}^Q[m_s | \mathcal{F}_t] \in M_{t,+} \) by \( M_t \supseteq M_s \cap \mathcal{L}_t^p \) and \( M_s = \mathcal{L}_s^p(M) \).
ii. Since $(Q, w) \in W_t$, in particular since $w \not\in M_t^\perp$ there exists $m_t \in M_t \subseteq M_s$ such that $\mathbb{E}[w^T m_t] \neq 0$. Then,

$$\mathbb{E} \left[ w_t^s(Q, w)^T m_t \right] = \mathbb{E} \left[ w^T \mathbb{E}^Q [m_t | F_i] \right] = \mathbb{E} \left[ w^T m_t \right] \neq 0.$$

(b) $w_t^T(Q^s, w_t^s(Q, w)) = w_t^T(Q, w) \in \mathcal{L}_+^q$ by $(Q, w) \in W_t$.

2. By lemma 2.2.6, for every $(\mathbb{R}, v) \in W_s$ there exists a $(Y, \tilde{v})$ with $Y \in \mathcal{L}_+^q$, $\tilde{v} \in (\mathbb{E}[Y | F_s] + M_s^\perp) \setminus M_s^\perp$ such that $F_{(\mathbb{R}, v)}^s = \tilde{F}_{(Y, \tilde{v})}^s$. And for every $(Y, \tilde{v})$ with $Y \in \mathcal{L}_+^q$ and $\tilde{v} \in (\mathbb{E}[Y | F_s] + M_s^\perp) \setminus M_s^\perp$ there exists $(\hat{Q}, w_s) \in W_s$ such that $\tilde{F}_{(Y, \tilde{v})}^s = F_{(\hat{Q}, w_s)}^s$ by setting $w_s = \mathbb{E}[Y | F_s]$ and

$$\bar{\xi}_{r_1, r_2}^i[\omega] = \begin{cases} \frac{\mathbb{E}[Y_i|F_{r_2}](\omega)}{\mathbb{E}[Y_i|F_{r_1}](\omega)} & \text{if } \mathbb{E}[Y_i|F_i](\omega) > 0 \\ 1 & \text{else} \end{cases}$$

for every $\omega \in \Omega$, and $\frac{dQ}{d\bar{F}} = \bar{\xi}_{s,T}^i$. Define $Q \in \mathcal{M}$ by $\frac{dQ}{d\bar{F}} = \bar{\xi}_{s,T}^i$, thus $Q^s = \hat{Q}$. Therefore it remains to show that there exists a $w_t \in \mathcal{L}_+^q$ such that $w_s = w_t^s(Q, w_t)$ and $(Q, w_t) \in W_t$. Let $w_t := \mathbb{E}[w_s | F_i] = \mathbb{E}[Y | F_i]$.

(a) Show $w_s = w_t^s(Q, w_t)$, i.e. show $(w_s)_i(\omega) = (w_t)_i(\omega)\bar{\xi}_{i,s}^i(\omega)$ for every $i = 1, \ldots, d$ and almost every $\omega \in \Omega$. We know if $\mathbb{E}[Y_i | F_i](\omega) = 0$ then $(w_t)_i(\omega) = 0$ and $(w_s)_i(\omega) = 0$ and thus $(w_s)_i(\omega) = (w_t^s(Q, w_t))_i[\omega]$. If $\mathbb{E}[Y_i | F_i](\omega) > 0$ then

$$w_t^s(Q, w_t)_i[\omega] = \mathbb{E}[Y_i | F_i](\omega) \frac{\mathbb{E}[Y_i | F_s](\omega)}{\mathbb{E}[Y_i | F_i](\omega)} = \mathbb{E}[Y_i | F_s](\omega) = (w_s)_i(\omega).$$

(b) Show $(Q, w_t) \in W_t$

i. Show $w_t \in M_{t,x}^+ \setminus M_t^\perp$.  

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A. Let \( m_t \in M_{t,+} \), then \( \mathbb{E}[w_t^T m_t] = \mathbb{E}[\mathbb{E}[w_s | \mathcal{F}_t]^T m_t] = \mathbb{E}[w_s^T m_t] \geq 0 \) by the tower property, \( M_{t,+} \subseteq M_{s,+} \) and \( w_s \in M_{s,+}^+ \).

B. Since \((Q_s, w_s) \in \mathcal{W}_s\), in particular since \( w_s / \in M_s^\perp \) there exists \( m_s \in M_s \) such that \( \mathbb{E}[w_s^T m_s] \neq 0 \). Then \( \mathbb{E}[m_s | \mathcal{F}_t] \in M_t \) by \( M_t \supseteq M_s \cap \mathcal{L}_s^p \) and \( M_s = \mathcal{L}_s^p(M) \). Therefore, \( \mathbb{E}[w_t^T \mathbb{E}[m_s | \mathcal{F}_t]] = \mathbb{E}[w_t^T m_s] = \mathbb{E}[w_s^T m_s] \neq 0 \).

ii. \( w_t^T (Q, w_t) = w_s^T (Q_s, w_s^*(Q, w_t)) = w_s^T (\hat{Q}, w_s) \in \mathcal{L}_s^q \).

\( \square \)

The following corollary of lemma A.2.1 uses the above result applied to penalty functions instead of the functionals \( F(\cdot, \cdot)[\cdot] \).

**Corollary A.2.2.** For any \((R, v) \in \mathcal{W}_s\) there exists \((Q, w) \in \mathcal{W}_t\) such that

\[-\beta_s(R, v) = -\beta_s(Q_s^*, w_s^*(Q, w))\]

for any times \( 0 \leq t < s \leq T \).

*Proof.*

\[-\beta_s(R, v) = \text{cl} \bigcup_{Z \in \mathcal{A}_s} F^s_{(R, v)}[Z] = \text{cl} \bigcup_{Z \in \mathcal{A}_s} F^s_{(Q_s^*, w_s^*(Q, w))}[Z] \]

(A.2.1)

\[-\beta_s(Q_s^*, w_s^*(Q, w)) = -\beta_s(Q_s^*, w_s^*(Q, w)),\]

where equation (A.2.1) is a result of lemma A.2.1.

\( \square \)

Lemma A.2.1 and corollary A.2.2 show that for any times \( t \leq s \) and for a given penalty function \( -\beta_s \) the set of dual variables \( \{(Q_s^*, w_s^*(Q, w)) : (Q, w) \in \mathcal{W}_t\} \) defines the same closed and convex risk measure at time \( s \) as the set of dual variables \( \mathcal{W}_s \),
that is

\[
R_s(X) = \bigcap_{(Q,w) \in \mathcal{W}_t} \left[ -\beta_s(Q^s, w^s_t(Q, w)) + (\mathbb{E}^Q [-X|\mathcal{F}_s] + G_s (w^s_t(Q, w))) \cap M_s \right].
\]

The following lemma, about the expectation of minimal penalty functions, is an extension of lemma 2.6 in [40]. As a set-valued operation, this theorem gives a set-valued version of when the conditional expectation of an infimum is equivalent to the infimum of the conditional expectation. The proof of the lemma is a simplified version of the proof of lemma 2.6 in [40] since the sets \( \{\mathbb{E}^Q[X|\mathcal{F}_t] + G_t(w)\} \) are shifted half spaces for any \( X \in A_t \) and a fixed \((Q, w) \in \mathcal{W}_t \) and thus are completely ordered, in contrast to the scalar case, where the points \( \mathbb{E}^Q[X|\mathcal{F}_t] \) under consideration are not completely ordered.

**Lemma A.2.3.** For any times \( 0 \leq t < s \leq T \), and if \( R_t \) is a closed convex risk measure, then for any \((Q, w) \in \mathcal{W}_t \), it follows that

\[
\mathbb{E}^Q [-\beta_s(Q^s, w^s_t(Q, w))|\mathcal{F}_t] = \text{cl} \bigcup_{X \in A_s} (\mathbb{E}^Q [X|\mathcal{F}_s] + G_t(w)) \cap M_s.
\]

**Proof.** Let \((Q, w) \in \mathcal{W}_t \). Then, by lemma A.2.1, \((Q^s, w^s_t(Q, w)) \in \mathcal{W}_s \). It holds

\[
-\beta_s(Q^s, w^s_t(Q, w)) = \text{cl} \bigcup_{X \in A_s} (\mathbb{E}^Q [X|\mathcal{F}_s] + G_s(w^s_t(Q, w))) \cap M_s
\]

\[
= \text{cl} \bigcup_{X \in A_s} \left\{ u \in M_s : \mathbb{E} [w^s_t(Q, w)^T \mathbb{E}^Q [X|\mathcal{F}_s]] \leq \mathbb{E} [w^s_t(Q, w)^T u] \right\}
\]

\[
= \text{cl} \bigcup_{X \in A_s} \left\{ u \in M_s : \mathbb{E} [w^T \mathbb{E}^Q [X|\mathcal{F}_t]] \leq \mathbb{E} [w^T \mathbb{E}^Q [u|\mathcal{F}_t]] \right\}
\]

\[
= \left\{ u \in M_s : \inf_{X \in A_s} \mathbb{E} [w^T \mathbb{E}^Q [X|\mathcal{F}_t]] \leq \mathbb{E} [w^T \mathbb{E}^Q [u|\mathcal{F}_t]] \right\}.
\]
Taking the conditional expectation on both sides yields

\[
\mathbb{E}_Q \left[ -\beta_s(Q^s, w^s_t(Q, w)) \mid \mathcal{F}_t \right] = \left\{ \mathbb{E}_Q \left[ u \mid \mathcal{F}_t \right] : u \in M_s, \right. \\
\left. \inf_{X \in A_s} \mathbb{E} [w^T \mathbb{E}_Q [X \mid \mathcal{F}_t]] \leq \mathbb{E} [w^T \mathbb{E}_Q [u \mid \mathcal{F}_t]] \right\} \\
= \left\{ u \in M_t : \inf_{X \in A_s} \mathbb{E} [w^T \mathbb{E}_Q [X \mid \mathcal{F}_t]] \leq \mathbb{E} [w^T u] \right\} \\
= \text{cl} \bigcup_{X \in A_s} (\mathbb{E}_Q [X \mid \mathcal{F}_t] + G_t(w)) \cap M_t.
\]

One can now show that the \(Q\)-conditional expectation (at time \(t\)) of the positive half-space defined by \(w^s_t(Q, w)\) is given by the positive half-space defined by \(w\).

**Corollary A.2.4.** Let \(0 \leq t < s \leq T\), \(Q \in \mathcal{M}\) where \(Q = \mathbb{P} \mid \mathcal{F}_t\) and \(w \in \mathcal{L}^Q_t\). Then,

\[
\mathbb{E}_Q \left[ G_s(w^s_t(Q, w)) \mid \mathcal{F}_t \right] = G_t(w).
\]

**Proof.** This is a special case of lemma A.2.3 obtained by setting \(M = \mathbb{R}^d\) and \(A_s = \mathcal{L}^p_t\).

We conclude our discussion on how dual variables across time are related by considering the conditional expectations of the \(\alpha_s\) and \(\Gamma_s\) functions used in the dual representation of conditionally convex risk measures (see corollary 2.2.12).

**Lemma A.2.5.** For any times \(0 \leq t < s \leq T\) and if \(R_t\) is a closed conditionally convex risk measure, then for any \((Q, w) \in \mathcal{W}_t\) with \(Q \in \mathcal{M}^c\), it follows that

\[
\text{cl} \mathbb{E}_Q \left[ -\alpha_s(Q^s, w^s_t(Q, w)) \mid \mathcal{F}_t \right] = \text{cl} \bigcup_{Z \in A_s} (\mathbb{E}_Q [Z \mid \mathcal{F}_t] + \Gamma_t(w)) \cap M_t.
\]
Proof. \( \subseteq \) For any \((Q, w) \in \mathcal{W}_t\) with \(Q \in \mathcal{M}^e\)

\[
\mathbb{E}^Q \left[ -\alpha_s(Q^*, w^*_t(Q, w)) \big| \mathcal{F}_t \right] \\
= \left\{ \mathbb{E}^Q [u_s] \big| \mathcal{F}_t : u_s \in M_s, w^*_t(Q, w)^T u_s \geq \text{ess inf}_{Z \in A_s} w^*_t(Q, w)^T E^Q [Z] \mathcal{F}_s \right\} \text{ P-a.s.} \\
\subseteq \left\{ \mathbb{E}^Q [u_s] \big| \mathcal{F}_t : u_s \in M_s, E \left[ w^*_t(Q, w)^T u_s \big| \mathcal{F}_t \right] \geq \mathbb{E} \left[ \text{ess inf}_{Z \in A_s} w^*_t(Q, w)^T E^Q [Z] \mathcal{F}_s \big| \mathcal{F}_t \right] \right\} \text{ P-a.s.} \\
= \left\{ u_t \in M_t : w^T u_t \geq \text{ess inf}_{Z \in A_s} w^T E^Q [Z] \mathcal{F}_t \right\} \text{ P-a.s.} \\
= \text{cl} \bigcup_{Z \in A_s} (\mathbb{E}^Q [Z] \mathcal{F}_t + \Gamma_t(w)) \cap M_t.
\]

And since \(\text{cl} \bigcup_{Z \in A_s} (\mathbb{E}^Q [Z] \mathcal{F}_t + \Gamma_t(w)) \cap M_t\) is closed, this direction is shown.

\( \supseteq \) Consider a point \(u \in \text{cl} \bigcup_{Z \in A_s} (\mathbb{E}^Q [Z] \mathcal{F}_t + \Gamma_t(w)) \cap M_t\) and assume \(u \notin \text{cl} \mathbb{E}^Q \left[ -\alpha_s(Q^*, w^*_t(Q, w)) \right| \mathcal{F}_t\). Since \(\text{cl} \mathbb{E}^Q \left[ -\alpha_s(Q^*, w^*_t(Q, w)) \right| \mathcal{F}_t\) is closed and convex, we can separate \(\{u\}\) and \(\text{cl} \mathbb{E}^Q \left[ -\alpha_s(Q^*, w^*_t(Q, w)) \right| \mathcal{F}_t\) by some \(v \in \mathcal{L}^d_t\), i.e. let \(v \in \mathcal{L}^d_t\) such that

\[
\mathbb{E} \left[ v^T u \right] = \inf_{z_t \in \text{cl} \mathbb{E}^Q \left[ -\alpha_s(Q^*, w^*_t(Q, w)) \right| \mathcal{F}_t} \mathbb{E} \left[ v^T z_t \right] = \inf_{z_s \in \alpha_s(Q^*, w^*_t(Q, w))} \mathbb{E} \left[ w^*_t(Q, v)^T z_s \right] = \mathbb{E} \left[ \text{ess inf}_{z_s \in \alpha_s(Q^*, w^*_t(Q, w))} w^*_t(Q, v)^T z_s \right].
\]

Note that in the last equality above we can interchange the expectation and infimum since \(-\alpha_s(Q^*, w^*_t(Q, w))\) is decomposable. By construction

\[
\text{ess inf}_{z_s \in \alpha_s(Q^*, w^*_t(Q, w))} w^*_t(Q, v)^T z_s = \begin{cases} \\
\text{ess inf}_{Z \in A_s} w^*_t(Q, v)^T E^Q [Z] \mathcal{F}_s \quad \text{on } D \\
-\infty \quad \text{on } D^c
\end{cases}
\]

where \(D = \{\omega \in \Omega : G_0(w^*_t(Q, v)[\omega]) = G_0(w^*_t(Q, w)[\omega])\}\). Since \(Q \in \mathcal{M}^e\), one has \(G_0(w^*_t(Q, v)[\omega]) = G_0(w^*_t(Q, w)[\omega])\) if and only if \(v(\omega) = \lambda(\omega)w(\omega)\) for some
\[ \lambda \in L_{t,++}^0 \text{ (such that } \lambda w \in \mathcal{L}_t^q) \text{. Therefore} \]

\[
\mathbb{E} \left[ \text{ess inf}_{z \in -\alpha_s(Q^s, w_t(Q,w))} w_t^s(Q,v)^T z_s \right] > -\infty
\]

if and only if

\[
\mathbb{E} \left[ \text{ess inf}_{z \in -\alpha_s(Q^s, w_t(Q,w))} w_t^s(Q,v)^T z_s \right] = \mathbb{E} \left[ \lambda \text{ess inf}_{z \in -\alpha_s(Q^s, w_t(Q,w))} w_t^s(Q,v)^T z_s \right] = \mathbb{E} \left[ \lambda \text{ess inf}_{Z \in A_s} w^T \mathbb{E}^Q[Z | \mathcal{F}_t] \right].
\]

But this implies

\[ \mathbb{E}[\lambda w^T u] < \mathbb{E}[\lambda \text{ess inf}_{Z \in A_s} w^T \mathbb{E}^Q[Z | \mathcal{F}_t]], \]

which is a contradiction to \( u \in \text{cl} \bigcup_{Z \in A_s} (\mathbb{E}^Q[Z | \mathcal{F}_t] + \Gamma_t(w)) \cap M_t. \)

\[ \square \]

**Corollary A.2.6.** Let \( 0 \leq t < s \leq T, (Q, w) \in \mathcal{W}_t \) with \( Q \in \mathcal{M}^e \). It follows that

\[ \text{cl} \mathbb{E}^Q[\Gamma_s(w_t^s(Q,w)) | \mathcal{F}_t] = \Gamma_t(w). \]

**Proof.** This is a special case of lemma A.2.5 obtained by setting \( M = \mathbb{R}^d \) and \( A_s = \mathcal{L}_t^q \).

\[ \square \]

### A.3 On the sum of closed acceptance sets and convex upper continuity

When considering multi-portfolio time consistency for closed risk measures we need to guarantee that the composed risk measures are closed, or else the recursive form would fail to hold. In particular, this would be true if the sum of acceptance sets are themselves closed. We will demonstrate the closedness of the sum of convex
acceptance sets when the associated dynamic risk measure is convex upper continuous, as was done in [36].

Recall that a function \( F : X \to \mathcal{P}(Y; C) \) is convex upper continuous (c.u.c.) if \( F^{-1}(D) := \{ x \in X : F(x) \cap D \neq \emptyset \} \) is closed for any closed set \( D \in \mathcal{G}(Y; -C) \).

**Proposition A.3.1.** Let \( F : X \to \mathcal{P}(Y; C_Y) \) and \( G : Y \to \mathcal{P}(Z; C_Z) \). If \( F,G \) are c.u.c. and \( G \) is convex and \(-C_Y\)-monotone, then \( H : X \to \mathcal{P}(Z; C_Z) \) defined by \( H(x) := \bigcup_{y \in F(x)} G(y) \) for any \( x \in X \) is c.u.c.

**Proof.** For any \( D \in 2^Z \), then

\[
H^{-1}(D) = \{ x \in X : H(x) \cap D \neq \emptyset \} = \left\{ x \in X : \bigcup_{y \in F(x)} G(y) \cap D \neq \emptyset \right\} = \{ x \in X : \exists y \in F(x) : G(y) \cap D \neq \emptyset \} = \{ x \in X : F(x) \cap G^{-1}(D) \neq \emptyset \} = F^{-1}(G^{-1}(D)).
\]

Additionally, if \( D \in \mathcal{G}(Z; -C_Z) \) then \( G^{-1}(D) \) is closed, if \( x, y \in G^{-1}(D) \) and \( \lambda \in [0, 1] \) then \( G(\lambda x + (1 - \lambda)y) \cap D \neq \emptyset \), and if \( x, y \in Y \) such that \( x - y \in C_Y \) with \( x \in G^{-1}(D) \) then \( y \in G^{-1}(D) \). This implies that \( G^{-1}(D) \in \mathcal{G}(Y, -C_Y) \), and thus \( F^{-1}(G^{-1}(D)) \) is closed for any \( D \in \mathcal{G}(Z; -C_Z) \).

**Lemma A.3.2.** Let \( M_t \) (\( M_s \)) be the set of eligible portfolios at time \( t \) (\( s \)) (a closed linear subspace of \( \mathcal{L}_{p_t}^p \) (\( \mathcal{L}_{p_s}^p \))). Let \( R_{t,s} \) be a c.u.c. convex stepped risk measure from \( t \) to \( s \) and \( R_s \) be a c.u.c. risk measure at time \( s \). Then, \( A_{t,s} + A_s \) is closed.

**Proof.** By lemma 3.2.6(1), \( A_{t,s} + A_s = \{ X \in \mathcal{L}^p : 0 \in \bigcup_{Z \in R_s(X)} R_{t,s}(-Z) \} \). Indeed,

\[
X \in A_{t,s} + A_s \iff -R_s(X) \cap A_t \neq \emptyset \\
\iff \exists Z \in R_s(X) \text{ s.t. } -Z \in A_t \text{ (i.e. } 0 \in R_t(-Z) = R_{t,s}(-Z)\) \\
\iff 0 \in \bigcup_{Z \in R_s(X)} R_{t,s}(-Z).
\]
Let \( \tilde{R}_t(X) := \bigcup_{Z \in R_t(X)} R_{t,s}(-Z) \) then \( A_{t,s} + A_s = \tilde{R}_t^{-1}(M_t,-) \). By proposition A.3.1, \( \tilde{R}_t \) is c.u.c., and thus \( \tilde{R}_t^{-1}(M_t,-) \) is closed.

**Remark A.3.3.** Let \( R_t \) be a conditional risk measure at time \( t \) and \( R_{t,s} := R_t|_{M_s} \) be the stepped risk measure from \( t \) to \( s \) associated with \( R_t \). If \( R_t \) is c.u.c. then, trivially, \( R_{t,s} \) is c.u.c.

Moreover, when applying lemma 3.3.1 to the proof of theorem 3.3.2 and corollary 3.3.3 we need not only the sum of closed convex acceptance sets to be closed, but also to be a (closed) convex acceptance set itself. This is given in the following lemma.

**Lemma A.3.4.** Let \( (A_t)_{t \in T} \) be a sequence of closed convex normalized acceptance sets. Assume \( A_{t,t+1} + A_{t+1} \subseteq A_t \), then \( A_{t,t+1} + A_{t+1} \) is a convex acceptance set at time \( t \). Furthermore, if \( (A_t)_{t \in T} \) is c.u.c., then \( A_{t,t+1} + A_{t+1} \) is closed.

**Proof.** Let us check the properties of acceptance sets (see definition 2.1.11).

1. \( A_{t,t+1} + A_{t+1} \subseteq L^p \) trivially.

2. \( M_t \cap (A_{t,t+1} + A_{t+1}) \supseteq M_t \cap M_{t+1} \cap A_t \neq \emptyset \) since \( 0 \in A_{t+1} \) (by \( A_{t+1} \) closed and normalized), \( M_t \cap A_t \neq \emptyset \), and \( M_t \cap M_{t+1} = M_t \).

3. \( M_t \cap (L^p \setminus \{A_{t,t+1} + A_{t+1}\}) \supseteq M_t \cap (L^p \setminus A_t) \neq \emptyset \) by \( A_{t,t+1} + A_{t+1} \subseteq A_t \).

4. \( A_{t,t+1} + A_{t+1} + L^p \subseteq A_{t,t+1} + A_{t+1} \) trivially.

\( A_{t,t+1} + A_{t+1} \) is convex since both \( A_{t,t+1} \) and \( A_{t+1} \) are convex. \( A_{t,t+1} + A_{t+1} \) is closed by lemma A.3.2 if \( (A_t)_{t \in T} \) is c.u.c.

We finish this section by considering a class of risk measures which are point plus cone and show that these risk measures are c.u.c. under \( p = +\infty \) and the weak* topology.
Proposition A.3.5. Fix some time $t$. Let $M_t = \mathcal{L}_t^\infty$ and let $p = +\infty$. Let $R_t(X) := \rho_t(X) + \mathcal{L}_{t,+}^\infty$ for some vector $\rho_t$ of scalar conditional risk measures, i.e. $\rho_t(X) := ((\rho_{t1}(X_1), ..., (\rho_{td}(X_d))^T$. If $\rho_t$ is (componentwise) lower semicontinuous and convex then $R_t$ is c.u.c.

Proof. Recall from the scalar literature that $\rho_t(X) \in \mathcal{L}_t^\infty$ for any $X \in \mathcal{L}_t^\infty$. Let $D \in \mathcal{G}(\mathcal{L}_t^\infty; \mathcal{L}_{t,-}^\infty)$. It follows that

$$R_t^{-1}(D) = \{X \in \mathcal{L}_t^\infty : R_t(X) \cap D \neq \emptyset\}$$

$$= \{X \in \mathcal{L}_t^\infty : \exists \hat{d} \in D, \rho_t(X) \leq \hat{d}\}$$

$$= \{X \in \mathcal{L}_t^\infty : \exists \hat{d} \in D, \rho_t(X) = \hat{d} \mathbb{P}\text{-a.s.}\}$$

$$= \{X \in \mathcal{L}_t^\infty : \rho_t(X) \in D\} = \rho_t^{-1}(D).$$

Therefore we wish to show that $\rho_t^{-1}(D)$ is weak* closed. From $\rho_t$ convex, it immediately follows that $\rho_t^{-1}(D)$ is convex, therefore $\rho_t^{-1}(D)$ is weak* closed if and only if $\rho_t^{-1}(D) \cap \{Z \in \mathcal{L}_t^\infty : \|Z\|_\infty \leq k\}$ is closed in probability for every $k$ by [60, proposition 5.5.1]. Pick any $k \geq 0$ and consider

$$(Z_n)_{n \in \mathbb{N}} \subseteq \rho_t^{-1}(D) \cap \{Z \in \mathcal{L}_t^\infty : \|Z\|_\infty \leq k\}$$

with $Z_n \to \tilde{Z}$ in probability (and thus $\tilde{Z} \in \{Z \in \mathcal{L}_t^\infty : \|Z\|_\infty \leq k\}$). Note that convergence in probability implies there exists a subsequence which converges almost surely, we denote this subsequence by $(Z_{n_m})_{m \in \mathbb{N}} \to \tilde{Z}$. For any sequence $(Y_n) \subseteq \mathcal{L}_t^\infty$, define $\liminf_{n \to \infty} \rho_t(Y_n) = \lim_{n \to \infty} \inf_{m \geq n} \rho_t(Y_m)$ where

$$\inf_{m \geq n} \rho_t(Y_m) = \begin{pmatrix} \inf_{m \geq n} (\rho_t)_1(Y_m)_1 \\ \vdots \\ \inf_{m \geq n} (\rho_t)_d(Y_m)_d \end{pmatrix}.$$
Since $D$ is a lower set and $\inf_{\hat{m} \geq m} \rho_t(Z_{n,\hat{m}}) \leq \rho_t(Z_{n,m})$ almost surely (and $\rho_t(Z_{n,m}) \in D$) for any $m \in \mathbb{N}$, then it follows that $\inf_{\hat{m} \geq m} \rho_t(Z_{n,\hat{m}}) \in D$ for any $m \in \mathbb{N}$. Note that $\| \inf_{\hat{m} \geq m} \rho_t(Z_{n,\hat{m}}) \|_{\infty} \leq \max(\|\rho_t(0)+k\|_{\infty},\|\rho_t(0)-k\|_{\infty}) =: \hat{k}$ by $\|Z_{n,\hat{m}}\|_{\infty} \leq k$ for every $\hat{m} \in \mathbb{N}$. Since $D \cap \{u \in L^\infty_t : \|u\|_{\infty} \leq \hat{k}\}$ is closed in probability (by [60, proposition 5.5.1]) it must contain all almost sure limit points, i.e. $\lim_{m \to \infty} \rho_t(Z_{n,m}) \in D \cap \{u \in L^\infty_t : \|u\|_{\infty} \leq \hat{k}\}$. Finally from componentwise lower semicontinuity we have $\lim_{m \to \infty} \rho_t(Z_{n,m}) \geq \rho_t(Z)$ almost surely, therefore by $D$ a lower set it follows that $\rho_t(\bar{Z}) \in D$, i.e. $\bar{Z} \in \rho_t^{-1}(D)$. \qed
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