THE FONTAINE-MAZUR CONJECTURE IN THE
RESIDUALLY REDUCIBLE CASE

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS
ADVISER: RICHARD LAWRENCE TAYLOR

JUNE 2018
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Abstract

In this thesis, we prove new cases of Fontaine-Mazur conjecture on two-dimensional Galois representations over \( \mathbb{Q} \) when the residual representation is reducible. Our approach is via a semi-simple local-global compatibility of the completed cohomology and a Taylor-Wiles patching argument for the completed homology in this case. As a key input, we also generalize works of Skinner-Wiles in the ordinary case.
Acknowledgements

I would like to express my great gratitude to my advisor, Professor Richard Taylor, for suggesting me to work on the topic of this thesis, for all he taught me throughout my time in Princeton, and for his patient guidance, support and encouragement. I would like to thank Professor Christopher Skinner for sharing his ideas on his thesis work and for agreeing to be a thesis reader. I would also like to thank Professor Shou-wu Zhang and Professor Akshay Venkatesh for being on my thesis committee.

It will be clear to the reader how much influence the work of Emerton [16], Paskunas [32] and Skinner-Wiles [39] had on this thesis. It is a pleasure to thank these people as well. My thanks also go to Ziquan Zhuang for his help on commutative algebra, Bao V. Le Hung for pointing out a mistake during the preparation of the thesis, and Yongquan Hu for some useful discussions.

Life in Princeton would be extremely hard for me without friends. I would like to thank my friends in the math department, especially my officemates Yuchen Liu, Charlie Stibitz, Anibal Velozo, Ziquan Zhuang and Amitesh Datta. Outside of Fine hall, I would like to thank Jiequn Han and Han Hao for many wonderful Mario Kart nights, and Zhiyuan Ding and Junliang Shen for their constant encouragement since we were undergraduates.

Part of this thesis was written when I was visiting the Morningside Center of Mathematics Chinese Academy of Sciences during the summer of 2017. I would like to thank the institution for its hospitality.

Finally, I thank my parents for their supportive love over the past 27 years.
To my parents.
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Chapter 1

Introduction

First we recall the following remarkable conjecture of Fontaine and Mazur made in [17].

\textbf{Conjecture 1.0.1 (Fontaine-Mazur).} Let $p$ be a prime number and

\[ \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\overline{\mathbb{Q}}_p) \]

be a continuous, irreducible representation such that

- \( \rho \) is only ramified at finitely many places,
- the restriction of \( \rho \) at the decomposition group at \( p \) is potentially semi-stable in the sense of Fontaine,
- \( \rho \) is odd, i.e. \( \det \rho(c) = -1 \) for any complex conjugation \( c \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \).

Then \( \rho \) arises from a cuspidal eigenform.

In [26], Kisin has proved this conjecture in many cases via the so-called Breuil-Mézard conjecture. Under slightly more restrictively conditions, Emerton in [16] gave another proof using his completed cohomology and local-global compatibility results. A common ingredient for both works is the \( p \)-adic local Langlands correspondence
for $\text{GL}_2(\mathbb{Q}_p)$, which was established by the work of many mathematicians including Breuil, Colmez, Berger, Kisin, Emerton, Paškūnas.

It should be pointed out that both works need assume the residual representation $\bar{\rho}$ to be irreducible when restricted to $\text{Gal}((\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_p))$. In particular, $\bar{\rho}$ has to be irreducible. Results for residually reducible $\rho$ are mostly known in the ordinary case (i.e. $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible) by the work of Skinner-Wiles [39]. They work with the $p$-adic family of ordinary forms (Hida family), which gives them more freedom and allows them to handle the residually reducible case.

A natural generalization of Hida family in the non-ordinary case is the completed cohomology. It is natural to ask whether we can establish new cases of Fontaine-Mazur conjecture in the non-ordinary, residually reducible case by working with completed cohomology. Here is our result (6.0.1):

**Theorem 1.0.2.** Let $p$ be an odd prime and $\rho$ be as in the conjecture. Assume furthermore that

- $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ has distinct Hodge-Tate weights,
- the semi-simplification of $\bar{\rho}$ is a sum of two characters $\bar{\chi}_1 \oplus \bar{\chi}_2$. Assume $\frac{\bar{\chi}_1}{\bar{\chi}_2}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \neq 1, \omega^{\pm 1}$. Here $\omega$ denotes the mod $p$ cyclotomic character.

Then $\rho$ comes from a cuspidal eigenform.

Some remarks on our conditions. As we just mentioned, when $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible, this is the result of Skinner-Wiles. Hence the only contribution here is the case where $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is irreducible. Our method will not yield a new proof in the ordinary case. In fact, our proof in the non-ordinary case uses the work of Skinner-Wiles (and its generalization) as a key ingredient.

The assumption $\frac{\bar{\chi}_1}{\bar{\chi}_2}|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \neq 1, \omega^{\pm 1}$ comes from the fact that the local deformation ring of pseudo-character $\bar{\chi}_1 + \bar{\chi}_2$ at $p$ is smooth in this case. I have some work in progress trying to remove this condition.
To prove the main result, we follow the strategy of Emerton. There are two steps:

1. Prove a ‘big’ $R = T$ result. Here $R$ denotes some global Galois deformation ring of pseudo-characters with no condition at $p$ and $T$ is a localization of some Hecke algebra for completed cohomology.

2. Prove a classicality result: if $\rho$ arises from completed cohomology and $\rho|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is irreducible, de Rham of distinct Hodge-Tate weights, then it comes from a classical eigenform.

In reality, we do not quite prove an $R = T$ result in the first step. What we actually proved is identifying some irreducible components of $\text{Spec } R$ as in $\text{Spec } T$, which is enough for our purpose. The basic idea is as follows. Let $p \in \text{Spec } R$ be the prime ideal given by $\rho$. Then a standard calculation of Galois cohomology shows that we may choose an irreducible component $C$ containing $p$ with large dimension (lemma [6.0.2]). Under our generic assumption on $\overline{\chi}_1|_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$, we can show that the ordinary locus $C^{\text{ord}}$ in $C$ that correspond to deformations reducible at $p$ also has large dimension (lemma [6.0.3]). Using the result of Skinner-Wiles and its generalization (theorem [5.1.2]), we are able to show that $C^{\text{ord}}$ is contained in $\text{Spec } T$. Now choose a ‘nice’ one-dimensional prime $q \in C^{\text{ord}}$ and apply Taylor-Wiles patching argument to this prime. We conclude that $C$ hence $p$ is contained in $\text{Spec } T$.

Here we need a version of patching argument for completed cohomology. For this part of the discussion, we will focus on patching at the maximal ideal as in the classical case. Although in our applications the patching will be at a one-dimensional prime (following [39]), there is no essential difference except several technical complications. A patching argument in this setting has already been considered by Gee and Newton in [18]. As far as I understand, there are two essential difficulties:

1. We need a correct lower bound on $\text{dim } T$. 

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2. The patched completed homology is not finitely generated over $R_\infty$, the product of some local deformation rings with several formal variables. This is equivalent with saying that the completed homology is not finitely generated over $T$.

We remark that in the classical case, since the Hecke algebra is a finite $\mathbb{Z}_p$-module, we have a natural and correct bound for its dimension (which is 1!). In the ordinary case, we get a bound on $\dim T$ from the Iwasawa algebra $\Lambda$ (see corollary 5.2.5 below). For the second difficulty, recall that the classical patching argument involves supports, which do not behave well once we are not in the finitely generated world. The failure of the completed homology to be finitely generated over $T$ is because there is an action of $\text{GL}_2(\mathbb{Q}_p)$ on it. How to translate the information coming from this action is crucial.

The solution to both problems is our local-global compatibility result and an application of the work of Paškūnas [32], which is the main technical innovation in this paper. Let $R_p^{\text{ps}}$ be the local deformation ring of some two-dimensional pseudo-representation at $p$ with fixed determinant. It will play the similar role here as the Iwasawa algebra in Hida’s theory. There are two actions of $R_p^{\text{ps}}$ on the completed homology: one comes from the associated Galois pseudo-representation over the Hecke algebra, the Galois side; the other one comes from the action of $\text{GL}_2(\mathbb{Q}_p)$, the spectral side. More precisely, by the work of Paškūnas, we know that $R_p^{\text{ps}}$ is in fact a component of the Bernstein centre of certain category of representations of $\text{GL}_2(\mathbb{Q}_p)$ (theorem 1.5 of [32]). Our main result is that both actions are the same (theorem 3.5.5).

This simple equality has surprisingly many important corollaries. For example,

**Corollary 1.0.3.** $T$ has dimension at least $1 + 2$.

In general, under certain assumptions, this will be $1 + 2[F : \mathbb{Q}]$ if we are working with some totally real field $F$ in which $p$ completely splits. We refer to theorem 3.6.1 for a precise statement. Also we get the classicality result (corollary 3.5.12) as we
need in the second step of the whole argument. For the patching argument: the local-
global compatibility result implies that a construction in the work of Paškūnas turns
the completed homology into a \textit{finitely generated, faithful} $\mathbb{T}$-module (corollary \textbf{3.5.10}).
Thus we can apply the usual patching argument to this module and everything works
well again.

A local-global compatibility conjecture for completed cohomology first appeared in
Emerton’s work \cite{Emerton} attacking the Fontaine-Mazur conjecture. Using our description
of the action of the Bernstein centre on the completed cohomology, we can show his
conjecture when the Galois representation at $p$ is irreducible. To prove our local-
global compatibility result, we follow the strategy of Emerton: the density result for
crystalline points reduces ourselves to the crystalline case, which follows from the
result of Berger-Breuil \cite{BB} and classical local-global compatibility.

Our patching argument for completed cohomology also gives a direct proof of the
results of Kisin, Emerton. I intend to provide the details in the future work.

The paper is organized as follows. We first recall some properties of deformation
rings of pseudo-representations in chapter \textbf{2}. This will only be used in our patching
argument at a one-dimensional prime. In chapter \textbf{3} we introduce completed coho-
mology and state and prove our local-global compatibility result. Chapter \textbf{4} is rather
technical. We give in detail our patching argument for completed homology at a
one-dimensional prime. In chapter \textbf{5} we follow the strategy of Skinner-Wiles and
generalize their work into the form we are using. In the last chapter, we put all these
results together and prove the main theorem.

\textbf{Notations}

Throughout the paper, we fix an odd prime $p$, a finite extension $E$ of $\mathbb{Q}_p$ and a
uniformizer $\varpi$ of $E$. We denote its ring of integer by $\mathcal{O}$ and residue field by $\mathbb{F}$. Also
we fix an embedding of $E$ into $\overline{\mathbb{Q}}_p$, some algebraic closure of $\mathbb{Q}_p$ and an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$.

We use $C^f_{\mathcal{O}}$ to denote the category of Artinian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$ and use $C_{\mathcal{O}}$ to denote the category of topological local $\mathcal{O}$-algebras which are isomorphic to inverse limits of objects of $C^f_{\mathcal{O}}$.

Following [32], we introduce some categories. See §2 of [32] for more precise definitions. Let $G$ be a topological group which is locally pro-$p$ and $(A, \mathfrak{m})$ be a complete local Noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$. We denote by $\text{Mod}_G(A)$ the category of $A[G]$-modules, $\text{Mod}_{G,\text{sm}}^\text{fin}(A)$ the full subcategory of $\text{Mod}_G(A)$ consisting of smooth $G$-representations, i.e. any element in the representation is killed by some power of $\mathfrak{m}$ and fixed by some open subgroup of $G$. We also denote by $\text{Mod}_{G,\text{fin}}^\text{fin}(A)$ the full subcategory of $\text{Mod}_{G,\text{fin}}^\text{sm}(A)$ consisting of representations locally of finite length and $\text{Mod}_{G,\text{adm}}^\text{adm}(A)$ the full subcategory consisting of locally admissible representations. Let $Z$ be the centre of $G$ and $\zeta : Z \to A^\times$ be a continuous character. Then we can define $\text{Mod}_{G,\zeta}^\text{sm}(A), \text{Mod}_{G,\zeta}^\text{fin}(A), \text{Mod}_{G,\zeta}^\text{adm}(A)$ similarly as subcategories with central character $\zeta$. All representations in these categories are considered to have discrete topology.

Let $H$ be a compact open subgroup of $G$ and $A[[H]]$ be the completed group algebra of $H$. Let $\text{Mod}_{G,\text{pro,aug}}^\text{pro,aug}(A)$ be the category of profinite linearly topological $A[[H]]$-modules with action of $A[G]$ such that both agree on $A[H]$, with morphisms $G$-equivariant continuous homomorphisms of topological $A[[H]]$-modules. This is independent of the choice of $H$. Taking Pontryagin duals (with the discrete topology on $E/\mathcal{O}$):

$$M \mapsto M^\vee := \text{Hom}^{\text{cont}}_{\mathcal{O}}(M, E/\mathcal{O})$$

induces an anti-equivalence of categories between $\text{Mod}_{G,\text{sm}}^\text{sm}(A)$ and $\text{Mod}_{G,\text{pro,aug}}^\text{pro,aug}(A)$. There is a natural isomorphism between $M^{\vee\vee}$ and $M$. Under this anti-equivalence, we define
$\mathcal{C}_{G,\zeta}(\mathcal{O})$ to be the full subcategory of $\text{Mod}^{\text{pro} \text{aug}}_G(A)$ with objects isomorphic to $\pi^\vee$ for some object $\pi$ in $\text{Mod}^{\text{lad}}_{G,\zeta}(A)$. Note that this is the $\mathcal{C}(\mathcal{O})$ used in §5 [32].

For a prime ideal $\mathfrak{p}$ of a commutative ring $R$, we denote its residue field by $k(\mathfrak{p})$. Let $R_\mathfrak{p}$ be the localization at $\mathfrak{p}$. We write $\hat{R}_\mathfrak{p}$ as its $\mathfrak{p}$-adic completion. For an ideal $I$, we denote its height by $\text{ht}(I)$ and its Krull dimension of $R/I$ by $\text{dim} R/I$.

For a representation $\rho : \Gamma \to \text{GL}_n(R)$, we use $\text{tr}(\rho)$ or $\text{tr} \rho$ to denote its trace and $\text{det} \rho$ to denote its determinant.

Suppose $F$ is a number field with maximal order $\mathcal{O}_F$. For any finite place $v$, we will write $F_v$ (resp. $\mathcal{O}_{F_v}$) for the completion of $F$ (resp. $\mathcal{O}_F$) at $v$, $\varpi_v$ for a uniformizer of $F_v$, $k(v)$ for the residue field, $N(v)$ for the norm of $v$ (in $\mathbb{Q}$), $G_{F_v}$ for a decomposition group above $v$, $I_{F_v}$ for its inertia group and $\text{Frob}_v$ for a geometric Frobenius element in $G_{k(v)} := G_{F_v}/I_{F_v}$. By abuse of notation, we will also fix a lifting of $\text{Frob}_v$ in $G_{F_v}$. If $l$ is a rational prime, then we denote $\mathcal{O}_F \otimes \mathbb{Z}_l$ by $O_{F,l}$. The adele ring of $F$ will be denoted by $\mathbb{A}_F$ and $| \cdot |_{\mathbb{A}_F} : \mathbb{A}_F \to \mathbb{R}$ denotes the adelic norm map. Suppose $S$ is a finite set of places of $F$. We use $\mathbb{A}^S_F$ to denote the set of adeles away from $S$ and $G_{F,S}$ for the Galois group of the maximal extension of $F$ unramified outside $S$ and all infinite places. We will view $(\mathbb{A}^S_F)^\times$ as a subgroup of $\mathbb{A}_F^\times$ with elements having 1 at $S$. The absolute Galois group of $F$ is denoted by $G_F = \text{Gal}(\overline{F}/F)$.

We use $\varepsilon$ to denote the $p$-adic cyclotomic character and $\omega$ to denote the mod $p$ cyclotomic character. Our convention for the Hodge-Tate weight of $\varepsilon$ is $-1$. We normalize local class field theory by sending uniformizers to geometric Frobenius elements.

For a finite set $Q$, we denote its cardinality by $|Q|$. 

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Chapter 2

Some results on pseudo-representations

In this chapter, we collect some results on the 2-dimensional pseudo-representation and its deformations. These results will be useful in our study of the patching argument at a one-dimensional prime.

2.1 Pseudo-representations

2.1.1. Suppose $\Gamma$ is a profinite, topologically finitely generated group and $R$ is a topological commutative ring with 1 in which 2 is invertible. A 2-dimensional pseudo-representation (or pseudo-character) of $\Gamma$ over $R$ is a continuous function $T: \Gamma \to R$ which behaves like a ‘trace’ in the sense that (see [40])

1. $T(1) = 2$, where $1 \in \Gamma$ is the identity element.

2. $T(\sigma \tau) = T(\tau \sigma)$ for any $\sigma, \tau \in \Gamma$.

3. $T(\gamma \delta \eta) + T(\gamma \eta \delta) - T(\gamma \eta)T(\delta) - T(\eta \delta)T(\gamma) - T(\delta \gamma)T(\eta) + T(\gamma)T(\delta)T(\eta) = 0$, for any $\delta, \gamma, \eta \in \Gamma$. 

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We define the determinant $\det(T)$ of a pseudo-representation $T$ by

$$\det(T)(\sigma) = \frac{1}{2}(T(\sigma)^2 - T(\sigma^2)), \ \sigma \in \Gamma.$$  

2.1.2 (Assumption). Though the following assumption on $\Gamma$ and $T$ can be weakened, it will simplify a lot of discussions below. We will keep this assumption in this chapter.

- There exists an order 2 element $\sigma^* \in \Gamma$ such that $T(\sigma^*) = 0$.

In our case, $\Gamma$ will be certain Galois group of a totally real field and $\sigma^*$ can be chosen as any complex conjugation in $\Gamma$.

2.1.3. Given a pseudo-representation $T$ which satisfies 2.1.2 and $\sigma, \tau \in \Gamma$, we can define:

- $a(\sigma) = \frac{1}{2}(T(\sigma^*\sigma) + T(\sigma)).$
- $d(\sigma) = \frac{1}{2}(-T(\sigma^*\sigma) + T(\sigma)) = T(\sigma) - a(\sigma)$.
- $x(\sigma, \tau) = a(\sigma\tau) - a(\sigma)a(\tau)$.

They satisfy the following identities appeared in the original definition of Wiles.

1. $x(\sigma, \tau) = d(\tau\sigma) - d(\tau)d(\sigma)$.
2. $x(\sigma\tau, \delta) = a(\sigma)x(\tau, \delta) + x(\sigma, \delta)d(\tau)$.
3. $x(\sigma, \tau\delta) = a(\delta)x(\sigma, \tau) + x(\sigma, \delta)d(\tau)$.
4. $x(\alpha, \beta)x(\sigma, \tau) = x(\alpha, \tau)x(\sigma, \beta)$.

Since there seems to be no good reference for these identities, we sketch a proof here.
Proof. Take $\gamma = \sigma, \delta = \tau, \eta = \sigma^*$ in the last axiom of the definition of pseudo-representation. This will give us the first identity.

Write $A(\gamma, \delta, \eta) = T(\gamma\delta\eta) + T(\gamma\eta\delta) - T(\gamma\eta)T(\delta) - T(\eta\delta)T(\gamma) - T(\delta\gamma)T(\eta) + T(\gamma)T(\delta)T(\eta)$. Consider

\[ A(\sigma, \tau, \delta\sigma^*) + A(\tau, \delta, \sigma^*) - A(\tau, \sigma\delta, \sigma^*) = 0. \]

A direct computation shows that this gives

\[ 2T(\sigma\tau\delta\sigma^*) = T(\sigma\tau)T(\delta\sigma^*) + T(\tau\delta)T(\sigma\sigma^*) + T(\sigma\delta)T(\tau\sigma^*) + T(\tau)T(\sigma\delta\sigma^*) + T(\sigma)T(\tau\delta\sigma^*) - T(\sigma)T(\tau)T(\delta\sigma^*) - T(\tau)T(\delta)T(\sigma\sigma^*). \]

Replacing $\delta$ by $\delta\sigma^*$ and taking the sum of this equality with the equality above, we obtain the second identity. The third identity can be proved similarly.

The last identity follows by applying the second and third identities repeatedly:

\[ x(\alpha, \beta)x(\sigma, \tau) = (a(\alpha\beta) - a(\alpha)a(\beta))x(\sigma, \tau) \]

\[ = (x(\alpha\beta\sigma, \tau) - x(\alpha\beta, \tau)d(\sigma)) - a(\alpha)a(\beta)x(\sigma, \tau) \]

\[ = (x(\alpha, \tau)d(\beta\sigma) + a(\alpha)x(\beta\sigma, \tau)) - x(\alpha\beta, \tau)d(\sigma) - a(\alpha)a(\beta)x(\sigma, \tau) \]

\[ = (x(\alpha, \tau)x(\sigma, \beta) + x(\alpha, \tau)d(\beta)d(\sigma)) + a(\alpha)x(\beta\sigma, \tau) - x(\alpha\beta, \tau)d(\sigma) \]

\[ - a(\alpha)a(\beta)x(\sigma, \tau) \]

Applying the second identity to $x(\beta\sigma, \tau), x(\alpha\beta, \tau)$, we get the desired identity. \qed

Remark 2.1.4. If $\rho : \Gamma \to \text{GL}_2(R)$ is a continuous representation, then $\text{tr}(\rho)$ is a pseudo-representation. If $\rho(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$ with $a(\sigma), d(\sigma)$ defined above and $x(\sigma, \tau) = b(\sigma)c(\tau)$. 

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2.1.5. Now assume $R$ is either a field or a discrete valuation ring. We can attach a 2-dimensional representation $\rho$ of $\Gamma$ over $R$ with trace $T$. There are two cases:

1. All $x(\sigma, \tau) = 0$. Then $a, d : \Gamma \to R^\times$ are two characters. Define

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & 0 \\ 0 & d(\sigma) \end{pmatrix}.$$ 

We call this case reducible.

2. $x(\sigma, \tau) \neq 0$ for some $\sigma, \tau$. Choose $\sigma_0, \tau_0$ such that $\frac{x(\sigma, \tau)}{x(\sigma_0, \tau_0)} \in R$ for any $\sigma, \tau$. Define

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & \frac{x(\sigma, \tau_0)}{x(\sigma_0, \tau_0)} \\ \frac{x(\sigma_0, \sigma)}{x(\sigma_0, \tau_0)} & d(\sigma) \end{pmatrix}.$$ 

One can check that this really defines a representation of $\Gamma$ using the identities above. It is clear that $\rho$ is absolutely irreducible. We call this case irreducible.

Note that in this case, if $R$ is a field, then $\rho$ is unique up to conjugation.

The determinant of $\rho$ is nothing but $\det(T)$ we defined before.

2.2 Deformation rings of pseudo-representations I

2.2.1. Let $T_\mathbb{F} : \Gamma \to \mathbb{F}$ be a 2-dimensional pseudo-representation. Then the functor sending each object $R$ of $C^f_\mathcal{O}$ to the set of pseudo-representations $T : \Gamma \to R$ which lift $T_\mathbb{F}$ is prorepresented by a complete Noetherian local $\mathcal{O}$-algebra $R^{ps}_{T_\mathbb{F}}$ (lemma 1.4.2 of [26]). Write $T^{univ} : \Gamma \to R^{ps}_{T_\mathbb{F}}$ as the universal pseudo-character. A standard argument shows that $R^{ps}_{T_\mathbb{F}}$ is topologically generated by $T^{univ}(\Gamma)$. By theorem 1 of [40], there exists a finite set $S \subset \Gamma$ such that the values of $T^{univ}$ on $S$ determine $T^{univ}$. Hence $T^{univ}(S)$ topologically generates $R^{ps}_{T_\mathbb{F}}$. 

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If $T_{\overline{F}}$ is irreducible, write $\overline{\rho}$ as its associated 2-dimensional representation. Then it is well-known that the natural map $R_{T_{\overline{F}}}^{\text{ps}} \rightarrow R_{\overline{\rho}}$ by evaluating the trace is an isomorphism. Here $R_{\overline{\rho}}$ is the universal deformation ring of $\overline{\rho}$.

From now on, we will assume our $T_{\overline{F}}$ is reducible.

Suppose $p$ is an one-dimensional prime of $R_{T_{\overline{F}}}^{\text{ps}}$. Consider $T(p) \overset{\text{def}}{=} T^{\text{univ}} \otimes k(p) : \Gamma \rightarrow k(p)$. Let $\rho(p) : \Gamma \rightarrow \text{GL}_2(k(p))$ be a representation with trace $T(p)$ as we discussed in the previous subsection. In this subsection, we assume

- $k(p) = E$.
- $\rho(p)$ is irreducible (hence absolutely irreducible).

We want to give a moduli interpretation of $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$, the $p$-adic completion of $(R_{T_{\overline{F}}}^{\text{ps}})_p$.

Consider the functor $D_p$ from the category of Artinian local $E$-algebras with residue field $E$ (equipped with $p$-adic topology) to the category of sets which sends $A$ to the set of 2-dimensional pseudo-representations over $A$ lifting $T(p)$. The main result here is

**Proposition 2.2.2.** The deformation problem $D_p$ is represented by $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$ with universal pseudo-representation $\Gamma \xrightarrow{T^{\text{univ}}} R_{T_{\overline{F}}}^{\text{ps}} \rightarrow \widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$.

**Proof.** Given an Artinian local $E$-algebra $A$ with residue field $E$ and a lifting $T_A$ of $T(p)$ to $A$, we need to define a map from $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$ to $A$. Let $S$ be a finite subset of $\Gamma$ such that $T^{\text{univ}}(S)$ topologically generates $R_{T_{\overline{F}}}^{\text{ps}}$. Consider the $\mathcal{O}$-algebra $A_0$ generated by $T_A(S)$ in $A$. It is easy to see that $A_0$ is a finite local $\mathcal{O}$-algebra. Moreover $T_A$ factors through $A_0$. Hence there exists a map $R_{T_{\overline{F}}}^{\text{ps}} \rightarrow A_0$ by the universal property. This map can be extended to a map $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p} \rightarrow A$. It is unique since the values of $T_A$ on $S$ determine the map. Therefore $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$ pro-represents $D_p$. 

**Corollary 2.2.3.** Let $D_{\rho(p)}$ be the functor from the category of Artinian local $E$-algebras with residue field $E$ to the category of sets which sends $A$ to the set of deformations of $\rho(p)$ to $A$. Then $D_{\rho(p)}$ is represented by $\widehat{(R_{T_{\overline{F}}}^{\text{ps}})_p}$. 

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Proof. The natural transformation from $D_{\rho(p)}$ to $D_{\rho}$ by evaluating the trace is an isomorphism since we assume $\rho(p)$ is absolutely irreducible. □

2.3 Deformation rings of pseudo-representations II

In this subsection, we are going to prove a similar but more precise result of corollary 2.2.3 for primes containing $p$. The setup is a bit complicated, but useful for our applications.

2.3.1 (Setup). We start with the following data:

- $A = \mathbb{F}[[T]]$, the formal power series ring over $\mathbb{F}$.
- $\Gamma_0$ is a profinite, topologically finitely generated group.
- $\rho_0 : \Gamma_0 \rightarrow \text{GL}_2(A), \sigma \mapsto \begin{pmatrix} a_0(\sigma) & b_0(\sigma) \\ c_0(\sigma) & d_0(\sigma) \end{pmatrix}$ is a continuous irreducible representation.

We assume that

- The reduction $\bar{\rho}_0$ of $\rho_0$ modulo $T$ has the form $\begin{pmatrix} \bar{a}_0 & \bar{b}_0 \\ 0 & \bar{d}_0 \end{pmatrix}$.
- Fix elements $\sigma_0, \tau_0 \in \Gamma_0$ such that $b_0(\sigma_0) = 1$ and $c_0(\tau_0) \neq 0$.
- Fix an order 2 element $\sigma_0^* \in \Gamma_0$ such that $\rho_0(\sigma_0^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $\Gamma$ be another profinite, topologically finitely generated group with a fixed surjective map

$$\pi : \Gamma \rightarrow \Gamma_0.$$
Moreover we assume there exists an order 2 element $\sigma^* \in \Gamma$ mapping to $\sigma_0^*$. In practice, $\Gamma_0$ will be $G_{F,S}$ for some totally real number field $F$ and some finite set $S$ of places of $F$. Group $\Gamma$ will be $G_{F,S,T}$ for some set $T$ of Taylor-Wiles primes and $\sigma^*$ will be some fixed complex conjugation.

Write $\rho$ (resp. $\bar{\rho}$) as the composite maps of $\pi$ and $\rho_0$ (resp. $\bar{\rho}_0$). Consider the following two universal deformation rings (pro-representing functors from $C^f_C$ to the category of sets):

- $R^{ps} = R^{ps}_{tr(\bar{\rho})}$, the universal deformation ring of the pseudo-representation $\text{tr}\bar{\rho} : \Gamma \to F$.

- $R_b$: the universal deformation ring of the representation $\bar{\rho} : \Gamma \to \text{GL}_2(F)$.

Note that the trace of $\rho$ and the representation $\rho$ give rise to prime ideals $q$ and $q_b$ of $R^{ps}$ and $R_b$. Both prime ideals contain $p$ and have to be one-dimensional since $\rho$ is irreducible while $\bar{\rho}$ is reducible. There exists a natural map:

$$\varphi : R^{ps} \to R_b$$

by evaluating the trace and $\varphi^*(q_b) = q$. It is natural to compare $(R^{ps})_q$ and $(R_b)_{q_b}$.

**Proposition 2.3.2.** There exists an element $c \in R^{ps} \setminus q$ such that

1. the image of $c$ in $R^{ps}/q \to A$ is $c_0(\tau_0) \neq 0$;

2. for any positive integer $n$, there exists an integer $N = N(n) \geq 0$, such that $c^N$ kills the kernel and the cokernel of the map $R^{ps}/q^n \to R_b/q_b^n$. Moreover $N$ only depends on $n$ and $\rho_0 : \Gamma_0 \to \text{GL}_2(A)$ (hence is independent of $\Gamma$).

**Proof.** Denote $R^{ps}/q$ by $B$. We may assume $B$ and $A$ have the same fraction fields by replacing $A$ with the normal closure of $B$. Note that $B$ is topologically generated by the traces of $\rho_0(\Gamma)$, hence independent of $\Gamma$. 

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Choose $\sigma_*, \tau_* \in \Gamma$ to be some liftings of $\sigma_0, \tau_0$. Recall that $\sigma^*$ is an order 2 element mapping to $\sigma_0^*$ in $\Gamma_0$. Since $\text{tr}(\bar{\rho})(\sigma_0^*) = 0$, by taking $\gamma = \delta = \eta = \sigma^*$ in the third condition of 2.1.1 we see that $T(\sigma^*) = 0$ for any lifting $T$ of pseudo-representation $\text{tr}(\bar{\rho})$. In particular, take $T$ to be $T^\text{univ} : \Gamma \to R^{\text{ps}}$, the universal pseudo-representation, and apply the construction in 2.1.3 for $\sigma^*$. We can define $a(\sigma), d(\sigma), x(\sigma, \tau) \in R^{\text{ps}}$ for $\sigma, \tau \in \Gamma$. We also define:

$$c(\tau) = x(\sigma_*, \tau) \in R^{\text{ps}}, \tau \in \Gamma$$

$$c = x(\sigma_*, \tau_*) \in \Gamma \text{ is the lifting of } \pi(\sigma_*) \text{ such that } b_0(\pi(\sigma_*)) = 1.$$  

Note that the reduction of $c(\tau)$ modulo $q$ is $c_0(\pi(\tau)) \neq 0$ as $b_0(\pi(\sigma_*)) = b_0(\sigma_0) = 1$.

We claim that $c$ actually works for the proposition.

**Special Case:** $R^{\text{ps}}/q = R_b/q_b$. First we need some auxiliary construction.

**Definition 2.3.3.** Let $R$ be a commutative ring with 1 and $I \in \text{Spec}(R)$ such that $I^n = 0$ for some integer $n$. Let $c \in R \setminus I$. We can define two new commutative rings:

$$R'' \overset{\text{def}}{=} (R \oplus I^1 \oplus \cdots \oplus I^{n-1})/J,$$

$$R' \overset{\text{def}}{=} R''/R''[c],$$

where

- $R \oplus I^1 \oplus \cdots \oplus I^{n-1}$ is viewed as a graded ring with $\text{gr}^i = I^i$.

- $J$ is an ideal of $R \oplus I^1 \oplus \cdots \oplus I^{n-1}$ generated by elements of the form

$$\left(a_0, \cdots, a_{n-1} \right) - \left(0, a_0 c, \cdots, a_{n-2} c \right)$$

for $a_i \in I^i$ such that $a_i c \in I^{i+1}$.  

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• $c \in R''$ by abuse of the notation is the image of $(c, 0, \cdots, 0)$ in $R''$.

• $R''[c]$ denotes the $c$-torsion part of $R''$.

**Lemma 2.3.4.**

1. The kernel of the natural map $R \to R'$ is killed by $c^{n+1}$.

2. The images of elements of the form $(0, a_1, \cdots, a_{n-1})$ in $R'$ define a prime ideal $I'$. We have $R/I = R'/I'$ and $(I')^n = 0$.

3. The cokernel of $R \to R'$ is killed by $c^{n-1}$.

**Proof.** Let $x \in \ker(R \to R'')$. By definition, there exists $a_i \in I^i$ with $a_{n-1}c = 0$ such that

$$(x, 0, \cdots, 0) = (a_0, a_1 - a_0c, \cdots, a_{n-1} - a_{n-2}c)$$

in $R \oplus I^1 \oplus \cdots \oplus I^{n-1}$. Hence $c^n x = ca_{n-1} = 0$. This proves the first part.

The second part is easy. We omit the proof here. The last part follows from

$$[c^{n-1}(a_0, \cdots, a_{n-1})] = [(a_0c^{n-1} + a_1c^{n-2} + \cdots + a_{n-1}, 0, \cdots, 0)]$$

in $R''$. \hfill $\square$

Apply this construction to $R_n^{ps} \overset{\text{def}}{=} R^{ps}/q^n$ with $I = q/q^n$ and $c = c(\tau_*)$. We get $(R_n^{ps})'$ with an ideal $I'$ and $(R_n^{ps})''$. Consider the composition of the following maps:

$$T'_n : \Gamma \xrightarrow{T_{n, \text{univ}}} R^{ps} \to R^{ps}/q^n \to (R_n^{ps})'.$$

Note that $(R_n^{ps})'/I' = R^{ps}/q = R_b/q_b$. The whole point of introducing $(R_n^{ps})'$ is

**Lemma 2.3.5.** There exists a lifting $\rho'_n$ of $\bar{\rho}$ from $(R_n^{ps})'/I'$ to $(R_n^{ps})'$ with trace $T'_n$. 

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Proof. By abuse of notation, we view \( a(\sigma), d(\sigma), x(\sigma, \tau) \) as elements in \( R_n^{ps}, (R_n^{ps})', (R_n^{ps})'' \). Recall that \( \rho : \Gamma \to \text{GL}_2(A) \) has images in \( \text{GL}_2(R_b/q_b) = \text{GL}_2(R_n^{ps}/q) \). Hence for any \( \sigma \),

\[
x(\sigma, \tau_*) \equiv b_0(\pi(\sigma))c_0(\tau_0) \mod q.
\]

in \( R_n^{ps} \). That is to say we can find \( \tilde{b}(\sigma) \in R_n^{ps} \) and \( y \in q/q^n \) such that

\[
x(\sigma, \tau_*) = \tilde{b}(\sigma)c(\tau_*) + y
\]

in \( R_n^{ps} \). Now let \( b''(\sigma) = [(\tilde{b}(\sigma), y, 0, \cdots, 0)] \) be an element in \( (R_n^{ps})'' \). It is easy to check that this is independent of the choice of \( \tilde{b}(\sigma) \) and \( y \). Note that we have

\[
b''(\sigma)c(\tau_*) = [(\tilde{b}(\sigma)c, yc, 0, \cdots, 0)] = [(\tilde{b}(\sigma)c + y, 0, \cdots, 0)] = x(\sigma, \tau_*) \in (R_n^{ps})''.
\]

Similarly,

\[
b''(\sigma)c(\tau)c(\tau_*) = x(\sigma, \tau)c(\tau_*)
\]

Let \( b(\sigma) \) be the image of \( b''(\sigma) \) in \( (R_n^{ps})' \). Using these identities and identities in \( 2.1.3 \) one can check that

\[
\rho' : \Gamma \to \text{GL}_2((R_n^{ps})'), \quad \sigma \mapsto \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}
\]

really defines a representation with trace \( T_n' \) which lifts \( \rho \).

Let \( \rho' \) be the representation constructed in the proof of the previous lemma. By the universal property, we get a map:

\[
j : R_b \to (R_n^{ps})'
\]

which necessarily factors through \( R_b/q_b^n \) by the second part of lemma \( 2.3.4 \).\]
On the other hand, we may apply the construction $2.3.3$ to $R_b/q^n_b$ with ideal $q_b/q^n_b$ and $c = c(\tau_\ast)$ and get a ring $(R_b/q^n_b)'$. It is easy to see that this construction is functorial and gives us a diagram (not necessarily commutative):

$$
\begin{array}{ccc}
R^\text{ps}/q^n & \xrightarrow{i^n} & (R^n_b)^{\prime} \\
\downarrow{\varphi_n} & \nearrow{j} & \downarrow{\varphi'_n} \\
R_b/q^n_b & \xrightarrow{i_b} & (R_b/q^n_b)^{\prime}
\end{array}
$$

The square and the upper left triangle are commutative by construction. As for the lower right triangle, we have

**Lemma 2.3.6.** For any $x \in R_b/q^n_b$,

$$
\varphi'_n \circ j(x) - i_b(x) \in (R_b/q^n_b)'[c].
$$

**Proof.** Let $\rho'_b : \Gamma \rightarrow \text{GL}_2((R_b/q^n_b)^{\prime})$ be a deformation of $\bar{\rho}$ induced by $i_b$. It is enough to show that modulo $(R_b/q^n_b)^{\prime}[c]$, this representation $\rho'_b$ is conjugate to $\varphi'_n \circ \rho'$ by an element in $1 + M_2(m')$, where $m'$ is the maximal ideal of $(R_b/q^n_b)^{\prime}$.

Note that $\text{tr}(\rho'_n) = \varphi'_n \circ \text{tr}(\rho')$. By conjugating with some element in $1 + M_2(m')$, we may assume

$$
\rho'_n(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho'_n(\sigma_\ast) = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}.
$$

Write $\rho'_n(\sigma) = \begin{pmatrix} a'(\sigma) & b'(\sigma) \\ c'(\sigma) & d'(\sigma) \end{pmatrix}$. Then

$$
\varphi'_n(a(\sigma)) = a'(\sigma), \quad \varphi'_n(d(\sigma)) = d'(\sigma),
$$

$$
\varphi'_n(b(\sigma)c(\tau)) = \varphi'_n(x(\sigma, \tau)) = b'(\sigma)c'(\tau),
$$
since they can be defined by the same formulae using the pseudo-representation. Thus

\[ c'(\tau) = b'(\sigma) c'(\tau) = \varphi_n'(b(\sigma) c(\tau)) = \varphi_n'(c(\tau)), \]

\[ c(b'(\sigma) - \varphi_n'(b(\sigma))) = 0. \]

In other words, \( \rho'_n \equiv \varphi'_n \circ \rho' \mod (R_b/q_b^n)[c] \). This is exactly what we want. \( \square \)

Now we can prove the proposition under the assumption \( R^{ps}/q = R_b/q_b \). Consider the upper left triangle of the previous diagram:

\[ \ker(\varphi_n) \subset \ker(i^{ps}) \subset (R^{ps}/q^n)[c^{n+1}], \]

where the last inclusion follows from the first part of lemma 2.3.4.

Suppose \( x \in R_b/q_b^n \). By the last part of lemma 2.3.4 we may write \( c^{n-1} j(x) = i^{ps}(y) = j \circ \varphi_n(y) \) for some \( y \in R^{ps}/q^n \). Thus \( z = \varphi_n(y) - c^{n-1} x \in \ker(j) \). Apply the previous lemma to \( z \). We get \( cz \in \ker(i_b) \). Now use lemma 2.3.4 again. We have \( c^{n+2} z = 0 \). Hence

\[ c^{2n+1} x = \varphi_n(c^{n+2} y) \in \varphi_n(R^{ps}_n). \]

This proves the proposition with \( N = 2n + 1 \).

**General Cases.** In general, we can use the following trick to reduce to the special case we treated before. Since \( B = R^{ps}/q \) have the same fraction field as \( A = \mathbb{F}[[T]] \), there exists a positive integer \( r \) such that \( T^r A \subset B \). We may assume \( r = p^m \). Hence \( R^{ps}/q \) can be viewed as an \( \mathbb{F}[[T^{p^m}]] \)-algebra. Choose a lifting \( \tilde{T} \) of \( T^{p^m} \) in \( R^{ps} \). This gives \( O[[T^{p^m}]] \)-algebra structures to \( R^{ps} \) and \( R_b \).

Define:

\[ \widetilde{R}^{ps} = R^{ps} \otimes_{O[[T^{p^m}]]} O[[T]], \]

\[ \widetilde{R}_b = R_b \otimes_{O[[T^{p^m}]]} O[[T]] \]
via the natural faithfully flat map $O[[T^{mp}]] \to O[[T]]$. Thus it suffices to prove the kernel and cokernel of 
\[
\tilde{\varphi}_n : \tilde{R}^{ps}/\tilde{q}^n \tilde{R}^{ps} \to \tilde{R}_b/\tilde{q}_b^n \tilde{R}_b
\]
are killed by some power, depending only on $n$, of $c$. Note that $\tilde{R}^{ps}$ and $\tilde{R}_b$ have natural surjective maps to $A$:
\[
\tilde{R}^{ps}/\tilde{q} \tilde{R}^{ps} = B \otimes_{F[[T^{mp}]]} F[[T]] \to A
\]
given by $b \otimes a \mapsto ab$. Let $\tilde{q}$ (resp. $\tilde{q}_b$) be the kernel of $\tilde{R}^{ps} \to A$ (resp. $\tilde{R}_b \to A$). It is clear that 
\[
\tilde{q}^n \subseteq \tilde{q} \tilde{R}^{ps} \subseteq \tilde{q}.
\]

Now one can check that the previous argument of the special case works here for $(\tilde{R}^{ps}, \tilde{q})$ and $(\tilde{R}_b, \tilde{q}_b)$. For example, lemma 2.3.5 still holds and we get a $O[[T]]$-algebra homomorphism $\tilde{R}_b \to (\tilde{R}^{ps}/\tilde{q}^n)'$, similar to the map $j$ in (2.1). As a consequence, the kernel and cokernel of 
\[
\theta_l : \tilde{R}^{ps}/\tilde{q}^l \to \tilde{R}_b/\tilde{q}_b^l
\]
are killed by $c^{2l+1}$. Take $l = np^m$ and write $\tilde{\varphi} : \tilde{R}^{ps} \to \tilde{R}_b$. We have 
\[
c^{2l+1} \tilde{R}_b \subseteq \tilde{\varphi}(\tilde{R}^{ps}) + \tilde{q}_b^l \subseteq \tilde{\varphi}(\tilde{R}^{ps}) + \tilde{q}_b^n \tilde{R}_b.
\]  
(2.2)

This proves that $c^{2np^m+1}$ kills the cokernel of $\tilde{\varphi}_n$.

Let $x$ be an element in $\tilde{R}^{ps}$ such that $\tilde{\varphi}(x) \in \tilde{q}_b^n \tilde{R}_b$. It is clear from (2.2) that 
\[
c^{2np^m+1} \tilde{q}_b \tilde{R}_b \subseteq \tilde{\varphi}(\tilde{q} \tilde{R}^{ps}) + \tilde{q}_b^n \tilde{R}_b.
\]
For any positive integer $k$, a simple induction on $k$ will give us

$$c^{(2np^m+1)(n+(2n-1)+\cdots+((n-1)k+1))} \tilde{\varphi}(x) \in \tilde{\varphi}(q^n \tilde{R}_{ps}) + q_b^{n+k(n-1)} \tilde{R}_b.$$ 

Take $k = 2p^m - 2$. Then $n + k(n - 1) \geq np^m$, hence

$$c^M \tilde{\varphi}(x) \in \tilde{\varphi}(q^n \tilde{R}_{ps}) + q_b^{np^m} \tilde{R}_b \subseteq \tilde{\varphi}(q^n \tilde{R}_{ps}) + \tilde{\varphi}^{np^m},$$

where $M = (2np^m + 1)(n + (2n - 1) + \cdots + ((n-1)k+1))$ only depends on $p^m, n$. This means that we may find $y \in q^n \tilde{R}_{ps}$ such that

$$\tilde{\varphi}(c^M x - y) \in \tilde{\varphi}^{np^m}.$$ 

But kernel of $\theta_{np^m} : \tilde{R}_{ps}/q^{np^m} \rightarrow \tilde{R}_b/q_b^{np^m}$ is killed by $c^{2np^m+1}$. Thus

$$c^{2np^m+1+M} x - c^{2np^m+1} y \in q^{np^m} \subseteq q^n \tilde{R}_{ps}.$$

In other words, $c^{2np^m+M+1}$ kills the kernel of $\tilde{\varphi}_n$. 

The following corollary generalizes Proposition 2.11 of [39].

**Corollary 2.3.7.**

1. For any one-dimensional prime ideal $p_b \in \text{Spec} R_b$ such that $\rho^{univ} \mod p_b$ is irreducible, the natural map $(R_{ps})_p \rightarrow (R_b)_{p_b}$ is an isomorphism and $\text{Spec}(R_b)_{p_b} \rightarrow \text{Spec}(R_{ps})_p$ is surjective. Here $\rho^{univ} : \Gamma \rightarrow \text{GL}_2(R_b)$ is a universal deformation and $p = p_b \cap R_{ps}$.

2. For any $\Omega \in \text{Spec} R_b$ such that $\rho^{univ} \mod \Omega$ is irreducible, we have

$$\dim R_b/\Omega \leq \dim R_{ps}/\Omega \cap R_{ps}.$$
Proof. Note that it is clear from the discussion in 2.1.5 that \( k(p_b) = k(p) \). If \( k(p) = E \), then the isomorphism between \( \widehat{(R^{ps})_p} \) and \( \widehat{(R_b)_{p_b}} \) is a direct consequence of corollary 2.2.3. If \( p \in p \) and the residue field of the normalization of \( R_b/p_b \) is \( F \), then the isomorphism follows from proposition 2.3.2.

In general, we can reduce the problem to these situations by the following trick: Let \( L \) be \( k(p) \) if \( p \notin p \) and a finite unramified extension of \( E \) with residue field same as the integral closure of \( F \) in \( k(p) \) if \( p \in p \). Let \( O_L \) be the ring of integers of \( L \). Then for any prime ideal \( p'_b \in \text{Spec} R_b \otimes O_L \) above \( p \), we may apply the results in the previous paragraph to deformation rings \( (R^{ps})' = R^{ps} \otimes O_L \), \( R'_b = R_b \otimes O_L \) and prime ideals \( p' = p'_b \cap (R^{ps})' \). Let \( O_{L'} \) be the ring of integers of \( L' \).

\[ \widehat{(R^{ps})'}_{p'} \cong \widehat{(R'_b)_{p'_b}}. \]

On the other hand, as \( (R'_b)_{p_b} = (R_b)_{p_b} \otimes O_L \) is a finite \((R_b)_{p_b}\)-algebra, we have

\[ \widehat{(R_b)_{p_b}} \otimes O_L \cong \prod_{p'_b} \widehat{(R'_b)_{p'_b}}, \]

where the product is taken over all primes \( p'_b \in \text{Spec} R'_b \) above \( p_b \). Similarly \( \widehat{(R^{ps})} \otimes O_L \cong \prod_{p'} \widehat{(R^{ps})'}_{p'} \). Here the product is taken over all \( p' \in \text{Spec}(R^{ps})' \) above \( p \). Note that there is a natural bijection between \{\( p' \}\} and \{\( p'_b \}\} as \( k(p) = k(p_b) \), which can be seen using the construction in 2.1.5. Thus

\[ \widehat{(R^{ps})} \otimes O_L \cong \prod_{p'} \widehat{(R^{ps})'}_{p'} \cong \prod_{p'_b} \widehat{(R'_b)_{p'_b}} \cong \widehat{(R_b)_{p_b}} \otimes O_L. \]

From this, we see that \( \widehat{(R^{ps})} \cong \widehat{(R_b)_{p_b}} \). Since \( \widehat{(R^{ps})} \) is faithfully flat over \( R^{ps} \) (and similar result holds for \( (R_b)_{p_b} \)), it is clear that \( \text{Spec}(R_b)_{p_b} \to \text{Spec}(R^{ps})_{p} \) is surjective.
As for the second part, we may find a prime ideal $p_b$ containing $\Omega$ such that $R_b/p_b$ is one-dimensional and $\rho^{\text{univ}} \mod p_b$ is irreducible. To see the existence of $p_b$, note that

$$\{p \in \text{Spec } R_b, \rho^{\text{univ}} \mod p \text{ is reducible}\}$$

is defined by all $x(\sigma, \tau)$ (see the beginning of proof of proposition 2.3.2 for notations here), hence closed in Spec $R_b$. Let $f$ be an element of the form $x(\sigma, \tau)$ not in $\Omega$. We can take $p_b$ to be any maximal ideal of $(R_b/\Omega)_f$. Let $p = R^p \cap p_b$. Our assertion follows from

$$\dim(R_b/\Omega)_{p_b} = \dim(R^p)_{p_b}/(\Omega) \leq \dim(R^p)_{p_b}/(\Omega \cap R^p) = \dim(R^p/\Omega \cap R^p)_{p_b}.$$ 

\[ \square \]

In application, we need the following version of proposition 2.3.2.

**Corollary 2.3.8.** Let $I$ be an ideal of $R^p$ contained in $q$ and $t$ be some positive integer. Write $R_1$ (resp. $R_2$) for $(R^p/I)[[y_1, \cdots, y_t]]$ (resp. $(R_b/IR_b)[[y_1, \cdots, y_t]]$) and $q_1$ (resp. $q_2$) for the ideal of $(R^p/I)[[y_1, \cdots, y_t]]$ (resp. $(R_b/IR_b)[[y_1, \cdots, y_t]]$) generated by $y_1, \cdots, y_t$ and $q$ (resp. $y_1, \cdots, y_t$ and $q_b$). Let $c$ be the element in proposition 2.3.2. Then the second part of proposition 2.3.2 holds for $R_1/q_1^n \to R_2/q_2^n$.

**Proof.** This is easy. We omit the proof here. \[ \square \]

**2.4 Some miscellaneous results**

We keep the same notations and setup as in the previous subsections.

**Lemma 2.4.1.** Let $p$ be a one-dimensional prime ideal of $R^p$ such that the associated representation $\rho(p)$ is irreducible. Then the fibre $\text{Spec}(R_b \otimes_{R^p} k(p))$ has at most one point.
Proof. For any $p' \in \text{Spec} \, R_b$ mapping to $p \in \text{Spec} \, R^{ps}$, we have a natural map:

$$R^{ps}/p \to R_b/p' \to k(p')$$

and a representation

$$\rho(p') : \Gamma \to \text{GL}_2(R_b/p')$$

that lifts $\bar{\rho}$. Note that $p'$ is one-dimensional as it cannot be the maximal ideal $m_b$ of $R_b$. From our previous discussion, we may conjugate $\rho(p')$ by some element in $1 + M_2(m_b)$ and assume all the entries of $\rho(p')$ are in $k(p)$ and $\rho(p')(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and $\rho(p')(\sigma_*)$ of the form $\begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$. Note that under these assumptions, since we assume $\rho(p')$ is irreducible, all the entries of $\rho(p')$ now are determined by $\text{tr}(\rho(p'))$. Thus this proves the uniqueness of $p'$.

**Proposition 2.4.2.** Let $a(\sigma), d(\sigma), x(\sigma, \tau), c(\sigma) \in R^{ps}$ be the elements as in the beginning of the proof of proposition 2.3.2. Denote by $m$ the maximal ideal of $R^{ps}$. Suppose $q_b$ is a prime of $\text{Spec} \, R_b$ such that $R_b/(q_b + mR_b)$ is not Artinian. Then

$$\text{ht}(I_c) \leq 1,$$

where $I_c \subseteq R^{ps}/\Omega$ is the ideal generated by $c(\sigma), \sigma \in \Gamma$ and $\Omega = q_b \cap R^{ps}$.

Proof. Let $\rho(q_b) : \Gamma \to \text{GL}_2(R_b/q_b)$ be a deformation induced from the universal deformation. As before, we may assume that it has the form $\sigma \mapsto \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$, where $a(\sigma), c(\sigma), d(\sigma) \in R^{ps}/\Omega$ are defined in the proposition.

Let $B_1$ be the integral closure of $R^{ps}/\Omega$ in its fraction field. Then by page 237 of [28] (since $R^{ps}/\Omega$ is complete and reduced), $B_1$ is finite over $R^{ps}/\Omega$. Denote the intersection of $B_1$ and $R_b/q_b$ (in the fraction field of $R_b/q_b$) by $B'_1$. Then $B'_1$ is also a finite $R^{ps}/\Omega$-algebra hence $B'_1/mB'_1$ is Artinian.
We claim that $c(\sigma), \sigma \in \Gamma$ generate an ideal of height at most one in $B_1$. Note that since $B_1$ is finite over $R^{\text{ps}}/\mathfrak{Q}$, this will imply the proposition. Suppose this is not true. Then for any height one prime ideal $\mathfrak{p}$ of $B_1$, there exists $\sigma \in \Gamma$ such that $c(\sigma) \notin \mathfrak{p}$, hence

$$b(\tau) = \frac{x(\tau, \sigma)}{c(\sigma)} \in (B_1)_{\mathfrak{p}}$$

for any $\tau \in \Gamma$.

Thus

$$b(\tau) \in \bigcap_{\mathfrak{p}}(B_1)_{\mathfrak{p}} = B_1,$$

where $\mathfrak{p}$ runs over all the height one prime ideals of $B_1$ and the equality is valid since $B_1$ is normal. In other words, all the entries of $\rho(q_b)$ lie in $B_1$ hence also in $B'_1$.

Consider the natural map given by the inclusion $B'_1 \rightarrow R_b/q_b$:

$$B'_1/MB'_1 \rightarrow R_b/(q_b + mR_b).$$

Recall that $B'_1/MB'_1$ is Artinian. The above map is surjective as the image contains all the entries of a universal deformation which topologically generate $R_b$. But this contradicts our assumption that $R_b/(q_b + mR_b)$ is not Artinian. \qed
Chapter 3

Local-global compatibility

In this section, we fix a totally field $F$ in which $p$ completely splits and a quaternion algebra $D$ with centre $F$ which is ramified at all infinite places of $F$ and unramified at all places above $p$. Also we fix isomorphisms $D \otimes F_v \simeq M_2(F_v)$ for any $v$ where $D$ is unramified. Under these isomorphisms, we may view $K_p = \prod_{v|p} \text{GL}_2(O_{F_v}), D^\times_p = \prod_{v|p} \text{GL}_2(F_v)$ as subgroups of $(D \otimes F \mathbb{A}_F)^\times$. We also write $N_{D/F} : (D \otimes F \mathbb{A}_F)^\times \to \mathbb{A}_F^\times$ as the reduced norm.

3.1 Quaternionic forms

3.1.1. We first recall some results on quaternionic forms. Reference is [11] and [16].

Let $A$ be a topological $\mathbb{Z}_p$-algebra and $U = \prod_v U_v$ be an open compact subgroup of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ such that $U_v \subseteq \text{GL}_2(O_{F_v})$ for $v|p$. Let $\psi : (\mathbb{A}_F^\infty)^\times/F_{>0}^\times \to A^\times$ be a continuous character, where $F_{>0}$ is the set of totally real elements in $F$. Also let $\tau : \prod_{v|p} U_v \to \text{Aut}(W_\tau)$ be a continuous representation on a finite $A$-module $W_\tau$. By abuse of notation, we also view $\tau$ as a representation of $U$ by projecting to $\prod_{v|p} U_v$.

Let $S_{\tau,\psi}(U, A)$ be the space of continuous functions:

$$f : D^\times \setminus (D \otimes_F \mathbb{A}_F^\infty)^\times \to W_\tau$$
such that for any $g \in (D \otimes_F \mathbb{A}_F^\infty)^\times$, $u \in U$, $z \in (\mathbb{A}_F^\infty)^\times$, we have

- $f(gu) = \tau(u^{-1})(f(g))$,
- $f(gz) = \psi(z)f(g)$.

Write $(D \otimes_F \mathbb{A}_F^\infty)^\times = \bigcup_{i \in I} D^\times t_i U(\mathbb{A}_F^\infty)^\times$ for some finite set $I$ and $t_i \in D^\times$. If $\tau^{-1}|_{U \cap (\mathbb{A}_F^\infty)^\times}$ acts as $\psi|_{U \cap (\mathbb{A}_F^\infty)^\times}$, then there is an isomorphism:

$$S_{\tau,\psi}(U, A) \simeq \bigoplus_{i \in I} W_\tau^{(t_i^{-1}D^\times t_i \cap U(\mathbb{A}_F^\infty)^\times)/F^\times},$$

by sending $f$ to $(f(t_i))_i$. We say $U$ is sufficiently small if $(t_i^{-1}D^\times t_i \cap U(\mathbb{A}_F^\infty)^\times)/F^\times$ is trivial for all $i \in I$. This can be achieved by shrinking $U_v$ for some $v$:

**Lemma 3.1.2.** Suppose that $D$ is unramified at $v$ and $\zeta + \zeta^{-1} \neq 2 \mod \pi^n_v$ for some $n$ and any root of unity $\zeta \neq 1$ in a quadratic extension of $F$. Then $U$ is sufficiently small if $U_v$ is contained in the subgroup of $\text{GL}_2(O_{F_v})$ whose elements are unipotent upper triangular modulo $\pi^n_v$.

**Proof.** Let $\gamma \in D^\times \cap t_i U(\mathbb{A}_F^\infty)^\times t_i^{-1}$ and $l$ be the order of $t_i^{-1}\gamma t_i$ in $(t_i^{-1}D^\times t_i \cap U(\mathbb{A}_F^\infty)^\times)/F^\times$. Let $\iota : D \to D$ be the main involution. Then $\gamma^l = \iota(\gamma)^l$ and $\frac{\gamma}{\iota(\gamma)}$ is a root of unity in a quadratic extension of $F$. This will contradict $t_i^{-1}\gamma t_i \in U_v F_v^\times$ unless $l = 1$. \qed

**Corollary 3.1.3.** Assume $\tau^{-1}|_{U \cap (\mathbb{A}_F^\infty)^\times} = \psi|_{U \cap (\mathbb{A}_F^\infty)^\times}$ and $U$ is sufficiently small. Then $S_{\tau,\psi}(U, A)$ is a finite free $A$-module and $S_{\tau \otimes B, \psi}(U, B) \simeq S_{\tau,\psi}(U, A) \otimes B$ for any $A$-algebra $B$.

**3.1.4.** The relationship of $S_{\tau,\psi}(U, A)$ with classical automorphic forms on $D^\times$ is as follows (see lemma 1.3 of [41]). Suppose $A = E$ and $(\tilde{k}, \tilde{w}) \in \mathbb{Z}_{>1}^{\text{Hom}(F, \overline{\mathbb{Q}}_p)} \times \mathbb{Z}^{\text{Hom}(F, \overline{\mathbb{Q}}_p)}$ such that $k_{\sigma} + 2w_{\sigma}$ is independent of $\sigma : F \to \overline{\mathbb{Q}}_p$. Write $w = k_{\sigma} + 2w_{\sigma} - 1$. We can
define the following algebraic representation $\tau_{(\vec{k}, \vec{w})}$ of $D_p^\times = (D \otimes \mathbb{Q}_p)^\times$ on

$$W_{(\vec{k}, \vec{w}), E} = \bigotimes_{\sigma : F \to E} (\text{Sym}^{k_\sigma - 2}(E^2) \otimes \text{det}^{w_\sigma}),$$

where $\text{Sym}^{k_\sigma - 2}$ denotes the space of homogeneous polynomials of degree $k_\sigma - 2$ in two variables with an action of $\text{GL}_2(F_v(\sigma))$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (f)(X, Y) = f(\sigma(a)X + \sigma(c)Y, \sigma(b)X + \sigma(d)Y).$$

Here $v(\sigma)$ is the place above $p$ given by $\sigma$. Let $\psi : (\mathbb{A}_F^\infty)^x / F_{> 0}^x \to E^x$ be a continuous character such that $\tau_{(\vec{k}, \vec{w})}^{-1}(\mathbb{A}_F^\infty) = \psi|_{U \cap (\mathbb{A}_F^\infty)^x}$ and write $\tau = \tau_{(\vec{k}, \vec{w})}$. Then there is an isomorphism:

$$S_{\tau_{(\vec{k}, \vec{w})}, \psi}(U, E) \otimes_{E, \iota_p} \mathbb{C} \xrightarrow{\sim} \text{Hom}_{D_\infty^\times}(W_{t_p(\vec{k}, \vec{w}), \mathbb{C}}^*(D_\infty \setminus (D \otimes \mathbb{A}_F)^x / U, \psi_C)),$$

$$f \otimes 1 \mapsto "w^* \mapsto (g \mapsto w^*(\tau_C(g_\infty)^{-1}\tau(g_p)f(g_\infty))\)"

where

- $D_\infty^\times = (D \otimes_{\mathbb{Q} \mathbb{R}})^\times$,

- $W_{t_p(\vec{k}, \vec{w}), \mathbb{C}} = W_{(\vec{k}, \vec{w}), E} \otimes_{E, \iota_p} \mathbb{C}$ is viewed as an algebraic representation $\tau_C$ of $D_\infty^\times$ (induced by $t_p$), and $W_{t_p(\vec{k}, \vec{w}), \mathbb{C}}^*$ is its $\mathbb{C}$-linear dual.

- $\psi_C : \mathbb{A}_F^x / F^x \to \mathbb{C}^x$ sends $g$ to $N_{F/\mathbb{Q}}(g_\infty)^{1-w}t_p(N_{F/\mathbb{Q}}(g_p)^{w-1}\psi(g_\infty)).$

- $C^\infty(D_\infty \setminus (D \otimes \mathbb{A}_F)^x / U, \psi_C)$ is the space of smooth $\mathbb{C}$-valued functions on $D_\infty \setminus (D \otimes \mathbb{A}_F)^x$, right invariant by $U$ and with central character $\psi_C$.

Note that the right hand side is a subspace of automorphic forms on $(D \otimes \mathbb{A}_F)^x$. 

Note that the right hand side is a subspace of automorphic forms on $(D \otimes \mathbb{A}_F)^x$. 

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Remark 3.1.5. This isomorphism is functorial in $U$. Fix $v|p$ and take the direct limit over all open compact subgroups $U_v$ in $\text{GL}_2(O_{F_v})$. We get

$$\lim_{U_v} S_{(\bar{k}, \bar{w})}^{\psi}(U, E) \otimes \mathbb{C} \simeq \text{Hom}_{D_{\infty}^{\times}}(W^*_{\bar{k}(\bar{w}), \mathbb{C}}, C^\infty(D^\times \setminus (D \otimes \mathbb{A}_F)^\times / U^v, \psi_{\mathbb{C}}))$$

where $U^v = \prod_{w \neq v} U_w$. Clearly there is an action of $\text{GL}_2(F_v)$ on the right hand side by right translation. On the left hand side, this is given by

$$g(f)(h) = \tau(g)(f(hg)), \ g \in \text{GL}_2(F_v), f \in S_{(\bar{k}, \bar{w})}^{\psi}(U, E), h \in (D \otimes_F \mathbb{A}_F^\times)^\times.$$

For simplicity, we will write $S_{(\bar{k}, \bar{w}), \psi}(U, E)$ for $S_{(\bar{k}, \bar{w})}^{\psi}(U, E)$ from now on.

### 3.2 Completed homology and cohomology

3.2.1. Now we introduce completed homology (and cohomology). We denote by $S_{\psi}(U, A)$ when $W_\tau = A$ with trivial $U_v$-actions. Note that the definition of $S_{\psi}(U, A)$ makes sense for any topological $\mathcal{O}$-module $A$. In the below we will use $S_{\psi}(U, A)$ for any topological $\mathcal{O}$-module $A$ by abuse of notation. Given $U^p = \prod_{v|p} U_v$ (a tame level) and a torsion $\mathcal{O}$-algebra $A$ with discrete topology, we define

**Definition 3.2.2.**

$$S_{\psi}(U^p, A) := \lim_{U_p} S_{\psi}(U^p U_p, A),$$

with discrete topology, where $U_p = \prod_{v|p} U_v$ runs over all open compact subgroups $U_v$ of $\text{GL}_2(F_v)$. The completed cohomology of tame level $U^p$ is defined to be

$$S_{\psi}(U^p) := \text{Hom}_{\mathcal{O}}(E/\mathcal{O}, S_{\psi}(U^p, E/\mathcal{O}))$$
equipped with $p$-adic topology, and the completed homology is defined to be

$$M_\psi(U^p) := S_\psi(U^p, E/O)^\vee = \text{Hom}_O(S_\psi(U^p, E/O), E/O)$$
equipped with compact-open topology.

**Remark 3.2.3.** It is easy to see that $S_\psi(U^p, A)$ is naturally isomorphic to the space of locally constant $A$-valued functions on $D^\times \setminus (D \otimes_F \mathbb{A}_F^\infty)^\times$ right invariant by $U^p$ with central character $\psi$. Using this equivalent definition, there is a natural action of $D^\times_p = \prod_{v|p} \text{GL}_2(F_v)$ on all spaces defined above by right translation. It is almost by definition that for any open compact subgroup $U_p = \prod_{v|p} U_v \subseteq K_p = \prod_{v|p} \text{GL}_2(O_{F_v})$,

$$S_\psi(U_p, A)^{U_p} = S_\psi(U_p U_p, A).$$

Hence $S_\psi(U^p, E/O)$ is a smooth admissible $O$-representation of $D^\times_p$ in the sense of §2 of [32]. It is also clear that $S_\psi(U^p)$ is $p$-torsion free and

$$S_\psi(U^p) \cong \lim_{\leftarrow n} S_\psi(U^p, O/p^n) \cong \text{Hom}_{O\leftarrow k}(M_\psi(U^p), O)$$

$$M_\psi(U^p) \cong \text{Hom}_O(S_\psi(U^p), O).$$

**Proposition 3.2.4.** Suppose $U^p$ is small enough such that $U^p K_p$ is sufficiently small and $\psi|_{N_{D/F}(U^p)}$ is trivial. Then $S_\psi(U, E/O)$ is an injective object in Mod_{k_p, \psi}^\text{sm}(O).

**Proof.** This is a variant of proposition 4.4.3 of [5]. We recall their proof here.

**Lemma 3.2.5.** Let $M$ be a finite torsion $O$-module with a smooth $K_p$-action. Then there is a natural isomorphism

$$S_{M, \psi}(U^p K_p, O) \sim \text{Hom}_{O[K_p]}(M^\vee, S_\psi(U^p, E/O))$$
Proof. The map is given by sending \( f \in S_{M,\psi}(U^p K_p, \mathcal{O}) \) to \( \ell^\vee \mapsto (g \mapsto \ell^\vee(f(g))) \). One can easily construct an inverse of this map. We omit the details here. \( \square \)

Now given \( 0 \to \pi' \to \pi \) in \( \text{Mod}^\text{sm}_{K_p,\psi}(\mathcal{O}) \), one needs to show \( \text{Hom}_{\mathcal{O}[[K_p]]}(\pi, S_{\psi}(U^p, E/\mathcal{O})) \to \text{Hom}_{\mathcal{O}[[K_p]]}(\pi', S_{\psi}(U^p, E/\mathcal{O})) \)

is surjective. By proposition 2.1.9. of [15], we may assume \( \pi \) is admissible. If \( \pi \) is a finite \( \mathcal{O} \)-module, we may apply the previous lemma to \( \pi^\vee, (\pi')^\vee \) and conclude the surjectivity from corollary 3.1.3. In general, we may write \( \pi = \bigcup_n \pi_n \) as an increasing union of representations of finite \( \mathcal{O} \)-length. Write \( \pi'_n = \pi_n \cap \pi', \pi''_n = \pi_n/\pi'_n \). The result follows from taking inverse limit of

\[ \text{Hom}_{\mathcal{O}[[K_p]]}(\pi''_n, S) \to \text{Hom}_{\mathcal{O}[[K_p]]}(\pi_n, S) \to \text{Hom}_{\mathcal{O}[[K_p]]}(\pi'_n, S) \]

over \( n \). Here for simplicity we write \( S = S_{\psi}(U^p, E/\mathcal{O}) \). Note that the first term is of finite \( \mathcal{O} \)-length, so it satisfies Mittag-Leffler conditions. \( \square \)

**Corollary 3.2.6.** Let \( S_{\psi}(U^p)_E \) be \( S_{\psi}(U^p) \otimes_{\mathcal{O}} E \), which is a Banach space with unit ball \( S_{\psi}(U^p) \). Then \( S_{\psi}(U^p)_E \) is a topological direct summand of \( \mathcal{C}_{\psi}(K_p, E)^{\otimes d} \) as an \( E[K_p] \)-module for some \( d \), where \( \mathcal{C}_{\psi}(K_p, E) \) is the space of continuous \( E \)-valued functions on \( K_p \) with central character \( \psi|_{O_{\bar{F},p}} \).

**Proof.** Taking the Pontryagin dual of the result in the proposition, we know that \( M_{\psi}(U^p) \) is a projective object in the category of profinite linearly topological \( \mathcal{O}[[K_p]] \)-modules with central character \( \psi^{-1}|_{O_{\bar{F},p}} \). Note that \( S_{\psi}(U^p, E/\mathcal{O}) \) is an admissible representation of \( K_p \). Hence \( M_{\psi}(U^p) \) is a finitely generated module over \( \mathcal{O}[[K_p]] \) and therefore a direct summand of

\[ (\mathcal{O}[[K_p]] \otimes_{\mathcal{O}[O_{\bar{F},p}], \psi^{-1}} \mathcal{O})^{\otimes d} \]

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for some $d$. Here we view $\mathcal{O}$ as a $\mathcal{O}[[O_{F,p}^\times]]$-module by $\psi^{-1}$. We get the corollary by taking the continuous Hom to $E$ since

$$\text{Hom}_{E}^\text{cont}(\mathcal{O}[[K_p]] \otimes_{\mathcal{O}[[O_{F,p}^\times]],\psi^{-1}} \mathcal{O}, E) \cong \mathcal{C}_\psi(K_p, E).$$

\[3.2.7\] (Twisting of a character). Given a continuous character

$$\theta : (\mathbb{A}_F^\times)^\times/N_{D/F}(U^p)F_F^\times \to \mathcal{O}^\times,$$

we can define a natural isomorphism by twisting with $\theta$:

$$S_\psi(U^p) \xrightarrow{\sim} S_{\psi\theta}(U^p)$$

by sending $f$ to $g \mapsto \theta(N_{D/F}(g))f(g)$. Here we identify $S_\psi(U^p)$ with functions on $D^\times \setminus (D \otimes_F \mathbb{A}_F^\times)^\times$. Using this isomorphism, we may sometimes assume $\psi$ is of finite order (not now).

\[3.2.8\] (A density result). We need a density result like [16] §5.4. This will reduce the local-global compatibility problem to the compatibility at \textit{crystalline points}. In this section, we fix a place $v$ above $p$ and we assume $\psi|_{N_{D/F}(U^p)}$ is trivial and $\psi|_{O_{F_v}^\times}$ is an \textit{algebraic character}. In particular, there exists an integer $m$ such that,

$$\psi(a_v) = \sigma_v(a_v)^m, \quad a_v \in O_{F_v}^\times,$$

where $\sigma_v : F \to E$ is the embedding induced by $v$. Consider the subspace $S_{\psi}(U^p)_{E \otimes_{\mathbb{F}_p} \mathbb{F}_{p'}}$ (see the definition below) of $\text{GL}_2(O_{F_v})$-algebraic, $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$-locally algebraic vectors of $S_{\psi}(U^p)_{E}$. The main result is
Proposition 3.2.9. Assume that $\psi|_{N_{D/P}(U^v)}$ is trivial and $\psi|_{O_{F,\psi}}$ is an algebraic character. Then $S_\psi(U^p)^{v-a,v'-la}_E$ is dense in $S_\psi(U^p)_E$.

Proof. We first recall the construction of $S_\psi(U^p)^{v-a,v'-la}_E$. For $(\vec{k}, \vec{w}) \in \mathbb{Z}_{\sigma=1}^{\text{Hom}(F, \mathbb{Q}_p)} \times \mathbb{Z}^{\text{Hom}(F, \mathbb{Q}_p)}$ such that $k_\sigma + 2w_\sigma = m + 2$ for any $\sigma$, we can associate an algebraic representation of $K_p$ (see 3.1.4):

$$W_{(\vec{k}, \vec{w})_E} = \bigotimes_{\sigma:F \rightarrow E} (\text{Sym}^{k_\sigma-2}(E^2) \otimes \text{det}^{w_\sigma}).$$

Note that these $W_{(\vec{k}, \vec{w})_E}$ exhaust all algebraic representations of $K_p$ with central character locally same as $\psi|_{O_{F,\psi}}$. Let $U^v$ be an open compact subgroup of $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$. Then the subspace of $\text{GL}_2(O_{F_v})U^v$-algebraic vectors of $S_\psi(U^p)_E$ is defined to be the image of the evaluation map:

$$\bigoplus_{(\vec{k}, \vec{w})} \text{Hom}_{E[\text{GL}_2(O_{F_v})U^v]}(W_{(\vec{k}, \vec{w})_E}, S_\psi(U^p)_E) \otimes E W_{(\vec{k}, \vec{w})_E} \rightarrow S_\psi(U^p)_E,$$

where the sum is taken over all $(\vec{k}, \vec{w})$ with $k_\sigma + 2w_\sigma = m + 2$ for any $\sigma$. The subspace $S_\psi(U^p)^{v-a,v'-la}_E$ of $\text{GL}_2(O_{F_v})$-algebraic, $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$-locally algebraic vectors of $S_\psi(U^p)_E$ is defined to be the union of all $\text{GL}_2(O_{F_v})U^v$-algebraic vectors where $U^v$ runs through all open compact subgroups of $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$.

Let $C_\psi^{v-a,v'-la}(K_p, E)$ be the subspace of $\text{GL}_2(O_{F_v})$-algebraic, $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$-locally algebraic vectors in $C_\psi(K_p, E)$. By corollary 3.2.6, it suffices to prove that $C_\psi^{v-a,v'-la}(K_p, E)$ is dense in $C_\psi(K_p, E)$. If $\psi|_{O_{F,\psi}}$ is trivial, write $PK_p = K_p/O_{F,p}^{\times} \cong \prod_{v|p} \text{PGL}_2(O_{F_v})$. Then $C_\psi(K_p, E) \cong C(PK_p, E)$, the space of continuous $E$-valued functions on $PK_p$. The density of $K_p$-algebraic vectors $C^{K_p-\text{alg}}(PK_p, E)$ in $C(PK_p, E)$ follows from Proposition A.3. of [33] with $G = \prod_{v|p} \text{PGL}_2$.

In general, note that $C(PK_p, E), C^{K_p-\text{alg}}(PK_p, E)$ are commutative rings and $C_\psi(K_p, E)$ (resp. $C_\psi^{v-a,v'-la}(K_p, E)$) is a $C(PK_p, E)$ (resp. $C^{K_p-\text{alg}}(PK_p, E)$)-module.
It suffices to prove

\[ C_\psi(K_p, E) = C(PK_p, E) \cdot C_{\psi}^{v - a, v' - la}(K_p, E). \]

i.e. \( C_\psi(K_p, E) \) is generated by its \( \text{GL}_2(O_{F_v}) \)-algebraic, \( \prod_{w \neq v, w \mid p} \text{GL}_2(O_{F_w}) \)-locally algebraic vectors as a \( C(PK_p, E) \)-module. We will prove this when \( K_p = \text{GL}_2(Z_p) \). The general case is similar. Write

\[ K_p \subseteq M_2(Z_p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in Z_p \}. \]

Then \( \psi(a), \psi(b), \psi(c), \psi(d) \) can be viewed as elements in \( C_{\psi}^{v - a, v' - la}(K_p, E) \). Let

\[
\begin{align*}
U_a &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(Z_p), a \notin pZ_p \} \\
U_b &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(Z_p), a \in pZ_p, b \notin pZ_p \}
\end{align*}
\]

It is clear that \( U_a, U_b \) are disjoint open sets of \( K_p \) and cover the whole group. Let \( 1_a, 1_b \) be the characteristic functions of \( U_a, U_b \). For any \( f \in C_\psi(K_p, E) \), we can write

\[ f = \frac{f1_a}{\psi(a)} \cdot \psi(a) + \frac{f1_b}{\psi(b)} \cdot \psi(b). \]

Note that \( \frac{f1_a}{\psi(a)}, \frac{f1_b}{\psi(b)} \) are well-defined functions on \( PK_p \). Hence we have expressed \( f \) as an element in \( C(PK_p, E) \cdot C_{\psi}^{v - a, v' - la}(K_p, E) \) and this is exactly what we want. \( \square \)

3.2.10 (Relation with classical automorphic forms). We can recover classical automorphic forms on \( D^\times \) from the completed cohomology in the following way. Suppose \( (\vec{k}, \vec{w}) \in Z_{>1}^{\text{Hom}(F_\infty p)} \times Z^{\text{Hom}(F_\infty p)} \) such that \( k_\sigma + 2w_\sigma \) is independent of \( \sigma \) and \( U_p \) is an
open subgroup of \( K_p \) such that \( \psi|_{U^p U_p \cap (\mathbb{A}_F^\times)^{\times}} = \tau^{-1}|_{U^p U_p \cap (\mathbb{A}_F^\times)^{\times}}. \) Then it is not too hard to deduce from lemma 3.2.5 that

\[
S_{(\overline{k},\overline{w}),\psi}(U^p U_p, E) \simeq \text{Hom}_{E[U_p]}(W_{(\overline{k},\overline{w}),E}^\times, S_\psi(U^p) E).
\]

In other words, locally algebraic vectors in \( S_\psi(U^p) E \) can be identified with automorphic forms on \((D \otimes \mathbb{A}_F^\infty)^\times\) (see 3.1.4).

### 3.3 Hecke algebras and Pseudo-representations

#### 3.3.1. First we introduce Hecke algebras on finite levels. Fix a topological \( \mathbb{Z}_p \)-algebra \( A \), a level \( U \), character \( \psi \) and a representation \( \tau : U_p \to \text{Aut}(W_\tau) \). Let \( S \) be a finite set of primes of \( F \) containing all places above \( p \) and places \( v \) where either \( D \) is ramified or \( U_v \) is not a maximal open subgroup. For any place \( v \not\in S \), we define the Hecke operator \( T_v \in \text{End}(S_{\tau,\psi}(U, A)) \) to be the double coset \([U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v]\). More precisely, write \( U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v = \bigsqcup_i \gamma_i U_v \), then

\[
(T_v \cdot f)(g) = \sum_i f(g \gamma_i), \quad f \in S_{\tau,\psi}(U, A).
\]

We define Hecke algebra \( \mathbb{T}_{\tau,\psi}^S(U, A) \subseteq \text{End}_A(S_{\tau,\psi}(U, A)) \) to be the \( A \)-subalgebra generated by all \( T_v, v \not\in S \). This is a finite commutative \( A \)-algebra. By definition, \( S_{\tau,\psi}(U, A) \) is a finitely-generated faithful \( \mathbb{T}_{\tau,\psi}^S(U, A) \)-module.

If \( U' \subseteq U \) is a smaller open subgroup, then we have a natural surjective map \( \mathbb{T}_{\tau,\psi}^S(U', A) \to \mathbb{T}_{\tau,\psi}^S(U, A) \) induced by \( S_{\tau,\psi}(U, A) \subseteq S_{\tau,\psi}(U', A) \).

If \( U \) is small enough and \( \psi|_{U \cap (\mathbb{A}_F^\times)^{\times}} = \tau^{-1}|_{U \cap (\mathbb{A}_F^\times)^{\times}} \), then for any ideal \( I \) of \( A \), there is a natural surjective map \( \mathbb{T}_{\tau,\psi}^S(U, A) \to \mathbb{T}_{\tau \otimes A/I,\psi}^S(U, A/I) \) induced by
\( S_{\tau \otimes A/I, \psi}(U, A/I) \simeq S_{\tau, \psi}(U, A) \otimes A/I \) (see 3.1.3). This also implies that \( T_{\tau \otimes A/I, \psi}(U, A/I) \to T_{\tau \otimes A/I, \psi}(U, A/I) \) has nilpotent kernel since \( S_{\tau, \psi}(U, A) \otimes A/I \) has full supports on both rings.

3.3.2. Now take \( \tau = \tau_{(\bar{k}, \bar{w})} \) (see 3.1.4) and \( A = \mathcal{O} \). Assume \( \psi|_{U \cap (A^\infty_F)^\times} = \tau_{(\bar{k}, \bar{w})}^{-1}|_{U \cap (A^\infty_F)^\times} \). It is well-known that there exists a two-dimensional pseudo-representation:

\[
T_{(\bar{k}, \bar{w}), \psi}(U) : G_{F,S} \to T_{\tau, \psi}(U, \mathcal{O}) =: T_{(\bar{k}, \bar{w}), \psi}(U, \mathcal{O})
\]

sending \( \text{Frob}_v \) to \( T_v \), \( v \notin S \). This pseudo-representation has determinant \( \psi \varepsilon^{-1} \).

Here by abuse of notation, we identify \( \psi \) with a character of \( G_{F,S} \) by the class field theory. Note that this map is surjective since all \( T_v, v \notin S \) are in the image. By Chebatorev density theorem, \( T_{(\bar{k}, \bar{w}), \psi}(U, \mathcal{O}) \) is independent of \( S \), so we may simply write \( T_{(\bar{k}, \bar{w}), \psi}(U, \mathcal{O}) \).

Under the same assumption plus \( U \) is sufficiently small, we have a natural surjective map \( T_{(\bar{k}, \bar{w}), \psi}(U, \mathcal{O})/\varpi^n \to T_{(\bar{k}, \bar{w}) \otimes \mathcal{O}/\varpi^n, \psi}(U, \mathcal{O}/\varpi^n) \) for any \( n \). Hence there also exists a two-dimensional pseudo-representation over \( T_{(\bar{k}, \bar{w}) \otimes \mathcal{O}/\varpi^n, \psi}(U, \mathcal{O}/\varpi^n) \) which is independent of \( S \). For simplicity, we write it as \( T_{(\bar{k}, \bar{w}), \psi}(U, \mathcal{O}/\varpi^n) \).

3.3.3. We will write \( T_{\psi}(U, A) \) for the Hecke algebra if \( \tau \) is the trivial action on \( A \). Suppose \( A = \mathcal{O}/\varpi^n \), \( U \) is sufficiently small and \( \psi|_{U \cap (A^\infty_F)^\times} \) is trivial modulo \( \varpi^n \), i.e. \( \psi(U \cap (A^\infty_F)^\times) \subseteq 1 + \varpi^n \mathcal{O} \). We are going to show the existence of pseudo-representation \( G_{F,S} \to T_{\psi}(U, \mathcal{O}/\varpi^n) \) with determinant \( \psi \varepsilon^{-1} \) sending \( \text{Frob}_v \) to \( T_v \) for \( v \notin S \) in this case.

It suffices to treat the case where \( U \) is small enough such that \( \psi|_{N_{D/F}(U)} \) is trivial modulo \( \varpi^n \). In general, we may shrink \( U \) to \( U' \) small enough and conclude from the surjective map \( T_{\psi}(U', \mathcal{O}/\varpi^n) \to T_{\psi}(U, \mathcal{O}/\varpi^n) \). Consider the Teichmüller lifting \( \tilde{\psi} \) of \( \psi \) modulo \( \varpi \). Then \( \psi^{-1}\tilde{\psi} \) is of pro-\( p \) order, hence can be written as \( \theta^2 \) for some
character $\theta : (\mathbb{A}_F^\infty)^\times/F_{>0}^\times \to \mathcal{O}^\times$. It is easy to see that we may choose $\theta$ to be trivial on $N_{D/F}(U)$. Then twisting with $\theta$ induces an isomorphism (3.2.7):

$$
\varphi_\theta : S_\psi(U, \mathcal{O}/\mathfrak{w}^n) \sim S_{\tilde{\psi}}(U, \mathcal{O}/\mathfrak{w}^n)
$$

sending $f$ to $g \mapsto \theta(N_{D/F}(g)) f(g)$. Note that this map also induces an isomorphism between Hecke algebras with

$$
T_v \circ \varphi_\theta = \theta(\text{Frob}_v) \varphi_\theta \circ T_v.
$$

Now since $\tilde{\psi}$ is of finite order, we may apply the results in 3.3.2 with $\tilde{k} = \tilde{w} = \tilde{0}$ and get the pseudo-representation over $T^S_{\tilde{\psi}}(U, \mathcal{O}/\mathfrak{w}^n)$ hence on $T^S_{\psi}(U, \mathcal{O}/\mathfrak{w}^n)$. The determinant is also easy to determine under this map. One direct corollary is that $T^S_{\tilde{\psi}}(U, \mathcal{O}/\mathfrak{w}^n)$ is independent of $S$. So we may drop $S$ in it from now on.

**Definition 3.3.4** (Big Hecke algebra). Let $U^p$ be a tame level and $\psi : (\mathbb{A}_F^\infty)^\times/(U^p \cap (\mathbb{A}_F^\infty)^\times)F_{>0}^\times \to \mathcal{O}^\times$ be a continuous character. Then we define the Hecke algebra

$$
T_\psi(U^p) = \lim_{\leftarrow (n,U^p)\in \mathcal{I}} T_\psi(U^p U_p, \mathcal{O}/\mathfrak{w}^n),
$$

where $\mathcal{I}$ is the set of pairs $(n,U^p)$ with $U^p \subseteq K^\times_p$ and $n$ a positive integer such that $\psi|_{U^p \cap \mathcal{O}_F^\times} \equiv 1 \mod \mathfrak{w}^n$. Equivalently, this is also

$$
\lim_{\leftarrow n} \lim_{U^p \subseteq K^\times_p} T_\psi(U^p U_p, \mathcal{O}/\mathfrak{w}^n).
$$

**3.3.5.** It is almost by definition that $T_\psi(U^p)$ acts faithfully on the completed cohomology $S_\psi(U^p)$ and commutes with the action of $D^\times_p$. By our previous discussion, there exists a two-dimensional pseudo-representation $T_\psi(U^p) : \text{G}_{F,S} \to T_\psi(U^p)$ sending Frobenius to $T_v$ with determinant $\psi\varepsilon^{-1}$. 37
From now on, we will always assume $\psi$ is trivial on $U^p \cap (A_F^\infty)^\times$.

**Proposition 3.3.6.** Let $U_p$ be a pro-$p$ subgroup of $K_p$. Hence $\psi|_{U_p \cap O_F^\times}$ is trivial modulo $\varpi$. Then the natural map

$$T_\psi(U^p) \to T_\psi(U^pU_p, \mathbb{F})$$

induces a bijection between the sets of maximal ideals. In particular, $T_\psi(U^p)$ is semi-local.

**Proof.** This is lemma 2.1.14 of [18]. We give a sketch here. It suffices to prove the map

$$T_\psi(U^pU'_p, O/\varpi^n) \to T_\psi(U^pU_p, \mathbb{F})$$

induces a bijection of maximal ideals, where $U'_p \subseteq U_p$ is an open normal subgroup small enough so that $U^pU'_p$ is sufficiently small and $\psi|_{U'_p \cap O_F^\times}$ is trivial modulo $\varpi^n$.

Let $m$ be a maximal ideal of the Artinian ring $T_\psi(U^pU'_p, O/\varpi^n)$. Since $T_\psi(U^pU'_p, O/\varpi^n)$ acts faithfully on $S_\psi(U^pU'_p, O/\varpi^n)$, we know that

$$S_\psi(U^pU'_p, O/\varpi^n)[m] \neq 0.$$

The $p$-group $U_p/U'_p$ acts naturally on this $\mathbb{F}$-vector space, hence has a non-zero fixed vector, which by definition belongs to $S_\psi(U^pU'_p, \mathbb{F})$. Thus $S_\psi(U^pU'_p, \mathbb{F})[m] \neq 0$ and $m$ is also a maximal ideal of $T_\psi(U^pU_p, \mathbb{F})$.

Let $U'_p, n$ be as in the proof. It follows from theorem 8.15 of [29] that

$$T_\psi(U^pU'_p, O/\varpi^n) \cong T_\psi(U^pU'_p, O/\varpi^n)|m_1 \times \cdots T_\psi(U^pU'_p, O/\varpi^n)|m_r$$

where $m_1, \ldots, m_r$ are the maximal ideals of $T_\psi(U^pU'_p, O/\varpi^n)$. Note that each $T_\psi(U^pU'_p, O/\varpi^n)|m_i$ is an Artinian local $O$-algebra. Passing to the limit, we get

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**Corollary 3.3.7.** Let \( m_1, \ldots, m_r \) be the maximal ideals of \( T_\psi(U^p) \). Then

1. \( T_\psi(U^p) \cong T_\psi(U^p)_{m_1} \times \cdots \times T_\psi(U^p)_{m_r} \) and each \( T_\psi(U^p)_{m_i} \) is \( m_i \)-adically complete and separated.

2. \( S_\psi(U^p) \cong S_\psi(U^p)_{m_1} \oplus \cdots \oplus S_\psi(U^p)_{m_r} \) and each \( S_\psi(U^p)_{m_i} \) is \( m_i \)-adically complete and separated.

**3.3.8.** One easily checks that \( T_\psi(U^p) \) commutes with base change. To be precise, let \( E' \) be a finite extension of \( E \) with ring of integers \( O' \). Replacing \( O \) by \( O' \) in all the definitions, we can define a Hecke algebra \( T'_\psi(U^p) \). Then \( T'_\psi(U^p) \cong T_\psi(U^p) \otimes_O O' \).

Suppose \( T_\psi(U^p) \) is non-zero and \( m \in \text{Spec} T_\psi(U^p) \) is a maximal ideal. Enlarge \( O \) if necessary, we may assume \( m \) has residue field \( \mathbb{F} \). Denote by \( T_m \) the two-dimensional pseudo-representation (over \( \mathbb{F} \)):

\[
G_{F,S} \xrightarrow{T_\psi(U^p)} T_\psi(U^p) \to T_\psi(U^p)/m = \mathbb{F}.
\]

For any \( v \mid p \), let \( R^{ps}_v \) be the universal deformation ring which parametrizes all two-dimensional pseudo-representations of \( G_{F_v} \) lifting \( T_m|_{G_{F_v}} \) with determinant \( \psi \varepsilon^{-1}|_{G_{F_v}} \). By the universal property, we have a natural map \( R^{ps,\psi\varepsilon^{-1}}_v \to T_\psi(U^p)_m \). Taking tensor products over all \( v \mid p \), we get

\[
R^{ps,\psi\varepsilon^{-1}}_p := \bigotimes_{v \mid p} R^{ps,\psi\varepsilon^{-1}}_v \to T_\psi(U^p)_m.
\]

Therefore, we have defined an action of \( R^{ps,\psi\varepsilon^{-1}}_p \) on the completed cohomology \( S_\psi(U^p)_m \). We denote this action by \( \tau_{\text{Gal}} \). It will appear in one side of our local-global compatibility result.
3.4 Paškūnas’ theory

In this subsection, we recall the main results of Paškūnas in [32] [34] in the case of GL$_2(\mathbb{Q}_p)$ and some generalizations in [18] in the case of products of GL$_2(\mathbb{Q}_p)$. The main application of his theory is to define another action of $R_p^{\psi,\psi^{-1}}$ on the completed cohomology. Reference is §3 and §5 of [32].

3.4.1. Let $G = \prod_{i=1}^{f} \text{GL}_2(\mathbb{Q}_p)$ and $Z(G) \simeq \prod_{i=1}^{f} \mathbb{Q}_p^\times$ be its centre. Fix a character $\zeta : Z(G) \to O^\times$. We may formulate categories $\text{Mod}^{\text{sm}}_{G,\zeta}(O)$, $\text{Mod}^{\text{fin}}_{G,\zeta}(O)$, $\text{Mod}^{\text{adm}}_{G,\zeta}(O)$, and $C_{G,\zeta}(O)$. See notations in the beginning of the paper. We note that the last two categories are anti-equivalent to each other. And in fact, the middle two are the same:

Lemma 3.4.2. Any locally admissible representation in $\text{Mod}^{\text{sm}}_{G,\zeta}(O)$ is locally of finite length.

Proof. This is Theorem 2.3.8 of [15] when $f = 1$ and Lemma B.7 of [18] in general. In fact, Gee and Newton work with products of PGL$_2(\mathbb{Q}_p)$, but their proof also works with products of GL$_2(\mathbb{Q}_p)$ with fixed determinant. \hfill \□

3.4.3 (Blocks). Let $\text{Irr}_{G,\zeta}$ be the set of irreducible representations in $\text{Mod}^{\text{sm}}_{G,\zeta}(O)$. We consider the following equivalence relation $\sim$ on $\text{Irr}_{G,\zeta}$: $\pi \sim \tau$ if there exists a sequence of irreducible representations $\pi_1 = \pi, \pi_2, \cdots, \pi_n = \tau$ such that $\text{Ext}^1_G(\pi_i, \pi_{i+1}) \neq 0$ or $\text{Ext}^1_G(\pi_{i+1}, \pi_i) \neq 0$ or $\pi_n \cong \pi_{i+1}$. An equivalence class is called a block.

There exists a natural decomposition of $\text{Mod}^{\text{adm}}_{G,\zeta}(O)$ with respect to the blocks:

$$\text{Mod}^{\text{adm}}_{G,\zeta}(O) \cong \prod_{\pi \in \text{Irr}_{G,\zeta}/\sim} \text{Mod}^{\text{adm}}_{G,\zeta}(O)^\pi,$$  \hspace{0.5cm} (3.2)
where $\text{Mod}^{\text{adm}}_{G,\zeta}(\mathcal{O})^\mathfrak{B}$ is the full subcategory of $\text{Mod}^{\text{adm}}_{G,\zeta}(\mathcal{O})$ consisting of representations with all irreducible subquotients in $\mathfrak{B}$. Taking Pontryagin dual, this gives:

$$\mathcal{E}_{G,\zeta}(\mathcal{O}) \cong \prod_{\mathfrak{B} \in \text{Irr}_{G,\zeta}/\sim} \mathcal{E}_{G,\zeta}(\mathcal{O})^\mathfrak{B}$$

For a block $\mathfrak{B}$, write $\pi_\mathfrak{B} = \bigoplus_{\pi \in \mathfrak{B}_i} \pi$, where $\mathfrak{B}_i$ is the set of isomorphism classes of elements of $\mathfrak{B}$. We will see that this is actually a finite set. Let $\pi_\mathfrak{B} \hookrightarrow J_\mathfrak{B}$ be an injective envelope of $\pi_\mathfrak{B}$ in $\text{Mod}^{\text{adm}}_{G,\zeta}(\mathcal{O})$. Its Pontryagin dual $P_\mathfrak{B} := J_\mathfrak{B}^\vee$ is a projective envelope of $\pi^\vee \cong \bigoplus_{\pi \in \mathfrak{B}_i} \pi^\vee$ in $\mathcal{E}_{G,\zeta}(\mathcal{O})$. Let

$$E_\mathfrak{B} := \text{End}_{\mathcal{E}_{G,\zeta}(\mathcal{O})}(P_\mathfrak{B}) \cong \text{End}_G(J_\mathfrak{B}).$$

This is a pseudo-compact ring. The topology is given as follows: for any quotient map $q : P_\mathfrak{B} \twoheadrightarrow M$ to some $M \in \mathcal{E}_{G,\zeta}(\mathcal{O})$ of finite length, we may define a right ideal:

$$\mathfrak{r}(M) = \{ \phi \in E_\mathfrak{B}, q \circ \phi = 0 \}.$$ 

Such $\{\mathfrak{r}(M)\}$ forms a basis of open neighborhood of 0 in $E_\mathfrak{B}$. We note that $P_\mathfrak{B}$ is a natural left $E_\mathfrak{B}$-module.

Suppose $M \in \mathcal{E}_{G,\zeta}(\mathcal{O})$, then $E_\mathfrak{B}$ acts naturally on $\text{Hom}_{\mathcal{E}_{G,\zeta}(\mathcal{O})}(P_\mathfrak{B}, M)$ (on the right). By writing $M = \varprojlim M_i$ with $M_i$ of finite length, we can equip the projective topology on $\text{Hom}(P_\mathfrak{B}, M) = \varprojlim \text{Hom}(P_\mathfrak{B}, M_i)$, which makes $\text{Hom}(P_\mathfrak{B}, M)$ into a pseudo-compact $E_\mathfrak{B}$-module. In fact, this functor

$$M \mapsto \text{Hom}_{\mathcal{E}_{G,\zeta}(\mathcal{O})}(P_\mathfrak{B}, M)$$
defines an anti-equivalence of categories between $\mathcal{C}_{G, \zeta}(O)_{\mathfrak{B}}$ and the category of right pseudo-compact $E_{\mathfrak{B}}$-modules. An inverse functor is given by

$$m \mapsto (m \hat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}}).$$

$P_{\mathfrak{B}}$ is called a projective generator of $\mathfrak{B}$.

3.4.4. In [32][34], Paškūnas computes $E_{\mathfrak{B}}$ and its centre when $G = \text{GL}_2(\mathbb{Q}_p)$ in almost all cases. We now recall his results.

Suppose $G = \text{GL}_2(\mathbb{Q}_p)$. In this case, blocks are computed in [33] Cor 1.2. For our purpose, we only list the results when $p \geq 3$.

1. $\mathfrak{B}_i = \{\pi\}$, $\pi$ is supersingular.

2. $\mathfrak{B}_i = \{(\text{Ind}^G_B \delta_1 \otimes \delta_2 \omega^{-1})_{\text{sm}}, (\text{Ind}^G_B \delta_2 \otimes \delta_1 \omega^{-1})_{\text{sm}}\}$ with $\delta_2 \delta_1^{-1} \neq \omega^{\pm 1}, 1$.

3. $\mathfrak{B}_i = \{(\text{Ind}^G_B \delta \otimes \delta \omega^{-1})_{\text{sm}}\}$.

4. If $p \geq 5$, $\mathfrak{B}_i = \{1, \text{Sp}, (\text{Ind}^G_B \omega \otimes \omega^{-1})_{\text{sm}}\} \otimes \delta \circ \det$.

5. If $p = 3$, $\mathfrak{B}_i = \{1, \text{Sp}, \omega \circ \det, \text{Sp} \otimes \omega \circ \det\} \otimes \delta \circ \det$.

for some smooth characters $\delta, \delta_1, \delta_2 : \mathbb{Q}_p^\times \rightarrow \mathbb{F}_p^\times$. Here $B$ denotes the upper triangular Borel subgroup of $B$, $(\text{Ind}^G_B)$ denotes the smooth induction and Sp denotes the mod $p$ special representation. See [33] for the precise definitions.

To describe (the centre of) $E_{\mathfrak{B}}$, we need to attach a semi-simple 2-dimensional representation $\bar{\rho}_{\mathfrak{B}}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ to each block. This is given by the following list:

1. $\bar{\rho}_{\mathfrak{B}} = \mathbf{V}(\pi)$ if a supersingular $\pi \in \mathfrak{B}$, where $\mathbf{V}$ is the Colmez’s functor normalized in §5.7 of [32].

2. $\bar{\rho}_{\mathfrak{B}} = \delta_1 \oplus \delta_2$ if $(\text{Ind}^G_B \delta_1 \otimes \delta_2 \omega^{-1})_{\text{sm}} \in \mathfrak{B}$ with $\delta_2 \delta_1^{-1} \neq \omega^{\pm 1}$.

3. $\bar{\rho}_{\mathfrak{B}} = \delta \oplus \delta \omega$ if $\delta \circ \det \in \mathfrak{B}$. 
Under this correspondence, the determinant of $\bar{\rho}_B$ is $\zeta \varepsilon \mod \varpi$. Note that this actually defines a bijection between the set of blocks and the set of two-dimensional semi-simple representations of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$.

**Theorem 3.4.5** (Paškūnas). Let $R_{G_{\mathbb{Q}_p}}^{\text{ps},\zeta \varepsilon}$ be the universal deformation ring which parametrizes all 2-dimensional pseudo-representations of $G_{\mathbb{Q}_p}$ lifting $\text{tr} \bar{\rho}_B$ with determinant $\zeta \varepsilon$. Then there exists a natural isomorphism between the centre of $E_B$ and $R_{G_{\mathbb{Q}_p}}^{\text{ps},\zeta \varepsilon}$ in case (1)(2)(3)(4).

**Proof.** This is Theorem 1.5 of [32] and Theorem 1.3 of [34].

**Theorem 3.4.6** (Paškūnas). $E_B$ is a finitely generated module over its centre in case (1)(2)(3)(4).

**Proof.** See Theorem 1.2 of [34] and Corollary 8.11, Corollary 9.25, Lemma 10.90 of [32].

3.4.7. Now we generalize the above results to the case $G = \prod_{i=1}^f \text{GL}_2(\mathbb{Q}_p)$. First we need to classify the blocks in this case.

**Lemma 3.4.8.**

1. Any irreducible representation $\pi$ in $\text{Mod}_{G_\mathcal{O},\zeta}(\mathcal{O})$ is isomorphic to some $\bigotimes_{i=1}^f \pi_i$, where $\pi_i$ are irreducible $\text{GL}_2(\mathbb{Q}_p)$ representations.

2. Let $P_r \to \pi_r^\vee$ be a projective envelope of $\pi_r^\vee$ for $r = 1, \ldots, f$. Then $P_1 \otimes \cdots \otimes P_f \to \bigotimes_{i=1}^f \pi_i^\vee$ is a projective envelope of $(\bigotimes_{i=1}^f \pi_i)^\vee$ in $\mathcal{C}_{G,\zeta}(\mathcal{O})$.

**Proof.** This is Lemma B.7, B.8 of [18].

**Lemma 3.4.9.** Write $G = G_1 \times G_2$, where each $G_r = \prod_{i=1}^f \text{GL}_2(\mathbb{Q}_p)$ with centre $Z_r$, $r = 1, 2$. Suppose $M_r, N_r \in \mathcal{C}_{G_r,\zeta|_{Z_r}}(\mathcal{O})$, $r = 1, 2$. Then there exists a natural
isomorphism:

\[ \text{Hom}_{G,(\mathcal{O})}(M_1 \hat{\otimes} M_2, N_1 \hat{\otimes} N_2) \cong \text{Hom}_{G_1,(\mathcal{O})}(M_1, N_1) \hat{\otimes} \text{Hom}_{G_2,(\mathcal{O})}(M_2, N_2) \]

**Proof.** The argument in proof of Lemma B.8 of [18] works here without any changes.

\[ \square \]

**Lemma 3.4.10.** Given \( f \) blocks \( \mathcal{B}_1, \ldots, \mathcal{B}_f \) of \( \text{GL}_2(\mathbb{Q}_p) \), then

\[ \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_f := \{ \pi_1 \otimes \cdots \otimes \pi_f | \pi_r \in \mathcal{B}_r, r = 1, \ldots, f \} \]

is a block of \( G \). In fact, any block arises in this way.

**Proof.** Since any irreducible representation is in \( \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_f \) for some blocks \( \mathcal{B}_1, \ldots, \mathcal{B}_f \) of \( \text{GL}_2(\mathbb{Q}_p) \), it suffices to prove that \( \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_f \) is indeed a block of \( G \).

We will only prove the case \( f = 2 \). The general case will follow directly by induction on \( f \). Let \( \pi = \pi_1 \otimes \pi_2, \pi' = \pi'_1 \otimes \pi'_2 \) be two irreducible representations of \( G \) such that \( \pi_1, \pi'_1 \) are in not the same block of \( \text{GL}_2(\mathbb{Q}_p) \). We claim that \( \text{Ext}^1_G(\pi, \pi') = \text{Ext}^1_G(\pi', \pi) = 0 \). To see this, let \( \pi_1 \hookrightarrow J_1, \pi_2 \hookrightarrow J_2 \) be injective envelopes of \( \pi_1, \pi_2 \). Suppose we have a non-split extension:

\[ 0 \to \pi \to \pi'' \to \pi' \to 0. \]

By lemma 3.4.8, \( J := (J'_1 \hat{\otimes} J'_2)' \) is an injective envelope of \( \pi \). It is easy to see that all irreducible subquotients of \( J \) must be a tensor product of irreducible subquotients of \( J_1, J_2 \). By universal property, we get a map \( \pi'' \to J \) which has to be injective. But \( \pi'_1 \) cannot appear as a subquotient of \( J_1 \) since \( \pi_1, \pi'_1 \) live in different blocks. Thus there is no such non-split extension. This proves that each block of \( G \) must be contained in some \( \mathcal{B}_1 \otimes \mathcal{B}_2 \).
It rests to prove that if Ext$_{GL^2(Q_p)}^1(\pi'_1, \pi_1) \neq 0$ and $\pi_1 \not\cong \pi'_1$, then Ext$_G^1(\pi'_1 \otimes \pi_2, \pi_1 \otimes \pi_2) \neq 0$ for irreducible representations $\pi_1, \pi'_1, \pi_2$ of GL$(2(Q_p))$. Choose a nonsplit extension $\pi'_3$ of $\pi'_1$ by $\pi_1$. Then Hom$_G(\pi_3 \otimes \pi_2, \pi_1 \otimes \pi_2) \cong$ Hom$_{GL^2(Q_p)}(\pi_3, \pi_1) \otimes$ Hom$_{GL^2(Q_p)}(\pi_2, \pi_2) = 0$ by the previous lemma. Thus $\pi_3 \otimes \pi_2$ is a nonsplit extension of $\pi'_1 \otimes \pi_2$ by $\pi_1 \otimes \pi_2$.

Corollary 3.4.11. Let $\mathfrak{B} = \mathfrak{B}_1 \otimes \cdots \otimes \mathfrak{B}_f$ be a block of $G$. If $p = 3$, we assume no $\mathfrak{B}_r$ contains $\delta \circ \det$ (case (5)). Then

1. $E_{\mathfrak{B}} \cong \bigotimes_{r=1}^f E_{\mathfrak{B}_r}$

2. The natural inclusion $R_{\mathfrak{B}_r}^{\text{ps}, \zeta} \to E_{\mathfrak{B}_r}$ in Theorem 3.4.5 induces a natural finite map

$$\bigotimes_{r=1}^f R_{\mathfrak{B}_r}^{\text{ps}, \zeta} \to E_{\mathfrak{B}},$$

which makes $E_{\mathfrak{B}}$ into a finitely generated module over $\bigotimes_{r=1}^f R_{\mathfrak{B}_r}^{\text{ps}, \zeta}$.

3. The centre of $E_{\mathfrak{B}}$ is Noetherian and $E_{\mathfrak{B}}$ is a finitely generate module over its centre.

Proof. Clearly the third part follows directly from the second since the image of $\bigotimes_{r=1}^f R_{\mathfrak{B}_r}^{\text{ps}, \zeta}$ is in the centre of $E_{\mathfrak{B}}$. By the previous lemmas, $P_{\mathfrak{B}} \cong \bigotimes P_{\mathfrak{B}_r}$ and End($P_{\mathfrak{B}}$) $\cong \bigotimes$ End($P_{\mathfrak{B}_r}$). The rest of the corollary all follows from Theorem 3.4.6.

Remark 3.4.12. It seems that $\bigotimes_{r=1}^f R_{\mathfrak{B}_r}^{\text{ps}, \zeta}$ is exactly the centre of $E_{\mathfrak{B}}$. This is at least true if all $\mathfrak{B}_r$ are in case (1)(2)(3) since $E_{\mathfrak{B}_r}$ is a free module over its centre in these cases.
3.5 Local-global compatibility

In this subsection, we use Paškūnas’ theory to define another action of \( R_p \) on the completed cohomology and prove that it is equal to the action defined from Galois side at the end of section 3.3.

3.5.1. Back to the setting in section 3.3. Let \( m \) be a maximal ideal of \( \mathbb{T}_\psi(U^p) \). We get a two-dimensional pseudo-representation with determinant \( \psi\varepsilon^{-1} \):

\[
T_m : G_{F,S} \to \mathbb{T}_\psi(U^p)_m.
\]

Restrict it to \( G_{F,v}, v|p \), we can attach a two-dimensional semi-simple representation \( \bar{\rho}_{m,v} \) of \( G_{F,v} \) over \( \mathbb{F} \). Let \( \bar{\rho}'_{m,v} = \bar{\rho}_{m,v} \otimes \varepsilon \). Using the recipe in 3.4.4, we can define a block \( \mathfrak{B}_{m,v} \) of \( GL_2(F_v) \) defined in lemma 3.4.10. Note that it has central character \( \psi \).

As before, we denote its projective generator by \( P_{\mathfrak{B}_m} \).

One of the central objects in our study is

**Definition 3.5.2.** \( m := Hom_{D^\times_p \otimes (\mathcal{O})}(P_{\mathfrak{B}_m}, M_\psi(U^p)_m) \).

**Remark 3.5.3.** The twist of cyclotomic character in \( \bar{\rho}'_{m,v} \) comes from the normalization of Colmez’s functor used in Paškūnas’ paper. For example, there would be no such twist if we are using the magical functor in [16].

3.5.4. There two actions of \( R_p^{ps, \psi^{-1}} := \bigotimes_{v|p} R_{v}^{ps, \psi^{-1}} \) on \( m \):

1. \( \tau_{\text{Gal}} \): defined in the last paragraph of section 3.3, which comes from the action of \( \mathbb{T}_\psi(U^p)_m \) on \( M_\psi(U^p)_m \).

2. \( \tau_{\text{Aut}} \): which comes from the action of \( \bigotimes_{v|p} R_{v}^{ps, \psi \varepsilon} \cong \bigotimes_{v|p} R_{v}^{ps, \psi^{-1}} \) on \( P_{\mathfrak{B}_m} \) via \( E_{\mathfrak{B}_m} \) in corollary 3.4.11. The natural isomorphisms between \( R_v^{ps, \psi} \) and \( R_v^{ps, \psi^{-1}} \) are given by twisting the inverse of cyclotomic character. We fix this isomorphism from now on.
Now we can state our main result of this section.

**Theorem 3.5.5** (Local-global compatibility). If $p = 3$, we assume that $\delta \circ \det \notin \mathcal{B}_{m,v}$ for any $v|p$. Then

1. $M_\psi(U^p)_m \in \mathfrak{C}_{D_p^*,\psi}(\mathcal{O})^{\mathfrak{B}^m}$ where $\mathfrak{B}_m$ is the block defined above.

2. Both actions $\tau_{\text{Gal}}, \tau_{\text{Aut}}$ of $R_{p,\psi_{e^{-1}}}$ on $P_{m,\psi_{e^{-1}}}$ on $\mathfrak{m}$ are the same.

We will follow the strategy of Emerton in [16]. The idea is that using the density result (proposition 3.2.9), it suffices to check the compatibility of both actions on classical crystalline points. But this is a consequence of the results and Berger-Breuil [3] and classical local-global compatibility. We also remark that the argument relies on the semi-simplicity of the Hecke actions at finite levels.

**Proof.** Let $v$ be a place above $p$. It suffices to prove

1. $M_\psi(U^p)_m \in \mathfrak{C}_{\text{GL}_2(F_v),\psi|_{F^\times_v}}(\mathcal{O})^{\mathfrak{B}_{m,v}}$ where $\mathfrak{B}_{m,v}$ is the block defined in 3.5.1.

2. Both actions $\tau_{\text{Gal}}, \tau_{\text{Aut}}$ of $R_{p,\psi_{e^{-1}}}$ on $\text{Hom}_{\text{GL}_2(F_v),\psi|_{F^\times_v}}(\mathcal{O})(P_{\mathfrak{B}_{m,v}}, M_\psi(U^p)_m)$ are the same.

Since the formulation of the theorem is compatible with twisting of characters, we may assume $\psi$ is crystalline at $v$ of Hodge-Tate weight $w_\psi$. Also it is clear that we may shrink $U^p$ so that $\psi|_{U^p \cap (A_F^\infty)\times}$ is trivial and $U^p K_p$ is sufficiently small.

Consider the isomorphism in 3.2.10: for any $(\vec{k}, \vec{w}) \in \mathbb{Z}^{\text{Hom}(F, \mathbb{C}_p)}_> \times \mathbb{Z}^{\text{Hom}(F, \mathbb{C}_p)}$ such that $k_\sigma + 2w_\sigma = w_\psi + 2$ independent of $\sigma$ and $U_p = \text{GL}_2(O_{F_v})U^v$ with $U^v$ an open subgroup of $\prod_{w \neq v, w|p} \text{GL}_2(O_{F_w})$, we have

$$S_{(\vec{k}, \vec{w}), \psi}(U^p U_p, E) \cong \text{Hom}_{E[U_p]}(W_{(\vec{k}, \vec{w}), E}^*, S_\psi(U^p)_E).$$

It is clear that this isomorphism is Hecke-equivariant. We get a natural surjective map $T_\psi(U^p)[\frac{1}{p}] \to T_{(\vec{k}, \vec{w}), \psi}(U^p K_p, E)$ sending $T_w$ to $T_w$ for $w \notin S$. Hence it follows from the theory of classical automorphic forms that the action of $T_\psi(U^p)[\frac{1}{p}]$ on
\( S(\vec{k}, \vec{w})(U^p U_p, E) \) is semi-simple. Let \( \mathfrak{p} \) be a prime ideal of \( \mathbb{T}(\vec{k}, \vec{w})(U^p U_p, E) \otimes E \mathbb{Q}_p \).

Then it corresponds to an automorphic representation \( \pi_p = \pi_\infty \bigotimes_{p} (\pi_p)_\infty \) on \( (D \otimes_F \mathbb{A}_F^\infty \bigotimes) \). From the discussion in 3.1.4 we know that

\[
(S(\vec{k}, \vec{w})(U^p U_p, E) \otimes_{E, p} \mathbb{C})[\mathfrak{p}] \cong (\pi_p^\infty)^{U^p U_p}.
\]

Fix \( U^v \) and take the limit over all open compact subgroups \( U_v \) of \( GL_2(O_{F_v}) \). We get

\[
\lim_{U_v} (S(\vec{k}, \vec{w})(U^p U^v U_v, E) \otimes_{E, p} \mathbb{C})[\mathfrak{p}] \cong (\pi_p^\infty)^{U^p U^v} \cong (\pi_p)_v^{\oplus d(p)},
\]

for some \( d(p) > 0 \). Here \( (\pi_p)_v \) is the local representation of \( \pi_p \) at place \( v \). Take a finite extension \( E(p) \) of \( E \) such that \( \mathfrak{p} \) is defined over \( E(p) \). There exists a model \( \pi_v^{E(p)} \) over \( E(p) \) of \( (\pi_p)_v \). Then we have

\[
\lim_{U_v} S(\vec{k}, \vec{w}, \psi)(U^p U^v U_v, E(p))[\mathfrak{p}] \cong (\pi_v^{E(p)})^{\oplus d(p)}.
\]

Combining with the isomorphism in 3.2.10 we get a map (by abuse of notation, \( \mathfrak{p} \) is viewed as a maximal ideal of \( \mathbb{T}_\psi(U^p) \otimes E(p) \))

\[
\Phi_p : W^*(\vec{k}, \vec{w}, E) \otimes E (\pi_v^{E(p)})^{\oplus d(p)} \to (S_\psi(U^p) \otimes E(p))[\mathfrak{p}].
\]

Using remark 3.1.5 it is easy to see that this map is actually \( GL_2(F_v)U^v \)-equivariant. Moreover the image contains the \( GL_2(O_{F_v})U^v \)-algebraic vectors of \( (S_\psi(U^p) \otimes E(E))(p) \) (see 3.2.9 for the precise definition here). This is essentially because

\[
\text{Hom}_{E[p]}(W^*(\vec{k}, \vec{w}, \psi), S_\psi(U^p) \otimes E(p)[\mathfrak{p}]) = 0
\]

unless \( \vec{k} = \vec{k}', \vec{w} = \vec{w}' \). We denote the closure of the image of \( \Phi_p \) in \( S_\psi(U^p) \otimes E(p) \) by \( \Pi(p) \). Recall that \( S_\psi(U^p) \otimes E(p) \) is a Banach space with a unit ball \( S_\psi(U^p) \otimes O_{E(p)} \).
Let $\Pi_{B_{m,v}} := \text{Hom}^\text{cont}_O(P_{B_{m,v}} E)$. This is a Banach space with unit ball $\text{Hom}^\text{cont}_O(P_{B_{m,v}} O)$.

**Lemma 3.5.6.** The inclusion map $\Pi(p) \hookrightarrow S_\psi(U^p) \otimes E(p)$ induces a natural injective map:

$$\text{Hom}_{E[\text{GL}_2(F_v)]}^\text{cont}(S_\psi(U^p)_m \otimes E, \Pi_{B_{m,v}}) \hookrightarrow \prod_p \text{Hom}_{E[\text{GL}_2(F_v)]}^\text{cont}(\Pi(p), \Pi_{B_{m,v}} \otimes E(p)),$$

where $p$ runs over all prime ideals of $\mathbb{T}_{(\hat{k},\hat{w}),\psi}(U^p \text{GL}_2(O_{F_v}U^v)_m \otimes \mathbb{Q}_p$ with $k_\sigma \geq 2$ and $k_\sigma + 2w_\sigma = w_\psi + 2$ for any $\sigma$ and $U^v$ runs through all open subgroups of $\prod_{w \neq v, w \mid p} \text{GL}_2(O_{F_w})$.

**Proof.** By proposition 3.2.9 any continuous map from $S_\psi(U^p)_m \otimes E$ is determined by its value on $(S_\psi(U^p)_m \otimes E)^{\psi-a,v'-la}$. But the action of Hecke algebra on this space is semi-simple, hence it follows from our previous discussion that it is contained in the space generated by all $\Pi(p) \cap S_\psi(U^p)_m \otimes E$. This clearly proves the lemma. \qed

Note that the left hand side of the previous lemma is nothing but:

$$\text{Hom}_{E[\text{GL}_2(F_v)]}^\text{cont}(S_\psi(U^p)_m \otimes E, \Pi_{B_{m,v}}) \cong \text{Hom}_{E[\text{GL}_2(F_v)]}^\text{cont}(P_{B_{m,v}}, M_\psi(U^p)_m \otimes E).$$

Since $S_\psi(U^p)$ is $p$-torsion free, it follows from the previous lemma that it suffices to prove

$$\tau_{\text{Gal}}|_{E[p]} = \tau_{\text{Aut}}|_{E[p]} \text{ on } \text{Hom}_{E[\text{GL}_2(F_v)]}^\text{cont}(\Pi(p), \Pi_{B_{m,v}} \otimes E(p))$$

for any $p$. Fix such a $p$ and suppose it comes from $\mathbb{T}_{(\hat{k},\hat{w}),\psi}(U^p \text{GL}_2(O_{F_v}U^v)_m$. Since all the formulations are compatible with base change, we may assume $E(p) = E$. 

There are two possibilities for $p$ depending on whether the automorphic representation $\pi_p$ factors through the reduced norm map $N_{D/F}$ or not. First we assume
that \( \pi_p \) does not factor through \( N_{D/F} \). Then the under Jacquet-Langlands correspondence, \( \pi_p \) corresponds to a regular algebraic cuspidal automorphic representation of \( GL_2(\mathbb{A}_F) \). See lemma 1.3. of [11] for more details.

**Galois side** The action \( \tau_{\text{Gal}} \) of \( R_{\psi, \psi^{-1}}^{\mathbf{ps}} \) on \( \text{Hom}_{\mathbf{E}}(\Pi(\mathbb{F}) \otimes \mathbf{E}, \Pi_{\mathbf{ps}, \psi} \otimes \mathbf{E}) \) is clear by our knowledge on the classical local-global compatibility at primes above \( p \).

**Lemma 3.5.7.** Let \( \mathfrak{p}_v = R_{\psi, \psi^{-1}}^{\mathbf{ps}}[\mathbf{ps}] \cap \mathfrak{p} \) and \( \rho(\mathfrak{p})_v : G_{\mathfrak{F}_v} \to GL_2(\mathbf{E}) \) be the semi-simple representation given by \( \mathfrak{p}_v \). Then

1. \( \rho(\mathfrak{p})_v \) is de Rham of Hodge-Tate weights \( (w_{\sigma_v}, w_{\sigma_v} + k_{\sigma_v} - 1) \), where \( \sigma_v : F \to \mathbf{E} \) is the embedding induced by \( v \). More precisely,

\[
\text{gr}^i(\rho(\mathfrak{p})_v \otimes \mathbf{B}_{\text{dR}})^{G_{\mathfrak{F}_v}} = 0
\]

unless \( i = w_{\sigma_v}, w_{\sigma_v} + k_{\sigma_v} - 1 \).

2. The semi-simple Weil-Deligne representation \( \text{WD}(\varepsilon \otimes \rho(\mathfrak{p})_v)^{ss} \) corresponds to \( \pi^{E(p)}_v \) under the Hecke correspondence in the sense of [12]. In our case, \( \text{WD}(\varepsilon \otimes \rho(\mathfrak{p})_v \otimes \mathbf{B}_{\text{cris}})^{G_{\mathfrak{F}_v}} \) with \( \text{Frob}_v \) acting via \( \varphi \).

**Proof.** This follows directly from Theorem 1.1 of [2]. \( \Box \)

**Remark 3.5.8.** Under the Hecke correspondence, \( \chi_1 \oplus \chi_2 \) will correspond to \( \text{Ind}_{B(\mathbb{F}_v)}^{GL_2(\mathbb{F}_v)} \chi_1 \otimes \chi_2 \cdot |^{-1} \) (generically). This is also the one used in [10].

**Automorphic side** Now we need to determine \( \Pi(\mathfrak{p}) \). Recall that this is the closure of

\[
W^*_{(k, \mathfrak{w}), E} \otimes (\pi^{E(p)}_v)^{\otimes d(p)} = [(\text{Sym}^{k_{\sigma_v} - 2}(E^2) \otimes \det w_{\sigma_v})^* \otimes \pi^{E(p)}_v]^{\otimes d(p)}
\]

in \( S_\psi(U^p)_E \). Here \( d(p)' \) is some multiple of \( d(p) \).

Let \( \Pi_v \) be the universal unitary completion of \( (\text{Sym}^{k_{\sigma_v} - 2}(E^2) \otimes \det w_{\sigma_v})^* \otimes \pi^{E(p)}_v \) as a \( E \)-representation of \( GL_2(\mathbb{F}_v) \). We note that \( \pi^{E(p)}_v \) is an irreducible principal series.
Otherwise it is one-dimensional and \( \pi_p \) has to factor through the reduced norm map by the approximation theorem, which we assume not the case. By the main results of [3], [30] in the non-ordinary case and Proposition 2.2.1 of [4] in the ordinary case, this is a topologically irreducible admissible unitary \( \text{GL}_2(F_v) \) representation.

**Lemma 3.5.9.** \( \Pi(p) \) is a quotient of \( \Pi_v^{\oplus d(p)'} \).

**Proof.** By the universal property, we get a continuous map \( \Pi_v^{\oplus d(p)'} \rightarrow \Pi(p) \) with dense image. Note that both \( \Pi_v^{\oplus d(p)'} \) and \( \Pi(p) \) are both admissible representations of \( \text{GL}_2(F_v) \). The surjectivity of this map follows from Proposition 3.1.3 of [16].

As a corollary, we get an injective map

\[
\text{Hom}^{\text{cont}}_{E[\text{GL}_2(F_v)]}(\Pi(p), \Pi_{\mathfrak{B}_{m,v}}) \hookrightarrow \text{Hom}^{\text{cont}}_{E[\text{GL}_2(F_v)]}(\Pi_v^{\oplus d(p)'} \oplus \mathfrak{p}_v, \Pi_{\mathfrak{B}_{m,v}}).
\]

Let \( \Pi_v^0 \) and an \( \text{GL}_2(F_v) \)-invariant open ball of \( \Pi_v \) and denote \( \text{Hom}_{\mathcal{O}}(\Pi_v^0, \mathcal{O}) \) by \( M_v \). Then

\[
\text{Hom}^{\text{cont}}_{E[\text{GL}_2(F_v)]}(\Pi_v^{\oplus d(p)'} \oplus \mathfrak{p}_v, \Pi_{\mathfrak{B}_{m,v}}) \cong E^{\oplus d(p)'} \otimes \text{Hom}_{\mathcal{O}}(\Pi_v^{\oplus d(p)'} \oplus \mathfrak{p}_v, \Pi_{\mathcal{O}}(\mathfrak{B}_{m,v}, M_v)).
\]

Now we only need prove that the action of the centre \( R_{\mathfrak{B}_{m,v}}^{\mathfrak{p}_v, \psi\epsilon^{-1}} \) on \( \text{Hom}(P_{\mathfrak{B}_{m,v}}, M_v) \otimes E \) also factors through \( R_{\mathfrak{B}_{m,v}}^{\mathfrak{p}_v, \psi\epsilon^{-1}}[1/p]/\mathfrak{p}_v \). Note that \( \Pi_v \) is topologically irreducible. By corollary 1.9 of [32], it suffices to show that \( M_v \) appears as a subquotient of \( P_{\mathfrak{B}_{m,v}} / \mathfrak{p}_v P_{\mathfrak{B}_{m,v}} \). Here we consider \( \mathfrak{p}_v \) as a prime ideal of \( R_{\mathfrak{B}_{m,v}}^{\mathfrak{p}_v, \psi\epsilon} \) by the isomorphism in [3.5.4].

If \( \rho(p)_v \) is absolutely irreducible, then by Theorem 1.10 of [32], up to isomorphism, there is only one irreducible Banach representation \( \Pi'_v \) appeared in the subquotient of \( \text{Hom}_{\mathcal{O}}^\text{cont}(P_{\mathfrak{B}_{m,v}} / \mathfrak{p}_v P_{\mathfrak{B}_{m,v}}, E) \), which is characterized by \( V(\Pi'_v) \cong \rho(p)_v \otimes \epsilon \). Here \( V \) is Colmez’s functor normalized as in [32]. By Theorem 1.3 of [10] (the convention for
Hodge-Tate weight of $\varepsilon$ is 1 there) and lemma 3.5.7 $\Pi'_v$ is a unitary completion of
\[ \text{Sym}^{k_{\sigma_v} - 2}(E^2) \otimes \det^{-(w_{\sigma_v} + k_{\sigma_v} - 2)} \otimes \pi_v^{E(p)} . \]

But the result of Berger and Breuil says that such unitary completion is unique. Hence $\Pi'_v \cong \Pi_v$ and $M_v \otimes E$ is even a quotient of $(P_{\mathfrak{B}_{m,v}}/p_v P_{\mathfrak{B}_{m,v}}) \otimes E$.

If $\rho(p)_v = \psi_1 \oplus \psi_2$ is reducible, we may assume Hodge-Tate weight of $\psi_1$ (resp. $\psi_2$) is $w_{\sigma_v}$ (resp. $w_{\sigma_v} + k_{\sigma_v} - 1$). Then
\[ \pi_v^{E(p)} \cong (\text{Ind}_{B(F_v)}^{GL_2(F_v)} \psi_1 \varepsilon^{w_{\sigma_v}} \otimes |1 - w_{\sigma_v} \otimes \psi_2 \varepsilon^{w_{\sigma_v} + k_{\sigma_v} - 1}| \cdot | - w_{\sigma_v} - k_{\sigma_v} + 1 )_{\text{sm}} \]
is irreducible by our assumption. Hence $\psi_1/\psi_2 \neq \varepsilon^{\pm 1}$. Proposition 2.2.1 of [4] tells us that $\Pi_v$ is the unitary parabolic induction $(\text{Ind}_{B(F_v)}^{GL_2(F_v)} \psi_2 \varepsilon \otimes \psi_1)_{\text{cont}}$. Compared with Theorem 1.11 of [32], we also conclude that $M_v$ appears in the subquotient of $P_{\mathfrak{B}_{m,v}}/p_v P_{\mathfrak{B}_{m,v}}$.

Finally we treat the case where $\pi_p$ factors through the reduced norm map. In this case, $\Pi(p)$ has the form $\eta \circ \det$ for some continuous character $\eta : F_v^\times \to \mathcal{O}^\times$ and the corresponding pseudo-character of $G_{F_v}$ is $\eta + \eta \varepsilon^{-1}$. Here as usual $\eta$ is also viewed as a character of $G_{F_v}$ by the class field theory. Our claim follows directly from proposition 10.107 of [32].

This finishes the proof of the second statement of the theorem. As for the first part, note that in lemma 3.5.6 we can replace $\mathfrak{B}_{m,v}$ by any other block $\mathfrak{B}'$. Since we have already seen that $\Pi(p)$ belongs to the block $\mathfrak{B}_{m,v}$, it is clear that $\text{Hom}_{E[GL_2(F_v)]}^\text{cont}(S_\psi(U^p)_m \otimes E, \Pi_{\mathfrak{B}'}) = 0$ unless $\mathfrak{B}' = \mathfrak{B}_{m,v}$. \hfill $\square$

**Corollary 3.5.10.** Under the same assumption as in the theorem,

1. $m$ (defined in 3.5.2) is a faithful, finitely generated $\mathbb{T}_\psi(U^p)_m$-module.

2. $\mathbb{T}_\psi(U^p)_m$ is a finite $R_p^{ps, \psi \varepsilon^{-1}}$-algebra.
Proof. The faithfulness follows from the first part of the theorem. Note that \( S_\psi(U^p, E/O) \) is an admissible representation of \( D_p^\times \). By Proposition 4.17 of \([32]\), \( m \) is a finitely generated \( E_{B_m} \)-module. But \( E_{B_m} \) is a finite algebra over \( \mathcal{R}_{ps, \psi}^{-1} \) (corollary 3.4.11), hence \( m \) is a finite module over \( \mathcal{R}_{ps, \psi}^{-1} \) via \( \tau_{\text{Aut}} \). On the other hand, \( \tau_{\text{Gal}} = \tau_{\text{Aut}} \) factors through \( T_\psi(U^p)_m \). This proves both finiteness assertions in the corollary.

**Corollary 3.5.11.** \( S_\psi(U^p, F)[m] \) is a representation of \( D_p^\times \) of finite length.

**Proof.** It follows from the first part of corollary 3.5.10 that \( m/\text{mm} \) is actually a finite dimensional \( F \)-vector space hence a finite length \( E_{B_m} \)-module.

**Corollary 3.5.12.** For any maximal ideal \( p \) of \( T_\psi(U^p)_m[\frac{1}{p}] \) such that for any \( v|p \),

- \( \rho(p)|_{G_{F,v}} \) is absolutely irreducible and de Rham with distinct Hodge-Tate weights,

where \( \rho(p) : G_{F,S} \to \text{GL}_2(k(p)) \) is the semi-simple representation associated to \( p \).

Then \( p \) is a pull-back of a maximal ideal of \( T_{(\tilde{k}, \tilde{w}), \psi}(U^p U_p)[\frac{1}{p}] \) for some weight \( (\tilde{k}, \tilde{w}) \in \mathbb{Z}_{>1}^{\text{Hom}(F, \mathbb{Q}_p^\times)} \times \mathbb{Z}^{\text{Hom}(F, \mathbb{Q}_p^\times)} \) and open compact subgroup \( U_p \subseteq K_p \). In other words, \( p \) comes from a classical automorphic representation on \( (D \otimes A_F^\times)^\times \). By Jacquet-Langlands correspondence, it also arises from a regular algebraic cuspidal automorphic representation of \( \text{GL}_2(A_F) \).

**Proof.** Let \( p_v \in \text{Spec} \mathcal{R}_{ps, \psi}^{-1} \) and \( \tilde{p} \in \text{Spec} T_\psi(U^p)_m \) be the pull-back of \( p \). Since \( \rho(p)|_{G_{F,v}} \) is absolutely irreducible, all the irreducible subquotients of

\[
\text{Hom}^\text{cont}_O(P_{\mathfrak{B}_m, v}/p_v P_{\mathfrak{B}_m, v}, E)
\]

are isomorphic (Theorem 1.10 of \([32]\)). We denote any such irreducible subquotient by \( \Pi_v \). Enlarge \( E \) if necessary, we may assume \( \Pi_v \) is absolutely irreducible.

**Lemma 3.5.13.** Under all the assumptions as in the proof,
1. The unitary representation $\Pi := \bigotimes_{v \mid p} \Pi_v$ of $D_p^\times$ is topologically irreducible.

2. Let $p_p = p \cap R_{p, \psi}^{\text{ps}, \psi^{-1}}$. Then $E_{B_m}[\frac{1}{p}]/(p_p)_{\psi}$ is a central simple $E$-algebra.

Proof. We are going to use the relations between $E_{B_m}[\frac{1}{p}]$-modules and Banach space representations. The reference here is §4 of [32].

To prove the first assertion, we will apply Theorem 4.34 of [32] with $G = D_p^\times$. The first condition in that theorem is clearly satisfied. The second condition follows from Proposition 4.20 by corollary 3.4.11. Thus it suffices to prove that

$$m(\Pi) := \text{Hom}_{C}^{GL_2(F_v)}(\Omega)(P_{B_m}, \Pi_0^d) \otimes E$$

is a simple $E_{B_m}[\frac{1}{p}]$-module, where $\Pi_0$ is an open $D_p^\times$-invariant unit ball of $\Pi$ and $\Pi_0^d := \text{Hom}_\mathcal{O}(\Pi_0, \mathcal{O})$. Using lemma 3.4.9 it is easy to see that

$$m(\Pi) \cong \bigotimes_{v \mid p} m(\Pi_v),$$

where $m(\Pi_v) := \text{Hom}_{C}^{GL_2(F_v)}(\Omega)(P_{B_m, v}, \Pi_{0, v}) \otimes E$ and $\Pi_{0, v}$ is an open $GL_2(F_v)$-invariant unit ball of $\Pi_v$. Let $E_v$ be the image of $E_{B_m, v}[\frac{1}{p}]$ in $\text{End}(\Pi_v)$. Since we assume $\Pi_v$ is absolutely irreducible, by Lemma 4.2 and Proposition 4.19 of [32], we know that $E_v$ is a central simple algebra over $E$ and $m(\Pi_v)$ is a simple $E_v$-module. Hence $\bigotimes_{v \mid p} m(\Pi_v) \cong \bigotimes_{v \mid p} m(\Pi_v)$ is a simple module of $\bigotimes E_v$. This is exactly what we need to show.

For the second claim, since we assume $\rho(p_v)$ is absolutely irreducible, it follows from Theorem 1.10 of [32] that $E_v$ is in fact $E_{B_m, v}[\frac{1}{p}]/(p_p)_{\psi}$. Hence our claim is clear by the discussion in the previous paragraph.

By Pontryagin duality, we can write

$$\text{Hom}_\mathcal{O}^{\text{cont}}(M_\psi(U^p)_m/\overline{p} M_\psi(U^p)_m, E) \cong S_\psi(U^p)_m[\widehat{\mathbf{p}}] \otimes E \cong (S_\psi(U^p)_m \otimes E)[\mathbf{p}].$$
As a consequence of our local-global compatibility result, m is a faithful, finitely generated \( T_\psi(U^p)_m \)-module. Therefore \( m/\hat{m} \otimes E \) is a non-zero \( E_2 m [\frac{1}{p}]/(p^p) \)-module. Hence \( (S_\psi(U^p)_m \otimes E)[p] \neq 0 \) and lemma 3.5.13 even implies that \( (S_\psi(U^p)_m \otimes E)[p] \cong (\bigotimes_{v|p} \Pi_v)^d \) for some positive integer \( d \). By Theorem 1.3 of [10], for each \( v\mid p \), \( \Pi_v \) has non-zero locally algebraic vectors of \( GL_2(F_v) \). Thus \( (S_\psi(U^p)_m \otimes E)[p] \) contains non-zero locally algebraic vectors of \( D_p^\times \). In view of the discussion in 3.2.10 this implies that \( p \) comes from some \( T_{(k, \vec{w}), \psi}(U^p) \).

\[ \square \]

3.6 A lower bound on the dimension of Hecke algebra

As another application of our local-global compatibility result, we prove

**Theorem 3.6.1.** Same assumption as in Theorem 3.5.5. Then each irreducible component of \( T_\psi(U^p)_m \) is of characteristic zero and of dimension at least \( 1 + 2[F:Q] \).

3.6.2. Since \( m \) is \( p \)-torsion free and is a faithful, finitely generated \( T_\psi(U^p)_m \)-module, it is clear that each irreducible component of \( T_\psi(U^p)_m \) is of characteristic zero.

We will establish a formula which relates the Gelfand-Kirillov dimension of \( M_\psi(U^p)_m/\varpi \) and \( M_\psi(U^p)_m/mM_\psi(U^p)_m \) as \( F[[K]] \)-modules (see definition below) with the usual dimension of \( m/\varpi m \) as a \( R_{p^\psi, \psi^{-1}}^\varpi \)-module. First we recall the definition and some basic properties of Gelfand-Kirillov dimension.

3.6.3. Let \( K_n = \prod_{v|p} (1 + p^n M_2(O_{F_v})) \) for some \( n > 0 \) large enough such that \( U^pK_n \) is sufficiently small and let \( Z_n \) be the center of \( K_n \). Denote \( K_n/Z_n \) by \( PK_n \). This is a \( p \)-adic Lie group of dimension \( 3[F:Q] \). It is clear that \( PK_n \) is torsion-free and \([PK_n, PK_n] \subseteq (PK_n)^p\), the subgroup generated by \( g^p, g \in PK_n \). Hence by Theorem 4.5 of [14], \( PK_n \) is uniform (see Definition 4.1 ibid.). Let \( \Lambda = F[[PK_n]] \) be the
completed group ring of $PK_n$ over $\mathbb{F}$ with maximal ideal $J_1$. It follows from Theorem 7.24 ibid. that

**Lemma 3.6.4.** $\text{gr}(\Lambda) := \bigoplus_{k \geq 0} J_1^k/J_1^{k+1}$ is isomorphic to $F[x_1, \ldots, x_{3[F:Q]}]$ as a graded ring.

**Corollary 3.6.5.** $\Lambda$ is left and right Noetherian and has no zero-divisors.

**Proof.** This is Corollary 7.25 of [14].

**Definition 3.6.6.** Let $m_p$ be the maximal ideal of $R_{p,\psi}^{\text{ps}}$. Define $RA$ to be the completed tensor product of $R_{p,\psi}^{\text{ps}}$ and $\Lambda$ over $\mathbb{F}$ with respect to $m_p$-adic and $J_1$-adic topology. This is a local ring. We denote its maximal by $J_2$.

**3.6.7.** By definition, $\text{gr}(RA) = \bigoplus_{k \geq 0} J_2^k/J_2^{k+1} \cong \text{gr}(R_{p,\psi}^{\text{ps}}/\mathcal{O}) \otimes F[x_1, \ldots, x_{3[F:Q]}]$ is Noetherian. Hence $RA$ is left and right Noetherian by Chapter II.1.2 Proposition 3 of [27]. We note that $RA$ acts on $P_{\mathfrak{B}_m}$ naturally (see the proof of Lemma 2.7 of [32]) and makes it into a finitely generated $RA$-module since $(P_{\mathfrak{B}_m}/m_p P_{\mathfrak{B}_m})^\vee$ is an admissible representation of $D_p^\times$ (of finite length).

Let $R$ be either $RA$ or $\Lambda$ with maximal ideal $J$. Let $M$ be a finitely generated left $R$-module. Consider $\text{gr}(M) := \bigoplus_{k \geq 0} J^k M/J^{k+1} M$. This is a finitely generated graded $\text{gr}(R)$-module. Its **Hilbert polynomial** $\varphi_M(t)$ (see [29] §13) is defined to be unique polynomial satisfying $\varphi_M(k) = \dim_F(J^k M/J^{k+1} M)$ for $k$ large enough. The **Gelfand-Kirillov dimension** $\dim_R(M)$ of $M$ over $R$ is defined to be the degree of the polynomial $t \varphi_M(t)$. Equivalently $\dim_R(M) = \lim \sup_k \log_k(\dim_F M/J^k M)$. For example, $\dim_\Lambda(\Lambda) = 3[F : \mathbb{Q}]$. The dimension is independent of the choice of the open compact subgroup $K_n$.

There is a natural map $\Lambda \to RA$ such that $J_1 J_2 = J_2 J_1$. The next result is well-known in commutative algebra and the proof is the same.

**Lemma 3.6.8.** Let $M$ be a finitely generated left $RA$-module. Assume it is also finitely generated as a $\Lambda$-module. Then $\dim_\Lambda(M) = \dim_{RA}(M)$. 

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Proof. Since $M/J_1M$ is a finite $\mathbb{F}$-vector space, there exists an integer $r$ such that $J_2^rM \subseteq J_1M$. Hence $J_1^rM \subseteq J_2^{rk}M \subseteq J_1^kM$. The desired result follows from the definition.

Another ingredient we need is Artin-Rees property.

**Lemma 3.6.9.** Let $R$ be either $\Lambda$ or $R\Lambda$ with maximal ideal $J$. Let $M$ be a finitely generated left $R$-module and $N \subseteq M$ be a submodule. Then there exists $c \in \mathbb{Z}_{>0}$ such that for any $k$,

$$J^{k+c}M \cap N \subseteq J^kN.$$

*Proof.* This follows from Chapter II 1.1 Proposition 3 of [27].

A lot of results in classical commutative algebra are also true in this setting.

**Lemma 3.6.10.** Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated left $R$-modules. Then $\dim_R(M) = \max(\dim_R(M'), \dim_R(M''))$.

*Proof.* Apply the previous lemma to $M' \subseteq M$ and get an integer $c$ as in the lemma. Then

$$\dim_{\mathbb{F}}(M/J^kM) \leq \dim_{\mathbb{F}}(M'/J^kM') + \dim_{\mathbb{F}}(M''/J^kM''),$$

$$\dim_{\mathbb{F}}(M/J^kM) \geq \dim_{\mathbb{F}}(M'/J^{k-c}M') + \dim_{\mathbb{F}}(M''/J^kM'').$$

The desired result is clear.

**Lemma 3.6.11.** Let $M$ be a finitely generated left $R$-module and $x \in J$ be an element in the centre of $R$. Suppose $M$ has no $x$-torsion, then $\dim_R(M/xM) = \dim_R(M) - 1$.

*Proof.* Note that

$$\dim_{\mathbb{F}}(M/(J^kM + xM)) = \dim_{\mathbb{F}}(M/J^kM) - \dim_{\mathbb{F}}(xM/(J^kM \cap xM))$$
By Artin-Rees lemma, there exists an integer $c$ such that for $k \geq c$, $J^k M \cap xM \subseteq J^{k-c} xM$.

$$\dim_R(M/(J^k M + xM)) \leq \dim_R(M/J^k M) - \dim_R(xM/J^{k-c} M) = \dim_R(M/J^k M) - \dim_R(M/J^{k-c} M)$$

Thus $\dim_R(M/xM) \leq \dim_R(M) - 1$. On the other hand, $J^k M \cap xM \supseteq xJ^{k-1} M$. We have

$$\dim_R(M/(J^k M + xM)) \geq \dim_R(M/J^k M) - \dim_R(M/J^{k-1} M).$$

This implies that $\dim_R(M/xM) \geq \dim_R(M) - 1$ and hence the equality. \qed

Now we can state our main result. Assuming it, we can give a proof of Theorem 3.6.1.

**Proposition 3.6.12 (Dimension formula).** Let $N$ be a finitely generated right $E_{\mathfrak{m}}/(\varpi)$-module with the induced topology. Then

$$\dim_{R_{\mathfrak{m}}} (N \otimes_{E_{\mathfrak{m}}} P_{\mathfrak{m}}) \leq \dim_{R_{p,\psi}} N + [F : \mathbb{Q}],$$

where $\mathfrak{m}_p$ is the maximal ideal of $R_{p,\psi}$. Note that $\dim_{R_{p,\psi}} N$ makes sense since $N$ is also a finitely generated $R_{p,\psi}$-module.

**Remark 3.6.13.** This formula roughly says ‘dimension of total space $\leq$ dimension of the base space + dimension of the special fibre’. This is exactly the heuristics in page 19 of [8].

Another remark is that it will be clear in the proof that if no $\mathfrak{m}_{m,v}$ belong to the last two blocks in 3.4.4 then the inequality in the proposition is in fact an equality. The problem of the last two blocks is that $\delta \circ \det$ has Gelfand-Kirillov dimension 0 rather than 1.
Proof of Theorem \ref{thm:proof_of_3.6.4}. Note that $\psi|_{1+\mathfrak{p}O_{F,p}}$ is trivial modulo $\varpi$. The same proof of Proposition \ref{prop:3.2.4} shows that $M_\psi(U^p)_m/\varpi$ is a projective, hence free module over $\Lambda = \mathbb{F}[[PK_n]]$ (defined in \ref{prop:3.6.3}). Let $\bar{m} = m/\varpi m$. Then it has full support on $\mathbb{T}_\psi(U^p)/(\varpi)$ and $\bar{m} \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m} = M_\psi(U^p)_m/\varpi \cong \Lambda^\oplus d$ for some $d$.

Let $\mathfrak{p}$ be a minimal prime ideal of $\mathbb{T}_\psi(U^p)_m$ and $m[\mathfrak{p}]$ be the set of elements of $m$ killed by $\mathfrak{p}$. Denote $m[\mathfrak{p}]/\varpi m[\mathfrak{p}]$ by $N$ and the image of $N \to \bar{m}$ by $N'$. Note that $m$ is torsion-free and has full support on $\mathbb{T}_\psi(U^p)_m$, hence $N' \neq 0$ and $0 \neq N' \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m} \subseteq \Lambda^\oplus d$. Since $\Lambda$ does not have zero-divisors, $N' \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m}$ has at least a copy of $\Lambda$ inside. Hence

$$\dim_\Lambda(N \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m}) \geq \dim_\Lambda(N' \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m}) \geq \dim_\Lambda \Lambda = 3[F:Q]$$

by lemma \ref{lem:3.6.10}. On the other hand, the other direction of the inequality is also true since $N \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m}$ is a finitely generated $\Lambda$-module. Thus

$$\dim_\Lambda(N \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m}) = 3[F:Q].$$

Note that this is also $\dim_{R\Lambda}(N \otimes_{E_{\varpi m}} \mathcal{P}_{\varpi m})$ by lemma \ref{lem:3.6.8}. Apply proposition \ref{prop:3.6.12} with $N = N$. We get

$$\dim_{R_p^{ps,\psi_\epsilon}} N \geq 2[F:Q].$$

Since the action of $R_p^{ps,\psi_\epsilon}$ on $N$ factors through $\mathbb{T}_\psi(U^p)_{m}/(\mathfrak{p}, \varpi)$, we have

$$\dim \mathbb{T}_\psi(U^p)_{m}/(\mathfrak{p}, \varpi) = \dim_{\mathbb{T}_\psi(U^p)_{m}/(\mathfrak{p}, \varpi)} N = \dim_{R_p^{ps,\psi_\epsilon}} N \geq 2[F:Q].$$

Since $\mathfrak{p}$ has characteristic zero, we have $\dim \mathbb{T}_\psi(U^p)_{m}/\mathfrak{p} - 1 = \dim \mathbb{T}_\psi(U^p)_{m}/(\mathfrak{p}, \varpi)$. This finishes the proof. \hfill \qed
Proof of Proposition 3.6.12. Write $d(N) = \dim_{\mathbb{R}}(N \otimes_{E_{\mathfrak{p}_m}} P_{\mathfrak{p}_m})$. The proof is by induction on the dimension of $N$ over $R_{\mathfrak{p},\psi\varepsilon}$. If $\dim_{R_{\mathfrak{p},\psi\varepsilon}} N = 0$, then $(N \otimes_{E_{\mathfrak{p}_m}} P_{\mathfrak{p}_m})^\vee$ is a smooth representation of $D^\times$ of finite length. Since each irreducible constituent has the form $\otimes_{v\mid p} \pi_v$, where $\pi_v$ is an irreducible representation of $GL_2(F_v)$, the result follows from

Lemma 3.6.14. Let $K$ be a pro-$p$ open subgroup of $GL_2(F_v)$ and $\pi$ be any smooth irreducible representation of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}$ with central character $\psi$. Then $\dim_{F[[K]]} \pi^\vee = 1$ unless $\pi$ is one-dimensional.

Proof. See the proof of Corollary 7.5 of [35]. Or one can prove this directly: the Gelfand-Kirillov dimension of principal series and special series can be computed by hand; the case of supersingular representations can be computed using Theorem 1.2 of [31].

Suppose we have proved for all $N$ of dimension at most $r$ over $R_{\mathfrak{p},\psi\varepsilon}$. Let $N$ be a finitely generated right $E_{\mathfrak{p}_m}$-module of dimension $r + 1$ over $R_{\mathfrak{p},\psi\varepsilon}$. Choose $x \in R_{\mathfrak{p},\psi\varepsilon}$ such that $\dim_{R_{\mathfrak{p},\psi\varepsilon}} N/xN = r$. Replace $x$ by its power, we may assume $N[x] = N[x^\infty]$. Let $N' = N/N[x]$. Then $N'$ has no $x$-torsion. $d(N') = d(N'/xN') + 1$ by lemma 3.6.11.

Since $\dim_{R_{\mathfrak{p},\psi\varepsilon}} N'/xN', \dim_{R_{\mathfrak{p},\psi\varepsilon}} \leq r$. By induction hypothesis,

$$d(N[x]) \leq \dim_{R_{\mathfrak{p},\psi\varepsilon}} N[x] + [F : \mathbb{Q}] \leq r + [F : \mathbb{Q}],$$

$$d(N') = d(N'/xN') + 1 \leq r + [F : \mathbb{Q}] + 1.$$

Hence $d(N) \leq r + 1 + [F : \mathbb{Q}]$ by lemma 3.6.10.

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3.7 A variant of completed homology

In order to apply Taylor’s Ihara avoidance in [42], we need to introduce a variant of completed homology considered before. Most of arguments in previous subsections still work here. We keep the notations as in the beginning of this section.

3.7.1. Let $U = \prod_v U_v$, where $U_v$ is an open compact subgroup $(D \otimes F_v)\times$. Write $U^p = \prod_{v \nmid p} U_v$ and $U_p = \prod_{v \mid p} U_v$. Let $\xi : U^p \to O^\times$ be a continuous smooth character and $\psi : (A_\infty^\times) / F_{>0}^\times \to O^\times$ be a continuous character such that $\psi|_{\prod_{v \nmid p} (O_{F_v}^\times \cap U_v)} = \xi|_{\prod_{v \nmid p} (O_{F_v}^\times \cap U_v)}$.

Given a topological $\mathbb{Z}_p$-algebra $A$ and a continuous representation $\tau : \prod_{v \mid p} U_v \to \text{Aut}(W_\tau)$, we may define $S_{\tau,\psi,\xi}(U, A)$ as the space of continuous functions:

$$f : D^\times \setminus (D \otimes F \mathbb{A}_F^\times) \to W_\tau,$$

such that for any $g \in (D \otimes F \mathbb{A}_F^\times)$, $z \in (A_\infty^\times)$, $u = u^p u_p \in U$, we have

$$f(guz) = \psi(z) \xi(u^p) \tau(u_p^{-1}) (f(g)).$$

If $\psi|_{U_p \cap O_{F_p}^\times} = \tau^{-1}|_{U_p \cap O_{F_p}^\times}$, then as in (3.1), we have

$$S_{\tau,\psi,\xi}(U, A) \simeq \bigoplus_{i \in I} W_{\tau}^{(t_i^{-1} D^\times \cap U(\mathbb{A}_F^\times)) / F^\times},$$

where $I = D^\times \setminus (D \otimes F \mathbb{A}_F^\times) / U(\mathbb{A}_F^\times)$ and $\{t_i\}_{i \in I}$ is a set of representatives. If $U$ is sufficiently small, corollary 3.1.3 is still valid. We will simply write $S_{\psi,\xi}(U, A)$ if $\tau$ is the trivial action on $A$.

3.7.2. We can introduce completed homology $M_{\psi,\xi}(U^p)$ and cohomology $S_{\psi,\xi}(U^p)$ similarly as in section 3.2 and Hecke algebra $T_{\psi,\xi}(U^p)$ as in section 3.3. Let $U^{np}$ be the kernel of $\xi$. Then we have a natural Hecke equivariant inclusion $S_{\psi,\xi}(U^{np}) \hookrightarrow S_{\psi}(U^{np})$. 

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From this, we can deduce our local-global compatibility (Theorem 3.5.5) for $M_{\psi,\xi}(U^p)$ directly from the case we have proved before. The same estimate of dimension of $T_{\psi,\xi}(U^p)$ can be obtained exactly the same as in section 3.6. We leave the details to interested readers.
Chapter 4

Patching at a one-dimensional prime

We are going to prove a ‘$R_q = \mathbb{T}_q$’ result, where $q$ is a one-dimensional prime ideal rather than the maximal ideal.

4.1 Setup and the statement of the main result

4.1.1. In this section, $F$ denotes a totally real field of even degree over $\mathbb{Q}$ in which $p$ splits completely. Write $\Sigma_p$ as the set of places of $F$ above $p$. Let $S$ be a finite set of finite places containing $\Sigma_p$ such that $p|N(v) - 1$, and let $\chi : G_{F,S} \to \mathcal{O}^\times$ be a continuous character such that

- $\chi$ is unramified at places outside of $\Sigma_p$.
- $\chi(\text{Frob}_v) \equiv 1 \mod \varpi$ for $v \in S \setminus \Sigma_p$.
- $\chi(c) = -1$ for any complex conjugation $c \in G_{F,S}$.

Denote by $\bar{\chi}$ the reduction of $\chi$ modulo $\varpi$. Let $\xi_v : k(v)^\times \to \mathcal{O}^\times$ be characters of $p$-power order for $v \in S \setminus \Sigma_p$. We will view $\xi_v$ as characters of $I_{F_v}$ by the local class field theory.
Consider the universal deformation ring $R_{ps,\{\xi_v\}}$ which pro-represents the functor from $C^\ell_O$ to the category of sets sending $R$ to the set of two-dimensional pseudo-representations $T$ of $G_{F,S}$ over $R$ such that $T$ is a lifting of $1 + \bar{\chi}$ with determinant $\chi$ and

$$T|_{I_{F_v}} = \xi_v + \xi_v^{-1}$$

for any $v \in S \setminus \Sigma_p$. If $\xi_v$ are all trivial, we will simply write $R_{ps,1}$.

We fix a complex conjugation $\sigma^* \in G_{F,S}$ so that we can associate a two-dimensional semi-simple representation $\rho(p) : G_{F,S} \to \text{GL}_2(k(p))$ with trace $T^{\text{univ}} \mod p$ for any $p \in \text{Spec } R_{ps,\{\xi_v\}}$ as in 2.1.5. Here, $T^{\text{univ}} : G_{F,S} \to R_{ps,\{\xi_v\}}$ is the pseudo-representation given by the universal property.

4.1.2. On the automorphic side, let $D$ be a quaternion algebra over $F$ ramified exactly at all infinite places. Fix an isomorphism between $(D \otimes_F \mathbb{A}_F^\infty)^\times$ and $\text{GL}_2(\mathbb{A}_F^\infty)$. By global class field theory, we may view $\psi = \chi \varepsilon$ as a character of $(\mathbb{A}_F^\infty)^\times / F^{\times}$. We also define a tame level $U_p = \prod_{v \nmid p} U_v$ as follows: $U_v = \text{GL}_2(O_{F_v})$ if $v \notin S$ and

$$U_v = \text{Iw}_v := \{g \in \text{GL}_2(O_{F_v}), g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \varpi_v\}$$

otherwise. For any $v \in S \setminus \Sigma_p$, the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \xi_v(\frac{a}{d} \mod \varpi_v)$ defines a character of $U_v$. The product of $\xi_v$ can be viewed as a character $\xi$ of $U^p$ by projecting to $\prod_{v \in S \setminus \Sigma_p} U_v$. As in the previous section, we denote $K_p = \prod_{v|p} \text{GL}_2(O_{F_v}), D_p^\times = \prod_{v|p} \text{GL}_2(F_v)$.

Using this, we can define a Hecke algebra $\mathbb{T} := \mathbb{T}_{\psi,\xi}(U^p)$ as in chapter 3.7. We also make the following assumption in this chapter (which defines a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$):

**Assumption:** $T_v - (1 + \chi(\text{Frob}_v)), v \notin S$ and $\varpi$ generate a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$. 64
4.1.3. By the discussion in 3.3.5, there is a natural pseudo-representation $T_m : G_{F,S} \to \mathbb{T}_m$ with determinant $\chi$ sending $\text{Frob}_v$ to $T_v$ for $v \notin S$. Let $R^{ps}$ be the universal object in $C_O$ which pro-represents the functor from $C^f_O$ to the category of sets sending $R$ to the set of two-dimensional pseudo-representations $T$ of $G_{F,S}$ over $R$ lifting $1 + \bar{\chi}$ with determinant $\chi$. By the universal property, there is a natural map $R^{ps} \to \mathbb{T}_m$, which is surjective since $\mathbb{T}_m$ is topologically generated by $T_v, v \notin S$.

We claim this map factors through $R^{ps,\{\xi_v\}}$, i.e. $T_m|_{I_{F_v}} = \xi_v + \xi_v^{-1}$ for $v \in S \setminus \Sigma_p$. If $\psi$ is of finite order, then this is a direct consequence of classical local-global compatibility at such $v$ at finite levels. In general, we can reduce to the previous case by twisting everything with a certain character (see for example the argument in 3.3.3).

We will say a prime $q \in \text{Spec} R^{ps,\{\xi_v\}}$ is modular if it comes from a prime of $\mathbb{T}_m$.

Using the construction for $R^{ps,\{\xi_v\}}$, we can define a two-dimensional semi-simple representation $\rho(p) : G_{F,S} \to GL_2(k(p))$ with trace $T_m \mod p$ for any $p \in \text{Spec} \mathbb{T}_m$.

**Definition 4.1.4.** Let $q$ be a prime ideal of $\mathbb{T}_m$ and $A$ be the normal closure of $\mathbb{T}_m/q$ in $k(q)$. We say $q$ is nice if $q$ contains $p$ and $\dim \mathbb{T}_m/q = 1$ and there exists a two-dimensional representation

$$\rho(q)^o : G_{F,S} \to GL_2(A)$$

satisfying the following properties:

1. $\rho(q)^o \otimes k(q) \cong \rho(q)$ is irreducible. In other words, $\rho(q)^o$ is a lattice of $\rho(q)$.

2. The mod $m_A$ reduction $\bar{\rho}_b$ of $\rho(q)^o$ is a non-split extension and has the form

$$\bar{\rho}_b(g) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad g \in G_{F,S}.$$

Here $m_A$ is the maximal ideal of $A$. 

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3. If $\rho(q)$ is dihedral, namely isomorphic to $\text{Ind}_{G_p}^{G_L} \theta$ for some quadratic extension $L$ of $F$ and continuous character $\theta : G_L \to k(q)^\times$, then $L \cap F(\zeta_p) = F$, where $\zeta_p \in \overline{F}$ is a primitive $p$-th root of unity.

4. $\rho(q)^o|_{G_{F_v}} = \bar{\rho}_b|_{G_{F_v}}$ for any $v \in S \setminus \Sigma_p$. Here we view $\text{GL}_2(F)$ as a subgroup of $\text{GL}_2(A)$ by the canonical embedding $F \to A$.

By abuse of notation, we say a prime $q^{ps} \in \text{Spec} R^{ps,\{\xi_v\}}$ is *nice* if it comes from a nice prime $q$ of $\mathbb{T}_m$ in the above sense.

**Remark 4.1.5.** This is different from the definition in the beginning of §6 of [39]. See the proof of lemma [4.3.11] below for an explanation.

Now we can state the main result of this section:

**Theorem 4.1.6.** Under the assumptions for $F, \chi$ as in this subsection, let $q \in \text{Spec} \mathbb{T}_m$ be a nice prime and $q^{ps} = q \cap R^{ps,\{\xi_v\}}$. If $p = 3$, we further assume $\bar{\chi}|_{G_{F_v}} \neq \omega$ for any $v|p$. Then the natural surjective map

$$(R^{ps,\{\xi_v\}})_{q^{ps}} \to \mathbb{T}_q$$

has nilpotent kernel.

**Corollary 4.1.7.** Under the same assumptions as in the previous theorem, let $p$ be a maximal ideal of $R^{ps,\{\xi_v\}}[\frac{1}{p}]$. Assume that

- For any $v|p$, $\rho(p)|_{G_{F_v}}$ is irreducible and de Rham with distinct Hodge-Tate weights.

- There exists an irreducible component of $R^{ps,\{\xi_v\}}$ containing both $p$ and a nice prime $q$.

Then $\rho(p)$ arises from a regular algebraic cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. 

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Proof. This follows directly from the previous theorem and corollary 3.5.12.

4.1.8. The rest of the chapter is to prove this theorem. Fix a nice $q$ and a choice of $\rho(q)^0$ as in the definition. We may assume $A$ has residue field $F$. If not, let $\mathcal{O}'$ be an unramified extension of $\mathcal{O}$ with the same residue field as $A$ and choose a prime ideal $q' \in \text{Spec} \mathbb{Z}_m \otimes \mathcal{O}'$ above $q$. Then the normal closure of $(\mathbb{Z}_m \otimes \mathcal{O}')/q'$ has residue field $F'$. Denote $q' \cap (R_{ps}(\xi_v) \otimes \mathcal{O}'_{q'})$ by $q'^{ps}$. Hence $(R_{ps}(\xi_v) \otimes \mathcal{O}')_{q'^{ps}} \to (\mathbb{Z}_m \otimes \mathcal{O}'_{q'})$ has nilpotent kernel by the theorem. Since the natural map $(R_{ps}(\xi_v))_{q^{ps}} \to (R_{ps}(\xi_v) \otimes \mathcal{O}'_{q'})$ is faithfully flat, this implies that $(R_{ps}(\xi_v))_{q^{ps}} \to \mathbb{T}_q$ has nilpotent kernel as well.

From now on, we fix an isomorphism $A \cong F[[T]]$.

The following lemma gives some sufficient conditions for the third condition in 4.1.4.

Lemma 4.1.9. Let $q$ be a prime ideal of $\text{Spec} R_{ps}(\xi_v)$ containing $p$ such that $R_{ps}(\xi_v)/q$ is one-dimensional. Suppose $\rho(q)$ is irreducible. Then the third condition in 4.1.4 holds for $q$ if one of the following conditions holds:

1. $\bar{\chi}$ is not quadratic. This includes the cases where $\bar{\chi}|_{G_{F_v}} = \omega$ or $\omega^{-1}$ for some $v|p$.

2. $\bar{\chi}|_{G_{F_v}} = 1$ for some $v|p$.

3. There exists a place $v|p$ such that $\bar{\chi}|_{G_{F_v}} \neq 1$ and $\rho(q)^0|_{G_{F_v}} \cong \begin{pmatrix} \chi_{v,1} & * \\ 0 & \chi_{v,2} \end{pmatrix}$ is reducible. Moreover $\chi_{v,1}/\chi_{v,2}$ is of infinite order, which is equivalent with saying $\chi_{v,1}$ is of infinite order as $\chi_{v,1}\chi_{v,2} = \bar{\chi}$ is of finite order.

Proof. Since the semi-simplification of the reduction of $\rho(q)$ is $1 + \bar{\chi}$, it is clear that if $\rho(q)$ is isomorphic to $\text{Ind}_{G_F}^{G_E} \theta$, then $\bar{\chi}$ is quadratic and $L = F(\bar{\chi}) = F^\ker(\bar{\chi})$. If $\bar{\chi}|_{G_{F_v}} = 1$ for some $v|p$, then $F(\bar{\chi}) \cap F(\mathcal{O}_p) = F$ as we assume $p$ completely splits in $F$. This proves the first two parts of the lemma.
As for the third part, suppose $\rho(q) \cong \text{Ind}_{G_{F(\chi)}}^{G_F} \theta$. Then by our assumption, $v$ is inert or ramified in $F(\bar{\chi})$. Hence $\rho(q)|_{G_{F_v}} \cong \text{Ind}_{G_{L_w}}^{G_{F_v}} \theta|_{G_{L_w}}$, where $w$ is the place above $v$ in $L$. Since $\rho(q)|_{G_{F_v}}$ is reducible, $\rho(q)|_{G_{L_w}} \cong \theta \oplus \theta$. This contradicts our assumption that $\chi_{v,1}/\chi_{v,2}$ is of infinite order. \qed

Remark 4.1.10. In view of the proof of lemma 4.1.9, it is enough to assume that $[F_v(\zeta_p) : F_v] = p - 1$ for any $v | p$ instead of that $p$ completely splits in $F$ as in 4.1.1.

4.2 Some local and global (framed) deformation rings

We introduce several universal lifting rings and recall some of their basic properties.

Definition 4.2.1. Let $v$ be a finite place of $F$.

- If $\bar{\rho}_b|_{G_{F_v}}$ is unramified, we define $R^{\square,ur}_v$ to be the universal object in $C_O$ that pro-represents the functor from $C^\text{fr}_O$ to the category of sets sending $R$ to the set of unramified liftings $\rho_R : G_{F_v}/I_{F_v} \to \text{GL}_2(R)$ of $\bar{\rho}_b|_{G_{F_v}}$ to $R$ with determinant $\chi$.

- If $v \in S \setminus \Sigma_p$, we define $R^{\square,\xi_v}_v$ to be the universal object in $C_O$ that pro-represents the functor from $C^\text{fr}_O$ to the category of sets sending $R$ to the set of liftings $\rho_R : G_{F_v} \to \text{GL}_2(R)$ of $\bar{\rho}_b|_{G_{F_v}}$ to $R$ with determinant $\chi$ such that

$$\text{tr}(\rho_R)|_{I_{F_v}} = \xi_v + \xi_v^{-1}.$$  

If $\xi_v$ is trivial, we write $R^{\square,\xi_v}_v$ as $R^{\square,1}_v$.

- We also define $R^{\square}_v$ to be the unrestricted (i.e. no condition on the liftings) universal lifting ring of $\bar{\rho}_b|_{G_{F_v}}$ with determinant $\chi$.  

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4.2.2. Recall that in the previous subsection, we defined $\rho(q) : G_{F,S} \to \text{GL}_2(A)$.

Here $A = \mathbb{F}[[T]]$ equipped with $T$-adic topology. Let $B$ be the topological closure of the $\mathbb{F}$-algebra generated by all the entries of $\rho(q) \circ (G_{F,S})$. By Chebotarev’s density theorem, we may find a finite set of primes $T'$ disjoint with $S$ such that the entries of $\rho(q) \circ (\text{Frob}_v), v \in T'$, topologically generate $B$. In fact, let $w \notin S$ be a place such that $\rho(q) \circ (\text{Frob}_w) \notin \text{GL}_2(\mathbb{F})$. We denote by $A'$ the topological closure of the $\mathbb{F}$-subalgebra in $A$ generated by the entries of $\rho(q) \circ (\text{Frob}_w)$. Then $B$ is a finite $A'$-module and by the ascending chain condition we can find such a finite set $T'$. We fix such a $T'$ from now on and denote

$$P = T' \cup S.$$  

For any Noetherian local $\mathcal{O}$-algebras $R_1, R_2$, we define their completed tensor product $R_1 \widehat{\otimes} \mathcal{O} R_2$ to be $\lim_{\leftarrow n} (R_1 / m_1^n \otimes \mathcal{O} R_2 / m_2^n)$, where $m_i$ is the maximal ideal of $R_i$. It follows from this definition that $R_1 \widehat{\otimes} \mathcal{O} R_2 \cong \widehat{R}_1 \widehat{\otimes} \mathcal{O} R_2$ and there is a natural map $\widehat{R}_1 \to R_1 \widehat{\otimes} \mathcal{O} R_2$, where $\widehat{R}_1$ is the $m_1$-adic completion of $R_1$. If $R_1$ has residue field $\mathbb{F}$, then $R_1 \widehat{\otimes} \mathcal{O} R_2$ is a complete Noetherian local $\mathcal{O}$-algebra as well (Lemma 1.3 of \cite{[43]}).

**Definition 4.2.3.** We define $R_{\text{loc}}^{\{\xi_v\}}$ to be

$$(\bigotimes_{v \in \Sigma_p} R_v^{\square}) \otimes (\bigotimes_{v \in S \setminus \Sigma_p} R_v^{\square, \xi_v}) \otimes (\bigotimes_{v \in T'} R_v^{\square, \text{ur}}),$$

where all the completed tensor products are taken over $\mathcal{O}$. By the universal property, $\rho(q)^{\circ}$ gives rise to a one-dimensional prime $q_{\text{loc}}^{\{\xi_v\}}$ of $R_{\text{loc}}^{\{\xi_v\}}$. Note that for any $v \in S \setminus \Sigma_p$, the pull-back of $q_{\text{loc}}^{\{\xi_v\}}$ to $R_v^{\square, \xi_v}$ is the maximal ideal of $R_v^{\square, \xi_v}$ by our assumption of $\rho(q)^{\circ}$.

The main result of this section is

**Proposition 4.2.4.** The $q_{\text{loc}}^{\{\xi_v\}}$-adic completion $(\widehat{R}_{\text{loc}}^{\{\xi_v\}})_{q_{\text{loc}}^{\{\xi_v\}}}$ of $(R_{\text{loc}}^{\{\xi_v\}})_{q_{\text{loc}}^{\{\xi_v\}}}$ is equidimensional of dimension $3[F : \mathbb{Q}] + 3|P|$. The generic point of each irreducible component has characteristic zero. Moreover,
1. If all $\xi_v$ are non-trivial, then $(R_{\text{loc}}^\wedge)_{q_{\text{loc}}}^{(\xi_v)}$ is integral.

2. In general, each minimal prime of $(R_{\text{loc}}^\wedge)_{q_{\text{loc}}}^{(\xi_v)}/(\varpi)$ contains a unique minimal prime of $(R_{\text{loc}}^\wedge)_{q_{\text{loc}}}^{(\xi_v)}$.

We first collect some useful results in commutative algebra (see also §1 of [43]).

**Lemma 4.2.5.** Let $R, S$ be complete Noetherian local $\mathcal{O}$-algebras with maximal ideals $m_R, m_S$ respectively. Suppose that $R/m_R = \mathbb{F}$ and $S$ is flat over $\mathcal{O}$. Then

1. The natural map $R \to R \hat{} \otimes_{\mathcal{O}} S$ is faithfully flat.

2. For any finitely generated $R$-module $M$, the $(m_R, m_S)$-adic completion $M \hat{} \otimes_{\mathcal{O}} S$ of $M \otimes_{\mathcal{O}} S$ as an $R \otimes S$-module is canonically isomorphic to its $m_R$-adic completion.

   Moreover there is a natural isomorphism:

   $$M \otimes_R (R \hat{} \otimes_{\mathcal{O}} S) \cong M \hat{} \otimes_{\mathcal{O}} S.$$  

   In particular, for any ideal $I$ of $R$, we have

   $$I(R \hat{} \otimes_{\mathcal{O}} S) \cong I \otimes_R (R \hat{} \otimes_{\mathcal{O}} S) \cong I \hat{} \otimes_{\mathcal{O}} S.$$  

3. For any ideal $p$ of $S$ such that $S/p$ is $\mathcal{O}$-flat, we have

   $$(R \hat{} \otimes_{\mathcal{O}} S)/(p) \cong R \hat{} \otimes_{\mathcal{O}} (S/p).$$

**Proof.** The flatness of $R \hat{} \otimes_{\mathcal{O}} S$ over $R$ is Lemma 1.3 of [43]. Also it is clear that $R \to R \hat{} \otimes_{\mathcal{O}} S$ is a local homomorphism, hence faithfully flat.

To prove the second part of the lemma, we note that $M/m_R^n M$ is of finite length as a $\mathcal{O}$-module for any $n > 0$. Thus $M/m_R^n M \otimes_{\mathcal{O}} S \cong \lim_{\leftarrow k} M/m_R^n M \otimes_{\mathcal{O}} S/m_S^k$. This proves the first assertion in the second part of the lemma. The second assertion is
trivially true if $M$ is a free $R$-module. The general case follows by writing $M$ as a finite presentation $R^\oplus r 	o R^\oplus s 	o M 	o 0$.

For the last part, by our assumption there is an exact sequence for any $n$:

$$0 	o R/m^n_R \otimes_O p \to R/m^n_R \otimes_O S \to R/m^n_R \otimes_O S/p \to 0.$$  

The inverse limit over $n$ remains exact. Now it suffices to show that

$$p(R \hat{\otimes}_OS) \to \varprojlim_n (R/m^n_R \otimes_O p)$$

is surjective. But this is clear as we can write $p$ as a quotient of $S^\oplus r$ for some $r$ and apply Theorem 8.1 of [29].

\begin{proof}
This is Lemma 1.5 of [43].
\end{proof}

**Lemma 4.2.6.** Let $R, S \in C_O$ with maximal ideals $m_R, m_S$ respectively. Let $p \in \text{Spec } R$ containing $p$ and $p' = (p, m_S) \in \text{Spec } R \hat{\otimes}_OS$. Then there is a canonical isomorphism

$$(R \hat{\otimes}_OS)_{p'} \cong \hat{R}_p \hat{\otimes}_OS.$$

\begin{proof}
This is Lemma 1.5 of [43].
\end{proof}

Later on, $S$ will be $(\bigotimes_{v \in S \setminus \Sigma_p} R_v^{\boxtimes \xi_v}) [[x_1, \ldots, x_g]]$ for some $g$ and $R$ will be $(\bigotimes_{v \in \Sigma_p} R_v^{\boxtimes}) \hat{\otimes} (\bigotimes_{v' \in \Sigma'} R_{v'}^{\boxtimes,ur})$.

**Lemma 4.2.7.** Let $S$ be a complete local Noetherian flat $O$-algebra of dimension $e$ and $R$ be a finitely generated $O$-algebra. Suppose that $S$ and $R$ satisfy the following conditions:

- $S \otimes_O O_L$ is integral for any finite extension $L/E$ with ring of integers $O_L$ (i.e. $S$ is geometrically integral over $O$). Moreover, we assume $S/pS \neq 0$.

- For each minimal prime $p$ of $R$, $R/p$ is $O$-flat of dimension $d + 1$ and $R/(p, \varpi)$ is generically reduced.

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Each minimal prime of \( R/(\wp) \) contains a unique minimal prime of \( R \).

Let \( m \) be a maximal ideal of \( R \) with residue field \( \mathbb{F} \). Then

1. If \( R \) is normal, then \( S \otimes_O R_m \) is integral of dimension \( d + e \).

2. For each minimal prime \( q \) of \( S \otimes_O R_m \), \( S \otimes_O R_m/q \) is \( O \)-flat of dimension \( d + e \).

3. Each minimal prime of \( (S \otimes_O R_m)/(\wp) \) contains a unique minimal prime of \( S \otimes_O R_m \).

Proof. Before giving the proof, we note the following useful fact: Let \( D \) be an excellent local ring and \( \hat{D} \) be its completion. Then \( D \to \hat{D} \) is regular and faithfully flat. Hence \( D \) is regular (normal, reduced) if and only if \( \hat{D} \) is regular (normal, reduced). See §32 of [29] for basic properties of excellent rings. In practice, all the rings below in this proof are excellent.

First we assume \( R \) is normal. Let \( m' \in \text{Spec} \ S \otimes_O R_m \) be the maximal ideal above \( m \). Then \( (S \otimes_O R_m)_{m'} \) is flat over \( R_m \). Hence

\[
\dim(S \otimes_O R_m)_{m'} = \dim S \otimes \mathbb{F} + \dim R_m = d + e.
\]

We conclude that \( \dim S \otimes_O R_m = d + e \) since taking completion preserves the dimension.

Note that \( S \otimes_O R_m \) is reduced (which is clear after inverting \( p \)) and excellent. \( S \otimes_O R_m \) is also reduced. It rests to prove that \( S \otimes_O R_m \) is irreducible.

For simplicity, we denote \( \text{Spec} S \otimes_O R_m \) by \( X \). By Corollary 10.5.8 of [19], the intersection of all maximal ideals of \( \hat{R}_m[1/p] \) is trivial. Let \( I \) be the set of pull-back of these maximal ideals to \( \hat{R}_m \). Then \( \bigcap_{p \in I} p = \{0\} \). Since \( \hat{R}_m \) is compact, for any \( n > 0 \), we may find a finite subset \( I_n \subseteq I \) such that \( \bigcap_{p \in I_n} p \subseteq m^n \hat{R}_m \). Let \( J = \bigcup_n I_n \). Then

\[
\bigcap_{p \in J} p(S \otimes_O \hat{R}_m) \subseteq \bigcap_n m^n(S \otimes_O \hat{R}_m) = \{0\}.
\]
In other words, $J$ is dense in $S \hat{\otimes}_O \widehat{R}_m$.

Now given two non-empty open sets $U_1, U_2$ of $X$, by the above equality, we can find $p_i \in J$ for $i = 1, 2$ such that $U_i \cap \text{Spec}(S \hat{\otimes}_O \widehat{R}_m)/(p_i)$ is non-empty. Note that

$$(S \hat{\otimes}_O \widehat{R}_m)/(p_i) \cong S \otimes_\mathcal{O} (\widehat{R}_m/p_i) \cong S \otimes_\mathcal{O} (\widehat{R}_m/p_i)$$

by Lemma \ref{4.2.5} and the fact that $\widehat{R}_m/p_i$ is a finite $\mathcal{O}$-domain. Hence $X_{p_i} \overset{\text{def}}{=} \text{Spec} S \otimes_\mathcal{O} k(p_i)$, the fibre of $X$ over $p_i$ in Spec $\widehat{R}_m$, is dense in Spec $S \otimes_\mathcal{O} (\widehat{R}_m/p_i)$. In particular, $X_{p_i} \cap U_i$ is a non-empty open set of $X_{p_i}$. From this and Corollary 10.5.8 of \cite{19} again, it is easy to see that we may find $q \in \text{Spec} S$ which is the pull-back of a maximal ideal of $S[\frac{1}{p}]$, such that $X_q \cap X_{p_i} \cap U_i$ is non-empty for $i = 1, 2$. Therefore, we only need to show $X_q$ is irreducible.

By the last part of Lemma \ref{4.2.5}, $X_q \cong \text{Spec}(S/q \hat{\otimes}_O \widehat{R}_m)[\frac{1}{p}]$. Let $\tilde{S}$ be the normal closure of $S/q$ in its fractional field. This is a complete discrete valuation ring with $\varpi$ contained in the maximal ideal. Fix a uniformizer $\lambda$ of $\tilde{S}$. Note that $\tilde{S}$ is a finite $S/q$-algebra since $S$ is excellent. It is easy to see that $(S/q \hat{\otimes}_O \widehat{R}_m)[\frac{1}{p}] \cong (\tilde{S} \hat{\otimes}_O \widehat{R}_m)[\frac{1}{p}] \cong (\tilde{S} \hat{\otimes}_O R_m)[\frac{1}{p}]$. We claim that $\tilde{S} \hat{\otimes}_O R_m$ is normal. Consequently, $X_q$ will be irreducible.

Let $m'$ be the unique maximal ideal of $\tilde{S} \hat{\otimes}_O R_m$ containing $m$. Note that $\tilde{S} \hat{\otimes}_O R_m$ is the $m'$-adically completion of $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}$. Hence to prove $\tilde{S} \hat{\otimes}_O R_m$ is normal, it suffices to show $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}$ is normal since $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}$ is excellent. We apply Serre’s criterion for normality: $(S_2)$ is true as $\tilde{S}$ is flat over $\mathcal{O}$. For $(R_1)$, note that the map $R_m[\frac{1}{p}] \to \tilde{S} \otimes_\mathcal{O} R_m[\frac{1}{p}]$, as a base change of $E \to \tilde{S}[\frac{1}{p}]$, is regular. Hence $\tilde{S} \otimes_\mathcal{O} R_m[\frac{1}{p}]$ is normal and any height one prime of it is regular. For height one prime of $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}$ containing $p$, since $R/(\varpi)$ is generically reduced, so is $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}/(\lambda) = (\tilde{S}/(\lambda) \otimes_F R_m/(\varpi))_{m'}$ as $\mathbb{F}$ is perfect. Thus any height one prime containing $\varpi$ is also regular. This proves that $(\tilde{S} \otimes_\mathcal{O} R_m)_{m'}$ is normal and finishes the proof of the first part of the lemma.
For the last two parts of the lemma, we follow the argument of the proof of lemma 2.7 in [42] (see also lemma 1.6 of [43]). First we note that for any finite extension $L/E$ with ring of integers $\mathcal{O}', S' = S \otimes_{\mathcal{O}} \mathcal{O}'$ and $R' = R \otimes_{\mathcal{O}} \mathcal{O}'$ also satisfy the assumptions in the lemma, with $\mathcal{O}$ replaced by $\mathcal{O}'$. Let $m' \in \text{Spec } R'$ be the unique maximal ideal above $m$. Using the isomorphism $(S \hat{\otimes}_{\mathcal{O}} R_m) \otimes_{\mathcal{O}} \mathcal{O}' \cong S' \hat{\otimes}_{\mathcal{O}} R'_m$, it is easy to see that it suffices to show the same assertions for $S', R', \mathcal{O}'$. Therefore, we may replace $\mathcal{O}$ by its extension if necessary.

We may assume $R$ is reduced. Denote the minimal primes of $R_m$ by $p_1, \cdots, p_r$ and $R_i = R_m/p_i$. Let $\hat{R}_i$ be the normalization of $R_i$ and $\hat{R} = \prod_i \hat{R}_i$ be the normalization of $R_m$. Hence $\hat{R}$ is a finite $R_m$-algebra. Let $Q$ be $R_m$-module $\hat{R}/R_m$. It follows from our assumptions that $(R_m)_{p_{i,j}}$ is a discrete valuation ring with maximal ideal $(\varpi)$ for any minimal prime $p_{i,j}$ of $R_m/(\varpi, p_i)$. Therefore $Q_{p_{i,j}} = 0$. Consider the exact sequence:

$$0 \rightarrow R_m \rightarrow \hat{R} \rightarrow Q \rightarrow 0.$$

and tensor it with $\hat{R}_m$ over $R_m$ (which is an exact functor):

$$0 \rightarrow \hat{R}_m \rightarrow \prod_{i,k} (\hat{R}_i)_{m_{i,k}} \rightarrow \hat{Q} \rightarrow 0,$$

where the product in the middle term is taken over all maximal ideals $m_{i,k}$ of $\hat{R}_i$ and $\hat{Q}$ is the $m$-adic completion of $Q$. Here we use the fact that the $m$-adic completion of $\hat{R}_i$ is isomorphic to $\prod_k (\hat{R}_i)_{m_{i,k}}$. Replace $E$ by some unramified extension if necessary.

We may assume $\hat{R}_i/m_{i,k} = \mathbb{F}$ for any $i,k$. Note that each term in the above short exact sequence is a finite $\hat{R}_m$-module. We can tensor it with $S \hat{\otimes}_{\mathcal{O}} \hat{R}_m$ (over $\hat{R}_m$) and apply lemma 4.2.5:

$$0 \rightarrow S \hat{\otimes}_{\mathcal{O}} \hat{R}_m \rightarrow \prod_{i,k} S \hat{\otimes}_{\mathcal{O}} (\hat{R}_i)_{m_{i,k}} \rightarrow S \hat{\otimes}_{\mathcal{O}} \hat{Q} \rightarrow 0.$$
Note that $(\hat{R}_i)_{m_{i,k}}$ is a normal local flat $\mathcal{O}$-algebra and $(\hat{R}_i)_{m_{i,k}}/(\varpi)$ is generically reduced as so is $R/(\varpi)$. We may apply the first part of the lemma to conclude that $S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}}$ is an integral domain with dimension $d + e$.

Let $q$ be a minimal prime ideal of $S \hat{\otimes}_\mathcal{O}\hat{R}_m/(\varpi)$. Then by going-down theorem, the pull-back of $q$ to $R_m/(\varpi)$ is also a minimal prime $p'$. Hence $(S \hat{\otimes}_\mathcal{O}\hat{Q})_q = 0$ since $Q_{p'} = 0$. In particular, we have

$$(S \hat{\otimes}_\mathcal{O}\hat{R}_m)_q \cong \prod_{i,k} (S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}})_q.$$ 

The left hand side is a local ring. Hence $(S \hat{\otimes}_\mathcal{O}\hat{R}_m)_q \neq 0$ for a unique pair $(i, k)$. This implies that $(S \hat{\otimes}_\mathcal{O}\hat{R}_m)_q$ is an integral domain. In other words, $q$ contains a unique minimal prime of $S \hat{\otimes}_\mathcal{O}\hat{R}_m$. This proves the third part of the lemma.

Now let $p$ be a minimal prime of $S \hat{\otimes}_\mathcal{O}\hat{R}_m$. Again by going-down theorem, its pull-back to $R_m$ defines a minimal prime and the same argument as in the previous paragraph shows that $(S \hat{\otimes}_\mathcal{O}\hat{R}_m)_p \cong (S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}})_p$ for a unique pair $(i, k)$. Hence $(S \hat{\otimes}_\mathcal{O}\hat{R}_m)/p$ maps injectively into $S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}}$. Note that $S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}}$ is a finite $S \hat{\otimes}_\mathcal{O}\hat{R}_m$-algebra. Thus

$$\dim(S \hat{\otimes}_\mathcal{O}\hat{R}_m)/p = \dim S \hat{\otimes}_\mathcal{O}(\hat{R}_i)_{m_{i,k}} = d + e.$$ 

The flatness of $(S \hat{\otimes}_\mathcal{O}\hat{R}_m)/p$ over $\mathcal{O}$ is also clear. This proves the second part of the lemma.

4.2.8. To prove proposition 4.2.4, we also need some basic properties of the local framed deformation rings defined in the beginning of this section here.

Lemma 4.2.9.

1. If $v \notin S$, then $\overline{\rho}_b|_{G_{F_v}}$ is unramified and $R_v^{\square,\text{ur}} \cong \mathcal{O}[[x_1, x_2, x_3]]$. 

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2. If \( v \mid p \), then \( R_v^{\Box} \) is either isomorphic to \( \mathcal{O}[[c_0, c_1, d_0, d_1, x_1, x_2, x_3]]/(c_0d_1 - c_1d_0) \) or formally smooth over \( \mathcal{O} \) of relative dimension 6. Hence \( R_v^{\Box} \) is a normal domain of dimension \( 1 + 6 \).

Proof. The first part is clear since we only need to lift \( \text{Frob}_v \). The second part is clear in the case \( \bar{\rho}_b \mid G_{F_v} \not\cong \left( \begin{array}{cc} 1 & * \\ 0 & \omega \end{array} \right) \otimes \eta \) for some character \( \eta \) as \( H^2(G_{F_v}, \text{ad}^0 \bar{\rho}_b) = 0 \) and \( R_v^{\Box} \) is smooth in this case. The computation of dimension is standard. If \( \bar{\rho}_b \mid G_{F_v} \) is a non-split extension of \( \omega \) by \( 1 \) up to some character, then it follows from Corollary B.5 of [32] that \( R_v^{\Box} \cong \mathcal{O}[[c_0, c_1, d_0, d_1, x_1, x_2, x_3]]/(c_0d_1 + c_1d_0 + pc_0) \) since \( R_v^{\Box} \) is formally smooth of relative dimension 3 over the universal deformation ring of \( \bar{\rho}_b \mid G_{F_v} \) with determinant \( \chi \). Note that we assume \( p > 3 \) in this case so that we can apply the quoted result. If \( \bar{\rho}_b \mid G_{F_v} \cong \eta \oplus \eta \omega \) for some \( \eta \), then its versal deformation ring \( R^{\text{ver}} \) is isomorphic to \( \mathcal{O}[[c_0, c_1, d_0, d_1, b]]/(c_0d_1 - c_1d_0) \) by Corollary 3.6, 3.7 of [23]. Note that in this case \( R_v^{\Box} \) is formally smooth of relative dimension 2 over \( R^{\text{ver}} \) by Proposition 2.1 of [21]. Hence we have exhausted all the cases of the second part of the lemma. \( \square \)

Lemma 4.2.10. For \( v \in S \setminus \Sigma_p \), there exists a finite type \( \mathcal{O} \)-algebra \( A^{\xi_v} \) and an \( \mathbb{F} \)-valued point \( x \in \text{Spec} \ A^{\xi_v} \) such that \( \hat{(A^{\xi_v})}_x \cong R_v^{\Box, \xi_v} \). Moreover,

1. If \( \xi_v \) is trivial, then for each minimal prime \( p \) of \( A^{\xi_v} \), \( A^{\xi_v}/p \) is \( \mathcal{O} \)-flat of dimension \( 3 + 1 \) and \( A^{\xi_v}/(p, \varpi) \) is generically reduced. Each minimal prime of \( A^{\xi_v}/(\varpi) \) contains a unique minimal prime of \( A^{\xi_v} \).

2. If \( \xi_v \) is non-trivial, then
   - \( R_v^{\Box, \xi_v} \) is irreducible of dimension \( 1 + 3 \) and flat over \( \mathcal{O} \).
   - \( R_v^{\Box, \xi_v}/(\varpi) \cong R_v^{\Box, 1}/(\varpi) \).
   - \( R_v^{\Box, \xi_v}[\frac{1}{p}] \) and \( (A^{\xi_v})_x[\frac{1}{p}] \) are regular.
   - Let \( S \) be a complete local Noetherian flat \( \mathcal{O} \)-algebra such that \( S/pS \neq 0 \) and \( S[\frac{1}{p}] \) is geometrically connected and geometrically normal, i.e. \( S[\frac{1}{p}] \otimes_{E} L \) is...
connected and normal for any finite extension $L/E$. Then $(S \hat{\otimes}_\mathcal{O} R_v^{\xi_v})[\frac{1}{p}]$ is geometrically connected and geometrically normal as well. The Krull dimension of $S \hat{\otimes}_\mathcal{O} R_v^{\xi_v}$ is $\dim S + 3$.

Proof. For any $R \in C_\mathcal{O}$, since $\bar{\rho}_b|_{G_{F_v}}$ has unipotent images, any lifting $\rho_R : G_{F_v} \to \text{GL}_2(R)$ factors through the pro-$p$ quotient $I_{F_v}(p)$ of $I_{F_v}$ when restricted to the inertia. Choose a topological generator $t$ of $I_{F_v}(p)$. Then $\rho_R$ is determined by the pair of matrices $(\rho_R(\text{Frob}_v), \rho_R(t))$. Hence $R_v^{\xi_v}$ represents the functor sending $R \in C_\mathcal{O}^t$ to the pair of $2 \times 2$ matrices $(\Phi, \Sigma)$ that lifts $(\bar{\rho}_b(\text{Frob}_v), \bar{\rho}_b(t))$ satisfying

- $\Phi \Sigma \Phi^{-1} = \Sigma^{N(v)}$

- $\Sigma$ has characteristic polynomial $(X - \xi_v(t))(X - \xi_v^{-1}(t))$.

- $\det \Phi = \chi(\text{Frob}_v)$.

See §3 of [42] for more details. Then we can take $\text{Spec } A^{\xi_v}$ to be the moduli space (over $\mathcal{O}$) of pair of matrices $(\Phi, \Sigma)$ satisfying the above conditions and $x$ to be the maximal ideal given by $(\bar{\rho}_b(\text{Frob}_v), \bar{\rho}_b(t))$.

If $\xi_v$ is trivial, then all the assertions follow from the first part of lemma 3.1 of [42]. Note that $\mathcal{M}((X - 1)^2, \chi(v))$ defined there is isomorphic to the spectrum of polynomial ring of one variable over $A^{\xi_v}$ since we require the determinant of $\Phi$ to be $\chi(\text{Frob}_v)$.

Now we assume $\xi_v$ is trivial. The isomorphism between $R_v^{\xi_v}/(\varpi)$ and $R_v^{\xi_v,1}/(\varpi)$ is clear since they represent the same deformation problem. For other assertions, note that the natural map $(A^{\xi_v})_x[\frac{1}{p}] \to R_v^{\xi_v}[\frac{1}{p}]$ is regular. Hence $R_v^{\xi_v}[\frac{1}{p}]$ is regular if and only so is $(A^{\xi_v})_x[\frac{1}{p}]$. Moreover, if this holds, all the maps in $S[\frac{1}{p}] \to S \otimes_E (A^{\xi_v})_x[\frac{1}{p}] \to (S \hat{\otimes}_\mathcal{O} R_v^{\xi_v})[\frac{1}{p}]$ would be regular. Thus $R_v^{\xi_v}[\frac{1}{p}]$ being regular implies that $(S \hat{\otimes}_\mathcal{O} R_v^{\xi_v})[\frac{1}{p}]$ is normal. Similar result holds for base change to finite extension of $E$. 

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For any finite extension $L/E$ with ring of integers $\mathcal{O}_L$, we may replace everything by its tensor with $\mathcal{O}_L$ over $\mathcal{O}$. Hence the connectedness of $(S \hat{\otimes}_\mathcal{O} R_v^{\square, \xi_v})[\frac{1}{p}]$ will imply that it is geometrically connected. If $\bar{\rho}|_{G_{F_v}}$ is trivial, in this case all the claims follow from proposition 3.15, 3.16 of [43]. If $\bar{\rho}|_{I_{F_v}}$ is non-trivial, then $R_v^{\square, \xi_v} \cong \mathcal{O}[[x_1, x_2, x_3]]$ by proposition 5.8 (1) of [38] and all the assertions are clear. If $\bar{\rho}|_{G_{F_v}}$ is trivial but $\bar{\rho}|_{G_{F_v}}$ is non-trivial, then $R_v^{\square, \xi_v} \cong \mathcal{O}[[x_1, x_2, x_3, x_4]]/(x_1^2 x_2 - c^2)$ with $c = \xi_v(t) - \xi_v(t)^{-1}$ (proposition 5.8 (2) of [38]). Clearly we are left to show $(S \hat{\otimes}_\mathcal{O} R_v^{\square, \xi_v})[\frac{1}{p}]$ is connected.

Consider the homomorphism
\[
\varphi : S[[x_1, x_2, x_3, x_4]]/(x_1^2 x_2 - c^2) \longrightarrow S[[x_1, y_2, x_3, x_4]]/(x_1 y_2 - c)
\]

sending $x_2$ to $y_2^2$. It is easy to see that this map is surjective after inverting $p$. Moreover if $f(x_1, x_2) \in S[[x_3, x_4]][[x_1, x_2]]$ is in the kernel of the map $\varphi$ i.e. $f(x_1, y_2^2) = g(x_1, y_2)(x_1 y_2 - c)$ for some $g(x_1, y_2) \in S[[x_3, x_4]][[x_1, y_2]]$, then
\[
f(x_1, y_2)^2 = -g(x_1, y_2)g(x_1, -y_2)(x_1 y_2^2 - c^2).
\]

But $g(x_1, y_2)g(x_1, -y_2) = h(x_1, y_2^2)$ for some $h(x_1, y_2) \in S[[x_3, x_4]][[x_1, y_2]]$. Hence $f(x_1, x_2)^2 = 0$ in $S[[x_1, x_2, x_3, x_4]]/(x_1^2 x_2 - c^2)$. Thus $\varphi$ has nilpotent kernel after inverting $p$ and it suffices to show that $\text{Spec}(S[[x_1, y_2, x_3, x_4]]/(x_1 y_2 - c))[\frac{1}{p}]$ is connected. This can be proved by the same argument as in the first part of the proof of lemma 4.2.7 since $\mathcal{O}[[x_1, y_2, x_3, x_4]]/(\varpi, x_1 y_2 - c)$ is generically reduced.

The dimension of $S \hat{\otimes}_\mathcal{O} R_v^{\square, \xi_v}$ is easy to compute as it is flat over $R_v^{\square, \xi_v}$. We remark that the local deformation problems considered in all these references have no condition on the determinants of the liftings. Hence their deformation rings have one more formal variable than our deformation rings.

Now we can prove proposition 4.2.4.

**Proof of proposition 4.2.4.** We denote $(\bigotimes_{v \in \Sigma_p} R_v^{\square}) \hat{\otimes} (\bigotimes_{v \in T_v} R_v^{\square, ur})$ by $R_1$ and the pull-back of $\mathfrak{q}_{\text{loc}}^{\{\xi_v\}}$ to $R_1$ by $\mathfrak{q}_1$. It follows from the explicit descriptions in lemma 4.2.9 that
$R_1$ is normal, hence $(\widehat{R_1})_{q_1}$, as a completion of $(R_1)_{q_1}$, is also normal. Moreover, using our assumption in the beginning of 4.1.8 we see that for any finite extension $L/E$ with ring of integers $\mathcal{O}_L$, $(\widehat{R_1})_{q_1} \otimes \mathcal{O}_L$ is a normal local ring. In particular, $(\widehat{R_1})_{q_1}[[\frac{1}{p}]]$ is geometrically normal and geometrically connected.

Let $\Sigma_1$ (resp. $\Sigma_2$) be the set of $v \in S \setminus \Sigma_p$ such that $\xi_v$ is non-trivial (resp. trivial). We may apply the second part of lemma 4.2.10 repeatedly and conclude that $(\widehat{R_1})_{q_1} \otimes \bigotimes_{v \in \Sigma_1} R_v^{[\xi_v]}[[\frac{1}{p}]]$ is geometrically normal and geometrically connected. Clearly $S_1 = (\widehat{R_1})_{q_1} \otimes \bigotimes_{v \in \Sigma_1} R_v^{[\xi_v]}$ is flat over $\mathcal{O}$ as $R_v^{[\xi_v]}$ and $R_1$ are all flat $\mathcal{O}$-algebras. Hence $S_1$ is geometrically integral. This proves the first part of the proposition. The dimension of $S_1$ is $3[F:Q] + 3|P| - 3|\Sigma_2|$. For each $v \in \Sigma_2$, we let $X_v^1$ be the finite type $\mathcal{O}$-algebra $A_v^\xi$ defined in lemma 4.2.10 and $x_v \in \text{Spec } X_v^1$ be the $x$ there. Then in order to apply 4.2.7 with $R = X, S = S_1$ and $m$ given by the product of all $x_v$, it suffices to show that $X = \bigotimes_{v \in \Sigma_2} X_v^1$ (over $\mathcal{O}$) satisfies:

- For each minimal prime $p$ of $X$, $X/p$ is $\mathcal{O}$-flat of dimension $1 + 3|\Sigma_2|$ and $X/(\varpi, p)$ is generically reduced.

- Each minimal prime of $X/(\varpi)$ contains a unique a minimal prime of $X$.

To see this, we may assume $X_v^1$ is reduced. Then $X$ is also reduced and $\mathcal{O}$-flat and we are left to show that

- $X$ is equidimensional of dimension $1 + 3|\Sigma_2|$.

- For each height one prime $p$ of $X$ with $\varpi \in p$, $X_p$ is a DVR with maximal ideal $(\varpi)$.

Let $p$ be a height one prime ideal containing $\varpi$ and $p_v$ be its pull-back to $X_v^1$. Then $p_v$ is a minimal prime of $X_v^1/(\varpi)$. In particular, $X_p$ is a localization of $\bigotimes_{v \in \Sigma_2} (X_v^1)_{p_v}$. By our assumption, $(X_v^1)_{p_v}$ is a DVR with maximal ideal $(\varpi)$. Hence the homomorphism
\( \mathcal{O} \to (X_v^1)_{p_v} \) is regular and an easy induction argument shows that \( \bigotimes_{v \in \Sigma_2} (X_v^1)_{p_v} \) is a regular ring. Therefore \( X_p \) is regular. Note that \( \bigotimes_{v \in \Sigma_2} (X_v^1)_{p_v}/(\varpi) \) is reduced as it is the tensor product of \( k(p_v) \) over \( \mathbb{F} \). This implies that \( X_p/(\varpi) \) is reduced. Hence \( X_p \) is a DVR with maximal ideal \( (\varpi) \).

Since \( p \) can be viewed as a minimal prime of \( \bigotimes_{v \in \Sigma_2} X_v^1/p_v \), it is easy to see that \( X/p \) has dimension \( \sum_{v \in \Sigma_2} \dim X_v^1/p_v = 3|\Sigma_2| \). Hence \( X \) is equidimensional of dimension \( 1 + 3|\Sigma_2| \) as \( X \) is \( \mathcal{O} \)-flat. This finishes the proof of proposition 4.2.4.

4.2.11. We also need to define some global framed deformation rings with certain local conditions.

Definition 4.2.12. Suppose \( M, Q \) are finite sets of primes of \( F \).

1. We define \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \) to be the universal object in \( C_\mathcal{O} \) pro-representing the functor \( \text{Def}_{\rho_b, Q}^{\square M, \{ \xi_v \}} \) from \( C_\mathcal{O} \) to the category of sets sending \( R \) to the set of tuples \( (\rho_R; \alpha_v)_{v \in M} \) modulo the equivalence relation \( \sim_M \) where
   
   - \( \rho_R : G_{F,S,\cup Q} \to \text{GL}_2(R) \) is a lifting of \( \bar{\rho}_b \) to \( R \) with determinant \( \chi \) such that \( \text{tr}(\rho_R)|_{I_{F_v}} = \xi_v + \xi_v^{-1} \) for any \( v \in S \setminus \Sigma_p \).
   
   - \( \alpha_v \in 1 + M_2(\mathfrak{m}_R), v \in M \). Here \( \mathfrak{m}_R \) is the maximal ideal of \( R \).
   
   - \( (\rho_R; \alpha_v)_{v \in M} \sim_M (\rho'_R; \alpha'_v)_{v \in M} \) if there exists an element \( \beta \in 1 + M_2(\mathfrak{m}_R) \) with \( \rho'_R = \beta \alpha_v \beta^{-1}, \alpha'_v = \beta \alpha_v \) for any \( v \in M \).

We define \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \) in the same way without the local conditions \( \text{tr}(\rho_R)|_{I_{F_v}} = \xi_v + \xi_v^{-1} \) for any \( v \in S \setminus \Sigma_p \). If \( M \) is empty, we will drop the superscript \( \square M \) in \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \) and \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \). If all \( \xi_v \) are trivial, we will simply write \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \) for \( R_{\rho_b, Q}^{\square M, \{ \xi_v \}} \).

2. We define \( R_Q^{\text{des}, \{ \xi_v \}} \) to be the universal deformation ring which pro-represents the functor from \( C_\mathcal{O}^f \) to the category of sets sending \( R \) to the set of two-dimensional pseudo-representations \( T \) of \( G_{F,S,\cup Q} \) over \( R \) such that \( T \) is a lifting of \( 1 + \bar{\chi} \) with
determinant \( \chi \) and

\[ T|_{I_{F_v}} = \xi_v + \xi_{v}^{-1} \]

for any \( v \in S \setminus \Sigma_p \). If \( \xi_v \) are all trivial, we will simply write \( R_Q^{\text{ps,1}} \).

We also define \( R_Q^{\text{ps}} \) in the same way without any condition for \( v \in S \setminus \Sigma_p \).

4.2.13. Suppose \( M \supseteq S \). Given a tuple \( (\rho_R; \alpha_v)_{v \in M} \) as in the above deformation, then for \( v \in M \), \( \alpha_v^{-1}\rho_R|_{G_{F_v}} \) is a well-defined lifting of \( \bar{\rho}_b|_{G_{F_v}} \). This induces natural maps \( R_{\bar{\rho}_b, Q}^{\square} \rightarrow R_M^{\square} \{ \xi_v \} \) and \( R_{\bar{\rho}_b, Q}^{\square} \rightarrow R_M^{\square} \{ \xi_v \} \). It is easy to see that

\[ R_M^{\square} \{ \xi_v \} \cong R_{\bar{\rho}_b, Q}^{\square} \otimes (\otimes_{v \in S \setminus \Sigma_p} R_{\bar{\rho}_b, Q}^{\square}) \otimes (\otimes_{v \in S \setminus \Sigma_p} R_{\bar{\rho}_b, Q}^{\square}). \]

Note that our local deformation constraints are all defined via the traces. So we can rewrite the above isomorphism as:

\[ R_{\bar{\rho}_b, Q}^{\square} \{ \xi_v \} \cong R_{\bar{\rho}_b, Q}^{\square} \otimes R_Q^{\text{ps}} \{ \xi_v \}, \]

where the map \( R_Q^{\text{ps}} \rightarrow R_M^{\square} \) is given by evaluating the trace of the universal lifting.

4.3 Existence of Taylor-Wiles primes

4.3.1. We will freely use the notation introduced in 4.2.2. In 4.2.3 we defined \( R_{\text{loc}}^{\{ \xi_v \}} \) to be

\[ (\otimes_{v \in \Sigma_p} R_v^{\square}) \otimes (\otimes_{v \in S \setminus \Sigma_p} R_v^{\square} \{ \xi_v \}) \otimes (\otimes_{v \in T_v} R_v^{\square} \{ \xi_v \}), \]

where all the completed tensor products are taken over \( \mathcal{O} \). Then there is a natural map

\[ R_{\text{loc}}^{\{ \xi_v \}} \rightarrow R_{\bar{\rho}_b, Q}^{\square} \{ \xi_v \} \]
for any finite set of primes $Q$. By the universal property, $(\rho(q)^0; 0)_{q \in P}$ gives rise to a prime $q_{b,Q}$ of $R_{\rho_b,Q}^{\square_P,\{\xi_v\}}$, whose pull-back to $R_{\rho_b}^{\{\xi_v\}}$ is $q_{\text{loc}}^{\{\xi_v\}}$. Note that by our choice of $T'$, we have $B = R_{\rho_b}^{\{\xi_v\}} / q_{b,Q} = R_{\text{loc}}^{\{\xi_v\}} / q_{\text{loc}}^{\{\xi_v\}}$.

For any $A$-module $M$ of finite length, we denote its length by $\ell(M)$. Now we can state the main result of this subsection. See also Proposition 6.10 of [39].

**Proposition 4.3.2.** Let $r = \dim_k (q) H^1(G_{F,P}, \text{ad}^0 \rho(q)(1))$, where $\text{ad}^0 \rho(q)$ denotes the trace 0 subspace of the adjoint representation $\text{ad} \rho(q)$ of $\rho(q)$ and (1) denotes the Tate twist. Then there exists an integer $C$ such that for any positive integer $N$, we can find a finite set of primes $Q_N$ disjoint with $P$ such that

1. $|Q_N| = r$.

2. $N(v) \equiv 1 \mod p^N$ for any $v \in Q_N$.

3. $\rho(q)(\text{Frob}_v)$ has distinct eigenvalues $\alpha_v, \beta_v$ with $\ell(A/(\alpha_v - \beta_v)^2) < C$ for any $v \in Q_N$.

4. There exists an $A$-module $M_N$ with $\ell(M_N) < C$ such that

$$ q_{b,Q_N} / (q_{b,Q_N}^{2}, q_{\text{loc}}^{\{\xi_v\}}) \otimes_B A \cong A^g \oplus M_N $$

as $A$-modules, where $g = r + |P| - [F : Q] - 1$.

5. There exists a map $R_{\text{loc}}^{\{\xi_v\}} [[x_1, \cdots, x_g]] \to R_{\rho_b,Q_N}^{\square_P,\{\xi_v\}}$ such that the images of $x_i$ are in $q_{b,Q_N}$ and $q_{b,Q_N} / (q_{b,Q_N}^{2}, q_{\text{loc}}^{\{\xi_v\}}, x_1, \cdots, x_g)$ is killed by some element $f \in B$ with $\ell(A/(f)) < C$.

**4.3.3.** Note that the last part is a direct consequence of the previous one since $T^M A \subseteq B$ for some $M > 0$ by our construction. Therefore it suffices to prove the existence of $Q_N$ satisfying the first four properties.

The following result is standard.
Lemma 4.3.4. Let $Q$ be a finite set of primes disjoint with $P$ such that $N(v) \equiv 1 \mod p$ and $\rho(q)^v(Frob_v)$ has distinct eigenvalues $\alpha_v, \beta_v$ for any $v \in Q$. Denote $q_{b, Q}/(q_{b, Q}^2, q_{\text{loc}}^{\xi_v}) \otimes_B A/(T^n)$ by $V_n$. Then

$$\ell(V_n) = \ell(H^1_{\mathcal{L}_{\mathfrak{p}, Q}}(G_F, W_n^*)) - \ell(H^0(G_F, W_n^*) + gn + \sum_{v \in Q} \ell(A/(T^{2n}, (\alpha_v - \beta_v)^2)),$$

where

- $W_n = \text{ad}^0 \rho(q)^v \otimes A/(T^n)$ and $W_n^* = \text{Hom}(W_n, \mathbb{F})(1)$.
- $H^1_{\mathcal{L}_{\mathfrak{p}, Q}}(G_F, W_n^*), W_n^* = \ker(H^1(G_F, W_n^*) \to \bigoplus_{v \in Q} H^1(G_{F_v}, W_n^*))$.
- $g = r + |P| - [F: \mathbb{Q}] - 1$ as defined in the proposition.

Proof. We give a sketch of proof here. By Pontryagin duality, $\ell(V_n)$ is equal to the length of

$$\text{Hom}_A(V_n, A/(T^n)) = \text{Hom}_B(q_{b, Q}/(q_{b, Q}^2, q_{\text{loc}}^{\xi_v}), A/(T^n)).$$

Since $R^{\xi_v}_{\text{loc}} + q_{b, Q} = R^{\text{alg}}_{\mathfrak{p}, Q}$, any element $h_0$ in $\text{Hom}_B(q_{b, Q}/(q_{b, Q}^2, q_{\text{loc}}^{\xi_v}), A/(T^n))$ can be uniquely extended to a $R^{\xi_v}_{\text{loc}}$-algebra homomorphism

$$h : R^{\text{alg}}_{\mathfrak{p}, Q} \to B \oplus (A/T^n)\epsilon$$

sending $x \in q_{b, Q}$ to $h_0(x)\epsilon$. Here the ring structure of $B \oplus (A/T^n)\epsilon$ is given by

$$(b + ae)(b' + a'd') = bb' + (ab' + a'b)\epsilon$$

and we view it as a $R^{\xi_v}_{\text{loc}}$-algebra by $R^{\xi_v}_{\text{loc}} \to R^{\xi_v}_{\text{loc}}/q_{\text{loc}}^{\xi_v} = B \hookrightarrow B \oplus (A/T^n)\epsilon$. Note that by construction, $h(q_{b, Q}) \subseteq (A/T^n)\epsilon$. In fact, this induces an isomorphism between $\text{Hom}_B(q_{b, Q}/(q_{b, Q}^2, q_{\text{loc}}^{\xi_v}), A/(T^n))$ and

$$\{h \in \text{Hom}_{R^{\xi_v}_{\text{loc}} \otimes_B} (R^{\text{alg}}_{\mathfrak{p}, Q}, B \oplus (A/T^n)\epsilon), h(q_{b, Q}) \subseteq (A/T^n)\epsilon\}.$$

By the universal property, this is the subset $\{[\rho, \alpha_v]_{v \in P}\}$ of $\text{Def}^{\text{alg}}_{\mathfrak{p}, Q}(B \oplus (A/T^n)\epsilon)$ with
• $\alpha_v \in \ker(GL_2(B \oplus (A/T^n)\epsilon) \to GL_2(B)), v \in P$.

• $\rho \mod (A/T^n)\epsilon = \rho(q)^o$ and $\alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v = \rho(q)^o|_{G_{F_v}}$ for $v \in P$.

Write $\rho(\sigma) = (1 + \phi(\sigma)\epsilon)\rho(q)^o(\sigma)$, $\alpha_v = 1 + m_v \epsilon$. It is easy to check that $\phi : G_{F,P \cup Q} \to \text{ad}^0 \rho(q)^o \otimes A/T^n$ defines a 1-cocycle in $Z^1(G_{F,P \cup Q}, \text{ad}^0 \rho(q)^o \otimes A/T^n)$ with $\phi|_{G_{F_v}} = \partial m_v$ for $v \in P$. Conversely, any $(\rho; \alpha_v)_{v \in P}$ arises from such a 1-cocycle and $m_v$. Using this, it is not hard to see that

$$\ell(V_n) = \sum_{v \in P} \ell(H^0(G_{F_v}, W_n)) + \ell(H^1_{L_{P \cup Q}}(G_F, W_n)) - \ell(H^0(G_{F,P \cup Q}, W_n)) + (|P| - 1)n,$$

where $H^1_{L_{P \cup Q}}(G_F, W_n) = \ker(H^1(G_{F,P \cup Q}, W_n)) \xrightarrow{\text{res}} \bigoplus_{v \in P \cup Q} H^1(G_{F_v}, W_n)/L_v$ and $L_v = 0$ if $v \in P$ and $L_v = H^1(G_{F_v}, W_n)$ otherwise. By Poitou-Tate long exact sequence and global Euler characteristic formula (see Theorem 2.19 of [11]), we may compute

$$\ell(H^1_{L_{P \cup Q}}(G_F, W_n)) = \ell(H^0(G_F, W_n)) - \ell(H^0(G_F, W_n^*)) + \ell(H^1_{L_{P \cup Q}}(G_F, W_n^*))$$

$$+ \sum_{v \in P \cup Q} (\ell(L_v) - \ell(H^0(G_{F_v}, W_n))) - \sum_{v|\infty} \ell(H^0(G_{F_v}, W_n)),$$

where $H^1_{L_{P \cup Q}}(G_F, W_n^*)$ and $W_n^*$ were defined in the lemma. By local Euler characteristic formula and local Tate duality, for $v \in Q$, it follows from our assumption on $v$ that

$$\ell(L_v) - \ell(H^0(G_{F_v}, W_n)) = \ell(H^0(G_{F_v}, W_n^*)) = \ell(H^0(G_{F_v}, W_n)) + n + \ell(A/(T^{2n}, (\alpha_v - \beta_v)^2)).$$

For any $v|\infty$, by our oddness assumption,

$$\ell(H^0(G_{F_v}, W_n)) = n.$$

The lemma is a direct consequence of all these formulae. \qed
4.3.5. Note that the perfect pairing $W_n \times W_n \to \mathbb{A}/T^n, (X,Y) \mapsto \text{tr}(XY)$ induces an isomorphism $W_n(1) \cong W_n^*$. It follows from the third condition in the definition of nice prime (4.1.4) that $\ell(H^0(G_F, W_n(1)))$ is bounded independent of $n$. Thus in view of the previous lemma, proposition 4.3.2 follows from

**Lemma 4.3.6.** There exists an integer $C$ such that for any $N > 0$, we can find a finite set of primes $Q_N$ disjoint with $P$ such that the first three parts of proposition 4.3.2 hold and

$$\ell(H^1_{L_{P\cup Q}}(G_F, W_n^*)) < C$$

for any $n$.

4.3.7. Let $Q$ be a finite set of primes given by lemma 4.3.4. Combine the following two exact sequences:

$$0 \to H^1_{L_{P\cup Q}}(G_F, W_n^*) \to H^1(G_F, P\cup Q, W_n^*) \to \bigoplus_{v \in Q} H^1(G_{F_v}, W_n^*),$$

$$0 \to H^1(G_F, W_n^*) \to H^1(G_F, P\cup Q, W_n^*) \to \bigoplus_{v \in Q} H^1(G_{F_v}, W_n^*)/H^1(G_{k(v)}, W_n^*).$$

We get that

$$0 \to H^1_{L_{P\cup Q}}(G_F, P\cup Q, W_n^*) \to H^1(G_F, P, W_n^*) \to \bigoplus_{v \in Q} H^1(G_{k(v)}, W_n^*),$$

where the last map is the restriction map. We note that $H^1(G_{k(v)}, W_n^*) \cong W_n^*/(\text{Frob}_v - 1)W_n^*$.

**Lemma 4.3.8.** Let $F(\zeta_p^{\infty}) = \bigcup_n F(\zeta_p^n)$, where $\zeta_p^n$ are primitive $p^n$-th roots of unity. Denote its absolute Galois group by $G_{F(\zeta_p^{\infty})} \subset G_{F}$. Then there exists $\sigma_1, \ldots, \sigma_r \in G_{F(\zeta_p^{\infty})}$ such that

- $\rho(q)(\sigma_i)$ has distinct eigenvalues for each $i$. 

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• The kernel and cokernel of the map \( H^1(G_{F,P}, W^*) \to \bigoplus_{i=1}^r W^*/(\sigma_i - 1)W^* \) have bounded lengths as \( A \)-modules, where \( W^* \) denotes \( \text{Hom}_F(\text{ad}^0 \rho(q)^o, F) \cong \text{ad}^0 \rho(q)^o(1) \).

4.3.9. Before giving the proof of this lemma, we first show that this implies lemma 4.3.6. Given a positive integer \( N \), we may choose places \( v_1, \ldots, v_r \) of \( F \) with \( \text{Frob}_{v_i} \in G_{F(\zeta_p^N)} \) close enough to \( \sigma_i \) so that \( (\sigma_i - 1)W^* = (\text{Frob}_{v_i} - 1)W^* \). Let \( Q_N = \{v_1, \ldots, v_r\} \). Consider the following diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(G_{F,P}, W^*)/T_n & \longrightarrow & \bigoplus_{i=1}^r H^1(G_{k(v)}, W^*)/T_n \\
& \downarrow & & \downarrow & \downarrow \\
& & 0 & \longrightarrow & H^1(G_{F,P,Q_N}, W_n^*) & \longrightarrow & H^1(G_{F,P}, W_n^*) & \longrightarrow & \bigoplus_{i=1}^r H^1(G_{k(v)}, W_n^*) \\
& & & \downarrow & & \downarrow & \downarrow \\
& & & & & H^2(G_{F,P}, W^*)^{\text{tor}} \\
\end{array}
\]

Here \( H^2(G_{F,P}, W^*)^{\text{tor}} \) denotes the \( T^\infty \)-torsion part of \( H^2(G_{F,P}, W^*) \). From this, it is easy to see that \( H^1(G_{F,P,Q_N}, W_n^*) \) are bounded independent of \( n \).

Now it suffices to prove the lemma below, which implies lemma 4.3.8 by induction on \( \sigma_i \).

**Lemma 4.3.10.** Fix an inclusion \( A^r \hookrightarrow H^1(G_{F,P}, W^*) \). For any \([\phi] \in A^r \setminus TA^r \) (viewed as an element in \( H^1(G_{F,P}, W^*) \)), there exists an element \( \sigma \in G_{F(\zeta_p^\infty)} \) such that \( \rho(q)(\sigma) \) has distinct eigenvalues and

\[
\ell(W^*/((\sigma - 1)W^* + A\phi(\sigma))) < +\infty.
\]

**Proof.** Let \( G_{F,P} = \ker(\varepsilon) \cap \ker(\rho(q)) \subseteq G_{F,P} \) and \( \Gamma = G_{F,P}/G_{F,P} \). Consider the following short exact sequence given by Hochschild-Serre spectral sequence:

\[
0 \to H^1(\Gamma, W^*) \to H^1(G_{F,P}, W^*) \to H^1(G_{F,P}, W^*)^\Gamma (= \text{Hom}_\Gamma(G_{F,P}, W^*)).
\]

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We quote the following result which is mainly due to Skinner-Wiles:

**Lemma 4.3.11.** $\ell(H^1(\Gamma, W^*)) < \infty$.

**Proof.** This is lemma 6.9 of [39] if $\rho(q)$ is not dihedral. Note that their paper requires $\rho(q)|G_{F_v} \cong \left( \begin{array}{cc} \chi_{v,1} & * \\ 0 & \chi_{v,2} \end{array} \right)$ with $\chi_{v,1}/\chi_{v,2}$ of infinite order for each $v|p$. However this assumption is only used to establish their lemma 6.1, which holds here as we assume $\rho(q)$ is irreducible and not dihedral.

If $\rho(q)$ is isomorphic to $\text{Ind}_{G_{F_v}}^{G}\theta$, then from the proof of lemma 4.1.9 we know that $\bar{\chi}$ is quadratic and $L = F(\bar{\chi})$. Hence $\rho(q)|G_L \cong \theta \oplus \theta^{-1}$. Therefore the only possible non-trivial abelian quotient of $\text{im}(\rho(q))$ is $\bar{\chi}$, where $\text{im}(\rho(q))$ denotes the image of $G_F$ in $\text{GL}_2(k(q))$ under $\rho(q)$. Since $L \cap F(\zeta_p) = F$, we have $F^{\ker(\rho(q))} \cap F(\zeta_p) = F$ and $\Gamma \cong \text{im}(\rho(q)) \times \text{Gal}(F(\zeta_p)/F)$. It is clear that $H^1(\Gamma, W^*) = 0$ as $\text{Gal}(F(\zeta_p)/F)$ acts non-trivially on $W^* \cong \text{ad}^0 \rho(q)^o(1)$.

Back to the proof of lemma 4.3.10, we first deal with the case that $\rho(q)$ is not dihedral. A direct consequence of the previous lemma is that $\phi|_{G_{F_v,p}} \in \text{Hom}_F(G_{F_v,p}, W^*)$ is non-zero. Since $\rho(q)$ is irreducible and not dihedral, $W^* \otimes k(q)$ is an irreducible representation of $\Gamma$ and the $A$-module generated by $\phi(G_{F,v})$ contains $T^C W^*$ for some $C > 0$.

Choose $\sigma_0 \in G_{F(\zeta_p)}$ such that $\rho(q)(\sigma_0)$ has distinct eigenvalues. This is possible, otherwise $\rho(q)(G_{F(\zeta_p)})$ would be a pro-$p$ group and $\rho(q)|_{G_{F(\zeta_p)}}$ would be reducible, which contradicts our third condition in 4.1.4. Choose $\tau \in G_{\tilde{F},p}$ so that

$$\ell(W^*/(A(\phi(\tau) + \phi(\sigma_0)) + (\sigma_0 - 1)W^*)) < +\infty.$$ 

It is clear that $\sigma = \sigma_0\tau$ satisfies all the requirements in the lemma. This proves lemma 4.3.10 and hence proposition 4.3.2 in the non-dihedral case.
Now assume that we are in the dihedral case, i.e. $\rho(q) \cong \text{Ind}_{G_p}^{G_L} \theta$. In view of the proof in the non-dihedral case, it is enough to show that for any non-zero irreducible subrepresentation $V$ of $W^* \otimes k(q)$ (viewed as a representation of $\Gamma$), we can find an element $\sigma_0 \in G_{F(\zeta_p \infty)}$ such that $\rho(q)(\sigma_0)$ has distinct eigenvalues and $V$ contains the $\sigma_0$-invariant subspace $(W^*)^{\sigma_0=1}$.

By our discussion in the second paragraph of the proof of lemma 4.3.11, $L = F(\bar{\chi})$ and $F^{\ker(\rho(q))} \cap F(\zeta_p \infty) = F$. It is clear that $W^* \cong \bar{\chi}(1) \oplus (\text{Ind}_{G_p}^{G_L} \theta^2)(1)$. For $\sigma_0 \in G_{F(\zeta_p \infty)}$ such that $\theta(\sigma_0) \neq 1$, we have $(W^*)^{\sigma_0=1} = \bar{\chi}(1)$. For $\sigma_0 \in G_{F(\zeta_p \infty)} \setminus G_{L(\zeta_p \infty)}$, we have $(W^*)^{\sigma_0=1}$ contained in $(\text{Ind}_{G_p}^{G_L} \theta^2)(1)$. Hence this finishes the proof of lemma 4.3.10. □

4.4 Completed homologies with auxiliary levels

4.4.1. In this subsection, we study the completed homologies with certain auxiliary levels. For any positive integer $N$, we fix a set of primes $Q_N$ as in proposition 4.3.2.

Denote the unique quotient of $k(v)^{\times}$ of order $p^N$ by $\Delta_v$ and $\bigoplus_{v \in Q_N} \Delta_v$ by $\Delta_N$.

Recall that $U^p = \prod_{v \nmid p} U_v$ where $U_v = \text{GL}_2(O_{F_v})$ if $v \notin S$ and $U_v = \text{Iw}_v$ otherwise.

Define tame levels $U^p_{Q_N,0} \subseteq U^p_{Q_N,v} = \prod_{v \nmid p} U_{Q_N,v}$ as follows:

- $U_{Q_N,0,v} = U_{Q_N,v} = U_v$ if $v \notin Q_N$.
- $U_{Q_N,0,v} = \text{Iw}_v$ if $v \in Q_N$.

- $U_{Q_N,v} = \ker(\text{Iw}_v \to \Delta_v)$ if $v \in Q_N$, where the map is the composite of $\text{Iw}_v \to \text{Iw}_v \to k(v)^{\times}$

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a}{d} \mod \varpi_v \quad \text{and the natural quotient map } k(v)^{\times} \to \Delta_v.
\]

We can define Hecke algebra $\mathbb{T}_{\psi,\xi}(U^p_{Q_N})$, $\mathbb{T}_{\psi,\xi}(U^p_{Q_N,0})$ as before. By abuse of notation, we also view $\mathfrak{m}$ as maximal ideals of these Hecke algebras.

It is clear that $\Delta_v = \text{Iw}_v/U_{Q_N,v}$ acts naturally on the completed homology $M_{\psi,\xi}(U^p_{Q_N})$ and $S_{\psi,\xi}(U^p_{Q_N} U_p, \mathcal{O}/\varpi^n)$ via the right translation of $\text{Iw}_w$ for any open
compact subgroup $U_p \subseteq K_p$ and positive integer $n$. Hence all these spaces are $O[\Delta_N]$-modules. Let $a_{Q_N}$ be the augmentation ideal of $O[\Delta_N]$.

**Lemma 4.4.2.** Suppose $U_p \subseteq K_p$ is an open compact subgroup such that $\psi|_{U_p \cap \Omega^\times}$ is trivial modulo $\varpi^n$ for some $n$ and $U^p U_p$ is sufficiently small. Then

1. $S_{\psi, \xi}(U^p_{Q_N} U_p, O/\varpi^n)$ and $S_{\psi, \xi}(U^p_{Q_N} U_p, O/\varpi^n)^\vee$ are finite flat $O/\varpi^n[\Delta_N]$-modules.

2. The natural map $S_{\psi, \xi}(U^p_{Q_N} U_p, O/\varpi^n)^\vee \to S_{\psi, \xi}(U^p_{Q_N,0} U_p, O/\varpi^n)^\vee$ induces a natural isomorphism:

$$S_{\psi, \xi}(U^p_{Q_N} U_p, O/\varpi^n)^\vee / a_{Q_N} S_{\psi, \xi}(U^p_{Q_N} U_p, O/\varpi^n)^\vee \cong S_{\psi, \xi}(U^p_{Q_N,0} U_p, O/\varpi^n)^\vee.$$

The same results hold with everything localized at $m$.

**Proof.** This is the Pontryagin dual of Lemma 2.1.4 of [26].

**Corollary 4.4.3.** $M_{\psi, \xi}(U^p_{Q_N})$ is a flat $O[\Delta_N]$-module and

$$M_{\psi, \xi}(U^p_{Q_N})/a_{Q_N} M_{\psi, \xi}(U^p_{Q_N}) \cong M_{\psi, \xi}(U^p_{Q_N,0}).$$

**Proof.** This follows from the lemma below.

**Lemma 4.4.4.** Let $R$ be a local complete Noetherian ring.

1. Suppose $\{M_i\}_{i \in N}$ is a projective system of flat $R$-modules. Then $M = \varprojlim M_i$ is also flat over $R$ and $M/JM \cong \varprojlim M_i/JM_i$ for any ideal $J$ of $R$.

2. Let $I$ be an ideal of $R$ and $\{N_i\}_{i \in N}$ be a projective system of $R$-modules such that $N_i$ is a flat $R/I^i$-module and the natural transition maps induce isomorphisms $N_i \cong N_{i+1}/I^i N_{i+1}$. Then $N = \varprojlim N_i$ is a flat $R$-module and $N_i \cong N/I^i N$. Moreover for any ideal $J$ of $R$, $N/JN \cong \varprojlim N_i/JN_i$. 

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Proof. These results are presumably well-known. We sketch a proof here. To prove
the flatness in the first part, it suffices to show $\text{Tor}_1^R(M, k) = 0$. Let $C_\bullet \to k$ be
a resolution of finite free $R$-modules. By our assumptions, $C_\bullet \otimes M_i \to k \otimes M_i$ is
exact and the projective limit of these exact sequences remains exact. Note that
$C_\bullet \otimes M \cong \varprojlim (C_\bullet \otimes M_i)$. This proves the flatness of $M$. The rest of the first part can
be proved by the same argument.

As for the second part, we first prove $N/JN \cong \varprojlim N_i/JN_i$ for any ideal $J$ of $R$. In particular, $N/I^iN \cong N_i$ for any $i$. Then the flatness of $N$ follows from Theorem
22.3 of [29]. Consider the following exact sequences

$\text{Tor}_1^R(R/J, N_i) \to J \otimes N_i \to N_i \to N_i/JN_i \to 0.$

We claim that the projective limit of these sequences over $i$ is still exact. It suffices
to check the Mittag-Leffler condition for $\{\text{Tor}_1^R(R/J, N_i)\}_i$. For any $k > i$, we have the diagram

$\begin{array}{ccc}
\text{Tor}_1^R(R/J, N_k) & \to & \text{Tor}_1^R(R/J, N_i) \\
\uparrow & & \| \\
\text{Tor}_1^R(R/J, N_{k+1}) & \to & \text{Tor}_1^R(R/J, N_i) \\
\end{array}
\xrightarrow{\alpha_1} \xrightarrow{\text{Tor}_1^R(R/J, N_i) \to I^iN_k/JI^iN_k} \xrightarrow{\alpha_2} \xrightarrow{N_k/JN_k} \xrightarrow{\text{Tor}_1^R(R/J, N_{k+1}) \to I^iN_{k+1}/JI^iN_{k+1}} \xrightarrow{N_{k+1}/JN_{k+1}}$.

To show the image of $\text{Tor}_1^R(R/J, N_k)$ in $\text{Tor}_1^R(R/J, N_i)$ stabilizes, it suffices to prove
the natural map between $\ker \alpha_1$ and $\ker \alpha_2$ is an isomorphism. Note that $\ker \alpha_1 = I^kN_{k+1}/(JI^iN_{k+1} \cap I^kN_{k+1})$ and $\ker \alpha_2 = I^kN_{k+1}/(JN_{k+1} \cap I^kN_{k+1})$. We only need to show that

$\text{Tor}_1^R(R/J, N_i) \to J \otimes N_i \to N_i \to N_i/JN_i \to 0.$

for $k$ large enough. There is a natural injective map from the LHS to the RHS. It is
enough to show that this is also surjective. Since $N_{k+1}$ is flat over $R/I^{k+1}$, we have
\[ JN_{k+1} \cap I^k N_{k+1} = (\bar{J} \cap I^k / I^{k+1}) \otimes_R N_{k+1}, \text{ where } \bar{J} \text{ is the image of } J \text{ in } R/I^{k+1}. \] Hence \( JN_{k+1} \cap I^k N_{k+1} \) is a quotient of \((J \cap I^k) \otimes_R N_{k+1}\). Similarly, \( JI^k N_{k+1} \cap I^k N_{k+1} \) is a quotient of \((JI^k \cap I^k) \otimes N_{k+1}\). By Artin-Rees lemma, \( JI^k \cap I^k = J \cap I^k \) for \( k \) large enough. This proves the equality (4.1).

Thus we have \( \lim_{\leftarrow} (J \otimes N_i) \to N \to \lim_{\leftarrow} (N_i/JN_i) \to 0 \) exact. Now it suffices to prove the natural map \( J \otimes N \to \lim_{\leftarrow} (J \otimes N_i) \) is surjective. But this is clear as \( J \) is a finitely presented \( R \)-module.

4.4.5. Let \( P(Q_N) \) be the power set of \( Q_N \). Now we consider the following map:

\[
\eta'_N : S_{\psi, \xi}(U^p U_p, \mathcal{O}/\varpi^n)^{P(Q_N)} \to S_{\psi, \xi}(U^p_{Q_N, 0} U_p, \mathcal{O}/\varpi^n) \quad \text{where} \quad (f_X)_{X \in P(Q_N)} \mapsto \sum_X \prod_{v \in X} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \cdot f_X,
\]

where \( \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \cdot f_X \) denotes the right translation of \( f_X \) by \( \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \). If \( X \) is the empty set, we set \( \prod_{v \in X} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \cdot f_X = f_X \). It is clear that this map commutes with the action of the Hecke operators. Hence we may localize this map at \( m \) and denote it by \( \eta_N \).

Recall that there is a map \( R^\text{ps}_{Q_N} \to T_{\psi, \xi}(U^p_{Q_N, 0})_m \) sending \( T^\text{univ}(\text{Frob}_v) \) to \( T_v \) for \( v \notin S \cup Q_N \), which factors through \( R^\text{ps}_{Q_N} \{ \xi_v \} \). Here \( T^\text{univ} : G_{F, S \cup Q_N} \to R^\text{ps}_{Q_N} \) is the universal trace.

**Proposition 4.4.6.** There exists a constant \( C \) and elements \( \tilde{f}_N \in R^\text{ps}_{Q_N} \{ \xi_v \} \) that satisfy the following properties: for any \( N \) and any open pro-\( p \) subgroup \( U_p \subseteq K_p \) such that \( \psi|_{N_{D/F}(U_p)} \) is trivial modulo \( \varpi^n \) and \( U^p U_p \) is sufficiently small, we have

1. \( \tilde{f}_N \) kills the kernel and cokernel of \( \eta_N \).
2. The image \(\tilde{f}_N\) of \(f_N\) in \(R_{Q_N}^{\text{bs},\{\xi\}}\rightarrow \mathbb{T}\psi,\xi(U_{Q_N,0}^p)_m \rightarrow \mathbb{T}\psi,\xi(U^p)_m \mod q\) is non-zero and the length \(\ell(A/(\tilde{f}_N)) < C\).

Proof. Define \(\tilde{f}_N\) to be \(\theta_{Q_N}^r\), where \(r\) is defined in proposition \(4.3.2\) \(\theta_{Q_N} = \prod_{w \in Q_N} \theta_w\) and

\[
\theta_w := (1 + N(w))^2 T^\text{univ}(Frob_w^2) - (1 + N(w)^2) T^\text{univ}(Frob_w)^2.
\]

It rests to check all the desired properties of \(\tilde{f}_N\).

**Kernel of \(\eta_N\):** We first identify the kernel of \(\eta_N\), following the proof of Lemma 2 of [13]. Let \((f_X)_{X \in P(Q_N)} \in \ker(\eta_N)\) and \(w\) be a place in \(Q_N\). Define

\[
f_1 = \sum_{X \ni w} \prod_{v \in X} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \cdot f_X,
\]

\[
f_2 = \sum_{X \not\ni w} \prod_{v \in X} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \cdot f_X.
\]

Since \(f_X\) is right invariant by \(U_w = \text{GL}_2(O_{F_w})\), it is clear that \(f_1 = -f_2\) is right invariant by \(\text{GL}_2(O_{F_w})\) and \(\begin{pmatrix} 1 & 0 \\ 0 & \pi_w \end{pmatrix}\). Hence by strong approximation theorem, \(f_1\) and \(f_2\) are both right invariant by \((D \otimes A_{F}^\infty)^{\times, \det = 1}\). Repeating the same argument, we can show that each \(f_X\) is right invariant by \((D \otimes A_{F}^\infty)^{\times, \det = 1}\). Hence \(f_X\) factors through the reduced norm map. Using this description, it is easy to check that any \(\theta_w\) kills the kernel.

**Cokernel of \(\eta_N\):** We will only show that \(\tilde{f}_N\) kills the cokernel of \(\eta_N\) in the case \(|Q_N| = 1\). The general case follows by induction on the number of primes in \(Q_N\).
Write $Q_N = \{w\}$. We want to show that the cokernel of

$$S_{\psi, \xi}(U^p U_p, \mathcal{O}/ \varpi^n)^{\oplus 2} \to S_{\psi, \xi}(U^p \{w\}, 0 U_p, \mathcal{O}/ \varpi^n)_m$$

$$(f_1, f_2) \mapsto f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi w \end{pmatrix} \cdot f_2$$

is killed by $\theta_3^w$.

Let $\tilde{\psi}$ be the Teichmüller lifting of $\psi \mod \varpi$ and write $\tilde{\psi} = \psi \theta^2$ for some continuous character $\theta : (\mathbb{A}^\infty_F)^\times / F_{>0}^\times \to \mathcal{O}^\times$ which is trivial on the kernel of $\psi$. By twisting with $\theta$ and arguing as in 3.3.3, it suffices to prove the case $\psi = \tilde{\psi}$ (note that the action of $\theta_w$ only differs by a unit in $\mathcal{O}^\times$).

Assume $\psi = \tilde{\psi}$ from now on. Then $\psi|_{U_{p,0}^\infty F_{p}}$ is trivial. Hence the natural map

$$S_{\psi, \xi}(U^p U_p, \mathcal{O})_m \to S_{\psi, \xi}(U^p U_p, \mathcal{O}/ \varpi^n)_m$$

and the similar map for $U^p \{w\}, 0 U_p$ are surjective. Thus it is enough to shows that the cokernel of

$$j : S_{\psi, \xi}(U^p U_p, \mathcal{O})^{\oplus 2} \to S_{\psi, \xi}(U^p \{w\}, 0 U_p, \mathcal{O})_m$$

$$(f_1, f_2) \mapsto f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \varpi w \end{pmatrix} \cdot f_2$$

is killed by $\theta_3^w$. Let $U$ be either $U^p U_p$ or $U^p \{w\}, 0 U_p$. We may define a perfect pairing:

$$\langle \cdot, \cdot \rangle_U : S_{\psi, \xi}(U, \mathcal{O})_m \times S_{\psi, \xi}(U, \mathcal{O})_m \to \mathcal{O}$$

by

$$\langle \varphi_1, \varphi_2 \rangle_U := \sum_{g \in D^\times \backslash (D \otimes F \mathbb{A}_F^\times)^\times / U(D^\times F \mathbb{A}_F^\times)} \varphi_1(g) \varphi_2(g) \psi(N_{D/F}(g))^{-1}$$

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for \( \varphi_1, \varphi_2 \in S_{\psi, \xi}(U, \mathcal{O})_m \). It is easy to check that \( \langle T_v \cdot \varphi_1, \varphi_2 \rangle_U = \langle \varphi_1, T_v \cdot \varphi_2 \rangle_U \) for \( v \notin S \cup Q \).

For simplicity, we write \( S_0, S_1 \) for \( S_{\psi, \xi}(U^p U_p, \mathcal{O})_m, S_{\psi, \xi}(U^p \{w\}, U_p, \mathcal{O})_m \). Elements in these spaces can be viewed as automorphic forms on \( D^\times \). Let \( j^+ : S_1 \to S_0^{\oplus 2} \) be the adjoint map of \( j \) and \( \tilde{j}^+ \) be the composite of \( S_1 \xrightarrow{j^+} S_0^{\oplus 2} \to S_0^{\oplus 2}/\ker j \). Consider the following diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & S_0^{\oplus 2}/\ker j & \overset{j}{\longrightarrow} & S_1 & \longrightarrow & \text{coker } j & \longrightarrow & 0 \\
& & \downarrow{\tilde{j}^+ \circ j} & & \downarrow{j^+} & & \downarrow & & \\
0 & \longrightarrow & S_0^{\oplus 2}/\ker j & \longrightarrow & S_0^{\oplus 2}/\ker \tilde{j} & \longrightarrow & 0.
\end{array}
\]

Then the snake lemma gives us a short exact sequence:

\[
\text{ker } \tilde{j}^+ \to \text{coker } j \to \text{coker } (\tilde{j}^+ \circ j).
\]

Note that the image of \( j \) contains exactly the automorphic forms whose corresponding automorphic representation is unramified at \( w \). Since the pairings are Hecke-equivariant, the automorphic representations generated by elements in \( \ker j^+ \) are either Steinberg at \( w \) or factor through the reduced norm map. In either case, the associated Galois representation at \( w \) is of the form \( \begin{pmatrix} \theta \varepsilon & * \\ 0 & \theta \end{pmatrix} \) for some unramified character \( \theta \). A direct computation shows that \( \theta_w \) kills \( \ker j^+ \). Hence \( \theta_w^2 \) kills \( \ker \tilde{j}^+ \).

We claim that \( \theta_w \) kills \( \text{coker}(\tilde{j}^+ \circ j) \). Combined with the previous paragraph, this will imply \( \theta_w^3 \) kills \( \text{coker } j \). Since \( \text{coker}(\tilde{j}^+ \circ j) \) is a quotient of \( \text{coker}(j^+ \circ j) \), it suffices to prove that \( \theta_w \) kills \( \text{coker}(j^+ \circ j) \). A direct computation shows that \( j^+ \circ j : S_0^{\oplus 2} \to S_0^{\oplus 2} \) is

\[
\begin{pmatrix}
N(w) + 1 & T_w \\
T_w & (N(w) + 1)\psi(\varpi_w)
\end{pmatrix}.
\]

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Thus coker\((j^+ \circ j)\) is killed by the determinant of this matrix. Using \(T^{univ}(\text{Frob}_w) = T_w\) and \(T^{univ}(\text{Frob}_w^2) = T_w^2 - 2\psi(\pi_w)N(w)\) on \(S_0\), it is easy to see that this determinant is simply \((-2N(w))^{-1}\theta_w\). Hence \(\theta_w\) kills coker\((j^+ \circ j)\). This finishes the proof of the first part of the proposition.

We still need to understand the image of \(\tilde{f}_N\) in \(A\). Let \(\pi_N\) be the map \(\mathcal{T}_\psi,\xi(U_{p, Q_N, 0})_m \to \mathcal{T}_\psi,\xi(U_p)_m \to \mathcal{T}_\psi,\xi(U_p)_{m \mod q} \to A\) and \(\alpha_w, \beta_w\) be the eigenvalues of \(\rho(q)\)(Frob\(_w\)) with \(w \in Q_N\). Then \(\pi_N(T^{univ}(\text{Frob}_w^i)) = \alpha_w^i + \beta_w^i\). Hence using the assumption \(N(w) \equiv 1 \mod p\), we have

\[
\pi_N(\theta_w) = (1 + N(w))^2(\alpha_w^2 + \beta_w^2) - (1 + N(w)^2)(\alpha_w + \beta_w)^2 = 2(\alpha_w - \beta_w)^2.
\]

Therefore it follows from our choice of \(Q_N\) that \(\ell(A/(\pi_N(\theta_w)))\) is uniformly bounded for any \(N, w\). This proves the second part of the proposition.

4.4.7. The usual patching argument requires a Galois-theoretic interpretation of the action of \(O[\Delta_{Q_N}]\) on \(M_{\psi, \xi}(U_{p, Q_N})_m\). We will only do it for a subring of \(O[\Delta_{Q_N}]\). More precisely, let \(O[\Delta_{Q_N}]' \subseteq O[\Delta_{Q_N}]\) be the \(O\)-subalgebra generated by elements of the form \(g + g^{-1}\) with \(g \in \Delta_v, v \in Q_N\). We may think \(O[\Delta_{Q_N}]'\) as a subring of \(\text{End}(M_{\psi, \xi}(U_{p, Q_N})_m)\).

**Proposition 4.4.8.** \(O[\Delta_{Q_N}]'\) is contained in the image of

\[
R^{ps, (\xi_v)}_{Q_N} \to \mathcal{T}_{\psi, \xi}(U_{Q_N})_m \to \text{End}(M_{\psi, \xi}(U_{p, Q_N})_m).
\]

Moreover let \(a'_{Q_N} = a_{Q_N} \cap O[\Delta_{Q_N}]'\). Then \(a'_{Q_N}\) is contained in the image of \(\ker(R^{ps, (\xi_v)}_{Q_N} \to R^{ps, (\xi_v)})\) in \(\text{End}(M_{\psi, \xi}(U_{p, Q_N})_m)\).

**Proof.** Consider the natural map \(I_{F_v} \to k(v) \times \to \Delta_v\) given by the local class field theory. For any \(\sigma_0 \in \Delta_v\), we choose a lifting \(\sigma \in I_{F_v}\). It suffices to show that \(T^{univ}(\sigma)\) acts as \(\sigma_0 + \sigma_0^{-1}\) on \(M_{\psi, \xi}(U_{p, Q_N})_m\). By definition, we only need to check this for the
action of $T^{\text{univ}}(\sigma)$ on $S_{\psi,\xi}(U_{Q_N}^p U_p, \mathcal{O}/\varpi^n)_m$ where $U_p$ is an open subgroup of $K_p$ such that $\psi|_{U_p \cap \mathcal{O}_{F_p}^\times} \mod \varpi^n$ is trivial and $U_{Q_N}^p U_p$ is sufficiently small.

Since the formulation commutes with twisting with a character unramified at $v$, we may argue as in the proof of proposition 4.4.6 and assume $\psi$ is equal to the Teichmüller lifting of its mod $\varpi$ reduction. Now it suffices to show that the action of $T^{\text{univ}}(\sigma)$ on $S_{\psi,\xi}(U_{Q_N}^p U_p, \mathcal{O})_m$ is equal to $\sigma_0 + \sigma_0^{-1}$. But this is a consequence of the local-global compatibility result at $v$: if $f \in S_{\psi,\xi}(U_{Q_N}^p U_p, \mathcal{O})_m$ generates an automorphic representation, then the local representation at $v$ is either Steinberg or a principal series since it has $U_{Q_N,v}$-fixed vectors (see for example Proposition 14.3 of [7]). In both cases, the desired result is clear.

\section{4.5 Patching I: patched completed homologies}

\subsection{4.5.1.} We summarize what we have done so far. For any $N$, we have

- a finite set of primes $Q_N$ with cardinality $r$ given by proposition 4.3.2
- $R_{Q_N}^{\text{ps}}(\xi_v) \rightarrow \mathcal{T}_{\psi,\xi}(U_{Q_N}^p) \rightarrow \text{End}_{E_{D^\times_p,v}(\mathcal{O})} (M_{\psi,\xi}(U_{Q_N}^p)_m)$, where $D^\times_p = \prod_{v \mid p} \text{GL}_2(F_v)$.
- $\Delta_{Q_N} = \prod_{v \in Q_N} \Delta_v$ and $\Delta_v$ is the unique quotient of $k(v)^\times$ of order $p^N$.
- $\mathcal{O}[\Delta_{Q_N}] \hookrightarrow \text{End}_{E_{D^\times_p,v}(\mathcal{O})} (M_{\psi,\xi}(U_{Q_N}^p)_m)$ that makes $M_{\psi,\xi}(U_{Q_N}^p)_m$ into a flat $\mathcal{O}[\Delta_{Q_N}]$-module. Also the image of $\mathcal{O}[\Delta'_{Q_N}]$ is contained in the image of $R_{Q_N}^{\text{ps}}(\xi_v)$.
- Elements $\tilde{f}_N \in R_{Q_N}^{\text{ps}}(\xi_v)$ and natural maps $\eta_N$ (see proposition 4.4.6).

We also set:

- $\Delta_{\infty} = Z_{p}^{\oplus r}$. Fix surjective maps $\mathbb{Z}_p \rightarrow \Delta_v$ for all $v \in Q_N$ and thus surjective maps $\Delta_{\infty} \rightarrow \Delta_{Q_N}$. 

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\( \mathcal{O}_\infty = \mathcal{O}[[y_1, \cdots, y_{4|P|-1}]] \) with maximal ideal \( b \) and prime ideal \( b_1 = (y_1, \cdots, y_{4|P|-1}) \).

\( S_\infty = \mathcal{O}_\infty[[\Delta_\infty]] \cong \mathcal{O}_\infty[[s_1, \cdots, s_r]] \). This is a local \( \mathcal{O}_\infty \)-algebra with maximal ideal \( a \).

\( \mathfrak{a}_0 = \ker(S_\infty \to \mathcal{O}_\infty) = (s_1, \cdots, s_r) \) the augmentation ideal, and \( \mathfrak{a}_1 = (\mathfrak{a}_0, b_1) = (y_1, \cdots, y_{4|P|-1}, s_1, \cdots, s_r) \). Hence \( a = (b) + \mathfrak{a}_0 \) and \( \mathfrak{a}_1 = (b_1) + \mathfrak{a}_0 \).

\( S'_\infty \subseteq \mathcal{O}_\infty[[\Delta_\infty]] \) is the closure (under the profinite topology) of the \( \mathcal{O}_\infty \)-subalgebra generated by all elements of the form \( g + g^{-1} \) with \( g = (0, \cdots, 0, a, 0, \cdots, 0) \in \Delta_\infty \) for some \( a \in \mathbb{Z}_p \). This is a regular local \( \mathcal{O}_\infty \)-algebra and \( S_\infty \) is a finite free \( S'_\infty \)-algebra.

\( \mathfrak{a}'_0 = \mathfrak{a}_0 \cap S'_\infty \), \( \mathfrak{a}'_1 = \mathfrak{a}_1 \cap S'_\infty \). We may find \( r \) elements \( s'_1, \cdots, s'_r \) that generate \( \mathfrak{a}'_0 \) and \( S'_\infty \cong \mathcal{O}_\infty[[s'_1, \cdots, s'_r]] \).

\textbf{4.5.2.} Following [36], we will use the language of ultrafilters to define patched completed homologies (see also [18]). This language seems to be essential here. For example, I don’t know how to rewrite the arguments below in the classical language (as in [39]).

Let \( \mathcal{I} \) be the set of positive integers and \( \mathbf{R} = \prod_{\mathcal{I}} \mathcal{O} \). From now on we fix a non-principal ultrafilter \( \mathfrak{F} \) on \( \mathcal{I} \). Then \( \mathfrak{F} \) gives rise to a multiplicative set \( S_\mathfrak{F} \subseteq \mathbf{R} \) consisting of all idempotents \( e_I \) with \( I \in \mathfrak{F} \) where \( e_I(i) = 1 \) if \( i \in I \), \( e_I(i) = 0 \) otherwise.

We define \( \mathbf{R}_{\mathfrak{F}} = S_\mathfrak{F}^{-1} \mathbf{R} \). This is a quotient of \( \mathbf{R} \) as all the elements in \( S_\mathfrak{F} \) are idempotents. Taking tensor product with \( \mathbf{R}_{\mathfrak{F}} \) over \( \mathbf{R} \) is an exact functor. Since \( \mathfrak{F} \) is non-principal, for any finite set \( T \subseteq \mathcal{I} \), we have \( \mathbf{R}_T \otimes_{\mathbf{R}} \mathbf{R}_{\mathfrak{F}} \cong \mathbf{R}_{\mathfrak{F}} \), where \( \mathbf{R}_T \) is the quotient of \( \mathbf{R} \) by elements of the form \( (a_i)_{i \in \mathcal{I}} \) with \( a_i = 0 \) for \( i \notin T \).

The following lemma is easy but extremely useful.

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Lemma 4.5.3. Suppose for any \( i \in I \), \( M_i \) is an \( \mathcal{O} \)-module with decreasing filtrations of \( \mathcal{O} \)-modules \( M_i \supseteq M_{i,1} \supseteq M_{i,2} \supseteq \cdots \). Then the natural map

\[
\prod_{i \in I} M_i \rightarrow \lim_{\rightarrow n} \left( \prod_{i \in I} M_i/M_{i,n} \otimes_{\mathcal{R}} \mathcal{R}_\mathfrak{F} \right)
\]

is surjective. The kernel contains all elements of the form \((a_i)_{i \in I}\) such that for any \( n \), there exists \( I_n \in \mathfrak{F} \) with \( a_i \in M_{i,n} \) for any \( i \in I_n \).

Proof. The description of the kernel follows from the definition of \( \mathcal{R}_\mathfrak{F} \). To prove the surjectivity, let \( \lim_{\rightarrow n} [(a_{i,n})_i] \) with \( a_{i,n} \in M_i/M_{i,n} \) be an element in the projective limit. This means that for any \( n \), there exists \( I_n \in \mathfrak{F} \) such that \( a_{i,n} \equiv a_{i,n+1} \mod M_{i,n} \) for any \( i \in I_n \). We may assume \( I_n \) contains \( I_{n+1} \) for any \( n \) and the intersection of all \( I_n \) is empty. Now for any \( i \in I_n \setminus I_{n+1} \), let \( a_i \in M_i \) be a lifting of \( a_{i,n+1} \). Set \( a_i = 0 \) for \( i \notin I_1 \). It is easy to see that \((a_i)_{i \in I}\) maps to \( \lim_{\rightarrow n} [(a_{i,n})_i] \). This finishes the proof of the lemma. \( \square \)

Note that since \( \mathcal{O}/(\varpi^n) \) has finite cardinality,

\[
\mathcal{R}_\mathfrak{F}/(\varpi^n) = \left( \prod_{I} \mathcal{O}/(\varpi^n) \otimes_{\mathcal{R}} \mathcal{R}_\mathfrak{F} \right) \cong \mathcal{O}/(\varpi^n).
\]

As a special case of the previous lemma, we have

Corollary 4.5.4. The inverse limit of (4.2) gives a surjective map \( \prod_I \mathcal{O} \rightarrow \mathcal{O} \). The kernel contains all \((a_i)_{i \in I}\) such that for any \( n \), there exists \( I_n \in \mathfrak{F} \) with \( a_i \in (\varpi^n) \) for any \( i \in I_n \).

4.5.5. We now define patched completed homologies. Recall that \( S_{\psi,\xi}(U_{Q(N)}^p U_p, \mathcal{O}/(\varpi^n))_m \) is an \( \mathcal{O}[\Delta_{Q,N}] \)-module. Hence \( S_{\psi,\xi}(U_{Q,N}^p U_p, \mathcal{O}/(\varpi^n))_m \otimes_{\mathcal{O}} \mathcal{O}_\infty \) is a natural \( S_{\infty} \)-module. For simplicity, we denote it by \( M(U_p, N, n) \). Then \( \prod_{N \in I} M(U_p, N, n) \) has a natural \( \mathcal{R} \)-module structure.
Definition 4.5.6 (Patched completed homologies). For any integer \( n > 0 \), we define

\[
\begin{align*}
M_n^{p,\xi} &:= \lim_{\xi \to U_p} (\prod_{N \in I} M(U_p, N, n)/a^n M(U_p, N, n) \otimes R \mathcal{R}_g) . \\
M_{n,0}^{p,\xi} &:= \lim_{\xi \to U_p} (\prod_{N \in I} (S_{\psi,\xi}(U_{p,N,0}U_p, \mathcal{O}/\mathcal{O}\bar{n})^\vee \otimes \mathcal{O}_{\infty}/b^n) \otimes R \mathcal{R}_g) . \\
M_n^{\{\xi\}} &:= \lim_{\xi \to U_p} (\prod_{I} (S_{\psi,\xi}(U_pU_p, \mathcal{O}/\mathcal{O}\bar{n})^\vee \otimes \mathcal{O}_{\infty}/b^n) \otimes R \mathcal{R}_g).
\end{align*}
\]

Here \( U_p \) runs through all open compact subgroups of \( D_p^\times = \prod_{v \mid p} \text{GL}_2(F_v) \) in all projective limits above. We note that \( M_n^{\{\xi\}} \) is nothing but

\[
\lim_{\xi \to U_p}(S_{\psi,\xi}(U_pU_p, \mathcal{O}/\mathcal{O}\bar{n})^\vee \otimes \mathcal{O}_{\infty}/b^n) = M_{\psi,\xi}(U_pU_p, \mathcal{O}/\mathcal{O}_{\infty}/b^n)
\]

since \( S_{\psi,\xi}(U_pU_p, \mathcal{O}/\mathcal{O}\bar{n})^\vee \otimes \mathcal{O}_{\infty}/b^n \) has finite cardinality.

4.5.7. In the definition of \( M_n^{p,\xi}, M_{n,0}^{p,\xi} \), we can replace the limits taken over all open compact subgroups of \( D_p^\times \) by over all open subgroups of \( K_p \). From this description, it is clear that both patched completed homologies are natural \( \mathcal{O}[[K_p]] \)-modules. The action of \( K_p \) can be extended to \( D_p^\times = \prod_{v \mid p} \text{GL}_2(F_v) \) in the usual way: the action of \( g \in D_p^\times \) induces isomorphisms \( M(U_p, N, n) \sim M(g^{-1}U_p g, N, n) \).

We prove some simple properties of \( M_n^{p,\xi} \). Note that the diagonal action of \( S_\infty \) on \( \prod_{N \in I} M(U_p, N, n)/a^n M(U_p, N, n) \) defines a natural \( S_\infty \)-module structure on \( M_n^{p,\xi} \).

**Proposition 4.5.8.** \( M_n^{p,\xi} \) is a flat \( S_\infty/a^n \)-module. Moreover the natural maps \( M_n^{p,\xi} \to M_{n-1}^{p,\xi} \) and \( M_{n,0}^{p,\xi} \) induce isomorphisms

\[
M_n^{p,\xi}/a^{n-1} M_n^{p,\xi} \cong M_{n-1}^{p,\xi}, \quad M_n^{p,\xi}/a_0 M_n^{p,\xi} \cong M_{n,0}^{p,\xi}.
\]

**Proof.** For \( U_p \) small enough and \( N > n \), \( M(U_p, N, n)/a^n M(U_p, N, n) \) is a flat \( S_\infty/a^n \)-module by lemma 4.4.2. Hence \( \prod_{N > n} M(U_p, N, n)/a^n M(U_p, N, n) \) is a flat \( S_\infty/a^n \)-module.
module. Since $\otimes R_\mathfrak{a}$ is an exact functor, applying lemma 4.4.4, we see that $M_{n,0}^{p,\{\xi_v\}}$ is a flat $S_\infty/\mathfrak{a}^n$-module.

Note that by lemma 4.4.2 again, we have

$$M(U_p, N, n)/(a^n + a_0)M(U_p, N, n) \cong S_{\psi, \xi}(U_{Q_N,0}^{p}U_p, \mathcal{O}/\mathfrak{a}^n)^\vee \otimes \mathcal{O}_\infty/b^n.$$  

Both isomorphisms in the proposition now follow from the first part of lemma 4.4.4 and the following easy lemma which we omit the proof here.

**Lemma 4.5.9.** Let $R$ be a Noetherian ring and $M = \prod_{i \in \mathcal{I}} M_i$ be a product of $R$-modules $M_i$. Then $M/IM \cong \prod_{i \in \mathcal{I}} M_i/IM_i$ for any ideal $I$ of $R$.

**4.5.10.** Fix isomorphisms between $P(Q_N)$ and $\{1, \cdots, 2^r\}$ for any $N$ from now on. Then the map $\eta_N$ defined in 4.4.5 induces a map:

$$\eta^p_n : M_{n,0}^{p,\{\xi_v\}} \to (M_{n}^{\{\xi_v\}})^{\oplus 2^r},$$

which commutes with the action of $D_p^{\times}$ and $\mathcal{O}[[K_p]]$.

It is clear that $\prod_N R_{Q_N}^{p,\{\xi_v\}}$ acts naturally on $M_{n,0}^{p,\{\xi_v\}}$ and $M_{n,0}^{p,\{\xi_v\}}$. Let $\tilde{f} = \prod_N \tilde{f}_N \in \prod_N R_{Q_N}^{p,\{\xi_v\}}$ where $\tilde{f}_N$ are defined in proposition 4.4.6. Here all the products are taken over all $N \in \mathcal{I}$ and we will keep this convention from now on in this section. The following result is a direct consequence of proposition 4.4.6 and lemma 4.5.12 below.

**Proposition 4.5.11.** $\tilde{f}^2$ kills the kernel and cokernel of $\eta^p_n$.

**Lemma 4.5.12.** Let $R$ be a Noetherian ring and $\{M_i\}_{i \in \mathbb{Z}_{>0}} \to \{N_i\}_{i \in \mathbb{Z}_{>0}}$ be a map between two projective systems of $R$-modules. Suppose there exists an element $f$ that kills the kernel and cokernel of $M_i \to N_i$ for any $i$. Then $f^2$ kills the kernel and cokernel of $\varprojlim M_i \to \varprojlim N_i$.  

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Proof. Suppose \((a_i)_i \in \lim \leftarrow N_i\). There exists \(b_i \in M_i\) mapping to \(fa_i \in N_i\) for any \(i\). Then \((fb_i)_i\) defines an element in \(\lim \leftarrow M_i\) and maps to \(f^2(a_i)_i\). The claim for the kernel is clear. \(\square\)

**Corollary 4.5.13.** Let \(\tilde{M}^{p,\{\xi_i\}}_n\) be the image of \(M^{p,\{\xi_i\}}_n\) into \((M^{p,\{\xi_i\}}_n)_f\), the localization of \(M^{p,\{\xi_i\}}_n\) by the powers of \(\tilde{f}\). Then \(\tilde{M}^{p,\{\xi_i\}}_n\) is a finitely generated module over \(\mathcal{O}[[K_p]]\). In other words, the Pontryagin dual of \(\tilde{M}^{p,\{\xi_i\}}_n\) is an admissible representation of \(D^\times_p\). In particular, \(\tilde{M}^{p,\{\xi_i\}}_n\) is an object in \(\mathfrak{C}_{D^\times_p,\psi}(\mathcal{O})\).

**Proof.** For simplicity, we write \(X\) for \(M^{p,\{\xi_i\}}_n\). We define a decreasing filtration on \(X\) with

\[
\text{Fil}^k X = X, \quad k \leq 0
\]

\[
\text{Fil}^k X = \{ x \in X, \tilde{f}^{2k} x \in a^k X \} \cap \text{Fil}^{k-1} X, \quad k > 0.
\]

Since \(a^n X = 0\), it is easy to see that \(\text{Fil}^k X \subseteq X[\tilde{f}^{2k}]\) when \(k \geq n\). We claim that \(\tilde{f}^{2n} \text{Fil}^k X / \tilde{f}^{2n} \text{Fil}^{k+1} X\) is a finite \(\mathcal{O}[[K_p]]\)-module for any \(k\). Note that this will imply \(\tilde{f}^{2n} X\) is a finite \(\mathcal{O}[[K_p]]\)-module and hence prove the corollary. In fact, we are going to prove a stronger result:

\[
\tilde{f}^{2k+2} \text{Fil}^k X / \tilde{f}^{2k+2} \text{Fil}^{k+1} X
\]

is finite over \(\mathcal{O}[[K_p]]\) for any \(k \leq n - 1\).

Consider the map:

\[
\varphi_k : \tilde{f}^{2k+2} \text{Fil}^k X \rightarrow \tilde{f}^2 (a^k X / a^{k+1} X) \subseteq a^k X / a^{k+1} X
\]

\[
\tilde{f}^{2k+2} x \mapsto \tilde{f}^2 (\tilde{f}^{2k} x).
\]
It is easy to see that \( \ker(\varphi_k) = \tilde{f}^{2k+2} \Fil^k \) \( X \). Hence we get an injective map:

\[
\tilde{f}^{2k+2} \Fil^k X / \tilde{f}^{2k+2} \Fil^{k+1} X \hookrightarrow \tilde{f}^2 (a^k X / a^{k+1} X).
\]

But \( X \) is flat over \( S_\infty / a^n \). Thus

\[
a^k X / a^{k+1} X \cong a^k / a^{k+1} \otimes X / aX \cong a^k / a^{k+1} \otimes M_{1,0}^{p,\{\xi_v\}}.
\]

By the previous proposition, the kernel of \( \eta_1^p : M_{1,0}^{p,\{\xi_v\}} \rightarrow M_1^{\{\xi_v\}} \) is killed by \( \tilde{f}^2 \). The result follows from the fact that \( M_1^{\{\xi_v\}} \cong M_\psi,\xi(U^p)_m / \varpi M_\psi,\xi(U^p)_m \) is finite over \( \mathcal{O}[[K_p]] \).

We note that the same argument shows that

**Lemma 4.5.14.** For any integer \( n > 0 \) and open subgroup \( U_p \) of \( D_p^\times \) small enough so that \( U_p U_p \), the cardinality of

\[
\tilde{f}_N^{2n} (M(U_p, N, n) / a^n M(U_p, N, n))
\]

has a uniform upper bound which is independent of \( N \).

### 4.6 Patching II: patched deformation rings

**4.6.1.** In this subsection, we are going to define a patched (local) deformation ring acting on our patched completed homologies. First we consider the action of \( R_{Q_N}^{ps,\{\xi_v\}} \).

Denote \( R_{Q_N}^{ps,\{\xi_v\}} \otimes \mathcal{O}_\infty \) by \( R_{N}^{ps} \) and its maximal ideal by \( m_N^{ps} \). Note that \( R_{N}^{ps} \) acts on \( M(U_p, N, n) \) \( \mathcal{O}_\infty \)-linearly. We will freely use the notation defined in the previous subsection.
Lemma 4.6.2. For any integer $n > 0$ and open subgroup $U_p \subseteq K_p$ small enough, there exists a constant $C = C_{n,U_p}$ (independent of $C$) such that $(m_N^{ps} \tilde{f}_N)^C$ annihilates $M(U_p, N, n)/a^n M(U_p, N, n)$ for any $N$.

**Proof.** Since $\mathcal{S}_{\psi, \xi}(U_p^p, \mathbb{F})_m$ is a finite dimensional $\mathbb{F}$-vector space, there exists $c'$ such that $m^{c'}$ acts trivially on $\mathcal{S}_{\psi, \xi}(U_p^p, \mathbb{F})_m$. Hence the case $n = 1$ follows from the properties of $\tilde{f}_N$ directly. In general, we can consider the $a$-adic filtration on $M(U_p, N, n)/a^n M(U_p, N, n)$ and reduce to the case $n = 1$.

**Definition 4.6.3.**

$$R^{ps,p} := \varprojlim_n \left( \prod_{N \in I} \left( R^{ps}_N / (m_N^{ps} \tilde{f}_N)^n \right) \otimes_{\mathbb{R}} R_{\tilde{\mathfrak{a}}} \right).$$

It follows from the previous lemma that $R^{ps,p}$ acts on $M^{p,\{\xi_v\}}$.

**4.6.4.** There is a natural surjective map $R^{ps,p} \to R^{ps,\square} := R^{ps,\{\xi_v\}} \otimes \mathcal{O}_{\infty}$ by taking the limits of

$$\prod_{N \in I} \left( R^{ps}_N / (m_N^{ps} \tilde{f}_N)^n \right) \otimes_{\mathbb{R}} R_{\tilde{\mathfrak{a}}} \to \prod_{N \in I} \left( R^{ps,\square} / (m_N^{ps,\square})^n \right) \otimes_{\mathbb{R}} R_{\tilde{\mathfrak{a}}} = R^{ps,\square} / (m_N^{ps,\square})^n.$$

Here $m_N^{ps,\square}$ denotes the maximal ideal of $R^{ps,\square}$. Denote the prime ideal $(q^{ps, \mathfrak{b}_1})$ of $R^{ps,\square}$ by $q^{ps,\square}$ and its pull-back to $R^{ps}_N$ by $q^{ps}_N$. We have

$$B' = R^{ps,\{\xi_v\}} / q^{ps} = R^{ps,\square} / q^{ps,\square} = R^{ps}_N / q^{ps}_N.$$

Recall that the integral closure of $B'$ in its fraction field is $A = \mathbb{F}[[T]]$ (see 4.1.8).

Consider the natural map $\prod_{N \in I} R^{ps}_N \to R^{ps,p}$, which is surjective by lemma 4.5.3.
Lemma 4.6.5 (Definition of $q^{ps,p}$). The image of $\prod_{N} q^{ps}_N \subseteq \prod_{N} R^{ps}_N$ in $R^{ps,p}$ defines a prime ideal $q^{ps,p}$, which is the pull-back of $q^{ps,\square}$ via the natural map $R^{ps,p} \rightarrow R^{ps,\square}$.

Proof. Let $I^{ps}$ be the kernel of $\prod_{N} R^{ps}_N \rightarrow R^{ps,p}$. It suffices to prove the composite $\prod_{N} R^{ps}_N \rightarrow R^{ps,p} \rightarrow R^{ps,\square}$ induces an isomorphism:

$$(\prod_{N} R^{ps}_N)/(I^{ps}, \prod_{N} q^{ps}_N) \sim \rightarrow R^{ps,\square}/q^{ps,\square} = R^{ps,\{\xi_v\}}/q^{ps} = B'.$$

By lemma 4.5.3, $I^{ps}$ is the set of elements $(x_N)_N \in \prod_{N} R^{ps}_N$ such that for any $n > 0$, there exists an element $I_n \in \mathfrak{S}$ with $x_N \in (m^{ps}_N f_N)^n$ for any $N \in I_n$. Note that the image of $\tilde{f}_N$ in $R^{ps}_N/q^{ps}_N = B' \subseteq A$ has a uniform bound on its $T$-adic valuation. It is easy to see that the image $I_{B'}$ of $I^{ps}$ in $R^{ps}_N/\prod_{N} q^{ps}_N = \prod_{N} B'$ consists of elements $\{b_N\}_N$ such that for any $n > 0$, there exists an element $I_n \in \mathfrak{S}$ with $b_N \in (m_{B'})^n$ for any $N \in I_n$, where $m_{B'}$ is the maximal ideal of $B'$. By lemma 4.5.3 again (see also corollary 4.5.4), $(\prod_{N} B')/I_{B'}$ is nothing but $B'$. This finishes the proof. \qed

4.6.6. In 4.2.11 we defined several global universal lifting rings. See the notations there and also in 4.3. For each $N$, we fix isomorphisms

$$R^{\square,p}_{\phi_b,q_N} \cong R^{\{\xi_v\}}_{\phi_b,q_N} \otimes_{O} O_{\infty}$$

such that $b_1 \subseteq O_{\infty}$ is contained in $q_{b,q_N}$ for each $i$. The natural map $R^{ps,\{\xi_v\}}_{Q_N} \rightarrow R^{\{\xi_v\}}_{\phi_b,q_N}$ given by evaluating the universal trace can be extended naturally to an $O_{\infty}$-algebra homomorphism $R^{ps}_N = R^{ps,\{\xi_v\}}_{Q_N} \otimes_{O} O_{\infty} \rightarrow R^{\square,p}_{\phi_b,q_N}$. For simplicity, we write $R_{b,N}$ for $R^{\square,p}_{\phi_b,q_N}$.

Definition 4.6.7.

$$R^p_b := (\prod_{N \in \mathcal{I}} R_{b,N}) \otimes_{\prod_{N \in \mathcal{I}}} R^{ps}_N R^{ps,p}.$$ 

This is a quotient of $\prod_{N \in \mathcal{I}} R_{b,N}$.
Lemma 4.6.8 (Definition of $q_b^p$). The image of $\prod_N q_{b,Q_N} \subseteq \prod_{N \in \mathcal{I}} R_{b,N}$ in $R_b^p$ defines a prime ideal $q_b^p$ of $R_b^p$. Its pull-back to $R_{b,p}$ is $q_{b,p}$. Moreover, $R_b^p/q_b^p$ is naturally isomorphic to $B = R_{b,N}/q_{b,Q_N}$ (defined in the beginning of section 4.3).

Proof. We use the notations in the proof of the previous lemma. Then

$$R_b^p/q_b^p = \left(\prod_{N \in \mathcal{I}} R_{b,N}\right)/(I_{q_{b,Q_N}}) = \left(\prod_{N} B\right)/(I_{B'}) \cong B.$$ 

The last isomorphism follows from the explicit description of $I_{B'}$ and the fact $B$ is a finite $B'$-algebra. \hfill $\Box$

Definition 4.6.9. For any positive integer $n$, we let $q_{ps,[n]}$ be the image of $\prod_N (q_{ps}^N)^n \subseteq \prod_N R_{ps}^N$ in $R_{ps,p}$. Similarly, we define $q_{b,[n]}$ as the image of $\prod_N q_{b,Q_N}^n \subseteq \prod_N R_{b,N}$ in $R_b^p$.

Remark 4.6.10. By definition, we have

- $q_{ps,[1]} = q_{ps,p}$, $q_{b,[1]} = q_b^p$.
- $(q_{ps,p})^n \subseteq q_{ps,[n]}$.

It is likely true that the inclusion above is an equality. However we don’t quite need this. For our purpose, the following lemma is enough.

Lemma 4.6.11. For any positive integers $n$, $j$ and open subgroup $U_p$ of $D_p^\times$, the image of $\prod_N (q_{ps}^N)^j$ and $(\prod_N q_{ps}^N)^j$ in the endomorphism ring of

$$\prod_{N \in \mathcal{I}} \tilde{f}_N^{-2n}(M(U_p,N,n)/a^n M(U_p,N,n)) \otimes_{R_{\mathfrak{S}}} R_{\mathfrak{S}} = \tilde{f}_N^{-2n}(\prod_{N \in \mathcal{I}} M(U_p,N,n)/a^n M(U_p,N,n) \otimes_{R_{\mathfrak{S}}} R_{\mathfrak{S}})$$

are the same.

Proof. By lemma 4.5.14 there is a uniform upper bound $C$ on the cardinality of

$$\tilde{f}_N^{-2n}(M(U_p,N,n)/a^n M(U_p,N,n))$$

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for all $N$. The lemma is clear as there are only finitely many isomorphism classes of triples $(M, R, I)$ where $M$ is a finite group of cardinality at most $C$, $R$ is a commutative subring of the endomorphism ring of $M$ and $I \subseteq R$ is an ideal.

4.6.12. Recall that in proposition 2.3.2 and corollary 2.3.8 we proved that there exist elements $c_N \in R_{ps}^N$ for $N > 0$ such that

- For any $n$, we may find a constant $k_n$ independent of $N$ (which is $N$ in proposition 2.3.2) such that $c_N^{k_n}$ kills the kernel and cokernel of the map $R_{ps}^N/(q_{ps}^N)^n \to R_{b,N}/(q_{b,Q_N})^n$.

- The image $c'$ of $c_N$ in $R_{ps}^N/q_N^N = B' \hookrightarrow A$ is non-zero and independent of $N$.

To be more precise, we take $\Gamma = G_{F,S \cup Q_N}, \Gamma_0 = G_{F,S}, \rho_0 = \rho(q)^0$ in the setup of proposition 2.3.2 and $I$ be the kernel of $R_{ps} \to R_{ps, \{\xi_v\}}$ in corollary 2.3.8.

We define $\tilde{c}$ to be image of $(c_N)_N \in \prod_N R_{ps}^N$ in $R_{ps,p}$. It is easy to see that $\tilde{c}$ mod $q_{ps,p}$ is equal to $c' \neq 0$. A direct corollary of the above discussion is

**Corollary 4.6.13.**

1. $\tilde{c} \notin q_{ps,p}.$

2. For any $n$, there exists an integer $k_n$ such that $\tilde{c}^{k_n}$ kills the kernel and cokernel of the natural map $R_{ps,p}/q_{ps,[n]} \to R_{b}/q_{b}^{[n]}$.

3. Let $(R_{ps,p})_{[q_{ps,p}]}$ (resp. $(R_{b}^{p})_{[q_{b}^{p}]}$) denote the completion of the localization $(R_{ps,p})_{q_{ps,p}}$ (resp. $(R_{b}^{p})_{q_{b}^{p}}$) with respect to the topology with a system of open neighborhoods of zero given by $(q_{ps,[n]})_{n>0}$ (resp. $(q_{b}^{[n]})_{n>0}$). Then

$$(R_{ps,p})_{[q_{ps,p}]} \cong (R_{b}^{p})_{[q_{b}^{p}]}.$$
Proof. The first part is clear and the last part follows directly from the first two parts. As for the second part, consider the natural map

$$\prod_N R^\text{ps}_N/(q_N^\text{ps})^n \to \prod_N R_{b,N}/(q_{b,Q_N})^n,$$

which is killed by the $k_n$-th power of $(c_N)_N$. We obtain the desired results by taking tensor product of this map with $R^\text{ps,p}$ over $\prod_N R^\text{ps}_N$.

4.6.14. We still need to relate these patched (global) deformation rings with some local deformation ring. By the construction in proposition 4.3.2 for any $N$, we have a map

$$R_{\text{loc}}^\{\xi\}[[x_1, \cdots, x_g]] \to R^\square_{b,\text{Q}} = R_{b,N}$$

sending $x_i$ to $q_{b,Q_N}$ and $q_{b,Q_N}/(q_{b,Q_N}^2, q_{\text{loc}}^\{\xi\}, x_1, \cdots, x_g)$ is killed by some element $f' \in B$ with $\ell(A/(f')) < C$. It is clear that we can take one $f' \neq 0$ which works for all $N$.

Definition 4.6.15. We define

$$R_\infty^\{\xi\} := R_{\text{loc}}^\{\xi\}[[x_1, \cdots, x_g]]$$

and $q_\infty^\{\xi\}$ to be its prime ideal generated by $q_{\text{loc}}^\{\xi\}, x_1, \cdots, x_g$.

Proposition 4.6.16. The diagonal map $R_\infty^\{\xi\} \to \prod_{N \in I} R_{b,N}$ induces a natural map

$$R_\infty^\{\xi\} \to R^p_{b}$$

which sends $q_\infty^\{\xi\}$ into $q^p_b$. Moreover, the $B(=R^p_b/q^p_b)$-module $q^p_b/(q^2_b, q_\infty^\{\xi\})$ is killed by $f' \neq 0$. 

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Proof. The first claim is clear. To show that \( q^p_b/(q^p_b)^2, q^\{\xi_v\}_\infty \) is killed by \( f' \), since the cokernel of

\[
\prod_N (q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2) \to \prod_N (q_b,Q_N/(q_b,Q_N)^2) \to q^p_b/q^2_b
\]

is killed by \( f' \), it suffices to show that the image of \( q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2 \to q^p_b/q^2_b \) is equal to the image of \( \prod_N R^\{\xi_v\}_\infty/q^\{\xi_v\}_\infty = \prod_N B \)-modules, and the action of \( \prod_N B \) on the target factors through \( R^p_b/q^p_b = B \). Thus it is enough to show that the composite map

\[
q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2 \to \prod_N (q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2) \to \prod_N (q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2) \otimes \prod_N B
\]

is surjective. Here the first map is given by the diagonal embedding and \( B \) is viewed as a \( \prod B \)-algebra by identifying them with \( R^p_b/q^p_b \) and \( \prod N R^\{\xi_v\}_\infty/q^\{\xi_v\}_\infty \). Explicitly, the map \( \prod_N B \to B \) is the one considered in lemma 4.5.3 with \( M_i = B \) and \( M_{i,n} = m^B_i \) where \( m_B \) is the maximal ideal of \( B \). Note that (4.3) is clearly true if \( q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2 \) is a finite free \( B \)-module. In general, we can get the surjectivity by writing \( q^\{\xi_v\}_\infty/(q^\{\xi_v\}_\infty)^2 \) as a quotient of a free \( B \)-module. 

\[ \square \]

**Corollary 4.6.17.** Let \( \widehat{(R^\{\xi_v\}_\infty)}_{q^\{\xi_v\}_\infty} \) be the \( q^\{\xi_v\}_\infty \)-adic completion of \( (R^\{\xi_v\}_\infty)_{q^\{\xi_v\}_\infty} \). Then we have a surjective local homomorphism:

\[
\widehat{(R^\{\xi_v\}_\infty)}_{q^\{\xi_v\}_\infty} \to \widehat{(R^p_b)}_{q^p_b} \cong \widehat{(R^{p\mathfrak{m}}_{\mathfrak{m}})}_{q^p_{\mathfrak{m}\mathfrak{p}}}.
\]

### 4.7 Patching III: An application of Paškūnas’ theory and the local-global compatibility result

#### 4.7.1. In section 3.5, we attach a block \( \mathfrak{B}_m \) of \( D_p^\times \) to the maximal ideal \( m \) of Hecke algebra. By theorem 3.5.5 \( M_{\psi,\xi}(U^p)m \in \mathfrak{C}_{D_p^\times,\psi}(\mathcal{O})^{\mathfrak{B}_m} \). From this, it is clear from the
proof of corollary 4.5.13 that $\bar{\mathcal{M}}^p_{\{\xi_v\}} \in \mathcal{C}_{D^+_p, \psi}(\mathcal{O})_{\mathfrak{B}_m}$. Recall that $\bar{\mathcal{M}}^p_{\{\xi_v\}}$ is defined to be the image of $\mathcal{M}^p_{\{\xi_v\}}$ into $(\mathcal{M}^p_{\{\xi_v\}})_{\bar{f}}$. Similarly, we denote the image of $\mathcal{M}^p_{\{\xi_v\}}$ into $(\mathcal{M}^p_{\{\xi_v\}})_{\bar{f}}$ by $\bar{\mathcal{M}}^p_{\{\xi_v\}}$. Let $P_{\mathfrak{B}_m}$ be the projective generator of $\mathfrak{B}_m$. See 3.5 for the notations here.

**Definition 4.7.2.** For any positive integer $n$, we define

- $\tilde{m}^p_{\{\xi_v\}} := \text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \bar{\mathcal{M}}^p_{\{\xi_v\}})$.
- $\tilde{m}^p_{\{\xi_v\}, 0} := \text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \bar{\mathcal{M}}^p_{\{\xi_v\}})$.
- $m_{\{\xi_v\}} := \text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \mathcal{M}^p_{\{\xi_v\}})$.
- $m^p_{\{\xi_v\}} := \text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \mathcal{M}^p_{\{\xi_v\}}) \cong m_{\{\xi_v\}} \otimes \mathcal{O}_n / \mathfrak{a}^n$.

**4.7.3.** By our local-global compatibility result (corollary 3.5.10), $m_{\{\xi_v\}}$ is a faithful, finitely generated $T_{\psi, \xi}(U^p)^m$-module. We note that $\text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \cdot)$ is an exact functor from $\mathcal{C}_{D^+_p, \psi}(\mathcal{O})_{\mathfrak{B}_m}$ to the category of right pseudo-compact $E_{\mathfrak{B}_m}$-modules (see 3.4.3).

**Lemma 4.7.4.** In the following statements, for any $R^p_{Q_N, \{\xi_v\}}$-module $M$, we use $M_{\bar{f}}$ to denote its localization by powers of $\bar{f}$.

1. $(\tilde{m}^p_{\{\xi_v\}})_{\bar{f}}$ is a flat $S_{\infty}/\mathfrak{a}^n$-module and there are natural isomorphisms

   $$(\tilde{m}^p_{\{\xi_v\}})_{\bar{f}} / \mathfrak{a}^n (\tilde{m}^p_{n+1})_{\bar{f}} \cong (\tilde{m}^p_{n})_{\bar{f}}.$$ 

2. The isomorphism in proposition 4.5.8 and $\eta^p$ in 4.5.10 induce isomorphisms:

   $$(\tilde{m}^p_{\{\xi_v\}, 0})_{\bar{f}} / \mathfrak{a}_0 (\tilde{m}^p_{n+1})_{\bar{f}} \cong (\tilde{m}^p_{n, 0})_{\bar{f}} \cong (m^p_{\{\xi_v\}})_{\bar{f}}^{2^r}.$$ 

**Proof.** For any finitely generated $S_{\infty}/\mathfrak{a}^n$-module $N$, it is easy to see that

$$\tilde{m}^p_{\{\xi_v\}} \otimes_{S_{\infty}/\mathfrak{a}^n} N \cong \text{Hom}_{\mathcal{C}_{D^+_p, \psi}(\mathcal{O})}(P_{\mathfrak{B}_m}, \bar{\mathcal{M}}^p_{\{\xi_v\}} \otimes_{S_{\infty}/\mathfrak{a}^n} N).$$
Using this, the first part follows from proposition 4.5.8.

By definition, the kernel of $\tilde{m}_{n+1}^{p,\{\xi_v\}}/a_0\tilde{m}_n^{p,\{\xi_v\}} \rightarrow \tilde{m}_{n,0}^{p,\{\xi_v\}}$ is killed by powers of $\tilde{f}$. This proves the first isomorphism in the second part. The second isomorphism follows directly from proposition 4.5.11.

**Corollary 4.7.5.** In definition 4.6.3, we defined a $R^{ps,p}$-module structure on $M_n^{p,\{\xi_v\}}$ hence also on $M_n^{p,\{\xi_v\}}, \tilde{m}_n^{p,\{\xi_v\}}$. Then the localization $(\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}}$ of $\tilde{m}_n^{p,\{\xi_v\}}$ at $q^{ps,p}$ is flat over $S/\mathfrak{a}^p$ and we have isomorphisms:

$$
(\tilde{m}_{n+1}^{p,\{\xi_v\}})_{q^{ps,p}}/a_n(\tilde{m}_{n+1}^{p,\{\xi_v\}})_{q^{ps,p}} \cong (\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}};
$$

$$
(\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}}/a_0(\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}} \cong (\tilde{m}_{n,0}^{p,\{\xi_v\}})_{q^{ps,p}} \cong (m_n^{\{\xi_v\}})_{q^{ps,p}}^{\oplus 2^r}.
$$

**Proof.** In view of the previous lemma, it suffices to prove that the image of $\tilde{f} = \prod \tilde{f}_N$ in $R^{ps,p}/q^{ps,p} = B'$ is non-zero. Let $\Pi_N \tilde{f}_N$ be the image of $\tilde{f}$ in $\prod_{N \in \mathcal{I}} (R_N^{ps,p}/q_N^{ps,p}) = \Pi_N B'$. By proposition 4.4.6, $A/(\tilde{f}_N)$ has a uniform bounded length for all $N$. Note that the natural map $\Pi_N B' \rightarrow R^{ps,p}/q^{ps,p} = B'$ is the one considered in the proof of lemma 4.6.5. The image of $\Pi_N \tilde{f}_N$ in $B'$ is non-zero by the explicit description of the kernel of this map there. \qed

**Definition 4.7.6.** For any $n > 0$, we define

1. $m_n^{p,\{\xi_v\}} := \lim_{\leftarrow n} (\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}}/q^{ps,\{\xi_v\}}_{n}(\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}}$, the completion of $(\tilde{m}_n^{p,\{\xi_v\}})_{q^{ps,p}}$ with respect to the system of ideals $(q^{ps,\{\xi_v\}})_{n>0}$. See lemma 4.7.8 below for an equivalent definition.

2. $m_{n,0}^{p,\{\xi_v\}} := \lim_{\leftarrow n} (\tilde{m}_{n,0}^{p,\{\xi_v\}})_{q^{ps,p}}/q^{ps,\{\xi_v\}}_{n}(\tilde{m}_{n,0}^{p,\{\xi_v\}})_{q^{ps,p}}$, the completion of $(\tilde{m}_{n,0}^{p,\{\xi_v\}})_{q^{ps,p}}$ with respect to the system of ideals $(q^{ps,\{\xi_v\}})_{n>0}$.

3. $m_\infty^{\{\xi_v\}} := \lim_{\leftarrow k} m_k^{\{\xi_v\}}$.

4. $m_0^{\{\xi_v\}} :=$ the $q$-adic completion of $(m^{\{\xi_v\}})_q$ as a $T_{\psi,\xi}(U^p)_q$-module. Recall that $q$ is a prime ideal of $T_{\psi,\xi}(U^p)_m$ defined in 4.1.8.
Clearly $(\hat{\mathcal{R}}_{\text{ps},p})_{q\text{ps},p}$ hence $\hat{\mathcal{R}}^{(\xi_v)}_{\text{ps},q\text{ps},p}$, (by corollary 4.6.17) act on these spaces.

The main result of this section is

**Proposition 4.7.7.** 1. $m^{(\xi_v)}_\infty$ is a finitely generated $(\hat{\mathcal{R}}^{(\xi_v)}_{\text{ps},q\text{ps},p})$-module.

2. $m^{(\xi_v)}_\infty$ is a flat $S_\infty$-module. Moreover, the isomorphisms in corollary 4.7.5 induce

$$m^{(\xi_v)}_\infty/\mathfrak{a}_1m^{(\xi_v)}_\infty \cong (m^{(\xi_v)}_0)_{\otimes 2^r}.$$

**Proof.** In view of the second part of lemma 4.4.4 and Theorem 8.4 of [29], the proposition is a direct consequence of the following lemma. □

**Lemma 4.7.8.**

1. $m^{p,(\xi_v)}_n$ is isomorphic to the $q\text{ps},p$-adic completion of $(\hat{m}^{p,(\xi_v)}_n)_{q\text{ps},p}$.

2. $m^{p,(\xi_v)}_1$ is a finitely generated $(\hat{\mathcal{R}}^{p,\chi}_{\text{ps},p})_{q\text{ps},p}$-module.

3. For each $n > 0$, $m^{p,(\xi_v)}_n$ is a flat $S_\infty/\mathfrak{a}^n$-module and there are natural isomorphisms

$$m^{p,(\xi_v)}_{n+1}/\mathfrak{a}^nm^{p,(\xi_v)}_{n+1} \cong m^{p,(\xi_v)}_n.$$

$$m^{p,(\xi_v)}_n/\mathfrak{a}_1m^{p,(\xi_v)}_n \cong (m^{p,(\xi_v)}_0)/\varpi^n m^{p,(\xi_v)}_0 \otimes 2^r.$$

**4.7.9.** To prove this lemma, we need some more notations. For $v|p$, let $R^{p,\chi}_v$ be the universal deformation ring which parametrizes all two-dimensional pseudo-representations of $G_{F_v}$ which lifts $1+\bar{\chi}|G_{F_v}$ with determinant $\chi$. Put $R^{p}_{p} = \bigotimes_{v|p} R^{p,\chi}_v$. By corollary 3.5.10 $m^{(\xi_v)}_\infty$ is a finitely generated $R^{p}_{p}$-module.

Note that there are two actions of $R^{p}_{p}$ on $\hat{m}^{p,(\xi_v)}_n$: one comes from the center of $E_{\mathfrak{a}_n}$ (see 3.4.3 for notations here), one comes from mapping $R^{p}_{p}$ diagonally into $\prod_{N} R^{p}_{N}$. We will always view $m^{p,(\xi_v)}_n$ as a $R^{p}_{p}$-module by the second action. It follows
from theorem 3.5.5 and the proof of corollary 4.5.13 that $\tilde{m}_n^{p,\xi}$ is finitely generated over $R_p^{ps}$.

**Definition 4.7.10.** We define $R_n$ to be the image of $R_p^{ps,p}$ in $\text{End}_{R_p^{ps}}(\tilde{m}_n^{p,\xi})$. This is a natural $R_p^{ps}$-algebra.

**4.7.11.** It is clear that $R_n$ is a finite $R_p^{ps}$-algebra hence a noetherian ring. We claim that the kernel of $R_p^{ps,p} \rightarrow \text{End}_{R_p^{ps}}(\tilde{m}_n^{p,\xi})$ is contained in $q_p^{ps,p}$, so that $q_p^{ps,p}$ defines a prime ideal $q_n$ of $R_n$. It suffices to show that

$$(\tilde{m}_n^{p,\xi})_{q_p^{ps,p}} \neq 0.$$

By corollary 4.7.5, we only need to check $(m_n^{\xi})_{q_p^{ps,p}} \neq 0$. Note that $m_n^{\xi}/ bm_n^{\xi} \cong m^{\xi}/ \omega m^{\xi}$ and the action of $R_p^{ps,p}$ on it factors through $T_{\psi,\xi}(U_p)_m$. Also it follows from lemma 4.6.5 that $q_p^{ps,p}$ is the pull-back of $q \in \text{Spec} T_{\psi,\xi}(U_p)_m$. Thus

$$(m_n^{\xi}/ bm_n^{\xi})_{q_p^{ps,p}} = (m^{\xi}/ \omega m^{\xi})_q \neq 0$$

because $q$ contains $\omega$ and $m$ is a faithful finitely generated $T_{\psi,\xi}(U_p)_m$-module.

**Proof of lemma 4.7.8.** For any positive integer $j$, let $q_n^{[j]}$ be the image of $q_n^{ps,[j]}$ in $R_n$. Recall that this is also the image of $\prod_N(q_n^{ps})^j$. We claim that $q_n^{[j]} = q_n^j$. This will imply our first assertion in the lemma.

Let $m_p$ be the maximal ideal of $R_p^{ps}$. Both $q_n^{[j]}$ and $q_n^j$ are finite $R_p^{ps}$-modules, hence naturally profinite groups by the $m_p$-adic topology. In particular, they are compact.

In the proof of corollary 4.5.13 we showed that $\tilde{f}^{2n} M_n^{p,\xi}$ is a finite $O[[K_p]]$-module. Therefore we can find an integer $C \geq 2n$ such that $\tilde{f}^C M_n^{p,\xi}$ has no $\tilde{f}$-torsion. There is a natural isomorphism between $\tilde{M}_n^{p,\xi}$ and $\tilde{f}^C M_n^{p,\xi}$ by multiplying $\tilde{f}^C$. We also note that the $O[[K_p]]$-module structure makes $\tilde{f}^C M_n^{p,\xi}$ into a natural profinite group.
Recall in \([4.5.6]\) we defined \(M_n^{p,\{\xi_v\}}\) as \(\lim_{\U_p} (\prod_{N \in \mathcal{I}} M(U_p, N, n)/a^n M(U_p, N, n) \otimes_{R_p} R_\delta)\). Let \(M(U_p, C)\) be the image of \(\tilde{f}C M_n^{p,\{\xi_v\}}\) in \(\prod_{N \in \mathcal{I}} M(U_p, N, n)/a^n M(U_p, N, n) \otimes_{R_p} R_\delta\). Clearly \(M(U_p, C)\) is contained in \(\tilde{f}C (\prod_{N \in \mathcal{I}} M(U_p, N, n)/a^n M(U_p, N, n) \otimes_{R_p} R_\delta)\), which is of finite cardinality by lemma \(4.5.14\). Since \(\tilde{f}C M_n^{p,\{\xi_v\}}\) is compact, we have

\[
\tilde{f}C M_n^{p,\{\xi_v\}} \cong \lim_{U_p} M(U_p, C).
\]

It follows from lemma \(4.6.11\) that \(q_n^{[j]}\) and \(q_n^{[l]}\) have the same images in the endomorphism rings of \(M(U_p, C)\). Taking the limits over \(U_p\) and using the compactness of \(q_n^{[j]}\) and \(q_n^{[l]}\), we conclude that they are equal.

As a consequence, \(m_n^{p,\{\xi_v\}}\) is isomorphic to the \(q_n\)-adic completion of \((\tilde{m}_n^{p,\{\xi_v\}})_{q_n}\) as a \((R_n)_{q_n}\)-module. Since \(\tilde{m}_n^{p,\{\xi_v\}}\) is a finitely generated module over \(R_p^{ps}\) hence also finitely generated over the noetherian ring \(R_n\), we have (Theorem 8.7 \[29\])

\[
m_n^{p,\{\xi_v\}} \cong \tilde{m}_n^{p,\{\xi_v\}} \otimes_{R_n} \left(\frac{R_n}{\mathfrak{q}_n}\right).
\]

Here \(\frac{R_n}{\mathfrak{q}_n}\) is the \(\mathfrak{q}_n\)-adic completion of \((R_n)_{q_n}\), which is flat over \(R_n\) (Theorem 8.8 ibid.). The third part of the lemma now follows directly from corollary \(4.7.5\). The second part holds as \(\frac{R_n}{\mathfrak{q}_n}\) is a quotient of \((R_p^{ps})_{[q^{ps}, p]}\) (Theorem 8.1 ibid.) and \(m_1^{p,\{\xi_v\}}\) is a finite \((R_1)_{q_1}\)-module. \(\square\)

**4.7.12.** Recall that in the beginning of section \(4.5\) we define \(S'_\infty \subseteq \mathcal{O}_\infty[[\Delta_\infty]]\) to be the closure (under the profinite topology) of the \(\mathcal{O}_\infty\)-subalgebra generated by all elements of the form \(g + g^{-1}, g = (0, \cdots, 0, a, 0 \cdots, 0) \in \Delta_\infty\) and ideal \(\mathfrak{a}'_1 = \mathfrak{a}_1 \cap S'_\infty\).

**Lemma 4.7.13.** \(S'_\infty\) is contained in the image of \((R_{\infty}^{[\xi_v]}_{q^{[\xi_v]}})_{q^{[\xi_v]}} \to \text{End}(m_{\infty}^{[\xi_v]}))\). Moreover \(\mathfrak{a}'_1\) is contained in the image of \(\ker((R_{\infty}^{[\xi_v]}_{q^{[\xi_v]}}) \to (R_{p^{[\xi_v]}_{q^{[\xi_v]}}_{q^{[\xi_v]}}}))\) induced by the map \(R_{p^{[\xi_v]}_{q^{[\xi_v]}}} \to R_{p^{[\xi_v]}_{q^{[\xi_v]}}\xi_v}\) (see \(4.6.4\)), where \((R_{p^{[\xi_v]}_{q^{[\xi_v]}}}^{[\xi_v]})_{q^{[\xi_v]}}\) denotes the \(\mathfrak{q}^{[\xi_v]}_{ps}\)-adic completion of \((R_{p^{[\xi_v]}_{q^{[\xi_v]}}^{[\xi_v]}})_{q^{[\xi_v]}}\).
Proof. This follows from proposition 4.4.8 ⋄

In other words, since \( S'_\infty \cong \mathcal{O}[[y_1, \cdots, y_4, p_1^{-1}, s'_1, \cdots, s'_r]] \), we may choose a lifting \( S'_\infty \to (\widehat{R_{\xi}}_{q(\xi)})_{q(\xi)} \) with \( a'_1 \) mapping into \( \ker((\widehat{R_{\xi}}_{\infty})_{q(\xi)} \to (\widehat{R_{ps}}_{\xi})_{q ps}) \).

4.8 Proof of theorem 4.1.6

4.8.1. We first summarize what we have done so far in the following commutative diagram:

\[
\begin{array}{ccc}
S'_\infty & \xrightarrow{\pi_R} & S_\infty \\
\downarrow & & \downarrow \\
(R(\xi))_{q(\xi)} & \xrightarrow{\pi_R} & \text{End}_{S_\infty}(m(\xi)) \\
\downarrow & & \downarrow \\
(R_{ps}(\xi))_{q ps} & \rightarrow & \widehat{T}_q \\
\end{array}
\]

where the dashed arrow exists by lemma 4.7.13 and the image of \( a'_1 \) is contained in \( \ker(\pi_R) \). In addition, we have

- \( m(\xi) \) is a finitely generated \((R(\xi))_{q(\xi)}\)-module.
- \( m(\xi) \) is a flat \( S_\infty \)-module and \( m(\xi)/a_1 m(\xi) \cong (m_0(\xi))^{2r} \).
- \( m_0(\xi) = (m(\xi))_q \cong (m(\xi))_q \otimes_\widehat{T}_q \widehat{T}_q \) is a finitely generated faithful \( \widehat{T}_q \)-module. By theorem 3.6.1, each irreducible component of \( \widehat{T}_q \) has dimension at least \( 2[F : Q] \).

Let \((R(\xi))' = (R(\xi))_{q(\xi)} \otimes S_\infty S_\infty\). This is also a local ring with \( \widehat{T}_q \) as a natural quotient by mapping \( S_\infty \) to \( S_\infty/a_1 = \mathcal{O} \). Hence \( \dim_{(R(\xi))'}(m(\xi)) = \dim_{\widehat{T}_q}(m_0(\xi)) \) is at least \( 2[F : Q] \). Note that \( y_1, \cdots, y_4, p_1^{-1}, s_1, \cdots, s_r \in a_1 \) form a regular sequence of \( m(\xi) \). We see immediately from these results that

Lemma 4.8.2. \( \dim_{(R(\xi))'}(m(\xi)) \geq 4|P| - 1 + r + 2[F : Q] \).

On the other hand, by proposition 4.2.4 and lemma 4.2.6, we have
Lemma 4.8.3. \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\) is equidimensional of dimension \(4|P| - 1 + r + 2[F : \mathbb{Q}]\).

As a corollary \((R^{(\xi_v)})'\) is also equidimensional of dimension \(4|P| - 1 + r + 2[F : \mathbb{Q}]\) since it is finite free over \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\). Combining this with lemma 4.8.2, we deduce that

Corollary 4.8.4. The support \(\text{Supp}_{(R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}}(m_{\infty}^{(\xi_v)})\) contains at least one irreducible component of \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\) and any minimal prime of \(\text{Supp}_{(R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}}(m_{\infty}^{(\xi_v)})\) has characteristic zero.

Proof. The first claim is clear as \((R^{(\xi_v)})'\) is finite over \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\), hence

\[
\dim_{(R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}}(m_{\infty}^{(\xi_v)}) = \dim_{(R^{(\xi_v)})'}(m_{\infty}^{(\xi_v)}) = 4|P| - 1 + r + 2[F : \mathbb{Q}] = \dim_{(R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}}(m_{\infty}^{(\xi_v)}).
\]

The second claim comes from the fact that \(m_{\infty}^{(\xi_v)}\) is flat over \(O\). \(\square\)

Lemma 4.8.5. Theorem 4.1.6 holds if \(m_{\infty}^{(\xi_v)}\) has full support on \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\).

Proof. As we mentioned before, the natural map \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}} \rightarrow (R_{\infty}^{(\xi_v)})_{q_0^{(\xi_v)}}\) extends to a map \((R^{(\xi_v)})' \rightarrow (R_{\infty}^{(\xi_v)})_{q_0^{(\xi_v)}}\) by sending \(S_{\infty}\) to \(S_{\infty}/a_1 = O\). Let \(p_0\) be a prime of \((R_{\infty}^{(\xi_v)})_{q_0^{(\xi_v)}}\) and denote its pull-back to \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\) by \(p_1\). It is easy to see that there is a unique prime \(p_1'\) of \((R^{(\xi_v)})'\) above \(p_1\), which is nothing but \(p_0 \cap (R^{(\xi_v)})'\). Since \(p_1\) is in the support of \(m_{\infty}^{(\xi_v)}\) over \((R_{\infty}^{(\xi_v)})_{q_\infty^{(\xi_v)}}\), we see that \(p_1' \in \text{Supp}_{(R^{(\xi_v)})'}(m_{\infty}^{(\xi_v)})\). Notice that \(a_1 \subseteq p_1'\). Thus

\[
p_1' \in \text{Supp}_{(R^{(\xi_v)})'}(m_{\infty}^{(\xi_v)}/a_1m_{\infty}^{(\xi_v)}) = \text{Supp}_{(R^{(\xi_v)})'}(m_{0}^{(\xi_v)}).
\]

But the action of \((R^{(\xi_v)})'\) on \(m_{0}^{(\xi_v)}\) factors through \((R_{\infty}^{(\xi_v)})_{q_0^{(\xi_v)}}\). Hence

\[
p_0 \in \text{Supp}_{(R_{\infty}^{(\xi_v)})_{q_0^{(\xi_v)}}}(m_{0}^{(\xi_v)}).
\]

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Since this works for all primes $p_0 \in \text{Spec} \left( R_{\text{ps}, \{\xi_v\}} \right)_{q^\infty}$, we conclude that $m_0^{\{\xi_v\}}$ has full support on $\left( R^\infty_{\text{ps}, \{\xi_v\}} \right)_{q^\infty}$. Thus $\left( R^\infty_{\text{ps}, \{\xi_v\}} \right)_{q^\infty} \to \mathbb{T}_q$ has nilpotent kernel. Note that this map is the $q^\infty$-adic completion of $\left( R^\infty_{\text{ps}, \{\xi_v\}} \right)_{q^\infty} \to \mathbb{T}_q$. Hence $\left( R^\infty_{\text{ps}, \{\xi_v\}} \right)_{q^\infty} \to \mathbb{T}_q$ also has nilpotent kernel.

4.8.6. Now we can prove theorem 4.1.6 when $\xi_v$ are all non-trivial. In this case, it follows from proposition 4.2.4 and lemma 4.2.6 that $\left( R^\infty_{\xi_v} \right)_{q^\infty}$ is irreducible. Hence $\left( m_\infty^{\{\xi_v\}} \right)$ has full support on $\left( R^\infty_{\xi_v} \right)_{q^\infty}$. The previous lemma implies theorem 4.1.6 directly.

4.8.7. In general, we use Taylor’s trick in [42]. Let $\xi'_v : k(v) \to \mathcal{O}^\times$ be non-trivial characters of $p$-power order for $v \notin S \setminus \Sigma_p$. Then the product of $\xi'_v$ can be viewed as a character $\xi'_v$ of $U^p$ (defined in 4.1.2) and we can define completed cohomology $S_{\psi, \xi'}(U^p), S_{\psi, \xi'}(U^p, E/\mathcal{O})$ and Hecke algebra $T' := T_{\psi, \xi'}(U^p)$ as in section 3.7.

Lemma 4.8.8. $T_v - (1 + \chi(\text{Frob}_v)), v \notin S \text{ and } \varpi \text{ generate a maximal ideal } m' \text{ of } T'$. 

Proof. This is equivalent with saying that $S_{\psi, \xi'}(U^p, \mathbb{F})[m']$ is non-zero. Since $\xi_v \equiv \xi'_v \mod \varpi$, we have $S_{\psi, \xi'}(U^p, \mathbb{F})[m'] = S_{\psi, \xi'}(U^p, \mathbb{F})[m]$, which is non-zero by our assumption. 

4.8.9. Therefore we get a non-zero surjective map $R^\infty_{\text{ps}, \{\xi'_v\}} \to T'_m$. Note that $R^\infty_{\text{ps}, \{\xi_v\}}/(\varpi) \cong R^\infty_{\text{ps}, \{\xi_v\}}/(\varpi)$ as both rings represent the same universal problem. Under this isomorphism, $q^\infty$ can be viewed as a prime ideal $q^\infty$ of $R^\infty_{\text{ps}, \{\xi_v\}}$ as $\varpi \in q^\infty$. The same argument of the previous lemma shows that $q^\infty$ comes from a prime ideal $q'$ of $T'$.

It is clear from the proof of proposition 4.3.2 that the same set of Taylor-Wiles primes $Q_N$ will also satisfy proposition 4.3.2 with $\xi_v$ replaced by $\xi'_v$, and we can choose the map $R^\{\xi'_v\}_{\text{loc}}[[x_1, \ldots, x_g]] \to R^\{\xi'_v\}_{\text{loc}}$ in proposition 4.3.2 to be the same as $R^\{\xi_v\}_{\text{loc}}[[x_1, \ldots, x_g]] \to R^\{\xi_v\}_{\text{loc}}$ after reducing mod $\varpi$, under the isomorphism $R^\{\xi'_v\}_{\text{loc}}/(\varpi) \cong R^\{\xi_v\}_{\text{loc}}/(\varpi)$ and the one similar for $R^\{\xi'_v\}_{\text{loc}}/(\varpi)$. Thus we can use the
same primes to patch our completed homology and get a diagram as in the beginning
of this subsection:

\[
\begin{array}{ccc}
(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}} & \longrightarrow & \text{End}_{S_\infty}(m_{\infty}^{\xi_v}) \\
\downarrow \pi'_{R} & & \downarrow \text{End}_{S_{\infty}}(m_{0}^{\xi_v}) \\
(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}} & \longrightarrow & \text{End}_{S_{\infty}}(m_{\infty}^{\xi_v})
\end{array}
\]

Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi) & \longrightarrow & \text{End}_{S_\infty}(m_{\infty}^{\xi_v}/\varpi m_{\infty}^{\xi_v}) \\
\downarrow \cong & & \downarrow \cong \\
(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi) & \longrightarrow & \text{End}_{S_{\infty}}(m_{\infty}^{\xi_v}/\varpi m_{\infty}^{\xi_v})
\end{array}
\]

under the natural isomorphisms $m_{\infty}^{\xi_v}/\varpi m_{\infty}^{\xi_v} \cong m_{\infty}^{\xi_v}/\varpi m_{\infty}^{\xi_v}$ and $(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi) \cong (R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi)$. Since $\xi_v'$ are all non-trivial, $m_{\infty}^{\xi_v}$ has full support on $(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi)$ by our previous result. Hence $m_{\infty}^{\xi_v}/\varpi m_{\infty}^{\xi_v}$ also has full support on $(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}/(\varpi)$ by the above diagram. By corollary 4.8.4, any minimal prime of the support of $m_{\infty}^{\xi_v}$ on $(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}$ has characteristic zero. It follows from proposition 4.2.4 that all minimal primes of $(R_{\infty}^{\xi_v})_{q_{\infty}^{\xi_v}}$ are in the support. Theorem 4.1.6 now follows from lemma 4.8.5.
Chapter 5

A generalization of results of Skinner-Wiles

In this section, we prove the modularity of some ordinary representations and some finiteness results, which partially generalize the work of Skinner-Wiles in [39]. We follow the beautiful method of [39] by establishing some “$R = T$” results for ordinary representations. One main difference in the proof is that we adopt Taylor’s Ihara avoidance trick [42] rather than the original level raising argument in [39]. Our main result (Theorem 5.1.2 below) completely removes the assumption in [39] that the reduction of $\psi_{v,1}$ modulo $\varpi$ is 1 for all $v | p$ (see the statement of Theorem 5.1.2 for the notation here). This requires some new results 5.6 on the existence of certain Eisenstein maximal ideal of the ordinary Hecke algebra.

The main result of this chapter will be a key ingredient of our proof of the modularity in the non-ordinary case (in chapter 6). In fact, we will use these ordinary points to find enough modular points to apply the result in the previous section. See chapter 6 for more details here. We note that the method in this section does not work (at least so far) in the non-ordinary case. This is because $p$-adic local Langlands correspondence is only established for $GL_2(Q_p)$. However one main step 5.7.6 in the
proof is to bound the $p$-part of the class group by taking suitable field extension, in which $p$ might not be split. On the other hand, Hida theory works well for all finite extensions of $\mathbb{Q}_p$ hence is more flexible with base change. This is why the ordinary case can be handled directly.

5.1 Statement of the main results

5.1.1. In this subsection, $F$ denotes an abelian totally real extension of $\mathbb{Q}$ in which $p$ is unramified. Let $S$ be a finite set of finite places containing all places above $p$. Let $\chi : G_{F,S} \to \mathcal{O}^\times$ be a continuous character such that

- $\chi(c) = -1$ for any complex conjugation $c \in G_{F,S}$.
- $\overline{\chi}$, the reduction of $\chi$ modulo $\varpi$, can be extended to a character of $G_{Q}$.
- $\overline{\chi}|_{G_{F,v}} \neq 1$ for any $v|p$.
- $\chi|_{G_{F,v}}$ is Hodge-Tate for any $v|p$. In other words, $\chi = \varepsilon^k \psi_0$ with $k$ an integer and $\psi_0$ a character of finite order.

Consider the universal deformation ring $R_{ps,ord}$ which pro-represents the functor from $C^\dagger_{O}$ to the category of sets sending $R$ to the set of two-dimensional pseudo-representations $T$ of $G_{F,S}$ over $R$ such that $T$ is a lifting of $1 + \overline{\chi}$ with determinant $\chi$ and $T|_{G_{F,v}}$ is reducible for any $v|p$. Denote the universal pseudo-representation by $T_{univ} : G_{F,S} \to R_{ps,ord}$.

Since $\overline{\chi}|_{G_{F,v}} \neq 1$, $T_{univ}|_{G_{F,v}} = \psi_{v,1}^{univ} + \psi_{v,2}^{univ}$ for some characters $\psi_{v,1}^{univ}, \psi_{v,2}^{univ} : G_{F,v} \to (R_{ps,ord})^\times$ which are liftings of $1, \overline{\chi}|_{G_{F,v}}$ respectively. by the class field theory, this induces a homomorphism $\mathcal{O}[[O_{F,v}^\times(p)]] \to R_{ps,ord}$ for any $v|p$. Here $O_{F,v}^\times(p)$ denotes the $p$-adic completion of $O_{F,v}^\times$. Taking the completed tensor product over $\mathcal{O}$ for all $v|p$, we get a map:

$$\Lambda_F := \bigotimes_{v|p} \mathcal{O}[[O_{F,v}^\times(p)]] \to R_{ps,ord}.$$
Now we can state the main results of this section:

**Theorem 5.1.2.** Under the assumptions for $F, \chi$ as above, we have

1. $R^{{\text{ps,ord}}}$ is a finite $\Lambda_F$-algebra.

2. For any maximal ideal $\mathfrak{p}$ of $R^{{\text{ps,ord}}}/(1, \mathfrak{p})$, we denote the associated semi-simple representation $G_{F,S} \to \text{GL}_2(k(\mathfrak{p}))$ by $\rho(\mathfrak{p})$ (see 2.1.5). Assume
   - $\rho(\mathfrak{p})$ is irreducible.
   - For any $v|\mathfrak{p}$, $\rho(\mathfrak{p})|_{G_{F_v}} \cong \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix}$ such that $\psi_{v,1}$ is Hodge-Tate and has strictly less Hodge-Tate number than $\psi_{v,2}$ for any embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$.

Then $\rho(\mathfrak{p})$ comes from a twist of a Hilbert modular form.

**Remark 5.1.3.** The condition that $p$ is unramified in $F$ can be weakened. However we decide to impose this condition here as this can simplify some arguments.

## 5.2 Hida families

As we remarked before, roughly speaking, the main results are proved by identifying $R^{{\text{ps,ord}}}$ with some ordinary Hecke algebra. We first collect some basic results for Hida families.

**5.2.1.** In this subsection, let $F$ be a totally real field of even degree over $\mathbb{Q}$ in which $p$ is unramified and $D$ be a totally definite quaternion algebra over $F$ which splits at all finite places. We fix isomorphisms $D \otimes_F F_v \cong M_2(F_v)$ for any finite place $v$.

Let $S$ be a finite set of places of $F$ that contains all places above $p$ and $U^p = \prod_{v|p} U_v$ be an open compact subgroup of $\prod_{v|p} \text{GL}_2(O_{F_v})$ such that $U_v = \text{GL}_2(O_{F_v})$ for $v \notin S$. 

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For any non-negative integer \(c\) and \(v\mid p\), we denote

\[
Iw_1(v^c) = \{ g \in \text{GL}_2(O_{F_v}), g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \varpi_v^c \},
\]

and \(U^p(c) = U^p \prod_{v\mid p} Iw_1(v^c)\), an open compact subgroup of \(\text{GL}_2(A_F)\).

Fix a continuous character \(\psi : (A_F^\infty)^\times / F^\times \to \mathcal{O}^\times\) such that \(\psi(a_p) = N_{F/Q}(a_p)^{-w}\) for some integer \(w\) and all \(a_p\) in some open subgroup of \(O_{F,p}^\times\). Here \(N_{F/Q} : O_{F,p}^\times \to \mathbb{Z}_p^\times \to \mathbb{E}^\times\) is the usual norm map. Recall that in 3.1.4, for \((\vec{k}, \vec{w}) \in \mathbb{Z}^\times \text{Hom}(F, \mathbb{Q}_p)^\times \times \mathbb{Z}^\times \text{Hom}(F, \mathbb{Q}_p)^\times\) such that \(k_\sigma + 2w_\sigma = w + 2\) independent of \(\sigma \in \text{Hom}(F, \mathbb{Q}_p)\), we defined an algebraic representation \(\tau(\vec{k}, \vec{w})\) of \(D_F = (D_F \otimes \mathbb{Q}_p)^\times\) on

\[
W(\vec{k}, \vec{w}), E = \bigotimes_{\sigma : F \to E} (\text{Sym}^{k_\sigma - 2}(E^2) \otimes \text{det}^{w_\sigma}).
\]

By abuse of notation, for any topological \(\mathcal{O}\)-algebra \(A\), we use \(\tau(\vec{k}, \vec{w})\) to denote the representation on \(W(\vec{k}, \vec{w}), A = \bigotimes_{\sigma : F \to E} (\text{Sym}^{k_\sigma - 2}(A^2) \otimes \text{det}^{w_\sigma})\). Then we can define \(S_{(\vec{k}, \vec{w}), \psi}(U^p(c), A)\) as in 3.1. Note that as we discussed in 3.1.4 \(S_{(\vec{k}, \vec{w}), \psi}(U^p(c), E)\) can be considered as a space of automorphic forms.

For any \(\gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times\), we define \((\gamma) \in \text{End}_A(S_{(\vec{k}, \vec{w}), \psi}(U^p(c), A))\) to be the double coset

\[
\langle \gamma \rangle = \prod_{\sigma \in \text{Hom}(F, \mathbb{Q}_p)} \sigma(\gamma)^{-w_\sigma} U^p(c) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} U^p(c).
\]

Explicitly, since \(U^p(c) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} U^p(c) = \bigsqcup_{\alpha \in O_{F,p}/(\gamma)} \begin{pmatrix} \gamma & \alpha \\ 0 & 1 \end{pmatrix} U^p(c),\)

\[
((\gamma) \cdot f)(g) = \prod_{\sigma \in \text{Hom}(F, \mathbb{Q}_p)} \sigma(\gamma)^{-w_\sigma} \sum_{\alpha \in O_{F,p}/(\gamma)} \tau_{(\vec{k}, \vec{w})}(\begin{pmatrix} \gamma & \alpha \\ 0 & 1 \end{pmatrix}) \cdot f(g \begin{pmatrix} \gamma & \alpha \\ 0 & 1 \end{pmatrix}).
\]
This action is independent of $c$ and defines a morphism of monoids $O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times \to \text{End}_A(S_{(\bar{k},\bar{w}),\psi}(U^p(c), A))$. Hida’s idempotent $e$ is defined to be

$$e := \lim_{n \to +\infty} \langle p \rangle^n \in \text{End}_A(S_{(\bar{k},\bar{w}),\psi}(U^p(c), A)).$$

For any topological $\mathcal{O}$-algebra $A$ and $k = (\bar{k}, \bar{w})$ as above, the space of ordinary forms is defined to be

$$S^\text{ord}_{k,\psi}(U^p(c), A) := eS_{k,\psi}(U^p(c), A).$$

Note that $\langle p \rangle$ acts by a unit on this space. Hence the morphism of monoids $O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times \to \text{End}_A(S_{k,\psi}(U^p(c), A))$ extends to a homomorphism

$$\langle \cdot \rangle : (F \otimes \mathbb{Q}_p)^\times \to \text{End}_A(S^\text{ord}_{k,\psi}(U^p(c), A)).$$

We define the ordinary Hecke algebra $T^\text{ord}_{k,\psi}(U^p(c), A)$ to be the $A$-subalgebra generated by $T_v$, $v \not\in S$ (see 3.3) and $\langle \gamma \rangle$, $\gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times$. Hence $\langle \cdot \rangle$ induces a smooth character $F^\times_v \to T^\text{ord}_{k,\psi}(U^p(c), A)^\times$. by the class field theory, this defines a character:

$$\psi_{v,1} : G_{F,v} \to T^\text{ord}_{k,\psi}(U^p(c), A)^\times.$$

If $A = \mathcal{O}$, there is a two-dimensional pseudo-representation with determinant $\psi^{-1}$

$$T_c : G_{F,S} \to T^\text{ord}_{k,\psi}(U^p(c), \mathcal{O})$$

sending $\text{Frob}_v$ to $T_v$.

5.2.2. Now we can introduce Hida families. We define

- $S^\text{ord}_{\psi}(U^p, E/\mathcal{O}) := \lim_{\gamma,c,n>0} S^\text{ord}_{k,\psi}(U^p(c), \omega^{-n}\mathcal{O}/\mathcal{O})$.
- $M^\text{ord}_{\psi}(U^p) := S^\text{ord}_{\psi}(U^p, E/\mathcal{O})^\vee$.
\[ \mathbb{T}_\psi^{\text{ord}}(U^p) := \lim_{\leftarrow c} \mathbb{T}_{k,\psi}^{\text{ord}}(U^p(c), \mathcal{O}) \] (Hecke algebra).

It is well-known (cf. theorem 2.3 of [21]) that these are all independent of the choice of \( k = (\vec{k}, \vec{w}) \). By abuse of notation, we also denote by \( \psi_v,1 : G_{F_v} \rightarrow \mathbb{T}_\psi^{\text{ord}}(U^p) \) the character induced by \( \langle \cdot \rangle \). Moreover there is a two-dimensional pseudo-representation with determinant \( \psi \varepsilon^{-1} \)

\[ T^{\text{ord}} : G_{F,S} \rightarrow \mathbb{T}_\psi^{\text{ord}}(U^p) \]

sending \( \text{Frob}_v \) to \( T_v \), such that for any \( p \in \text{Spec} T^{\text{ord}}(U^p) \), its associated two-dimensional semi-simple representation \( \rho(p) \) has the property

\[ \rho(p)|_{G_{F_v}} \cong \begin{pmatrix} \psi_v,1 & * \\ 0 & * \end{pmatrix}, v|p. \]

See proposition 2.3 of [20], though our convention for Hecke algebra is slightly different.

Let \( \Lambda_F = \bigotimes_{v \mid p} \mathcal{O}[[O_{F_v}(p)]] \) defined in the previous subsection. Then there is a natural map

\[ \Lambda_F \rightarrow \mathbb{T}_\psi^{\text{ord}}(U^p(c)) \]

induced by \( \psi_v,1 \). Hence \( M_\psi^{\text{ord}}(U^p) \) has a \( \Lambda_F \)-module structure. It is well-known (cf. theorem 3.8 of [21], though the determinant is not fixed there) that \( M_\psi^{\text{ord}}(U^p) \) is in fact finite free over \( \Lambda_F \). Hence \( \mathbb{T}_\psi^{\text{ord}}(U^p) \) is a finite torsion-free \( \Lambda_F \)-algebra with same dimensions \([F : \mathbb{Q}] + 1\).

5.2.3. The ordinary forms at finite level \( \mathbb{T}_k,\psi(U^p(c), \mathcal{O}) \) can be recovered in the following way (cf. theorem 2.4 [21]): for \( k = (\vec{k}, \vec{w}) \in \mathbb{Z}_{\geq 1}^{\text{Hom}(F, \overline{\mathbb{Q}}_p)} \times \mathbb{Z}^{\text{Hom}(F, \overline{\mathbb{Q}}_p)} \) such that \( k_\sigma + 2w_\sigma = w + 2 \) independent of \( \sigma \in \text{Hom}(F, \overline{\mathbb{Q}}_p) \), let \( a_{k,c} \) be the ideal of \( \Lambda_F \) generated by \( \langle \gamma \rangle - \prod_\sigma \sigma(\gamma)^{-w_\sigma}, \gamma \in 1 + p^rO_{F,p} \). Then

\[ M_\psi^{\text{ord}}(U^p)/a_{k,c}M_\psi^{\text{ord}}(U^p) \cong S_{k,\psi}(U^p(c), E/\mathcal{O})^\vee. \]
Roughly speaking, finite level ordinary forms correspond to the locus where $\psi_{v,1}$ are locally algebraic characters with certain weights.

5.2.4. By the same argument as in the proof of proposition 3.3.6, we know that $T^\text{ord}_\psi(U^p)$ is a complete semi-local ring. Let $m_1, \ldots, m_s$ be its maximal ideals. Then $T^\text{ord}_\psi(U^p) = T^\text{ord}_\psi(U^p)_{m_1} \times \cdots \times T^\text{ord}_\psi(U^p)_{m_s}$ and for any maximal ideal $m$, $M^\text{ord}_\psi(U^p)_m$ is a direct summand of $M^\text{ord}_\psi(U^p)$ and hence finite free over $\Lambda_F$. Since $\Lambda_F$ is regular, we have

$$\text{depth}_{T^\text{ord}_\psi(U^p)_m}(M^\text{ord}_\psi(U^p)_m) \geq \text{depth}_{\Lambda_F}(M^\text{ord}_\psi(U^p)_m) = \dim \Lambda_F = \dim T^\text{ord}_\psi(U^p)_m.$$ 

Therefore $M^\text{ord}_\psi(U^p)_m$ is a Cohen-Macaulay $T^\text{ord}_\psi(U^p)_m$-module. Note that $M^\text{ord}_\psi(U^p)_m$ is also a faithful $T^\text{ord}_\psi(U^p)_m$-module. By Theorem 17.3 of [29], for any minimal prime $p$ of $T^\text{ord}_\psi(U^p)_m$

$$\dim T^\text{ord}_\psi(U^p)_m/p = \dim T^\text{ord}_\psi(U^p)_m = \dim \Lambda_F = [F : \mathbb{Q}] + 1.$$ 

$p$ must be of characteristic zero as $\varpi \in T^\text{ord}_\psi(U^p)_m$ is a regular element. Thus we get (compare this with theorem 3.6.1)

**Corollary 5.2.5.** $T^\text{ord}_\psi(U^p)_m$ is equidimensional of dimension $[F : \mathbb{Q}] + 1$. Moreover, any minimal prime ideal has characteristic zero.

5.2.6. As in section 3.7, we also need a variant of the space of ordinary forms. Let $\xi_v : U_v \to O^\times$ be smooth characters for $v \in S \setminus \Sigma_p$. Then product of $\xi_v$ defines a character $\xi$ of $U^p$. We define $S^\text{ord}_{k,\psi,\xi}(U^p(c), A)$ to be $eS_{k,\psi,\xi}(U^p(c), A)$. Similarly we can define $S^\text{ord}_{\psi,\xi}(U^p, E/O), M^\text{ord}_{\psi,\xi}(U^p), T^\text{ord}_{\psi,\xi}(U^p)$ as above and have $\dim T^\text{ord}_{\psi,\xi}(U^p)_m = [F : \mathbb{Q}] + 1$ for any maximal ideal $m$ of $T^\text{ord}_{\psi,\xi}(U^p)$. 

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5.3 Patching at a one-dimensional prime: ordinary case

In this subsection, we are going to prove a result similar to Theorem 4.1.6 in the ordinary setting, which basically says an irreducible component of $R^{\text{ord}}_{\Sigma^p,\Sigma^\circ}(\xi_v)$ (see below for the definition) is modular if it contains a nice prime.

5.3.1. Setup (Compare with 4.1) Let $F, D$ be as in the previous section and $S$ be a finite set of places of $F$ containing all places above $p$ such that for any $v \in S \setminus \Sigma_p$,

$$N(v) \equiv 1 \mod p.$$ 

Let $\xi_v : k(v)^\times \to \mathcal{O}^\times$ be a character of $p$-power order for each $v \in S \setminus \Sigma_p$. We will view $\xi_v$ as characters of $I_{F_v}$ by the class field theory. We also fix a complex conjugation $\sigma^* \in G_F$.

Fix a continuous character $\chi : G_{F,S} \to \mathcal{O}^\times$ such that

- $\chi$ is unramified at places outside of $\Sigma_p$.
- $\chi(\text{Frob}_v) \equiv 1 \mod \varpi$ for $v \in \Sigma_p \setminus S$.
- $\chi(c) = -1$ for any complex conjugation $c \in G_F$.
- For any $v | p$, $\bar{\chi}|_{G_{F_v}} \neq 1$ and $\chi|_{G_{F_v}}$ is Hodge-Tate. Here $\bar{\chi}$ denotes the reduction of $\chi$ modulo $\varpi$.
- $[F_v : \mathbb{Q}_p] \geq 2$ for any $v | p$.

Let $\Sigma^\circ$ be a subset of $\Sigma_p$. We define $\bar{\psi}_{v,1} : G_{F_v} \to \mathbb{F}^\times$ to be $1$ if $v \in \Sigma^\circ$ and $\bar{\chi}|_{G_{F_v}}$ if $v \not\in \Sigma^\circ$.

For a pseudo-representation, we have the following definition for it being ordinary.
Definition 5.3.2. Let $T : G_F \to R$ be a two-dimensional pseudo-representation over some ring $R$ such that for some place $v$, $T|_{G_{F_v}} = \psi_1 + \psi_2$ is a sum of two characters. We say $T$ is $\psi_1$-ordinary if for any $\sigma, \tau \in G_{F_v}$, $\eta \in G_F$,

$$T(\sigma \tau \eta) - \psi_1(\sigma)T(\tau \eta) - \psi_2(\tau)T(\sigma \eta) + \psi_1(\sigma)\psi_2(\tau)T(\eta) = 0.$$ 

Lemma 5.3.3. Let $\rho : G_F \to \text{GL}_2(R)$ be a two-dimensional representation over some ring $R$ with trace $T$ such that $T|_{G_{F_v}} = \psi_1 + \psi_2$, a sum of two characters. Suppose $\rho|_{G_{F_v}}$ has the form

$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$ 

Then $T$ is $\psi_1$-ordinary. Conversely, if $R$ is a field and $T$ is $\psi_1$-ordinary, then after possibly enlarging $R$,

$$\rho|_{G_{F_v}} \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

Proof. The first claim follows from some simple computation. For the second claim, we may assume $\psi_1 \neq \psi_2$. Enlarge $R$ if possible, then under suitable basis, we may assume $\rho|_{G_{F_v}} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ or $\rho|_{G_{F_v}} = \begin{pmatrix} \psi_2 & b \\ 0 & \psi_1 \end{pmatrix}$ for some $b : G_{F_v} \to R$. Suppose we are in the second case, the ordinary condition is the same as

$$\text{tr}[(\rho(\sigma) - \psi_1(\sigma))(\rho(\tau) - \psi_2(\tau))\rho(\eta)] = 0$$

for any $\sigma, \tau \in G_{F_v}, \eta \in G_F$. Equivalently, if we write $\rho(\eta) = \begin{pmatrix} * & * \\ c & * \end{pmatrix}$ and $\phi = \psi_1 - \psi_2$, then

$$(b(\sigma)\phi(\tau) - b(\tau)\phi(\sigma))c = 0.$$ 

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Since $\rho$ is irreducible, we can always choose $\eta$ so that $c \neq 0$, hence $b(\sigma)\phi(\tau) - b(\tau)\phi(\sigma) = 0$. Choose $\sigma \in G_{F_v}$ such that $\phi(\sigma) \neq 0$, then for any $\tau \in G_{F_v}$,

$$
\begin{pmatrix}
0 & 1 \\
1 & \frac{b(\sigma)}{\phi(\sigma)}
\end{pmatrix}
\rho(\tau)
\begin{pmatrix}
-\frac{b(\sigma)}{\phi(\sigma)} & 1 \\
1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
\psi_1(\tau) & 0 \\
0 & \psi_2(\tau)
\end{pmatrix}.
$$

This is exactly what we want. \qed

5.3.4. Now consider the universal deformation ring $R^{ps,ord}_{\Sigma_0}$ which pro-represents the functor from $C^0_F$ to the category of sets sending $R$ to the set of tuples $(T, \{\psi_v, 1\}_{v \in \Sigma_p})$ where

- $T$ is a two-dimensional pseudo-representation of $G_{F,S}$ over $R$ lifting $1 + \bar{\chi}$ with determinant $\chi$ and for any $v \in S \setminus \Sigma_p$,

$$
T|_{I_{F_v}} = \xi_v + \xi_v^{-1}.
$$

- For any $v|p$, $\psi_{v,1} : G_{F_v} \to R^\times$ is a character which lifts $\bar{\psi}_{v,1}$ and satisfies

$$
T|_{G_{F_v}} = \psi_{v,1} + \psi_{v,1}^{-1}\chi.
$$

Moreover $T$ is $\psi_{v,1}$-ordinary.

We remark that $R^{ps,ord}_{\Sigma_0,\{\xi_v\}}$ is a quotient of $R^{ps,ord}_{\Sigma_0}$ (defined in 5.1) as all $\psi_{v,1}$ already take values over $R^{ps,ord}$. There is a natural map $\Lambda_F \to R^{ps,ord}_{\Sigma_0,\{\xi_v\}}$ coming from the universal lifting of $\bar{\psi}_{v,1}$ when restricted to the inertia. Note that this might be different from the one considered in 5.1 unless $\Sigma_0$ is empty. Since we have fixed a complex conjugation $\sigma^* \in G_{F,S}$, we can attach a semi-simple representation $\rho(p) : G_{F,S} \to \text{GL}_2(k(p))$ for any $p \in \text{Spec}R^{ps,ord}_{\Sigma_0,\{\xi_v\}}$ with trace $T^{\text{univ}} \mod p$ as in 2.1.5. Here $T^{\text{univ}} : G_{F,S} \to R^{ps,ord}_{\Sigma_0,\{\xi_v\}}$ is the universal trace.

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5.3.5. On the automorphic side, we put tame level \( U^p = \prod_{v \notin \Sigma_p} U_v \) with \( U_v = \text{GL}_2(\mathcal{O}_{F_v}) \) if \( v \notin S \) and \( U_v = \text{Iw}_v \) if \( v \in S \setminus \Sigma_p \) as in [4.1]. For any \( v \in S \setminus \Sigma_p \), the map \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \xi_v(\frac{a}{d} \mod \varpi_v) \) defines a character of \( U_v \). The product of \( \xi_v \) can be viewed as a character \( \xi \) of \( U^p \) by projecting to \( \prod_{v \notin S \setminus \Sigma_p} U_v \). By global class field theory, we may view \( \psi = \chi \varepsilon \) as a character of \( (\mathbb{A}_F^\infty)^\times / F^\times_{>0} \). Then we can consider a space of ordinary forms \( M_{\psi, \xi}(U^p) \) and ordinary Hecke algebra \( T_{\text{ord}} := \prod_{\psi, \xi} (U^p) \) as in the previous subsection. Recall that \( \langle \rangle \) induces characters \( \psi_{v, 1} : G_{F_v} \to (T_{\text{ord}})^\times \) for \( v | p \) and homomorphism \( \Lambda_F \to T_{\text{ord}} \).

We make the following assumption here. We will see later that this assumption holds after making suitable field extension of \( F \).

- **Assumption**: \( T_v - (1 + \chi(\text{Frob}_v)), v \notin S \) and \( \psi_{v, 1}(\gamma) - \bar{\psi}_{v, 1}(\gamma), \gamma \in F_v^\times, v | p \) and \( \varpi \) generate a maximal ideal \( m \) of \( T_{\text{ord}} \). Here \( \bar{\psi}_{v, 1}(\gamma) \in \mathcal{O} \) is any lifting of \( \bar{\psi}_{v, 1}(\gamma) \in \mathbb{F} \).

As we recalled before, there is a two-dimensional pseudo-representation \( T_{m, \text{ord}} : G_{F,S} \to T_{m, \text{ord}} \) with determinant \( \chi \) sending \( \text{Frob}_v \) to \( T_v \). Moreover \( T_{m, \text{ord}} \) is \( \psi_{v, 1} \)-ordinary and \( T_v|_{I_{F_v}} = \xi_v + \xi_v^{-1} \). These results can be checked on the finite level. Therefore we get a map

\[
R_{\Sigma_0}^{\text{ps,ord}, \{ \xi_v \}} \to T_{m, \text{ord}}
\]

which is necessarily surjective since the topological generators \( T_v, v \notin S \) and \( \psi_{v, 1}(F_v^\times), v | p \) are in the image. This map is also a \( \Lambda_F \)-algebra homomorphism.

5.3.6. We say a prime \( p \in \text{Spec} R_{\Sigma_0}^{\text{ps,ord}, \{ \xi_v \}} \) is \textit{modular} if it comes from a prime of \( T_{m, \text{ord}} \). Recall that in [4.1] we call a prime \( q \in \text{Spec} T_{m, \text{ord}} \) \textit{nice} if \( T_{m, \text{ord}}/q \) is one-dimensional of characteristic \( p \) and there exists a two-dimensional representation

\[
\rho(q)^{\circ} : G_{F,S} \to \text{GL}_2(A)
\]

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satisfying the following properties:

- \( \rho(q) \otimes k(q) \cong \rho(q) \) is irreducible. In other words, \( \rho(q) \otimes \) is a lattice of \( \rho(q) \).

- The mod \( m_A \) reduction \( \bar{\rho}_b \) of \( \rho(q) \) is a non-split extension and has the form

\[
\bar{\rho}_b(g) = \begin{pmatrix}
* & * \\
0 & *
\end{pmatrix}, \quad g \in G_{F,S}.
\]

Here \( m_A \) is the maximal ideal of \( A \).

- \( \rho(q)|_{G_{F_v}} = \bar{\rho}_b|_{G_{F_v}} \) for any \( v \in S \setminus \Sigma_p \) under the canonical inclusion map \( GL_2(F) \to GL_2(k(q)) \).

Note that this is slightly different from 4.1.4: we drop the third condition in the original definition. This is because that condition is automatically true here: by definition, the image of \( q \) under the finite map \( \text{Spec} T_{\text{ord}} \to \text{Spec} \Lambda_F \) is not the maximal ideal. Hence \( \psi_v \mod q \) is of infinite order for at least one \( v|p \). It then follows from the third part of lemma 4.1.9 that \( \rho(q) \) is not dihedral.

The main technical result we are going to prove in this section is:

**Proposition 5.3.7.** Under all the assumptions in this subsection, if \( q \in \text{Spec} T_{\text{ord}} \) is a nice prime, then

\[
(R_{\Sigma_0}^{p,s,\text{ord},\{\xi_v\}})_q \to \tau_{q}^{\text{ord}}
\]

has nilpotent kernel. Here \( q^{ps} = q \cap R_{\Sigma_0}^{p,s,\text{ord},\{\xi_v\}} \).

**5.3.8.** The proof of proposition 5.3.7 is almost the same as the proof of theorem 4.1.6 with completed cohomology replaced by Hida family. We decide to omit the details here. Rather, we would like to point out some technical differences between these two cases.
1. In the definition 4.2.3 of $R_{\text{loc}}^{\{\xi_v\}}$, we need to replace the unrestricted deformation rings $R_{v|p}$ by *ordinary deformation rings* $R_{v}^\Delta$ which pro-represents the functor from $C^\Delta_{\text{loc}}$ to the category of sets sending $R$ to the set of pairs $(\rho_R, \psi_R)$ such that

- $\rho_R : G_{F_v} \to \GL_2(R)$ is a lifting of $\bar{\rho}_{b|G_{F_v}}$ with determinant $\chi|_{G_{F_v}}$.
- $\psi_R : G_{F_v} \to R^\times$ is a lifting of $\bar{\psi}_{\psi,1}$ such that $\rho_R$ has a (necessarily unique) $G_{F_v}$-stable rank one $R$-submodule, which is a direct summand of $R^\oplus 2$ as a $R$-module, with $G_{F_v}$ acting via $\psi_R$.

The second part of lemma 4.2.9 is now replaced by:

- $R_{v}^\Delta$ is a *normal* domain of dimension $1 + 2[F_v : \Q_p] + 3$ and flat over $\mathcal{O}$.

The proof will be given below (5.3.9). Once this is proved, proposition 4.2.4 in this case can be established in exactly the same way.

2. As for global deformation rings considered in 4.2.12, we also need to take quotients of them by ordinary conditions. More precisely, instead of using $R_{\bar{\rho}_b, Q}^{\square, \{\xi_v\}}$, we should consider

$$R_{\bar{\rho}_b, Q}^{\Delta, \{\xi_v\}} := R_{\bar{\rho}_b, Q}^{\square, \{\xi_v\}} \otimes_{\bigotimes_{v|p} R_v^\square} \left( \bigotimes_{v|p} R_v^\Delta \right)$$

and unframed deformation rings

$$R_{\bar{\rho}_b, Q}^{\Delta, \{\xi_v\}} := R_{\bar{\rho}_b, Q}^{\{\xi_v\}} \otimes_{\bigotimes_{v|p} R_v^\square} \left( \bigotimes_{v|p} R_v^\Delta \right).$$

We will omit $Q$ in the subscript if $Q$ is empty. Note that such ordinary conditions are not defined via traces, hence the natural surjective map

$$R_{\bar{\rho}_b}^{\{\xi_v\}} \otimes_{R^{\square, \{\xi_v\}}} R_{\Sigma^\text{ord}}^{\text{ps}, \{\xi_v\}} \to R_{\bar{\rho}_b}^{\Delta, \{\xi_v\}}.$$
might not be an isomorphism. Thus in the proof of corollary 4.6.13 we cannot apply corollary 2.3.8 which is a consequence of proposition 2.3.2 directly. However, it follows from lemma 5.3.3 that the kernel is nilpotent and an easy computation shows that the kernel of the above map is killed by the element \( c \) in proposition 2.3.2. Therefore corollary 4.6.13 still can be proved in the same way.

3. In the patching argument, since \( M_{\psi,\xi}^{\text{ord}}(U_p)_m \) is already finite over the Hecke algebra, we can work with the patched Hida family directly. In fact, the whole point of using Paškūnas’ theory in 4.7 is to make completed homology into some module which is finitely generated over the Hecke algebra. The role of \( \Lambda_F \) in Hida’s theory is replaced by \( R_{\psi,\varepsilon}^{-1} \) in the completed cohomology side (see theorem 3.5.5).

**Lemma 5.3.9.** \( R^\Delta_v \) is a normal domain of dimension \( 1 + 2[F_v : \mathbb{Q}_p] + 3 \) and flat over \( \mathcal{O} \) if \( [F_v : \mathbb{Q}_p] \geq 2 \).

**Proof.** We will only deal with the case \( \bar{\psi}_{v,1} = 1 \) (the other case \( \bar{\psi}_{v,1} = \bar{\chi} \) is similar). Since we assume \( \bar{\chi}|_{G_{F_v}} \neq 1 \), for any lifting \( (\rho_R, \psi_R) \) to some \( R \in \mathcal{C}_\mathcal{O} \) as in the definition of \( R^\Delta_v \), we may find a unique matrix \( U = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \) such that \( U^{-1}\rho_RU \) is of the form

\[
\begin{pmatrix}
\psi_R & * \\
0 & *
\end{pmatrix}
\]. Hence \( R^\Delta_v \cong R^B_v[[n]] \), where \( R^B_v \) parametrizes all liftings of \( \bar{\rho}_b|_{G_{F_v}} \) into the Borel subgroup \( B \) of upper triangular matrices with determinant \( \chi|_{G_{F_v}} \).

It is well known that we can write \( R^B_v \cong \mathcal{O}[[x_1, \cdots, x_g]]/(f_1, \cdots, f_r) \), where \( g - r = 2[F_v : \mathbb{Q}_p] + 2 \) and \( r = \dim H^2(G_{F_v}, \text{ad}^0 \bar{\rho}_b^B) \). Here \( \bar{\rho}_b^B \) is \( \bar{\rho}_b|_{G_{F_v}} \) viewed as a representation of \( G_{F_v} \) into the Borel subgroup \( B(\mathbb{E}) \) and \( \text{ad}^0 \bar{\rho}_b^B \) denotes the trace zero subspace of its adjoint representation. There is a canonical exact sequence as \( G_{F_v} \)-representations:

\[
0 \to \bar{\chi}^{-1} \to \text{ad}^0 \bar{\rho}_b^B \to 1 \to 0.
\]
Hence $r \leq 1$. The lemma is clear if $\tilde{\chi} \neq \omega^{-1}$ as $r = 0$ in this case.

Now we assume $\tilde{\chi} = \omega^{-1}$. From $R_v^B \cong O[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$, it is easy to see that each irreducible component of $R_v^B$ has dimension at least $2[F_v : \mathbb{Q}_p] + 3$. We claim that $R_v^B/(\varpi)$ is normal of dimension $2[F_v : \mathbb{Q}_p] + 2$. Assuming this, it is clear from comparing the dimensions that $\varpi$ is a regular element in $R_v^B$. This implies that $R_v^B$ is an $O$-flat normal algebra of dimension $2[F_v : \mathbb{Q}_p] + 3$ by Serre’s criterion for normality. Hence we are left to show our claim for $R_v^B/(\varpi)$. Since $R_v^B/(\varpi) \cong \mathbb{F}[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$ with $r \leq 1$, we see that $R_v^B/(\varpi)$ is a complete intersection ring. It suffices to show that the non-regular locus has codimension at least 2 and $\dim R_v^B/(\varpi) = 2[F : \mathbb{Q}_p] + 2$.

Write $R_1 = R_v^B/(\varpi)$. Given a prime ideal $\mathfrak{q}$ of Spec $R_1$ such that $R_1/\mathfrak{q}$ has dimension one, we use $\rho(\mathfrak{q})^B : G_{F_v} \to B(k(\mathfrak{q}))$ to denote the push-forward of the universal lifting over $R_v^B$ to $k(\mathfrak{q})$. We need the following result concerning the smoothness of $(R_1)_\mathfrak{q}$.

**Lemma 5.3.10.** $(R_1)_\mathfrak{q}$ is regular of dimension $2[F_v : \mathbb{Q}_p] + 1$ if $\rho(\mathfrak{q})^B$ is not of the form

$$
\begin{pmatrix}
1 & * \\
0 & \omega^{-1}
\end{pmatrix}
$$

**Proof.** We will relate the $\mathfrak{q}$-adic completion of $(R_1)_\mathfrak{q}$ with some other universal lifting ring. First by enlarging $\mathbb{F}$ if necessary, we may assume the normal closure of $R_1/\mathfrak{q}$ is isomorphic to $A = \mathbb{F}[[T]]$ (as $\mathbb{F}$-algebras). Fix such an isomorphism and hence an embedding $A \hookrightarrow k(\mathfrak{q})$.

Clearly $R_2 := R_1 \otimes_{\mathbb{F}} A \cong R_1[[T_1]]$ pro-represents the functor assigning each Artinian local $A$-algebra $R$ with residue field $\mathbb{F}$ to the set of continuous homomorphisms of $G_{F_v}$ to $B(R)$ with determinant $\chi$ that lift $\tilde{\rho}_B^B$. Viewed as taking values in $\text{GL}_2(A)$, the representation $\rho(\mathfrak{q})^B$ induces a prime ideal $\mathfrak{p}$ of $R_2$. We claim that the $\mathfrak{p}$-adic completion $(\hat{R}_2)_\mathfrak{p}$ is the universal lifting ring for $\rho(\mathfrak{q})^B : G_{F_v} \to \text{GL}_2(k(\mathfrak{q}))$ (with determinant $\chi$), i.e. if $R$ is an Artinian local $k(\mathfrak{q})$-algebra with residue field $k(\mathfrak{q})$ and
if $\rho^B : G_{F_v} \to B(R)$ is a continuous lifting of $\rho(q)^B$ with determinant $\chi$, then there exists a unique map of $k(q)$-algebras $\widehat{(R_2)_p} \to R$ such that the push forward of the universal lifting on $R_2$ is $\rho^B$. The proof is standard: let $R^0$ be the $A$-subalgebra of $R$ generated by the matrix entries of $\rho^B$. This is a finite local $A$-algebra with residue field $\mathbb{F}$ and $\rho^B$ can be viewed as taking values in $B(R^0)$. Hence we get a natural map $R_2 \to R^0$ and our claim follows easily. In particular, under our assumption on $\rho(q)^B$, the deformation ring $\widehat{(R_2)_p}$ is formally smooth of relative dimension $2[F_v : \mathbb{Q}_p] + 2$ over $k(q)$.

Let $a \in (R_1)_q$ be a lifting of $T \in A \hookrightarrow (R_1)_q/\mathfrak{q}$. Using the natural map $(R_1)_q \to (R_2)_p$, we also view $a$ as an element of $\widehat{(R_2)_p}$ by abuse of notation. We claim that

1. The map sending $T_1$ to $a$ induces an isomorphism $\widehat{(R_2)_p}/(T_1 - a) \sim \to (R_1)_q$.

2. $T_1 - a \notin (\mathfrak{p}(R_2)_p)^2$.

Note that these two claims imply lemma 5.3.10. For the second assertion, we have

$$(R_2/qR_2) \otimes_{R_1} k(q) \cong (R_1/q)[[T_1]] \otimes_{R_1} k(q) \cong \mathbb{F}[[T, T_1]][\frac{1}{T}]$$

Under the above isomorphism, $T_1 - a$ is identified with $T_1 - T$ in $\mathbb{F}[[T, T_1]][\frac{1}{T}]$ and the image of $\mathfrak{p}$ is generated by $T_1 - T \neq 0$. From this explicit description, the assertion is clear.

For the first claim, we use $B_q$ to denote $B \otimes_{R_1} (R_1)_q$ for any $R_1$-algebra $B$. It follows from the previous paragraph that $\mathfrak{p}(R_1[[T_1]])_q = (\mathfrak{q}, T_1 - a)$ is a maximal ideal in $(R_1[[T_1]])_q$. Hence it suffices to prove that for any integer $n > 0$,

$$(R_1[[T_1]])_p/(\mathfrak{q}^n, T_1 - a) = (R_1[[T_1]])_q/(\mathfrak{q}^n, T_1 - a) \sim \to (R_1/q^n)_q.$$

It is easy to see that $T^k \in R_1/q$ for some integer $k$ as $T \in k(q)$. Hence there exists some element $b$ in the maximal ideal of $R_1$ such that $a^k - b \in q(R_1)_q$. Thus
\[(T^k_1 - b)^n = 0 \text{ in } (R_1[[T_1]])_q/(q^n, T_1 - a). \] Using this, we get
\[
(R_1[[T_1]])_q/(q^n, T_1 - a) \cong (R_1[T_1])_q/(q^n, T_1 - a) \cong (R_1/q^n)_q.
\]

Back to the proof of lemma 5.3.9. The locus where \(\rho(q)^B\) is of the form
\[
\begin{pmatrix}
1 & * \\
0 & \omega^{-1}
\end{pmatrix}
\]
has dimension \(\dim_Z Z^1(G_{F_v}, \omega) = [F_v : \mathbb{Q}_p] + 2 \leq 2[F_v : \mathbb{Q}_p]\). Since each irreducible component of \(R_1\) has dimension at least \(2[F_v : \mathbb{Q}_p] + 2\), we conclude from lemma 5.3.10 that the non-regular locus has codimension at least 2. This finishes the proof of the lemma.

\[\square\]

5.4 \(R^{\text{ord}}_b\) is modular for one \(b\)

It follows from the main result of the previous section that if we could show any irreducible component of \(R^{\text{ps,ord},\{\xi_v\}}\) has a nice prime, then any prime of \(R^{\text{ps,ord},\{\xi_v\}}\) would be modular. However, the geometry of \(\text{Spec } R^{\text{ps,ord},\{\xi_v\}}\) seems a bit hard to control. On the other hand, the ring-theoretic property of the usual global deformation ring \(R^{\text{ord}}_b\) (see the precise definition below) can be obtained by Galois cohomologies. The goal of this section is to show that under certain assumptions, the image of \(\text{Spec } R^{\text{ord}}_b\) in \(\text{Spec } R^{\text{ps,ord},\{\xi_v\}}\) is modular for one extension class \(b\).

5.4.1. Keep all the notations \(F, D, \chi, S, \Sigma^o, \sigma^*\) as in the previous subsection. Also we fix a complex conjugation \(\sigma^* \in G_{F,S}\). We will define some global deformation ring \(R^{\text{ord}}_b\) and discuss its connectedness dimension and the dimension of its reducible locus. We first introduce the following Selmer group. Recall that \(\Sigma^o\) is a subset of \(\Sigma_p\).
Definition 5.4.2.

\[ H_{\Sigma^o}^1(F) := \ker(H^1(G_{F,S}, \mathbb{F}(\bar{\chi}^{-1})) \xrightarrow{\text{res}} \bigoplus_{v \in \Sigma_p \setminus \Sigma^o} H^1(G_{F_v}, \mathbb{F}(\bar{\chi}^{-1}))). \]

It is easy to see that there is a bijection between \( H_{\Sigma^o}^1(F) \) and the group of extensions

\[ 0 \to 1 \to \bar{\rho} \to \bar{\chi} \to 0 \]

as \( \mathbb{F}[G_{F,S}] \)-modules such that \( \bar{\rho} \) is \( \bar{\psi}_{v,1} \)-ordinary for \( v|p \). The bijection is given as follows: any cohomology class \( b = [\phi_b] \in H_{\Sigma^o}^1(F) \) defines a two-dimensional representation

\[ \bar{\rho}_b : G_{F,S} \to \text{GL}_2(\mathbb{F}), \ g \mapsto \begin{pmatrix} 1 & \phi_b(g)\bar{\chi}(g) \\ 0 & \bar{\chi}(g) \end{pmatrix}. \]

Since \( \bar{\chi}(\sigma^*) = -1 \), we can always choose \( \phi_b \) so that \( \phi_b(\sigma^*) = 0 \). We will keep this assumption for \( \bar{\rho}_b \) from now on.

5.4.3. For any non-zero \( b \in H_{\Sigma^o}^1(F) \), we will write \( R^\text{ord}_b \) for \( R^{\Delta,(\xi_v)}_{\bar{\rho}_b} \) introduced in the second part of 5.3.8. More precisely, it represents the functor from \( C^f_O \) to the category of sets sending \( R \) to the set of tuples \( (\rho_R, \{\psi_{v,1}\}_{\Sigma_p}) \) where

- \( \rho_R : G_{F,S} \to \text{GL}_2(R) \) is a lifting of \( \bar{\rho}_b \) with determinant \( \chi \).

- For \( v|p, \psi_{v,1} : G_{F_v} \to R^\times \) is a lifting of \( \bar{\psi}_{v,1} \) such that \( \rho_R \) has a (necessarily unique) \( G_{F_v} \)-stable rank one \( R \)-submodule, which is a direct summand of \( \rho_R \) as a \( R \)-module, with \( G_{F_v} \) acting via \( \psi_R \).

Recall that in 5.3.8 we also introduce the framed ordinary deformation ring \( R^{\Delta_S,(\xi_v)}_{\bar{\rho}_b} \) with framing at \( S \). Then there is a non-canonical isomorphism by choosing a universal framing

\[ R^{\Delta_S,(\xi_v)}_{\bar{\rho}_b} \cong R^\text{ord}_b[[y_1, \cdots, y_4|S|-1]]. \]
Let $R_{\text{ord}}^\Delta$ be the following completed tensor product over $\mathcal{O}$:

$$(\bigotimes_{v \in \Sigma_p} R^\Delta_v) \hat{\otimes} (\bigotimes_{v \in S \setminus \Sigma_p} R^\square_v \xi_v),$$

where $R^\Delta_v$ is defined in the first part of 5.3.8 and $R^\square_v \xi_v$ is defined in 4.2.1. It is well-known (for example see corollary 2.3.5 of [9]) that $R^\Delta_{\bar{\rho}_b}(\xi_v)$ can be written as the form

$$R^\Delta_{\bar{\rho}_b}(\xi_v) \cong R_{\text{ord}}^\Delta [[x_1, \ldots, x_{g_1}]]/(h_1, \ldots, h_{r_1})$$

where $g_1 = \dim_{\mathbb{F}} H^1(G_{F,S}, \text{ad}^0 \bar{\rho}_b(1)) + |S| - 1 - [F : \mathbb{Q}] - \dim_{\mathbb{F}} H^0(G_{F,S}, \text{ad}^0 \bar{\rho}_b(1))$ and $r_1 = \dim_{\mathbb{F}} H^1(G_{F,S}, \text{ad}^0 \bar{\rho}_b(1)).$

**Lemma 5.4.4.** Each irreducible component of $R^\text{ord}_b$ has dimension at least $[F : \mathbb{Q}]$. Moreover, $R^\text{ord}_b$ can be written as the form $R_p[[x_1, \ldots, x_{g_2}]]/(f_1, \ldots, f_{r_2})$ with

$$g_2 - r_2 \geq 3|S| - 2|\Sigma_p| - [F : \mathbb{Q}] - 2$$

and $R_p$ a domain of dimension $1 + 2[F : \mathbb{Q}] + 3|\Sigma_p|$.

**Proof.** By the first part of 5.3.8, lemma 4.2.10 and lemma 3.3 of [2], we know that $R_{\text{loc}}^\text{ord}$ is equidimensional of dimension $2[F : \mathbb{Q}] + 3|S| + 1$. Since $\dim H^0(G_{F,S}, \text{ad}^0 \bar{\rho}_b(1)) \leq 1$. The first claim now follows from a simple computation.

For the second part, let $R_p = \bigotimes_{v | p} R^\Delta_v$. This is a domain of dimension $1 + 2[F : \mathbb{Q}] + 3|\Sigma_p|$ by lemma 5.3.9. Note that for $v \in S \setminus \Sigma_p$, $R^\square_v \xi_v$ is a quotient of the unrestricted deformation ring $R^\square_v$ by one equation $\text{tr}(t) = \xi(t) + \xi^{-1}(t)$, where $t$ is a topological generator of of the pro-$p$ quotient of $I_{F_v}$. On the other hand, it is well-known that $R^\square_v$ has the form $\mathcal{O}[[x_1, \ldots, x_{g_v}]]/(f_1, \ldots, f_{r_v})$. Here $g_v = \dim_{\mathbb{F}} H^1(G_{F_v}, \text{ad}^0 \bar{\rho}_b) + 3 - \dim_{\mathbb{F}} H^0(G_{F_v}, \text{ad}^0 \bar{\rho}_b)$ and $r_v = \dim_{\mathbb{F}} H^2(G_{F_v}, \text{ad}^0 \bar{\rho}_b)$. It follows from local Euler characteristic formula that $g_v - r_v = 3$. From this, it is easy to deduce the second claim in the lemma. \qed
5.4.5 **(Connectedness dimension)**. In order to show that there are enough ‘nice’ primes, we need the notion of *connectedness dimension*. Reference here is [6].

**Definition 5.4.6.** Let $R$ be a complete noetherian local ring. The connectedness dimension of $R$ is defined to be

$$c(R) = \min_{Z_1, Z_2} \{\dim(Z_1 \cap Z_2)\}$$

where $Z_1, Z_2$ are non-empty and are unions of irreducible components of $\text{Spec } R$ such that $Z_1 \cup Z_2 = \text{Spec } R$.

Suppose $I$ is an ideal of a complete noetherian local ring $R$. The arithmetic rank $r(I)$ is defined as the minimal number of elements in $I$ which span an ideal with the same radical as $I$. The following result is Theorem 2.4 of [6].

**Proposition 5.4.7.** Let $I, R$ be as above. Then $c(R/I) \geq c(R) - r(I) - 1$.

Combining this with lemma 5.4.4, we get

**Corollary 5.4.8.** $c(R_{b, R}^{\text{ord}}/\wp) \geq [F : \mathbb{Q}] - |S| + |\Sigma_p| - 2$.

5.4.9 **(Reducible locus of $R_{b, R}^{\text{ord}}$).** Let $c \in H_{\Sigma_0}^1(F) \setminus \{0\}$. We choose a universal deformation

$$\rho^{\text{univ}} : G_{F,S} \to \text{GL}_2(R_{b, R}^{\text{ord}}) \ g \mapsto \begin{pmatrix} \tilde{a}(g) & \tilde{b}(g) \\ \tilde{c}(g) & \tilde{d}(g) \end{pmatrix}$$

so that $\rho^{\text{univ}}(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It is clear that the set of $p \in \text{Spec } R_{b, R}^{\text{ord}}$ such that $\rho^{\text{univ}} \otimes k(p)$ is reducible is closed in $\text{Spec } R_{b, R}^{\text{ord}}$. In fact, it is generated by the set of $x(\sigma, \tau)$ (see the discussion and notation in 2.1.5). We denote its reduced subscheme by $\text{Spec } R_{b, R}^{\text{ord}}$ and call it the reducible locus of $\text{Spec } R_{b, R}^{\text{ord}}$. 

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The dimension of reducible locus is crucial in our later analysis of Spec $R_b^{ord}$. An upper bound is given in lemma 2.7 of [39]. We recall their argument here.

Let $G_{F,S}(p)$ be the maximal pro-$p$ abelian quotient of $G_{F,S}$. We denote its $\mathbb{Z}_p$-rank by $\delta_F + 1$ and choose a set of elements $\tau_0, \cdots, \tau_{\delta_F} \in G_{F,S}$ whose images in $G_{F,S}^\text{ab}(p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ form a basis. Since $F \neq \mathbb{Q}$, by the class field theory we have a trivial bound $\delta_F \leq [F : \mathbb{Q}] - 2$.

On the other hand, let $F(\bar{\chi})$ be the splitting field of $\bar{\chi}$ and $\tilde{S}$ be the set of places of $F(\bar{\chi})$ above $S$. Then

$$H^1(G_{F,S}, \mathbb{F}(\bar{\chi}^{-1})) = \text{Hom}_{\text{Gal}(F(\bar{\chi})/F)}(G_{F(\bar{\chi})}, \mathbb{F}(\bar{\chi}^{-1})).$$

In other words, we get a pairing $H^1(G_{F,S}, \mathbb{F}(\bar{\chi}^{-1})) \times G_{F(\bar{\chi})}, \tilde{S} \to \mathbb{F}$. Denote $\text{dim}_F H^1_{\Sigma_{\varphi}}(F)$ by $r_s$. Then we can choose elements $\sigma_1, \cdots, \sigma_{r_s} \in G_{F(\bar{\chi})}, \tilde{S}$ such that they form a basis of $H^1_{\Sigma_{\varphi}}(F)^\vee$ under this pairing.

**Proposition 5.4.10.** Let $I$ be the ideal of $R_b^{\text{red}}$ generated by $\varpi$ and the image of elements

- $\bar{a}(\tau_i) + \bar{d}(\tau_i) - 2$, $i = 0, \cdots, \delta_F$,
- $\bar{b}(\sigma_i) - \bar{b}_i$, $i = 1, \cdots, r_s$, where $\bar{b}_i \in \mathcal{O}$ is a lifting of the reduction of $\bar{b}(\sigma_i)$ modulo the maximal ideal of $R_b^{\text{ord}}$.

Then $R_b^{\text{red}}/I$ has finite length. In particular, $\text{dim } R_b^{\text{red}} \leq \delta_F + \text{dim}_F H^1_{\Sigma_{\varphi}}(F) + 2$.

**Proof.** Let $p$ be a prime of $R_b^{\text{red}}/I$. It suffices to prove $\rho(p) = \rho^{\text{univ}} \mod p$ is a trivial deformation. First note that the semi-simplification of $\rho(p)$ is a sum of two characters $\chi_1, \chi_2$ of $G_{F,S}$. By the definition of $I$, $\chi_1(\tau_i) + \chi_2(\tau_i) = 2$ and $\chi_1(\tau_i)\chi_2(\tau_i) = \bar{\chi}(\tau_i) = 1$. Hence $\chi_1(\tau_i) = \chi_2(\tau_i) = 1$. Therefore $\chi_1, \chi_2$ are of finite orders and have to be $1, \bar{\chi}$.  

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Thus $\rho(p)|_{G_{F(\bar{\chi}),S}}$ is unipotent. Since $\rho^{univ}(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we must have

$$\rho(p)|_{G_{F(\bar{\chi}),S}} = \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix}.$$ 

Note that $\ast$ is an element in $H^1_{\Sigma_o}(F) \otimes k(p) \subseteq \text{Hom}_{\text{Gal}(F(\bar{\chi})/F)}(G_{F(\bar{\chi}),\bar{S}}, k(p)(\bar{\chi}^{-1})).$ It follows from our construction of $I$ that $\rho(p) = \bar{\rho}_b$. □

5.4.11 (Reducible locus of $R^{ps,ord,\{\xi_v\}}_{\Sigma_o}$). Similarly the set of $p \in \text{Spec } R^{ps,ord,\{\xi_v\}}_{\Sigma_o}$ such that $\rho(p)$ is reducible is closed in $\text{Spec } R^{ps,ord,\{\xi_v\}}_{\Sigma_o}$. We denote its reduced subscheme by $\text{Spec } R^{ps,red}_{\Sigma_o}$. Note that the universal trace becomes a sum of two characters when restricted on $\text{Spec } R^{ps,red}_{\Sigma_o}$. One is a lifting of $\bar{\chi}$ and the other one is a lifting of $1$. The latter one induces a surjective map $\mathcal{O}[[G_{F,S}^{ab}(p)]] \to R^{ps,red}_{\Sigma_o}$. Hence $\dim R^{ps,red}_{\Sigma_o} \leq \delta_F + 2$.

5.4.12. Now we can state and prove the main result of this subsection. For any non-zero $c \in H^1_{\Sigma_o}(F)$, we say a prime $p \in \text{Spec } R^{ord}_{b}$ is modular if its image in $\text{Spec } R^{ps,ord,\{\xi_v\}}_{\Sigma_o}$ is modular. We say a subset $Z \subseteq \text{Spec } R^{ord}_{b}$ is modular if any prime of $Z$ is modular. Note that the set of modular primes is closed in $\text{Spec } R^{ord}_{b}$.

Similarly, we say a prime $p \in \text{Spec } R^{ord}_{b}$ is nice if its image in $\text{Spec } R^{ps,ord,\{\xi_v\}}_{\Sigma_o}$ is nice (defined after proposition 5.3.7). Then proposition 5.3.7 implies that an irreducible component of $\text{Spec } R^{ord}_{b}$ is modular if it contains a nice prime.

Proposition 5.4.13. Let $b \in H^1_{\Sigma_o}(F) \setminus \{0\}$. Assume

- There exists a modular prime $p \in \text{Spec } R^{ord}_{b}$ such that $R^{ord}_{b}/p$ is one-dimensional and $\rho^{univ} \otimes k(p)$ is irreducible. In particular, the assumption in 5.3.5 holds.
- $[F : \mathbb{Q}] - 4|S| + 4|\Sigma_p| - 3 > \delta_F + \dim_F H^1_{\Sigma_o}(F)$.

Then $R^{ord}_{b}$ is modular.
Proof. Let \( p' \in \text{Spec} \, R^\text{ps,ord}_{\Sigma_0}^{\xi_v} \) be the image of \( p \). This is still one-dimensional as \( \rho^\text{univ} \otimes k(p) \) is irreducible. By corollary 5.2.5, the modular locus in \( \text{Spec} \, R^\text{ps,ord}_{\Sigma_0}^{\xi_v} \) is equidimensional of dimension \([F : \mathbb{Q}] + 1\). In particular, we can choose a modular prime \( \mathfrak{P} \) contained in \( p' \) with \( \dim(R^\text{ps,ord}_{\Sigma_0}^{\xi_v}/\mathfrak{P})_{p'} = [F : \mathbb{Q}] \). Notice that

\[
\dim(R^\text{ps,ord}_{\Sigma_0}^{\xi_v}/\mathfrak{P})_{p'} = \dim \left( R^\text{ps,ord}_{\Sigma_0}^{\xi_v}/\mathfrak{P} \right)_{p'} = \dim \left( \widehat{R^\text{ord}}_{\Sigma_0}/\mathfrak{P} \right) = \dim \left( R^\text{ord}_{b}/\mathfrak{P} \right)
\]

where the equality in the middle follows from corollary 2.3.8 and the discussion in the second part of 5.3.8. Thus we may find a modular prime \( \mathfrak{Q} \) of \( R^\text{ord}_{b} \) such that \( \dim(R^\text{ord}_{b}/\mathfrak{Q}) = 1 + [F : \mathbb{Q}] \). I claim we can find one nice prime containing \( \mathfrak{Q} \). Hence at least one irreducible component of \( R^\text{ord}_{b} \) is modular.

For \( v \in S \setminus \Sigma_p \), we know that \( R^\text{ps,ord}_{v}^{\xi_v}/(\varpi) \) is 3-dimensional by lemma 4.2.10. Hence we may find \( f_{v,1}, f_{v,2}, f_{v,3} \) in \( R^\text{ps,ord}_{v}^{\xi_v} \) that form a system of parameters of \( R^\text{ps,ord}_{v}^{\xi_v}/(\varpi) \). Consider the quotient \( R' \) of \( R^\text{ord}_{b}/\mathfrak{Q} \) by \( \varpi \) and all such \( f_{v,1}, f_{v,2}, f_{v,3}, v \in S \setminus \Sigma_p \). It has dimension at least

\[
[F : \mathbb{Q}] - 3(|S| - |\Sigma_p|).
\]

On the other hand, it follows from proposition 5.4.10 that the reducible locus in the special fibre \( R^\text{red}_{b}/(\varpi) \) has dimension at most \( \delta_F + \dim_{\mathbb{F}} H^1_{\Sigma_0}(F) + 1 \), which is less than \([F : \mathbb{Q}] - 3(|S| - |\Sigma_p|)\) by our assumption. Hence there must exist a one-dimensional prime \( \mathfrak{q} \in \text{Spec} \, R' \) such that \( \rho^\text{univ} \otimes k(\mathfrak{q}) \) is irreducible. It is easy to see that \( \mathfrak{q} \) is nice in view of the definition of ‘nice’ above proposition 5.3.7.

Now let \( Z_1 \) be the union of modular irreducible components of the special fibre \( R^\text{ord}_{b}/(\varpi) \) and \( Z_2 \) be the union of other irreducible components of the special fibre. It suffices to show \( Z_2 \) is empty. Suppose not, we have already seen \( Z_1 \) is non-empty, hence by corollary 5.4.8,

\[
\dim Z_1 \cap Z_2 \geq [F : \mathbb{Q}] - |S| + |\Sigma_p| - 2.
\]
Note that by our assumption, this is larger than $3(|S| - |\Sigma_p|) + \dim R^\text{red}_b/(\varpi)$. Therefore arguing as above, we can find a nice prime in $Z_1 \cap Z_2$. This implies that some irreducible component in $Z_2$ is also modular. Contradiction. Thus $R^\text{ord}_b$ is modular.

\begin{corollary}
Assume
\begin{itemize}
  \item (Assumption in 5.3.5) $T_v - (1 + \chi(\text{Frob}_v)), v \notin S$ and $\psi_v, 1(\gamma) - \tilde{\psi}_v, 1(\gamma), \gamma \in F_v^\times, v | p$ and $\varpi$ generate a maximal ideal $m$ of $\mathbb{T}^\text{ord}$. Here $\tilde{\psi}_v, 1(\gamma) \in \mathcal{O}$ is any lifting of $\bar{\psi}_v, 1(\gamma) \in \mathbb{F}$.
  \item $[F : \mathbb{Q}] - 4|S| + 4|\Sigma_p| - 3 > \delta_F + \dim \mathbb{F} H^1_{\Sigma_0}(F)$.
\end{itemize}

Enlarge $\mathcal{O}$ if necessary, then there exists a non-zero $b \in H^1_{\Sigma_0}(F)$ such that $R^\text{ord}_b$ is modular.

\begin{proof}
It is clear that we only need to find an element $b \in H^1_{\Sigma_0}(F)$ that satisfies the conditions in the previous proposition. By our first assumption and corollary 5.2.5, the modular locus in Spec $R^{\text{ps,ord}, \xi_v}_{\Sigma_0}$ has dimension $[F : \mathbb{Q}] + 1$. Since the reducible locus has dimension at most $\delta_F + 2 \leq [F : \mathbb{Q}]$, we may choose a one-dimensional modular prime $p'$ such that $\rho(p)$ is irreducible. Then after enlarging $\mathcal{O}$ if necessary, we can find a representation $G_{F,S} \to \text{GL}_2(A)$, where $A$ is normalization of $R^{\text{ps,ord}, \xi_v}_{\Sigma_0}/p'$, such that its residual representation gives rise to some non-zero $b \in H^1_{\Sigma_0}(F)$, which clearly has the properties we want.
\end{proof}

5.5 $R^\text{ord}_b$ is modular for any $b$

The goal of this section is to extend the result in the previous section to any non-zero extension class $b \in H^1_{\Sigma_0}(F)$ (under certain assumptions).

\begin{proposition}
Keep all the notations as in the previous section and all the assumptions in corollary 5.4.14.
\end{proposition}
• (Assumption in 5.3.5) $T_v - (1 + \chi(\text{Frob}_v)), v \notin S$ and $\psi_v,\gamma - \tilde{\psi}_v,\gamma, \gamma \in F_v^\times, v | p$ and $\varpi$ generate a maximal ideal $\mathfrak{m}$ of $T^\text{ord}$. Here $\tilde{\psi}_v,\gamma \in \mathcal{O}$ is any lifting of $\psi_v,\gamma \in \mathbb{F}$.

• $[F : \mathbb{Q}] - 4|S| + 4|\Sigma_\mathfrak{p}| - 3 > \delta_F + \text{dim}_F H^1_{\Sigma_o}(F)$.

Then after replacing $E$ by some finite extension, $R^\text{ord}_b$ is modular for any non-zero $b \in H^1_{\Sigma_o}(F)$.

Proof. First by corollary 5.4.14 we may assume there exists a non-zero class $b_1 \in H^1_{\Sigma_o}(F)$ such that $R^\text{ord}_{b_1}$ is modular (after possibly enlarging $\mathcal{O}$). Now let $b_2 \in H^1_{\Sigma_o}(F)$ be another class in $H^1_{\Sigma_o}(F) \setminus \{F b_1\}$. In view of proposition 5.4.13 it suffices to find a modular one-dimensional prime of $R^\text{ps,ord}_{\Sigma_o}(\xi_v)$ whose associated representation $\rho(\mathfrak{p})$ is irreducible with residue representation corresponding to the extension class $b_2 \in H^1_{\Sigma_o}(F)$.

By our assumption, we may extend $b_1, b_2$ to a basis $b_1, \cdots, b_s$ of $H^1_{\Sigma_o}(F)$. Write $b_i = [\phi_{b_i}]$. Also we can choose elements $\sigma_1, \cdots, \sigma_s \in G_{F(\bar{\chi}),\bar{S}}$ such that

$$\phi_{b_i}(\sigma_j) = \delta_{ij}$$

under the pairing before proposition 5.4.10.

Consider the following deformation of $\tilde{\rho}_{b_1}$:

$$\rho_{12} : G_{F,S} \to \text{GL}_2(\mathbb{F}[[T]]), \quad \sigma \mapsto \begin{pmatrix} 1 & \bar{\chi}(\sigma)(\phi_{b_1}(\sigma) + \phi_{b_2}(\sigma)T) \\ 0 & \bar{\chi}(\sigma) \end{pmatrix}.$$ 

This gives rise to a one-dimensional prime $q_{12}$ of $R^\text{ord}_{b_1}$. As in 5.4.9 we may choose a universal deformation

$$\rho^{\text{univ}} : G_{F,S} \to \text{GL}_2(R^\text{ord}_{b_1}) \quad g \mapsto \begin{pmatrix} \tilde{a}(g) & \tilde{b}(g) \\ \tilde{c}(g) & \tilde{d}(g) \end{pmatrix}.$$
so that $\tilde{b}(\sigma_1) = 1$, $\rho^{univ}(\sigma^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the reduction of $\rho^{univ}$ modulo $q_{12}$ is $\rho_{12}$. It is clear that $\tilde{b}(\sigma_i) \in q_{12}, i = 3, \ldots, r_s$. Let $\mathfrak{Q}$ be a minimal prime of $R_{b_1}^{ord}/(\tilde{b}(\sigma_3), \ldots, \tilde{b}(\sigma_{r_s}))$ contained in $q_{12}$. Then by lemma 5.4.4,

$$\dim R_{b_1}^{ord}/\mathfrak{Q} \geq [F : \mathbb{Q}] + 1 - (r_s - 2) = [F : \mathbb{Q}] + 3 - \dim_F H^1_{\Sigma_S}(F).$$

We claim that $\rho^{univ} \otimes k(\mathfrak{Q})$ is irreducible. If not, since $\tilde{b}(\sigma_i), i = 3, \ldots, r_s$ are already contained in $\mathfrak{Q}$, it follows from proposition 5.4.10 that $R_{b_1}^{ord}/\mathfrak{Q}$ has dimension at most $1 + \delta_F + 3$. But by our second assumption

$$[F : \mathbb{Q}] + 3 - \dim_F H^1_{\Sigma_S}(F) > 4|S| - 4|\Sigma_p| + 1 + \delta_F + 4 \geq \delta_F + 5.$$

In other words, this upper bound for $\dim R_{b_1}^{ord}/\mathfrak{Q}$ contradicts the previous lower bound. Therefore $\mathfrak{Q}$ is not in the reducible locus. Hence let $\mathfrak{Q}' = \mathfrak{Q} \cap R_{\Sigma_S}^{ps, ord, \{\xi_v\}}$. We may apply the second part of corollary 2.3.7 and conclude

$$\dim R_{\Sigma_S}^{ps, ord, \{\xi_v\}}/\mathfrak{Q}' \geq \dim R_{b_1}^{ord}/\mathfrak{Q} \geq [F : \mathbb{Q}] + 3 - \dim_F H^1_{\Sigma_S}(F).$$

Consider the ideal $I_{b_1}$ of $R_{\Sigma_S}^{ps, ord, \{\xi_v\}}/\mathfrak{Q}'$ generated by elements $x(\sigma_1, \tau), \tau \in G_{F,S}$ defined by the universal pseudo-character $T^{univ} : G_{F,S} \to R_{\Sigma_S}^{ps, ord, \{\xi_v\}}$. See 2.1.3 and the beginning of the proof of proposition 2.3.2 for the precise definition. These elements have the property that $x(\sigma_1, \tau)$ maps to $\tilde{b}(\sigma_1)\tilde{c}(\tau) = \tilde{c}(\tau)$ in $R_{b_1}^{ord}$. It follows from proposition 2.4.2 that $ht(I_{b_1}) \geq 1$, hence

$$\dim R_{\Sigma_S}^{ps, ord, \{\xi_v\}}/(\varpi, \mathfrak{Q}', I_{b_1}) \geq [F : \mathbb{Q}] + 1 - \dim_F H^1_{\Sigma_S}(F).$$
Note that we may apply proposition 2.4.2 here as $\Omega$ is contained in $q_{12}$ which is a one-dimensional prime ideal of $R_{b_1}^{\text{ord}}$ mapping to the maximal ideal of $R_{\Sigma_0}^{\text{ps,ord},\{\xi_{\nu}\}}$. Notice that

$$[F : \mathbb{Q}] + 1 - \dim_{\mathbb{F}} H^1_{\Sigma_0}(F) > \delta_F + 2 \geq \dim R_{\text{ps,red}}^\text{ord}$$

by our assumption. Hence we can find a one-dimensional prime $p$ containing $\varpi, \Omega', I_{b_1}$ such that $\rho(p)$ is irreducible. It has to be modular as $\Omega'$, the image of $\Omega$, is modular. We claim that there is a lattice of $\rho(p)$ such that its residue representation belongs to the extension class $b_2 \in H^1_{\Sigma_0}(F)$.

Let $A = \tilde{F}[[T]]$ be the normalization of $R_{\Sigma_0}^{\text{ps,ord},\{\xi_{\nu}\}}/p$, where $\tilde{F}$ is a finite extension of $\mathbb{F}$. Then we may find a lattice of $\rho(p)$: $\rho^\sigma : G_{F,S} \to \text{GL}_2(A)$ such that the reduction of $\rho^\sigma \mod T$ has the form

$$\begin{pmatrix} 1 & * \\ 0 & \bar{\chi} \end{pmatrix}, * \neq 0.$$ 

and $\rho^\sigma(\sigma^* g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Write $\rho^\sigma(g) = \begin{pmatrix} a'(g) & b'(g) \\ c'(g) & d'(g) \end{pmatrix}$. Since $\rho^\sigma$ is irreducible, we may find some $\tau' \in G_{F,S}$ such that $c'(\tau) \neq 0$. Now for $i = 3, \cdots, r_s$, the image of $x(\sigma_i, \tau')$ in $R_{b_1}^{\text{ord}}$ is

$$\tilde{b}(\sigma_i) c'(\tau') \in \Omega,$$

hence $x(\sigma_i, \tau') \in \Omega'$ and $b'(\sigma_i)c'(\tau') = x(\sigma_i, \tau') = 0$ in $A$. Therefore $b'(\sigma_i) = 0, i = 3, \cdots, r_s$.

Similarly $b'(\sigma_1)c'(\tau) = 0$ in $A$ as $x(\sigma_1, \tau') \in I_{b_1}$ by our construction. Thus $b'(\sigma_1) = 0$.

Hence $b'(\sigma_i) = 0$ unless $i \neq 2$. Since we assume the reduction $\tilde{\rho}$ of $\rho^\sigma$ is non-split, it follows from the discussion below definition 5.4.2 that after possibly conjugating
\( \rho^o \) by element of the form \( \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \), \( n \in A^\times \), the reduction \( \bar{\rho} \) has to belong to the extension class \( b_2 \in H_{\Sigma_0}^1(F) \). This is exactly what we want. \( \square \)

**Corollary 5.5.2.** Keep the same assumption as in the previous proposition, then

1. A prime \( p \in \text{Spec} R_{\Sigma_0}^{\text{ps,ord},\{\xi_v\}} \) is modular if \( \rho(p) \) is irreducible.

2. \( R_{\Sigma_0}^{\text{ps,ord},\{\xi_v\}} \) is a finite \( \Lambda_F \)-algebra.

3. If \( p \) is a maximal ideal of \( R_{\Sigma_0}^{\text{ps,ord},\{\xi_v\}} \big|_{F_v} \) such that

- \( \rho(p) \) is irreducible.

- Write \( \rho(p)|_{G_{F_v}} \cong \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix} \). We assume \( \psi_{v,1} \) is Hodge-Tate and has strictly less Hodge-Tate number than \( \psi_{v,2} \) for any \( v|p \) and any embedding \( F_v \hookrightarrow \overline{\mathbb{Q}}_p \).

Then \( \rho(p) \) comes from a twist of a Hilbert modular form.

**Proof.** If \( p \) in the first claim is one-dimensional, then enlarging \( E \) if necessary, we can find a non-zerp extension class \( b \in H_{\Sigma_0}^1(F) \) such that \( p \) is in the image of \( \text{Spec} R_b^{\text{ord}} \). And the assertion follows from the previous proposition directly. In general, we may find a one-dimensional prime \( p' \) containing \( p \) such that \( \rho(p') \) is irreducible. Suppose \( p' \) is in the image of \( \text{Spec} R_b^{\text{ord}} \), then by the first part of corollary 2.3.7, \( p \) is also in the image hence modular as well. This proves the first assertion.

A direct consequence of this is that the natural \( \Lambda_F \)-equivariant map

\[
R_{\Sigma_0}^{\text{ps,ord},\{\xi_v\}} \to T_{m}^{\text{ord}} \times R^{\text{ps,red}}
\]

has nilpotent kernel. Hence it suffices to prove that the image is a finite \( \Lambda_F \)-algebra. This is clearly true for \( T_{m}^{\text{ord}} \). As for \( R^{\text{ps,red}} \), as we discussed in 5.4.11, \( R^{\text{ps,red}} \) is a quotient of \( \mathcal{O}[[G_{F,S}^b(p)]] \). Its finiteness over \( \Lambda_F \) follows from global class field theory.
5.6 Existence of Eisenstein maximal ideal

In this subsection, we give sufficient conditions for the existence of Eisenstein maximal ideal in assumption 5.3.5. We keep the same notations as in the previous subsection. In particular, \( \psi = \chi \varepsilon \) and \( F \) is a totally real field of even degree over \( \mathbb{Q} \) in which \( p \) is unramified.

**Proposition 5.6.1.** Assume

- \( \text{ord}_{\varpi} L_p(F, -1, \tilde{\chi} \omega) > 0 \) if \( \Sigma^o \) is \( \Sigma_p \) or empty, where \( \tilde{\chi} \omega \) is the Teichmüller lifting of \( \chi \omega \) and \( L_p(F, s, \tilde{\chi} \omega) \) is the \( p \)-adic L-function associated to the character \( \tilde{\chi} \omega \).

Then enlarging \( \mathcal{O} \) if necessary, there exists an open subgroup \( U^p_e = \prod_{v \nmid p} U_{e,v} \subseteq \prod_{v \nmid p} \text{GL}_2(\mathcal{O}_{F,v}) \) such that \( \varpi \) and

- \( T_v - (1 + \chi(\text{Frob}_v)) \) for \( v \nmid p \) such that \( U_{e,v} = \text{GL}_2(\mathcal{O}_{F,v}) \);
- \( \psi_{v,1}(\gamma) - \tilde{\psi}_{v,1}(\gamma), \gamma \in F_v^\times, v \mid p \), where \( \tilde{\psi}_{v,1}(\gamma) \in \mathcal{O} \) is any lifting of \( \tilde{\psi}_{v,1}(\gamma) \in \mathbb{F} \);

generate a maximal ideal of \( T^\text{ord}_\psi(U^p_e) \).

**Remark 5.6.2.** If \( \Sigma^o \) is \( \Sigma_p \), this is proposition 3.18 of [39]. If \( \Sigma^o \) is empty, we can twist everything by \( \chi^{-1} \) and reduce to the previous case.

As in [39], we will use Eisenstein series and congruences to find a cuspidal ordinary eigenform with desired Hecke eigenvalues. To do this, we first review some basics of the theory of Hilbert modular forms (of parallel weight). References here are [37] and \S2 of [22], though some conventions here are different.

5.6.3. Let \( \delta \) be the different of \( F \). For an ideal \( n \) of \( \mathcal{O}_F \), we denote

\[
U^1(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}) : b \equiv 0 \mod \delta^{-1}, c \equiv 0 \mod \delta n, d - 1 \equiv 0 \mod n \}. 
\]
Let $\Sigma_{\infty}$ be the set of infinite places of $F$ and $\mathfrak{H}$ be the usual upper half plane. There is a natural action of $GL^+_2(F \otimes \mathbb{Q}) \cong GL^+_2(\mathbb{R})^{\Sigma_{\infty}}$ on $\mathfrak{H}^{\Sigma_{\infty}}$:

\[
u_{\infty}(z) = (a_{\tau}z_{\tau} + b_{\tau})_{\tau \in \Sigma_{\infty}}, \quad \nu_{\infty} = \begin{pmatrix} a_{\tau} & b_{\tau} \\ c_{\tau} & d_{\tau} \end{pmatrix} \in GL^+_2(\mathbb{R})^{\Sigma_{\infty}}.
\]

Here $GL^+_2(\mathbb{R}) \subseteq GL_2(\mathbb{R})$ denotes the subgroup with positive determinant. Define $j : GL_2(F \otimes \mathbb{Q}) \times \mathfrak{H}^{\Sigma_{\infty}} \to \mathbb{C}^{\Sigma_{\infty}}$

\[
j(\nu_{\infty}, z) = (c_{\tau}z_{\tau} + d_{\tau}), \quad \nu_{\infty} = \begin{pmatrix} a_{\tau} & b_{\tau} \\ c_{\tau} & d_{\tau} \end{pmatrix}.
\]

Denote by $z_0$ the point $(i, \cdots, i) \in \mathfrak{H}^{\Sigma_{\infty}}$. It is fixed by

\[
C_{\infty} := (SO_2(\mathbb{R})^{\Sigma_{\infty}}) \subseteq GL^+_2(\mathbb{R})^{\Sigma_{\infty}} \subseteq GL_2(\mathbb{A}_F).
\]

5.6.4 (Adelic holomorphic modular forms on $GL_2/F$ of parallel weight $k$).

Let $k$ be a positive integer. For any $x \in \mathbb{C}^{\Sigma_{\infty}}$, we write $x^k$ for the product $\prod_{\tau \in \Sigma_{\infty}} x_{\tau}^k$.

If a function $f : GL_2(\mathbb{A}_F) \to \mathbb{C}$ satisfies $f|_{\nu_{\infty}} = f$ for any $\nu_{\infty} \in C_{\infty}$, where

\[
(f|_{\nu_{\infty}})(g) = j(\nu_{\infty}, z_0)^{-k} \det(\nu_{\infty})^{1/2} f(g \nu_{\infty}^{-1}),
\]

then we can define a function $f_x : \mathfrak{H}^{\Sigma_{\infty}} \to \mathbb{C}$ for any $x \in GL_2(\mathbb{A}_F^{\infty})$ by

\[
f_x(z) = j(\nu_{\infty}, z_0)^k \det(\nu_{\infty})^{-1} f(x \nu_{\infty}), \quad z = \nu_{\infty}(z_0)
\]

with $\nu_{\infty} \in GL^+_2(F \otimes \mathbb{R})$.

Let $n$ be an ideal of $\mathcal{O}_F$. A function $f : GL_2(\mathbb{A}_F) \to \mathbb{C}$ is called a modular form of weight $k$ and level $U^1(n)$ if
• $f(axu) = f(x)$ for any $a \in \text{GL}_2(F), u \in U^1(n)$,

• $f|_k u = f$ for any $u \in C^\infty$,

• $f_x$ is homomorphic on $\mathfrak{H}^\Sigma$ for any $x \in \text{GL}_2(A_F^\infty)$.

Such a function $f$ is called a cusp form if the constant form of the Fourier expansion of $f_x$ (see §5.6.6 below) vanishes for all $x$. We denote the space of modular forms and cusp forms by $M_k(n)$ and $S_k(n)$. Let $\theta : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ be a Hecke character such that $\theta(x) = N_{F/Q}(x)^{-k+2}, x \in (F \otimes \mathbb{R})^\times$. We denote by $M_k(n, \theta) \subseteq M_k(n), S_k(n, \theta) \subseteq S_k(n)$ the subspace of functions with the centre $A_F^\times$ acting via $\theta$.

We remark that the relation between functions considered here and those in [37] is as follows: given $f \in M_k(n, \theta)$, then we can define another function $f_0 : \text{GL}_2(A_F) \to \mathbb{C}$ by

$$f_0(g) = f(g) \mid \text{det}(g) \mid_{A_F^\times}^{k-1}.$$ 

This is an element of $M_k(n, \theta) \cdot |\cdot|_{A_F^\times}^{k-2}$ defined in §2 of [37]. Here $|\cdot|_{A_F^\times} : \mathbb{A}_F^\times \to \mathbb{R}^\times$ denotes the adelic norm.

5.6.5 (Relation with classical Hilbert modular forms). Let $t^{(i)} \in \mathbb{A}_F^\times, i = 1, \cdots, h$ be a set of representatives of the class group $\mathbb{R}^\times \backslash \mathbb{A}_F^\times/\hat{O}_F^\times (F \otimes \mathbb{R})^\times$ such that $t^{(i)}_w = 1$ for $w|np^\infty$. Fix such a choice from now on. Write

$$\Gamma_i = \text{GL}_2(F) \cap t^{(i)}U^1(n)\text{GL}_2^+(F \otimes \mathbb{R})(t^{(i)})^{-1}.$$ 

Let $f \in M_k(n)$ and write $x_i = \begin{pmatrix} t^{(i)} & 0 \\ 0 & 1 \end{pmatrix}$. Then $f_i = f_{x_i} : \mathfrak{H}^\Sigma \to \mathbb{C}$ satisfies

$$f_i(z) = j(\gamma, z)^{-k} \text{det}(\gamma)^1 f_i(\gamma z), \gamma \in \Gamma_i.$$ 

Note that $\text{det}(\gamma)^1 = 1$ here as $\gamma \in \Gamma_i$. Hence $f_i$ is a Hilbert modular form of level $\Gamma_i$ and parallel weight $k$ in the usual sense. We denote by $M_k(\Gamma_i)$ the space
of holomorphic functions on $\mathcal{H}^{\Sigma}$ satisfying the above condition. Then the map $f \mapsto (f_i)_{i=1,\ldots,h}$ induces an isomorphism $M_k(n) \cong \bigoplus_{i=1}^h M_k(\Gamma_i)$.

5.6.6 (Fourier expansion and Dirichlet series). Each $f_i(z)$ has a Fourier expansion at $\infty$:

$$f_i(z) = a_i(0, f) + \sum_{\mu \in (t^{(i)})_{>0}^{-1}} a_i(\mu, f) e(\mu \cdot z)$$

where

- $(t^{(i)})$ is the fractional ideal associated to $t^{(i)}$ and $(t^{(i)})_{>0}^{-1} \subseteq (t^{(i)})^{-1}$ is the subset of totally real elements.
- $e(\mu \cdot z) = e^{2\pi \sqrt{-1} \sum_{\tau \in \Sigma^{\infty}} \mu_\tau \tau}$.

For any ideal $a$, we put $C(a, f) = a_i(\mu, f)|t^{(i)}|^{-1}_{A_F}$ if $a$ is of the form $(\mu(t^{(i)})), \mu \in (t^{(i)})_{>0}^{-1}$ and zero otherwise. We attach a Dirichlet series to $f$ by ($N(a)$ denotes the norm of $a$)

$$D(f, s) = \sum_{a \in \mathcal{O}_F} \frac{C(a, f)}{N(a)^s}.$$ 

Remark 5.6.7. Any modular form $f \in M_k(n)$ has an adelic Fourier expansion (see proposition 2.26 of [22], note that $[\kappa_1] = [0]$ here):

$$f \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = |y|_{A_F}(a_0(y, f) + \sum_{\xi \in \mathcal{O}_F} a_\infty(\xi y, f)e(\sqrt{-1}\xi \cdot y) e_F(\xi x),$$

where $e_F : \mathbb{A}_F/F \to \mathbb{C}^\times$ denotes the standard additive character determined by the condition $e_F(x_\infty) = e^{2\pi \sqrt{-1} \sum_{\tau} x_\tau}$ for $x_\infty = (x_\tau) \in F \otimes \mathbb{R}$. The relation between $a_\infty$ and $a_i$ is given by

$$a_\infty(\mu(t^{(i)}), f) = a_i(\mu, f)|t^{(i)}|^{-1}_{A_F} = C(\mu(t^{(i)}), f).$$

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One can check that $C(a, f)$ is the same as $C(a, f_0)$ considered in (2.24) of [37], where $f_0$ is defined at the end of 5.6.4. Hence $D(f, s) = D(s, f_0)$ defined in (2.25) of [37].

5.6.8 (Hecke operators). For place $v \nmid pn$, we define Hecke operator $T_v \in \text{End}(M_k(n))$ in the usual way:

$$(T_v f)(g) = \sum_{\gamma_i} f(g \gamma_i), \quad U^1(n) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U^1(n) = \bigsqcup_{\gamma_i} \gamma_i U^1(n).$$

For $\gamma \in O_{F_p} \cap (F \otimes \mathbb{Q}_p)^\times$, we can define Hecke operator $\langle \gamma \rangle$ as in 5.2.1 (with $\bar{w} = \bar{0}$ here)

$$(T_v f)(g) = \sum_{\gamma_i} f(g \gamma_i), \quad U^1(n) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} U^1(n) = \bigsqcup_{\gamma_i} \gamma_i U^1(n).$$

In terms of the Fourier expansion, let $f \in M_k(n, \theta)$, then a simple computation shows that

- $C(a, T_v f) = C(a(\varpi_v), f) + N(\varpi_v)\theta(\varpi_v)C(\varpi_v^{-1}, f)$ if $v \nmid pn$.
- $C(a, \langle \varpi_v \rangle f) = C(a(\varpi_v), f)$ if $v|p$.
- $C(a, \langle \gamma \rangle f) = C(a, f)$ for $\gamma \in O_{F_v}^\times$.

For any subring $R \subset \mathbb{C}$, let $M_k(n, R) = \{ f \in M_k(n), a_i(\mu, f) \in R \text{ for any } i, \mu \}$. We view $\mathcal{O}$ as a subring of $\mathbb{C}$ via the isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. Suppose $\mathcal{O}$ is large enough. It is clear that all the Hecke operators above preserve $M_k(n, \theta, \mathcal{O}) := M_k(n, \mathcal{O}) \cap M_k(n, \theta)$. In fact, $M_k(n, \theta, \mathcal{O})$ defines an integral structure of $M_k(n, \theta)$ (2.3.18 of [22]):

$$M_k(n, \theta, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \cong M_k(n, \theta).$$

The same result holds with space of cusp forms: let $S_k(n, \theta, \mathcal{O}) := M_k(n, \mathcal{O}) \cap S_k(n, \theta)$, then

$$S_k(n, \theta, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C} \cong S_k(n, \theta).$$
We define Hecke algebras $\tilde{T}(n, \theta) \subseteq \text{End}_O(M_k(n, \theta, O))$ and similarly $T(n, \theta) \subseteq \text{End}_O(S_k(n, \theta, O))$ as the $O$-subalgebras generated by $T_v, v \nmid pn, \langle \gamma \rangle, \gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times$. We can define Hida’s idempotent $e$ in $\text{End}_O(M_k(n, \theta, O))$ and $\text{End}_O(S_k(n, \theta, O))$ as in 5.2.1 and the ordinary Hecke algebra $T_{\text{ord}}(n, \theta) := eT(n, \theta)$ similarly for cusp forms.

Proof of proposition 3.6.1. As we remarked in 5.6.2, we may assume $\Sigma^o$ is not empty or $\Sigma_p$.

By global class field theory (see theorem 5 of chapter 10 of [1]), there exists a continuous character $\tilde{\theta} : G_F \to \mathbb{F}^\times$ such that

- For $v|p$, $\tilde{\theta}|_{G_{F_v}} = (\tilde{\psi}_v, 1)^{-1}$ (defined before 5.3.2).
- $\tilde{\theta}$ is ramified at somewhere outside of $\Sigma_p$.

Since we assume that $\tilde{\chi}|_{G_{F_v}} \neq 1$ for any $v|p$, it follows from $\Sigma^o \neq \Sigma_p$ that $\tilde{\theta}|_{G_{F_v}} \neq 1$ for some $v|p$. Similarly, $\tilde{\theta}\tilde{\chi}|_{G_{F_v}} \neq 1$ for some $v|p$. Let $\theta_1$ (resp. $\theta_2$) be the Teichmüller lifting of $\tilde{\theta}$ (resp. $\tilde{\theta}\tilde{\chi}$) and $n_1$ (resp. $n_2$) be its conductor. Hence $\theta_1|_{G_{F_v}}$ (resp. $\theta_2|_{G_{F_v}}$) is trivial for $v \in \Sigma^o$ (resp. $v \in \Sigma_p \setminus \Sigma^o$). We remark that for any $i \in \{1, 2\}$, there always exists a place $v|p$ such that $\theta_i|_{G_{F_v}} \neq 1$. Put $\theta = \theta_1\theta_2|_{A_F}$ and $n = n_1n_2$. Consider the weight one Eisenstein series $E_1 = E_1(\theta_1, \theta_2) \in M_1(np, \theta)$ with Dirichlet series (proposition 3.4 of [37])

$$L^{(\Sigma_p \setminus \Sigma^o)}(F, s, \theta_1)L^{(\Sigma^o)}(F, s, \theta_2)$$

where for a finite set of places $S$, $L^{(S)}(F, s, \theta_i)$ denotes the usual $L$-function associated to $\theta_i$ with Euler factors at $v \in S$ removed. It is clear that

- $T_v \cdot E_1 = (\theta_1(\text{Frob}_v) + \theta_2(\text{Frob}_v))E_1$ for $v \nmid pn$.
- $\langle \gamma \rangle \cdot E_1 = E_1$ for $\gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times$. 

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We note that $a_i(0, E_1) = 0$ for any $i$ since $\theta_1, \theta_2$ are ramified by our assumptions. This can be checked by a direct computation using (3.7) and proposition 3.4 of [37].

Enlarge $\mathcal{O}$ if necessary, we may assume $E_1 \in M_1(n_p, \mathcal{O}) \setminus \varpi M_1(n_p, \mathcal{O})$.

Next we want to find a modular form of higher weight congruent to $E_1$. I thank Professor Richard Taylor for showing me the following lemma.

**Lemma 5.6.9.** There exists a Hilbert modular form

$$\Theta \in M_{(p-1)2j}(p^{c-1}, | \cdot |_{\mathbb{A}_F}^{-(p-1)2j+2}, \mathcal{O})$$

for some positive integers $j$ and $c$, such that

- $C_i(0, \Theta) = 1$ for any $i$;
- $a_i(\mu, \Theta) \equiv 0 \mod p$ for any $\mu \neq 0$ and $i$.

**Proof.** This follows from lemma 1.4.2. of [46]. \qed

Now consider the function $E : \text{GL}_2(\mathbb{A}_F) \to \mathbb{C}$,

$$E(u) = E_1(u)\Theta(u)|\det(u)|_{\mathbb{A}_F}^{-1}$$

It is easy to see that $E \in M_{k_0}(np^c, \tilde{\theta}, \mathcal{O})$ with $k_0 = (p - 1)2j + 1, \tilde{\theta} = \theta|\cdot|_{\mathbb{A}_F}^{-(p-1)2j}$ and

$$C_i(\mu, E) \equiv C_i(\mu, E_1) \mod \varpi.$$

Hence $E \notin \varpi M_{k_0}(np^c, \tilde{\theta}, \mathcal{O})$ and

- $T_v \cdot E \equiv (\theta_1(\text{Frob}_v) + \theta_2(\text{Frob}_v))E \mod (\varpi)$ for $v \nmid pn$.
- $\langle \gamma \rangle \cdot E \equiv E \mod (\varpi)$ for $\gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times$.

Therefore $\varpi, T_v - (\theta_1(\text{Frob}) + \theta_2(\text{Frob})), v \nmid pn$ and $\langle \gamma \rangle - 1, \gamma \in O_{F,p} \cap (F \otimes \mathbb{Q}_p)^\times$ generate a maximal ideal $m_1$ of $\mathbb{T}(np^c, \tilde{\theta})$. Note that $\mathbb{T}(np^c, \tilde{\theta})_{m_1}$ is a direct summand of
\[ T(n \mathfrak{p}, \tilde{\theta}). \] We denote the idempotent associated to \( m_1 \) by \( \epsilon_1 \). Let \( \epsilon_0 \) be the composite of \( \epsilon_1 \) with Hida's idempotent \( e \). We claim that \( E^{\text{ord}} := \epsilon_0(E) \) is in fact a cusp form.

By proposition 1.5 of [45], a complement of \( S_{k_0}(np, \tilde{\theta}) \) in \( M_{k_0}(np, \tilde{\theta}) \) is spanned by some automorphic forms inside the automorphic representations \( \pi(k_0, \psi_1, \psi_2) \) generated by the Eisenstein series \( E_{k_0}(\psi_1, \psi_2) \) (see the discussion before proposition 1.5 [45] for the precise statement) with Dirichlet series \( L(F, s, \psi_1) L(F, s - k_0 + 1, \psi_2) \), where \( \psi_1, \psi_2 \) are Hecke characters of finite orders such that

- \( \psi_1 \psi_2 \mid \theta_{\mathfrak{p}}^{-k_0 + 2} = \tilde{\theta} \).
- \( p^c \mathfrak{n} \) divides the product of conductors of \( \psi_1 \) and \( \psi_2 \).

Strictly speaking, the result in [45] only proved the case of weight two. However the same argument works for higher weight. Suppose there exists some automorphic form \( F \in \pi(k_0, \psi_1, \psi_2) \cap M_{k_0}(np, \tilde{\theta}) \) such that \( \epsilon_0(F) \neq 0 \). It follows from \( e(F) \neq 0 \) that \( F \) is an ordinary form and \( \psi_1 \mid F_{\mathfrak{p}}^\times \) is unramified for \( \mathfrak{v} \mid p \). Moreover since \( \epsilon_1(F) \neq 0 \), we must have

- \( \psi_1 + \psi_2 \equiv \theta_1 + \theta_2 \mod \varpi \).
- \( \psi_1(\varpi \mathfrak{v}) \equiv 1 \mod \varpi, \mathfrak{v} \mid p \).

From the first identity, it is easy to see that \( \psi_1 \equiv \theta_1 \) or \( \theta_2 \mod \varpi \). In either case, we may find a \( \mathfrak{v} \mid p \) such that \( \psi_1 \mid G_{F_{\mathfrak{v}}} \neq 1 \mod \varpi \). This contradicts the second equality above. Thus \( \epsilon_0(F) = 0 \) and \( E^{\text{ord}} = \epsilon_0(E) \) has to be a cusp form.

Thus \( \varpi, T_v - (\theta_1(\text{Frob}) + \theta_2(\text{Frob}_v)), v \nmid \mathfrak{p} \mathfrak{n} \) and \( \langle \gamma \rangle - 1, \gamma \in O_{F, \mathfrak{p}} \cap (F \otimes \mathbb{Q}_p)^\times \) in fact generate a maximal ideal \( \mathfrak{m}_2 \) of \( \mathbb{T}^{\text{ord}}(np, \tilde{\theta}) \), the ordinary Hecke algebra of cusp forms. Hence we may find an ordinary cuspidal eigenform \( F' \in S_{k_0}(np, \tilde{\theta}, O)_{\mathfrak{m}_2} \). Using the Jacquet-Langlands correspondence, we can transfer \( F' \) from \( \text{GL}_2 / F \) to \( D^\times \) and get an ordinary eigenform \( F'' \in S_{k_0}(np, \tilde{\theta}_p, U^\times_F(c), O) \) with the same \( T_v, \langle \gamma \rangle \) eigenvalues. Here \( U^\times_F = \prod_{\mathfrak{p} \mid \mathfrak{l}p} U_{e, \mathfrak{v}} \) is an open subgroup of \( \prod_{\mathfrak{p} \mid \mathfrak{l}p} \text{GL}_2(O_{F, \mathfrak{v}}) \) such that
\( U_{e,v} = \text{GL}_2(O_{F_v}), v \nmid \mathfrak{p}n \) and \( \tilde{\theta}_p : (\mathbb{A}_F^\infty)^{\times}/F_{>0}^\times \to \mathcal{O}^\times \) is the \( p \)-adic character associated to \( \tilde{\theta} \):

\[
\tilde{\theta}_p(g) = \tilde{\theta}(g)N_{F/Q}(g_p)^{k_0-2}.
\]

Shrink \( U^p_e \) if necessary, we may assume \( F'' \otimes \theta_1^{-1} : (D \otimes_F \mathbb{A}_F^\infty)^{\times} \rightarrow \mathcal{O} \),

\[
g \mapsto F''(g) \theta_1^{-1}(N_{D/F}(g))
\]
is an element in \( S^{\text{ord}}_{(k_0,0),\tilde{\theta}_p \theta_1^{-1}}(U^p_e(c), \mathcal{O}) \). It follows from the definition of \( \theta_1 \) that \( \varpi, T_v - (1+\chi(\text{Frob}_b)), v \nmid \mathfrak{p}n \) and \( \psi_{v,1}(\gamma) - \tilde{\psi}_{v,1}(\gamma), \gamma \in F^\times_v, v|p \) (as in the proposition) generate a maximal ideal of \( T^{\text{ord}}_{(k_0,0),\tilde{\theta}_p \theta_1^{-1}}(U^p_e(c)) \), hence also a maximal ideal of \( T^{\text{ord}}_{\tilde{\theta}_p \theta_1^{-1}}(U^p_e) \). Using the weight independence result (Thm. 2.3 of [21]), we conclude that these elements generate a maximal ideal of \( T^{\text{ord}}_{\psi}(U^p_e) \). \( \Box \)

## 5.7 Proof of theorem 5.1.2

We will freely use the notation introduced in section 5.1. The strategy is roughly as follows: we will make certain soluble base change and invoke proposition 5.5.1 and soluble base change results. First we do some reduction work.

5.7.1. Let \( \tilde{\chi}, \tilde{\omega} : G_{F,S} \rightarrow \mathcal{O}^\times \) be the Teichmüller lifting of \( \tilde{\chi}, \omega \). Then we may write

\[
\chi = \tilde{\chi}^k \tilde{\omega}^{-k} \theta^2
\]

for some integer \( k \) and some smooth character \( \theta : G_{F,S} \rightarrow \mathcal{O}^\times \) of \( p \)-power order. It is easy to see that we only need to prove theorem 5.1.2 with \( \chi \) replaced by \( \chi \theta^{-2} \). Hence we may assume \( \chi \) and \( \tilde{\chi} \) ramify at the same set of places outside of \( \Sigma_{F,p} \).

It suffices to prove theorem 5.1.2 with \( R^{\text{ns,ord}} \) replaced by \( R^{\text{ns,ord}}/\mathfrak{p} \) for any minimal prime \( \mathfrak{p} \) of \( R^{\text{ns,ord}} \). Fix a minimal prime \( \mathfrak{p} \). Let \( \rho(\mathfrak{p}) : G_{F,S} \rightarrow \text{GL}_2(k(\mathfrak{p})) \) be the associated semi-simple representation and let \( \Sigma_{F,2}^0 \) be the set of places \( v \) above \( p \) such
that \( \rho(p)|_{G_{F_v}} \cong \begin{pmatrix} \psi_{v,2}^{\text{univ}} \mod p & \ast \\ 0 & \psi_{v,1}^{\text{univ}} \mod p \end{pmatrix} \). We denote the set of places of \( F \) above \( p \) by \( \Sigma_{F,p} \). Let \( \Sigma^o_F = \Sigma_{F,p} \setminus \Sigma_{F}^2 \). Then

\[
\rho(p)|_{G_{F_v}} \cong \begin{pmatrix} \psi_{v,1}^{\text{univ}} \mod p & \ast \\ 0 & \psi_{v,2}^{\text{univ}} \mod p \end{pmatrix}
\]

is a non-split extension for any \( v \in \Sigma^o_F \). By twisting with \( \chi^{-1} \) if necessary, we may assume

\[
|\Sigma^o_F| \leq \frac{1}{2} |\Sigma_{F,p}|. \tag{5.1}
\]

Later on we will use this to bound the Selmer group in \[5.4.2\].

We denote the trace of \( \rho(p) \) by \( T(p) \). Then \( T(p) \) is \( (\psi_{v,1}^{\text{univ}} \mod p) \)-ordinary (resp. \( (\psi_{v,2}^{\text{univ}} \mod p) \)-ordinary) if \( v \in \Sigma^o_F \) (resp. \( v \in \Sigma_{F,p} \setminus \Sigma^o_F \)).

The following simple lemma and corollary will be quite useful.

**Lemma 5.7.2.** Let \( R \in C_O \) be a domain with maximal ideal \( \mathfrak{m} \) and fraction field \( K \). Let \( \rho : G_{F,S} \rightarrow \text{GL}_2(R) \) be a continuous representation such that

\[
\text{tr } \rho \equiv 1 + \bar{\chi} \mod \mathfrak{m}.
\]

Then for any \( v \in S \setminus \Sigma_{F,p} \), there exists a finite extension \( K'/K \) such that either

- \( \rho \otimes K'|_{G_{F_v}} \) is reducible, or

- \( \rho \otimes K'|_{G_{F_v}} \cong \text{Ind}_{G_{F_{v,2}}}^{G_{F_v}} \theta \), where \( F_{v,2} \) denotes the unramified quadratic extension of \( F_v \) and \( \theta : G_{F_{v,2}} \rightarrow k(p)^\times \) is a character.

**Proof.** If \( \bar{\chi}|_{I_{F_v}} \) is non-trivial, then \( \rho \otimes K'|_{I'_{F_v}} \) is a direct sum of distinct characters where \( I'_{F_v} \) denotes the prime to \( p \) subgroup of \( I_{F_v} \). Hence \( \rho \otimes K'|_{G_{F_v}} \) is reducible
in this case. Now suppose $\bar{\chi}|_{I_{F_v}} = 1$. In this case, $\rho|_{I_{F_v}}$ has to factor through its pro-$p$ quotient $I_{F_v}(p) \cong \mathbb{Z}_p$. Hence there exists a quadratic extension $K'/K$ such that $\rho \otimes K'|_{I_{F_v}}$ is reducible. If $\rho \otimes K'|_{I_{F_v}}$ is a non-split extension, then $\rho \otimes K'|_{G_{F_v}}$ is reducible. Otherwise $\rho \otimes K'|_{I_{F_v}} \cong \theta_1 \oplus \theta_2$ and $\rho \otimes K'|_{G_{F_v}}$ is either reducible or dihedral.

\begin{proof}
Let $R \in C_{\mathcal{O}}$ be a domain with maximal ideal $m$ and fraction field $K$. Let $T : G_{F,S} \to R$ be a continuous 2-dimensional pseudo-representation with determinant $\chi$ such that $T \equiv 1 + \bar{\chi}$ mod $m$. Then for any $v \in S \setminus \Sigma_{F,p}$, there exists a finite extension $K'/K$ and two characters $\theta_1, \theta_2 : I_{F_v} \to (K')^\times$ of finite orders which can be extended to $G_{F,v}$ such that

$$T|_{I_{F_v}} = \theta_1 + \theta_2.$$ 

\end{proof}

**Corollary 5.7.3.** Let $R \in C_{\mathcal{O}}$ be a domain with maximal ideal $m$ and fraction field $K$. Let $T : G_{F,S} \to R$ be a continuous 2-dimensional pseudo-representation with determinant $\chi$ such that $T \equiv 1 + \bar{\chi}$ mod $m$. Then for any $v \in S \setminus \Sigma_{F,p}$, there exists a finite extension $K'/K$ and two characters $\theta_1, \theta_2 : I_{F_v} \to (K')^\times$ of finite orders which can be extended to $G_{F,v}$ such that

$$T|_{I_{F_v}} = \theta_1 + \theta_2.$$ 

**Proof.** Let $R^{ps}$ be the universal deformation ring which parametrizes all two-dimensional pseudo-representations with determinant $\chi$ that lift $1 + \bar{\chi}$. Denote the universal pseudo-representation by $T^{univ}$. It suffices to treat the case $R = R^{ps}/q, T = T^{univ}$ mod $q$ for some $q \in \text{Spec } R^{ps}$. Our assertion is clear if the semi-simple representation $\rho(q)$ associate to $q$ is reducible. Now assume $\rho(q)$ is irreducible. Then arguing as in the proof of corollary 5.5.2, we conclude that $q$ is in the image of $\text{Spec } R_b \to \text{Spec } R^{ps}$ where $R_b$ parametrizes all deformations of $\bar{\rho}_b : G_{F,S} \to \text{GL}_2(\mathbb{F})$ with determinant $\chi$ for some representation $\bar{\rho}_b$ of the form

$$\begin{pmatrix}
1 & * \\
0 & \bar{\chi}
\end{pmatrix}, * \neq 0.$$ 

Our assertion follows from the previous lemma.

\begin{proof}
5.7.4 (Existence of Eisenstein ideal). Arguing as in (4.16), (4.17) of [39], we can find a finite totally real field extension $F_1/F$ such that

- $F_1/\mathbb{Q}$ is abelian with even degree.
- $F(\bar{\chi}) \cap F_1 = F.$

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• \( p \) is unramified in \( F_1 \).

• \( \text{ord}_\wp L_p(F_1, -1, \tilde{\chi}) > 0 \) as in proposition 5.6.1.

Let \( \Sigma_{F_1}^o \) be the set of places of \( F_1 \) above \( \Sigma_F^o \). Then it follows from proposition 5.6.1 that there exists an open subgroup \( U^p_e = \prod_{v \not\mid p} U_{e,v} \subseteq \prod_{v \not\mid p} \text{GL}_2(O_{F_1,v}) \) such that \( \mathbb{T}^\text{ord}(U^p_e) \) has the desired maximal ideal (see 5.7.9 below for details here). Denote by \( S_1 \) the union of places of \( F_1 \) above \( S \) and places \( v \) such that \( U_{e,v} \neq \text{GL}_2(O_{F_1,v}) \).

5.7.5. Let \( F_2/F_1 \) be a finite totally real field extension with following properties:

• \( F_2/Q \) is abelian.

• \( p \) is unramified in \( F_2 \).

• \( F_1(\tilde{\chi}) \cap F_2 = F_1 \).

• For any place \( w \not\mid p \) above \( v \in S_1 \), we have \( 2[[k(w) : k(v)] \) and \( p|N(w) - 1 \) and

\[ \bar{\chi}|_{G_{F_2,w}} = 1. \]

In particular, \( \bar{\chi}|_{G_{F_2}} \) and \( \chi|_{G_{F_2}} \) are unramified outside of places above \( p \).

The only non-trivial requirement is the fourth one. This is possible since we assume \( \bar{\chi} : G_F \to \mathbb{F}^\times \) can be extended to \( G_Q \).

5.7.6. Finally, let \( l_0 \) be a rational prime congruent to 1 modulo the order of \( \bar{\chi} \) and larger than the norm of any place in \( S_1 \). We choose a finite extension \( F_3/F_2 \) contained in the cyclotomic \( \mathbb{Z}_{l_0} \)-extension of \( F_2 \) of sufficiently large \( l_0 \)-power degree such that:

• \( |S_3| \leq \frac{1}{12}[F_3 : Q] \), where \( S_3 \) denotes the set of places of \( F_3 \) above \( S_1 \).

• The \( p \)-part of class group of \( F_3(\bar{\chi}) \) satisfies \( |\text{Cl}(F_3(\bar{\chi}))[p]| \leq \frac{1}{12}[F_3 : Q] \).

This is possible in view of the result of Washington [44].
Denote the set of primes of $F_3$ above $p$ (resp. $\Sigma^o_p$) by $\Sigma_p$ (resp. $\Sigma^o$). Then $|\Sigma^o| = |\Sigma_p^o| \le \frac{1}{2}$ by our assumption (5.1). In particular, $|\Sigma^o| \le |\Sigma_p| - 1$. We introduced the following Selmer group as in 5.4.2:

$$H^1_{\Sigma^o}(F_3) := \ker(H^1(G_{F_3,S_3},\mathbb{F}(\bar{\chi}^{-1})) \rightarrow \bigoplus_{v \in \Sigma_p \setminus \Sigma^o} H^1(G_{F_3,v},\mathbb{F}(\bar{\chi}^{-1}))).$$

**Lemma 5.7.7.** $F_3$ is an abelian extension of $\mathbb{Q}$. Moreover it satisfies the following properties:

- $F_2(\bar{\chi}) \cap F_3 = F_2$.
- For any $v | p$ in $F_3$, $F_{3,v}$ is an unramified extension of $\mathbb{Q}_p$ and $\bar{\chi}|_{G_{F_3,v}} \neq 1$.
- $\dim_{\mathbb{F}} H^1_{\Sigma^o}(F_3) \leq \frac{2}{3}[F_3 : \mathbb{Q}] - 1$.

**Proof.** The first two claims are clear as $l_0$ is larger than the order of $\bar{\chi}$ and $p$. For the last one, let $K$ be the kernel of

$$H^1_{\Sigma^o}(F_3) \rightarrow \bigoplus_{v \in \Sigma^o} H^1(G_{F_3,v},\mathbb{F}(\bar{\chi}^{-1})) \oplus \bigoplus_{v \in S_3 \setminus \Sigma_p} H^1(G_{F_3,v},\mathbb{F}(\bar{\chi}^{-1}))/H^1(G_{k(v)},\mathbb{F}(\bar{\chi}^{-1}))).$$

Since $p$ is prime to the order $\bar{\chi}$, we have a natural injection

$$K \hookrightarrow H^1(G_{F_3(\bar{\chi})},\mathbb{F}(\bar{\chi}^{-1})) \cong \text{Hom}(G_{F_3(\bar{\chi})},\mathbb{F})(\bar{\chi}^{-1}).$$

It is easy to see that the image of $K$ lies inside the subspace of characters which are unramified everywhere. Hence its dimension is bounded by $\dim_{\mathbb{F}}[\text{Cl}(F_3(\bar{\chi}))[p]]$. Note that $\bar{\chi}|_{G_{F_3,v}}$ is trivial for $v \in S_3 \setminus \Sigma_p$, hence

$$H^1(G_{F_3,v},\mathbb{F}(\bar{\chi}^{-1}))/H^1(G_{k(v)},\mathbb{F}(\bar{\chi}^{-1}))) \cong \text{Hom}(O_{F_{3,v}}^\times,\mathbb{F}),$$

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which is one-dimensional. For \( v \in \Sigma_p \), it follows from our second claim in the lemma that

\[
\dim_G H^1(G_{F_{3,v}}, \mathbb{F}(\bar{\chi}^{-1})) = [F_{3,v} : \mathbb{Q}_p] + e_\chi \leq [F_{3,v} : \mathbb{Q}_p] + 1,
\]

where \( e_\chi = 1 \) if \( \bar{\chi}|G_{F_{3,v}} = \omega^{-1} \) and 0 otherwise. Put all these results together:

\[
\dim_G H^1_{\Sigma_0}(F_3) \leq \dim K + \sum_{v \in \Sigma_0} \dim H^1(G_{F_{3,v}}, \mathbb{F}(\bar{\chi}^{-1})) + \sum_{v \in S_3 \setminus \Sigma_p} \dim \text{Hom}(\mathcal{O}_{F_{3,v}}^\times, \mathbb{F})
\]

\[
\leq \dim_{\mathbb{F}_p} |\text{Cl}(F_3(\bar{\chi}))[p]| + \sum_{v \in \Sigma_0} [F_{3,v} : \mathbb{Q}_p] + |\Sigma_0| + |S_3| - |\Sigma_p|
\]

\[
\leq \frac{1}{12} [F_3 : \mathbb{Q}] + \frac{|\Sigma_0|}{|\Sigma_p|} [F_3 : \mathbb{Q}] + \frac{1}{12} [F_3 : \mathbb{Q}] - 1
\]

\[
\leq \frac{2}{3} [F_3 : \mathbb{Q}] - 1.
\]

5.7.8. For \( v \in S_3 \setminus \Sigma_p \), we define a \( p \)-power character \( \xi_v : k(v)^\times \to \mathcal{O}^\times \) as follows: let \( p \) be the prime of \( \mathcal{R}_{\text{ps,ord}} \) defined in the beginning of this subsection. By corollary 5.7.3 and our construction of \( F_2 \) in 5.7.5, enlarging \( E \) if necessary, there exists a character \( \xi_v : k(v)^\times \to \mathcal{O}^\times \), viewed as a character of \( I_{F_{3,v}} \) by the class field theory, such that

\[
T(p)|_{I_{F_{3,v}}} = \xi_v + \xi_v^{-1}.
\]

If \( p \in \mathfrak{p} \), then we can simply take \( \xi_v \) to be trivial. Then using the data \( \{\xi_v\}_v, \Sigma_0^\alpha \), we can define \( R_{\Sigma_0}^{\text{ps,ord},(\xi_v)} \) as in 5.3.4 with \( F = F_3, S = S_3 \). We are going to apply corollary 5.5.2. First we need check the assumptions in proposition 5.5.1.

Now the second assumption in proposition 5.5.1 becomes:

\[
[F_3 : \mathbb{Q}] - 4|S_3| + 4|\Sigma_p| - 3 > \delta_{F_3} + \dim_G H^1_{\Sigma_0}(F_3).
\]

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Since $F_3$ is abelian over $\mathbb{Q}$, Leopoldt’s conjecture is known in this case. Hence $\delta_{F_3} = 0$. Then the inequality follows easily from lemma 5.7.7 and our assumption on $|S_3|$.

5.7.9. To see the first assumption (Assumption in 5.3.5) is satisfied, it follows from the existence of the Eisenstein maximal ideal as in 5.7.4 that we can find a two-dimensional Galois representation (after possibly replacing $E$ by some extension) $\rho_1 : G_{F_1,S_1} \to \text{GL}_2(\mathcal{O})$ of determinant $\chi$ which comes from an automorphic representation of $\text{GL}_2(\mathbb{A}_{F_1})$ such that

- $\text{tr}\rho_1 \equiv 1 + \bar{\chi} \mod \varpi$.

- For any $v|p$, $\rho_1|_{G_{F_1,v}} \cong \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix}$ with $\psi_{v,1} \equiv 1 \mod \varpi$ if $v \in \Sigma_{F_1}$ and $\psi_{v,1} \equiv \bar{\chi}|_{G_{F_1,v}} \mod \varpi$ otherwise. Moreover, $\psi_{v,1}$ is Hodge-Tate and has strictly less Hodge-Tate number than $\psi_{v,2}$ for any embedding $F_{1,v} \hookrightarrow \mathbb{Q}_p$.

Note that $\rho_1|_{G_{F_3}}$ has to be irreducible. Otherwise suppose $G_{F_3}$ acts as a character $\theta$ on some one-dimensional subspace $L$. Then the reduction of $\theta$ modulo $\varpi$ is 1 or $\bar{\chi}$. Hence $\theta(c)$ has to be a constant for any complex conjugation $c \in G_{F_3}$. This implies that $G_{F_1}$ also fixes $L$ and $\rho_1$ is reducible, which contradicts the cuspidal condition.

Now it follows from the soluble base change (see for example 1.4 [2]) that $\rho_1|_{G_{F_3}}$ also comes from an automorphic representation of $\text{GL}_2(\mathbb{A}_{F_3})$. By lemma 5.7.2 for any $w \in S_3 \setminus \Sigma_p$, there exists a character $\xi'_w : k(w)^\times \to \mathcal{O}^\times$ of $p$-power order such that $\rho_1|_{I_{F_3,w}} = \xi'_w + (\xi'_w)^{-1}$. Hence the elements in Assumption 5.3.5 generate a maximal ideal of $\mathcal{T}^{\text{ord}}_{\psi,\xi'}(U^p)$ with $\xi' = \prod \xi'_v$, a character of $U^p$. See 5.3.5 for the precise meanings of these notations. Arguing as in the proof of lemma 4.8.8, we conclude that these elements also generate a maximal ideal of $\mathcal{T}^{\text{ord}}_{\psi,\xi}(U^p)$.

Thus we can apply corollary 5.5.2 to $R^{\text{ps,ord},\{\xi_v\}}_{\Sigma^0}$. 

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5.7.10. It is easy to see that $T(p)|_{G_{F_3}}$ (defined in the beginning of this subsection) induces a natural map by the universal property:

$$R_{\Sigma_0}^{ps,ord,\{\xi_v\}} \rightarrow R^{ps,ord}/p.$$ 

Lemma 5.7.11. This is a finite map.

Proof. It suffices to prove $R^{ps,ord}/(p, m_1)$ is of finite length, where $m_1$ is the maximal ideal of $R_{\Sigma_0}^{ps,ord,\{\xi_v\}}$. Let $q$ be a prime ideal $R^{ps,ord}/(p, m_1)$. We need to show that $q$ is in fact the maximal ideal. Denote by $\rho(q) : G_F \rightarrow GL_2(k(q))$ the associated two-dimensional semi-simple representation. Then

$$tr\rho(q)|_{G_{F_3}} = 1 + \bar{\chi}|_{G_{F_3}}.$$ 

Hence $\rho(q)|_{G_{F_3}}$ is reducible. Arguing as in the second paragraph of 5.7.9 we see that $\rho(q)$ is also reducible. It follows easily from this that $tr\rho(q) = 1 + \bar{\chi}$. Therefore $q$ is the maximal ideal of $R^{ps,ord}/(p, m_1)$.

5.7.12. Now we can prove theorem 5.1.2. As in 5.3.4 there is a map

$$\iota_{F_3} : \Lambda_{F_3} := \hat{\bigotimes}_{v|p} O[[O_{F_3,v}^{x}(p)]] \rightarrow R_{\Sigma_0}^{ps,ord,\{\xi_v\}}.$$ 

Similarly there is a map $\iota_F : \Lambda_F \rightarrow R^{ps,ord}/p$ coming from the universal deformations of $1|_{G_{F_v}}$ if $v \in \Sigma_F$ and $\bar{\chi}|_{G_{F_v}}$ if $v \in \Sigma_{F,p} \setminus \Sigma_F$. Note that this map is different from the one in theorem 5.1.2 which comes from the universal deformations of $1|_{G_{F_v}}$ for any $v|p$. However, it is easy to see that both maps have the same images in $R^{ps,ord}/p$. 

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By the universal property, we have the commutative diagram:

\[
\begin{array}{ccc}
\Lambda_{F_3} & \xrightarrow{\iota_{F_3}} & R^{ps,ord,\{\xi_v\}}_{\Sigma_0} \\
\downarrow{N_{F_3/F}} & & \downarrow{N_{F_3/F}} \\
\Lambda_F & \xrightarrow{\iota_F} & R^{ps,ord}/p
\end{array}
\]

where the left vertical map arises from the norm map and the right vertical map is the one we considered in the previous lemma. Note that both maps are finite: the left one is in fact surjective and the other one follows from lemma 5.7.11. By corollary 5.5.2 \(\iota_{F_3}\) is also finite. Therefore \(\iota_F\) also has to be finite. This proves the first part of theorem 5.1.2.

For the first part of theorem 5.1.2, let \(\rho(p)\) be a Galois representation as in the theorem. Then \(\rho(p)|_{G_{F_3}}\) is irreducible by the same argument in the second paragraph of 5.7.9. Hence we can apply the third part of corollary 5.5.2 to \(\rho(p)|_{G_{F_3}}\). Theorem 5.1.2 now follows from the theory of soluble base change.
Chapter 6

The main theorem

Theorem 6.0.1. Let $F$ be a totally real abelian extension of $\mathbb{Q}$ in which $p$ completely splits. Suppose

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\mathcal{O})$$

is a continuous irreducible representation with the following properties

- $\rho$ ramifies at only finitely many places.

- Let $\overline{\rho}$ be the reduction of $\rho$ modulo $\varpi$. We assume its semi-simplification has the form $\overline{\chi}_1 \oplus \overline{\chi}_2$ and $\overline{\chi}_1/\overline{\chi}_2$ can be extended to a character of $G_{\overline{\mathbb{Q}}}$. 

- $\rho|_{G_{F_v}}$ is irreducible and de Rham of distinct Hodge-Tate weights for any $v|p$ and any embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$. Moreover,

$$\left(\overline{\chi}_1/\overline{\chi}_2\right)|_{G_{F_v}} \neq 1, \omega^{\pm1}.$$ 

- $\left(\overline{\chi}_1/\overline{\chi}_2\right)(c) = -1$ for any complex conjugation $c \in \text{Gal}(\overline{F}/F)$.

Then $\rho$ arises from a twist of a Hilbert modular form, i.e. a regular algebraic cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. 

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Proof. We may always twist $\rho$ with some smooth character and assume $\overline{\chi}_1$ is trivial. Denote the set of primes where $\rho$ is ramified by $S$ and the set of primes above $p$ by $\Sigma_{F,p}$. By soluble base change and Theorem 5 of chapter 10 of [1], we may always assume

$$|S \setminus \Sigma_{F,p}| + 1 < [F : \mathbb{Q}].$$

Let $\chi$ be the determinant of $\rho$ and $\overline{\chi}$ be the reduction of $\chi$ modulo $\varpi$. Consider the functor from $\mathcal{C}_f\mathcal{O}$ to the category of sets sending $R$ to the set of 2-dimensional pseudo-representations of $G_{F,S}$ which lift $1 + \overline{\chi}$ with determinant $\chi$. This is pro-represented by a complete local noetherian ring $R^{\text{ps}}$ and $\text{tr} \rho$ gives rise to a prime ideal $p \in \text{Spec } R^{\text{ps}}$.

**Lemma 6.0.2.** $\dim R^{\text{ps}}_p \geq 2[F : \mathbb{Q}]$.

**Proof.** By corollary 2.2.3, $(\overline{R^{\text{ps}}})_p$ parametrizes all deformations with determinant $\chi$ of $\rho_E = \rho \otimes E : G_{F,S} \to \text{GL}_2(E)$. See the precise deformation problem in [2.2.3] except here we put an extra condition on the determinant. It follows from this description that

$$(\overline{R^{\text{ps}}})_p \cong E[[x_1, \cdots, x_{h_1}]]/(f_1, \cdots, f_{h_2}),$$

where $h_i = \dim_E H^i(G_{F,S}, \text{ad}^0 \rho_E)$, $i = 1, 2$ and $\text{ad}^0 \rho_E$ as usual denotes the subspace of $\text{End}_E(\rho_E)$ with trace 0. Then by the global Euler characteristic formula (for example see lemma 9.7 of [25]),

$$h_1 - h_2 = (\dim_E \text{ad}^0 \rho_E)[F : \mathbb{Q}] - \sum_{v \mid \infty} \dim_E(\text{ad}^0 \rho_E)^{G_{F_v}} + h_0 = 2[F : \mathbb{Q}].$$

since we assume $\rho_E$ is irreducible and $\dim_E(\text{ad}^0 \rho_E)^{G_{F_v}} = 1$ for any $v \mid \infty$. This proves the dimension inequality in the lemma. \(\square\)

Choose an irreducible component $C = \text{Spec } R^{\text{ps}}/\mathfrak{q}$ of $\text{Spec } R^{\text{ps}}$ that contains $p$ and has dimension at least $2[F : \mathbb{Q}] + 1$. Let $T^{\text{univ}} : G_{F,S} \to R^{\text{ps}}$ denote the universal
pseudo-character. By lemma 5.7.3, for any $v \in S \setminus \Sigma_p$, we can write $T_{\text{univ}}^{|I_{F_v}} \equiv \theta_v,1 + \theta_v,2 \mod \wp$ for some characters $\theta_v,1, \theta_v,2$ of finite orders after possible enlarging $\mathcal{O}$.

Now consider the following map given by the universal property:

$$R_{p}^{ps} \to R_{p}^{ps}$$

where $R_{p}^{ps}$ denotes the completed tensor product $\bigotimes_{v \mid p} R_{v}^{ps}$ over $\mathcal{O}$ and $R_{v}^{ps}$ denotes the universal deformation ring which parametrizes all 2-dimensional pseudo-representations of $G_{F_v}$ that lift $(1 + \bar{\chi})|_{G_{F_v}}$ with determinant $\chi|_{G_{F_v}}$. Let $R_{v}^{ps,ord}$ be the quotient of $R_{v}^{ps}$ that parametrizes all reducible liftings (i.e. liftings that are sum of two characters) and $R_{p}^{ps,ord}$ be the completed tensor product of $\bigotimes_{v \mid p} R_{v}^{ps,ord}$. We denote

$$R_{p}^{ps,ord} = R_{p}^{ps} \otimes_{R_{p}^{ps}} R_{p}^{ps,ord};\ C^{ord} = \text{Spec } R_{p}^{ps,ord} \cap C.$$ 

Note that a direct definition of $R_{p}^{ps,ord}$ was given in 5.1.1 with the same notation.

**Lemma 6.0.3.** $\dim C^{ord} \geq [F : Q] + 1$.

**Proof.** For any $v \mid p$, since we assume $\bar{\chi}|_{G_{F_v}} \neq 1, \omega^\pm 1$, it follows from corollary B.20 of [32] that the kernel of $R_{v}^{ps} \to R_{v}^{ps,ord}$ is a principal ideal. Hence

$$\dim C^{ord} \geq \dim C - |\Sigma_p| \geq [F : Q] + 1.$$

We remark that this is the essential reason that we rule out the case $\bar{\chi}|_{G_{F_v}} = \omega^\pm 1$ as in this case, the kernel is generated by two elements.

On the other hand, there is a map $\Lambda_F \to R_{p}^{ps,ord}$ as in 5.1.1, which comes from the universal deformation of $1|_{I_{F_v}}, v \mid p$. Theorem 5.1.2 tells us that this is a finite map. Combining this with the previous lemma, we see that
Corollary 6.0.4. There exists an irreducible component $C_{1}^{\text{ord}}$ of $C^{\text{ord}}$ such that

1. $\dim C_{1}^{\text{ord}} = [F : \mathbb{Q}] + 1$.

2. $C^{\text{ord}} \to \text{Spec } \Lambda_{F}$ is a finite surjective map.

3. Let $C_{1}^{\text{ord,aut}}$ be the set of primes $q \in C_{1}^{\text{ord}}$ such that

   - $p \notin q$ and $R^{\text{ps,ord}}/q$ is one-dimensional.
   - Denote the associated semi-simple representation $G_{F, S} \to \text{GL}_{2}(k(q))$ by $\rho(q)$ (see 2.1.5). We assume $\rho(q)$ is irreducible.
   - For any $v|p$, $\rho(p)|_{G_{Fv}} \cong \begin{pmatrix} \psi_{v,1} & * \\ 0 & \psi_{v,2} \end{pmatrix}$ such that $\psi_{v,1}$ is Hodge-Tate and has strictly less Hodge-Tate number than $\psi_{v,2}$ for any embedding $F_{v} \hookrightarrow \overline{\mathbb{Q}}_{p}$.

Then $C_{1}^{\text{ord,aut}}$ is dense in $C_{1}^{\text{ord}}$.

Proof. The existence of an irreducible component that satisfies the first condition is clear. Fix one and denote it by $C_{1}^{\text{ord}}$. The second claim follows from the first claim and the fact that $\Lambda_{F}$ is a domain. As for the last one, first note that the set of $q$ such that $\rho(q)$ is reducible has dimension at most $2 + \delta_{F} = 2 < 1 + [F : \mathbb{Q}]$ (see 5.4.11). Hence we may ignore the second condition in the definition of $C_{1}^{\text{ord,aut}}$.

Let $\psi_{v,1}^{\text{univ}}, \psi_{v,2}^{\text{univ}} : G_{F_{v}} \to (R^{\text{ps,ord}})^{\times}$ be the liftings of $1, \overline{\chi}|_{G_{Fv}}$ respectively. It follows from lemma 5.3.3 that for any $v|p$, there exists $n_{v} \in \{1, 2\}$ such that for any $q \in C_{1}^{\text{ord}}$,

$$\rho(q)|_{G_{Fv}} \cong \begin{pmatrix} \psi_{v,n_{v}}^{\text{univ}} \mod q & * \\ 0 & \psi_{v,3-n_{v}}^{\text{univ}} \mod q \end{pmatrix}.$$  

Using this and the surjectivity of the map $C_{1}^{\text{ord}} \to \text{Spec } \Lambda_{F}$, we see that the image of $C_{1}^{\text{ord,aut}}$ in $\text{Spec } \Lambda_{F}$ is dense. Thus $C_{1}^{\text{ord,aut}}$ is also dense in $C_{1}^{\text{ord}}$ as the map to $\text{Spec } \Lambda_{F}$ is finite. \qed
Now we choose a prime \( q \in C_{1}^{\text{ord}} \) that is “potentially nice” in the following sense:

- \( p \in q \) and \( R^{\text{ps}}/q \) is one-dimensional. In particular, the image of \( q \) in \( \text{Spec } \Lambda_{F} \) is not the closed point so that we can apply the third part of lemma 4.1.9.
- \( \rho(q) \) is irreducible.
- For any \( v \in S \setminus \Sigma_{p} \), \( \rho(q)|_{G_{F_{v}}} \) has finite image.

To see the existence of such a prime, denote the generic point of \( C_{1}^{\text{ord}} \) by \( \mathcal{Q} \in \text{Spec } R^{\text{ps}} \).

Let \( I_{S} \subseteq R^{\text{ps}} \) be the ideal generated by \( T^{\text{univ}}(\text{Frob}_{v}) - 1 - \chi(\text{Frob}_{v}), v \in S \setminus \Sigma_{p} \).

Consider \( R^{\text{ps}}/(\varpi, \mathcal{Q}, I_{S}) \). By our assumptions, its dimension is larger than \( 1 = \delta_{F} + 1 \).

Hence we can choose a prime \( q \) of \( R^{\text{ps}}/(\varpi, \mathcal{Q}, I_{S}) \) such that \( \rho(q) \) is irreducible. We claim that \( q \) is potentially nice, i.e. \( \rho(q)|_{G_{F_{v}}} \) has finite image for any \( v \in S \setminus \Sigma_{p} \).

By lemma 5.7.2, \( \rho(q)|_{G_{F_{v}}} \) is either reducible or induced from a character \( \theta \) of \( G_{F_{v}} \). In the second case, it suffices to prove \( \theta \) has finite orders. This follows from \( \theta(\text{Frob}_{v}^{2}) = -\overline{\chi(\text{Frob}_{v})} \in \mathbb{F}^{\times} \). In the first case, note that \( \text{tr}\rho(q)(\text{Frob}_{v}) = 1 + \overline{\chi(\text{Frob}_{v})} \).

It is easy to see that the semi-simplification of \( \rho(q)|_{G_{F_{v}}} \) is a sum of two characters of finite orders. Hence \( \rho(q)|_{H} \) is unipotent for some subgroup \( H \) of finite index in \( G_{F_{v}} \).

Clearly \( \rho(q)(H) \) is finite. Therefore in either case, our claim is clear.

We fix such a choice of potentially nice prime \( q \).

Finally choose a finite totally real soluble extension \( F_{1} \) of \( F \) in which \( p \) splits completely such that for any place \( w \) of \( F_{1} \) above some place \( v \in S \setminus \Sigma_{p} \),

- \( \rho(q)|_{G_{F_{1},v}} \) is trivial.
- \( N(w) \equiv 1 \mod p \).
- \( T^{\text{univ}}|_{I_{F_{1},w}} \equiv 2 \mod \mathfrak{P} \). See the paragraph below the proof of lemma 6.0.2 for the notations here and for the existence of such \( F_{1,w} \).

Let \( S_{1} \) be the set of places of \( F_{1} \) above \( S \). Consider \( R^{\text{ps},1} \) defined in 4.1.1 More precisely, it pro-represents the functor from \( C_{1}^{\text{f}} \) to the category of sets sending \( R \) to
the set of two-dimensional pseudo-representations $T$ of $G_{F_1,S_1}$ over $R$ such that $T$ is a lifting of $1 + \bar{\chi}|_{G_{F_1,S_1}}$ with determinant $\chi|_{G_{F_1,S_1}}$ and

$$T|_{I_{F_1,v}} = 2$$

for any $v \in S_1, v \nmid p$.

It follows from the construction that there is a map $R_{ps,1}^{ps} \to R_{ps}^p/\mathfrak{p}$. Let $p', q' \in \text{Spec } R_{ps,1}^{ps}$ be the images of $p, q$. We claim that $q'$ is a nice prime in the sense of 4.1.4. All the conditions are clear except that $\rho(q)|_{G_{F_1}}$ is irreducible and $q'$ is modular.

The irreducibility can be proved in the same way as in the second paragraph of 5.7.9. To see that $q'$ is modular, we will actually prove that the image of $C_1^{\text{ord}}$ in $\text{Spec } R_{ps,1}^{ps}$ is modular. First notice that for any $q_1 \in C_1^{\text{ord,aut}}$, $\rho(q_1)$ comes from a regular algebraic cuspidal automorphic representation of $GL_2(\mathbb{A}_{F_1})$. Hence by soluble base change and the same irreducibility argument, the image of any prime of $C_1^{\text{ord,aut}}$ in $\text{Spec } R_{ps,1}^{ps}$ is modular. By the density result in corollary 6.0.4, we see that the image of $C_1^{\text{ord}}$ is in fact modular. In particular, $q'$ is modular and hence nice.

Note that we can find an irreducible component of $R_{ps,1}^{ps}$ that contains the image of $C$. Therefore we can apply corollary 4.1.7 with $p = p'$ and $q = q'$ and conclude that $\rho(p') = \rho|_{G_{F_1}}$ comes from a regular algebraic cuspidal automorphic representation of $GL_2(\mathbb{A}_{F_1})$. Theorem 6.0.1 now follows from soluble base change.

\[\square\]
Bibliography


