INTEGRATED ASSET ALLOCATION

STRATEGIES: APPLICATION TO

INSTITUTIONAL INVESTORS

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Abstract

Investors with incomes from businesses need to make investment decisions in face of business decisions. Prominent examples include: sovereign wealth funds with state businesses, pension funds with sponsor companies, family offices with family businesses, households with salary incomes, etc. This paper establishes a general theoretical framework to analyze optimal investment decisions for these entities. The machinery for solving the integrated asset allocation problem is developed. And the theoretical proof is given. We also develop applications to oil-based sovereign wealth funds and family offices to illustrate the usage of the framework. Oil-based sovereign wealth funds (SWF) are set up by oil-producing countries. The SWF gets transfers from oil business incomes. Examples include Norway’s Government Pension Fund Global (GPFG) and Abu Dhabi Investment Authority (ADIA). Family offices are set up by wealthy families. The family office generally gets transfers from family business incomes. Applications to SWFs and family offices involve two elements: first, the asset allocation problem is modeled according to the general framework in a stochastic control setting; second, we invent a real option pricing technique to value all future claims on business incomes. The integrated optimal asset allocation is then solved with both elements. The superiority of integrated optimal decisions are demonstrated via these applications.
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To my parents.
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Chapter 1

Introduction

1.1 Asset Allocation Model

The last five decades have seen great progress on the asset allocation theory. Since the classical one-period Markowitz mean-variance portfolio optimization introduced by Markowitz (1952) in the 1950s, this research area has been flourishing. Harry Markowitz won the Nobel Prize in Economics in 1990 for his pioneering works. Markowitz (1952), Levy and Markowitz (1979) and Markowitz (1991) enrich his portfolio choice theory. Markowitz’s theory is viewed as the foundation of asset allocation models. And it lays the grounds for all modern finance theory. Here we give a simple version of Markowitz portfolio selection problem: the investor’s objectives include maximization of the expected one-period investment return and minimization the expected one-period portfolio return variance. Given a weighting parameter $\lambda$ for the two competing objectives, the portfolio choice problem is as follows:

$$\max_w r + w^T(R - r) - \lambda w^T\Sigma w \quad (1.1)$$
$w$ is the portfolio choice vector for investable risky assets (assuming there are N assets). $r$ is the risk-free asset return. $R$ is the expected one-period return vector of the N risky assets. $\Sigma$ is the expected covariance of asset returns.

The solution to this problem (1.1) is: $w^* = \frac{1}{\lambda} \Sigma^{-1} (R - r)$. $\Sigma^{-1} (R - r)$ is called the Markowitz portfolio.

In this simplest case, the investor’s utility function is the expectation of a quadratic function on asset returns. All forms of utility function, like isoelastic, exponential, hyperbolic, can be substituted.

Aside from being static and single-period, the solution of Markowitz portfolio choice theory is prone to errors of input parameters, namely expected mean and covariance of risky assets. However, the real market parameters are very challenging to accurately estimate. Another strand of research deals with this issue: how to use due judgment and optimization techniques to ameliorate the impact of estimation errors. Black and Litterman (1991, 1992) show a methodology of combining investor’s views with portfolio optimization with Bayesian methods. Jorion (1992) proposes a resampling method to construct the efficient frontier of optimal portfolios. Lauprete and etc. (2002), Fabozzi and etc. (2007) and Kim and etc. (2015) advance robust optimization methodologies applied to asset allocation problems.

To illustrate the essential role of asset allocation in investment management, researches have been done to show empirically and theoretically that asset allocation decision accounts for more than 90 percent of portfolio performance. Examples include Hensel and etc. (1991), Grinold and Easton (1998), Ibbotson and Kaplan (2000), Ibbotson (2010) and Lee and Kim (2016).

Pure investment decisions on financial assets do not incorporate all of real-world investors’ needs. So researchers and practitioners extend the theory of asset allocation to integrate investor-specific decisions with investment management decisions. There is abundance of literature in this area, with certain categories of investors researched on more than others. Some papers set out to study asset allocation for personal investors. Merton (1969) incorporates both personal investors’ lifetime consumption decisions and portfolio selection decisions in the model. Campell and Viceira (2001) integrates the investor’s human capital and life-cycle spending with long-term portfolio choices. Dammon and etc. (2004) considers both taxable and tax-deferred investing for personal investors. Berger and Mulvey (1998) includes
household assets (savings, house, salary, etc.) and household liabilities (spending, mortgage, etc.) altogether and builds an integrated asset and liability management system for individual investors with tax considerations. For investment management of pension plan, there is numerous literature as well. Harrison and Sharpe (1983) discusses optimal funding and asset allocation rules for pension plans. Sundaresan and Zapatero (1997) gives a model for valuation and optimal asset allocation of pension plans. Cairns, Blake and Dowd (2006) derives optimal dynamic asset allocation for defined contribution plans, with human capital and life-cycle styles taken into account. Ang, Chen and Sundaresan (2013) analyzes the optionality effect in asset allocation problem arising from the pension plan’s incentive to minimize downside risk of covering its liabilities. Mulvey (1996), Mulvey and etc. (1998), Mulvey and etc. (2000), Mulvey, Simsek and Zhang (2008) build a holistic system for asset allocation decisions and make a lot of impact in the pension plan industry. The system built on their papers has been used by Towers Watson to generate optimal asset allocation decisions for consulting pension plans around the globe. Asset allocation models and systems have been built for insurance companies too. Mulvey and Ziemba (1998), Mulvey and Shetty (2004) show how to use multi-period asset allocation models to build integrated risk management systems for insurance companies. Carino and Ziemba (1998) and Carino, Myers and Ziemba (1998) show how to formulate the financial planning system for Yasuda Kasai insurance company for generating optimal asset allocation decisions. There are only a few relevant papers for university endowments. Merton (1993) models university’s financial liabilities related to university activities and builds a model to solve the optimal asset allocation for supporting the university’s financial objectives. Asset allocation models for sovereign wealth funds are also rare. Since SWF is one of the central topics of this paper, we summarize the relevant literature in this sub-field in a separate section. For family offices, as far as
we know, there is no relevant paper yet to build an asset allocation model for them. Family offices are another important subject of this paper.

The papers listed above only represents a droplet in the sea of voluminous literature of asset allocation models. Theoretical researches generally formulate the decision problem in a mathematical framework and draw critical relationships to provide insights. On the other hand, researches of real-world applications have been building implementable systems and making industry impacts. The financial market has been evolving and getting increasingly sophisticated. There are more and more investible assets to increase the degree of market completeness. At the same time, more advanced methodologies, like machine learning, advanced financial econometrics and optimization are being developed to upgrade investment management practices. Moreover, the astonishing development of information system, big data analytics, infrastructures makes it easier to collect relevant data for both investment and investor-specific decisions. Thus, to meet the needs of better investment management in an integrated fashion, we set out to develop a framework to incorporate both asset allocation decisions and investor-specific decisions. The ambition of this paper is to develop a general framework that can be applied to as many types of investors as possible, with a mathematically solid machinery to solve the integrated asset allocation problems. We want to advance the asset allocation theory to incorporate investor-specific decisions. This integrated asset allocation framework can provide guidelines for a specific investor to make integrated optimal decisions, just as modern portfolio theory has done for investors to make optimal investment decisions for merely financial assets.

For institutional investors with huge amount of assets to manage, complicated liabilities to bear and competing goals to fulfill, how to accordingly manage asset allocation is a critical question. For personal investors with less assets to manage,
how to manage asset allocation according to risk tolerance, earnings, family structure and other personal characteristics is also a very important problem: good decisions can make big differences in their living standards. In the portfolio choice theory, the decision-maker is only concerned with investment decisions of financial assets. In our extended framework, the decision-maker needs to consider investor-specific conditions in addition to investment decisions. We can separate the two sets of decisions into two entities: one entity for investment management and the other one for investor-specific decisions. The investor-specific entity makes cash flow transfers to and from the investment entities, hence relevant to be included in the model. Here are some examples: sovereign wealth funds with state businesses, pension plans with sponsoring companies, foundations with sponsoring companies, family offices with family businesses, the former as investment entity and latter as investor-specific entity. Personal investors with salary incomes can be viewed as similar to family offices with businesses. Because all the investor-specific entities are income-generating businesses, we name the generic investor-specific entity "business entity". In a way, university endowments and universities can also be thought to have the same structure, even though we hardly think university as a "business" per se. We then name the dual structure of investment entities with business entities as the I-B Complex, short for Investment-Business Complex. Typically, these entities need to make integrated investment decisions with "business" decisions taken into account. In our framework, the scope of investor-specific entities only include those that generate incomes. The incomes may be negative though, since businesses can also take funds from investment entities sometimes. Spending are not considered to be in the scope of investor-specific entities in our framework. For example, central banks and governments can spend some funds in sovereign wealth funds. Families can spend the money in family offices. We believe spendings should be incorporated in the objectives. In some cases, the income-generating, spending and investing may
be from a single natural entity, for example, the case of personal investors. In such cases, one can think of separate imagery business, investing and spending roles that are fulfilled by the same natural entity.

In this paper, we firstly build a general framework of integrated decision-making for these entities. To accomplish this goal, we use stochastic control to set the stage for the problem. We set framework to be abstract enough for characterizing any specific decision-making problem for any entity in the category. The general framework provides generic insights and serves as a tool to analyze different problems related to asset allocation decisions. When it comes to modeling a specific entity, a specific investor needs specific formulations, which is a case-by-case study. To illustrate the application of our theory to a realistic problem, we develop two applications: one to oil-based sovereign wealth funds and another to family offices. In the next section, we give a brief introduction about sovereign wealth funds and the asset allocation problem they face.

1.2 Sovereign Wealth Funds

Sovereign wealth funds are set up by state authorities to pool specified funds together for investment management. Al-Hassan and etc. (2013) categorizes SWFs into five types: stabilization funds, savings funds, development funds, pension reserve funds and reserve investment corporations. This IMF paper defines the scope and roles of different SWFs comprehensively and clearly. Stabilization funds are set up to insulate government budget and regional economy from commodity price volatility and external shocks (e.g., Chile (Economic and Social Stability Fund), Timor-Leste, Iran and Russian (Oil Stabilization Fund)). The investment horizon
and liquidity needs are similar to central bank reserve managers, as their role in countercyclical fiscal policies to smooth economy cycles. As a result, they invest largely in liquid assets (over 80 percent in fixed income and instruments negatively correlated with the source of risks like commodity price risks. Savings fund’s objective is to share wealth across generations by transforming nonrenewable assets into diversified financial assets (Abu Dhabi Investment Authority, Libya, Russia (National Wealth Fund)). Their investment horizon is very long and their investment mandate emphasizes high risk-return profile. So allocating high portfolio percentage to equities and alternative investments (over 70 percent) are normal for savings funds. Development funds are intended to support priority socio-economic projects, usually infrastructure construction (e.g. UAE (Mubadala) and Iran (National Development Fund)). Pension reserve funds are established to meet identified future cash outflows of pension-related contingent-type liabilities on the government balance sheet (e.g. Australia, Ireland and New Zealand). They typically allocate large shares to equities and alternative assets to meet rising pension payouts. Reserve investment corporations are established to reduce negative carry costs of holding cash reserves or to earn higher return on enormous assets. The funds within them are still counted as reserves (e.g. China, South Korea, and Singapore). They normally allocate large shares in equities and alternatives (up to 50 percent for South Korea and 75 percent for Singapore’s GIC).

The objectives of SWFs depend on country-specific circumstances and evolve over time. Many resource-based funds have multiple objectives. For Azerbaijan, Botswana, Triniad & Tobago, and also our case study – Norway, they have both objectives of stabilization and savings. Australia’s SWF has objectives of saving and pension reserve. For Kazakhstan, the objectives are stabilization, saving and
Al-Hassan and etc. (2013) also identifies three types of institutional frameworks for SWFs: 1. central bank manages the assets under a mandate given by the ministry of finance (e.g. Norwegian Government Pension Fund Global, Botswana and Chile); 2. a separate fund management entity, owned by the government, is set up to manage assets under a mandate given by the ministry of finance, like Government Investment Corporation of Singapore; 3. the ministry of finance gives mandates directly to one or more external private fund managers. Which form of institutional framework the government chooses depends on country-specific conditions. Though, the investment management process should be the same. The owner of the SWF (usually the ministry of finance) is responsible for setting the investment mandates. The investment mandate should clarify the objective of the SWF and the risk-bearing capacity of the SWF. Based on the investment mandate, combinations of candidate asset classes can be established. The combinations are typically expressed as an SAA, short for Strategic Asset Allocation, which sets a target allocation to asset classes within the investment universe. Then the investment management process can be implemented according to the SAA. Active management of tactical tilts and active investment within each asset class can be done within a pre-specified risk budget.

The objectives of sovereign wealth funds can only be achieved if the SWFs are managed within a sound governance structure with appropriate regulatory policies. The International Working Group of Sovereign Wealth Funds, which has members of 23 countries, permanent observers of five countries and organizations including the OECD and the World Bank, published the General Accepted Principles and Practices (GAPP) for SWFs, the so-called "Santiago Principles". Ang (2010) summarizes the principles of SWFs into four benchmarks: Legitimacy – it requires
SWFs’ capital to be not immediately spent because the essential purpose of SWFs
is to transfer wealth from the present to future; Integrated Policy and Liabilities –
it requires SWFs to take into count the broader environment of natural resource
wealth, taxation policy, government budgeting, contingent-type liabilities, develop-
ment policy and etc. in the process of formulating investment policies; Performance
– requirements need to be set forth for managers to track or beat a well-defined
financial index, after the first two principles are met; Endurance – SWFs must have
long-run perspectives, for example, having well-functioning capital markets, free
flow of capital across countries, good corporate government and preservation and
enhancement of shareholder rights. This last principle is considered secondary to the
first three.

according to these principles, especially in regards to performance benchmark
and governance structure, since Norway has already done very well in the first two
principles. These two papers also propose a factor-based approach to enhance the
performance benchmarking and investment management process of Norway’s SWF.

According to Sovereign Fund Institute, there are more than 7 trillion dollars
managed by SWFs (see figure 1.1) by September, 2015. SWFs that are funded by
oil and gas exports totalled 4.29 trillion dollars by the end of 2014, which accounts
for about 60 percent of the AUM of SWFs. The two largest SWFs, Norway’s GPFG
($873 billion) and United Arab Emirates’ ADIA ($773 billion) are all oil-based
SWFs. Between the years of 2003 to 2013, SWFs’ assets have skyrocketed because of
rising commodity prices, mainly driven by oil/gas prices (see figure 1.2). Oil-based
sovereign wealth funds get their fundings from state oil-producing businesses and
taxation of non-state oil companies. The charters of Norway’s Government Pension
Fund Global (GPFG) and United Arab Emirates’ Abu Dhabi Investment Authority (ADIA) state their goals. Essentially, their objective is to transform oil wealth into diversified portfolios, and to sustain the wealth of their countries by investing in financial markets. According to Norges Bank (Norway’s central bank), “the Government Pension Fund Global is saving for future generations in Norway. One day the oil will run out, but the return on the fund will continue to benefit the Norwegian population”. The budgetary rule for the fund spending is 4 percent annually. “This helps phase oil revenue into the economy gradually, and spending just the return on the fund rather than eating into its capital means that fund will also benefit future generations.”

Regarding the spending rule of resource-based SWFs, Leigh and Olters (2006) does a good study on the case of Gabon, an oil-rich country that set up an institution to manage their oil wealth. The paper develops a model to calculate the sustainable spending rule and also takes into account habit formation issues. SWFs should have solid analysis of sustainable spending rate and not be mislead by the illusion that binding budgetary constraints disappear when oil revenues are large. Also, there is the well-known “Dutch disease” discussed by Corden and Neary (1982): generous endowment of natural resources lead to foreign currency inflows and domestic currency appreciation, resulting in the country’s other industry less price competitive and eventually lagging behind. The term is coined because of the Dutch economic crisis of the 1960s following the discovery of North Sea natural gas. However, spending rules are not the focus of our study. One can refer to relevant papers listed above for thorough discussion. But one can definitely use our framework to analyze the effects of different spending policies on optimal asset allocation and perform wealth growth simulations.
It is clear that good investment management of SWFs is essential to a lot of resource-based countries. In 2014, the petroleum and natural gas sector accounted for 45 percent of Norway’s export revenues and more than 20 percent of GDP ( $ 0.5 trillion in 2013). The energy industry employs about one third of the country’s labor force. The largest energy company, state-owned Statoil ASA, controls 70 percent of Norway’s oil and natural gas production. According to statistics published by the Ministry of Finance, the government transfers about 60 percent of annual oil incomes to the SWF. Norway has a very high standard of living compared to other European counties, ranking the fifth in GDP ( PPP ) per capital ( $ 68430 in 2015 ), and a strongly integrated welfare system. Norway’s modern manufacturing and welfare system rely hugely on the exploitation and investment management of natural resource wealth.

Managing oil-based SWFs is a very challenging task, due to several reasons. The investment horizon of SWFs can be seen as infinite. Oil-based SWFs need to navigate through boom/bust cycles and extreme events like 08 crisis to achieve the long-term financial goals within budgetary policies. Also, the SWFs must take into account the risks of commodity price volatility and the stabilization responsibilities of shielding away these negative effects. The recent collapse of oil prices in 2014 and 2015 hit some oil-producing countries and their SWFs badly. With oil prices plunging, the economic growth is stalled. Stabilization purposes need to fulfilled by SWFs then: funds are withdrew from SWFs to ease temporary fiscal hardship and stabilize foreign reserves. Norwegian government announced its first ever capital withdrawals from its SWF in 2016, which may amount to $ 10 billion dollars. Saudi Arabia’s foreign reserves fell more than $ 100 billion from mid-2015 to early 2016. The head of SWF in Kazakhstan said the veichle would run out of money in seven years as oil price drop cuts its revenue. Algeria burned through all the reserves it
took nearly four years to accumulate in just one year. The sheer amount of SWFs’ assets worsen the situation. JP Morgan estimates SWFs may sell $ 700 billion of European stocks in 2016. A withdrawal of assets by SWFs against the background of liquidity concerns may lead to large price movements. The liquidity issues erode the value of SWFs even more. Thus, hedging the risks of oil prices is an essential task in managing oil-based SWFs through a very long time horizon of economy cycles. An integrated asset allocation model needs to be developed to incorporate risks from oil reserves into the long-term investment management process. Oil-based SWFs should view financial assets and oil reserves as a whole portfolio in the process of investment management, as opposed to treating them separately. Only focusing on financial assets will inevitably generate suboptimal investment decisions.

An important question to ask is whether oil-based SWFs should invest in the oil industry. From the hedging perspective, the SWFs should underweight oil industry and invest in asset negatively correlated with energy sector. Investing in oil industry may cause serious issues when oil prices plummet. For example, in October 2015, the Kazakhstan SWF "borrowed $ 1.5 billion in its first syndicated loan to help a cash-strapped subsidiary saddled with a troubled oil-field investment", according to a WSJ article. On the other hand, investing in oil extraction technology may lower the cost of production and increase the industry efficiency, which in turn increases the value exploited from oil reserves and thus the long-term value of oil-based SWF. In addition, exploration and development of new oil reserves will increase the total oil wealth level. We will analyze both of the effects in our applications.

The integrated asset allocation problem of SWFs has been discussed in a few papers. But hardly is there any literature that builds a comprehensive model to deeply analyze and solve the problem. Gintschel and Scherer (2008) and Scherer...
(2009a, 2009b) include oil reserves as financial assets into the financial portfolio and then optimize the whole portfolio in a Markowitz’s mean-variance optimization fashion. However, Markowitz’s MVO is only good for investors with single period horizon under static market. It is not sufficient to apply to SWFs, where time horizon is very long and market conditions are dynamic. Moreover, the value of oil reserves is not simply oil price times oil reserve amount. The oil reserve cannot be all produced at once. Also, there are a lot of complex features like optionality of waiting, nonlinearity in costs, minimum production constraints and minimal producable levels. The oil production process greatly affects the wealth accumulation and investment process throughout the course. So oil production and oil reserve value function need to be modeled in details. Bodie and Briere (2012, 2013) uses a similar approach of including oil reserve assets into the financial portfolio. They identify and explicitly model the assets and liabilities on a SWF’s balance sheet and use a CRRA utility function for objective function. However, the approach still lacks multi-period formulation and precise valuation of oil reserves, which are the two most important features for oil-based SWFs. Martellini and Milhau (2010) models the integrated asset allocation problem in a multi-period dynamic setting and derives generic optimal allocation solutions. But they do not provide an approach to value the oil reserves explicitly nor the linkage to asset allocation.

In this paper, we develop a general framework for all I-B complexes and apply it to oil-based SWFs. We use the abstract solutions from the stochastic control framework as guidance for applications. To develop an application for oil-based sovereign wealth fund, we need at least one of the two elements: assumptions on oil production decisions or a valuation methodology to value oil reserves. In the first approach, we impose assumptions on production policies of the oil business. The imposition specifies income flows from the oil business to the SWF. For example, we
can assume constant oil production rate. Since the variation of oil production for
countries like Norway is rather small compared to the total production level, this
assumption is not unrealistic. In the second approach, we can model the state oil
business explicitly. Real option theory is a nice choice for modeling the business
entity in an I-B complex. It can capture the essential features of owning a business:
various types of decisions in a business, just like execution decisions in options.
Real options can be used to approximate the decision-contingent payoffs from state
business. Thus, the portfolio of the SWF can be modeled as financial assets with
contingency claims of real options. The asset allocation problem is then a portfolio
optimization problem with options. We use both approaches in this paper to model
the integrated asset allocation problem faced by oil-based SWFs.

Family business is another example of application of the general framework.
Similar approaches can be used to model the asset allocation problem of family
offices. There is literally no literature of the asset allocation problem faced by family
offices, to our knowledge. The lack of research for these two types of investors is the
reason why we select them for application.

The generic integrated decision-making framework for I-B complexes incorporates
both investment and business decisions. Under this generic setting, investment
decisions and business decisions affect each other. The entanglement of investment
and business decisions makes it very difficult, if not impossible, to solve the inte-
grated decision-making problem. Moreover, allowing both decisions to affect each
other is not the case of I-B complexes. First of all, there is very few, if not none,
organizations that can decide both decisions without conflict of interests. Let it be
a sovereign wealth fund with a state business, a pension plan with a sponsoring
company or university with an endowment. There is always separated governance
structures operating independently for the investment entity and the business entity. One can easily imagine there will be insolvable conflicts of interests and difficulties of incentive design, if the two governance structures are merged into one. Second of all, it is unwise to tie business decisions to investment decisions or the other way around. For investment decisions to be optimized, parameters of market dynamics need to be estimated. As commonly known, market parameters (like expectation and covariances of asset returns) are very difficult to estimate accurately. Thus, entangling business decisions with investment decisions allows the business decision-making process to be affected by the errors in estimated market parameters. Investment management industry has been dealing with market parameter errors for a long time. A lot of research and practices have been done regarding the issue. One can refer the discussion of papers addressing this issue in the introduction of asset allocation models, such as Black-Litterman model and robust optimization techniques. However, there is little available knowledge about how to deal with errors from market parameters in the business decision-making process. Plus, every business type has a unique decision-making process, unlike the rather standard setup of maximizing return and minimizing risks in the investment decision-making process. In conclusion, entangling business and investment decisions together is neither advisable nor realistic. To address this issue, some assumptions should be imposed upon the generic I-B complex framework in order to model real-world I-B complexes. We introduce in Chapter 4 the Disentanglement Conditions. We also prove the Separation Theorem: if I-B complexes and market conditions satisfy the Disentanglement Conditions, business decisions can be separated from and then linked to investment decisions. One can think of the Disentanglement Conditions as a set of regularity conditions that confine the generic framework to mostly resembling real-world integrated asset allocation problems faced by I-B complexes.
Equipped with the generic framework and regularity conditions of Distentanglement, entities with the similar I-B complex structure can be modeled and integrated asset allocation problems solved for. To do this, one needs to make efforts in identifying and modeling state variables for the business incomes. The level of modeling difficulty depends upon the complexity of investor-specific circumstances. There are two approaches to model the income flows from business entities. In the first approach, a fixed income flow policy can be assumed. One can think of this policy as the reduced form of already optimized business incomes. In the second approach, one need to model business entities explicitly and solve for the optimized business income. In this paper, we utilize the theory of real option pricing to model the state business of oil-producing. To our knowledge, this is the first paper that introduce real option theory into asset allocation model. We argue the technique has the potential for applications to other entities. After business incomes are optimized, the investment entity can take these incomes as given and solve for the optimal asset allocation decisions. The Separation Theorem guarantees the solved investment decisions are still optimal for the integrated asset allocation problem.

The following sections are organized as such: in Chapter 2, we build the general framework of asset allocation for I-B Complexes and explain components of the abstract framework. In Chapter 3, we apply the general framework to an oil-based sovereign wealth fund and specify elements in the framework. In Chapter 4, we solve the integrated optimal asset allocation problem for an oil-based sovereign wealth fund by the approach of assuming a fixed income policy. We take Norway’s Government Pension Fund Global as an example for case study. In Chapter 5, we show how to model oil prices and use real option theory to value the business entity – oil-producing state businesses. In Chapter 6, we link the results from real option techniques with the asset allocation problem and solve the integrated optimal
asset allocation problem. In Chapter 7, we apply the framework to a hypothetical U.S.-based family office. In Chapter 8, we conclude and make suggestions for future work.
Chapter 2

General Framework for Integrated Asset Allocation

I-B Complexes face decisions from both investment side and business side. The investment entity and business entity both have their own goals and constraints. For each of the entities, we use a stochastic control problem to model the decision-making problem. The linkage between the two stochastic control model is the income transfer from the business entity to the investment entity. There is abundance of literature that focuses on the asset allocation problem for investment management. But little previous research has been done to model the decision-making problem of the supporting business. We will devote the first part of this section to formulating the decision-making problem of the business entity. We denote $K$ as state variables for the business entity. $K$ can be a single variable or a vector variable. $K$ characterizes the state of business entities. If the business entity is a company, the state variables may be the financial capital, human capital, balance sheet variables (asset, liability, equity), technology state and etc. of the company. For an oil-producing business, oil price and the state of oil reserve are the prevailing state variables. There are two sets of decisions that affect the business state: $u$ – business decisions and $v$ –
transfer decisions from the business entity to the investment entity. For a company, $u$ may include: production, sales, operations, expansion or downsizing, purchasing or selling, R&D, merger and acquisition, IPO and bankruptcy, etc. The abstract business decisions can represent a lot of types of decisions. If the business entity is the oil-producing business, the oil production decisions are the main business decisions. $v$ represents the transfer of income from the business entity to the investment entity. Controlled by these two sets of decisions, the business state evolves according to the dynamics below:

$$dK_t^u = b(K_t^u, u_t, v_t, t)dt + \sum_{j=1}^{n} \sigma^K_j(t, K_t^u, u_t, v_t)dB^K_{j,t} \quad (2.1)$$

Where $t$ is time; the growth rate function $b(t, K_t^u, u_t, v_t)$ depends on state variables $K$ and decision variables $(u, v)$. The stochastic element represents the various risk factors affecting the business entity. The volatility function $\sigma^K_j(t, K_t^u, u_t)$ is the $j$th risk factor and depends on state variable $K$ and decision variables $u$. Due to the nature of transfer decision $v$, the volatility factors generally do not depend on it. $(B^K_{j,t})_j$ is standard multi-dimensional Brownian motion.

We intentionally make our model abstract enough so that it can fit various types of business entities. The goal of the business is to optimize a certain utility function of profits. We denote the instant profit function of the on-going business to be $p(K, u, v, t)$. And we denote the final payoff of the business to be $Z(K, T)$. The on-going profit function is viewed as the discounted profit flow at each instant. The final payoff is viewed as the cash flow when the business stops to be existent, like going into default.

The stochastic control problem is as follows:
Business Objective Function:

\[ J^1[u, v] = E[\int_{0}^{\tau_{u,v}} p(K_{s}^{u,v}, u_s, v_s, s)ds + Z(K_{\tau_{u,v}}^{u,v}, \tau_{u,v})] \quad (2.2) \]

\( \tau_{u,v} \) is a stopping time. \( \tau \) can be a fixed investment horizon \( T \). \( \tau \) can also be infinite: for certain entities, like endowments, the investment horizon is so long that it can be viewed as infinite; and the corresponding business entity can also be viewed to be everlasting. \( \tau \) can also be indefinite: a stopping time that depends on decisions. For example, it can be the uncertain default or exit time for a company, which is subject to business and transfer decisions.

Next, we formulate the asset allocation problem for the investment entity. The state variable for an investment entity is obviously the wealth level \( W \). The decision variables are asset allocation decisions at each instant. Suppose there are \( m \) investable assets on the market. The weight of portfolio on the \( j \) th asset at time \( t \) is \( w_{j,t} \). The expected return and volatility of \( j \) th asset are denoted as \( r_j \) and \( \sigma_j \). The correlation between \( i \) th and \( j \) th asset is \( \rho_{ij} \). We consider constant return and covariance parameters in this paper. But it is easy to extend the framework to incorporate time-varying market dynamics. One merely needs to include more state variables of market and dynamics equations. The other decision variable for the investment entity is consumption or spending decisions. We denote the spending decision at \( t \) as \( q_t \). The dynamics equation of wealth is:

\[ dW_t^{w,q} = \sum_{j=1}^{m} w_j r_j W_t^{w,q} dt + (1 - \sum_{j=1}^{m}) r W_t^{w,q} dt + w_j \sigma_j W_t^{w,q} dB_{j,t}^W + v_t dt - q_t dt \quad (2.3) \]
The investment goal can be generalized as maximizing a certain utility of on-going consumption plus the payoff utility of wealth level at the endpoint. We denote the time-discounted utility function of on-going consumption at time $t$ to be $U(q, W, K, t)$. It is generalized as a function of consumption $q$, business state $K$, wealth state $W$ and time $t$. The investment objective function is:

$$J^2[w, q] = \mathbb{E}\left[\int_0^{\tau_{u,v}} U(q_s, W_s, K_s, s) ds + Y(W_{\tau_{u,v}}, \tau_{u,v})\right] \quad (2.4)$$

$J^2[w, q]$ is the objective function of the investment entity that depends on decisions $(w, q)$. The final horizon $\tau_{u,v}$ is the same as the horizon for the business entity, since the I-B Complex should have one consistent time horizon. The first term in the objective function is the integral of time-discounted utility. The scope of this term is broader than merely represents consumptions as in a lot of literature. First, it can represent the utility from on-going consumption. Moreover, it can also characterize any path-dependent objectives of the investment entity. For example, in the case of a family office, the investor is concerned with drawdowns and wants the wealth level to be above a threshold level as much as possible to sustain the lifestyle. Then, $U$ can quantify this requirement using a certain functional form: deeply negative for values below the lifestyle threshold and mildly positive for above the threshold. Another example is pension plan: the funding ratio of pension fund needs to be above a certain threshold. $U$ can quantify this utility by putting large penalty on wealth levels below a certain threshold percentage of liabilities.

The second part of the objective function is the utility of wealth at the endpoint: $Y(W, \tau)$. 


Now that we have all the building blocks from business and investment entities, we can formulate the integrated optimization problem for the I-B complex. We characterize the objective of an I-B complex as optimizing the weighted sum of (2.2) and (2.4).

**Overall Objective Function:**

\[
J[u,v,w,q] = \lambda J^1[u,v] + (1 - \lambda) J^2[w,q] = \mathbb{E}[\int_0^T \lambda p(K_{u,v}^t, u,v,s)ds + \lambda Z(K_{u,v}^{\tau_{u,v}}, \tau_{u,v}) + (1 - \lambda) Y(W_{w,q}^{\tau_{u,v}}, \tau_{u,v})]
\]  

(2.5)

The overall objective function \(J[u,v,w,q]\) depends on decision variables \((u,v,w,q)\). The value of weighting parameter \(\lambda\) depends on two aspects: the level of control of business entity v.s. investment entity by the decision-maker; the trade-off between business and investment entities. If the decision-maker has control over only investment entity, then \(\lambda\) is 1. However, it does not mean business decisions shall not to be optimized in this case. The first part of the expectation is integral of on-going utility: \(\lambda p(K_{u,v}^t, u,v,s) + (1 - \lambda) U(q_s, W_s, s)\). The second part is the weighted sum of two final payoffs: \(\lambda Z(K_{u,v}^{\tau_{u,v}}, \tau_{u,v}) + (1 - \lambda) Y(W_{w,q}^{\tau_{u,v}}, \tau_{u,v})\). To be more general, we set the endpoint payoff as \(R(K_{u,v}^{\tau_{u,v}}, W_{w,q}^{\tau_{u,v}}, \tau_{u,v})\) instead of a weighted sum.

Thus, the whole optimization problem can be written as:

\[
\max_{u,v,w,q} J[u,v,w,q] = \mathbb{E}_0[\int_0^T \lambda p(K_{u,v}^t, u,v,t)dt + (1 - \lambda) U(q_t, W_t, t)dt + R(K_{u,v}^{\tau_{u,v}}, W_{w,q}^{\tau_{u,v}}, \tau_{u,v})]
\]  

(2.6)

\[dK_t^u = b(K_t^u, u_t, v_t, t)dt + \sum_{j=1}^n \sigma^K_j(t, K_t^u, u_t)dB_{j,t}^K\]
\[
dW_t^{w,q} = \sum_{j=1}^{m} w_j r_j W_t^{w,q} dt + (1 - \sum_{j=1}^{m}) r W_t^{w,q} dt + w_j \sigma_j W_t^{w,q} dB_{j,t}^W + v_t dt - q_t dt
\]

To solve this stochastic control problem, we can use the verification theorem for indefinite horizon optimal control:

The indefinite time stochastic control problem is stated as follows. Define a probability space with filtration and measure \((\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbb{P})\) and a progressively adapted M-dimensional \(\mathcal{F}_t\)-Brownian motion \(B_t\). The correlation between \(B_p\) and \(B_q\) is \(\rho_{p,q}\). The \(N\)-dimensional state variable is \(X\), and decision variable \(\alpha\). The domain for \(X\) is \(S\) and the domain for \(\alpha\) is \(U\). The controlled dynamics of state variable are characterized as stochastic differential equations with controls:

\[
dX_t^\alpha = b(t, X_t^\alpha, \alpha_t) dt + \sigma(t, X_t^\alpha, \alpha_t) dB_t, \quad X_0 = x \quad (2.7)
\]

Here \(b\) and \(\sigma\) are functions \(b : [0, \infty] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^N\) and \(\sigma : [0, \infty] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^N \times M\).

The objective function is:

\[
J[\alpha] = E\left[\int_0^{\tau^\alpha} w(X_s^\alpha, \alpha_s) ds + R(X_{\tau^\alpha})\right] \quad (2.8)
\]

where \(w : S \times U \rightarrow \mathbb{R}\) and \(R : \partial S \rightarrow \mathbb{R}\) are measurable functions and the stopping time \(\tau^\alpha\) is the first exit time of \(X_t^\alpha\) from \(S \subset \mathbb{R}^n\). The boundary of \(S\) is \(\partial S\).

The verification theorem (Theorem 2.1) for this problem is as follows:
Assume that $S$ has a compact closure $\overline{S}$ and $X_0 \in S$ a.s. Suppose there is a function $V : \overline{S} \to \mathbb{R}$ that is $C^2$ on $\overline{S}$ and satisfies:

Define the operator $\Omega_t^\alpha V(X,t) := \sum_{i=1}^N b^i(t, X, \alpha) \frac{\partial V}{\partial X_i} + \frac{1}{2} \sum_{i,j} \sum_{p,q} \sigma^i \sigma^j \rho_{pq} \frac{\partial^2 V}{\partial X_i \partial X_j}$.

$$\max_{\alpha \in U} \Omega_t^\alpha V(x,t) + w(X,\alpha,t) = 0, \quad X \in S \quad (2.9) \text{ Bellman equation}$$

$$V(X,\tau) = R(X) \quad (2.10) \text{ Boundary condition}$$

Find the maximum (which we have implicitly assumed to exist):$$\alpha^* \in \text{argmax}_{\alpha \in U} \Omega_t^\alpha V(x,t) + w(X,\alpha,t).$$

Denote by $\mathcal{R}$ the class of admissible strategies such that:

$$\mathcal{R} := \{ \alpha \in U | E[\sum_{i=1}^n \sum_{k=1}^m \int_0^{\tau_\alpha} \frac{\partial V}{\partial X^\alpha_s}(X^\alpha_s, s)\sigma^{ik}(X^\alpha_s, \alpha_s)dB^k_s] = 0 \}$$

If $\alpha_t^*$ is an admissible Markovian strategy ($\alpha_t^* = \alpha^*(X_\tau^\alpha)$, the strategy is a function of only current state regardless of previous path) and belongs in $\mathcal{R}$, then $J[\alpha^*] \leq J[\alpha]$ for any $u \in \mathcal{R}$, and the optimal objective can be expressed as $J[\alpha^*] = E(V(X_0))$.

Then we can apply this verification theorem to our problem (2.6). First, we need to generalize the state, control variables and dynamics to be the same format as in the verification theorem (Theorem2.1):

**State variables:** $X := (K, W)$. The domain $S$ for state variable $X$ is the set of all possible states for $(K, W)$: $S := \{(K,W)|K \text{ and } W \text{ are achievable under initial values and market with non-zero probability}\}$. As mentioned before, we now only consider constant market parameters. Time-varying market conditions are not among the main topics in this paper. Though, it is easy to extend to time-varying market conditions. We
just need to include market state \( \mathcal{M} \) in state variables: \( X := (K, W, \mathcal{M}) \).

**Control variables**: \( \alpha := (u, v, w, q) \). Control variables include business decisions \( u \), transfer decisions \( v \), investment decisions \( w \) and spending decisions \( q \). Decision variables are denoted as \( \alpha \). The domain for \( \alpha \) is \( U := \{(u, v, w, q)| \text{all possible joint decisions of } (u, v, w, q) \text{ satisfying regularization conditions } \mathcal{R}\} \).

We also need to generalize the dynamics specifications for state variables:

**Drift of \( X = (K, W) \):**

The drift of \( K \) is \( b(K^u_t, u_t, v_t, t) \). The drift of \( W \) is \( (1 - \sum_{j=1}^{m} rW^{w,q}_t + v_t dt - q_t) \). We denote the generalized drift parameter to be:

\[
\mathcal{B} := \{b^i(t, X, \alpha)\}_{i=1,...,N} = (b(K^u_t, u_t, v_t, t), \sum_{j=1}^{m} w_j r_j W^{w,q}_t dt + (1 - \sum_{j=1}^{m} rW^{w,q}_t + v_t dt - q_t).
\]

In this case, \( N = \text{number of business state variables} + 1 \).

**Covariance structure of \( X = (K, W) \):**

Denote \( \{B_{l,t}\}_{l=1,...,\mathcal{M}} := \{B^K_{l,t}, B^W_{l,t}\} \). The Brownian motions in dynamics of business and investable assets are re-ordered as a single set of joint Brownian motions. The total dimension of Brownian motions in the model is denoted as \( \mathcal{M} \).

Denote \( \{\sigma^i\}_{l=1,...,\mathcal{M}} := \{\sigma^{K,i}, \sigma^{W,j}\} \). The volatility parameters in the dynamic equations are also re-ordered and denoted as a single set of volatility parameters corresponding to respective \( B_{l,t} \).

The correlation between \( B_p \) and \( B_q \) is denoted as \( \rho_{pq} \).

Next we apply the verification theorem. In the remainder of this section, we derive results from applying the theorem in a generic and heuristic way. These derivations
provide insights into our integrated decision-making problem, and give us guidelines for applications. We view the business state variable $K$ as a generic state variable for business. Even though $K$ is a vector which may consist of many state variables as mentioned before: financial capital, human capital, technology, etc., we view $K$ as a single variable that represents the state of business for simplicity. Without loss of generality, this simplification makes our analyses more straightforward. We impose the similar abstraction on business decision variable $u$. Even though $u$ may be multi-dimensional, we only view it as one-dimensional variable in the following derivations. It is easy to extend to the case of multi-dimensional state variables, and the conclusions are essentially the same. After all the specifications above, the Bellman equation (2.9) is as follows:

$$\max_{(u,v,w,q) \in U} \lambda p(t, u, v, K) + (1-\lambda)U(q, W, t) + \frac{\partial V}{\partial K}b(t, K, u, v) + \frac{\partial V}{\partial W} \left[ \sum_{j=1}^{m} w_j(r_j-r) + r \right] W + v - q$$

$$+ \frac{1}{2} \frac{\partial^2 V}{\partial K^2} \sum_{j=1}^{n} (\sigma_j^K(t, K, u))^2 + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sum_{i,j} \sigma_i^W w_i w_j + \frac{1}{2} \frac{\partial^2 V}{\partial K \partial W} \sum_{i,j} w_j \sigma_i^K(t, K, u) \sigma_j^W \rho_{i,j}^K \sigma_{i,j}^W + \frac{\partial V}{\partial t} = 0 \quad (2.11)$$

The boundary equation (2.10) is as follows:

$$V(K, W, \tau) = R(K, W, \tau) \quad (2.12)$$

To solve equation (2.11), we first need to take partial derivative of the term in bracket of the left-hand side of this equation to each decision variable $(u, v, w, q)$. In the following paragraphs, we derive the decision solutions heuristically. To make the derivation rigorous in the mathematical sense, we need to include technical conditions. However, what we want from the general framework is just insights and guidelines for application. Hence, we omit the technicalities here and proceed with
heuristics.

Below are the calculated partial derivatives for decision variables:

For $u$:  
\[
\lambda \frac{\partial p}{\partial u} + \frac{\partial b}{\partial u} \frac{\partial V}{\partial K} + \frac{\partial^2 V}{\partial K^2} W \sum_{i,j} w_i \frac{\partial \sigma_i^K}{\partial u} \sigma_i^K \rho_{i,j}^{K,W} = 0 \quad (2.13)
\]
or the solution $(u,...)$ hits the boundary $\partial U$.

For $v$:  
\[
\lambda \frac{\partial p}{\partial v} + \frac{\partial b}{\partial v} + \frac{\partial V}{\partial W} = 0 \quad (2.14)
\]
or the solution $(...,v,...)$ hits the boundary $\partial U$.

For $w$:  
\[
\frac{\partial V}{\partial W} (r_j - r) W + \frac{\partial^2 V}{\partial W^2} W^2 \sum_i w_i \sigma_i^W \sigma_j^W \rho_{i,j} + \frac{\partial^2 V}{\partial K \partial W} W \sum_i \sigma_i^K \sigma_j^W \rho_{i,j}^{K,W} = 0 , \quad j = 1,...,m \quad (2.15)
\]

For $q$:  
\[
(1 - \lambda) \frac{\partial U}{\partial q} - \frac{\partial V}{\partial W} = 0 \quad (2.16)
\]
or the solution $(...,q,...)$ hits the boundary $\partial U$.

We now analyze each of these solutions one by one.

For $u$: Equation (2.13) provides insights into the trade-off of the business decision. The first term $\lambda \frac{\partial p}{\partial u}$ is the marginal ”profit” increase by one additional unit of business decision $u$ times the weighting parameter $\lambda$. If $u$ represents production decisions, then $\frac{\partial p}{\partial u}$ is the marginal profit increase from one unit of additional production. If $u$ represents sales decisions like marketing, then $\frac{\partial p}{\partial u}$ is the marginal profit from one additional unit of marketing effort. If $u$ represents expansion or downsizing decisions, $\frac{\partial p}{\partial u}$ is the marginal profit from one unit more or less resources. If $u$ represents R&D decisions, $\frac{\partial p}{\partial u}$ is the marginal profit from one additional unit of R&D input. Since the on-going profit at each instant is affected mostly by the current short-term decisions of production, sales, operations, etc., and less affected by long-term decisions like
expansion or R&D, the short-term business decisions may play a more important role in this trade-off. However, the short-term and long-term decisions need to be taken into account together. The domain constraint for business decisions is \( \mathbb{U} \). It represents the decision budget constraints of the business entity. Normally, the more resources are allocated to one decision or set of decisions, the less resources are left for other decisions. For example, if more resources are allocated to short-term activities like production, marketing, etc., less resources are available for long-term devotions like R&D.

The sum of second, third and fourth terms in (2.13) represents the marginal increase of value function by one additional decision variable \( u \). This marginal increase comes from three sources: the first one, \( \frac{\partial b}{\partial u} \frac{\partial V}{\partial K} \) is the marginal increase of value function from accumulation of business state variables (or we can interpret as “business capital”) \( K \) through marginal increase in “business capital” growth rate \( b \) due to one additional input \( u \). This effect channel is through business capital growth. The additional input of \( u \) affects the accumulation rate of \( K \). In turn, the business capital \( K \) affects the value function. The direction of \( \frac{\partial b}{\partial u} \) depends on the nature of the business decision. Some business decisions, like production, decrease the business capital. So the additional intensity of such activities may lower the accumulation rate of capital. Some business decisions increase business capital or transform one kind of business capital to another. Reinvestment decision increases instantaneous capital growth. R&D transforms financial and human capital into technological capital. Application of more technology increases profitability and in turn increase capital accumulation. The business capital \( K \) increases the value function in two ways: first it can increase future profitability; second it can increase the final payoff. The wealth \( W \) also increases the value function in two ways: first it can increase future consumptions or withdraws; second it can increase the final payoff. The second and third
term are marginal changes of expected value due to volatility effects. Equation (2.13) captures the trade-off between short-term instant benefit and long-term value growth.

We summarize the meaning of equation (2.13) here: the decision-maker makes trade-off decisions of short-term profitability and long-term value growth of business and wealth. The optimal point for each business decision \( u \) within decision domain, if not a corner solution, is the interior point where the marginal increase (or decrease) of short-term profitability is equal to the marginal decrease (or increase) of long-term added-value from business and wealth growth.

Above is the case for interior point solution for \( u \). If the Bellman equation is concave in \( u \) and the domain \( \mathbb{U} \) satisfy certain regularity conditions, we can have a nice interior optimal solution. However, the optimal decision of \( u \) may hit the boundary in some cases. For example, when the equation is linear in decision variable, the optimal point will be a corner solution. This is exactly the case for our oil-based sovereign wealth fund example. We will show the natural linkage between the corner-solution and optionality in chapter 3.

For \( v \): \( v \) is the transfer decision from business to investment entity. In a continuous time setting, it represents the transfer rate. The more incomes are transferred from business to investment, the less capital is retained in the business entity. As a result, the growth rate of business capital \( K \) will be lower. Hence, partial derivatives: \( \frac{\partial p}{\partial v} \) (marginal profit change due to one unit transfer change) and \( \frac{\partial b}{\partial v} \) (marginal business capital growth rate change due to one unit transfer change) are both negative. The lowered profit and business growth rate lead to a decrease in value function. On the other hand, higher transfer leads to more capital for investment and hence results in an increase of the value function. Equation (2.14) characterizes this trade-off.
The optimal point is where the increase and decrease effects of an additional unit transfer balance out. The discussion only applies to interior point solution. The optimal solution may be a corner solution. For some I-B complexes, the transfer policy is stipulated in the charter as a fixed policy. In a lot of cases, the transfer decision is not subject to the investment manager. Also, in most of the cases, we cannot quantify the trade-off between business and investment entities. The value of $\lambda$ is hard to determine. So taking transfer as a predetermined policy is often a wise approach. It may be seen as that the decision-maker has already optimized this decision. In most cases, there is no need to solve for equation (2.14). The transfer policy should be taken as an input in the optimization problem. For example, in our study of Norway’s Government Pension Fund Global, we take a predetermined policy of the oil income transfer rate.

For $w$: Equation (2.15) provides insights into the integrated optimal portfolio. We denote the $m \times m$-dimension covariance matrix of investable assets as: $
abla. \Sigma_{ij} = \sigma_i^W \sigma_j^W \rho_{ij}$. And we denote the $m$-dimensional vector of covariance between business states and $j$th investable asset returns as $H_j$: $H_j = \sum_i \sigma_i^K \sigma_j^W \rho_{i,j}^K$ when $j = 1, ..., m$. $H = (H_1, ..., H_m)^T$. Denote $H_j^i = \sigma_i^K \sigma_j^W \rho_{i,j}^K$. $H^i = (H_1^i, ..., H_m^i)^T$. Denote the $m$-dimensional risk premium vector as $\xi$: $\xi = \{r_j - r\}$. From (2.15), we can derive the optimal asset allocation $w^*$:

$$w^* = -\frac{\partial V}{\partial W} \Sigma^{-1} \xi - \frac{\partial^2 V}{\partial W^2} \Sigma^{-1} H = -\frac{\partial V}{\partial W} \Sigma^{-1} \xi - \sum_i \frac{\partial^2 V}{\partial W^2} \Sigma^{-1} H^i = (2.17)$$

The optimal portfolio $w^*$ is comprised of two parts: the first part is a scaled Markowitz optimal portfolio, namely the efficient portfolio of mean-variance optimization. The second part is a series of hedging portfolios for the business risks. The optimal portfolio is the weighted sum of the two portfolios. $\Sigma^{-1} \xi$ is the Markowitz
optimal portfolio. The coefficient before it: \(-\frac{\partial V}{W \partial W^2}\) is positive, because the first-order derivative \(\frac{\partial V}{\partial W}\) is obviously positive and the second-order derivative \(\frac{\partial^2 V}{\partial W^2}\) is negative for the concavity of utility function. In the second term, the \(i\)th element \(\frac{\partial^2 V}{W \partial K \partial W} \Sigma^{-1} H^i\) is a hedging portfolio for the \(i\)th business decision. To illustrate this point, assume all investable asset are uncorrelated with each other. This assumption cause no loss of generality since we can always use Principle Component Analysis (PCA) to decompose all assets into orthogonal assets and invest in these assets instead. In such case, \(\Sigma^{-1}\) is a diagonal matrix. With volatilities \(\sigma^K_i\) and \(\sigma^W_j\) at a certain level, the higher the correlation between \(i\)th business state and \(j\)th asset, the larger value is \((\Sigma^{-1} H^i)_j\) (this is the \(j\)th asset weight in the element hedging portfolio without the coefficient). Now we look at its coefficient \(\frac{\partial^2 V}{W \partial K \partial W} \Sigma^{-1} H^i\). Normally, the value function should be concave in \((K,W)\): concavity of utility in \(K\) and \(W\). So \(\frac{\partial^2 V}{\partial K \partial W}\) is negative. Hence, the coefficient \(-\frac{\partial^2 V}{W \partial K \partial W}\) is negative. As a result, the more positive (negative) the correlation between \(j\)th investable asset and \(i\)th business state, the more negative (positive) is the weight of \(j\)th asset in the hedging portfolio for \(i\)th business state. That’s why we can interpret \(\frac{\partial^2 V}{W \partial K \partial W} \Sigma^{-1} H^i\) as the hedging portfolio for risks of the \(i\)th business state. If in some unusual cases where \(\frac{\partial^2 V}{\partial K \partial W}\) is positive, then the ”hedging portfolio” is ”hedging” of an alternative meaning. One example is for pension fund. The state variable ”pension liability” has a hedging portfolio of positive coefficient: the higher correlation between an investable asset and pension liability, the more weight of this asset is in hedging portfolio, namely, the ”liability hedging portfolio”. One can refer to the body of literature on liability-driven investing. Examples include Rauh (2006), Amenc and etc. (2010), Ang, Chen and Sundaresan (2013), Das, Kim and Statman (2014).

We can see the derivation of optimal asset allocation \(w^*\) is not entangled with other decision variables. Equation (2.17) holds true in a generic way. In cases where
other decision variables are not subject to the control of decision-maker, equation (2.17) still provides the optimal asset allocation. The generic insights from (2.17) are summarized as follows:

**Given other decision policies**: business decisions, transfer decision and consumption decisions (either optimized by the decision-maker, or not subject to control), the optimal asset allocation is a weighted combination of the mean–variance efficient portfolio and hedging portfolios for all business risks.

This result enables us to solve for an integrated optimal asset allocation solution without solving for other decisions. In cases like sovereign wealth funds, pension plans or family offices where the management structure does not allow an integrated decision optimization process for business and investment together, we can still apply equation (2.17). Moreover, even if other decisions are not "optimal" in their own merits, we can still solve for the integrated optimal asset allocation based on (2.17) under given policies of other decisions. Hence, (2.17) is a very powerful tool with universal application.

We can easily extend equation (2.17) to the case where market conditions are time-varying. Under such conditions, the optimal asset allocation are comprised of three elements: one scaled efficient mean-variance portfolio, a series of hedging portfolio for business risks and hedging portfolios for changes in investment opportunity set. Please refer to Merton (1973) for the hedging portfolio of investment opportunity set change, namely, the ICAPM model. The discussion above is for interior optimal solutions. If domain $U$ has constraints on asset allocation, the optimal solution may be a corner solution. For example, we can impose no shorting
For $q$: Equation (2.16) captures the trade-off between consumption and accumulation of wealth. The higher the consumption rate, the higher utility of consumption will be. On the other hand, higher consumption lowers the wealth growth. These two effects change the value function in opposite ways. The optimal point is where the marginal increase and decrease effects of one additional unit consumption rate cancel out.

The derivations above provide economic insights for optimal decisions and guidelines of applications. The solutions from the general framework are very generic. They hold true for all types of I-B complexes. Next, we apply this framework to a specific example of oil-based SWFs for concrete illustrations of its use. We choose this example for two reasons: 1. oil-based SWFs have clear governance structures. They are set up by oil-producing countries’ government and are subject to government controls. For example, Norway’s Government Pension Fund Global is set up by Norwegian government and under supervision of Ministry of Finance. The oil-based SWF is managed by an investment management unit, e.g. Norges Bank Investment Management. The government has control over the transfer from oil-producing business to the SWF. The oil-producing companies are mostly state-owned. Other companies need to buy permits and pay oil taxes to the government. The clear structure of governance makes modeling relatively easy. 2. Oil-producing business model is simple. The dimensions of state variables and decision variables are relatively small. Business state variables include oil reserve level, production capacity, oil price and production costs. In our study, these state variables are sufficient to characterize the oil-producing business. Decision variables include production, exploration and abandonment. The main decision of our study is production. We do not take into account
exploration and abandonment decisions for these decisions are second-order decisions. We use Norway’s Government Pension Fund Global as our reference for modeling.
Chapter 3

Application to Oil-based Sovereign Wealth Fund

This chapter is devoted to application of our general framework to oil-based sovereign wealth funds. We use Norway’s Government Pension Fund Global as the modeling reference. As discussed in chapter 2, it is impossible to tie business decisions to investment (SWF) decisions: oil-producing companies make business decisions; government decides on transfer and spending policies; SWF investment management chooses asset allocation. It is very hard, if not impossible, to circumvent all the conflicts of interests and design a governance structure to make business and investment decisions simultaneously. Moreover, our paper’s focus is on the asset allocation decisions. So we focus on asset allocation decisions, taking other decisions as given or already optimized. As said in chapter 2, this approach can be viewed as the government and business have already chosen or optimized these decisions in the decision domain \( U \). As discussed in chapter 2, equation (2.17) of the optimal asset allocation decision does not involve other decision variables. The equation always holds true regardless of how other decisions are chosen. Even though other decisions are not subject to control or even not "optimal" in their own merits, we
can still solve for the integrated optimal asset allocation. The main goal for us is to solve for this integrated optimal asset allocation. For other decisions (business, transfer and consumption), there are two approaches to model them: 1. we can use exogenous specifications to model them; 2. we model the optimization problem for them separately and use the solutions as an exogenous inputs for the integrated asset allocation optimization problem. Which approach to take depends upon the specific circumstances of each I-B complex. In the following paragraphs, we specify the state variables and decision variables. We also provide modeling specifications for them. Finally, we use the case study of oil-based SWFs to provide some context for equilibrium equations (2.13)-(2.17) of optimal decisions. We examine these equations one by one.

**Business state variables**: The essential state variables of the oil-producing business are oil reserve level, oil price, production cost and production capability. These variables are sufficient enough to characterize oil-production business.

As the decision-maker for the investment entity, we take business decisions from the oil business entity as inputs. From the SWF’s macro-level point of view, the oil-producing business can be seen as a single business entity. The aggregate states of oil-producing industry should be used to characterize the business entity. Detailed features of specific companies, like financial and human capital, balance sheet, technology, are irrelevant from the aggregate level. Thus, we can use production capacity and production cost function to characterize the aggregate industry state. We specify the two variables: the projected production capacity time series and estimated production costs, and use them as inputs instead of variables in the optimization problem. This specification means we consider the integrated decision-making problem under a certain technology state and production capacity. We can still change the
values of the technology state and production capacity for analysis. But modeling the dynamics of these two subjects are out of scope for this study. So the state variables are oil reserve level and oil price. We denote the oil reserve level as $k$ and oil price as $\psi$.

Investment state variables: $W$ the wealth level is the only state variable for investment management.

Business decision variables: We specify aggregate production capacity and average cost function. So detailed business decisions of specific companies, like sales, operations, expansion or downsizing, purchasing or selling, merger and acquisition, financing and exit, etc., do not need to be considered. The business decision is production at the aggregate industry level. As discussed, there are two approaches to incorporate the production decision: 1. we can treat the decision as exogenous and use specifications to model them, assuming they have already been chosen; 2. we can model the business optimization explicitly and solve for optimal production decisions. We use both modeling approaches in chapter 5 and chapter 6. In the first approach, we use projected production as producing decisions in this approach. By this approach, we essentially calculate the intrinsic value of oil reserves. The second approach models the oil-production problem explicitly and solve for optimal production policies. This approach calculates the intrinsic value plus option premiums from production flexibility. We denote the production rate at time $t$ to be $u_t$.

Transfer decision variable: Transfer decisions are made by the government. The government makes the trade-off decision of oil business sustainability and wealth accumulations to decide on transfers. With no knowledge of the decision-making process, we take the average historical transfer policy as inputs. Denote the transfer
rate as $\gamma$. Transfer $v$ is transfer rate times oil profit.

Investment decision variables Asset allocation decisions: $w_t$.

Consumption decision variables Consumption choices $q_t$ is decided by the government. The charter of SWFs usually specify the proper spending policy as a certain percentage of total SWF wealth. Without further knowledge, we use this spending policy as consumption decisions.

Next, we examine equilibrium equations (2.13)-(2.17) of optimal decisions. Even though some of the decisions are specified as inputs instead of solved in the optimization problem, we examine them anyway to gain insights. These discussion provides a thinking framework for policy-makers: when related decisions are to be made, the right trade-off should be analyzed and useful information should be collected. We look at the specified decisions $v$ and $q$ first:

For $v$:
As discussed in chapter 2, (2.14) represents the optimal trade-off of the increase (decrease) effect on business profitability v.s. decrease (increase) effect on wealth value caused by the marginal transfer. For oil-based SWFs, it means the trade-off of oil-production capability v.s. wealth accumulation. Too much transfer hurts companies’ profitability, sustainability and R & D capability. Too little transfer slows down wealth accumulation. The government policy maker needs to make a decision for this trade-off to determine the transfer policy. If the decision $v$ is not an interior point of policy domain $U$, it hits the policy boundary $\partial U$. The equilibrium equation for $v$ (2.14) means:
To make the transfer decision, the policy-maker needs to know: the importance of oil companies’ profitability $\lambda$; the effect of additional transfer on companies’ profitability $\frac{\partial P}{\partial v}$; the effect on the long-term value through the business entity’s capital growth $\frac{\partial V}{\partial K} \frac{\partial b}{\partial v}$; the effect of additional transfer on the long-term wealth accumulation $\frac{\partial V}{\partial W}$.

To understand the first, second and third effects, the policy-maker needs to understand oil business. To understand the fourth effect, the policy-maker needs to understand wealth accumulation outlooks in financial markets. Overall, the policy-maker needs to understand oil companies’ incentives and national utility to make this trade-off.

In modeling specifications, we take given policies as transfer decisions. The historical average is used as estimate for transfer policy. We denote it to be $\gamma$. Denote oil profit rate as $R_{oil}^t$. Transfer decision $v$ is: $v_t = \gamma R_{oil}^t$ (3.1).

For $q$:
As discussed in chapter 2, (2.16) represents the optimal trade-off decision for the temporary consumption v.s. the long-term wealth accumulation. Too much current consumption slows down long-term wealth accumulation, which is future generation’s consumption. Too little spending limits the government’s measures to sustain current generation’s living standards, like wealth distribution, healthcare benefits, infrastructure building, etc. The policy-maker needs to make the trade-off between contemporary and future generations’ living standards, which borders upon philosophical discussion of fairness. If the decision $q$ is not an interior point of policy domain $\mathbb{U}$, it hits the policy boundary $\partial \mathbb{U}$. The equilibrium equation of optimal
decisions $q(2.16)$ provide insights:

To make the spending decision, the policy-maker needs to understand:

- the effect on the national utility of an additional unit of current spending $\frac{\partial U}{\partial q}$;
- the effect on long-term wealth accumulation of an additional unit of current spending $\frac{\partial V}{\partial W}$.

To evaluate the first effect, the policy-maker needs to understand current national needs and government spending efficiency. To understand the second effect, the policy-maker needs to understand wealth accumulation outlooks in financial markets.

In our modeling specifications, we take consumption or spending policy as given inputs. In fact, the charter of Norway’s Government Pension Fund Global states the withdraw policies as a certain percentage of total SWF wealth. Denote the consumption policy rate as $\rho$. Then the consumption decision $q$ is: $q_t = \rho W_t$ (3.2).

Next, we look at business decision $u$:

For $u$:

As discussed in chapter 2, (2.13) represents the trade-off decision for contemporary profitability and long-term business value. In the oil-based SWF case, the business decision is current oil production. Too much contemporary production decreases the oil reserve level too quickly, and therefore wastes the option to wait and produce at higher oil price. Too little contemporary production hurts the business sustainability, loses the current profitability and opportunities to use profits for investment now. The decision-maker needs to make an optimal decision between current production and future production. If the decision is not an interior point of policy domain $\mathbb{U}$, it
will hit the policy boundary $\partial U$. Equation (2.13) tells us:

To make this decision, the decision - maker needs to understand: the marginal effect of production on current profitability and importance of the business entity’s profitability $\lambda \frac{\partial p}{\partial u}$; the marginal effect of production on long - term expected value of oil reserve and wealth through level effect: $\frac{\partial b}{\partial u} \frac{\partial V}{\partial K} + \lambda \frac{\partial p}{\partial v} \frac{\partial v}{\partial u} + \frac{\partial V}{\partial W} \frac{\partial w}{\partial u}$; and volatility effect: $\rho_{i,j} W K^i W^j \sum_{j} \sigma_{K} K_{j} \frac{\partial \sigma_{K}}{\partial u}$. 

Next, we parametrize the model for the oil-based SWF. In this case, business state variable $K$ represents the oil reserve. We do not consider the “volatility” of oil reserve level. So the dynamics of oil reserve is deterministic and controlled by production decision $u$. The reason for this specification is: even though there is uncertainty in observed oil reserve level, the true amount is basically certain. The uncertainty of oil reserve level comes from observation errors rather than innate uncertain natures. Also, the uncertainty in observed oil reserve is hardly related to any financial assets ($\rho_{i,j} W K^i W^j$ is zero). These reasons justify a deterministic dynamics for oil reserve. Other than the effects in (2.13), the production decision $u$ also affects transfer decision because we specified that transfer $v$ is in proportion to oil revenue (3.1)]. As a result, we need to consider (2.13) and (2.14) together since they are both affected by $u$. Thus, the equation for optimal production solution is:

$$\lambda \frac{\partial p}{\partial u} + \frac{\partial b}{\partial u} \frac{\partial V}{\partial K} + \lambda \frac{\partial p}{\partial v} \frac{\partial v}{\partial u} + \frac{\partial V}{\partial W} \frac{\partial w}{\partial u} = 0 \ (3.3)$$

We should keep in mind that this equation is for interior point solution in $U$. If we cannot find a solution for equation (3.3), it means the increase and decrease effects of marginal production never reaches equilibrium inside $U$. For this case, the optimal production is either the upside or downside boundary of decision domain $U$: 42
producing at the maximal capacity or minimal requirement.

Next we specify the functional form for each term in equation (3.3): 

**profit function** $p$: profit at time $t$ is oil price $\psi$ times production units $u$ then minus production cost $C_t$: $R^{oil}_t = (\psi u - C_t)$ (3.4). According to oil-production literature, the production cost $C$ depends on production amount $u$, current oil reserve states (oil reserve level $K$, geological features) and available production technology. Given the particular features of the country’s oil reserve endowment and technology state, the average unit cost is just a function of oil reserve level $K$: $C = C(u, k)$. This function increases in regards to $u$ and decreases with $K$. The effect of oil reserve level $K$ is easy to understand: the higher oil reserve level, the easier to produce and the lower the production cost, and vice versa. Thus, the profit function $p$ for a future time $t^*$ is oil revenue $R^{oil}_t$ minus transfer rate to the SWF $v$ times discounted factor: $e^{-\int_{t^*}^{t} r^o ds}$. $r^o_s = r_s + \lambda^o t \sigma^o_t$ is the market risk-adjusted return for oil income ($\sigma^o_t$ is risk factors in oil price, $\lambda^o_t$ is risk premiums for these factors). So $p_t(t^*, u, v, K) = e^{-\int_{t^*}^{t} r^o ds}(\psi u_t - C(u, k) - v)$. We assume $C(u, k)$ is linear in $u$ below production capacity and infinite beyond production capacity: $C(u, k) = uc(k)$. This is a close approximation to the actual situation when current production amount is insignificant compared to the total reserve level. The profit function $p$ is:

$$p_{t,t^*} = e^{-\int_{t}^{t^*} r^o ds}(1 - \gamma)(\psi - c(k))u_t$$ (3.5)

**transfer** $v$: transfer is a fixed percentage of oil revenue: $v_t = \gamma R^{oil}_t = \gamma(\psi - c(k))u_t$ (3.6).

**drift for k**: the change rate of oil reserve $k$ is negative production rate: $b = -u$. 

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drift for $\psi$: the drift of $\psi$ is $\alpha(\psi, t)$, a mean-reversion drift part. We discuss the oil price dynamics in details in chapter 6. Oil price does not depend on production decision $u$. Here we considers the state businesses as price takers. Market manipulation through business decisions is out of scope for our study.

Plug in the specifications above. Then equation (3.3) is:

$$\lambda(1 - \gamma)(\psi - c(k)) - \frac{\partial V}{\partial k} + \frac{\partial V}{\partial W} \gamma(\psi - c(k)) = 0 \ (3.7)$$

Equation (3.7) is for the interior point solution in domain $\mathbb{U}$. However, we can see the left-hand side of equation (3.7) does not depend on the production decision $u$. Equation (3.7) does not hold true except for when the left-hand side happens to be zero. As a result, we generally cannot find an interior optimal point solution for $u$. The optimal decision hit the boundary of decision domain $\mathbb{U}$: either produce at the maximal capacity or minimal requirement. Whether the oil production rate $u$ reaches the upper or lower boundary depends the sign of the left-hand side of equation (3.7). Remember the optimal solution is solved via HJB equation (2.11).

The left-hand side of equation (3.7) is the coefficient before production decision $u$ in the bracket of maximization in equation (2.11). As a result, if the coefficient $\lambda(1 - \rho)(\psi - c(k)) - \frac{\partial V}{\partial k} + \frac{\partial V}{\partial W} \rho(\psi - c(k))$ is positive, the decision-maker should continue increasing production until production hits the maximal capacity (denote as $u_{\text{max},t}$); otherwise, if the coefficient is negative, the decision-maker should only produce at the minimal required amount, as the lower bound for $u$ in decision domain $\mathbb{U}$ (denote as $u_{\text{min},t}$). We can write the condition as below:

If $\psi \geq c(k) + \frac{\partial V}{\partial k} + \frac{\partial V}{\partial W} \gamma$, then $u^* = u_{\text{max}}$.

If $\psi < c(k) + \frac{\partial V}{\partial k} + \frac{\partial V}{\partial W} \gamma$, then $u^* = u_{\text{min}}$. (3.8)
The conditions of (3.8) can be interpreted as follows: if the oil price is higher than the production cost plus the ratio of the marginal effect on long-term value by additional saved oil reserve over the marginal effect by additional produced profit and additional accumulated wealth, the optimal decision is to increase production up to full capacity; otherwise, the optimal decision is to produce at the minimal requirement. The decision rule is very similar to the optimal execution rule of thresholding for American options. The threshold separates regions of execution and waiting. This insight provides the basis why we use real option theory to model, price and generate production decisions for the oil business entity. We show the details in chapter 6.

Next we analyze the focus of this study: integrate optimal asset allocations \( \{w_j\} \)

For \( w_j \):

We show in chapter 3 the integrated asset allocation (2.17) is:

\[
 w^* = -\frac{\partial V}{\partial W} \Sigma^{-1} \xi - \sum_i \frac{\partial^2 V}{\partial W \partial^2 W} \Sigma^{-1} H^i = -\frac{\partial V}{\partial W} \Sigma^{-1} \xi - \frac{\partial^2 V}{\partial W \partial^2 W} \Sigma^{-1} H^{oil} \tag{3.9}
\]

In the case of oil-based SWFs, the business risks come from oil price uncertainties. The covariance vector in hedging portfolio is \( H^{oil} \): \( H^{oil}_j = \sigma^{oil} \sigma_j \rho_{oil,j} \). \( \sigma^{oil} \) is the volatility of oil price. \( \sigma_j \) is the \( j \)th asset’s volatility. \( \rho_{oil,j} \) is the correlation between oil price and the \( j \)th asset. The integrated optimal asset allocation is the weighted sum of a mean-variance efficient portfolio and a hedging portfolio for the risks in oil-producing business.

Some other specifications for oil-based SWFs For our modelling reference: the Norwegian SWF, we make some specifications for parameter values and functional forms. First of all, as the decision-maker of the SWF, we do not take into account utility of the business entity: \( \lambda = 0 \). Also, we treat the oil business has already chosen
or optimized their structures. These decisions are reflected in the specifications of production capacity, minimal production level and transfer policy. In the base case (chapter 4), we also take production decision $u$ as chosen. Second of all, we do not take into account the consumption utility along the way: $U(q) = 0$. Since the consumption or spending policy is fixed (equation (3.2)), the consumption decision is not subject to choose. To add in the utility of consumption along the way $\int_0^\tau (1-\lambda)U(q_t)$ contributes to nothing more than just complicating the computations. Hence, this term is not included in the objective function. We only focus on optimizing the final utility function: $R(K, W, \tau)$. Finally, the final utility function should be only a function of wealth $W$: $R(K, W, \tau) = R(W)$. The terminal point is chosen to be the endpoint of business. For a business, $\tau$ is the time when the business stops (bankrupt or liquidated). In the case of oil-based SWF, $\tau$ means when the oil reserve get depleted. As a result, the final utility function should be a function of only $W$. We also assume market is complete and investment opportunity set is constant. Moreover, we assume all the functions in the problem have enough order of smoothness. And the domain for decision variables are convex, closed and bounded.
Chapter 4

Asset Allocation Model for
Oil-based Sovereign Wealth Funds

In this chapter, we explicitly model the optimal asset allocation problem for the sovereign wealth fund using the Norwegian SWF as a reference. One can see the HJB equation (2.11) is very complicated. The integrated asset allocation problem is very challenging to solve. We need to make some modeling specifications to make the integrated asset allocation problem realistic yet solvable. In this chapter, we model the simplified base case where the optionality properties of oil production is not taken into account. Constant oil production rate is assumed. This is consistent with Norway’s own projection of deterministic future projection rate. The uncertainty of income only stems from the uncertainty of oil price. We show in chapter 3 that the integrated optimal asset allocation has a component for hedging the oil price uncertainty. We can compare the optimal solution with the ”simple optimal solution”, which is solved by only considering financial assets and not taking into account oil reserve assets. Certainty equivalent of final wealth is used to calculate the wealth loss of this suboptimal portfolio choice. In this base case, the gain in certainty equivalent of final wealth comes from hedging the risk of oil price. We also assume constant
parameters in the return dynamics for all assets and oil price. In the base case, we can solve the stochastic control problem and get semi-analytical solution. And the optimal portfolio choice can be calculated explicitly. We use this base case to study both the general framework and the asset allocation problem for an oil-based SWF. After examining the general framework applied to the oil-based SWFs, we come up with the set of conditions under which the integrated asset allocation problem are solvable for all types of I-B complexes.

Now, we parametrize the general framework for application to oil-based SWFs. First of all, we assume constant relative risk aversion utility function. And the sovereign wealth fund optimize the expected utility at a specified future date $T$. Optimizing future generation’s living standards is the essential goal of SWFs, as stated in the charter of Norway’s GFPG.

\[
\max_{w_j} E_t[U(W_T)] \quad (4.1)
\]
\[
U(W) = \frac{W^{1-\eta}-1}{1-\eta}, \eta > 1
\]
\[
U(W) = \ln(W), \eta = 1
\]

To simplify the calculation, we just use $U(W) = -W^{1-\eta}$ ($\eta > 1$), since it is equivalent to the utility function $\frac{W^{1-\eta}-1}{1-\eta}$ for $\eta > 1$.

Assume there are $N$ risk factors in the market. They are represented by $N$ orthogonal Brownian motions: $\{Z_1, Z_2, ..., Z_N\}$. There are $M$ assets in the market. And for the $i$th asset, its price $R_i$ follows the return dynamics as below:

\[
dR_i = R_i[(r + \sum \xi_j \sigma_{ij}) + \sum \sigma_{ij}dZ_j] \quad (4.2)
\]

$\sigma_{ij}$ is the volatility component on the $j$th factor of the $i$th asset. $\xi_j$ is the risk premium for the $j$th risk factor. Denote $R$ to be the price vector, $C$ to be the matrix
\{\sigma_{ij}\}, \xi to be risk premium vector and Z to be risk factors. Then the asset dynamics can be written as:

\[ dR = R[(r + C\xi) + CdZ] \] (4.3)

The dynamics of oil price \( \psi \) is as follows:

\[ d\psi = \alpha(\psi)dt + \psi \sum \sigma_{\psi j}dZ_j + \psi \sigma_0 dZ_0 \] (4.4)

\( \alpha(\psi) \) is the drift function of oil price. In the case of oil price, \( \alpha(\psi) = \alpha_0(\psi_0 - ln(\psi))\psi \).

This is the drift of standard mean-reverting process. The mean-reverting process is a commonly used dynamics to model commodity prices. One can refer to Bessembinder and etc. (1995), Schwartz (1997) and Pindyck (2004) among the abundance of commodity research. \( \sigma_{\psi j} \) is the volatility component of oil price on the \( j \)th risk factor. \( Z_0 \) and \( \sigma_0 \) are the idiosyncratic risk factor and its volatility component. \( Z_0 \) is orthogonal to other risk factors. If \( \sigma_0 \) is zero, then the market is complete. Under complete market conditions, we can solve analytically the integrated asset allocation problem. Oil industry ETF, oil company equity and oil futures are all available to invest in the market. They are all closely correlated with oil price. And the nearest date futures are a good proxy for spot. Hence, we assume the market is complete. Denote \( \{\sigma_{\psi j}\} \) as \( \sigma_{\psi} \).

Under the constant production assumption, we also assume the income stream from oil business is constantly proportional to oil business revenue: \( m(\psi - c) \). \( m \) equals production rate multiplied by the constant transfer rate \( \gamma \). \( c(t) \) is the production cost. Unit production cost is a function of oil reserve level. Given the constant production and the current oil reserve, production cost should be a deterministic function of time \( t \). Denote portfolio choice \( \{w_j\} \) by vector \( w \). Denote the spending
ratio as $\rho$. The controlled dynamics for wealth is:

$$dW = W[(r + w^TC\xi)dt + w^TCdZ] + m(\psi - c)dt - \rho Wdt \quad (4.5)$$

Since the production rate is constant, then the oil reserve level is dependent on the current time $t$. As a result, the state variables $(k, \psi, W, t)$ can be reduced to $(\psi, W, t)$. We denote the value function by $J(W, \psi, t)$. $J$ is a function of wealth $W$, oil price $\psi$ and time $t$. The Hamilton-Jacobi-Bellman equation of this case is:

$$\max_w J_W[r - \rho + w^TC\xi] + J_Wm(\psi - c) + \frac{1}{2} J_{WW}W^2w^TCC^Tw + J_\psi \alpha(\psi) + \frac{1}{2} J_{\psi\psi}\psi^2(\sigma_\psi^2 + \sigma_0^2) + J_{W\psi}Ww^TCS_\psi^T + J_t = 0 \quad (4.6)$$

Solve for the portfolio choice $w$:

$$J_WWC\xi + J_{WW}W^2CC^Tw + J_{W\psi}WC_\psi^T = 0 \quad (4.7)$$

The optimal portfolio choice is $w^* = -\frac{J_W}{J_{WW}}(CC^T)^{-1}C\xi - \frac{J_{W\psi}}{J_{WW}}(CC^T)^{-1}C_\psi^T$ (4.8). We can see there is two separate funds in the optimal portfolio. The first term is an efficient portfolio in terms of Markowitz’s mean-variance optimization. The second one is a hedging portfolio for the oil business income risks. Plug in the optimal portfolio choice, we can get the HJB partial differential equation. The HJB equation is:

$$J_WW(r - \rho) + J_WM(\psi - c) - \frac{1}{2} \theta_1 \frac{J_W}{J_{WW}} - \theta_2 \frac{J_{W\psi}}{J_{WW}} \psi - \frac{1}{2} \theta_3 \frac{J_{W\psi}^2}{J_{WW}} + J_\psi \alpha(\psi) + \frac{1}{2} \theta_4 J_{\psi\psi} \psi^2 + J_t = 0 \quad (4.9)$$

Where $\theta_1 = \xi^TC^T(CC^T)^{-1}C\xi$, $\theta_2 = \xi^TC^T(CC^T)^{-1}C_\psi^T$, $\theta_3 = \sigma_\psi^T(CC^T)^{-1}C_\psi^T$, $\theta_4 = \sigma_\psi^2 + \sigma_0^2$. Based on the functional form of utility function, we try the
parametrized function \( J(W, \psi, t) = -A(t)(\delta(t)W + g(\psi, t))^{1-\eta} \) (4.10). Plug in (4.10) in (4.9)

\[
A(r - \rho)\delta(1 - \eta)(\delta W + g)^{-\eta}W + A\delta(1 - \eta)(\delta W + g)^{-\eta}m(\psi - c) + \frac{1}{2}A\theta \frac{1-\eta}{\eta}(\delta W + g)^{1-\eta} - A(1 - \eta)\theta g + \frac{1}{2}A\theta_2(1 - \eta)\eta(\delta W + g)^{-\eta-1}g^2 = + A(1 - \eta)(\delta W + g)^{-\eta}A_{g}\psi^2 + A_t(\delta W + g)^{1-\eta} + A(1 - \eta)(\delta W + g)^{-\eta}(\lambda_t W + g_t) = 0
\]

(4.11)

To make the coefficients of (\( \delta W + g \))\(^{-\eta}W\), (\( \delta W + g \))\(^{1-\eta}\), (\( \delta W + g \))\(^{-\eta}\) to be zeros, we have the following equations:

\[
(r - \rho)\delta + \delta_t = 0
\]

(4.12)

\[
\frac{1}{2}\theta \frac{1-\eta}{\eta}A = 0
\]

(4.13)

\[
\delta m(\psi - c(t)) - \theta g g(\psi) + \frac{1}{2}\theta_4 g^2 \psi^2 + g_t + \alpha_0(\psi_0 - ln(\psi))\psi g = 0
\]

(4.14)

We also have the boundary condition: \( J(W, \phi, T) = -W^{1-\eta} \). Thus, the boundary condition for \( \delta(t), A(t), g(\psi, t) \) are:

\[
\delta(T) = 1
\]

(4.15)

\[
A(T) = 1
\]

\[
g(\psi, T) = 0
\]

To make the coefficient of (\( \delta W + g \))\(^{-\eta-1}\) to be zero, we have the equation:

\[
(\sigma_0 C C^T - (\sigma_0^T + \sigma_0^2))g^2 = 0
\]

(4.16) For this equation to hold true, we need to have either \( \sigma_0 C C^T - (\sigma_0^T + \sigma_0^2) = 0 \) (4.16.1) or \( g = 0 \) (4.16.2). For equation (4.16.1) to hold, we need the idiosyncratic risk factor loading \( \sigma_0 \) to be zero, namely, the market to be complete. Under the market completeness condition, the appropriate risk factor components can be selected and reorganize to
prove $\sigma_\psi C^T(CC^T)^{-1}C\sigma_\psi^T - \sigma_\psi \sigma_\psi^T = 0$. For (4.16.2) to hold, $g$ needs to be constant in $\psi$. This condition does not generally hold true. For our problem, there are quite a few assets to track oil price closely. Thus, market completeness is assumed. As a result, (4.16) holds. Idiosyncratic risks are always difficult to take into account. Cairns, Blake and Dowd (2006) encounter a similar issue when they assume unhedgeable idiosyncratic risks exist in personal incomes. When considering the existence of unhedgeable idiosyncratic risks, the problem gets too complicated to solve. Equation (4.12) and (4.13) are easy to solve given the boundary conditions (4.15). We can get the following solutions:

$$\delta = e^{(r-\rho)(T-t)}$$  \hfill (4.12 result)

$$A(t) = \exp\left(\frac{1}{2} \theta 1 - \eta \eta (T - t)\right)$$  \hfill (4.13 result)

Solving equation (4.14) is, in fact, equivalent to calculating the expectation of a function of the sum of a geometric Ornstein-Uhlenbeck process. There is no easy analytical solution to Equation (4.14) as there is no analytical solution to price Asian options, which is also the expectations of a function of weighted sums of geometric Brownian motions. Another approach, Feynman-Kac formula, enables us to calculate the solution to equation (5.14). Feynman-Kac formula converts solving partial differential equations into calculating probabilistic expectation or the other way around. According to Feynman-Kac formula (see Appendix A.1), we can show:

$g(\psi, t)$ can be represented as an expectation under probability measure $Q^*$ conditional on time $t$:

$$g(\psi, t) = E_t^{Q^*} \left[ \int_t^T \delta m(\psi - c(s)) ds \right]$$  \hfill (4.17)

If $\psi$ is the solution to the partial differential equation (4.14):
\[ \delta m(\psi - c(t)) + g_t + \alpha_0(\psi_0 - \frac{\theta_2}{\alpha_0} - ln(\psi))\psi g_\psi + \frac{1}{2} \theta_4 g_\psi \psi^2 = 0 \]

And satisfies the boundary condition: \( g(\psi, T) = 0. \)

If the expectation (4.17) is calculated, then the PDE (4.14) is solved. We need to define the probability measure \( Q^* \) as well. According to Feynman-Kac formula, \( \psi \) must follow the geometric Ornstein-Uhlenbeck process below under probability measure \( Q^* \):

\[ d\psi = \alpha_0(\psi_0 - \frac{\theta_2}{\alpha_0} - ln(\psi))\psi dt + \sqrt{\theta_4} \psi dW_t^{Q^*} \quad (4.18) \]

\( W_t^{Q^*} \) is a Brownian motion under \( Q^* \) measure. (4.18) can be shown to be exactly the oil price process under risk neutral measure. So \( Q^* \) is the risk neutral measure. The proof is as below:

\[ dR_i = R_i[(r + \sum \xi_j \sigma_{ij}) + \sum \sigma_{ij} dZ_j] \]

\[ d\psi = \alpha(\psi) dt + \psi \sum \sigma_{ij} dZ_j \]

\( \sigma_0 \) is assumed to be zero, namely no unhedgeable idiosyncratic risks. There exists a unique risk neutral measure for asset and oil. Denote the risk-neutral measure as \( Q \). Then:

\[ \frac{dQ}{dP} = exp\left(-\int_0^t \frac{1}{2} \xi^T \xi dt - \int_0^t \xi dt\right) \quad (4.19.1) \]

\[ dZ_t^Q = dZ_t + \xi dt. \quad Z_t^Q \] are joint Brownian motions under \( Q \) measure. As a result, under the risk neutral measure \( Q \), asset returns are:
\[ dR_i = R_i [r + \sum \sigma_{ij} dZ_j^Q] \quad (4.19.2) \]

And the oil price under risk-neutral measure \( Q \) is:

\[
\begin{align*}
d\psi_t &= \alpha_0 (\psi_0 - \ln(\psi)) \psi t dt + \psi \sum \sigma_{\psi j} (dZ_j^Q - \xi_j dt) \\
&= \alpha_0 (\psi_0 - \frac{\sum \sigma_{\psi j} \xi_j}{\alpha_0} - \ln(\psi)) \psi t dt + \psi \sum \sigma_{\psi j} dZ_j^Q \\
&= \alpha_0 (\psi_0 - \frac{\theta_2}{\alpha_0} - \ln(\psi)) \psi t dt + \psi \sum \sigma_{\psi j} dZ_j^Q \quad (4.19.3)
\end{align*}
\]

We can see (4.19.3) matches exactly the oil price motion in equation (4.18). Thus, \( Q^* \) is the risk-neutral measure: \( Q^* = Q \). Proof completes.

Equation (4.17) is the expectation under risk neutral measure of future oil income cash flows, given the assumed constant production policy. An interesting item of equation (4.17) is the risk-free forward rate \( \delta(t) \). As in (5.12 result), the interest rate is \((r - \rho)\) instead of \( r \). This result stems from the fact that wealth is consumed at a constant rate of \( \rho \). Hence, all cash flows should be applied with interest rate \((r - \rho)\) instead of \( r \). We can calculate the expectation (4.17) with Monte Carlo simulation method. Write (4.17) as below:

\[
g(\psi, t) = E_\psi^Q \left[ \int_t^T \delta m \psi ds \right] - \int_t^T m c(s) ds \quad (4.17.1)
\]

First of all, we can easily calculate the deterministic part: \(- \int_t^T m c(s) ds\). Let \( f(t) \) be the solution of: \( f(t)' + mc(t) = 0 \) (5.20) with boundary condition \( f(T) = 0 \). Then \( f(t) = - \int_t^T m c(s) ds \). Next, we specifies the functional form of the cost function \( c(t) \). As introduced in section 3, the production cost is a function of oil reserve. Under a certain technology level, unit production cost is negatively affected by oil reserve level \( k \): \( c = c_0 k^{-\xi} \). With constant depletion of oil reserve, \( k(t) = (K_o + \frac{m}{\gamma}(T-t)) \). \( K_o \)
is the level of depletion where production of oil does not make a profit anymore. Denote $\gamma$ to be the transfer rate. Then $c(t) = c_0(K_0 + \frac{m}{\gamma}(T-t))^{-\zeta}$ (4.21). So $f(t)$ follows:

$$f(t) = -m \int_t^T c_0(K_0 + \frac{m}{\gamma}(T-s))^{-\zeta}e^{(r-\rho)(T-s)}ds \quad (4.17.2)$$

Secondly, we need to calculate the expectation term $E_t^Q[\int_t^T e^{(r-\rho)(T-s)}m\psi ds]$. The integral inside the bracket is a weighted arithmetic sum of a geometric Orstein-Uhlenbeck process. If we use Monte Carlo simulation approach with brute force, we need to simulate a very large number of paths for $\psi$ in order to get an accurate estimation of the expectation. However, we can explicitly calculate the expectation of the geometric sum of a geometric Orstein-Uhlenbeck process, which is closely correlated with the arithmetic sum. So a smarter way of Monte Carlo simulation is to use simulated geometric summations as control variates for arithmetic ones. Computation shows the arithmetic sums and geometric sums are highly correlated and this control variate method can reduce the standard deviation by a factor greater than 100.

Let $X_0 = ln(\psi_0)$. It can be shown:

$$\psi_t = exp(X_0e^{-\alpha_0t} + (\phi_0 - \frac{\theta_2}{\alpha_0} - \frac{1}{2}\frac{\theta_4}{\alpha_0^2})(1 - e^{-\alpha_0t}) + \int_t^0 e^{\alpha_0(s-t)}\sqrt{\theta_4}dW^Q_s) \quad (4.17.3)$$

Define the following two variables $V_{oil,t}$ and $\tilde{V}_{oil,t}$:

$$V_{oil,t} := \int_t^T e^{-(r-\rho)(T-s)}\psi ds \quad (4.17.4)$$

Namely, the weighted sum of oil prices. $g$ can be represented as: $g(\psi,t) = me^{(r-\rho)(T-t)}E_t^Q[V_{oil,t}] + f(t)$. 

55
And the control variate $\tilde{V}_{oil,t}$:

$$\tilde{V}_{oil,t} = (T - t)\exp \left( \int_t^T \frac{ds}{T-t} \right) \left( -(r - \rho)(s - t) + \ln \psi \right) \quad (4.17.5)$$

Namely, the geometric sum of oil prices.

From the dynamics equation (4.19.3), we can derive:

$$\tilde{V}_{oil,t} = (T - t)\exp \left( \int_t^T -(r - \rho)(s - t) \frac{dt}{T-t} + \int_t^T x_t e^{-\alpha_0(s-t)} + \left( \phi_0 - \frac{\theta_2}{\alpha_0} - \frac{1}{2} \frac{\theta_4}{\alpha_0} \right) (1 - e^{-\alpha_0(s-t)}) \frac{dt}{T-t} + \int_t^T \int_t^s e^{\alpha_0(r-s)} \sqrt{\theta_4} dW^Q_r \frac{ds}{T-t} \right) \quad (4.17.6)$$

After calculation, we can derive (see Appendix A.2):

$$\tilde{V}_{oil,t} = (T - t)\exp(\tilde{H})$$

$$\tilde{H} \sim N(\mu, \sigma^2)$$

$$\mu = -(r - \rho) \frac{T-t}{2} + \frac{x_t}{\alpha_0(T-t)} \left( e^{-\alpha_0 t} - e^{-\alpha_0 T} \right) + \left( \phi_0 - \frac{\theta_2}{\alpha_0} - \frac{1}{2} \frac{\theta_4}{\alpha_0} \right) \left( 1 - e^{-\alpha_0 (t-\frac{T-t}{2})} \right)$$

$$\sigma^2 = \frac{\theta_4}{(T-t)^2} \left( \frac{1}{4\alpha_0^2} \left( e^{2\alpha_0(T-t)} - 1 \right) - \frac{T-t}{2\alpha_0} - \frac{(T-t)^2}{2\alpha_0} \right)$$

(4.17.7)

The Monte Carlo simulation scheme works as follows. First of all, we discretize total time horizon $T - t$ into $M$ equal space of length $\Delta t$: $t_0, t_1, ..., t_j, ..., t_M$ and simulate $N$ discretized paths of oil price $\hat{\psi}$ according to the Euler-Maruyama method:

$$\hat{\psi}_{i,j} = \hat{\psi}_{i,j-1} + \alpha_0(\phi_0 - \frac{\theta_2}{\alpha_0} - \ln \hat{\psi}_{i,j-1}) \hat{\psi}_{i,j-1} \Delta t + \hat{\psi}_{i,j-1} \sqrt{\theta_4} \Delta z_{i,j}^Q \quad i = 1, 2, ..., N$$

$$j = t_1, t_2, ..., t_M.$$
And we calculate the estimates of \( V_{oil,t}, \tilde{V}_{oil,t} \) on each path: we denote them by \( \hat{V}_{oil,t} \) and \( \hat{\tilde{V}}_{oil,t} \) respectively.

\[
\hat{V}_{oil,t,i} = \sum_j e^{-(\rho)\Delta t} \psi_{i,j} \Delta t \quad (4.17.9)
\]

\[
\hat{\tilde{V}}_{oil,t,i} = (T - t) exp(\sum_j \Delta t (-(\rho)(t_j - t) + ln \psi_{i,j})) \quad (4.17.10)
\]

From (4.17.7), we can calculate \( E^Q_t[\tilde{V}_{oil,t}] \) analytically:

\[
E^Q_t[\tilde{V}_{oil,t}] = (T - t) exp(\mu + \sigma^2/2) \quad (4.17.11)
\]

Then we can calculate the estimate of \( E^Q_t[V_{oil,t}] \) using the approach of control variate:

1. First we regress \( \hat{V}_{oil,t,i} \) on \( \hat{\tilde{V}}_{oil,t,i} \) to estimate the slope \( \beta \).

2. Then we calculate the estimate of \( E^Q_t[V_{oil,t}] \) with the formula:

\[
\hat{V}_{oil,t} = \hat{V}_{oil,t,i} - \beta(\hat{\tilde{V}}_{oil,t,i} - E^Q_t[\tilde{V}_{oil,t}])
\]

Namely, the mean of \( \hat{V}_{oil,t,i} - \beta(\hat{\tilde{V}}_{oil,t,i} - E^Q_t[\tilde{V}_{oil,t}]) \). With control variate, the higher correlation (\( \rho_{oil} \)) between \( \hat{V}_{oil,t,i} \) and \( \hat{\tilde{V}}_{oil,t,i} \), the more accurate is the estimate. We can calculate the empirical standard deviation of \( \hat{V}_{oil,t} \):

\[
Var(\hat{V}_{oil,t}) = Var(\hat{V}_{oil,t,i}) + \beta^2 Var(\hat{\tilde{V}}_{oil,t,i}) - 2\beta Cov(\hat{V}_{oil,t,i}, \hat{\tilde{V}}_{oil,t,i}) = (1 - \rho^2_{oil}) Var(\hat{V}_{oil,t,i}) \quad (4.17.12)
\]

Where \( \rho_{oil} \) is the correlation between \( \hat{V}_{oil,t,i} \) and \( \hat{\tilde{V}}_{oil,t,i} \).
After the calculations, we can calculate the value function:

\[ J(W, \psi, t) = -e^{\frac{1}{2}\xi^T C^T (C C^T)^{-1} C \xi \frac{1-\eta}{\eta} (T - t)}(e^{(r-\rho)(T-t)}W + g(\psi, t))^{1-\eta} \]

\[ g(\psi, t) = me^{(r-\rho)(T-t)}V_{oil, t} + f(t) \]

(4.20)

Where we substitute \( \hat{V}_{oil, t} \) for \( V_{oil, t} \). If we transform the solution back to the case where the utility function is the conventional form: \( U(W) = \frac{W^{1-\eta}-1}{1-\eta} \) instead of the simplified form: \(-W^{1-\eta}\), the solution is:

\[ \frac{1}{1-\eta}(e^{\frac{1}{2}\xi^T C^T (C C^T)^{-1} C \xi \frac{1-\eta}{\eta} (T - t)}(e^{(r-\rho)(T-t)}W + g(\psi, t))^{1-\eta} - 1) \] (4.20.1).

Now that we have the analytical solution of the value function, we can calculate the optimal portfolio choice according to (4.8):

\[ w^* = -\frac{J_W}{J_{WW} W} (C C^T)^{-1} C \xi - \frac{J_{W \psi}}{J_{WW} W} (C C^T)^{-1} C \sigma^T \]

\[ = \frac{\delta(t)W + g(\psi, t)}{\lambda(t) W} (C C^T)^{-1} C \xi - \frac{g_\psi}{\lambda(t) W} (C C^T)^{-1} C \sigma^T \] (4.21)

Where \( g_\psi \) can be calculated by perturbing the value of \( \psi \) andcalculate the finite difference divided by perturbation:

\[ g_\psi = \frac{g(\psi + \Delta \psi, t) - g(\psi, t)}{\Delta \psi} \] (4.22)

The values of \( g(\psi + \Delta \psi, t) \) and \( g(\psi, t) \) are calculated using the Monte Carlo simulation scheme described above.

The method we use in this chapter calculates the value function (4.20) in a straightforward manner. However, using the arbitrage logic often applied in deriva-
tive pricing, we can arrive at the answer (4.20) more smartly. The term in the inner bracket of (5.20): $e^{(r-\rho)(T-t)}W + g(\psi, t)$ is equal to $e^{(r-\rho)(T-t)}(W + e^{-(r-\rho)(T-t)}g(\psi, t))$. $(W + e^{-(r-\rho)(T-t)}g(\psi, t))$ is the value of initial wealth plus the present value of cash flows from oil reserve. At every instantaneous moment, consumption of the rate $\rho$ is spent. As a result, the effective risk-free rate is $(r - \rho)$. Thus, we use $(r - \rho)$ instead of $r$ to discount all cash flows. One can use the following trading strategy to replicate the result in (4.20):

1. Short sell a portfolio of the value $e^{-(r-\rho)(T-t)}g(\psi, t)$ at time $t$. The portfolio is constructed with securities that can replicate all the cash flows from oil reserve. This can be done because none of the risk factors in oil price is idiosyncratic according to market completeness assumption. So we can construct the infinitesimal portfolio to replicate each instantaneous cash flow from oil reserve. Add all the infinitesimal portfolios together we can get the replicating portfolio for all cash flows. Denote the portfolio weight as $w_{oil}$. The elements of $w_{oil}$ consist of each asset’s value divided by the wealth $W_t$.

2. Construct the efficient portfolio $w_{efficient}$ for the total of assets: $W^{total}_t = (W_t + e^{-(r-\rho)(T-t)}g(\psi, t))$. The objective is to maximize expected terminal utility: $E[U(W^{total}_T)]$. Deduct the portfolio sold in step 1 from this efficient portfolio to calculate the overall optimal portfolio.

3. Use cashflows from the oil reserve to fund the borrowed portfolio. Dynamically update the shorted portfolio and the efficient portfolio as total asset value $W^{total}$ changes.
This dynamic trading strategy replicates cash flows in the integrated optimal solution we have just solved. Thus, instead of solving the complicated differential equations, we can combine financial assets with oil reserve assets and maximize the final expected utility of total asset value \( E[U(W_{\text{total}})] \) to get the solution. The solution is also easy to get: 

\[
J(W_{\text{total}}, t) = \exp\left(\frac{1}{2} \xi^T C^T (CC^T)^{-1} C \xi \frac{1-e^{-\eta}}{\eta} (T-t) \right) \left( e^{(r_{\text{effective}} - \rho)(T-t)} W_{\text{total}} \right)^{1-\eta}.
\]

We simply need to replace the effective risk-free rate \( r_{\text{effective}} \) with \( r - \rho \), and \( W_{\text{total}} \) with the total wealth of initial wealth plus discounted cash flows from oil reserve: 

\[
W_{\text{total}} = W_t + e^{-(r-\rho)(T-t)} g(\psi, t).
\]

\( g(\psi, t) \) can be calculated using risk-neutral measure pricing and Monte Carlo simulation approach. The optimal trading strategy is comprised of two parts: one is the efficient portfolio to maximize expected utility: \( w_{\text{efficient}} \); the other is the short-selling portfolio \( w_{\text{oil}} \). \( w_{\text{oil}} \) is exactly the opposite of hedging portfolio \( w_{\text{hedge}} \): 

\[
w_{\text{hedge}} = -w_{\text{oil}}.
\]

The optimal asset allocation is: 

\[
w_{\text{opt}} = w_{\text{efficient}} - w_{\text{oil}} = w_{\text{efficient}} + w_{\text{hedge}}.
\]

The reason that the optimal strategy can be replicated and the reason that the integrated optimal asset allocation can be solved are the same: there is "disentanglement" between investment and oil business decisions. The oil reserve is evaluated separately because oil business decisions can be solved as a standalone problem. We propose a set of general Disentanglement Conditions in chapter 5 to fully explore this elegant structure.

Under the same assumptions and model specifications, we can easily extend the results to the case of multiple income sources. For example, the Sovereign Wealth Fund has incomes from two resources instead of one. Let the prices of two resources to be \( \psi_1 \) and \( \psi_2 \). Their production cost unit rate is \( c_1 \) and \( c_2 \) respectively. Their production rate times transfer rate are \( m_1 \) and \( m_2 \) respectively. And with market completeness, their dynamics follows:

\[
d\psi_1 = \alpha_1(\psi_1)dt + \sum \sigma_{\psi_1j} dZ_j
\]
\[d\psi_2 = \alpha_2(\psi_2)dt + \sum \sigma_{\psi,j}dZ_j\]

Then, the optimal portfolio choice is:

\[
w^* = -\frac{J_W}{J_W J_W W} (C C^T)^{-1} C \xi - \frac{J_{W\psi_1}}{J_W J_W W} (C C^T)^{-1} C \sigma_{\psi,1}^T - \frac{J_{W\psi_2}}{J_W J_W W} (C C^T)^{-1} C \sigma_{\psi,1}^T
\]

And the value function is:

\[
J(W, \psi, t) = \frac{1}{1-\eta} \left( \exp\left(\frac{1}{2} \xi^T (C C^T)^{-1} \xi (T-t) \right)(e^{(r-\rho) (T-t)W} + g_1(\psi_1, t) + g_2(\psi_2, t))^{1-\eta} - 1 \right)
\]

\(g_1, g_2\) satisfy the equations below respectively:

\[
\delta m(\psi_1 - c_1(t)) + g_t + \alpha_0(\psi_0 - \frac{\theta_2}{\alpha_0} - \ln(\psi_1))\psi_1 g_{1,\psi} + \frac{1}{2} g_{1,\psi^2} = 0
\]

\[
\delta m(\psi_2 - c_2(t)) + g_t + \alpha_0(\psi_0 - \frac{\theta_2}{\alpha_0} - \ln(\psi_1))\psi_1 g_{1,\psi} + \frac{1}{2} g_{1,\psi^2} = 0
\]

The number of income resources can be easily extended to \(N\).

Next, we can analyze the case for Norwegian Sovereign Wealth Fund with real data. The parameters can be estimated. We define the risky asset universe to include the U.S. stock, developed markets stock, emerging markets stock, the U.S. bond, Europe bond and commodities. The risk-free asset return is approximated by 3-month Treasury bill return. The estimated parameters in table 4.1.

To illustrate the added value of integrated optimal solution, we compare the integrated asset allocation strategy with the “simple optimal” strategy which is calculated without taking into account oil reserve assets. If we do not take into account
oil income risks, the investor is simple solving a typical optimization problem in asset allocation at each time point $t$: $\max_w E_t[W_T]$. The “simple optimal” asset allocation is just a mean-variance optimal portfolio according to investor’s risk aversion parameter:

$$w^* = \frac{1}{\eta}(CC^T)^{-1}C\xi \quad (4.23)$$

We call this policy the simple asset allocation strategy. Under such policy, the wealth $W'$ of the sovereign wealth fund follows the dynamics below:

$$dW' = (r - \rho + \frac{1}{\eta}\xi^T C^T (CC^T)^{-1}C\xi)W' dt + W' \frac{1}{\eta}\xi^T C^T (CC^T)^{-1}CdZ + m(\psi_t - c(t))dt$$

(4.24)

Denote $r - \rho + \frac{1}{\eta}\xi^T C^T (CC^T)^{-1}C\xi := r_p; \frac{1}{\eta}\xi^T C^T (CC^T)^{-1}C := v$. After calculation (see Appendix A.3 ), we can show:

$$W'_T = W'_0 \exp(r_p T - \frac{1}{2}v\xi^T T + vZ_T) + \int_0^T e^{r_p(T-s)-\frac{1}{2}v\xi^T (T-s)+v(Z_T-Z_s)}m(\psi_s - c(s))ds$$

(4.24.1)

To show the added value of integrated optimal solution, we calculate constant equivalents of final wealth by two strategies for comparison. In constant equivalent calculation, we first calculate the expected final utility values $E(U(W_T))$ and $E(U(W'_T))$. And then invert them using the utility function: $W^e_T = U^{-1}(E(U(W_T)))$; $W'^{e,e}_T = U^{-1}(E(U(W'^e_T)))$. $W^e_T$ and $W'^{e,e}_T$ are called constant equivalents. They are the constant wealth values that have the same utility values as the uncertain random variables $W_T$ and $W'_T$ do respectively. Hence, they are suitable for comparing the two strategies. We write down the calculation procedures below:
First of all, we simulate discretized asset returns and oil paths in a consistent way according to their covariance matrix $CC^T$ and covariance $\sigma_\psi$. Then we calculate $\hat{W}'$ accordingly. Let $i$ denote the path number and $j$ denote the time interval number with time intervals $t_0, t_1, ..., t_j, ...$

$$
\hat{W}'_i = W'_0 \exp(r_p T - \frac{1}{2} \nu \nu^T T + \nu Z_{i,T}) + \sum_j e^{r_p(T-t_j)-\frac{1}{2} \nu \nu^T(T-t_j)+\nu(Z_{i,T}-Z_{i,t_j})} m(\psi_{i,j} - c(t_j)) \Delta s
$$

Finally we can calculate the utility value for each path: $\hat{U}'_i := U(\hat{W}'_i)$ and calculate the constant equivalent $\hat{U}' = \frac{1}{N} \sum_i [\hat{U}'_i]$.

Next, we show the computational results.

We use 6 asset classes to represent global risky asset universe: U.S. Stock, Developed Markets Stock, Emerging Markets Stock, U.S. Bond, Developed Markets Bond, Emerging Markets Bond and Commodity. The proxy indices for each of the asset classes are S & P 500 (Standard & Poor’s 500 Index), STOXX 600 (STOXX European 600 Index), MSCI EM Index (MSCI Emerging Markets Index), Barclays U.S. Aggregate Bond Index, Barclays Euro Aggregate Bond Index and GSCI Index (Goldman Sachs Commodity Index) respectively. We use Treasury 3-Month bill as the proxy for risk-free asset. For oil price, we use the Brent Oil spot price data.

For all the proxy indices and treasury bill, we use monthly total return index data (the last trading day of each month) from April 2007 to March 2015. For oil price, we use Brent Oil price during the same time range. The data are used to estimate mean excess return over risk-free return and covariance matrix for asset
classes and oil price. The estimation results are shown as in Table 4.1.

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P 500</th>
<th>STOXX 600</th>
<th>MSCI EM</th>
<th>Barclays Euro Agg</th>
<th>Barclays U.S. Agg</th>
<th>GSCI</th>
</tr>
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<tbody>
<tr>
<td>Excess Return</td>
<td>7.23%</td>
<td>0.71%</td>
<td>3.03%</td>
<td>5.70%</td>
<td>7.48%</td>
<td>0.68%</td>
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<td>Volatility</td>
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<td>14.70%</td>
<td>17.98%</td>
<td>8.62%</td>
<td>12.63%</td>
<td>17.48%</td>
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<tr>
<td>Correlation</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>S&amp;P 500</td>
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<td>0.1301</td>
<td>0.1428</td>
<td>0.1214</td>
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<tr>
<td>STOXX 600</td>
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<td>-0.2978</td>
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<tr>
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<td>-0.3069</td>
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<tr>
<td>Barclays Euro Agg</td>
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<td>-0.3364</td>
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<tr>
<td>Barclays U.S. Agg</td>
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<td>0.3869</td>
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</tbody>
</table>

All the relevant parameters are calculated according to the statistics of Norway’s oil industry and Norway’s Government Pension Fund Global. They are listed in Table 4.2.

The parameters in oil price dynamics are estimated using maximal likelihood method. Because the mean-reversion parameters include the long-term equilibrium price, we use longer data for the estimation of mean-reversion parameters: daily Brent crude spot data from May.20, 1987 to April.21, 2014. We use data of Brent crude spot and the proxies of asset classes to calculate the correlation between oil
<table>
<thead>
<tr>
<th>Table 4.2: Relevant Parameters</th>
</tr>
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<tbody>
<tr>
<td>Intial Wealth Level</td>
</tr>
<tr>
<td>Time Horizon</td>
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<tr>
<td>Risk-free Rate</td>
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<tr>
<td>Consumption Rate</td>
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<tr>
<td>Cost Function Coefficient</td>
</tr>
<tr>
<td>Long-term Log Oil Price Mean</td>
</tr>
<tr>
<td>Mean Reversion Velocity</td>
</tr>
<tr>
<td>Log Oil Price Volatility</td>
</tr>
</tbody>
</table>

price and returns of assets. The results are shown in Table 4.3.

<table>
<thead>
<tr>
<th>Table 4.3: Correlation Between Oil Price and Assets</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500 STOXX 600 MSCI EM Barclays Euro Agg Barclays U.S. Agg GSCI</td>
</tr>
<tr>
<td>---------------------------------------------------</td>
</tr>
<tr>
<td>0.028 0.0686 0.1912 -0.3989 -0.3090 0.8313</td>
</tr>
</tbody>
</table>
Then we can follow equations (4.17.1)-(4.17.12), (4.20)-(4.22) to calculate the optimal asset allocation strategies and value functions. We can also compare the constant equivalents of final wealth by our integrated optimal strategy: \( W^e_T \) and by “simple optimal” strategy: \( W'^e_T \).

For different numbers of risk tolerance \( \eta \), the value of constant equivalent by integrated optimal strategy: \( W^e_T \) and that by “simple optimal” strategy \( W'^e_T \) are listed below in Table 4.4.

<table>
<thead>
<tr>
<th>Risk Tolerance ( \eta )</th>
<th>Integrated (NOK bil.)</th>
<th>Simple (NOK bil.)</th>
<th>Wealth Diff</th>
<th>TR Diff %</th>
<th>Ann Return Diff %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,750,500</td>
<td>1,490,000</td>
<td>260,500</td>
<td>17.5</td>
<td>1.6</td>
</tr>
<tr>
<td>2</td>
<td>123,030</td>
<td>112,570</td>
<td>10,460</td>
<td>9.3</td>
<td>0.9</td>
</tr>
<tr>
<td>3</td>
<td>49,860</td>
<td>43,660</td>
<td>6,200</td>
<td>14.2</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>31,750</td>
<td>27,611</td>
<td>4,139</td>
<td>15.0</td>
<td>1.4</td>
</tr>
<tr>
<td>5</td>
<td>24,214</td>
<td>21,367</td>
<td>2,847</td>
<td>13.3</td>
<td>1.3</td>
</tr>
<tr>
<td>6</td>
<td>20,210</td>
<td>18,037</td>
<td>2,173</td>
<td>12.1</td>
<td>1.1</td>
</tr>
<tr>
<td>7</td>
<td>17,768</td>
<td>15,888</td>
<td>1,880</td>
<td>11.8</td>
<td>1.1</td>
</tr>
<tr>
<td>8</td>
<td>16,129</td>
<td>13,696</td>
<td>2,433</td>
<td>17.8</td>
<td>1.6</td>
</tr>
<tr>
<td>9</td>
<td>14,956</td>
<td>13,297</td>
<td>1,659</td>
<td>12.5</td>
<td>1.2</td>
</tr>
<tr>
<td>10</td>
<td>14,084</td>
<td>12,364</td>
<td>1,720</td>
<td>13.9</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Figure 4.1 shows the constant equivalent of final wealth by integrated optimal strategy and that by “simple” optimal strategy for different risk tolerance parameters.

Since the risk tolerance of 1 is too extreme for the institutional investors, we hereby show the figure of constant equivalent comparison of risk tolerance only from \( \eta = 2 \) to \( \eta = 10 \), as in Figure 4.2.
To further illustrate the effect of integrated optimal asset allocation, we use different initial values of wealth and risk tolerance numbers to calculate the constant equivalent of final wealth by integrated optimal strategy and that by “simple optimal” strategy. The comparison of constant equivalents of final wealth is shown in Figure 4.3. The comparison of total return loss of constant equivalent of final wealth is shown in Figure 4.4. The comparison of total loss percentage of constant equivalents of final wealth is shown in Figure 4.5. The comparison of annualized return loss of constant equivalents of wealth is shown in Figure 4.6.

We can also use this analytical framework to analyze critical relationships. We list three questions of interests here:

1. How does transfer policy affect final wealth?
2. What is the valuation of an oil production technology to the SWF? Should the state invest in oil production technology R&D?
3. What is the valuation of additional oil reserves to the SWF? Should the state fund oil reserve exploration?

These are all important questions of concern to the SWF and the government. Without the analytical framework we propose, one may find them hard to answer and come up with a quantitative estimate. We study these questions one by one using the framework.

For the first question, to study the effect of transfer policy, we change the value of parameter $\gamma$ and re-calculate constant equivalents of wealth at different risk aversions. Other parameters are kept the same as shown in Table 4.2. The results are shown in Figure 4.7.
At risk aversion of 5, the constant equivalents of final wealth at different transfer rates are shown in Figure 4.8.

One can see the differences in constant equivalents of wealth caused by changes in transfer policy. The more transfer is made, the higher future wealth level the SWF will have. We can calculate the sensitivity of transfer policy at the current status.

At risk aversion of 5, one additional percentage of transfer increases the constant equivalent of future wealth by 5.2570 billion NOK.

On the other hand, changing the transfer policy will also have effects on oil industry. The government can use our analytical framework as a decision support tool to conduct the cost-benefit analysis. Similarly, the SWF can estimate the effects of different spending policies by changing \( \rho \). The results are shown in figure 4.9 and 4.10.

For the second question, the oil production technology is reflected in the production cost function. Advances in oil production technology lower costs. So the parameter in cost function: \( c_0 \) is decreased due to technological advances in production. To study the effect of technology advances on the long-term wealth of the SWF, we change the value of parameter \( c_0 \) and re-calculate the constant equivalents of wealth. The results are shown in Figure 4.11 and 4.12. The government and SWF can use these results to decide whether and how much investments should be made in production technologies.

One can see the different constant equivalents of wealth caused by different levels of technology advances. The more advanced production technology is developed, the higher future wealth level the SWF will have. We can calculate the sensitivity of
technology advances at the current status of the SWF:

At risk aversion of 5, one additional unit of production cost parameter decrease caused by technology advances increases the constant equivalent of future wealth by 1.5 billion NOK.

Consider a R&D project of production technology. The project is estimated to decrease $c_0$ by $y$ if successful. The project is estimated to cost $C_{c_0}$. And the the successful rate is $rate_s$. The analytical framework we propose can help the government decide whether to fund this R&D project using cost-benefit analysis:

Suppose the sensitivity of production cost parameter to the constant equivalent of future SWF wealth is $rate_{c_0}$. And suppose the sensitivity of current wealth to the constant equivalent of future wealth is $rate_W$. If the expected added value $rate_s y rate_{zeta}$ is larger than the costs $rate_W C_{c_0}$, the R&D project is worth funding. Otherwise, it is not.

For the third question, the value of additional oil reserves to the SWF can be calculated by increasing the oil reserve level parameter $k$. The results are shown in Figure 4.13 and 4.14. The government and SWF can use these results to decide whether and how much investments should be made in oil field explorations.

One can see the different constant equivalents of wealth caused by different oil reserve levels. The more oil reserve is available, the higher future wealth level the SWF will have. We can calculate the sensitivity of the oil reserve level change at the current status:
At risk aversion of 5, one additional unit of oil reserves increases the constant equivalent of future wealth by 1.2e+03 billion NOK.

Consider an oil reserve field exploration project. The project will find $k_{field}$ amount of additional oil reserves. $k_{field}$ is a random variable. The project is estimated to cost $C_{field}$. The analytical framework we propose can help the government decide whether to fund this exploration project using cost-benefit analysis:

Suppose the sensitivity of additional oil reserves to the constant equivalent of future SWF wealth is $rate_{oil}$. If the expected added value $rate_{oil}E[k_{field}]$ is larger than the opportunity cost of wealth: $rate_{W}C_{field}$, the exploration project is worth funding. Otherwise, it is not.

We here analyze two case studies of funding oil-related projects. Even though the oil-based SWF should hedge oil risks from an asset allocation perspectives and avoid related assets, funding oil production technology R&D and oil explorations may be beneficial to the SWF. The government can use our analytical framework as a decision support tool to analyze these “invest-in-your-own-business” type of decisions.

In this chapter, we make some modeling assumptions to apply the general framework to oil-based SWFs. In general, the HJB equation (2.11) of the integrated asset allocation problem (2.6) with boundary conditions (2.12) is very hard to solve analytically or numerically. However, in this chapter, we are able to solve the integrated asset allocation problem for oil-based SWFs semi-analytically with the right amount of modeling assumptions. In chapter 5, we propose a list of general conditions that make the general integrated asset allocation problem (2.6) solvable. These conditions are universally true for I-B complexes under our study. We
also develop a machinery to solve the general problem (2.6) under these conditions in chapter 5. Then, we can apply the general framework to all types of I-B complexes.

An important assumption we make in this chapter is on business decisions: oil production. We assume deterministic oil production. So the analysis in the chapter incorporates the intrinsic value of oil reserve assets. There is also an option premium in addition to the intrinsic value of oil reserves due to flexible production decisions. To incorporate the option premium, we need to solve for optimal business decisions (oil production decisions) rather than assuming deterministic production policies. In chapter 6, we introduce real option pricing techniques to value the option premium of oil reserves. Using the machinery developed in chapter 5 and the option valuation of oil reserves, we incorporate the optionality of business decision-making into the integrated asset allocation decisions for oil-based SWFs in chapter 7.
Figure 4.1: Constant Equivalent of Final Wealth
Figure 4.2: Constant Equivalent of Final Wealth 2
Figure 4.3: Constant Equivalent of Final Wealth: Initial Wealth Level, Risk Tolerance
Figure 4.4: Constant Equivalent of Final Wealth: Total Loss
Figure 4.5: Constant Equivalent of Final Wealth: Total Loss Percentage
Figure 4.6: Constant Equivalent of Final Wealth: Annual Return Loss
Figure 4.7: Constant Equivalents of Final Wealth at Different Transfer Rates and Risk Aversions
Figure 4.8: Constant Equivalents of Final Wealth at Different Transfer Rates (Risk Aversion = 5)
Figure 4.9: Constant Equivalents of Final Wealth at Different Spending Rates and Risk Aversions
Figure 4.10: Constant Equivalents of Final Wealth at Different Spending Rates (Risk Aversion = 5)
Figure 4.11: Constant Equivalents of Final Wealth at Different Cost Coefficients and Risk Aversions
Figure 4.12: Constant Equivalents of Final Wealth at Different Cost Coefficients (Risk Aversions = 5)
Figure 4.13: Constant Equivalents of Final Wealth at Different Reserve Levels and Risk Aversions
Figure 4.14: Constant Equivalents of Final Wealth at Different Reserve Levels (Risk Aversions = 5)
Chapter 5
Separating Business Decisions
From Investment Decisions

In chapter 2 and 3, we show the structure of general framework and generic derivations. In chapter 4, we make some assumptions to make the framework both realistic and solvable for oil-based SWFs. To apply the framework to various types of real-world case studies, we need to solve the integrated asset allocation problem explicitly. That means we need to solve equation (2.11)-(2.12). Equation (2.11):

$$\max_{(u,v,w,q) \in U} \lambda p(t, u, v, K) + (1-\lambda) U(q, W, t) + \frac{\partial V}{\partial K} b(t, K, u, v) + \frac{\partial V}{\partial W} \left[ \sum_{j=1}^{m} w_j (r_j - r) W + v - q \right] + \frac{1}{2} \frac{\partial^2 V}{\partial K^2} \sum_{j=1}^{n} \left( \sigma_j^K(t, K, u) \right)^2 + \frac{1}{2} \frac{\partial^2 V}{W^2} \sum_{i,j} \sigma_i^W w_i w_j + \frac{1}{2} \frac{\partial^2 V}{\partial K \partial W} W \sum_{i,j} w_j \sigma_i^K(t, K, u) \sigma_j^W \rho_{i,j}^W + \frac{\partial V}{\partial t} = 0$$

Equation (2.12):
\[ V(K, W, \tau) = R(K, W, \tau) \]

To find a solution to these equations is very hard if not impossible. One can supposedly discretize the stochastic optimal control problem and use finite-different approximations for the differentials (see van Handel (2007) chapter 6.6 for reference):

\[
\frac{\partial^2 V(x)}{\partial x^2} \approx \frac{V(x+\delta)-2V(x)+V(x-\delta)}{\delta^2}, \quad \frac{\partial V(x)}{\partial x} \approx \frac{V(x+\delta)-V(x-\delta)}{2\delta}
\]

And then, one can use the method of Markov chain approximation to get approximate numerical solutions to the HJB equation. However, due to the high dimension of decision and state variable space, this method is hardly applicable.

On the other hand, it has been shown in literature that, with the proper functional forms, optimal investment decisions with consumption (Merton (1969)), optimal investment decisions with consumption and incomes can be solved with analytical (Cairns, Blake and Dowd (2006)) or approximate solutions (Campbell and Viceira (2001) chapter 6).

So, if we can somehow separate business decisions from investment decisions, we may be able to solve for business decisions first, and then solve the integrated asset allocation problem with incomes explicitly. Also, this two-stage approach seems to be the most natural way to solve the integrated asset allocation problem. As discussed in the introduction chapter, certain assumptions should be imposed on the generic I-B complex framework to make it most resemble the real-world I-B complexes. These assumptions can be viewed as regularity conditions for I-B complexes.

Next, we provide a list of conditions that makes business decisions separate from investment decisions in the integrated asset allocation problem (2.6). These
conditions serve two purposes: 1. make the general framework closely resemble the structure of real-world I-B complexes; 2. make the general integrated asset allocation problem solvable via a generalized machinery. One can view these “disentanglement” conditions as the “maximum set” of real-world I-B complexes and, at the same time, the “minimum set” of assumptions to make the integrated asset allocation problem solvable.

Following the notations in chapter 4, we denote the vector of business risks \( (\sigma^K_j) \) to be \( \sigma_K \). The Disentanglement Conditions are listed as below:

**Disentanglement Conditions:**

1. **Preference weighting parameter** \( \lambda \) is 0

2. **Transfer decision** \( v \) is a percentage of the business payoff \( v = \gamma P^K(K^u,v, u_s, s) \) (5.1.1). Business payoff \( P^K \) depends only on business state, business decision and time. \( \gamma \) is a function of payoff \( P^K \). Profit function is \( p = P^K - v = (1 - \gamma)P^K \).

3. **Instant utility** \( U(q,s) \) depends only on consumption.

4. **Utility function at the endpoint** has the following semi-linear property:

\[
R(K,W) = F(W + \nu(K,T)) \quad (5.1.2)
\]

5. **Volatility of business variable(s)** \( \sigma_K \) is not subject to control: \( \frac{\partial \sigma_K}{\partial u} = 0 \)

6. **Market is complete and the investment opportunity set is constant.**

7. **All the functional forms in the problem** has enough order of smoothness.

The domain for decision variables are convex, closed and bounded.
Here, we state and prove the Separation Theorem:

**Separation Theorem**

If Disentanglement Conditions 1 – 7 are satisfied by the integrated asset allocation problem (as in (2.6)) and suppose the value function is smooth enough, then:

1. The business decisions \( u \) is separable from investment decisions and can be separately optimized.
2. The solved business decisions can be taken as inputs for solving the integrated asset allocation problem. The solution investment decisions solved by this two-stage approach is the same as the true optimal decisions of the integrated asset allocation problem (2.6).

**Proof:**

First, we solve the asset allocation optimization problem without business incomes:

\[
\max_{u,v,w,q} J[w] = E_0[\int_0^T U(q_t, t) dt + R(W_{\tau_{u,v}}, \tau_{u,v})]
\]

\[
dW_t^{w,q} = \sum_{j=1}^m w_j r_j W_t^{w,q} dt + (1 - \sum_{j=1}^m) r W_t^{w,q} dt + w_j \sigma_j W_t^{w,q} dB_{j,t} - q_t dt
\]

This problem (5.2.1) is an optimal asset allocation and consumption problem without complications of business incomes. One can refer to Merton (1969) for analytical solutions to the problem with certain functional forms. Denote the value function solution to this problem (5.2.1) as \( J(W, t) \). Then \( J(W, t) \) satisfies the HJB equation and boundary conditions of problem (5.3.1). Here, we follow the notations in chapter 4: risk premium vector \((r_j - r)\) is \( C \xi \).
The HJB equation for (5.2.1) is:

\[
\max_{w,q} U(q,s) + J_W \{ W[r + w^T C \xi] - q \} + \frac{1}{2} J_{WW} W^2 w^T C C^T w + J_t = 0 \quad (5.2.2)
\]

Boundary condition: \( J(W,T) = R(W) \).

We try the function of the form \( J(W + \nu(K,t),t) \) for the integrated asset allocation problem (2.6). First, we prove the first part of the Separation Theorem if the solution to (2.6) is of this functional form. For simplicity of notations, we assume the business state variable \( K \) and business decision \( u \) are both one-dimensional. It is not hard to extend to multi-dimensional case.

Due to Disentanglement Condition 2, the transfer decision is tied to the profit. Thus, we need to solve for \( u \) and \( v \) together as in deriving the optimal solution (3.8). Plug in equation (5.1.1) from Disentanglement condition 2 and substitute for \( v \). In addition, use Disentanglement Condition 1: \( \lambda = 0 \), and Disentanglement Condition 5: \( \frac{\partial \sigma_K}{\partial u} = 0 \). Hence, the equilibrium equation to solve for optimal business decision \( u^* \) is:

\[
\frac{\partial V}{\partial K} b_u(K, u^*, v^*, t) + \frac{\partial V}{\partial W} (\gamma P^K)_u(K, u^*, t) = 0 \quad (5.2.3)
\]

where \( v^* = \gamma P^K(K, u^*, t) \).

(5.2.3) is for interior solution of \( u^* \). The optimal solution may hit the boundary of decision domain. If the functional form of the solution is \( J(W + \nu(K,t),t) \), then we can prove that business decisions are separable from investment decisions.

First of all, if \( u^* \) is an interior point solution, then (5.2.3) is satisfied. We know \( \frac{\partial V}{\partial W} \) is positive due to the nature of utility function: more wealth is better. So we can
divide both sides of equation (5.2.3) by $\frac{\partial V}{\partial W}$ and get:

$$\frac{\partial V}{\partial W} b_u(K, u^*, v^*, t) + (\gamma P^K)_u(K, u^*, t) = 0 \quad (5.2.4)$$

If the functional form of the solution is $J(W + \nu(K, t), t)$, then:

$$\frac{\partial V}{\partial W} = J' = \nu(K, t) \quad (5.2.5)$$

Thus, (5.2.4) is: $\nu(K, t)b_u(K, u^*, v^*, t) + (\gamma P^K)_u(K, u^*, t) = 0$ (5.2.6). As a result, the optimal business decision only depend on $K$ and $t$.

Secondly, if $u^*$ is a corner solution, $u^*$ still depends only on $K$ and $t$. The left-hand side of (5.2.3) is the partial differential of $u$ derived from the HJB equation (2.11) under the Disentanglement Conditions. Which boundary the optimal business decision lies on depends on the sign of the left-hand side term of (5.2.3). Since $\frac{\partial V}{\partial W}$ is positive, if the solution has the form $J(W + \nu(K, t), t)$, then the left-hand side term of (5.2.3) is $\frac{\partial V}{\partial W}(\nu(K, t)b(K, u^*, v^*, t) + (\gamma P^K)_u(K, u^*, t))$, the sign of which only depends on $K$, $u^*$ and $t$. Hence, the optimal business decision rule is similar to conditions (3.8):

If $\nu(K, t)b(K, u, v, t) + (\gamma P^K)_u(K, u, t) > 0$ everywhere, then $u^* = u_{\text{max,t}}$.

If $\nu(K, t)b(K, u, v, t) + (\gamma P^K)_u(K, u, t) < 0$ everywhere, then $u^* = u_{\text{min,t}}$.

(5.3.1)

where $v = \gamma P^K(K, u, t)$. We assume sufficient smoothness for all the functions and regularity on decision domains. So all the possible outcomes are the two in (5.3.1), in case no interior-point solution is found. Otherwise, we can find an interior-point solution of $u^*$. 

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From Distentanglement Condition 6, we know that at the endpoint $V = R(K,W) = F(W + \nu(K))$. We know the functional form of $J(W + \nu(K,t),t)$ is satisfied at the endpoint:

$$
\frac{\partial V}{\partial K} = \frac{F' \cdot \nu'(K)}{F} = \nu'(K) \quad (5.3.2)
$$

where we pick $\nu(K,T) = \nu(K)$.

Thus, at the endpoint, business decision $u$ does not depend on wealth level $W$. Business decision and investment decision are separated at the final moment. If the value function $V$ satisfy the semi-linear condition as in Disentanglement Condition 6 everywhere, by the functional form of $J(W + \nu(K,t),T)$, the optimal business decisions only depends on business state variable(s) $K$ at each time point $t$. In conclusion, we prove that business decisions are separable from investment decisions if the functional form of $J$ is semi-linear: $J(W + \nu(K,t),T)$. Our next mission is to find a solution to the integrated asset allocation problem (2.6).

To illustrate the nature of dynamic programming in business decisions and valuation, we conceptually construct a valuation function for the business entity as follows:

1. At the endpoint $T$, we have shown in (5.3.2) that the optimal business decision only depends on the business state variable $K$. So we can solve the optimal business decision at the last instant $T - dt$ to $T$ according to optimal business decision rule (5.3.3):

$$
\nu_K(K)b_u(K,u^*,v^*,T) + (\gamma P^K)_u(K,u^*,T) = 0
$$

or:

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If $\nu_K(K)b(K,u,v,T) + (\gamma P^K)u(K,u,T) > 0$ everywhere, then $u^* = u_{\text{max},T}$.

If $\nu_K(K)b(K,u,v,T) + (\gamma P^K)u(K,u,T) < 0$ everywhere, then $u^* = u_{\text{min},T}$.

2. Due to Disentanglement Condition 7, the market is complete. Thus, there exists a unique risk-neutral measure $Q$ that can be used to price the business cash flows. At the instant $T - dt$, calculate the expected final business value $\nu(K_T)$ discounted by the effective risk-free rate $r_f$: $E_t^Q\left[\frac{\nu(K)}{1 + r_f dt}\right]$. Add the value of payoff at this instant: $P(K_T, u^*, T)dt$ to the expected final value to get the business value function at $T - dt$: $\nu(K, T - dt) = P(K_T, u^*, T)dt + E_t^Q\left[\frac{\nu(K)}{1 + r_f dt}\right]$. Obviously, the function only depends upon the value of $K$.

3. After we have solved for the business value at time $t$ and show the value function only depends upon the business state $K_t$ ($\nu(K,t)$), we can determine the optimal business decision at instant $t - dt$ to $t$ next according to the optimal decision rule (5.3.4):

$$\nu_K(K,t)b_u(K, u^*, v^*, t) + (\gamma P^K)_u(K, u^*, t) = 0$$

or:

If $\nu_K(K,t)b(K, u, v, t) + (\gamma P^K)_u(K, u, t) > 0$ everywhere, then $u^* = u_{\text{max},t}$.

If $\nu_K(K,t)b(K, u, v, t) + (\gamma P^K)_u(K, u, t) < 0$ everywhere, then $u^* = u_{\text{min},t}$.

And calculate the business value function at $t - dt$: $\nu(K, t - dt) = P(K_t, u^*, t)dt + E_t^Q\left[\frac{\nu(K,t)}{1 + r_f dt}\right]$ (5.3.5). The value function at $t - dt$ is obviously dependent on $K$ only.

4. Repeat the procedure at each instant and calculate the value function of business at all possible state $K$ and time $t$: $\nu(K,t)$. 

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In conclusion, we prove that the business decisions $u_t$ only depends upon the business state variable at each time point $t$. Business decisions are separable from investment decisions, if the functional form of solution is semi-linear in $W$ and $K$.

Next, we construct a machinery to link the separately solved business decisions to investment decisions, and then solve for the value function by the functional form of $J(W + \nu(K,t),t)$. Then we can prove the business and asset allocation decisions are truly the optimal solutions to the integrated decision-making problem (2.6) by the Verification Theorem of stochastic control (See Theorem 2.1).

We plug in the function: $J(W + \nu(K,t),t)$ into the HJB equation (2.11) and boundary condition (2.12).

$$
\begin{align*}
\max_{w,q} U(q,s) + J_W \{ W[r + w^T C\xi] - q \} + J_W \gamma P^K(K, u^*, t) + \frac{1}{2} J_{WW} W^2 w^T C C^T w + J_K b(K, u^*, v^*, t) + \frac{1}{2} \theta_1 J_{wW} J_{WW} - \frac{1}{2} \theta_2 J_{wW} W^T C \sigma_K^T + J_t &= 0 \\
\end{align*}
$$

where $u^*$ is the optimal business decisions computed by the dynamic programming procedure (5.3.4). $v^*$ is the transfer decision tied to the business decisions $u^*$.

$$
\begin{align*}
w^* &= -\frac{J_W}{J_{ww} W} (CC^T)^{-1} C \xi - \frac{J_{ww} K}{J_{ww} W} (CC^T)^{-1} C \sigma_K^T \\
q^* &= U^{-1}(J_W, s)
\end{align*}
$$

Plug in (5.4.2) and (5.4.3) into (5.4.1), we get:

$$
\begin{align*}
U(q^*, t) + J_W (rW - q^*) + J_W \gamma P^K(K, u^*, t) - \frac{1}{2} \theta_1 J_{wW} - \theta_2 J_{wW} J_{ww} K - \frac{1}{2} \theta_3 J_{ww} K + J_t &= 0
\end{align*}
$$
Following notations in chapter 4, \( \theta_1 = \xi^T C (CT)^{-1} C \xi, \theta_2 = \xi^T C (CT)^{-1} C \sigma_K^T, \theta_3 = \sigma_K^T (CT)^{-1} C \sigma_K^T, \theta_4 = \sigma_K \sigma_K^T. \)

Plug in the functional form of \( J(W + \nu(K, t), t) \) into (5.4.4), we get:

\[
U(q^*, t) + J_W(rW - q^*) + J_W \gamma P^K(K, u^*, t) - \frac{1}{2} \theta_1 \frac{J_W}{J_{WW}} - \theta_2 J_W \nu_K - \frac{1}{2} \theta_3 J_{WW} \nu_K^2 + J_W \nu_K b(K, u^*, v^*, t) + \frac{1}{2} \theta_4 J_{WW} \nu_K^2 + \frac{1}{2} \theta_4 J_W \nu_K + J_t + J_W \nu_t = 0 \tag{5.4.5}
\]

Since \( J(W, t) \) is the solution to (5.2.1), then \( J(W, t) \) satisfies:

\[
U(q^*, t) + J_W(rW - q^*) - \frac{1}{2} \theta_1 J_{WW} + J_t = 0 \tag{5.4.6}
\]

Now substitute \( W + \nu(K, t) \) for \( W \) and change the optimal consumption decision accordingly: \( q^* = U^{-1}(J_W(W + \nu, s), s) \). Moreover, \( \theta_3 = \theta_4 \) due to Disentanglement Condition 7 of market is completeness. (5.4.6) is transformed into:

\[
J_W(-r\nu + \gamma P^K(K, u^*, t) - \theta_2 \nu_K + \nu_K b(K, u^*, v^*, t) + \frac{1}{2} \theta_4 \nu_K + \nu_t) = 0 \tag{5.4.7}
\]

(5.4.7) is equivalent to the equation below:

\[
\nu_t - r\nu + \nu_K b^Q(K, u^*, v^*, t) + \frac{1}{2} \nu_K \sigma_K \sigma_K^T + \gamma P^K(K, u^*, t) = 0 \tag{5.4.8}
\]

where \( b^Q(K, u^*, v, t) = b(K, u^*, v^*, t) - \theta_2 \) is the business growth rate under risk-neutral measure \( Q \).

Hence, as long as we can find a solution to the partial differential equation (5.4.8) and the boundary condition \( \nu(K, T) = \nu(K) \), the function \( J(W + \nu(K, t), t) \) is a solution to the HJB equation (2.11) and boundary condition (2.12).
By Feynman-Kac theorem, the solution to (5.4.8) and boundary condition $\nu(K, T) = \nu(K)$ can be calculated as an expectation under the risk-neutral measure:

$$
\nu(K, t) = E^Q_t \left[ \int_t^T e^{-r(T-s)} \gamma P^K(K, u^*, s) ds + e^{-r(T-t)} \nu(K) \right] (5.4.9)
$$

Under the risk-neutral measure, the business state variable follows the dynamics:

$$
dK = b^Q(K, u^*, v^*, t) dt + \sigma_K dW^Q_t (5.4.10)
$$

One can see $\nu(K, t)$ is exactly the sum of fair values of future business income transfers. Thus, the solution to (5.4.8) and boundary condition $\nu(K, T) = \nu(K)$ is found.

Hence, the function $J(W + \nu(K, t), t)$ satisfies the Bellman equation (2.11) under the Disentanglement Conditions. In addition, $J(W + \nu(K, T), T) = R(W + \nu(K), T)$, hence satisfying Boundary condition (2.12) as well. From Verification Theorem (Theorem 2.1), we can prove $J(W + \nu(K, t), t)$ is the solution to the integrated asset allocation problem (2.6) under Disentanglement Conditions. Both parts of the Separation Theorem are proved. Proof completes.

Now that we have proved the Separation Theorem, we can discuss the universality of the Disentanglement Conditions. Condition 1. preference weighting parameter $\lambda$ being equal to zero means the decision-maker’s role is the investment manager. After all, the focus of our problem is integrated asset allocation problem. The decision-maker under study is the investment manager. Condition 2 means the transfer decision is a predetermined policy of business payoff. Condition 3 assumes consumption utility only depends on consumption itself and time. Condition 4 of
semi-linear property of utility function means the business asset value and wealth value are treated as the same by the decision-maker at the endpoint. After all, the business gets liquidated (converted to liquid cash) or goes obsolete (zero value) at the endpoint. So the value from business should be indistinguishable from the wealth value. Condition 5 assumes the business state variables’ volatility can not be controlled. Condition 6 assumes market completeness. Condition 7 is regularity assumption on functional forms and decision domains.

One can see that Conditions 1. 3. 4. are very natural assumptions for a decision-maker of the investment entity in an I-B Complex. Condition 2 is commonly used in practice. Condition 6 of market completeness is often a necessary condition for financial research. Otherwise, the problem is often unsolvable. Consumption 7 includes necessary regularity conditions for a mathematical problem. Only Condition 5 needs some further discussions about its generality. For example, one may argue that equity volatility of the business can be controlled by the business of how much debt to raise, in case that business state variables include equity. However, if we change the state variables to business asset and debt instead of business equity, the volatilities of these business variables is hardly subject to control. The prevailing effects on volatilities of asset and debt are often at the industry level. So we argue for the generality of Condition 5 if the business state variables are chosen appropriately. As long as the business is in a fairly competitive industry and affected by the overall industry conditions, which is not subject to control by any individual company, condition 5 is commonly satisfied. For example, the oil-producing business of Norway is more of a price-taker than can manipulate the market. The oil price volatility is not controllable by Norway’s oil-producing business. So Disentanglement Condition 5 is satisfied in this case.
The Separation Theorem is a powerful tool for solving integrated asset allocation problems. It allows the business entities of I-B Complexes to make business decisions as they see fit with no concern over decisions of investment entities. And the investment entities of I-B Complexes can simply take business decisions as given and optimize the integrated asset allocation problem without any concern over business side. This kind of separation in management is both practical and necessary for I-B Complexes. And in reality, sovereign wealth funds, pensions plans, foundations and endowments follow this kind of structure. As long as the general conditions of dis-entanglement apply, the Separation Theorem guarantees the solved asset allocation decisions are truly optimal in an integrated fashion. The integrated asset allocation problem can be modeled following the general framework we introduce in this paper. At the same time, the proof of the Separation Theorem provides the procedures to solve for the optimal integrated asset allocation:

Step 1. Solve the optimal business decisions $u^*$ independently. Calculate the business value function $\nu(K,t)$.

Step 2. Solve the asset allocation optimization problem without business income: (5.2.1). Calculate the investment value function $J(W,t)$.

Step 3. Integrate business with investment value function: $J(W + \nu(K, t), t)$. This integrated value function is the solution to the integrated asset allocation problem. Moreover, the integrated optimal asset allocation decisions can be calculated using equation (5.4.2):

$$w^* = -\frac{J_W}{J_{WW}W} (CC^T)^{-1}C\xi - \frac{J_{WK}}{J_{WW}W} (CC^T)^{-1}C\sigma_K^T$$
Now we can try the Separation Theorem on the integrated asset allocation problem in chapter 4. We can easily verify that the modeling specifications of the integrated asset allocation problem in chapter 4 (4.1)-(4.5) satisfy all the Disentanglement Conditions. Thus, we can follow the machinery in the Separation Theorem and solve the problem in a two-stage approach.

First, we solve the “pure” asset allocation problem without business incomes:

\[
\max_{w} J[w] = E_0[U(W_T)] \tag{5.5.1}
\]

\[
dW_t = W_t[r - \rho + w^T C \xi] dt + W_t w^T C \xi dZ
\]

The HJB equation for this problem is:

\[
\max_w J^*_W W[r - \rho + w^T C \xi] + \frac{1}{2} J_{WW} W^2 w^T C C^T w + J_t = 0 \tag{5.5.2}
\]

The boundary condition is

\[
J^*_W(W, T) = U(W) = -W^{1-\eta} \tag{5.5.3}
\]

The solution for (5.5.2)-(5.5.3) can be easily solved analytically:

\[
J^*(W, \psi, t) = -\exp((r - \rho + \frac{1}{2} \xi^T C^T (CC^T)^{-1} C \xi \frac{1-\eta}{\eta})(T - t)) W^{1-\eta} \tag{5.5.4}
\]

(5.5.4) is the similar to the solution Merton (1992) derives for an investor with CRRA utility function under continuous time.

Next, we calculate the business value function \(\nu^*(\psi, t)\). From (5.4.9), we get:

\[
\nu^*(K, t) = E_t^Q[\int_t^T e^{-(r-\rho)(T-s)} m(\psi - c(s)) ds] \tag{5.5.5}
\]

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where $Q$ is the risk-neutral measure. We plug in $(r - \rho)$ instead of $r$ as discount rate since the parameter in place of interest rate in the dynamics of wealth in problem (5.5.1) is $(r - \rho)$. $\nu^*(K,T) = \nu(K) = 0$. $\gamma P = m(\psi - c(t))$. From (5.4.10), the dynamics of $\psi$ under $Q$ is:

$$d\psi = \alpha_0(\psi_0 - \frac{\theta}{\alpha_0} - ln(\psi))\psi dt + \sqrt{\theta_4}\psi dW_t^Q \quad (5.5.6)$$

The integrated value function solution to problem (5.5.1) is:

$$J^*(W + \nu^*(\psi,t),t) =$$

$$-\exp((r-\rho + \frac{1}{2}\xi^TC(C^T)^{-1}C\xi^{1-\eta})(T-t))(W + E_t^Q[\int_t^T e^{-(r-\rho)(T-s)}m(\psi-c(s))ds])^{1-\eta}$$

(5.5.7)

One can see $J^*(W,\psi,t) = J(W,\psi,t)$ of (4.20). The solution we get using the Separation Theorem is exactly the same as what we get in chapter 4, where we use brute force to solve the HJB equation (4.6). If we transform the solution to the case where the utility function is the full form: $U(W) = \frac{W^{1-\eta}}{1-\eta}$ instead of the simplified form: $-W^{1-\eta}$, the solution is: $\frac{1}{1-\eta}(\exp(\frac{1}{2}\xi^TC(C^T)^{-1}C\xi^{1-\eta})(T-t))(e^{(r-\rho)(T-t)}W + g(\psi,t))^{1-\eta} - 1)$, exactly the same as (4.20.1). One minor note is that $\nu^*$ is not the same as $g$ in chapter 4 (4.17), since we use a slightly different calculation approach in chapter 4.

The machinery of two-stage calculation is much easier to follow than the brute force approach. And it is applicable to all cases that satisfy the Disentanglement Conditions. The Separation Theorem is a powerful tool to solve the integrated asset allocation problems of I-B complexes.
In chapter 7, we apply the Separation Theorem to the SWF’s integrated asset allocation problem where the production decisions are to be solved instead of pre-specified. Before we do that, we first need to introduce the real option technique for valuing oil reserves.
Chapter 6

Oil Production and Real Option Pricing

In this chapter, we show how to solve the business decision and price the oil reserve numerically using real option pricing theory. The approach introduced here can be used within the integrated asset allocation model of oil-based SWFs when production decision $u$ is to be optimized instead of specified.

As showed in chapter 3, the optimal oil production decision is either producing at the maximal capacity, or producing at the minimal requirement. The decision depends the sign of left-hand side of (3.7): $\lambda(1 - \gamma)(\psi_t - c(k)) - \frac{\partial V}{\partial k} + \frac{\partial V}{\partial W} \rho(\psi_t - c(k))$.

The conditions for optimal production decision follows (3.8). The decision depends on the state variables value: $(k, \psi, W, t)$. As discussed in chapter 3, this optimal decision rule is very similar to optimal American call option execution rule. Let us look at the right-hand side of conditions in (3.8).
The right-hand side terms in condition (3.8) are the threshold level for "executing" the American option, which is one unit of oil reserve. This threshold means: if the profit of executing one unit of oil reserve \( \psi_t - c(k) \) is greater than the value added by keeping one unit of oil reserve, then produce at the maximal capacity; otherwise, if the profit of executing one unit of oil production is less than keeping the unit of oil reserve for future production, then produce at the minimal requirement. The production cost, namely the strike price of the American option, is a function of oil reserve level. Production cost should gets increasingly high if at overcapacity level. Here we assume production cost at overcapacity is infinity.

To calculate the optimal execution (production) policy, one needs to calculate the threshold profit level \( \frac{\partial V}{\partial k} \lambda (1 - \gamma) + \frac{\partial V}{\partial W} \gamma \). According to our specification in section 3 (refer to section 3 last paragraph for reason): \( \lambda = 0 \), the profit threshold is \( \frac{\partial V}{\partial k} \). Also, the value function at terminal point \( \tau \) is a function of wealth \( W \): \( R(W, k) = R(W) \). As a result, the profit threshold is zero since \( \frac{\partial V}{\partial k} = 0 \) at the terminal point. The optimal production policy at terminal point is to produce as much as possible, as long as the oil price is above production cost. This terminal point is our starting point to solve for optimal production policy from backwards. However, if the threshold \( \frac{\partial V}{\partial W} \) is a function of both financial wealth \( W \) and reserve capital \( k \), the computation for solving this problem will be extremely difficult if not impossible. In this case, there exists an "entanglement" between business decisions and investment decisions. Both decisions need to solved together. With the numerous investable assets in financial markets, the dimension of decision variables (portfolio weights on assets and production decisions) is very high. It makes the dynamic programming problem very difficult to solve.

On the other hand, if the threshold is just a function of business state variable \( K = (k, \psi) \), the problem is much easier to solve. One can solve the business
decision-making problem as a standalone problem without concerns over investment decisions. The two sets of decisions are “separable”. Here, we can see the power of the Disentanglement Conditions and the Separation Theorem. We can easily verify the Disentanglement Conditions are satisfied by the general specifications of oil-based SWFs in chapter 3.

For condition 1, $\lambda = 0$ is satisfied. For condition 2, transfer policy is a percentage of the business income as in (3.6). And the business payoff $u(\psi - c(k))$ is a function of oil price, oil reserve level and oil production decision. For condition 3, instant utility is omitted in the SWF application. For condition 4, the semi-linear property is satisfied in the specifications. For condition 5, we assume oil price dynamics is not affected by production decisions and oil reserve’s dynamics does not have an uncertainty element. Condition 6 and 7 are satisfied too. Thus, we can apply the Separation Theorem and solve the oil production decisions as a standalone problem.

From the first part of the Separation Theorem, we know that the threshold level for oil price: $c + \frac{\partial V}{\partial W} \gamma + \frac{\partial V}{\partial W} \lambda (1 - \gamma) = c(k) + \frac{\partial V}{\partial W} \gamma \frac{\partial V}{\partial W}$ is a function of only business state variables $K = (k, \psi)$.

As discussed before, the oil reserve asset is very similar to a portfolio of American call options. Condition (3.8) is very similar to the optimal execution rule of American options. This type of option-like assets are not really financial options trades in the market. But they are coined as “real options”. The real option assets are typified by managerial options. Different managerial decisions may result in different payoffs. And the managerial choices are often done in an uncertain environment. Typical examples include natural resources (with production choices), factories (with production choices), land (with building choices), real estate (with renting
or sale options), business (with business strategies), R&D (with abandon option) and etc. Dixit and Pindyck (1994) gives a comprehensive treatment of “real options” in the context of option premiums in investment opportunities. There are a lot of literature studying real option theoretically and empirically.


However, the existing models are not perfectly suitable for valuation of oil reserve assets in our case. And the valuation techniques are not sophisticated enough to handle the exotic features of oil reserves. In this study, we borrow valuation approach from exotic option pricing theory and optimization techniques from machine learning area in order to solve the oil reserve valuation problem. Next, we examine the unique
features of oil reserve assets first.

To numerically calculate the valuation and the optimal execution policy for the option portfolio of oil reserves, we need to discretize the continuous-time set-up. Time is discretized into periods. The oil reserve is also discretized into small production units. Denote the production unit to be \( U_{\text{unit}} \). One can only choose to produce integral number of the unit. One unit of oil can be viewed as one American call option on oil. Unlike normal American call options, there are several exotic features for these “real options” of oil reserve. First of all, there are upper and lower constraints on the number of contracts that can be executed in each time period. The maximal amount \( U_{\text{max}} \) is the production capacity for a single time period. The minimal amount \( U_{\text{min}} \) is the lowest oil production level required to maintain the oil business running. Thus, choices for \( u \) are integral multiple of \( U_{\text{unit}} \) in the range of \([U_{\text{min}}, U_{\text{max}}]\). Secondly, the strike price of the option contract changes with the oil reserve level. The strike price for one unit of oil reserves is the unit production cost: \( c(K) \). The unit production cost increases when oil reserve level decreases, and vice versa. So the strike prices increases when the number of real options decreases. The contracts that get executed in earlier periods can be executed at a lower strike price than those executed in later periods.

These exotic features make optimal execution rule for the real options of oil reserve a lot different than the normal American call options. For a normal American call option on a non-dividend-paying asset, early execution is never optimal. One should always wait until the expiration date to execute. However, for the real options of oil reserve, the combination of maximal execution constraint and increasing strike prices makes early execution desirable. To compute the optimal execution rule, we need to
follow the Bellman equation and solve the problem backwards from the terminal point.

Thus, we need to specify a terminal point. This terminal point represents the time when oil reserve gets depleted. We use the estimated oil reserve level of Norway divided by the projected oil production by Norway’s government agency to get the total production time horizon. In theory, one can hold the real option portfolio of oil reserve indefinitely. By the argument of never executing American call option before expiration, one should put off oil production into the future indefinitely. However, there are two reasons that this hold-off strategy is not a desirable one in our case. First of all, there is a minimal required production level at each time period to keep the oil industry running. At least, the minimal amount $u_{min}$ needs to be executed. So the oil reserve level definitely gets lower throughout time. As a result, the production cost, namely strike price, gets higher. This feature may make early execution desirable. The other reason against putting off production indefinitely is: the conclusion of not executing American call option before expiration is derived under the Black-Scholes assumptions, where there is no limit on short selling of the underlying assets. In our case, short selling the underlying asset oil is surely constrained. Short selling oil spot is hard if not impossible. Even short selling oil futures is constrained by marginal requirement and credit limit. So the conclusion that never short selling American call option before expiration is lack of necessary conditions and not necessarily true in our case.

Based on the characteristics of oil reserve assets, we choose an approach similar to Longstaff and Schwartz’s method, which is widely used in pricing options with exotic features. Longstaff and Schwartz (2001) uses a simulation-based approach to price American options. The approach generates simulated price paths to estimate the conditional payoff of continuation at each time step with a regression-based
approximation method. The continuation payoff is to be compared with execution payoff to determine the optimal execution rule at each step. The algorithm solves the optimal execution policy backwards step by step from the terminal point.

Here, we briefly describe their approach. Denote the value of option at time $t$ to be $V_t^{\text{option}}$. Denote the risk-neutral probability measure for pricing assets to be $Q$ and risk-free rate to be $r$. The decision of whether or not to execute one option depends on the comparison of its execution value and continuation value. For the oil reserve, the payoff from executing one unit of real option is $(\psi - c(K))U_{\text{unit}}$. The continuation value is the expectation value of the unexecuted real option under risk-neutral measure discounted by risk-free rate: $E^{Q}\left[ V_{t+1}^{\text{option}} \right] \bigg/ (1+r)$. If $(\psi - c(K))U_{\text{unit}} > E^{Q}\left[ V_{t+1}^{\text{option}} \right] \bigg/ (1+r)$, it is better to execute the option contract. On the other hand, if execution payoff is less than continuation value $(\psi - c(K))U_{\text{unit}} < E^{Q}\left[ V_{t+1}^{\text{option}} \right] \bigg/ (1+r)$, it is better to wait and not execute the option contract immediately. Longstaff and Schwartz (2001) uses a series of basis functions of state variables at each time point to approximate the continuation value. The state variables normally involve underlying assets’ prices, market states and etc. Denote state variables to be $(X, K)$. The state variables represent all relevant information about the derivative. Thus, Markovian condition applies as shown in the equation below:

$$
E^{Q}\left[ V_{t+1}^{\text{option}} \right] \bigg/ (1+r) = E^{Q}\left[ V_{t+1}^{\text{option}} \right] \bigg/ (1+r) | F_t \\
= E^{Q}\left[ V_{t+1}^{\text{option}} \right] | X, K
$$

(6.1)

One can see the continuation value is a function of state variables $X, K$. Longstaff and Schwartz’s method uses simulation to generate a lot of paths for the state variables and then use linear combination of basis functions of state variables to
approximate the continuation value function (6.1) at each time point. For example, the state variables for an American-type Asian option are the underlying asset price and the current average price level. Let $\mathcal{X}, \mathcal{K}$ represent the state variables for this exotic derivative. If we choose polynomial functions as basis functions and approximate to the second order with $1, \mathcal{X}, \mathcal{K}, \mathcal{X}^2, \mathcal{K}^2, \mathcal{X}\mathcal{K}$, the approximated continuation function at time $t$ is:

$$\tilde{E}[\frac{V_{t+1}^{\text{option}}}{1+r} | \mathcal{X}, \mathcal{K}] = a^t_0 + a^t_1 \mathcal{X} + a^t_2 \mathcal{K} + a^t_3 \mathcal{X}^2 + a^t_4 \mathcal{K}^2 + a^t_5 \mathcal{X}\mathcal{K} \quad (6.2)$$

$(a^t_0, a^t_1, a^t_2, a^t_3, a^t_4, a^t_5)$ are calculated from regression in each time step. The data of $V_{t+1}^{\text{option}}$ and $\mathcal{X}, \mathcal{K}$ are obtained from simulations.

The exotic features of oil reserves require pricing this portfolio of real options altogether, since the strike price of any single option depends on the total number of real options, namely the total oil reserve level. At each time period, not only do we need to decide whether to execute the option, but also need to decide the optimal number of options to execute. When it is optimal to execute, we execute to the amount where marginal profit of execution is less than the continuation value of option or to the maximal capacity. The depletion state is defined as the point where oil reserve reduces to a pre-specified level. Beyond this level, the oil reserve is no longer economical for production.

Longstaff and Schwartz’s approach gives a flexible enough framework to formulate the exotic option pricing problem. However, the approach has some shortcomings in computation. The approach is parametric and uses an affine form to combine features, namely linear combination of basis functions. This kind of parametric approach works best when one has some prior knowledge into the structure of derivative. An obstacle
to this parametric approach is that the algorithm may fail to converge to a solution. One can refer to Boyan and Moore (1995) and Tsitsiklis and Van Roy (1996) for detailed discussion. Thus, though we use the Longstaff and Schwartz’s approach to formulate our oil reserve valuation problem, an improved algorithm is used to solve it.

The computational approach we use is nonparametric, by the name of kernel-based reinforcement learning. Instead of using parametric functions to approximate the continuation value function $E_t^Q \left[ \frac{V^{option}_{t+1}}{1+r} \right]$ at each time step, this approach uses a kernel-based estimation. One can refer to Ormoneit and Sen (2001) for technical details.

We discretize time into intervals and state variables into units. The generic algorithm is as follows:

1. Simulation Step: simulate hundreds of thousands of paths for oil prices for the whole time horizon according to the oil price dynamics under risk-neutral measure. The maximal number of periods is a specified number $T$. The oil price dynamics is estimated from historical data.

2. Policy Step:
   Case a: Time stage $t$ equals $T$. On each path, if options have not been all executed before final stage $T$, the optimal strategy at $T$ is to exercise as many options as possible if they are in the money. Then we can get the option value on each path for the final stage.

   Case b: Time stage $t$ is less than $T$. On each path, if the execution value of the real option is greater than continuation value estimated in time step $t$, the optimal policy is to execute as many options as the execution value is still greater than continuation value. Otherwise, if the execution value is smaller than continuation value even for a single unit of option, do not execute at all. Then we can get the option value on
each path for this time stage $t$.

3. Estimation Step: use kernel-based approach to estimate the value function of oil price $\psi$ and oil reserve level $K$ at each step $t$.

4. Iterate backwards until $t = 0$.

This approach requires estimating the oil price dynamics. From the oil log return time series in Figure 6.1, we can see the oil log return exhibits the mean-reverting behavior. The mean-reversion fashion of commodity prices stems from the fact that demand-supply relationships revert commodity prices to long-term mean after short-term shocks. Following the literature of oil price dynamics, we use OU (Ornstein-Uhlenbeck) process to model and fit the oil log return as before. OU process captures the exponential decay behavior of short-term shocks and mean reversion to long-term mean.

Ornstein-Uhlenbeck process of return is described by the dynamics below in equation (6.3):

$$dr_\psi = \beta(\mu - r_\psi)dt + \sigma dW_t \quad (6.3)$$

where $r$ is the log return; parameter $\mu$ is the long-term average of log return; $\beta$ is the mean-reverting velocity; $\sigma$ is the volatility; $W_t$ is a standard Brownian motion. Denote the oil price to be $\psi$. The dynamics in log return space is equivalent to the dynamics in price space:

$$d\psi = \alpha_0(\psi_0 - \ln(\psi))\psi dt + \sigma\psi dW_t \quad (6.4)$$

Where $\alpha_0 = \beta$, $\psi_0 = \mu + \frac{\sigma^2}{2\alpha_0}$.
Figure 6.1: Time Series of Oil Price and Oil Log Return
The parameters in the dynamics equation (6.3) can be fitted using the maximum likelihood estimation method as in chapter 4. And we use daily time series of the same period from May 20, 1987 to Mar 27, 2015 of oil price data. The fitted annualized parameters are:

\[ \mu = 6.13, \beta = 0.32, \sigma = 0.32 \]

Next, we can apply the pricing approach to Norway’s oil reserve. In the application, we use real data from Norway for the parameters as shown in table 4.4. We simulate 100000 paths with the fitted oil price dynamics. Then we use the reinforcement learning algorithm described above to price the oil reserve as a portfolio of real options while generating optimal extraction decisions at the same time. Figure 6.2 shows the value of oil reserve in regards to various levels of oil prices and oil reserve levels. The parameters are specified according to data of Norway’s oil reserves as in table 4.4. For the unit cost function \( c(K) \), namely the state-dependent strike price, we use a parametrized functional form as in chapter 4: \( c(K) = aK^{-\zeta} \) (\( a,\alpha \) are parameters to be estimated). We use the time series yearly data of average unit oil production cost and identified oil reserve to estimate the two parameter. Both data series are retrieved from the online database of U.S. Energy Information Administration (EIA). We find \( \zeta \) is about 1.16. The \( R \)-square of this estimation is as high as over 80 percent. Thus, the parametrization of unit cost function is close to reality.

In chapter 7, we integrate this value function of oil reserves in the asset allocation model of oil-based SWFs. This approach captures both the intrinsic value (as in Chapter 4) and option premiums of oil reserves.
Figure 6.2: Value function of oil reserves in regards to oil reserve level and oil price
Chapter 7

Integrate Real Option Pricing with Asset Allocation Model

In chapter 4, we solve the asset allocation problem for the oil-based sovereign wealth fund assuming a constant oil production rate. This approach captures the intrinsic value of oil reserves but not the option premiums from production choices. In chapter 6, we formulate and solve the valuation and optimal production problem of oil reserve assets using real option techniques. To provide an integrated optimal decisions of production and asset allocation, we need to integrate valuation of oil reserve assets into the asset allocation problem. We show the integration in this chapter and the detailed procedures of oil reserve valuation.

We denote the value of real options to be $G(\psi_t, k, t)$. As introduced in chapter 6, we solve the real option pricing and optimal execution problem with a nonparametric dynamic programming technique: reinforcement learning. By the reinforcement learning approach, one first simulates a large number of paths for the underlying dynamics. Then the value function is approximated on each time step based on the realized outcomes in the simulation starting from the terminal point. The
approximated value function of the previous step is used for searching the optimal solution in the current step. The algorithm repeats this procedure to solve the whole problem backwards step by step until it reaches the beginning of time horizon. In the approximation step, a kernel-based approach is used to approximate the value function. The kernel-based approach uses a specified kernel function to assign weights to all the realized values in simulations, and then approximate the value function of every point with the weighted sum. The idea is like interpolation and extrapolation.

We describe the approach in details here:

First of all, we discretize the time horizon into intervals and the total oil reserve $K$ into small units. We set the time interval to be one year. Namely, how much oil to produce is decided on a yearly basis. We discretize oil reserve into units: $\delta K$ is the smallest unit to produce in one year. Secondly, we specify the maximal production capacity for each year as $P_1, P_2, ..., P_T$. These are the maximal amount of oil that can be produced each year. Thirdly, we set the minimal required production level: $k_{\text{min}}$. This amount is the minimal level to sustain the oil production business running, e.g. retaining personnel and keeping equipments from obsolescence. In addition, as in previous chapters, we set a certain minimal level: $K_o$, below which level production activity discontinues. Finally, we assume the time horizon is the total available time for producing oil. If there are any leftover oil resources at the end of production horizon, it goes wasted. The algorithm is as follows:

Step 0. We discretize the total time horizon into $T$ yearly intervals and simulate $N_o$ paths for the oil prices. Let $i$ be path number and $h$ be year number:

$$\hat{\psi}_{i,t} = \hat{\psi}_{i,t} + \alpha_0 (\phi_0 - \frac{\theta_2}{\alpha_0} - ln \hat{\psi}_{i,t}) \psi_{i,t} \Delta t + \hat{\psi}_{i,t} \sqrt{\theta_1 \Delta z_{i,t}}$$
\[
\psi_{i,0} = \psi_0
\]

Step 1. We discretize the oil reserve space into \( M_o = \left\lfloor \frac{K - K_0}{\delta K} \right\rfloor + 1 \) points. \([\ldots]\) means the maximal integer smaller than the number between \([\ldots]\). The discretized reserve space \( K_0, K_1, \ldots, K_{M_o} \) is constructed as follows: \( K_0 = K, K_j = K_0 - j \times \delta K \). The estimates of the value function \( G(\psi, k, t) \) on \((\psi_{i,t}, K_j, t)\) is denoted by \( \hat{G}_{t,i,j} \).

Step 2. Choose a specified kernel function and a bandwidth \( b \) for the kernel function: \( kn(x, y) = \text{kernel}\left(\frac{||x - y||}{b}\right) \). \(|| . ||\) is the Euclidean norm. There is a collection of kernel function forms we can choose from, for example, the gaussian function: \( \text{kernel}(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \).

Step 3. Let time \( t = T \). We solve the optimization problem on all paths and oil reserve levels \((\psi_{i,T}, K_j)\). Let \( u_{\text{min}} = \left\lfloor \frac{k_{\text{min}}}{\delta K} \right\rfloor \) if \( k_{\text{min}} \) is integral. Otherwise, let it be \( \left\lfloor \frac{k_{\text{min}}}{\delta K} \right\rfloor + 1 \). Let \( u_{\text{max},T} = \left\lfloor \frac{P_T}{\delta K} \right\rfloor \). Let \( u \) denote the production decision, namely how many units of oil reserve to produce. \( c(K_j) \) is the unit production cost at \( K_j \) level. \( u \times (\psi_{i,T} - c(K_j)) \) is the payoff for producing \( u \) unit of oil. For each \((\psi_{i,T}, K_j)\), search for the optimal value of \( u \): \( u^T_{t,i,j} \) that makes \( u \times (\psi_{i,T} - c(K_j)) \) maximal in the range of \([u_{\text{min}}, u_{\text{max},T}]\). Then assign this optimal value to \( \hat{G}_{t,i,j} \).

Step 4. Approximate the value function \( G(\psi, k, t) \) at each reserve level \( K_j \) and at time \( t \) with the kernel-based method.

\[
\hat{G}(\psi, K_j, t) = \frac{\sum_i kn(\psi_{i,t}, \psi) \hat{G}_{i,j}}{\sum_i kn(\psi_{i,t}, \psi)} \tag{7.2}
\]

Step 5. Move one time step forward from \( t \) to \( t - 1 \). \( u \) must be larger than the minimal requirement \( u_{\text{min}} \) and smaller than production capacity \( u_{\text{max},t} \). Also, \( K_j - u \)
must be larger than the minimal reserve level \( K_0 \). Namely, \( u < \frac{K_j - K_0}{\delta K} \). \( c(K_j) \) is the unit production cost at \( K_j \) reserve level. For \( (\psi_{i,t-1}, K_j) \), the payoff for producing \( u \) unit of oil is \( u * (\psi_{i,t-1} - c(K_j)) \). After producing \( u \) units at this step, we are left with \( K_j - u \) units of oil reserve for the remainder of time. Let \( r \) be the risk-free rate. Then the approximate continuation value for this path is \( \hat{G}(\psi_{i,t}, K_j - u, t) \frac{1}{1+ r} = \hat{G}_{t_i,j}^{-t-1,u} \). The total payoff is: \( u * (\psi_{i,t-1} - c(K_j)) + \hat{G}_{t_i,j}^{-t-1,u} \frac{1}{1+ r} \).

For each \( (\psi_{i,t-1}, K_j) \), we search for the optimal value of \( u: u_{t_i,j}^{-1} \) that makes \( u * (\psi_{i,t-1} - c(K_j)) + \hat{G}_{t_i,j}^{-t-1,u} \maximal \) in the range \([u_{min}, u_{max,t-1} \wedge \frac{K_j - K_0}{\delta K}]\). Assign this optimal value to \( \hat{G}_{t_i,j}^{-t-1,u} \).

Step 6. Go to step 4. Repeat until \( t = 0 \).

At the end of the algorithm, we get the estimate of the value function \( G(\psi, k, t) \): \( \hat{G}(\psi, k, t) \). The results are in figure 6.2.

With the estimate of \( G(\psi, k, t) \), we can calculate the total value function for the sovereign wealth fund:

\[
J(W, \psi, T) = -exp \left( \frac{1}{2} \xi^T C^T(CC^T)^{-1}C \xi \frac{1-\eta}{\eta} T \right) \left( e^{(r-\rho)T} W + G(\psi, K, T) \right)^{1-\eta} \tag{7.3}
\]

Where \( \hat{G}(\psi, K, T) \) is substituted for \( G(\psi, K, T) \).
We can also calculate the optimal portfolio:

\[ w^* = -\frac{J_W}{J_{WW}W} (CC^T)^{-1}C\xi - \frac{J_{W\psi}}{J_{WW}W} (CC^T)^{-1}C\sigma^T \]

\[ = \frac{\lambda(t)W + G(\psi, K_t, t)}{\lambda(t)\eta W} (CC^T)^{-1}C\xi - \frac{G_{\psi}}{\lambda(t)W} (CC^T)^{-1}C\sigma^T \]  

(7.4)

Where \( K_t \) is the oil reserve level at time \( t \). \( G(\psi, k, t) \) is substituted with \( \hat{G}(\psi, k, t) \).

And \( G_{\psi} \) is calculated by:

\[ G_{\psi}(\psi, k, t) = \frac{G(\psi + \Delta\psi, k, t) - G(\psi, k, t)}{\Delta\psi} \]

Where \( G(\psi + \Delta\psi, k, t) \) and \( G(\psi, k, t) \) are substituted with \( \hat{G}(\psi + \Delta\psi, k, t) \) and \( \hat{G}(\psi, k, t) \) respectively.

One can follow the same arbitrage logic in chapter 7 to construct an equivalent trading strategy by short selling an “oil portfolio” and dynamically trading:

1. Short sell a portfolio of the value \( e^{-(r-\rho)(T-t)}G(\psi, t) \) at time \( t \). The portfolio is constructed with securities that can replicate all the cash flows from oil reserve. This can be done because the market is complete according to assumption. So we can construct the infinitesimal portfolio to replicate each instantaneous cash flow from oil reserve. Adding all the infinitesimal portfolios together, we can get the replicating portfolio for all future cash flows. Denote the portfolio as \( w_{oil} \). The elements of \( w_{oil} \) are equal to the asset values in the replicating portfolio divided by the wealth \( W_t \).

2. Construct the efficient portfolio \( w_{efficient} \) to maximize expected terminal utility: \( E[U(W_T^{total})] \) with the total wealth value equal to \( W_T^{total} = (W_t + e^{-(r-\rho)(T-t)}G(\psi, t)) \).
Deduct the portfolio sold in step 1 from this efficient portfolio to calculate the overall portfolio.

3. Use cashflows from the oil reserve to fund the shorted portfolio. Dynamically update the shorted portfolio and the efficient portfolio as the total wealth value $W_{total}$ changes.

This dynamic trading strategy replicates the integrated optimal asset allocation strategy exactly. Instead of solving the complicated differential equations, we can use the optimal solution to maximize the final expected utility of total assets. The value function is simply $J(W, t) = -\exp\left(\frac{1}{2}\xi^T C T (C C^T)^{-1} C \xi \frac{1-\eta}{\eta} (T - t) \right) \left( e^{r(T-t)} W \right)^{1-\eta}$.

We just need to replace risk-free rate $r$ with the effective risk-free rate $r - \rho$, and $W$ with the total assets of initial wealth and discounted cash flows from oil reserve: $(W_t + e^{-(r-\rho)(T-t)} G(\psi, k, t))$. $G(\psi, k, t)$ can be calculated with the reinforcement learning method described above. The optimal portfolio is comprised of two parts: the efficient portfolio to maximize expected utility: $w_{efficient}$; and the short-selling portfolio $w_{oil}$. $w_{oil}$ is exactly the opposite of hedging portfolio $w_{hedge}$: $w_{hedge} = -w_{oil}$. As a result, the optimal asset allocation is: $w_{opt} = w_{efficient} - w_{oil} = w_{efficient} + w_{hedge}$.

The reason that this replication strategy works and the integrated optimal asset allocation can be easily solved is that there is the ”disentanglement” between investment and oil business decisions. The oil reserve can be valued separately.

Next, we can follow the same analysis in chapter 4 to illustrate the added value of integrating real option pricing with asset allocation. We compare constant equivalents of final wealth by the integrated asset allocation with real option pricing with that by the integrated asset allocation with constant production policy (as in chapter 4). The value added comes from the option premiums from production choices and the
integration of real options of oil reserves with the financial asset allocation. Table 7.1 shows the value added for different risk aversion levels.

Figure 7.1 shows the constant equivalents of final wealth with real option pricing and those with only intrinsic valuation of oil reserves (constant production rate as in chapter 4) for different risk tolerance parameters.

Since the risk tolerance of 1 is too extreme for the institutional investor, Figure 7.2 shows the comparison with risk tolerance only from $\eta = 2$ to $\eta = 10$.

To further illustrate the option premium of oil reserves, we use different initial values of wealth and risk tolerance numbers to calculate the constant equivalents of final wealth with real option valuation and those with just intrinsic valuation of oil reserves (constant production rate as in chapter 4). The comparison of constant equivalents of final wealth is shown in Figure 7.3. The comparison of total return loss of constant equivalent of final wealth is shown in Figure 7.4. The comparison of total loss percentage of constant equivalents of final wealth is shown in Figure 7.5. The comparison of annualized return loss of constant equivalents of wealth is shown in Figure 7.6.
### Table 7.1: Constant Equivalents of Final Wealth

<table>
<thead>
<tr>
<th>Risk Tolerance $\eta$</th>
<th>Real Option (NOK bil.)</th>
<th>Intrinsic (NOK bil.)</th>
<th>Wealth Diff</th>
<th>TR Diff %</th>
<th>Ann Return Diff %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,768,900</td>
<td>1,750,500</td>
<td>18,400</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>124,320</td>
<td>123,030</td>
<td>1,290</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>50,393</td>
<td>49,860</td>
<td>533</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>32,084</td>
<td>31,750</td>
<td>334</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>5</td>
<td>24,470</td>
<td>24,214</td>
<td>256</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>6</td>
<td>20,427</td>
<td>20,210</td>
<td>217</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>7</td>
<td>17,955</td>
<td>17,768</td>
<td>187</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>8</td>
<td>16,299</td>
<td>16,129</td>
<td>170</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>9</td>
<td>15,118</td>
<td>14,956</td>
<td>162</td>
<td>1.1</td>
<td>0.1</td>
</tr>
<tr>
<td>10</td>
<td>14,235</td>
<td>14,084</td>
<td>151</td>
<td>1.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**Figure 7.1: Constant Equivalent of Final Wealth**
Figure 7.2: Constant Equivalent of Final Wealth 2
Figure 7.3: Constant Equivalent of Final Wealth: Initial Wealth Level, Risk Tolerance
Figure 7.4: Constant Equivalent of Final Wealth: Total Loss
Figure 7.5: Constant Equivalent of Final Wealth: Total Loss Percentage
Figure 7.6: Constant Equivalent of Final Wealth: Annual Return Loss
Chapter 8

Asset Allocation Model for Family Offices

In this chapter, we introduce another application of the integrated asset allocation framework. The application is to family offices. Family offices are investment institutions set up by high-net-worth families to manage their family wealth. The high-net-worth families often have prominent family businesses which generate and transfer profits to family offices. We can apply the integrated asset allocation to this type of I-B complexes.

Asset allocation for family offices is within the realm of private wealth management. Private wealth management has some unique features than investment management in general. Unlike general investment management, financial planning is an essential element of private wealth management. Financial planning is translating clients personal goals into identifiable future cash flows. It is often intertwined with asset allocation. For private wealth management, there is an asset allocation framework coined as goal-based investing. The framework stems from the behavior of mental accounting in the financial planning process. Mental accounting
refers to the investors ignorance of the fungibility of money and creating separate mental compartments within a portfolio. For example, a family may hold separate portfolios designated for different spending goals, such as lifestyle, vacations, college expenditures, retirement and wealth transfer.

There is a series of literature covering the behavior asset allocation framework. Brunel (2003) divides the portfolio into components designed to provide for a specific objective: growth, capital preservation, income and liquidity. A number of articles consider behavioral asset allocation, including Nevins (2004), Chhabra (2005), and Fraser and Jennings (2006). Nevins bases his sub-portfolios on client goals, whereas other behavioral asset allocation approaches use timeline-based sub-portfolios. Brunel (2005) includes a behavioral asset allocation case study with a $50-million-in-assets family.

However, goal-based investing has the shortcoming of sub-optimality. The process of optimizing sub-portfolios does not produce the globally optimal portfolio when all sub-portfolios are considered together; Horvitz and Wilcox (2007) shows this issue of sub-optimization. The efficacy of goal-based investing is that the framework facilitates wealth managers to communicate the asset allocation process to clients based on the mental accounting behaviors. As Brunel (2003) shows, the explanation based on the behavioral framework of an asset allocation can help high-net-worth clients understand and accept alternative investments. Even though behavioral asset allocation is more of a communication tool than a portfolio optimization theory, it is useful for identifying clients objectives.

First of all, the objectives of clients are meaningful personal goals. Merrill Lynch Wealth Management classifies clients goals into seven categories: family, health,
work, home, finances, leisure and giving. Personal goals are then translated into identifiable future cash flows. For example, a client may have a family goal of supporting a 7-years-old child’s college education. This goal can be translated into four-year cash flows of the yearly college tuition and living support, which begins 11 years into the future.

Second of all, personal goals have rankings. Goals like basic lifestyle consumption may have highest ranking for securing. Goals of luxurious spending, like buying a boat or vacation house may have lower rankings. Chhabra (2005) classifies clients’ goals into three categories: essential, important and aspirational goals. Deguest, Lionel, Milhau, Suri and Wang (2015) develops a comprehensive framework of goal-based investing for Merrill Lynch Wealth Management.

The subject of this chapter – family offices – are generally created by ultra-high-net-worth families to manage their family wealth. The objectives of these ultra-high-net-worth are most focused on long-term goals. Brunel (2003) classifies the objectives of wealth management into three categories: liquidity, income and capital preservation. From the perspective of goal-based investing, the goals of family offices are mainly long-term capital preservation and growth over multi-generational horizons, since the needs of liquidity and income only accounts for a small portion of the huge wealth owned by ultra-high-net-worth families. Pompian (2006) also addresses the unique characteristics of ultra-high-net-worth families by focusing on the aspirations, attitudes and behaviors in a multi-generational context.

After the introduction above, let us model the investment objective of family offices. We set the long-term wealth preservation and growth as the main objective for family offices. The liquidity needs and income goals are insignificant compared
to the long-term goal for a family office. Thus, we treat them as spending policies instead of objectives. We set a fixed long-term horizon and optimize the utility of wealth at the end of the time horizon. The objective is:

$$\max E_t[R(W_T, K_T)] \quad (8.1.1)$$

where $W_T$, $K_T$ are family wealth and family business value at time $T$. $R(.,.)$ is the utility function.

We treat family business value and family wealth the same and use CRRA type of utility function. Then $R(W, K) = \frac{(W+K)^{1-\eta}-1}{1-\eta}$ for $\eta > 1$. We simplify the utility function to be $R(W, K) = -(W+K)^{1-\eta}$ for it is equivalent to the CRRA function.

The dynamics of family business value is as follows:

$$dK = \mu_K K dt + K \sigma_K dZ \quad (8.1.2)$$

where $\mu_K$, $\sigma_K$ are the mean and volatility of the business capital return.

The profit function is $p = \gamma \mu_p K$ (8.1.3), where $\gamma$ is the payout ratio, and $\mu_p$ is return on equity (ROE). It is a well-known fact that ROE has the mean-reversion fashion through business cycles. Thus, we model it as a mean-reverting process:

$$d\mu_p = \alpha_p (\bar{\mu}_p - \mu_p) dt + \sigma_p dZ \quad (8.1.4)$$

where $\alpha_p$ is the mean-reverting velocity, and $\bar{\mu}_p$ is the long-term mean of ROE. $\sigma_p$ is the volatility of ROE growth.
The consumption or spending policy is specified as a percentage of wealth in each period: $q = \rho W$. $\rho$ is the spending rate.

We assume the same market dynamics for risky asset as in chapter 4 (equation 4.2, 4.3). As a result, the wealth dynamics is:

$$dW = [r + w^T C\xi - \rho]W dt + W w^T C dZ + \gamma \mu_p K dt \quad (8.1.5)$$

The definition of variables and parameters are the same as in previous chapters: $w$ is the vector of portfolio weights on risky assets. $C$ is the volatility component matrix. $Z$ is the vector of risk factors. $\xi$ is the risk premium vector.

Consider the Disentanglement Conditions 1-7. It can be easily verified that they are all satisfied by the family office problem. As a result, by the Separation Theorem, the value function solution to the family office (8.1) is: $J(W + \nu(K, \mu_p, t), t)$. $J(W, t)$ is the value function solution to the optimization problem below:

$$\max_w J[w] = E_t[R(W_T)] \quad (8.2.1)$$

$$dW = [r + w^T C\xi - \rho]W dt + W w^T C dZ \quad (8.2.2)$$

and $J(W, T) = R(W) \quad (8.2.3)$.

And $\nu(K, \mu_p, t)$ is the solution to the following problem:

$$-(r - \rho)\nu + \frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial K} (\mu_K - \xi^T C^T (CC^T)^{-1} C \sigma_K^T) + \frac{1}{2} \frac{\partial^2 \nu}{\partial K^2} \sigma_K^T \sigma_K + \frac{\partial \nu}{\partial \mu_p} (\alpha_p (\mu_p - \mu_p)) -$$

$$\xi^T C^T (CC^T)^{-1} C \sigma_p^T + \frac{1}{2} \frac{\partial^2 \nu}{\partial \mu_p \partial \sigma_p} \sigma_p^T + \frac{\partial^2 \nu}{\partial \mu_p \partial K} \sigma_K \sigma_p^T + \gamma (\mu_p K) = 0 \quad (8.2.4)$$

and $\nu(K, \mu_p, T) = K \quad (8.2.5)$.
From the condition of market completeness, we have \( \mu_K - \xi^T C C^T (CC^T)^{-1} C \sigma_K^T = r \) (8.2.6).

From (8.2.4), (8.2.5) and (8.2.6), we can derive the following formula for \( \nu \):

\[
\nu(K, \mu_p, t) = E^Q \left[ \int_t^T e^{-(r-\rho)(s-t)} \gamma(\mu_p, s) K_s + e^{-(r-\rho)(T-t)} K_T \right] \quad (8.2.7)
\]

and under the risk-neutral probability measure \( Q^* \):

\[
dK = rKdt + K\sigma_K^* dZ^* \quad (8.2.8)
\]

\[
d\mu_p = \left[ \alpha_p(\mu_p - \mu_p) - \xi^T C C^T (CC^T)^{-1} C \sigma_p^T \right] dt + \sigma_p dZ^* \quad (8.2.9)
\]

where \( Z^* \) is the standard multi-dimensional Brownian motion under probability measure \( Q^* \).

The solution to problem (8.2.1)-(8.2.3) is exactly (5.5.4):

\[
J(W, t) = -\exp\left( \frac{1}{2} \xi^T C C^T (CC^T)^{-1} C \xi \frac{1}{\eta} (T - t) \right) (e^{(r-\rho)(T-t)} W)^{1-\eta} \quad (8.2.10)
\]

By the Separation Theorem, the overall value function solution to problem (8.1.1)-(8.1.5) is:

\[
J(W, K, \mu_p, t) = -\exp\left( \frac{1}{2} \xi^T C C^T (CC^T)^{-1} C \xi \frac{1}{\eta} (T - t) \right) (e^{(r-\rho)(T-t)} (W + \nu(K, \mu_p, t)))^{1-\eta} \quad (8.3.1)
\]

The optimal asset allocation solution is:

\[
w^* = -\frac{\partial \nu}{W} - \frac{\partial^2 \nu}{W^2} \sum^{-1} - \frac{\partial^2 \nu}{W^2} \sum^{-1} H
\]

\[
= \frac{W + \nu(K,t)}{\eta W} (CC^T)^{-1} C \xi - \frac{\nu}{W} (CC^T)^{-1} C \sigma_K^T - \frac{\nu_{\mu p}}{W} (CC^T)^{-1} C \sigma_p^T \quad (8.3.2)
\]

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We can calculate $\nu(K, \mu_p, t)$ using equation (8.2.7) by Monte Carlo simulation and use similar control variate approach as in chapter 4 to reduce computational complexity. And we can calculate $\nu_K$ by $\nu_K(K, \mu_p) = \frac{\nu(K+\delta K, \mu_p) - \nu(K, \mu_p)}{\delta K}$ and calculate $\nu_{\mu_p}$ by $\nu_{\mu_p}(K, \mu_p) = \frac{\nu(K, \mu_p + \delta \mu_p) - \nu(K, \mu_p)}{\delta \mu_p}$.

Then we can use real data to calculate the constant equivalent and asset allocation under the integrated optimization setting, and those under simple optimization setting. Then, we can compare the value added of the integrated asset allocation framework.

In this application, we use a hypothetical U.S.-based family office as the modeling reference. We can use a factor-based asset allocation model instead of an asset-based asset allocation model, since factors in the U.S. financial markets are heavily researched on and evidenced. We first introduce the concept of risk factors and factor investing.

8.1 Factor-based Asset Allocation

Risk factors are the risk drivers of assets. Risk factors provide clearer return/risk profiles than asset classes. The factor-based asset allocation has a lot of benefits.

Nobel Prize Laureate, William Sharpe (1964) pioneers the Capital Asset Pricing Model (CAPM). It is a single factor model to explain asset returns. But it has restrictive assumptions on normality and utility function. To address these restrictions, Ross (1976) proposes an alternative model—Arbitrage Pricing Theory (APT). It is a multi-factor model to explain asset returns and serves as a theoretical foundation
of asset pricing.

There is the strand of literature that studies different assets exposures to a single factor or several factors. Jaffe and Mandelker (1976) and Fama and Schwert (1977) examine various assets exposures to inflation. The famous papers of Nobel Prize Winners Fama and French (1992, 1993) pioneer the three factor (market, value and size) model to capture the average market movement and cross-sectional variations in equities. There are also strands of literature that studies the explanatory and predictive risk factors for a single asset class. For example:


For bond: Elton, Gruber and Blake (1995) develops the factor model for bonds using four and six factors, including equity index, bond index, credit spread, option spread, GNP and inflation. Ludvigson and Ng (2009) examines predictive macro factors for bond risk premia.

exposures of hedge funds using buy-and-hold and option-based strategies. Jakub and Stafford (forthcoming) studies a systematic risk factor of writing put options as a major component of hedge fund returns.


There is also some literature covering more complex and dynamic relationships, both in prediction and explanation, for risk factors and asset returns. Brandt and Wang (2002) explores the implications of consumption growth and inflation factors, for the term structure of interest rates and the cross-section variation of stock returns. Campbell and Viceira (2005) develops a model to characterize the dynamics of stock and bond, with factors of real interest rate, nominal interest rate, dividend yield and term spread. The asset allocation based on the dynamics of assets is then solved.

Applications of factor-based asset allocation also appear in papers. Asl and Etula (2012) develop a factor-based approach for strategic asset allocation. Ang (2014) is a comprehensive treatment of a systematic approach to factor investing. It includes the methodology of factor investing for institutional investors and case studies.

There are a lot of advantages of factor-based asset allocation:

1. First of all, it is more efficient to get a diversified portfolio by using risk factors. Correlations across risk factors are much lower than those across asset classes. Also, it is easier to expand the opportunity set to include more potential exposures using factor-based approach than the traditional asset-based approach. Briand, Nielsen and Stefek (2009) compares a typical 60/40 equity/fixed income allocation v.s. a simple equal-weighted factor-based allocation. They find the factor-based approach
can generate returns similar to the traditional allocation with 65% less volatility.

2. Second of all, risk factors provide a more accurate classification of return/risk profiles than asset classes. Using factor-based approach, one can take only the risk exposure one intends to. For example, equity risk plagues other asset classes, like high yield bond, real estate, hedge funds and private equity. Allocating to high yield bonds, which is classified as fixed income among asset classes, one actually takes on a lot of equity risk exposures of companies that issue those bonds. The issue get more severe during crisis. Page and Taborsky (2011) gives an example of carry trade exhibiting equity risks during crisis times.

3. Finally, factor-based approach gives a more accurate framework for expressing views and risk management. Market has shifting regimes. Asset class returns are driven by common factors, and the risk factor returns are highly regime-specific. Risk factors include interest rate, long-short bond term spread, credit spread, inflation, etc. They provide a more direct framework for investors to express views, and enable investors to make the most of insights and diversify their portfolios accordingly.

One can see that from equation (4.2)-(4.3), we already have used a factor-based approach to model the market dynamics of risky assets. Next, we apply a factor-based approach to parametrize the market.

8.2 Parametrization of A Family Office

We take a U.S.-based family office as the subject. We assume the family office has $10 billion under management. The family business has a market value of $20 billion. We take return and profitability data from a public company in technology
sector – Apple, for statistics of the family business. The spending rate $\rho$ is assumed
to be 4 percent. The transfer rate is assumed to be 5 percent. This number is
in line with the average ratio of dividend yield over ROE for Apple Inc. We only
consider the positive cash flows from business since dividends are always positive. So
$\gamma(x) = max(5x, 0)$. Because we can use a series of smooth functions to approximate
$\gamma(x)$ as close as needed, we don’t need to worry about the smoothness condition
in the Disentanglement Condition 7. The Separation Theorem is applicable. The
investment horizon is assumed to be 10 years.

For the market dynamics parameters, we use factor-level data instead of asset-
level data. We identify these factors for a U.S.-biased investor: the U.S. equity
premium, size premium, value premium, term spread, credit spread, international
stock premium and international bond premium. We use the proxies below to
measure the factor returns: the U.S. equity premium – Ibbotson Associates U.S.
Equity Premium Index return, size premium – Ibbotson Associates U.S. Size Pre-
mium Index return, value premium – Fama French Value Factor return, term
spread – GF 10 year Total Return minus MS 30 day Total Return, credit spread
– MS Long-term Corporate Bond Total Return minus GF 10 year Total Return,
international stock – MSCI EAFE Index return minus S&P 500 Total Return,
international bond premium – ML Global Government Bond excluding U.S. Gov-
ernment Bond Index Total Return and technology sector – S&P 500 Information
Technology index Total Return. We use monthly data of these indices from Jan.
1991 to Dec. 2015 for calculation of mean return and covariance matrix of the factors.

The annulized mean return and covariance matrix estimates are listed in table 8.1
We can repeat the analysis procedures as in SWFs. To show the added value of integrated asset allocation, we compare the constant equivalents of final wealth by integrated optimal strategy and “simple optimal” strategy.

For the “simple optimal” strategy, we can calculate the constant equivalent of final wealth by the simulation approach.

First of all, we simulate discretized asset returns and family business returns and ROE in a consistent way according to their covariance matrix $CC^T$, $C\sigma_k^T$, $C\sigma_p^T$ and
Then we calculate $\hat{W}^\prime$ accordingly. Let $i$ denote the path number and $j$ denote the time interval number with time intervals $t_0, t_1, ..., t_j, ...$.

$$\hat{W}^\prime_i = W^\prime_0 \exp(r_p T - \frac{1}{2} \nu \nu^T T + \nu Z_{i,T})$$
$$+ \sum_{j} e^{r_p (T-t_j)} - \frac{1}{2} \nu \nu^T (T-t_j) + \nu (Z_{i,T} - Z_{i,t_j}) \gamma_{\mu | i,j} K_{i,j} \Delta s$$

where $r_p = r - \rho + \frac{1}{\eta} \xi^T C^T (CC^T)^{-1} C \xi$; $\nu = \frac{1}{\eta} \xi^T C^T (CC^T)^{-1} C$.

Then we can calculate the utility value for each path: $\hat{U}^\prime_i := U(\hat{W}^\prime_i)$ and calculate the estimate $\hat{U}^\prime = \frac{1}{N} \sum_i [\hat{U}^\prime_i]$. Then, constant equivalent can be calculated by inversion.

For different number of risk tolerance $\eta$, the value of constant equivalents by integrated optimal strategy: $W_T^{\xi}$ and the “simple optimal” strategy $W_T^{\tau e}$ are listed below in Table 8.2.

Figure 8.1 shows the constant equivalents of final wealth by integrated optimal strategy and “simple optimal” strategy for different risk tolerance parameters.

To further illustrate the effect of integrated optimal asset allocation, we use different initial values of wealth and risk tolerance numbers to calculate the constant equivalents of final wealth by integrated optimal strategy and those by “simple optimal” strategy. The comparison of constant equivalents of final wealth is shown in Figure 8.2. The comparison of total return loss of constant equivalent of final wealth is shown in Figure 8.3. The comparison of total loss percentage of constant equivalents of final wealth is shown in Figure 8.4. The comparison of annualized return loss of constant equivalents of wealth is shown in Figure 8.5.
Table 8.2: Constant Equivalents of Final Wealth

<table>
<thead>
<tr>
<th>Risk Tolerance ( \eta )</th>
<th>Integrated ($ bil.)</th>
<th>Simple ($ bil.)</th>
<th>Wealth Diff</th>
<th>TR Diff %</th>
<th>Ann Return Diff %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>871</td>
<td>466</td>
<td>405</td>
<td>86.9</td>
<td>6.3</td>
</tr>
<tr>
<td>3</td>
<td>257</td>
<td>156</td>
<td>100</td>
<td>64.0</td>
<td>5.0</td>
</tr>
<tr>
<td>4</td>
<td>139</td>
<td>89</td>
<td>50</td>
<td>56.2</td>
<td>4.5</td>
</tr>
<tr>
<td>5</td>
<td>97</td>
<td>58</td>
<td>39</td>
<td>66.5</td>
<td>5.1</td>
</tr>
<tr>
<td>6</td>
<td>76</td>
<td>48</td>
<td>28</td>
<td>58.3</td>
<td>4.6</td>
</tr>
<tr>
<td>7</td>
<td>64</td>
<td>39</td>
<td>24</td>
<td>62.4</td>
<td>4.9</td>
</tr>
<tr>
<td>8</td>
<td>56</td>
<td>33</td>
<td>23</td>
<td>68.7</td>
<td>5.2</td>
</tr>
<tr>
<td>9</td>
<td>50</td>
<td>29</td>
<td>21</td>
<td>73.7</td>
<td>5.5</td>
</tr>
<tr>
<td>10</td>
<td>46</td>
<td>29</td>
<td>18</td>
<td>61.8</td>
<td>4.8</td>
</tr>
</tbody>
</table>

We can also use this analytical framework to analyze critical relationships. We list three questions of interests here:

1. How does the spending policy affect the long-term wealth?
2. What is the value of an increase in long-term family business profitability for the family office? Under what conditions should the family spend money to increase the business profitability, for example, hiring better management, funding R&D projects, purchasing better technology?
3. What is the value of additional business capital for the family office? Under what conditions should the family office invest in the family business?

These are all important questions of concern to the family office and the wealthy family. Without the analytical framework, one may find them hard to answer and come up with a quantitative estimate. We study these questions one by one.
For the first question, to study the effect of spending policy, we change the value of parameter $\rho$ and re-calculate constant equivalents of wealth at different risk aversions while keeping all other parameters the same. The results are shown in Figure 8.6 and Figure 8.7

One can see the different constant equivalents of wealth caused by different spending policies. The more spending, the lower future wealth level will be. We can calculate the sensitivity of transfer policy at the current status:

At risk aversion of 8, one additional percentage of spending decreases the constant equivalent of future wealth by 5.2570 billion dollars.

For the second question, the increase in long-term business profitability is the increase in the long-term mean of ROE. Increases in long-term ROE mean enables the family business to generate more profits for the family. To study the effect of increases in long-term profitability on the wealth of family office, we change the value of parameter $\mu_p$ and re-calculate the constant equivalents of final wealth. The results are shown in Figure 8.8 and Figure 8.9. The family and family office can use these results to decide whether and how much investments should be made to increase the business profitability.

One can see the different constant equivalents of wealth caused by different levels of business profitability. The higher long-term profitability is, the higher future family wealth level will be. We can calculate the sensitivity of business profitability at the current status:
At risk aversion of 8, one additional unit of business profitability increases the constant equivalent of future wealth by 0.1565 billion dollars.

Consider a case of purchasing better technology for the family business. The newer technology is estimated to increase long-term profitability by $y$. The project is estimated to cost $C_{tech}$. The analytical framework we propose can help the family decide whether to fund the purchase using cost-benefit analysis:

Suppose the sensitivity of long-term profitability to the constant equivalent of future family wealth is $\mu_p$. And suppose the sensitivity of current wealth to the constant equivalent of future family wealth is $\mu_W$. If the expected value added $\mu_p y$ is larger than the costs $\mu_W C_{tech}$, the purchase is worthwhile. Otherwise, it is not.

For the third question, the value of additional business capital to the family wealth can be calculated by increasing the business capital parameter $K$. The results are shown in Figure 8.10 and Figure 8.11. The family and family office can use these results to decide whether and how much investments should be made to increase the business capital.

One can see the different constant equivalents of wealth caused by different levels of family business capital. The more capital business has, the higher future family wealth will have. We can calculate the sensitivity of the business capital at the current status:
At risk aversion of 8, one additional unit of business capital increases the constant equivalent of future wealth by 3.5169 billion

The analytical framework can help the family decide whether to invest in its own business using cost-benefit analysis:

Suppose the sensitivity of additional business capital to the constant equivalent of future family wealth is \( \text{rate}_{\text{cap}} \). And suppose the sensitivity of current wealth to the constant equivalent of future family wealth is \( \text{rate}_{W} \). If the expected added value \( \text{rate}_{\text{cap}} \) is larger than the cost \( \text{rate}_{W} \), the investment is worth funding. Otherwise, it is not.

Above, two case studies of investments in the family’s own business are analyzed. Even though the family office should hedge business income risks from the asset allocation perspectives and avoid related assets, investments to increase business profitability and business capital may be beneficial to the family. The family can use this analytical framework as a decision support tool to analyze these “invest-in-your-own-business” type of decisions.
Figure 8.1: Constant Equivalent of Final Wealth
Figure 8.2: Constant Equivalent of Final Wealth: Initial Wealth Level, Risk Tolerance
Figure 8.3: Constant Equivalent of Final Wealth: Total Loss
Figure 8.4: Constant Equivalent of Final Wealth: Total Loss Percentage
Figure 8.5: Constant Equivalent of Final Wealth: Annual Return Loss
Figure 8.6: Constant Equivalents of Final Wealth at Different Spending Rates and Risk Aversions
Figure 8.7: Constant Equivalents of Final Wealth at Different Spending Rates (Risk Aversion = 8)
Figure 8.8: Constant Equivalents of Final Wealth at Different Long-term ROE Levels and Risk Aversions
Figure 8.9: Constant Equivalents of Final Wealth at Different Long-term ROE Levels (Risk Aversion = 8)
Figure 8.10: Constant Equivalents of Final Wealth at Different Business Capital Levels and Risk Aversions
Figure 8.11: Constant Equivalents of Final Wealth at Different Business Capital Levels (Risk Aversion = 8)
Chapter 9

Conclusion

9.1 Conclusion

In this paper, we propose a general framework to analyze the asset allocation problem of investment entities with supporting businesses, “I-B Complexes” as we coin them. Examples include sovereign wealth funds with state businesses, pension plans with sponsoring companies, family offices with family businesses and individuals with personal incomes. We generalize them into a integrated decision-making framework. We solve the integrated asset allocation problem in a generic way and provide insights about optimal asset allocation and critical relationships. An important discovery is the Disentanglement Conditions. Under these conditions, integrated asset allocation can be solved without entanglement of business decisions. We identify these conditions and prove the Separation Theorem that can separate business and investment decisions. The conditions are of generality: most (if not all) asset allocation problems of I-B complexes satisfy these conditions. The proof of the theorem is provided. And a machinery for solving the integrated asset allocation problem is developed.
We then apply the framework to two specific examples: oil-based sovereign wealth funds and family offices. In the SWF case, the similarity between oil reserve and real options is drew. Option pricing technique is used to calculate the oil reserve value and optimal production policy. Then, the optimal integrated asset allocation is solved for both the constant oil production policy and the optimal production policy. The Norway’s Sovereign Wealth Fund (GPFG) is used as the modeling reference. The gains in constant equivalents of final wealth are computed to illustrate the benefits of integrated asset allocation over “simple optimal” allocation. In addition, the differences between constant equivalents of wealth under optimal production policy and those under constant production policy are calculated to show the option premiums of oil reserves. The framework is also used for analyzing several critical questions relevant to the SWF. They include the effect of transfer policy, whether to invest in oil production technology and whether to engage in oil field exploration activities.

Another application is to family offices. We develop a model for a hypothetical U.S.-based family offices with the general framework. The Separation Theorem is applied to solving the integrated asset allocation problem. We use factor-based asset allocation instead of asset-based asset allocation for this application. Similarly, differences between constant equivalents of wealth by the integrated asset allocation and those by the “simple optimal” allocation are computed to show benefits of integrated asset allocation. And then, several questions of interests in regards to the family office are analyzed. They include how to decide on transfer policy, whether to invest to increase business profitability and whether to invest into the family’s own business.
9.2 Future Work

The general framework we develop, together with the Disentanglement Conditions and the Separation Theorem, establish a powerful foundation for studying asset allocation problems and other related problems of I-B complexes. But there are some limitations of this integrated asset allocation framework. Also, extensions can be developed for broader application of the framework. Future researches of addressing the limitations and extensions can further empower the framework. Future research directions include:

1. One limitation of the general framework is the hardship of adding constraints into the asset allocation problem. For example, a lot of institutions have non-negativity constraints on asset allocations. Adding non-negativity constraints to portfolio weights in the stochastic control setup can make the problem very difficult to solve. One potential idea of addressing this limitation is to follow a hybrid approach of optimization and simulation:

   Firstly, solve the optimal business decisions separately. And calculate the business value function $\nu$ according to (5.4.9)-(5.4.10).

   Secondly, use the functional form $J(W + \nu(K,t), t)$ for value function and plug in the HJB equation (5.4.1). Collect all terms relevant to asset allocation in the left-hand-side term under the maximization bracket. Then the optimization problem for solving asset allocation decisions is:

   $\max_w J_W W w^T C \xi + \frac{1}{2} J_{WW} W^2 w^T C C^T w + J_{WK} W w^T C \sigma_K^T$ (9.1)

   Devide the term inside optimization by $J_W W$. Note $J_W$ is positive due to properties of utility function ($\text{increasing with } W$). So a negative sign needs to be added:

   $\max_w w^T C \xi + \frac{1}{2} \frac{J_{WW} W}{J_W} w^T C C^T w + \frac{J_{WK} W}{J_W} \nu_K(K,t) w^T C \sigma_K^T$ (9.2)
If we use CRRA as the utility function, we can assume \(-\frac{J ww}{w} = \eta\), the risk-aversion parameter. Then (9.2) is:

\[
\max_w \ w^T C \xi - \frac{1}{2} \eta w^T CC^T w - \eta \nu_K(K, t) w^T \sigma_K^T \quad (9.3)
\]

This optimization problem is basically the problem Gintschel and Scherer (2008) solves. Their model set-up is a special case of (9.3).

For the optimization problem (9.3), we can add the non-negativity constraints for asset allocations: \(w \geq 0\). But to solve the value function in the HJB equation (5.4.1) and verify it to be of the form \(J(W + \nu(K, t), t)\) may be hard. However, we can still use numerical results from (9.3) with non-negativity constraints to compare with the “simple optimal” asset allocation with non-negative constraints.

Another example of limitation is incorporating transaction costs. To incorporate linear transaction costs is to put linear constraints on intertemporal asset allocation. The intertemporal constraints makes the problem even harder if not impossible to solve. In general, some efforts need to be made for the framework to incorporate flexible constraints.

2. Investing in the integrated optimal portfolio may be very difficult. For example, the SWF is so large that there are severe limitations on purchases and sales in the futures market. Moreover, the futures market includes contango and backwardation and thereby roll returns must be considered. Stochastic control is a good academic framework for analyzing the critical relationships between important state and decision variables. However, it is not always practical for implementation. Stochastic programming can address these issues above easily. It allows flexible modeling structure and constrains. But stochastic programming has its own limitations. The approach
involves the step of scenario generation. For an asset allocation problem with a time horizon as long as ten years or above, the scenario tree for asset returns will grow exponentially into a huge one. The curse of dimension may render the problem very difficult to solve. How to combine stochastic control and stochastic programming together is a direction for future research. One may generate scenarios at initial steps and then apply approximated values solved from stochastic control at each end node. One can refer to Konicz and Mulvey (2015) for an example of linking stochastic control and stochastic programming.

3. Another limitation is the assumption of market completeness. If this condition is not satisfied, then the Separation Theorem does not apply. For example, in chapter 4, equation (4.16) does not hold true if there is idiosyncratic risk in oil prices. In general, the term $-\frac{1}{2}\theta_3 J_{KK}^2$ and the term $\frac{1}{2}\theta_4 J_{KK}$ cannot cancel out since $\theta_3$ is not equal to $\theta_4$. As a result, we cannot follow the machinery to get the solution to the HJB equation (2.11). If market completeness is not satisfied, we need to think of alternative approaches to solve the asset allocation problem. One potential solution is to simply leave out the idiosyncratic risk factors. We just solve the hypothetical optimal integrated asset allocation without idiosyncratic business risks: $w^*_h$. This hypothetical optimal asset allocation partially hedges the business risks—the hedgeable part. Then we add back the idiosyncratic business risks to the dynamics. We use simulations to calculate the constant equivalents of final wealth by the hypothetical asset allocation $w^*_h$ and the "simple optimal" asset allocation. Since $w^*_h$ partially hedges business risks, its constant equivalent of final wealth should be higher. Thus, even though the strategy cannot hedge all the business risks, it is a better solution than "simple optimal" asset allocation. The business risks cannot be fully hedged anyway.
4. A third limitation is on market parameters. In the framework of this paper, we assume constant parameters for market dynamics, especially risk-free rate, risk premiums and covariance matrices. However, financial markets have shifting regimes. And these parameters may be time-varying and stochastic. It should not be hard to extend the framework to time-varying market conditions. One can follow the methodology in Merton (1973). Besides an efficient portfolio element and a hedging element for business risks, the optimal integrated asset allocation will have a third element of hedging changes in investment opportunity set. Since the asset allocation is a multi-period problem, one needs to consider allocation decisions on both the asset dimensions and the time dimension. The covariance of asset returns of different time period is as important as covariance between asset returns during the same time period. One may model the asset return dynamics as Merton (1973). To do so, one needs to identify the right state variables to characterize the changes in investment opportunity sets. Bae, Kim and Mulvey (2012) gives an alternative method of solving for optimal asset allocation under regime-switching market conditions. They use hidden markov model to characterize asset return dynamics and use stochastic programming to solve the asset allocation problem.

5. Another extension is about the governance structure assumption. The Disentanglement Condition 1 assumes $\lambda = 1$. In this case, the decision-maker is the investment entity in the I-B complex. Normally, the business entity and investment entity have separate governance structure of decision-makers. But if the objectives of both the business and investment entities are considered, then $\lambda$ is not 0. The Disentanglement Condition 1 is not satisfied. Business decisions and investment decisions are “entangled” together. Just like Quantum Entanglement is a very hard issue in physics, the "entanglement" of business and investment decisions makes the physics of integrated asset allocation extremely hard to solve. Here we propose one possible
solution:

Step 0. Pretend the Disentanglement Conditions still hold true and solve for an intial solution for business decisions $u_0$, investment decisions $w_0$, consumption decisions $q_0$ and value function $V_0$.

Step 1. Follow the machinery again, use equation (5.2.3) and plug in $V_0$ to solve for an iteration of “optimal” business decisions:

$$
\frac{\partial V_0}{\partial K} b_u(K, u_1, v_1, t) + \frac{\partial V_0}{\partial W} (\gamma P^K) u(K, u_1, t) = 0 \quad (9.4)
$$

where $v_1 = \gamma P^K(K, u_1, t)$.

Or else, if it is a corner solution:

If $\frac{\partial V_0}{\partial K} b_u(K, u_1, v_1, t) + \frac{\partial V_0}{\partial W} (\gamma P^K) u(K, u_1, t) > 0$ everywhere, then $u_1 = u_{\text{max},t}$.

If $\frac{\partial V_0}{\partial K} b_u(K, u_1, v_1, t) + \frac{\partial V_0}{\partial W} (\gamma P^K) u(K, u_1, t) < 0$ everywhere, then $u_1 = u_{\text{min},t}$.

(9.5)

Step 2. Use equation (5.4.4) to solve for an iteration of “optimal” investment $w_1$ and consumption $q_1$ decisions as well as the value function $V_1$:

$$
U(q_1, t) + \frac{\partial V_1}{\partial W}(rW - q_1) + \frac{\partial V_1}{\partial W} \gamma P^K(K, u_1, t) - \frac{1}{2} \theta_1 \frac{V_{1,w}}{V_{1,ww}} - \theta_2 \frac{V_{1,w}V_{1,ww}}{V_{1,ww}} - \frac{1}{2} \theta_3 \frac{V_{1,ww}^2}{V_{1,ww}} + \frac{V_{1,K}b(K, u_1, v_1, t) + \frac{1}{2} \theta_4 V_{1,KK} + V_{1,t}}{V_{1,ww}} = 0 
$$

$$
w_1 = -\frac{V_{1,w}}{V_{1,ww}W} (CC^T)^{-1}C\xi - \frac{V_{1,ww}}{V_{1,ww}W} (CC^T)^{-1}C\sigma_K^T \quad (9.6)
$$

$$
q_1 = U^{-1}(V_{1,w}, s) \quad (9.7)
$$

Step 3. Iterate Step 1 and Step 2 until the value function solution converges to a final solution $V_{it}^*$. We can take the final solution $V_{it}^*$ as the value function solution. Using this value function, we can solve for optimal business $u^*$, investment $w^*$ and consumption $q^*$ decisions. However, it will take some efforts to find the appropriate conditions for it to be convergent.
6. The final direction for future research we propose is on more applications. We can apply the general framework to a broad class of “I-B Complexes”. In this paper, we mainly focus on the asset side of I-B complexes. The externalities to asset allocation are all “businesses”, whether it is state business or family business. However, we can apply the framework to incorporate asset & liability management too. Liabilities can be viewed as “negative businesses”. While normal businesses, in general, generate positive cash flows for the investment entity, liabilities generate negative cash flows for the investment entity. One potential application is to pension plans. The payoff function of liabilities is the annual pension payout of pension plan. Assume the market is complete, we can follow the machinery to calculate the value function $V = J(W - \nu_L(L, t), t)$. In that, $\nu_L(L, t)$ is the value of liabilities. $L$ is the state variable of pension liabilities. The optimal integrated asset allocation is:

$$w^* = -\frac{Jw}{JwW} (CC^T)^{-1}C\xi - \frac{JwL}{JwW} (CC^T)^{-1}C\sigma_{L}^T \quad (9.9)$$

where the second component is a liability-driven investment (LDI) portfolio to hedge the liability risks. Because of the negative sign before the hedging portfolio, it invests more heavily in assets with more positive correlations with liabilities. Thus, the hedging portfolio is actually a liability-driven investment (LDI).

Another potential application is personal investing. Personal goals can be viewed as liabilities. We use $\nu_i^G(K_i^G, t)$ to represent the value function of the $i$th goal out of $N_g$ goals. And denote the state variable of the $i$th goal to be $K_i^G$. For example, if the $i$th goal is a living standard goal, then the state variable is inflation. If the goal is an college-education goal, then the state variable is HEPI index (Higher Education Price Index). Then, the value function for personal investor is

$$V = J(W - \sum_{i=1}^{N_g} \nu_i^G(K_i^G, t), t).$$

And the optimal integrated asset allocation is:

$$w^* = -\frac{Jw}{JwW} (CC^T)^{-1}C\xi - \sum_{i=1}^{N_g} \frac{JwK_i^G}{JwW} (CC^T)^{-1}C\sigma_{K_i^G}^T \quad (9.10)$$
where the second to last components are hedging portfolios for each individual goals. Because of the negative sign before the hedging portfolios, each hedging portfolio invests more in assets with positive correlations with the corresponding goal. Thus, each hedging portfolio is a goal-hedging portfolio (GHP) as coined by Deguest, Martellini, Milhau, Suri and Wang (2015).

However, this application can only treat all liabilities equally. The rankings of goals cannot be reflected in the application. An alternative approach is to characterize the goals in consumption utility $U(q)$ or payoff functions $P_{K_i}^{G}$. If the goals are characterized in consumption utility, our framework will be not so useful. If the goals are characterized in the payoff functions, the goals are set up as competing goals with each other and with the long-term wealth growth. We can use the relative values of $\lambda : \lambda_1, \ldots, \lambda_{N_g}$ for $P_{K_1}^{G}, \ldots, P_{K_N}^{N_g}$ to characterize the rankings for different goals.

However, to solve the asset allocation in this set-up needs a solution to question 4 first: how to solve the integrated asset allocation problem when $\lambda$ is not zero. Notice that, for (9.9) and (9.10), the hedging portfolios will disappear if the volatilities of liabilities and goals are zero. So the hedging portfolios are just for uncertainties in liabilities and goals. On the other hand, the asymmetry of utility function for meeting liabilities and goals results in nonlinearity of optimal asset allocation. Ang, Chen and Sundaresan (2013) shows pension plans’ liability-meeting objective is equivalent to shorting a put option on its own assets. This feature causes the nonlinearity of optimal asset allocation. This nonlinearity is reflected in the coefficient of the efficient portfolio, namely the first component of (9.10), instead of the hedging portfolios.

The scope of the business entities is also to be further researched on. In the framework, we differentiate spending with cash flow transfers with business entities. The distinction may be blurred in some cases. For example, Temasek uses the spending to help with Singapore government budget. This type of spending will increase the profitability of the sponsor and should be distinguished from pure consumption. However,
it is definitely unwise to include the Singapore government in the model if we were to develop one for Temasek. But at least part of the spending should be counted as "invest-in-your-own-business" type of investment. How to quantitatively incorporate the effect is a topic worthy of discussion. Another example is regional pension plan. There are times when specific investments in the fund that can help with liabilities. CalPERS (California Public Employees’ Retirement System), the largest public pension plan in the U.S., could invest some of their $300 Billion AUM in infrastructure in California – which would generate an increase in the number of active State workers compared with CalPERS investing exclusively outside California. Thus, expenditures of the investment entity sometimes have purposes of both spending and "investment in your own business". The identification and quantification process need to be done on a case-by-case basis.
Appendix A

Technical Details

This chapter is for documenting the technical details of theorems, proofs and calculations.

A.1 Feynman-Kac Formula

Consider the PDE:

\[ \frac{\partial u}{\partial t}(x,t) + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) - V(x,t)u(x,t) + f(x,t) = 0 \]  (A.1.1)

The equation is defined for all \( x \) in \( \mathbb{R} \) and \( t \) in \([0, T]\), subject to the boundary condition:

\[ u(x,T) = \phi(x), \quad (A.1.2) \]

where \( \mu, \sigma, \phi, V, f \) are smooth functions, \( T \) is a parameter. To solve for the function \( u : \mathbb{R} \times [0, T] \to \mathbb{R} \), the Feynman-Kac formula theorem tells us that the solution can be written as a conditional expectation:

\[ u(x,t) = E^Q[\int_t^T e^{-\int_t^T V(X_r,r)dr} + e^{-\int_t^T V(X_r,r)dr}\phi(X_T)|X_t = x] \]  (A.1.3)
under the probability measure $Q$ such that $X$ is an Ito process driven by the equation:

$$dX = \mu(X, t)dt + \sigma(X, t)dW^Q,$$  \hspace{1cm} (A.1.4)

where $W^Q_t$ is a Brownian motion under $Q$, and the initial condition for $X(t)$ is $x$.


### A.2 Distribution for Control Variate $\tilde{V}_{oil,t}$

In equation (4.19.3), we can see that the term inside exponential is a normally distributed random variable for any time $t$. So the control variate $\tilde{V}_{oil,t}$ is a geometric normal random variable. We only need to calculate the mean and variance of the term inside exponential to get the distribution of $\tilde{V}_{oil,t}$.

For mean $\mu$:

$$\mu = \int_t^T -(r - \rho)(s - t) \frac{dt}{T-t} + \int_t^T x_t e^{-\alpha_0(s-t)} + (\phi_0 - \frac{\theta_4}{\alpha_0} - \frac{1}{2} \frac{\theta_4}{\alpha_0^2})(1 - e^{-\alpha_0(s-t)}) \frac{ds}{T-t}$$

$$= -(r - \rho) \frac{T-t}{2} + \frac{x_t}{\alpha_0(T-t)}(e^{-\alpha_0 t} - e^{-\alpha_0 T}) + (\phi_0 - \frac{\theta_4}{\alpha_0} - \frac{1}{2} \frac{\theta_4}{\alpha_0^2})(1 - \frac{e^{-\alpha_0 t} - e^{-\alpha_0 T}}{\alpha_0(T-t)})$$

since the expectation for the term of stochastic calculus is 0.

For variance $\sigma$:

$$\text{Var}(\int_t^T \int_t^s e^{\alpha_0(\tau-s)} \sqrt{\theta_4} dW^Q Q d\tau) = E[(\int_t^T \int_t^s e^{\alpha_0(\tau-s)} \sqrt{\theta_4} dW^Q Q d\tau)^2]$$

$$= \frac{\theta_4}{(T-t)^2} \left( \frac{1}{4\alpha_0} (e^{2\alpha_0(T-t)} - 1) - \frac{T-t}{2\alpha_0} - \frac{(T-t)^2}{2\alpha_0} \right)$$
A.3 Calculation of Control Variate $W'$

Calculate the differential of $\exp(-r_p t - \frac{1}{2} \nu \nu^T t + \nu Z_t) W'$. Plug in (4.24). We can get:

$$d(\exp(-r_p t + \frac{1}{2} \nu \nu^T t - \nu Z_t)W') = \exp(-r_p t + \frac{1}{2} \nu \nu^T t - \nu Z_t) m(\phi_t - c(t)) dt \quad (A.3.1)$$

Hence, we can get:

$$\exp(-r_p T + \frac{1}{2} \nu \nu^T T - \nu Z_T) W_T - W_0 = \int_0^T \exp(-r_p s + \frac{1}{2} \nu \nu^T s - \nu Z_s) m(\phi_s - c(s)) ds$$

(A.3.2)

Hence, we can get (4.24.1) by moving $W_0$ from the left-hand side to the right-hand side and dividing both sides of (A.3.2) by $\exp(-r_p T + \frac{1}{2} \nu \nu^T T - \nu Z_T)$. 

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