AN EXAMINATION OF THE SYSTEMATIC RISKS
IN THE MULTI-NAME CREDIT AND EQUITY
MARKETS

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Abstract

The focus of this dissertation is to examine the systematic risks in the multi-name credit and equity markets during the recent financial crisis. We first consider a hybrid intensity-based model that can price both credit and equity instruments, and here, we analyze the multi-name credit spreads and the option-implied volatility skews. Secondly, we develop a top-down utility-indifference model for valuing credit and equity instruments, and in this case, we compare the implied risk aversions of investors in the two markets.

In the first part, we present an intensity-based common factor model that can be used to link the credit market to the equity market. In particular, we use the hybrid intensity model to price single-name credit instruments such as credit default swaps (CDSs), multi-name credit derivatives such as collateralized debt obligations (CDOs), and equity index options such as calls and puts on the S&P 500. The CDS prices have analytical expressions; the CDO prices have to be computed numerically using a recursion algorithm and Fourier transform methods; and the equity index option prices have semi-analytical expressions that are computed using numerical integration. Once we have the expressions for the model prices of these instruments, we then calibrate the model parameters to fit the market data. We study two problems, the “forward” and “backward” problems: in the former, we start from equity index options and then compute the CDO tranche spreads, while in the latter, we fit the parameters to the CDO tranche spreads and then back out the equity index option prices and implied volatilities. In both cases, we analyze the systematic risks inherent in the credit and equity markets by examining the tranche spreads and implied volatility skews from the model and the market. We find that based on our hybrid model, the systematic risks in the two markets were similar from 2004 to 2007, while the credit market incorporated far greater systematic risk than the equity market during the financial crisis from 2008 to 2010.
In the second part, we consider a top-down utility-indifference model that incorporates the investor’s risk aversion for valuing both multi-name credit derivatives and equity index options. In particular, we assume that the default loss process is a self-exciting counting process in which the intensity is mean-reverting with feedback from defaults, while the stock index process has stochastic variance which also contains feedback from defaults. Now, for the optimal control problem, the investor can invest in the equity/credit derivative, the money market and the stock index. The indifference prices then arise from the Merton and tranche holder’s value functions, which are the solutions to systems of multi-variable Hamilton-Jacobi-Bellman PDEs. Here, the differential equations can be solved recursively using either finite differences or trinomial trees. From our numerical tests, we find that the investors’ implied risk aversions for CDOs were increasing with seniority, whereas the risk aversions for equity put options increased as moneyness levels decreased. In addition, we find that over the 16-month period from June 2009 to September 2010, the largest risk aversions from the credit market far exceeded those from the equity market, indicating greater systematic risk in the credit market during the crisis, consistent with the conclusions from the first part of the thesis.

Lastly, we also discuss two extensions to the top-down model, first where we allow for correlation between the equity, variance, and intensity processes and secondly, where we impose that the stock index drops by proportional amounts at times of default.
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To my wife and my parents.
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Chapter 1

Introduction

The focus of this dissertation is to examine the systematic risks in the multi-name credit and equity markets during the recent financial crisis. In the first part of this thesis, we consider a hybrid intensity-based model for pricing collateralized debt obligations (CDOs) and equity index options, and then we analyze the CDO tranche spreads and the option implied volatility skews. In the second part, we develop a top-down utility-indifference model for valuing both credit and equity instruments, and then we compare the implied risk aversions of investors in the two markets.

In this introductory chapter, we first discuss the history of the multi-name credit derivatives market and provide an overview of CDOs and the causes of the recent credit crisis. Then, we examine various hybrid equity-credit models and introduce our intensity-based model for comparing the systematic risk profiles in the credit and equity markets. Next, we discuss the idea of top-down utility-indifference valuation for pricing both credit and equity derivatives and we present our approach for exploring the impact of default feedback and investors’ risk aversion. We conclude this chapter by highlighting the main contributions of our work.
1.1 Multi-Name Credit Derivatives

The market in credit derivative products has grown significantly over the past decade, with a drop after the financial crisis. Table 1.1 shows the size of the credit derivatives market in recent years, courtesy of the International Swaps and Derivatives Association (ISDA)\(^1\) here, the size of the credit market accounts for the outstanding notional of credit default swaps referencing single names, indexes, baskets, and portfolios. The credit market size rapidly increased from around $900 billion in 2001 to $62 trillion at the end of 2007, and then dropped back down to below $30 trillion in 2010. Also, ISDA reports that since the end of 2003, the size of the credit derivatives market has exceeded that of the equity derivatives market.

<table>
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<th>Year</th>
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<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
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<tbody>
<tr>
<td>Volume</td>
<td>0.9</td>
<td>2.2</td>
<td>3.8</td>
<td>8.4</td>
<td>17.1</td>
<td>34.4</td>
<td>62.3</td>
<td>38.6</td>
<td>30.4</td>
<td>26.3</td>
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At the center of this financial revolution was the creation and increased popularity of structured products such as the asset backed security (ABS) and the collateralized debt obligation (CDO). An ABS is a security whose value and income payments are derived from a specified pool of underlying assets. Meanwhile, a CDO is a financial contract which depends on the default events of a basket portfolio (e.g., 125 firms) over periods from three to ten years in length. As illustrated in Figure 1.1, a CDO is divided into several tranches, including equity, mezzanine, and senior. The equity tranche is the riskiest and requires the highest premium; the mezzanine tranches are the intermediate tranches with medium-level risk; and the senior (or super senior) tranche is generally of the highest credit quality and requires the lowest premium. Each tranche has two counterparties, the protection buyer and the protection seller (also termed the tranche holder). The buyer pays to the seller a yield or tranche

\(^1\)See http://www.isda.org/statistics/recent.html.
spread (the insurance premium) on the remaining amount that the protection seller is still responsible for. As losses begin to hit his tranche, the protection seller pays those claims to the protection buyer. The precise details of the payments are described in Chapter 2.

**Figure 1.1: CDO Tranches**

![CDO Tranches Diagram](image)

An important development in the CDO market was the creation, in 2003, of the *Dow Jones CDX* indices for North America and the *Dow Jones iTraxx* indices for Europe. These benchmark indices reference a standardized pool of credit default swaps (CDSs), have standardized maturities, and standardized tranches. The result has been a substantial increase in liquidity and transparency in what was primarily an over-the-counter market. Reliable market quotes (spreads) now exist for CDOs, written on the CDX or iTraxx portfolios consisting of 100 or 125 investment grade companies, with maturities of 3, 5, 7, and 10 years. In Chapter 2, we will calibrate our model to these CDO tranche spreads, along with the underlying CDS spreads.

A number of academics, analysts and investors such as Warren Buffett and the IMF’s former chief economist Raghuram Rajan warned that CDOs, ABSs and other credit derivatives spread risk and uncertainty about the value of the underlying assets more widely, rather than reduce risk through diversification. Many of the assets held
by these CDOs had been subprime mortgage-backed bonds. Hence, as housing prices dropped, global investors began to stop funding CDOs in 2007, contributing to the collapse of certain structured investments held by major investment banks and the bankruptcy of several subprime lenders. This led to the subprime mortgage crisis in 2007 and the resulting financial crisis in 2008. Table 1.2 below shows the global CDO issuance volume before, during, and after the crisis, as obtained from SIFMA. We observe that the volume grew from below $80 billion in 2001 to its peak of over $520 billion in 2006, and then declined dramatically in the wake of the crisis in 2007, settling in at under $8 billion in both 2009 and 2010.

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<th>Year</th>
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<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
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<tbody>
<tr>
<td>Volume</td>
<td>78.4</td>
<td>83.1</td>
<td>86.6</td>
<td>157.8</td>
<td>251.3</td>
<td>520.6</td>
<td>481.6</td>
<td>61.9</td>
<td>4.3</td>
<td>7.7</td>
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In hindsight, many market participants, including credit rating agencies, failed to adequately account for large risks such as a nationwide plunge in housing values. Coval, Jurek, and Stafford (2009b) mention that 60 percent of structured products were given the highest possible grade of AAA, while only 1 percent of bonds were rated AAA. Essentially, the market neglected the systematic risk, which represents the potential of different assets to all move downward in tandem. In a financial crisis, many assets tend to be correlated and fall together, signifying large systematic risk. For example, Bhansali, Gingrich, and Longstaff (2008) showed that under their three-factor top-down model, the credit crisis (as of Dec. 2007) had more than twice the systematic risk of the May 2005 automotive-downgrade crisis. In addition, Hull (2009) and Coval, Jurek, and Stafford (2009b), among others, have argued that investors did not recognize that senior tranches of the CDOs were more susceptible to systematic risk than junior tranches, and they discuss how the losses for the CDO-squareds (e.g., ABSs on CDOs) were amplified compared to CDOs. A lesson to be learned from the

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credit crisis is that one should always be prepared for the potentially large impact of systematic risk, particularly in the structured credit market.

In this thesis, we explore two methods for measuring the systematic risks in the credit and equity markets. In particular, we first use a hybrid intensity-based model to examine the link between the CDO tranche spreads and implied volatilities from equity index options. Then, we develop a top-down utility-based model to analyze the investor’s risk aversion across tranches and moneyness levels.

1.2 Equity-Credit Intensity-Based Models

In the first part of the thesis, we examine the link between the equity and credit markets by considering an intensity-based model that can price both CDOs and equity index options. In this section, we start by reviewing the literature on hybrid equity-credit intensity models, and then we describe our model and our approach for examining the systematic risks in the two markets.

We first examine the recent literature on the joint valuation of stock options and single-name credit derivatives such as CDSs. In particular, [Carr and Wu (2010)] discuss a joint equity-credit framework in which the stock has CIR stochastic volatility and jumps to default, while the intensity of default is an affine function of the stochastic variance process and an idiosyncratic process. In [Carr and Linetsky (2006)], the stock price has constant elasticity of variance (CEV) and jumps to default, while the intensity is an affine function of the instantaneous variance of the stock. Meanwhile, [Bayraktar (2008)] and [Bayraktar and Yang (2011)] consider a jump-to-default model for the stock, where the intensity is an affine function of the stochastic variance and an idiosyncratic stochastic component, both of which are driven by fast and slow mean-reverting processes. In all of the above models, the intensity is an affine function of the stock’s stochastic variance and an idiosyncratic component, but only for
a single name; we wish to extend this formulation to the multi-name case.

Recently, there have emerged structural-based hybrid models for pricing equity derivatives and multi-name credit derivatives such as \( n^{th} \)-to-default swaps and CDOs. For example, Coval, Jurek, and Stafford (2009a) consider a one-period structural model, based on Merton (1974) and incorporated within a CAPM framework, in order to price both index options and CDOs using the Arrow-Debreu state-contingent approach. The state prices are derived from options using an analog of Breeden and Litzenberger (1978) in the presence of an implied volatility smile, while the model parameters are fitted to the 5-year CDX index spread. The authors conclude that the large skew from out-of-the-money put options leads to the overpricing of senior tranches, and hence, the catastrophe risk from the equity market is not fully accounted for in the credit market.

On the other hand, a few authors recently have argued that the equity index options market and multi-name credit derivatives market are indeed consistent. In particular, Collin-Dufresne, Goldstein, and Yang (2010) use Black-Cox’s dynamic structural model within CAPM to jointly price long-dated S&P options and CDO tranches. The authors match the entire term structure of CDX index spreads instead of just the 5-year spread in order to accurately capture information regarding the timing of expected defaults and the specification of idiosyncratic dynamics. Finally, they show that the pricing of CDO tranches is consistent with the pricing of equity index options, thus offering a resolution to the puzzle reported by Coval, Jurek, and Stafford (2009a). In support of Collin-Dufresne, Goldstein, and Yang (2010), Luo and Carverhill (2011) employ the three-factor portfolio intensity model of Longstaff and Rajan (2008), which allows for firm-specific, industry, and economywide default events, and conclude that the CDO tranche market is well integrated with the S&P 500 index option market. Also, Li and Zhao (2011) argue that the model of Coval, Jurek, and Stafford (2009a), which relies on out-of-sample analysis, is mis-specified.
because it is not flexible enough to even capture CDX tranche prices \textit{in sample}. In addition, Li and Zhao use a skewed-$t$ copula model to jointly price index options and CDO tranches, and they conclude that the CDO market is actually efficient.

In Chapter 2, we consider a dynamic hybrid \textit{intensity-based} model for valuing equity index options and multi-name credit derivatives. We assume that the intensity is an affine function of the stock index variance and an idiosyncratic (firm) component, following the idea of Carr and Wu (2010) but where we model the stock market index rather than the individual stock. This modification allows us to price options on the stock index in a closed form because the index and its variance follow the stochastic volatility with jumps in variance (SVJ-V) model. In addition, we suppose that the common factor (i.e., the variance) and idiosyncratic processes are affine jump diffusions (AJDs), as in Mortensen (2006). Then, in this \textit{bottom-up} approach, we can analytically price single-name credit derivatives such as CDSs and also efficiently price multi-name credit derivatives such as CDOs using recursion and Fourier inversion. The semi-analytical valuations allow us to calibrate our model to equity index options, CDSs, and CDOs and to assess the systematic risks inherent in the equity and credit markets. Finally, we compare our results to related works such as Coval, Jurek, and Stafford (2009a) and Collin-Dufresne, Goldstein, and Yang (2010), and we explain how our results differ from the existing literature.

1.3 Top-Down Utility-Based Indifference Valuation

In the second part of the thesis, we consider a top-down model for the utility-indifference valuation of CDOs and equity index options, and we wish to examine systematic risk factors such as feedback effects and risk aversion on the optimal strategy and indifference prices. In this section, we first review the literature on top-down intensity based models and utility-indifference valuation, and then we motivate our
modeling framework and our approach to measuring the systematic risks.

Many multi-name credit derivatives, such as CDOs, are path-dependent contingent claims on the number of defaults and the aggregate loss in a credit portfolio. Therefore, an alternative to the bottom-up approach is the top-down approach, which focuses on describing the portfolio loss counting process without specifying the constituent firm’s default dynamics; see, for example, Giesecke, Goldberg, and Ding (2011). Default dependency can be captured by specifying dependence of the default intensity on the portfolio loss or number of defaults; see, for example, Errais, Giesecke, and Goldberg (2010), Lopatin and Misirpashaev (2008), and Arnsdorff and Halperin (2008). More precisely, Azizpour and Giesecke (2008) argue in favour of a feedback effect, a form of systematic risk whereby the default of one firm adversely affects other firms in the portfolio. Hence, the portfolio default process is “self-exciting” in the sense that each default increases the total default intensity, or the likelihood of more defaults. Accordingly, in Chapter 3 we use a top-down model with feedback for the portfolio loss process.

For valuation, we note that the standard no-arbitrage pricing theory, as developed by Black and Scholes (1973) and Merton (1973), relies on the ability to construct a dynamic portfolio that replicates the payoff of the derivative in question. In these complete market models, all risks associated with a derivative position can be hedged perfectly by trading the opposite of this replicating portfolio. However, market frictions, such as non-tradability of assets and other hedging constraints, render perfect replication infeasible. In these incomplete markets, the derivative holder is inevitably exposed to some unhedgeable risks, and the value of the derivative naturally depends on how he partially hedges the risks, as well as his tradeoff between risk and return.

One common approach to derivative pricing in incomplete markets is to formulate the problem in the context of utility maximization. The simplest form of such problems is to determine expected utility of the random payoff from a contingent claim,
and derive the certainty equivalent for holding the claim. When the holder can trade other assets in addition to the claim, then it is natural to compare investments with and without the claim. Hodges and Neuberger (1989) extend the certainty equivalent concept from a static setting to a dynamic financial market, and they develop a methodology now commonly known as utility indifference pricing. This mechanism involves maximizing the investor’s expected utility of an investment with the claim in question. Indeed, the utility indifference price of a claim is defined as the amount of money that the holder is willing to pay so that his maximal expected utility is the same as that from an investment without the claim.

In the context of credit derivatives pricing, Sircar and Zariphopoulou (2007) use the technology of utility-indifference valuation in an intensity-based model to analyze the yield spreads for single-name defaultable bonds and a representative two-name credit derivative. Furthermore, Sircar and Zariphopoulou (2010) extend this model to the multi-name case by considering a bottom-up approach for the indifference valuation of CDOs. They find that due to the non-linearity of the utility-indifference valuation mechanism, increasing the investor’s risk aversion indeed enhances the effective correlation between the times of the credit events of the various firms. This leads to non-trivial senior tranche spreads that are in line with the market data. Motivated by these results, we examine the risk aversion of investors in both the credit and equity markets under a top-down utility-indifference valuation framework that incorporates feedback from defaults.

In Chapter 3, we consider a top-down intensity-based approach for the indifference pricing of multi-name credit derivatives such as CDOs and of equity derivatives. In particular, we model the portfolio loss process as a self-exciting counting process, and we assume that the stock index has a stochastic variance process that is mean-reverting and jumps upwards at default times. For the utility maximization, we assume that the investor can invest in the money market, the stock index, and the

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credit or equity derivative. Then, to obtain the indifference price of the derivative, we solve systems of Hamilton-Jacobi-Bellman PDEs recursively on a grid. Next, we numerically assess the impact of the default feedback and risk aversion on the optimal strategies and the credit/equity derivative prices. Lastly, we examine the systematic risks in the two markets by comparing the implied risk aversions from credit derivatives to the risk aversions from equity derivatives over a 16-month period from 2009 to 2010.

1.4 Overview

The rest of the thesis is organized as follows. In Chapter 2, we first discuss our intensity-based common factor model and explain the valuation methodology for pricing both CDO tranches and equity index options. We then describe the “forward” and “backward” problems, in which we calibrate to equity index options and CDOs, respectively. Lastly, we provide numerical test results and we analyze the systematic risks in the credit and equity markets.

In Chapter 3, we discuss our top-down indifference valuation methodology for pricing both multi-name credit derivatives and equity derivatives. Based on our top-down model with event feedback, we use the technology of utility-indifference valuation to price CDOs, European claims, and equity index options. Next, we assess the impact of default feedback and risk aversion on the optimal strategies and the credit/equity derivative prices. Lastly, we examine the systematic risks in the credit and equity markets during the credit crisis.

Chapter 4 concludes with a summary of the systematic risks in the credit and equity markets during the financial crisis. This chapter also discusses two extensions for future work, first where we allow for correlation between the variance and intensity processes and secondly where we impose that the stock index drops by proportional
amounts at times of default.

Lastly, we provide some technical results in the appendices. In particular, Appendix A provides the moment generating and characteristic functions for affine jump diffusions, as used for the model in Chapter 2. In Appendix B we describe a variation of the top-down model of Chapter 3 in which the volatility is an inverse CIR process with jumps.
Chapter 2

Intensity-Based Valuation of CDOs and Equity Index Options

In this chapter, we present an intensity-based common factor model that can be used to link the credit market to the equity market, as motivated in Section 1.2. In particular, we use our hybrid model to price single-name credit instruments such as credit default swaps (CDSs), multi-name credit derivatives such as CDOs, and equity index options such as calls and puts on the S&P 500, and then we analyze the systematic risks inherent in the credit and equity markets.

Here is an outline of this chapter. In Section 2.1, we introduce the bottom-up intensity-based common-factor model that we will use for pricing both credit derivatives and equity index options. Next, in Section 2.2, we discuss the CDO framework, and in particular, the tranche structure, the payment streams, and the fair values; we follow up in Section 2.3 with the algorithm for valuing CDO tranche spreads and upfront fees under our model. In Sections 2.4 and 2.5, we describe the valuation of credit default swaps (CDSs) and equity index options under our model. Section 2.6 describes the sources of the market data and various market conventions. In Section 2.7, we consider two problems: first, we examine the forward problem
where we calibrate our model parameters from equity index options and CDS spreads and then price CDO tranches; secondly, we examine the backward problem where we fit the model parameters to CDO tranche spreads and CDS spreads and then price equity index options. We conclude the chapter in Section 2.8 by providing numerical results for the two problems above and analyzing the systematic risks in the credit and equity markets.

2.1 Intensity-Based Common-Factor Model

Suppose we are given a probability space \((\Omega, \mathcal{F}, \mathbb{P}^*)\), where \(\mathbb{P}^*\) is the risk-neutral probability measure reflected by market prices of credit derivatives. We consider a CDO on a portfolio of \(\hat{N}\) underlying firms, labelled 1 to \(\hat{N}\). For \(i = 1, \ldots, \hat{N}\), we let \(\tau_i\) be the default time of the \(i^{th}\) firm and we let \((\lambda^i_t)_{t \geq 0}\) be the intensity process of firm \(i\). As in Mortensen (2006), we suppose that \(\lambda^i_t\) has the form

\[
\lambda^i_t = X^i_t + c_i Y_t, \tag{2.1}
\]

where \(c_i > 0\) is a constant, \(X^i = (X^i_t)_{t \geq 0}\) is the idiosyncratic process for the \(i^{th}\) firm, and \(Y = (Y_t)_{t \geq 0}\) is a market factor process that is independent of the \(X^i\) but affects all of the firms. Here, \(c_i\) is a spread level parameter; a firm with high (low) credit spreads will have a high (low) \(c_i\), and \(c_i = 1\) represents a firm with an average credit spread. We let \(X^i\) and \(Y\) be affine jump diffusions (AJDs) that satisfy the stochastic differential equations (SDEs)

\[
dX^i_t = \kappa_i (\bar{x}_i - X^i_t) dt + \sigma_i \sqrt{X^i_t} dW^i_t + dJ^i_t, \tag{2.2}
\]

\[
dY_t = \kappa_Y (\bar{y} - Y_t) dt + \sigma_Y \sqrt{Y_t} dW^Y_t + dJ^Y_t, \tag{2.3}
\]

where
\( W^Y, W^1, \ldots, W^N \) are independent \( \mathbb{P}^* \)-Brownian motions,

\( J^Y, J^1, \ldots, J^N \) are independent compound Poisson processes under \( \mathbb{P}^* \) with respective jump intensities \( l_Y, l_1, \ldots, l_N \) and exponentially distributed jump sizes with respective means \( \xi_Y, \xi_1, \ldots, \xi_N \),

\( \kappa_i, \bar{x}_i, \) and \( \sigma_i \) are the respective mean-reversion speed, mean-reverting level\(^1\) and volatility parameters for the process \( X^i \),

\( \kappa_Y, \bar{y}, \) and \( \sigma_Y \) are the respective mean-reversion speed, mean-reverting level, and volatility parameters for the process \( Y \), and

the Feller condition\(^2\) holds for the \( X^i \) and for \( Y \):

\[
\sigma_i^2 \leq 2 \kappa_i \bar{x}_i, \quad \sigma_Y^2 \leq 2 \kappa_Y \bar{y},
\]

so that the processes \( X^i \) and \( Y \) will stay positive almost surely.

Note that (2.1)-(2.3) is a bottom-up model for credit derivative valuation, since we model the intensity for each underlying firm and then build up the loss distribution for the portfolio\(^3\). Mortensen (2006) observes that due to the jump components in \( X^i \) and \( Y \), this model is able to capture the high correlation embedded in the senior CDO tranche spreads. Through an empirical analysis, Feldhütter and Nielsen (2010) showed that this model is able to capture both the level and the time series dynamics of CDO tranche spreads.

In addition to the intensity process defined above, we incorporate the price process for the stock market index (e.g., the S&P 500 index), as denoted by \( S = (S_t)_{t \geq 0} \). As

---

\(^1\)Due to the jump terms, the long-term means of the idiosyncratic and market factor processes are larger than their respective mean-reverting levels. See below for the computations.

\(^2\)This condition was first studied by Feller (1957). A proof of the positivity of the processes under the Feller condition is given in Proposition 6.2.4 of Lamberton and Lapeyre (1996).

\(^3\)In Chapter 3 we consider a top-down model for valuing CDOs under a utility-indifference approach.
motivated by Carr and Wu (2010), we assume that $S$ has a stochastic variance process which is a constant multiple of the market factor process $Y$:

\[
dS_t = (r - q)S_t dt + \sqrt{b_Y}S_t dW^S_t,
\]

(2.4)

where $r$ is the risk-free interest rate, $q$ is the continuous dividend yield, $W^S$ is a $\mathcal{P}^*$-Brownian motion and $b_Y$ is a positive constant. We suppose that the Brownian motions $W^S$ and $W^Y$ (from (2.3)) are correlated via

\[
\mathbb{E}^*[dW^S_t \cdot dW^Y_t] = \rho dt,
\]

where $\mathbb{E}^*$ represents the expectation operator under $\mathbb{P}^*$. Empirical evidence suggests that the market return process and its volatility are negatively correlated (this is termed the leverage effect); hence, we assume that $\rho < 0$. In addition, there is evidence that credit spreads are positively correlated to the market return volatilities; see, for example, Collin-Dufresne, Goldstein, and Martin (2001). We capture this relation through the positive coefficients $c_i$ in (2.1) and $b_Y$ in (2.4).

We suppose that the initial values of the processes $X^i$, $Y$, and $S$ are given, respectively, by

\[
X^i_0 = x^i_0, \quad Y_0 = y_0, \quad S_0 = \bar{S}_0.
\]

(2.5)

To help estimate the sizes of the model parameters that would fit the market data, we compute the expected values of the idiosyncratic process $X^i$ and market factor process $Y$ at fixed times; we also show that the long-term means are indeed larger than the respective mean-reverting levels $\bar{x}_i$ and $\bar{y}$. For the process $Y$, we integrate the SDE (2.3) from 0 to $t$, take expectations, and differentiate to obtain the following ODE for $M(t) := \mathbb{E}^*Y_t$:

\[
M'(t) = \kappa_Y (\bar{y} - M(t)) + l_Y \xi_Y,
\]
with the initial condition $M(0) = y_0$. Using the integrating factor $e^{\kappa_Y t}$, we obtain the following solution:

$$E^*Y_t = y_0 e^{-\kappa_Y t} + \left( \bar{y} + \frac{l_Y \xi_Y}{\kappa_Y} \right) \left( 1 - e^{-\kappa_Y t} \right), \quad t \geq 0.$$  

Hence, the long-term mean for the market factor process is

$$\lim_{t \to \infty} E^*Y_t = \bar{y} + \frac{l_Y \xi_Y}{\kappa_Y} > \bar{y}.$$  

Similarly, for the idiosyncratic process $X^i$, we have

$$E^*X^i_t = x_0 e^{-\kappa_i t} + \left( \bar{x}_i + \frac{l_i \xi_i}{\kappa_i} \right) \left( 1 - e^{-\kappa_i t} \right), \quad t \geq 0,$$

and hence, the long-term mean is

$$\lim_{t \to \infty} E^*X^i_t = \bar{x}_i + \frac{l_i \xi_i}{\kappa_i} > \bar{x}_i.$$  

### 2.2 CDO Framework

Let us describe the framework of a collateralized debt obligation (CDO) by describing the tranche structure, the payment legs, and the fair value of the legs. First, we make the following simplifying assumption.

**Assumption 2.2.1.** The total notional is 1 unit and all $\hat{N}$ firms of the CDO have the same weight of $1/\hat{N}$.

We let $N = (N_t)_{t \geq 0}$ be the portfolio loss process, that is,

$$N_t = \sum_{i=1}^{\hat{N}} 1_{\{\tau_i \leq t\}},$$
where \( \tau_i \) denotes the default time of the \( i^{th} \) firm. Usually defaults do not incur full loss of the value of the underlying bonds, as there is some recovery. Suppose \( \delta_r \in [0, 1) \) is the fraction recovered, so \( (1 - \delta_r) \) is the fractional loss incurred on each default, sometimes called the loss-given-default. Then, the fractional portfolio loss is

\[
L_t = \frac{(1 - \delta_r)N_i}{N}.
\]

(2.6)

### 2.2.1 CDO Tranches

A typical CDO consists of several tranches, such as equity, mezzanine and senior. Each tranche is characterized by an attachment \( (K_L) \) and detachment \( (K_U) \) point; an investor who buys this tranche is responsible for all losses occurring in the interval \([K_L, K_U]\). The equity tranche is the riskiest and requires the highest premium; the mezzanine tranches are the intermediate tranches with medium-level risk; the senior (or super senior) tranche is of the highest credit quality and requires the lowest premium.

Table 2.1 displays the attachment and detachment points for each of the 6 tranches, namely the Equity, Mezzanine 1, Mezzanine 2, Mezzanine 3, Senior, and Super Senior tranches, for the two standardized credit indices, the Dow Jones CDX (comprising North American companies) and the Dow Jones iTraxx (comprising European companies). In our analysis, we focus on the CDX tranches because the firms in the CDX appear to be more representative of the firms comprising the S&P 500 index, on which we compute option prices.

We now define the tranche loss process, which will be used to define the payment legs.

---

4It is standard practice to assume a constant recovery rate \( \delta_r \), despite evidence that a more sophisticated treatment of recovery rates, such as assuming a beta distribution, is warranted; see, for example, [Altman, Resti, and Sironi (2001)], [Bakshi, Madan, and Zhang (2001)], and [Guo, Jarrow, and Lin (2008)].
Table 2.1: Attachment \((K_L)\) and Detachment \((K_U)\) Points for the Tranches of the Dow Jones CDX and Dow Jones iTraxx indices.

<table>
<thead>
<tr>
<th></th>
<th>Equity</th>
<th>Mezz 1</th>
<th>Mezz 2</th>
<th>Mezz 3</th>
<th>Senior</th>
<th>SuperSen</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(K_L)</td>
<td>(K_U)</td>
<td>(K_L)</td>
<td>(K_U)</td>
<td>(K_L)</td>
<td>(K_U)</td>
</tr>
<tr>
<td>CDX</td>
<td>0%</td>
<td>3%</td>
<td>3%</td>
<td>7%</td>
<td>10%</td>
<td>15%</td>
</tr>
<tr>
<td>iTraxx</td>
<td>0%</td>
<td>3%</td>
<td>3%</td>
<td>6%</td>
<td>9%</td>
<td>12%</td>
</tr>
</tbody>
</table>

Definition 2.2.1. For the tranche characterized by \([K_L, K_U]\), the tranche loss at time \(t\) is defined by \(F(L_t)\), where \(L_t\) is the fractional portfolio loss at time \(t\), given by (2.6), and the function \(F\) is given by

\[
F(x) = (x - K_L)^+ - (x - K_U)^+, \quad x \in [0, 1].
\]

Observe that \(F\) is equivalent to the payoff of a call spread on the loss fraction, with “strikes” given by the attachment and detachment points.

2.2.2 Payment Legs

Each CDO tranche has two legs: the premium or fixed leg comprising an upfront payment along with periodic payments from the protection buyer to the tranche holder (i.e., the protection seller) on the remaining notional for the tranche; and the protection or floating leg where payments are made by the tranche holder to the protection buyer as the losses impact the tranche. The premium payments are made at the \(K\) regular\(^5\) payment dates \(T_1, T_2, \ldots, T_K = T\), with

\[
T_k = k\Delta \tau, \quad k = 1, \ldots, K, \quad \text{and} \quad \Delta \tau = \frac{T}{K}.
\]

We make the mild assumption that insurance payouts in the protection leg are also made at the same payment dates, covering all the losses since the previous payment.

\(^5\)Here, we suppose that the payment dates are equally spaced while ignoring market issues such as accrual periods and daycount conventions.
date. To be precise, here are the payment streams:

(a) For the *premium leg*, the tranche holder receives from the protection buyer the following:

- an upfront payment of $U(K_U - K_L)$ at time 0, and
- $R(T_k - T_{k-1})[K_U - K_L - F(L_{T_k})]$ at each payment date $T_k$, $k = 1, 2, \ldots, K$,

where $U$ is the upfront fee and $R$ is the annualized running spread.$^6$ The amount $K_U - K_L - F(L_{T_k})$ is the remaining notional for the tranche at time $T_k$.

(b) For the *protection leg*, the tranche holder pays the protection buyer

- $F(L_{T_k}) - F(L_{T_{k-1}})$ at each payment date $T_k$, $k = 1, 2, \ldots, K$.

Here, the difference $F(L_{T_k}) - F(L_{T_{k-1}})$ represents the loss incurred between the previous payment date $T_{k-1}$ and the current payment date $T_k$ as it impacts the tranche that the holder is responsible for.

### 2.2.3 Fair Value

Under the risk-neutral probability $\mathbb{P}^*$, the fair value of the premium leg is given by

$$\text{Prem} = U(K_U - K_L) + R \sum_{k=1}^{K} (T_k - T_{k-1}) e^{-rT_k} \mathbb{E}^* \left[ K_U - K_L - F(L_{T_k}) \right], \hspace{1cm} (2.7)$$

and the fair value of the protection leg is given by

$$\text{Prot} = \sum_{k=1}^{K} e^{-rT_k} \mathbb{E}^* \left[ F(L_{T_k}) - F(L_{T_{k-1}}) \right]. \hspace{1cm} (2.8)$$

Once we know the fair values of the premium and protection legs, we can compute the running spread or the upfront fee, depending on which one is given in the market.

$^6$Note that some tranches pay only the running spread, while other tranches pay the upfront fee in addition to the running spread. See Section 2.6.2 for more details.
Definition 2.2.2. (i) When the upfront fee $U$ is given, the running spread is the number $R$ that equates the fair values of the two legs (2.7) and (2.8) and is hence given by

$$R = \frac{\sum_{k=1}^{K} e^{-rT_k} \mathbb{E}^*[F(L_{T_k}) - F(L_{T_{k-1}})] - U(K_U - K_L)}{\sum_{k=1}^{K} (T_k - T_{k-1}) e^{-rT_k} \mathbb{E}^*[K_U - K_L - F(L_{T_k})]}.$$  (2.9)

(ii) On the other hand, when the running spread $R$ is given, the upfront fee is the number $U$ that equates the fair values of the two legs (2.7) and (2.8) and is hence given by

$$U = \frac{1}{K_U - K_L} \left[ \sum_{k=1}^{K} e^{-rT_k} \mathbb{E}^*[F(L_{T_k}) - F(L_{T_{k-1}})] - R \sum_{k=1}^{K} (T_k - T_{k-1}) e^{-rT_k} \mathbb{E}^*[K_U - K_L - F(L_{T_k})] \right].$$  (2.10)

To compute the expectations $\mathbb{E}^*[F(L_{T_k})]$, $k = 1, \ldots, K$, in (2.9) and (2.10), we need to determine the portfolio loss distribution, as explained next in Section 2.3.

### 2.3 Intensity-Based Valuation of CDOs

In this section, we describe our algorithm for valuing CDO tranches under the intensity-based common factor model of Section 2.1. Recall that $\tau_i$ is the default time of the $i^{th}$ firm. Let $U = (U_t)_{t \geq 0}$ be the integrated market factor process, defined by

$$U_t = \int_0^t Y_s ds, \quad t \geq 0,$$

where $Y$ satisfies the SDE (2.3). The following is an outline of our algorithm.

1) We first compute the single-name conditional default probabilities given the integrated market factor $U_t$, that is, $\mathbb{P}^*\{\tau_i \leq t \mid U_t\}$. 

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2) For the CDO, we build up the conditional loss distribution, \( P^* \{ N_t = k \mid U_t \} \), using a recursive algorithm.

3) We use the fast Fourier Transform (FFT) algorithm to obtain the density of the integrated market factor \( U_t \).

4) We average over \( U_t \) to obtain the portfolio loss distribution, \( P^* \{ N_t = k \} \).

5) Lastly, we compute the expected tranche losses at the payment dates and substitute them into the formula for the CDO tranche spreads.

### 2.3.1 Single-Name Conditional Default Probabilities

Given the market factor \( U_t = u \), the conditional default probability for the \( i^{th} \) firm is equal to

\[
p_{i}(t\mid u) := P^* \{ \tau_i \leq t \mid U_t = u \} = 1 - \mathbb{E}^*[e^{-\int_0^t \lambda_i s \, ds} \mid U_t = u] = 1 - e^{-c_i u} \mathbb{E}^*[e^{-\int_0^t X^i_s ds}] \quad \text{since} \quad \lambda^i_s = X^i_s + c_i Y_s \quad \text{with} \quad X^i \perp \perp Y
\]

\[
= 1 - e^{-c_i u} e^{\alpha_i(t) + \beta_i(t) x^0_s}, \quad (2.11)
\]

where the functions \( \alpha_i(\cdot) \) and \( \beta_i(\cdot) \) are given explicitly by Equation (A.4) in Appendix A.1, using \( q = -1 \) and the suitable parameters for the process \( X^i \).

### 2.3.2 Conditional Loss Distribution

Let us determine the conditional loss distribution for the portfolio. As the CDO contains a heterogeneous pool of firms, we use the recursive algorithm suggested by [Andersen, Sidenius, and Basu (2003)] and employed by [Mortensen (2006)]. With \( t > 0 \) fixed, we define \( P_k(m\mid u) \) as the probability of exactly \( m \) defaults by time \( t \) among
the first $k$ entities, conditional on the integrated market factor taking the value $u$.
Explicitly, we have

$$P_k(m|u) = \mathbb{P}^* \{ N_t^k = m \mid U_t = u \},$$

where $N_t^k$ denotes the number of defaults by time $t$ among the first $k$ entities. For the recursion, we start from $P_0(m|u) = 1_{\{m=0\}}$, $m = 0, \ldots, \hat{N}$, and then we use conditional independence to obtain the following recurrence relation, for $k = 1, \ldots, \hat{N}$:

$$P_k(m|u) = P_{k-1}(m|u)(1 - p_k(t|u)) + P_{k-1}(m-1|u)p_k(t|u), \quad m = 1, \ldots, k,$$

$$P_k(0|u) = P_{k-1}(0|u)(1 - p_k(t|u)),$$

with the convention $P_k(m|u) = 0$ for $m > k$, and where $p_k(t|u)$ is given by (2.11).
Finally, the conditional loss distribution for the total portfolio, given $U_t = u$, is

$$\mathbb{P}^* \{ N_t = m \mid U_t = u \} = \mathbb{P}^* \{ N_t^{\hat{N}} = m \mid U_t = u \} = P_{\hat{N}}(m|u), \quad m = 0, 1, \ldots, \hat{N}. \quad (2.13)$$

### 2.3.3 Density of Integrated Market Factor

Let us determine the density $f_{U_t}$ of the integrated market factor $U_t = \int_0^t Y_s ds$ by fast Fourier transform (FFT) methods. To that end, the characteristic function of $U_t$ is defined by

$$\Phi_{U_t}(r) := \mathbb{E}^* [e^{irU_t}] = \int_0^\infty e^{iru} f_{U_t}(u) du, \quad r \in \mathbb{R}, \quad (2.14)$$

which is known in closed form via the solution of Riccati ODEs, as described in Appendix A.2. By Fourier inversion, the density is given by

$$f_{U_t}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iru} \Phi_{U_t}(r) dr, \quad u > 0.$$

---

7See Appendix C.1 of Papageorgiou (2007) for a more thorough treatment of Fourier inversion and the FFT algorithm.
Now, for some fixed integer $N_U$, we employ the Fourier grid points $r_j = (j-1)\Delta r$, $j = 1, \ldots, N_U$, and $u_k = (k-1)\Delta u$, $k = 1, 2, \ldots, N_U$, to obtain the approximation

$$f_{U_t}(u_k) \approx \frac{1}{\pi} \sum_{j=1}^{N_U} \text{Re} \left( e^{-i\Delta r \Delta u(j-1)(k-1)} \Phi_{U_t}(\Delta r(j-1)) \Delta r \right).$$

Since the discrete Fourier transform (DFT) requires that $\Delta r \cdot \Delta u = \frac{2\pi}{N_U}$, we can re-write the density as

$$f_{U_t}(u_k) \approx \sum_{j=1}^{N_U} \text{Re} \left( e^{-i\frac{2\pi}{N_U}(j-1)(k-1)} \frac{\Delta r}{\pi} \Phi_{U_t}(\Delta r(j-1)) \right), \quad k = 1, \ldots, N_U. \quad (2.15)$$

Since we are only concerned with the real part of the summation in (2.15), the approximation relies on finding a suitable value $\bar{r}$ such that $\text{Re}(\Phi_{U_t}(r)) \approx 0$ for $r > \bar{r}$; then, from the boundary condition, we set $\Delta r = \bar{r}/(N_U - 1)$.

### 2.3.4 Portfolio Loss Distribution

The portfolio loss distribution is computed by integrating the conditional probability over the range of the integrated market factor. In particular, we have

$$\mathbb{P}^*\{N_t = k\} = \int_0^\infty \mathbb{P}^*\{N_t = k \mid U(t) = u\} f_{U_t}(u) du, \quad k = 0, \ldots, \hat{N}, \quad (2.16)$$

where the conditional probability $\mathbb{P}^*\{N_t = k \mid U_t = u\}$ is given by (2.13) and the density $f_{U_t}$ is given by (2.15).

### 2.3.5 CDO Tranche Spread

Once we have computed the loss distribution $\mathbb{P}^*\{N_t = k\}, k = 0, \ldots, \hat{N}$, from (2.16), we can calculate the expected tranche loss $\mathbb{E}^*[F(L_t)]$ at time $t$ using the relation (2.6).

\footnote{See also Eckner (2009), Section 3.2, which uses cubic spline interpolation to determine the upper limit of integration in (2.14), from which one can back out $\Delta u$ and $\Delta r$.}
and a simple weighted average:

\[
E^*[F(L_t)] = \sum_{k=0}^{\hat{N}} F \left( \frac{(1 - \delta r) k}{\hat{N}} \right) P^*\{N_t = k\}.
\]  

(2.17)

Finally, to obtain the running spread \( R \) in (2.9) and the upfront fee \( U \) in (2.10), we substitute the expected tranche loss (2.17) at the payment dates \( T_1, \ldots, T_K \). At time \( T_0 = 0 \), the expected tranche loss is simply \( E^*[F(L_0)] = E^*[F(0)] = 0 \) since \( N_0 = 0 \).

### 2.4 Credit Default Swap Valuation

We now give a brief description of a credit default swap (CDS) and its risk-neutral valuation, following Appendix B of Mortensen (2006); for a more detailed description of CDSs, see, for example, Duffie (1999).

#### 2.4.1 Product Description

A CDS is an insurance contract between two counterparties written on the event of default of a third reference entity. In the event of default before maturity of the contract, the protection seller pays the loss-given-default to the protection buyer. That is, at default, the protection buyer delivers a defaulted bond to the protection seller in return for face value. To compensate for that, the protection buyer pays fixed premium payments periodically until the default of the reference entity or the maturity of the contract, whichever occurs first. CDSs are among the most liquid single-name credit derivatives and provide a good indication of the market’s view towards the credit risk of the underlying firm.
2.4.2 Survival Probability

In order to value the CDS, we need to compute the survival probability for each firm. Recall that \( \tau_i \) is the default time of the \( i^{th} \) firm. Then, since \( X^i \) and \( Y \) are independent processes, the survival probability for the \( i^{th} \) firm is given by

\[
P^* \{ \tau_i > t \} = \mathbb{E}^* \left[ e^{-\int_0^t \lambda^i_s ds} \right]
\]

\[
= \mathbb{E}^* \left[ e^{-\int_0^t X^i_s ds} \right] \cdot \mathbb{E}^* \left[ e^{-c_i \int_0^t Y_s ds} \right]
\]

\[
= e^{\alpha_i(t) + \beta_i(t)x_0^i} \cdot e^{\alpha(t) + \beta(t)y_0}.
\]  

(2.18)

Here, the functions \( \alpha_i(\cdot) \) and \( \beta_i(\cdot) \) are given explicitly by Equation \((A.4)\) in Appendix \[A.1\] with \( q = -1 \) and the suitable parameters for the process \( X^i \), while \( \alpha(\cdot) \) and \( \beta(\cdot) \) are given by the same equation, with \( q = -c_i \) and the suitable parameters for the process \( Y \).

2.4.3 CDS Pricing

Let us describe the valuation of a CDS for the \( i^{th} \) firm. We suppose that the CDS has maturity \( \bar{T} \), notional of one dollar, payment dates \( \bar{T}_1, \ldots, \bar{T}_K = \bar{T} \), a constant recovery rate of \( \delta_r \), and a spread of \( \bar{S}_i \). We make the assumption that losses on average occur in the middle of the intervals. Then, the value of the premium leg, based on
the fixed premium payments by the protection buyer, is given by

\[ \text{Prem}(0, \bar{T}; \bar{S}_i) = \mathbb{E}^* \left[ \sum_{k=1}^{K} \left( e^{-r \bar{T}_k} \mathbf{1}_{\{\tau_i > \bar{T}_k\}} \bar{S}_i (T_k - T_{k-1}) + e^{-r \tau_i} \mathbf{1}_{\{\bar{T}_{k-1} < \tau_i \leq \bar{T}_k\}} \bar{S}_i (\tau_i - T_{k-1}) \right) \right] \]

\[ = \bar{S}_i \sum_{k=1}^{K} \left( (\bar{T}_k - \bar{T}_{k-1}) e^{-r \bar{T}_k} \mathbb{P}^* \{ \tau_i > \bar{T}_k \} \right. \]

\[ + \frac{\bar{T}_k - \bar{T}_{k-1}}{2} e^{-r(T_k + T_{k-1})/2} \left[ \mathbb{P}^* \{ \tau_i > \bar{T}_{k-1} \} - \mathbb{P}^* \{ \tau_i > \bar{T}_k \} \right] \right), \tag{2.19} \]

where \( \mathbb{P}^* \{ \tau_i > t \} \) is the survival probability \((2.18)\) for the \(i\)th firm. The value of the protection leg, based on the conditional loss-given-default payment by the protection seller, is given by

\[ \text{Prot}(0, \bar{T}) = \mathbb{E}^* \left[ e^{-r \tau_i} \mathbf{1}_{\{\tau_i \leq \bar{T}\}} (1 - \delta_r) \right] \]

\[ = (1 - \delta_r) \sum_{k=1}^{K} e^{-r(T_k + T_{k-1})/2} \left[ \mathbb{P}^* \{ \tau_i > \bar{T}_{k-1} \} - \mathbb{P}^* \{ \tau_i > \bar{T}_k \} \right]. \tag{2.20} \]

The CDS spread is then given by the value \( \bar{S}_i \) that equates the premium and protection legs, that is, \( \bar{S}_i \) satisfies

\[ \text{Prem}(0, \bar{T}; \bar{S}_i) = \text{Prot}(0, \bar{T}), \tag{2.21} \]

and is hence given by the closed-form expression

\[ \bar{S}_i = (1 - \delta_r) \sum_{k=1}^{K} e^{-r(T_k + T_{k-1})/2} \left[ \mathbb{P}^* \{ \tau_i > \bar{T}_{k-1} \} - \mathbb{P}^* \{ \tau_i > \bar{T}_k \} \right] / \]

\[ \sum_{k=1}^{K} \left( (\bar{T}_k - \bar{T}_{k-1}) e^{-r \bar{T}_k} \mathbb{P}^* \{ \tau_i > \bar{T}_k \} + \frac{\bar{T}_k - \bar{T}_{k-1}}{2} e^{-r(T_k + T_{k-1})/2} \left[ \mathbb{P}^* \{ \tau_i > \bar{T}_{k-1} \} - \mathbb{P}^* \{ \tau_i > \bar{T}_k \} \right] \right). \tag{2.22} \]
2.5 Pricing of Equity Index Options

Recall that from the SDEs (2.3) and (2.4), the stock index \((S_t)\) and the market factor \((Y_t)\) satisfy the respective SDEs

\[
\begin{align*}
\text{d}S_t &= (r - q)S_t \text{d}t + \sqrt{b_Y Y_t} S_t \text{d}W^S_t, \\
\text{d}Y_t &= \kappa_Y (\bar{y} - Y_t) \text{d}t + \sigma_Y \sqrt{Y_t} \text{d}W^Y_t + \text{d}J^Y_t,
\end{align*}
\]

where \(W^S\) and \(W^Y\) are correlated Brownian motions with \(\mathbb{E}^*[\text{d}W^S_t \cdot \text{d}W^Y_t] = \rho \text{d}t\), and \(J^Y\) is a compound Poisson process with jump intensity \(l_Y\) and exponentially distributed jump sizes with mean \(\xi_Y\). Noting that \(Z_t := b_Y Y_t\), \(t \geq 0\), defines the variance process for \((S_t)\), we have the stochastic volatility with jumps in volatility (SVJ-V) model\(^9\) that is,

\[
\begin{align*}
\text{d}S_t &= (r - q)S_t \text{d}t + \sqrt{Z_t} S_t \text{d}W^S_t, \\
\text{d}Z_t &= \kappa_Z (\bar{z} - Z_t) \text{d}t + \sigma_Z \sqrt{Z_t} \text{d}W^Z_t + \text{d}J^Z_t, \tag{2.23}
\end{align*}
\]

where \(W^Z = W^Y\), \(J^Z\) is a compound Poisson process with intensity \(l_Z\) and exponentially distributed jump sizes with mean \(\xi_Z\), and the parameters for \(Z\) are given by \(\kappa_Z = \kappa_Y\), \(\sigma_Z = \sqrt{b_Y \sigma_Y}\), \(\bar{z} = b_Y \bar{y}\), \(z_0 = b_Y y_0\), \(l_Z = l_Y\), and \(\xi_Z = b_Y \xi_Y\). Hence, the variance parameters in the model are given by

\[
\chi = \{\kappa_Z, \sigma_Z, \bar{z}, z_0, l_Z, \xi_Z, \rho\}. \tag{2.24}
\]

The following proposition gives the price of a call option on the stock index under the SVJ-V model; we note that put option prices can be computed simply by the put-call parity relation, \(P = C - S_0 e^{-qT} + K e^{-rT}\).

\(^9\)See Section 4 of Duffie, Pan, and Singleton (2000) for further analysis on the SVJ-V model.
Proposition 2.5.1. Let \((S_t, Z_t)\) be the stock index process and variance process satisfying \((2.23)\), and let \(\chi\) be the set of variance parameters \((2.24)\). Denote \(\psi^\chi\) as the discounted transform of the log-state price \(s_t = \log S_t\):

\[
\psi^\chi(u, (s, z), t, T) = E^{t, \chi} \left[ e^{-r(T-t)}e^{us_T} \mid s_t = s, Z_t = z \right].
\]

Then, the price \(C\) at time 0 of a European call option with strike \(K\) and maturity \(T\) can be written as

\[
C(0, K, T, (\bar{S}_0, z_0), \chi) = E^{t, \chi} \left[ e^{-rT}(S_T - K)^+ \mid S_0 = \bar{S}_0, Z_0 = z_0 \right]
= G_{1,-1}(-\log K; (\log \bar{S}_0, z_0), T, \chi) - KG_{0,-1}(\log \bar{S}_0, T, \chi),
\]

where \(G_{a,b}\), for \(a, b \in \mathbb{R}\), is given by

\[
G_{a,b}(w; (s, z), T, \chi) = \frac{\psi^\chi(a, (s, z), 0, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im} \left( \psi^\chi(a + i\xi b, (s, z), 0, T)e^{-i\xi w} \right)}{\xi} d\xi,
\]

with \(\text{Im}(c)\) denoting the imaginary part of \(c \in \mathbb{C}\). In addition, \(\psi^\chi\) simplifies to

\[
\psi^\chi(u, (s, z), t, T) = \exp \left( \bar{\alpha}(T - t, u) + us + \bar{\beta}(T - t, u)z, \right),
\]

where, letting \(\bar{b} = \sigma_Z \rho_u - \kappa_Z\), \(\bar{a} = u(1 - u)\), and \(\gamma = \sqrt{\bar{b}^2 + \bar{a} \sigma_Z^2}\) (where \(\bar{b}, \bar{a}, \gamma\) all depend on \(u\), we have

\[
\bar{\beta}(\tau, u) = -\frac{\bar{a}(1 - e^{-\gamma \tau})}{2\gamma - (\gamma + \bar{b})(1 - e^{-\gamma \tau})}
\]

and

\[
\bar{\alpha}(\tau, u) = \alpha_0(\tau, u) - l_Z \tau + l_Z \int_0^\tau \theta(u, \bar{\beta}(s, u)) ds.
\]
Here, \( \alpha_0 \) is given by

\[
\alpha_0(\tau, u) = -r\tau + (r - q)u\tau - \kappa Z \bar{z} \left( \frac{\gamma + \bar{b}}{\sigma_Z^2} \tau + \frac{2}{\sigma_Z^2} \log \left[ 1 - \frac{\gamma + \bar{b}}{2\gamma} (1 - e^{-\gamma \tau}) \right] \right)
\]

and \( \theta \) is the transform of the jump-size distribution \( \nu \):

\[
\theta(c_1, c_2) = \int_{\mathbb{R}^2} e^{c_1 z_1 + c_2 z_2} d\nu(z_1, z_2), \quad c_1, c_2 \in \mathbb{C}.
\]

Proof. Here is an outline of the proof. First, the form of \( C \) in (2.25) arises from the payoff of a standard call option; see, for example, Heston (1993). Then, the expression of \( G_{a,b} \) arises from Fourier inversion as stated in Proposition 2 (and proven in Appendix A) of Duffie, Pan, and Singleton (2000). Finally, the form of \( \psi^\chi \) is provided in Section 4 of Duffie, Pan, and Singleton (2000).

Remark 2.5.1. In the SVJ-V model, the first component (i.e., the stock process) does not have jumps, while the second component (i.e., the variance process) has an exponential jump distribution with mean \( \xi_Z \). Hence, the jump-size distribution \( \nu \) is identified by

\[
d\nu(z_1, z_2) = \delta_0(dz_1)1_{(0,\infty)}(z_2) \frac{1}{\xi_Z} e^{-\frac{z_2}{\xi_Z}} dz_2, \quad z_1, z_2 \in \mathbb{R},
\]

and the transform of \( \nu \) is given by

\[
\theta(c_1, c_2) = \int_{\mathbb{R}^2} e^{c_1 z_1 + c_2 z_2} \frac{1}{\xi_Z} e^{-\frac{z_2}{\xi_Z}} dz_2 = \frac{1}{1 - c_2 \xi_Z}, \quad \text{when } \text{Re}(c_2) < \frac{1}{\xi_Z}.
\]

Integrating \( \theta(u, \bar{\beta}(s, u)) \) with respect to \( s \), we obtain the integral in the expression for \( \bar{\alpha} \).
above:

\[
\int_0^\tau \theta(u, \bar{\beta}(s, u)) ds = \int_0^\tau \frac{1}{1 - \beta(s, u) \xi Z} ds = \int_0^\tau \frac{2\gamma - (\gamma + \bar{b})(1 - e^{-\gamma s})}{2\gamma + [\xi Z \bar{a} - (\gamma + b)](1 - e^{-\gamma s})} ds
\]

\[
= \frac{\gamma - \bar{b}}{\gamma - \bar{b} + \xi Z \bar{a}} \tau - \frac{2\xi Z \bar{a}}{\gamma^2 - (b - \xi Z \bar{a})^2} \log \left(1 - \frac{(\gamma + b) - \xi Z \bar{a}}{2\gamma}(1 - e^{-\gamma \tau})\right).
\]

2.6 Market Data and Conventions

In this section, we discuss the source of the market data, the CDX Series information, and the CDS conventions and adjustments that will be used in our empirical testing of the model.

2.6.1 Data

Our primary data are equity index options, CDS spreads, and CDO tranche spreads. In particular, we consider European put options on the S&P 500 index (SPX), CDS spreads based on the underlying firms of the CDX North American Investment Grade Index (CDX.NA.IG)\textsuperscript{10} and CDO tranche spreads written on the CDX.NA.IG. We obtained the SPX option data from OptionMetrics via Wharton Research Data Services (WRDS), and we retrieved the CDS and CDO tranche spreads from Bloomberg.

To compare risks across the equity and credit markets, it seems plausible to examine instruments with similar maturities. However, we note that the credit instruments in the market, in general, had longer maturities than the equity index options. In this case, we selected the SPX options with the longest maturities of just over two years, we considered CDSs with maturities of 1 and 5 years, and we examined CDOs with maturities of 5 years.

\textsuperscript{10}Note that the CDX.NA.IG consists of an equally-weighted portfolio of 125 firms with investment grade corporate debt.
2.6.2 CDX Series Information

Table 2.2 below has the general information for the CDX North American Investment Grade (CDX.NA.IG) series which rolls over every 6 months, on March 20 and September 20 of each year, with a possible one- or two-day settlement lag. The table also lists the tranches that pay upfront fees and the particular running spread $R$ in those cases.

Table 2.2: CDX.NA.IG Series Information

<table>
<thead>
<tr>
<th>Series</th>
<th>Effective Date</th>
<th>Maturity Date</th>
<th>Upfront Tranches</th>
<th>$R$ (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Oct. 21, 2003</td>
<td>Dec. 22, 2008</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>Mar. 23, 2004</td>
<td>Jun. 22, 2009</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>Sep. 21, 2004</td>
<td>Dec. 21, 2009</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>4</td>
<td>Mar. 21, 2005</td>
<td>Jun. 21, 2010</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>5</td>
<td>Sep. 21, 2005</td>
<td>Dec. 20, 2010</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>6</td>
<td>Mar. 21, 2006</td>
<td>Jun. 20, 2011</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>7</td>
<td>Sep. 21, 2006</td>
<td>Dec. 20, 2011</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>8</td>
<td>Mar. 21, 2007</td>
<td>Jun. 20, 2012</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>9</td>
<td>Sep. 21, 2007</td>
<td>Dec. 20, 2012</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>10</td>
<td>Mar. 21, 2008</td>
<td>Jun. 20, 2013</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>11</td>
<td>Sep. 21, 2008</td>
<td>Dec. 20, 2013</td>
<td>Equity</td>
<td>500</td>
</tr>
<tr>
<td>12</td>
<td>Mar. 20, 2009</td>
<td>Jun. 20, 2014</td>
<td>All</td>
<td>100</td>
</tr>
<tr>
<td>13</td>
<td>Sep. 21, 2009</td>
<td>Dec. 22, 2014</td>
<td>All</td>
<td>100</td>
</tr>
<tr>
<td>14</td>
<td>Mar. 22, 2010</td>
<td>Jun. 22, 2015</td>
<td>All</td>
<td>100</td>
</tr>
<tr>
<td>15</td>
<td>Sep. 20, 2010</td>
<td>Dec. 21, 2015</td>
<td>All</td>
<td>500, 100, 25</td>
</tr>
<tr>
<td>16</td>
<td>Mar. 21, 2011</td>
<td>Jun. 20, 2016</td>
<td>All</td>
<td>500, 100, 25</td>
</tr>
<tr>
<td>17</td>
<td>Sep. 20, 2011</td>
<td>Dec. 20, 2016</td>
<td>All</td>
<td>500, 100, 25</td>
</tr>
</tbody>
</table>

We observe that for Series 1 to 11, that is, up until early 2009, only the equity tranche (0-3%) paid an upfront fee with a running spread of $R = 500$ bps, while the other tranches had no upfront fee\(^\text{11}\). For Series 12 to 14, coinciding with the tail end of the credit crisis, we note that all of the tranches paid upfront fees but had a

\(^{11}\text{We note, however, that these conventions changed for }\text{off-the-run}\text{ (i.e., not current) series beginning in mid-2009. In particular, as of June 20, 2009, for Series 11 and earlier, the equity (0-3%), mezzanine 1 (3-7%), and mezzanine 2 (7-10%) tranche spreads are now quoted as upfront fees with 500-bp coupons, while the mezzanine 3 (10-15%), senior (15-30%), and super senior (30-100%) tranche spreads in the market are quoted as upfront fees with 100-bp coupons. For reference, see http://www.iiderivatives.com/pdf/DW052509.pdf.}\)
lower running spread of $R = 100$ bps. Lastly, we note that the tranche structure was altered for Series 15, which rolled in September 2010, and for all subsequent series.\textsuperscript{12} In particular, for these newer series, the 4 tranches are 0-3%, 3-7%, 7-15%, and 15-100%, and the market spreads are quoted as upfront fees with respective running spreads of 500 bps, 100 bps, 100 bps, and 25 bps.

In our calibration, we ignore the super senior tranche\textsuperscript{13} of the CDO as it is very illiquid and difficult to fit to an intensity model. Note that Mortensen (2006), Paggeorgiou and Sircar (2009), Collin-Dufresne et al. (2010), and others also do not consider the super senior tranche. In particular, Collin-Dufresne et al. (2010) argue that calibrating market dynamics to match the super senior tranche impacts equity state prices significantly only for moneyness levels around 0.2 or lower (corresponding to very illiquid products); this indicates that one cannot ‘extrapolate’ the information in option prices to deduce information regarding the super senior tranche.

2.6.3 CDS Conventions and Adjustments

The maturity of the credit default swaps fall on the International Monetary Market (IMM) Dates, in particular, March 20, June 20, September 20 and December 20 of each calendar year. Thus, for example, a ‘five-year’ contract traded any time between September 21, 2005 and December 20, 2005 would have termination date of December 20, 2010, implying that the actual maturity would be more than 5 years.

The CDX spread is the coupon for the CDS index and is equivalent to the running spread for the [0%, 100%] tranche of the CDX. To avoid a theoretical arbitrage, the CDX spread should equal the average of the CDS spreads, but in practice, this is not the case; the difference is termed the CDS-CDX basis. As the markets went haywire after September 2008, the CDX.NA.IG spread collapsed, for various reasons, such as:

\textsuperscript{12}See http://www.markit.com/cds/documentation/resource/Credit_Indices_Primer_October%202010.pdf.
\textsuperscript{13}The super senior tranche corresponds to the 30-100% tranche for the older series and the 15-100% tranche for the newer series since Series 15.
• Dispersion between specific single names and IG index

• Abnormal unwind costs and jump to default risk

• Upfront cash payment on index due to low index strike

• Illiquidity in certain single names

• Mod-R vs Ex-R restructuring language

• Single name technicals.

Table 2.3 below shows the CDX spreads compared with the average CDS spreads from the market for 15 selected dates from 2004 to 2010 that we will use for our testing. We observe that for all of the dates since March 2008, the average CDS spreads are indeed larger than the CDX spreads observed in the market.

<table>
<thead>
<tr>
<th>Date</th>
<th>Series</th>
<th>CDX</th>
<th>Avg. CDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>Series 2</td>
<td>70</td>
<td>69.44</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>Series 5</td>
<td>49</td>
<td>45.94</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>Series 7</td>
<td>34</td>
<td>36.75</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>Series 8</td>
<td>45</td>
<td>41.10</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>Series 9</td>
<td>182</td>
<td>191.71</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>Series 10</td>
<td>115</td>
<td>157.07</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>Series 11</td>
<td>173</td>
<td>279.03</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>Series 9</td>
<td>220</td>
<td>306.66</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>Series 9</td>
<td>258</td>
<td>353.28</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>Series 12</td>
<td>139</td>
<td>173.65</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>Series 12</td>
<td>121</td>
<td>160.15</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>Series 13</td>
<td>98</td>
<td>104.33</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>Series 13</td>
<td>89</td>
<td>92.28</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>Series 9</td>
<td>141</td>
<td>168.32</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>Series 9</td>
<td>118</td>
<td>140.56</td>
</tr>
</tbody>
</table>

To correct this basis on each date, we first scale all of the 5-year CDS spreads in the market by a constant factor so that the average of the new 5-year spreads matches
the 5-year CDX spread. Then, we multiply the 1-year CDS spreads in the market by the same factor (i.e., the ratio of the new 5-year CDS spreads to the old 5-year CDS spreads) in order to keep the ratios between the 1-year and 5-year CDS spreads unchanged. In particular, we set, for $i = 1, \ldots, \tilde{N}$,

$$\bar{S}_{new}(5) = \frac{\bar{S}_{CDX}(5)}{\bar{S}_{market}(5)} \cdot \bar{S}_{new}(5) \cdot \bar{S}_{CDX}(5),$$

$$\bar{S}_{new}(1) = \frac{\bar{S}_{market}(1)}{\bar{S}_{market}(5)} \cdot \frac{\bar{S}_{new}(5)}{\bar{S}_{market}(5)} \cdot \frac{\bar{S}_{CDX}(5)}{\bar{S}_{market}(5)}.$$  (2.26)

We note that these adjustments are similar to those done by Feldhütter (2008), who points out the maturity mismatch between the CDO and the underlying CDS contracts and then adjusts all the CDS spreads for a particular maturity by a constant factor such that the average CDS spread matches the CDX level reported by MarkIt.\footnote{On the other hand, O’Kane (2011) discusses three alternative methods to fix the CDS-CDX basis: Spread Adjustment, Default Rate Multiplier, and Survival Probability Exponentiation.}

2.7 Calibration

In this section, we describe two calibration methods, via the “forward” and “backward” problems, that allow us to analyze the systematic risks inherent in the credit and equity markets. First, in Section 2.7.1 we consider the forward problem, in which we start from equity index options, fit the CDS spreads and then compare the CDO tranche spreads from the model and the market.\footnote{We denote this problem the “forward” problem because we are starting from the equity market, which is more established than the credit market.} Then, in Section 2.7.2 we consider the backward problem, in which we fit the model parameters to the CDO tranche spreads and CDS spreads and then compare the equity index option volatilities from the model and the market. Subsequently, in Section 2.8 we present relevant numerical results for the calibration procedures in both the forward and backward directions.
2.7.1 Forward Problem: From Equity Index Options to CDO Tranche Spreads

In this section, we consider the forward problem, going from equity index options to CDO tranche spreads. In particular, we first calibrate the market factor parameters to the S&P 500 option data. Then, using the market parameters as input, we calibrate the idiosyncratic parameters to the CDS spread data. Next, using the calibrated market and idiosyncratic parameters, we compute the CDO tranche spreads from the intensity-based model, and we compare the model spreads to the market-observed spreads. This idea is illustrated in Figure 2.1 below.

![Diagram](image)

Figure 2.1: Forward Problem: From Equity Index Options to CDO Tranche Spreads
Calibration of Market Factor

We wish to calibrate the market factor parameters,

\[ \chi_Y = \{ \kappa_Y, \sigma_Y, \bar{y}, y_0, l_Y, \xi_Y, \rho, b_Y \}, \]  

(2.27)
to the implied volatilities of S&P 500 options, as motivated by Coval, Jurek, and Stafford (2009a).

From Proposition 2.5.1 in Section 2.5 above, we know the analytical call price \( C \) under the SVJ-V model and hence, we can uniquely determine the Black-Scholes implied volatility, \( \sigma_{\text{imp, model}}(\chi_Y) \), for any set of market factor parameters \( \chi_Y \). We also know the implied volatilities of the S&P 500 options from the market, say \( \sigma_{\text{imp, market}}^j, j = 1 \ldots, J \), for some integer \( J \). Then, to calibrate the market factor parameters \( \chi_Y \), we run an optimization scheme to minimize the sum of squared errors (SSE),

\[ \text{SSE} = \sum_{j=1}^{J} (\sigma_{\text{imp, model}}(\chi_Y) - \sigma_{\text{imp, market}}^j)^2. \]  

(2.28)

This can be implemented, for example, by using the \textit{lsqlin} function in MATLAB.

Calibration of Idiosyncratic Components

For \( i = 1, \ldots, \hat{N} \), we wish to calibrate the idiosyncratic parameters,

\[ \eta_i = \{ \kappa_i, \sigma_i, \bar{x}_i, x_0^i, l_i, \xi_i, c_i \}, \]  

(2.29)
to the adjusted market CDS spreads for the \( i^{th} \) firm\(^{16} \). Here is our calibration procedure for the \( i^{th} \) firm, \( i = 1, \ldots, \hat{N} \).

1. We first define \( \bar{w} \in [0, 1] \) as a correlation parameter that represents the systematic share of the intensities, and hence, \( 1 - \bar{w} \) represents the idiosyncratic

---

\(^{16}\)Recall that the market spreads are adjusted according to Equation (2.26) in Section 2.6.3.
share of the intensities. This correlation parameter can be chosen freely; for example, we can set \( \bar{w} \) to be the calibrated value from the backward problem in Section 2.7.2.

2. We make the following parsimonious assumptions for the idiosyncratic parameters:

\[
\kappa_i = \kappa_Y, \quad l_i = l_Y \frac{1 - \bar{w}}{\bar{w}},
\]

\[
c_i = \frac{\bar{w} \bar{x}_i}{(1 - \bar{w}) \bar{y}}, \quad \sigma_i = \sqrt{c_i \sigma_Y}, \quad \xi_i = c_i \xi_Y,
\]

where the market factor parameters \( \kappa_Y, \sigma_Y, \bar{y}, l_Y, \) and \( \xi_Y \) were calibrated above in Section 2.7.1. For the idiosyncratic parameters, we observe that \( \kappa_i \) and \( l_i \) are fixed; \( c_i, \sigma_i, \) and \( \xi_i \) are dependent on \( \bar{x}_i \); and \( \bar{x}_i \) and \( x^i_0 \) are to be explicitly calibrated.

3. We calibrate \( \{\bar{x}_i, x^i_0\} \) to the 1-year and 5-year CDS spreads from the market, while enforcing the above conditions for the idiosyncratic parameters \( \kappa_i, l_i, c_i, \sigma_i, \) and \( \xi_i \).

We now provide more precise details of the calibration. We let \( \bar{S}^{i,j}_{model}(\chi_Y, \eta_i) \) and \( \bar{S}^{i,j}_{market} \) be the respective market and model CDS spreads for maturity \( T_j \), with \( T_1 = 1 \) and \( T_2 = 5 \). Here, the model spread is computed from the closed-form expression (2.22), using the set of market factor parameters \( \chi_Y \) and the set of idiosyncratic parameters \( \eta_i \). Then, for the calibration, we minimize the sum of squared differences between the market and model-implied CDS spreads, that is,

\[
\sum_{j=1}^{2} \left( \bar{S}^{i,j}_{model}(\chi_Y, \eta_i) - \bar{S}^{i,j}_{market} \right)^2.
\]

This optimization can be implemented, for example, by using the \texttt{lsqnonlin} function.
function in MATLAB.

**Market vs. Model CDO Tranche Spreads**

Once we have calibrated the market factor parameters $\chi_Y$ and the idiosyncratic parameters $\eta_i$, for $i = 1, \ldots, \hat{N}$, we price the CDO tranche spreads in our model by following the algorithm in Section 2.3. Then, by comparing the model and market tranche spreads, we analyze the systematic risks in the equity and credit markets. In particular, since the model is calibrated to equity index options, we observe that if the senior spreads in the market are larger (smaller) than those from the model, then this indicates that the systematic risk in the credit market is greater (smaller) than that in the equity market.

**2.7.2 Backward Problem: From CDO Tranche Spreads to Equity Index Options**

In this section, we analyze the systematic risks inherent in the credit and equity markets by considering the backward problem, going from CDO tranche spreads to equity index options. In particular, we first calibrate the market factor and idiosyncratic parameters to both the CDO tranche spreads and the CDS spreads from the market. Then, using the calibrated parameters, we compute the equity index option prices from the SVJ-V model and obtain the corresponding Black-Scholes implied volatilities. Finally, we compare the model implied volatilities to the market implied volatilities, and by examining the volatility skews, we measure the systematic tail risks inherent in the credit and equity markets. This idea is illustrated in Figure 2.2 below.
Fitting Parameters to CDO Tranches and CDS Spreads

We wish to fit the market factor parameters $\chi_Y$ in (2.27) and the idiosyncratic parameters $\eta_i$ in (2.29), for $i = 1, \ldots, \hat{N}$, to both the CDO tranche spreads and CDS spreads. Note that the market factor parameters $\rho$ and $b_Y$ do not impact the pricing of the credit derivatives and hence are free to be chosen later on. Let us explain how we calibrate the rest of the market factor parameters,

$$\eta_Y := \{\kappa_Y, \sigma_Y, \bar{y}, y_0, l_Y, \xi_Y\},$$

along with the idiosyncratic parameters

$$\eta_i = \{\kappa_i, \sigma_i, \bar{x}_i, x_{0i}, l_i, \xi_i, c_i\}.$$
Due to the large number of parameters \((6 + 7\hat{N})\) that need to be calibrated, we use a parsimonious version of the model that is both practical and flexible. Here, we follow Section 4.2 of [Mortensen (2006)] and we minimize the deviations from the market CDO tranche spreads while matching the individual credit curves.

We recall from (2.1) that the intensity is given by \(\lambda_t^i = X_t^i + c_i Y_t, \ t \geq 0\). To achieve similar correlation across the firms, we let \(c_i\) be a spread level parameter, where large (small) \(c_i\) represents a firm with high (low) credit spreads, and \(c_i = 1\) represents an average firm. We now make a few simplifying assumptions for the model.

First, we assume
\[
\kappa_i = \kappa_Y, \ \sigma_i = \sqrt{c_i \sigma_Y}, \ \xi_i = c_i \xi_Y, \quad (2.30)
\]
and we rename the parameters
\[
\bar{\kappa} = \kappa_Y, \ \bar{\sigma} = \sigma_Y, \ \bar{\xi} = \xi_Y. \quad (2.31)
\]

Secondly, we assume that all names have the same aggregate jump intensity
\[
\bar{l} = l_Y + l_i. \quad (2.32)
\]

Third, as in [Duffie and Gârleanu (2001)], we assume that the systematic parts of the mean-reversion level and of the jump intensity are identical and given by the same fraction for all underlying names,
\[
\bar{w} := \frac{c_i \bar{y}}{c_i \bar{y} + \bar{x}_i} = \frac{l_Y}{l_Y + l_i} \in [0, 1]. \quad (2.33)
\]

Thereby, \(\bar{w}\) is a correlation parameter representing the systematic share of the intensities: a low \(\bar{w}\) implies that most of the intensity is idiosyncratic, whereas a high \(\bar{w}\)

---

\(^{18}\)See [Eckner (2009)], Section 4.4, for a more intricate calibration scheme that minimizes a weighted average of the RMSEs for the CDS spreads, CDX spreads, and CDO tranche spreads.
implies a high systematic share. In addition, we ensure that the initial value of an average (hypothetical) name satisfies the same division.[19]

\[ \bar{w} = \frac{y_0}{y_0 + x_0^{avg}}. \] (2.34)

The remaining free parameters are \((\bar{\kappa}, \bar{\sigma}, \bar{l}, \bar{\xi}, \bar{w})\). We will select them in a way to minimize the CDO tranche price deviations while maintaining exact fit to the single-name CDS spreads. In particular, for each set of the parameters, the procedure is the following:

1. For a hypothetical average firm \((c_i = 1)\), whose intensity \(\bar{\lambda}_i = \bar{Y}_t + \bar{X}_t\) has parameters

\[ \{ \bar{\kappa}, \bar{\sigma}, \bar{\bar{y}} + \bar{x}_{avg}, y_0 + x_0^{avg}, \bar{l}, \bar{\xi} \}, \]

we calibrate the mean-reversion level \(\bar{\bar{y}} + \bar{x}_{avg}\) and the initial level \(y_0 + x_0^{avg}\) to fit the 1-year and 5-year spreads of the (adjusted) average CDS curve. Then, using the assumptions \((2.31)-(2.34)\), we can specify all of the market factor parameters \(\eta_Y\), as follows:

\[ \kappa_Y = \bar{\kappa}, \sigma_Y = \bar{\sigma}, \bar{\bar{y}} = \bar{w}(\bar{\bar{y}} + \bar{x}_{avg}), y_0 = \bar{w}(y_0 + x_0^{avg}), l_Y = \bar{w}\bar{l}, \xi_Y = \bar{\xi}. \] (2.35)

2. For each underlying firm \(i\), whose intensity \(\lambda_i^i = c_iY_t + X_t^i\) has parameters

\[ \{ \bar{\kappa}, \sqrt{c_i\bar{\sigma}}, c_i\bar{\bar{y}} + \bar{x}_i, c_iy_0 + x_0^i, \bar{l}, c_i\bar{\xi} \}, \]

we calibrate \((\bar{x}_i, x_0^i)\) to fit the firm-specific 1-year and 5-year CDS spreads. In doing so, the scaling factor \(c_i\) is always chosen to ensure that the systematic

[19] The same division cannot be ensured for all individual initial values since the CDS slopes differ, but in practice this will almost be satisfied if the CDS shapes only differ slightly.
part of the mean-reversion level remains equal to $\bar{w}$, via (2.33):

$$c_i = \frac{\bar{w} \bar{x}_i}{(1 - \bar{w}) \bar{y}}.$$

Using the assumptions (2.30)-(2.33) above, we can specify the rest of the idiosyncratic parameters $\eta_i$ (in addition to $\bar{x}_i, x_i^0, c_i$), as follows:

$$\kappa_i = \bar{\kappa}, \quad \sigma_i = \sqrt{c_i \bar{\sigma}}, \quad l_i = (1 - \bar{w}) \bar{l}, \quad \xi_i = c_i \bar{\xi}.$$

3. Using the calibrated parameters from Steps 1 and 2 above, we compute the model CDO tranche spreads via the algorithm in Section 2.3, and then we compare the model spreads to the market spreads. In particular, we denote by $R_{k,\text{model}}$ the model CDO tranche spread for tranche $k$, $k = 1, \ldots, 5$, and we denote by $R_{k,\text{market mid}}, R_{k,\text{market bid}}, R_{k,\text{market ask}}$ the corresponding mid, bid, and ask tranche spreads from the market. Then, the criteria function is given by the root-mean square price errors (RMSE) relative to bid/ask CDO tranche spreads,

$$\text{RMSE} = \sqrt{\frac{1}{5} \sum_{k=1}^{5} \left( \frac{R_{k,\text{model}} - R_{k,\text{market mid}}}{R_{k,\text{market ask}} - R_{k,\text{market bid}}} \right)^2}. \quad (2.36)$$

Since liquidity varies across tranches, deviations from the most reliable prices have the highest weights.

Now, we repeat Steps 1 to 3 above until the RMSE in (2.36) is less than a pre-specified tolerance level $\epsilon$. Note that at the end of this calibration, we have obtained the model parameters $\eta_Y$ and $\eta_i$, $i = 1, \ldots, \hat{N}$.

**Market vs. Model Implied Volatilities**

Finally, once we have calibrated the model parameters $\eta_Y$ and $\eta_i$, $i = 1, \ldots, \hat{N}$, to the CDO tranche spreads and CDS spreads from the market, we can compare the
Black-Scholes implied volatility skews from the market to those from the fitted model.

Let us now describe our procedure. For the market, we select put options of the longest maturity, that is, approximately two years, for the 15 selected dates shown in Table 2.3 in Section 2.6.3. Meanwhile, for the model, we use the fitted parameters from above and we choose a range of values for the free parameters \( b_Y \) and \( \rho \) from (2.27). We note that under this SVJ-V model\(^{20}\), these parameters have the following impact on the Black-Scholes implied volatility curve:

- \( b_Y \) determines the level of the curve, and
- \( \rho \) determines the skew/smile of the curve.

Hence, we select \( b_Y \) to match the level of the curve, and then we vary \( \rho \) to try to match the skew. In this backward problem, we observe that if the model skew is greater (less) than the market skew for a wide range of \( \rho \), then this indicates that the risk in the credit market is greater (less) than that in the equity market.

### 2.8 Numerical Results

Here we present the numerical results for both the forward problem, where we start with the equity index options and then compute the CDO tranche spreads, and the backward problem, where we start from the CDO tranche data and then determine the behavior of the implied volatilities from equity index options.

Throughout this section, we use the following set of parameters for the CDO:

\[
\hat{N} = 125; \quad r = 0.04; \quad \delta_r = 0.35; \quad T = 5; \quad \Delta \tau = T_k - T_{k-1} = 0.25, \quad k = 1, \ldots, K;
\]

and for the CDS, we suppose that \( \bar{T} \in \{1, 5\} \) and \( \bar{T}_k - \bar{T}_{k-1} = 0.25, \quad k = 1, \ldots, \bar{K} \).

\(^{20}\)Recall the specification of the SVJ-V model in Section 2.5.
2.8.1 Forward Problem

In this section, we present the numerical results for the forward problem, starting from the equity index options, calibrating to the CDS spreads, and comparing the values for the CDO tranche spreads.

Market Factor Parameters and Implied Volatility

Table 2.4 shows the calibrated parameters $\chi_Y = \{\kappa_Y, \sigma_Y, \bar{y}, y_0, l_Y, \xi_Y, \rho, b_Y\}$ for the market factor process $Y$, as obtained from the procedure in Section 2.7.1, for a set of 15 dates ranging from Aug. 2004 to Sep. 2010. In addition, the table displays the sum of squared errors (SSE) as defined in Equation (2.28).

Table 2.4: Calibrated Market Factor Parameters $\chi_Y$ and SSE

<table>
<thead>
<tr>
<th>Date</th>
<th>$\kappa_Y$</th>
<th>$\sigma_Y$</th>
<th>$\bar{y}$</th>
<th>$y_0$</th>
<th>$l_Y$</th>
<th>$\xi_Y$</th>
<th>$\rho$</th>
<th>$b_Y$</th>
<th>SSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>0.0884</td>
<td>0.0676</td>
<td>0.0331</td>
<td>0.0008</td>
<td>0.0465</td>
<td>0.0372</td>
<td>-0.5439</td>
<td>12.4988</td>
<td>1.22E-04</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>0.0030</td>
<td>0.0369</td>
<td>0.1471</td>
<td>0.0005</td>
<td>0.0057</td>
<td>0.0822</td>
<td>-0.9004</td>
<td>23.4574</td>
<td>5.67E-06</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>0.1472</td>
<td>0.1020</td>
<td>0.0294</td>
<td>0.0007</td>
<td>0.0121</td>
<td>0.0308</td>
<td>-0.8855</td>
<td>3.8749</td>
<td>6.53E-05</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>0.1904</td>
<td>0.0620</td>
<td>0.0100</td>
<td>0.0008</td>
<td>0.0093</td>
<td>0.0001</td>
<td>-0.9026</td>
<td>10.9793</td>
<td>1.92E-04</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>0.1155</td>
<td>0.1163</td>
<td>0.0695</td>
<td>0.0018</td>
<td>0.0093</td>
<td>0.0421</td>
<td>-0.2977</td>
<td>9.9442</td>
<td>6.72E-05</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>0.0669</td>
<td>0.1008</td>
<td>0.0703</td>
<td>0.0072</td>
<td>0.0002</td>
<td>0.0121</td>
<td>-0.8772</td>
<td>5.4990</td>
<td>7.19E-05</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>0.8477</td>
<td>0.2760</td>
<td>0.0237</td>
<td>0.0133</td>
<td>0.0202</td>
<td>0.0308</td>
<td>-0.3856</td>
<td>7.9995</td>
<td>1.17E-04</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>0.9869</td>
<td>0.3729</td>
<td>0.0362</td>
<td>0.0002</td>
<td>0.0943</td>
<td>0.0943</td>
<td>-0.3259</td>
<td>7.3869</td>
<td>2.10E-05</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>1.6089</td>
<td>0.6307</td>
<td>0.0518</td>
<td>0.0002</td>
<td>0.0431</td>
<td>0.0698</td>
<td>-0.1981</td>
<td>4.9165</td>
<td>2.40E-05</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>0.0840</td>
<td>0.2692</td>
<td>0.0325</td>
<td>0.0001</td>
<td>0.0371</td>
<td>0.0254</td>
<td>-0.2553</td>
<td>8.6026</td>
<td>1.37E-04</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>0.4975</td>
<td>0.3339</td>
<td>0.0528</td>
<td>0.0024</td>
<td>0.0310</td>
<td>0.0814</td>
<td>-0.1604</td>
<td>5.8887</td>
<td>1.44E-05</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>0.6566</td>
<td>0.2371</td>
<td>0.0167</td>
<td>0.0024</td>
<td>0.0489</td>
<td>0.0122</td>
<td>-0.0642</td>
<td>14.1382</td>
<td>4.56E-05</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>0.7500</td>
<td>0.3100</td>
<td>0.0182</td>
<td>0.0002</td>
<td>0.0070</td>
<td>0.0380</td>
<td>-0.1000</td>
<td>9.0100</td>
<td>9.72E-06</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>0.0726</td>
<td>0.2249</td>
<td>0.1696</td>
<td>0.0060</td>
<td>0.0129</td>
<td>0.0677</td>
<td>-0.3454</td>
<td>8.1916</td>
<td>3.38E-05</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>0.0869</td>
<td>0.2721</td>
<td>0.1931</td>
<td>0.0031</td>
<td>0.0101</td>
<td>0.0215</td>
<td>-0.3185</td>
<td>7.8973</td>
<td>2.92E-05</td>
</tr>
</tbody>
</table>

Figures 2.3 to 2.5 display the market implied volatility and the model calibrated volatility for the 15 dates above, where for each date, we considered the options with maturity closest to two years.
Figure 2.3: Implied Volatility, Aug. 23, 2004 to Jul. 17, 2007
Figure 2.4: Implied Volatility, Mar. 14, 2008 to Jul. 8, 2009

- Implied volatilities for $T = 2.7844$ on 2008-03-14
- Implied volatilities for $T = 2.4959$ on 2008-06-20
- Implied volatilities for $T = 2.1726$ on 2008-10-16
- Implied volatilities for $T = 2.9397$ on 2009-01-08
- Implied volatilities for $T = 2.6932$ on 2009-04-08
- Implied volatilities for $T = 2.4438$ on 2009-07-08
Figure 2.5: Implied Volatility, Sep. 8, 2009 to Sep. 8, 2010

- Implied volatilities for $T = 2.274$ on 2009-09-08
- Implied volatilities for $T = 2.0247$ on 2009-12-08

- Implied volatilities for $T = 2.7945$ on 2010-03-08
- Implied volatilities for $T = 2.5425$ on 2010-06-08

- Implied volatilities for $T = 2.2904$ on 2010-09-08
CDS Spreads

Once we have calibrated the market factor parameters, we then calibrate the idiosyncratic parameters $\eta_i, i = 1, \ldots, \hat{N}$, to the (adjusted) market CDS spreads with maturities of 1 and 5 years, following the procedure described in Section 2.7.1. Table 2.5 displays the average of the 1-year and 5-year spreads from the market and the calibrated model. We generally obtained a good fit for the CDS spreads except for the 1-year spreads on Oct. 31, 2006 and Jul. 17, 2007, but in those cases, the absolute differences were small.

Table 2.5: Forward Problem: Average 1-yr and 5-yr CDS Spreads (bps)

<table>
<thead>
<tr>
<th></th>
<th>1-yr</th>
<th>5-yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>24.62</td>
<td>25.17</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>14.22</td>
<td>15.53</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>8.71</td>
<td>11.62</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>16.93</td>
<td>19.14</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>138.86</td>
<td>135.98</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>87.52</td>
<td>88.57</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>155.52</td>
<td>163.64</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>240.18</td>
<td>240.19</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>295.37</td>
<td>295.54</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>131.39</td>
<td>131.40</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>135.31</td>
<td>129.53</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>59.09</td>
<td>61.13</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>45.75</td>
<td>47.65</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>111.49</td>
<td>112.06</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>104.86</td>
<td>103.52</td>
</tr>
</tbody>
</table>

CDO Tranche Spreads

Table 2.6 compares the 5-year CDO tranche spreads from the market (CDX.NA.IG) and the model for various dates from Aug. 2004 to Sep. 2010. The model spreads are computed from the intensity-based model via the algorithm in Section 2.3 using the calibrated parameters $\chi_Y$ and $\eta_i, i = 1, \ldots, \hat{N}$, above. Here, the relative difference is
defined as

\[
\text{Rel. Diff.} = \frac{\text{Model} - \text{Market}}{|\text{Market}|},
\]

where we note that the absolute value in the denominator is used to indicate that the model spread is greater than the market spread when the relative difference is positive whereas the market spread is greater when the relative difference is negative. In terms of the outputs in the table, the tranche spreads that are quoted in percentages (%) are upfront fees while those quoted in bps (no percentage signs) are running spreads; see Table 2.2 and the subsequent comments in Section 2.6.2 for details on the CDX series.

Table 2.6: 5-year CDO Tranche Spreads (bps)

<table>
<thead>
<tr>
<th>Date (Series)</th>
<th>Source</th>
<th>0-3%</th>
<th>3-7%</th>
<th>7-10%</th>
<th>10-15%</th>
<th>15-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>Market</td>
<td>40.00%</td>
<td>312.50</td>
<td>122.50</td>
<td>42.50</td>
<td>12.50</td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>45.84%</td>
<td>394.38</td>
<td>130.05</td>
<td>59.21</td>
<td>13.17</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>14.59%</td>
<td>26.20%</td>
<td>6.16%</td>
<td>39.31%</td>
<td>5.37%</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>Market</td>
<td>41.10%</td>
<td>117.50</td>
<td>32.90</td>
<td>15.80</td>
<td>7.90</td>
</tr>
<tr>
<td>(Series 5)</td>
<td>Model</td>
<td>44.53%</td>
<td>128.84</td>
<td>31.37</td>
<td>19.63</td>
<td>7.24</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>8.34%</td>
<td>9.65%</td>
<td>-4.66%</td>
<td>24.24%</td>
<td>-8.32%</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>Market</td>
<td>23.66%</td>
<td>88.26</td>
<td>18.75</td>
<td>7.25</td>
<td>3.43</td>
</tr>
<tr>
<td>(Series 7)</td>
<td>Model</td>
<td>23.71%</td>
<td>112.17</td>
<td>14.70</td>
<td>4.60</td>
<td>2.79</td>
</tr>
<tr>
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<td>Rel. Diff.</td>
<td>0.19%</td>
<td>27.09%</td>
<td>-21.61%</td>
<td>-36.58%</td>
<td>-18.53%</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>Market</td>
<td>31.76%</td>
<td>156.36</td>
<td>33.95</td>
<td>17.23</td>
<td>5.59</td>
</tr>
<tr>
<td>(Series 8)</td>
<td>Model</td>
<td>36.48%</td>
<td>151.89</td>
<td>14.70</td>
<td>4.73</td>
<td>4.27</td>
</tr>
<tr>
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<td>Rel. Diff.</td>
<td>14.88%</td>
<td>-2.86%</td>
<td>-61.35%</td>
<td>-72.55%</td>
<td>-23.62%</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>Market</td>
<td>67.92%</td>
<td>836.90</td>
<td>462.22</td>
<td>265.56</td>
<td>129.45</td>
</tr>
<tr>
<td>(Series 9)</td>
<td>Model</td>
<td>88.60%</td>
<td>2,235.78</td>
<td>735.12</td>
<td>186.26</td>
<td>12.59</td>
</tr>
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<td>-29.86%</td>
<td>-90.27%</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>Market</td>
<td>51.55%</td>
<td>447.32</td>
<td>240.75</td>
<td>124.01</td>
<td>66.66</td>
</tr>
<tr>
<td>(Series 10)</td>
<td>Model</td>
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<td>888.82</td>
<td>165.36</td>
<td>25.12</td>
<td>2.48</td>
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<tr>
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<td>48.73%</td>
<td>98.70%</td>
<td>-31.31%</td>
<td>-79.75%</td>
<td>-96.28%</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>Market</td>
<td>71.50%</td>
<td>1,297.00</td>
<td>676.67</td>
<td>209.34</td>
<td>66.50</td>
</tr>
<tr>
<td>(Series 11)</td>
<td>Model</td>
<td>83.52%</td>
<td>1,637.50</td>
<td>611.51</td>
<td>222.54</td>
<td>25.03</td>
</tr>
<tr>
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<td>Rel. Diff.</td>
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<td>26.25%</td>
<td>-9.63%</td>
<td>6.30%</td>
<td>-62.36%</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>Market</td>
<td>74.91%</td>
<td>39.75%</td>
<td>811.03</td>
<td>446.13</td>
<td>111.25</td>
</tr>
<tr>
<td>(Series 9)</td>
<td>Model</td>
<td>91.90%</td>
<td>51.08%</td>
<td>785.96</td>
<td>255.26</td>
<td>24.86</td>
</tr>
<tr>
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<td>-3.09%</td>
<td>-42.78%</td>
<td>-77.66%</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>Market</td>
<td>76.76%</td>
<td>45.80%</td>
<td>15.25%</td>
<td>520.21</td>
<td>137.00</td>
</tr>
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</table>

Continued on next page
Table 2.6 – continued from previous page

<table>
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<tr>
<th>Date (Series)</th>
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<th>0-3%</th>
<th>3-7%</th>
<th>7-10%</th>
<th>10-15%</th>
<th>15-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Series 9)</td>
<td>Model</td>
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<td>58.38%</td>
<td>16.43%</td>
<td>339.31</td>
<td>31.12</td>
</tr>
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<td>27.46%</td>
<td>7.71%</td>
<td>-34.77%</td>
<td>-77.29%</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>Market</td>
<td>64.00%</td>
<td>34.89%</td>
<td>16.73%</td>
<td>6.80%</td>
<td>-0.83%</td>
</tr>
<tr>
<td>(Series 12)</td>
<td>Model</td>
<td>88.61%</td>
<td>44.61%</td>
<td>12.94%</td>
<td>0.84%</td>
<td>-4.05%</td>
</tr>
<tr>
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<td>Rel. Diff.</td>
<td>38.45%</td>
<td>27.85%</td>
<td>-22.63%</td>
<td>-87.68%</td>
<td>-387.63%</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>Market</td>
<td>62.38%</td>
<td>27.31%</td>
<td>10.88%</td>
<td>5.21%</td>
<td>-1.84%</td>
</tr>
<tr>
<td>(Series 12)</td>
<td>Model</td>
<td>82.59%</td>
<td>28.28%</td>
<td>5.12%</td>
<td>-1.59%</td>
<td>-4.10%</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>32.40%</td>
<td>3.56%</td>
<td>-52.92%</td>
<td>-130.51%</td>
<td>-122.62%</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>Market</td>
<td>53.42%</td>
<td>22.54%</td>
<td>8.57%</td>
<td>1.75%</td>
<td>-2.44%</td>
</tr>
<tr>
<td>(Series 13)</td>
<td>Model</td>
<td>72.47%</td>
<td>26.67%</td>
<td>7.67%</td>
<td>0.21%</td>
<td>-3.85%</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>35.67%</td>
<td>18.34%</td>
<td>-10.48%</td>
<td>-87.93%</td>
<td>-57.90%</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>Market</td>
<td>53.81%</td>
<td>19.75%</td>
<td>7.38%</td>
<td>0.88%</td>
<td>-2.60%</td>
</tr>
<tr>
<td>(Series 13)</td>
<td>Model</td>
<td>64.85%</td>
<td>20.74%</td>
<td>5.91%</td>
<td>0.05%</td>
<td>-3.56%</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>20.51%</td>
<td>5.01%</td>
<td>-19.97%</td>
<td>-94.69%</td>
<td>-36.83%</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>Market</td>
<td>52.95%</td>
<td>14.95%</td>
<td>-1.61%</td>
<td>0.81%</td>
<td>-1.48%</td>
</tr>
<tr>
<td>(Series 9)</td>
<td>Model</td>
<td>72.36%</td>
<td>8.62%</td>
<td>-9.14%</td>
<td>-1.92%</td>
<td>-2.29%</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>36.65%</td>
<td>-42.31%</td>
<td>-467.58%</td>
<td>-336.60%</td>
<td>-54.89%</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>Market</td>
<td>47.98%</td>
<td>7.23%</td>
<td>-5.78%</td>
<td>-0.52%</td>
<td>-1.71%</td>
</tr>
<tr>
<td>(Series 9)</td>
<td>Model</td>
<td>66.26%</td>
<td>2.92%</td>
<td>-9.55%</td>
<td>-1.97%</td>
<td>-2.08%</td>
</tr>
<tr>
<td></td>
<td>Rel. Diff.</td>
<td>38.10%</td>
<td>-59.57%</td>
<td>-65.14%</td>
<td>-279.63%</td>
<td>-21.63%</td>
</tr>
</tbody>
</table>

For a clearer illustration, in Figure 2.6 below, we plot for each tranche the time series of relative differences between the market and model CDO tranche spreads. Note that for the first 4 data points, that is, until July 2007, there were reasonably small differences, but since 2008, there was more divergence between the model and market spreads.

For further examination, Figures 2.7 and 2.8 below show separate time series plots for the equity (0-3%) and senior (15-30%) tranches. We observe that for the dates between 2004 and 2007, the equity and senior tranche spreads in the market are similar to those in the calibrated model. Hence, this signifies that the systematic risks in the equity and credit markets were similar up until 2007. On the other hand, for all of the newer dates since March 2008, the equity tranche spreads observed in the market are much lower than the model spreads, while the senior tranche spreads...
from the market are significantly larger than the corresponding model spreads. In particular, the credit market accounts for larger losses impacting the senior tranches than as predicted by the model, which is calibrated to equity index options. Hence, we find that in this forward problem, the CDO market contained more systematic risk than was seen in the equity index options market since 2008.
Figure 2.7: Time Series of Relative Differences for Equity CDO Tranche

Figure 2.8: Time Series of Relative Differences for Senior CDO Tranche
2.8.2 Backward Problem

In this section, we present the numerical results for the backward problem, where we calibrate our model to CDO tranche spreads and CDS spreads and then compare the implied volatilities from equity index options.

Calibrating to CDO and CDS Spreads

Table 2.7 displays the average 1-year and 5-year CDS spreads from the market and the calibrated model, following the calibration procedure in Section 2.7.2. Note that the market and model CDS spreads are close in most cases, with the exception of the 1-year spreads on Oct. 31, 2006, in which case the absolute difference is small.

Table 2.7: Backward Problem: Average 1-yr and 5-yr CDS Spreads (bps)

<table>
<thead>
<tr>
<th>Date</th>
<th>1-yr Market</th>
<th>1-yr Model</th>
<th>Rel. Diff.</th>
<th>5-yr Market</th>
<th>5-yr Model</th>
<th>Rel. Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>24.62</td>
<td>25.93</td>
<td>5.31%</td>
<td>70.00</td>
<td>69.38</td>
<td>-0.88%</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>14.22</td>
<td>15.03</td>
<td>5.67%</td>
<td>49.00</td>
<td>48.19</td>
<td>-1.66%</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>8.71</td>
<td>11.26</td>
<td>29.28%</td>
<td>34.00</td>
<td>33.20</td>
<td>-2.37%</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>16.93</td>
<td>17.57</td>
<td>3.77%</td>
<td>45.00</td>
<td>44.47</td>
<td>-1.18%</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>138.86</td>
<td>150.55</td>
<td>8.43%</td>
<td>182.00</td>
<td>167.54</td>
<td>-7.95%</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>87.52</td>
<td>98.47</td>
<td>12.52%</td>
<td>115.00</td>
<td>102.64</td>
<td>-10.75%</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>155.52</td>
<td>167.27</td>
<td>7.55%</td>
<td>173.00</td>
<td>162.14</td>
<td>-6.28%</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>240.18</td>
<td>252.93</td>
<td>5.31%</td>
<td>220.00</td>
<td>204.23</td>
<td>-6.71%</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>295.37</td>
<td>309.91</td>
<td>4.92%</td>
<td>258.00</td>
<td>240.20</td>
<td>-6.90%</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>131.39</td>
<td>139.20</td>
<td>5.94%</td>
<td>139.00</td>
<td>130.06</td>
<td>-6.43%</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>135.31</td>
<td>146.93</td>
<td>8.59%</td>
<td>121.00</td>
<td>108.33</td>
<td>-10.47%</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>59.09</td>
<td>63.22</td>
<td>7.00%</td>
<td>98.00</td>
<td>95.11</td>
<td>-2.95%</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>45.75</td>
<td>49.89</td>
<td>9.06%</td>
<td>89.00</td>
<td>86.65</td>
<td>-2.64%</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>111.49</td>
<td>123.77</td>
<td>11.01%</td>
<td>141.00</td>
<td>127.57</td>
<td>-9.53%</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>104.86</td>
<td>112.04</td>
<td>6.84%</td>
<td>118.00</td>
<td>110.28</td>
<td>-6.54%</td>
</tr>
</tbody>
</table>

In Table 2.8 we show the market tranche spreads and the model tranche spreads fitted to the data for the period of Aug. 2004 to Sep. 2010, following the procedure described in Section 2.7.2. Here, the RMSE is computed according to Equation (2.36) and is generally less than 10 except for one of the dates (i.e., Apr. 8, 2009) during the crisis.
Table 2.8: Fitting the CDO Tranche Spreads

<table>
<thead>
<tr>
<th>Date (Series)</th>
<th>Source</th>
<th>0-3%</th>
<th>3-7%</th>
<th>7-10%</th>
<th>10-15%</th>
<th>15-30%</th>
<th>RMSE</th>
</tr>
</thead>
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<td>Aug. 23, 2004 (Series 2)</td>
<td>Market</td>
<td>40.00%</td>
<td>312.50</td>
<td>122.50</td>
<td>42.50</td>
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<td></td>
<td>Bid-Ask</td>
<td>2.00%</td>
<td>15.00</td>
<td>7.00</td>
<td>7.00</td>
<td>3.00</td>
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</tr>
<tr>
<td></td>
<td>Model</td>
<td>48.04%</td>
<td>349.62</td>
<td>120.25</td>
<td>60.36</td>
<td>14.96</td>
<td>2.43</td>
</tr>
<tr>
<td>Dec. 5, 2005 (Series 5)</td>
<td>Market</td>
<td>41.10%</td>
<td>117.50</td>
<td>32.90</td>
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<td>6.80</td>
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<td>1.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>42.47%</td>
<td>112.59</td>
<td>30.63</td>
<td>21.48</td>
<td>8.95</td>
<td>1.29</td>
</tr>
<tr>
<td>Oct. 31, 2006 (Series 7)</td>
<td>Market</td>
<td>23.66%</td>
<td>88.26</td>
<td>18.75</td>
<td>7.25</td>
<td>3.43</td>
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<tr>
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<td>Bid-Ask</td>
<td>0.31%</td>
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<td>1.11</td>
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<td>17.00</td>
<td>5.79</td>
<td>0.92</td>
</tr>
<tr>
<td>Jul. 17, 2007 (Series 8)</td>
<td>Market</td>
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<td>156.36</td>
<td>33.95</td>
<td>17.23</td>
<td>5.59</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>1.61</td>
<td>1.50</td>
<td></td>
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<td>157.53</td>
<td>35.60</td>
<td>21.54</td>
<td>8.95</td>
<td>1.29</td>
</tr>
<tr>
<td>Mar. 14, 2008 (Series 9)</td>
<td>Market</td>
<td>67.92%</td>
<td>836.90</td>
<td>462.22</td>
<td>265.56</td>
<td>129.45</td>
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</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
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<td>9.21</td>
<td>9.07</td>
<td>9.11</td>
<td>4.97</td>
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<td>1.63</td>
</tr>
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<td>240.75</td>
<td>124.01</td>
<td>66.66</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.78%</td>
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<td>6.50</td>
<td>4.75</td>
<td>3.37</td>
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</tr>
<tr>
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<td>459.27</td>
<td>226.01</td>
<td>139.51</td>
<td>51.56</td>
<td>2.78</td>
</tr>
<tr>
<td>Oct. 16, 2008 (Series 11)</td>
<td>Market</td>
<td>71.50%</td>
<td>1,297.00</td>
<td>676.67</td>
<td>209.34</td>
<td>66.50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>1.50%</td>
<td>50.00</td>
<td>26.67</td>
<td>11.33</td>
<td>10.00</td>
<td></td>
</tr>
<tr>
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<td>Model</td>
<td>77.84%</td>
<td>1,386.39</td>
<td>588.96</td>
<td>269.74</td>
<td>53.74</td>
<td>3.52</td>
</tr>
<tr>
<td>Jan. 8, 2009 (Series 9)</td>
<td>Market</td>
<td>74.91%</td>
<td>39.75%</td>
<td>811.03</td>
<td>446.13</td>
<td>111.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.53%</td>
<td>0.41%</td>
<td>8.79</td>
<td>8.79</td>
<td>3.60</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>77.51%</td>
<td>33.94%</td>
<td>830.44</td>
<td>413.34</td>
<td>82.60</td>
<td>7.84</td>
</tr>
<tr>
<td>Apr. 8, 2009 (Series 9)</td>
<td>Market</td>
<td>76.76%</td>
<td>45.80%</td>
<td>15.25%</td>
<td>520.21</td>
<td>137.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.34%</td>
<td>0.42%</td>
<td>0.35%</td>
<td>5.45</td>
<td>1.82</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>81.38%</td>
<td>37.37%</td>
<td>12.86%</td>
<td>494.26</td>
<td>132.64</td>
<td>11.51</td>
</tr>
<tr>
<td>Jul. 8, 2009 (Series 12)</td>
<td>Market</td>
<td>64.00%</td>
<td>34.89%</td>
<td>16.73%</td>
<td>6.80%</td>
<td>-0.83%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.52%</td>
<td>0.53%</td>
<td>0.63%</td>
<td>0.48%</td>
<td>0.25%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>66.67%</td>
<td>29.97%</td>
<td>15.73%</td>
<td>7.61%</td>
<td>-0.94%</td>
<td>4.86</td>
</tr>
<tr>
<td>Sep. 8, 2009 (Series 12)</td>
<td>Market</td>
<td>62.38%</td>
<td>27.31%</td>
<td>10.88%</td>
<td>5.21%</td>
<td>-1.84%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.50%</td>
<td>0.50%</td>
<td>0.50%</td>
<td>0.50%</td>
<td>0.29%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>61.48%</td>
<td>20.77%</td>
<td>7.75%</td>
<td>1.27%</td>
<td>-3.45%</td>
<td>7.83</td>
</tr>
<tr>
<td>Dec. 8, 2009 (Series 13)</td>
<td>Market</td>
<td>53.42%</td>
<td>22.54%</td>
<td>8.57%</td>
<td>1.75%</td>
<td>-2.44%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>1.00%</td>
<td>0.78%</td>
<td>0.62%</td>
<td>0.50%</td>
<td>0.40%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>54.74%</td>
<td>19.45%</td>
<td>9.05%</td>
<td>3.88%</td>
<td>-1.26%</td>
<td>2.99</td>
</tr>
<tr>
<td>Mar. 8, 2010 (Series 13)</td>
<td>Market</td>
<td>53.81%</td>
<td>19.75%</td>
<td>7.38%</td>
<td>0.88%</td>
<td>-2.60%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>1.00%</td>
<td>1.00%</td>
<td>1.13%</td>
<td>0.75%</td>
<td>0.50%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>54.70%</td>
<td>17.92%</td>
<td>7.30%</td>
<td>2.19%</td>
<td>-2.32%</td>
<td>1.22</td>
</tr>
<tr>
<td>Jun. 8, 2010 (Series 9)</td>
<td>Market</td>
<td>52.95%</td>
<td>14.95%</td>
<td>-1.61%</td>
<td>0.81%</td>
<td>-1.48%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.52%</td>
<td>0.45%</td>
<td>0.43%</td>
<td>0.28%</td>
<td>0.09%</td>
<td></td>
</tr>
</tbody>
</table>

Continued on next page
Table 2.8 – continued from previous page

<table>
<thead>
<tr>
<th>Date</th>
<th>Source</th>
<th>0-3%</th>
<th>3-7%</th>
<th>7-10%</th>
<th>10-15%</th>
<th>15-30%</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>52.27%</td>
<td>8.38%</td>
<td>-3.62%</td>
<td>0.96%</td>
<td>-1.98%</td>
<td>7.32</td>
</tr>
<tr>
<td>Sep. 8, 2010 (Series 9)</td>
<td>Market</td>
<td>47.98%</td>
<td>7.23%</td>
<td>-5.78%</td>
<td>-0.52%</td>
<td>-1.71%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bid-Ask</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.25%</td>
<td>0.26%</td>
<td>0.05%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Model</td>
<td>47.75%</td>
<td>5.25%</td>
<td>-6.18%</td>
<td>-1.03%</td>
<td>-2.10%</td>
<td>5.10</td>
</tr>
</tbody>
</table>

Table 2.9 gives the set of calibrated parameters \( \{ \bar{\kappa}, \bar{\sigma}, \bar{l}, \bar{\xi}, \bar{w}, y_0 + \bar{x}_{avg}, \bar{y} + \bar{x}_{avg} \} \) that yield the fitted model prices from Table 2.8. Based on these parameters and Equation (2.35) in Section 2.7.2, we can obtain \( \eta_Y = \{ \kappa_Y, \sigma_Y, \bar{y}, y_0, l_Y, \xi_Y \} \).

**Table 2.9: Parameters Fitted to CDO and CDS Spreads**

<table>
<thead>
<tr>
<th>Date</th>
<th>( \bar{\kappa} )</th>
<th>( \bar{\sigma} )</th>
<th>( \bar{l} )</th>
<th>( \bar{\xi} )</th>
<th>( \bar{w} )</th>
<th>( y_0 + x_{0 avg} )</th>
<th>( \bar{y} + \bar{x}_{avg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>0.07</td>
<td>0.05</td>
<td>0.05</td>
<td>0.04</td>
<td>0.86</td>
<td>14.59 bps</td>
<td>370.44 bps</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
<td>0.09</td>
<td>0.61</td>
<td>6.10 bps</td>
<td>2,200.85 bps</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>0.20</td>
<td>0.06</td>
<td>0.02</td>
<td>0.03</td>
<td>0.69</td>
<td>0.65 bps</td>
<td>110.34 bps</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>0.01</td>
<td>0.07</td>
<td>0.01</td>
<td>0.06</td>
<td>0.67</td>
<td>11.66 bps</td>
<td>1,881.90 bps</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>0.03</td>
<td>0.92</td>
<td>0.01</td>
<td>0.05</td>
<td>0.92</td>
<td>136.03 bps</td>
<td>6,612.07 bps</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>0.08</td>
<td>0.52</td>
<td>0.01</td>
<td>0.05</td>
<td>0.85</td>
<td>95.72 bps</td>
<td>979.97 bps</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>1.00</td>
<td>0.50</td>
<td>0.02</td>
<td>0.05</td>
<td>0.97</td>
<td>198.58 bps</td>
<td>299.27 bps</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>0.88</td>
<td>0.48</td>
<td>0.27</td>
<td>0.08</td>
<td>0.97</td>
<td>386.96 bps</td>
<td>125.51 bps</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>1.20</td>
<td>0.95</td>
<td>0.05</td>
<td>0.12</td>
<td>0.99</td>
<td>520.10 bps</td>
<td>405.13 bps</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>0.35</td>
<td>0.28</td>
<td>0.05</td>
<td>0.10</td>
<td>0.97</td>
<td>185.14 bps</td>
<td>153.59 bps</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>0.27</td>
<td>0.22</td>
<td>0.05</td>
<td>0.05</td>
<td>0.93</td>
<td>211.22 bps</td>
<td>99.70 bps</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>0.50</td>
<td>0.50</td>
<td>0.02</td>
<td>0.02</td>
<td>0.96</td>
<td>47.49 bps</td>
<td>246.79 bps</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>0.55</td>
<td>0.40</td>
<td>0.01</td>
<td>0.05</td>
<td>0.95</td>
<td>25.98 bps</td>
<td>209.70 bps</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>0.20</td>
<td>0.37</td>
<td>0.02</td>
<td>0.10</td>
<td>0.98</td>
<td>131.23 bps</td>
<td>481.07 bps</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>0.07</td>
<td>0.29</td>
<td>0.01</td>
<td>0.02</td>
<td>0.96</td>
<td>134.01 bps</td>
<td>922.99 bps</td>
</tr>
</tbody>
</table>

**Market vs. Model Implied Volatilities**

We observe that the market factor parameters \( b_Y \) and \( \rho \) in (2.27) do not impact the pricing of the credit instruments, so they are free to be chosen. First, Table 2.10 shows the set of scaling factors \( b_Y \), which are selected to match the level of the implied volatility curve for each date.
Table 2.10: Scaling Factor to Fit Implied Volatility Curve

<table>
<thead>
<tr>
<th>Date</th>
<th>$b_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aug. 23, 2004</td>
<td>12.5</td>
</tr>
<tr>
<td>Dec. 5, 2005</td>
<td>13.0</td>
</tr>
<tr>
<td>Oct. 31, 2006</td>
<td>12.0</td>
</tr>
<tr>
<td>Jul. 17, 2007</td>
<td>10.7</td>
</tr>
<tr>
<td>Mar. 14, 2008</td>
<td>10.0</td>
</tr>
<tr>
<td>Jun. 20, 2008</td>
<td>7.7</td>
</tr>
<tr>
<td>Oct. 16, 2008</td>
<td>6.7</td>
</tr>
<tr>
<td>Jan. 8, 2009</td>
<td>7.0</td>
</tr>
<tr>
<td>Apr. 8, 2009</td>
<td>5.4</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>8.5</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>7.8</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>15.0</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>9.4</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>10.0</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>10.1</td>
</tr>
</tbody>
</table>

Next, we choose a range of values for the correlation parameter $\rho$, which impacts the skew of the curve. In particular, Figures 2.9 to 2.11 display the implied volatility plots from the market and the model for $\rho \in \{-0.30, -0.45, -0.60, -0.75, -0.90\}$. For the dates in 2004, 2006, and 2007, we find that the market curve has similar skew to the model curve, and for Dec. 5, 2005, the market curve actually has greater skew. Hence, this signifies that the equity risk was equal to or greater than the credit risk up until 2007. Meanwhile, for all of the newer dates since 2008, we find that the model curves have greater skew than the market curves, and this indicates that the systematic risk in the credit market was greater than that in the equity market for this recent period.
Figure 2.9: Market vs. Model Implied Vols, Aug. 23, 2004 to Jul. 17, 2007

- Implied volatilities for $T = 1.8164$ on 2004-08-23
- Implied volatilities for $T = 2.0466$ on 2005-12-05
- Implied volatilities for $T = 2.1397$ on 2006-10-31
- Implied volatilities for $T = 2.4274$ on 2007-07-17
Figure 2.10: Market vs. Model Implied Vols, Mar. 14, 2008 to Jul. 8, 2009

- Implied volatilities for T = 2.7644 on 2008-03-14
- Implied volatilities for T = 2.4989 on 2008-06-20
- Implied volatilities for T = 2.1726 on 2008-10-16
- Implied volatilities for T = 2.9397 on 2009-01-08
- Implied volatilities for T = 2.6932 on 2009-04-08
- Implied volatilities for T = 2.4438 on 2009-07-08
Figure 2.11: Market vs. Model Implied Vols, Sep. 8, 2009 to Sep. 8, 2010
2.8.3 Analysis of Systematic Risks

In our hybrid multi-name equity-credit model, we analyzed the systematic risk in the two markets, in the forward and backward directions. In both directions, we found that the systematic risks in the two markets were similar from 2004 to 2007, while the credit market incorporated far greater systematic risk than the equity market during the financial crisis from 2008 to 2010. Let us review the latter results in more detail.

For the forward problem in Section 2.8.1, we find that for the dates since 2008, after fitting the model parameters to the equity index options and CDS spreads, the junior CDO tranche spreads from the model are much larger than those observed in the market, while the senior CDO tranche spreads from the model are smaller than the corresponding market spreads. Hence, the credit market accounts for larger losses impacting the senior CDO tranches than as predicted by the model, and this suggests that the credit market contained more systematic risk than was seen in the equity market.

For the backward problem in Section 2.8.2, we observed that for the more recent dates since 2008, after fitting the parameters to the CDO and CDS spreads, the implied volatility curves from the model had greater skew than those from the market. This indicates that based on the framework of our model, the systematic risk in the credit market was greater than that in the equity market.

Combining the results from the forward and backward cases above, we conclude with the following: our hybrid model indicates that the systematic risk in the credit market was greater than that in the equity market during the credit crisis. This result is the opposite of that found in Coval, Jurek, and Stafford (2009a), and also differs from the conclusions of Collin-Dufresne, Goldstein, and Yang (2010), Luo and Carverhill (2011), and Li and Zhao (2011).

Finally, we provide some economic rationale for why the systematic risk in the credit market was larger than that in the equity market during the credit crisis. First,
we note that the S&P 500 index option market is more developed than the CDX market and that investors generally have a better understanding of the systematic risks embedded in the equity market, especially during harsh times such as crises. We speculate that investors of credit derivatives would hence be more risk-averse and perhaps pay extra to hedge against the systematic risks in the credit market during the crisis. In addition, we note that in 2008, the Securities and Exchange Commission (SEC) started to crack down on fraud and market manipulation in the multi-name credit derivatives markets. In light of the fraudulent behaviour in the market, investors would then be more cautious when investing in sophisticated derivatives such as CDOs and thus would be willing to pay an extra premium to protect against massive defaults. This excess insurance premium would signify larger embedded systematic risk in the credit market relative to the equity market.
Chapter 3

Top-Down Indifference Valuation of CDOs and Equity Index Options

In contrast to Chapter 2 where we considered a bottom-up model for the valuation of CDOs and equity index options, in this chapter we consider a top-down model with event feedback. We use the technology of utility-indifference pricing in top-down intensity-based models of default risk to price both equity index options and multi-name credit derivatives such as collateralized debt obligations (CDOs). This utility-indifference approach is partly motivated by Sircar and Zariphopoulou (2010), who consider a bottom-up indifference pricing model for valuing CDOs and conclude that the tranche holder’s risk aversion helps to explain the phenomena of high premiums in credit derivatives. We extend this utility-indifference approach to value equity derivatives on the stock index in addition to valuing CDOs.

Here is an outline of Chapter 3. In Section 3.1, we describe the general top-down model used for the utility valuation of CDOs and equity index options. In particular, we assume that the loss process is a self-exciting counting process in which the intensity is mean-reverting with feedback from defaults. For the control problem, the investor can invest in the derivative, the money market and the stock index, whose
process has stochastic variance that contains feedback from defaults. In Section 3.2, we describe the maximal expected utility problems, in particular, the Merton problem and the European claim holder’s problem, and then we examine the indifference valuation of European claims. Section 3.3 discusses the indifference pricing of CDO tranches, with a careful examination of the utility problem and the timing of the cash flows. Section 3.4 discusses the equity derivative holder’s problem, which involves an extra dimension to explicitly account for the stock index process.

In Section 3.5, we discuss two implementation methodologies for computing the indifference prices of European claims, CDO tranches, and equity derivatives. First, we solve a system of three- or four-dimensional non-linear PDEs numerically by use of explicit finite difference methods. Secondly, we write the PDEs as expectations and then discretize the underlying processes using trinomial trees.

In Section 3.6, we present the numerical results based on these implementations, while also calibrating the model parameters to match the vanilla products in the market. We observe that the feedback from defaults and the risk aversion parameter are important factors in the indifference prices. We also find that the implied risk aversion increases with seniority of the CDO tranche and with out-of-the-moneyness for equity put options, reflecting greater relative fear of extreme losses that these options represent. Finally, comparing the implied risk aversions from the two markets in 2009-10, we observed greater systematic risk in the credit market than in the equity market.

3.1 General Model

We build our model on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathbb{P}\) is the real-world probability measure. We consider a collateralized debt obligation (CDO) on a portfolio of \(\hat{N}\) underlying firms. We let \(N = (N_t)_{t \geq 0}\) be the portfolio loss process, where
$N_t \in \{0,1,\ldots,\hat{N}\}$ represents the number of defaults in the portfolio by time $t$. As in Errais, Giesecke, and Goldberg (2010), we suppose $N$ is a self-exciting counting process; in particular, its intensity $\lambda = (\lambda_t)_{t \geq 0}$ is a mean-reverting diffusion process with feedback from defaults. In particular, the intensity process jumps by the constant amount $\delta$ at each default time:

$$
\begin{cases}
    d\lambda_t = \alpha(\Lambda - \lambda_t)dt + \sigma \sqrt{\lambda_t}dW^\lambda_t + \delta dN_t, & \text{when } N_t < \hat{N}, \\
    \lambda_t = 0, & \text{when } N_t = \hat{N},
\end{cases}
$$

where

- $(W^\lambda_t)_{t \geq 0}$ is a Brownian motion under $\mathbb{P}$,
- $\alpha, \Lambda, \sigma, \delta$ are the respective mean-reversion speed, mean-reverting level, volatility, and feedback parameters for the intensity process, and
- we impose the restriction $\delta < \alpha$ to ensure that the mean of the intensity process does not diverge as time goes to infinity (this growth condition is shown at the bottom of this section), and
- we enforce the Feller condition $\sigma^2 \leq 2\alpha \Lambda$ to ensure that the intensity process remains positive almost surely.

We let $S = (S_t)_{t \geq 0}$ denote the price process for stock market index, for example, the S&P 500 index. We suppose that $S$ has stochastic variance $v = (v_t)_{t \geq 0}$ that follows a mean-reverting diffusion process with feedback from defaults. With regard to the feedback, we suppose that $v$ increases by the constant $a$ at each jump time of $N$; this assumption is motivated by empirical studies that a firm’s volatility generally increases when its credit risk increases; see Carr and Wu (2010) for a discussion. In
particular, we assume that \((S, v)\) satisfies the following set of SDEs:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{v_t} dW_t, \quad (3.2) \\
dv_t &= \kappa(\theta - v_t) dt + \eta \sqrt{v_t} dW_t^v + adN_t, \quad (3.3)
\end{align*}
\]

where

- \(\mu\) is the drift rate for the stock index,
- \(\kappa, \theta, \eta,\) and \(a\) are the respective mean-reversion speed, mean-reverting level, volatility, and feedback parameters for the variance process,
- we enforce the Feller condition \(\eta^2 \leq 2\kappa \theta\) to ensure that the variance process stays positive almost surely, and
- \((W_t)_{t \geq 0}, (W_t^v)_{t \geq 0},\) and \((W_t^\lambda)_{t \geq 0}\) are correlated \(\mathbb{P}\)-Brownian motions:

\[
\begin{align*}
\mathbb{E}\{dW_t \cdot dW_t^v\} &= \rho_{Sv} dt, \quad \mathbb{E}\{dW_t \cdot dW_t^\lambda\} = \rho_{S\lambda} dt, \quad \mathbb{E}\{dW_t^v \cdot dW_t^\lambda\} = \rho_{v\lambda} dt.
\end{align*}
\]

The leverage effect says that the stock index and its variance should be negatively correlated (i.e., \(\rho_{Sv} < 0\)). In addition, we expect the stock index and the intensity to be negatively correlated (i.e., \(\rho_{S\lambda} < 0\)), while the variance and the intensity should be positively correlated (i.e., \(\rho_{v\lambda} > 0\)). However, for analytical tractability and computational efficiency, we only account for correlation between the stock index and its variance, that is, we take \(\rho_{Sv} < 0\) but set \(\rho_{v\lambda} = \rho_{S\lambda} = 0\).

Let us consider the situation when all the firms in the portfolio have defaulted (i.e., when \(N_t = \hat{N}\)). At the time of the \(\hat{N}^{th}\) default, the intensity process \((\lambda_t)\) drops to zero and remains there since no more defaults can occur. In addition, the stock index’s variance process \((v_t)\) instantaneously jumps up by \(a\), the feedback effect disappears thereafter, and the variance decays exponentially (with a Brownian noise)
to the mean-reverting level \( \theta \). A more realistic assumption may be to impose that the variance remains large when all the firms in the portfolio have defaulted, since the stock index should be even more volatile during a financial crisis. Theoretically, one could model this assumption by increasing the mean-reverting level to some \( \bar{\theta} > \theta \) or by simply fixing the variance at some large constant value \( \nu_{\text{max}} \). Another related issue is that the stock index in our model does not jump downwards to some near-zero level at times of default (as only the variance increases but not the stock level); see Chapter 4 for a discussion of the case where the stock index drops by a fractional amount upon a default.

To help estimate the sizes of the model parameters that would match the market data, we compute the expected values of the intensity and variance processes at fixed times. We show that, because of the feedback effects, the long-term means of the processes are larger than the respective mean-reverting levels \( \Lambda \) and \( \theta \). Note that we require the intensity process’ feedback parameter \( \delta \) to be smaller than its mean-reverting speed \( \alpha \) in order to prevent the means of the intensity and variance processes from growing to infinity.

(i) By integrating the SDE (3.1) for the intensity process from 0 to \( t \) and taking expectations, we obtain the following ODE for \( M(t) := \mathbb{E}\lambda_t \):

\[
M'(t) = \alpha(\Lambda - M(t)) + \delta M(t),
\]

with the initial condition \( M(0) = \lambda_0 \). For \( \delta < \alpha \), we use the integrating factor \( e^{(\alpha - \delta)t} \) to obtain the following solution:

\[
\mathbb{E}\lambda_t = \frac{\alpha \Lambda}{\alpha - \delta} + \left( \lambda_0 - \frac{\alpha \Lambda}{\alpha - \delta} \right) e^{-(\alpha - \delta)t}, \quad t \geq 0. \tag{3.4}
\]
Hence, the long-term mean for the intensity process is

\[
\lim_{t \to \infty} \mathbb{E}\lambda_t = \frac{\alpha \Lambda}{\alpha - \delta} > \Lambda.
\]

Note that when \( \delta = \alpha \), we have the simple differential equation

\[ M'(t) = \alpha \Lambda, \]

with the initial condition \( M(0) = \lambda_0 \), which yields the solution

\[ \mathbb{E}\lambda_t = \lambda_0 + \alpha \Lambda t, \quad t \geq 0, \]

which is affine in \( t \) and hence grows to infinity as \( t \to \infty \).

(ii) Similarly, by integrating the SDE (3.3) for the variance process, we find that the mean, \( m(t) := \mathbb{E}v_t \), satisfies the ODE

\[ m'(t) = \kappa(\theta - m(t)) + a \mathbb{E}\lambda_t, \]

with the initial condition \( m(0) = v_0 \). For \( \delta < \alpha \), we use the integrating factor \( e^{\kappa t} \) and apply the result for \( \mathbb{E}\lambda_t \) above to obtain the following solution:

\[
\mathbb{E}v_t = v_0 e^{-\kappa t} + \left( \theta + \frac{a}{\kappa} \cdot \frac{\alpha \Lambda}{\alpha - \delta} \right) \left( 1 - e^{-\kappa t} \right) + \frac{a}{\alpha - \delta - \kappa} \left( \lambda_0 - \frac{\alpha \Lambda}{\alpha - \delta} \right) \left[ e^{-\kappa t} - e^{-(\alpha-\delta)t} \right], \quad t \geq 0. \tag{3.5}
\]

Hence, the long-term mean for the variance process is

\[
\lim_{t \to \infty} \mathbb{E}v_t = \theta + \frac{a}{\kappa} \left( \frac{\alpha \Lambda}{\alpha - \delta} \right) > \theta.
\]
Note that when $\delta = \alpha$, we have the simpler differential equation

$$m'(t) = \kappa(\theta - m(t)) + a(\lambda_0 + \alpha \Lambda t),$$

with the same initial condition $m(0) = v_0$. Using the same integrating factor $e^{\kappa t}$ as above, we obtain the solution

$$Ev_t = v_0 e^{-\kappa t} + \left( \theta + \frac{a \lambda_0}{\kappa} - \frac{aa \Lambda}{\kappa^2} \right) \left( 1 - e^{-\kappa t} \right) + \frac{aa \Lambda}{\kappa} t,$$

which grows to infinity as $t \to \infty$.

### 3.2 Maximal Expected Utility Problems

In this section, we consider the maximal expected utility problems, in particular, the Merton problem and the European claim holder’s problem, and then we examine the indifference valuation of European claims.

Let $T < \infty$ denote our finite fixed horizon, chosen to coincide with the expiration date of the derivatives contracts of interest. Both the money market and the stock index are available for trading by the investor until time $T$. We denote the investor’s control process by $\pi = (\pi_t)_{t \geq 0}$, where $\pi_t$ is the dollar amount held in the stock index at time $t$. The control process $\pi$ is said to be admissible if it is non-anticipating and satisfies the integrability constraint $\mathbb{E} \left[ \int_0^T \pi_s^2 ds \right] < \infty$; we denote the set of admissible policies by $\mathcal{A}$. When there is no credit derivative available, we assume that the money that is not invested in the stock index goes directly into the money market, which grows at the risk-free rate $r$. Then, the investor’s wealth process $X = (X_t)_{t \geq 0}$, evolves
according to

\[ dX_t = \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)dt \]

\[ = ((\mu - r)\pi_t + rX_t)dt + \sqrt{\nu_t}\pi_t dW_t. \]  

(3.6)

Throughout this chapter, we suppose that the investor has risk preferences described by the exponential utility function \( U : \mathbb{R} \to \mathbb{R}_- \) defined by

\[ U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}, \]

where \( \gamma > 0 \) is the coefficient of absolute risk aversion. In the case of exponential utility, the optimal control is always independent of the initial wealth \( x \).

### 3.2.1 Merton Problem

We first consider the Merton problem, where the investor wishes to maximize his expected utility of terminal wealth when there is no credit derivative available. The Merton value function is defined by

\[ M(t, x, v, n, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\{-e^{-\gamma X_T} | X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \}. \]  

(3.7)

Under certain mild regularity conditions (see, e.g., Theorem 11.2.3 of Øksendal (2007)), it suffices to consider Markov controls. Then, by Theorem 3.1 of Øksendal and Sulem (2007), the Merton value function solves the following HJB PDE:

\[
\begin{aligned}
\partial_t M &+ \sup_{\pi \in \mathbb{R}} \mathcal{L}^*_{(X,v,N,\lambda)} M = 0, \\
M(T, x, v, n, \lambda) & = -e^{-\gamma x},
\end{aligned}
\]

(3.8)

\[ \mathcal{L}^*_{(X,v,N,\lambda)} \] denotes the Hamilton-Jacobi-Bellman operator.

\[ 1 \text{We remark that exponential utility is an example of a utility function that has constant absolute risk aversion (CARA): } \frac{U''(x)}{U'(x)} = \gamma. \]
where $\mathcal{L}^\pi_{(X,v,N,\lambda)}$ denotes the infinitesimal generator of the joint Markov process $(X, v, N, \lambda)$:

$$\mathcal{L}^\pi_{(X,v,N,\lambda)} M = \left( (\mu - r)x + r x \right) \partial_x M + \frac{1}{2} v \pi^2 \partial_{xx} M + \kappa (\theta - v) \partial_v M + \frac{1}{2} \eta^2 v \partial_{vv} M$$

$$+ \rho_S v \pi \partial_{v\pi} M + \kappa (\theta - v) \partial_v \lambda M + \frac{1}{2} \sigma^2 \lambda \partial_{\lambda\lambda} M$$

$$+ \lambda \left[ M(t, x, v + a, n + 1, \lambda + \delta) - M(t, x, v, n, \lambda) \right].$$

Note that for the boundary case when all firms have defaulted ($n = \hat{N}$), the Merton problem is independent of $\lambda$ and hence, $\mathcal{L}^\pi_{(X,v,N,\lambda)}$ reduces to the generator $\mathcal{L}^\pi_{(X,v)}$ for $(X, v)$:

$$\mathcal{L}^\pi_{(X,v)} M = \left( (\mu - r)x + r x \right) \partial_x M + \frac{1}{2} v \pi^2 \partial_{xx} M + \kappa (\theta - v) \partial_v M$$

$$+ \frac{1}{2} \eta^2 v \partial_{vv} M + \rho_S v \pi \partial_{v\pi} M.$$

In both cases, the optimal control $\pi^*$ is the maximizer of the quadratic in $\pi$ and is thus given by

$$\pi^* = \frac{(\mu - r) \partial_x M + \rho_S v \pi \partial_{v\pi} M}{-v \partial_{xx} M}.$$

Upon substitution of the control $\pi^*$, the HJB PDE (3.8) reduces to

$$\begin{cases}
\partial_t M - \frac{1}{2v \partial_{xx} M} \left[ \left( (\mu - r)x + r x \right) \partial_x M + \rho_S v \pi \partial_{v\pi} M \right]^2 + r x \partial_x M \\
+ \kappa (\theta - v) \partial_v M + \frac{1}{2} \eta^2 v \partial_{vv} M + \alpha (\Lambda - \lambda) \partial_\lambda M + \frac{1}{2} \sigma^2 \lambda \partial_{\lambda\lambda} M \\
+ \lambda \left[ M(t, x, v + a, n + 1, \lambda + \delta) - M(t, x, v, n, \lambda) \right] = 0,
\end{cases}$$

(3.9)

To simplify the PDE, we make the substitution

$$M(t, x, v, n, \lambda) = -e^{-\gamma x} g^{(n)}(t, v, \lambda),$$

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and then the optimal control becomes

\[
\pi^*(t, v, n, \lambda) = \frac{e^{-r(T-t)}}{\gamma} \left( \frac{\mu - r}{v} + \rho_{sv} \frac{g_v(n)}{g(n)} \right).
\]  

(3.10)

Note that the optimal strategy contains a “volatility hedge” (i.e., \(\rho_{sv} \frac{g_v(n)}{g(n)}\)) that arises because of the stochastic volatility.

For \(n < \tilde{N}\), we see that \(g^{(n)}\) satisfies the PDE

\[
\begin{cases}
  g^{(n)}_t + \kappa(\tilde{\theta} - v)g^{(n)}_v + \frac{1}{2}\eta^2 v g^{(n)}_{vv} + \alpha(\Lambda - \lambda)g^{(n)}_\lambda + \frac{1}{2}\sigma^2 \lambda g^{(n)}_{\lambda\lambda} \\
  -\frac{(\mu - r)^2}{2v} g^{(n)} - \frac{1}{2}\rho_{sv}^2 \eta^2 v \left(\frac{g_v(n)}{g(n)}\right)^2 + \lambda \left[ g^{(n+1)}(t, v + a, \lambda + \delta) - g^{(n)}(t, v, \lambda) \right] = 0,
\end{cases}
\]

(3.11)

\[g^{(n)}(T, v, \lambda) = 1,\]

with \(\tilde{\theta} = \theta - \frac{\rho_{sv} \eta (\mu - r)}{\kappa}\). For the boundary case \(n = \tilde{N}\), when all firms have defaulted, the intensity remains at zero and so the solution must be independent of \(\lambda\). In particular, \(g^{(\tilde{N})}\) satisfies

\[
\begin{cases}
  g^{(\tilde{N})}_t + \kappa(\tilde{\theta} - v)g^{(\tilde{N})}_v + \frac{1}{2}\eta^2 v g^{(\tilde{N})}_{vv} - \frac{(\mu - r)^2}{2v} g^{(\tilde{N})} - \frac{1}{2}\rho_{sv}^2 \eta^2 v \left(\frac{g_v(\tilde{N})}{g(\tilde{N})}\right)^2 = 0, \\
  g^{(\tilde{N})}(T, v) = 1.
\end{cases}
\]

(3.12)

To eliminate the non-linear term \(\left(\frac{g_v(\tilde{N})}{g(\tilde{N})}\right)^2\), we make the power transformation

\[
g^{(\tilde{N})}(t, v) = u(t, v)^\beta,
\]

(3.13)

where \(\beta = (1 - \rho_{sv}^2)^{-1}\). Upon substitution into (3.12), we find that \(u\) satisfies the PDE

\[
\begin{cases}
  u_t + \kappa(\tilde{\theta} - v)u_v + \frac{1}{2}\eta^2 v u_{vv} - \frac{(\mu - r)^2}{2\beta v} u = 0, \\
  u(T, v) = 1.
\end{cases}
\]

(3.14)
We remark that the PDE (3.14) for $u$ yields the Feynman-Kac representation
\[ u(t, v) = \mathbb{E}\left[ \exp\left\{ -\int_t^T (\mu - r)^2 \frac{ds}{2\beta \tilde{v}_s} \right\} \left| \tilde{v}_t = v \right. \right], \tag{3.15} \]
where $\mathbb{E}$ is the expectation operator under an equivalent martingale measure $\tilde{\mathbb{P}}$ (that is equivalent to $\mathbb{P}$) and $(\tilde{v})_{t \geq 0}$ satisfies the SDE
\[ d\tilde{v}_t = \kappa(\tilde{\theta} - \tilde{v}_t) \, dt + \eta \sqrt{\tilde{v}_t} \, d\tilde{W}^v_t, \]
where $\tilde{W}^v$ is a $\tilde{\mathbb{P}}$-Brownian motion. Hurd and Kuznetsov (2008) provide explicit formulae for expectations of the type (3.15) in terms of confluent hypergeometric functions.

In Appendix B we consider an alternative model where the stochastic variance follows an inverse Cox-Ingersoll-Ross (CIR) process with jumps. Then, for the boundary case $n = \hat{N}$, the variance process in that model is simply an inverse CIR process without jumps, and hence, the expectation in (3.15) is just the Laplace transform of an integrated CIR process and is known in closed form (i.e., the defaultable bond price when the interest rate evolves as a CIR process).

### 3.2.2 European Claim Holder’s Problem

Suppose we have a European claim with payoff $P(N_T)$ at maturity $T$. The claim holder’s problem is to maximize his expected utility of terminal wealth when he can invest in the money market, the stock index, and the European claim. We define the claim holder’s value function by
\[ V(t, x, v, n, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left\{ -e^{-\gamma(X_T + P(N_T))} | X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right\}. \tag{3.16} \]

Ahn and Gao (1999) first studied the inverse CIR process (without jumps) for interest rate modeling.
Under the same regularity conditions as for the Merton problem, the value function $V$ solves the following HJB PDE:

$$
\begin{aligned}
V_t + \sup_{\pi \in \mathbb{R}} \mathcal{L}_t^{\pi, (x,v,N,\lambda)} V &= 0, \\
V(T, x, v, n, \lambda) &= -e^{-\gamma(x+P(n))},
\end{aligned}
$$

(3.17)

with $\mathcal{L}_t^{\pi, (x,v,N,\lambda)}$ specified above. Making the substitution

$$
V(t, x, v, n, \lambda) = -e^{-\gamma xe^{(T-t)} h(n)}(t, v, \lambda)
$$

yields the optimal control

$$
\pi^*(t, v, n, \lambda) = \frac{e^{-r(T-t)}}{\gamma} \left( \frac{\mu - r}{v} + \rho S_v \frac{h_v(n)}{h(n)} \right)
$$

(3.18)

Note that once again the optimal strategy contains a “volatility hedge” that arises because of the stochastic volatility.

For $n < \hat{N}$, we have the following PDE problem for $h(n)$:

$$
\begin{aligned}
&\begin{cases}
    h_t^{(n)} + \kappa(\tilde{\theta} - v) h_v^{(n)} + \frac{1}{2} \eta^2 v h_{vv}^{(n)} + \alpha(N - \lambda) h_\lambda^{(n)} + \frac{1}{2} \sigma^2 \lambda h_{\lambda\lambda}^{(n)} \\
    \frac{-(\mu - r)^2}{2v} h^{(n)} - \frac{1}{2} \rho S \eta^2 v \left( \frac{h_v^{(n)}}{h^{(n)}} \right)^2 + \lambda \left[ h^{(n+1)}(t, v + a, \lambda + \delta) - h^{(n)}(t, v, \lambda) \right] = 0
\end{cases}
\end{aligned}
$$

(3.19)

For the boundary case $n = \hat{N}$, when all firms have defaulted, the intensity remains at zero and so the solution must be independent of $\lambda$. In particular, $h^{(\hat{N})}$ satisfies the same PDE (3.12) as $g^{(\hat{N})}$, but with a different terminal condition:

$$
\begin{aligned}
&\begin{cases}
    h_t^{(\hat{N})} + \kappa(\tilde{\theta} - v) h_v^{(\hat{N})} + \frac{1}{2} \eta^2 v h_{vv}^{(\hat{N})} - \frac{1}{2} \rho S \eta^2 v \left( \frac{h_v^{(\hat{N})}}{h^{(\hat{N})}} \right)^2 = 0, \\
    h^{(\hat{N})}(T, v) = e^{-\gamma P(\hat{N})},
\end{cases}
\end{aligned}
$$

(3.20)
It is easy to verify that $h^{(N)}$ is simply equal to its terminal condition multiplied by $g^{(N)}$, that is,

$$h^{(N)}(t, v) = e^{-\gamma P^{(N)}(N)} g^{(N)}(t, v).$$ \hfill (3.21)

### 3.2.3 Indifference Pricing of European Claims

The buyer’s indifference price is the value $p$ at which the investor is indifferent, in terms of maximum expected utility, between two scenarios: i) not holding the claim and ii) holding the claim but having his initial wealth reduced by $p$. Meanwhile, the seller’s indifference price is defined similarly as the value $\tilde{p}$ at which the seller is indifferent between i) not selling the claim and ii) selling the claim and having his initial wealth increased by $\tilde{p}$. In this thesis, however, we focus only on the buyer’s side, and henceforth, “indifference price” will always refer to the buyer’s indifference price.

We now define the indifference price precisely in terms of the Merton and claim holder’s value functions.

**Definition 3.2.1.** The indifference price for a European claim that pays $P(N_T)$ at maturity $T$ is defined as the function $p \equiv p(t, v, n, \lambda)$ such that

$$M(t, x, v, n, \lambda) = V(t, x - p, v, n, \lambda),$$

where $M$ and $V$ are defined by (3.7) and (3.16), respectively. In terms of the corresponding transformed value functions $g^{(n)}$ and $h^{(n)}$, the indifference price is given by

$$p(t, v, n, \lambda) = \frac{e^{-r(T-t)}}{\gamma} \log \left( \frac{g^{(n)}(t, v, \lambda)}{h^{(n)}(t, v, \lambda)} \right). \hfill (3.22)$$

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Expansion in Risk-Aversion Parameter

In this part, we wish to express the indifference price of a European claim using an asymptotic expansion in the risk aversion parameter $\gamma$, which will yield good approximations for small $\gamma$. First, we define the non-discounted indifference pricing function

$$
\phi^{(n)}(t, v, \lambda) = \frac{1}{\gamma} \log \left( \frac{g^{(n)}(t, v, \lambda)}{h^{(n)}(t, v, \lambda)} \right).
$$

(3.23)

For the boundary case $n = \hat{N}$, we see that from (3.21), we have

$$
\phi^{(\hat{N})}(t, v, \lambda) \equiv P(\hat{N}).
$$

For $n < \hat{N}$, applying the PDEs (3.11) and (3.19) leads to the following PDE for $\phi^{(n)}$:

$$
\begin{cases}
\gamma \left[ \partial_t \phi^{(n)} + \tilde{L}_{v,\lambda}^{(n)} \phi^{(n)} \right] - \frac{\gamma^2}{2} \left[ \frac{\eta^2 v}{\beta} \left( \partial_v \phi^{(n)} \right)^2 + \sigma^2 \lambda \left( \partial_\lambda \phi^{(n)} \right)^2 \right] \\
+ \lambda \left( \frac{g^{(n+1)}(t, v + a, \lambda + \delta)}{g^{(n)}(t, v, \lambda)} \right) \left[ 1 - e^{-\gamma (\phi^{(n+1)}(t, v + a, \lambda + \delta) - \phi^{(n)}(t, v, \lambda))} \right] = 0,
\end{cases}
$$

(3.24)

where $\tilde{L}_{v,\lambda}^{(n)}$ is the following linear operator:

$$
\tilde{L}_{v,\lambda}^{(n)} = \left[ \kappa (\tilde{\theta} - v) + \frac{\eta^2 v}{\beta} \left( \frac{g_v^{(n)}(t, v, \lambda)}{g^{(n)}(t, v, \lambda)} \right) \right] \partial_v + \frac{1}{2} \eta^2 v \partial_v v \\
+ \left[ \alpha (\Lambda - \lambda) + \sigma^2 \lambda \left( \frac{g_\lambda^{(n)}(t, v, \lambda)}{g^{(n)}(t, v, \lambda)} \right) \right] \partial_\lambda + \frac{1}{2} \sigma^2 \lambda \partial_\lambda \lambda,
$$

with $\tilde{\theta} = \theta - \frac{\rho v \sigma (\mu - r)}{\kappa}$ as before. Here, we note that $\tilde{L}_{v,\lambda}^{(n)}$ does not depend on $\gamma$ because $g^{(n)}$ is independent of $\gamma$.

To simplify the PDE (3.24), we construct an expansion for $\phi^{(n)}$ in powers of $\gamma$:

$$
\phi^{(n)} = \phi_0^{(n)} + \gamma \phi_1^{(n)} + \gamma^2 \phi_2^{(n)} + \ldots.
$$

(3.25)

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and we do a Taylor expansion (around $\gamma = 0$) for the term $e^{-\gamma [\phi^{(n+1)}(t,v+a,\lambda+\delta) - \phi^{(n)}(t,v,\lambda)]}$.

Then, by considering the order $\gamma$ terms, we obtain the following PDE for $\phi_0^{(n)}$:

\[
\begin{align*}
\partial_t \phi_0^{(n)} + \tilde{L}_{v,\lambda}^{(n)} \phi_0^{(n)} + \\
+ \lambda \left( \frac{g^{(n+1)}(t,v+a,\lambda+\delta)}{g^{(n)}(t,v,\lambda)} \right) \left( \phi_0^{(n+1)}(t,v+a,\lambda+\delta) - \phi_0^{(n)} \right) = 0,
\end{align*}
\]

(3.26)

\[
\phi_0^{(n)}(T,v,\lambda) = P(n),
\]

\[
\phi_0^{(\tilde{N})}(t,v,\lambda) = P(\tilde{N}),
\]

and by considering the order $\gamma^2$ terms, we obtain the following PDE for $\phi_1^{(n)}$:

\[
\begin{align*}
\partial_t \phi_1^{(n)} + \tilde{L}_{v,\lambda}^{(n)} \phi_1^{(n)} + \Phi^{(n)}(t,v,\lambda) + \\
+ \lambda \left( \frac{g^{(n+1)}(t,v+a,\lambda+\delta)}{g^{(n)}(t,v,\lambda)} \right) \left( \phi_1^{(n+1)}(t,v+a,\lambda+\delta) - \phi_1^{(n)} \right) = 0,
\end{align*}
\]

(3.27)

\[
\phi_1^{(n)}(T,v,\lambda) = 0,
\]

\[
\phi_1^{(\tilde{N})}(t,v,\lambda) = 0,
\]

where $\Phi^{(n)}$ is a source term defined by

\[
\Phi^{(n)}(t,v,\lambda) = \frac{-1}{2} \left[ \frac{\eta^2 v}{\beta} \left( \partial_v \phi_0^{(n)} \right)^2 + \sigma^2 \lambda \left( \partial_{\lambda} \phi_0^{(n)} \right)^2, \\
+ \lambda \left( \frac{g^{(n+1)}(t,v+a,\lambda+\delta)}{g^{(n)}(t,v,\lambda)} \right) \left( \phi_0^{(n+1)}(t,v+a,\lambda+\delta) - \phi_0^{(n)} \right) \right].
\]

The PDEs (3.26) and (3.27) yield the following Feynman-Kac representations for $\phi_0^{(n)}$ and $\phi_1^{(n)}$, respectively:

\[
\phi_0^{(n)} = \tilde{E} \left[ P(\tilde{N}_T) \bigg| \tilde{v}_t = v, \tilde{\lambda}_t = \lambda, \tilde{N}_t = n \right]
\]

(3.28)

and

\[
\phi_1^{(n)} = \tilde{E} \left[ \int_t^T \Phi^{(\tilde{N}_s)}(s,\tilde{v}_s,\tilde{\lambda}_s) \, ds \bigg| \tilde{v}_t = v, \tilde{\lambda}_t = \lambda, \tilde{N}_t = n \right],
\]

(3.29)
where $\tilde{E}$ is the expectation operator under an equivalent martingale measure (EMM) $\tilde{\mathbb{P}}$. Here, $(\tilde{N}_u)_{u \geq t}$ is a $\tilde{\mathbb{P}}$-self-exciting counting process with intensity $(\tilde{h}_u)_{u \geq t}$ given by

$$
\tilde{h}_u = \tilde{\lambda}_u \left( \frac{g(\tilde{N}_{u+1})(u, \tilde{v}_u|\tilde{v}_t=v, \tilde{\lambda}_u|\tilde{\lambda}_t=\lambda)}{g(\tilde{N}_u)(u, \tilde{v}_u|\tilde{v}_t=v, \tilde{\lambda}_u|\tilde{\lambda}_t=\lambda)} \right),
$$

where $(\tilde{v}_u)$ and $(\tilde{\lambda}_u)$ satisfy

$$
d\tilde{v}_u = \left[ \kappa(\tilde{\theta} - \tilde{v}_u) + \frac{\eta^2}{\beta}(\partial_u g(\tilde{N}_u)(u, \tilde{v}_u, \tilde{\lambda}_u)) \right] du + \eta \sqrt{\tilde{v}_u} d\tilde{W}^v_u + ad\tilde{N}_u,
$$

$$
d\tilde{\lambda}_u = \left[ \alpha(\Lambda - \tilde{\lambda}_u) + \sigma^2 \tilde{\lambda}_u \left( \partial_\lambda g(\tilde{N}_u)(u, \tilde{v}_u, \tilde{\lambda}_u) \right) \right] du + \sigma \sqrt{\tilde{\lambda}_u} d\tilde{W}^\lambda_u + \delta d\tilde{N}_u,
$$

where $\tilde{\lambda}_u = 0$, when $\tilde{N}_u < \tilde{N}$,

where $(\tilde{W}^v_u)_{u \geq t}$ and $(\tilde{W}^\lambda_u)_{u \geq t}$ are uncorrelated $\tilde{\mathbb{P}}$-Brownian motions.

Summarizing, we have the following result for the indifference price.

**Summary 3.2.1.** The indifference price of the European claim is given by

$$
p(t, v, n, \lambda) = e^{-r(T-t)} \left( \phi^{(n)}_0 + \gamma \phi^{(n)}_1 + \gamma^2 \phi^{(n)}_2 + \ldots \right),
$$

where $\phi_0$ and $\phi_1$ are the solutions to the respective PDEs (3.26) and (3.27), which have the Feynman-Kac representations (3.28) and (3.29), respectively.

**Remark 3.2.1.** Taking the limit as the risk aversion parameter $\gamma \to 0$, we obtain the Davis (1997) price, that is, the discounted expectation of the payoff under the EMM $\tilde{\mathbb{P}}$:

$$
\lim_{\gamma \to 0} p^{(n)}(t, v, \lambda) = e^{-r(T-t)} \phi^{(n)}_0 = e^{-r(T-t)} \tilde{E} \left[ P(\tilde{N}_T) \mid \tilde{v}_t = v, \tilde{\lambda}_t = \lambda, \tilde{N}_t = n \right].
$$

\footnote{Since the processes $\tilde{v}$ and $\tilde{\lambda}$ in the Feynman Kac representations (3.28) and (3.29) are highly convoluted, it may be more practical numerically to solve the PDEs (3.26) and (3.27) instead.}
3.3 CDO Pricing

We now consider the top-down indifference pricing for a CDO tranche with \( \hat{N} \) firms, total notional of 1 unit, attachment point \( K_L \) and detachment point \( K_U \). We suppose the payment dates are \( T_1, T_2, \ldots, T_K \), where

\[
T_k = k\Delta \tau, \quad k = 1, \ldots, K, \quad \text{with} \quad \Delta \tau = \frac{T}{K}.
\]

As in Chapter 2, we assume a constant fractional recovery rate of \( \delta_r \in (0, 1) \). Recalling that \( N_t \) represents the portfolio loss at time \( t \), then the fractional portfolio loss at time \( t \) is given by

\[
L_t = (1 - \delta_r)\frac{N_t}{\hat{N}},
\]

and the tranche loss at time \( t \) is equal to

\[
F(L_t) = (L_t - K_L)^+ - (L_t - K_U)^+.
\]

We refer to Section 2.2.2 for a detailed description of the payment legs.

Next, we determine the tranche holder’s value function based on the cash flows for the premium and protection legs. Because of the discrepancy between the times when the cash flows are first known and when they are actually paid out, we require shifts in the payment times in our optimization. To shift the payments, we rely on the following result: under exponential utility, the optimal control \( \pi^* \) is independent of the wealth level, and hence, any additional wealth received just goes into the bank account (rather than into the stock index). This result is stated more formally in the following lemma.

**Lemma 3.3.1.** Suppose that we have exponential utility, and we wish to maximize the expected utility of wealth at time \( T \), given the information at time \( t < T \). Then, for any additional cash flow \( C_t \) that is determined at time \( t \) but paid out at the later
date $T$, it is equivalent in our optimization to assume that the discounted amount $C_t e^{-r(T-t)}$ is paid out at time $t$.

**Proof.** Let $V$ be the value function defined as in (3.16):

$$V(t, x, v, n, \lambda) = \sup_{\pi \in A} \mathbb{E}\left\{ -e^{-\gamma(X_T + P(N_T))} \mid X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right\},$$

and let $h$ be the corresponding transformed value function

$$h(t, v, n, \lambda) = -e^{\gamma xe^{r(T-t)}} V(t, x, v, n, \lambda).$$

Suppose that the cash flow $C_t$ is a deterministic function of $(t, x, v, n, \lambda)$ and is paid at time $T$. Then, the maximal expected utility of terminal wealth is given by

$$\sup_{\pi \in A} \mathbb{E}\left[ -e^{-\gamma(X_T + P(N_T) + C_t)} \mid X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right]$$

$$= \sup_{\pi \in A} \mathbb{E}\left[ -e^{-\gamma(X_T + P(N_T))} \mid X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right] e^{-\gamma C_t}$$

$$= V(t, x, v, n, \lambda) e^{-\gamma C_t}$$

$$= -e^{\gamma xe^{r(T-t)}} h(t, v, n, \lambda) e^{-\gamma C_t}$$

$$= -e^{\gamma (x + C_t e^{-r(T-t)}) e^{r(T-t)}} h(t, v, n, \lambda)$$

$$= V\left(t, x + C_t e^{-r(T-t)}, v, n, \lambda\right)$$

$$= \sup_{\pi \in A} \mathbb{E}\left[ -e^{-\gamma(X_T + P(N_T))} \mid X_t = x + C_t e^{-r(T-t)}, v_t = v, N_t = n, \lambda_t = \lambda \right].$$

Hence, when the additional cash flow $C_t$ is paid out at time $T$, we may assume instead that the discounted amount $C_t e^{-r(T-t)}$ is paid out at time $t$. \hfill \square

**Remark 3.3.1.** In the above proof, we have shown that when the cash flow $C_t$ is determined at time $t$ and paid out at time $T$, or equivalently, when the discounted cash flow $C_t e^{-r(T-t)}$ is both determined and paid out at time $t$, then the maximal
expected utility is simply the original value function multiplied by $e^{-\gamma C_t}$, that is,

$$
\sup_{\pi \in A} \mathbb{E} \left[ -e^{-\gamma (X_T + P(N_T) + C_t)} \mid X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right] 
$$

$$
= \sup_{\pi \in A} \mathbb{E} \left[ -e^{-\gamma (X_T + P(N_T))} \mid X_t = x + C_t e^{-r(T-t)}, v_t = v, N_t = n, \lambda_t = \lambda \right] 
$$

$$
= V(t, x, v, n, \lambda) e^{-\gamma C_t}. 
$$

Next, we determine the CDO tranche holder’s value function, first in the case where the CDO only has a single payment date and then in the general case when the CDO has multiple payments.

### 3.3.1 Single Payment Date

We first consider a CDO tranche with a single payment date at the maturity $T$. Given an initial wealth level of $x$, the terminal wealth for the tranche holder is

$$
X_T(x + U(K_U - K_L)) + RT[K_U - K_L - F(L_T)] - [F(L_T) - F(L_0)],
$$

where $U(K_U - K_L)$ is the upfront payment received by the tranche holder and added to his initial wealth, $RT[K_U - K_L - F(L_T)]$ is the premium leg payment received at maturity $T$, and $F(L_T) - F(L_0)$ is the protection leg payment paid at maturity $T$. Noting that $L_0 = \frac{(1-\delta_r)N_0}{N}$ and $L_T = \frac{(1-\delta_r)N_T}{N}$, we can re-write the terminal wealth as

$$
X_T(x + U(K_U - K_L)) + F \left( \frac{(1-\delta_r)N_0}{N} \right) + P_T(N_T),
$$

where $F \left( \frac{(1-\delta_r)N_0}{N} \right)$ is a cash flow that is known at time 0 but paid out at time $T$, and $P_T(N_T)$ represents the netted cash flows that are determined and paid out at time $T$:

$$
P_T(N_T) = RT \left[ K_U - K_L - F \left( \frac{(1-\delta_r)N_T}{N} \right) \right] - F \left( \frac{(1-\delta_r)N_T}{N} \right).
$$
Then, the tranche holder’s problem, which is to maximize his expected utility of terminal wealth when holding the CDO tranche, can be expressed as the following:

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma \left[ X_T + F \left( \frac{(1-\delta_r)N_0}{N} \right) + P_T(N_T) \right]} \left| X_0 = x + U(K_U - K_L), v_0 = v, N_0 = n, \lambda_0 = \lambda \right. \right\}$$

$$= \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma \left[ X_T + P_T(N_T) \right]} \left| X_0 = x + U(K_U - K_L), v_0 = v, N_0 = n, \lambda_0 = \lambda \right. \right\} \cdot e^{-\gamma P_0(n)} \cdot e^{-\gamma P_0(n)},$$

(3.30)

where $P_0(n) = F \left( \frac{(1-\delta_r)N_0}{N} \right)$. Let us solve this problem explicitly by considering the tranche holder’s value function, first for $t \in (0, T]$ and then for $t = 0$.

For $t \in (0, T]$, we define the value function

$$V(t, x, v, n, \lambda; R) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma \left[ X_T + P_T(N_T) \right]} \left| X_t = x, v_t = v, N_t = n, \lambda_t = \lambda \right. \right\}. \quad (3.31)$$

Note that we have not yet accounted for the upfront payment $U(K_U - K_L)$ and the cash flow $F \left( \frac{(1-\delta_r)n}{N} \right)$ because these amounts are determined at time 0 and we are only considering $t > 0$ for now. Now, in the time interval $(0, T]$, the value function $V$ is equivalent to that for the European claim of Section 3.2.2. Hence, making the substitution

$$V(t, x, v, n, \lambda; R) = -e^{-\gamma x e^{(T-t)}} h^{(n)}(t, v, \lambda; R)$$

leads to the PDEs (3.19) and (3.20) for $h^{(n)}$, but with adjusted terminal conditions. In particular, for $t \in (0, T]$, we see that $h^{(n)}$, $n < \hat{N}$, satisfies the PDE

$$\begin{cases}
\left( \mu - \rho \right)^2 h^{(n)}(t, v, \lambda) - 2\rho \sigma^2 \nu^2 v \frac{h^{(n)}(t, v, \lambda)^2}{h^{(n)}} + \lambda \left[ \frac{h^{(n+1)}(t, v + a, \lambda + \delta)}{h^{(n)}} - h^{(n)}(t, v, \lambda) \right] = 0. \quad (3.32)
\end{cases}$$

$$h^{(n)}(T, v, \lambda) = e^{-\gamma P_T(n)}.$$
and \( h(\hat{N}) \) satisfies the PDE

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial h}{\partial t} + \kappa (\bar{\theta} - v) h - \frac{1}{2} \eta^2 v \frac{\partial^2 h}{\partial v^2} - \frac{(\mu - r)^2}{2v} h + \frac{1}{2} \rho^2 \gamma^2 v \frac{(h)^2}{h} = 0, \\
\hat{N}(T,v) = e^{-\gamma P_T(\hat{N})},
\end{array} \right.
\end{align*}
\]

where \( P_T(n) \), for \( n = 0, \ldots, \hat{N} \), represents the netted cash flows that are both determined and paid out at maturity \( T \) when \( N_T = n \), that is,

\[
P_T(n) = RT \left[ K_U - K_L - F \left( \frac{(1 - \delta_r)n}{N} \right) \right] - F \left( \frac{(1 - \delta_r)n}{N} \right).
\]

At time \( t = 0 \), we have to adjust the value function to account for the netted cash flow \( P_0(n) = F \left( \frac{(1 - \delta_r)n}{N} \right) \) (when \( N_0 = n \)), and we need to add the upfront payment \( U(K_U - K_L) \) to the initial wealth. By Remark \[3.3.1\] receiving the cash flow \( P_0(n) \) at time \( T \) is equivalent to multiplying the value function at time 0 by \( e^{-\gamma P_0(n)} \):

\[
V(0, x, v, n, \lambda; R) = V(0^+, x, v, n, \lambda; R)e^{-\gamma P_0(n)};
\]

correspondingly, for the transformed value function \( h(n) \), we have:

\[
h(n)(0, v, \lambda; R) = h(n)(0^+, v, \lambda; R)e^{-\gamma P_0(n)}.
\]

In addition, we add the upfront payment \( U(K_U - K_L) \) to the initial wealth \( x \), which gives the following value function at time 0:

\[
V(0, x + U(K_U - K_L), v, n, \lambda; R) = -e^{-\gamma (x + U(K_U - K_L))} e^T h(n)(0, v, \lambda; R).
\]  

We now verify that this adjusted value function \[3.34\] corresponds to the tranche
holder’s problem (3.30) by applying the definition (3.31) for \( V \) when \( t > 0 \):

\[
V(0, x + U(K_U - K_L), v, n, \lambda; R) = V(0^+, x + U(K_U - K_L), v, n, \lambda; R) e^{-\gamma P_0(n)}
\]

\[
= \lim_{t \to 0} V(t, x + U(K_U - K_L), v, n, \lambda; R) e^{-\gamma P_0(n)}
\]

\[
= \lim_{t \to 0} \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma(X_T + P_T(N_T))} \mid X_t = x + U(K_U - K_L), v_t = v, N_t = n, \lambda_t = \lambda \right\} e^{-\gamma P_0(n)},
\]

\[
= \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma(X_T + P_T(N_T))} \mid X_0 = x + U(K_U - K_L), v_0 = v, N_0 = n, \lambda_0 = \lambda \right\} e^{-\gamma P_0(n)},
\]

where the last equality follows by continuity. Having determined the value function for the tranche holder, we are now ready to define the \textit{indifference upfront fee} for a CDO tranche.

\textbf{Definition 3.3.1.} For a CDO tranche with attachment and detachment points \( K_L \) and \( K_U \), respectively, the indifference upfront fee is defined as the value \( U \) that solves the equation

\[
M(0, x, v, n, \lambda) = V(0, x + U(K_U - K_L), v, n, \lambda; R),
\]

where \( M \) is defined by (3.7) and \( V \) is defined by (3.34). In terms of the corresponding transformed value functions \( g^{(n)} \) and \( h^{(n)} \), the indifference price is given by

\[
U = \frac{e^{-rT}}{\gamma(K_U - K_L)} \log \left( \frac{h^{(n)}(0, v, \lambda; R; \gamma)}{g^{(n)}(0, v, \lambda)} \right). \tag{3.35}
\]

Note that the indifference upfront fee depends on the running spread \( R \) and the risk aversion parameter \( \gamma \).

Thus, if we know the risk aversion parameter \( \gamma \), we can compute the indifference

\footnote{Note that we have already explicitly accounted for jumps in wealth from the external cash flows.}
\footnote{We note that the following definition, corollary, and claim also hold in the general case with multiple payment dates (see Section 3.3.2).}
upfront fee explicitly. On other hand, if we are given the upfront fee $U$ from the market, we can determine the implied risk aversion parameter $\gamma$ implicitly, as described next.

**Corollary 3.3.1.** Given the upfront fee $U$ and running spread $R$, as observed from the market data, the tranche holder’s implied risk aversion $\gamma$ is the value of $\gamma$ that solves (3.35) implicitly.

We expect the upfront fee to be increasing in the risk aversion parameter $\gamma$, because the more risk averse the investor is, the greater insurance (as measured by the size of the upfront fee) that the investor will require in order to protect against his losses. This result is also verified numerically in Section 3.6.2.

**Claim 3.3.1.** The upfront fee $U$ is a strictly increasing function of the risk aversion parameter $\gamma$. Hence, the implied risk aversion $\gamma$ is unique, if it exists.

**Proof.** This result follows by examining the PDE for the transformed function

$$
\phi^{(n)}(t, v, \lambda) = \frac{e^{-r(T-t)}}{\gamma(K_U - K_L)} \log \left( \frac{h^{(n)}(t, v, \lambda; R; \gamma)}{g^{(n)}(t, v, \lambda)} \right),
$$

whence $U = \phi^{(n)}(0, v, \lambda)$, observing that the terminal condition $\phi^{(n)}(T, v, \lambda) = -\frac{P_T(n)}{K_U - K_L}$ is independent of $\gamma$, and using comparison principles, as in Theorem 4.3 of [Musiela and Zariphopoulou (2001)].

### 3.3.2 Multiple Payment Dates

Here, we consider the general case, where the payment dates are $T_1, T_2, \ldots, T_K$. In order to maximize terminal wealth, we do a recursive optimization, starting from maturity $T = T_K$ and going back to time $T_{K-1}$ and so on until the initial time 0. Using the same reasoning as in the single payment case and repeatedly applying Lemma [3.3.1], we obtain the following result for the transformed value function $h^{(n)}$ for the CDO tranche holder.
Proposition 3.3.1. In the last payment interval \((T_{K-1}, T]\), we have that \(h^{(n)}\), for \(n < \hat{N}\), satisfies the same PDE as \((3.32)\) but with a different terminal condition:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{h^{(n)}_t}{n} + \kappa(\tilde{\theta} - v)h^{(n)}_v + \frac{1}{2}\eta^2 v h^{(n)}_{vv} + \alpha(\Lambda - \lambda)h^{(n)}_\lambda + \frac{1}{2}\sigma^2 \lambda h^{(n)}_{\lambda\lambda} \\
- \frac{(\mu - r)^2}{2v} h^{(n)} - \frac{1}{2}\rho^2_S v^2 \frac{(h^{(n)}_v)^2}{h^{(n)}} + \lambda \left[ h^{(n+1)}(t, v + a, \lambda + \delta) - h^{(n)}(t, v, \lambda) \right] = 0, \quad (3.36)
\end{array} \right.
\end{align*}
\]

and \(h^{(\hat{N})}\) satisfies the PDE \((3.33)\) with a modified terminal condition:

\[
\begin{align*}
\left\{ \begin{array}{l}
h^{(\hat{N})}_t + \kappa(\tilde{\theta} - v)h^{(\hat{N})}_v + \frac{1}{2}\eta^2 v h^{(\hat{N})}_{vv} - \frac{(\mu - r)^2}{2v} h^{(\hat{N})} - \frac{1}{2}\rho^2_S v^2 \frac{(h^{(\hat{N})}_v)^2}{h^{(\hat{N})}} = 0, \quad (3.37)
\end{array} \right.
\end{align*}
\]

where \(P_T(n)\), for \(n = 0, \ldots, \hat{N}\), represents the net of the cash flows that are determined at the maturity time \(T\) when \(N_T = n\), that is,

\[
P_T(n) = R(T - T_{K-1}) \left[ K_U - K_L - F \left( \frac{(1 - \delta_r)n}{\hat{N}} \right) \right] - F \left( \frac{(1 - \delta_r)n}{\hat{N}} \right).
\]

In the interval \((T_{k-1}, T_k]\), for \(k = 1, \ldots, K - 1\), we have that \(h^{(n)}\), for \(n < \hat{N}\), and \(h^{(\hat{N})}\) satisfy the same PDEs \((3.36)\) and \((3.37)\), respectively, but with the following adjusted terminal condition at time \(T_k\):

\[
h^{(n)}(T_k, v, \lambda) = h^{(n)}(T_k^+, v, \lambda) e^{-\gamma P_{T_k}(n)} e^{(T_k+1 - T_k)}, \quad (3.38)
\]

where \(P_{T_k}(n)\), for \(n = 0, \ldots, \hat{N}\), represents the net of all the cash flows that are
determined (and in essence, paid out) at time $T_k$ when $N_{T_k} = n$.

$$P_{T_k}(n) = R(T_k - T_{k-1}) \left[ K_U - K_L - F \left( \frac{(1 - \delta_r)n}{N} \right) \right] - F \left( \frac{(1 - \delta_r)n}{N} \right) + F \left( \frac{(1 - \delta_r)n}{N} \right) e^{-r(T_{k+1} - T_k)},$$

and the exponential factor $e^{r(T_{k+1} - T_k)}$ in (3.38) is used to bring the net cash flow $P_{T_k}(n)$ forward from $T_k$ to $T_{k+1}$. \[7\]

Lastly, at time $t = 0$, we adjust the value function to account for the first cash flow from the protection leg:

$$h^{(n)}(0, v, \lambda) = h^{(n)}(0^+, v, \lambda) e^{-\gamma P_0(n)},$$

where $P_0(n) = F \left( \frac{(1 - \delta_r)n}{N} \right)$, for $n = 0, \ldots, \hat{N}$, is known at time 0 but paid out at time $T_1$. Note that this adjustment to the value function is equivalent to increasing the wealth at time 0 by the discounted amount $P_0(n) e^{-rT_1}$.

### 3.4 Equity Derivative Pricing

We now consider the indifference pricing of equity derivatives with payoffs of the form $P(S_T)$ at maturity $T$. For example, this includes calls and puts on the stock index, in which $P(S_T) = (S_T - K)^+$ and $P(S_T) = (K - S_T)^+$, respectively. In this equity derivative problem, we consider the Merton problem from Section 3.2.1 and the derivative holder’s problem, whose optimal control depends explicitly on the stock index.\[8\]

\[6\]We note that the cash flow $F \left( \frac{(1 - \delta_r)n}{N} \right)$ is known at time $T_k$ and paid out at time $T_{k+1}$, and so by Lemma 3.3.1, we may assume that the discounted cash flow $F \left( \frac{(1 - \delta_r)n}{N} \right) e^{-r(T_{k+1} - T_k)}$ is actually paid out at the earlier time $T_k$.

\[7\]By Remark 3.3.1, accounting for the net cash flow $P_{T_k}(n)$ that is determined and paid at $T_k$ is equivalent to multiplying the value function by the factor $e^{-\gamma P_{T_k}(n)} e^{r(T_{k+1} - T_k)}$.

\[8\]Recall that for the Merton problem, the optimal control is independent of the initial value of the stock index.
Derivative Holder’s Problem

The derivative holder’s problem is to maximize his expected utility of terminal wealth when he can invest in the money market, the stock index, and the equity derivative. We define the derivative holder’s value function by

\[
V(t, x, S, v, n, \lambda) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left\{ -e^{-\gamma(X_T + P(S_T))} \middle| X_t = x, S_t = S, v_t = v, N_t = n, \lambda_t = \lambda \right\}. \tag{3.39}
\]

Hence, the value function solves the following HJB PDE:

\[
\begin{cases}
  V_t + \sup_{\pi \in \mathcal{A}} \mathcal{L}^\pi_{(X,S,v,N,\lambda)} V = 0, \\
  V(T, x, S, v, n, \lambda) = -e^{-\gamma(x + P(S))},
\end{cases}
\tag{3.40}
\]

where \(\mathcal{L}^\pi_{(X,S,v,N,\lambda)}\) denotes the infinitesimal generator of the joint Markov process \((X, S, v, N, \lambda)\):

\[
\mathcal{L}^\pi_{(X,S,v,N,\lambda)} V = \left( (\mu - r) \pi + rx \right) \partial_x V + \frac{1}{2} v \pi^2 \partial_{xx} V + \mu S \partial_s V + \frac{1}{2} v S^2 \partial_{ss} V + Sv \pi \partial_s V + \kappa (\theta - v) \partial_v V + \frac{1}{2} \eta^2 v \partial_{vv} V + \rho S \eta v \partial_v S \partial_v V + \rho S \eta v \pi \partial_v \pi V + \alpha (\Lambda - \lambda) \partial_\lambda V + \frac{1}{2} \sigma^2 \lambda \partial_\lambda \lambda V + \lambda \left[ V(t, x, S, v + a, n + 1, \lambda + \delta) - V(t, x, S, v, n, \lambda) \right],
\]

with the simplifications \(\partial_\lambda = 0\) and \(\lambda = 0\) when \(n = \hat{N}\). Making the substitution

\[
V(t, x, S, v, n, \lambda) = -e^{-\gamma x \epsilon(T-t)} h(t, S, v, n, \lambda)
\]

yields the optimal control

\[
\pi^\ast(t, S, v, n, \lambda) = \frac{e^{-\gamma(T-t)}}{\gamma} \left( \frac{\mu - r}{v} + S \frac{h_s}{h} + \rho S \eta v \frac{h_v}{h} \right). \tag{3.41}
\]
Hence, we obtain the following non-linear PDE for $h$ when $n < \hat{N}$:

$$
\begin{aligned}
\begin{cases}
  h_t - \frac{(\mu - r)^2}{2v} h + r S h_S + \frac{1}{2} S^2 v h_{SS} - \frac{1}{2} S^2 v \frac{h_S^2}{h} - \frac{1}{2} \rho_S v^2 \eta v^2 \frac{h_v^2}{h} \\
  + \kappa (\tilde{\theta} - v) h_v + \frac{1}{2} \eta^2 vh_{vv} + \rho_S \eta S v \left( h_{Sv} - \frac{h_{Sv}}{h} \right) + \alpha (\Lambda - \lambda) h_\lambda \\
  + \frac{1}{2} \sigma^2 \lambda h_{\lambda \lambda} + \lambda [h(t, S, v + a, n + 1, \lambda + \delta) - h(t, S, v, n, \lambda)] = 0,
\end{cases}
\end{aligned}
$$

(3.42)

and the following PDE for $h$ when $n = \hat{N}$,

$$
\begin{aligned}
\begin{cases}
  h_t - \frac{(\mu - r)^2}{2v} h + r S h_S + \frac{1}{2} S^2 v h_{SS} - \frac{1}{2} S^2 v \frac{h_S^2}{h} - \frac{1}{2} \rho_S v^2 \eta v^2 \frac{h_v^2}{h} \\
  + \kappa (\tilde{\theta} - v) h_v + \frac{1}{2} \eta^2 vh_{vv} + \rho_S \eta S v \left( h_{Sv} - \frac{h_{Sv}}{h} \right) = 0,
\end{cases}
\end{aligned}
$$

(3.43)

Indifference Price

The buyer’s indifference price is the value $p$ at which the investor is indifferent, in terms of maximum expected utility, between not holding the derivative and holding the derivative but having his initial wealth reduced by $p$. We now define the indifference price precisely in terms of the Merton and derivative holder’s value functions.

**Definition 3.4.1.** The indifference price for an equity derivative that pays $P(S_T)$ at maturity $T$ is defined as the function $p \equiv p(t, S, v, n, \lambda)$ such that

$$
M(t, x, v, n, \lambda) = V(t, x - p, S, v, n, \lambda),
$$

where $M$ and $V$ are defined by (3.7) and (3.39), respectively. In terms of the corre-
sponding transformed value functions \( g \) and \( h \), the indifference price is given by

\[
p(t, S, v, n, \lambda) = \frac{e^{-r(T-t)}}{\gamma} \log \left( \frac{g(t, v, n, \lambda)}{h(t, S, v, n, \lambda)} \right).
\] (3.44)

For a put option on the stock index, we expect the buyer’s price to be increasing in the risk aversion parameter \( \gamma \), because the more risk averse the investor is, the greater insurance (as measured by the price of the put option) that the investor will require in order to protect against his losses when he is holding the stock. This result is shown below and also verified numerically in Section 3.6.3.

**Claim 3.4.1.** In the case of a put option, where \( P(S_T) = (K - S_T)^+ \), the indifference price \( p \) is a strictly increasing function of the risk aversion parameter \( \gamma \). Hence, the implied risk aversion \( \tilde{\gamma} \) is unique, if it exists.

**Proof.** This result follows by examining the PDE for \( p(t, S, v, n, \lambda) \), observing that the terminal condition \( p(T, S, v, n, \lambda) = P(S) \) is independent of \( \gamma \), and using comparison principles, as in Theorem 4.3 of [Musiela and Zariphopoulou 2001].

### 3.5 Implementation

In this section, we discuss two implementations for solving the maximal expected utility problems of Sections 3.2 and 3.3. We note that the implementation for the equity derivatives problem in Section 3.4 is similar, but it involves an extra dimension for the stock index. First, we describe our methodology for solving the valuation PDEs with explicit finite differences, and then we describe an alternate approach where we write the PDEs as expectations and discretize the underlying processes using trinomial trees.
3.5.1 Finite Differences

Here we describe our approach for solving the transformed PDEs \( g^{(n)} \) and \( h^{(n)} \) by use of explicit finite differences. We first look at the implementation for the Merton PDE, and then we examine the claim holder’s PDE, which follows a similar implementation but with modifications at the payment dates. In each case, we start with the boundary case \( n = \hat{N} \), where all firms have defaulted, and then we explain the middle case where \( n < \hat{N} \).

**Merton PDE**

**Boundary Case**

For \( n = \hat{N} \) (i.e., all firms have defaulted), we recall that

\[
g^{(\hat{N})} = u(t, v)^{\beta},
\]

where \( u \) satisfies the PDE (3.14) and \( \beta = (1 - \rho_{Sv}^2)^{-1} \).

We solve the PDE (3.14) for \( u \) by using an explicit finite difference scheme. First, we discretize the time dimension into \( I \) equally spaced steps \( t_0, t_1, \ldots, t_I = T \), that is,

\[
t_i = i\Delta t, \quad i = 0, \ldots, I, \quad \text{with } \Delta t = T/I.
\]

Similarly, we discretize the spatial dimension into \( J \) equally spaced steps \( v_0, v_1, v_2, \ldots, v_J \), where

\[
v_j = v_{\min} + j\Delta v, \quad j = 0, \ldots, J, \quad \text{with } \Delta v = (v_{\max} - v_{\min})/J.
\]

Then, we solve the PDE backwards in time: setting \( s_i = T - t_i \) for \( i = 0, \ldots, I \), we

\[\text{footnote}{9}\text{For an introduction to finite difference methods for solving PDEs, see, e.g., Wilmott, Howison, and Dewynne (1995)}\]
start from \( s_0 = T \), proceed to \( s_1 \) and continue all the way until we reach \( s_I = 0 \). Hence, we have a two-dimensional grid in (reverse) time and variance for the value function \( u \), as follows:

\[
u^i_j := u(s_i, v_j), \quad 0 \leq i \leq I, \quad 0 \leq j \leq J.\]

On this grid, we have the following initial condition (in time), which corresponds to the terminal condition in the PDE:

\[
u^0_j = 1, \quad 0 \leq j \leq J.
\]

Then, we discretize the PDE (3.14) on the grid, as follows: for \( 0 \leq i \leq I - 1 \) and \( 1 \leq j \leq J - 1 \) (excluding the boundaries for the variance), we have

\[
u^{i+1}_j = \nu^i_j + \frac{\Delta t}{2\Delta v} \kappa (\bar{\theta} - v_j) (\nu^i_{j+1} - \nu^i_{j-1})
+ \frac{\Delta t}{(\Delta v)^2} 2\eta^2 v_j (\nu^i_{j+1} - 2\nu^i_j + \nu^i_{j-1})
- \Delta t \frac{(\mu - r)^2}{2\beta v_j}\nu^i_j.
\]

(3.45)

Observing that \( v_{\min} \leq v \leq v_{\max} \), we impose the following boundary conditions in the variance dimension:

\[
\begin{align*}
\{ \quad & u(s, v) = u_0(s, v) \quad \text{at } v = v_{\min}, \\
& u_{vv}(s, v) = 0 \quad \text{at } v = v_{\max},
\end{align*}
\]

where \( u_0 \) is the solution for \( u \) in the no-diffusion case (when \( \eta = 0 \) in (3.14)) and is given explicitly by\(^{10}\)

\[
u_0(s, v) = \left( \frac{\tilde{\theta} e^{\kappa(T-s)} + v - \tilde{\theta}}{v} \right) \frac{(\mu - r)^2}{2\beta \kappa \theta}.
\]

\(^{10}\)This formula can be derived by carefully evaluating the integral in (3.15) when \( \eta = 0 \).
Indeed, for large \( v \), we set the second derivative with respect to \( v \) to be zero, that is, we assume the value function to be linear at the right endpoint. For small \( v \), we set \( u(t, v) \) equal to the no-diffusion solution because we expect the diffusion term \( \frac{1}{2} \eta^2 v u_{vv} \) in the PDE (3.14) to be small for \( v \) near zero. On the grid, these boundary conditions are specified by

\[
\begin{cases}
  u^i_0 = u_0(s_i, v_{\min}), & 1 \leq i \leq I, \\
  u^i_J = 2u^i_{J-1} - u^i_{J-2}, & 1 \leq i \leq I.
\end{cases}
\]  

(3.46)

Middle Case

Let us describe our explicit finite difference scheme for the transformed Merton value function \( g^{(n)} \), which we solve recursively from \( n = \hat{N} - 1 \) down to \( n = 0 \). Recall that \( g^{(n)} \), for \( n < \hat{N} \), satisfies the PDE (3.11).

Fixing \( n \in \{0, 1, \ldots, \hat{N} - 1\} \), we solve the PDE (3.11) for \( g^{(n)} \) by using an explicit finite difference scheme. As in the boundary case above, we discretize the time dimension into \( I \) equally spaced steps and the variance dimension into \( J \) equally spaced steps. In addition, we now discretize the intensity dimension into \( L \) equally spaced steps \( \lambda_0, \lambda_1, \ldots, \lambda_L \), where

\[
\lambda_l = \lambda_{\min} + l \Delta \lambda, \quad l = 0, \ldots, L, \quad \text{with} \quad \Delta \lambda = (\lambda_{\max} - \lambda_{\min})/L.
\]

Again, we solve the PDE backwards in time: setting \( s_i = T - t_i \), \( i = 0, \ldots, I \), we start from \( s_0 = T \), proceed to \( s_1 \) and continue all the way until we reach \( s_I = 0 \). Then, we define the value function \( g^{(n)} \) on the three-dimensional grid in time, variance, and intensity, as follows:

\[
g_{j,l}^{(n),i} := g^{(n)}(s_i, v_j, \lambda_l), \quad 0 \leq i \leq I, \quad 0 \leq j \leq J, \quad 0 \leq l \leq L.
\]
On this grid, we have the following initial condition (in time), which corresponds to the terminal condition in the PDE:

\[ g^{(n),0}_{j,l} = 1, \quad 0 \leq j \leq J, \quad 0 \leq l \leq L. \]

Then, we discretize the PDE (3.11) on the grid, as follows: for \( 0 \leq i \leq I - 1, 1 \leq j \leq J - 1 \) and \( 1 \leq l \leq L - 1 \) (excluding the boundaries for the variance and intensity), we have

\[
\begin{align*}
g^{(n),i+1}_{j,l} &= g^{(n),i}_{j,l} + \frac{\Delta t}{2\Delta v} \kappa (\bar{\theta} - v_j) \left( g^{(n),i}_{j+1,l} - g^{(n),i}_{j-1,l} \right) + \frac{\Delta t}{(\Delta v)^2} \frac{1}{2} \eta^2 v_j \left( g^{(n),i}_{j+1,l} - 2g^{(n),i}_{j,l} + g^{(n),i}_{j-1,l} \right) \\
&\quad + \frac{\Delta t}{2\Delta \lambda} \alpha (\Lambda - \lambda_l) \left( g^{(n),i}_{j,l+1} - g^{(n),i}_{j,l-1} \right) + \frac{\Delta t}{(\Delta \lambda)^2} \frac{1}{2} \sigma^2 \lambda_l \left( g^{(n),i}_{j+1,l} - 2g^{(n),i}_{j,l} + g^{(n),i}_{j-1,l} \right) \\
&\quad - \Delta t \frac{(\mu - r)^2}{2v_j} g^{(n),i}_{j,l} - \frac{\Delta t}{2} \rho_S v \eta^2 v_j \left( \frac{g^{(n),i}_{j+1,l} - g^{(n),i}_{j-1,l}}{2\Delta v} \right) \frac{1}{2} g^{(n),i}_{j,l} \\
&\quad + \Delta t \lambda_l \left[ g^{(n+1)}(s_i, v_j + a, \lambda_l + \delta) - g^{(n),i}_{j,l} \right].
\end{align*}
\]

Note that the shifted term \( g^{(n+1)}(s_i, v_j + a, \lambda_l + \delta) \) in (3.47) is computed by linear interpolation of the two-dimensional grid (with time fixed) for variance and intensity\(^{11}\)

Observing the bounds \( v_{\min} \leq v \leq v_{\max} \) and \( \lambda_{\min} \leq \lambda \leq \lambda_{\max} \), we impose the following boundary conditions for \( g^{(n)} \):

\[
\begin{align*}
\partial_v g^{(n)}(s, v, \lambda) &= 0 \quad \text{at } v = v_{\min}, \\
\partial_v g^{(n)}(s, v, \lambda) &= 0 \quad \text{at } v = v_{\max}, \\
\partial_{\lambda\lambda} g^{(n)}(s, v, \lambda) &= 0 \quad \text{at } \lambda = \lambda_{\min}, \\
g^{(n)}(s, v, \lambda) &= g^{(n+1)}(s, v + a, \lambda) \quad \text{at } \lambda = \lambda_{\max}.
\end{align*}
\]

In other words, we set the second derivative with respect to \( v \) to be zero at both variance endpoints, that is, we assume the value function is linear in \( v \) at the endpoints.

\(^{11}\)For fast two-dimensional interpolation, we use the MATLAB function \texttt{qinterp2}, which is available for download at http://www.mathworks.com/matlabcentral/fileexchange/10772-fast-2-dimensional-interpolation.
For the boundary conditions in the intensity dimension, we assume the value function is linear at the left endpoint and takes a shifted value at the right endpoint. On the grid, the boundary conditions for time step $i$, $1 \leq i \leq I$, are specified by

$$
\begin{align*}
&g^{(n),i}_{0,l} = 2g^{(n),i}_{1,l} - g^{(n),i}_{2,l}, \quad l = 1, \ldots, L - 1, \\
&g^{(n),i}_{J,l} = 2g^{(n),i}_{J-1,l} - g^{(n),i}_{J-2,l}, \quad l = 1, \ldots, L - 1, \\
&g^{(n),i}_{j,0} = 2g^{(n),i}_{j,1} - g^{(n),i}_{j,2}, \quad j = 0, \ldots, J, \\
&g^{(n),i}_{j,L} = g^{(n+1)}(s_i, v_j + a, \lambda_L), \quad j = 0, \ldots, J,
\end{align*}
$$

(3.48)

where the $g^{(n+1)}$ term in the last boundary condition is computed by linear interpolation of the grid. Note that in the case of the “corner” boundary conditions, that is, when $(j, l) \in \{(0, 0), (J, 0), (0, L), (J, L)\}$, we give preference to the intensity boundary condition since the intensity, in general, has a greater impact than the variance has on the indifference price.

**Claim Holder’s PDE**

**Boundary Case**

For the *European claim* in the boundary case when $n = \hat{N}$, we write $h^{(\hat{N})}$ in the form (3.21) and make the power transformation (3.13) to get

$$
h^{(\hat{N})}(t, v) = e^{-\gamma P(\hat{N})} u(t, v)^\beta,
$$

where $u$ is computed on the grid by finite differences via (3.45) with the boundary conditions (3.46).

For *CDO tranche*, there does not appear to be such a simple formula for $h^{(\hat{N})}$ in terms of $u$ because of the multiple adjustments at the payment dates. Instead, we follow the procedure in Proposition 3.3.1 as explained in the next part for the middle case $n < \hat{N}$.
Middle Case

Regarding the PDE for \( h^{(n)} \), we use the same finite difference scheme as that for \( g^{(n)} \), as described above, with the only change being the modified payoffs at the payment dates. We first consider a European claim with a single payment at maturity, and then we describe the modifications for CDO tranches with multiple payment dates.

For a European claim that pays out \( P(n) \) at maturity, we have the following initial condition (since time is reversed) on the grid:

\[
h_{j,l}^{(n),0} = e^{-\gamma P(n)}, \quad 0 \leq j \leq J, \ 0 \leq l \leq L.
\]

We employ the finite difference scheme (3.47) and the boundary conditions (3.48), with \( h^{(n)} \) in place of \( g^{(n)} \).

For a CDO tranche with multiple payment dates, we also use the finite difference scheme (3.47) and the boundary conditions (3.48), with \( h^{(n)} \) in place of \( g^{(n)} \), but we must account for the modifications at the payment dates \( T_0, T_1, \ldots, T_K = T \), according to Proposition 3.3.1. In particular, for the terminal time \( T \), which corresponds to the initial time step \( i = 0 \) on the grid, we have

\[
h_{j,l}^{(n),0} = e^{-\gamma P_T(n)}, \quad 0 \leq j \leq J, \ 0 \leq l \leq L,
\]

where

\[
P_T(n) = R(T - T_{K-1}) \left[ K_U - K_L - F \left( \frac{(1 - \delta_r)n}{N} \right) \right] - F \left( \frac{(1 - \delta_r)n}{N} \right).
\]

For the payment date \( T_k, k = K - 1, \ldots, 1 \), corresponding to time step \( i = \frac{T - T_k}{\Delta t} \) on the grid, we set

\[
h_{j,l}^{(n),i} = h_{j,l}^{(n),i} \cdot e^{-\gamma \left[ P_{T_k}(n) e^{\gamma(T_{k+1} - T_k)} \right]}, \quad 0 \leq j \leq J, \ 0 \leq l \leq L,
\]
where
\[ P_{T_k}(n) = R(T_k - T_{k-1}) \left[ K_U - K_L - F\left(\frac{(1-\delta_{r})n}{N}\right) - F\left(\frac{(1-\delta_{r})n}{N}\right) + F\left(\frac{(1-\delta_{r})n}{N}\right) e^{-r(T_{k+1} - T_k)} \right]. \]

Lastly, at time 0, which corresponds to the final time step \( i = I \) on the grid, we modify the value function as follows:
\[ h^{(n),I}_{j,l} = h^{(n),I}_{j,l} e^{-\gamma P_{0}(n)}, \quad 0 \leq j \leq J, \quad 0 \leq l \leq L, \]
where
\[ P_{0}(n) = F\left(\frac{(1-\delta_{r})n}{N}\right). \]

### 3.5.2 Trinomial Trees

Here we describe an alternative approach for solving the PDEs for \( g^{(n)} \) and \( h^{(n)} \), using trinomial trees for the underlying volatility and intensity processes. We first apply the Feynman-Kac formula to write the PDEs as expectations under an equivalent probability measure, and then, to evaluate the expectations, we use trinomial trees to model the volatility and intensity processes. This “tree approach” allows us to avoid directly specifying the boundary conditions that were required in the finite-difference PDE approach. We note, however, that the computation time for the tree approach is longer because the final result only applies for a single choice of the initial parameters \( (v, \lambda) \) as opposed to a grid of values for the finite difference approach.

Below we describe in detail our approach for computing the transformed Merton value function \( g^{(n)} \), and we note that our approach for the transformed European claim holder’s or tranche holder’s value function \( h^{(n)} \) is similar, except that we also need to account for the cash flows at the payment dates, as described in Proposition 3.3.1.
Merton PDE

We note that from Equation (3.11), in the case \( n < \hat{N} \), the transformed Merton value function \( g^{(n)} \) satisfies the PDE

\[
\begin{align*}
(\partial_t + \mathcal{L}_v + \mathcal{L}_\lambda)g^{(n)}(t, v, \lambda) + \Psi^{(n)}(t, v, \lambda) - \left( \frac{(\mu - r)^2}{2v} + \lambda \right) g^{(n)}(t, v, \lambda) &= 0, \\
g^{(n)}(T, v, \lambda) &= 1,
\end{align*}
\]

(3.49)

where the differential operators \( \mathcal{L}_v \) and \( \mathcal{L}_\lambda \) are defined, respectively, by

\[
\begin{align*}
\mathcal{L}_v &= \kappa (\tilde{\theta} - v) \partial_v + \frac{1}{2} \eta^2 v \partial_{vv}, \\
\mathcal{L}_\lambda &= \alpha (\Lambda - \lambda) \partial_\lambda + \frac{1}{2} \sigma^2 \lambda \partial_{\lambda\lambda},
\end{align*}
\]

and the source term \( \Psi^{(n)} \) is given by

\[
\Psi^{(n)}(t, v, \lambda) = -\frac{1}{2} \rho^2 S v \eta^2 v \left( \frac{g^{(n)}(t, v, \lambda)}{g^{(n)}(t, v, \lambda)} \right)^2 + \lambda g^{(n+1)}(t, v + a, \lambda + \delta).
\]

(3.50)

For the boundary case \( n = \hat{N} \), \( g^{(\hat{N})} \) satisfies the simplified PDE

\[
\begin{align*}
(\partial_t + \mathcal{L}_v)g^{(\hat{N})}(t, v) + \Psi^{(\hat{N})}(t, v) - \frac{(\mu - r)^2}{2v} g^{(\hat{N})}(t, v) &= 0, \\
g^{(\hat{N})}(T, v) &= 1,
\end{align*}
\]

(3.51)

where the source term \( \Psi^{(\hat{N})} \) is given by the simpler expression

\[
\Psi^{(\hat{N})}(t, v) = -\frac{1}{2} \rho^2 S v \eta^2 v \left( \frac{g^{(\hat{N})}(t, v)}{g^{(\hat{N})}(t, v)} \right)^2.
\]

(3.52)

Feynman-Kac Representation

We wish to apply the Feynman-Kac formula to represent the PDEs (3.49) and (3.51) as expectations under an equivalent martingale measure. First, we discretize the time
dimension into \( I \) equally spaced steps \( t_1, t_2, \ldots, t_I = T \), that is,

\[
t_i = i \Delta t, \quad i = 1, \ldots, I, \quad \text{with} \quad \Delta t = T/I.
\]

Then, starting from the maturity \( T = t_I \), we can write the value function \( g^{(n)} \) at the earlier time \( t_{I-1} \) as an expectation of a functional of the variance and intensity processes at times \( u \in (t_{I-1}, t_I] \). Then we proceed to time \( t_{I-2} \) and compute a similar expectation, and we repeat this procedure until \( t_0 = 0 \). In particular, for successive times \( t_i \) and \( t_{i+1} \), we apply the following proposition, which follows by the Feynman-Kac formula.

**Proposition 3.5.1.** Let \( 0 \leq t_i < t_{i+1} \leq T \), and suppose \( g^{(n)} \), for \( n < \hat{N} \), satisfies (3.49) and \( g^{(\hat{N})} \) satisfies (3.51). Then, there exists a probability measure \( \tilde{P} \) such that, for \( n < \hat{N} \),

\[
g^{(n)}(t_i, v, \lambda) = \tilde{E}\left\{ \exp\left( -\int_{t_i}^{t_{i+1}} \left( \frac{(\mu - r)^2}{2v_s} + \tilde{\lambda}_s \right) ds \right) g^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) + \int_{t_i}^{t_{i+1}} \exp\left( -\int_{t_i}^{u} \left( \frac{(\mu - r)^2}{2v_s} + \tilde{\lambda}_s \right) ds \right) \Psi^{(n)}(u, \tilde{v}_u, \tilde{\lambda}_u)du \bigg| \tilde{v}_{t_i} = v, \tilde{\lambda}_{t_i} = \lambda \right\},
\]

(3.53)

and, for \( n = \hat{N} \),

\[
g^{(\hat{N})}(t_i, v) = \tilde{E}\left\{ \exp\left( -\int_{t_i}^{t_{i+1}} \frac{(\mu - r)^2}{2v_s} ds \right) g^{(\hat{N})}(t_{i+1}, \tilde{v}_{t_{i+1}}) + \int_{t_i}^{t_{i+1}} \exp\left( -\int_{t_i}^{u} \frac{(\mu - r)^2}{2v_s} ds \right) \Psi^{(\hat{N})}(u, \tilde{v}_u)du \bigg| \tilde{v}_{t_i} = v \right\}.
\]

(3.54)

Here, \( \tilde{E} \) is the expectation operator under \( \tilde{P} \), and \( (\tilde{v}_t) \) and \( (\tilde{\lambda}_t) \) satisfy the respective SDEs

\[
d\tilde{v}_t = \kappa(\tilde{\theta} - \tilde{v}_t)dt + \eta \sqrt{\tilde{v}_t}d\tilde{W}_t^v,
\]

\[
d\tilde{\lambda}_t = \lambda(\Lambda - \tilde{\lambda}_t)dt + \sigma \sqrt{\tilde{\lambda}_t}d\tilde{W}_t^\lambda,
\]

(3.55)
where $\tilde{W}^v$ and $\tilde{W}^\lambda$ are independent $\tilde{P}$-Brownian motions.

Note that the $(\tilde{v}_t)$ and $(\tilde{\lambda}_t)$ defined above in (3.55) are independent Cox-Ingersoll-Ross (CIR) processes without jumps and hence can be simulated using trinomial trees, as we will discuss below. First, we do an approximation of the integrals that appear in the above expectations.

**Integral Approximation**

To evaluate the expectations (3.53) and (3.54), we approximate the integrals there by taking the values of the processes at the right endpoints of the time interval; in particular, for $s \in (t_i, t_{i+1}]$, we take

$$\tilde{v}_s \approx \tilde{v}_{t_{i+1}}, \quad \tilde{\lambda}_s \approx \tilde{\lambda}_{t_{i+1}}, \quad \text{and} \quad g^{(n)}(s, v, \lambda) \approx g^{(n)}(t_{i+1}, v, \lambda) \quad \forall v, \lambda.$$

Then, for $n < \hat{N}$, setting $F_{t_{i+1}} = \frac{(\mu-\sigma)^2}{2\tilde{v}_{t_{i+1}}} + \tilde{\lambda}_{t_{i+1}}$, we have the approximation

$$g^{(n)}(t_i, v, \lambda) \approx \tilde{E}\left\{ e^{-(t_{i+1}-t_i)F_{t_{i+1}}} g^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) + \int_{t_i}^{t_{i+1}} e^{-(u-t_i)F_{t_{i+1}}} du \cdot \Psi^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) \mid \tilde{v}_{t_i} = v, \tilde{\lambda}_{t_i} = \lambda \right\}$$

$$= \tilde{E}\left\{ e^{-(t_{i+1}-t_i)F_{t_{i+1}}} g^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) + \frac{1 - e^{-(t_{i+1}-t_i)F_{t_{i+1}}}}{F_{t_{i+1}}} \Psi^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) \mid \tilde{v}_{t_i} = v, \tilde{\lambda}_{t_i} = \lambda \right\}. \quad (3.56)$$
For the case \( n = \hat{N} \), setting \( \bar{F}_{t_{i+1}} = \frac{(\mu-r)^2}{2\bar{v}_{t_{i+1}}} \), we have a similar approximation

\[
g^{(\hat{N})}(t_i, v) \approx \mathbb{E} \left\{ e^{-(t_{i+1} - t_i)\bar{F}_{t_{i+1}}} g^{(\hat{N})}(t_{i+1}, \bar{v}_{t_{i+1}}) + \int_{t_i}^{t_{i+1}} e^{-(u-t_i)\bar{F}_{t_{i+1}}} du \cdot \Psi^{(\hat{N})}(t_{i+1}, \bar{v}_{t_{i+1}}) \bigg| \bar{v}_{t_i} = v \right\} = \mathbb{E} \left\{ e^{-(t_{i+1} - t_i)\bar{F}_{t_{i+1}}} g^{(\hat{N})}(t_{i+1}, \bar{v}_{t_{i+1}}) + \frac{1 - e^{-(t_{i+1} - t_i)\bar{F}_{t_{i+1}}}}{\bar{F}_{t_{i+1}}} \Psi^{(\hat{N})}(t_{i+1}, \bar{v}_{t_{i+1}}) \bigg| \bar{v}_{t_i} = v \right\}. \tag{3.57}
\]

With these right-endpoint approximations, we see that \( g^{(n)} \) in (3.56) and \( g^{(\hat{N})} \) in (3.57) are simply expectations of functions of the random variables \( \bar{v}_{t_{i+1}} \) and \( \bar{\lambda}_{t_{i+1}} \) (which are both determined at time \( t_{i+1} \)) rather than expectations of functionals of the two processes \( \bar{v} \) and \( \bar{\lambda} \) on \( (t_i, t_{i+1}] \). Since we know the values of \( g^{(n)}(t_{i+1}, v, \lambda) \) for the entire grid of \( v \) and \( \lambda \) from the previous iteration (i.e., time \( t_{i+1} \)), all that remains is to determine the possible values and probabilities for \( \bar{v}_{t_{i+1}} \) and \( \bar{\lambda}_{t_{i+1}} \), and we do this by branching out the processes from the nodes \( \bar{v}_{t_i} = v \) and \( \bar{\lambda}_{t_i} = \lambda \), according to the trinomial tree. The trinomial tree construction for these processes is explained in the next part of this subsection.

**Remark 3.5.1.** For the boundary case \( n = \hat{N} \), we can obtain a simpler expression than (3.57) by first making the power transformation (3.13), that is, \( g^{(\hat{N})}(t, v) = u(t, v)^\beta \), with \( \beta = (1 - \rho_S^2)^{-1} \). Recall that, then, \( u \) satisfies the PDE (3.14), which yields the following Feynman-Kac representation for \( t_i < t_{i+1}, 0 \leq i \leq I - 1 \):

\[
u(t_i, v) = \mathbb{E} \left\{ \exp \left\{ - \int_{t_i}^{t_{i+1}} \frac{(\mu-r)^2}{2\beta\bar{v}_s} ds \right\} u(t_{i+1}, \bar{v}_{t_{i+1}}) \bigg| \bar{v}_{t_i} = v \right\}.
\]

An approximation of the above integral, by taking the value of the integrand at the
right endpoint, yields

\[ u(t_i, v) \approx \tilde{E} \left[ \exp \left( -(t_{i+1} - t_i) \frac{(\mu - r)^2}{2\beta \tilde{v}_{t_{i+1}}} \right) u(t_{i+1}, \tilde{v}_{t_{i+1}} \mid \tilde{v}_{t_i} = v) \right], \quad 0 \leq i \leq I - 1, \]

which can be computed recursively from the trinomial tree for \((\tilde{v}_t)\). Substituting back, we obtain the value function \(g^{(N)}\) at all of the tree nodes.

**Tree Construction**

To evaluate the expectations in (3.56) and (3.57), we use trinomial trees to model the volatility and intensity processes \((\tilde{v}_t)\) and \((\tilde{\lambda}_t)\) that satisfy the set of SDEs (3.55) in Proposition 3.5.1. Each tree is constructed by imposing that the conditional local mean and variance at each node are equal to those of the continuous-time process, and the geometry of the tree is then designed to ensure that the branching probabilities are positive. This methodology closely follows that described in Appendix F and Section 3.9.1 of [Brigo and Mercurio (2006)](https://doi.org/10.1007/978-3-540-34651-1), but with some variation in the truncations of the trees.

We now explain the algorithm for the volatility tree; the algorithm for the intensity tree is identical but uses different model parameters. Since the volatility of \((\tilde{v}_t)\) diverges, we first make a transformation to another process \(Y = (Y_t)_{t \geq 0}\) that has constant volatility. Defining \(Y_t = \sqrt{\tilde{v}_t}, \ t \geq 0\), we obtain by Itô’s Lemma,

\[
dY_t = \left[ \left( \frac{\kappa \tilde{\theta}}{2} - \frac{1}{8} \eta^2 \right) \frac{1}{Y_t} - \frac{\kappa}{2} Y_t \right] dt + \frac{\eta}{2} d\tilde{W}^v_t, \quad Y_0 = \sqrt{\tilde{v}_0},
\]

where \(\tilde{W}^v\) is a \(\tilde{P}\)-Brownian motion as before.

Let us now explain the structure of the tree for \(Y\). As described above, we discretize the time dimension into \(I\) equally sized steps \(t_0 = 0, t_1 = \Delta t, \ldots, t_I = \)
\[ I \Delta t = T, \text{ with } \Delta t = T/I. \] Then, at each time \( t_i \), we discretize the spatial dimension into equally sized steps, as follows:

\[ Y_{t_i} \in \{ y_j := j \Delta y \mid j = j^i_{\text{min}}, \ldots, j^i_{\text{max}} \}, \tag{3.59} \]

with \( \Delta y, j^i_{\text{max}}, \text{ and } j^i_{\text{min}} \) to be determined. To describe the branching of the tree, we assume that at time \( t_i \), when we are on the \( j^{th} \) node with associated value \( y_j \), the process can move to \( y_{k+1} \), \( y_k \), or \( y_{k-1} \) with probabilities \( p_u \), \( p_m \), and \( p_d \), respectively. The central node is therefore the \( k^{th} \) node at time \( t_{i+1} \), where \( k \) is to be suitably determined.\(^{12}\) This tree structure is illustrated in Figure 3.1 below.

\(^{12}\)We omit to express the explicit dependence of \( k \) on the index \( j \), for notational convenience.
Integrating the SDE \([3.58]\) from \(t_i\) to \(t_{i+1}\), we have

\[
Y_{t_{i+1}} = Y_{t_i} + \int_{t_i}^{t_{i+1}} \left[ \left( \frac{\kappa \tilde{\theta}}{2} - \frac{1}{8} \eta^2 \right) \frac{1}{Y_s} \right] ds + \frac{\eta}{2} \left( \tilde{W}_{t_{i+1}} - \tilde{W}_{t_i} \right) + \kappa Y_s.
\]

Hence, when \(\Delta t = t_{i+1} - t_i\) is small, we can approximate the conditional mean and variance of \(Y_{t_{i+1}}\), given \(Y_{t_i} = y_j\), as follows:

\[
\begin{align*}
\tilde{E}[Y_{t_{i+1}} | Y_{t_i} = y_j] &\approx y_j + \left[ \left( \frac{\kappa \tilde{\theta}}{2} - \frac{1}{8} \eta^2 \right) \frac{1}{y_j} - \frac{\kappa}{2} y_j \right] \Delta t =: M_j, \\
\tilde{\text{Var}}(Y_{t_{i+1}} | Y_{t_i} = y_j) &\approx \frac{\eta^2}{4} \Delta t =: V^2.
\end{align*}
\]

We wish to find the tree branching probabilities \(p_u, p_m,\) and \(p_d\) such that the conditional mean and variance implied by the tree match the approximated moments in \((3.60)\). Precisely, noting that \(y_{k+1} = y_k + \Delta y\) and \(y_{k-1} = y_k - \Delta y\), we look for positive constants \(p_u, p_m,\) and \(p_d\) summing up to one and satisfying

\[
\begin{align*}
p_u(y_k + \Delta y) + p_m y_k + p_d(y_k - \Delta y) &= M_j, \\
p_u(y_k + \Delta y)^2 + p_m y_k^2 + p_d(y_k - \Delta y)^2 &= V^2 + M_j^2.
\end{align*}
\]

Simple algebra leads to

\[
\begin{align*}
y_k + (p_u - p_d) \Delta y &= M_j, \\
y_k^2 + 2y_k \Delta y (p_u - p_d) + \Delta y^2 (p_u + p_d) &= V^2 + M_j^2,
\end{align*}
\]

and then setting \(\zeta_{j,k} = M_j - y_k\) yields

\[
\begin{align*}
(p_u - p_d) \Delta y &= \zeta_{j,k}, \\
(p_u + p_d) \Delta y^2 &= V^2 + \zeta_{j,k}^2.
\end{align*}
\]
Using $p_m = 1 - p_u - p_d$, the probabilities are given by

$$
\begin{align*}
    p_u &= \frac{V^2}{2\Delta y^2} + \frac{\zeta_{j,k}^2}{2\Delta y^2} + \frac{\zeta_{j,k}}{2\Delta y}, \\
    p_m &= 1 - \frac{V^2}{\Delta y^2} - \frac{\zeta_{j,k}^2}{\Delta y^2}, \\
    p_d &= \frac{V^2}{2\Delta y^2} + \frac{\zeta_{j,k}^2}{2\Delta y^2} - \frac{\zeta_{j,k}}{2\Delta y}.
\end{align*}
$$

(3.61)

Next, as argued by Hull and White (1993), for convergence purposes, we set

$$
\Delta y = V\sqrt{3}.
$$

which yields $y_j = j\Delta y = j\sqrt{3}V$. Then, the branching probabilities reduce to

$$
\begin{align*}
    p_u &= \frac{1}{6} + \frac{\zeta_{j,k}^2}{6V^2} + \frac{\zeta_{j,k}}{2\sqrt{3}V}, \\
    p_m &= \frac{2}{3} - \frac{\zeta_{j,k}^2}{3V^2}, \\
    p_d &= \frac{1}{6} + \frac{\zeta_{j,k}^2}{6V^2} - \frac{\zeta_{j,k}}{2\sqrt{3}V}.
\end{align*}
$$

(3.62)

Brigo and Mercurio (2006) suggest choosing the value of $y_k = k\Delta y$ at the central node to be as close as possible to $M_j$, taking

$$
k = \text{round} \left( \frac{M_j}{\Delta y} \right).$$

(3.63)

This approach ensures that the branching probabilities in (3.62) remain positive. Indeed, $p_u$ and $p_d$ are positive for every value of $\zeta_{j,k}$: we see this by viewing $p_u$ and $p_d$ as quadratic functions in $x := \frac{\zeta_{j,k}}{\sqrt{3}V}$, from which it follows that $p_u \geq 1/24$ and $p_d \geq 1/24$. Meanwhile, $p_m$ is positive if and only if $|\zeta_{j,k}| \leq V\sqrt{2}$; however, the choice (3.63) implies that $M_j \in \left[ \left( k - \frac{1}{2} \right) \Delta y, \left( k + \frac{1}{2} \right) \Delta y \right]$, and since $\zeta_{j,k} = M_j - y_k$.
and $\Delta y = V\sqrt{3}$, we have

$$|\zeta_{j,k}| = |M_j - k\Delta y| \leq \frac{1}{2}\Delta y = V\frac{\sqrt{3}}{2} < V\sqrt{2},$$

and hence, the positivity condition for $p_m$ is also satisfied.

We note, however, that taking (3.63) for the central node may force the tree to hit zero (or go negative in general). For example, if we are at node $y_1 = \Delta y$ and we have round $\left(\frac{M_1}{\Delta y}\right) = 1$, then according to the proposed branching with $k = 1$, we would move to $y_2 = 2\Delta y$, $y_1 = \Delta y$, or $y_0 = 0$ with positive probabilities $p_u$, $p_m$, and $p_d$, respectively. However, moving to level zero would be problematic because the mean at that node would be infinite ($M_0 = +\infty$) and hence, we would not be able to match the moments at the subsequent step.\footnote{Nawalkha and Beliaeva (2007) suggest that when the tree hits the level zero, one should match the mean of the original CIR process ($\tilde{v}_t$) but ignore the variance since it is zero, and then because of the extra degree of freedom, one should simply set the middle probability $p_m$ to zero. In our implementation, we instead choose to truncate the tree below some positive level.}

Also, we cannot allow the tree to go negative because it is approximating the positive process ($Y_t$).

We choose to truncate the tree below the level $y_{j_{\min}} > 0$, for some $j_{\min} > 0$ to be determined, and then we check that the branching probabilities at the bottom of the tree are positive. For computational efficiency, which we need to consider when the number of time steps is large, we also truncate the tree above the level $y_{j_{\max}}$, for some $j_{\max}$ to be determined. We assume that the tree starts at time 0 at the spatial node $j_0 = \text{round}(y_0/\Delta y)$ with $j_{\min} < j_0 < j_{\max}$; note that we can choose the $\Delta t$ and hence $\Delta y$ such that $j_0\Delta y$ is sufficiently close to $y_0$.\footnote{Recall that $\Delta y = V\sqrt{3} = \frac{2}{3}\sqrt{3}\Delta t$ from (3.60).} Then, the branching at the maximum and minimum nodes is as follows:

- Once we reach the maximum node $j_{\max}$, we move to nodes $j_{\max}$, $j_{\max} - 1$, and $j_{\max} - 2$ with respective probabilities $p_u$, $p_m$, and $p_d$, in which the central node $k$ is set to $j_{\max} - 1$.\footnote{Recall that $\Delta y = V\sqrt{3} = \frac{2}{3}\sqrt{3}\Delta t$ from (3.60).}
• Similarly, once we reach the minimum node $j_{\text{min}}$, we branch to nodes $j_{\text{min}} + 2$, $j_{\text{min}} + 1$, and $j_{\text{min}}$ with respective probabilities $p_u$, $p_m$, and $p_d$, in which the central node $k$ is set to $j_{\text{min}} + 1$.

Let us now describe the selection of the maximum node $j_{\text{max}}$ and minimum node $j_{\text{min}}$ in our model implementation. We observe that the variance process ($\tilde{v}_t$) is mean-reverting, and hence, the square-root process ($Y_t$) is also mean-reverting. Hence, we expect that for $y_j$ near the mean of the process, the central branching node will be at the same level as the current node, that is, $\text{round} \left( \frac{M_j}{\Delta y} \right) = j$; for large $y_j$, we expect the central branching node to be lower than the current node, that is, $\text{round} \left( \frac{M_j}{\Delta y} \right) < j$; and for small $y_j$, we expect the central branching node to be higher than the current node, that is, $\text{round} \left( \frac{M_j}{\Delta y} \right) > j$. This motivates us to choose the maximum node as

$$j_{\text{max}} = \min \left\{ j > j_0 : \text{round} \left( \frac{M_j}{\Delta y} \right) = j - 1 \right\},$$  

(3.64)

that is, $j_{\text{max}}$ is the smallest index $j$ larger than the starting node $j_0$ such that the conditional mean is closest to a lower node. Similarly, for the minimum node, we choose

$$j_{\text{min}} = \max \left\{ 1, \max \left\{ j < j_0 : \text{round} \left( \frac{M_j}{\Delta y} \right) = j + 1 \right\} \right\},$$  

(3.65)

that is, $j_{\text{min}}$ is the largest index $j$ less than $j_0$ such that the conditional mean is closest to a higher node, with the restriction $j_{\text{min}} \geq 1$, which ensures that the process does not hit zero. In this special case where $j_{\text{min}} = 1$ and $\text{round} \left( \frac{M_j}{\Delta y} \right) \leq j$ for all $j \geq 1$, we need to verify numerically that the probabilities of branching from node $y_1$ to nodes $y_1$, $y_2$, and $y_3$ are all positive, since positivity is not guaranteed from the analytical expressions for the probabilities. In our numerical analysis, all of the branching probabilities at the bottom of the tree stayed positive so we did not follow an intricate truncation procedure, such as that recommended by Nawalkha and...
To recap, our branching is as follows: from the $i^{th}$ time step and the $j^{th}$ spatial node, we move to the $k^{th}$ spatial node at time step $i + 1$, with

$$k = \begin{cases} 
  j - 1 & \text{if } j = j_{\text{max}}, \\
  j & \text{if } j_{\text{min}} < j < j_{\text{max}}, \\
  j + 1 & \text{if } j = j_{\text{min}},
\end{cases} \quad (3.66)$$

where the maximum and minimum nodes are determined by (3.64) and (3.65), respectively. Consequently, we have also determined that the set of values for the process $Y$ at time $t_i$, $0 \leq i \leq I$, is given by \{$y_j : j_{i_{\text{min}}}^i \leq j \leq j_{i_{\text{max}}}^i$\}, with

$$j_{i_{\text{min}}}^i = \max\{j_0 - i, j_{\text{min}}\} \quad \text{and} \quad j_{i_{\text{max}}}^i = \min\{j_0 + i, j_{\text{max}}\}.$$ 

Lastly, we reverse the initial transformation, via $\tilde{v}_t = Y_t^2$, to obtain the values of the variance process at the corresponding tree nodes. In particular, at time $t_i$, $0 \leq i \leq I$, we have

$$\tilde{v}_{t_i} \in \{\hat{v}_j := y_j^2 \mid j_{\text{min}}^i \leq j \leq j_{\text{max}}^i\}.$$ 

Having constructed the trinomial tree for the volatility process, we now repeat the above procedure to construct the trinomial tree for the intensity process ($\tilde{\lambda}_t$), after making the transformation

$$Z_t = \sqrt{\tilde{\lambda}_t}, \quad t \geq 0.$$

\footnote{Nawalkha and Beliaeva (2007) provide a detailed analysis for the truncation for the trinomial tree when the underlying process nears zero. To ensure the process stays positive, they allow for jumps of \textit{multiple steps} at the bottom of the tree (and at the top of the tree), and they show the choice $\Delta y = V \sqrt{3}$ (suggested by Hull and White (1993)) is not sufficient to guarantee positive probabilities. Indeed, Nawalkha and Beliaeva (2007) show that when choosing $\Delta y = bV$ for some constant $b$, a necessary condition is $1 \leq b \leq \sqrt{2}$. In particular, they choose the value of $b = \sqrt{1.5 + Adj}$, where Adj is a small adjustment term that keeps $b$ between 1 and $\sqrt{2}$ and allows for expansion ($Adj > 0$) or contraction ($Adj < 0$) of the step sizes.}
We assume that at time step $i$ and spatial node $l$, the tree for $(Z_i)$ branches out from $z_l$ to $z_{l+1}, z_k,$ and $z_{l-1}$, with respective probabilities $q_u, q_m,$ and $q_d$, where $k$ is the central node at time $i + 1$ determined analogously to (3.66). Once we have determined the tree nodes for $(Z_i)$, we can obtain the values of $(\lambda_i)$ on the tree: for $0 \leq i \leq I$, we have

$$\lambda_i \in \{ \lambda_i := z_l^2 \mid l_{\text{min}} \leq l \leq l_{\text{max}} \},$$

where $l_{\text{min}}$ and $l_{\text{max}}$ are defined analogously to the variance case.

Combining the trees for the two independent processes $(\tilde{v}_t)$ and $(\tilde{\lambda}_t)$, we obtain a two-dimensional tree with nodes marked by the ordered triples $(i, j, l)$, corresponding to the $i$th time step $t_i$, the $j$th volatility spatial step $\tilde{v}_j$, and the $l$th intensity spatial step $\tilde{\lambda}_l$. From node $(i, j, l)$, the tree can branch to 9 possible nodes at time step $i + 1$:

- up-up: $(i + 1, k + 1, \tilde{k} + 1)$
- mid-up: $(i + 1, k, \tilde{k} + 1)$
- down-up: $(i + 1, k - 1, \tilde{k} + 1)$
- up-mid: $(i + 1, k + 1, \tilde{k})$
- mid-mid: $(i + 1, k, \tilde{k})$
- down-mid: $(i + 1, k - 1, \tilde{k})$
- up-down: $(i + 1, k + 1, \tilde{k} - 1)$
- mid-down: $(i + 1, k, \tilde{k} - 1)$
- down-down: $(i + 1, k - 1, \tilde{k} - 1)$

with respective probabilities

$$\pi_{u,u}, \pi_{m,u}, \pi_{d,u},$$
$$\pi_{u,m}, \pi_{m,m}, \pi_{d,m},$$
$$\pi_{u,d}, \pi_{m,d}, \pi_{d,d},$$

where $\pi_{a,b} = p_a q_b$ for $a, b \in \{u, m, d\}$.

**Computing Expectations via the Trinomial Trees**

After building the trees for $(\tilde{v}_t)$ and $(\tilde{\lambda}_t)$, we are now able to compute the expectations (3.56) and (3.57) for $g(n)$ and $g(\tilde{N})$, respectively. For $n < \tilde{N}$, by defining the term inside the expectation as

$$\phi^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) = e^{-(t_{i+1} - t_i)F_{t_{i+1}} g^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}})} + \frac{1 - e^{-(t_{i+1} - t_i)F_{t_{i+1}}}}{F_{t_{i+1}}} \Psi^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}),$$

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we have the following approximation for $g^{(n)}$ at node $(i, j, l)$:

$$g^{(n)}(t_i, \tilde{v}_j, \tilde{\lambda}_l) \approx \widetilde{E} \left[ \phi^{(n)}(t_{i+1}, \tilde{v}_{i+1}, \tilde{\lambda}_{i+1}) \mid \tilde{v}_{t_i} = \tilde{v}_j, \tilde{\lambda}_i = \tilde{\lambda}_l \right]$$

$$\approx \pi_{u, u} \phi^{(n)}(t_{i+1}, \hat{v}_{k+1}, \hat{\lambda}_{k+1}) + \pi_{m, u} \phi^{(n)}(t_{i+1}, \hat{v}_{k}, \hat{\lambda}_k) + \pi_{d, u} \phi^{(n)}(t_{i+1}, \hat{v}_{k-1}, \hat{\lambda}_{k-1})$$

$$+ \pi_{u, m} \phi^{(n)}(t_{i+1}, \hat{v}_{k+1}, \hat{\lambda}_k) + \pi_{m, m} \phi^{(n)}(t_{i+1}, \hat{v}_{k}, \hat{\lambda}_k) + \pi_{d, m} \phi^{(n)}(t_{i+1}, \hat{v}_{k-1}, \hat{\lambda}_{k-1})$$

$$+ \pi_{u, d} \phi^{(n)}(t_{i+1}, \hat{v}_{k+1}, \hat{\lambda}_k) + \pi_{m, d} \phi^{(n)}(t_{i+1}, \hat{v}_{k}, \hat{\lambda}_k) + \pi_{d, d} \phi^{(n)}(t_{i+1}, \hat{v}_{k-1}, \hat{\lambda}_{k-1}).$$

For the boundary case $n = \hat{N}$, we define the term inside the expectation as

$$\phi^{(\hat{N})}(t_{i+1}, \tilde{v}_{i+1}) = e^{-(t_{i+1} - t_i)\hat{F}_{i+1}} g^{(\hat{N})}(t_{i+1}, \tilde{v}_{i+1}) + \frac{1 - e^{-(t_{i+1} - t_i)\hat{F}_{i+1}}}{\hat{F}_{i+1}} \Psi^{(\hat{N})}(t_{i+1}, \tilde{v}_{i+1}),$$

which leads to the following simpler approximation for $g^{(\hat{N})}$ at node $(i, j)$:

$$g^{(\hat{N})}(t_i, \hat{v}_j) \approx \widetilde{E} \left[ \phi^{(\hat{N})}(t_{i+1}, \tilde{v}_{i+1}) \mid \tilde{v}_{t_i} = \hat{v}_j \right]$$

$$\approx p_u \phi^{(\hat{N})}(t_{i+1}, \hat{v}_{k+1}) + p_m \phi^{(\hat{N})}(t_{i+1}, \hat{v}_k) + p_d \phi^{(\hat{N})}(t_{i+1}, \hat{v}_{k-1}).$$

Now, let us further examine the source term $\Psi^{(n)}$, for $n < \hat{N}$, that appears in the expression for $\phi^{(n)}$ above and is defined in (3.50), that is,

$$\Psi^{(n)}(t_{i+1}, \tilde{v}_{i+1}, \tilde{\lambda}_{i+1}) = -\frac{1}{2} \rho_{yv} \eta^2 \tilde{v}_{i+1} \left( \frac{g^{(n)}(t, \tilde{v}_{i+1}, \tilde{\lambda}_{i+1})}{g^{(n)}(t_{i+1}, \tilde{v}_{i+1}, \tilde{\lambda}_{i+1})} \right)^2 + \tilde{\lambda}_{i+1} g^{(n+1)}(t_{i+1}, \tilde{v}_{i+1} + a, \tilde{\lambda}_{i+1} + \delta).$$

For $y = \sqrt{\nu}$ and $z = \sqrt{\lambda}$, we define

$$\tilde{g}^{(n)}(t, y, z) = g^{(n)}(t, v, \lambda).$$

Then, we compute the partial derivative term $g_v^{(n)}$ above by using the chain rule and

\[\text{We note that the source term } \Psi^{(\hat{N})} \text{ for the boundary case } n = \hat{N}, \text{ as defined in (3.52), is a simpler version of the case } n < \hat{N}, \text{ so we do not explain it further.}\]
a central finite difference, as follows:

\[ g_v^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) = \left. \frac{\partial}{\partial v} g^{(n)}(t, v, \lambda) \right|_{t=t_{i+1}, v=\tilde{v}_{t_{i+1}}, \lambda=\tilde{\lambda}_{t_{i+1}}} \]

\[ = \left. \left( \frac{1}{2y} \cdot \frac{\partial}{\partial y} g^{(n)}(t, y, z) \right) \right|_{t=t_{i+1}, y=Y_{t_{i+1}}, z=Z_{t_{i+1}}} \]

\[ \approx \frac{1}{2Y_{t_{i+1}}} \left( \frac{\tilde{g}^{(n)}(t_{i+1} + \Delta y, Z_{t_{i+1}}) - \tilde{g}^{(n)}(t_{i+1}, Y_{t_{i+1}} - \Delta y, Z_{t_{i+1}})}{2\Delta y} \right). \]

As a further approximation (and to avoid issues at the endpoints), for each of the 9 cases \((\tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) \in \{\hat{v}_{k+1}, \hat{v}_k, \hat{v}_{k-1}\} \times \{\hat{\lambda}_{\tilde{k}+1}, \hat{\lambda}_\tilde{k}, \hat{\lambda}_{\tilde{k}-1}\},\) we take the partial derivative term to be the one based on the central node \((i + 1, k, \tilde{k}),\) that is,

\[ g_v^{(n)}(t_{i+1}, \tilde{v}_{t_{i+1}}, \tilde{\lambda}_{t_{i+1}}) \approx \frac{1}{2y_k} \left( \frac{\tilde{g}^{(n)}(t_{i+1}, y_{k+1}, z_{\tilde{k}}) - \tilde{g}^{(n)}(t_{i+1}, y_{k-1}, z_{\tilde{k}})}{2\Delta y} \right). \]

As for the shifted term that appears in \(\Psi^{(n)}\), we have

\[ g^{(n+1)}(t_{i+1}, \tilde{v}_{t_{i+1}} + a, \tilde{\lambda}_{t_{i+1}} + \delta) = \tilde{g}^{(n+1)}(t_{i+1}, \sqrt{Y_{t_{i+1}}^2 + a}, \sqrt{Z_{t_{i+1}}^2 + \delta}), \]

which we compute, for each of the 9 cases, by two-dimensional linear interpolation of the grid

\[ \{\tilde{g}^{(n+1)}(t_{i+1}, y_j, z_l) : j_{\min}^{i+1} \leq j \leq j_{\max}^{i+1}, l_{\min}^{i+1} \leq l \leq l_{\max}^{i+1}\}. \]

### 3.6 Numerical Results

In this section, we present numerical results for the indifference pricing of European claims, CDO tranches, and equity index options. For the latter two instruments, we also compute the implied risk aversions corresponding to market data, and finally, we compare the systematic risks in the two markets.

Throughout this section, we use the following set of parameters, unless otherwise
noted:

\[ \hat{N} = 25, n = 0, \gamma = 0.1, \mu = 0.05, r = 0, T = 1, t = 0; \]

observe that we consider a reduced number of firms (\( \hat{N} = 25 \) instead of 125) for computational efficiency. Also, we use the following variance parameters:

\[ v_t = 0.0535, \theta = 0.04, \kappa = 0.5, \eta = 0.2, \rho_{sv} = -0.6, a = 0.1; \]

and the following intensity parameters:

\[ \lambda_t = 2.5, \Lambda = 2, \alpha = 1.5, \sigma = 0.2, \delta = 0.2, \]

along with a fixed recovery rate of \( \delta_r = 0.35. \)

Our numerical computations are based on the implementation in Section 3.5 and in particular, the finite difference scheme of Section 3.5.1. We note that the alternative approach using trinomial trees, as described in Section 3.5.2, yields similar results and hence, we do not display them here.

### 3.6.1 European Claims

To illustrate the indifference pricing of European claims, we consider the simple linear payoff \( P(n) = n, \) that is, the payoff at maturity is the number of firms that have defaulted by that time.

Figure 3.2 plots the indifference price, which was computed using the finite difference scheme with the size of the three-dimensional grid chosen as \( I = 1000, J = 40 \) and \( L = 50. \) Figure 3.3 shows the indifference price as functions of both \( v \) and \( \lambda; \) note that the initial variance \( v \) has a slight impact on the price, while the initial intensity \( \lambda \) has a large impact, as we would expect in this case.
Impact of Default Feedback

In Figure 3.4, we plot the indifference price as functions of the feedback parameters $\delta$ and $a$ for the intensity and variance processes, respectively. We note that the impact of $\delta$ is significant, with nearly a 40% increase in the price when $\delta$ increases from 0
to 1, while the impact of \(a\) is minimal, with less than a 1% change in price when \(a\) goes from 0 to 0.2.

**Figure 3.4: Indifference Price as Functions of Feedback Parameters**

![Indifference Price as Functions of Feedback Parameters](image)

**Function of Risk Aversion**

In Figure 3.5 we test the impact of the risk aversion parameter \(\gamma\) on the indifference price. As expected, the indifference price is a decreasing function of \(\gamma\) because when the investor’s risk aversion increases, he is only willing to pay less to enter the contract.

**Optimal Strategy**

Recall that from (3.10) and (3.18), the optimal strategies for the Merton and claim holder problems (when \(n\) firms have defaulted) are given respectively by

\[
\pi^*(n)(t, v, \lambda) = \frac{e^{-r(T-t)} \left( \frac{\mu - r}{v} + \rho \sigma \eta g_v^{(n)}(t, v, \lambda) \right)}{g^{(n)}(t, v, \lambda)}
\]
Figure 3.5: Indifference Price as Function of Risk Aversion

\[ H(t, v, \lambda) = e^{-r(T-t)} \gamma \left( \frac{\mu - r}{v} + \rho_{SV} \eta g(v(t, v, \lambda)) \right). \]

As a comparison, in the non-diffusive case with \( \eta = 0 \), the optimal strategies are independent of \( \lambda \) and \( n \):

\[ \pi_M^{*(n)}(t, v, \lambda) = \pi_H^{*(n)}(t, v, \lambda) = e^{-r(T-t)} \frac{\mu - r}{v}. \]

The difference is that in the diffusive case, the optimal strategy contains the “volatility hedge”, \( \rho_{SV} \eta g^{(n)}(t, v, \lambda) \), which arises from the stochastic variance of the stock index. This term is negative because \( \rho_{SV} < 0 \) and the other terms \( g_v \) and \( g \) are positive, as we see from the definition of \( g \) and as verified numerically.

Figure 3.6 plots the optimal strategy \( \pi_M^{*(n)} \) on the \((v, \lambda)\) grid for the Merton problem in the diffusive case. Since the volatility hedge is minimal, the optimal strategy is nearly identical to the optimal strategy in the non-diffusive case, in which \( \pi^* \) is
inversely proportional to $\nu$ and constant in $\lambda$.

**Figure 3.6: Optimal Strategy for Merton Problem**

In addition, we note that due to the minimal volatility hedge, the optimal strategy is approximately inversely proportional to the risk aversion parameter $\gamma$. Thus, we have the intuitive result that the higher the risk aversion, the less money is held in the stock index and the more money is invested in the (risk-free) money market.

### 3.6.2 CDO Upfront Fees

In this subsection, we test the indifference valuation of upfront fees for CDO tranches by following the finite difference scheme of Section 3.5.1. In particular, we test the impact of the feedback parameters, the initial intensity level, and the risk aversion parameter. Then, based on market data from 2009 to 2010, we estimate the model parameters and compute the implied risk aversions.
We consider the following tranches (in %):


\[ [0, 3], [3, 7], [7, 10], [10, 15], [15, 30], [30, 100], \]

which correspond, respectively, to the Equity, Mezzanine 1, Mezzanine 2, Mezzanine 3, Senior, and Super Senior tranches of the CDX.NA.IG. Note that we do in fact consider the super senior tranche here in the indifference valuation model, as opposed to the intensity model of Chapter 2 where we considered only the first 5 tranches. Now, for each tranche, we compute the indifference upfront fee when the running spread is \( R = 100 \) bps or 500 bps, depending on the market convention; for details on the conventions, see Table 2.2 in Section 2.6.

**Impact of Feedback**

We now test the impact of the default feedback parameter \( \delta \) on the indifference upfront fee. Table 3.1 displays the upfront fee for \( \delta \in \{0, 0.1, 0.2, 0.5, 0.8, 1, 1.2\} \), with the other parameters given in the beginning of this section. Observe that by increasing the feedback parameter \( \delta \), the upfront fees rise because a larger feedback implies more defaults on average over the long term. In particular, from Equation (3.4), the long-term mean of the intensity process is given by

\[
\lim_{t \to \infty} \mathbb{E} \lambda_t = \frac{\alpha \Lambda}{\alpha - \delta},
\]

which is increasing for \( \delta \in (0, \alpha) \). Also, note that the feedback has greater impact for the senior tranches than for the junior tranches due to the tranche structure, that is, how the attachment and detachment points both increase with seniority.

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Table 3.1: Upfront Fee as Function of $\delta$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>0.8</th>
<th>1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>0.8830</td>
<td>0.8831</td>
<td>0.8831</td>
<td>0.8834</td>
<td>0.8836</td>
<td>0.8838</td>
<td>0.8840</td>
</tr>
<tr>
<td>Mezz 1</td>
<td>0.6286</td>
<td>0.6368</td>
<td>0.6450</td>
<td>0.6678</td>
<td>0.6879</td>
<td>0.7004</td>
<td>0.7117</td>
</tr>
<tr>
<td>Mezz 2</td>
<td>0.3662</td>
<td>0.3834</td>
<td>0.4005</td>
<td>0.4486</td>
<td>0.4919</td>
<td>0.5185</td>
<td>0.5430</td>
</tr>
<tr>
<td>Mezz 3</td>
<td>0.1384</td>
<td>0.1566</td>
<td>0.1751</td>
<td>0.2319</td>
<td>0.2884</td>
<td>0.3253</td>
<td>0.3606</td>
</tr>
<tr>
<td>Senior</td>
<td>0.0057</td>
<td>0.0106</td>
<td>0.0164</td>
<td>0.0396</td>
<td>0.0722</td>
<td>0.0987</td>
<td>0.1285</td>
</tr>
<tr>
<td>SuperSen</td>
<td>-0.0098</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0095</td>
<td>-0.0085</td>
<td>-0.0071</td>
<td>-0.0048</td>
</tr>
</tbody>
</table>

Function of Intensity

In Figure 3.7, we plot the upfront fees as functions of the initial intensity $\lambda$ for all of the tranches. As expected, the upfront fees are increasing in intensity, and they increase at a faster rate for the junior tranches than for the senior tranches as a result of the tranche structure.

![Figure 3.7: Upfront Fees as Functions of Intensity](image)

Function of Risk Aversion

Table 3.2 shows the results for the upfront fee $U$ as a function of the risk aversion parameter $\gamma$. In particular, we take $\gamma \in \{0.01, 0.1, 0.5, 1, 3, 5, 10, 20\}$, and we find
that the upfront fee is strictly increasing in the risk aversion parameter $\gamma$, as claimed in Section 3.3.1.

### Table 3.2: Upfront Fee as Function of $\gamma$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.01</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>0.8821</td>
<td>0.8822</td>
<td>0.8827</td>
<td>0.8834</td>
<td>0.8862</td>
<td>0.8888</td>
<td>0.8951</td>
<td>0.9061</td>
</tr>
<tr>
<td>Mezz 1</td>
<td>0.6416</td>
<td>0.6419</td>
<td>0.6434</td>
<td>0.6452</td>
<td>0.6526</td>
<td>0.6597</td>
<td>0.6771</td>
<td>0.7089</td>
</tr>
<tr>
<td>Mezz 2</td>
<td>0.3960</td>
<td>0.3963</td>
<td>0.3977</td>
<td>0.3994</td>
<td>0.4062</td>
<td>0.4131</td>
<td>0.4303</td>
<td>0.4650</td>
</tr>
<tr>
<td>Mezz 3</td>
<td>0.1715</td>
<td>0.1718</td>
<td>0.1730</td>
<td>0.1745</td>
<td>0.1807</td>
<td>0.1872</td>
<td>0.2043</td>
<td>0.2424</td>
</tr>
<tr>
<td>Senior</td>
<td>0.0155</td>
<td>0.0156</td>
<td>0.0159</td>
<td>0.0163</td>
<td>0.0179</td>
<td>0.0198</td>
<td>0.0257</td>
<td>0.0450</td>
</tr>
<tr>
<td>SuperSen</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
<td>-0.0097</td>
</tr>
</tbody>
</table>

### Parameter Estimation

Here, we briefly describe a crude procedure for estimating the intensity parameters from market data. Fixing the parameters $\alpha, \sigma, \delta$, we choose the parameters $\lambda_0$ and $\Lambda$ such that the expected intensities match those implied from the 1-year and 5-year CDX spreads. In particular, we note that from (3.4), the mean intensity of the portfolio is given by

$$\mathbb{E} \lambda_t = \frac{\alpha \Lambda}{\alpha - \delta} + \left( \frac{\lambda_0}{\alpha - \delta} - \frac{\alpha \Lambda}{\alpha - \delta} \right) e^{-(\alpha - \delta)t}, \quad t \geq 0.$$ 

On the other hand, we observe that a $n$-year CDX spread of $s_n$ corresponds approximately to an average constant intensity of $\frac{s_n}{1 - \delta r}$ for each firm, or $\frac{s_n}{1 - \delta r} \hat{N}$ for the portfolio. With $\alpha, \sigma, \delta$ fixed arbitrarily, we then choose $\lambda_0$ and $\Lambda$ so that

$$\mathbb{E} \lambda_1 = \frac{s_1}{1 - \delta r} \hat{N}, \quad \mathbb{E} \lambda_5 = \frac{s_5}{1 - \delta r} \hat{N}.$$ 

\[17\] For a more detailed fitting procedure involving historical data and maximum likelihood estimation, see Azizpour, Giesecke, and Kim (2011).

\[18\] As stated at the beginning of this section, we assume that the portfolio has $\hat{N} = 25$ firms for computational efficiency.
As for the variance parameters, we use the parameters listed at the beginning of this section for each date in the sample.

Table 3.3 below shows the fitted intensity parameters along with the 1-yr and 5-yr CDX spreads for each of the 16 dates in our sample set, which range from Jun. 8, 2009 to Sep. 8, 2010. Here, the CDX spreads for the first two sets of dates correspond to the on-the-run (i.e., current) series, Series 12 and Series 13, respectively, while the CDX spreads for the last set of dates correspond to the off-the-run (i.e., not current) series, Series 9. For the parameter fitting, we use the above procedure to match the expected intensities, but only for the set of dates corresponding to Series 9. In fact, for Series 12 and 13, this procedure actually yields negative implied risk aversions, an unrealistic result; hence, for these dates, we choose the intensity parameters (e.g., with $\lambda_0 = \Lambda$) such that the implied risk aversions are positive for all of the tranches.

Table 3.3: Intensity Parameters, Jun. 8, 2009 to Sep. 8, 2010

<table>
<thead>
<tr>
<th>Date</th>
<th>$\lambda_0$</th>
<th>$\Lambda$</th>
<th>$\alpha$</th>
<th>$\sigma$</th>
<th>$\delta$</th>
<th>$s_1$ (bps)</th>
<th>$s_5$ (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jun. 8, 2009</td>
<td>0.21</td>
<td>0.21</td>
<td>0.50</td>
<td>0.20</td>
<td>0.02</td>
<td>118.16</td>
<td>125</td>
</tr>
<tr>
<td>Jul. 8, 2009</td>
<td>0.23</td>
<td>0.23</td>
<td>1.50</td>
<td>0.20</td>
<td>0.10</td>
<td>131.39</td>
<td>139</td>
</tr>
<tr>
<td>Aug. 7, 2009</td>
<td>0.21</td>
<td>0.21</td>
<td>0.50</td>
<td>0.20</td>
<td>0.02</td>
<td>104.92</td>
<td>111</td>
</tr>
<tr>
<td>Sep. 8, 2009</td>
<td>0.22</td>
<td>0.22</td>
<td>1.50</td>
<td>0.20</td>
<td>0.10</td>
<td>135.31</td>
<td>121</td>
</tr>
<tr>
<td>Oct. 8, 2009</td>
<td>0.18</td>
<td>0.18</td>
<td>0.75</td>
<td>0.20</td>
<td>0.07</td>
<td>95.47</td>
<td>101</td>
</tr>
<tr>
<td>Nov. 9, 2009</td>
<td>0.21</td>
<td>0.21</td>
<td>0.50</td>
<td>0.20</td>
<td>0.02</td>
<td>99.25</td>
<td>105</td>
</tr>
<tr>
<td>Dec. 8, 2009</td>
<td>0.18</td>
<td>0.18</td>
<td>0.75</td>
<td>0.20</td>
<td>0.07</td>
<td>59.09</td>
<td>98</td>
</tr>
<tr>
<td>Jan. 8, 2010</td>
<td>0.14</td>
<td>0.14</td>
<td>0.75</td>
<td>0.20</td>
<td>0.08</td>
<td>46.43</td>
<td>77</td>
</tr>
<tr>
<td>Feb. 8, 2010</td>
<td>0.20</td>
<td>0.20</td>
<td>0.75</td>
<td>0.20</td>
<td>0.05</td>
<td>60.30</td>
<td>100</td>
</tr>
<tr>
<td>Mar. 8, 2010</td>
<td>0.18</td>
<td>0.18</td>
<td>0.75</td>
<td>0.20</td>
<td>0.07</td>
<td>45.75</td>
<td>89</td>
</tr>
<tr>
<td>Apr. 8, 2010</td>
<td>0.23</td>
<td>0.39</td>
<td>0.75</td>
<td>0.30</td>
<td>0.03</td>
<td>93.00</td>
<td>99</td>
</tr>
<tr>
<td>May 7, 2010</td>
<td>0.23</td>
<td>0.54</td>
<td>0.75</td>
<td>0.30</td>
<td>0.02</td>
<td>115.53</td>
<td>130</td>
</tr>
<tr>
<td>Jun. 8, 2010</td>
<td>0.27</td>
<td>2.90</td>
<td>0.04</td>
<td>0.30</td>
<td>0.02</td>
<td>111.49</td>
<td>141</td>
</tr>
<tr>
<td>Jul. 8, 2010</td>
<td>0.26</td>
<td>1.70</td>
<td>0.06</td>
<td>0.30</td>
<td>0.03</td>
<td>101.22</td>
<td>128</td>
</tr>
<tr>
<td>Aug. 9, 2010</td>
<td>0.30</td>
<td>0.44</td>
<td>0.75</td>
<td>0.30</td>
<td>0.02</td>
<td>96.87</td>
<td>109</td>
</tr>
<tr>
<td>Sep. 8, 2010</td>
<td>0.31</td>
<td>0.48</td>
<td>0.75</td>
<td>0.30</td>
<td>0.02</td>
<td>104.86</td>
<td>118</td>
</tr>
</tbody>
</table>
Note that we have selected this range of dates during the credit crisis because this is the earliest period in which all of the CDX.NA.IG tranches pay upfront fees. Indeed, for the dates preceding June 2009, at least one of the tranches did not pay an upfront fee and thus, the tranche spread was quoted as a running spread. We note that, in this indifference valuation model, it is highly inefficient to compute the running spread \( R \) as it is the \textit{implicit} solution of a system of PDEs; on the other hand, we are able to compute the upfront fee \( U \) directly via the numerical schemes described in Section 3.5.

\textbf{Market-Implied Risk Aversion}

Next, we determine the implied risk aversion parameter \( \tilde{\gamma} \) that yields the market upfront fee \( U \) for 5-year CDO tranches, using the model parameters from Table 3.3. Since the upfront fee is increasing in \( \gamma \), we are able to use a bisection method to compute \( \tilde{\gamma} \) numerically. To compare with the results in relevant literature, such as Sircar and Zariphopoulou (2010) and Sirignano (2010), we suppose that the total notional for the CDO is \( \hat{N} \), that is, each firm has a notional of one unit.

Tables 3.4 to 3.6 show the implied risk aversion parameters \( \tilde{\gamma} \) for 16 dates ranging from June 2009 to September 2010\(^{19}\). For each date, we observe that, with some exceptions for the mezzanine tranches, the implied risk aversion \( \tilde{\gamma} \) is generally increasing across tranches. This increasing trend reflects greater relative fear of extreme losses that the higher tranches represent, a result that is consistent with Sircar and Zariphopoulou (2010). In addition, we note that an investor who holds a more senior tranche would have higher risk aversion, and hence, his optimal control \( \pi^* \) would be smaller and he would thus invest less money in the stock index.

\(^{19}\)Note that in Tables 3.4 and 3.5 the upfront fees correspond to the on-the-run Series 12 and Series 13, respectively, where all of the tranches pay running spreads of 100 bps. On the other hand, in Table 3.6, the upfront fees correspond to the off-the-run Series 9, where the first three tranches now pay running spreads of 500 bps while the last three tranches pay running spreads of 100 bps. The market conventions for the upfront fees and the running spreads are explained in more detail in Section 2.6.2.
Table 3.4: Implied Risk Aversion for Jun. 8, 2009 to Sep. 8, 2009

<table>
<thead>
<tr>
<th></th>
<th>8-Jun-09</th>
<th>8-Jul-09</th>
<th>7-Aug-09</th>
<th>8-Sep-09</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Series 12)</td>
<td>(Series 12)</td>
<td>(Series 12)</td>
<td>(Series 12)</td>
</tr>
<tr>
<td><strong>Equity</strong></td>
<td>58.89% 0.18</td>
<td>64.00% 0.26</td>
<td>58.84% 0.17</td>
<td>62.38% 0.26</td>
</tr>
<tr>
<td><strong>Mezz 1</strong></td>
<td>24.52% 0.99</td>
<td>34.89% 1.57</td>
<td>23.59% 0.87</td>
<td>27.31% 0.93</td>
</tr>
<tr>
<td><strong>Mezz 2</strong></td>
<td>9.78% 3.09</td>
<td>16.73% 3.51</td>
<td>10.19% 3.18</td>
<td>10.88% 2.78</td>
</tr>
<tr>
<td><strong>Mezz 3</strong></td>
<td>2.61% 3.74</td>
<td>6.80% 3.85</td>
<td>3.25% 3.86</td>
<td>5.21% 3.81</td>
</tr>
<tr>
<td><strong>Senior</strong></td>
<td>-0.43% 3.66</td>
<td>-0.83% 3.35</td>
<td>-2.05% 3.46</td>
<td>-1.84% 3.32</td>
</tr>
<tr>
<td><strong>SuperSen</strong></td>
<td>-2.72% 4.38</td>
<td>-2.99% 3.97</td>
<td>-3.48% 4.28</td>
<td>-3.35% 4.04</td>
</tr>
</tbody>
</table>

Table 3.5: Implied Risk Aversion for Oct. 8, 2009 to Mar. 8, 2010

<table>
<thead>
<tr>
<th></th>
<th>8-Oct-09</th>
<th>9-Nov-09</th>
<th>8-Dec-09</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Series 13)</td>
<td>(Series 13)</td>
<td>(Series 13)</td>
</tr>
<tr>
<td><strong>Equity</strong></td>
<td>54.62% 0.28</td>
<td>61.75% 0.56</td>
<td>53.42% 0.13</td>
</tr>
<tr>
<td><strong>Mezz 1</strong></td>
<td>21.01% 0.99</td>
<td>23.40% 0.85</td>
<td>22.54% 1.18</td>
</tr>
<tr>
<td><strong>Mezz 2</strong></td>
<td>8.60% 3.20</td>
<td>8.57% 2.84</td>
<td>8.57% 3.19</td>
</tr>
<tr>
<td><strong>Mezz 3</strong></td>
<td>2.30% 3.72</td>
<td>1.03% 3.39</td>
<td>1.75% 3.61</td>
</tr>
<tr>
<td><strong>Senior</strong></td>
<td>-2.43% 3.20</td>
<td>-2.55% 3.38</td>
<td>-2.44% 3.20</td>
</tr>
<tr>
<td><strong>SuperSen</strong></td>
<td>-3.48% 4.16</td>
<td>-3.55% 4.23</td>
<td>-3.65% 4.04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>8-Jan-10</th>
<th>8-Feb-10</th>
<th>8-Mar-10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Series 13)</td>
<td>(Series 13)</td>
<td>(Series 13)</td>
</tr>
<tr>
<td><strong>Equity</strong></td>
<td>44.75% 0.16</td>
<td>57.80% 0.20</td>
<td>53.81% 0.18</td>
</tr>
<tr>
<td><strong>Mezz 1</strong></td>
<td>14.51% 1.08</td>
<td>24.31% 1.05</td>
<td>19.75% 0.82</td>
</tr>
<tr>
<td><strong>Mezz 2</strong></td>
<td>4.25% 3.37</td>
<td>8.79% 2.94</td>
<td>7.38% 2.93</td>
</tr>
<tr>
<td><strong>Mezz 3</strong></td>
<td>0.29% 4.03</td>
<td>0.75% 3.30</td>
<td>0.88% 3.41</td>
</tr>
<tr>
<td><strong>Senior</strong></td>
<td>-2.95% 3.36</td>
<td>-2.65% 3.28</td>
<td>-2.60% 3.17</td>
</tr>
<tr>
<td><strong>SuperSen</strong></td>
<td>-3.82% 4.12</td>
<td>-3.72% 4.06</td>
<td>-3.63% 4.04</td>
</tr>
</tbody>
</table>
Table 3.6: Implied Risk Aversion for Apr. 8, 2010 to Sep. 8, 2010

<table>
<thead>
<tr>
<th></th>
<th>8-Apr-10 (Series 9)</th>
<th>7-May-10 (Series 9)</th>
<th>8-Jun-10 (Series 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( U )</td>
<td>( \gamma )</td>
<td>( U )</td>
</tr>
<tr>
<td>Equity</td>
<td>45.17%</td>
<td>0.05</td>
<td>51.15%</td>
</tr>
<tr>
<td>Mezz 1</td>
<td>4.45%</td>
<td>0.16</td>
<td>10.60%</td>
</tr>
<tr>
<td>Mezz 2</td>
<td>-6.76%</td>
<td>2.15</td>
<td>-2.48%</td>
</tr>
<tr>
<td>Mezz 3</td>
<td>-0.56%</td>
<td>3.00</td>
<td>0.78%</td>
</tr>
<tr>
<td>Senior</td>
<td>-1.96%</td>
<td>3.54</td>
<td>-1.31%</td>
</tr>
<tr>
<td>SuperSen</td>
<td>-2.30%</td>
<td>4.28</td>
<td>-2.11%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>8-Jul-10 (Series 9)</th>
<th>9-Aug-10 (Series 9)</th>
<th>8-Sep-10 (Series 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( U )</td>
<td>( \gamma )</td>
<td>( U )</td>
</tr>
<tr>
<td>Equity</td>
<td>47.33%</td>
<td>0.25</td>
<td>46.69%</td>
</tr>
<tr>
<td>Mezz 1</td>
<td>11.60%</td>
<td>0.88</td>
<td>6.99%</td>
</tr>
<tr>
<td>Mezz 2</td>
<td>-2.88%</td>
<td>2.30</td>
<td>-5.60%</td>
</tr>
<tr>
<td>Mezz 3</td>
<td>0.62%</td>
<td>2.46</td>
<td>-0.54%</td>
</tr>
<tr>
<td>Senior</td>
<td>-1.59%</td>
<td>2.50</td>
<td>-1.88%</td>
</tr>
<tr>
<td>SuperSen</td>
<td>-1.99%</td>
<td>3.20</td>
<td>-2.08%</td>
</tr>
</tbody>
</table>

As we shall see next in Section 3.6.3, the observed behaviour of the investors in the credit market is qualitatively similar to that observed in the equity market, where both the implied risk aversions and the implied volatilities are increasing for out-of-the-money put options.

### 3.6.3 Equity Index Options

In this subsection, we test the indifference valuation of equity index options. In particular, we focus on the pricing of put options on the S&P 500, with payoff of the form

\[
P(S_T) = (K - S_T)^+,
\]

where \( K \) is the strike price. Since the payoff depends on the underlying stock index, there is an extra dimensionality involved in the PDE problem, as we have the parameter \( S \) to go along with \((t, v, \lambda, n)\). For the grid spacing, we choose the following
discretization for the stock price bracketing the strike \( K \):

\[
S_{\text{min}} = 0.80K, \ S_{\text{max}} = 2.30K, \ W = 50,
\]

where we focus mainly on out-of-the-money put options \((S > K)\) since they help to identify the systematic risk in the market. Meanwhile, for the variance and intensity dimensions, we choose the following discretization:

\[
v_{\text{min}} = 0.004, \ v_{\text{max}} = 0.4, \ J = 40, \ \lambda_{\text{min}} = 0, \ \lambda_{\text{max}} = 2, \ L = 20.
\]

Finally, for the model parameters, we use the intensity parameters from Table 3.3 and the variance parameters from the beginning of this section:

\[
v_t = 0.0535, \ \theta = 0.04, \ \kappa = 0.5, \ \eta = 0.2, \ \rho_{Sv} = -0.6, \ a = 0.1.
\]

In our numerical testing, we examine the impact of parameters such as risk aversion, the initial variance level, and the number of firms, and then we compute the implied risk aversions based on the market prices of equity options. Armed with this information, we are then able to compare the systematic risks in the multi-name credit and equity markets under our indifference valuation model.

**Market Data**

We consider put options on the S&P 500 with maturities between 2 and 3 years, with the same start dates, ranging from mid 2009 to late 2010, as in Section 3.6.2.
Optimal Strategy

Recall that from \((3.41)\), the optimal strategy for the holder of the equity index option is given by

\[
\pi^*(t, S, v, n, \lambda) = e^{-r(T-t)} \left( \frac{\mu - r}{v} + \frac{S h_s}{h} + \frac{\rho sv \eta h_v}{h} \right);
\]

hence, as the risk aversion \((\gamma)\) increases, the amount \((\pi^*)\) held in the stock decreases.

Note that the amount held in the stock may increase or decrease as the stock price increases, as there is no straightforward relationship here.

Pricing

For the indifference pricing of put options, we consider a more efficient scheme that allows us to compute the prices for all of the strikes at the same time. In particular, we convert all of the strikes to $1 and then we scale the spot value and put option price by the same factor. For example, considering the market data on June 8, 2009, for the option with original strike of 500, we convert the strike to 1, the spot from 939.14 to 939.14/500 = 1.87828 and the put price from 28.10 to 28.10/500 = 0.0562, while for the option with original strike of 1000, we convert the strike to 1, the spot from 939.14 to 939.14/1000 = 0.93914 and the put price from 190.25 to 190.25/1000 = 0.19025. Note that by fixing the strike at $1 and then varying the spot, we are able to compute the indifference put prices for a range of moneyness levels simultaneously on the grid. Hence, this also yields a more efficient computation of the implied risk aversions corresponding to the set of put options in the market.

Here, we test the impact of some of the model parameters. In particular, Table \(3.7\) below shows the buyer’s indifference put price as a function of risk aversion, for different moneyness levels, and we note that the put price is increasing in \(\gamma\), as

\[124\]
claimed in Section 3.4. We also find that the indifference price is increasing in the initial variance $v_0$ and the number of firms $\hat{N}$, but for brevity, we do not display these results here.

**Table 3.7: Put Price as Function of Risk Aversion $\gamma$**

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>0.01</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5000</td>
<td>0.0284</td>
<td>0.0287</td>
<td>0.0305</td>
<td>0.0331</td>
<td>0.0365</td>
<td>0.0416</td>
<td>0.0506</td>
</tr>
<tr>
<td>0.5988</td>
<td>0.0532</td>
<td>0.0537</td>
<td>0.0563</td>
<td>0.0603</td>
<td>0.0656</td>
<td>0.0736</td>
<td>0.0883</td>
</tr>
<tr>
<td>0.7463</td>
<td>0.0837</td>
<td>0.0845</td>
<td>0.0884</td>
<td>0.0945</td>
<td>0.1029</td>
<td>0.1156</td>
<td>0.1396</td>
</tr>
<tr>
<td>0.8621</td>
<td>0.1046</td>
<td>0.1056</td>
<td>0.1106</td>
<td>0.1185</td>
<td>0.1293</td>
<td>0.1459</td>
<td>0.1778</td>
</tr>
<tr>
<td>0.9901</td>
<td>0.1249</td>
<td>0.1262</td>
<td>0.1324</td>
<td>0.1422</td>
<td>0.1558</td>
<td>0.1768</td>
<td>0.2178</td>
</tr>
<tr>
<td>1.2500</td>
<td>0.1574</td>
<td>0.1591</td>
<td>0.1674</td>
<td>0.1808</td>
<td>0.1995</td>
<td>0.2289</td>
<td>0.2884</td>
</tr>
</tbody>
</table>

**Implied Risk Aversion**

Next, we determine the implied risk aversion parameter $\bar{\gamma}$, that is, the value of $\gamma$ that equates the model price with the market price. Since we have verified that the put price is increasing with respect to $\gamma$, we can then determine $\bar{\gamma}$ by a bisection method. In the event that the model price is larger than the market price for $\gamma$ near zero, the implied risk aversion $\bar{\gamma}$ would be a negative value, meaning that the investor is *risk-loving* as opposed to risk-averse. However, we deem negative risk aversions to be acceptable here because we are only concerned with *relative* values for comparison. In particular, here we are comparing risk aversions across moneyness levels, and later we will compare the largest risk aversions from the equity market with those from the credit market.

In Figures 3.8 to 3.10 we plot the implied risk aversion as a function of moneyness $K/S$ for the selected dates between Jun. 2009 and Sep. 2010. Here, we used the intensity parameters from Table 3.3 and the variance parameters from the beginning of the section. For each date, we observe that the implied risk aversion is generally

---

20We note that the intensity parameters have minimal impact on the indifference valuation of equity index put options and, hence, minor effect on the implied risk aversion $\bar{\gamma}$ corresponding to
larger for deep out-of-the-money (Moneyness ≪ 1) and for in-the-money (Moneyness > 1) put options. This “smile” reflects greater relative fear of market downturn or upturn, as investors are willing to pay greater insurance for large movements in the equity index. This behaviour of the investors is consistent with that inferred from the equity implied volatility smile, that is, larger implied volatilities for both out-of-the-money and in-the-money put options.

Figure 3.8: Equity-Implied Risk Aversion, Jun. 8, 2009 to Sep. 8, 2009

---

market put prices. Thus, for computational efficiency, we used the same intensity parameters for all 16 dates in this section, with the results displayed in Figures 3.8 to 3.10.
Figure 3.9: Equity-Implied Risk Aversion, Oct. 8, 2009 to Mar. 8, 2010
Figure 3.10: Equity-Implied Risk Aversion, Apr. 8, 2010 to Sep. 8, 2010

- Implied Gamma, Apr. 8, 2010
- Implied Gamma, May 7, 2010
- Implied Gamma, June 8, 2010
- Implied Gamma, July 8, 2010
- Implied Gamma, Aug. 9, 2010
- Implied Gamma, Sep. 8, 2010
Finally, we observe that the behaviour of the equity investors is qualitatively similar to that observed in the credit market, where we recall that the implied risk aversion was larger for the more senior tranches. Indeed, under both the equity and credit markets, the risk aversion was larger for market-crash scenarios that accounted for systematic risk, and this motivates us to do a quantitative comparison of the two markets next in Section 3.6.4.

### 3.6.4 Analysis of Systematic Risks

Based on the results from Sections 3.6.2 and 3.6.3, we can now compare the systematic risks in the credit and equity markets. In particular, we analyze the systematic risks in the two markets by comparing the largest implied risk aversions from the CDO tranches with those from the index put options. We find that the largest risk aversions occur at the super senior tranches for the CDO and at the deep out-of-the-money levels for the S&P 500 index options. This signifies that the investors of these derivatives are the most risk averse and are willing to pay extra to hedge against the worst-case scenario of a market crash.

Figure 3.11 below shows the largest implied risk aversions from the CDO tranches (Credit) and the S&P 500 index options (Equity) on the 16 selected dates between June 2009 and September 2010, as extracted from Tables 3.4 to 3.6 and Figures 3.8 to 3.10. We find that for each of the 16 specified dates during the credit crisis, the largest implied risk aversion from the CDO tranches far exceeded that from the S&P 500 index options, thus indicating that the systematic risk in the credit market was greater than that in the equity market. This result is consistent with the conclusions from the bottom-up intensity-based model of Chapter 2.
Figure 3.11: Credit vs. Equity – Largest Risk Aversions
Chapter 4

Conclusions and Extensions

In this thesis, we have analyzed two approaches for comparing the systematic risks in the credit and equity markets. In summary, we found that the systematic risks in the two markets were similar from 2004 to 2007, while the credit market incorporated far greater systematic risk than the equity market during the financial crisis from 2008 to 2010. Let us recap the two approaches and the noteworthy results during the crisis.

In Chapter 2, we examined a hybrid equity-credit intensity model and we analyzed the systematic risk by considering two cases. First, in the forward case, going from S&P options to CDO tranche spreads, we found that for recent dates since the credit crisis, the senior tranche spreads from the market were larger than the corresponding model spreads, and hence, the credit market appeared to account for more systematic risk than the equity market. In the backward case, going from CDO tranche spreads to S&P options, we obtained the same result for recent data. In particular, we found that the market implied volatility skew was less than the skew inferred from the fitted model, and hence, the equity market accounted for less systematic risk than the credit market. From these two cases, we conclude that under our hybrid equity-credit model, the systematic risk in the credit market was greater than that in the equity market during the crisis period of 2008 to 2010. This result is the opposite of that found in
Coval, Jurek, and Stafford (2009a) and differs notably from the conclusions of Collin-Dufresne, Goldstein, and Yang (2010), Luo and Carverhill (2011), and Li and Zhao (2011).

In Chapter 3, we considered a top-down indifference valuation model and we analyzed the risk aversions of investors in multi-name credit derivatives and equity derivatives. We found that the implied risk aversion was generally increasing with seniority of the CDO tranche, reflecting greater relative fear of extreme losses that the higher tranches represent. We also found that the implied risk aversion for put options on the equity index was increasing as moneyness levels decreased, a behaviour that is consistent with the implied volatility skew observed in equity index options. Using the same model parameters for the two markets, we found that over the 16-month period from June 2009 to September 2010, the largest risk aversions from the credit market were noticeably higher than those from the equity market. This indicates that there was larger systematic risk in the credit market over this period, a result that is consistent with the conclusions of Chapter 2.

In the rest of this chapter, we discuss two different extensions to the top-down indifference model of Chapter 3.

- We can consider the general model introduced in Section 3.1, where we allow the equity, volatility, and intensity processes to all be correlated. In particular, the parameters $\rho_{SV}$, $\rho_{SL}$, and $\rho_{V\lambda}$ are not necessarily zero. Then, we can again derive the optimal controls and the HJB PDEs for the Merton problem and the European claim holder’s problem. These PDEs involve an extra correlation term but can be solved numerically, leading to the indifference price.

- Another extension is to allow the stock index to drop whenever a constituent firms defaults. In particular, the default of a firm in the portfolio causes a downward proportional jump in the stock market index in addition to the upward
jumps in the volatility and intensity. This scenario is more realistic than that considered in Section 3.1 but is more challenging to model. Indeed, the optimal control and the HJB PDEs both involve multiple Lambert-W functions, and thus, we do not have straightforward numerical schemes for solving the PDEs and obtaining the indifference prices. Hence, this approach is primarily of theoretical interest.
Appendix A

Technical Results for an Integrated Affine Jump Diffusion

In this appendix, we give technical results for an integrated affine jump diffusion (AJD) as it relates to the pricing of CDOs and equity index options for the model in Chapter 2. In particular, we first give the moment generating function and then we provide the characteristic function of an integrated AJD.

We define the integrated AJD by

\[
U_t \equiv \int_0^t Y_s ds,
\]

where \(Y = (Y_t)_{t \geq 0}\) is an AJD as defined in Section 2.1. Recall that \(Y\) satisfies the SDE

\[
dY_t = \kappa(\bar{y} - Y_t)dt + \sigma \sqrt{Y_t} dW_Y^t + dJ_t, \quad Y_0 = y_0,
\]

(A.1)

where \(W_Y^t\) is a standard \(\mathbb{P}^\ast\)-Brownian motion, and \(J\) is an independent compound Poisson process with jump intensity \(l\) and exponentially distributed jumps with mean \(\xi\). The Laplace transform of the jump size distribution \(\nu\) is

\[
\psi(c) = \int_{\mathbb{R}^+} e^{cz} d\nu(z) = \int_{\mathbb{R}^+} e^{cz} \frac{1}{\xi} e^{-\frac{1}{\xi} z} dz = \frac{1}{1 - c\xi},
\]

for \(c \in \mathbb{C}\) and \(\text{Re}(c) < 1/\xi\).
A.1 Moment Generating Function

By Duffie, Pan, and Singleton (2000), it follows that for \( t > 0 \) and \( q \in \mathbb{R} \),

\[
E^* \left[ e^{qU_t} \right] = E^* \left[ e^{q \int_0^t Y_s \, ds} \right] = e^{\alpha(t) + \beta(t)y_0},
\]

(A.2)

where \( \alpha(\cdot) \) and \( \beta(\cdot) \) solve the pair of Riccati ODEs

\[
\begin{align*}
\alpha'(t) &= \kappa \bar{y} \beta(t) + l \frac{\xi \beta(t)}{1 - \xi \beta(t)}, \\
\beta'(t) &= -\kappa \beta(t) + \frac{1}{2} \sigma^2 \beta(t)^2 + q,
\end{align*}
\]

(A.3)

with boundary conditions \( \alpha(0) = \beta(0) = 0 \). Appendix B of Duffie and Gârleanu (2001) provides an explicit solution, as follows:

\[
\begin{align*}
\alpha(t) &= -\frac{2 \kappa \bar{y}}{c_1 + d_1 e^{-\gamma t}} \log \left( \frac{c_1 + d_1 e^{-\gamma t}}{e_1 e^{-\gamma t}} \right) + \frac{\kappa \bar{y} t}{c_1} + l \left( \frac{d_1/c_1 - d_2/c_2}{-\gamma d_2} \right) \log \left( \frac{c_2 + d_2 e^{-\gamma t}}{c_2 + d_2} \right) + l \frac{1 - c_2}{c_2} t, \\
\beta(t) &= \frac{1 - e^{-\gamma t}}{c_1 + d_1 e^{-\gamma t}},
\end{align*}
\]

(A.4)

where

\[
\begin{align*}
\gamma &= \sqrt{\kappa^2 - 2\sigma^2 q}, \\
c_1 &= (\kappa + \gamma)/(2q), \\
c_2 &= 1 - \xi/c_1, \\
d_1 &= (-\kappa + \gamma)/(2q), \\
d_2 &= (d_1 + \xi)/c_1.
\end{align*}
\]

(A.5)

Hence, the moment generating function of an integrated AJD is known in closed form.
A.2 Characteristic Function

Setting \( q = iv \) for \( v \in \mathbb{R} \) in (A.2) gives an explicit formula for the characteristic function of the integrated AJD at time \( t \):

\[
E^* \left[ e^{ivU_t} \right] = e^{\alpha(t) + \beta(t)y_0},
\]

(A.6)

where \( \alpha(\cdot) \) and \( \beta(\cdot) \) solve the pair of complex-valued ODEs in (A.3). Repeating the derivation by Duffie and Gârleanu (2001) here for the complex-valued case, we arrive at the solution (A.4). In this case, we interpret \( \gamma \) in (A.5) as

\[
|\gamma|^2 \exp(i \arg(\gamma^2)/2),
\]

where for any \( z \in \mathbb{C} \), \( \arg(z) \) is defined such that \( z = |z| \exp(i \arg(z)) \) with \( -\pi < \arg(z) \leq \pi \). Moreover, we take \( \log(z) = \log(|z|) + i \arg(z) \), although any other branch of the complex logarithm would work as well, since the logarithm of \( \gamma \) shows up only in the exponent of (A.6). See Lord and Kahl (2010) for a discussion on evaluating transforms of the form (A.6) with a complex-valued exponent.
Appendix B

Variation for Top-Down Model

In this appendix, we discuss a variation for the top-down model of Chapter 3 in which the volatility is an inverse CIR process with jumps. In particular, we suppose that the stochastic variance process \((v_t)_{t \geq 0}\) evolves as a particular inverse CIR process with jumps, that is, the reciprocal of a CIR process \((x_t)_{t \geq 0}\) that has jumps\(^\dagger\). Starting from the SDE for \((x_t)\), we derive the SDE that \((v_t)\) must satisfy.

**Proposition B.0.1.** Suppose that \((x_t)_{t \geq 0}\) is a CIR process with the downward jump function \(g(x_t) = \frac{ax_t^2}{1+ax_t}\), that is,

\[
dx_t = \kappa(\theta - x_t)dt + \eta \sqrt{x_t}dW_t - g(x_t-)dN_t,
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion and \((N_t)_{t \geq 0}\) is a counting process that is independent of \((W_t)_{t \geq 0}\). Then, the inverse process \((v_t)\), defined by \(v_t = 1/x_t\), \(t \geq 0\), satisfies the SDE

\[
dv_t = v_t \left[ \kappa \theta' \left( \frac{1}{\theta'} - v_t \right) dt - \eta \sqrt{v_t}dW_t \right] + adN_t,
\]

with \(\theta' = \theta - \frac{\eta^2}{\kappa}\).

\(^\dagger\)Ahn and Gao (1999) first studied the inverse CIR process (without jumps) for interest rates.
Proof. First, we consider the jump part: if a default occurs at time $t$, then

$$x_t = x_{t-} - g(x_{t-}) = x_{t-} - \frac{ax_{t-}^2}{1 + ax_{t-}} = \frac{x_{t-}}{1 + ax_{t-}},$$

and so

$$v_t = \frac{1}{x_t} = \frac{1 + ax_{t-}}{x_{t-}} = \frac{1}{x_{t-}} + a = v_{t-} + a.$$

Thus, $v$ jumps by the constant $a$ whenever a default occurs:

$$dv^j_t = adN_t.$$ 

For the continuous part, we apply Itô’s Lemma to get

$$dv^c_t = d\left(\frac{1}{x_t}\right)^c = \left(-\frac{1}{x_t^2}\right) dx^c_t + \frac{1}{2}\left(\frac{2}{x_t^3}\right) d\langle x^c, x^c \rangle_t$$

$$= -\frac{1}{x_t^2} \left[ \kappa (\theta - x_t) dt + \eta \sqrt{x_t} dW_t \right] + \frac{\eta^2}{x_t^2} dt \quad \text{since } d\langle x^c, x^c \rangle_t = \eta^2 x_t dt$$

$$= -v_t^2 \left[ \kappa \left( \theta - \frac{1}{v_t} \right) dt + \frac{\eta}{\sqrt{v_t}} dW_t \right] + \eta^2 v_t^2 dt$$

$$= -v_t^2 \left[ \kappa \left( \theta - \frac{\eta^2}{\kappa} - \frac{1}{v_t} \right) dt + \frac{\eta}{\sqrt{v_t}} dW_t \right]$$

$$= -v_t \left[ \kappa v_t \left( \theta' - \frac{1}{v_t} \right) dt + \eta \sqrt{v_t} dW_t \right] \quad \text{where } \theta' = \theta - \frac{\eta^2}{\kappa}$$

$$= -v_t \left[ \kappa \theta' \left( v_t - \frac{1}{\theta'} \right) dt + \eta \sqrt{v_t} dW_t \right]$$

$$= v_t \left[ \kappa \theta' \left( \frac{1}{\theta'} - v_t \right) dt - \eta \sqrt{v_t} dW_t \right].$$

Combining the jump and continuous parts, we obtain

$$dv_t = dv^c_t + dv^j_t = v_t \left[ \kappa \theta' \left( \frac{1}{\theta'} - v_t \right) dt - \eta \sqrt{v_t} dW_t \right] + adN_t,$$

as desired.
References


