ROBUST PORTFOLIO OPTIMIZATION
WITH APPLICATIONS IN CURRENCIES
AND PRIVATE EQUITY

Lorenzo Reus Heredia

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Abstract

This dissertation presents new insights and variations of the Robust Counterpart problem defined by Ben-Tal and Nemirovski. These models are applied to a set of practical portfolio problems.

First, we focus on the uncertainty set. We establish the volume to compare solutions of the same original problem, but with differing sizes. Using various shapes, we generate and compare robust currency carry trade strategies. In order to combine previous strategies, we build a hidden Markov model to classify carry trade performance under economic regimes. Backtests results on the major developed world currencies show that our dynamic strategy can outperform benchmark carry trade indexes, by a 8.5% increase in annual returns and 13% decrease in ulcer.

Second, we present a methodology to find less conservative robust solutions by applying transformations on the original problem. The transformations aim to group uncertain and correlated parameters into fewer constraints. We test the methodology in a multistage portfolio problem, using a factor-based market and exchange traded funds prices. Simulations show that allocation strategies using this methodology are in fact less conservative than solutions without it. Portfolios present lower cash holding ratios and higher final values at most risk aversion levels.

Third, we apply the previous methodology in a portfolio problem that includes private equity investments in the pool of assets. We model private equity as investments with uncertain future commitments and distributions flows. Investments in this area are considered highly illiquid and can produce cash flow burdens on the sponsoring organization. Employing the same previous data set for liquid assets and using historical cash flows information, we construct allocation strategies with alternative risk aversion levels. Simulation results depict that only more conservative strategies can avoid liquidity issues. In those cases, including private equity assets increase annual return by 5% to 7% without increasing risk.
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Chapter 1
Dynamic Robust Carry Trade Based on Uncertainty
Set Selection

1.1. Introduction

Robust Optimization is a concept that is used in problems that deal with uncertain parameters. There is no unique definition of what a robust optimization problem is. However, all approaches and formulations aim a similar objective: Find a solution(s) that can still be optimal (near optimal) under changes in the parameters. For example Mulvey et al. (1995) propose an optimization model that penalizes a solution when it becomes infeasible under each of the possible outcomes (scenarios) of the uncertain data. In this work we will take the definition proposed in Ben-Tal and Nemirovski (1998):

If we have the following problem:

$$\min_{x \in \mathbb{R}^N} f_0(x, r) : f(x, r) \in K \quad (P)$$

with $r$ uncertain data and $K$ a convex cone. The robust counterpart of $(P)$ is defined as:

$$\min_{x \in \mathbb{R}^N} \left\{ \sup_{r \in U} f_0(x, r) : f(x, r) \in K, \forall r \in U \right\} \quad (P^*)$$

With $U \in \mathbb{R}^M$ called the uncertainty set for the uncertain data. Therefore, we required feasibility for every realization of $r$ under the uncertainty set. The optimal solution we looked for is the best guarantee value of $(P)$. The complexity of solving $(P^*)$ will depend on the type functions mappings $f(x, r)$ and the $U$ chosen. The idea is to have a convex $(P^*)$, which we know we can solve efficiently.
The previous approach has been widely used in the portfolio optimization problem, introduced by Markowitz (1952). The purpose to add robustness is to reduce the solution sensibility of the expected mean and covariance matrix, explicitly shown in Best and Grauer (1991). For example Goldfarb and Iyengar (2003) assume a factor model for the returns and derive SOCP counterparts with ellipsoid uncertainty set on the mean and factor loading matrix. Recently, Ye et al (2012) derived a SDP with uncertainty sets on the mean and the covariance matrix under the same probability measure, making the sets dependent of each other.

Robust counterpart derivations also appear when dealing with risk measures. El Ghaoui et al. (2002) propose a conic approach to find the worst case VaR when the mean and covariance are uncertain but bounded. Quaranta and Zaffaroni (2008) solves the minimization of the CVaR introduced by Rockafellar and Uryasev (2000), but using confidence intervals on the returns, as oppose to the stochastic programming approach propose by the latter.

Finally robust formulation also shows up in currency portfolios. For example, Fonseca et al. (2011) build up a counterpart for a portfolio, which includes an ellipsoidal uncertainty set for the returns of each currency (which is $\Delta c$) and a box uncertainty set for the cross exchange rates.

In the previous research mentioned, different types of uncertainty sets have been employed. For uncertain vectors, the most used one is the ellipsoid given its suitability to handle correlated components. But there are other used uncertainty sets, such as the polyhedral or the box set. These sets can be represented by the following family of uncertainty sets, which are induced by the $p$-norm:

$$U_{p \geq 1} = \{ \mu + \Sigma^{1/2}v, \|v\|_p \leq \theta \} \quad (1.1)$$

$\mu$ represents the geometric center and $\Sigma$ is related with the orientation and length of the semi-axes. Parameter $\theta \geq 0$ is linked with the size. Is not hard to see that when $p = 1,2$ and $\infty$, $U_p$ has the shape of a polyhedron, ellipsoid and box respectively.

Despite of all the research done with the formulation defined in Ben-Tal and Nemirovski (1998), there is almost no work that focuses on saying which uncertainty set and size should we use for
a determined portfolio problem. Bertsimas et al. (2004) give probability guarantees in keeping feasibility of linear constraints under uncertainty sets induced by different norms. However the works miss to answer how this information can be used to determine what type of uncertainty set and size to choose, or how to apply it in a particular problem. Li et al. (2012) also find probability bounds for different uncertainty sets. In this case, they provide a guideline to select the size of the uncertainty set for a scheduling and production planning problem. The size should be selected in order to have the best compromise between the probability of keeping feasibility and getting high optimal values (case of maximization). However, this chance constraint method needs information about the distribution of the uncertain variables, which not always is possible to obtain accurately.

Therefore the main objectives of this present work are the following:

1. Show an alternative way to compare solutions of different uncertainty sets, which doesn’t need to assume a distribution for the parameters as in Li et al. (2012). Instead we will use the volume of the uncertainty set to make a fair comparison.
2. Design a strategy on a particular real world allocation problem, based in the selection of the appropriate uncertainty set. The problem to work with is the known exchange carry trade problem.

Thus, the work is organized as follows. In the second section we derive the counterparts of several possible formulations that can be made when solving a portfolio problem, under the family of uncertainty sets in (1.1). We will show that these counterparts, under the same uncertainty set, will produce the same sets of solutions when changing the user defined parameters. The purpose of this section is not to find the best formulation, but to make sure we are not taking the first formulation that comes to mind.

The third section shows how to formulate the chosen counterpart from section one in terms of the volume of an uncertainty set, instead of leaving it in terms of θ. We also show, with empirical experimentation, that sparsity in the solution changes depending in the norm used (uncertainty set).
The fourth section explains the carry trade problem and how to model it under the formulation chosen from previous sections. For some uncertainty sets and different volumes, we design carry trade strategies and show their performance using real data.

The fifth section shows how to improve previous results by selecting uncertainty sets according to the prediction in carry trade performance in the next period. For that prediction we will use an HMM model constructed with one of the strategies design previously and the VIX index. Finally in section six, we present conclusions and possible extensions of this work.
1.2. Portfolio Formulations

Consider the following case of \((P)\):

\[
\max_{x \in \mathcal{P}} r^T x \quad (1.2)
\]

Where \(\mathcal{P}\) describes a bounded, closed and non-empty polyhedron. If \(r \in \mathcal{U}_p\) then is easy to show that the counterpart \((P^*)\) is

\[
\max_{x \in \mathcal{P}} \left\{ \mu^T x - \theta \left\| \Sigma^{1/2} x \right\|_q \right\} \quad (1.3)
\]

where \(q\) such that \(\frac{1}{p} + \frac{1}{q} = 1\). In fact \((P^*)\):

\[
\max_{x \in \mathcal{P}} \left\{ \inf_{r \in \mathcal{U}_p} r^T x \right\} = \max_{x \in \mathcal{P}} \left\{ x^T \mu + \theta \inf_{\|v\|_{p^{1/2}}} \left( \Sigma^{1/2} x \right)^T v \right\} = \max_{x \in \mathcal{P}} \left\{ x^T \mu - \theta \left\| \Sigma^{1/2} x \right\|_p^* \right\}
\]

with \(\|\cdot\|_p^*\) as the dual norm of the \(p\)-norm. \(^{1}\) By holder inequality\(^{2}\), is easy to get that \(\|\cdot\|_p^* = \|\cdot\|_q\).

One might consider that the previous formulation, which is basically a max-min approach, is not precisely what we think of what a robust solution is. A robust solution should be linked with solutions that are not so sensitive to changes in the parameters. In that direction, we can write another formulation. Imagine that now \((P)\), is the following:\n
\[
\max_{x \in \mathcal{P}} \left\{ u^T x : |r^T x - s^T x| \leq \lambda, \forall r, s \in \mathcal{U}_p \right\} \quad (1.4)
\]

In words, we choose the portfolio with best compromise between expected return and its return variation with returns changes.

\(^{1}\) The dual norm of \(x\) w.r.t the \(p\)-norm is defined as \(\|x\|_p^* = \sup_{\|v\|_{p^{1/2}}} |x^Tv|\)

\(^{2}\) \(|x^Ty| \leq \|x\|_p \|y\|_q \quad \forall x, y \quad \frac{1}{p} + \frac{1}{q} = 1\)
**Proposition 1.1:** The counterpart \((P^*)\) of (1.4) is equivalent to

\[
\max_{x \in \mathcal{P}} \left\{ u^T x : 2\theta \left\| \Sigma^{1/2} x \right\|_q \leq \lambda \right\} \quad (1.5)
\]

**Proof:** For \(r, s \in \mathcal{U}_p\):

\[
|r^T x - s^T x| = \theta \left| \left( \Sigma^{1/2} x \right)^T v - \left( \Sigma^{1/2} x \right)^T w \right| \text{ with } \|v\|_p \leq 1, \|w\|_p \leq 1
\]

Then:

\[
\sup_{r, s \in \mathcal{U}_p} |r^T x - s^T x| = \theta \sup_{\|v\|_p \leq 1, \|w\|_p \leq 1} \left| \left( \Sigma^{1/2} x \right)^T v - \left( \Sigma^{1/2} x \right)^T w \right| = 2\theta \sup_{\|v\|_p \leq 1} \left\| \Sigma^{1/2} x \right\|_q
\]

**Remark:** When \(p = 2\), we have that (1.5) is the known mean-variance Markowitz problem. However we have not made an assumption about the distribution of the returns.

Another way to reduce the variations of the portfolio return is to bound the variance for an arbitrary family of returns \(\{r_i\}_{i=1}^N\) taken from \(\mathcal{U}_p\). Hence the formulation turns to be:

\[
\max_{x \in \mathcal{P}} \left\{ u^T x : \frac{1}{N} \sum_{i=1}^N \left( r_i^T x - \frac{1}{N} \sum_{j=1}^N r_j^T x \right)^2 \leq \lambda^2, \quad \forall \{r_i\}_{i=1}^N \in \mathcal{U}_p \right\} \quad (1.6)
\]

**Proposition 1.2:** The counterpart \((P^*)\) of (1.6) is equivalent to

\[
\max_{x \in \mathcal{P}} \left\{ u^T x : \sqrt{2}\theta \left\| \Sigma^{1/2} x \right\|_q \leq \lambda \right\} \quad (1.7)
\]
Proof:

\[
\sup_{\{r_i\}_{i=1}^N \in \mathcal{U}_p} \sum_{i=1}^N \left( r_i^T x - \frac{1}{N} \sum_{j=1}^N r_i^T x \right)^2 = \sup_{\{r_i\}_{i=1}^N \in \mathcal{U}_p} \sum_{i=1}^N \left( \Sigma^{1/2} x \right)^T v_i
\]

\[
- \frac{1}{N} \sum_{j=1}^N \left( \Sigma^{1/2} x \right)^T v_j
\]

\[
= \theta^2 \left( \sup_{\|v\|_p \leq 1} \left\{ \sum_{i=1}^N v_i^T \Sigma^{1/2} x x^T \Sigma^{1/2} v_i - \frac{1}{N} \sum_{i,j=1}^N v_i^T \Sigma^{1/2} x x^T \Sigma^{1/2} v_j \right\} \right)
\]

\[
= \theta^2 \left( N \sup_{\|v\|_p \leq 1} \left\{ v^T \Sigma^{1/2} x x^T \Sigma^{1/2} v \right\} + \frac{N^2}{N} \sup_{\|v\|_p \leq 1} \left\{ v^T \Sigma^{1/2} x x^T \Sigma^{1/2} v \right\} \right)
\]

\[
= 2N \theta^2 \sup_{\|v\|_p \leq 1, \|v\|_p \leq 1} \left\{ u^T \Sigma^{1/2} x x^T \Sigma^{1/2} u \right\} = 2N \theta^2 \left\| \Sigma^{1/2} x x^T \Sigma^{1/2} \right\|_{q,p}
\]

Where \(\|A\|_{q,p} \equiv \sup_{\|v\|_p \leq 1} \left\{ \|Av\|_q \right\} \). Then

\[
\sup_{\|v\|_p \leq 1} \left\| \Sigma^{1/2} x x^T \Sigma^{1/2} v \right\|_q = \left\| \Sigma^{1/2} x \right\|_q \sup_{\|v\|_p \leq 1} \left\| x^T \Sigma^{1/2} v \right\|_q
\]

\[
= \left\| \Sigma^{1/2} x \right\|_q \sup_{\|v\|_p \leq 1} \left( \left( \Sigma^{1/2} x \right)^T v \right) = \left\| \Sigma^{1/2} x \right\|_q^2
\]

It is clear that (1.5) and (1.7) are equivalent in the sense that we get the same optimal solution by adjusting the user-defined parameter \(\theta\). If you solve (1.5) with \(\theta = \bar{\theta}\) then you use \(\theta = 2\bar{\theta}/\sqrt{2}\) in (1.7). Analogously, if you solve (1.7) with \(\theta = \bar{\theta}\) then you use \(\theta = \sqrt{2\bar{\theta}}/2\) in (1.5). Therefore the set of optimal solutions (characterize by \(\theta\)) for both problems are the same. Thus, risk/return frontiers curves will be the same too. We will show now that the same happens with (1.3), at least for the \(\mathcal{U}_p\) we are more interested in, which are \(p \in \{1,2,\infty\}\). The proof is shown in the appendix.

So no matter what problem we want to solve, the performance/risk solutions will be the same. Therefore, it seems reasonable to be more concern about other issues other than different formulations, for example the uncertainty set chosen (p).
1.3. Uncertainty Set Selection

As said before, parameter $\theta$ adjust the level of uncertainty for $\mu$. With bigger $\theta$ (more uncertainty) more conservative/less risk solutions will be optimal. Notice that with $\theta = 0$ (no uncertainty) we solve the deterministic version of (1.3), using the expected values of the returns.

The idea is to compare (1.3) under different norms. To make a fair comparison it will be appropriate to describe the sets by a measure of size, like volume. Although $\theta$ is proportional to the size $\mathcal{U}_p$, it is easy to show that the volume between each $\mathcal{U}_p$ is different under the same value of $\theta$. Hence, we will describe (1.3) in terms of the volume of each set.

**Proposition 1.3:**

Denoting $V_p$ as the volume of $\mathcal{U}_p$, there exist an $\alpha_p$, increasing in $p$, with $\frac{2^N}{N!} \leq \alpha_p \leq 2^N$ such that:

$$V_p = \alpha_p \sqrt{|\Sigma|} \theta^N \quad (1.8)$$

where $|\Sigma|$ is the determinant of $\Sigma$ and $N$ the number of assets.

**Proof:**

Notice that $\mathcal{U}_p = \left\{ \mu + (\theta^2 \Sigma)^{\frac{1}{2}} v, \|v\|_p \leq 1 \right\} = \left\{ \mu + \Sigma v, \|v\|_p \leq 1 \right\}$. By the norm equivalence property $\|v\|_p \leq \|v\|_\beta$ if $p \leq \beta$, thus $\mathcal{U}_p \subseteq \mathcal{U}_{\beta}$ and $V_p \leq V_\beta$.

So let’s analyze the volume when $p = 1$ and $p = \infty$. We center our system of coordinates on $\mu$ and rotate in order to have the axis in the same direction as the eigenvectors of $\Sigma$. Of course, the volume is not altered. Under these new coordinates $\mathcal{U}_p = \{ Dv, \|v\|_p \leq 1 \}$, where $D$ is the diagonal matrix containing the eigenvalues $\{\lambda_i\}_{i=1}^N$ of $\Sigma$.

For $p = 1$, we can show that $\mathcal{U}_1$ is equal to the set defined by $A = \{ \sum_{i=1}^N t_i (\lambda_i e_i), \|t\|_1 \leq 1 \}$ with $e_i$ as the i-th canonical vector. In fact:
x ∈ U₁ implies x = Dv for some v, ∥v∥₁ ≤ 1.

But x = ∑ᵢ₌₁ᴺ xᵢ eᵢ + ∑ᵢ₌₁ᴺ vᵢ(λᵢ eᵢ) with ∥v∥₁ ≤ 1. So x ∈ A

Now if x ∈ A, then x = ∑ᵢ₌₁ᴺ tᵢ(λᵢ eᵢ) = ∑ᵢ₌₁ᴺ(λᵢ tᵢ) eᵢ for some t with ∥t∥₁ ≤ 1. In order to have x ∈ U₁, is enough to show that the vector defined as v = D⁻¹x satisfies ∥v∥₁ ≤ 1:

\[∥D⁻¹x∥₁ = ∑ᵢ₌₁ᴺ \frac{1}{λᵢ} |xᵢ| = ∑ᵢ₌₁ᴺ \frac{1}{λᵢ} |λᵢ tᵢ| = ∑ᵢ₌₁ᴺ |tᵢ| ≤ 1\]

Then, defining the simplex ̃A = \{ ∑ᵢ₌₁ᴺ tᵢ(λᵢ eᵢ) + tᵢ₋¹₀, ∑ᵢ₌₁ᴺ tᵢ = 1, tᵢ ≥ 0 \}, is not hard to see that V₁ = 2ᴺ Vol( ̃A). But applying the formula shown in Stein (1966):

\[Vol( ̃A) = \frac{1}{N!} |(-1)^{N+1} \det(λ₁ e₁, λ₂ e₂, ..., λᴺ eᴺ)| = \frac{1}{N!} \prodᵢ₌₁ᴺ λᵢ\]

So

\[V₁ = \frac{2ᴺ}{N!} \prodᵢ₌₁ᴺ λᵢ = \frac{2ᴺ}{N!} \prodᵢ₌₁ᴺ \frac{λᵢ}{|Σ|} = \frac{2ᴺ}{N!} |Σ| \left(\frac{1}{N!} \right) \left(\frac{1}{|Σ|} \right)^{N} \left(\frac{1}{N!} \right)^{N} = \frac{2ᴺ}{N!} \theta N \sqrt{|Σ|}\]

For p = ∞, we can show that U∞ is equal to the box set B = \{ x | -λᵢ ≤ xᵢ ≤ λᵢ, ∀i ∈ {1, ..., N} \}. In fact:

x ∈ U∞ implies x = Dv for some v, ∥v∥∞ ≤ 1. Now

\[|(Dv)ᵢ| = λᵢ |vᵢ| ≤ λᵢ ∀i. \text{ Then } x ∈ B\]

Now if x ∈ B, then \(|xᵢ| / λᵢ| ≤ 1 ∀i. \text{ In order to have } x ∈ U∞, \text{ is enough to show that the vector defined as } v = D⁻¹x \text{ satisfies } ∥v∥∞ ≤ 1:\n
\[∥D⁻¹x∥∞ = \maxᵢ \left(\frac{1}{λᵢ} |xᵢ| \right) ≤ 1. \text{ By looking at } B, \text{ is easy to see that:}\]

\[V∞ = \prodᵢ₌₁ᴺ 2λᵢ = 2ᴺ \prodᵢ₌₁ᴺ λᵢ = 2ᴺ \theta N \sqrt{|Σ|}\]
Even if we don’t have an exact formula for $V_p$ for some $p$, we still now the existence of $\alpha_p$

$$\frac{2^N}{N!} \leq \alpha_p \leq 2^N.$$ In fact as:

$$V_p = V_p \frac{V_p}{V_\infty} = \frac{V_p}{V_\infty} 2^N \delta^N \sqrt{|\Sigma|} \text{ then } \alpha_p = \frac{V_p}{V_\infty} 2^N.$$ So $\alpha_p \leq 2^N$. In the same way:

$$V_p = V_p \frac{V_1}{V_1} = \frac{V_p}{V_1} \frac{2^N}{N!} \delta^N \sqrt{|\Sigma|} \text{ then } \alpha_p = \frac{V_p \frac{2^N}{N!}}{V_1}.$$ So $\alpha_p \geq \frac{2^N}{N!}$.

Remark: As shown in Sykora (2005), $\alpha_2 = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ with $\Gamma(\cdot)$ as the gamma function

Using proposition 3, we can write (1.3) in terms of the volume $V$:

$$F_p(V) \equiv \max_{x \in \mathcal{P}} \mu^T x - K_p \left\| \Sigma^{1/2} x \right\|_q V^{1/N}$$ (1.9)

With $K_p = \left( \sqrt{|\Sigma|} \alpha_p \right)^{-1/N}$

By solving (1.9), we have a fair way to determine when is better to use the solution induced by one uncertainty set than another. We next show some properties for $F_p(V)$.

1. $F_p(0) = F_{\bar{p}}(0) \forall p \neq \bar{p}$. As $\mathcal{P}$ is not empty and bounded $-\infty < F_p(0) < \infty$

2. $F_p(\infty) = \begin{cases} 0 & \text{if } 0 \in \mathcal{P} \\ \infty & \forall p \end{cases}$

To see this, let’s redo the proof for (1.3) with $\mathcal{U}_p = \mathbb{R}^N$. For that case $\inf_{r \in \mathbb{R}^N} r^T x = -\infty$ unless $x \neq 0$. If $0 \in \mathcal{P}$ then $x = 0$ is the only feasible solution and therefore optimal.

3. $F_p(V)$ is strictly decreasing $\forall p \geq 1$

When $V_1 < V_2$, the function defined as $g_p(x, V) \equiv \mu^T x - K_p V^{1/N} \left\| \Sigma^{1/2} x \right\|_q$ satisfies $g_p(x, V_1) > g_p(x, V_2) \forall x \in \mathcal{P}$. So $F_p(V_1) > F_p(V_2)$

4. $F_p(V)$ is convex $\forall p \geq 1$: 

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Is clear that \( V^{1/N} \) is concave \( \forall \, V \in \mathcal{R}_+ \). Then \( g_p(x, V) \) is convex in \( V \) for each \( x \in P \). Thus \( \text{Max}_{x \in P} \, g_p(x, V) \) is convex.

5. If \( N > 1 \) then

\[
\partial F_p(\infty) \equiv \lim_{V \to \infty} \partial F_p(V) = \{0\} \quad \forall p
\]

where \( \partial F_p(V) \) is the subdifferential set of \( F_p \) at \( V \).

Proof:

As proved in Zalinescu (2002), if \( g(x) = \max_{t \in T} f_t(x) \), then

\[
\partial g(x) = \text{cl} \left\{ \text{Cov} \bigcup \{ \partial f_t(x) : f_t(x) = g(x) \} \right\}
\]

So \( \partial F_p(V) \) is the interval \( [a_p(V), b_p(V)] \) where

\[
a_p(V) = \min \left\{ \frac{-K_p}{V^{1-1/N}} \| \Sigma^{1/2} x \|_q \mid x \in P: g_p(x, V) = F_p(V) \right\}
\]

\[
b_p(V) = \max \left\{ \frac{-K_p}{V^{1-1/N}} \| \Sigma^{1/2} x \|_q \mid x \in P: g_p(x, V) = F_p(V) \right\}
\]

\([a_p(V), b_p(V)] \neq \emptyset \) since \( P \) is non empty and \( [a_p(V), b_p(V)] \subseteq \mathcal{R}_- \). Clearly \([a_p(V), b_p(V)] \to \{0\} \) as \( V \to \infty \).

1.3.1. Solution Sparsity

If we compare \( F_p(V) \) for different \( p \), we know differences between them are produced by the penalization term \( K_p \| \Sigma^{1/2} x \|_q \). Therefore the difference in the solutions characteristics can be compared to the ones found in the norm approximation problem. For example, when \( q < \infty \) \( \| \Sigma^{1/2} x \|_q \) changes with any change in one of the component of the vector \( \Sigma^{1/2} x \). However this doesn’t happen with \( q = \infty \), since there can be a changes in the vector \( \Sigma^{1/2} x \) without changing the absolute value of the maximum element. Hence when \( q = \infty \), the components of \( \Sigma^{1/2} x \) will have more freedom to be different than 0, and hence \( x \) can be less sparse. Let’s show this with a simple example:
Suppose $N = 2$ and for simplicity $\Sigma^{1/2}$ is diagonal with entries $\lambda_1$ and $\lambda_2$. Assume $P = \{x = (x_1, x_2) | x_1 + x_2 = C, x_1 \geq 0, x_2 \geq 0\}$. Without loss of generality $\mu_1 > \mu_2$.

$$F_1(V) = \max_{x \in \mathcal{P}} \{\mu^T x - K_1V^{1/N} \max[\lambda_1|x_1|, \lambda_2|x_2|]\}$$

$$= \max_{0 \leq x_1 \leq C} \left\{ (\mu_1 - \mu_2)x_1 + C\mu_2 - K_1V^{1/N} \max[\lambda_1|x_1|, \lambda_2(C - x_1)] \right\}$$

$$= \max_{0 \leq x_1 \leq C} \left\{ \begin{cases} (\mu_1 - \mu_2 - K_1V^{1/N}\lambda_1)x_1 + C\mu_2 & \text{if } x_1 \geq \frac{\lambda_2 C}{\lambda_1 + \lambda_2} \\ (\mu_1 - \mu_2 + K_1V^{1/N}\lambda_2)x_1 + C\mu_2 - K_1V^{1/N}\lambda_2 C & \text{~} \end{cases} \right. $$

Hence $x_1 = \left\{ \begin{cases} C & \text{if } \mu_1 - \mu_2 - K_1V^{1/N}\lambda_1 \geq 0 \\ \frac{\lambda_2 C}{\lambda_1 + \lambda_2} & \text{~} \end{cases} \right. $ with

$$F_1(V) = \left\{ \begin{cases} (\mu_1 - K_1V^{1/N}\lambda_1)C & \text{if } \mu_1 - \mu_2 - K_1V^{1/N}\lambda_1 \geq 0 \\ C & \text{if } \mu_1 - \mu_2 - K_1V^{1/N}\lambda_2 \geq 0 \\ 0 & \text{~} \end{cases} \right. $ 

$$F_{\infty}(V) = \max_{x \in \mathcal{P}} \{\mu^T x - K_{\infty}V^{1/N}(\lambda_1|x_1| + \lambda_2|x_2|)\}$$

$$= \max_{0 \leq x_1 \leq C} \left\{ (\mu_1 - \mu_2)x_1 + C\mu_2 - K_{\infty}V^{1/N}(\lambda_1|x_1| + \lambda_2(C - x_1)) \right\}$$

$$= \max_{0 \leq x_1 \leq C} \left\{ (\mu_1 - \mu_2 - (\lambda_1 - \lambda_2)K_{\infty}V^{1/N})x_1 + C\mu_2 - K_{\infty}V^{1/N}C\lambda_2 \right\}$$

Hence $x_1 = \left\{ \begin{cases} C & \text{if } \mu_1 - \mu_2 - (\lambda_1 - \lambda_2)K_{\infty}V^{1/N} \geq 0 \\ 0 & \text{~} \end{cases} \right. $ with

$$F_{\infty}(V) = \left\{ \begin{cases} (\mu_1 - \lambda_1 K_{\infty}V^{1/N})C & \text{if } \mu_1 - \mu_2 - K_{\infty}V^{1/N}\lambda_1 \geq 0 \\ C\mu_2 - K_{\infty}V^{1/N}C\lambda_2 & \text{~} \end{cases} \right. $$

For $p = 1$ we have two possible answers $(x_1, x_2) = (C, 0)$ or $(\frac{\lambda_2 C}{\lambda_1 + \lambda_2}, \frac{\lambda_1 C}{\lambda_1 + \lambda_2})$. For $p = 2$ we don’t have an explicit solution, but we can show it behaves like $p = 1$: There are cases were we have corner solution but other that don’t. For instance if $(\lambda_1, \lambda_2) = (0.3, 0.4)$, $(\mu_1, \mu_2) =$
(1,1.25) and $C = 1$ we get $x_1 = 0.3629$. However for $p = \infty$ we always have a corner solution \((x_1, x_2) = (C, 0)\) or \((0, C)\). Thus we can have less sparsity with $p = 1$ or $p = 2$.

We can also show this with some empirical examples. Fixing $N = 10$ and $V = 100$, we construct 100 random instances for $\mu, \Sigma$, taken from data that was generated from random numbers between 1 and 10. $P = \{x \geq 0, 1^T x = 1\}$. The box plot in Figure 1.1 shows that sparsity, measured by the number of 0 in $x$, is clearly bigger when $p = \infty$ and smaller when $p = 1$.

Figure 1.1: Solutions sparsity under $p=\{1,2,\infty\}$.

Sparsity is directly linked with diversification of each solution. Obviously the more sparse, the less diversified. So it can be used to determine when to use one norm or the other, according to how much diversification we want for the solution.
1.4. Carry Trade

The carry trade is one of the strategies that deals with international assets, i.e. with assets that are traded in a different currency (in a foreign market). International assets give new opportunities to enhance portfolio returns and/or obtain more diversification. However, investing in those assets involves currency trading. The total return $\tilde{R}_{ij}^{t+1}$ of investing in asset $j$ of currency $c$ at time $t$ is defined as:

$$\tilde{R}_{cj}^{t+1} \equiv \frac{R_{cj}^{t+1} S_{c}^{t+1}}{S_{c}^{t}}$$  \hspace{1cm} (1.10)$$

Where $R_{cj}^{t}$ is the total asset return and $S_{c}^{t}$ is the exchange rate of currency $c$ at time $t$, measured in units of the numeraire (national currency) in one unit of currency $c$. By applying log and approximation of the log at (1.10), we have that the return $\tilde{r}_{cj}^{t}$ (total return -1) of the investment is

$$\tilde{r}_{cj}^{t+1} \approx r_{cj}^{t+1} + \frac{S_{c}^{t+1}}{S_{c}^{t}} - 1 \approx r_{cj}^{t+1} + s_{c}^{t+1} - s_{c}^{t} \equiv r_{cj}^{t+1} + \Delta s_{c}^{t+1}$$ \hspace{1cm} (1.11)$$

where $s_{c}^{t} \equiv \log(S_{c}^{t})$

Now have two sources of risk: The risk of the asset and the risk of exchange rate movement ($r_{cj}^{t+1}$ and $s_{c}^{t+1}$ are uncertain in(1.11)). As seen by Eun and Resnick (1988), empirical evidence shows that correlations among exchange rates are higher than the correlation among stock market correlation. They show cross-correlations among the stock and exchange market is positive. Therefore special care is needed to allocate and lower risk by diversification. One way to get rid of currency risk is to use currency hedging. For example in Eun and Resnick (1988), they use a forward contract on the expected return of a stock. Black (1989) presents a universal hedging formula for the proportion each investor wants to hedge. This formula depends on three values:
1. The average across investors of the expected excess return on the world market portfolio
2. The average across investors of the volatility of the world market portfolio
3. The average exchange rate volatility (averaged variances) across all pairs of countries.

More hedging strategies can be found in Glen & Jorion (1993) or Topaloglou et al. (2008).

The carry trade strategy is a currency strategy driven by foreign interest rates. It consists of taking short (long) positions in currency contracts from low (high)-yielding currencies. Hence, this strategy expects that a currency will appreciate (depreciate) when the interest is high (low) in relation to the rest of the currencies. The strategy is net zero, i.e. the total value of short positions is equal to the total value of long positions.

The previous idea goes against what it we called the uncovered interest parity. The parity says that the national (US) interest rate $i^t_{\text{US}}$ should follow the following relationship:

$$i^t_{\text{US}} = i^t_c + \frac{E(S^{t+1}_c)}{S^t_c} - 1 \quad (1.12)$$

where $i^t_c$ is the interest rate of currency c at time t. With (1.12), (low)high-yielding currencies are expected to appreciate (depreciate), until the no-arbitrage condition is reached. However, there is empirical evidence showing deviations from (1.12), which is known as the forward premium puzzle. Empirical evidence can be found in Bansal (2000), (1984) and Pissinger (2011).

Possible explanations for the forward puzzle are named by Chinn (2009) which are:

1. The invalidity of the rational expectations hypothesis
2. Issues of econometric implementation
3. The existence of an exchange risk premium
1.4.1. Equal Weighted Carry Trade Portfolio

Many known carry trade indexes have equal weighted positions in the currencies where they go long and short. For example the Deutsche Bank G10 Currency index is composed of currency futures contracts from the G10 currencies. It has equally-weighted long (short) positions in the highest (lowest) three interest rates currencies\(^3\). The same rule is applied for the JPMorgan’s Income FX and Income EM strategies\(^4\).

The performance shown by the mentioned indexes have two main things in common: We can have profit in the long term, but also large drawdown. For example for the Deutsche Bank G10 Currency, the 1989-2010 annualized return is 7.49%, but a volatility of 9.7% and a maximum drawdown of 34.55%\(^5\).

Given that we don’t have the history of contracts transactions from previous indexes, we will construct our own benchmark portfolio. Just like the other index, we will use the same rule for weighting positions: Rank interests rates and go long (short) in the best (worst) K currencies. Notice that benchmark portfolio can also be constructed with forward rates. By the Covered Interest rate parity:

\[
i_{US}^t = i_c^t + f_c^t - s_c^t
\]

where \(f_c^t\) is the log of the forward rate of currency c at time t. So ranking currencies by its interest rates is equivalent to rank them by \(s_c^t - f_c^t\).

The return for each currency at each time period will be the one derived from (1.11), i.e. \(i_{c}^{t+1} + \Delta s_{c}^{t+1}\), just like defined in Brunnermeier et al. (2009). Using the 3-month LIBOR rate and the G10 currencies\(^6\), the performance for the Feb-1994 to Sep-2012 (daily rebalancing) is presented in Table 1.1:

\(^3\) http://dbfunds.db.com/dbv/index.aspx
\(^4\) http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/research/FXStrategy
\(^5\) https://index.db.com/htmlPages/DBCR_brochure.pdf
\(^6\) US Dollar, EURO, British Pound, Swiss Franc, Canadian Dollar, Japanese Yen, Norwegian Krone, Swedish Krone, Australian Dollar, New Zeland Dollar
Table 1.1: Annualized Performance of benchmark carry trade portfolio returns.

<table>
<thead>
<tr>
<th>K</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.6%</td>
<td>15.3%</td>
<td>43.4%</td>
<td>16.8%</td>
<td>51.6%</td>
<td>-0.4</td>
<td>11.3</td>
</tr>
<tr>
<td>2</td>
<td>6.4%</td>
<td>12.4%</td>
<td>38.7%</td>
<td>10.1%</td>
<td>62.9%</td>
<td>-0.6</td>
<td>10.6</td>
</tr>
<tr>
<td>3</td>
<td>6.1%</td>
<td>10.4%</td>
<td>33.5%</td>
<td>8.6%</td>
<td>71.5%</td>
<td>-0.6</td>
<td>10.0</td>
</tr>
<tr>
<td>4</td>
<td>4.3%</td>
<td>8.6%</td>
<td>28.3%</td>
<td>8.8%</td>
<td>49.4%</td>
<td>-0.6</td>
<td>9.6</td>
</tr>
<tr>
<td>5</td>
<td>3.3%</td>
<td>7.0%</td>
<td>28.2%</td>
<td>8.5%</td>
<td>38.4%</td>
<td>-0.6</td>
<td>8.7</td>
</tr>
</tbody>
</table>

Table 1.1 shows that carry trade idea is profitable on average. If we increase K, we observe a diversification effect. However we always have fat tails, meaning that there can be huge losses in many periods. To have some insight of the most active currencies we look at Figure 1.2, which show some metrics about the weights participation in the K=3 benchmark portfolio in time.

Figure 1.2: Currency participation in the benchmark carry trade portfolio.

Weights Mean represent the average weights for each currency for period 1994-2012. Time Used is the fraction of periods where the currency is used.
Is clear that the New Zealand Dollar (NZ D) and the Australian Dollar (AUS D) are the main target currencies (go long). Oppositely, the Japanese Yen (JPY Y) and Swiss Franc (SWISS F) are the main funding currencies. Currencies like the Norwegian Krone (NOR K) and the UK are also quite used to long, while the American Dollar (US D) and EURO are quite used to fund. These results align with empirical evidence of the most used carry trade pairs (Galati, Heath, & McGuire, 2007)
1.4.2. Exchange Rate Prediction

The risk in the carry trade comes from exchange rate movement. So if we have a good model for predicting exchange rates for each currency, we can dynamically change the weights of our portfolio. Exchange rates dynamics and factors have been extensively studied. For example, it’s well known that macroeconomics indicators such as PPP, interest rates, inflation, public debt, GDP and balance of payments play an important role on exchange rates. Mulvey (1998) builds a global scenario system for future capital markets, based on a cascade of differential equations. One of this DE is built to predict exchange rates, which depends on several economics factors like the mentioned above. The problem with economic factors is that they play role in longer horizons. For shorter horizons (we will work on daily rates) it is hard to find an accurate point estimator for future exchange rates, as explained in Kilian and Taylor (2003). In this work they show that you need to build a nonlinear, exponential smooth transition autoregressive (ESTAR) model to slightly improve random walk prediction. In the same direction Cheung et al. (2005) tested five known models and found that no one could consistently beat random walk forecasts. According to them, US exchange traders perceive that economic fundamentals are more important at longer horizons, while short-run deviations from the fundamentals are attributed to excess speculation and institutional customer/hedge fund manipulation.

Using the data of the benchmark portfolio, we know build up some simple models and compare their prediction with the random walk. The models are the following:

<table>
<thead>
<tr>
<th></th>
<th>$\Delta s^{t+1}_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random Walk</td>
<td>$\Delta s^{t+1}_c = 0$</td>
</tr>
<tr>
<td>Uncovered interest parity</td>
<td>$\Delta s^{t+1}<em>c = i^{t}</em>{US} - i^{t}_{c}$</td>
</tr>
<tr>
<td>AR</td>
<td>$\Delta s^{t+1}_c = \beta_0 + \beta_1 s^t_c + \epsilon^t$</td>
</tr>
</tbody>
</table>

Where $\Delta s^{t+1}_c \equiv s^{t+1}_c - s^t_c$ and $\epsilon^t$ is an i.i.d standard normal noise. The third model is called AR since it is equivalent to the AR(1) regression $s^{t+1}_c = \beta_0 + \beta_1 s^t_c + \epsilon^t$ with $\beta_1 = 1 + \beta_1$. Hence the prediction error $e^{t+1}_c = s^{t+1}_c - \hat{s}^{t+1}_c$ for each model is the following:
Random Walk \[ e_c^{t+1} = \Delta s_c^{t+1} \]

Uncovered interest parity \[ e_c^{t+1} = \Delta s_c^{t+1} + i_c^t - i_d^t \]

AR \[ e_c^{t+1} = \Delta s_c^{t+1} - \beta_0 - \beta_1 s_c^t \]

Betas are obtained by doing a regression on the same equation with all the previous information available. We re-run the regression (update betas) every year. As a reference, Table 1.2 shows regression results for the entire testing period (Feb-1995 to Sep-2012). All the information, such as R^2 and the p-value for the whole model, is pointing out that the model does not fit the data properly. Furthermore, we can’t reject \( \beta_0 = 0 \) and \( \beta_1 = 0 \) at a 5% level for any currency.

<table>
<thead>
<tr>
<th>Currency</th>
<th>Beta 0</th>
<th>Beta 1</th>
<th>p_value</th>
<th>p_value</th>
<th>R^2</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>NDRK</td>
<td>0.0022</td>
<td>-0.0011</td>
<td>0.121</td>
<td>0.112</td>
<td>0.1%</td>
<td>0.11</td>
</tr>
<tr>
<td>SWISS F</td>
<td>0.0001</td>
<td>-0.0006</td>
<td>0.757</td>
<td>0.329</td>
<td>0.0%</td>
<td>0.33</td>
</tr>
<tr>
<td>JPY Y</td>
<td>0.0033</td>
<td>-0.0007</td>
<td>0.315</td>
<td>0.307</td>
<td>0.0%</td>
<td>0.31</td>
</tr>
<tr>
<td>NZD</td>
<td>0.0003</td>
<td>-0.0007</td>
<td>0.361</td>
<td>0.325</td>
<td>0.0%</td>
<td>0.23</td>
</tr>
<tr>
<td>UK P</td>
<td>-0.0009</td>
<td>-0.0018</td>
<td>0.035</td>
<td>0.033</td>
<td>0.1%</td>
<td>0.03</td>
</tr>
<tr>
<td>CAN D</td>
<td>0.0000</td>
<td>-0.0003</td>
<td>0.973</td>
<td>0.521</td>
<td>0.0%</td>
<td>0.52</td>
</tr>
<tr>
<td>SWE K</td>
<td>0.0030</td>
<td>-0.0015</td>
<td>0.071</td>
<td>0.068</td>
<td>0.1%</td>
<td>0.07</td>
</tr>
<tr>
<td>EURO</td>
<td>-0.0002</td>
<td>-0.0009</td>
<td>0.212</td>
<td>0.139</td>
<td>0.0%</td>
<td>0.14</td>
</tr>
<tr>
<td>AUS D</td>
<td>0.0001</td>
<td>-0.0006</td>
<td>0.663</td>
<td>0.357</td>
<td>0.0%</td>
<td>0.36</td>
</tr>
</tbody>
</table>

Table 1.3 shows that root mean square error is slightly lower with the Random Walk, for all the currencies tested. Statistical tests show that we can’t reject the hypothesis \( \overline{\text{err}}_{RW} = \overline{\text{err}}_{UIP} \) and \( \overline{\text{err}}_{RW} = \overline{\text{err}}_{AR} \), where \( \overline{\text{err}} \) is the mean error of each model. The results are shown in the appendix.

Figure 1.3 shows the cumulative errors of each method in time (we plot cumulative errors instead of errors just to have a clearer picture) for two currencies\(^7\). Thus there is no clear advantage of using a particular method depending on time. We also observe pronounced ups and downs, which supports the evidence of currency big swings.

\(^7\) The same behavior can be seen in the rest of the currencies, so plots are shown in the appendix.
Table 1.3: RMSE for the Random Walk, UIP and AR models.

<table>
<thead>
<tr>
<th>Currency</th>
<th>RW</th>
<th>UIP</th>
<th>AR</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOR K</td>
<td>73.72</td>
<td>73.74</td>
<td>73.88</td>
</tr>
<tr>
<td>SWISS F</td>
<td>70.72</td>
<td>70.74</td>
<td>70.89</td>
</tr>
<tr>
<td>JPY Y</td>
<td>70.34</td>
<td>70.38</td>
<td>70.88</td>
</tr>
<tr>
<td>NZ D</td>
<td>82.67</td>
<td>82.69</td>
<td>82.74</td>
</tr>
<tr>
<td>UK P</td>
<td>55.80</td>
<td>55.80</td>
<td>55.92</td>
</tr>
<tr>
<td>CAN D</td>
<td>53.43</td>
<td>53.44</td>
<td>53.57</td>
</tr>
<tr>
<td>SWE K</td>
<td>76.75</td>
<td>76.77</td>
<td>77.00</td>
</tr>
<tr>
<td>EURO</td>
<td>63.43</td>
<td>63.44</td>
<td>63.55</td>
</tr>
<tr>
<td>AUS D</td>
<td>81.28</td>
<td>81.31</td>
<td>81.37</td>
</tr>
</tbody>
</table>

Numbers are amplified by $10^4$

Figure 1.3: Cumulative errors of the 3 models for the NOR Krone and SWISS Franc.

All these results, lead us to the conclusion that no model that can predict better than the random walk.
1.4.3. Robust Carry Trade (RCT)

Given the complexity in predicting exchange rate movements, we prefer to assume that carry trade returns can take any value in a certain set, like the ones defined in (1.1). Just like as benchmark portfolios, the returns are defined by the RHS of (1.12). The robust carry trade problem (RCT) can be formulated as the RC in (1.9), i.e.:

\[
RCT_p(V) = \max_{w \in P} \mu^T w - K_p \left\| \Sigma^{1/2} w \right\|_q V^{1/N}
\]

\[
P = \{(w, w_+, w_-): 1^T w_+ = 1, \quad 1^T w_- = 1, \quad w_+ \geq 0, \quad w_- \geq 0, \quad w = w_+ - w_- \}
\]

\(\mu\) and \(\Sigma\) are the mean value and covariance matrix for the returns. Given what we discussed before:

\[
\hat{\mu} = \mu^t + \Delta \hat{s}^{t+1} = \mu^t
\]

\(\Sigma\) is estimated with the sample covariance. The variables \(w_+, w_-\) are the long and short decomposition of a position. The first two constraints ensure the basic rule of the carry trade: We have to borrow (short) and invest (long) everything from the set of currencies and the total amount should be the same.

It is easy to see that \(P\) is bounded, closed and non-empty polyhedron. All constraints are linear. By the first four constraints, \(w_+, w_-\) are bounded between 0 and 1. Then by the last constraint, \(-1 \leq w \leq 1\). Hence \(RCT_p(V)\) satisfies all the properties seen in section 2.

Is important to clarify that RCT looks to improve the way of selecting and assigning the target and funding currencies. It doesn’t address the issue to which assets in that currency to invest. Given the objective function of RCT, target (funding) currencies will be the ones with best (worst) compromise between interest rates and exchange rate volatility. The difference with benchmark portfolio is that weights here are only chosen by looking the interest rates.
1.4.4. Results

We solve \( RCT_p(V) \) with the same data that we use to determine the performance of benchmark carry trade. We use CVX software\(^6\), because it is designed for convex programming and is easy to use under Matlab environment. For simplicity, we will call the resulting series with the same name, i.e. \( RCT_p(V) \). Using different volumes \( V \) and \( p = \{1, 2, \infty\} \), the performance is the shown in Table 1.4.

Table 1.4: Annualized Performance of robust carry trade returns.

<table>
<thead>
<tr>
<th>log(V)</th>
<th>Norm</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>-\infty</td>
<td>-</td>
<td>8.7%</td>
<td>15.3%</td>
<td>43.4%</td>
<td>16.7%</td>
<td>51.9%</td>
<td>-0.4</td>
<td>11.4</td>
</tr>
<tr>
<td>-94</td>
<td>\infty</td>
<td>9.1%</td>
<td>14.0%</td>
<td>43.3%</td>
<td>13.6%</td>
<td>67.3%</td>
<td>-0.7</td>
<td>11.5</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6.5%</td>
<td>13.4%</td>
<td>53.4%</td>
<td>21.3%</td>
<td>30.8%</td>
<td>-0.8</td>
<td>10.9</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8.5%</td>
<td>14.3%</td>
<td>47.4%</td>
<td>16.4%</td>
<td>51.7%</td>
<td>-0.7</td>
<td>11.0</td>
</tr>
<tr>
<td>-88</td>
<td>\infty</td>
<td>8.2%</td>
<td>12.5%</td>
<td>44.5%</td>
<td>15.2%</td>
<td>54.2%</td>
<td>-0.9</td>
<td>10.7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>5.0%</td>
<td>10.2%</td>
<td>46.9%</td>
<td>19.7%</td>
<td>25.7%</td>
<td>-0.8</td>
<td>10.3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8.1%</td>
<td>13.0%</td>
<td>50.4%</td>
<td>17.5%</td>
<td>46.6%</td>
<td>-0.9</td>
<td>11.1</td>
</tr>
<tr>
<td>-84</td>
<td>\infty</td>
<td>7.2%</td>
<td>10.6%</td>
<td>42.7%</td>
<td>16.7%</td>
<td>43.3%</td>
<td>-0.7</td>
<td>10.6</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>4.1%</td>
<td>7.7%</td>
<td>28.8%</td>
<td>11.4%</td>
<td>35.8%</td>
<td>-0.6</td>
<td>7.7</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7.2%</td>
<td>10.8%</td>
<td>47.6%</td>
<td>18.8%</td>
<td>38.3%</td>
<td>-1.0</td>
<td>11.9</td>
</tr>
<tr>
<td>-82</td>
<td>\infty</td>
<td>6.3%</td>
<td>9.0%</td>
<td>37.5%</td>
<td>14.2%</td>
<td>44.5%</td>
<td>-0.3</td>
<td>8.8</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3.4%</td>
<td>6.3%</td>
<td>31.4%</td>
<td>11.7%</td>
<td>28.9%</td>
<td>-0.5</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.4%</td>
<td>8.8%</td>
<td>38.1%</td>
<td>16.0%</td>
<td>39.7%</td>
<td>-0.8</td>
<td>8.4</td>
</tr>
<tr>
<td>-75</td>
<td>\infty</td>
<td>5.1%</td>
<td>6.9%</td>
<td>28.9%</td>
<td>11.3%</td>
<td>45.2%</td>
<td>-0.7</td>
<td>10.0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3.3%</td>
<td>4.4%</td>
<td>18.5%</td>
<td>8.8%</td>
<td>37.9%</td>
<td>-0.6</td>
<td>8.5</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.0%</td>
<td>5.8%</td>
<td>34.2%</td>
<td>14.7%</td>
<td>27.0%</td>
<td>-0.7</td>
<td>8.3</td>
</tr>
<tr>
<td>-73</td>
<td>\infty</td>
<td>3.5%</td>
<td>3.8%</td>
<td>8.9%</td>
<td>2.9%</td>
<td>121.1%</td>
<td>-0.8</td>
<td>28.0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.4%</td>
<td>2.0%</td>
<td>7.3%</td>
<td>2.7%</td>
<td>88.8%</td>
<td>-1.6</td>
<td>43.8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.3%</td>
<td>2.2%</td>
<td>10.0%</td>
<td>3.2%</td>
<td>70.9%</td>
<td>-1.7</td>
<td>47.5</td>
</tr>
</tbody>
</table>

By looking at the Ulcer performance Index\(^9\) (UPI), \( RCT_\infty(V) \) gives the best compromise between return and risk, for all volumes. \( RCT_\infty(V) \) dominates \( RCT_2(V) \) in almost every \( V \), if we look the return and risk measures excepting standard deviation. So we’ve found a strategy that

---

\(^6\) http://cvxr.com/cvx/

\(^9\) UPI is the ratio between Average Return and Ulcer
performed better than Markowitz (Remember, from section 1, we showed that $RCT_2(V)$ is equivalent to solve Markowitz). As expected, there is a tendency to decrease return and risk as we increase $V$.

Figure 1.5 can give us more information about the results obtained. It shows the profit/loss time series of returns for $RCT_p(V)$, at extremes values of $V$. The first thing to notice is that bigger volumes tend to have more days with no carry trade transaction. (The solution $w = 0$ in the RCT is feasible). The other thing to notice is that biggest drawdown coincides with the 07-08 global financial crash. This phenomenon make sense, since investors in crisis period prefer to keep their capital in “safer” currencies like those from developed countries (US dollar, Japanese Yen, Euro, Swiss Franc). Recall from Figure 1.2 that those currencies were mainly funding currencies. So what happens is that these currencies appreciate with respect to the target currencies, which go against the carry trade bet.

Figure 1.4: Profit of RCT strategies with log(V)=-94 and log(V)=-73
The problem comes when we compare the performance of $RCT_p(V)$ with the benchmark portfolio performance from Table 1. We can’t say we have improved performance. Recalling the benchmark portfolio performance when $K=3$ (which has the best performance in terms of UPI):

<table>
<thead>
<tr>
<th>K</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6.1%</td>
<td>10.4%</td>
<td>33.5%</td>
<td>8.6%</td>
<td>71.5%</td>
<td>-0.6</td>
<td>10.0</td>
</tr>
</tbody>
</table>

There are $RCT_p(V)$ strategies that have more than 6.1% mean return and less than 10.4% of standard deviation. Moreover, some of these configurations have less kurtosis and skewness too. However, all configurations above 6.1% mean return have also more max drawdown and more ulcer. Figure 1.5 can give us more information about the results obtained. It shows the profit/loss time series of returns for the benchmark carry trade (when $K=3$) and $RCT_p(\exp(-82))$. We select this specific volume because benchmark carry trade ends up with the same level or mean return as the $RCT_p$ strategies.

Figure 1.5: Profit of benchmark carry trade portfolio v/s RCT. log(V)=-82
From this figure, we can clearly see that performance results depend highly on the time window chosen. For example, RCT$_{\infty}$ and RCT$_2$ are considerable better than benchmark until the 2008 financial crisis. Although RCT$_1$ seems to be the worst strategy at the end, we should not discard it. First of all, RCT$_1$ do better than the rest of RCT$_p$ strategies in some time periods, like Figure 1.6 shows.

Figure 1.6: Profit loss for strategies during 2008 financial crisis. The left plot is for log(V)=–84 and right log(V)=–82

Moreover, Table 1.5 shows the % of periods each RCT$_p$ strategy has the best return among the rest. We can clearly see that, no matter the volume we choose, RCT$_1$ gives the best return in at least 30% of the time. The correlation matrix in Table 1.6 confirms what we see in previous results: RCT$_{\infty}$ and RCT$_2$ are more alike strategies, compared to RCT$_1$.

Table 1.5: % of periods each RCT had the best returns.

<table>
<thead>
<tr>
<th>log V</th>
<th>$\infty$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>–94</td>
<td>42%</td>
<td>48%</td>
<td>25%</td>
</tr>
<tr>
<td>–88</td>
<td>36%</td>
<td>42%</td>
<td>23%</td>
</tr>
<tr>
<td>–84</td>
<td>35%</td>
<td>40%</td>
<td>25%</td>
</tr>
<tr>
<td>–82</td>
<td>38%</td>
<td>38%</td>
<td>25%</td>
</tr>
<tr>
<td>–75</td>
<td>46%</td>
<td>41%</td>
<td>30%</td>
</tr>
<tr>
<td>–73</td>
<td>36%</td>
<td>30%</td>
<td>34%</td>
</tr>
</tbody>
</table>
Table 1.6: Correlation Matrix of RCT returns

<table>
<thead>
<tr>
<th></th>
<th>∞</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>∞</td>
<td>1.00</td>
<td>0.60</td>
<td>0.84</td>
</tr>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

The cause the latter behavior can be partially explained by the sparsity effect explained in section 2. More sparsity leads to less diversification. Figure 1.7 shows the sparsity box-plot for three values of $V$.

Figure 1.7: Sparsity box plot for RCT returns. $\log(v) = \{88, 84, 82\}$

All previous results are pointing out that it will be better to dynamically select the uncertainty sets. We still have strategies that do poorly in some time windows. Therefore we can increase their performance dramatically if we can take some action in those time windows. What comes next is to find good triggers for $RCT_p$ strategies.
1.5. Dynamic Robust Carry Trade Strategy

As we have seen in section 3, carry trade performance deteriorates with market crisis. To graph this correlation, we plot the benchmark portfolio (K=3) and the SPY index in Figure 1.8.

Figure 1.8: SPY index and Benchmark Carry Trade Portfolio

The previous statement can be explained by empirical findings in Brunnermeier et al. (2009). We know carry trade performance is sensible to unexpected exchange movements. According to them, this occurs when investors near funding constraints. In order to identify funding liquidity they use two measures:

1. The implied volatility of the S&P500, called VIX\textsuperscript{10}. As they say VIX is not only a proxy to measure “global risk appetite” in equity market, but also in credit market.
2. The TED spread, which is the difference between the LIBOR interbank interest rate and the T-Bill rate. TED rises when banks face liquidity problems.

They show that weekly increases in VIX and TED come along with carry trade losses (Although the results with TED have less statistical power). Hence, we will use these measures to identify

\textsuperscript{10}http://www.cboe.com/micro/VIX/vixintro.aspx
periods of time when the carry trade will probably not work as expected.

1.5.1. Hidden Markov Model (HMM)

One known way to classify a strategy according to its performance and also to predict its future behavior is to use a HMM. When we apply a HMM to a strategy we assume its returns are induced by a Markov chain, whose states (regimes) are not observable. We assume that returns have a specific distribution depending on the regime. Thus, the idea is to identify those regimes with the observable data, like strategy returns. But in fact one can include any information that can help in the identification.

Thus we calibrate the Markov chain, i.e. we find the best fit for the following parameters:

1. Transition probability matrix of the chain
2. The initial probabilities of the chain
3. The distribution of the returns under each regime

After this step, we are able to know the next period regime and therefore know the strategy performance with more accuracy. One known way to calibrate a HMM is to use the Baum-Welch Algorithm (1970), which is a particular case of the Expectation Maximization Algorithm. For details and explanations see Prajogo (2011).

What’s next is to build up a new strategy, based on the previous $RCT_p$ strategies. The steps are the following:

1. HMM Fitting: We first calibrate HMM using previous returns of $RCT_{\infty}(V)$ (for a specific value of V) and the VIX index.
2. Design Dynamic Carry Trade Strategy: With the fitted HMM, we then choose what strategy or strategies to use, among $RCT_1(V)$ and $RCT_{\infty}(V)$ and the benchmark portfolio. Notice than don’t do the carry trade is also a possibility. The decision depends in the performance of each strategy in the regime with more probabilities to be in the next period.
1.5.2. HMM Fitting

We assume parameters of the HMM can change, since new information is added as we move over time. Hence we recalibrate the HMM every 2 years (500 days), using all the returns available at that period. To fit the HMM, we use the R package, called RHmm\textsuperscript{11}.

First we have to define the number of states. For this purpose we use the Bayesian Information Criterion (BIC). Table 1.7 shows the BIC each period the HMM is fitted, using 2, 3 and 4 states and for a particular volume. The BIC values are slightly in favor of using three regimes. Results for other volumes also show that BIC is generally lower with 3 regimes, and they are shown in the appendix.

Table 1.7: Bayesian Information Criteria values for HMM with 2,3 and 4 regimes,

<table>
<thead>
<tr>
<th>log(V)=-82</th>
<th>Number of States</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date</td>
<td>2</td>
</tr>
<tr>
<td>Jun-00</td>
<td>-16077</td>
</tr>
<tr>
<td>Jun-02</td>
<td>-21440</td>
</tr>
<tr>
<td>Jun-04</td>
<td>-26975</td>
</tr>
<tr>
<td>Jun-06</td>
<td>-32503</td>
</tr>
<tr>
<td>Jun-08</td>
<td>-37807</td>
</tr>
<tr>
<td>Jun-10</td>
<td>-42744</td>
</tr>
<tr>
<td>Jun-12</td>
<td>-47918</td>
</tr>
</tbody>
</table>

Table 1.8 shows the parameters of the last HMM fit (the one using all dates until Jun-2012), using 3 regimes, for the same volume of Table 1.7. \textsuperscript{12}

---

\textsuperscript{11} http://cran.r-project.org/web/packages/RHmm/index.html
\textsuperscript{12} The 2-regime HMM fitting for previous periods are shown in appendix
Table 1.8: 3-Regime Hidden Markov Model fitting for RCT, and VIX returns.

<table>
<thead>
<tr>
<th>T. Matrix</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>Cov R1</th>
<th>RCT,</th>
<th>R_VIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>68%</td>
<td>32%</td>
<td>0%</td>
<td>RCT,</td>
<td>0.01%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>R2</td>
<td>7%</td>
<td>86%</td>
<td>7%</td>
<td>R_VIX</td>
<td>-0.01%</td>
<td>1.37%</td>
</tr>
<tr>
<td>R3</td>
<td>0%</td>
<td>10%</td>
<td>90%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mean</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>Cov R2</th>
<th>RCT,</th>
<th>R_VIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCT,</td>
<td>-0.14%</td>
<td>0.03%</td>
<td>0.06%</td>
<td>RCT,</td>
<td>0.003%</td>
<td>-0.001%</td>
</tr>
<tr>
<td>R_VIX</td>
<td>3.26%</td>
<td>-0.41%</td>
<td>-0.05%</td>
<td>R_VIX</td>
<td>-0.001%</td>
<td>0.221%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cov R3</th>
<th>RCT,</th>
<th>R_VIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCT,</td>
<td>0.001%</td>
<td>-0.002%</td>
</tr>
<tr>
<td>R_VIX</td>
<td>0.229%</td>
<td></td>
</tr>
</tbody>
</table>

**Remark:** T. Matrix shows the transition matrix for regimes one to three (R1, R2 and R3). Mean show the expected return of RCT, (log(V)=exp(-82)) and VIX for each regime. Cov R1, Cov R2 and Cov R3 are the covariance matrices for each regime.

Clearly the first regime is the one which VIX has more volatility and coincides with poor performance in the carry trade. In the second regime the VIX is more stable and carry trade have slight positive returns. The third regime has the smallest VIX returns and the highest carry trade returns in average. Looking at the transition matrix, the market is regularly in the second or third regime and there is no change of going from regime 1 to 3 or vice versa. Hence, we will label regime one as “Unstable”, regime two as “Transition” and regime three as “Stable”. This regime labeling can be also done for the HMM fitting under other volumes. The characteristics are similar to the HMM fitting shown above, thus we show the rest of the results in the Appendix. Figure 1.9 shows the most probable sequence of states after applying Viterbi algorithm to all returns from Feb-94 to Sep-2012.
1.5.3. Results

Given previous analysis, the following strategy will be made:

- If we are in the Stable Regime, we will use $RCT_{\infty}(V)$ weights since this is our most aggressive strategy available.
- If we are in the Transition Regime, we do a 50/50 mix between $RCT_{\infty}(V)$ and the benchmark portfolio (when $K = 3$). Since mean return in this regime is slightly positive, we want to build a less aggressive portfolio. We will see the effects of changing the mix later.
- If we are in Unstable Regime, we do $-RCT_1(V)$, with $-RCT_p(V)$ as the problem in which the optimal weights are exactly the same weights of $RCT_p(V)$ with the opposite sign. Thus it can be thought as a reverse carry trade strategy. We will also analyze the case of doing $-RCT_{\infty}(V)$ or just don’t apply any strategy later.
To implement the strategy we left the first third of the data to train the first period. The data is then updated as we go over time. Figure 1.9 shows the results of this dynamic strategy compared to $RCT_\infty(V)$ strategy for the test period Jun-2000 to Sep-2012.

Table 1.9: Performance of dynamic versus $RCT_\infty$ between Jun-2000 to Sep-2012.

<table>
<thead>
<tr>
<th>log(V)</th>
<th>Norm</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>-∞</td>
<td>Dynamic</td>
<td>8.8%</td>
<td>15.9%</td>
<td>43.4%</td>
<td>17.0%</td>
<td>51.7%</td>
<td>-0.37</td>
<td>12.50</td>
</tr>
<tr>
<td>-94</td>
<td>Dynamic</td>
<td>9.4%</td>
<td>14.4%</td>
<td>43%</td>
<td>13%</td>
<td>70%</td>
<td>-0.76</td>
<td>13.07</td>
</tr>
<tr>
<td>-88</td>
<td>Dynamic</td>
<td>7.9%</td>
<td>12.4%</td>
<td>44%</td>
<td>16%</td>
<td>50%</td>
<td>-1.17</td>
<td>13.22</td>
</tr>
<tr>
<td>-84</td>
<td>Dynamic</td>
<td>5.7%</td>
<td>10.0%</td>
<td>43%</td>
<td>20%</td>
<td>29%</td>
<td>-0.79</td>
<td>13.94</td>
</tr>
<tr>
<td>-82</td>
<td>Dynamic</td>
<td>5.5%</td>
<td>8.2%</td>
<td>37%</td>
<td>16%</td>
<td>35%</td>
<td>-0.04</td>
<td>10.05</td>
</tr>
<tr>
<td>-75</td>
<td>Dynamic</td>
<td>4.4%</td>
<td>6.4%</td>
<td>29%</td>
<td>13%</td>
<td>35%</td>
<td>-0.39</td>
<td>5.62</td>
</tr>
<tr>
<td>-73</td>
<td>Dynamic</td>
<td>3.8%</td>
<td>3.6%</td>
<td>6%</td>
<td>2%</td>
<td>186%</td>
<td>0.11</td>
<td>13.71</td>
</tr>
</tbody>
</table>

| Benchmark | K=3 | 5.5% | 11.3% | 33% | 10% | 55% | -0.55 | 10.02 |

For most volumes, we observe a clear improvement with respect to $RCT_\infty(V)$ and the benchmark portfolio returns. We have increased mean returns and decrease all risk measures. The skewness is bigger (positive for $\log(V) = -94$ and $(V) = -88$) and the kurtosis is dramatically reduced when $\log(V) \geq -84$. The UPI index always increases in a 5%-230% range.

Figure 1.10 and Figure 1.11 show the profit path of $RCT_\infty(V)$, the dynamic strategy and the benchmark portfolio, for the entire testing period and for some values of V. Notice that the dynamic strategy doesn’t dominate $RCT_\infty(V)$ for every time window. In fact, before the 2008 crisis, $RCT_\infty(V)$ is more profitable in average that the dynamic strategy. However the dynamic strategy outperform during and after this crash. The prediction accuracy of the signal, in the Stable and Unstable states, is shown in Table 1.10. Almost for every V, the prediction is in the range 55-63% which is quite good.
Figure 1.10: Profit of benchmark carry trade portfolio, RCT, and dynamic RCT. Left side is for log(V)=−75 and right side for log(V)=−82.

Figure 1.11: Profit of benchmark carry trade portfolio, RCT, and dynamic RCT. Left side is for log(V)=−88 and right side for log(V)=−94.
Table 1.10: Prediction Accuracy of the HMM signal.

<table>
<thead>
<tr>
<th>log V</th>
<th>Table</th>
<th>Prediction Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>-94</td>
<td>698</td>
<td>220</td>
</tr>
<tr>
<td></td>
<td>516</td>
<td>195</td>
</tr>
<tr>
<td>-88</td>
<td>883</td>
<td>149</td>
</tr>
<tr>
<td></td>
<td>620</td>
<td>146</td>
</tr>
<tr>
<td>-84</td>
<td>1103</td>
<td>172</td>
</tr>
<tr>
<td></td>
<td>700</td>
<td>172</td>
</tr>
<tr>
<td>-82</td>
<td>865</td>
<td>186</td>
</tr>
<tr>
<td></td>
<td>639</td>
<td>184</td>
</tr>
<tr>
<td>-75</td>
<td>747</td>
<td>164</td>
</tr>
<tr>
<td></td>
<td>460</td>
<td>313</td>
</tr>
<tr>
<td>-73</td>
<td>451</td>
<td>967</td>
</tr>
<tr>
<td></td>
<td>341</td>
<td>1053</td>
</tr>
</tbody>
</table>

Table 1.1 shows the number of periods the signal says Stable (Unstable) and the return of the strategy is positive. Table 2.1 (Table 1.2) shows the number of periods the signal says Stable (Unstable) and the return of the strategy is negative. The prediction accuracy is the ratio between Table 1.1 + Table 2.2 and the total of periods in the table.

Is important to mention that the design of the dynamic strategy can be changed and the results will still outperform $RCT_p(V)$ strategies. For example, we can change the 50%-50% mix when the signal says “Transition”. If we increase (decrease) the weight of $RCT_\infty(V)$ in that regime, thedynamic strategy will tend to have more (less) return and risk. But it will still dominate $RCT_\infty(V)$, as shown in the appendix. The same happens if we change the dynamic strategy when the signal says “Unstable”. We can do $-RCT_\infty(V)$ or just do nothing. In the first case, the dynamic strategy performance almost doesn’t change. In the second case, the dynamic strategy tends to decrease the return and volatility slightly. However it has more drawdown and Ulcer. Therefore applying the reverse carry trade allows faster recover when drawdown occurs.
1.6. Conclusions and Future Directions

The objectives proposed in the present chapter have been satisfied. On the first two sections we successfully parameterized the robust counterparts (RC) of a general portfolio problem by the volume of their respective uncertainty set. This parameterization allowed us to compare the performance of RC with uncertainty sets that can differ in shapes, such as an ellipsoid, a box and a polyhedron. We show that the RC induced by the ellipsoidal uncertainty set is equivalent to the Markowitz portfolio problem in the sense that they generate the same set of optimal solutions. Thus we gave the option to measure the performance of different RC with respect to this benchmark problem.

By an example and simulated data, we show that solutions of these RC can differ in sparsity, which is directly related to the level of diversification and hence the return/risk performance. Specifically we found out that the RC induced by the polyhedral uncertainty set has the smaller sparsity, while the RC induced by the box uncertainty tends to have more corner solutions. These empirical findings opens the possibility of choosing one particular RC depending on the level of risk the investor wants to take, or on the level of uncertainty that the investor has on future returns.

On the following section we apply the previous RC for the exchange carry trade strategy (CT). As we showed through benchmark CT portfolios, this strategy is exposed to currency risk and can perform poorly with big exchange swings. We also discuss that short term currency movement is difficult to model. Therefore we found suitable to solve the carry trade portfolio problem with the robust methodology.

The performance obtained with real data also shows the same sparsity pattern found with simulated data. So as expected the strategy RCT₁(V) tend to have less mean return and risk than RCT₂(V) or RCT∞(V). Comparing the latter strategies it was clear that RCT∞(V) have the best compromise between return and risk. However this strategy still didn’t outperform one of the benchmark carry trade strategy. There were still large drawdowns, some of them with slow recovery. The strategy only offered robustness when we take a large uncertainty box set, which made the strategy too conservative.
In the last section, we show a way to overcome the problem of using only one uncertainty set for all the periods. The methodology consisted in building a portfolio based in the weights of one or a mix of the \( RCT_p(V) \) and benchmark strategies. The construction of that portfolio depended on the signal we received about the market condition for the carry trade strategy on the next period. The signal was generated by an HMM model on the \( RCT_\infty(V) \) and VIX returns. The hidden states could be naturally identified in the fitting and had similar characteristic if we change the volume. Basically the three states classified \( RCT_\infty(V) \) returns as bad, zero or good respectively. The signal prediction accuracy was always above 55% across all volume tested. So with this new signal information, we were able to design the dynamic strategy. The rule for building this strategy was the following: Be aggressive, i.e. take \( RCT_\infty(V) \), when the signal is in favor of CT, diversify your portfolio, by adding the benchmark portfolio, when the signal indicates transition and do the anti carry strategy when signal is against carry trade strategy.

The dynamic strategy considerably improved previous performance. Depending on the volume selected, mean annualized return increased between 2% to 4%, as compared to the best fix strategy. Risk measure decreased at the same time. For example, ulcer decreased in a range of 5%-10%. The dynamic strategy also outperformed the equal weighted benchmark carry trade portfolio, when the volume is below a certain threshold. For The performance difference was mainly produced because of 2008 crisis. The HMM signal helped to create a dynamic strategy that reduced carry trade losses in that period.

In terms of the \( F_p(V) \) problem, one possible extension of this work is to test \( F_p(V) \) on other problems, rather than to apply it for the carry trade problem. It will be interesting to see what uncertainty set offers the best performance when we mix different asset classes in a portfolio. We can also apply HMM to improve results.

In terms of the carry trade strategy, one possible extension is to increase the number of currencies in the portfolio. Like seen in Kim (2013), it will be interesting to include currencies from emerging countries and see how much benefit this brings. The problem to include more countries comes with data availability. Carry trade is based in ranking currency yields. In our case, we use the LIBOR rate, which is not necessary available for every country. One way to overcome this problem is by using forward rates and the covered interest rate parity.
Using the weights obtained by the RCT, the next step might be to address the issue of which asset do we invest in the selected currencies. In the present work, we tested the performance of the weighting process by assuming we get the LIBOR rate. However, we can test the performance with future contracts, just like Deutsche Bank G10 Currency index do. We can also try debt instruments, such as bonds or t-bills.
Chapter 2

Grouping Uncertainty Method to Get Less Conservative Solution in Robust Counterparts

2.1. Introduction

One of the drawbacks attributed to robust optimization definitions, like the one defined in Ben-Tal and Nemirovski (1998), is that solutions can be too conservative. The reason is given by construction: such formulations give the “best of the worst case” optimal solution. It might be appropriate to use these techniques when is difficult to have some information about the uncertainty. As an example we have the carry trade problem seen in chapter one, where we discuss the difficulty to predict future exchange rates. However when there is valid information about the distribution (probability) of the uncertainty, then it might be better to use solutions derived by other optimization tools, like Stochastic Programming. The problem with the latter is the tractability that arises when exponential scenarios are generated, for example in multistage portfolio problems.

There is previous research that attempts to find less conservative solutions from robust optimization framework. Mulvey et al. (1995) use the fact that uncertain data can take values on a finite number of scenarios, with known probability. Given this, their model can efficiently penalize infeasibilities in the constraints. So solutions that are profitable for the majority of scenarios and bad in few scenarios can be better than conservative solutions. Ceria and Stubbs (2006) introduce the concept of Zero Net Alpha-Adjustment into the robust counterpart of the Markowitz portfolio with expected mean uncertainty. When there is symmetry in the expected return, they add this information by adding a constraint, when solving the inner problem (the problem that finds the worst case expected return over the uncertainty set). The constraint is
basically imposing that the net difference between positive and negative deviations from the mean estimate should be zero. Geometrically this constraint reduces the uncertainty set, taking out all the cases where total deviations from the mean estimate are bias to be negative or positive. Hence the worst case value under this new set is more “optimistic” than the original one, making the optimal solution less conservative.

In the previous chapter we have seen how changing the size and shape of the uncertainty set, can make robust counterpart solution less conservative. In the following chapter, we want to get less conservative by taking advantage of the correlation of the uncertain data. To do this, we will change the formulation of the original problem \((P)\) into an equivalent problem. The transformations applied to \((P)\) must align with the following concept: Group uncertain and correlated parameters of different constraints into the same constraints. The reason to do this comes from this simple fact: The worst case scenario of two or more correlated parameters is better than the worst case scenario of each individual parameter. Mathematically, suppose \(a_1\) and \(a_2\) are two uncertain and correlated parameters. Then, for any function \(f_1(x, a_1)\) and \(f_2(x, a_1)\):

\[
\inf_{a_1,a_2} \{f_1(x, a_1) + f_2(x, a_2)\} \geq \inf_{a_1} f_1(x, a_1) + \inf_{a_2} f_2(x, a_2) \quad (2.1)
\]

So the strategy is to find equivalent problems for \((P)\), such that we have as many correlated variables in the same constraint as we can. The transformations to do such things can even reduce the number of constraints with at least one uncertain value. We expect to get rid of worst cases, which barely happen because of the correlation of the uncertain data.

The latter concept has been implicitly used by Ben-Tal et al. (2000) for a multi-stage portfolio problem. In this work, they model a problem that originally has the returns of each asset and each time period in a different constraint. By a change of variables (transformation) they derived a new formulation, which grouped the return of all the assets and previous periods in one constraint (the cash constraint). Hence, they derived the Robust Counterpart of that formulation, using ellipsoidal uncertainty sets, and compare its performance with the Stochastic Programming version of the same problem. The input used assumes lognormal distributed returns, which were driven by some created factors. The returns between each asset were correlated in a same time period, but independent across different time periods.
Bertsimas and Pachamanova (2008), based on Ben-Tal et al. (2000) approach, construct a robust multistage problem with polyhedral uncertainty sets. The benefit of using such type of uncertainty sets is that the RCs turn out to be linear problems (instead of the SOCP derived with ellipsoids). Hence it becomes more suitable to use this model when adding constraints with integer variables, since there are more methods for integer programming with a linear framework than with nonlinear.

The main objective of the present chapter is to show how to apply GUM to a problem and show how you can increase the optimal value of its Robust Counterpart with it. Hence the chapter is organized as follows:

In section two we start explaining GUM with a simple example. Then we define a GUM transformation and apply it to several types of constraints. Using these constraints, we build up a toy example. We derive the RC of the original problem and the RC of the GUM-transformed problem, assuming ellipsoidal uncertainty sets on the uncertain parameters. Finally, we show the performance difference between same instances of both problems.

In section three we introduce the multistage portfolio problem (MPP) built by Ben-Tal et al. (2000). We derive the RC of the MPP and the RC of the problem obtained after applying GUM transformations.

Section four describes the data to solve the MPP: The first one is a simulated market, based on a factor model for returns used by Ben-Tal et al. (2000). The second market is based on real ETF returns. Section five measures and analyze the performance of the final portfolio value derived by both RC. Finally in section six, we present conclusions and possible extensions of this chapter.
2.2. GUM Examples and Transformations

We will start explaining this idea of GUM with a simple example. Suppose we have the following problem \((E)\):

\[
\begin{align*}
\text{maximize} \quad & \{ y_1 + y_2 : \quad y_1 \leq a_1 x_1, \quad y_2 \leq a_2 x_2 \} \\
\text{subject to} \quad & (x_1, x_2) \in \mathcal{P}, \\
& (y_1, y_2) \in \mathbb{R}^2
\end{align*}
\]

Where \(\mathcal{P}\) is a bounded set in \(\mathbb{R}_+^2\), \(a_1\) and \(a_2\) are non-zero parameters. Now let’s write \((E)\) in an equivalent but different way. The idea GUM to group \(a_1\) and \(a_2\) into the same constraint or objective. It is obvious to see that \((E)\) can be written as \((\tilde{E})\):

\[
\begin{align*}
\text{maximize} \quad & a_1 x_1 + a_2 x_2 \\
\text{subject to} \quad & (x_1, x_2) \in \mathcal{P}, \\
& (y_1, y_2) \in \mathbb{R}^2
\end{align*}
\]

Notice that \((\tilde{E})\) has only one equation with the parameters, while \((E)\) has 2 constraints. Now consider \(a_1\) and \(a_2\) moves in the uncertainty set \(\mathcal{U} \subseteq \mathbb{R}_+^2\). Then the RC \((E^*)\) for formulation \((E)\) is:

\[
\begin{align*}
\text{maximize} \quad & \left\{ y_1 + y_2 : \quad y_1 \leq \inf_{(a_1, a_2) \in \mathcal{U}} \{ a_1 x_1 \}, \quad y_2 \leq \inf_{(a_1, a_2) \in \mathcal{U}} \{ a_2 x_2 \} \right\} \\
\text{subject to} \quad & (x_1, x_2) \in \mathcal{P}, \\
& (y_1, y_2) \in \mathbb{R}^2
\end{align*}
\]

Which is equivalent to:

\[
\begin{align*}
\text{maximize} \quad & \left\{ x_1 \inf_{a_1 \in \mathcal{U}} a_1 + x_2 \inf_{a_2 \in \mathcal{U}} a_2 \right\} \\
\text{subject to} \quad & (x_1, x_2) \in \mathcal{P}, \\
& (y_1, y_2) \in \mathbb{R}^2
\end{align*}
\]

The robust counterpart \((E^*)\) for formulation \((\tilde{E})\) is equivalent to

\[
\begin{align*}
\text{maximize} \quad & \left\{ \inf_{(a_1, a_2) \in \mathcal{U}} \{ a_1 x_1 + a_2 x_2 \} \right\} \\
\text{subject to} \quad & (x_1, x_2) \in \mathcal{P}, \\
& (y_1, y_2) \in \mathbb{R}^2
\end{align*}
\]

By (2.1)

\[
\inf_{a_1, a_2 \in \mathcal{U}} \{ a_1 x_1 + a_2 x_2 \} \geq \inf_{a_1 \in \mathcal{U}} a_1 x_1 + \inf_{a_2 \in \mathcal{U}} a_2 x_2
\]

The equality holds when parameters are independent. Therefore the optimal value of \((E^*)\) is greater or equal to the optimal value of \((\tilde{E})\).
Let’s see this inequality explicitly when $\mathcal{U}$ is an ellipsoid defined by:

$$
\mathcal{U}(\hat{a}, \Sigma, \delta) = \left\{ \hat{a} + \Sigma^{1/2}v, \|v\|_2 \leq \theta \right\} \quad (2.2)
$$

where $\Sigma^{1/2}$ is a positive definite matrix. Now:

$$
\inf_{a_k \in \mathcal{U}} a_k = \hat{a}_k - \theta \sqrt{\left(\Sigma^{1/2}\right)^2_{k1} + \left(\Sigma^{1/2}\right)^2_{k2}} = \hat{a}_k - \theta \sqrt{\Sigma_{kk}} \quad (2.3)
$$

$\Sigma$ is such that $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. Therefore the optimal value of $(E^*)$ is given by:

$$
\max_{(x_1,x_2) \in \mathcal{P}} \{\hat{a}_1 x_1 + \hat{a}_2 x_2 - \theta(x_1 \sqrt{\Sigma_{11}} + x_2 \sqrt{\Sigma_{22}})\}
$$

We know from chapter one that:

$$
\inf_{a \in \mathcal{U}} x^T a = x^T \hat{a} - \theta \left\|\Sigma^{1/2}x\right\|_2 \quad (2.4)
$$

Then $(E^*)$ is:

$$
\max_{x \in \mathcal{P}} \hat{a}_1 x_1 + \hat{a}_2 x_2 - \theta \left\|\Sigma^{1/2}x\right\|_2
$$

Is easy to see that $\left\|\Sigma^{1/2}x\right\|_2 \leq (x_1 \sqrt{\Sigma_{11}} + x_2 \sqrt{\Sigma_{22}})$. In fact:

$$
\left\|\Sigma^{1/2}x\right\|_2^2 = \Sigma_{11} x_1^2 + \Sigma_{22} x_2^2 + 2 \Sigma_{12} x_1 x_2 \leq \Sigma_{11} x_1^2 + \Sigma_{22} x_2^2 + 2 \sqrt{\Sigma_{11} \Sigma_{22}} x_1 x_2
$$

$$
= (x_1 \sqrt{\Sigma_{11}} + x_2 \sqrt{\Sigma_{22}})^2
$$

The inequality comes from the fact that $\Sigma$ is definite positive.
The transformation from \((E^*)\) to \((E^*)\) in the previous example didn’t need any change of variable. However in more complex problems, this might be required in order to move uncertain parameters from one constraint to another. In general if \(P\) has the following structure:

\[
\min_{x^k \in \mathbb{R}^N} \phi_o(x^1, x^2, \ldots, x^K) \\
\phi_k(a^k, x^k) \leq b \quad \forall k \in \{1, \ldots, K\}
\]

\(\{\phi_k\}_{k=1}^K\) are convex functions and at least one entry of each \(a^k\) is uncertain. \(b\) is certain

**Definition 2.1**

A GUM transformation on a variable \(x \in \mathbb{R}^N\) and parameter \(a \in \mathbb{R}^M\) is a function \(T : \mathbb{R}^M \times \mathbb{R}^L \rightarrow \mathbb{R}^N\) such that:

- \(T(a, \bar{x}) = x\) for some \(\bar{x} \in \mathbb{R}^L\)
- \(\phi(a, T(a, \bar{x}))\) is not dependent of \(a\)

Then all the uncertain parameters are moved to the objective function. The problem \(\bar{P}\) is then

\[
\min_{\bar{x}^k \in \mathbb{R}^N} \phi_o(T(a^1, \bar{x}^1), T(a^2, \bar{x}^2), \ldots, T(a^K, \bar{x}^K)) \\
\phi_k(a, T(a, \bar{x})) \leq b \quad \forall k \in \{1, \ldots, K\}
\]

Let’s see the cases of linear and quadratic constraints.

1. Bounded: \(\phi(a, x) = Dx + a\)

Here \(D\) is a diagonal certain matrix and suppose all entries of \(a\) are uncertain. These constraints appear, for example, when the variables are bounded above and/or below buy an uncertain box.

The transformation in this case is straightforward: \(\bar{x} = Dx + a\). Hence

\[
T(a, \bar{x}) = D^{-1}(\bar{x} - a) \quad (2.5)
\]

Which makes \(\phi_k(a, T(a, \bar{x})) \leq b\) into \(\bar{x} \leq b\)
2. Linear: \( \phi(a, x) = a^T x \)

Assume \( \{a_i\}_{i=1}^{r} \) are non-zero and positive random parameters and assume \( \{a_i\}_{i=r+1}^{N} \) are certain. Transformations are not unique. Which one is handier will depend on how the transformation affects the rest of the problem and the type of uncertainty set used. The most intuitive change of variables is to define:

\[
\tilde{x}_i = \begin{cases} 
  a_i x_i & 1 \leq i \leq r \\
  x_i & r + 1 \leq i \leq N
\end{cases}
\]

Then:

\[
T(a, \tilde{x}) = D^{-1}(a)\tilde{x} \quad (2.6)
\]

With \( D(a) \equiv \begin{bmatrix} \text{diag}(\{a_i\}_{i=1}^{r}) & 0 \\
0 & I_{N-r} \end{bmatrix} \)

Then \( \phi(a, T(a, \tilde{x})) \leq b \) is equivalent to:

\[
a^T T(a, \tilde{x}) = \sum_{i=1}^{r} a_i \frac{x_i}{a_i} + \sum_{i=r+1}^{N} a_i \tilde{x}_i = \sum_{i=1}^{r} \tilde{x}_i + \sum_{i=r+1}^{N} a_i \tilde{x}_i \leq b
\]

However we can do other sorts of transformations. For example if \( b = 0 \), we can do the following:

\[
\tilde{x}_i = \begin{cases} 
  \frac{x_i}{\prod_{j=1, j \neq i}^{r} a_j} & 1 \leq i \leq r \\
  \frac{x_i}{\prod_{j=1}^{r} a_j} & r + 1 \leq i \leq N
\end{cases}
\]

\[
\phi(a, T(a, \tilde{x})) = \left( \prod_{j=1}^{r} a_j \right) a^T D^{-1}(a)\tilde{x} = \left( \prod_{j=1}^{r} a_j \right) \left( \sum_{i=1}^{r} \tilde{x}_i + \sum_{i=r+1}^{N} a_i \tilde{x}_i \right)
\]

Then \( \phi(a, T(a, \tilde{x})) \leq 0 \) is equivalent to:
\[ \sum_{i=1}^{r} \tilde{x}_i + \sum_{i=r+1}^{N} a_i \tilde{x}_i \leq 0 \]

3. Quadratic: \( \phi(a, x) = x^T \text{diag}(a)x \)

For simplicity, we assume all entries for \( a \) are uncertain and positive to make \( \phi(a, x) \) convex.

Then possible transformations are:

\[ T(a, \tilde{x}) = \sqrt{\text{diag}^{-1}(a)}\tilde{x} \quad (2.7a) \]
\[ T(a, \tilde{x}) = \text{diag}^{-1/2}(a)\tilde{x} \quad (2.7b) \]

Here \( \sqrt{\text{diag}^{-1}(a)} \tilde{x} \) means \( T(a, \tilde{x})_k = \frac{\tilde{x}_k}{\sqrt{a_k}} \)

Hence with (2.7a) we get:

\[ \phi(a, T(a, \tilde{x})) = \sum_{i=1}^{N} a_i T(a, \tilde{x})_i^2 = \sum_{i=1}^{N} a_i \left( \frac{\tilde{x}_i}{\sqrt{a_i}} \right)^2 = \sum_{i=1}^{N} \tilde{x}_i \]

Then \( \phi(a, T(a, \tilde{x})) \leq b \) is equivalent to \( \sum_{i=1}^{N} \tilde{x}_i \leq b \)

With (2.7b) you get:

\[ \phi(a, T(a, \tilde{x})) = \left( \text{diag}^{-1/2}(a)\tilde{x} \right)^T \text{diag}(a)\text{diag}^{-1/2}(a)\tilde{x} = \tilde{x}^T \tilde{x} \]

Then \( \phi(a, T(a, \tilde{x})) \leq 0 \) is equivalent to \( \tilde{x}^T \tilde{x} \leq b \)
2.2.1. GUM Transformation Test

Now we will test the difference of applying the robust counterpart of a toy problem $P$ and $\bar{P}$.
We will use all the types of constraints described above. Suppose $P$ is the following:

$$
\begin{align*}
\max_{x^1 \geq 0, x^2 \geq 0, x^3 \geq 0} c^1.x^1 + c^2.x^2 - (x^3)^T diag(c^3)x^3 \\
Dx^1 + a^1 & \leq b^1 \\
diag^{-1}(a^2)x^2 & \leq b^2 \\
diag^{-1}(a^3)diag(x^3)x^3 & \leq b^3
\end{align*}
$$

Each $x^k \in \mathbb{R}^N$. $D$ is a certain and diagonal. Each $b^k$ and $c^k \geq 0$ are certain and $\in \mathbb{R}^N$ while $a^k \in \mathbb{R}^N$ are uncertain and positive (hence $b^k \geq 0$). The operation $a.b$ is the dot product of vectors $a$ and $b$. We then have $N$ constraints of each type (bounded, linear and quadratic).

Applying the GUM transformations given by (2.5) – (2.7), is easy to see that $\bar{P}$ is:

$$
\begin{align*}
\max_{\tilde{x}^1 \geq 0, \tilde{x}^2 \geq 0, \tilde{x}^3 \geq 0} \tilde{c}^1.(\tilde{x}^1 - a^1) + \tilde{c}^2.\tilde{x}^2 - \tilde{c}^3.\tilde{x}^3 \\
\tilde{x}^1 & \leq b^1 \\
\tilde{x}^2 & \leq b^2 \\
\tilde{x}^3 & \leq b^3
\end{align*}
$$

With $\tilde{c}^1 = D^{-1}c^1$, $\tilde{c}^2 = diag(a^2)c^2$ and $\tilde{c}^3 = diag(a^3)c$

To get the RC we assume the following ellipsoidal uncertainty set $\mathcal{U}^k$ for each vector $a^k \in \mathbb{R}^N$:

$$
\mathcal{U}^k = \left\{ \hat{a}^k + \left( V^k \right)^k w, \|w\|_2 \leq \theta \right\} \quad k:1,2,3
$$

with $\hat{a}^k$ and $V^k = \left( V^k \right)^k \left( V^k \right)^k$ the expectation and covariance matrix of $a$ respectively. We assume $a^k$ is independent of $a_j$ if $j \neq k$. 

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To get the RC $P^*$ of $P$, notice that just like (2.3):

\[
\inf_{a^k \in U^k} a^k_j = \hat{a}^k_j - \theta \sqrt{v^k_j} \equiv \hat{a}^k_j - \theta \bar{v}^k_j \quad \text{sup}_{a^k \in U^k} a^k_j = \hat{a}^k_j + \theta \sqrt{v^k_j} \equiv \hat{a}^k_j + \theta \bar{v}^k_j
\]

Then $P^*$:

\[
\begin{align*}
\max_{x^1 \geq 0, x^2 \geq 0, x^3 \geq 0} & \quad c^1. x^1 + c^2. x^2 - (x^3)^T diag(c^3) x^3 \\
& \quad D x^1 + \hat{a}^1 + \theta \bar{v}^1 \leq b^1 \\
& \quad \text{diag}^{-1}(\hat{a}^2 - \theta \bar{v}^2)x^2 \leq b^2 \\
& \quad \text{diag}^{-1}(\hat{a}^3 - \theta \bar{v}^3)\text{diag}(x^3)x^3 \leq b^3
\end{align*}
\]

To get robust counterpart $\bar{P}^*$ of $\bar{P}$ notice that:

\[
\inf_{a^1 \in U^1} \begin{bmatrix} \bar{c}^1. (\bar{x}^1 - a^1) \\ \bar{c}^2. \bar{x}^2 \\ \bar{c}^3. \bar{x}^3 \end{bmatrix} = \inf_{a^1 \in U^1} \begin{bmatrix} \bar{c}^1. (\bar{x}^1 - a^1) \\ \bar{c}^2. \bar{x}^2 \\ \sup_{a^3 \in U^3} \begin{bmatrix} \bar{c}^3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \bar{c}^1. (\bar{x}^1 - a^1) \\ \bar{c}^2. \bar{x}^2 \\ \sup_{a^3 \in U^3} \begin{bmatrix} \bar{c}^3 \end{bmatrix} \end{bmatrix}
\]

By (2.4):

\[
\begin{align*}
\inf_{a^1 \in U^1} \begin{bmatrix} \bar{c}^1. (\bar{x}^1 - a^1) \\ \bar{c}^2. \bar{x}^2 \end{bmatrix} &= \begin{bmatrix} \bar{c}^1. \bar{x}^1 - \sup_{a^1 \in U^1} \bar{c}^1. a^1 \end{bmatrix} = \begin{bmatrix} \bar{c}^1. (\bar{x}^1 - \hat{a}^1) - \theta \sqrt{(\bar{c}^1)^T \bar{V}^1 \bar{c}^1} \end{bmatrix} \\
\inf_{a^2 \in U^2} \begin{bmatrix} \bar{c}^2. \bar{x}^2 \end{bmatrix} &= \inf_{a^2 \in U^2} \sum_j \left[ (\bar{a}^2_j c^2_j) \bar{x}^2_j \right] = a^2. [\text{diag}(c^2) \bar{x}^2] \\
&= \bar{a}^2. [\text{diag}(c^2) \bar{x}^2] - \theta \sqrt{(\bar{x}^2)^T \text{diag}(c^2) \bar{V}^2 \text{diag}(c^2) \bar{x}^2} \\
&= \bar{c}^2. \bar{x}^2 - \theta \sqrt{(\bar{x}^2)^T \bar{V}^2 \bar{x}^2}
\end{align*}
\]

With $\bar{c}^2 = \text{diag}(c^2) \bar{a}^2$ and $\bar{V}^2 = \text{diag}(c^2) \bar{V}^2 \text{diag}(c^2)$.

We apply the last procedure for $\sup_{a^3 \in U^3} \bar{c}^3. \bar{x}^3$ too. Then $\bar{P}^*$ is the following SOCP:
\[-c^1, \hat{a}^1 - \theta \sqrt{(\hat{c}^1, \hat{b}^1 \hat{c}^1 + \max_{\hat{x}^1 \geq 0, \hat{x}^2 \geq 0, \hat{x}^3 \geq 0} \hat{c}^1, \hat{x}^1 + \hat{c}^2, \hat{x}^2 - \theta \sqrt{(\hat{x}^2, \hat{b}^2 \hat{x}^2 - \hat{c}^3, \hat{x}^3 - \theta \sqrt{(\hat{x}^3, \hat{b}^3 \hat{x}^3})}\]

\[\hat{x}^1 \leq b^1 \]
\[\hat{x}^2 \leq b^2 \]
\[\hat{x}^3 \leq b^3 \]
2.2.2. Results

An experiment will consist of the simulation of 100 different values for \( N, c^k, b^k, \hat{\alpha}^k, V^k, D \). \( N \) is an integer random number between 2 and 10. Then we generate \( c^k \) and \( D \) with random numbers between 0 and 1. For each \( k \), we generate a random 100 by \( N \) matrix with numbers between 0 and 1 and set \( \hat{\alpha}^k \) and \( V^k \) as the mean and covariance of this matrix. For \( k > 1 \), \( b^k \) is also generated with random number between 0 and 1. For \( k > 1 \) we have to make sure that \( b^1 \geq \hat{\alpha}^1 + \theta \hat{\theta}^1 \), since in that way we make sure \( P^* \) is feasible. With these settings, \( x^k = 0 \ \forall k \) is feasible in \( \tilde{P}^* \) and \( P^* \).

In each simulation we compute the optimal value of \( P^* \), which we call \( z(P^*) \), and the objective value of \( P^* \) with the optimal solution obtained in \( \tilde{P}^* \). The latter one will be called \( z(\tilde{P}^*) \). We keep track of the ratio \( \frac{z(P^*)}{z(\tilde{P}^*)} \). Table 2.1 shows the statistics of the above ratio using different values of \( \theta \). Since both problems are convex, we use CVX to solve them. Figure 2. 1 shows the frequency plot for some values of \( \theta \).

Table 2.1: Statistics of ratio between objective values of \( P^{**} \) and \( P^* \).

<table>
<thead>
<tr>
<th>Theta</th>
<th>1</th>
<th>0.85</th>
<th>0.75</th>
<th>0.5</th>
<th>0.25</th>
<th>0.15</th>
<th>0.01</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.36</td>
<td>2.13</td>
<td>1.99</td>
<td>1.64</td>
<td>1.32</td>
<td>1.20</td>
<td>1.03</td>
<td>1.02</td>
</tr>
<tr>
<td>Std</td>
<td>0.49</td>
<td>0.41</td>
<td>0.36</td>
<td>0.23</td>
<td>0.12</td>
<td>0.07</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>Left CI (95%)</td>
<td>2.26</td>
<td>2.05</td>
<td>1.92</td>
<td>1.60</td>
<td>1.30</td>
<td>1.18</td>
<td>1.02</td>
<td>1.01</td>
</tr>
<tr>
<td>Right CI (95%)</td>
<td>2.45</td>
<td>2.21</td>
<td>2.06</td>
<td>1.69</td>
<td>1.34</td>
<td>1.21</td>
<td>1.04</td>
<td>1.02</td>
</tr>
<tr>
<td>Skew</td>
<td>1.61</td>
<td>1.60</td>
<td>1.60</td>
<td>1.57</td>
<td>1.48</td>
<td>1.36</td>
<td>2.38</td>
<td>2.44</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3.63</td>
<td>3.46</td>
<td>3.35</td>
<td>3.08</td>
<td>2.60</td>
<td>2.04</td>
<td>6.01</td>
<td>6.35</td>
</tr>
<tr>
<td>% times ratio=1</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>17%</td>
<td>59%</td>
</tr>
</tbody>
</table>
By looking at the table and figures, we clearly see that when there is little uncertainty, the optimal values $z(P^*)$ and $z(\bar{P}^*)$ are almost the same. But as uncertainty grows the difference in the optimal values becomes larger on average. The $z$ value with $\bar{P}^*$ solution can be 2.36 time larger than with $P^*$ solution, as shown with $\theta = 1$. From $\theta \geq 0.15$ we have that $z(P^*) < z(\bar{P}^*)$ at every simulation. Moreover, the left side of the 95\% confidence interval (left CI 95\%) is greater than 1.

Now we will apply GUM in a well known problem in finance, which is the multistage portfolio allocation problem.
2.3. Multistage Portfolio Problem

As previously said, we will apply the transformation to a Multistage Portfolio Problem (MPP), based on the model developed in Ben-Tal et al. (2000). The MPP is following:

There is a pool of $L$ public traded assets (such as stocks or ETF) and we have to determine the positions of each asset from now to some horizon. Each asset will have an unknown return and a known transaction cost, in the case we want to buy or sell positions from the asset. We include cash as a possible asset (number $L + 1$). The goal is to increase the portfolio value after the horizon $H$.

Thus, the decision variables are:

$x^t_l$: Position (monetary value) for asset $l$ at time $t$

$sell^t_l$: Amount to sell of asset $l$ at time $t$. $l \leq L$

$buy^t_l$: Amount to buy of asset $l$ at time $t$. $l \leq L$

All the variables are positive. The input is:

$r^t_l$: return for asset $l$ at time $t$ (unknown at time $t$).

$u^t_l$: transaction cost percentage for selling asset $l$ at time $t$. $l \leq L$

$v^t_l$: transaction cost percentage for buying asset $l$ at time $t$. $l \leq L$

$x^0_l$: Initial position of asset $l$ the portfolio

The MPP is the following:

$$\text{maximize} \sum_{l \in \{1, \ldots, L\}} r^H_l x^H_l$$  \hspace{1cm} (2.8)

$$x^t_l = r^{t-1}_l x^{t-1}_l - sell^t_l + buy^t_l \hspace{1cm} \forall l \leq L, t \geq 1$$  \hspace{1cm} (2.9)

$$x^{t+1}_l = r^{t+1}_l x^{t+1}_l + \sum_{l \leq L} (1 - u^t_l) sell^t_l - \sum_{l \leq L} (1 + v^t_l) buy^t_l \hspace{1cm} \forall t \hspace{1cm} (2.10)$$
Equation (2.8) is the value of the portfolio after the horizon. Constraint (2.9) describes the dynamics of non-cash asset. The position hold at one period is the value obtained by investing in the previous period, minus the amount we sell and plus the amount to buy. Observe that \( r_t^f \geq 0 \) \( (r_t^f = 0 \) if the asset looses all its value). (2.10) describes the dynamics for cash. The cash at each time period is the cash from previous period, plus the total money collected by selling assets minus the total money spent in buying assets.

We assume that transaction costs and cash return are deterministic. However the returns \( r_{isl}^{t+1} \) are not known with certainty when the problem is solved. Therefore, one possibility of solving MPP is using the robust optimization approach.

### 2.3.1. GUM Transformation on MPP

It is not hard to see that MPP is an optimal control problem (Bellman, 1959). In fact, defining \( x^t = [x_1^t, x_2^t, \ldots, x_L^t] \), \( sell^t = [sell_1^t, sell_2^t, \ldots, sell_L^t] \) and \( buy^t = [buy_1^t, buy_2^t, \ldots, buy_L^t] \). Then MPP is

\[
\begin{align*}
\text{maximize} & \quad \phi(x^H, x_{L+1}^H) \\
\text{subject to} & \quad x^t = A^{t-1}x^{t-1} - sell^t + buy^t \quad \forall l \leq L, t \geq 1 \\
& \quad x_{L+1}^t = r_{L+1}^{t+1}x_{L+1}^{t+1} + (1 - u^t).sell^t - (1 + v^t).buy^t \quad \forall t
\end{align*}
\]

With \( A_{lm}^t = \begin{cases} r_t^l & \text{if } l = m \\ 0 & \text{if } l < m \end{cases} \), \( (1 - u^t) = [(1 - u_1^t), (1 - u_2^t), \ldots, (1 - u_L^t)] \),

\[
1 + v^t = [(1 + v_1^t), (1 + v_2^t), \ldots, (1 + v_L^t)] \quad \text{and} \quad \phi(x^H, x_{L+1}^H) = (r^H).x^H + r_{L+1}^{H}x_{L+1}^H
\]

Moreover, matrices \( A^{t+1} \) hold all the uncertain parameters and are diagonal. Denoting:

\[
R_t^l \equiv \prod_{h=0}^{t-1} A_{ll}^h = \prod_{h=0}^{t-1} r_t^h
\]
\[ D^t \equiv \mathrm{diag}(R^t_1, R^t_2, \ldots, R^t_L) \quad \text{for } t \geq 1 \text{ and } D^0 \equiv I \]

Then we will apply the following GUM transformations:

\[ x^t = T(\tilde{x}^t) = D^t \tilde{x}^t \quad \text{sell}^t = T(\tilde{y}^t) = D^t \tilde{sell}^t \quad \text{buy}^t = T(\tilde{z}^t) = D^t \tilde{buy}^t \]

\[ x^t_{L+1} = T(\tilde{x}^t_{L+1}) = R^t_{L+1} \tilde{x}^t_{L+1} \]

Hence we can write \( MPP \) problem into the following equivalent form \( \overline{MPP} \):

\[
\begin{align*}
\text{maximize} & \quad \beta^t \leq 0, \text{sell}^t \geq 0, \text{buy}^t \geq 0 \\
& \quad \left( r^H \right)^t D^H \tilde{x}^H + r^H_{L+1} R^H_{L+1} \tilde{x}^H_{L+1} \\
\end{align*}
\tag{2.11}
\]

\[
\begin{align*}
D^t \tilde{x}^t & = A^t D^{t-1} \tilde{x}^{t-1} - D^t \tilde{sell}^t + D^t \tilde{buy}^t \quad \forall t \geq 1 \\
R^t_{L+1} \tilde{x}^t_{L+1} & = r^t_{L+1} R^t_{L+1} \tilde{x}^t_{L+1} + (1 - u^t). (D^t \tilde{sell}^t) - (1 + v^t). (D^t \tilde{buy}^t) \quad \forall t \geq 1 \\
\end{align*}
\tag{2.12}
\tag{2.13}
\]

As \( A^{t-1} D^{t-1} = D^t \), then assuming \( D^t \) is invertible (which holds if \( r^h_t > 0 \) \( \forall h \leq t \)), we have that (2.12) is: \( \tilde{x}^t = \tilde{x}^{t-1} - \tilde{sell}^t + \tilde{buy}^t \)

(2.13) is: \( \tilde{x}^t_{L+1} = \tilde{x}^{t-1}_{L+1} + \frac{1}{r^t_{L+1}} ((1 - u^t) \cdot R^t) \cdot \tilde{sell}^t + \frac{1}{R^t_{L+1}} ((1 + v^t) \cdot R^t) \cdot \tilde{buy}^t \) with \( a \cdot b \) as the component wise multiplication between two vectors

(2.11) is: \( (r^H)^t D^H \tilde{x}^H + r^H_{L+1} R^H_{L+1} \tilde{x}^H_{L+1} = R^{H+1} \tilde{x}^H + R^{H+1}_{L+1} \tilde{x}^H_{L+1} \)

So \( \overline{MPP} \) is:

\[
\begin{align*}
\text{maximize} & \quad \beta^t \leq 0, \text{sell}^t \geq 0, \text{buy}^t \geq 0 \\
& \quad \left( R^{H+1}_{L+1} \right) \cdot \left( \tilde{x}^H_{L+1} \right) \\
\end{align*}
\]

\[
\begin{align*}
\tilde{x}^t & = \tilde{x}^{t-1} - \tilde{sell}^t + \tilde{buy}^t \quad \forall t \geq 1 \\
\tilde{x}^t_{L+1} & = \tilde{x}^{t-1}_{L+1} + \frac{1}{R^t_{L+1}} ((1 - u^t) \cdot R^t) \cdot \tilde{sell}^t + \frac{1}{R^t_{L+1}} ((1 + v^t) \cdot R^t) \cdot \tilde{buy}^t \quad \forall t \geq 1 \\
\end{align*}
\tag{2.12}
\tag{2.13}
\]

As we can see, \( \overline{MPP} \) has one \( H + 1 \) equations with uncertain parameters, while \( MPP \) has \( HL + 1 \).
2.3.2. Robust Counterpart for MPP

Now let’s write the $MPP^*$ and $\overline{MPP}^*$ using the following ellipsoidal uncertainty set $\mathcal{U}^{t\in\mathbb{Z}}$ for each $R^{t\in\mathbb{Z}}$

$$\mathcal{U}^t = \{\rho^t + \begin{pmatrix} V^t \end{pmatrix}^t w, \|w\|_2 \leq \theta^t\}$$

With $\rho^t$ and $V^t$ the expectation and covariance matrix of $R^t$ respectively. Since $R^t$ is a positive random variable we can impose a range for $\theta^t$:

$$R^t = \rho^t + \theta^t \begin{pmatrix} V^t \end{pmatrix}^t w \text{ with } \|w\|_2 \leq 1.$$ Following equation (1.3):

$$\inf R^t = \inf_{\|w\|_2 \leq \theta^t} \rho^t + \begin{pmatrix} V^t \end{pmatrix}^t w = \rho^t - \theta^t \begin{pmatrix} V^t \end{pmatrix}^t \|w\|_2 = \rho^t - \theta^t \sqrt{V^t}$$

Defining

$$\theta^t_{\text{max}} = \min_{i} \frac{\rho^t}{\sqrt{V^t_{ii}}} \quad (2.14)$$

we have that $R^t \geq 0$ with any $\theta^t \in [0, \theta^t_{\text{max}}]$. 

To formulate the RC, first we turn constraints with uncertain parameters into inequalities in the $\leq$ direction (the cash constraint for $\forall t > 1$). We can do that because although the feasible set changes, the optimal solution doesn’t. In fact, inequalities are always active in the optimal solution. To see the latter, suppose we have an inactive cash inequality. This means that the money we should have now (LHS) is not the cash we actually have (RHS). In other words we are throwing or giving money away. The same applies to stock positions.
The formulation of the robust counterpart $\overline{MPP}^*$ is:

$$
\begin{align*}
\text{maximize} & \quad x^t = x^{t-1} - \text{sell}^t + \text{buy}^t \quad \forall t \geq 1 \\
\inf_{R^H+1 \in U^H+1} \left\{ \left( \frac{R^H+1}{R^H_{L+1}} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) \right\} \\
x^t & \leq \inf_{R^t \in U^t} \left\{ x^t_{L+1} + \frac{1}{R^t_{L+1}} \left( (1 - u^t)^t \cdot R^t \right) \cdot \left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right) \right\} \quad \forall t > 1 \\
x^1_{L+1} & = x^0_{L+1} + \frac{1}{r^0_{L+1}} \left( (1 - u^1)^0 \cdot r^0 \right) \cdot \left( \begin{array}{c} \text{sell}^1 \\ \text{buy}^1 \end{array} \right) \quad t = 1
\end{align*}
$$

To solve the inner problem we use equation (1.3). According to it:

- \( \inf_{R^H+1 \in U^H+1} \left\{ \left( \frac{R^H+1}{R^H_{L+1}} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) \right\} = \left( \frac{\rho^H+1}{R^H_{L+1}} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) - \theta^{H+1} \sqrt{\gamma^{H+1} H_{X^H}} \)

- \( \inf_{R^t \in U^t} \left\{ x^t_{L+1} + \frac{1}{R^t_{L+1}} \left( (1 - u^t)^t \cdot R^t \right) \cdot \left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right) \right\} = x^t_{L+1} + \left( \begin{array}{c} (1 - u^t)^t \cdot \rho^t \\ -(1 + v^t)^t \cdot \rho^t \\ \text{sell}^t \\ \text{buy}^t \end{array} \right) - \frac{\theta^t}{R^t_{L+1}} \sqrt{\left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right) \cdot \left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right)} \)

With \( V^t = \left( \begin{array}{c} (1 - u^t)(1 - u^t)^t \cdot V^t \\ -(1 - u^t)(1 + v^t)^t \cdot V^t \\ (1 + v^t)(1 + v^t)^t \cdot V^t \end{array} \right) \). \( A \cdot B \) is the componentwise multiplication of matrix \( A \) and \( B \). Then, $\overline{MPP}^*$ is the following SOCP problem:

$$
\begin{align*}
\text{maximize} & \quad \left( \rho^H_{L+1} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) - \theta^{H+1} \sqrt{\gamma^{H+1} H_{X^H}} \\
\inf_{R^H+1 \in U^H+1} \left\{ \left( \frac{R^H+1}{R^H_{L+1}} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) \right\} = \left( \frac{\rho^H_{L+1}}{R^H_{L+1}} \right) \cdot \left( \begin{array}{c} x^H \\ x^H_{L+1} \end{array} \right) - \theta^{H+1} \sqrt{\gamma^{H+1} H_{X^H}} \\
x^t & \leq x^{t-1} + \frac{\theta^t}{R^t_{L+1}} \sqrt{\left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right) \cdot \left( \begin{array}{c} \text{sell}^t \\ \text{buy}^t \end{array} \right)} \\
x^t_{L+1} + \frac{1}{R^t_{L+1}} \cdot \left( \begin{array}{c} (1 - u^t)^t \cdot \rho^t \\ -(1 + v^t)^t \cdot \rho^t \\ \text{sell}^t \\ \text{buy}^t \end{array} \right) \right\} \leq x^t_{L+1} + \frac{1}{R^t_{L+1}} \cdot \left( \begin{array}{c} (1 - u^t)^t \cdot \rho^t \\ -(1 + v^t)^t \cdot \rho^t \\ \text{sell}^t \\ \text{buy}^t \end{array} \right) \right\} \quad \forall t > 1 \\
x^1_{L+1} = x^0_{L+1} + \frac{1}{r^0_{L+1}} \cdot \left( \begin{array}{c} (1 - u^1)^0 \cdot r^0 \\ -(1 + v^1)^0 \cdot r^0 \\ \text{sell}^1 \\ \text{buy}^1 \end{array} \right) \right\} \quad t = 1
\end{align*}
$$

Remark: We omit the ~ accent on the variables to simplify notation.
Is important to notice that \( \overrightarrow{MPP}^* \) is always feasible. In fact the “hold” solution, which is to keep the initial positions, is feasible. To make it clear, this solution is:

- \( sell_t^i = buy_t^i = 0 \) \( \forall l, t \)
- \( x_t^i = x_{t-1}^i \) \( \forall l, t \geq 1 \)

By having transaction costs, optimal solution satisfies the property:

\[
sell_t^i > 0 \implies buy_t^i = 0 \quad (2.17)
\]

Proof:

Suppose there is an optimal solution with \( sell_{t^*}^i > 0 \) and \( buy_{t^*}^i > 0 \), for some \( l^* \) and \( t^* \).

If \( sell_{t^*}^i > buy_{t^*}^i \), we define a new solution, which is the same as optimal solution, except with:

- \( \overline{sell}_{t^*}^i = sell_{t^*}^i - buy_{t^*}^i \)
- \( buy_{t^*}^i = 0 \)
- \( \overline{x}_{t+1}^i = \overline{x}_{t+1}^i + \frac{1}{R_{t+1}} \left[ (u_{t^*}^i + v_{t^*}^i) \rho_{t^*}^i \cdot buy_{t^*}^i \right] \quad \forall t \geq t^* \)

Notice \( \overline{x}_{t+1}^i > x_{t+1}^i \) \( \forall t \geq t^* \). The new solution satisfies (2.15). The new solution also satisfy (2.16a) for \( t > t^* \). We have to see if (2.16a) (or (2.16b) if \( t^* = 1 \)) holds for \( t = t^* \). In fact:

\[
\frac{1}{R_{t+1}} \left( (1 - u^i)^* \cdot \rho^i \right) \cdot \overline{sell}^i + \frac{1}{R_{t+1}} \left( -(1 + v^i)^* \cdot \rho^i \right) \cdot \overline{buy}^i
\]

\[
- \left( \frac{1}{R_{t+1}} \left( (1 - u^i)^* \cdot \rho^i \right) \cdot sell^i + \frac{1}{R_{t+1}} \left( -(1 + v^i)^* \cdot \rho^i \right) \cdot buy^i \right)
\]

\[
= \frac{1}{R_{t+1}} \left[ (1 - u_{t^*}^i) \rho_{t^*}^i \left( sell_{t^*}^i - sell_{t^*}^i \right) - (1 + v_{t^*}^i) \rho_{t^*}^i \left( buy_{t^*}^i - buy_{t^*}^i \right) \right]
\]

\[
= \frac{1}{R_{t+1}} \left[ (1 - u_{t^*}^i) \rho_{t^*}^i \left( sell_{t^*}^i - sell_{t^*}^i \right) - (1 + v_{t^*}^i) \rho_{t^*}^i \left( buy_{t^*}^i - buy_{t^*}^i \right) \right]
\]

Therefore the new solution satisfies (2.16a) for \( t = t^* \). As \( \overline{x}_{t+1}^i > x_{t+1}^i \), then the optimal value of the new solution is better than the optimal value of the optimal solution.

If \( sell_{t^*}^i < buy_{t^*}^i \):
We proceed exactly as before to construct the contradiction.

Remark: Since the variables are non-negative, (2.17) is equivalent too: \( \text{buy}_t > 0 \implies \text{sell}_t = 0 \)

### 2.3.3. Robust Counterpart for \( \overline{MPP} \)

To obtain the RC for \( \overline{MPP} \) notice that \( r_t^x = \frac{R_{t+1}^x}{R_t^x} \). Then

If \( t > 1 \):

\[
\inf (r_t^x x_t^x) = x_t^x \inf_{R_t^x \in \mathcal{U}^t} \left\{ \frac{R_{t+1}^x}{R_t^x} \right\} = x_t^x \inf_{\|w_x\|_2 \leq \theta^t} \left\{ \frac{\rho^{t+1}_x + \left( \frac{1}{V^1} \right)^{t+1} \cdot w_1}{\rho^t_x + \left( \frac{1}{V^1} \right)^t \cdot w_2} \right\}
\]

With \( \left( \frac{V^1}{V^2} \right)^{t+1} \) as the column \( x \) of \( \left( \frac{V^1}{V^2} \right)^{t+1} \) then

\[
= x_t^x \left\{ \inf_{\|w_x\|_2 \leq \theta^t} \rho^{t+1}_x + \left( \frac{1}{V^1} \right)^{t+1} \cdot w_1 \right\} = x_t^x \left\{ \frac{\rho^{t+1}_x - \theta^{t+1} \left\| \left( \frac{V^1}{V^2} \right)^{t+1} \right\|_2}{\rho^t_x + \theta^t \left\| \left( \frac{V^1}{V^2} \right)^t \right\|_2} \right\}
\]

\[
= x_t^x \left\{ \frac{\rho^{t+1}_x - \theta^{t+1} \sqrt{V^t}}{\rho^t_x + \theta^t \sqrt{V^t}} \right\}
\]

If \( t = 1 \):
\[
\inf \{ r_t x_t^t \} = x_t^t \left( \frac{\rho_l^{t+1} - \theta^{t+1} \sqrt{V_{ll}^{t+1}}}{r_t^0} \right)
\]

If \( H > 1 \):

\[
\inf \sum_{l \leq L} r_l^H x_l^H = \inf_{\|w_1\|_2 \leq 1} \sum_{l \leq L} \left( \frac{\rho_l^{H+1} + \theta^{H+1} \left( V_{l}^1 \right)_1^{H+1}}{\rho_l^H + \theta^H \left( V_{l}^1 \right)_1^H} . w_1 \right) x_l^H
\]

We first solve the optimal \( w_1 \) (in terms of \( w_2 \)). This is equivalent as solving:

\[
\inf \inf_{\|w_1\|_2 \leq 1} \hat{\rho}_1^{H+1} x^H + \theta^{H+1} \hat{\rho}_2^{H+1} . w_1
\]

With:

- \( \hat{\rho}_1^{H+1} = \text{diag}^{-1}(a) \rho_1^{H+1} \)
- \( \hat{\rho}_2^{H+1} = \sum_{l \leq L} \text{diag}^{-1}(a) \left( V_{l}^1 \right)_1^{H+1} x_l^H \)
- \( a_l = \rho_l^H + \theta^H \left( V_{l}^1 \right)_1^H . w_2 \)

Hence by (2.4):

\[
= \hat{\rho}_1^{H+1} x^H - \theta^{H+1} \| \hat{\rho}_2^{H+1} \|_2
\]

But:

\[
\| \hat{\rho}_2^{H+1} \|_2 = \sqrt{\sum_k \left( \sum_{l} x_l^H \left( V_{l}^1 \right)_1^{H+1} \right)^2} \]

\[
= \sqrt{\sum_k \sum_{l,m} x_l^H x_m^H \left( V_{l}^1 \right)_1^{H+1} \left( V_{m}^1 \right)_1^{H+1}} = \sqrt{\sum_{l,m} \sum_{a_l a_m} x_l^H x_m^H V_{l}^m \left( V_{l}^1 \right)_1^{H+1} \left( V_{m}^1 \right)_1^{H+1}}
\]

\[
= (x^H)' \text{diag}^{-1}(a) V_{H+1} \text{diag}^{-1}(a) x^H
\]
Then:

$$
\inf_{\|w_2\|_2 \leq 1} \left\{ \text{diag}^{-1}(a)\rho^{H+1}.x^H - \theta^{H+1}\sqrt{(x^H)^T[\text{diag}^{-1}(a)\text{V}^{H+1}\text{diag}^{-1}(a)]x^H} \right\}
$$

(2.18)

Where $a = a(w_2)$. However, it is hard to get a close solution to this problem. Since we want to show that optimal values of $MPP^*$ are lower than $\overline{MPP}^*$, then it is enough to show it for an upper bound of (2.18). Notice that $w_2^i = \frac{\left( v^2 \right)^H_m}{\|v^2\|_2}$ is the maximum value for $a_i$. We will select one of the $w_2^i$, which of course will give an upper bound.

When $\theta^{H+1}$ is small, the idea is to select $w_2^i$ such that:

$$
\varepsilon_{i,m} \equiv \rho^H_m + \theta^H \left( v^2 \right)^H_m . w_2^i - \left( \rho^H_m + \theta^H \left( v^2 \right)^H_m . w_2^i \right) = \theta^H \left( \sqrt{\text{V}^{H}_{mm}} - \frac{(\text{V}^H_{ml})}{\sqrt{\text{V}^{H}_{ll}}} \right)
$$

is as small as possible. So we will select the $l^* = \arg\min_l \sum_m |\varepsilon_{i,m}|$. Then $a(w_2^i)_{m} = \rho^H_m + \theta^H \left( v^2 \right)^H_m . w_2^i$, such that:

When $\theta^{H+1}$ is big, the idea is to select $w_2^i = -\frac{\left( v^2 \right)^H_m}{\|v^2\|_2}$ such that:

$$
\varepsilon_{i,m} \equiv \rho^H_m - \theta^H \sqrt{\text{V}^{H}_{mm}} - \left( \rho^H_m + \theta^H \left( v^2 \right)^H_m . w_2^i \right) = \theta^H \left( -\sqrt{\text{V}^{H}_{mm}} + \frac{(\text{V}^H_{ml})}{\sqrt{\text{V}^{H}_{ll}}} \right)
$$

is as small as possible. So we will select the $l^* = \arg\min_l \sum_m |\varepsilon_{i,m}|$. Then:

$$
a(w_2^i)_{m} = \rho^H_m - \theta^H \left( v^2 \right)^H_{ml^*} \frac{1}{\sqrt{\text{V}^{H}_{ll}}}
$$
Therefore the (2.18) bound becomes

\[ B(x^H) \equiv \text{diag}^{-1}(a(w_2^H))\rho^{H+1}, x^H - \theta^{H+1} \left( x^H \right)^T \left[ \text{diag}^{-1}(a(w_1^H))Y^{H+1} \text{diag}^{-1}(a(w_1^H)) \right] x^H \]

If \( H = 1 \): In this case, \( a = r^0 \), so:

\[ \inf \sum_{l \leq L} r_l^H x_l^H = (\text{diag}^{-1}(r^0)\rho^{H+1}), x^H - \theta^{H+1} \sqrt{(x^H)^T [\text{diag}^{-1}(r^0)Y^{H+1} \text{diag}^{-1}(r^0)] x^H} \]

Hence \( MPP^* \) is the following (just showing case \( H > 1 \)):

| \hline
| maximize & \( B(x^H) + r_{L+1}^H x_{L+1}^H \) & \\
| \hline
| \( x_l^t \leq x_l^{t-1} \left\{ \frac{\rho_l^t - \theta^t \sqrt{V_l^t}}{\rho_l^{t-1} + \theta^{t-1} \sqrt{V_l^{t-1}}} \right\} - \text{sell}_l^t + \text{buy}_l^t \) & \( \forall l \leq L, t \geq 2 \) & \\
| \hline
| \( x_1^1 = x_1^0 r_1^0 - \text{sell}_1^1 + \text{buy}_1^1 \) & \( \forall l \leq L \) & \\
| \hline
| \( x_{L+1}^t = r_{L+1}^{t-1} x_{L+1}^{t-1} + (1 - u^t) \cdot \text{sell}^t - (1 + v^t) \cdot \text{buy}^t \) & \( \forall t \) & \\
| \hline

Is important to notice that \( MPP^* \) is always feasible. In fact we can set:

- \( \text{sell}_l^t = \text{buy}_l^t = 0 \) \( \forall l \leq L, t \)
- \( x_l^t = x_l^{t-1} \left\{ \frac{\rho_l^t - \theta^t \sqrt{V_l^t}}{\rho_l^{t-1} + \theta^{t-1} \sqrt{V_l^{t-1}}} \right\} \) \( \forall l \leq L, t \geq 2 \)
- \( x_1^t = x_1^0 r_1^0 \) \( \forall l \leq L \)
- \( x_{L+1}^t = r_{L+1}^{t-1} x_{L+1}^{t-1} \) \( \forall t \)
2.4. Assets Markets

Now we will test the performance of $MPP^*$ and $\overline{MPP}^*$ using different settings for the asset market. The first setting is the model used in Ben-Tal (2000). This model builds up a market, in which returns are driven by factors. The second setting uses real world asset information.

2.4.1. Factor-Based market Model

The returns $r_t^l$ are log normally distributed and follow the simple factor model:

- $\ln r_t^l = (\kappa \Omega_l^T e) + \sigma \Omega_l^T Z_t \quad \forall t \geq 1, l \leq L$
- $\ln r_{l+1}^t = \kappa \quad \forall t \geq 0$

Where

- $\Omega_l = [\Omega_{l1}, \ldots, \Omega_{lk}, \ldots, \Omega_{lk}]$ is the vector which measures the impact of each factor $k$ into asset $l$.
- $e$ is the K-dim ones vector.
- $Z_t$ is a normal random vector. $Z_1, Z_2 \ldots Z^H$ are i.i.d., hence $r^t = [r_1^t, \ldots, r_L^t]$ are i.i.d in $t$. $r^0$ is known.

Now let’s compute the data needed for $MPP^*$ and $\overline{MPP}^*$

Proposition 2.1:

- $\rho_t^l = E(R_t^l) = r_t^0 \exp \left( \left( \kappa \Omega_l^T e + \frac{1}{2} \sigma^2 \Omega_l^T \Omega_l \right) (t - 1) \right)$
- $V_{lm}^t = \text{Cov}(R_t^l, R_m^t) = r_t^0 r_m^0 \exp \left( \left( \kappa (\Omega_l + \Omega_m)^T e + \frac{1}{2} \sigma^2 (\Omega_l^T \Omega_l + \Omega_m^T \Omega_m) \right) (t - 1) \right) \left( \exp \left( \sigma^2 (\Omega_l^T \Omega_m) (t - 1) \right) - 1 \right)$
Proof:

\[ E(R_t^i) = E\left( \prod_{h=0}^{t-1} r_t^h \right) \text{ by independence of } r^1, r^2, ... r^{t-1} \]

\[ = r_t^0 \prod_{h=1}^{t-1} E \left( \exp \left( (\kappa \Omega_t^T e) + \sigma \Omega_t^T Z^h \right) \right) = r_t^0 \exp \left( \kappa \Omega_t^T e (t - 1) \right) \prod_{h=1}^{t-1} E \left( \exp \left( \sigma \Omega_t^T Z^h \right) \right) \]

Now by independence of the component of each component of \( Z^h \):

\[ E \left( \exp \left( \sigma \Omega_t^T Z^h \right) \right) = E \left( \prod_k \exp \left( \sigma \Omega_{tk} Z_k^h \right) \right) = \prod_k \exp \left( \frac{1}{2} \sigma^2 \Omega_{tk}^2 \right) = \exp \left( \frac{1}{2} \sigma^2 \Omega_t \Omega_i \right) \]

For \( V_{tm}^t \):

\[ E(R_t^i R_m^t) = r_t^0 r_m^0 \prod_{h=1}^{t-1} E \left( \exp \left( \kappa \Omega_t^T e + \sigma \Omega_t^T Z^h \right) \exp \left( \kappa \Omega_m^T e + \sigma \Omega_m^T Z^h \right) \right) \]

\[ = r_t^0 r_m^0 \exp \left( \kappa (\Omega_t + \Omega_m)^T e (t - 1) \right) \prod_{h=1}^{t-1} E \left( \exp \left( \sigma \Omega_t^T Z^h + \sigma \Omega_m^T Z^h \right) \right) \]

Now:

\[ E \left( \exp \left( \sigma \Omega_t^T Z^h + \sigma \Omega_m^T Z^h \right) \right) = E \left( \prod_k \exp \left( \sigma (\Omega_{tk} + \Omega_{mk}) Z_k^h \right) \right) = \prod_k \exp \left( \frac{1}{2} \sigma^2 (\Omega_{tk} + \Omega_{mk})^2 \right) \]

\[ = \exp \left( \frac{1}{2} \sigma^2 (\Omega_t + \Omega_m)^T (\Omega_t + \Omega_m) \right) \]

Market Setup

We will use the setting proposed in (2000). Notice that:

\[ E(r_t^i) = \exp \left( \kappa \Omega_t^T e + \frac{1}{2} \sigma^2 \Omega_t^T \Omega_i \right) \]
According to reality $\kappa$ and $\sigma$ are of the same order and $\ll 1$. Expected returns of stocks is higher than cash return, hence we must set $\Omega^T_t$ so that $\Omega^T_t e \geq 1$.

The more attractive the asset is, the more risky should be. Hence the probability $P(\ln r^t_t < \kappa)$ (asset returns below cash returns) should be considerable for more risky assets. A way to make this happen is to set:

$$\sigma = \frac{\kappa}{\gamma} \quad K = \left(\frac{w_{max}}{w_{max} - 1}\right)^2 \quad \Omega^T_t e = \frac{L - 1}{L} + \frac{1}{L} w_{max}$$

with $\gamma$ moving in interval $[0.5, 1.5]$ and $w_{max} \equiv \max_t \Omega^T_t e \in [1.5, 2]$. Each $\Omega_t$ will be set to have $k_t$ non-zeros entries, according to the formula:

$$k_t = \min[K, \left\lfloor \frac{L - 1}{L} K + \frac{1}{L} + 1 \right\rfloor]$$

The non-zero entries are randomly generated, but they need to add $\Omega^T_t e$.

By (2.14) we have that:

$$\theta^t_{max} = \min \left\{ \frac{r^0_t \exp\left(\kappa \Omega^T_t e + \frac{1}{2} \sigma^2 \Omega^T_t \Omega_t \right) (t - 1)}{\sqrt{r^0_t r^0_t \exp\left(\left(2\kappa (\Omega_t)^T e + \sigma^2 (\Omega^T_t \Omega_t)\right)(t - 1)\right) \left(\exp\left(\sigma^2 (\Omega^T_t \Omega_t)(t - 1)\right) - 1\right)}} \right\}$$

$$= \min \left\{ \frac{1}{\sqrt{\left(\exp\left(\sigma^2 (\Omega^T_t \Omega_t)(t - 1)\right) - 1\right)}} \right\}$$

So if we want to set $\theta = \theta^t \forall t$, then $\theta \in \left[0, \min_t \theta^t_{max}\right] = \left[0, \min_t \frac{1}{\sqrt{\left(\exp\left(\sigma^2 (\Omega^T_t \Omega_t)(t - 1)\right) - 1\right)}}\right]$
2.4.2. Real World Assets Model

In this setting the market is composed by a set of ETFs, which represent different asset classes. Each of the chosen ETF tracks index that are composed by securities of the same asset class. The ETF are the following:

<table>
<thead>
<tr>
<th>ETF</th>
<th>Asset Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>BND</td>
<td>Intermediate term corporate and government bonds.</td>
</tr>
<tr>
<td>DBC</td>
<td>Futures Commodities</td>
</tr>
<tr>
<td>DBV</td>
<td>Carry Trade: Short term Futures on G-10 Currencies</td>
</tr>
<tr>
<td>EEM</td>
<td>Stocks from Emerging Markets</td>
</tr>
<tr>
<td>EFA</td>
<td>Stocks from Developed and Non-American Markets</td>
</tr>
<tr>
<td>SPY</td>
<td>Stock from US market</td>
</tr>
<tr>
<td>TLT</td>
<td>Long term US government Bonds</td>
</tr>
</tbody>
</table>

The model also assumes that $r^t_l$ (l is an ETF) are log normally distributed:

- $\ln r^t_l = \mu + Lz^t \quad \forall t \geq 1$
- $\ln r^t_{l+1} = \kappa \quad \forall t \geq 0$

Where $\mu$ and $C = LL^T$ are its expected value and covariance matrix respectively. Following the same steps of proposition 2.1:

$$\rho^t_l = E(R^t_l) = r^0_l \exp \left( \left( \mu_l + \frac{1}{2} C_{ll} \right) (t-1) \right)$$

$$V_{lm} = r^0_l r^0_m \exp \left( \left( \mu_l + \mu_m + \frac{1}{2} (C_{ll} + C_{mm}) \right) (t-1) \right) \left( \exp(C_{lm}(t-1)) - 1 \right)$$

$$\min_t \theta^t_{\text{max}} = \min_l \frac{1}{\sqrt{\exp(HC_{ll}) - 1}}$$
The historical ETFs annual mean return and volatility are shown in Figure 2.2\textsuperscript{13}:

Figure 2.2: Annual Mean Return and Volatility of ETFs

\[
\kappa = 0.0079. \text{ The correlation between the ETF is shown in Table 2.2} \textsuperscript{14}
\]

Table 2.2: Correlation of ETFs

<table>
<thead>
<tr>
<th></th>
<th>BND</th>
<th>DBC</th>
<th>DBV</th>
<th>EEM</th>
<th>EFA</th>
<th>SPY</th>
</tr>
</thead>
<tbody>
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<td>BND</td>
<td></td>
<td>0.1</td>
<td></td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
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<td></td>
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<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>DBV</td>
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<td>0.8</td>
<td>0.8</td>
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<td>0.9</td>
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<tr>
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<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td></td>
<td>0.9</td>
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<tr>
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<tr>
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<td>-0.3</td>
<td>-0.3</td>
<td>-0.2</td>
<td>-0.3</td>
</tr>
</tbody>
</table>

\textsuperscript{13} Data source: http://finance.yahoo.com/. We use the data from the beginning of each ETF to January 2013.
\textsuperscript{14} We use the data from the period we have prices for all ETF, which was April 2003.
2.4.3. Simulations

We define the $MPP^*$ ($\overline{MPP}^*$) strategy as the allocations chosen by the solution of $MPP^*$ ($\overline{MPP}^*$) in a rolling horizon procedure. The latter is basically to take, from the solution, only the allocations of the present period and rebalance the portfolio according to the realized returns. Then solve again, but starting from the next period and iterate until horizon is reached.

A simulation is equivalent to do a rolling horizon procedure on the $MPP^*$ and $\overline{MPP}^*$. So for each simulation we must generate a initial portfolio allocation $x^0$ and initial cash $x^0_{L+1}$, which are random numbers between 0 and 1. We also generate the returns $r_t^t$, by generating the vectors $Z^t$. In each simulation we keep track of:

The fraction between the actual and initial value of our portfolio (including cash):

$$PV^t = \frac{\sum_{t=L+1}^{L+H} x^t}{\sum_{t=L+1}^{L+H} x^0_t}$$

The cash holding ratio, which is the proportion of the portfolio value in cash:

$$\text{CashR}^t = \frac{x^t_{L+1}}{\sum_{t=L+1}^{L+H} x^t}$$

The annual return of the portfolio in one simulation will be defined as the geometric return of the final and initial value of the portfolio, i.e.:

$$\text{ret} \equiv \sqrt[H+1]{PV^{H+1}} - 1$$

The maximum drawdown and the ulcer of one simulation will be defined as:

$$\text{mdd} \equiv \max_{1 \leq t \leq H+1} \left(1 - \frac{PV^t}{\max_{s \leq t} PV^s}\right)$$

$$\text{ulcer} \equiv \sqrt{\frac{1}{H+1} \sum_{t=1}^{H+1} \left(1 - \frac{PV^t}{\max_{s \leq t} PV^s}\right)^2}$$
2.5. Results

2.5.1. Factor-Based Market

For the factor-based market model, we use the following setting:

- 30 assets: $L = 30$
- 10 year horizon: $H = 10$
- Transaction cost of 10%: $u_t^f = v_t^f = 0.1$
- Cash return 10%: $\kappa = 0.1$
- 36 factors: $K = 36$ (i.e. $w_{max} = 1.2$)

Hence at year $t$, both $\text{MP}^*_{\text{MPP}}$ and $\overline{\text{MP}}^*_{\text{MPP}}$ will have the following number of variables: $3L(H - t + 1) + (H - t + 1) = 910 - 91(t - 1)$

At period $t$, $\text{MP}^*_{\text{MPP}}$ has $L(H - t + 1) + (H - t + 1) = 310 - 31(t - 1)$ constraints, while $\overline{\text{MP}}^*_{\text{MPP}}$ has $2(H - t + 1) = 20 - 2(t - 1)$ constraints.

$\text{MP}^*_{\text{MPP}}$ and $\overline{\text{MP}}^*_{\text{MPP}}$ are written on AMPL modeling language and they are solved by CPLEX 12.3.

An experiment is the realization of 100 simulations for a determined parameter setting. We’ll create different experiments by changing the volatility $\sigma$ of the assets. We will generate different strategies by changing the volume of the uncertainty sets, by changing $\theta$. Higher theta means the strategy is more conservative. The performance statistics are shown in Table 2.3 and Table 2.4.
Table 2.3: Performance of MPP*~ and MPP* strategies, using $\sigma=0.3$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>RC</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>MPP~</td>
<td>8.51%</td>
<td>0.02%</td>
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<td>168.03</td>
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<td>9.32%</td>
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<td>19.74</td>
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<td>124.60</td>
<td>0.32</td>
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<td></td>
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<td>10.51%</td>
<td>1.71%</td>
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<td>0.5%</td>
<td>21.48</td>
<td>0.25</td>
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</tr>
<tr>
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<td>11.80%</td>
<td>1.78%</td>
<td>0.5%</td>
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<td>71.78</td>
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<td>165.18</td>
<td>-0.31</td>
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<td>21.55</td>
<td>0.29</td>
<td>2.54</td>
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<td>0.3%</td>
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<td>0.47</td>
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<td>1.72%</td>
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<td>0.5%</td>
<td>21.55</td>
<td>0.29</td>
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<td>0.5%</td>
<td>21.73</td>
<td>0.31</td>
<td>2.56</td>
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<td>MPP~</td>
<td>14.81%</td>
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<td>3.3%</td>
<td>1.1%</td>
<td>12.59</td>
<td>0.33</td>
<td>2.48</td>
</tr>
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<td>MPP</td>
<td>10.77%</td>
<td>3.03%</td>
<td>10.1%</td>
<td>3.6%</td>
<td>2.97</td>
<td>0.28</td>
<td>2.43</td>
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<td>15.61%</td>
<td>3.62%</td>
<td>5.2%</td>
<td>1.7%</td>
<td>8.22</td>
<td>0.31</td>
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<td>0.36</td>
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</table>

Remark: With $\sigma = 0.3$, then $\min_t \theta_t^{\max} = 0.9196$

Table 2.4: Performance of MPP*~ and MPP* strategies, using $\sigma=0.5$

<table>
<thead>
<tr>
<th>Theta</th>
<th>RC</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPP~</td>
<td>8.51%</td>
<td>2.97%</td>
<td>9.9%</td>
<td>3.5%</td>
<td>2.71</td>
<td>0.35</td>
<td>2.37</td>
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<tr>
<td></td>
<td>MPP</td>
<td>9.56%</td>
<td>0.02%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>165.18</td>
<td>-0.31</td>
<td>1.91</td>
</tr>
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<td></td>
<td>MPP~</td>
<td>10.17%</td>
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<td>10.71%</td>
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<td>0.30</td>
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<td>13.30%</td>
<td>2.97%</td>
<td>3.3%</td>
<td>1.1%</td>
<td>12.59</td>
<td>0.33</td>
<td>2.48</td>
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<td></td>
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<td>10.77%</td>
<td>3.03%</td>
<td>10.1%</td>
<td>3.6%</td>
<td>2.97</td>
<td>0.28</td>
<td>2.43</td>
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<td>5.2%</td>
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<td>2.98</td>
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<td>4.2%</td>
<td>3.55</td>
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<tr>
<td></td>
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<td>12.52%</td>
<td>13.69%</td>
<td>48.5%</td>
<td>25.1%</td>
<td>0.50</td>
<td>0.65</td>
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<tr>
<td></td>
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<td>12.52%</td>
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<td>48.5%</td>
<td>25.1%</td>
<td>0.50</td>
<td>0.65</td>
<td>3.37</td>
</tr>
</tbody>
</table>

Remark: With $\sigma = 0.5$, then $\min_t \theta_t^{\max} = 0.3683$
On most experiment settings, $\overline{MPP}^*$ strategies presented higher returns than $MPP^*$ strategies on average. For more conservative thetas, $\overline{MPP}^*$ strategies show lower risk than $MPP^*$ strategies (see std, mdd and ulcer). That’s why the UPI of $\overline{MPP}^*$ strategies is much better in those cases ($\theta \geq 0.6$ in the case $\sigma = 0.3$ and $\theta \geq 0.4$ in the case $\sigma = 0.5$). However less conservative $\overline{MPP}^*$ strategies show higher risk as compared to the same $MPP^*$ ones.

As expected, $\overline{MPP}^*$ show more return and volatility as the level of uncertainty assumed decrease. However that’s not the case for $MPP^*$ strategies, which show little sensibility to theta changes. For the most conservative strategy ($\theta = 2.5$), the final portfolio value for $\overline{MPP}^*$ has no volatility, indicating that capital is left as cash. For the less conservative strategy ($\theta = 0$), investments go to the asset with better mean return. In this case both strategies give the same performance, since both models are equivalent. Obviously, when the market risk $\sigma$ is bigger, the performance return and volatility is also higher and the difference between both strategies becomes larger.

If we look at the mean values of $PV^t$ when $\sigma = 0.5$ (The cause of the performance differences is explained by how conservative allocations are. Figure 2.4 shows that the mean Cash$R^t$ in $\overline{MPP}^*$ strategies is practically zero in most cases ($\theta \leq 0.6$). However, the ratio for $MPP^*$ strategies stays near 5% in time, even for small thetas. The cash ratio pattern is the similar when $\sigma = 0.3$, as shown in the appendix.
Figure 2.3), we can see that our portfolio value increases exponentially for $MPP^*$ as we move towards the horizon. As expected, more conservative strategies have lower $PV_t$. However $MPP^*$ strategies seem to increase linearly and show the same $PV_t$. The latter is consistent with the performance statistics results. The same happens when $\sigma = 0.3$, as shown in the appendix.

The cause of the performance differences is explained by how conservative allocations are. Figure 2.4 shows that the mean CashR in $MPP^*$ strategies is practically zero in most cases ($\theta \leq 0.6$). However, the ratio for $MPP^*$ strategies stays near 5% in time, even for small thetas. The cash ratio pattern is the similar when $\sigma = 0.3$, as shown in the appendix.
Figure 2.3: Mean PV(t) when $\sigma=0.5$. The left (right) plot shows MPP* (MPP*) strategies.

Figure 2.4: Mean CashR(t) when $\sigma=0.5$. The left (right) plot shows MPP* (MPP*) strategies.
As expected, the ratio decreases when the strategy is less conservative. We confirm that when $\theta = 2.5$, everything is held in cash and when $\theta = 0$, everything is invested in assets. Notice that for many $MPP^*$ strategies there is a big ratio increase on the last period. This happens because the upper bound that we maximize when solving $MPP^*$ is exactly (2.18) when $H = 1$. Remember that maximizing the upper bound will produce less conservative solutions that the exact Robust Counterpart would do.

The composition pattern of the portfolio in time, for each strategy, can be seen in Table 2.5. Notice that assets in the factor-based market are increasing in return and risk (i.e. asset one has the lowest mean return and risk and asset N the highest). The main conclusion is the following: At the same level of $\theta$, $MPP^*$ allocations are less conservative than $MPP^*$ allocations.

Is important to say that weights in time increase slightly towards risky assets. This makes sense, since there is more certainty as we get closer to the horizon. However, there are no significant weights changes in time, for each strategy and value of theta. The buy and sell transactions in time, which are shown in the appendix (Table A.2.2) reveal that $MPP^*$ strategies are closer to a buy-hold strategy. Except for less conservative strategies, the mean buy and sell values are zero in many periods. However $MPP^*$ strategies buy and sell positions more frequently. The same conclusion can be obtained when $\sigma = 0.3$, as shown in the appendix.
Table 2.5: Allocation of MPP*~ and MPP* in time, for some assets and σ=0.5

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<th>MPP</th>
<th>Theta=1</th>
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</table>
In an attempt to say something about what strategy is better, we analyze the difference between the $PV_{H+1}$ of $\overline{MP}$ and $MPP^*$ strategies. The results are shown in Table 2.6.

Table 2.6: $PV(H+1)$ difference statistics between $MPP^*$ and $MPP^*$ strategies, when $\sigma = 0.3$ and $\sigma = 0.5$

<table>
<thead>
<tr>
<th>Theta</th>
<th>Mean</th>
<th>Left CI</th>
<th>Right CI</th>
<th>$MPP^* \Rightarrow MPP$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-0.24</td>
<td>-0.36</td>
<td>-0.12</td>
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<td>1</td>
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<td>0.27</td>
<td>0.35</td>
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<tr>
<td>0.8</td>
<td>0.41</td>
<td>0.36</td>
<td>0.47</td>
<td>100%</td>
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<tr>
<td>0.6</td>
<td>0.49</td>
<td>0.39</td>
<td>0.59</td>
<td>98%</td>
</tr>
<tr>
<td>0.4</td>
<td>0.63</td>
<td>0.43</td>
<td>0.83</td>
<td>84%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.87</td>
<td>0.44</td>
<td>1.29</td>
<td>68%</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Theta</th>
<th>Mean</th>
<th>Left CI</th>
<th>Right CI</th>
<th>$MPP^* \Rightarrow MPP$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-0.39</td>
<td>-0.63</td>
<td>-0.14</td>
<td>42%</td>
</tr>
<tr>
<td>1</td>
<td>-0.27</td>
<td>-0.48</td>
<td>-0.06</td>
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<tr>
<td>0.8</td>
<td>0.89</td>
<td>0.75</td>
<td>1.03</td>
<td>96%</td>
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<tr>
<td>0.6</td>
<td>1.25</td>
<td>1.00</td>
<td>1.50</td>
<td>100%</td>
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<td>0.4</td>
<td>1.80</td>
<td>1.28</td>
<td>2.32</td>
<td>80%</td>
</tr>
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<td>0.2</td>
<td>3.01</td>
<td>1.63</td>
<td>4.39</td>
<td>72%</td>
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</table>

When $\sigma = 0.3$, there is evidence that $\overline{MP}$ strategies clearly outperform $MPP^*$ strategies in most cases. When $\theta < 2.5$, the mean and even the left side of the 95% confidence interval (Left CI) is positive. The fraction of times the difference is positive (last column) is at least 68%. And when $\theta = 2.5$, $MPP^*$ strategy is just slightly better, not clearly dominant. The conclusions for $\sigma = 0.5$ are almost the same as with $\sigma = 0.3$. 
2.5.2. Results with ETF

The horizon $H$, the transaction costs $u_t^k$ and $v_t^k$ and the initial portfolio values $x^0$ and $x^0_{L+1}$ will have the same value as on the factor-based market. The performance of $\overline{MPP}^*$ and $MPP^*$ strategies, using ETF historical prices, are shown in Table 2.7.

<table>
<thead>
<tr>
<th>Theta</th>
<th>RC</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
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<td>0.68%</td>
<td>0.53%</td>
<td>6.3%</td>
<td>2.9%</td>
<td>0.23</td>
<td>0.69</td>
<td>2.92</td>
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<td>6.47%</td>
<td>3.69%</td>
<td>17.1%</td>
<td>7.2%</td>
<td>0.90</td>
<td>0.68</td>
<td>3.13</td>
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<tr>
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<td>MPP~</td>
<td>8.48%</td>
<td>2.84%</td>
<td>9.1%</td>
<td>3.5%</td>
<td>2.42</td>
<td>-0.33</td>
<td>3.05</td>
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<td>7.39%</td>
<td>3.69%</td>
<td>15.8%</td>
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<td>1.13</td>
<td>0.42</td>
<td>3.18</td>
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<td>9.73%</td>
<td>3.60%</td>
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<td>-0.06</td>
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<td>7.81%</td>
<td>4.20%</td>
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<td>14.03%</td>
<td>7.60%</td>
<td>26.6%</td>
<td>11.1%</td>
<td>1.27</td>
<td>0.07</td>
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<td>9.83%</td>
<td>5.20%</td>
<td>18.8%</td>
<td>7.7%</td>
<td>1.28</td>
<td>0.46</td>
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<td>MPP~</td>
<td>14.37%</td>
<td>8.50%</td>
<td>31.6%</td>
<td>13.6%</td>
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<td>12.78%</td>
<td>6.59%</td>
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<td>8.50%</td>
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<td>1.11</td>
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<td>MPP~</td>
<td>14.37%</td>
<td>8.50%</td>
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<td>13.6%</td>
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<td>14.37%</td>
<td>8.50%</td>
<td>31.6%</td>
<td>13.6%</td>
<td>1.06</td>
<td>-0.19</td>
<td>2.63</td>
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</table>

One of the differences with respect the factor-based market is that $MPP^*$ strategies performance is sensitive to theta. However, many of the patterns are like before. The mean return of $\overline{MPP}^*$ strategies can be 1%-4% more than the annual return of $MPP^*$ strategies. For more conservative strategies we also have lower risk with $MPP^*$ strategies and otherwise as we become less conservative. The UPI is only higher for $MPP^*$ strategies at ($\theta \in \{1,0.8\}$). For the rest of strategies UPI values similar. Even the magnitudes of the statistics are of the same order with the factor-based model.
Cash holding behavior differences between both types of strategies are similar to the differences seen with the factor-based market. As seen in Figure 2.5, $MPP^*$ strategies invests more in cash than $\overline{MPP}^*$. Just like in Figure 2.4, we see an increase in cash position in the last period of $MPP^*$ strategies. If we look at the mean values of $PV^t$ on Figure 2.6, the portfolio value increases steadily in time, for both types of strategies.

Figure 2.5: Mean CashR(t). The left (right) plot shows $MPP^*$ (MPP*) strategies
The composition of the portfolio in time for each strategy can be seen in Table 2.8. The main conclusion is exactly the same as in the factor-based market: At the same level of uncertainty assumed, \( \overline{MP}P^* \) allocations in ETF are less conservative than \( MP^* \) allocations. First thing to notice is that \( \overline{MP}P^* \) strategies never hold position in the following ETFs: DBC, DBV and EFA. This is reasonable, since these ETFs are dominated by one more assets, in terms of mean return and volatility (look Figure 2.2). However \( MP^* \) strategies have a much more diversified portfolio, so all ETFs are used. As we get less conservative, \( \overline{MP}P^* \) moves investments mainly from bonds (TLT and BND) and US stocks (SPY) to emerging market stocks (EMM). \( MP^* \) strategies also follows this pattern, but with less conservative thetas. Second thing to notice is that allocations weights increase slightly in EMM as we move towards horizon. As we explain before, this is expected since there is more certainty as we get closer to the horizon. The buy and sell transactions in time show also the same behavior as in the factor-based market. As show in the appendix (Table A.2.3), \( MP^* \) strategies are closer to a buy-hold strategy. \( \overline{MP}P^* \) strategies buy and sell positions almost at every time.
Table 2.8: Allocation composition of MPP∗∗ and MPP∗ strategies in time.

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<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>0%</td>
<td>100%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>
Just like we did with the factor-based market, we want to see if there is dominance of $\overline{MPP}^*$ over $MPP^*$ strategies. Table 2.9 shows the results:

Table 2.9: PV(H+1) difference statistics between MPP** and MPP* strategies.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Mean</th>
<th>Left CI</th>
<th>Right CI</th>
<th>MPP**&gt;MPP</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>-1.06</td>
<td>-1.31</td>
<td>-0.80</td>
<td>2%</td>
</tr>
<tr>
<td>1</td>
<td>0.20</td>
<td>-0.01</td>
<td>0.40</td>
<td>70%</td>
</tr>
<tr>
<td>0.8</td>
<td>0.44</td>
<td>0.19</td>
<td>0.68</td>
<td>72%</td>
</tr>
<tr>
<td>0.6</td>
<td>2.15</td>
<td>1.51</td>
<td>2.80</td>
<td>92%</td>
</tr>
<tr>
<td>0.4</td>
<td>1.23</td>
<td>0.78</td>
<td>1.68</td>
<td>86%</td>
</tr>
<tr>
<td>0.2</td>
<td>0.40</td>
<td>0.19</td>
<td>0.60</td>
<td>90%</td>
</tr>
</tbody>
</table>

The pattern is almost the same to what we have in the factor-based market. But now the evidence of dominance is stronger. For each $\theta \leq 1$ tested, the values of $\overline{MPP}^*$ strategies are bigger than $MPP^*$ values, for around 70%-90% of the simulations.
2.6. Conclusions and Future Directions

We have shown, throughout several examples, how the concept of grouping correlated parameters into fewer constraints can make the Robust Counterpart to have less conservative solutions. We have shown that the grouping can be achieved by applying transformations (the GUM transformations) into the original problem and before deriving its robust counterpart.

In the toy problem example, we could see the magnitude of the optimal value increment when the RC is derived from the GUM transformation. Depending on the level of uncertainty of the data (measured by \( \theta \)), the optimal value increased consistently in almost every generated instance. The increase reached a 2.35 factor on average.

Then we showed how to apply GUM transformations into a portfolio problem, and how to build strategies based on the allocations given by \( \overline{\text{MPP}}^* \) and \( MPP^* \). In this problem we know beforehand that the optimal value of \( \overline{\text{MPP}}^* \) strategies will be higher than \( MPP^* \) ones, just like the toy problem. In both markets, the factor-based and the one formed with ETFs, we showed that allocations of \( \overline{\text{MPP}}^* \) strategies are in fact less conservative than allocations of \( MPP^* \) strategies. The cash holding ratio in \( \overline{\text{MPP}}^* \) strategies is lower than \( MPP^* \) strategy too. What we didn’t know beforehand, given the uncertainty of the true returns in the simulations, if that allocations made by \( \overline{\text{MPP}}^* \) strategies improve the performance of the portfolio. By measuring the final values of the portfolio for both types of strategies, we obtained that \( \overline{\text{MPP}}^* \) strategies gave better returns and even lower risk with bigger values of theta. By taking the difference of the final portfolio values, we showed that \( \overline{\text{MPP}}^* \) strategies statistically dominate \( MPP^* \) strategies. In the factor-based market the dominance is not sensible with the volatility in the market. For the market composed on ETFs, results also depicted strong evidence that \( \overline{\text{MPP}}^* \) strategies statistically dominate \( MPP^* \) strategies.

In terms of the GUM transformations, one possible extension is to derive new transformations for different kind of functions. In that way, we might apply GUM transformations for more optimization problems. It will be interesting to derive a bound of the ratio between the optimal values of the two RC (the RC of the original problem and the RC of the GUM transform problem),
for different type of problems. With the bound, we can measure the improvement of using GUM transformation, without depending on the instances.

For the MPP problem, we can derive RC with other uncertainty set, like the polyhedron or box used in Chapter 1, to see if there is a performance improvement. There is also the possibility to test the performance of the RC with other settings. Many parameters we can be changed, like the horizon, transaction costs, number and magnitude for the factors etc. We can also adjust the uncertainty level depending on the time left to horizon.
Chapter 3
GUM Applied to MPP with Private Equity Asset Class

3.1. Introduction

Private Equity (PE), according to what it is stated by European Private Equity & Venture Capital Association (EVCA\textsuperscript{15}), is equity capital provided to enterprises not quoted on a stock market. It is an important asset class for many institutional investors, such as endowments, family offices, pension plans etc. The reason they work with PE is the same reason we include an asset in a portfolio: Obtain better returns and/or lower risk by diversification.

However, PE returns can’t be obtained like other assets classes that are traded in public markets, such as stocks and the ETF we’ve seen in the previous chapter. First, private companies don’t need to reveal public information, like companies in the public market do. So in many cases, there is incomplete data to determine gains and any value increase in time. Many assets traded in public markets are highly liquid, thus you can make any gains effective at any period of time. However PE is illiquid in the sense that gains are not necessarily available as soon as an investor wants. As we will see in a following section, PE gains are effective after many years, when the firm goes public or when a buyer appears. When there is illiquidity, valuation of a company gets hard and inaccurate, since there is no market that is indirectly pricing its value when traded.

There is previous work that refers to the amount an investor should invest on PE. Lamm and Ghaleb-Harter (2001) construct a portfolio using the IRR (internal rate of return) quarterly returns from the Venture Economics Database, periods 1986 to 2001. To fairly compare and add PE as an asset class, they employ a ten period mean variance portfolio approach. They figure out that PE allocations should be around 19\% to 51\%, depending on mean return targets. The other asset classes where represented by the SP500, EAFE and EACM 100 (for hedge funds) indexes. Cash was represented with 10 year treasury securities. Chen et al. (2002) take Venture Capital

\textsuperscript{15} http://evca.eu/
(VC) data (periods 1968-1999) also from Venture Economics and model the returns with a sort of CAPM: VC returns depends on stock market returns. The term that doesn’t depend on stock market corresponds to specific VC investments returns, which follows a multinormal normal distribution with some mean and a covariance matrix. The latter can depend on the duration of the VC and its period of existence. The parameters needed in previous model are calibrated with a maximum likelihood technique. The low correlation of 0.04 between VC returns and SP500 index in their results, show that VC allocation can be in the range of 2%-9%, depending on risk targets. Allocations were derived with the Markowitz model. The assets involved are SP500, EAFE, T-bills, T-Bonds and US inflation.

Ennis and Sebastian (2005) focus on how allocations in PE should depend in the type of investor. They mention factors like the access investors have to the best funds or their skills, which can determine the final performance of the PE. They also explain how risk tolerance, liquidity requirement and even the capacity of confidential dealing can play an important role in the success of the investment. Using Pincus/Venture Economics Post-Venture Capital Index (PVCI) as a proxy for PE, they realize that mean-variance optimal portfolios allocate in PE only when the portfolio has predominant allocations in equity (US and international). Hence, their conclusion is: PE investments will only benefit investors that positively align with previous factors and that have equity based portfolio. The allocation suggested in these cases is around 10%.

Cumming et al. (2012) analyze the role of alternative assets (one of them is PE) in portfolio allocation. They highlight the fact that mean variance approach for this type of assets is not proper, since risk and return must be covered with higher moments. To do that, they fit returns with a mix of two normal distributions. They also include investor preference by setting an objective function that combines the probability of outperforming and underperforming some benchmark thresholds. The allocations with this model outperform the Markowitz mean variance approach in out-of-sample data and show that alternative assets are complementary to traditional assets. Depending on the investor, allocations in US Buyouts fall in the 10% - 20% range.

Finally, we mention Ling (2010) that mimics PE returns with publicly traded assets. The idea is to replicate IRR with a mix of sector-level ETF’s returns. To obtain the weights of the mix, she
employs a regression that minimizes mean square errors of the dollar wealth path between ETF’s portfolio and the average PE fund. The methodology is applied to study the optimal PE allocation in university endowment.

In the previous chapter, we derive the Robust Counterpart of the GUM transformed version of MPP. We test its performance using a factor-based market of liquid assets and using some ETFs historical prices. In this chapter we will want to extend the MPP model and include the option to invest in PE investments. The structure of PE investments, which require time to make effective gains, fits properly in a multistage setting. Hence MPP is adequate to include this asset class in the portfolio choice. The aim of the chapter is the following:

1. Include PE assets using explicitly the information about cash flows, rather than convert information about PE investments into some returns or use the IRR returns.

2. Address possible misleading conclusions that might arise in just measuring PE with the IRR return and not considering the timing structure of PE cash flows. We want to show that PE commitments can lead to liquidity problems, which is something that can’t be shown if we treat PE as another liquid asset class.

3. We want to see how much PE to allocate in a portfolio and the advantages of including them in terms of performance. To do this, we derived the same RC using the GUM transformation on the previous chapter. But now we include the uncertainty of PE cash flows. We will analyze performance and composition of the portfolio, using the same type of factor-based market and ETF in chapter two.

The chapter is organized as follows:

In the second section, we explain the main concepts about PE characteristics and structure. Then we give information about PE historical returns, cash flows and performance from different sources. In the third section we show the new model after including PE investments. We apply the GUM transformations to it and derive its RC from this new problem. In section
four we show the performance of the strategy based on the RC. We analyze the composition of the portfolio in time and the effect in adding PE, using different uncertainty levels. Finally in section five, we present conclusions and possible extensions of this chapter.
3.2. Private Equity Investments

3.2.1. Participants, Levered Buy-Outs and Venture Capital

According to Fenn et.al (1997) or Kraft (2001), the participants can be categorized in three:

- **The issuers:** The firms for which a PE funds invest in.
- **The investors:** The ones who contribute capital to the PE funds. They are generally corporate or public pension funds, investments banks, insurance companies or wealth private investors.
- **The intermediaries:** The PE funds, who manages the capital from investors and decide to what firms they should invest. In order to meet the performances required by the investors, PE funds participate actively in the management decisions of the issuers.

Figure 3.1 describes how the world’s investors have allocated collective wealth. At the end of 2008, PE investments are just to 1.2% of the total capital, which is equivalent to $1.14 trillion.

*Figure 3.1: Invested Capital Allocation at the end of 2008*

*Source: USB Global Asset Management Venture Economics, EnnisKnupp. December 2008*
However, there are institutional investors with greater allocations in PE. As an example we have university endowments. Figure 3.2 shows the portfolio allocation of Princeton University Endowment. PE constitutes 38% of the endowment by June 2011.

Figure 3.2: Investment Allocation of Princeton University Endowment

```
```

The most known type PE investments are the Venture Capital (VC) and Leverage Buy-Out (LBO). VC is associated with firms in their early stages, which need investments to develop and expand. Therefore PE funds are actively supporting the management and executing the business plan. Venture capital has been used in famous tech companies, such as Facebook or Google. For more details about VC look at Gompers and Lerner (2001) or Hellmann and Puri (2002).

LBO is associated with more established companies. Based on the definition made in Loos (2005), LBO is a transaction in which the intermediary purchases a significant equity stake of a firm, using debt financing. The reasons to do an LBO can vary, but mainly they are the following:
• Fund Growth Projects
• Turn Around: In cases when the firm is having difficulties, the firm needs new management that can reestablish prosperity.
• Management BO: Funds provided to management teams that want to acquire a product line or business

Figure 3.3 shows the investments amounts and the number of companies for VC and LBO in the EVCA 2011 survey\textsuperscript{16}. There are more VC companies than BO companies (some years twice more), but the funds go mainly to BO transactions (around 90\% in 2010 and 2011).

Figure 3.3: (Left) Investment amount in VC and LBO for 07-13. (Right) Number of companies

\textsuperscript{16} Figure is taken from the 2011 EVCA survey found in: http://www.evca.eu/knowledgecenter/statisticsdetail.aspx?id=454
3.2.2. Stages

According to Loos (2005) a buyout process can be distinguished in three phases:

1. The acquisition:
   This is the selection of the target firm and the structuring of the transaction. Here the firm is valued and the debt financing is planned.

2. Post-Acquisition Management:
   In this stage the buyout must create value from the investment. According to Kester and Luehrman (1995), in the first BO, intermediaries only create value in the target firm by being active shareholders and getting involved in the financial engineering. But after the 1990, PE firms started to participate in operational changes of the purchased firm, in order to reduced inefficiencies. Nowadays PE firm develop strategic growth platforms and have an active role in changing the industry landscape.

3. Divestment:
   Stage when the investment (with its gain/losses) is taken out from the PE target firm. Kaplan and Stroemberg’s (2008) show that the most common way of exiting is by an initial public offering (IPO) or by a trade sale of the portfolio company to a strategic buyer. It is also common to see a second LBO. IPO is decreasing popularity over time. Ljungqvist and Richardson (2003) show that divestment occur after 10 years in average.
3.2.3. Historical Performance

Performance of PE investment is highly dependent of the database used. Given the confidentiality of private companies’ results, reports are based on the data available for each PE fund. Each fund manages a particular set of PE companies, which can be substantially different. As pointed out in Ennis and Sebastian (2005), PE performance depends a lot of the investor preferences and skills. Table 3.1 depicts that the best PE have huge profits, while the worst can produce huge losses. This pattern is consistent with two different sources which uses data from different time periods.

Table 3.1: Distribution of IRR for PE investments, taken from two different database

<table>
<thead>
<tr>
<th>Quartile</th>
<th>Annual IRR(%) 1990-2006(*)</th>
<th>Annual IRR(%) 2011(**)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35.7</td>
<td>22.54</td>
</tr>
<tr>
<td>2</td>
<td>16.3</td>
<td>6.16</td>
</tr>
<tr>
<td>3</td>
<td>6.8</td>
<td>-2.08</td>
</tr>
<tr>
<td>4</td>
<td>-11</td>
<td>-12.4</td>
</tr>
</tbody>
</table>

(*) is taken from Klier (2009). (***) taken from 2011 EVCA survey

There are metrics that are used to measure the performance of a PE investment. One of the most used ones is the internal rate of return (IRR), which is the return that makes the present value of all cash flows and unrealized gains equal to 0. Figure 3.4 shows that the 5-year IRR from private equity index (compose by European private equity investments) compare to 5 year public market indexes returns. Taken from Cumming et al. (2012), Figure 3.5 shows the annualized returns, volatility and correlation matrix of different asset class indexes.

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17 Figure is taken from the 2011 EVCA survey found in: http://www.evca.eu/knowledgecenter/statisticsdetail.aspx?id=454
Figure 3.4: Returns of PE index compared to other asset class returns.

The HSBC Smaller European Company Index is a total return index covers 1350 companies in developed markets in Europe. The Morgan Stanley Euro Index (MSCI) is a free float-adjusted market capitalization weighted index that is designed to measure the equity market performance of the developed markets in Europe. EMBI+ includes liquid US Dollar denominated Brady bonds, Euro bonds and Sovereign Loans from emerging market countries.

Figure 3.5: Mean Return, volatility and correlation of different asset classes. Period 1999-2009.

HFRI correspond to the hedge fund research inc index. Both PE indices (Buyout and VCO comes from the Thompson Reuters VentureXpert database. US Bonds correspond to JP Morgan US Government Bond. S&P GCSI is the MSCI S&P Commodity.
From both figures, we can clearly see that PE returns have low correlation with many of the other assets. This means new opportunities to diversify a portfolio. We can also see a big difference between the US Buyout and VC index performance. What is most remarkable is that both private equity indexes present lower risk-mean returns, as compared to other classes, such as the Emerging Markets and Commodity indices. It is assumed that PE investments should have higher returns than equity from public markets, due to the illiquidity and patience needed to obtain gains. The explanation to this sort of contradiction is explained by the dependence of the data source. As pointed out before, best PE performance is substantially different from the worst ones.

Cumming et al. (2012) reveal that risk can’t be captured only with second moments. Table 3.2 exhibits the skew and kurtosis of the indexes from Figure 3.5. The Jarque-Bera test rejects normality for VC at 1%, but not for Buyout.

Table 3.2: Higher moments measure and normality test for different asset classes.

<table>
<thead>
<tr>
<th></th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>Jarque-Bera</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP500</td>
<td>4.48</td>
<td>-0.46</td>
<td>16.707</td>
</tr>
<tr>
<td>EEM</td>
<td>2.98</td>
<td>-0.32</td>
<td>2.189</td>
</tr>
<tr>
<td>US Bonds</td>
<td>4.79</td>
<td>0.00</td>
<td>17.646</td>
</tr>
<tr>
<td>S&amp;P GCSI</td>
<td>4.25</td>
<td>-0.51</td>
<td>14.337</td>
</tr>
<tr>
<td>HFRI</td>
<td>6.73</td>
<td>-0.52</td>
<td>82.39</td>
</tr>
<tr>
<td>US Buyout</td>
<td>3.24</td>
<td>-0.19</td>
<td>1.085</td>
</tr>
<tr>
<td>US VC</td>
<td>6.91</td>
<td>1.63</td>
<td>142.293</td>
</tr>
</tbody>
</table>

The problem of using IRR is that it doesn’t give information the timing of cash flow of a PE investment. According to the data handled by Ljungqvist and Richardson (2003), the cash inflow (drawdowns) pattern is very different from the out-flow pattern (distributions). Figure 3.6 shows that PE investments require a huge amount of the committed capital (72%) on the first 4 years. Cumulative distributions only exceed cumulative drawdowns from year seven. At divestment stage (10 years or more) total investment is 93% of the committed capital and distribution.
Another way of comparing PE performance with public market is to use the Public Market Equivalent (PME). This measure compares how much a PE fund investor actually earned to what the investor would have earned in an equivalent investment in the S&P 500. Results with PME metric can be found in Harris et.al (2012). Using Burgiss\textsuperscript{18} private capital fund database on the 1984-2008 periods, they show that PME with different public market indexes (S&P500, Russell 200 and 300, Nasdaq) is on average 1.22, which is equivalent to a difference of 3% annual in favor of PE. Details of the results are shown in the appendix.

\textsuperscript{18} http://www.burgiss.com/
3.3. MPP with Private Equity Investments

Now we will add the option to invest in PE to the MPP. We will model a PE investment as a project which have the cash flow dynamic and duration as shown in Figure 3.6. We assume we have a pool of $N$ PE investments, which start at different times. For this purpose, we will introduce the following parameters and variables:

**Variable**

- $tc_i$: Total Commitment for PE $i$.

This is the amount of capital that we commit to give for the PE through time. At the end, the total real commitment can differ from $tc_i$. We’ll implicitly assume that we can be one of the many investors raising money on PE $i$. Hence, we’ll not include a minimal amount to commit nor an upper bound to commit (To make the problem bounded we will add a bound $M$, with $M$ big). As we will see later, the utilities will be based on our total commitment. Obviously, if $tc_i = 0$, it means we don’t invest in PE $i$.

**Parameters**

- $s(i)$: Starting period of PE $i$. Hence, cash flows will appear in periods $s(i) + 1, s(i) + 2$ and so on. Note that $s(i)$ can be negative, which means that the investment starts before the initial period of the model.

- $nf_i^t$: Net flow of PE $i$ at time $t$. This is the difference between distributions and commitments per unit of total commitment. So at each time period we will gain (loss) the amount $nf_i^t tc_i$. Of course, $nf_i^t = 0$ if $t \leq s(i)$.

The value $put_i^t$ of each PE investment at time $t$ will correspond to its present and future cash flows, in time $t$ value. The flows are brought to period $t$, at a discount rate equivalent to cash returns. Therefore is a risk neutral evaluation:

$$put_i^t \equiv \sum_{s \geq t} nf_i^s \exp(-\kappa(s - t)) \quad (3.1)$$
We want to include PE investments committed previously \( (s(i) < 0) \), since they can affect the decisions in the planning horizon. Therefore we keep track of this with:

- \( tc_i^0 \): Total Commitment amount with PE i, when \( s(i) < 0 \). This is a parameter, since we know if we’ve committed to i or not.

To make the problem bounded, we’ll bound \( tc_i \leq M \). The MPP with PE, which we’ll call PEMPP, is the following:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i} r_i x_i^H + \sum_{i} put_{i}^{H+1} tc_i \\
\text{subject to} & \quad x_i^t = v_i^{t-1} x_i^{t-1} - sell_i^t + buy_i^t \quad \forall l \leq L, t \geq 1 \\
& \quad x_{L+1}^t = r_{L+1} x_{L+1}^{t-1} + \sum_{i \leq L} (1 - u_i^t) sell_i^t - \sum_{i \leq L} (1 + v_i^t) buy_i^t + \sum_{i \leq N} n_{i}^t tc_i \quad \forall t \\
& \quad tc_i = tc_i^0 \quad \forall i, s(i) < 0
\end{align*}
\]

Applying the same transformations than in MPP, the PEMPP is the following:

\[
\begin{align*}
\text{maximize} & \quad \left( \begin{array}{c}
R_i^{H+1} \\
R_{L+1}^{H+1} \\
put_{i}^{H+1}
\end{array} \right) \cdot \left( \begin{array}{c}
x_i^H \\
x_{L+1}^H \\
tc
\end{array} \right) \\
\text{subject to} & \quad \tilde{x}^t = \tilde{x}^{t-1} - \tilde{sell}^t + \tilde{buy}^t \quad \forall t \geq 1 \\
& \quad \tilde{x}_{L+1}^t = \frac{1}{r_{L+1}} \left( \begin{array}{c}
(1 - u_i^t) \cdot R_i^t \\
-(1 + v_i^t) \cdot R_i^t
\end{array} \right) \cdot \left( \begin{array}{c}
sell_i^t \\
\frac{buy_i^t}{tc}
\end{array} \right) \quad \forall t \geq 1 \\
& \quad tc_i = tc_i^0 \quad \forall i, s(i) < 0
\end{align*}
\]

For the PEMPP*, we will consider the fact that \( nf^{t \geq 2} \) is uncertain in practice. Assume the same ellipsoidal uncertainty, but now including \( R^{t \geq 2} \) and \( ut^{t \geq 2} \). Then \( U^t \) is the following:
\[ \mathcal{U}^t = \{ \rho^t + (V^t_1)^t \ w, \|w\|_2 \leq \theta^t \} \]

Now \( \rho^t = (\rho^t_R, \rho^t_F) \) and \( V^t = \begin{pmatrix} V^t_R & V^t_{RF} \\ V^t_{RF} & V^t_F \end{pmatrix} \) the expectation and covariance matrix of the vector \( (R^t_{nf}) \). In the case \( t = H + 1 \), the vector is \( (R^H_{n}) R^{H+1}_{n} \). Just as (2.9), with \( \theta_{\text{max}} = \min_t \frac{(\rho_R)^t}{\sqrt{(V_R)^t}} \) we have that \( R^t \geq 0 \) for any \( \theta^t \in [0, \theta_{\text{max}}^t] \).

Then, just like we derive \( \overline{MPP}^* \), the \( P\overline{EMPP}^* \) is:

\[
\begin{align*}
\text{maximize} & \quad x^{t \geq 0, \text{sell}^t \geq 0, \text{buy}^t \geq 0, 0 \leq t_c \leq M} \\
& \quad \begin{pmatrix} \rho^t_R \\ R^t_{H+1} \\ \theta^t \\ \rho^t_F \end{pmatrix} \begin{pmatrix} X^t_H \\ x^t_{H+1} \\ tc \end{pmatrix} \quad \text{subject to}
\end{align*}
\]

\[
\begin{align*}
x^t &= x^{t-1} - \text{sell}^t + \text{buy}^t \quad \forall t \geq 1 \quad (3.2)
\end{align*}
\]

\[
\begin{align*}
x^t_{L+1} &= x^t_{L+1} - \theta^t \\
& \quad \frac{\text{sell}^t \text{buy}^t}{tc} \sqrt{\left( \frac{\text{sell}^t \text{buy}^t}{tc} \right)^t} \\
& \quad \leq x^t_{L+1} + \frac{1}{R^t_{L+1}} \left( (1 - u^t)_R \rho^t_R \right) \left( \text{sell}^t \text{buy}^t \right) \\
& \quad \forall t > 1 \quad (3.3a)
\end{align*}
\]

\[
\begin{align*}
x^t_{L+1} &= x^t_{L+1} + \frac{1}{R^t_{L+1}} \left( \frac{(1 - u^t)_R \ r^t}{n^t} \right) \left( \text{sell}^t \text{buy}^t \right) \quad t = 1 \quad (3.3b)
\end{align*}
\]

\[
\begin{align*}
tc_i &= tc^0_i \quad \forall i, s(i) < 0 \quad (3.4)
\end{align*}
\]

With \( V^t = \begin{pmatrix} (1 - u^t)(1 - u^t)_R \ V^t_R & -(1 - u^t)(1 + v^t)_R \ V^t_R \\ -(1 - u^t)(1 + v^t)_R \ V^t_R & (1 + v^t)(1 + v^t)_R \ V^t_R \\ (1 - u^t)_R \ V^t_R & -(1 + v^t)_R \ V^t_R \end{pmatrix} \)

98
3.3.1. Liquidity Insolvency

In contrast to $\overline{MPP}^*$, $PE\overline{MPP}^*$ can be ineasible. It can happen that we need to pay commitments from previous PE investments and there is not enough cash for it, even if we sell all our positions in the liquid market. This liquidity insolvency problem can happen at any time.

For $t = 1$, notice that $\{n f_i^1\}^1_{s(i)=0}$ is negative under the cash flow dynamic explained in Ljungqvist and Richardson (2003). It means during the first year it is more likely to have drawdown than distribution in the first year. If $\{\rho_F^1\}^1_{s(i)=0}$ is negative, a sufficient and necessary condition for liquidity insolvency problem:

$$
\frac{r_{L+1}^0}{r_{L+1}^0} x_{L+1}^0 + \left( (1 - u^1) \cdot r^0 \right) x^0 + \sum_{i,s(i)<0} n f_i^1 t c_i^0 < 0
$$

Proof

For $t = 1$, liquidity insolvency is equivalent not to satisfy (3.3b):

$$
\frac{x_{L+1}^0}{r_{L+1}^0} \left( (1 - u^1) \cdot r^0 \right) \cdot sell^1 - \frac{x_{L+1}^0}{r_{L+1}^0} \left( (1 + v^1) \cdot r^0 \right) \cdot buy^1 + \sum_{i,s(i)<0} n f_i^1 t c_i^0 < 0 \quad \forall sell^1, buy^1, tc \ (3.5)
$$

We will find an upper bound for the LHS of (3.5). Since $x^1 \geq 0$, by (3.2) we have $0 \leq sell^1 \leq x^1$. As $-(1 + v^1) \leq 0$, then $\frac{(1 - u^1) \cdot r^0 \cdot sell^1}{r_{L+1}^0} - \frac{(1 + v^1) \cdot r^0 \cdot buy^1}{r_{L+1}^0} \leq \frac{(1 - u^1) \cdot r^0 \cdot x^0}{r_{L+1}^0}$

By the (3.4) and the fact that we have $u t_i^1 = 0$ if $1 \leq s(i)$.

$$
\rho_F t c = \sum_{i,s(i)<0} n f_i^1 t c_i^0 + \sum_{i,s(i) \geq 0} n f_i^1 t c_i = \sum_{i,s(i)<0} n f_i^1 t c_i^0 + \sum_{i,s(i)=0} n f_i^1 t c_i
\leq \sum_{i,s(i)<0} n f_i^1 t c_i^0
$$

Then (3.5) is satisfied if only if:

$$
x_{L+1}^0 + \frac{(1 - u^1) \cdot r^0}{r_{L+1}^0} x^0 + \frac{1}{r_{L+1}^0} \sum_{i,s(i)<0} n f_i^1 t c_i^0 < 0
$$
3.3.2. Factor-Based Market Model

To test the performance of $PEMP^*$, we will use the same two settings as in chapter 2. However we need to extend the setting to include PE. We’ll assume that PE $i$ starts at a random period $s(i) < H$ and has a duration (life) of $D$. For the factor-based market we will assume that $nf^t_i$ is normally distributed and follow the simple factor model:

Defining: $\mathbb{1}^t_i = \begin{cases} 1 & \text{if } s(i) \leq t \leq s(i) + D \\ 0 & \text{otherwise} \end{cases}$ we have:

$$nf^t_i = \mathbb{1}^t_i \left( \bar{n}f^{t-s(i)} + \sigma_{PE} \Omega_t Z^{t-1} \right)$$

Where $\bar{n}f^h$ is the expected net flow that any PE should have in its $h$-th period of life. This data can be directly taken from Ljungqvist and Richardson (2003), which is shown in Figure 3.6. $\Omega_t$ and $Z_t$ are the same as defined in chapter 2. Notice that under this model, the vectors $ut^t$ with $nf^t = [nf^t_1, \ldots, nf^t_i, \ldots, nf^t_N]$ are i.i.d. in t. Notice that $nf^1_i$ is known. Now let’s compute the data needed for $PEMP^*$.

Is clear that for $t \leq H$:

$$(\rho^F)^t_i = E(nf^t_i) = \mathbb{1}^t_i \bar{n}f^{t-s(i)}$$

Now by (3.1):

$$(\rho^F)^{H+1}_i = E(put^{H+1}_i) = \sum_{t \geq H+1} E(nf^t_i) \exp(-\kappa(t-(H+1)))$$

$$= \sum_{t \geq H+1} \mathbb{1}^t_i \bar{n}f^{t-s(i)} \exp(-\kappa(t-(H+1)))$$
Now let’s compute the covariance matrices. For this purpose we will use the following equality.

**Proposition 3.1:**

\[
\sum_k \text{Cov} \left( \prod_{h=1}^{t-1} \exp \left( (\kappa \Omega_i e) + \sigma \Omega_i^T Z_h \right), \Omega_{ik} Z_{k-1} \right) = \sigma \exp \left( (\kappa \Omega_i e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t - 1) \right) \Omega_i^T \Omega_i
\]

The proof is left in the appendix.

**Proposition 3.2:**

\[
(V_F)_{ij}^t = \begin{cases} 
\mathbb{1}_i \mathbb{1}_j \sigma_{PE}^2 \Omega_i^T \Omega_i & t < H + 1 \\
\sum_{t \geq H + 1} \mathbb{1}_i \mathbb{1}_j \exp \left( -2\kappa (t - (H + 1)) \right) \sigma_{PE}^2 \Omega_i^T \Omega_i & t = H + 1 
\end{cases}
\]

\[
(V_{RF})_{ii}^t = r_{i}^t \mathbb{1}_i \sigma_{PE} \exp \left( (\kappa \Omega_i e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t - 1) \right) \Omega_i^T \Omega_i
\]

Proof:

For \( V_F^t, t \leq H \):

\[
(V_F)_{ij}^t = \text{Cov}(nf_t, nf_j^t) = \text{Cov} \left( \mathbb{1}_i \left( nf^{t-s(i)} + \sigma_{PE} \Omega_i^T Z^{t-1} \right), \mathbb{1}_j \left( nf^{t-s(j)} + \sigma_{PE} \Omega_i^T Z^{t-1} \right) \right)
\]

\[
= \mathbb{1}_i \mathbb{1}_j \sigma_{PE}^2 \text{Cov}(\Omega_i^T Z^{t-1}, \Omega_i^T Z^{t-1}) = \mathbb{1}_i \mathbb{1}_j \sigma_{PE}^2 E \left( (\Omega_i^T Z^{t-1}) (\Omega_i^T Z^{t-1}) \right) = \mathbb{1}_i \mathbb{1}_j \sigma_{PE}^2 \Omega_i^T \Omega_i
\]

For \( V_F^{H+1}, V_{RF}^H \):

\[
(V_F)^{H+1} \cdot (V_F)_{ij}^{H+1} = \text{Cov}(put_i^{H+1}, put_j^{H+1})
\]

\[
= \text{Cov} \left( \sum_{t \geq H + 1} nf_j \exp (-\kappa (t - (H + 1))), \sum_{t \geq H + 1} nf_j \exp (-\kappa (t - (H + 1))) \right)
\]
\[
\sum_{t \geq H + 1} \exp \left( -\kappa(t - (H + 1)) \right) \exp \left( -\kappa(v - (H + 1)) \right) \text{Cov}(n f^t, n f^v)
\]

By independence of \(n f^t\) with \(n f^v\), \(\text{Cov}(n f^t, n f^v) = 0\) if \(t \neq v\). Then

\[
\sum_{t \geq H + 1} \exp \left( -2\kappa(t - (H + 1)) \right) \text{Cov}(n f^t, n f^t)
\]

\[
\sum_{t \geq H + 1} \exp \left( -2\kappa(t - (H + 1)) \right) \text{Cov}(n f^t, n f^t)
\]

\[
\sum_{t \geq H + 1} \mathbb{I}_t^t \exp \left( -2\kappa(t - (H + 1)) \right) \sigma_{\text{PE}}^2 \Omega_i^T \Omega_i
\]

For \(V_{RF}^t, t \leq H\):

\[
(V_{RF})_{ii}^t = \text{Cov}(R_i^t, n f_i^t) = \text{Cov} \left( \int_1^{t-1} \prod_{h=1}^{t-1} \exp \left( (\kappa \Omega_i^T e + \sigma \Omega_i^T Z^h) \right), \mathbb{I}_t^t \left( \eta f^t \mathbf{s} + \sigma \Omega_i^T Z^{t-1} \right) \right)
\]

\[
= n_i^0 \mathbb{I}_t^t \sigma_{\text{PE}} \sum_h \text{Cov} \left( \prod_{h=1}^{t-1} \exp \left( (\kappa \Omega_i^T e + \sigma \Omega_i^T Z^h) \right), \Omega_i^T Z^{t-1} \right)
\]

By Proposition 3.1

\[
= n_i^0 \mathbb{I}_t^t \sigma_{\text{PE}} \exp \left( (\kappa \Omega_i^T e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t - 1) \right) \Omega_i^T \Omega_i
\]

For \(V_{RF}^{H+1}\)

\[
(V_{RF})_{ii}^t = \text{Cov}(R_i^{H+1}, p u_i^{H+1})
\]

\[
= \text{Cov} \left( \int_1^{H} \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i^T e + \sigma \Omega_i^T Z^h) \right), \sum_{t \geq H + 1} n f^t \exp \left( -\kappa(t - (H + 1)) \right) \right)
\]

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\[ = r_i^0 \sum_{t \geq H+1} \ll_i^i \sigma_{\text{PE}} \exp \left( -\kappa (t - (H + 1)) \right) \text{Cov} \left( \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i \text{e}) + \sigma \Omega_i \text{Z}^h \right), \Omega_i \text{Z}^{t-1} \right) \]

\[ = r_i^0 \sum_{t \geq H+1} \ll_i^i \sigma_{\text{PE}} \exp \left( -\kappa (t - (H + 1)) \right) \sum_k \text{Cov} \left( \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i \text{e}) + \sigma \Omega_i \text{Z}^h \right), \Omega_{ik} \text{Z}^{t-1} \right) \]

\[ = r_i^0 \sum_k \sum_{t \geq H+1} \ll_i^i \sigma_{\text{PE}} \exp \left( -\kappa (t - (H + 1)) \right) \text{Cov} \left( \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i \text{e}) + \sigma \Omega_i \text{Z}^h \right), \Omega_{ik} \text{Z}^{t-1} \right) \]

\[ \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i \text{e}) + \sigma \Omega_i \text{Z}^h \right) \text{ is independent of } \text{Z}^{t-1} \text{ when } t \geq H + 2. \text{ Then} \]

\[ = r_i^0 \sum_k \ll_i^{H+1} \sigma_{\text{PE}} \text{Cov} \left( \prod_{h=1}^{H} \exp \left( (\kappa \Omega_i \text{e}) + \sigma \Omega_i \text{Z}^h \right), \Omega_{ik} \text{Z}^{H} \right) \]

By Proposition 3.1:

\[ = r_i^0 \ll_i^{H+1} \sigma_{\text{PE}} \exp \left( (\kappa \Omega_i \text{e} + \frac{1}{2} \sigma^2 \Omega_i \Omega_i) \text{H} \right) \Omega_i \text{H} \Omega_i \]

---

**Market Setup**

To build the \( \Omega_i \) rows for each PE \( i \), we apply the same procedure explained in chapter 2. However the sum \( \Omega_i \text{e} \) of the non-zero entries will be greater than the same sum we have for liquid assets. (Recall the greatest sum for liquid asset is given by \( w_{\text{max}} \equiv \max_i \Omega_i \text{e} \)). With this setting, we are assuming that PE investment have more volatility than liquid assets. Therefore we set:

\[ \Omega_i \text{e} = w_{\text{max}} + \frac{0.5i}{N} \quad \forall i \]

The number of non-zero entries is just like for liquid assets, i.e.:

\[ k_i = \min \left\{ K, \left\lfloor \frac{N - i}{N} K + \frac{i}{N} + 1 \right\rfloor \right\} \]
To keep things as real as possible, we set $\sigma_{\text{PE}}$ such that $n_{i}^{\text{PE}}$, i.e. the first cash flow of a PE investment, is negative with a high probability. Under normal circumstances this value should be negative. Then $\sigma_{\text{PE}}$ is such that

$$P \left( n_{i}^{\text{PE}} < 0 \right) \geq 0.95 \quad \forall i \quad (3.6)$$

This is equivalent to: $\sigma_{\text{PE}} \leq \frac{-n_{i}^{\text{PE}}}{1.96 \sqrt{\Omega_{i}^{T} \Omega_{i}}} \forall i$ or $\sigma_{\text{PE}} \leq \min_{\Omega_{i}^{T} \Omega_{i} = a_{i}^2} \max_{\Omega_{i}^{T} \Omega_{i} = a_{i}^2}$. Now $\max_{\Omega_{i}^{T} \Omega_{i} = a_{i}^2} \min_{\Omega_{i}^{T} \Omega_{i} = a_{i}^2} = a_{i}^2$.

Since $a_{i} = w_{\text{max}} + \frac{0.5i}{N}$ increasing in $i$, then we make sure we hold (3.6) with:

$$\sigma_{\text{PE}} \leq \sigma_{\text{PE, max}} \equiv \frac{-n_{i}^{\text{PE}}}{1.96 \left( w_{\text{max}} + 0.5 \right)} \quad (3.7)$$

### 3.3.3. Real World Assets Model

For the market composed by ETF, we will assume that $n_{i}^{\text{ETF}}$ has the same structure than with the factor-based market. The difference is that we align the volatility of the PE with the volatility of the ETF. Following the model from chapter 2:

- $\ln r_{t} = \mu + LZ_{t} \quad \forall t \geq 1$
- $\ln r_{t+1} = \kappa \quad \forall t \geq 0$

The difference with chapter 2 is that $L$ is the cholesky decomposition of the covariance matrix $C$ for liquid assets and PE investments. $C$ has now dimensions $(N + L) \times (N + L)$. Obviously each $Z_{t}$ are i.i.d. normal vector of dimesion $(N + L)$. Then:

- $n_{i}^{\text{ETF}} = \mathbb{E} \left[ (n_{i}^{\text{ETF}})^{\mu} + \sigma_{\text{PE}} L_{i}^{T} Z_{t-1} \right]$

All the expectations needed on $\mathbb{E} \mathbb{E}^{\text{MPP}}$ don’t change with respect the formulas shown, either in chapter 2 or in the previous factor-based market. For the covariance calculations, we replace the following from Proposition 3.1 and Proposition 3.2:

When $l$ is an ETF:
• \( \mu_i \) by \( \kappa \Omega_i^T e \)
• \( \sigma \Omega_i \) by \( L_i \) where \( L_i \) is a row from matrix \( L \)

When \( i \) is a PE investment:

• \( \Omega_i \) by \( L_i \) where \( L_i \) is a row from matrix \( L \)

Then Proposition 3.1 can be expressed as:

\[
\sum_k \text{Cov} \left( \prod_{h=1}^{t-1} \exp(\mu_i + L_i^T Z^h), \Omega_{ik} Z_{k}^{t-1} \right) = \exp \left( \left( \mu_i + \frac{1}{2} C_{ii} \right)(t - 1) \right) C_{ii}
\]

Proposition 3.2:

\[
(V_F)_{ij}^t = \begin{cases} 
I_i^T I_j \sigma_{PE}^2 C_{ij} & t < H + 1 \\
\sum_{t \geq H + 1} I_i^T I_j \exp(-2\kappa (t - (H + 1))) \sigma_{PE}^2 C_{ij} & t = H + 1
\end{cases}
\]

\[
(V_{RF})_{ii}^t = r_{it}^0 I_i^T \sigma_{PE} \exp \left( \left( \mu_i + \frac{1}{2} C_{ii} \right)(t - 1) \right) C_{ii}
\]

Market Setup

Let’s divide the covariance matrix \( C \) in the the blocks \( C = \begin{pmatrix} C_L & C_{LN} \\ C_{LN}^T & C_N \end{pmatrix} \)

\( C_L \) represents the covariance between ETFs, which is known from chapter 2. We’ll assume no correlation between ETFs and PE investment flows, hence \( C_{LN} = 0 \). Finally, we’ll assume positive correlation between PE flows. The correlation matrix \( p_N \) correspond to the following matrix:

\[
p_N = \frac{1}{N} e e^T + \left( 1 - \frac{1}{N} \right) I
\]

Recall \( e \) is the vector of ones, here a N-dimension one. \( I \) is the N by N identity matrix. Is easy to see \( p_N \) semi positive definite. Then \( C_N \) and hence \( C \) are semi positive definite too.
The variance values of PE investment flows are based on the maximum variance of the ETF (called as $C_{i,\text{max}}$), in the following way:

$$C_{ii} = C_{i,\text{max}} \left(1 + \frac{0.5i}{N}\right)$$

If $\sigma_{PE} = 1$, the variance of the PE investments flows $nf_i^f$ will be between $C_{i,\text{max}}$ and $1.5C_{i,\text{max}}$, increasing in $i$. With the variance and correlation, we can compute the block $C_N$. Equation (3.7) in the ETF market is:

$$\sigma^* = \min_i \frac{-nf_i^0}{\frac{1.96\sqrt{C_{ii}}}{1.96\sqrt{1.5C_{i,\text{max}}}}} = \frac{-nf_i^0}{\frac{1.96\sqrt{C_{i,\text{max}}}}{1.96\sqrt{1.5C_{i,\text{max}}}}}$$
3.4. Results

Now we present the performance given by $\mathbf{P \hat{E} M PP^*}$ allocations using the factor-based market and the ETF presented in the previous section.

The test on each market is based on simulations just like in chapter 2. In each simulation, we allocate according to $\mathbf{P \hat{E} M PP^*}$ solution and proceed with the same rolling horizon procedure used before. In order to compare simulations results with the ones from chapter 2, all values generated for liquid assets will be exactly the same. So we just need to generate the following:

- Initial PE investments positions: There will be no previous PE investments when we start the rolling horizon, so $t_c^0 = 0$
- Stating times of PE investments: Random integers between 0 and $\left\lfloor \frac{H}{2} \right\rfloor$.
- Cash Flows of PE investments: As a remark, for cash flows between 1 and $H+1$, we employ the same values of $Z^t$ used for liquid assets returns.

With PE investments, the metric $PV^t$ and $\text{CashR}^t$ defined in chapter 2 is now:

$$PV^t = \frac{\sum_{i \leq L+1} x_i^t + \sum_{i \leq N} \text{put}_i^t t_c_i}{\sum_{i \leq L+1} x_i^0 + \sum_{i \leq N} t_c_i^0}$$

$$\text{CashR}^t = \frac{x_{L+1}^t}{\sum_{i \leq L+1} x_i^t + \sum_{i \leq N} \text{put}_i^t t_c_i}$$

Now we keep track of the PE investment ratio, which is defined by:

$$\text{PER}^t = \frac{\sum_{i \leq N} \text{put}_i^t t_c_i}{\sum_{i \leq L+1} x_i^t + \sum_{i \leq N} \text{put}_i^t t_c_i}$$

We also want to measure how many times there is liquidity insolvency. So we define the measure:

$$\text{LIR} = \frac{\# \text{liquidity insolvency}}{\# \text{Simulations}}$$
When liquidity insolvency happens (say time $t^*$), we stop the simulation (stop investments). All the cash and positions in liquid assets are given to pay the insolvency. This means that $x_i^t = 0 \ \forall l, t \geq t^*$. Hence, the portfolio will be the PE cash flows. We assume that we can pay any commitment with present or future distributions.

We have to say that these actions underestimate the costs of LI. We assume that insolvency is fully paid by giving all the liquid positions. But in fact the debt can be more. We also assume we can pay present commitments with future distributions.

### 3.4.1. Results with Factor-Based Market Model

For PE investments we use the following:

- 10 PE assets: $N = 10$
- Duration: 10 years: $D = 10$

As a consequence, $P_{EMPP}^*$ size at time $t$ will be:

**Variables:**

$$3L(H - t + 1) + (H - t + 1) + N = 910 - 91(t - 1) + 10$$

**Constraints:** The exact number will depend on the starting time $s$ of each PE (look at (3.4)). An upper bound is:

$$L(H - t + 1) + (H - t + 1) + N = 310 - 31(t - 1) + 10$$

For the factor-based market model, we fix the volatility of the liquid market $\sigma = 0.5$. We use the same parameters values for liquid assets as in Chapter 2. Hence, we have that:

$$\sigma^* = -\frac{-0.1614}{1.96(1.2+0.5)} = 0.0484.$$
We will run simulation changing the values for the volatility of the illiquid market $\sigma_{PE}$, based on $\sigma^*$. First we present the returns statistics performance, without considering simulations with liquidity insolvency. The results are shown in Table 3.3:

Table 3.3: Performance of PEMPP$^*$ strategies conditioned on non-LI cases, using $\sigma_{PE} = \sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>15.5%</td>
<td>0.8%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>2182.65</td>
<td>0.1</td>
<td>2.3</td>
</tr>
<tr>
<td>1</td>
<td>16.8%</td>
<td>1.1%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>-0.2</td>
<td>2.5</td>
</tr>
<tr>
<td>0.8</td>
<td>17.3%</td>
<td>1.3%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>0.0</td>
<td>2.6</td>
</tr>
<tr>
<td>0.6</td>
<td>18.0%</td>
<td>1.5%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>0.7</td>
<td>2.9</td>
</tr>
<tr>
<td>0.4</td>
<td>18.9%</td>
<td>1.9%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>0.9</td>
<td>3.3</td>
</tr>
<tr>
<td>0.2</td>
<td>19.5%</td>
<td>3.8%</td>
<td>0.5%</td>
<td>0.2%</td>
<td>93.67</td>
<td>-0.3</td>
<td>4.7</td>
</tr>
<tr>
<td>0</td>
<td>16.1%</td>
<td>13.4%</td>
<td>35.8%</td>
<td>17.8%</td>
<td>0.90</td>
<td>0.6</td>
<td>2.9</td>
</tr>
</tbody>
</table>

As expected, less conservative strategies have more return and volatility. However when $\theta = 0$, return decreases and all risk measures increase significantly. It is remarkable that risk measures (std deviation, mean mdd, mean ulcer) are zero in many strategies. The same happens with $\sigma_{PE} = \{0.1,2\}\sigma^*$, as seen in appendix (Table A.3.3 and Table A.3.4).

To make the comparison of PEMPP$^*$ with MPP$^*$ strategies and see the benefit of having PE investments, we’ll show the performance of MPP$^*$ strategies on Table 3.4, but just using the simulations with no liquidity problems.

Table 3.4: Performance of MPP$^*$ strategies condition on non-LI cases, using $\sigma_{PE} = \sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>8.51%</td>
<td>0.02%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>165.18</td>
<td>-0.31</td>
<td>1.91</td>
</tr>
<tr>
<td>1</td>
<td>10.26%</td>
<td>1.09%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>3699.98</td>
<td>1.15</td>
<td>4.50</td>
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<tr>
<td>0.8</td>
<td>13.83%</td>
<td>2.78%</td>
<td>2.7%</td>
<td>0.9%</td>
<td>16.12</td>
<td>0.32</td>
<td>2.80</td>
</tr>
<tr>
<td>0.6</td>
<td>14.95%</td>
<td>3.56%</td>
<td>3.8%</td>
<td>1.2%</td>
<td>12.49</td>
<td>-0.05</td>
<td>2.58</td>
</tr>
<tr>
<td>0.4</td>
<td>15.99%</td>
<td>5.51%</td>
<td>10.0%</td>
<td>3.4%</td>
<td>4.70</td>
<td>-0.16</td>
<td>2.41</td>
</tr>
<tr>
<td>0.2</td>
<td>17.35%</td>
<td>7.86%</td>
<td>14.3%</td>
<td>5.7%</td>
<td>3.05</td>
<td>0.45</td>
<td>3.05</td>
</tr>
<tr>
<td>0</td>
<td>15.06%</td>
<td>12.72%</td>
<td>45.7%</td>
<td>22.8%</td>
<td>0.66</td>
<td>0.33</td>
<td>2.39</td>
</tr>
</tbody>
</table>
The performance of $\overline{P\overline{E}MPP^*}$ strategies is far better than for $\overline{MPP^*}$ strategies. This means that in the cases where no liquidity insolvency occurs, PE investments improve the returns by a range of 1%-7% (difference is higher for more conservative strategies) and the risk measures (std, max drawdown and ulcer) decrease significantly. We get similar conclusions in the case $\sigma_{PE} = \{0.1,2\sigma^*\}$, as shown in the appendix.

Now we will see the asset class composition of our portfolio in time. Just like before, we only consider simulations without liquidity problems (we will analyze performance with them later). Figure 3.7 shows the $PV^t$, CashR$^t$ and PER$^t$ measures.

Figure 3.7: Mean PV(t) (Left) and CashR(t) (Right) of $P\overline{E}MPP^*$ strategies, using $\sigma_{PE} = \sigma^*$

We observe that $PV^t$ increases similarly across thetas, except for the least conservative strategy ($\theta = 0$), and at different rates across time. It increases slower at the beginning and at the end. That seems to match with the huge CashR$^t$ ratio amount on these periods. As expected, the % of cash hold is decreasing as strategies become less conservative, and is zero when $\theta = 0$. Notice that CashR$^t$ ratio can be very different across thetas at the first and last periods. For example, between $\theta = 2.5$ and $\theta = 1$, there is approx. 25% difference at period 2 or 10% between $\theta = 1$ and $\theta = 0.8$. If we look at the PER$^t$ (Figure 3.8) in the first six years across
thetas, we surprisingly see less illiquidity in the portfolio as we get less conservative (from $\theta = 1$ to $\theta = 0.2$). This difference can go to 15% in some periods. However after year seven, $\text{PER}^t$ is slightly higher as strategies becomes less conservative.

Figure 3.8: Mean $\text{PER}(t)$ of PEMPP$^{**}$ strategies, using $\sigma_{\text{PE}} = \sigma^*$

With the last two figures, we can understand the composition of the portfolio value in time. Notice that $1 - \text{CashR}^t + \text{PER}^t$ corresponds to the fraction invested in liquid assets. Except for one strategy ($\theta = 0$), the pattern is the following:

From the start, the portfolio composition quickly changes towards PE investments. It takes only a couple of years to get $\text{PER}^t$ above 50%, which means that the value of the portfolio at that time comes mainly from PE investments. The liquid part of the portfolio is composed mainly as cash for more conservative strategies (higher thetas) and of liquid assets for less conservative ones (lower thetas). This pattern holds until the middle of the path. At this point, approx. 95% of the portfolio value derives from PE investments. Finally, as we get closer to horizon, the pattern moves in the opposite direction. Given that there are no new PE opportunities, investments move towards liquid assets (for lower thetas) or cash (for higher thetas). That also explains the decrease in growth rate for $\text{PV}^t$ in the last half too.
Is interesting that for the least conservative strategy ($\theta = 0$), investments go mainly to the more attractive liquid assets (more risky too) instead of going to the more attractive PE investments. This means that if we were risk neutral, then the best thing is not to invest in PE. In fact, if we decrease $\sigma_{PE}$, then the $\theta = 0$ strategy becomes like the solution found in chapter 2, i.e. PE investments are not considered. However, if we increase $\sigma_{PE}$, then the portfolio composition of the $\theta = 0$ strategy becomes closer to the rest of the strategies. The portfolio composition in time for the cases $\sigma_{PE} = \{0.1, 2\}\sigma^*$ is shown in the appendix.

Previous results have shown that PE investments are an important part of the portfolio value in time, even for more conservative strategies. That’s why we can frequently have LI, like shown in Figure 3.9. As expected, more conservative strategies have fewer cases of LI, no matter the volatility of PE flows. In fact, the most conservative strategy never has LI problems. Except for $\theta > 0$, LI is not sensitive to changes in volatility of PE flows.

Figure 3.9: LIR of PEMPP*~ strategies for different PE market volatility.

The performance including LI is shown in Table 3.5. Results can drop to the point that it is preferable not to include PE investments. To see this, we compare the previous results with the ones shown in Table 2.4 for the $\overline{MPP}^*$ strategy.
Table 3.5: Performance of PEMPP∗~, using \( \sigma_{PE} = \sigma^* \).

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>15.5%</td>
<td>0.8%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>2182.65</td>
<td>0.1</td>
<td>2.3</td>
</tr>
<tr>
<td>1</td>
<td>14.4%</td>
<td>4.7%</td>
<td>10.1%</td>
<td>3.6%</td>
<td>4.02</td>
<td>-1.7</td>
<td>5.2</td>
</tr>
<tr>
<td>0.8</td>
<td>14.5%</td>
<td>5.1%</td>
<td>11.7%</td>
<td>4.2%</td>
<td>3.49</td>
<td>-1.5</td>
<td>4.8</td>
</tr>
<tr>
<td>0.6</td>
<td>14.4%</td>
<td>5.7%</td>
<td>14.3%</td>
<td>5.1%</td>
<td>2.83</td>
<td>-1.1</td>
<td>3.6</td>
</tr>
<tr>
<td>0.4</td>
<td>14.2%</td>
<td>5.9%</td>
<td>18.1%</td>
<td>6.3%</td>
<td>2.24</td>
<td>-0.7</td>
<td>3.2</td>
</tr>
<tr>
<td>0.2</td>
<td>14.5%</td>
<td>6.4%</td>
<td>19.1%</td>
<td>6.7%</td>
<td>2.15</td>
<td>-0.2</td>
<td>2.4</td>
</tr>
<tr>
<td>0</td>
<td>7.3%</td>
<td>30.0%</td>
<td>41.1%</td>
<td>18.9%</td>
<td>0.39</td>
<td>-2.7</td>
<td>10.5</td>
</tr>
</tbody>
</table>

Figure 3.10 shows the mean return and ulcer for PEMPP∗~ and MPP∗~ strategies. Although the best compromise between returns and ulcer is obtained by the most conservative PEMPP∗ strategy, most of the PEMPP∗ strategies are dominated by one or many of the MPP∗ ones. What it is remarkable is that the pattern doesn’t change if we increase or decrease the volatility, as seen in the appendix for \( \sigma_{PE} = \{0.1,2\}\sigma^* \) (Figure A.3.5).

Figure 3.10: Mean Ulcer and Mean Return of PEMPP∗~ and MPP∗~ strategies, using \( \sigma_{PE} = \sigma^* \).
The bottom line is we must be really conservative (set $\theta = 2.5$ in our market) if we want to include PE investments to our portfolio. If that’s the case, our mean return increases by 7% without increasing risk measures. If we get less conservative, robustness is lost. We end up with LI problems, which make $PEMPP^*$ strategies worse than $MPP^*$ ones.

### 3.4.2. Results with ETF

We include PE investments just like we did on the factor-based market, using the same setting. The ETF are the same used in Chapter 2. Given that $L = 7$, we only include two PE investments ($N = 2$) in the pool, with the characteristics described in the market setup. The performance of $PEMPP^*$ and $MPP^*$ strategies, without simulations ending in LI, are shown in Table 3.6 and Table 3.7. In this market $\sigma^* = 0.257$.

Table 3.6: Performance of PEMPP$^*$ strategies condition on non-LI cases, using $\sigma_{PE} = \sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>1.4%</td>
<td>2.1%</td>
<td>5.8%</td>
<td>2.6%</td>
<td>0.54</td>
<td>2.6</td>
<td>9.1</td>
</tr>
<tr>
<td>1</td>
<td>14.6%</td>
<td>4.2%</td>
<td>6.6%</td>
<td>2.4%</td>
<td>6.07</td>
<td>-0.1</td>
<td>2.5</td>
</tr>
<tr>
<td>0.8</td>
<td>15.7%</td>
<td>4.7%</td>
<td>8.7%</td>
<td>3.2%</td>
<td>4.90</td>
<td>-0.2</td>
<td>2.4</td>
</tr>
<tr>
<td>0.6</td>
<td>16.4%</td>
<td>6.1%</td>
<td>16.1%</td>
<td>6.4%</td>
<td>2.55</td>
<td>-0.2</td>
<td>2.5</td>
</tr>
<tr>
<td>0.4</td>
<td>16.8%</td>
<td>8.5%</td>
<td>24.4%</td>
<td>10.5%</td>
<td>1.60</td>
<td>-0.7</td>
<td>3.2</td>
</tr>
<tr>
<td>0.2</td>
<td>17.3%</td>
<td>8.9%</td>
<td>24.8%</td>
<td>10.8%</td>
<td>1.60</td>
<td>-0.6</td>
<td>3.2</td>
</tr>
<tr>
<td>0</td>
<td>18.5%</td>
<td>9.1%</td>
<td>25.0%</td>
<td>10.5%</td>
<td>1.76</td>
<td>-0.9</td>
<td>3.9</td>
</tr>
</tbody>
</table>

The pattern of the results is similar to the result of the factor-based market, i.e. private equity helps to achieve far better performance if we don’t consider LI cases. For each theta, there is an increase in return and decrease in all risk measures. Mean return increase between 3% and 6%, and UPI can be doubled in many cases. Notice there’s a huge increase in risk measures when $\theta < 0.6$. The same pattern of results is obtained with $\sigma_{PE} = \{0.1,2\}\sigma^*$ as seen in appendix (Table A.3.8 - Table A.3.9).
Table 3.7: Performance of MPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = \sigma^*$. 

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.7%</td>
<td>0.5%</td>
<td>6.3%</td>
<td>2.9%</td>
<td>0.2</td>
<td>0.7</td>
<td>2.9</td>
</tr>
<tr>
<td>1</td>
<td>8.7%</td>
<td>2.7%</td>
<td>9.0%</td>
<td>3.4%</td>
<td>2.6</td>
<td>-0.4</td>
<td>3.3</td>
</tr>
<tr>
<td>0.8</td>
<td>10.0%</td>
<td>3.6%</td>
<td>11.3%</td>
<td>4.3%</td>
<td>2.3</td>
<td>-0.1</td>
<td>2.7</td>
</tr>
<tr>
<td>0.6</td>
<td>14.5%</td>
<td>7.7%</td>
<td>26.3%</td>
<td>11.1%</td>
<td>1.3</td>
<td>0.0</td>
<td>2.4</td>
</tr>
<tr>
<td>0.4</td>
<td>14.7%</td>
<td>8.8%</td>
<td>31.6%</td>
<td>13.8%</td>
<td>1.1</td>
<td>-0.2</td>
<td>2.6</td>
</tr>
<tr>
<td>0.2</td>
<td>15.1%</td>
<td>8.9%</td>
<td>30.9%</td>
<td>13.4%</td>
<td>1.1</td>
<td>-0.3</td>
<td>2.7</td>
</tr>
<tr>
<td>0</td>
<td>15.7%</td>
<td>9.4%</td>
<td>31.3%</td>
<td>13.3%</td>
<td>1.2</td>
<td>-0.5</td>
<td>2.9</td>
</tr>
</tbody>
</table>

Now we will see the asset class composition of our portfolio in time. Just like in the factor-based model, we only consider simulations without liquidity problems. Figure 3.11 shows the $PV^t$, CashR$^t$ and PER$^t$ measures.

Figure 3.11: Mean $PV(t)$ (Left) and CashR(t) (Right) of PEMPP*~ strategies, using $\sigma_{PE} = \sigma^*$

The CashR$^t$ measure shows some difference with respect to the factor model market. The most conservative strategy ($\theta = 2.5$) holds mainly cash in time. For the rest, cash is hold only at the
end of the path. This means that during the first periods, these strategies prefer to put liquid allocations (if any) in ETFs instead on cash. The reason might be the low cash return, relative to ETFs returns. Therefore, strategies prefer to sell positions in ETF and pay transaction costs than to hold cash if they want to invest in PE investments.

Given the high cash ratio for the most conservative strategy, we expect a low final portfolio value $\text{PV}^t$. For less conservative strategies, we observe that $\text{PV}^t$ increases similarly across thetas and at different rates across time, just like with the factor-based market. The exponential increase in the first half is given by the considerable allocation in PE investment, clearly shown in Figure 3.12. This includes future positive flows and hence more value to the portfolio.

In the second half of the path, there is a decrease in the growth rate, given there are no opportunities for new PE investments. In the second half, we also have a group strategies $(\theta \in \{0.6,0.8,1\})$ whose $\text{PV}^t$ increase linearly in time, while other group of strategies $(\theta \in \{0,0.2,0.4\})$ have a $\text{PV}^t$ that increase with more oscillations. The risk gap seen in Table 3.6 aligns with this outcome.

---

Figure 3.12: Mean PER(t) of PEMPP*~ strategies, using $\sigma_{PE} = \sigma^*$
One of the reasons of the latter may be explained in the illiquidity of the portfolios. Figure 3.12 shows that strategies within each group have similar PE\textsuperscript{L} in time. However, the illiquidity of the first cluster is much higher than the second one. The difference can reach 30% in the middle of the path. Notice that having a more conservative strategy with higher composition in illiquid assets is not new, as seen in the in the factor base model. Therefore, in the ETF market there are ETFs that are more attractive than PE investments when we look only at the expected gains.

We can confirm the latter when we analyze the results by changing the PE volatility (σ\textsubscript{PE}). Less conservative solutions have almost the same composition when we change σ\textsubscript{PE}. Decisions should not change, since expected values haven’t changed. However the composition of more conservative strategies suffers some changes if we change σ\textsubscript{PE}. For lower σ\textsubscript{PE}, composition start to looks like the one found in the factor-based market. Cash holding increases in the first and last periods and there is more illiquidity for every strategy in time. This means that conservative strategies consider more PE investments if their flows are more stable. The opposite happens when PE flows volatility increases. The results obtained with σ\textsubscript{PE} = \{0.1,2\}σ\textsuperscript{*} can be seen in the appendix (Figure A.3.6- Figure A.3.7).

Liquidity problems also appear in the ETF market. Figure 3.13 shows the LIR measure for different PE market volatilities. Just like the ETF market, more conservative strategies reduce LI cases, for each σ\textsubscript{PE}. In fact, the most conservative strategy has no LI at all. Again, LIR is not sensitive to changes in volatility of PE flows. However, notice that LI issues are on average less frequent in this market, except for θ = 0. For example, when θ = 1, LIR is at most 10%, while in the other market is around 25%. For θ = 0.4 LIR is around 20%, while in the other market is around 50%. We have said previously that conservative strategies decreases illiquidity when we increase σ\textsubscript{PE}. That’s the reason why LI is low for these strategies when σ\textsubscript{PE} = 2σ\textsuperscript{*}.

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The performance including LI is shown in Table 3.8. Again, results can be lower than without including PE investments if we are not conservative enough. To see this, we compare the previous results with the ones shown in Table 2.7 for $\mathcal{MPP}^*$ strategies.

**Table 3.8: Performance of PEMPP**, using $\sigma_{PE} = \sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>1.4%</td>
<td>2.1%</td>
<td>5.8%</td>
<td>2.6%</td>
<td>0.54</td>
<td>2.6</td>
<td>9.1</td>
</tr>
<tr>
<td>1</td>
<td>13.5%</td>
<td>7.3%</td>
<td>9.8%</td>
<td>3.7%</td>
<td>3.67</td>
<td>-3.6</td>
<td>21.3</td>
</tr>
<tr>
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<td>13.6%</td>
<td>10.3%</td>
<td>13.6%</td>
<td>5.2%</td>
<td>2.62</td>
<td>-3.3</td>
<td>15.0</td>
</tr>
<tr>
<td>0.6</td>
<td>14.1%</td>
<td>11.3%</td>
<td>20.3%</td>
<td>8.2%</td>
<td>1.73</td>
<td>-2.9</td>
<td>12.9</td>
</tr>
<tr>
<td>0.4</td>
<td>13.3%</td>
<td>19.0%</td>
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<td>11.7%</td>
<td>1.14</td>
<td>-4.5</td>
<td>26.5</td>
</tr>
<tr>
<td>0.2</td>
<td>11.2%</td>
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<td>30.6%</td>
<td>13.2%</td>
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<td>-3.6</td>
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</tr>
<tr>
<td>0</td>
<td>-3.8%</td>
<td>43.7%</td>
<td>45.1%</td>
<td>19.7%</td>
<td>-0.19</td>
<td>-1.6</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Figure 3.14 shows the mean return and ulcer for $\mathcal{PMP}^*$ and $\mathcal{MPP}^*$ strategies. What’s different from the factor-based market is that many $\mathcal{PMP}^*$ strategies are not dominated by any $\mathcal{MPP}^*$. Since we have less LI in this market, $\mathcal{PEMP}^*$ strategies don’t deteriorate like before (in relation to the $\mathcal{MPP}^*$ strategies in the same market). What is similar to the previous
market is that we have to choose conservative strategies to incorporate PE investments successfully. In fact, the \( \text{PEMP}\) strategies with \( \theta = \{0.8, 1\} \), although they don’t dominate any \( \text{MPP}\) strategy, they have much better UPI than all of them. For example, we can keep the same level of risk obtained by some \( \text{MPP}\) strategies, but increase the returns by 5% on average. Conversely, we can fix the same level of return obtained by some \( \text{MPP}\) strategies, but decrease the mean ulcer by at least 6%. It is clear that less conservative \( \text{PEMP}\) strategies have worst performance than many \( \text{MPP}\). The same figure for decrease and increase in PE flows volatility can be deduced by all the information we know previously. If we decrease \( \sigma_{PE} \), the portfolios will tend to appear like the ones from the factor-based market. Hence, many \( \text{PEMP}\) strategies will be dominated by more than one \( \text{MPP}\) strategy. Only conservative strategies enhance performance. If we increase \( \sigma_{PE} \), then conservative \( \text{PEMP}\) strategies decreases LI problem as seen in Figure 3.13. Therefore, the latter will not only have a better UPI than all the \( \text{MPP}\) strategies. \( \text{PEMP}\) strategies will dominate them. The figures for \( \sigma_{PE} = \{0.1, 2\} \) are shown in the appendix (Figure A.3.10).

Figure 3.14: Mean Ulcer and Mean Return of \( \text{PEMP}^{\ast}\) and \( \text{MPP}^{\ast}\) strategies, using \( \sigma_{PE} = \sigma^{\ast} \).
3.5. Conclusions and Future Directions

We’ve accomplished all the objectives proposed in the chapter. Modeling PE investment as flows allowed us to see liquidity issues that can be presented after committing to it. This is something that can’t be observed if we condense PE performance in a return, just like a liquid asset. Results showed that taking LI into account can dramatically affect the decision of including PE investments in the portfolio.

When we didn’t consider LI cases, annual returns increased by 1% to 7% in the factor-based model and 3% to 6% in the ETF market. Risk measures decreased substantially for every risk aversion level, with values closer to zero in the case of the factor-based model. Moreover, these differences are almost insensible to volatility changes in PE market.

However when we considered LI, results depicted that including PE can deteriorate performance for low risk aversion strategies, especially for the factor-based model. The decrease in performance was of course correlated with the LI ratios. For the factor-based model, less conservative strategies presented ratios from 40% to 60%, whereas in the ETF market ratios moved from 20% to 40%. Therefore we can increase performance by selecting more conservative strategies. In those cases, there are no LI issues and we can increase annual returns by 5% in the factor-based market and up to 7% in the ETF market, without worsening ulcer.

We could also observe the portfolio illiquidity for each strategy. For more conservative strategies, the value of the portfolio belonging to PE (measure by PER) increased as more investments were offered in time and decreased when close to horizon. The PER peak could be very high, depending on the market and the PE volatility. For the factor-based market, we had peaks of approx. 80%-90% and ratios of 40% at horizon, for each level of PE market volatility. In the ETF market, we had peaks of 80%-90% only when the PE volatility is small. When PE volatility increased, more conservative strategies presented peaks around 30%-40% and ratios of 10%-20%. The difference in the results between markets depended in the return/risk profile off the liquid assets.
To obtain new conclusions, there is a lot of experimentation and sensitivity analysis that can be done. For example, we can change some settings, like the lifetime and number of PE investments in relation to the pool of liquid assets. We can change the correlation structure between PE investment and liquid assets and see how performance is affected.

It would be insightful to compare performance between our methodology with the ones mentioned in the literature. To do it, we have to compare everything with the same data set and testing procedure. Data preprocessing is needed prior testing. For example, we have to find the liquid returns that replicate PE investments like in Ling (2010), or fit the returns with the mix of normal distributions like in Cumming et al. (2012).

With our multistage portfolio model, we are able to test any PE database and obtain results for that particular situation. Making slight modifications to it, we can plug any type of investment with uncertain cash flows, such as fix income products or standard projects. For example, this model can be perfectly used in capital budgeting problems. Therefore, a next step could be to test the methodology in problems involving those investments.
Bibliography


Appendix

Chapter 1

Proposition: When $p \in \{1, 2, \infty\}$, (1.3) is equivalent with (1.5) and (1.7). Equivalent in the sense that we get the same optimal solutions by adjusting the user-defined parameters.

Proof:

i. For $p = 2$

Suppose $(x^*, \delta^*)$ are the primal and dual optimal for (1.5), where $\theta$ is the dual variable associated with the constraint $x^T \Sigma x \leq \left(\frac{\lambda}{2\theta}\right)^2$. The dual variables associated with the feasible $\mathcal{P}$ are omitted since they don’t affect the proof procedure\(^{19}\). The KKT conditions for (1.5) satisfy:

$$u - 2\delta^* \Sigma x^* = 0, \quad \delta^* \left(x^* \Sigma x^* - \left(\frac{\lambda}{2\theta}\right)^2\right) = 0, \quad \delta^* \geq 0, \quad x^* \Sigma x^* \leq \left(\frac{\lambda}{2\theta}\right)^2$$

\(^{19}\) Suppose $\kappa$ are the dual variables associated with $x \in \mathcal{P}$ and $G(\kappa, x) = \kappa^T F(x)$ is the lagrangean penalization term associated with $\kappa$ and $x \in \mathcal{P}$. We have two set of KKT conditions where $\kappa$ appears, for both (1.3) and (1.6):

1. $\nabla L = 0$: We always have the terms $\nabla u^T x - \nabla G(\kappa, x) = u - \nabla G(\kappa, x)$. As $x \in \mathcal{P}$, constraints in $G(\kappa, x)$ are linear and therefore $\nabla G(\kappa, x)$ only depends on $\kappa$.

2. Complementary slackness: $\kappa_i F_i(x) = 0$ and $\kappa_i \geq 0$

Hence we can use the optimal $\kappa^*$ in (1.5) or (1.7) and they will satisfy the KKT conditions for conditions in (1.3). For the first KKT condition we replace $u$ with $\tilde{u} = u - \nabla G(\kappa, x)$ without affecting any of the analysis required for the proof.
We want to see if there is a $\theta_3$ such that $x^*$ satisfies KKT conditions for (1.3), i.e.:

$$u - \theta_3 \frac{\Sigma x^*}{\sqrt{x^T \Sigma x^*}} = 0 \quad (A. 1.1)$$

By convexity of (1.3), condition A.1.1 is sufficient to make $x^*$ optimal for (1.3).

If $\delta^* = 0$, then $u = 0$ and thus with $\theta_3 = 0$ we satisfy (A. 1.1).

If $\delta^* > 0$, then $x^* = \frac{\Sigma^{-1} u}{2 \delta^*}$ and $x^* \Sigma x^* = \left(\frac{\lambda}{2 \theta}\right)^2$. So: $\frac{\sqrt{u \Sigma^{-1} u}}{\left(\frac{\lambda}{2 \theta}\right)} = 2 \delta^*$

Then with $\theta_3 = \frac{\sqrt{x^* \Sigma x^* \sqrt{u \Sigma^{-1} u}}}{\left(\frac{\lambda}{2 \theta}\right)}$

$$u - \theta_3 \frac{\Sigma x^*}{\sqrt{x^T \Sigma x^*}} = \frac{\sqrt{u \Sigma^{-1} u}}{\left(\frac{\lambda}{2 \theta}\right)} \Sigma x^* - \theta_3 \frac{\Sigma x^*}{\sqrt{x^T \Sigma x^*}} = 0$$

For the other direction, i.e. getting KKT conditions of (1.5) from KKT of (1.3), we need:

If $\theta_3 = 0$ then $\delta^* = 0$ and $\frac{\lambda}{\theta} = 2 \sqrt{x^* \Sigma x^*}$

If $\theta_3 > 0$ then $\delta^* = \frac{\theta_3}{2 \sqrt{x^* \Sigma x^*}}$ and $\frac{\lambda}{\theta} = 2 \frac{\sqrt{x^* \Sigma x^* \sqrt{u \Sigma^{-1} u}}}{\theta_3}$

ii. For $p = 1$:

Notice than (1.3) and (1.5) can be written as LP. For example (1.3)

$$\max_{z,x \in P} \left\{ \mu^T x - \theta \max_{1 \leq i \leq N} \left(\Sigma^{1/2} x\right)_i \right\} = \max_{x \in P} \left\{ \mu^T x - \theta z : z \geq \left(\Sigma^{1/2} x\right)_i \forall 1 \leq i \leq N \right\}$$

$$= \max_{z,x \in P} \left\{ \mu^T x - \theta z : z1 \geq \Sigma^{1/2} x \geq -z1 \forall 1 \leq i \leq N \right\}$$

Following the same steps for (1.5) we have:

$$= \max_{z,x \in P} \left\{ \mu^T x : \left(\frac{\lambda}{2 \theta}\right) 1 \geq \Sigma^{1/2} x \geq -\left(\frac{\lambda}{2 \theta}\right) 1 \forall 1 \leq i \leq N \right\}$$
Suppose \((x^*, \delta_1^*, \delta_2^*)\) are the primal and dual optimal for (1.5), where \(\delta_1^*\) and \(\delta_2^*\) are the dual variable associate with the constraints \(\sum^{1/2} x \geq - \left(\frac{\lambda}{2\theta}\right) 1\) and \(\left(\frac{\lambda}{2\theta}\right) 1 \geq \sum^{1/2} x\) respectively. The KKT conditions for (1.5) satisfy:

\[
\begin{align*}
    u + \sum^{1/2} \delta_1^* - \sum^{1/2} \delta_2^* &= 0, \\
    (\delta_1^*)_i \left(\left(\sum^{1/2} x^*_i\right) + \left(\frac{\lambda}{2\theta}\right)\right) &= 0, \\
    (\delta_2^*)_i \left(\left(\sum^{1/2} x^*_i\right) - \left(\frac{\lambda}{2\theta}\right)\right) &= 0, \\
    \delta_1^*, \delta_2^* &\geq 0, \\
    (\delta^*)_i \left(\sum^{1/2} x^*_i\right) + z^* &= 0, \\
    (\delta^*)_i \left(\sum^{1/2} x^*_i\right) - z^* &= 0, \\
    \delta_1^*, \delta_2^* &\geq 0, \\
    z^* &\geq \sum^{1/2} x^* \geq -z^*
\end{align*}
\]

We want to see if there is a \(\theta_3\) and \(z\) such that \((x^*, \delta_1^*, \delta_2^*)\) satisfies KKT conditions for (3), i.e:

\[
\begin{align*}
    u + \sum^{1/2} \delta_1^* - \sum^{1/2} \delta_2^* &= 0, \\
    -\theta_3 + 1^T \delta_1^* + 1^T \delta_2^* &= 0, \\
    (\delta_1^*)_i \left(\left(\sum^{1/2} x^*_i\right) + z^*\right) &= 0, \\
    (\delta_2^*)_i \left(\left(\sum^{1/2} x^*_i\right) - z^*\right) &= 0, \\
    \delta_1^*, \delta_2^* &\geq 0, \\
    z^* &\geq \sum^{1/2} x^* \geq -z^*
\end{align*}
\]

Clearly setting \(\theta_3 = 1^T \delta_1^* + 1^T \delta_2^*\) and \(z = \frac{\lambda}{2\theta}\) we satisfy all the KKT conditions for (1.3).

Analogously, on the other way around we need \(\frac{\lambda}{\theta} = 2 z^*\) to make \((z^*, x^*, \delta_1^*, \delta_2^*)\) also a solution for (1.5).

iii. For \(p = \infty\):

Notice than (1.3) and (1.5) can be written as LP. For example (1.3):

\[
\text{Max}_{z, x \in P} \left\{ \mu^T x - \theta \sum_{i=1}^{N} \left(\sum^{1/2} x\right)_i \right\} = \text{Max}_{x \in P} \left\{ \mu^T x - \theta 1^T z: z \geq \sum^{1/2} x \geq -z, \quad \forall 1 \leq i \leq N \right\}
\]

For (1.5) we have:

\[
\text{Max}_{z, x \in P} \left\{ \mu^T x: z \geq \sum^{1/2} x \geq -z, \quad 1^T z \leq \left(\frac{\lambda}{2\theta}\right), \quad \forall 1 \leq i \leq N \right\}
\]
Here $z$ is a vector. Suppose $(x^*, z^*, \delta_1^*, \delta_2^*, \delta_3^*)$ are the primal and dual optimal for (1.5), where $\delta_1^*, \delta_2^*$ and $\delta_3^*$ are the dual variables associated with the constraints $\Sigma^{1/2}x \geq z, z \geq \Sigma^{1/2}x$ and $1^Tz \leq \left(\frac{\lambda}{2\theta}\right)$ respectively. The KKT conditions for (1.5) satisfy:

$$u + \Sigma^{1/2}\delta_1^* - \Sigma^{1/2}\delta_2^* = 0, \quad \delta_1^* + \delta_2^* - \delta_3^*1 = 0, \quad (\delta_1^*)_i\left((\Sigma^{1/2}x^*)_i + z_i^*\right) = 0,$$

$$(\delta_2^*)_i\left((\Sigma^{1/2}x^*)_i - z_i^*\right) = 0, \quad \delta_3^*\left(1^Tz - \left(\frac{\lambda}{2\theta}\right)\right) = 0, \quad \theta_1^*, \theta_2^*, \theta_3^* \geq 0,$$

$$z^* \geq \Sigma^{1/2}x^* \geq -z^*, \quad 1^Tz^* \leq \left(\frac{\lambda}{2\theta}\right).$$

We want to see if there is a $\theta_3$ such that $(x^*, z^*, \delta_1^*, \delta_2^*)$ satisfies KKT conditions for (1.3), i.e.:

$$u + \Sigma^{1/2}\delta_1^* - \Sigma^{1/2}\delta_2^* = 0, \quad -\delta_3^*1 + \delta_1^* + \delta_2^* = 0, \quad (\delta_1^*)_i\left((\Sigma^{1/2}x^*)_i + z_i^*\right) = 0,$$

$$(\delta_2^*)_i\left((\Sigma^{1/2}x^*)_i - z_i^*\right) = 0, \quad \delta_1^*, \delta_2^* \geq 0, \quad z^* \geq \Sigma^{1/2}x^* \geq -z^*$$

Clearly setting $\theta_3 = \delta_3^*$ we satisfy all the KKT conditions for (1.3).

Analogously, on the other way around we need $\delta_3^* = \theta_3$ and $\frac{\lambda}{\theta} = 21^Tz^*$ to make $(z^*, x^*, \delta_1^*, \delta_2^*)$ also a solution for (1.5).

Table A.1.1: One-way Anova results between mean error of the random walk and the uip models.

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Table A.1.2: One-way Anova results between mean error of the random walk and the AR models.

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Figure A.1.1: Cumulative errors of the 3 models for different currencies.
Table A.1.3: BIC results for each volume, starting from Feb-1994.

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Table A.1.4: HMM fitting for each volume between period Feb-1994 to Sep-2012

| log(V)=∞  | Cov R1 | RCT∞ | RVIX | R1    | 86%  | 14%  | 0%   | R2    | 2%   | 85%  | 13%  | R3    | 0%   | 7%   | 93%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 86%  | 14%  | 0%   | RCT∞ | -0.07% | -0.13% | RVIX | -0.13% | 1.999 |
| RCT∞      | -0.4% | 0.0% | 0.1% | RVIX | 3.3% | 0.7%  | -0.3% |
| RVIX      | 0.00% | -0.00% | 0.167% |

| log(V)=-94 | Cov R1 | RCT∞ | RVIX | R1    | 90%  | 9%   | 1%   | R2    | 1%   | 93%  | 5%   | R3    | 2%   | 15%  | 83%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 90%  | 9%   | 1%   | RCT∞ | -0.03% | -0.06% | RVIX | -0.06% | 1.16% |
| RCT∞      | -0.3% | 0.1% | 0.1% | RVIX | 1.5% | -0.4% | 1.2% |
| RVIX      | 0.00% | -0.00% | 0.00% |

| log(V)=-88 | Cov R1 | RCT∞ | RVIX | R1    | 80%  | 7%   | 13%  | R2    | 2%   | 95%  | 3%   | R3    | 4%   | 4%   | 92%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 80%  | 7%   | 13%  | RCT∞ | -0.03% | 1.40%  | RVIX | -0.03% | 1.40% |
| RCT∞      | -0.3% | 0.1% | 0.1% | RVIX | 2.6% | -0.4% | 0.2% |
| RVIX      | 0.00% | 0.16% |

| log(V)=75  | Cov R1 | RCT∞ | RVIX | R1    | 68%  | 32%  | 0%   | R2    | 7%   | 91%  | 1%   | R3    | 0%   | 3%   | 97%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 68%  | 32%  | 0%   | RCT∞ | 0.01%  | -0.01% | RVIX | -0.01% | 1.37% |
| RCT∞      | -0.1% | 0.0% | 0.1% | RVIX | 3.3% | -0.4% | -0.1% |
| RVIX      | 0.00% | -0.00% | 0.383% |

| log(V)=73  | Cov R1 | RCT∞ | RVIX | R1    | 71%  | 23%  | 6%   | R2    | 11%  | 88%  | 2%   | R3    | 1%   | 1%   | 98%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 71%  | 23%  | 6%   | RCT∞ | 0.003% | -0.003% | RVIX | 0.003% | 0.383% |
| RCT∞      | -0.1% | 0.0% | 0.1% | RVIX | 3.3% | -0.4% | -0.1% |
| RVIX      | 0.00% | -0.00% | 0.221% |

| log(V)=73  | Cov R1 | RCT∞ | RVIX | R1    | 71%  | 23%  | 6%   | R2    | 11%  | 88%  | 2%   | R3    | 1%   | 1%   | 98%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 71%  | 23%  | 6%   | RCT∞ | -0.003% | 0.001% | RVIX | -0.003% | 0.383% |
| RCT∞      | -0.1% | 0.0% | 0.1% | RVIX | 3.3% | -0.4% | -0.1% |
| RVIX      | 0.00% | -0.00% | 0.221% |

| log(V)=73  | Cov R1 | RCT∞ | RVIX | R1    | 71%  | 23%  | 6%   | R2    | 11%  | 88%  | 2%   | R3    | 1%   | 1%   | 98%  |
|           |        |      |      |       |      |      |      |       |      |      |      |       |      |      |      |
| Mean      | R1    | 71%  | 23%  | 6%   | RCT∞ | 0.003%  | -0.003% | RVIX | 0.003%  | 0.383% |
| RCT∞      | -0.1% | 0.0% | 0.1% | RVIX | 3.3% | -0.4% | -0.1% |
| RVIX      | 0.00% | -0.00% | 0.221% |
Table A.1.5: Sensibility analysis performance for Dynamic Strategy

**Decision change in Transition Regime: 80% in RCT**

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<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
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**Decision change in Transition Regime: 20% in RCT**

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**Decision change in Unstable Regime: use - RCT**

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**Decision change in Unstable Regime: do nothing**

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Chapter 2

Figure A.2.1: Mean values of PV(t) when $\sigma=0.3$. The left (right) plot shows MPP*~ (MPP*) strategies

Figure A.2.2: Mean values of CashR(t) with $\sigma=0.3$. The left (right) plot shows MPP*~ (MPP*) strategies
Table A.2.1: Allocation composition of MPP\textsuperscript{**} and MPP\textsuperscript{*} strategies in the factor-based market. Assets are 1,5,10,15,20,25 and 30 and $\sigma=0.3$.

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Table A.2.2: Transactions MPP** and MPP* strategies in the factor-based market. Assets are 1,5,10,15,20,25 and 30 and σ=0.3. The left table shows the bought amount and the right table the sold amount.

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Table A.2.3: Transactions MPP** and MPP* for the real world market in time. The left table shows the bought amount and the right table the sold amount.
Chapter 3

Table A.3.1: Private Equity Public Market Equivalent (PME).

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Average - Overall Sample: 1.10
Median - Overall Sample: 1.05
Proposition 3.1 Proof:

\[
\sum_k Cov \left( \prod_{h=1}^{t-1} \exp \left( (\kappa \Omega_i^T e + \sigma \Omega_i^T Z^h), \Omega_{ik} Z_{k}^{t-1} \right) \right) \\
= \sum_k E \left( \prod_{h=1}^{t-1} \exp \left( (\kappa \Omega_i^T e + \sigma \Omega_i^T Z^h), \Omega_{ik} Z_{k}^{t-1} \right) \right) \\
= \exp((t-1)\kappa \Omega_i^T e) \sum_k \Omega_{ik} E \left( \prod_{h=1}^{t-2} \exp \left( (\sigma \Omega_i^T Z^h) \right) \right) E \left( \exp(\sigma \Omega_i^T Z_{l-1}^{t-1}) \right) \\
= \exp((t-1)\kappa \Omega_i^T e(t-1)) \exp \left( (t-2) \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i \right) \sum_k \Omega_{ik} E \left( \exp(\sigma \Omega_i^T Z_{l-1}^{t-1}) \right) \\
= \exp((t-1)\kappa \Omega_i^T e(t-1)) \exp \left( (t-2) \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i \right) \\
\sum_k \Omega_{ik} E \left( \exp \left( \sigma \sum_{m \neq k} \Omega_{im} Z_{m}^{t-1} \right) \right) E \left( Z_{k}^{t-1} \exp(\sigma \Omega_k Z_{k}^{t-1}) \right) \\
= \exp \left( (\kappa \Omega_i^T e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t-2) \right) \exp(\kappa \Omega_i^T e) \\
\sum_k \Omega_{ik} \prod_{m \neq k} \exp \left( \frac{1}{2} \sigma^2 \Omega_{im}^2 \right) \sigma \Omega_{ik} \exp \left( \frac{1}{2} \sigma^2 \Omega_{ik}^2 \right) \\
= \exp \left( (\kappa \Omega_i^T e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t-2) \right) \exp(\kappa \Omega_i^T e) \sum_k \Omega_{ik} \Omega_{ik} \exp \left( \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i \right) \\
= \exp \left( (\kappa \Omega_i^T e + \frac{1}{2} \sigma^2 \Omega_i^T \Omega_i) (t-1) \right) \Omega_i^T \Omega_i
Table A.3.2: Performance of PEMPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 0.1\sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
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<td>-0.97</td>
<td>2.62</td>
</tr>
<tr>
<td>0.8</td>
<td>16.6%</td>
<td>0.7%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>-0.95</td>
<td>3.29</td>
</tr>
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<td>0.1%</td>
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<td>425.33</td>
<td>-0.86</td>
<td>5.24</td>
</tr>
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<td>2.7%</td>
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<td>81.58</td>
<td>-1.28</td>
<td>7.26</td>
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<tr>
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<td>12.5%</td>
<td>13.7%</td>
<td>48.1%</td>
<td>25.0%</td>
<td>0.50</td>
<td>0.66</td>
<td>3.38</td>
</tr>
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</table>

Table A.3.3: Performance of MPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 0.1\sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>8.51%</td>
<td>0.02%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>165.18</td>
<td>-0.31</td>
<td>1.91</td>
</tr>
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<td>1</td>
<td>10.38%</td>
<td>1.14%</td>
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<td>0.0%</td>
<td>3841.25</td>
<td>0.97</td>
<td>3.75</td>
</tr>
<tr>
<td>0.8</td>
<td>13.93%</td>
<td>2.80%</td>
<td>2.9%</td>
<td>0.9%</td>
<td>15.15</td>
<td>0.30</td>
<td>2.69</td>
</tr>
<tr>
<td>0.6</td>
<td>14.63%</td>
<td>3.62%</td>
<td>4.3%</td>
<td>1.4%</td>
<td>10.55</td>
<td>0.11</td>
<td>2.40</td>
</tr>
<tr>
<td>0.4</td>
<td>15.40%</td>
<td>5.35%</td>
<td>11.0%</td>
<td>3.8%</td>
<td>4.05</td>
<td>0.11</td>
<td>2.40</td>
</tr>
<tr>
<td>0.2</td>
<td>19.06%</td>
<td>7.21%</td>
<td>18.0%</td>
<td>6.9%</td>
<td>2.78</td>
<td>0.75</td>
<td>2.98</td>
</tr>
<tr>
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<td>12.52%</td>
<td>13.69%</td>
<td>48.5%</td>
<td>25.1%</td>
<td>0.50</td>
<td>0.65</td>
<td>3.37</td>
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</table>

Table A.3.4: Performance of PEMPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 2\sigma^*$.

<table>
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<tr>
<th>Theta</th>
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<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
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<td>0.0%</td>
<td>2436.76</td>
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<td>2.25</td>
</tr>
<tr>
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<td>17.2%</td>
<td>1.6%</td>
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<td>0.0%</td>
<td>$\infty$</td>
<td>-0.05</td>
<td>2.32</td>
</tr>
<tr>
<td>0.8</td>
<td>17.9%</td>
<td>1.7%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>0.18</td>
<td>2.45</td>
</tr>
<tr>
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<td>18.9%</td>
<td>2.4%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>$\infty$</td>
<td>0.45</td>
<td>2.89</td>
</tr>
<tr>
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<td>19.7%</td>
<td>3.2%</td>
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<td>56.83</td>
<td>0.66</td>
<td>5.17</td>
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<td>26.4%</td>
<td>13.4%</td>
<td>1.46</td>
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<td>2.37</td>
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</table>

Table A.3.5: Performance of MPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 2\sigma^*$.

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>8.51%</td>
<td>0.02%</td>
<td>0.2%</td>
<td>0.1%</td>
<td>165.18</td>
<td>-0.31</td>
<td>1.91</td>
</tr>
<tr>
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<td>10.37%</td>
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<td>0.0%</td>
<td>3737.16</td>
<td>1.08</td>
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<tr>
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<td>0.28</td>
<td>2.72</td>
</tr>
<tr>
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<td>3.53%</td>
<td>3.9%</td>
<td>1.3%</td>
<td>11.39</td>
<td>0.00</td>
<td>2.65</td>
</tr>
<tr>
<td>0.4</td>
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<td>5.37%</td>
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<td>4.56</td>
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<td>2.51</td>
</tr>
<tr>
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<td>7.42%</td>
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<td>6.6%</td>
<td>2.50</td>
<td>0.62</td>
<td>3.67</td>
</tr>
<tr>
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<td>15.65%</td>
<td>45.0%</td>
<td>21.9%</td>
<td>0.73</td>
<td>0.78</td>
<td>2.84</td>
</tr>
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</table>
Figure A.3.1: Mean PV(t) (Left) and CashR(t) (Right) of PEMPP*~ strategies, using $\sigma_{PE} = 0.1\sigma^*$

Figure A.3.2: Mean PER(t) of PEMPP*~ strategies, using $\sigma_{PE} = 0.1\sigma^*$
Figure A.3.3: Mean PV(t) (Left) and CashR(t) (Right) of PEMPP*~ strategies, using $\sigma_{PE} = 2\sigma^*$

Figure A.3.4: Mean PER(t) of PEMPP*~ strategies, using $\sigma_{PE} = 2\sigma^*$
Figure A.3.5: Mean Ulcer and Mean Return of PEMPP* and MPP* strategies, using $\sigma_{PE} = 0.1\sigma^*$ (top) and $\sigma_{PE} = 2\sigma^*$ (bottom).
## ETF Market

Table A.3.6: Performance of PEMPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 0.1 \sigma^*$.  

<table>
<thead>
<tr>
<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.7%</td>
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<td>1.63</td>
<td>5.36</td>
</tr>
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<td>14.0%</td>
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<td>12.64</td>
<td>0.24</td>
<td>2.95</td>
</tr>
<tr>
<td>0.8</td>
<td>14.4%</td>
<td>2.1%</td>
<td>3.4%</td>
<td>1.2%</td>
<td>11.76</td>
<td>0.32</td>
<td>2.56</td>
</tr>
<tr>
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<td>16.0%</td>
<td>4.9%</td>
<td>14.1%</td>
<td>5.5%</td>
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Table A.3.7: Performance of MPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 0.1 \sigma^*$.  

<table>
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<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
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<td>0.23</td>
<td>0.69</td>
<td>2.92</td>
</tr>
<tr>
<td>1</td>
<td>8.65%</td>
<td>2.69%</td>
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<td>3.4%</td>
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<td>3.37</td>
</tr>
<tr>
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<td>4.2%</td>
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<td>2.91</td>
</tr>
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<td>1.10</td>
<td>-0.29</td>
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<td>31.3%</td>
<td>13.6%</td>
<td>1.16</td>
<td>-0.55</td>
<td>15.75%</td>
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</table>

Table A.3.8: Performance of PEMPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 2 \sigma^*$.  

<table>
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<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
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<td>6.3%</td>
<td>2.9%</td>
<td>0.23</td>
<td>0.69</td>
<td>2.92</td>
</tr>
<tr>
<td>1</td>
<td>9.7%</td>
<td>3.5%</td>
<td>8.6%</td>
<td>3.2%</td>
<td>3.01</td>
<td>0.27</td>
<td>3.49</td>
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<td>4.5%</td>
<td>2.65</td>
<td>0.00</td>
<td>2.49</td>
</tr>
<tr>
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<td>12.2%</td>
<td>1.40</td>
<td>-0.62</td>
<td>3.22</td>
</tr>
<tr>
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<td>11.6%</td>
<td>1.54</td>
<td>-0.63</td>
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</tr>
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<td>27.6%</td>
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<td>-0.50</td>
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</table>

Table A.3.9: Performance of MPP*~ strategies condition on non-LI cases, using $\sigma_{PE} = 2 \sigma^*$.  

<table>
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<th>Theta</th>
<th>Average Return</th>
<th>Standard Deviation</th>
<th>Maximum Drawdown</th>
<th>Ulcer</th>
<th>UPI</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
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<td>2.5</td>
<td>0.68%</td>
<td>0.53%</td>
<td>6.3%</td>
<td>2.9%</td>
<td>0.23</td>
<td>0.69</td>
<td>2.92</td>
</tr>
<tr>
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<td>2.84%</td>
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<td>2.42</td>
<td>-0.33</td>
<td>3.05</td>
</tr>
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<td>-0.06</td>
<td>2.78</td>
</tr>
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<td>13.3%</td>
<td>1.14</td>
<td>-0.36</td>
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<td>-0.33</td>
<td>2.50</td>
</tr>
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<td>1.16</td>
<td>-0.57</td>
<td>2.82</td>
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</table>
Figure A.3.6: Mean PV(t) (Left) and CashR(t) (Right) of PEMPP*~ strategies, using $\sigma_{PE} = 0.1\sigma^*$

Figure A.3.7: Mean PER(t) of PEMPP*~ strategies, using $\sigma_{PE} = 0.1\sigma^*$
Figure A.3.8: Mean PV(t) (Left) and CashR(t) (Right) of PEMPP*~ strategies, using $\sigma_{PE} = 2\sigma^*$

Figure A.3.9: Mean PER(t) of PEMPP*~ strategies, using $\sigma_{PE} = 2\sigma^*$
Figure A.3.10: Mean Ulcer and Mean Return of PEMPP\textsuperscript{*} and MPP\textsuperscript{*} strategies, using $\sigma_{PE} = 0.1\sigma^*$ (top) and $\sigma_{PE} = 2\sigma^*$ (bottom).