INTERBANK MARKETS AND
THEIR OPTIMAL REGULATION

VOLUME ONE

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A DISSERTATION PRESENTED TO THE
FACULTY OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE BY
THE DEPARTMENT OF ECONOMICS

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MAY 2016
Abstract

Recent years have seen interest in interbank market regulation. Unfortunately, a review of the literature in this area identifies two gaps. The first is that previous studies have tended to take the form of the interbank contract as given, making it difficult to generate convincing policy prescriptions. The second gap is that previous studies have tended to focus on ex-post policies which kick in after an interbank disruption has occurred. It’s thus natural to ask if ex-ante policies have a role to play.

This dissertation takes steps toward addressing these gaps. Chapter 1 focuses on endogenizing the form of banks’ interbank contracts, namely by developing a model in which banks choose this form as part of an optimal contracting problem. I argue that the model can account for interbank disruptions like those witnessed during the recent crisis. I also explore its predictions on the circumstances under which the economy becomes vulnerable to these disruptions, along with some policy implications. With respect to the first of these issues, I show that the in-model risk of interbank disruptions is procyclical, consistent with the view that endogenous risk tends to accumulate during good times. I then derive a “no-go” result on the policy front. In particular, I show that the aforementioned disruptions are constrained efficient, with no role for policy.

Chapter 2 extends the baseline model from chapter 1 to include a standard source of constrained inefficiency, namely a pecuniary externality in the capital market. In this setting, a planner’s solution can be implemented using ex-ante prudential policies to regulate banks initial leverage and liquidity, supplemented with ex-post monetary stimulus in the event that a disruption occurs. This highlights the need for coordination between monetary policy and financial regulation. I furthermore argue that the ex-ante components do most of the heavy lifting.

Chapter 3 emphasizes the importance of ex-ante intervention more starkly, namely by developing an alternate extension in which pecuniary externalities arise in the deposit market. In this case, policy must take the form of an ex-ante intervention, since the interest rate on deposits is locked-in if and when interbank disruptions occur.
Acknowledgements

I’d like to begin by thanking my adviser, Nobuhiro Kiyotaki. This thesis would not have been possible without his constant guidance over the past three years. Nobu’s research is well-known throughout the profession, but only his advisees can really appreciate just how generously he shares his time and experience with the younger generation. For this, and many other things, I am very grateful to him.

Several other members of the faculty deserve special thanks, both for their input on this project and more generally for their support at points throughout my graduate career. These include my readers, Mark Aguiar and Oleg Itskhoki, along with Chris Sims, Ben Moll, Richard Rogerson, Mike Golosov, Juan-Pablo Xandri, Hyun Song Shin, Gabe Chodorow-Reich, Markus Brunnermeier, Harvey Rosen, Roland Benabou, Anne Case, and Stephen Redding. I feel privileged to have had access to such fine minds.

I’ve also been privileged to have made some great friends at Princeton, including Cristian Alonso, Qingqing Cao, Cheng Chen, Benjamin Connault, Judd Cramer, Arnab Datta, Nikhil Gupta, Ji Huang, Alex Jakobsen, Caitlin McGugan, Delwin Olivan, Ishita Rajani, Doron Ravid, Gabriele La Spada, Kai Steverson, Neel Sukhatme, Justin Weidner, Tom Winberry, and Christian vom Lehn. To all of you: it’s been a pleasure sharing the ups-and-downs of the past six years with you, and I’m now delighted to see you heading off to great careers in academia, policy, and industry.

I’d next like to thank Laura Hedden, the department’s graduate administrator. As far as I can tell, managing a building full of economics PhD’s is like herding some very well-caffeinated cats, but Laura always pulls it off with a cheerful professionalism. I’d also like to thank Keiko Kiyotaki for being such a reliably friendly face on campus and for taking such interest in how her husband’s students are doing.

Finally, I’d like to thank my parents, Timothy and Eva, and my little sister, Stephanie. Words simply cannot express the extent of my gratitude for their support throughout my time at Princeton.
To my parents and sister, with love and deep gratitude.
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Chapter 1

Optimal interbank contracts

1.1 Introduction

Recent years have seen renewed interest in the regulation of interbank markets, both among policymakers and among academics. This interest stems mainly from the events of the 2007–9 financial crisis, during which many interbank markets experienced especially severe contractions. However, it’s more generally understood that interbank markets play a key role in liquidity allocation, so it’s natural to worry that disruptions in these markets could have spillovers into the larger economy. Indeed, some of the linkages between interbank markets and real economic outcomes have now been documented in Iyer et al. (2014).

A review of the current literature on interbank markets and their regulation identifies a potentially important gap, namely that previous studies have tended to take the form of the interbank contract as given, rather than allowing banks to choose this contract optimally. Since the famous Lucas critique (1976), it’s well-known that this can make it difficult to generate convincing policy prescriptions. In fact, the problem is especially acute in the

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1 The best-known interbank market is the federal funds market, in which banks lend and borrow reserve balances at the Federal Reserve on an uncollateralized basis, generally at overnight maturity. See section 1 in Ashcraft and Duffie (2007) for a quick overview.

2 Among others, see section 2.3 in Bech and Monnet (2013) for details on the drop in overnight interbank volumes in the US, UK, Europe, and Australia.

3 Iyer et al. (2014) collect detailed data on firm and bank balance sheets in Portugal during an August 2007 freeze in European interbank markets. They show (i) that banks with greater exposure to interbank markets during this period withdrew credit from firms, and (ii) that these firms were unable to make up for these withdrawals either by switching to other banks or by tapping other non-bank sources of credit.

4 Moreover, if we consider wholesale funding markets in general, rather than interbank markets in particular, then it’s also useful to note that Goetz and Gozzi (2010) document greater drops in employment and establishments in metropolitan areas where banks had greater exposure to wholesale funding sources, while Dagher and Kazimov (2012) show that these areas also experienced greater reductions in mortgage credit.
case of interbank markets, since banks’ relative sophistication makes it all the more likely that they will adjust the institutional arrangements surrounding these markets in response to policy changes. As a result, progress in interbank regulation hinges in part on our developing models in which the form of the interbank contract emerges endogenously.

In this chapter, I take a first step toward addressing these concerns. More specifically, I build a simple model for the interbank market which endogenizes the form of the interbank contract as part of an optimal contracting problem. I then show that the model can account for interbank disruptions like those witnessed during the recent crisis. I also explore its predictions on the circumstances under which the economy becomes vulnerable to these disruptions, along with some implications for policy.

To be more specific about my model, its bare bones come from a canonical model of liquidity, namely Holmström and Tirole (1998), which I herein denote HT. There are three periods, $t \in \{0, 1, 2\}$. At $t = 0$, banks collect deposits and allocate funds between a riskless storage technology and a risky investment technology. At $t = 1$, they experience aggregate and idiosyncratic shocks, conditional on which they choose between maintaining or liquidating their investments. Maintained investments then mature at $t = 2$, though payouts are subject to limited pledgeability.

I depart from HT in two respects. The first has to do with my treatment of depositors. In HT and the literature following it, banks are able to negotiate state-contingent contracts with their depositors. It’s thus implicitly assumed that banks are able to revisit their depositors at $t = 1$ if shocks necessitate their raising additional funds. Though fruitful from a modeling perspective, this assumption is difficult to reconcile with one of the stronger stylized facts emerging from the empirical banking literature, namely that banks’ deposit liabilities are highly inertial. I therefore introduce a form of limited participation in the spirit of Allen and Gale (1994), which precludes depositors’ acting as a source of funds at $t = 1$. In particular, I assume that depositors lack the sophistication needed to sign and enforce state-contingent contracts, and instead suppose that these contracts are only available when banks negotiate with other banks. As a result, banks’ maintenance plans at $t = 1$ must be financed using some combination of storage, liquidations, and interbank lending.

My second departure from HT is that I introduce some information asymmetry into the economy. More specifically, I assume that the idiosyncratic shocks arriving at $t = 1$ convey private information about the productivity of banks’ investments. This assumption is natu-

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5 The sluggishness in banks’ deposit liabilities has been noted by Feldman and Schmidt (2001), Dinger and Craig (2014), and Hahm et al. (2012), among others. This is due in part to deposit insurance and geographic limits, but also due to switching costs, as documented by Sharpe (1997), Kim et al. (2003), and Hannah and Roberts (2011). See also Billett and Garfinkel (2004), Song and Thakor (2007), and Huang and Ratnovski (2011), among others.
rally motivated by experience during the recent crisis, when uncertainty on the location of sub-prime risk inside the financial system was a first-order issue.

Together, these ingredients give rise to three key results, two positive and one normative. On the positive front, I find that the model delivers episodes which qualitatively resemble the interbank disruptions witnessed during the recent crisis (proposition 1.1 and the discussion thereafter). More specifically, if the balance sheets that banks select at \( t = 0 \) exhibit sufficiently high deposits and sufficiently low storage, then the interaction between the information asymmetry and limited pledgeability gives rise to episodes during which banks with highly productive investments are unable to maintain those investments, despite the fact that some storage is left sitting idle on other banks’ balance sheets. This helps to rationalize banks’ accumulation of excess reserves and other liquid assets during the crisis, as documented by Heider et al. (2009), Ashcraft et al. (2011), Ennis and Wolman (2012), and Acharya and Merrouche (2012), among others.\(^6\) It also resonates with statements from policymakers suggesting that they did not trust interbank markets to allocate liquidity to its most urgent uses.\(^7\) I also show that the episodes in question also have the property that the set of interbank debtors is polluted by a range of very low-productivity types, which helps to rationalize evidence of heightened counterparty fears during the crisis, including a widening of the LIBOR-OIS spread,\(^8\) along with complementary evidence from Afonso et al. (2011) and Benos et al. (2014).\(^9\)\(^10\)

My next positive result concerns comparative statics at \( t = 0 \). In particular, since the episodes described in my previous paragraph only arise when banks select relatively fragile balance sheets at \( t = 0 \), it’s natural to ask about the circumstances under which banks would

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6 To be clear, Heider et al. (2009), Ashcraft et al. (2011), and Acharya and Merrouche (2012) focus on excess reserves, while Ennis and Wolman (2012) include broader measures of liquidity in their analysis.

7 For example, during hearings before the UK House of Commons Treasury Committee on September 20, 2007, the Chancellor of the Exchequer eschewed the use of broad open market operations as a potential solution for the Northern Rock crisis, namely in favour of a more targeted liquidity injection: “[Northern Rock] told the [Monetary Policy] Committee they have had to borrow about £13 or £14 billion from the Bank. To get that sort of money into the hands of one institution you would have to put many more billions of pounds into the market generally.” In a post-mortem, the Treasury Committee concluded that “[i]t is most unlikely that any such lending operation in September...could have been of a sufficient scale to ensure that Northern Rock could have received the liquidity it then required.” See House of Commons Treasury Committee (2008).

8 See Taylor and Williams (2009) for details on the widening of the LIBOR-OIS spread, along with an argument for its interpretation as a measure of interbank counterparty risk.

9 Broadly speaking, Afonso et al. (2011) find evidence that the amount and terms of lending in the federal funds market became more sensitive to borrowing banks’ risk profiles following the Lehman Brothers bankruptcy.

10 The Benos et al. (2014) data come from CHAPS, the UK’s large-value payment system. Following the Lehman Brothers bankruptcy, they show that banks were more likely to delay payments due to higher-risk counterparties and argue that this behaviour was motivated by a desire for insulation against intraday defaults.
be willing to make such choices ex ante. On this front, I first show that banks are more likely to select balance sheets of this sort when expected productivity at $t = 0$ is high, making the risk of interbank disruptions procyclical (proposition 1.2). This finding complements a growing literature highlighting the tendency for endogenous risk to accumulate inside the financial system during good times (e.g., Borio and Drehmann 2009; Schularick and Taylor 2012). I also show that interbank disruptions are more likely to occur when banks are poorly capitalized (proposition 1.3), which complements an extensive literature on the potentially stabilizing effects of bank equity (e.g., He and Krishnamurthy 2012; Bigio 2014).

On the normative front, I report a strong “no-go” result. In particular, I find that the interbank disruptions described above are constrained efficient and admit no role for policy in the economy (proposition 1.4). Broadly speaking, this constrained efficiency arises because the wider interbank contract space creates greater potential for coordination among banks, relative to previous literature. Though the model is too stylized to interpret this finding as a literal prescription that policymakers abstain from intervention in the interbank market, it does serve as a warning that interbank disruptions of the sort witnessed during the recent crisis do not necessarily count as evidence of a pathology. Policies which aim to eliminate them from the economy may thus constitute a bridge too far. Moreover, this “no-go” result provides a useful benchmark for the policy exercises in my next chapter, all of which I carry out in the context of a more ambitious model that’s been expanded to include a fire-sale externality.

In terms of related literature, the literature on interbank disruptions is too vast to do it much justice here. Some of the more closely related work in this area include Heider et al. (2009), who show how asymmetric information on the quality of banks’ investments can give rise to episodes during which banks substitute out of the market in favour of liquidity hoarding, potentially up to the point where the market shuts down completely. Similarly, Freixas and Holthausen (2005) show that banks operating in two distinct regions can only form a common interbank market so long that enough information flows between those regions. Moreover, this market will subsequently collapse if information asymmetry later spikes.

In related work, Bruche and Suarez (2010) consider a setting in which an interbank market is needed to distribute resources from banks with relatively large depositor bases to those with relatively small one and identify an interaction between deposit insurance and interbank counterparty risk that may impede this flow. They also conduct quantitative exercises indicating that the resulting allocative inefficiencies could be quite sizeable when counterparty risk reaches levels commensurate with those tested during the recent crisis. In a different vein, Acharya et al. (2012) argue that the few banks remaining in good health
after a large-scale crisis may enjoy enough market power to prevent the interbank market from efficiently allocating liquidity to their distressed peers, forcing the latter to rely instead on costly fire-sales.

Since all these papers make ad-hoc assumptions on the form of the interbank contract, my main contribution has to do with my emphasis on optimal contracts. To some extent, Bhattacharya and Gale (1986) and Bhattacharya and Fulghieri (1994) share this emphasis, since they take a mechanism-design approach to the interbank market. However, their models fail to deliver episodes of the sort described above.

My findings also connect with a growing literature on the composition of banks’ liabilities, a topic which has received substantial attention since the recent crisis, during which wholesale lending markets were affected much more severely than retail deposit markets. In particular, since my model includes both deposits and interbank loans, it provides a simple and relatively tractable framework for understanding how banks trade-off between these two funding sources. Moreover, my finding that state-contingent contracts allow enough interbank coordination to achieve constrained efficiency admits a natural connection with a sub-literature emphasizing the possibility that depositor runs can be eliminated from Diamond/Dybvig-style models when agents have access to more general contracts, including Green and Lin (2003), Cavalcanti and Monteiro (2011), and Andolfatto et al. (2014).

The remainder of the chapter is organized as follows. In section 1.2 I outline the model. In sections 1.3 and 1.4 I then solve the model and discuss its key properties. Section 1.5 concludes and provides some motivation for my next chapter.

1.2 Model

1.2.1 Environment

There are three periods, \( t \in \{0, 1, 2\} \), and a single homogeneous good. The economy contains a unit measure of islands, each populated by a single bank and a single household. Banks receive an endowment \( E^b \) at \( t = 0 \) and aim to maximize expected consumption at \( t = 2 \).

\[ \text{On the empirical front, Altunbas et al. (2011) and Bolgona (2015) provide evidence that banks exhibiting greater reliance on wholesale funding tended to fare worse during the recent crisis, while Lopez-Espinosa et al. (2012) show that these banks make greater contributions to overall systemic risk. In related work, Jeong (2009), Damar et al. (2013), and Dewally and Shao (2013) document some potentially dangerous linkages between wholesale funding and procyclical credit growth, while Hahn et al. (2013) show that an accumulation of wholesale liabilities inside the financial system may serve as a useful early warning sign for currency and credit crises. On the theoretical front, important contributions include Huang and Ratnovski (2011), a seminal model for the potential “dark side” of wholesale funding, and Gertler et al. (2015), which explores the macroeconomic implications of banks’ choice over their mix of wholesale and retail liabilities.} \]
Households receive an endowment $E^h$ at $t = 0$ and aim to maximize
\[ \mu(c_0) + \mathbb{E}(c_2), \]
where $c_t$ denotes consumption at time $t$, with $\mu'(\cdot) > 0 > \mu''(\cdot)$, and $\mu'(E^h) = 1$.

Technologies. Banks have access to two technologies: a storage technology, and an investment technology. The storage technology allows them to store goods at a one-to-one rate between periods. On the other hand, the investment technology is similar to that in HT. It’s outlined in figure [I.1].

The details on the investment technology are as follows. At $t = 0$, banks make some initial investment $I_0 \geq 0$. This investment will eventually mature at $t = 2$, when banks either succeed, meaning that they receive a positive payout, or fail, meaning that their payout is zero. However, some of the uncertainty regarding terminal payouts is resolved at $t = 1$. More specifically, at the beginning of $t = 1$, nature reveals two pieces of information: a bank-specific fundamental $\theta$, and an aggregate state $\omega$. The bank-specific fundamental $\theta$ is private and gives the probability with which a particular bank will succeed at $t = 2$. It’s distributed over $[0, 1]$ on an i.i.d. basis, namely with some cumulative $F$ admitting some positive density $f$. On the other hand, the aggregate state is public and can be interpreted as an aggregate productivity shock. More specifically, it determines the payout that successful banks receive at $t = 2$. It can take one of two values, $\omega \in \{B, G\}$, where $\omega = G$ denotes a good state in which payouts are relatively high, while $\omega = B$ denotes a bad state in which payouts are relatively low. The former occurs with probability $\alpha_G \in (1/2, 1)$, while the latter occurs with probability $\alpha_B$.

After observing the pair $(\theta, \omega)$, banks must choose how much of their initial investments they wish to keep running, $I_{\omega}(\theta) \in [0, I_0]$. As in HT, continuation entails some maintenance cost $\rho$ — more specifically, banks must reinvest $\rho$ goods for each unit of investment that they wish to keep on line. Any units of investment on which they opt to forego maintenance must be liquidated. Liquidation in state $\omega$ generates some small per-unit payout $\ell \in [0, 1)$.

Once banks have made their maintenance decisions, we move on to $t = 2$, when banks learn if they’ve succeeded or failed. These outcomes are independently distributed across banks, and the identity of the economy’s successful banks is public. Successful banks receive some payout $\chi_\omega$ per unit of investment maintained at $t = 1$, with $\chi_G > \chi_B$. However, as in HT, this payout cannot be pledged in its entirety to outsiders — rather, contracting

\[ \text{That is, } f(\theta) > 0 \ \forall \theta \in [0, 1]. \]  
It would also suffice if $f(\theta) > 0 \ \forall \theta \in (0, 1)$, so long that
\[ \lim_{\theta \searrow 0} \{F(\theta)/f(\theta)\} = 0. \]
Figure 1.1: Banks’ investment technology
frictions oblige successful banks to retain some fraction $\gamma \in (0,1)$. The particular frictions underlying this requirement are not important for the sequel, so I remain agnostic over the various microfoundations that the literature offers, including shirking, absconding, and so forth. See Holmström and Tirole (2010) for a few examples.

**Markets.** It’s natural to allow for two markets in this economy: a market for deposits, and a market for interbank claims.

The details on the market for deposits are as follows. At the beginning of $t = 0$, banks post contracts under which local households can make deposits, namely on a take-or-leave-it basis. However, households suffer from a form of limited participation in spirit of Allen and Gale (1994), and this constrains the form of the deposit contracts that they can accept. Broadly speaking, the idea is that households are relatively unsophisticated and thus unable to use state-contingent contracts. More specifically, I assume that contracts cannot be made contingent on information revealed at $t \in \{1,2\}$. As a result, deposit contracts can be summarized by a pair $(D,R)$. The interpretation is that households make an initial deposit $D$ at $t = 0$ and then receive interest and principal $DR$ at $t = 2$ on a non-contingent basis. Though stark, this approach precludes banks’ quickly adjusting their deposits in response to new information and is thus consistent with the inertia observed in banks’ real-world deposit liabilities, as discussed in my introduction.

On the other hand, the details on the interbank market are as follows. At the beginning of $t = 0$, banks have an opportunity to meet with one another so as to negotiate mutual insurance contracts in anticipation of the idiosyncratic shock $\theta$. In contrast with my approach to the market for deposits, I eschew any restriction on the form that these contracts might take and instead allow for state-contingent contracts; the idea is that banks’ relative sophistication makes it possible for them to use these contracts when dealing with other banks. The contracting problem which then arises is the subject of my next subsection.

### 1.2.2 Banks’ optimal contracting problem

With state-contingent contracts in the interbank market, we can think of an interbank contract as an agreement within a group of banks about how its members will behave going forward, much as in Bhattacharya and Gale (1986) and Bhattacharya and Fulghieri (1994). More precisely, we can think of an interbank contract as an object of the form

$$C := \left[ (D, R), I_0, \{S_\omega(\theta), I_\omega(\theta), T_{\omega,f}(\theta), \Delta T_\omega(\theta) \}_{(\theta,\omega) \in [0,1] \times \{B,G\}} \right],$$

where the interpretation is as follows:
• at $t = 0$, the contract specifies the offer $(D, R)$ that members should make to local households, along with the way that their funds should be distributed between initial investment $I_0$ and storage $E + D - I_0$;

• at $t = 1$, the contract specifies the portfolio of storage and investment $[S_\omega(\theta), I_\omega(\theta)]$ to which members should adjust, conditional on the pair $(\theta, \omega)$;

• finally, at $t = 2$, the contract specifies the transfer $T_\omega f(\theta)$ that failed banks should send back to their interbank creditors, along with the additional transfer $\Delta T_\omega(\theta)$ that should be extracted from successful banks.

See figure 1.2 for a visual summary.

Now, given that banks are homogeneous at the time that negotiations take place, these negotiations should settle on a contract which maximizes the average bank’s expected profits — that is, banks choose $C$ so as to maximize

$$\sum_{\omega \in \{B, G\}} \alpha_\omega \int_0^1 \left[ S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta) \right] dF(\theta) - RD. $$

Of course, several constraints attend this optimization. The first is that contracts must induce banks to reveal their draws on the fundamental $\theta$ — that is,

$$S_\omega(\theta) - T_\omega f(\theta) - RD + \theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)]$$

$$\geq S_\omega(\theta') - T_\omega f(\theta') - RD + \theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')] , $$

$$\forall (\theta, \theta', \omega) \in [0, 1]^2 \times \{B, G\} , \quad (TT)$$

where the label $\text{(TT)}$ stands for “truth telling”. The second constraint has to do with limited pledgability:

$$\Delta T_\omega(\theta) \leq (1 - \gamma) \chi_\omega I_\omega(\theta) , \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\} . \quad (LP)$$

The next constraint is the individual rationality constraint for depositors:

$$RD = \mu(E^h) - \mu(E^h - D) =: \Delta \mu(D) . \quad (IR)$$

The remaining constraints then have to do with physical feasibility. For example:

$$(E^h + D - I_0) + \ell I_0 = \int_0^1 \left[ S_\omega(\theta) + (\rho + \ell) I_\omega(\theta) \right] dF(\theta) , \quad \forall \omega \in \{B, G\} \quad \text{(F1a)}$$
\[0 = \int_0^1 \left[T_{\omega f}(\theta) + \theta \Delta T_{\omega}(\theta)\right] dF(\theta), \ \forall \omega \in \{B, G\}\] (F2a)

Here (F1a) says that the banks entering into the contract \(C\) must carry enough liquidity into \(t = 1\) to cover the adjustments to which they’ve committed, while (F2a) says that the transfers that they make among themselves at \(t = 2\) must net to zero. Finally:

\[(D, I_0) \in [0, \bar{E}^h] \times [0, \bar{E}^b + D]\] (F0)

\[S_\omega(\theta) \geq 0, \ \forall (\theta, \omega) \in [0, 1] \times \{B, G\}\] (F1b)

\[S_\omega(\theta) \geq T_{\omega f}(\theta) + RD, \ \forall (\theta, \omega) \in [0, 1] \times \{B, G\}\] (F2b)

\[S_\omega(\theta) + \chi_\omega I_\omega(\theta) \geq T_{\omega f}(\theta) + \Delta T_{\omega}(\theta) + RD, \ \forall (\theta, \omega) \in [0, 1] \times \{B, G\}\] (F2c)

\[I_\omega(\theta) \in [0, I_0], \ \forall (\theta, \omega) \in [0, 1] \times \{B, G\}\] (F1c)

\[I_\omega(\theta) \implies \Delta T_{\omega}(\theta) = 0, \ \forall (\theta, \omega) \in [0, 1] \times \{B, G\}\] (F2d)

Here (F0) through (F2c) are non-negativity constraints, while (F1c) reminds us that banks’ investments can’t be scaled up at \(t = 1\). (F2d) reminds us that we can’t condition on success or failure in the case of banks whose investments are fully liquidated at \(t = 1\).

### 1.2.3 Parametric assumptions

I close this section with my parametric assumptions. The first three read as follows:

**Assumption 1.1.** \(\sum_{\omega \in \{B, G\}} \alpha_\omega \mathbb{E} \left[\max \{\ell, \theta \chi_\omega - \rho\}\right] > 1\).

**Assumption 1.2.** \(\chi_B > \rho + \ell\).

Here assumption \([1.1]\) ensures that it’s profitable for banks to engage in some investment at \(t = 0\), while assumption \([1.2]\) ensures that continuation is profitable for some types in both states.

Next, it will be useful to impose the following restriction on the form of the distribution from which banks draw their types:
Figure 1.2: Interbank contract
Assumption 1.3. The cumulative function $F$ is strictly log-concave.\textsuperscript{13}

This assumption is relatively common in models with mechanism-design components and holds for a wide range of distributions over $[0,1]$, including the uniform, truncated (log-)

 normal, and truncated exponential cases.\textsuperscript{14,15} It will also be useful to impose a lower bound on the productivity differential across states:

Assumption 1.4. The bad state is “sufficiently bad”, namely in the sense that the payout $\chi_B$ satisfies an upper bound given in the appendix. Conversely, the good state is “sufficiently good”, namely in the sense that the payout $\chi_G$ satisfies a lower bound given in the appendix.

Finally, I follow HT in assuming that households are deep-pocketed:

Assumption 1.5. The household endowment $E^h$ is relatively large in comparison with the endowment $E^b$ received by banks — specifically, $E^b + E^h - \Delta \mu(E^h) < 0$.

1.3 Solution at $t = 1$

I'll now begin solving the model and characterizing its key properties. Due to the model’s relative simplicity, this amounts to solving the optimal contracting problem outlined in subsection 1.2.2. On this front, the game plan is as follows. The present section focuses on banks’ behaviour at $t = 1$, taking as given the initial balance sheets selected at $t = 0$, as summarized by the initial investment $I_0$ and the deposit contract $(D,R)$. More specifically, subsection 1.3.1 shows that banks’ optimal contracting problem admits an intuitive reformulation, which subsection 1.3.2 then exploits to solve for banks’ behaviour in the bad state. Subsection 1.3.2 repeats for the good state. In section 1.3, I’ll then step back to $t = 0$ and focus on endogenizing banks’ initial balance-sheet choices.

\textsuperscript{13} That is, $\frac{d^2 \log F(\theta)}{d \theta^2} < 0$, $\forall \theta \in [0,1]$.

\textsuperscript{14} See table one in \textit{Mohtashami Borzadarn and Mohtashami Borzadaran (2011)} for a relatively comprehensive list of examples.

\textsuperscript{15} At the risk of being a bit pedantic, I note that there is a distinction between log-concavity of a given \textit{cumulative function} and log-concavity of the underlying \textit{distribution}, since the latter refers to a situation in which the \textit{density function} is log-concave. The latter assumption is a bit more common in mechanism design, but also constitutes a stronger condition, since any log-concave density function is known to admit a log-concave cumulative function, though the converse is untrue. (See theorem one in \textit{Bagnoli and Bergstrom (2005)}. An important example of the distinction in question would be the (truncated) log-normal distribution, which is not log-concave but still admits a log-concave cumulative function.
1.3.1 Reformulation of banks’ optimal contracting problem

Lemma 1.3.1. Fix some initial balance sheet \((D, I_0, R)\), and define a subcontract

\[ C_\omega := \{S_\omega(\theta), I_\omega(\theta), T_{\omega f}(\theta), \Delta T_{\omega}(\theta)\}_{\theta \in [0,1]}, \]

which collects all those terms in the interbank contract obtaining in state \(\omega\). Banks’ optimal choice on this subcontract can be summarized by two numbers: a threshold fundamental \(\theta_\omega \in [0,1]\), and an investment scale \(I_\omega \in [0, I_0]\). The interpretation is that (almost) all types in \([\theta_\omega, 1]\) maintain exactly \(I_\omega\) units of investments, whereas all types in \([0, \theta_\omega)\) liquidate completely — that is,

\[ I_\omega(\theta) = \begin{cases} I_\omega & \text{for (almost) all } \theta \in [\theta_\omega, 1] \\ 0 & \text{for all } \theta \in [0, \theta_\omega) \end{cases}. \]

More specifically, banks choose \((\theta_\omega, I_\omega)\) so as to maximize the surplus generated by their maintained investments,

\[ I_\omega \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell) dF(\theta) =: I_\omega \Pi_\omega(\theta_\omega), \]

subject to two constraints. The first is a physical constraint stipulating that banks must carry enough liquidity into \(t = 1\) to cover the maintenance plans to which they’ve committed:

\[ (E^b + D - I_0) + \ell I_0 \geq \int_{\theta_\omega}^{1} (\rho + \ell) I_\omega dF(\theta) =: I_\omega \Psi_\omega(\theta_\omega). \quad \text{(PC}_\omega \text{)} \]

The second is a financial constraint on which I’ll elaborate in a moment:

\[ (E^b + D - I_0) + \ell I_0 \geq RD - I_\omega \left[ \Pi_\omega(\theta_\omega) - \theta_\omega \gamma \chi_\omega F(\theta_\omega) - \int_{\theta_\omega}^{1} \theta \gamma \chi_\omega dF(\theta) \right]. \quad \text{(FC}_\omega \text{)} \]

Proof. All proofs are in the appendix. ■

Here the financial constraint \((\text{FC}_\omega)\) captures the combined effects of the economy’s information and pledgability frictions. Its interpretation hinges on our recognizing the term

\[ I_\omega \theta_\omega \gamma \chi_\omega F(\theta_\omega) \]
as the total present value which must be promised to the types in \([0, \theta_\omega]\), whom we can think of as interbank creditors; were this promise any smaller, creditors would be tempted to mimic the types in \([\theta_\omega, 1]\), whom we can think of as interbank debtors. Similarly, the term

\[
I_\omega \int_{\theta_\omega}^{1} \theta \gamma_{\omega} dF(\theta)
\]

gives the total present value which must be retained by interbank debtors due to limited pledgability. Therefore, the term

\[
I_\omega \left[ \Pi_{\omega}(\theta_\omega) - \theta_\omega \gamma_{\omega} F(\theta_\omega) - \int_{\theta_\omega}^{1} \theta \gamma_{\omega} dF(\theta) \right] =: I_\omega \Delta_{\omega}(\theta_\omega)
\]

gives the portion of total surplus that can be pledged to depositors, which I’ll herein refer to as pledgable surplus The financial constraint can then be interpreted as a statement about the way that liquidity can help to “grease the wheels” at \(t = 1\). In particular, a buffer of liquidity is needed to fill any gap between the promise to depositors, \(RD\), and pledgable surplus, \(I_\omega \Delta_{\omega}(\theta_\omega)\) — that is,

\[
(FC_{\omega}) \iff (E^{b} + D - I_{0}) + \ell I_{0} \geq RD - I_\omega \Delta_{\omega}(\theta_\omega).
\]

In light of these findings, we now see that banks’ optimal contracting problem can be reformulated as a choice over

\[
[(D, R), I_{0}, \{\theta_\omega, I_\omega\}_{\omega \in \{B, G\}}],
\]

where the goal is to maximize

\[
\sum_{\omega \in \{B, G\}} \alpha_\omega \left[ (E^{b} + D - I_{0}) + \ell I_{0} + I_\omega \Pi_{\omega}(\theta_\omega) - RD \right],
\]

subject only to the physical and financial constraints described above, along with (IR), (FC0), and the requirement that \((\theta_\omega, I_\omega) \in [0, 1] \times [0, I_0] \forall \omega \in \{B, G\}\). Denote this program \((P)\).

Now, it’s natural to make certain conjectures on the form of the solution that this program takes. In particular, it’s natural to conjecture that the physical constraint is lax in the bad state, though the financial constraint associated with this state might bind. The intuition is that the bad state is one in which relatively few types draw fundamentals which justify maintenance, so the physical constraint should be relatively easy to satisfy. At the same time, all those types whose fundamentals warrant liquidation must be bribed to reveal themselves, so satisfying the financial constraint may be difficult. Similar reasoning leads to
a conjecture that the financial constraint should be lax in the good state, though the physical constraint associated with this state might bind. I also conjecture that the non-negativity constraint (F0) should be lax. Let ($\mathbb{P}$-rex) denote the relaxed program implied by these three conjectures. In the sequel, I’ll focus on solving this relaxed program before verifying my conjectures in section 1.4.

1.3.2 Details on the bad state

Under program ($\mathbb{P}$-rex), the constraint that banks have to worry about in the bad state is the financial constraint. To get a sense for how this constraint might distort their behaviour, I’ve used figure 1.3 to plot the per-unit surplus function $\Pi_B(\theta_B)$, along with the per-unit pledgable surplus function $\Delta_B(\theta_B)$. The former is in dashed red, while the latter is in solid blue.

Figure 1.3 has several key properties, all of which I establish formally in the appendix. The first is that the per-unit surplus function naturally peaks around the type for whom maintenance is NPV-neutral, $\theta_B^\Pi := (\rho + \ell)/\chi_B$. This means that banks would prefer a subcontract of the form $(\theta_B, I_B) = (\theta_B^\Pi, I_0)$, all else equal, namely because this subcontract keeps all NPV-positive types operating at full scale.

On the other hand, the per-unit pledgable surplus function peaks around a lower type, $\theta_B^\Delta < \theta_B^\Pi$. The intuition for this lower type is that a reduction in the threshold $\theta_B$ lowers the bribe that must be paid to inframarginal creditors so as to discourage their mimicking debtors. As a result, allowing some NPV-negative types into the set of interbank debtors may nonetheless increase pledgable surplus so long that their expected losses are offset by the corresponding savings on the aforementioned bribe — indeed, $\theta_B^\Delta$ has the property that these two effects offset each other exactly:

$$\left(\rho + \ell - \chi_B \theta_B^\Delta\right) f(\theta_B^\Delta) = \gamma \chi_B F(\theta_B^\Delta)$$

where $f(\theta_B^\Delta)$ represents the expected losses for type $\theta_B^\Delta$ and $F(\theta_B^\Delta)$ represents the savings on the bribe to inframarginal creditors.

It will also be useful to note that the per-unit pledgable surplus function is strictly negative over $[0, 1]$, namely due to the scarcity of pledgable income in the bad state (assumption 1.4). On the other hand, per-unit surplus either exhibits single-crossing from below over the interval $[0, 1]$, namely at some type $\theta_B^\Pi < \theta_B^\Pi$, or otherwise is positive over all of this interval, in which case I adopt a convention that $\theta_B^\Pi = 0$.

With these points in mind, we have a few cases to consider, depending on banks’ initial balance sheets. Suppose first that initial balance sheets are relatively liquid, namely in the
sense

\[(E^b + D - I_0) + \ell I_0 \geq RD - I_0 \Delta_B(\theta^\Pi_B),\]

(1.1)

In this case, banks will be able to implement the unconstrained optimum \((\theta_B, I_B) = (\theta^\Pi_B, I_0)\), as described above. Next, suppose that initial initial balance sheets instead exhibit relatively low liquidity and high leverage, namely in the sense that

\[(E^b + D - I_0) + \ell I_0 < RD.\]

(1.2)

Since the per-unit pledgable surplus function is always negative, this situation has the property that there’s no choice on the subcontract \((\theta_B, I_B)\) which balances the financial constraint. As a result, banks would be forced to default on their deposits. Since banks are obliged to keep their deposits non-contingent, we can conclude that initial balance sheets in this range cannot be selected at \(t = 0\). In effect, the problem is that debt-overhang viz-à-viz depositors prohibits banks’ maintaining any investments at \(t = 1\); at the same time, the proceeds from liquidating these investments are too low to cover the promise to depositors.

Of course, the bounds in (1.1) and (1.2) are not mutually exclusive. The most interesting case arises when initial balance sheets exhibit moderate liquidity and moderate leverage, namely in the sense that

\[(E^b + D - I_0) + \ell I_0 \in [RD, RD - I_0 \Delta_B(\theta^\Pi_B)].\]

For initial balance sheets in this range, banks can’t implement their unconstrained optimum and instead adjust the subcontract \((\theta_B, I_B)\) so as to re-balance the financial constraint. Now, from figure 1.3 we see that there are two options for how banks might go about re-balancing the financial constraint. The first is to reduce the threshold \(\theta_B\) to some point in \([\theta^\Delta_B, \theta^\Pi_B]\). The alternative is to reduce the investment scale \(I_B\). For brevity, I’ll respectively refer to these two options as the extensive and intensive margins. Both margins have their drawbacks: reliance on the intensive margin distorts the distribution of liquidity across the set of interbank debtors — in particular, types near the top of the interval \([\theta_B, 1]\) receive relatively too little liquidity, whereas types near the bottom receive relatively too much; on the other hand, reliance on the extensive margin introduces relatively unprofitable types into the set of debtors.

When banks trade off between these margins, it turns out that they obey a strict pecking order. More specifically, in the appendix, I show that the interval \((\max \{\theta^\Pi_B, \theta^\Delta_B\}, \theta^\Pi_B)\) admits a critical type \(\theta^\Xi_B\), with the special property that banks prefer to rely on the extensive margin until they reach a point at which further reliance would require that they let \(\theta^\Xi_B\) into the set.
Figure 1.3: Per-unit surplus functions
of interbank debtors. At this point, banks prefer to revert to the intensive margin rather
than allow any more inferior types into the set. The intuition for this pecking order should be
relatively clear from figure 1.3: types near $\theta^{\Pi}_B$ are very close to breaking even in NPV terms,
so the costs associated with the extensive margin are second-order in this neighbourhood;
however, as we lean on the extensive margin, each successive type that we let into the set of
debtors generates less pledgable surplus at the cost of greater expected losses.

To get a bit more precise about this pecking order, I note that the rate of transformation
along the financial constraint is given by

$$
\frac{dI_B}{d\theta_B} \Delta_B(\theta_B) + I_B' \Delta_B'(\theta_B) = 0 \iff \frac{dI_B}{d\theta_B} = \frac{-1}{I_B'} \frac{\Pi_B'(\theta_B) \Delta_B'(\theta_B)}{\Delta_B(\theta_B)},
$$

so banks prefer the extensive margin whenever

$$
(\frac{dI_B}{d\theta_B} \Pi_B(\theta_B) + I_B' \Pi'_B(\theta_B) = I_B \left[ \Pi'_B(\theta_B) - \frac{\Pi_B(\theta_B) \Delta_B'(\theta_B)}{\Delta_B(\theta_B)} \right] \leq 0.
$$

Now, it should be clear that the starred term is strictly negative at $\theta_B = \theta^{\Pi}_B$ but strictly
positive at $\theta_B = \max \{ \theta^{\Pi}_B, \theta^{\Delta}_B \}$. A sufficient condition for the pecking order that I’ve just
described would then be that the starred term also exhibit single-crossing, which I verify in
the appendix.

To summarize, the situation is as follows:

**Proposition 1.1** (banks’ behaviour in the bad state). Under program $(\mathbb{P}$-rex), the subcon-
tract $(\theta_B, I_B)$ exhibits the following dependence on banks’ initial balance sheets:

- if initial balance sheets exhibit low leverage and high liquidity, namely in the sense that

  $$(E^b + D - I_0) + \ell I_0 \geq RD - I_0 \Delta_B(\theta^{\Pi}_B),$$

then $(\theta_B, I_B) = (\theta^{\Pi}_B, I_0)$. Since all NPV-positive types thus receive enough liquidity to
keep operating at full scale, I’ll refer to this situation as one in which banks experience a
liquidity surplus;

- if initial balance sheets exhibit moderate leverage and moderate liquidity, namely in the
sense that

  $$
  (E^b + D - I_0) + \ell I_0 \in \left[ RD - I_0 \Delta_B(\theta^{\Delta}_B), RD - I_0 \Delta_B(\theta^{\Pi}_B) \right],
  $$

then $(\theta_B, I_B) = (\theta^{\Delta}_B, I_0)$. Since all NPV-negative types thus lose enough liquidity to
not keep operating at full scale, I’ll refer to this situation as one in which banks experience a
liquidity deficit;
then banks rely only on the extensive margin — that is, \( I_B = I_0 \), with \( \theta_B \) set s.t. the financial constraint holds with equality. I’ll thus refer to this situation as one in which banks experience an extensive distortion;

- if initial balance sheets exhibit high leverage and low liquidity, namely in the sense that

\[
(E^b + D - I_0) + \ell I_0 \in \left[ RD, RD - I_0 \Delta_B(\theta_B^\infty) \right),
\]

then banks rely on both margins — specifically, \( \theta_B = \theta_B^\infty \), with \( I_B \) set s.t. the financial constraint holds with equality. I’ll thus refer to this situation as one in which banks experience a dual distortion;

- finally, if initial balance sheets exhibit very high leverage and very low liquidity, namely in the sense that

\[
(E^b + D - I_0) + \ell I_0 < RD,
\]

then the subcontract \( (\theta_B, I_B) \) cannot be chosen to satisfy the financial constraint.

See figure 1.4 for an illustration. When constructing this figure, I’ve used the individual rationality constraint for depositors, (IR), to eliminate the interest rate \( R \) as choice variable, so the pair \( (D, I_0) \) will herein suffice as a summary of banks’ initial balance-sheet choices.

Proposition 1.1 constitutes one of my main results, namely because the dual distortion regime exhibits three key properties suggestive of an interbank disruption like those witnessed during the recent crisis:

- the first such property is that an active intensive margin causes liquidity to be misallocated across the set of interbank debtors. In particular, as mentioned earlier, types near the top of the interval \([\theta_B, 1]\) receive relatively too little liquidity, while types near the bottom receive relatively too much;

- the second property is that liquidity is also being misallocated between interbank debtors and interbank creditors. More specifically, with the intensive margin active but the physical constraint lax, the dual distortion regime has the property that banks with strong fundamentals are unable to keep operating at full scale, despite the fact that some liquidity is still sitting idle inside the banking system. As mentioned in my introduction, this helps to rationalize banks’ accumulation of excess reserves and other liquid assets during the crisis, as documented by Heider et al. (2009), Ashcraft et al. (2011), Ennis and Wolman (2012), Acharya and Merrouche (2012), and others;

- finally, because the extensive margin is active, the dual distortion regime also has the property that the set of interbank debtors is polluted by a subset of negative-NPV types.
Figure 1.4: Banks’ behaviour in the bad state as a function of their initial balance-sheet choices.
As mentioned in my introduction, this rationalizes reports of heightened counterparty fears during the crisis, as documented by \cite{TayWil2009, Afonso2011, Benos2014}, and others.

Due to their importance, I’ll briefly provide some intuition for how the model’s ingredients give rise to these key properties. In particular, I’ll argue that a simpler model which abstracts from imperfect information would fail to deliver all three of these properties. Of course, with perfect information, there’s no longer any need to bribe interbank creditors to reveal their types, so the per-unit pledgable surplus function would instead be given by

\[
\Pi_B(\theta_B) - \int_{\theta_B}^{1} \theta \gamma \chi_B dF(\theta) =: \Delta^\dagger_B(\theta_B).
\]

I’ve plotted this function in figure 1.5, namely in dash-dotted purple. What’s crucial is that per-unit pledgable surplus now peaks around the very high type \( \min\{1, (\rho + \ell)/(1 - \gamma)\chi_B\} > \theta_B^\Pi \). As a result, if banks found themselves in a situation where the unconstrained optimum \((\theta_B, I_B) = (\theta_B^\Pi, I_0)\) violates the financial constraint, then a reduction in \(\theta_B\) would prove counterproductive, while an increase would help in generating pledgable surplus. The model would thus deliver a counterfactual prediction that the bad state of the world should be one in which the set of interbank debtors is saturated with very good types.

Now, before closing this subsection, it will be useful to record banks’ value functions under each of the regimes above, namely as a first step toward eventually pinning down their initial balance-sheet choices:

- under the liquidity surplus regime, banks’ payout is given by

\[
v^{LS}_B(D, I_0) := (E^b + D - I_0) + \ell I_0 + I_0 \Pi_B(\theta_B^\Pi) - \Delta\mu(D);
\]

- on the other hand, their payout under the extensive distortion regime is given by

\[
v^{ED}_B(D, I_0) := (E^b + D - I_0) + \ell I_0 + I_0 \Pi_B[\theta_B^{ED}(D, I_0)] - \Delta\mu(D),
\]

where the threshold \(\theta_B^{ED}(D, I_0)\) is chosen to satisfy the financial constraint — i.e.,

\[
(E^b + D - I_0) + \ell I_0 = \Delta\mu(D) - I_0 \Delta_B[\theta_B^{ED}(D, I_0)];
\]

- finally, their payout under the dual distortion regime is given by

\[
v^{DD}_B(D, I_0) := (E^b + D - I_0) + \ell I_0 + I^{DD}_B(D, I_0) \Pi_B(\theta^\Xi_B) - \Delta\mu(D),
\]
Figure 1.5: Consequences of perfect information

\[
\begin{align*}
\Pi_B(\theta_B) & = \text{per - unit surplus,} \\
\Delta_B^i(\theta_B) & = \text{per - unit pledgeable surplus under perfect info,} \\
\Delta_B(\theta_B) & = \text{per - unit pledgeable surplus,}
\end{align*}
\]

\[
\min \left\{ 1, \frac{\rho + \ell}{(1 - \gamma)X_B} \right\}
\]
where the scale $I_B^{DD}(D, I_0)$ is chosen to satisfy the financial constraint — i.e.,

\[(E^b + D - I_0) + \ell I_0 = \Delta \mu(D) - I_B^{DD}(D, I_0) \Delta_B \left( \theta_B^2 \right). \]  \hspace{1cm} (1.5)

### 1.3.3 Details on the good state

I now turn my attention to the good state, in which $(\mathbb{P}$-rex) has the property that the constraint banks have to worry about is now the physical constraint, rather than the financial constraint. As a result, their behavior in this state is relatively mechanical. Suppose for example that initial balance sheets exhibit high liquidity, namely in the sense that

\[(E^b + D - I_0) + \ell I_0 \geq I_0 \Psi_G \left( \theta_G^II \right). \]

In this case, banks are able to implement the unconstrained optimum described in my previous subsection, $(\theta_G, I_G) = (\theta_G^II, I_0)$. I’ll thus refer to this situation as one in which banks experience another \textit{liquidity surplus}. If instead

\[(E^b + D - I_0) + \ell I_0 \in \left[ 0, I_0 \Psi_G \left( \theta_G^II \right) \right), \]

then banks are forced to ration liquidity. With the financial constraint lax, they’re able to do so in a way which sends each unit of liquidity to its best possible use, namely by setting $I_G = I_0$ while increasing $\theta_G$ until the physical constraint binds. I’ll thus refer to this situation as one in which banks experience \textit{liquidity rationing}. Finally, if both of the conditions above fail, then it should be clear that the subcontract $(\theta_G, I_G)$ cannot be chosen to balance the physical constraint. As a result, initial balance sheets in this last range cannot be selected at $t = 0$.

Now, much as in my previous subsection, it will be useful to record banks’ value functions under each of the regimes above before finally shifting our attention back to $t = 0$:  

- under the liquidity surplus regime, banks’ payout is given by

\[v_G^{LS}(D, I_0) := (E^b + D - I_0) + \ell I_0 + I_0 \Pi_G \left( \theta_G^II \right) - \Delta \mu(D); \]

- on the other hand, their payout under the liquidity rationing regime is given by

\[v_G^{LR}(D, I_0) := (E^b + D - I_0) + \ell I_0 + I_0 \Pi_G \left[ \theta_G^{LR}(D, I_0) \right] - \Delta \mu(D), \]
where the threshold \(\theta^{LR}_{G}(D, I_0)\) is chosen to satisfy the physical constraint — i.e.,

\[
(E^b + D - I_0) + \ell I_0 = I_0 \Psi_{G} [\theta^{LR}_{G}(D, I_0)].
\] (1.6)

1.4 Solution at \(t = 0\)

We’re now ready to solve for banks’ initial balance-sheet choices under program (\(\mathbb{P}\)-rex). Based on the analysis in my last section, we know that there’s a total of six cases for us to consider, since there are three possible regimes associated with the bad state, \(r_B \in \{LS, ED, DD\}\), and two possible regimes associated with the good state, \(r_G \in \{LS, LR\}\). See figure 1.6 for a visual summary, and the appendix for details on this figure’s construction.

Fortunately, some cases can be ruled out almost immediately. Suppose for example that \((r_G, r_B) = (LS, DD)\). In this case, the marginal deposit generates return

\[
\alpha_G(v^{LS}_G)_{D}(D, I_0) + \alpha_B(v^{DD}_B)_{D}(D, I_0) = [1 - \Delta \mu'(D)] - \alpha_B[1 - \Delta \mu'(D)] \frac{\Pi_{B}(\theta^{\Xi}_{B})}{\Delta_{B}(\theta^{\Xi}_{B})} < 0.
\]

The interpretation is relatively straightforward once we recognize that increasing \(D\) while holding \(I_0\) constant amounts to taking the marginal deposit, raised at cost \(\mu > 1\), and placing it in storage, where it only generates a unit return. In the good state, this deposit ends up sitting idle, since all NPV-positive projects are already operating at full scale. On the other hand, it contributes to a tighter financial constraint in the bad state. As a result, it’s strictly optimal for banks to select out of this case in favour of one involving lower leverage. A similar argument can be used to rule out the case where \((r_G, r_B) = (LS, ED)\), under which

\[\alpha_G(v^{LS}_G)_{D}(D, I_0) + \alpha_B(v^{ED}_B)_{D}(D, I_0)\]

\[\text{To be clear, the derivation is as follows. (1.5) } \implies (I^{DD}_B)_{D}(D, I_0) = (-1)[1 - \Delta \mu'(D)]/\Delta_{B}(\theta^{\Xi}_{B}), \text{ so}
\]

\[
\alpha_G(v^{LS}_G)_{D}(D, I_0) + \alpha_B(v^{DD}_B)_{D}(D, I_0) = \alpha_G[1 - \Delta \mu'(D)] + \alpha_B[1 - \Delta \mu'(D)] \frac{\Pi_{B}(\theta^{\Xi}_{B})}{\Delta_{B}(\theta^{\Xi}_{B})}
\]

\[
= [1 - \Delta \mu'(D)] - \alpha_B[1 - \Delta \mu'(D)] \frac{\Pi_{B}(\theta^{\Xi}_{B})}{\Delta_{B}(\theta^{\Xi}_{B})}.
\]

All of this section’s marginal returns can be derived in similar ways.
\[= \left[1 - \Delta \mu'(D)\right] - \alpha_B\left[1 - \Delta \mu'(D)\right] \frac{\Pi_B[\theta_B^{ED}(D, I_0)]}{\Delta_B[\theta_B^{ED}(D, I_0)]} < 0.\]

The case where \((r_G, r_B) = (LS, LS)\) can also be ruled out, though the argument is a bit different. Under this case, we have

\[
\alpha_G(v_G^{LS})_I(D, I_0) + \alpha_B(v_B^{LS})_I(D, I_0) = \sum_{\omega \in \{B, G\}} \alpha_\omega [\ell_\omega + \Pi(\theta_\omega^H)] - 1 > 0,
\]

where the inequality follows from the fact that investment is relatively profitable from an ex-ante perspective (assumption 1.1). In effect, with the financial constraint lax in both states, all of the surplus generated by the marginal unit of investment accrues directly to banks, rather than depositors, leaving banks with a strict incentive to select out this case in favour of one involving higher investment.

So, we can now restrict attention to initial balance sheets under which the good state is associated with liquidity rationing — that is,

\[
(E^b + D - I_0) + \ell I_0 < I_0 \Psi_G(\theta_G^H),
\]

but

\[
(E^b + D - I_0) + \ell I_0 \geq \Delta \mu(D). \tag{1.7}
\]

For balance sheets in this range, marginal returns are given by

\[
\alpha_G(v_G^{LR})_D(D, I_0) + \alpha_B(v_B^{rB})_D(D, I_0)
\]

\[= \alpha_G \left[1 - \Delta \mu'(D) + \frac{\Pi_G[\theta_G^{LR}(D, I_0)]}{\Psi_G(\theta_G^{LR}(D, I_0))}\right].\]
Figure 1.6: Banks’ behaviour in both states as a function of their initial balance-sheet choices
\[
\begin{align*}
\alpha_B \begin{cases}
1 - \Delta \mu'(D) & \text{if } r_B = LS \\
[1 - \Delta \mu'(D)] \left[ 1 - \frac{\Pi_B[\theta_B^{ED}(D, I_0)]}{\Delta_B[\theta_B^{ED}(D, I_0)]} \right] & \text{if } r_B = ED \\
[1 - \Delta \mu'(D)] \left[ 1 - \frac{\Pi_B[\theta_B^\Xi]}{\Delta_B[\theta_B^\Xi]} \right] & \text{if } r_B = DD,
\end{cases}
\end{align*}
\]

and

\[
\alpha_G(v_{LR}^G)I(D, I_0) + \alpha_B(v_{RB}^B)I(D, I_0) = \alpha_G \left[ \ell + \Pi_G[\theta_G^{LR}(D, I_0)] - 1 \right] - \left[ 1 - \ell + \Psi_G[\theta_G^{LR}(D, I_0)] \right] \left[ \frac{\Pi'_G[\theta_G^{LR}(D, I_0)]}{\Psi'_G[\theta_G^{LR}(D, I_0)]} \right]
\]

\[
\begin{align*}
\alpha_B \begin{cases}
\ell + \Pi_B[\theta_B^{II}] - 1 & \text{if } r_B = LS \\
\ell + \Pi_B[\theta_B^{ED}(D, I_0)] - 1 & \text{if } r_B = ED \\
-1(1 - \ell) \left[ 1 - \frac{\Pi_B[\theta_B^\Xi]}{\Delta_B[\theta_B^\Xi]} \right] & \text{if } r_B = DD.
\end{cases}
\end{align*}
\]

Now, the envelope theorem ensures that these expressions are continuous in \((D, I_0)\), even around the boundaries separating the various regimes associated with the bad state of the world. As a result, solutions for program \((P\text{-}rex)\) must take one of three distinct forms. The first is an interior solution under which both of the expressions above reach zero. The second in principle would be a corner solution under which \(D = 0\), but this possibility is precluded by the fact that depositors’ outside options satisfy \(\mu'(E^h) = 1\), meaning that they lack a good use to which they can put the last unit of their endowments.
The third and final possibility is a corner solution under which the “no-default” constraint on line (1.7) binds — that is, banks select initial balance sheets so levered and illiquid that they’re barely able to pay their depositors in the bad state. Under a solution of this form, banks anticipate that they won’t collect any profits in the bad state and instead focus exclusively on the good state, namely by choosing \((D, I_0)\) to satisfy the first-order condition

\[
(v_G^{LR})_I(D, I_0) + \left[ \frac{1 - \ell}{1 - \Delta \mu'(D)} \right] (v_G^{LR})_D(D, I_0) = 0,
\]

where the starred term gives the rate of transformation along the “no-default” constraint. Since this solution has the property that no interbank transfers occur, I’ll refer to it as a situation in which the interbank market collapses.

To summarize:

**Lemma 1.4.1.** Solutions for \((\mathbb{P}-\text{rex})\) must have the property that banks experience liquidity rationing in the good state. As for their behaviour in the bad state, there are two possible cases: one under which the “no-default” constraint is lax, with

\[
\alpha_G(v_G^{LR})_x(D, I_0) + \alpha_B(v_B^{RP})_x(D, I_0) = 0, \quad \forall x \in \{D, I_0\},
\]

and another under which the “no-default” constraint binds, with (1.8) holding.

Moreover:

**Lemma 1.4.2.** A solution of the aforementioned form exists, is unique, and generalizes to the more constrained program \((\mathbb{P})\).

In light of these two lemmata, it’s now natural to ask about the conditions under which the model’s solution takes a corner form, along with the regime that this solution associates with the bad state of the world in non-corner cases. For now, I focus on how the answers to these questions vary over the business cycle, namely by taking comparative statics with respect to two parameters governing the productivity of banks’ investment technology: the payout that successful projects generate in the good state, \(\chi_G\), and the probability on this state, \(\alpha_G\). Now, intuitively speaking, we should expect that banks are only comfortable exposing themselves to some risk of distortion in the bad state if expected productivity at \(t = 0\) is relatively high. In the appendix, I confirm this intuition:

**Proposition 1.2** (procyclical risk in the interbank market.). Fix all parameters save for the payout \(\chi_G\) and let \(\chi_G^*\) denote the lower bound on this payout at which my parametric
assumptions begin to fail. The range of potential values for this payout then admits a partition
\( \chi_G \leq \chi_G^{LS} \leq \chi_G^{ED} \leq \chi_G^{DD} \leq \infty \) such that the solution for program program (\( \mathbb{P} \)) exhibits the following properties:

- if \( \chi_G \in (\chi_G^{LS}, \chi_G^{ED}] \), then banks experience a liquidity surplus in the bad state;
- if \( \chi_G \in (\chi_G^{ED}, \chi_G^{DD}] \), then banks experience an extensive distortion in the bad state;
- if \( \chi_G \in (\chi_G^{DD}, \infty) \), then an interbank collapse occurs in the bad state.

Similar results obtain if the parameter being varied is instead the probability on the good state, \( \alpha_G \).

See figure [1.7] for an illustration.

Here proposition 1.2 implies that the risk of interbank distortions is procyclical. It also associates higher levels of expected productivity at \( t = 0 \) with qualitatively more severe distortions, namely as these distortions spread from the extensive margin to the intensive margin and eventually give rise to a full interbank collapse. These findings complement a growing literature on endogenous risk inside the financial system, and more specifically reinforce the view that financial crises represent “booms gone bad”, as argued by Borio and Drehmann (2009), Schularick and Taylor (2012), and others.

Derivations very similar to those underlying proposition 1.2 also allow us to characterize the economy’s response to changes in banks’ net worth:

**Proposition 1.3** (stabilizing effect of bank equity.). Fix all parameters save for the endowment \( E^b \) received by banks, and let \( \bar{E} \) denote the upper bound on this endowment implied by the requirement that households be deep-pocketed (assumption 1.5). The interval \([0, \bar{E}]\) then admits a partition \( 0 \leq E^{DD} \leq E^{ED} \leq E^{LS} \leq \bar{E} \) such that the solution for program program (\( \mathbb{P} \)) exhibits the following properties:

- if \( E^b \in [E^{LS}, \bar{E}] \), then banks experience a liquidity surplus in the bad state;
- if \( E^b \in [E^{ED}, E^{LS}) \), then banks experience an extensive distortion in the bad state;
- if \( E^b \in [E^{DD}, E^{ED}) \), then banks experience a dual distortion in the bad state, but the “no-default” constraint remains lax;
- if \( E^b \in (0, E^{DD}) \), then an interbank collapse occurs in the bad state.
See figure 1.7 for an illustration. That low capitalization in the banking sector thus leaves
the economy more vulnerable to interbank distortions complements an extensive literature
on the potentially stabilizing benefits of bank equity, including recent work by He and Krishnamurthy [2012], Bigio [2014], and many others. Moreover, in the context of a more
ambitious model which endogenized banks’ endowments, lemma 1.3 could provide a ration-
ale for capital injections of the sort witnessed during the recent crisis, though I leave this
issue as a topic for future research.

Apart from these mainly positive findings, propositions 1.2 and 1.3 also have some im-
portant normative implications. On this front, I note that the model at hand has been
constructed in a way which precludes constrained inefficiencies. In fact, it can easily be
shown that banks’ solution for program (P) coincides with the contract on which a utilitar-
ian planner would settle if he faced the same information and pledgeability frictions as do
private agents. That is:

**Proposition 1.4** (“no-go” for policy.) The interbank contract $C$ selected by banks also
maximizes utilitarian welfare,

$$
\sum_{\omega \in \{B, G\}} \alpha_{\omega} \int_{0}^{1} \left[ S_{\omega}(\theta) + \theta \chi_{\omega} I_{\omega}(\theta) \right] dF(\theta) + \mu(E^h - D),
$$

subject to the truth-telling constraint, (TT); limited pledgability constraint, (LP); individual
rationality constraint for depositors, (IR); and feasibility constraints (F0) through (F2d).

Though propositions 1.2 and 1.3 imply that the parameter space includes regions in which
the economy is vulnerable to interbank distortions, we can thus conclude that these distortions
do not give rise to a need for some kind of policy intervention, neither at $t = 0$, nor at $t = 1$.
Intuitively speaking, this constrained efficiency is a consequence of banks’ having access to
a larger contract space, which creates greater potential for coordination among banks. As
mentioned in my introduction, this result thus admits a natural parallel with Green and Lin
(2003), Cavalcanti and Monteiro (2011), Andolfatto et al. (2014), and other papers showing
that depositor runs can be eliminated from Diamond/Dybvig-style models when agents have
access to state-contingent contracts.

### 1.5 Conclusion

I’ll now close with a brief summary of this chapter’s findings. In this chapter, I identified
a potentially important gap in the literature on interbank regulation, namely that previous
literature has tended to take the form of the interbank contract as given, and took a first
Figure 1.7: Banks’ behaviour at $t = 0$ as a function of parameters

NB: $\alpha_G$ denotes the lower bound on $\alpha_G$ implied by my parametric assumptions.
step toward addressing this gap, namely by developing a simple model which endogenizes the form of the interbank contract as part of an optimal contracting problem. I argued that the model can account for interbank disruptions like those witnessed during the recent crisis and explored its implications for the risk factors predicting these disruptions. In particular, I showed that the in-model risk of interbank disruptions is procyclical, consistent with the view that endogenous risks tend to accumulate inside the financial system during good times (e.g., Borio and Drehmann 2009, Schularick and Taylor 2012). I also showed that interbank disruptions are more likely to occur when banks are poorly capitalized.

The chapter’s last findings then had to do with optimal policy. In particular, I showed that the aforementioned interbank disruptions are constrained efficient and admit no role for policy. In light of this strong “no-go” result, it’s now natural to ask about the role for policy that might emerge if we expanded the model to include a potential source of constrained inefficiency. I take up this question in my next chapter, namely by introducing a fire-sale externality into the economy.
1.A Appendix

1.A.1 Proof of lemma 1.3.1

Given some initial balance sheet \((D, I_0, R)\), along with some state \(\omega\), banks choose the subcontract \(C_\omega\) so as to maximize

\[
\int_0^1 \left[ S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta) \right] dF(\theta),
\]

subject to (TT), (LP), and (F1a) through (F2d). Call this program \((Q_0)\). I’ll then proceed in steps.

Step one. I first note that (F2c) can be dropped, namely because it’s sure to hold whenever (LP) and (F2b) both hold. I next note that standard mechanism-design arguments can be used to show that (TT) holds if and only if the following two conditions hold:

\[
S_\omega(\theta) - T_{\omega f}(\theta) + \theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)]
= S_\omega(0) - T_{\omega f}(0) + \int_0^\theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] d\theta, \ \forall \theta \in [0, 1] \quad \text{(TTa)}
\]

the difference \(\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)\) is weakly increasing in \(\theta\), \(\forall \theta \in [0, 1]\) \quad \text{(TTb)}

Now, if we use (TTa) to eliminate \(T_{\omega f}(\theta)\) as a choice variable \(\forall \theta \in (0, 1]\), then (F2a) and (F2b) can respectively be re-written as follows:

\[
\int_0^1 \left[ S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta) - S_\omega(0) + T_{\omega f}(0)
- \int_0^\theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')] d\theta' \right] dF(\theta) = 0 \quad \text{(F2a')}\]

\[
S_\omega(0) - T_{\omega f}(0)
\geq RD + \theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] - \int_0^\theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')] d\theta', \ \forall \theta \in [0, 1] \quad \text{(F2b')}\]
Of course, if \((LP)\) and \((TTb)\) both hold, then it should be clear that \((F2b')\) holds for all types i.f.f. it holds for \(\theta = 1\) — that is,

\[
S_\omega(0) - T_{\omega f}(0) \geq RD + \chi_\omega I_\omega(1) - \Delta T_\omega(1) - \int_0^1 [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] d\theta.
\]

If we then use \((F2a')\) to eliminate \(T_{\omega f}(0)\) from this last equation, we get

\[
\int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) \geq RD + \chi_\omega I_\omega(1) - \Delta T_\omega(1) - \int_0^1 [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] d\theta
\]

\[
+ \int_0^1 \int_0^\theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')] d\theta' dF(\theta)
\]

\[
= RD + \chi_\omega I_\omega(1) - \Delta T_\omega(1) - \int_0^1 F(\theta) [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] d\theta,
\]

where last line follows from a change in the order of integration.

So, the situation is as follows. If we define \(H_\omega(\theta) := \chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)\) \(\forall \theta \in [0,1]\), then \((Q_0)\) simplifies to choosing

\[
\{S_\omega(\theta), I_\omega(\theta), H_\omega(\theta)\}_{\theta \in [0,1]},
\]

so as to maximize \((\text{Obj})\), subject to \((F1a), (F1b), (F1c)\), and the following four constraints:

\[
\int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) \geq RD + H_\omega(1) - \int_0^1 F(\theta) H_\omega(\theta) d\theta \quad (TTa')
\]

\[
H_\omega(\theta) \text{ is weakly increasing, } \forall \theta \in [0,1] \quad (TTb')
\]

\[
H_\omega(\theta) \geq \gamma \chi_\omega I_\omega(\theta), \forall \theta \in [0,1] \quad (LP')
\]

\[
I_\omega(\theta) = 0 \implies H_\omega(\theta) = 0, \forall \theta \in [0,1] \quad (F2d')
\]

Call this program \((Q_1)\).
Step two. Under \((Q_1)\), it should be clear that the distribution of storage across types is irrelevant, so long that storage remains non-negative for all types. So, if we use \((F1a)\) to eliminate total storage \(\int_0^1 S_\omega(\theta)dF(\theta)\), then \((Q_1)\) simplifies to choosing
\[
\{I_\omega(\theta), H_\omega(\theta)\}_{\theta \in [0,1]},
\]
so as to maximize
\[
(E^b + D - I_0) + \ell I_0 + \int_0^1 (\theta \chi_\omega - \rho - \ell) I_\omega(\theta)dF(\theta),
\] (Obj')
subject to \((F1c), (TTb'), (LP'), (F2d')\), and the following two constraints:
\[
(E^b + D - I_0) + \ell I_0 \geq \int_0^1 (\rho + \ell) I_\omega(\theta)dF(\theta) \quad (F1a')
\]
\[
(E^b + D - I_0) + \ell I_0 + \int_0^1 (\theta \chi_\omega - \rho - \ell) I_\omega(\theta)dF(\theta)
\]
\[
\geq RD + H_\omega(1) - \int_0^1 F(\theta)H_\omega(\theta)d\theta \quad (TTa'')
\]
Call this program \((Q_2)\).

Step three. It’s now useful to make three observations about \((Q_2)\). The first is that any solution for this program must have the property that the set of types for whom \(I_\omega(\theta) > 0\) must constitute an upper interval — i.e., \(\exists \theta_\omega \in [0, 1] \text{ s.t. } I_\omega(\theta) > 0 \forall \theta \in (\theta_\omega, 1], \text{ and } I_\omega(\theta) = 0 \forall \theta \in [0, \theta_\omega)\). The argument is straightforward. First of all, \((F2d')\) ensures that \(H_\omega(\theta) = 0\) for all types satisfying \(I_\omega(\theta) = 0\). On the other hand, \((LP')\) ensures that \(H_\omega(\theta) > 0\) for all types satisfying \(I_\omega(\theta) > 0\). Finally, \((TTb')\) ensures that the latter case applies to higher types than does the former.

My next observation is that it’s weakly optimal to keep \(H_\omega(\theta)\) constant over the interval \((\theta_\omega, 1]\). The argument is as follows. Suppose that \((Q_2)\) admits a solution under which \(H_\omega(\theta)\) is non-constant over the interval \((\theta_\omega, 1]\). I’ll then construct an alternate contract under which all quantities are unchanged, save for the schedule \(\{\hat{H}_\omega(\theta)\}_{\theta \in (\theta_\omega, 1]}\), which I’ll replace with some alternate schedule \(\{\tilde{H}_\omega(\theta)\}_{\theta \in (\theta_\omega, 1)}\) under which the desired property obtains. Specifically, I set \(\hat{H}_\omega(\theta) = H(1) \forall \theta \in (\theta_\omega, 1]\). That this alternate contract delivers the same value for \((Obj')\) without violating any constraints should be obvious, so it also qualifies as a solution.
My last observation is that it’s strictly optimal to keep $I_\omega(\theta)$ constant over almost all of the interval $(\theta_\omega, 1]$. The argument is as follows. Suppose that $(Q_2)$ admits a solution under which this property fails — i.e., the interval $(\theta_\omega, 1]$ admits at least two subsets of positive measure within which $I_\omega(\theta)$ takes different values. I’ll then construct an alternate contract under which all quantities are unchanged, save for the schedule \{\hat{I}_\omega(\theta)\}_{\theta \in (\theta_\omega, 1)}$, which I’ll adjust to a new schedule \{\hat{\hat{I}}_\omega(\theta)\}_{\theta \in (\theta_\omega, 1)}$ under which the desired property obtains. More specifically, given some $\theta^*_\omega \in (\theta_\omega, 1)$, I set

$$
\hat{\hat{I}}_\omega(\theta) = \begin{cases} 
0.5 \left[ I_\omega(\theta) + \sup_{\hat{\theta} \in (\theta_\omega, 1)} I_\omega(\hat{\theta}) \right] & \text{if } \theta \geq \theta^*_\omega \\
0.5I_\omega(\theta) & \text{if } \theta < \theta^*_\omega
\end{cases}, \forall \theta \in (\theta_\omega, 1).
$$

I then adjust $\theta^*_\omega$ s.t.

$$
\int_{\theta_\omega}^{1} I_\omega(\theta)dF(\theta) = \int_{\theta_\omega}^{1} \hat{\hat{I}}_\omega(\theta)dF(\theta),
$$

which can indeed be done, namely because the cumulative $F$ admits a positive density. That this alternate contract deliver a strictly higher value for $(\text{Obj})^\prime$ without violating any constraints should be obvious, #!

In light of these observations, I’ll now restrict attention to candidate solutions under which $H_\omega(\theta)$ ($I_\omega(\theta)$) is constant over at some value $H_\omega$ ($I_\omega$) over (almost all of) the interval $(\theta_\omega, 1]$, if non-empty. If instead $\theta_\omega = 1$, I adopt a convention that $(H_\omega, I_\omega) = [H_\omega(1), I_\omega(1)]$. $(Q_2)$ then simplifies to a choice over the triplet $(\theta_\omega, I_\omega, H_\omega)$, where the goal is to maximize

$$
(E^b + D - I_0) + \ell I_0 + \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell) I_\omega dF(\theta),
$$

(Obj")

subject to the following constraints:

$$
(E^b + D - I_0) + \ell I_0 \geq \int_{0}^{1} (\rho + \ell) I_\omega dF_\omega(\theta)
$$

(F1a")

$$
(E^b + D - I_0) + \ell I_0 + \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell) I_\omega dF(\theta)
$$

$$
\geq RD + H_\omega \left[ 1 - \int_{\theta_\omega}^{1} F(\theta)d\theta \right]
$$

(TTa")

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(\theta_\omega, I_\omega) \in [0, 1] \times [0, I_0] \quad \text{(F1c')}

H_\omega \geq \gamma \chi_\omega I_\omega \quad \text{(LP'')}

I_\omega = 0 \implies H_\omega = 0 \quad \text{(F2d'')}

Call this program \((Q_3)\).

**Step four.** At this point, it’s useful to note that \((TTa'')\) and \((LP'')\) combine as follows:

\[
\frac{(E^b + D - I_0) + \ell I_0 + \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell) I_\omega dF(\theta) - RD}{1 - \int_{\theta_\omega}^{1} F(\theta) d\theta} \geq H_\omega \geq \gamma \chi_\omega I_\omega
\]

Combining these two constraints in this way makes clear the fact that \(H_\omega\) can always be chosen to satisfy \((TTa'')\), \((LP'')\), and \((F2d'')\), given any choice on \((\theta_\omega, I_\omega) \in [0, 1] \times [0, I_0]\) satisfying \((*) \geq \gamma \chi_\omega I_\omega\). So, \((Q_3)\) finally simplifies to choosing \((\theta_\omega, I_\omega)\) so as to maximize \((Obj'')\), subject to \((F1a'')\), \((F1c)\), and the inequality \((*) \geq \gamma \chi_\omega I_\omega\), which can be re-written as

\[
(E^b + D - I_0) + \ell I_0 + \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell) I_\omega dF(\theta)
\]

\[
\geq RD + \gamma \chi_\omega I_\omega \left[1 - \int_{\theta_\omega}^{1} F(\theta) d\theta\right] = RD + \gamma \chi_\omega I_\omega \left[\theta_\omega F(\theta_\omega) + \int_{\theta_\omega}^{1} \theta dF(\theta)\right],
\]

where the last line follows from integration by parts. Written in this way, we recognize this expression as the financial constraint described in lemma 1.3.1. All that remains is then to observe that \((F1a'')\) corresponds to the physical constraint.

**1.A.2 Construction of figure 1.3**

All the claims being made on the shape of the per-unit surplus function should be obvious. As for my claims on the shape of the per-unit pledgable surplus function, I first note that

\[
\Delta_B'(\theta_B) = \Pi_B'(\theta_B) - \gamma \chi_B F(\theta_B) = (\rho + \ell - \chi_B \theta_B) f(\theta_B) - \gamma \chi_B F(\theta_B)
\]

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\[ \propto \rho + \ell - \chi_B \left[ \theta_B + \frac{\gamma F(\theta_B)}{f(\theta_B)} \right] \]

where \( \propto \) denotes a situation in which two expressions share a common sign. Now, log-concavity of the cumulative function \( F \) (assumption 1.3) ensures that the ratio \( F(\theta_B)/f(\theta_B) \) is strictly increasing, so the starred term above is strictly decreasing, namely with

\[ \left[ \rho + \ell - \chi_B \left[ \theta_B + \frac{\gamma F(\theta_B)}{f(\theta_B)} \right] \right]_{\theta=\theta_B^I} < 0 < \left[ \rho + \ell - \chi_B \left[ \theta_B + \frac{\gamma F(\theta_B)}{f(\theta_B)} \right] \right]_{\theta=0}. \]

These observations suffice to verify my claim that per-unit surplus peaks at \( \theta_B^\Delta \), along with my claim that this peak satisfies \( \theta_B^\Delta < \theta_B^I \). So, all that remains is to confirm that per-unit surplus is negative over \([0, 1]\). On this front, it suffices for assumption 1.4 to include a requirement that \( \chi_B \) be small enough that the following condition obtains:

**Subassumption 1.4.1.** \( \Delta_B(\theta_B^\Delta) < 0. \)

### 1.A.3 Derivation of banks’ pecking order (proposition 1.1)

In the main text, I argued that the extensive margin dominates so long that

\[ \Pi_B'(\theta_B) - \frac{\Pi_B(\theta_B) \Delta_B'(\theta_B)}{\Delta_B(\theta_B)} \leq 0. \]

These observations suffice to verify my claim that per-unit surplus peaks at \( \theta_B^\Delta \), along with my claim that this peak satisfies \( \theta_B^\Delta < \theta_B^I \). So, all that remains is to confirm that per-unit surplus is negative over \([0, 1]\). On this front, it suffices for assumption 1.4 to include a requirement that \( \chi_B \) be small enough that the following condition obtains:

**Subassumption 1.4.1.** \( \Delta_B(\theta_B^\Delta) < 0. \)

I also argued that the starred term is strictly positive at \( \theta_B = \max \{ \theta_B, \theta_B^\Delta \} \), but strictly negative at \( \theta_B = \theta_B^I \), so a sufficient condition for the pecking order described in the main text would be that the starred term also exhibits single-crossing. To verify single-crossing, I suggest that we proceed as follows. First, note that

\[ (*) \propto \Pi_B(\theta_B) \Delta_B'(\theta_B) - \Pi_B'(\theta_B) \Delta_B(\theta_B) \]

\[ \propto \Pi_B(\theta_B) \left[ \Pi_B'(\theta_B) - \gamma \chi_B F(\theta_B) \right] \]

\[ - \Pi_B'(\theta_B) \left[ \Pi(\theta_B) - \gamma \chi_B \theta_B F(\theta_B) - \gamma \chi_B \int_{\theta_B}^{1} \theta dF(\theta) \right] \]
\[ \propto \Pi'_B(\theta_B) \left[ \theta_B F(\theta_B) + \int_{\theta_B}^{1} \theta dF(\theta) \right] - \Pi_B(\theta_B) F(\theta_B) \]

\[ = \Pi'_B(\theta_B) \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] - \Pi_B(\theta_B) F(\theta_B) \]

\[ = \left( \rho + \ell - \theta_B \chi_B \right) f(\theta_B) \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] - \Pi_B(\theta_B) F(\theta_B) \]

\[ \propto (\rho + \ell - \theta_B \chi_B) \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] - \frac{\Pi_B(\theta_B) F(\theta_B)}{f(\theta_B)} \]

\[ = (\rho + \ell - \theta_B \chi_B) \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] - \frac{F(\theta_B)}{f(\theta_B)} \int_{\theta_B}^{1} (\theta \chi_B - \rho - \ell) dF(\theta) \]

\[ = (\rho + \ell) \left[ \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)} \right] \]

\[ - \chi_B \left[ \theta_B \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)}{f(\theta_B)} \int_{\theta_B}^{1} \theta dF(\theta) \right] \]

\[ \propto \frac{\rho + \ell}{\chi_B} - \frac{\theta_B \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)} \int_{\theta_B}^{1} \theta dF(\theta)}{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)}} \] (1.9)

\[ = \frac{\rho + \ell}{\chi_B} - \frac{\theta_B \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)} \mathbb{E}(\theta | \theta \geq \theta_B)}{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)}} \] (1.10)

\[ = \frac{\rho + \ell}{\chi_B} - \frac{\mathbb{E}(\theta | \theta \geq \theta_B) + \frac{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] \left[ \mathbb{E}(\theta | \theta \geq \theta_B) - \theta \right]}{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)}}}{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right] + \frac{F(\theta_B)[1 - F(\theta_B)]}{f(\theta_B)}} \] (1.11)

\[ = \frac{\rho + \ell}{\chi_B} \]
\[- \left[ \mathbb{E}(\theta | \theta \geq \theta_B) - \frac{\mathbb{E} \left[ \max \{ \theta, \theta_B \} \right]}{1 - F(\theta_B)} \int_{\theta_B}^{1} \theta dF(\theta) - \theta_B \left[ 1 - F(\theta_B) \right] \right] \cdot (1.12) \]

Re-writing in this way is useful because

\[
\xi'(\theta_B) = \frac{f(\theta_B) \left[ \int_{\theta_B}^{1} \theta dF(\theta) - \theta_B \left[ 1 - F(\theta_B) \right] \right]}{[1 - F(\theta_B)]^2},
\]

(1.13)

and

\[
\xi'_2(\theta_B) = \frac{F(\theta_B) \left[ 1 - F(\theta_B) \right] + f(\theta_B) \mathbb{E} \left[ \max \{ \theta, \theta_B \} \right]}{[1 - F(\theta_B)]^2},
\]

(1.14)

so \( \xi_3(\theta_B) = \xi'_1(\theta_B)/\xi'_2(\theta_B) \). In turn,

\[
\Xi'(\theta_B) = \xi'_1(\theta_B) - \xi'_2(\theta_B)\xi_3(\theta_B) - \xi_2(\theta_B)\xi'_3(\theta_B) = -\xi_2(\theta_B)\xi'_3(\theta_B) > 0,
\]

where the inequality follows from the fact that the numerator in (1.13) is strictly decreasing, while the numerator in (1.14) is strictly increasing, namely due to log-concavity of the cumulative \( F \) (assumption 1.3). This suffices to verify single-crossing.

1.A.4 Construction of figure 1.6

The only potentially ambiguous aspect of figure 1.6 has to do with the relative placement of the loci

\[
\{(D, I_0) \in \mathbb{R}^2_+ \text{ s.t. } (E^b + D - I_0) + \ell I_0 = \Delta \mu(D) - I_0 \Delta_B(\theta_B) \Pi_B \}\}
\]

and

\[
\{(D, I_0) \in \mathbb{R}^2_+ \text{ s.t. } (E^b + D - I_0) + \ell I_0 = I_0 \Psi_G(\theta_B) \Pi_G \}\}
\]

In particular, I’ve drawn the figure s.t. the former admits a higher vertical intercept than the latter — i.e.,

\[
\frac{E^b}{1 - \ell - \Delta_B(\theta_B) \Pi_B} > \frac{E^b}{1 - \ell + \Psi_G(\theta_B) \Pi_G}
\]

So, it would suffice for assumption 1.4 to include a requirement that \( \chi_G \) be large enough that the following condition obtains:
Subassumption 1.4.2. $\Delta_B(\theta_B^H) > (-1)\Psi_G(\theta_G^H)$.

1.A.5 Solution for programs $(\mathbb{P}$-rex) and $(\mathbb{P})$ (lemmata 1.4.1-1.4.2, propositions 1.2-1.3)

1.A.5.1 Notation

I begin by introducing some notation:

- For all $I_0 \in [0, E_b]/[1 - \ell - \Delta_B(\theta_B^H)]$, let $D_B^H(I_0) \in [0, \infty)$ denote the level of deposits at which the financial constraint associated with the bad state begins to bind — i.e.,
  
  $$[E^b + D_B^H(I_0) - I_0] + \ell I_0 + I_0 \Delta_B(\theta_B^H) = \Delta \mu [D_B^H(I_0)]$$;

- For all $I_0 \in [0, E_b]/[1 - \ell - \Delta_B(\theta_B^G)]$, let $D_B^G(I_0) \in [0, \infty)$ denote the level of deposits at which the “no-default” constraint binds — i.e.,
  
  $$[E^b + D_B^G(I_0) - I_0] + \ell I_0 + I_0 \Delta_B(\theta_B^G) = \Delta \mu [D_B^G(I_0)]$$;

- For all $I_0 \in [0, E_b]/(1 - \ell)$, let $\overline{D}_B(I_0) \in [0, \infty)$ denote the level of deposits at which the physical constraint associated with the good state binds — i.e.,
  
  $$[E^b + \overline{D}_B(I_0) - I_0] + \ell I_0 = \Delta \mu [\overline{D}_B(I_0)]$$;

- For all $I_0 \in \mathbb{R}$, let $D_G^H(I_0)$ denote the level of deposits at which the physical constraint associated with the good state binds — i.e.,
  
  $$[E^b + D_G^H(I_0) - I_0] + \ell I_0 = I_0 \Psi_G(\theta_G^H)$$;

- Finally, for all $(D, I_0) \in \mathbb{R}_+^2$ satisfying $(E^b + D - I_0) + \ell I_0 \geq \Delta \mu (D)$, let $v(D, I_0)$ give banks’ expected payout, computed on an unconditional basis at $t = 0$, after taking account of proposition 1.1 and the analysis in subsection 1.3.3 — e.g.,
  
  $$D \leq \min \{D_G^H(I_0), D_B^H(I_0)\} \implies v(D, I_0) = \alpha_G v_G^{LR}(D, I_0) + \alpha_B v_B^{LS}(D, I_0).$$

Remark 1. It can easily be verified that the payout function $v(D, I_0)$ is $C^1$ in both its arguments, even around the boundaries separating regimes. This is a consequence of the envelope theorem.
Remark 2. For clarity, figure L.8 illustrates some of the notation used in this subsection.

1.A.5.2 Some preliminary results

Sublemma 1.A.1. The derivative $v_D(D, I_0)$ is strictly decreasing in its first argument.

Proof. I’ll take cases on states and regimes.

Case one: $r_B = ED$. Under this case, we know that

$$(v_B^{ED})_D(D, I_0) = [1 - \Delta \mu'(D)] \left[ 1 - \frac{\Pi_B'[\theta_B^{ED}(D, I_0)]}{\Delta_B'\theta_B^{ED}(D, I_0)} \right],$$

where

$$(\theta_B^{ED})_D(D, I_0) = \frac{(-1)[1 - \Delta \mu'(D)]}{I_0 \Delta_B'[\theta_B^{ED}(D, I_0)]} < 0.$$

Moreover,

$$(\Pi_B'/\Delta_B')(\theta) = \frac{\chi B^{\theta - \rho - \ell}}{\chi B^{\theta + \frac{\gamma F(\theta)}{f(\theta)} - \rho - \ell}},$$

so, $\forall \theta \in (\theta^\Delta_B, \theta^\Pi_B)$,

$$(\Pi_B'/\Delta_B')(\theta) \propto \chi B^{\theta + \frac{\gamma F(\theta)}{f(\theta)} - \rho - \ell}$$

$$> 0 \iff \theta > \theta^\Pi_B$$

$$< 0 \iff \theta < \theta^\Delta_B$$

Conclude that

$$(v_B^{ED})_{DD}(D, I_0) = (-1)\Delta \mu''(D)[1 - (\Pi_B'/\Delta_B')[\theta_B^{ED}(D, I_0)]]$$

$$- [1 - \Delta \mu'(D)](\theta_B^{ED})_D(D, I_0)(\Pi_B'/\Delta_B')'\theta_B^{ED}(D, I_0) < 0.$$
Case two: \( r_B = DD \). Under this case, we know that
\[
(v^{DD}_B)_D(D, I_0) = [1 - \Delta \mu'(D)] \left[ 1 - \frac{\Pi_B(\theta_B^\Delta)}{\Delta_B(\theta_B^\Delta)} \right],
\]
so
\[
(v^{DD}_B)_{DD}(D, I_0) = (-1)\Delta \mu''(D)[1 - (\Pi_B/\Delta_B)(\theta_B^\Xi)] < 0.
\]

Case three: \( r_G = LR \). Under this case, we know that
\[
(v^{LR}_G)_D(D, I_0) = 1 - \Delta \mu'(D) + \frac{\Pi_G[\theta^{LR}_G(D, I_0)]}{\Psi_G[\theta^{LR}_G(D, I_0)]},
\]
where
\[
(\theta^{LR}_G)_D(D, I_0) = \frac{1}{I_0 \Psi'_G[\theta^{LR}_G(D, I_0)]} < 0. \tag{1.16}
\]
Moreover,
\[
(\Pi_G/\Psi'_G)(\theta) = \frac{\theta \chi_G - \rho - \ell}{\rho + \ell}, \tag{1.17}
\]
so
\[
(v^{LR}_G)_{DD}(D, I_0) = (-1)\Delta \mu''(D) + (\theta^{LR}_G)_D(D, I_0)(\Pi'_G/\Psi'_G)'[\theta^{LR}_G(D, I_0)] < 0.
\]

Case four: \( r_\omega = LS \ (\omega \in \{B, G\}) \). Under this case, we know that
\[
(v^{LS}_\omega)_D(D, I_0) = 1 - \Delta \mu'(D),
\]
so
\[
(v^{LS}_\omega)_{DD}(D, I_0) = (-1)\Delta \mu''(D) < 0. \]

**Sublemma 1.A.2.** The composition \( v_D[\overline{D}_B(I_0), I_0] \) exhibits single-crossing from below over the interval \([0, E^b/(1 - \ell)]\), namely at some interior point \( I_B \) satisfying \( \overline{D}_B(I_B) < D^I_B(I_B) \).

**Proof.** It should be clear that the function \( \overline{D}_B(I_0) \) intersects the function \( D^I_B(I_0) \) once from above over the interval \([0, E^b/(1 - \ell)]\), namely at some interior point which I’ll denote \( \tilde{I} \) for the purposes of this sublemma. So, \( \forall I_0 \in [0, \tilde{I}] \), we have
\[
v_D[\overline{D}_B(I_0), I_0] = \alpha_G(v^{LS}_G)_D[\overline{D}_B(I_0), I_0] + \alpha_B(v^{DD}_B)_D[\overline{D}_B(I_0), I_0]
\]
\[
\alpha_G \left[ 1 - \Delta \mu' \left[ \overline{D}_B(I_0) \right] \right] \\
+ \alpha_B \left[ 1 - \Delta \mu' \left[ \overline{D}_B(I_0) \right] \right] \left[ 1 - (\Pi_B/\Delta_B)(\theta_{\overline{B}}^\infty) \right] < 0.
\]

On the other hand, it should be clear that
\[
v_D \left[ \overline{D}_B[E^b/(1 - \ell)], E^b/(1 - \ell) \right] = \alpha_G(v_G^{LR})_D \left[ 0, E^b/(1 - \ell) \right] \\
+ \alpha_B(v_B^{DD})_D \left[ 0, E^b/(1 - \ell) \right] \\
= \alpha_G \left[ 1 - \Delta \mu'(0) + (\Pi_G'/\Psi_G') \left[ \theta_G^{LR} \left[ 0, E^b/(1 - \ell) \right] \right] \right] \\
+ \alpha_B \left[ 1 - \Delta \mu'(0) \right] \left[ 1 - (\Pi_B/\Delta_B)(\theta_{\overline{B}}^\infty) \right] \\
= \alpha_G(\Pi_G'/\Psi_G') \left[ \theta_G^{LR} \left[ 0, E^b/(1 - \ell) \right] \right] > 0,
\]

where the last equality follows from the fact that \( \mu'(E^b) = 1 \).

So, our goal will now be to show that the composition in question is strictly increasing over the interval \((\tilde{I}, E^b/(1 - \ell))\). Now, in light of my previous sublemma, along with the fact that the function \( \overline{D}_B(I_0) \) is strictly decreasing, it would suffice if we could show that
\[
v_D \left[ \overline{D}_B(I_0), I_0 \right] > 0 \quad \text{whenever} \quad I_0 \in (\tilde{I}, E^b/(1 - \ell)).
\]
To this end, I note that all such choices on \( I_0 \) satisfy
\[
v_D \left[ \overline{D}_B(I_0), I_0 \right] = \alpha_G(v_G^{LR})_D \left[ \overline{D}_B(I_0), I_0 \right] + \alpha_B(v_B^{DD})_D \left[ \overline{D}_B(I_0), I_0 \right],
\]

where
\[
(v_G^{LR})_D \left[ \overline{D}_B(I_0), I_0 \right] = 1 - \Delta \mu' \left[ \overline{D}_B(I_0) \right] + (\Pi_G'/\Psi_G') \left[ \theta_G^{LR} \left[ \overline{D}_B(I_0), I_0 \right] \right],
\]
and
\[
(v_B^{DD})_D \left[ \overline{D}_B(I_0), I_0 \right] = \left[ 1 - \Delta \mu' \left[ \overline{D}_B(I_0) \right] \right] \left[ 1 - (\Pi_B/\Delta_B)(\theta_{\overline{B}}^\infty) \right],
\]

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with
\[
(\theta_G^{LR})_I \left[ \bar{D}_B(I_0), I_0 \right] = \frac{(-1) \left( 1 - \ell + \Psi_G \left[ \theta_G^{LR} \left[ \bar{D}_B(I_0), I_0 \right] \right] \right)}{I_0 \Psi'_G \left[ \theta_G^{LR} \left[ \bar{D}_B(I_0), I_0 \right] \right]} > 0.
\]

So,
\[
v_{DI_0} \left[ \bar{D}_B(I_0), I_0 \right] = \alpha_G(v_G^{LR})_{DI} \left[ \bar{D}_B(I_0), I_0 \right]
\]
\[
= \alpha_G(\theta_G^{LR})_I \left[ \bar{D}_B(I_0), I_0 \right] \left( \Pi'_G / \Psi'_G \right)' \left[ \theta_G^{LR} \left[ \bar{D}_B(I_0), I_0 \right] \right] > 0,
\]
as desired. ■

Sublemma 1.A.3. \(\forall x \in \{\Pi, \Xi\}\), the composition \(v_D[D_B^x(I_0), I_0]\) exhibits single-crossing from below over the interval \(I_0 \in [0, E^b/[1 - \ell - \Delta_B(\theta_B^x)]]\), namely at some interior point \(I_B^x\) satisfying \(D_B^x(I_B^x) < D_B^x(I_B^x)\).

Proof. This can be verified using essentially the same argument as for sublemma 1.A.2. ■

1.A.5.3 Solution for program (P-rex): existence and uniqueness

In this subsubsection, I argue that program (P-rex) admits a unique solution.

Now, in light of the discussion in the main text, we know that candidate solutions must have the property that liquidity rationing occurs in the good state, so \(I_0\) must lie somewhere in the interval \((E^b/[1 - \ell + \Psi_G(\theta_G^x)], E^b/(1 - \ell)]\). For the time being, I’ll fix some \(I_0\) in this range and focus on the choice on \(D\), which must lie somewhere in the interval \([0, \min\{D_G^I(I_0), \bar{D}_B(I_0)\}]\). In fact, we can rule out the bottom of this interval, namely because

\[
v_D(0, I_0) = \alpha_G \left[ 1 - \Delta \mu'(0) + (\Pi'_G / \Psi'_G) \left[ \theta_G^{LR}(0, I_0) \right] \right]
\]

\[
\begin{cases}
1 - \Delta \mu'(0) & \text{if } D \leq D_B^I(I_0) \\
[1 - \Delta \mu'(0)] \left[ 1 - (\Pi_B'/\Delta_B') \left[ \theta_B^{EP}(D, I_0) \right] \right] & \text{if } D \in (D_B^I(I_0), \bar{D}_B^I(I_0)] \\
[1 - \Delta \mu'(0)] \left[ 1 - (\Pi_B/\Delta_B)(\theta_B^x) \right] & \text{if } D \in (\bar{D}_B^I(I_0), \bar{D}_B(I_0)]
\end{cases}
\]
\[ \alpha_G (\Omega'_G / \Psi'_G) \left[ \theta^L_G (0, I_0) \right] > 0, \]

where the last equality follows from the fact that \( \mu'(E^h) = 1 \). Moreover, in light of sublemma 1.A.1, we know that \( v_D (0, I_0) \) is strictly decreasing in its first argument. As a result, there are two cases for us to consider, depending on \( I_0 \). If \( I_0 \) satisfies

\[ v_D (\min \{ D^G_H (I_0), \overline{D}_B (I_0) \}, I_0) < 0, \]

then \( \exists ! D^* (I_0) \in (0, \min \{ D^G_H (I_0), \overline{D}_B (I_0) \}) \) s.t. \( v_D (D, I_0) = 0 \), and this point represents the uniquely optimal choice on \( D \). Otherwise, it’s strictly optimal to make \( D \) as large as possible, in which case I adopt a convention that \( D^* (I_0) = \min \{ D^G_H (I_0), \overline{D}_B (I_0) \} \). Moreover, sublemma 1.A.2 provides us with a clean separation of these two cases — specifically, \( D^* (I_0) = \overline{D}_B (I_0) \forall I_0 \in [I_B, E^h / (1 - \ell)] \), and otherwise \( D^* (I_0) < \min \{ D^G_H (I_0), \overline{D}_B (I_0) \} \).

So, I’ll now restrict attention to pairs of the form \( (D, I_0) = [D^* (I_0), I_0] \). Given any such pair, I’ll now check if banks have an incentive to deviate in their initial balance-sheet choices, namely by making some small adjustment to their choice on \( I_0 \). Now, if \( I_0 < I_B \), then the “no-default” constraint is lax, and the return to a marginal increase in \( I_0 \) is given by \( v_I [D^* (I_0), I_0] \). Otherwise, the “no-default” constraint binds, and increases in \( I_0 \) must be offset by decreases in \( D \), so the relevant return reads as

\[ v_I [\overline{D}_B (I_0), I_0] + \overline{D}'_B (I_0) v_D [\overline{D}_B (I_0), I_0] \]

\[ = \alpha_G \left[ (v^L_G)_I [\overline{D}_B (I_0), I_0] + \overline{D}'_B (I_0) (v^L_G)_D [\overline{D}_B (I_0), I_0] \right] \]

\[ + \alpha_B \left[ (v^D_B)_I [\overline{D}_B (I_0), I_0] + \overline{D}'_B (I_0) (v^D_B)_D [\overline{D}_B (I_0), I_0] \right]. \]

\[ = 0, \text{ b/c all resources go to depositors in the bad state} \]

\[ = \alpha_G \left[ (v^L_G)_I [\overline{D}_B (I_0), I_0] + \frac{1 - \ell}{1 - \Delta \mu' [\overline{D}_B (I_0)]} (v^L_G)_D [\overline{D}_B (I_0), I_0] \right]. \]
So, \( \forall I_0 \in [E^b/[1 \ell + \Psi G(\theta G)]], E^b/(1 \ell) \), I define a function

\[
h(I_0) := \begin{cases} 
  v_I[D^*(I_0), I_0] & \text{if } I_0 \leq I_B \\
  \alpha_G \left( (v_{LR} G) I \left[ D^*(I_0), I_0 \right] + \left[ \frac{1 - \ell}{1 - \Delta \mu'[D^*(I_0)]} \right] (v_{LR}^G) D \left[ D^*(I_0), I_0 \right] \right) & \text{if } I_0 > I_B.
\end{cases}
\]

Now, it should be clear that the function \( h(\cdot) \) is continuous. It should also be clear that it satisfies

\[
h[E^b/[1 \ell + \Psi G(\theta G)]] = \sum_{\omega \in \{B, G\}} \alpha_\omega (v_{LS}^\omega)_I[0, E^b/[1 \ell + \Psi G(\theta G)] > 0,
\]

namely due assumption 1.1. On the other hand,

\[
\lim_{I_0 \searrow E^b/(1 \ell)} \{h(I_0)\} = (v_{LR} G)_I[I_0, E^b/(1 \ell)] + \lim_{I_0 \searrow E^b/(1 \ell)} \left\{ \frac{1 - \ell}{1 - \Delta \mu'[D^*(I_0)]} (v_{LR}^G) D \left[ D^*(I_0), I_0 \right] \right\}
\]

\[
= (v_{LR} G)_I[I_0, E^b/(1 \ell)] + \lim_{I_0 \searrow E^b/(1 \ell)} \left\{ \frac{1 - \ell}{1 - \Delta \mu'[D^*(I_0)]} \right\} \times \cdots
\]

\[
+ \lim_{I_0 \searrow E^b/(1 \ell)} \left\{ \cdots \times \left[ 1 - \Delta \mu'[D^*(I_0)] + \frac{\theta_{LR} G}{\Psi G} \left[ D^*(I_0), I_0 \right] \right] \right\}
\]

\[
= -\infty,
\]

namely due to the fact that \( \mu'(E^b) = 1 \). So, it would suffice if we could show that the function \( h(\cdot) \) is strictly decreasing. I’ll use the remainder of this subsubsection to establish this monotonicity, namely by taking cases on \( I_0 \).

Case one. Suppose first that \( I_0 \) has the property that the pair \( [D^*(I_0), I_0] \) lies in the region where \( r_B = LS \). In this case, we want to confirm that

\[
h'(I_0) = v_{II}[D^*(I_0), I_0] + (D^*)'(I_0) v_{ID}[D^*(I_0), I_0]
\]
\[ v_{II}[D^*(I_0), I_0] - \frac{v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0]}{v_{DD}[D^*(I_0), I_0]} < 0, \]

or equivalently

\[ v_{II}[D^*(I_0), I_0]v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0, \]

which we recognize as the usual second-order condition. Now, in the good state, we know that

\[ (v_{LR}^G)_{II}[D^*(I_0), I_0] = \ell + \Pi_G[\theta_{LR}^G[D^*(I_0), I_0]] - 1 \]

\[ - [1 - \ell + \Psi_G[\theta_{LR}^G[D^*(I_0), I_0]](\Pi_G'/\Psi_G')[\theta_{LR}^G[D^*(I_0), I_0]], \]

and

\[ (v_{LR}^G)_{ID}[D^*(I_0), I_0] = 1 - \Delta_{\mu'}[D^*(I_0)] + (\Pi_G'/\Psi_G')[\theta_{LR}^G[D^*(I_0), I_0]], \]

with

\[ (\theta_{LR}^G)_{I}[D^*(I_0), I_0] = \frac{(-1)[1 - \ell + \Psi_G[\theta_{LR}^G[D^*(I_0), I_0]]]}{I_0 \Psi_G'[\theta_{LR}^G[D^*(I_0), I_0]]} > 0, \]

and

\[ (\theta_{LR}^G)_{D}[D^*(I_0), I_0] = \frac{1}{I_0 \Psi_G'[\theta_{LR}^G[D^*(I_0), I_0]]} < 0. \]

So,

\[ (v_{LR}^G)_{II}[D^*(I_0), I_0] = (-1)[1 - \ell + \Psi_G[\theta_{LR}^G[D^*(I_0), I_0]]] \times \cdots \]

\[ \cdots \times (\theta_{LR}^G)_{I}[D^*(I_0), I_0] (\Pi_G'/\Psi_G')[\theta_{LR}^G[D^*(I_0), I_0]] > 0, \]

\[ (v_{LR}^G)_{ID}[D^*(I_0), I_0] = (-1)[1 - \ell + \Psi_G[\theta_{LR}^G[D^*(I_0), I_0]]] \times \cdots \]
\[
\cdots \times (\theta^\text{LR}_G)_D[D^*(I_0), I_0](\Pi'_G/\Psi'_G)[\theta^\text{LR}_G[D^*(I_0), I_0]] > 0,
\]

\[
(v^\text{LR}_G)_{DD}[D^*(I_0), I_0] = (-1)\Delta\mu''[D^*(I_0)]
\]

\[
+ (\theta^\text{LR}_G)_D[D^*(I_0), I_0](\Pi'_G/\Psi'_G)[\theta^\text{LR}_G[D^*(I_0), I_0]] < 0,
\]

and

\[
(v^\text{LR}_G)_{DI}[D^*(I_0), I_0] = (\theta^\text{LR}_G)_I[D^*(I_0), I_0](\Pi'_G/\Psi'_G)[\theta^\text{LR}_G[D^*(I_0), I_0]] > 0.
\]

On the other hand, in the bad state, we have

\[
(v^\text{LS}_B)_{D}[D^*(I_0), I_0] = 1 - \Delta\mu'[D^*(I_0)],
\]

and

\[
(v^\text{LS}_B)_{I}[D^*(I_0), I_0] = \ell + \Pi_B(\theta^\text{II}_B) - 1,
\]

so

\[
(v^\text{LS}_B)_{II}[D^*(I_0), I_0] = (v^\text{LS}_B)_{DI}[D^*(I_0), I_0] = 0,
\]

and

\[
(v^\text{LS}_B)_{DD}[D^*(I_0), I_0] = (-1)\Delta\mu''[D^*(I_0)] < 0.
\]

So, using the symbol \(\cdot\) to suppress arguments, we have

\[
v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
= v_{II}(\cdot)[\alpha_G(\theta^\text{LR}_G)_D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot) - \Delta\mu''(\cdot)] - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
> v_{II}(\cdot)\alpha_G(\theta^\text{LR}_G)_D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
= \alpha_G(-1)[1 - \ell + \Psi_G(\cdot)](\theta^\text{LR}_G)_I(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)\alpha_G(\theta^\text{LR}_G)_D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)
\]
\[-G(\theta_G^{LR})_I(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)\alpha_G(-1)[1 - \ell + \Psi_G(\cdot)](\theta_G^{LR})_D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)\]

\[= 0,\]

as desired.

**Case two.** Suppose next that $I_0$ has the property that the pair $[D^*(I_0), I_0]$ lies in the region where $r_B = ED$. In this case, the desired monotonicity is again equivalent to the second-order condition

\[v_{II}[D^*(I_0), I_0]v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0,\]

and banks’ behaviour in the good state is unchanged. However, in the bad state, we now have

\[v_{BI}[D^*(I_0), I_0]v_{D'B'}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0,\]

and banks’ behaviour in the good state is unchanged. However, in the bad state, we now have

\[(v_{B I}^{ED})_I[D^*(I_0), I_0] = \ell + \Pi_B[\theta_B^{ED}[D^*(I_0), I_0]] - 1 \]

\[+ [1 - \ell - \Delta_B[\theta_B^{ED}[D^*(I_0), I_0]](\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]],\]

and

\[(v_{B D}^{ED})_D[D^*(I_0), I_0] = [1 - \Delta\mu'[\theta_B^{ED}[D^*(I_0), I_0]]][1 - (\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]],\]

with

\[(\theta_B^{ED})_I[D^*(I_0), I_0] = \frac{1 - \ell - \Delta_B[\theta_B^{ED}[D^*(I_0), I_0]]}{I_0\Delta'_B[\theta_B^{ED}[D^*(I_0), I_0]]} \times 0,\]

and

\[(\theta_B^{ED})_D[D^*(I_0), I_0] = \frac{(-1)[1 - \Delta\mu'[\theta_B^{ED}[D^*(I_0), I_0]]}{I_0\Delta'_B[\theta_B^{ED}[D^*(I_0), I_0]]} \times 0.\]

So,

\[(v_{B I}^{ED})_I[D^*(I_0), I_0] = [1 - \ell - \Delta_B[\theta_B^{ED}[D^*(I_0), I_0]]] \times \cdots \]

\[\cdots \times (\theta_B^{ED})_I[D^*(I_0), I_0] (\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]] = 0,\]
\[(v_E^{ED})_{ID}[D^*(I_0), I_0] = [1 - \ell - \Delta_B[\theta_B^{ED}[D^*(I_0), I_0]]] \times \cdots \]

\[\cdots \times (\theta_B^{ED})_D[D^*(I_0), I_0](\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]] < 0,\]

\[(v_E^{ED})_{DD}[D^*(I_0), I_0] = (-1)\Delta \mu''[D^*(I_0)][1 - \underbrace{(\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]]}_{<0}]

- [1 - \Delta \mu'[D^*(I_0)]] \times \cdots \]

\[\cdots \times (\theta_B^{ED})_D[D^*(I_0), I_0](\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]] < 0,\]

and

\[(v_E^{ED})_{DI}[D^*(I_0), I_0] = (-1)[1 - \Delta \mu'[D^*(I_0)]] \times \cdots \]

\[\cdots \times (\theta_B^{ED})_I[D^*(I_0), I_0](\Pi'_B/\Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0]] < 0.\]

Now,

\[v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot) = [\alpha_G(v_G^{LR})_{II}(\cdot) + \alpha_B(v_E^{ED})_{II}(\cdot)] \times \cdots \]

\[\cdots \times [\alpha_G(v_G^{LR})_{DD}(\cdot) + \alpha_B(v_E^{ED})_{DD}(\cdot)]

- [\alpha_G(v_G^{LR})_{DI}(\cdot) + \alpha_B(v_E^{ED})_{DI}(\cdot)] \times \cdots \]

\[\cdots \times [\alpha_G(v_G^{LR})_{ID}(\cdot) + \alpha_B(v_E^{ED})_{ID}(\cdot)]

= \alpha_G^2[(v_G^{LR})_{II}(\cdot)(v_G^{LR})_{DD}(\cdot) - (v_G^{LR})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)]

+ \alpha_B^2[(v_G^{LR})_{II}(\cdot)(v_G^{LR})_{DD}(\cdot) - (v_G^{LR})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)]

+ \alpha_G\alpha_B[(v_G^{LR})_{II}(\cdot)(v_E^{ED})_{DD}(\cdot) - (v_E^{ED})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)]

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are strictly positive. On this front, I first note that so it will suffice if we can show that all of the terms in square brackets after the last equality are strictly positive. On this front, I first note that

\[
(v_G^{LR})_{II}(\cdot)(v_G^{LR})_{DD}(\cdot) - (v_G^{LR})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)
= (v_G^{LR})_{II}(\cdot)[\theta_G^{LR}D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot) - \Delta \mu''(\cdot)] - (v_G^{LR})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)
> (v_G^{LR})_{II}(\cdot)[\theta_G^{LR}D(\cdot)(\Pi'_G/\Psi'_G)'(\cdot) - (v_G^{LR})_{DI}(\cdot)(v_G^{LR})_{ID}(\cdot)
= (-1)[1 - \ell + \Psi_G(\cdot)][(\theta_G^{LR})_{II}(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)(\theta_G^{LR})_{DI}(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)
- (\theta_G^{LR})_{II}(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)(1 - \ell + \Psi_G(\cdot))(\theta_G^{LR})_{DI}(\cdot)(\Pi'_G/\Psi'_G)'(\cdot)
= 0.
\]

Similarly,

\[
(v_B^{ED})_{II}(\cdot)(v_B^{ED})_{DD}(\cdot) - (v_B^{ED})_{DI}(\cdot)(v_B^{ED})_{ID}(\cdot)
= (v_B^{ED})_{II}(\cdot)(-1)\Delta \mu''(\cdot)[1 - (\Pi'_B/\Delta'_B)(\cdot)]
- (v_B^{ED})_{II}(\cdot)[1 - \Delta \mu'(\cdot)](\theta_B^{ED})_{DI}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot) - (v_B^{ED})_{DI}(\cdot)(v_B^{ED})_{ID}(\cdot)
> (-1)(v_B^{ED})_{II}(\cdot)[1 - \Delta \mu'(\cdot)](\theta_B^{ED})_{DI}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot) - (v_B^{ED})_{DI}(\cdot)(v_B^{ED})_{ID}(\cdot)
= (-1)[1 - \ell - \Delta_B(\cdot)][(\theta_B^{ED})_{II}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)[1 - \Delta \mu'(\cdot)](\theta_B^{ED})_{DI}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)
+ [1 - \Delta \mu'(\cdot)](\theta_B^{ED})_{II}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)[1 - \ell - \Delta_B(\cdot)](\theta_B^{ED})_{DI}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)
= 0
\]
Finally,

\[
(v_{LR}^{II}(\cdot)) (v_{ED}^{DD}(\cdot)) - (v_{BD}^{ED}(\cdot)) (v_{LR}^{ID}(\cdot)) > 0,
\]

and

\[
(v_{LR}^{DD}(\cdot)) (v_{BD}^{ED}(\cdot)) - (v_{BD}^{ED}(\cdot)) (v_{LR}^{ID}(\cdot)) > 0,
\]

as desired.

**Case three.** Now suppose that \(I_0\) has the property that the pair \([D^*(I_0), I_0]\) lies in the region where \(r_B = DD\), but the “no-default” constraint is lax. In this case, the desired monotonicity is still equivalent to the second-order condition

\[
v_{II}[D^*(I_0), I_0] v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0] v_{ID}[D^*(I_0), I_0] > 0,
\]

and banks’ behaviour in the good state is still unchanged. However, in the bad state, we now have

\[
(v_{BD}^{DD})_I[D^*(I_0), I_0] = (-1)(1 - \ell)[1 - (\Pi_B/\Delta_B)(\theta_B^\Xi)] < 0,
\]

and

\[
(v_{BD}^{DD})_I[D^*(I_0), I_0] = [1 - \Delta \mu'[D^*(I_0)][1 - (\Pi_B/\Delta_B)(\theta_B^\Xi)],
\]

so

\[
(v_{BD}^{DD})_{II}[D^*(I_0), I_0] = (v_{BD}^{DD})_{ID}[D^*(I_0), I_0] = 0,
\]

and

\[
(v_{BD}^{DD})_{DD}[D^*(I_0), I_0] = (-1)\Delta \mu''[D^*(I_0)][1 - (\Pi_B/\Delta_B)(\theta_B^\Xi)] < 0 = (v_{BD}^{DD})_{DI}[D^*(I_0), I_0].
\]

So,

\[
v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
= v_{II}(\cdot)[\alpha_G(v_{LR}^{GG})_{DD}(\cdot) + \alpha_B(v_{BD}^{DD})_{DD}(\cdot)] - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
> v_{II}(\cdot)\alpha_G(v_{LR}^{LR})_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
= v_{II}(\cdot)\alpha_G(\theta_{LR}^{\Gamma}D(\cdot)(\Pi_G^\prime/\Psi_G)(\cdot) - \Delta \mu''(\cdot)) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

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\[ v_H(\cdot)\alpha_G(\theta_{LR}^G)_{D}(\cdot)(\Pi'_{G}/\Psi'_{G})'(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot) \]

\[ = \alpha_G(-1)[1 - \ell + \Psi_G(\cdot)](\theta_{LR}^G)_{I}(\cdot)(\Pi'_{G}/\Psi'_{G})'(\cdot)\alpha_G(\theta_{LR}^G)_{D}(\cdot)(\Pi'_{G}/\Psi'_{G})'(\cdot) \]

\[ - \alpha_G(\theta_{LR}^G)_{I}(\cdot)(\Pi'_{G}/\Psi'_{G})'(\cdot)\alpha_G(-1)[1 - \ell + \Psi_G(\cdot)](\theta_{LR}^G)_{D}(\cdot)(\Pi'_{G}/\Psi'_{G})'(\cdot) \]

\[ = 0, \]

as desired.

**Case four.** Suppose finally that \( I_0 \) has the property that “no-default” constraint binds at \([D^*(I_0), I_0]\). In this case, we have

\[ h(I_0) = \alpha_G \left[ (v_{LR}^G)_I \left[ D_B(I_0), I_0 \right] + D'_B(I_0)(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right] \right] \]

and need to confirm that this object is strictly decreasing in \( I_0 \). Now,

\[ h'(I_0) \propto (v_{LR}^G)_I \left[ D_B(I_0), I_0 \right] + D'_B(I_0)(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right]\]

\[ + D''_B(I_0)(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right] + D'_B(I_0)(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right] \]

\[ + (D'_B(I_0))^2(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right]\]

\[ < D''_B(I_0)(v_{LR}^G)_D \left[ D_B(I_0), I_0 \right] \]

\[ = \frac{D'_B(I_0)\Delta\mu'' \left[ D_B(I_0) \right] (1 - \ell)}{[1 - \Delta\mu' \left[ D_B(I_0) \right]]} (v_{LR}^G)_D \left[ D_B(I_0), I_0 \right] \]
so it would suffice if we could show that $(v_G^{LR})_D [D_B(I_0), I_0] > 0$. To see that this is indeed the case, recall that a binding “no-default” constraint implies

$$v_D [D_B(I_0), I_0] = \alpha_G (v_G^{LR})_D [D_B(I_0), I_0] + \alpha_B (v_B^{DD})_D [D_B(I_0), I_0] > 0.$$  

1.A.5.4 Dependence on parameters

In this last subsection, I show how the solution for \((\mathbb{P}\text{-}rex)\) responds to changes in the model’s parameters. As explained in the main text, the particular parameters on which I focus are the probability on the good state, \(\alpha_G\); the payout that successful projects generate in this state, \(\chi_G\); and the endowment that banks receive at \(t = 0\), \(E^b\). I let \(\beta := (\alpha_G, \chi_G, E^b)\) collect these parameters and will now include it as an explicit argument in any functions into which these parameters enter.

Now, based on the analysis in my previous subsection, it should be clear that

$$h [I_B^H(\beta), \beta] > [I_B^E(\beta), \beta] > h [I_B(\beta), \beta],$$

and also that the form of the solution for \((\mathbb{P}\text{-}rex)\) depends on where zero lies in this chain of inequalities. More specifically, the solution has the property that a liquidity surplus occurs in the bad state i.f.f. \(h [I_B^H(\beta), \beta] \leq 0\), while the extensive distortion regime obtains i.f.f. \(h [I_B^E(\beta), \beta] \leq 0 < h [I_B^H(\beta), \beta]\). Moreover, in the case of dual distortions — i.e., \(h [I_B^E(\beta), \beta] > 0\) —, we know that the “no-default” constraint binds i.f.f. \(h [I_B(\beta), \beta] > 0\).

So, our goal in this subsection will be to determine how parameters influence the signs on \(h [I_B^H(\beta), \beta], h [I_B^E(\beta), \beta]\), and \(h [I_B(\beta), \beta]\).

I’ll begin with my attention on \(h [I_B^H(\beta), \beta]\). In particular, I define \(D_B^H(\beta) := D^* [I_B^H(\beta), \beta],\) so that the pair \([D_B^H(\beta), I_B^H(\beta)]\) satisfies

$$[E^b + D_B^H(\beta) - I_B^H(\beta)] + \ell I_B^H(\beta) + I_B^H(\beta) \Delta_B (\theta_B^H) = \Delta \mu [D_B^H(\beta)]$$

\[\iff E^b + D_B^H(\beta) - \Delta \mu [D_B^H(\beta)] = I_B^H(\beta) [1 - \ell - \Delta_B (\theta_B^H)], \quad (1.18)\]

and

$$\alpha_G (v_G^{LR})_D [D_B^H(\beta), I_B^H(\beta), \beta] + (1 - \alpha_G) (v_G^{LS})_D [D_B^H(\beta), I_B^H(\beta), \beta] = 0$$

\[\iff 1 - \Delta \mu' [D_B^H(\beta)] + \alpha_G (\Pi_G/\Psi_G) [\theta_G^{LR} [D_B^H(\beta), I_B^H(\beta), \beta], \beta] = 0, \quad (1.19)\]
with

\[
E^b + D_B^H(\beta) - I_B^H(\beta)] + \ell I_B^H(\beta) = I_B^H(\beta)\Psi_G \left[ \theta_G^{LR} \left[ D_B^H(\beta), I_B^H(\beta), \beta \right] \right]
\]

\[\iff E^b + D_B^H(\beta) = I_B^H(\beta) [1 - \ell + \Psi_G \left[ \theta_G^{LR} [D_B^H(\beta), I_B^H(\beta), \beta] \right] \right] \cdot (1.20)\]

Now, dividing 1.18 by 1.20 yields

\[
\left[ 1 - \frac{\Delta \mu \left[ D_B^H(\beta) \right]}{E^b + D_B^H(\beta)} \right] \left[ 1 - \ell + \Psi_G \left[ \theta_G^{LR} [D_B^H(\beta), I_B^H(\beta), \beta] \right] \right] = 1 - \ell - \Delta_B (\theta_B^H), \quad (1.21)
\]

with

\[
g_D \left[ D_B^H(\beta), \beta \right] \propto \Delta \mu' [E^h - D_B^H(\beta)][E^b + D_B^H(\beta)] - \Delta \mu \left[ D_B^H(\beta) \right]
\]

\[
= \mu' [E^h - D_B^H(\beta)][E^b + D_B^H(\beta)] - [\mu(E^h) - \mu(E^h - D_B^H(\beta))]
\]

\[
> \mu' [E^h - D_B^H(\beta)] D_B^H(\beta) - [\mu(E^h) - \mu(E^h - D_B^H(\beta))]
\]

\[
\propto \mu' [E^h - D_B^H(\beta)] - \frac{\mu(E^h) - \mu(E^h - D_B^H(\beta))}{D_B^H(\beta)}
\]

\[
> 0,
\]

where the last inequality follows from concavity of the function \( \mu(\cdot) \). Differentiating lines 1.19 and 1.21 w.r.t. \( \chi_G \) then yields the following system, where I use the symbol \( \cdot \) to suppress arguments:

\[
\frac{d}{d\chi_G} \left[ \theta_G^{LR} (\cdot) \right] \left[ (\Pi/G/\Psi_G)_\theta^H (\cdot) \right] + \left[ (\Pi/G/\Psi_G)_\chi_G^H (\cdot) \right] = (D_B^H)_{\chi_G} (\beta) (1/\alpha_G) \Delta \mu'' (\cdot)
\]

\[
\frac{d}{d\chi_G} \left[ \theta_G^{LR} (\cdot) \right] \left[ 1 - g(\cdot) \right] \Psi_G (\cdot) = (D_B^H)_{\chi_G} (\beta) g_D (\cdot) [1 - \ell + \Psi_G (\cdot)]
\]
Figure 1.8: Visual aid for proving existence and uniqueness
An application of Cramer’s rule then yields

\[
\frac{d}{d\chi_G} \left[ \theta^{LR}_G (\cdot) \right] < 0.
\]

Similar arguments yield

\[
\frac{d}{d\alpha_G} \left[ \theta^{LR}_G (\cdot) \right] < 0 < \frac{d}{dE'} \left[ \theta^{LR}_G (\cdot) \right].
\]

Moreover,

\[
h \left[ I^H_B(\beta), \beta \right] = \alpha_G (v^{LR}_G)_I \left[ D^H_B(\beta), I^H_B(\beta), \beta \right] \\
+ (1 - \alpha_G) (v^{LS}_B)_I \left[ D^H_B(\beta), I^H_B(\beta), \beta \right]
\]

\[
\propto \ell + \Pi_G \left[ \theta^{LR}_G \left[ D^H_B(\beta), I^H_B(\beta), \beta \right], \beta \right] - 1
\]

\[
- \left[ 1 - \ell + \Pi_G \left[ \theta^{LR}_G \left[ D^H_B(\beta), I^H_B(\beta), \beta \right] \right] \right] \times \cdots
\]

\[
\cdots \times (\Pi'_G/\Psi'_G) \left[ \theta^{LR}_G \left[ D^H_B(\beta), I^H_B(\beta), \beta \right], \beta \right]
\]

\[
+ [(1 - \alpha_G)/\alpha_G] (v^{LS}_B)_I \left[ D^H_B(\beta), I^H_B(\beta), \beta \right]
\]

\[
= \ell + \int_{\theta^{LR}_G[D^H_B(\beta), I^H_B(\beta), \beta]}^1 (\chi_G \theta - \rho - \ell) dF(\theta) - 1
\]

\[
- \left[ 1 - \ell + \int_{\theta^{LR}_G[D^H_B(\beta), I^H_B(\beta), \beta]}^1 (\rho + \ell) dF(\theta) \right] \times \cdots
\]

\[
\cdots \times \left[ \frac{\chi_G \theta^{LR}_G[D^H_B(\beta), I^H_B(\beta), \beta]}{\rho + \ell} - 1 \right]
\]

\[
+ [(1 - \alpha_G)/\alpha_G] (v^{LS}_B)_I \left[ D^H_B(\beta), I^H_B(\beta), \beta \right]
\]
\[
\begin{align*}
\int_{\theta_G}^{\theta_R} & \left[ \theta \chi_G \right] \, \theta dF(\theta) \\
& - \left[ 1 - \ell + \int_{\theta_G}^{\theta_R} \left( \rho + \ell \right) dF(\theta) \right] \times \ldots \\
& \ldots \times \left( \frac{\theta_G^{LR} \left[ D_B^H(\beta), \Pi_B^H(\beta), \beta \right]}{\rho + \ell} \right) \\
& + \left[ \left( 1 - \alpha_G \right) / \alpha_G \right] \left( v_{LS}^B \right) I \left[ D_B^H(\beta), \Pi_B^H(\beta), \beta \right] \tag{1.25}
\end{align*}
\]
\[ \tilde{h}(\beta). \quad (1.28) \]

It’s then useful to include in assumption 1.4 a requirement that \( \chi_B \) be small enough that the average bank fails to break even in the bad state:

**Subassumption 1.4.3.** \( \ell + \Pi_B(\theta_B^H) < 1. \)

Under this condition, we get

\[
\tilde{h}_{\chi_G}(\beta) = \frac{d}{d\chi_G} \left[ \theta^L_G \left[ D_B^H(\beta), I_B^H(\beta), \beta \right] \right] \left( \frac{-1}{\rho + \ell} \right) \times \ldots
\]

\[
\ldots \times \left[ 1 - \ell + \int_{\theta_G}^{1} \frac{\rho + \ell}{\theta_G} dF(\theta) \right]
\]

\[
= \frac{(1 - \alpha_G) \left[ \ell + \Pi_B(\theta_B^H) - 1 \right]}{\alpha G \chi_G^2}
\]

\[
> 0.
\]

Repeating for \( \alpha_G \) and \( E_b^b \) then yields

\[
\tilde{h}_{\alpha_G}(\beta) > 0 > \tilde{h}_{E_b}(\beta),
\]

so we can conclude that \( h \left[ I_B^H(\beta), \beta \right] \) is more likely to be positive the greater is \( \chi_G \), the greater is \( \alpha_G \), and the lesser is \( E_b \). That these tendencies also hold for \( h \left[ I_B^H(\beta), \beta \right] \) and \( h \left[ \Pi_B(\beta), \beta \right] \) can be shown using essentially the same arguments. The partitions in propositions 1.2 and 1.3 can then be constructed as follows. Fix all parameters save for the payout \( \chi_G \), and recall that \( \chi_G \) gives the lower bound on this payout at which my parametric assumptions begin to fail. In light of the analysis above, we know that \( h \left[ I_B^H(\beta), \beta \right] \) either is single-signed over the interval \((\chi_G, \infty)\) or otherwise exhibits single-crossing from below over this interval. In the latter case, I let \( \chi_G^2 \) denote the point the crossing point. If instead we have uniform positivity (negativity), I set \( \chi_G^2 = \chi_G \) \((\chi_G = \infty)\). The rest of the partition can be constructed on a mutatis mutandis basis, and similar constructions go through w.r.t. \( \alpha_G \) and \( E_b \).
1.A.5.5 Generalization to program \((P)\)

I next argue that the solution for program \((\mathbb{P}\text{-rex})\) also solves program \((P)\). To do this, I’ll verify three constraints: the physical constraint associated with the bad state, \((PC_B)\); the financial constraint associated with the good state, \((FC_G)\); and non-negativity constraint \((F0)\).

Details on \((PC_B)\). In the bad state, the financial and physical constraints respectively read as follows:

\[
(E^b + D - I_0) + \ell I_0 + I_B\Delta_B(\theta_B) \geq \Delta \mu(D)
\]

\[
\iff (E^b + D - I_0) + \ell I_0 \geq \Delta \mu(D) - I_B\Delta_B(\theta_B)
\]

\[
(E^b + D - I_0) + \ell I_0 \geq I_B\Psi(\theta_B)
\]

Comparing these two inequalities, we see that a sufficient condition for the financial constraint’s being stronger would be that \(\Psi(\theta_B) + \Delta_B(\theta_B) < 0\). Now,

\[
\Psi(\theta_B) + \Delta_B(\theta_B) = \int_{\theta_B}^{1} (\rho + \ell) dF(\theta) + \int_{\theta_B}^{1} (\theta \chi_B - \rho - \ell) dF(\theta)
\]

\[
- \gamma \chi_B \left[ \theta_B F(\theta_B) + \int_{\theta_B}^{1} \theta dF(\theta) \right]
\]

\[
= \chi_B \left[ \int_{\theta_B}^{1} (1 - \gamma) \theta dF(\theta) - \gamma \theta_B F(\theta_B) \right]. \quad (1.29)
\]

It should be clear that the expression on line 1.29 exhibits single-crossing from above over the interval \([0, 1]\), namely at an interior point which I’ll denote \(\theta_B^{\Psi+\Delta}\). Since banks’ choices on \(\theta_B\) under program \((\mathbb{P}\text{-rex})\) are bounded from below by \(\theta_B^{\Xi}\), it would thus suffice for assumption 1.4 to include a requirement that \(\chi_B\) be small enough that the following condition obtains:

**Subassumption 1.4.4.** \(\theta_B^{\Xi} > \theta_B^{\Psi+\Delta}\).

Details on \((FC_G)\) and \((F0)\). Suppose that the solution for program \((\mathbb{P}\text{-rex})\) has the property that \(I_0 \leq \bar{I}_B\). In this case, we know that the choice on \(I_0\) satisfies the following first-order
condition:

\[ \alpha_G(v^L_G)I(D, I_0) + \alpha_B(v^R_B)I(D, I_0) \]

\[ = \alpha_G(v^L_G)I(D, I_0) + \alpha_B \begin{cases} 
\ell + \Pi_B(\theta^R_B) - 1 & \text{if } r_B = LS \\
\ell + \Pi_B[\theta^E_B(D, I_0)] - 1 & \text{if } r_B = ED \\
(-1)(1 - \ell)\left[1 - (\Pi_B/\Delta_B)(\theta^R_B)\right] & \text{if } r_B = DD
\end{cases} \]

Now, in light of subassumption 1.4.3, it should be clear that \((v^R_B)I(D, I_0) < 0\), so it must be the case that \((v^L_G)I(D, I_0) > 0\). In fact, this last inequality also holds when \(I_0 > I_B\), namely because the relevant first-order condition then reads as

\[ (v^L_G)I(D, I_0) + \left[\frac{1 - \ell}{1 - \Delta \mu'(D)}\right](v^L_G)D(D, I_0) = 0, \]

and I’ve already argued that the starred term is strictly negative (see case four in subsubsection 1.A.5.3).

At this point, it’s useful to recall from the analysis on lines 1.22 through 1.27 that

\[ (v^L_G)I(D, I_0) \propto \int_{\theta^L_G(D, I_0)}^1 \theta dF(\theta) - \left[1 - \ell + \int_{\theta^L_G(D, I_0)}^1 (\rho + \ell) dF(\theta)\right] \left[\frac{\theta^L_G(D, I_0)}{\rho + \ell}\right] \]

\[ =: \zeta_G[\theta^L_G(D, I_0)] \]

Re-writing in this way is useful because it should be clear that the function \(\zeta_G(\cdot)\) exhibits single-crossing from above over the the interval \([0, 1]\), namely at some interior point which I’ll denote \(\theta^\zeta_G\). Conclude that \(\theta^L_G(D, I_0) < \theta^\zeta_G\).

Now, I claim that it will suffice if assumption 1.4 includes a requirement that \(\chi_G\) be large enough that the following condition obtains:
**Subassumption 1.4.5.** \( \min \{ \Delta_G(\theta^c_G), \rho - (\rho + \ell)F(\theta^c_G) \} > 0. \)

The argument w.r.t. (FC\(_G\)) is as follows. We know that the per-unit pledgable surplus function \( \Delta_G(\cdot) \) achieves a peak at \( \theta^\Delta_G \) and strictly decreases thereafter. Moreover, based on the analysis in my previous paragraph, we know that \( \theta^\Delta_G(D, I_0) \in (\theta^H_G, \theta^c_G) \). Since \( \theta^\Delta_G < \theta^H_G \), the subassumption above ensures that \( \Delta_G[\theta^L R_G(D, I_0)] > 0. \) On the other hand, subassumption 1.4.1 ensures that pledgable surplus is negative in the bad state, so it must be the case that

\[
(E^b + D - I_0) + \ell I_0 + I_0 \Delta_G \left[ \theta^L R_G(D, I_0) \right] > (E^b + D - I_0) + \ell I_0 + I_B \Delta_B(\theta_B) \geq \Delta \mu(D),
\]

where the last inequality follows from the fact that the financial constraint associated with the bad state is sure to hold under program (P-rex).

On the other hand, the argument w.r.t \( (F0) \) is as follows. In light of assumption 1.5 combined with the fact the \( \mu'(E^b) = 1 \), it should be clear that there’s only one inequality in \( (F0) \) that we have to worry about, namely the non-negativity of storage: \( I_0 \leq E^b + D. \) To see that this inequality indeed holds, recall that the physical constraint in the good state always binds under program (P-rex) and can be re-written as

\[
\frac{E^b + D - I_0}{I_0} = \Psi_G \left[ \theta^L R_G(D, I_0) \right] - \ell = \int_{\theta^L R_G(D, I_0)}^{1} (\rho + \ell) dF(\theta) - \ell = \rho - (\rho + \ell) F \left[ \theta^L R_G(D, I_0) \right] > 0,
\]

where the inequality follows from subassumption 1.4.5.

**1.A.6 Proof of proposition 1.4**

This should now be obvious.
Chapter 2

Rethinking interbank regulation

2.1 Introduction

In this next chapter, I turn my attention to policy, namely by expanding the model presented in chapter 1 to include a fire-sale externality. Externalities of this form have emerged as a standard source of constrained inefficiency in the post-crisis policy literature (e.g., Lorenzoni, 2008; Stein, 2012; Korinek, 2012; Gersbach and Rochet, 2012a,b), largely due to evidence that banks “crowded the exit” in various asset markets at the height of the crisis (e.g., Brunnermeier, 2009; Krishnamurthy, 2010; Merrill et al., 2012). Moreover, recent simulations by Greenwood et al. (2012) and Duarte and Eisenbach (2013) indicate that fire sales, once under way, can have a substantial effect on banks’ financial health. For these reasons, the expanded model developed in this chapter serves as a useful laboratory for studying optimal policy in interbank markets.

The main motivation for this exercise has to do with a gap in the current literature on interbank regulation. In particular, this literature has to date focused mostly on ex-post policies which kick in after an interbank disruption has come under way — e.g., open-market operations, lender-of-last-resort interventions, bail-outs. In contrast, relatively little attention has been paid to policies which kick in at an ex-ante stage, including the new prudential policies now coming online as part of Basel III. Since interbank markets represent a major source of liquidity for banks, this omission is especially problematic in the case of prudential policies which constrain banks’ liquidity-management practices, such as the Basel III liquidity coverage ratio, which places a lower bound on banks’ liquid asset holdings. A full understanding of how these policies should be designed and implemented thus hinges in part on our anticipating their potential impact on interbank markets.

For an overview of Basel III, see Basel Committee on Banking Supervision (2010, 2013).
To get a bit more specific about this model, its overall structure comes from chapter 1 with the only point of departure having to do with the liquidation of banks’ unmaintained investments at the interim date \( t = 1 \). In particular, chapter 1 assumed that banks had access to a second-best technology with which they could extract some small payout \( \ell \) from each of their unmaintained investments. In contrast, I now suppose that banks have no use to which they can put their unmaintained investments. Instead, I endow households with a salvaging technology with which these investments can be converted back into consumption goods. I also allow for a secondary market in which these investments change hands on a competitive basis, so \( \ell \) is now endogenized as a market-clearing price.

Despite the fact that these changes make the model significantly more difficult to solve, several of the basic results established in chapter 1 still hold. In particular, I show that the model still admits episodes which qualitatively resemble the interbank disruptions witnessed during the recent crisis, with liquidity locked up on unproductive banks’ balance sheets and the set of interbank debtors polluted by a subset of very low-productivity types. I also show that these episodes are more likely to occur when expected productivity at \( t = 0 \) is relatively high, making the overall risk of interbank disruptions procyclical. Moreover, the normative “no-go” result established in chapter 1 still holds in some sense. More specifically, I show that the parameter space stills admits regions in which the occurrence of interbank disruptions is constrained efficient, though banks’ failure to internalize their influence on secondary-market prices implies that policy now has an important role to play in limiting the severity of these disruptions.

The details on this role for policy constitute this chapter’s main findings. On this front, I show that banks tend to liquidate too heavily at \( t = 1 \), relative to the allocation preferred by a utilitarian planner, and also that the initial balance sheets on which banks settle at \( t = 0 \) are too “fragile” in a sense that I make precise in the sequel. Correcting these two tendencies jointly requires careful coordination between an ex-post intervention at \( t = 1 \) and an ex-ante intervention at \( t = 0 \). In particular, I show that the appropriate ex-post intervention admits a natural interpretation as a monetary stimulus aiming to reduce the effective rate at which banks store goods between \( t = 1 \) and \( t = 2 \), thus discouraging them from liquidating their investments in favour of storage (proposition 2.1). In contrast, two instruments are needed at \( t = 0 \), since this period presents banks with distinct choices on the size and composition of their initial balance sheets. In particular, I show that an appropriately specified liquidity coverage ratio will do the trick if supplemented with a limit on the leverage that banks are able to take on in the deposit market (propositions 2.2 and 2.3).

Crucially, it turns out that the ex-ante component does most of the heavy lifting. More specifically, I show that the aforementioned ex-ante interventions are always necessary and
sometimes even sufficient for implementation of a utilitarian planner’s solution (also propositions 2.2 and 2.3). The intuition for this somewhat surprising sufficiency is relatively straightforward once we recall from chapter 1 that banks tend to obey a pecking order during interbank disruptions. More specifically, the financial constraint that binds during these disruptions can be balanced along one of two margins: an intensive margin, which reduces the scale at which interbank debtors are allowed to keep operating, and an extensive margin, which allows more and more low-productivity types into the set of interbank debtors. In chapter 1, I showed that banks strictly prefer the latter until they reach a critical type at which they revert to the former rather than letting any more inferior types into the set of interbank debtors. In the sequel, I furthermore show that the planner obeys a similar pecking order. However, he opts to switch around a lower type since he understands that letting more types into the set of interbank debtors has the added benefit of reducing the volume of liquidations in the secondary market, thus taking some pressure off the price set therein. As a result, ex-post interventions of the sort described above are only needed in the case of an interbank disruption so severe that banks are forced to activate both margins, since in this case the planner must take steps to regulate the trade-off between these two margins. However, in the case of a less severe disruption during which banks only make moderate use of the extensive margin, their behaviour at $t = 1$ mechanically coincides with the planner’s. The latter case thus has the property that the role for policy is confined to $t = 0$, while the former case — if anything — leaves the need for ex-ante intervention enhanced by moral-hazard concerns. For these reasons, ex-ante interventions emerge as a qualitatively more important part of the policy mix.

In terms of related literature, the literature on interbank regulation is too vast to do it much justice here. Some recent contributions include Acharya et al. (2012), who consider an environment in which interbank inefficiencies stem from an assumption that banks have some market power, which leaves liquidity-rich banks with an incentive to engage in predatory hoarding so as to force liquidity-poor banks into fire sales. They show that an appropriate lender-of-last-resort policy can eliminate this incentive by providing liquidity-poor banks with a better bargaining position. On the other hand, Allen et al. (2009) show how constraints on the interbank contract space can give rise to excessive volatility in the interbank interest rate, which the central bank can correct using state-contingent open-market operations. Similar results hold in an extension by Freixas et al. (2011). Bruche and Suarez (2010) show how deposit insurance can give rise to distortions in an interbank market prone to counterparty risk. They argue that guarantees or subsidies for interbank lending can be used to correct these distortions, while Heider et al. (2009) identify a similar intervention as a potential
solution for inefficiencies stemming from asymmetric information between banks, along with a few alternative policies.

Since all these examples make ad-hoc assumptions on the interbank contract space and focus mostly on ex-post interventions in the interbank market, this chapter’s main contributions have to do with its emphasis on optimal contracts and ex-ante interventions. To some extent, the aforementioned work by Freixas et al. (2011) and related work by Kharroubi and Vidon (2009) also share the latter emphasis, since their policy analyses include a role for the policy rate between periods $t = 0$ and $t = 1$. However, these frameworks have the property that an appropriate choice on this rate need not be supplemented with any kind of prudential policies, leaving open the questions raised above concerning the potential usefulness of these policies and the task of coordinating them with the other parts of the policy framework. Moreover, the particular inefficiency on which Kharroubi and Vidon (2009) focus has to do with a coordination failure that leads banks to abstain from storage at $t = 0$, implying that there’s no liquidity for the interbank market to allocate at $t = 1$. In contrast, I focus on episodes during which liquidity exists inside the banking system but fails to reach its most productive use.

In addition to the literature on interbank regulation, this chapter’s findings also connect with three more parts of the post-crisis policy literature. First of all, my finding that the policy rate has a key role to play in ensuring financial stability reinforces an emerging view that financial stability and price stability cannot cleanly be separated as policy objectives (e.g., Stein, 2012, 2013; Brunnermeier and Sannikov, 2012, 2013, 2015). Secondly, my characterization of the optimal coordination between ex-post and ex-ante interventions extends a growing sub-literature on the relative merits of these two forms of intervention (e.g., Keister, 2010; Fahri and Tirole, 2012; Jeanne and Korinek, 2013; Stravrakeva, 2013; Chari and Kehoe, 2013). In particular, since this sub-literature has tended to the view main task for ex-ante intervention as the correction of the moral hazards arising from expectations of ex-post intervention, this chapter provides a potentially important counterexample in which

---

2 More specifically, Kharroubi and Vidon (2009) note that the liquid assets on banks’ balance sheets serve as a natural candidate to collateralize interbank lending. As a result, a self-fulfilling scenario can emerge in which a particular bank expects that his peers won’t hold any liquid assets and thus anticipates that he won’t be able to lend them any excess liquidity of his own due to lack of collateral, reducing his expected return on liquid assets to the point that he too refuses to hold them.

3 See Smets (2014) for an overview of the debate on explicitly incorporating financial-stability concerns into central banks’ mandates. Svensson (2014a,b) articulates a strongly opposing view, while Woodford (2013) advocates “flexible inflation-targeting” as a middle ground. See also Adrian and Shin (2008, 2009, 2010) for evidence on the links between the policy rate and the size of banks’ balance sheets, one of the key stylized facts underlying this debate. Complementary evidence comes from de Nicolo et al. (2010), Altunbas et al. (2011), Paligorova and Santos (2012), Dell’Arricia et al. (2013), Dreschler et al. (2014), and Jimenez et al. (2014), among others.
ex-ante interventions sometimes suffice to ensure constrained efficiency. Finally, this chapter’s findings connect with the current debate on introducing liquidity-based rules into the prudential toolkit, an area where regulatory practice is currently far out ahead of theory. More specifically, since a liquidity coverage ratio emerges as a key part of my implementation of the planner’s solution, this chapter provides a framework that can help to rationalize policies of this sort while providing some insight into how they might fit into the overall policy infrastructure.

The remainder of the chapter is organized as follows. In section 2.2 I present the model. In section 2.3 I then focus on the interim date $t = 1$, solving for banks’ behaviour in this period, along with the potential implications for policy. Section 2.4 repeats for the initial date $t = 0$ and provides a full characterization of the optimal policy mix. Section 2.5 concludes.

2.2 Model

2.2.1 Changes to the economic environment

The structure of the economy is unchanged relative to the baseline model from [1] with the exception that the liquidation value $\ell$ that banks are able to extract from their unmaintained investments is now endogenous and state-specific. To endogenize this payout, I take an approach similar to that in Lorenzoni (2008). More specifically, I assume that unmaintained investments are useless to banks, but may still be useful to households, each of whom owns a firm operating in a “traditional sector”, as distinct from the sector in which banks deploy their funds. At $t = 1$, these firms have access to a salvaging technology with which $L_\omega$ units of unmaintained investments can be converted back to into $\Lambda(L_\omega)$ units of consumption, where $\Lambda'(\cdot) > 0 > \Lambda''(\cdot)$, with $\Lambda'(0) \leq 1$, and $\lim_{L \to \infty} \{\Lambda(L)\} = 0$. The proceeds are then remitted to households as a dividend. Under the assumption that unmaintained investments trade hands in a competitive secondary market, liquidation values are thus pinned down by the first-order condition

$$\ell_\omega = \Lambda'(L_\omega).$$

See figure 2.1 for an illustration.

2.2.2 Definitions

In this environment, it will be useful for us to follow Freixas et al. (2011) in adopting a notion of “ex-ante” versus “ex-post” equilibrium, where the latter isolates banks’ behaviour
Figure 2.1: Banks’ investment technology (top) and firms’ salvaging technology (bottom) under the expanded model.
at $t = 1$ from their behaviour at $t = 0$. This will allow us to address the questions raised in my introduction concerning the relative merits of intervention at $t = 0$ versus $t = 1$.

Intuitively speaking, the relevant definitions are as follows. Given some state $\omega$, along with some initial balance sheet $(D, I_0, R)$, an ex-post equilibrium is a subcontract $C_\omega$ and secondary-market price $\ell_\omega$ such that (i) banks find it optimal to use the subcontract $C_\omega$ in state $\omega$, taking the price $\ell_\omega$ as given; (ii) the subcontract $C_\omega$ dictates liquidation decisions which cause the secondary market to clear at price $\ell_\omega$ in state $\omega$. More formally:

**Definition.** Given some state $\omega$ and an initial balance sheet $(D, I_0, R)$ satisfying the participation constraint for depositors, $(IR)$, an *ex-post equilibrium* is a subcontract $C_\omega$ and a secondary-market price $\ell_\omega$ such that the following two conditions hold:

- the subcontract $C_\omega$ maximizes

$$\int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta),$$

taking $\ell_\omega$ as given, subject to the following constraints:

$$S_\omega(\theta) - T_\omega f(\theta) - RD + \theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)]$$

$$\geq S_\omega(\theta') - T_\omega f(\theta') - RD + \theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')], \quad \forall (\theta, \theta') \in [0, 1]^2 \quad (TT_\omega)$$

$$\Delta T_\omega(\theta) \leq (1 - \gamma) \chi_\omega I_\omega(\theta), \quad \forall \theta \in [0, 1] \quad (LP_\omega)$$

$$(E^b + D - I_0) + \ell_\omega I_0 = \int_0^1 [S_\omega(\theta) + (\rho + \ell_\omega) I_\omega(\theta)] dF(\theta) \quad (F1a_\omega)$$

$$0 = \int_0^1 [T_\omega f(\theta) + \theta \Delta T_\omega(\theta)] dF(\theta) \quad (F2a_\omega)$$

$$S_\omega(\theta) \geq 0, \quad \forall \theta \in [0, 1] \quad (F1b_\omega)$$

$$S_\omega(\theta) \geq T_\omega f(\theta) + RD, \quad \forall \theta \in [0, 1] \quad (F2b_\omega)$$

$$S_\omega(\theta) + \chi_\omega I_\omega(\theta) \geq T_\omega f(\theta) + \Delta T_\omega(\theta) + RD, \quad \forall \theta \in [0, 1] \quad (F2c_\omega)$$
\[ I_\omega(\theta) \in [0, I_0], \ \forall \theta \in [0, 1] \quad \text{(F1c}_\omega) \]

\[ I_\omega(\theta) \implies \Delta T_\omega(\theta) = 0, \ \forall \theta \in [0, 1] \quad \text{(F2d}_\omega) \]

- the secondary market clears — i.e., \( \ell_\omega = \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]]. \)

In contrast, an ex-ante equilibrium is a full interbank contract \( C \) and a schedule of secondary-market prices \((\ell_B, \ell_G)\) such that (i) banks find it optimal to select the contract \( C \), taking the price schedule \((\ell_B, \ell_G)\) as given; (ii) in each state \( \omega \), the contract \( C \) dictates liquidation decisions which cause the secondary market to clear at price \( \ell_\omega \). More formally:

**Definition.** An *ex-ante equilibrium* is a contract \( C \) and a schedule of secondary-market prices \((\ell_B, \ell_G)\) such that

- the contract \( C \) maximizes
  
  \[ \sum_{\omega \in \{B,G\}} \alpha_\omega \int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) - RD, \]

  taking \((\ell_B, \ell_G)\) as given, subject to (TT), (LP), (IR), (F0), (F1a_B), (F1a_G), and (F1b) through (F2d);

- the secondary market clears in both states — i.e., \( \ell_\omega = \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]], \ \forall \omega \in \{B, G\}. \)

Moreover, an ex-ante equilibrium is *monotonic* if it satisfies \( \ell_G \geq \ell_B \) — i.e., more liquidations occur in the bad state.

When evaluating the efficiency of a particular equilibrium, the benchmark on which I focus is one under which the social planner internalizes the price-setting process but otherwise faces the same information and contracting frictions as do banks. More specifically:

**Definition.** The planner’s *ex-ante problem* involves choosing an interbank contract \( C \) and a price schedule \((\ell_B, \ell_G)\) so as to maximize utilitarian welfare,

\[ \sum_{\omega \in \{B,G\}} \alpha_\omega \left[ \Lambda \left[ I_0 - \int_0^1 I_\omega(\theta)dF(\theta) \right] - \ell_\omega \left[ I_0 - \int_0^1 I_\omega(\theta)dF(\theta) \right] \right. \]

\[ \left. + \int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) + \mu(E^h - D) \right], \]

subject to the same constraints facing banks in ex-ante equilibrium, along with the market-clearing condition \( \ell_\omega = \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]], \ \forall \omega \in \{B, G\}. \)
Similarly:

**Definition.** Given some state $\omega$, along with some initial balance sheet $(D, I_0, R)$ satisfying the participation constraint for depositors, $(IR)$, the planner’s ex-post problem involves choosing a subcontract $C_\omega$ and price $\ell_\omega$ so as to maximize

$$
\Lambda \left[ I_0 - \int_0^1 I_\omega(\theta)dF(\theta) \right] - \ell_\omega \left[ I_0 - \int_0^1 I_\omega(\theta)dF(\theta) \right] + \int_0^1 \left[ S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta) \right] dF(\theta),
$$

subject to the same constraints facing banks in ex-post equilibrium, along with the market-condition $\ell_\omega = \Lambda'\left[I_0 - I_\omega[1 - F(\theta_\omega)]\right]$.

**2.2.3 Parametric assumptions**

I’ll now close this section with my parametric assumptions. Most are inherited from chapter [1] with adjustments ensuring that they now hold over the full range of potential liquidation values:

**Assumption 2.1.** Investment at $t = 0$ is profitable — i.e.,

$$
\sum_{\omega \in \{B,G\}} \alpha_\omega \mathbb{E} \left[ \max \{0, \theta \chi_\omega - \rho\} \right] > 1.
$$

**Assumption 2.2.** Continuation at $t = 1$ is always profitable for some types — i.e., $\chi_B > \rho + \Lambda'(0)$.

**Assumption 2.3.** The bad state is “sufficiently bad”, namely in the sense that the payout $\chi_B$ satisfies an upper bound given in the appendix. Conversely, the good state is “sufficiently good”, namely in the sense that the payout $\chi_G$ satisfies a lower bound given in the appendix.

**Assumption 2.4.** Households are deep-pocketed in comparison with banks — specifically, $E^h + E^h - \Delta \mu(E^h) < 0$.

Apart from these assumptions, it will be useful to make some functional-form assumptions, beginning with the form of the salvaging technology:

**Assumption 2.5.** \( \frac{d}{dt} \left[ L\Lambda'(L) \right] > 0 > \frac{d^2}{dt^2} \left[ L\Lambda'(L) \right], \forall L \in \mathbb{R}_+ \).

In light of firms’ first-order condition, (2.1), we can read this assumption as a stipulation that the revenues raised in the secondary market should increase with the total volume of liquidations, but do so at a decreasing rate — e.g., $\Lambda(L) = \Lambda_0 \log(1 + L)$, with $\Lambda_0 \in (0, 1]$. It will also be useful to make a simplifying assumption on the distribution from which banks draw their types:
Assumption 2.6. The distribution $F$ is standard uniform.

2.3 Solution and optimal policy at $t = 1$

In this section, I fix banks’ initial balance-sheet choices and solve for the ex-post equilibria emerging in each state at $t = 1$. I also solve the planner’s ex-post problem in each state and explore the implications for policy. More specifically, subsection 2.3.1 focuses on the bad state, while subsection 2.3.2 repeats for the good state.

2.3.1 Details on the bad state

Since banks take prices as given, lemma 1.3.1 still holds as a description of their behaviour at $t = 1$. Their choices on the subcontract $C_\omega$ can thus be summarized by a pair $(\theta_\omega, I_\omega) \in [0, 1] \times [0, I_\omega]$, with the usual interpretation that all types in $[0, \theta_\omega)$ liquidate completely, whereas (almost) all others keep operating at scale $I_\omega$. More specifically, banks choose this pair to maximize the surplus generated by their maintained investments,

$$I_\omega \int_{\theta_\omega}^{1} (\theta \chi_\omega - \rho - \ell_\omega) dF(\theta) =: I_\omega \Pi_\omega(\theta_\omega, \ell_\omega),$$

subject to the usual physical and financial constraints:

$$(E^b + D - I_0) + \ell_\omega I_0 \geq \int_{\theta_\omega}^{1} (\rho + \ell_\omega) I_\omega dF(\theta) =: I_\omega \Psi_\omega(\theta_\omega, \ell_\omega) \quad \text{(PC}_\omega)$$

$$(E^b + D - I_0) + \ell_\omega I_0 \geq \Delta \mu(D) - I_\omega \left[ \Pi_\omega(\theta_\omega, \ell_\omega) - \theta_\omega \gamma \chi_\omega F(\theta) - \int_{\theta_\omega}^{1} \theta \gamma \chi_\omega dF(\theta) \right] \quad \text{(FC}_\omega)$$

To be clear, assumption 2.6 is stronger than necessary for all of my positive results and some of my normative results, most importantly including my implementation of the planner’s solution. More specifically, lemmata 2.3.1 through 2.3.8 and proposition 2.1 all go through so long that the cumulative function $F(\cdot)$ is log-concave, while lemmata 2.4.1-2.4.3 and propositions 2.2-2.3 go through if we further assume that $f(1) \geq 1$, which holds for a wide range of distributions, including the uniform distribution and appropriate truncations of the (log-)normal distribution. All of this section’s proofs have been constructed in a way which makes these weaker dependencies clear.
Moreover, the productivity differential across states (assumption 2.3) can be shown to ensure that the constraint banks have to worry about in the bad state is the financial constraint, rather than the physical one. As a result, when $\omega = B$, solutions for the program above must fall under one of the three regimes listed in proposition 1.1, namely liquidity surplus ($r_B = LS$), extensive distortion ($r_B = ED$), and dual distortion ($r_B = DD$).

So, when searching for ex-post equilibria in the bad state, we can restrict attention to candidates under which one of these three regimes obtains. I’ll begin with the case of a liquidity surplus. Under an ex-post equilibrium of this type, it should be clear that banks settle on a subcontract of the form $(\theta_B, I_B) = [\theta_B^{LS}(I_0), I_0]$, where $\theta_B^{LS}(I_0)$ satisfies

$$
\chi_B \theta_B^{LS}(I_0) = \rho + \Lambda' \left[I_0 F[\theta_B^{LS}(I_0)]\right] \quad (2.2)
$$

— that is, $\theta_B^{LS}(I_0)$ gives the type for whom continuation is NPV-neutral, after taking market-clearing into account. That this equation admits a unique solution should be obvious, so all that remains is to verify the financial constraint,

$$
(E^b + D - I_0) + I_0 \Lambda' \left[I_0 F[\theta_B^{LS}(I_0)]\right] \geq \Delta \mu(D) - I_0 \Delta_B \left[\theta_B^{LS}(I_0), \Lambda' \left[I_0 F[\theta_B^{LS}(I_0)]\right]\right]. \quad (2.3)
$$

In the appendix, I show that this inequality is more likely to hold when initial balance sheets are especially conservative, specifically in the sense that they exhibit high liquidity and low leverage. More precisely, the situation is as follows:

**Lemma 2.3.1.** $\forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0$, $\exists \theta_B^{LS}(D) \in \mathbb{R}_+$ s.t. (2.3) holds i.f.f. $I_0 \leq \overline{T}_B^{LS}(D)$, and in this case the bad state admits a unique ex-post equilibrium, namely of the “liquidity surplus” type. Moreover, this function is strictly decreasing, with $E^b + D - \Delta \mu(D) = 0 \implies \overline{T}_B^{LS}(D) = 0$.

Things are a bit more complicated when the distorted regimes obtain, namely because the financial constraint binds. Now, in chapter 1, I argued that banks have two ways to go about balancing a binding financial constraint: one option would be to reduce the marginal type $\theta_B$ (the extensive margin), and the alternative would be to reduce the investment scale $I_B$ (the intensive margin). I also argued that these two margins admit a strict pecking order, with banks preferring the former until $\theta_B$ falls below some critical type. As a result, an ex-post equilibrium of the “extensive distortion” type should have the property that banks are able to balance the financial constraint without driving $\theta_B$ so low that the intensive margin comes
on line — that is, \((\theta_B, I_B) = [\theta^E_B(D, I_0), I_0]\), with

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 F[\theta^E_B(D, I_0)]]
\]

\[
= \Delta \mu(D) - I_0 \Delta_B[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]],
\]

and

\[
(\Pi_B)_0[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]]
\]

\[
\leq \Pi_B[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]] \times \ldots
\]

\[
\ldots \times \left(\frac{\Delta_B}[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]}{\Delta_B[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]]}\right),
\]

where the inequality follows from the analysis in chapter [1]. On the other hand, an ex-post equilibrium of the “dual distortion” type would have \((\theta_B, I_B) = [\theta^D_B(D, I_0), I^D_B(D, I_0)]\), with

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]]
\]

\[
= \Delta \mu(D) - I^D_B(D, I_0) \Delta_B[\theta^D_B(D, I_0), \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]],
\]

and

\[
(\Pi_B)_0[\theta^D_B(D, I_0), \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]]]
\]

\[
= \Pi_B[\theta^D_B(D, I_0), \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]]] \times \ldots
\]

\[
\ldots \times \left(\frac{\Delta_B}[\theta^D_B(D, I_0), \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]]}{\Delta_B[\theta^D_B(D, I_0), \Lambda'[I_0 - I^D_B(D, I_0)[1 - F[\theta^D_B(D, I_0)]]]}\right).
\]

Now, in the appendix, I show that the former case is most likely to obtain when initial balance sheets exhibit moderate leverage and liquidity. More specifically:

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Lemma 2.3.2. \( \forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0, \exists! \mathcal{T}^{ED}_B(D) \in [I^{LS}_B(D), \infty) \text{ s.t. } (2.4) \) holds with equality when \( I_0 = \mathcal{T}^{ED}_B(D) \). Moreover, this function has the property that the bad state admits a unique ex-post equilibrium, namely of the “extensive distortion” type, whenever \( I_0 \in (T^{LS}_B(D), T^{ED}_B(D)] \). It’s also strictly decreasing, with \( E^b + D - \Delta \mu(D) = 0 \implies \mathcal{T}^{ED}_B(D) = 0. \)

On the other hand, if initial balance sheets exhibit higher leverage and lower liquidity, then the bad state either admits an ex-post equilibrium of the “dual distortion” type or otherwise fails to admit any ex-post equilibria, namely because the financial constraint fails under all candidates:

Lemma 2.3.3. \( \forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0, \exists! \mathcal{T}^{DD}_B(D) \in [I^{ED}_B(D), \infty) \text{ s.t. } \)

\[
[E^b + D - \mathcal{T}^{DD}_B(D)] + \mathcal{T}^{DD}_B(D) \Lambda'[\mathcal{T}^{DD}_B(D)] = \Delta \mu(D).
\]

Moreover, this function has the property that the bad state admits a unique ex-post equilibrium, namely of the “dual distortion” type, whenever \( I_0 \in (T^{ED}_B(D), T^{DD}_B(D)] \). It’s also strictly decreasing, with \( E^b + D - \Delta \mu(D) = 0 \implies \mathcal{T}^{DD}_B(D) = 0. \)

Lemma 2.3.4. If instead \( E^b + D - \Delta \mu(D) < 0 \), or \( E^b + D - \Delta \mu(D) \geq 0 \) with \( I_0 > \mathcal{T}^{DD}_B(D) \), then the bad state admits no ex-post equilibria of any type.

Efficiency and optimal policy. I next turn my attention to the economy’s efficiency in the bad state. Now, whenever the financial constraint is lax, it can easily be shown that the allocation obtaining in ex-post equilibrium also solves the planner’s ex-post problem, namely because the fire-sale externality remains dormant. However, once the financial constraint binds, this coincidence begins to break down. The intuition for this break-down hinges on our recognizing that the extensive and intensive margins discussed above have opposite implications for the price at which the secondary market clears and thus, by extension, the tightness of the financial constraint. In particular, since the extensive margin involves letting more types into the set of interbank debtors, it’s associated with less liquidations, a higher price, and greater slack in the financial constraint, while the intensive margin works in the opposite direction. As a result, we should expect the planner to lean more heavily on the extensive margin, relative to banks’ behaviour in ex-post equilibrium.

As a first step toward verifying this intuition, I note that the planner’s ex-post problem admits a reformulation similar to that obtaining for banks. In particular, the arguments underlying lemma 1.3.1 can be used to show that the planner’s ex-post problem in state
Figure 2.2: Banks’ behaviour in the bad state as a function of their initial balance-sheet choices
\( \omega \) amounts to a choice over the pair \((\theta_\omega, I_\omega) \in [0, 1] \times [0, I_0]\) with the usual interpretation that all types in \([0, \theta_\omega)\) shut down, whereas (almost) all others operate at scale \(I_\omega\). More specifically, the planner chooses this pair to maximize the surplus generated by both the economy’s technologies:

\[
\int_{\theta_\omega}^{1} (\theta \chi - \rho) I_\omega dF(\theta) + \Lambda[I_0 - I_\omega[1 - F(\theta_\omega)]],
\]

subject to the usual physical and financial constraints, evaluated at market-clearing prices:

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]] \geq I_\omega \Psi_\omega[\theta_\omega, \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]]] \quad (PC_{\omega}^{SP})
\]

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]] \geq \Delta \mu(D) - I_\omega \Delta \omega[\theta_\omega, \Lambda'[I_0 - I_\omega[1 - F(\theta_\omega)]]] \quad (FC_{\omega}^{SP})
\]

Moreover, if \(\omega = B\), then the productivity differential across states (assumption 2.3) ensures that the constraint that the planner has to worry about is the financial constraint, rather than the physical constraint.

Now, when the financial constraint binds, the rate of transformation along this constraint is given by

\[
dI_B \over d\theta_B = -I_B \left[ (\Delta_B)_{\theta}[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] + f(\theta_B)\delta(\theta_B, I_B, I_0) \right] \over \Delta_B[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] - [1 - F(\theta_B)]\Lambda(\theta_B, I_B, I_0) ,
\]

where

\[
\delta(\theta_B, I_B, I_0) := [I_0 + I_B(\Delta_B)_{\theta} \theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] \Lambda''[I_0 - I_B[1 - F(\theta_B)]]
\]

\[
= [I_0 - I_B[1 - F(\theta_B)]) \Lambda''[I_0 - I_B[1 - F(\theta_B)] < 0
\]

is a wedge distinguishing this rate from that perceived by banks — c.f. (1.3). The planner thus prefers the extensive margin so long that

---

5 The superscript “SP” stands for “social planner”.

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\[
\frac{dI_B}{d\theta_B} \left[ \int_{\theta_B}^{1} (\theta \chi_B - \rho) dF(\theta) - [1 - F(\theta_B)] \Lambda'[I_0 - I_B[1 - F(\theta_B)]] \right] = \Pi_B[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]]
\]

or equivalently

\[
\Pi_B[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] 
\leq \Pi_B[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] \times \cdots 
\]

\[
\cdots \times \frac{(\Delta_B)\theta[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] + f(\theta_B)\delta(\theta_B, I_B, I_0)}{\Delta_B[\theta_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] - [1 - F(\theta_B)]\delta(\theta_B, I_B, I_0)}. 
\tag{2.5}
\]

In the appendix, I show that this condition is weaker than the analogous condition for banks, confirming our intuition the planner should be inclined to rely more heavily on the extensive margin. More specifically, it can be shown that the situation is as follows:

**Lemma 2.3.5.** \(\forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0, \exists! T_B^{ED|SP}(D) \in [T_B^{ED}(D), T_B^{DD}(D)] \text{ s.t. } (2.5) \) holds with equality when \(\theta_B = \theta_B^{ED|SP}(D, I_0), \text{ with } I_B = I_0 = T_B^{ED|SP}(D). \) Moreover, this function has the following properties. If \(I_0 \in (T_B^{LS}(D), T_B^{ED|SP}(D)], \) then the planner’s ex-post problem in the bad state admits a unique solution, namely under which an extensive distortion occurs. On the other hand, if \(I_0 \in (T_B^{ED|SP}(D), T_B^{DD}(D)], \) then the planner’s ex-post problem still admits a unique solution, but this solution now has the property that a dual distortion occurs — specifically, \((\theta_B, I_B) = [\theta_B^{DD|SP}(D, I_0), I_B^{DD|SP}(D, I_0)], \) where this pair is pinned down by binding versions of \((2.5)\) and \((FC_B^{SP}). \) Also, \(T_B^{ED|SP}(D) \) is strictly decreasing, with \(E^b + D - \Delta \mu(D) = 0 \implies T_B^{ED|SP}(D) = 0.\)

**Lemma 2.3.6.** If instead \(E^b + D - \Delta \mu(D) < 0, \) or \(E^b + D - \Delta \mu(D) \geq 0 \text{ with } I_0 > T_B^{DD}(D), \) then the planner’s ex-post problem in the bad state is insoluble.

See figure 2.3 for an illustration.

In light of these last two lemmata, we see that there’s no need for policy intervention in the bad state when initial balance sheets are relatively liquid and unlevered. More specifically, if \(I_0 \leq T_B^{ED}(D), \) then the allocation obtaining in ex-post equilibrium coincides with the solution for the planner’s ex-post problem, either because the financial constraint is lax

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\( I_0 \leq T^L_B(D) \), or because it binds so weakly that the planner and banks both prefer to rely exclusively on the extensive margin \((T^L_B(D) < I_0 \leq T^E_B(D))\).

On the other hand, if \( I_0 \in (T^E_B(D), T^{DD}_B(D)] \), then one of two problems arises: it’s either the case that banks activate the intensive margin despite the planner’s preferring to rely only on the extensive margin \((T^E_B(D) < I_0 \leq T^{ED|SP}_B(D))\) or otherwise that both margins are active, but the planner prefers an allocation under which the extensive margin does more of the work \((T^{ED|SP}_B(D) < I_0 \leq T^{DD}_B(D))\). Either way, some kind of policy intervention is now needed. The appropriate intervention should have the property that it reduces the private return from liquidation and thus disincentivizes banks’ reliance on the intensive margin. On this front, I follow Fahri and Tirole (2012) in assuming that the policymaker is able to levy some tax \( \tau_B \) on banks’ storage at \( t = 1 \), and then remits the proceeds as a lump sum \( T_B \) at \( t = 2 \). We can think of this as a simple metaphor for monetary policies aiming to drive the interest rate beneath its natural level.\(^6\) In the appendix, I confirm that a policy of this form will indeed do the trick:

**Proposition 2.1** (optimal ex-post intervention). \( \forall (D,I_0) \in \mathbb{R}_+^2 \) s.t. \( E^b + D - D\mu'(E^h - D) \geq 0 \) and \( I_0 \leq T^{DD}_B(D) \), the tax \( \tau_B \) and transfer \( T_B \) can jointly be chosen to implement the solution for the planner’s ex-post problem in the bad state. Moreover, the appropriate choice satisfies \( \tau_B \geq 0 \), with strict inequality whenever \( I_0 > T^{ED}_B(D) \).

This proposition constitutes one of this chapter’s main results, specifically due to its implication that an instrument normally associated with price stability also has an important role to play in ensuring financial stability. As mentioned in the introduction, this reinforces a key theme in the post-crisis policy literature, namely that these two objectives cannot be separated cleanly.

### 2.3.2 Details on the good state

I now turn my attention to the good state. In this state, the productivity differential across states (assumption 2.3) now ensures that the constraint banks have to worry about is the physical constraint, rather than the financial one. As a result, ex-post equilibria must take one of two forms. The first would be the usual liquidity surplus scenario \((r_G = LS)\) under which banks are able to keep all NPV-positive types operating at full scale — i.e., \((I_G, \theta_G) = [I_0, \theta_{LS}^G(I_0)]\), with

\[
(E^b + D - I_0) + I_0\Lambda'[I_0 F[\theta_{LS}^G(I_0)]] \geq I_0\Psi_G[\theta_{LS}^G(I_0)], \Lambda'[I_0 F[\theta_{LS}^G(I_0)]]]. \tag{2.6}
\]

\(^6\) See section I.B in Fahri and Tirole (2012) for three concrete interpretations along these lines.
Figure 2.3: Planner’s behaviour in the bad state as a function of his initial balance-sheet choices
The alternative would be a liquidity rationing scenario \((r_G = LR)\) under which the physical constraint binds — specifically, \((I_G, \theta_G) = [I_0, \theta_{LR}^G(D, I_0)]\), where \(\theta_{LR}^G(D, I_0)\) solves

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 F[\theta_{LR}^G(D, I_0)]] = I_0 \Psi_G[\theta_{LR}^G(D, I_0), \Lambda'[I_0 F[\theta_{LR}^G(D, I_0)]]].
\]

Now, it should be clear that (2.6) is more likely to hold the lesser is \(I_0\) and the greater is \(D\), so we should expect the former case to obtain when banks raise lots of deposits but allocate most of them to storage. More specifically, it can be shown that the situation is as follows:

**Lemma 2.3.7.** For any initial balance sheet on to which banks would be willing to select in a monotonic equilibrium, the financial constraint associated with the good state is lax. As for the physical constraint, I note the following. \(\forall D \in \mathbb{R}_+, \exists ! \bar{T}_G^{LS}(D) \in \mathbb{R}_+\) s.t. (2.6) holds with equality when \(I_0 = \bar{T}_G^{LS}(D)\). If the balance sheet in question satisfies \(I_0 \leq \bar{T}_G^{LS}(D)\), then the good state admits a unique ex-post equilibrium, namely under which banks experience a liquidity surplus. If instead \(I_0 > \bar{T}_G^{LS}(D)\), then the good state still admits a unique ex-post equilibrium, but this ex-post equilibrium now has the property that banks experience liquidity rationing. Moreover, \((\bar{T}_G^{LS})'(D) > 0\), with \(\bar{T}_G^{LS}(0) < T_B^{LS}(0)\).

See figure 2.4 for an illustration.

Efficiency and optimal policy. As for the economy’s efficiency in this state, matters are somewhat simpler than in my previous subsection. In particular, it can be shown that the allocation described in lemma 2.3.7 also solves the planner’s ex-post problem. More precisely:

**Lemma 2.3.8.** For any initial balance sheet on to which the planner would be willing to select at \(t = 0\), the financial constraint associated with the good state is lax. As for the physical constraint, one of two cases must obtain. If the balance sheet in question satisfies \(I_0 \leq \bar{T}_G^{LS}(D)\), then the planner’s ex-post problem admits a unique solution, namely under which a liquidity surplus occurs — i.e., \((I_G, \theta_G) = [I_0, \theta_{LS}^G(I_0)]\). If instead \(I_0 > \bar{T}_G^{LS}(D)\), then the planner’s ex-post problem admits a unique solution, namely under which liquidity rationing occurs — i.e., \((I_G, \theta_G) = [I_0, \theta_{LR}^G(D, I_0)]\).

In the case of a liquidity surplus, this coincidence with the ex-post equilibrium allocation naturally follows from the physical constraint’s being lax. On the other hand, in the case of liquidity rationing, the intuition for this coincidence hinges on our recognizing that a binding physical constraint pins down the total volume of investments being maintained and thus, by extension, the total volume of liquidations taking place. This eliminates the only potential source of disagreement between the planner and banks. As a result, the model admits no role for policy in the good state: to the extent that interventions are needed at \(t = 1\), they should be confined to the bad state.
Figure 2.4: Banks’ and planner’s behaviour in the good state as a function of their initial balance-sheet choices
2.4 Solution and optimal policy at \( t = 0 \)

In this section, I finally shift my attention to banks’ behaviour at \( t = 0 \). On this front, most of my previous chapter’s insights still hold despite the present model’s being more complicated, including the general form of the solution on which banks settle, along with their tendency to accept a greater risk of interbank distortions when expected productivity is relatively high. More specifically, the situation is as follows:

**Lemma 2.4.1.** A monotonic equilibrium exists, is unique, and has the property that banks experience liquidity rationing in the good state. As for their behaviour in the bad state, one of four cases must obtain:

- the first has a liquidity surplus occurring in the bad state, with banks’ choices on \( D \) and \( I_0 \) respectively pinned down by the first-order conditions

\[
\alpha_G \left[ 1 - \Delta \mu'(D) + \frac{(\Pi_G \theta)[\theta^L_G(D, I_0), \Lambda'[I_0 F[\theta^L_G(D, I_0)]]]}{(\Psi_G \theta)(\cdot)} \right] + \alpha_B[1 - \Delta \mu'(D)] = 0,
\]

and

\[
\alpha_G \left[ \Lambda'(\cdot) + \Pi_G(\cdot) - 1 - [1 - \Lambda'(\cdot) + \Psi_G(\cdot)(\Pi_G \theta)(\cdot)] \right]
\]

\[
+ \alpha_B[\Lambda'[I_0 F[\theta^L_B(I_0)]] + \Pi_B[\theta^L_B(I_0), \Lambda'[I_0 F[\theta^L_B(I_0)]] - 1] = 0, \tag{2.7}
\]

where I use \( \cdot \) to suppress obvious arguments;

- the second case has an extensive distortion occurring in the bad state, so banks’ first-order conditions instead read as

\[
\alpha_G \left[ 1 - \Delta \mu'(D) + \frac{(\Pi_G \theta)[\theta^L_G(D, I_0), \Lambda'[I_0 F[\theta^L_G(D, I_0)]]]}{(\Psi_G \theta)(\cdot)} \right]
\]

\[
+ \alpha_B[1 - \Delta \mu'(D)] \left[ 1 - \frac{(\Pi_B \theta)[\theta^E_B(D, I_0), \Lambda'[I_0 F[\theta^E_B(D, I_0)]]]}{(\Delta_B \theta)(\cdot)} \right] = 0,
\]

---

NB: since banks take secondary-market prices as given, all of the first-order conditions given in this lemma take the same form as under the baseline model, except that we now evaluate at market-clearing prices. Compare with section 1.4 in particular.
and
\[
\alpha_G \left[ \Lambda' - \Pi_G - 1 - [1 - \Lambda' + \Psi_G] \left( \frac{\Pi_G}{\Psi_G} \right) \right] \\
\quad + \alpha_B \left[ \Lambda' - \Pi_B - 1 + [1 - \Lambda' - \Delta_B(\cdot)] \left( \frac{\Pi_B}{\Delta_B(\cdot)} \right) \right] = 0; \quad (2.8)
\]

- the third case has the property that a dual distortion occurs in the bad state, but the “no-default” constraint remains lax, so banks’ first-order conditions read as

\[
\alpha_G \left[ 1 - \Delta \mu'(D) + \left( \frac{\Pi_G}{\Psi_G} \right) \theta(\cdot) \right] + \alpha_B \left[ 1 - \Delta \mu'(D) \right] \times \cdots \\
\cdots \times \left[ 1 - \frac{\Pi_B}{\Delta_B(\cdot)} \right] = 0,
\]

and
\[
\alpha_G \left[ \Lambda' - \Pi_G - 1 - [1 - \Lambda' + \Psi_G] \left( \frac{\Pi_G}{\Psi_G} \right) \right] = \alpha_B \left[ 1 - \Lambda' \right] \left[ 1 - \frac{\Pi_B(\cdot)}{\Delta_B(\cdot)} \right];
\]

- the final case is an “interbank collapse” scenario under which a dual distortion occurs in the bad state, with the “no-default” constraint now binding, so banks’ initial balance-sheet choices are pinned down by this constraint, along with a first-order condition given in the appendix.

Moreover, non-monotonic equilibria do not exist.

**Lemma 2.4.2.** Fix all parameters save for the payout that successful banks generate in the good state, \( \chi_G \), and let \( \chi_G \) denote the lower bound on this payout at which my parametric assumptions begin to fail. The range of potential values for this payout then admits a partition
\[
\chi_G \leq \chi_G^{LS} \leq \chi_G^{ED} \leq \chi_G^{DD} \leq \infty
\]
with the property that banks behave as follows in ex-ante equilibrium:

- if \( \chi_G \in (\chi_G^{LS}, \chi_G^{DD}] \), then banks experience a liquidity surplus in the bad state;
- if \( \chi_G \in (\chi_G^{LS}, \chi_G^{ED}] \), then banks experience an extensive distortion in the bad state;
- if \( \chi_G \in (\chi_G^{ED}, \chi_G^{DD}] \), then banks experience a dual distortion in the bad state, but the “no-default” constraint remains lax;
• if \( \chi_G \in (\chi_G^{DD}, \infty) \), then an interbank collapse occurs in the bad state.

Similar results obtain if the parameter being varied is instead the probability on the good state, \( \alpha_G \).

See figure 2.5 for an illustration.

Efficiency and optimal policy. As for the planner’s ex-ante problem, its solution takes a similar form, though he adjusts his first-order conditions to take account of the price-setting process:

**Lemma 2.4.3.** The planner’s ex-ante problem admits a unique solution, namely under which liquidity rationing occurs in the good state. As for the bad state, one of four cases must obtain:

• the first has a liquidity surplus occurring in the bad state, with the planner’s choices on \( D \) and \( I_0 \) respectively pinned down by the first-order conditions

\[
\alpha_G \left[ 1 - \Delta \mu'(D) + \frac{(\Pi_G)_\theta[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]]]}{(\Psi_G)_\theta(\cdot) - [f(\theta^L_R(D, I_0))\delta(\theta^L_R(D, I_0), I_0, I_0)]} \right] + \alpha_B[1 - \Delta \mu'(D)] = 0,
\]

and

\[
\alpha_G \left[ \Lambda'(\cdot) + \Pi_G(\cdot) - 1 - \frac{[1 - \Lambda'(\cdot) + \Psi_G(\cdot) - F(\cdot)\delta(\cdot)](\Pi_G)_\theta(\cdot)}{(\Psi_G)_\theta(\cdot) - [f(\cdot)\delta(\cdot)]} \right]
\]

\[
+ \alpha_B[\Lambda'[I_0 F[\theta^L_S(I_0)]] + \Pi_B[\theta^L_S(I_0)], \Lambda'[I_0 F[\theta^L_S(I_0)]] - 1] = 0,
\] (2.9)

where I’ve highlighted the wedges that distinguish the planner’s first-order conditions from those obtaining in equilibrium\(^8\).

---

\(^8\) That wedges emerge in the good state may be surprising at first glance, since we know that the relevant constraint in this state has the flavour of a budget constraint, and budget constraints generally fail to effect pecuniary externalities. However, banks’ inability go short in the secondary market causes the usual logic to break down on this front.
Figure 2.5: Banks’ behaviour at $t = 0$ as a function of parameters

NB: $\alpha_G$ denotes the lower bound on $\alpha_G$ implied by my parametric assumptions.
the second case has an extensive distortion occurring in the bad state, with first-order conditions

$$\alpha_G \left[ 1 - \Delta \mu'(D) + \frac{(\Pi G)_\theta[\theta_{GR}^L(D, I_0), \Lambda'[I_0 F[\theta_{GR}^L(D, I_0)]]]}{(\Psi G)_\theta[\cdot] - f[\theta_{GR}^L(D, I_0)] \delta[\theta_{GR}^L(D, I_0), I_0, I_0]} \right] + \alpha_B [1 - \Delta \mu'(D)] \left[ 1 - \frac{(\Pi B)_\theta[\theta_{BD}^E(D, I_0), \Lambda'[I_0 F[\theta_{BD}^E(D, I_0)]]]}{(\Delta B)_\theta[\cdot] + f[\theta_{BD}^E(D, I_0)] \delta[\theta_{BD}^E(D, I_0), I_0, I_0]} \right] = 0,$$

and

$$\alpha_G \left[ \Lambda'(\cdot) + \Pi G(\cdot) - 1 - \frac{[1 - \Lambda'(\cdot) + \Psi G(\cdot)[\cdot; F(\cdot) \delta(\cdot)](\Pi G)_\theta(\cdot)]}{(\Psi G)_\theta[\cdot] - f(\cdot) \delta(\cdot)} \right] + \alpha_B \left[ \Lambda'(\cdot) + \Pi B(\cdot) - 1 - \frac{[1 - \Lambda'(\cdot) - \Delta B(\cdot)[\cdot; F(\cdot) \delta(\cdot)](\Pi B)_\theta(\cdot)]}{(\Delta B)_\theta[\cdot] + f(\cdot) \delta(\cdot)} \right] = 0; \ (2.10)$$

the third case has the property that a dual distortion occurs in the bad state, but the “no-default” constraint remains lax, with first-order conditions

$$\alpha_G \left[ 1 - \Delta \mu'(D) + \frac{(\Pi G)_\theta[\theta_{GR}^L(D, I_0), \Lambda'[I_0 F[\theta_{GR}^L(D, I_0)]]]}{(\Psi G)_\theta[\cdot] - f[\theta_{GR}^L(D, I_0)] \delta[\theta_{GR}^L(D, I_0), I_0, I_0]} \right] + \alpha_B [1 - \Delta \mu'(D)] \times \cdots \times \left[ \frac{1 - \Pi B[\theta_{BD}^{DD|SP}(D, I_0), \Lambda'[I_0 - I_{BD|SP}^{DD|SP}(D, I_0)][1 - F[\theta_{BD|SP}^{DD|SP}(D, I_0), I_0]]]}{\Delta B(\cdot)[1 - F[\theta_{BD|SP}^{DD|SP}(D, I_0)] \delta[\theta_{BD|SP}^{DD|SP}(D, I_0), I_0, I_0]]} \right] = 0,$$

and
\[
\alpha_G \left[ \Lambda'(\cdot) + \Pi_G(\cdot) - 1 - \left( 1 - \Lambda'(\cdot) + \Psi_G(\cdot) \right) \frac{\ast F(\cdot) \delta(\cdot) (\Pi_G)_{\theta}(\cdot)}{(\Psi_G)_{\theta}(\cdot) - f(\cdot) \delta(\cdot)} \right] \\
= \alpha_B \left[ 1 - \Lambda'(\cdot) - \left( 1 - \Lambda'(\cdot) - \delta(\cdot) \ast \right) \frac{\Pi_B(\cdot)}{\Delta_B(\cdot) - [1 - F(\cdot)] \delta(\cdot)} \right];
\]

- the final case is an “interbank collapse” scenario under which a dual distortion occurs in the bad state, with the “no-default” constraint now binding, so the planner’s initial balance-sheet choices are pinned down by this constraint, along with a first-order condition given in the appendix.

Broadly speaking, we should that the wedges highlighted in this last lemma lead the planner to settle on an initial balance sheet that’s in some sense more conservative than the initial balance sheet preferred by banks. One way to formalize this intuition is as follows:

**Lemma 2.4.4.** The solution for the planner’s ex-ante problem has the property that one of the distorted regimes obtains in the bad state only if this is also true in ex-ante equilibrium — that is, the parameter space in which the ex-ante equilibrium is vulnerable to the emergence of a distorted regime is a superset of the parameter space in which the planner’s ex-ante solution exhibits this vulnerability.

At this point, it’s now natural to ask about the policies that can be used to close the gap between the solution for the planner’s ex-ante problem and the allocation arising in ex-ante equilibrium. Based on the analysis in my last two subsections, it should be clear that the answers to these questions hinge critically on where the initial balance sheet preferred by the planner lies in relation to the locus

\[
\{(D, I_0) \in \mathbb{R}_+ \text{ s.t. } I_0 = T_B^{ED|SP}(D), \text{ with } E^b + D \geq \Delta \mu(D) \}. \tag{2.11}
\]

Suppose for example that the solution for the planner’s ex-ante problem places the initial balance sheet \((D, I_0)\) beneath this locus. In this case, the results reported in subsection 2.3.1 and 2.3.2 imply that the ex-post equilibrium to which this initial balance sheet gives rise in the bad (good) state will yield an allocation that also solves the corresponding ex-post problem facing the planner. As a result, the only role for policy is to discipline banks’ initial balance-sheet choices at \(t = 0\); conditional on banks’ being directed to the right initial
balance sheet, the secondary market will subsequently take care of itself. Now, with a fire-sale externality at work, a natural candidate for disciplining banks’ choice on \( I_0 \) would be a liquidity coverage ratio. As explained in \( \text{Basel Committee on Banking Supervision} \) (2013), the liquidity coverage ratio requires that banks project their liquidity needs over a thirty-day stress-test scenario and then hold enough liquid assets to cover a certain portion of these needs. Since in-model liquidity needs stem from the maintenance requirement \( \rho \), we can think of a policy of this sort as a requirement of the form

\[
E^b + D - I_0 \geq s \rho I_0,
\]

with \( s \in [0, 1] \). As for disciplining banks’ choice on \( D \), it can be shown that a leverage limit of the form

\[
\frac{D}{E^b} \leq \overline{d}
\]

will do the trick:

**Proposition 2.2** (optimal ex-ante intervention). *If the solution for the planner’s ex-ante problem places the initial balance sheet \((D, I_0)\) beneath the locus on line 2.11, then the ex-ante intervention \((s, \overline{d})\) can be chosen to implement this solution as an ex-ante equilibrium, with no subsequent need for ex-post intervention. Moreover, the appropriate choice on \((s, \overline{d})\) has the property that (2.12) and (2.13) both bind."

On the other hand, for initial balance sheets lying above the locus on line 2.11, the analysis in subsection 2.3.1 implies that an ex-post intervention \((\tau_B, T_B)\) is needed in the bad state to bring the allocation obtaining in ex-post equilibrium into alignment with the solution for the planner’s ex-post problem. If anything, this fact enhances the need for ex-ante intervention, since the expectation that policymakers will reduce the effective interest rate in the bad state creates a further disincentive against storage at \( t = 0 \):

**Proposition 2.3** (optimal ex-ante intervention, cont.). *If the solution for the planner’s ex-ante problem instead places the initial balance sheet \((D, I_0)\) above the locus on line 2.11, then the ex-ante intervention \((s, \overline{d})\) and ex-post intervention \((\tau_B, T_B)\) can jointly be chosen to implement this solution as an ex-ante equilibrium. Moreover, the appropriate choice on \((s, \overline{d}, \tau)\) has the property that (2.12) and (2.13) both bind."

These last two lemmata constitute some of this section’s main findings, mainly due to their implication that the ex-ante intervention is always necessary and sometimes even sufficient for implementation of the planner’s solution. As a result, a planner who focuses exclusively on ex-post intervention will not be able to implement this solution — likewise one who
views ex-ante intervention merely as a corrective for moral hazard. As explained in my introduction, this suggests that previous literature may have underestimated the role that ex-ante interventions have to play in the optimal regulation of interbank markets.

2.5 Conclusion

I’ll now close with a brief summary of this chapter’s findings. In this chapter, I extended the baseline model from chapter[1] to include a standard source of constrained inefficiency, namely a fire-sale externality. In this setting, I showed that the normative “no-go” result established in chapter[1] still holds in some sense. More specifically, I showed that the parameter space stills admits regions in which the occurrence of interbank disruptions is constrained efficient, though policy now has an important role to play in limiting the severity of these disruptions.

I then provided a full characterization of this role for policy. In particular, I showed that the planner’s solution can be implemented through careful coordination between a set of ex-ante prudential policies and an ex-post monetary stimulus. Moreover, I argued that the ex-ante component does most of the heavy lifting in the overall policy mix, specifically in the sense that it’s always necessary and sometimes even sufficient for implementation. In my next chapter, I’ll make this last point even more emphatically, namely by developing an alternate framework in which an ex-ante prudential policy always suffices to ensure constrained efficiency.
2.A Appendix

2.A.1 Proof of lemmata 2.3.1 through 2.3.4

I begin by reviewing some basic results from chapter 1 and introducing some notation. First, given some \( \ell_B \in [0, \Lambda'(0)] \), we know that the per-unit surplus function \( \Pi_B(\theta_B, \ell_B) \) peaks around the type \( (\rho + \ell_B)/\chi_B =: \theta_B^\Pi(\ell_B) \) and either exhibits single-crossing from below over \([0, 1), \) namely at some type \( \theta_B^\Pi(\ell_B) < \theta_B^\Pi(\ell_B) \), or otherwise is positive over all of \([0, 1), \) in which case I set \( \theta_B^\Pi(\ell_B) = 0. \)

Next, we know that the per-unit pledgable surplus function \( \Delta_B(\theta_B, \ell_B) \) peaks at a relatively low type \( \theta_B^\Delta(\ell_B) < \theta_B^\Pi(\ell_B) \) satisfying

\[
\rho + \ell_B = \chi_B \left[ \theta_B^\Delta(\ell_B) + \gamma(F/f) \left[ \theta_B^\Delta(\ell_B) \right] \right],
\]

where \((F/f)(\theta) := F(\theta)/f(\theta). \) Moreover, if the upper bound on \( \chi_B \) imposed by assumption 2.3 is sufficiently strict, we know that \( \Delta_B(\theta_B, \ell_B) \) is uniformly negative on \([0, 1), \)

Subassumption 2.3.1. \( \Delta_B[\theta_B^\Delta(0), 0] < 0. \)

Finally, we know that the interval \((\max \{ \theta_B^\Pi(\ell_B), \theta_B^\Delta(\ell_B) \}, \theta_B^\Pi(\ell_B) ) \) admits a critical type \( \theta_B^{\Xi}(\ell_B) \) satisfying

\[
\theta_B \geq \theta_B^{\Xi}(\ell_B) \iff \Pi_B(\theta_B, \ell_B) (\Delta_B)_{\theta} (\theta_B, \ell_B) - (\Pi_B)_{\theta} (\theta_B, \ell_B) \Delta_B(\theta_B, \ell_B) \lesssim 0
\]

\[
\iff \rho + \ell_B - \chi_B \Xi(\theta_B) \lesssim 0, \quad \forall \theta \in [0, 1],
\]

where \( \Xi'(\theta_B) > 0, \) and, in turn, \((\theta_B^{\Xi})'(\ell_B) > 0. \)

With these points in mind, we’re now ready to begin proving lemmata 2.3.1 through 2.3.4. As a first step, I import the following two subassumptions from chapter 1, with adjustments ensuring that they now hold over the full range of potential liquidation values:

Subassumption 2.3.2. \( \Lambda'(0) + \Pi_B[\theta_B^\Pi[\Lambda'(0)], \Lambda'(0)] < 1. \)

Subassumption 2.3.3. \( \theta_B^{\Xi}(0) > \theta_B^{\Psi+\Delta}, \) where \( \theta_B^{\Psi+\Delta} \) is defined in subsection 1.A.5.5.

It can easily be verified that both these conditions are more likely to obtain the lower is \( \chi_B. \) Moreover, it should be clear from the analysis in subsection 1.A.5.5 that subassumption 2.3.3
makes the financial constraint associated with the bad state stronger than the physical constraint associated with the bad state, so we can ignore the latter constraint going forward.

Sublemma 2.A.1. ∀D ∈ ℝ⁺ s.t. E^b + D − Δµ(D) ≥ 0, ∃T^LS_B(D) ∈ ℝ⁺ s.t.

\[(E^b + D - I_0) + I_0Λ' [I_0F [\theta^LS_B (I_0)]]\]

\[+ I_0Δ_B [\theta^LS_B (I_0), Λ' [I_0F [\theta^LS_B (I_0)]]] - Δµ(D) ≥ 0 \quad (2.14)\]

i.f.f. I_0 ≤ T^LS_B(D), and in this case the bad state admits a unique ex-post equilibrium of the “liquidity surplus” type, whereas no ex-post equilibria of this type exist otherwise. Moreover, this function is strictly decreasing, with E^b + D − Δµ(D) = 0 ⇒ T^LS_B (D) = 0.

Proof. In the main text, I argued that the bad state admits a unique ex-post equilibrium of the “liquidity surplus” type if initial balance sheets satisfy (2.14), and otherwise admits no ex-post equilibria of this type. Now, if D is large enough that E^b + D − Δµ(D) < 0, then it should be clear that (2.14) fails for all I_0 ∈ ℝ⁺, so I’ll restrict attention to choices on D satisfying E^b + D − Δµ(D) ≥ 0. Given any such choice on D, it should be clear that the left-hand side of (2.14) (i) is strictly decreasing in D; (ii) is non-negative at I_0 = 0; and (iii) goes to −∞ as I_0 ↗ ∞. I furthermore claim that it’s strictly decreasing in I_0. To see this, note that all I_0 ∈ ℝ++ satisfy (θ^LS_B)'(I_0) < 0,

\[= I_0 \int_{\theta^LS_B (I_0)}^{\theta^LS_B (I_0)} [\theta_X - ρ - Λ' [I_0F [\theta^LS_B (I_0)]]] dF(θ)\]

\[- I_0γ_X [\theta^LS_B (I_0)F [\theta^LS_B (I_0)] + \int_{\theta^LS_B (I_0)}^{\theta^LS_B (I_0)} θdF(θ)]\]

\[= \mathbb{E}[\max\{θ, θ^LS_B (I_0)] = \mathbb{E} \max[θ^LS_B (I_0)]\]

\[+ I_0Λ' [I_0F [\theta^LS_B (I_0)]] - I_0\]

\[= I_0(1 - γ)X_B \mathbb{E} \max[θ^LS_B (I_0)] - I_0X_Bθ^LS_B (I_0)F [\theta^LS_B (I_0)]\]
\[- I_0 \int_{\theta_{B}^{LS}(I_0)}^{1} \left[ \rho + \mathcal{N}[I_0 F[\theta_{B}^{LS}(I_0)]] \right] dF(\theta) \]

\[+ I_0 \mathcal{N}[I_0 F[\theta_{B}^{LS}(I_0)]] - I_0 \]

\[= I_0 (1 - \gamma) \chi_B \mathbb{E} \max[\theta_{B}^{LS}(I_0)] \]

\[- I_0 F[\theta_{B}^{LS}(I_0)] \left[ \chi_B \theta_{B}^{LS}(I_0) - \rho - \mathcal{N}[\theta_{B}^{LS}(I_0)] \right] - (1 + \rho) I_0 \]

\[= I_0 (1 - \gamma) \chi_B \mathbb{E} \max[\theta_{B}^{LS}(I_0)] - (1 + \rho) I_0, \]

and, in turn,

\[\frac{d}{dI_0} \left[ (E^b + D - I_0) + I_0 \mathcal{N}[I_0 F[\theta_{B}^{LS}(I_0)]] + I_0 \Delta_B \left[ \theta_{B}^{LS}(I_0), \mathcal{N}[I_0 F[\theta_{B}^{LS}(I_0)]] \right] \right] \]

\[= \frac{d}{dI_0} \left[ I_0 (1 - \gamma) \chi_B \mathbb{E} \max[\theta_{B}^{LS}(I_0)] - (1 + \rho) I_0 \right] \]

\[= (1 - \gamma) \chi_B \mathbb{E} \max[\theta_{B}^{LS}(I_0)] - (1 + \rho) + (\theta_{B}^{LS})'(I_0) I_0 (1 - \gamma) \chi_B F[\theta_{B}^{LS}(I_0)] < 0. \]

Conclude the following: \(\forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0, \exists \tilde{T}_{B}^{LS}(D) \in \mathbb{R}_+ \text{ s.t. } (2.14) \text{ holds} \) i.f.f. \(I_0 \leq \tilde{T}_{B}^{LS}(D)\), with \(\tilde{T}_{B}^{LS}\) strictly decreasing. The sublemma’s claims on existence and uniqueness, along with the claim that \(E^b + D - \Delta \mu(D) = 0 \implies \tilde{T}_{B}^{LS}(D) = 0\), then follow immediately. \(\blacksquare\)

**Sublemma 2.A.2.** \(\forall I_0 \in \mathbb{R}_+, \exists \hat{\theta}_B^{\Xi}(I_0) \in [0, 1] \text{ s.t.} \)

\[\theta_B \overset{\geq}{\lesssim} \hat{\theta}_B^{\Xi}(I_0) \iff \chi_B^{\Xi}(\theta_B) \overset{\geq}{\lesssim} \rho + \mathcal{N}[I_0 F(\theta_B)], \forall \theta_B \in [0, 1], \quad (2.15) \]

with \((\hat{\theta}_B^{\Xi})'(I_0) < 0\), and \(\hat{\theta}_B^{\Xi}(I_0) < \theta_{B}^{LS}(I_0)\)

**Proof.** This should be obvious. \(\blacksquare\)
In an ex-post equilibrium of the “extensive distortion” type, banks set \( \theta_B \) such that:

\[
\forall \theta_B \in \mathbb{R}_+ \quad \exists I_0 \in [L^D_B(D), \infty) \quad \text{s.t.} \quad I_0 \geq T^E_B(D) \quad \iff \quad (E^b + D - I_0) + I_0 \Lambda'[I_0F[\hat{\theta}_B^2(I_0)]] + I_0 \Delta_B[\hat{\theta}_B^2(I_0), \Lambda'[I_0F[\hat{\theta}_B^2(I_0)]]] \leq \Delta \mu(D),
\]

\( \forall I_0 \in \mathbb{R}_+ \). Moreover, this function has the property that the bad state admits a unique ex-post equilibrium of the “extensive distortion” type when \( I_0 \in (T^L_B(D), T^E_B(D)) \) and otherwise admits no ex-post equilibria of this type. It’s also strictly decreasing, with \( E^b + D - \Delta \mu(D) = 0 \implies T^E_B(D) = 0. \)

**Proof.** In an ex-post equilibrium of the “extensive distortion” type, banks set \( I_B = I_0 \), while \( \theta_B \) satisfies the following conditions:

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0F(\theta_B)] + I_0 \Delta_B[\theta_B, \Lambda'[I_0F(\theta_B)]] - \Delta \mu(D) = 0 \quad (2.16)
\]

\[
\rho + \Lambda'[I_0F(\theta_B)] \in (\chi_B \theta_B, \chi_B \Xi(\theta_B)) \quad (2.17)
\]

I’ll begin with three observations about this system. First of all, it should be clear that the lower bound on line 2.17 holds i.f.f. \( \theta_B < \theta_B^L(I_0) \). Secondly, in light of my previous sublemma, it should be clear that the upper bound on line 2.17 holds i.f.f. \( \theta_B \geq \hat{\theta}_B^2(I_0) \). Finally, when \( \theta_B \in [\hat{\theta}_B^2(I_0), \theta_B^L(I_0)] \), the left-hand side of line 2.16 admits the following derivative w.r.t. \( \theta_B \), where \( \cdot \) suppresses obvious arguments:

\[
I_0^2 f(\theta_B) \Lambda''[I_0F(\theta_B)] + I_0(\Delta_B)\theta[\theta_B, \Lambda'(\cdot)] + I_0^3 f(\theta_B)\Lambda''(\cdot)(\Delta_B)\epsilon[\theta_B, \Lambda'(\cdot)] = I_0^3 f(\theta_B) F(\theta_B) \Lambda''(\cdot) + I_0(\Delta_B)\theta[\theta_B, \Lambda'(\cdot)]
\]

\[
< 0, \text{ b/c } \underbrace{2.17}_{\theta_B \geq \theta_B^L(I_0)} \implies \theta_B \geq \theta_B^L(\Lambda'(\cdot)) > \theta_B^L(\Lambda'(\cdot))
\]

\[
< 0.
\]

Conclude that the system on lines 2.16 and 2.17 admits a unique solution if the following two conditions hold:

\[
(E^b + D - I_0) + I_0 \Lambda[I_0F[\hat{\theta}_B^2(I_0)]] + I_0 \Delta_B[\hat{\theta}_B^2(I_0), \Lambda[I_0F[\hat{\theta}_B^2(I_0)]]] - \Delta \mu(D) \geq 0 \quad (2.18)
\]

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\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 F[\theta_B^{LS}(I_0)]] \\
+ I_0 \Delta_B[\theta_B^{LS}(I_0), \Lambda'[I_0 F[\theta_B^{LS}(I_0)]], - \Delta \mu(D) < 0 \tag{2.19}
\]

On the other hand, if either of these conditions fails, then no solution exists.

Now, \((2.18)\) clearly fails whenever \(D\) is large enough that \(E^b + D - \Delta \mu(D) < 0\), so I’ll restrict attention to choices on \(D\) satisfying \(E^b + D - \Delta \mu(D) \geq 0\). Under this restriction, it should be clear that \((2.19)\) holds i.f.f. \(I_0 > T_B^{LS}(D)\). As for \((2.18)\), it should now be clear that its left-hand side (i) is non-negative at \(I_0 = T_B^{LS}(D)\); (ii) goes to \(-\infty\) as \(I_0 \nearrow \infty\); and (iii) is strictly decreasing in \(D\). I furthermore claim that it’s strictly decreasing in \(I_0\) as well, though the argument establishing this monotonicity is a bit involved. As a first step, note that all \(I_0 \in \mathbb{R}_{++}\) satisfy

\[
I_0 \Delta_B[\hat{\theta}_B^{\infty}(I_0), \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]]] + I_0 \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] - I_0
\]

\[
= I_0 \int_{\hat{\theta}_B^{\infty}(I_0)}^{1} [\theta_{\chi B} - \rho] - \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]]dF(\theta) - I_0 \gamma_{\chi B} E \max[\hat{\theta}_B^{\infty}(I_0)]
\]

\[
+ I_0 \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] - I_0
\]

\[
= I_0 \int_{\hat{\theta}_B^{\infty}(I_0)}^{1} (\theta_{\chi B} - \rho) dF(\theta) - I_0 \gamma_{\chi B} E \max[\hat{\theta}_B^{\infty}(I_0)] + I_0 F[\hat{\theta}_B^{\infty}(I_0)] \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] - I_0
\]

where \(\lambda^{Rev}(\cdot)\) gives the revenues raised in the secondary market as a function of the total volume of liquidations taking place. In turn, using \(\cdot\) to suppress obvious arguments,

\[
\frac{d}{dI_0} \left( (E^b + D - I_0) + I_0 \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] + I_0 \Delta_B[\hat{\theta}_B^{\infty}(I_0), \Lambda'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]], - \Delta \mu(D) \right)
\]

\[
= \int_{\hat{\theta}_B^{\infty}(I_0)}^{1} (\theta_{\chi B} - \rho) dF(\theta) - \gamma_{\chi B} E \max[\hat{\theta}_B^{\infty}(I_0)] + F[\hat{\theta}_B^{\infty}(I_0)] (\lambda^{Rev})'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] - 1
\]

\[
+ (\hat{\theta}_B^{\infty})'[I_0 F[\hat{\theta}_B^{\infty}(I_0)][\rho + (\lambda^{Rev})'[I_0 F[\hat{\theta}_B^{\infty}(I_0)]] - \chi B \hat{\theta}_B^{\infty}(I_0) - \gamma \chi B (F/f)[\hat{\theta}_B^{\infty}(I_0)]]
\]
\( \propto I_0 f(\cdot) [\chi_B \hat{\theta}^B_0 (I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \\
- \left[ \frac{1}{(\hat{\theta}^B_0)'(I_0)} \right] \left[ \int_{\hat{\theta}^B_0 (I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \xi \max(\cdot) + F(\cdot) (\lambda^{\text{Rev}})'(\cdot) - 1 \right] \\
= I_0 f(\cdot) [\chi_B \hat{\theta}^B_0 (I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \\
- \left[ \frac{\chi_B \xi'(\cdot) - I_0 f(\cdot) \Lambda''(\cdot)}{F(\cdot) \Lambda''(\cdot)} \right] \times \ldots \\
\ldots \times \left[ \int_{\hat{\theta}^B_0 (I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \xi \max(\cdot) + F(\cdot) (\lambda^{\text{Rev}})'(\cdot) - 1 \right] \\
= I_0 f(\cdot) [\chi_B \hat{\theta}^B_0 (I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \\
- \left[ \frac{\chi_B \xi'(\cdot)}{F(\cdot) \Lambda''(\cdot)} - \frac{I_0 f(\cdot)}{F(\cdot)} \right] \times \ldots \\
\ldots \times \left[ \int_{\hat{\theta}^B_0 (I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \xi \max(\cdot) + F(\cdot) (\lambda^{\text{Rev}})'(\cdot) - 1 \right] \\
< \int_{\hat{\theta}^B_0 (I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \xi \max(\cdot) + F(\cdot) \Lambda' - 1 \\
= \int_{\hat{\theta}^B_0 (I_0)}^1 [\theta \chi_B - \rho - \Lambda'(\cdot)] - \gamma \chi_B \xi \max(\cdot) + \Lambda'(\cdot) - 1 \\
< \int_{\hat{\theta}^B_0 (I_0)}^1 [\theta \chi_B - \rho - \Lambda'(\cdot)] - \gamma \chi_B \xi \max(\cdot) \\
= \Delta_B [\hat{\theta}^B_0 (I_0), \Lambda'(\cdot)] \\
< 0 \\
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\[ < I_0 f(\cdot) [\chi_B \hat{\theta}_B^\Xi(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] + \left[ \frac{I_0 f(\cdot)}{F(\cdot)} \right] \times \cdots \]

\[ \cdots \times \left[ \int_{\hat{\theta}_B(I_0)}^{1} (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max(\cdot) + F(\cdot)(\lambda^{\text{Rev}})'(\cdot) - 1 \right] \]

\[ \propto F(\cdot) [\chi_B \hat{\theta}_B^\Xi(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] + \int_{\hat{\theta}_B(I_0)}^{1} (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max(\cdot) + F(\hat{\theta}_B(I_0))(\lambda^{\text{Rev}})'(\cdot) - 1 \]

\[ = (1 - \gamma) \chi_B \mathbb{E} \max(\cdot) + \gamma \chi_B (F(\cdot)(F/f)(\cdot) - (1 + \rho) \]

\[ = \frac{\chi_B \mathbb{E} \max(\cdot) + \gamma \chi_B [F(\cdot)(F/f)(\cdot) - \mathbb{E} \max(\cdot)] - (1 + \rho)}{\Theta}. \]

The monotonicity being claimed would thus hold if we could show that the starred term is non-positive. Now, if \( F[\hat{\theta}_B^\Xi(I_0)](F/f)[\hat{\theta}_B^\Xi(I_0)] - \mathbb{E} \max[\hat{\theta}_B^\Xi(I_0)] \leq 0, \) then it should be clear that this follows directly from subassumption 2.3.2, so it would suffice if we could rule out the case where \( F[\hat{\theta}_B^\Xi(I_0)](F/f)[\hat{\theta}_B^\Xi(I_0)] - \mathbb{E} \max[\hat{\theta}_B^\Xi(I_0)] > 0. \) To show that this case can indeed be ruled out, I note from lines 1.9 through 1.12 in subsection I.A.3 that

\[ \Xi(\theta_B) = \frac{\theta_B \mathbb{E} \max(\theta_B) + (F/f)(\theta_B) \int_{\theta_B}^{1} \theta dF(\theta)}{\mathbb{E} \max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]} \]

\[ = \theta_B + \frac{(F/f)(\theta_B) \left[ \int_{\theta_B}^{1} \theta dF(\theta) - \theta_B[1 - F(\theta_B)] \right]}{\mathbb{E} \max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]} \]

\[ = \theta_B + \frac{(F/f)(\theta_B) [\mathbb{E} \max(\theta_B) - \theta_B]}{\mathbb{E} \max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]}. \]
Re-writing $\Xi(\theta_B)$ in this way is useful because it makes clear the following: if indeed $F[\hat{\theta}_B(I_0)](F/f)[\hat{\theta}_F(I_0)] - E \max[\hat{\theta}_B(I_0)] > 0$, then

$$\Xi[\hat{\theta}_B(I_0)] = \hat{\theta}_B(I_0) + \frac{(F/f)[\hat{\theta}_B(I_0)][E \max[\hat{\theta}_B(I_0)] - \hat{\theta}_B(I_0)]}{(F/f)[\hat{\theta}_B(I_0)] - [F[\hat{\theta}_B(I_0)](F/f)[\hat{\theta}_B(I_0)] - E \max[\hat{\theta}_B(I_0)]]}$$

$$> E \max[\hat{\theta}_B(I_0)]$$

$$> \frac{\rho + \Lambda'[I_0F[\hat{\theta}_B(I_0)]]}{\chi_B},$$

where the last inequality follows from the fact that $\hat{\theta}_B(I_0) = \theta_B[\Lambda'[I_0F[\hat{\theta}_B(I_0)]]] > \hat{\theta}_B[I_0F[\hat{\theta}_B(I_0)]]$ and clearly contradicts (2.15).

We can finally conclude the following: $\forall D \in \mathbb{R}_+$ s.t. $E^b + D - \Delta \mu(D) > 0$, $\exists T_B^E(\hat{\theta}_B, (D), \infty)$ s.t.

$$I_0 \geq T_B^E(D) \iff (E^b + D - I_0) + I_0 \Lambda'[I_0F[\hat{\theta}_B(I_0)]]$$

$$+ I_0 \Delta_B[\hat{\theta}_B(I_0), \Lambda'[I_0F[\hat{\theta}_B(I_0)]]] \leq \Delta \mu(D), \ \forall I_0 \in \mathbb{R}_+,$$

with $T_B^E(D)$ strictly decreasing. The sublemma’s claims on existence and uniqueness, along with the claim that $E^b + D - \Delta \mu(D) = 0 \implies T_B^E(D) = 0$, then follow immediately. ■

**Sublemma 2.A.4.** $\forall D \in \mathbb{R}_+$ s.t. $E^b + D - \Delta \mu(D) \geq 0$, $\exists T_B^D(D) \in [T_B^E(D), \infty)$ s.t.

$$I_0 \geq T_B^D(D) \iff (E^b + D - I_0) + I_0 \Lambda'(I_0) - \Delta \mu(D) \leq 0, \ \forall I_0 \in \mathbb{R}_+.$$

Moreover, this function has the property that the bad state admits a unique ex-post equilibrium of the “dual distortion” type when $I_0 \in [T_B^E(D), T_B^D(D)]$ and otherwise admits no ex-post equilibria of this type. It’s also strictly decreasing, with $E^b + D - \Delta \mu(D) = 0 \implies T_B^D(D) = 0$.

**Proof.** In an ex-post equilibrium of the “dual distortion” type, the pair $(I_B, \theta_B)$ satisfies the following conditions:

$$(E^b + D - I_0) + I_0 \Lambda'[I_0 - I_B[1 - F(\theta_B)]]$$

$$+ I_B \Delta_B[\hat{\theta}_B, \Lambda'[I_0 - I_B[1 - F(\theta_B)]]] - \Delta \mu(D) = 0$$
\[
\chi_B \Xi(\theta_B) = \rho + \Lambda'[I_0 - I_B[1 - F(\theta_B)]]
\]

\[
I_B \in [0, I_0)
\]

Now, if we define \( M_B := I_B[1 - F(\theta_B)] \) as the total volume of investments being maintained, then the following system in \((M_B, \theta_B)\) is equivalent, where \( \xi_1(\cdot) \) and \( \xi_2(\cdot) \) are defined in subsection 1.A.3.

\[
(E^b + D - I_0) + I_0 \Lambda'(I_0 - M_B)
\]

\[ + M_B[\chi_B \xi_1(\theta_B) - \rho - \Lambda'(I_0 - M_B) - \gamma \chi_B \xi_2(\theta_B)] - \Delta \mu(D) = 0 \tag{2.23}
\]

\[
\chi_B \Xi(\theta_B) = \rho + \Lambda'(I_0 - M_B) \tag{2.24}
\]

\[
M_B/[1 - F(\theta_B)] \in [0, I_0) \tag{2.25}
\]

Moreover, we can re-write \( (2.24) \) as \( \theta_B = \theta_B^\Xi[\Lambda'(I_0 - M_B)] \). Re-writing in this way is useful because it can be argued that the composition

\[
(E^b + D - I_0) + I_0 \Lambda'(I_0 - M_B)
\]

\[ + M_B[\chi_B \xi_1(\theta_B) - \rho - \Lambda'(I_0 - M_B) - \gamma \chi_B \xi_2(\theta_B)] - \Delta \mu(D) = 0 \tag{2.23}
\]

is strictly decreasing in \( M_B \). The argument is as follows. First, note that this composition admits the following derivative w.r.t. \( M_B \):

\[
( -1 )I_0 \Lambda''(I_0 - M_B) + \chi_B \xi_1[\theta_B^\Xi[\Lambda'(I_0 - M_B)]] - \rho - \Lambda'(I_0 - M_B)
\]

\[- \gamma \chi_B \xi_2[\theta_B^\Xi[\Lambda'(I_0 - M_B)]]
\]

\[- M_B \chi_B(\theta_B^\Xi)'[\Lambda'(I_0 - M_B)] \Lambda''(I_0 - M_B) \xi_1'[\theta_B^\Xi[\Lambda'(I_0 - M_B)]]
\]

\[ + M_B \Lambda''(I_0 - M_B)
\]
\[
+ M_B \chi_B (\theta_B^\xi'[\Lambda'(I_0 - M_B)] \Lambda''(I_0 - M_B)) \gamma \xi'_2[\theta_B^\xi'[\Lambda'(I_0 - M_B)]]
\]
\[
\geq 0
\]
\[
= (-1) (\Lambda^{\text{Rev}})'(I_0 - M_B) + \chi_B \xi_1[\theta_B^\xi'[\Lambda'(I_0 - M_B)] - \rho - \gamma \chi_B \xi_2[\theta_B^\xi'[\Lambda'(I_0 - M_B)]
\]
\[
- M_B \chi_B (\theta_B^\xi'[\Lambda'(I_0 - M_B)] \Lambda''(I_0 - M_B)) \xi'_1[\theta_B^\xi'[\Lambda'(I_0 - M_B)]
\]
\[
- \gamma \xi'_2[\theta_B^\xi'[\Lambda'(I_0 - M_B)]]
\]
\[
< (-1) M_B \chi_B (\theta_B^\xi'[\Lambda'(I_0 - M_B)] \Lambda''(I_0 - M_B)) \xi'_1[\theta_B^\xi'[\Lambda'(I_0 - M_B)]
\]
\[
\leq 0
\]
\[
- \gamma \xi'_2[\theta_B^\xi'[\Lambda'(I_0 - M_B)]].
\]

So, the monotonicity being claimed would hold if we could show that \(\xi'_1[\theta_B^\xi'[\Lambda'(I_0 - M_B)] < \gamma \xi'_2[\theta_B^\xi'[\Lambda'(I_0 - M_B)]\). To see that this is indeed the case, recall that

\[
\xi'_1(\theta_B) = \frac{f(\theta_B) \left[ \int_{\theta_B}^{\Lambda'(I_0 - M_B)} \theta dF(\theta) - \theta_B \left[ 1 - F(\theta_B) \right] \right]}{\left[ 1 - F(\theta_B) \right]^2},
\]

and

\[
\xi'_2(\theta_B) = \frac{F(\theta_B) \left[ 1 - F(\theta_B) \right] + f(\theta_B) E \max \{ \theta, \theta_B \}}{\left[ 1 - F(\theta_B) \right]^2}.
\]

In turn, using \(\cdot\) to suppress obvious arguments,

\[
\Xi[\theta_B^\xi'[\Lambda'(I_0 - M_B)] = \left[ \theta_B + \frac{(F/f)(\theta_B) \left[ \int_{\theta_B}^{\Lambda'(I_0 - M_B)} \theta dF(\theta) - \theta_B \left[ 1 - F(\theta_B) \right] \right]}{E \max(\theta_B) + (F/f)(\theta_B) \left[ 1 - F(\theta_B) \right]} \right]_{\theta_B=\theta_B^\xi'[\Lambda']} = \left[ \theta_B + \frac{(F/f)(\theta_B) \xi'_1(\theta_B)}{\xi'_2(\theta_B)} \right]_{\theta_B=\theta_B^\xi'[\Lambda']} (2.26)
\]

\[
= \frac{\rho + \Lambda'[\theta_B^\xi'[\Lambda'(\cdot)]]}{\chi_B}
\]
\[< \theta_B^\Xi [\Lambda' (\cdot)] + \gamma (F/f) [\theta_B^\Xi [\Lambda' (\cdot)]] , \]  
\hspace{1cm} (2.27)

where the first line follows from (2.20)-(2.21), while the last line follows from the fact that 
\[\theta_B^\Xi [\Lambda'(I_0 - M_B)] > \theta_B^\Xi [\Lambda'(I_0 - M_B)].\] Comparing (2.26) and (2.27) then yields 
\[\xi'_1 [\theta_B^\Xi [\Lambda'(I_0 - M_B)]] < \gamma \xi'_2 [\theta_B^\Xi [\Lambda'(I_0 - M_B)]] , \] as desired.

We can finally conclude that the system on lines 2.23 through 2.25 admits a unique solution if the following two conditions hold:

\[ (E^b + D - I_0) + I_0 \Lambda'(I_0) - \Delta \mu (D) \geq 0 \]  
\hspace{1cm} (2.28)

\[ (E^b + D - I_0) + I_0 \Lambda'[\hat{\theta}_B^\Xi (I_0)] \]  
\hspace{1cm} + I_0 \Delta \mu [\hat{\theta}_B^\Xi (I_0), \Lambda'[\hat{\theta}_B^\Xi (I_0)]] - \Delta \mu (D) < 0 \]  
\hspace{1cm} (2.29)

On the other hand, if either of these conditions fails, then no solution exists. Now, (2.28) clearly fails whenever \( D \) is large enough that \( E^b + D - \Delta \mu (D) < 0 \), so I'll restrict attention to choices on \( D \) satisfying \( E^b + D - \Delta \mu (D) \geq 0 \). Under this restriction, it should be clear that (2.29) holds i.f.f. \( I_0 > \bar{I}_B^{ED} (D) \). As for (2.28), it should now be clear that its left-hand side (i) is non-negative at \( I_0 = \bar{I}_B^{ED} (D) \); (ii) goes to \(-\infty\) as \( I_0 \nearrow \infty \); and (iii) is strictly decreasing in \( D \) and \( I_0 \). Conclude that \( \exists \bar{T}_B^{DD} (D) \in [\bar{I}_B^{ED} (D), \infty) \)

\[ I_0 \geq \bar{T}_B^{DD} (D) \iff (E^b + D - I_0) + I_0 \Lambda' (I_0) - \Delta \mu (D) \leq 0, \ \forall I_0 \in \mathbb{R}_+ , \]

with \( \bar{T}_B^{DD} (D) \) strictly decreasing. All of the sublemma’s claims on existence and uniqueness, along with the claim that \( E^b + D - \Delta \mu (D) = 0 \iff \bar{T}_B^{DD} (D) = 0 \), then follow immediately.

\section*{2.A.2 Proof of lemmata 2.3.5 and 2.3.6}

Let \( (\mathbb{P}_B^{SP}) \) denote the task of choosing \((\theta_B, I_B) \in [0, 1] \times [0, I_0]\) so as to maximize

\[ \int_{\theta_\omega}^{1} (\theta_{\chi_\omega} - \rho) I_\omega dF (\theta) + \Lambda [I_0 - I_\omega [1 - F (\theta_\omega)]] := \Pi_B^{SP} (\theta_\omega, I_\omega, I_0) , \]

subject to \( (PC_B^{SP}) \) and \( (FC_B^{SP}) \). Now, much as in chapter \( 1 \) it’s natural to conjecture that the physical constraint is lax, so I let \( (\mathbb{P}_B^{SP} \text{-rex}) \) denote a relaxed program from which I drop
(PC\textsuperscript{SP}{B}). In the sequel, I’ll focus on solving (P\textsuperscript{SP}{B}-rex) before finally arguing that its solution generalizes to (P\textsuperscript{SP}{B}).

As a first step in this direction, I make a few basic observations about (P\textsuperscript{SP}{B}-rex) and introduce some notation. In particular, I note that the surplus function \( \Pi|_{\text{SP}{B}}(\theta, I_B, I_0) \) peaks around the type \( \theta_B^{\Pi|_{\text{SP}{B}}}(I_B, I_0) \) for whom maintenance is NPV-neutral — i.e.,

\[
\rho + N'[I_0 - I_B][1 - F[\theta_B^{\Pi|_{\text{SP}}}(I_B, I_0)]] = \chi_B\theta_B^{\Pi|_{\text{SP}}}(I_B, I_0).
\]

Moreover, the derivative \((\Pi_{\text{SP}{B}})'_\theta(I_B, I_B, I_0)\), viewed as a function of the choice on \( \theta_B \), either exhibits single-crossing from below over \([0, 1)\), namely at some relatively low type \( \theta_B^{\Pi|_{\text{SP}}}(I_B, I_0) < \theta_B^{\Pi|_{\text{SP}}}(I_B, I_0) \), or otherwise is positive over all of this interval, in which case I set \( \theta_B^{\Pi|_{\text{SP}}}(I_B, I_0) = 0 \).

Next, I define

\[
\Delta_B^{\text{SP}}(\theta_B, I_B, I_0) := I_0N'[I_0 - I_B[1 - F(\theta_B)]]
\]

\[
+ I_B \int_{\theta_B}^{1} [\theta_B \chi_B - \rho - N'[I_0 - I_B[1 - F(\theta_B)]]]dF(\theta)
\]

\[
- I_B \gamma \chi_B \mathbb{E} \max(\theta_B)
\]

\[
= \lambda^{\text{Rev}}[I_0 - I_B[1 - F(\theta_B)]]
\]

\[
+ I_B \left[ \int_{\theta_B}^{1} (\theta_B \chi_B - \rho)dF(\theta) - \gamma \chi_B \mathbb{E} \max(\theta_B) \right],
\]

so \( \Delta_B^{\text{SP}}(\theta_B, I_B, I_0) \) peaks around some relatively low type \( \theta_B^{\Delta|_{\text{SP}}}(I_B, I_0) < \theta_B^{\Pi|_{\text{SP}}}(I_B, I_0) \) satisfying

\[
\rho + (\lambda^{\text{Rev}})'[I_0 - I_B[1 - F[\theta_B^{\Delta|_{\text{SP}}}(I_B, I_0)]]] = \chi_B\theta_B^{\Delta|_{\text{SP}}}(I_B, I_0) + \gamma \chi_B(F/f)[\theta_B^{\Delta|_{\text{SP}}}(I_B, I_0)].
\]

Moreover,

\[(\Delta_B^{\text{SP}})'_\theta(I_B, I_B, I_0)\]
\[
\int_{\theta_B}^{1} \left[ \theta \chi_B - \rho - (\lambda_{\text{Rev}})^{\gamma} [I_0 - I_B [1 - F(\theta_B)]] \right] dF(\theta) - \gamma \chi_B \mathbb{E} \max(\theta_B)
\]

\[
< \int_{\theta_B}^{1} (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max(\theta_B)
\]

(2.30)

\[
= \Delta_B(\theta_B, 0)
\]

(2.31)

< 0,

(2.32)

where (2.30) and (2.32) respectively follow from assumption 2.5 and subassumption 2.3.1.

In light of these observations, we see that the planner’s unconstrained optimum has \((\theta_B, I_B) = [\theta_B^{\Pi_{SP}}(I_0, I_0), I_0]\), which clearly coincides with the subcontract \((\theta_B, I_B) = [\theta_B^{LS}(I_0), I_0]\) derived in my previous subsection. As a result, we know that this contract solves \((\Pi_{SP}^{\text{SP}} - \text{rex})\) so long that 

\[
E_b + D - \Delta \mu(D) \geq 0, \quad \text{with} \quad I_0 \leq T_B^{LS}(D).
\]

On the other hand, if either of these conditions fails, then \((\Pi_{SP}^{\text{SP}} - \text{rex})\) either is insoluble or otherwise has the property that the financial constraint binds. In the latter case, the rate of transformation along the financial constraint is given by

\[
\frac{(dI_B/d\theta_B)(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0) + (\Delta^{SP})_{\theta_B}(\theta_B, I_B, I_0)}{\Delta_{SP}^{B}(\theta_B, I_B, I_0)} = 0
\]

\[
\Longleftrightarrow dI_B/d\theta_B = (-1)(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0)/(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0),
\]

so the planner prefers the extensive margin long that

\[
(dI_B/d\theta_B)(\Pi_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0) + (\Pi_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0)
\]

\[
= (\Pi_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0) - \frac{(\Pi_{SP}^{B})_{I_B}(\theta_B, I_B, I_0)(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0)}{(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0)} \leq 0,
\]

or equivalently

\[
\left(\Pi_{SP}^{B}\right)_{I_B}(\theta_B, I_B, I_0)(\Delta_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0) - (\Delta_{SP}^{B})_{I_B}(\theta_B, I_B, I_0)(\Pi_{SP}^{B})_{\theta_B}(\theta_B, I_B, I_0) \leq 0.
\]

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Here we see that the starred term is strictly negative when \( \theta_B = \theta_B^{\Pi|SP}(I_B, I_0) \), but strictly positive at \( \theta_B = \max\{\theta_B^{\Delta|SP}(I_B, I_0), \theta_B^{\Pi|SP}(I_B, I_0)\} \), so it must reach zero at some \( \theta_B^{\Xi|SP}(I_B, I_0) \in (\max\{\theta_B^{\Delta|SP}(I_B, I_0), \theta_B^{\Pi|SP}(I_B, I_0)\}, \theta_B^{\Pi|SP}(I_B, I_0)) \). Moreover:

**Sublemma 2.A.5.** \( \theta_B^{\Xi|SP}(I_B, I_0) \) is unique — i.e., \( \theta_B \gtrless \theta_B^{\Xi|SP}(I_B, I_0) \iff (\ast) \qquad \leq \quad 0, \quad \forall \theta_B \in [\max\{\theta_B^{\Pi|SP}(I_B, I_0), \theta_B^{\Pi|SP}(I_B, I_0)\}, \theta_B^{\Pi|SP}(I_B, I_0)] \).

**Proof.** Using \( \cdot \) to suppress obvious arguments, I note that

\[
(\ast) \propto \int_{\theta_B}^{\ast} [\theta \chi_B - \rho - \Lambda'[I_0 - I_B[1 - F(\theta_B)]]]dF(\theta) \times \ldots
\]

\[
\ldots \times \left[ \int_{\theta_B}^{\ast} \left[ \theta \chi_B - \rho - (\lambda^{\text{Rev}})'[I_0 - I_B[1 - F(\theta_B)]] - \chi_B \theta_B - \chi_B \gamma(F/f)(\theta_B) \right] \right]
\]

\[
= \int_{\theta_B}^{\ast} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta) [\rho + \Lambda'(\cdot) - \chi_B \theta_B]
\]

\[
+ \int_{\theta_B}^{\ast} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta) \times \ldots
\]

\[
\ldots \times [(\lambda^{\text{Rev}})'(\cdot) - \Lambda'(\cdot) - \gamma \chi_B(F/f)(\theta_B)]
\]

\[
= [\rho + \Lambda'(\cdot) - \chi_B \theta_B] \int_{\theta_B}^{\ast} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta)
\]

\[
+ [\rho + \Lambda'(\cdot) - \chi_B \theta_B] \times \ldots
\]

\[
\ldots \times \left[ \int_{\theta_B}^{\ast} [(\lambda^{\text{Rev}})'(\cdot) - \Lambda'[I_0 - I_B[1 - F(\theta_B)]]]dF(\theta) + \gamma \chi_B \mathbb{E} \max(\theta_B) \right]
\]
\[
= \int_{\theta_B}^{1} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta) \times \ldots
\]
\[
\ldots \times [(\lambda^{\text{Rev}})'(\cdot) - \Lambda'(\cdot) - \gamma \chi_B (F/f)(\theta_B)]
\]
\[
+ [\rho + \Lambda'(\cdot) - \chi_B \theta_B] \times \ldots
\]
\[
\ldots \times \left[ \int_{\theta_B}^{1} [(\lambda^{\text{Rev}})'(\cdot) - \Lambda'(\cdot)]dF(\theta) + \gamma \chi_B \mathbb{E} \max(\theta_B) \right]
\]
\[
= \gamma \chi_B \mathbb{E} \max(\theta_B)[\rho + \Lambda'(\cdot) - \chi_B \theta_B] - (F/f)(\theta_B) \times \ldots
\]
\[
\ldots \times \int_{\theta_B}^{1} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta)
\]
\[
+ [(\lambda^{\text{Rev}})'(\cdot) - \Lambda'(\cdot)] \times \ldots
\]
\[
\ldots \times \left[ \int_{\theta_B}^{1} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta) + [1 - F(\theta_B)] \times \ldots \right]
\]
\[
\ldots \times [\rho + \Lambda'(\cdot) - \chi_B \theta_B][1 - F(\theta_B)]
\]
\[
= \chi_B [\mathbb{E} \max(\theta_B) - \theta_B]
\]
\[
\propto \mathbb{E} \max(\theta_B)[\rho + \Lambda'(\cdot) - \chi_B \theta_B]
\]
\[
- (F/f)(\theta_B) \int_{\theta_B}^{1} [\theta \chi_B - \rho - \Lambda'(\cdot)]dF(\theta)
\]
\[
+ (1/\gamma)[(\lambda^{\text{Rev}})'(\cdot) - \Lambda'(\cdot)][\mathbb{E} \max(\theta_B) - \theta_B]
\]
\[
= [\rho + \Lambda'(\cdot)][\mathbb{E} \max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]]
\]
\[
- \chi_B \left[ \theta_B \mathbb{E} \max(\theta_B) + (F/f)(\theta_B) \int_{\theta_B}^{1} \theta dF(\theta) \right]
\]
\[ + (1/\gamma)[(\lambda^\text{Rev})'(-) - \Lambda'(-)][\mathbb{E}\max(\theta_B) - \theta_B] \]

\[ \propto \frac{\rho + \Lambda'(-)}{\chi_B} - \frac{\theta_B\mathbb{E}\max(\theta_B) + (F/f)(\theta_B)\int_{\theta_B}^1 \theta dF(\theta)}{\mathbb{E}\max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]} = \Xi(\theta_B) \text{ (see subsection 1.A.3)} \]

\[ + \frac{(\lambda^\text{Rev})'(-) - \Lambda'(-)}{\chi_B} \frac{\gamma\chi_B}{\mathbb{E}\max(\theta_B) - \theta_B} \frac{\mathbb{E}\max(\theta_B) - \theta_B}{\mathbb{E}\max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)]} = \xi_3(\theta_B) \text{ (ditto)} \]

\[ \propto \frac{1}{\xi_3(\theta_B)} \left[ \frac{\rho + \Lambda'(-)}{\chi_B} - \Xi(\theta_B) \right] + \frac{(\lambda^\text{Rev})'(-) - \Lambda'(-)}{\gamma\chi_B} \mathbb{E}\max(\theta_B) + (F/f)(\theta_B)[1 - F(\theta_B)] \]

\[ =: \Xi^\text{SP}(\theta_B, I_B, I_0). \quad (2.33) \]

Now, from {1.12}-(1.14), we know that

\[ \frac{\Xi(\theta_B)}{\xi_3(\theta_B)} = \frac{\xi_1(\theta_B)}{\xi_3(\theta_B)} - \xi_2(\theta_B), \]

with \( \xi_3(\theta_B) = \xi_1(\theta_B)/\xi_2(\theta_B) \), \( \xi'_3(\theta_B) < 0 \),

\[ \frac{d}{d\theta_B} \left[ \frac{\Xi(\theta_B)}{\xi_3(\theta_B)} \right] = \xi_1(\theta_B) \frac{d}{d\theta_B} \left[ \frac{1}{\xi_3(\theta_B)} \right]. \]

In turn, again using \( \cdot \) to suppress obvious arguments, and, in turn,

\[ (\Xi^\text{SP})_{\theta}[\theta_B^\text{SP}(I_B, I_0), I_B, I_0] = \frac{-\xi'_3[I_B^\text{SP}(I_B, I_0)]}{[\xi_3(\cdot)]^2} \times \ldots \]

\[ < 0, \text{ namely } b/c \theta_B^\text{SP}(I_B, I_0) > \theta_B^\text{SP}(I_B, I_0) \]

\[ \ldots \times \left[ \frac{\rho + \Lambda'[I_0 - I_B[1 - F(\cdot)]]}{\chi_B} - \xi_1(\cdot) \right] \]
\[ + \frac{I_B f(\cdot) \Lambda''(\cdot)}{\chi_B \gamma} \left( \frac{1}{\xi_3(\cdot)} - \frac{1}{\gamma} \right) + \frac{I_B f(\cdot)}{\gamma \chi_B \lambda^{Rev}(\cdot)} \right] < 0 \]

So, uniqueness would obtain if we could show that \((**) > 0\). To see that this is indeed the case, note that

\[
\Xi^{SP}_{\infty} \left[ \theta^I_B(I_B, I_0), I_B, I_0 \right] \propto \rho + \left[ 1 - (1/\gamma) \xi_3[\theta^\infty_B(I_B, I_0)] \right] \times \cdots
\]

\[
\cdots \times \Lambda'[I_0 - I_B[1 - F[\theta^\Xi_{\infty}B(I_B, I_0)]]]
\]

\[
- \chi_B \Xi[\theta^\Xi_{\infty}B(I_B, I_0)] + (1/\gamma) \xi_3[\theta^\Xi_{\infty}B(I_B, I_0)] \times \cdots
\]

\[
\cdots \times (\lambda^{Rev})'[I_0 - I_B[1 - F[\theta^\Xi_{\infty}B(I_B, I_0)]]]
\]

\[
= \rho + \left[ 1 - (1/\gamma) \xi_3[\theta^\Xi_{\infty}B(I_B, I_0)] \right] \times \cdots
\]

\[
\cdots \times \Lambda'[I_0 - I_B[1 - F[\theta^\Xi_{\infty}B(I_B, I_0)]]]
\]

\[
- \chi_B \theta^\Xi_{\infty}B(I_B, I_0)
\]

\[
- \chi_B(F/f)[\theta^\Xi_{\infty}B(I_B, I_0)] \xi_3[\theta^\Xi_{\infty}B(I_B, I_0)]
\]

\[
+ (1/\gamma) \xi_3[\theta^\Xi_{\infty}B(I_B, I_0)] \times \cdots
\]

\[
\cdots \times (\lambda^{Rev})'[I_0 - I_B[1 - F[\theta^\Xi_{\infty}B(I_B, I_0)]]]
\]

\[
= 0
\]

\[
> \rho + (\lambda^{Rev})'[I_0 - I_B[1 - F[\theta^\Xi_{\infty}B(I_B, I_0)]]]
\]

\[
- \chi_B \theta^\Xi_{\infty}B(I_B, I_0) - \gamma \chi_B (F/f)[\theta^\Xi_{\infty}B(I_B, I_0)],
\]
where (2.37) follows from (2.20)-(2.21), while (2.39) follows from the fact that \( \hat{\theta}^{|SP|}_B(I_B, I_0) > \hat{\theta}^{| Δ|SP}_B(I_B, I_0) \). Comparing (2.37) and (2.39) then yields

\[
0 > [\xi_3[\hat{\theta}^{|SP|}_B(I_B, I_0)] - \gamma] \chi_B (F/f)[\hat{\theta}^{|SP|}_B(I_B, I_0)] + [1 - (1/\gamma)] \xi_3[\hat{\theta}^{|SP|}_B(I_B, I_0)] \times \cdots
\]

\[
\cdots \times [(\lambda^{\text{Rev}}')^T[I_0 - I_B[1 - F[\hat{\theta}^{|SP|}_B(I_B, I_0)]]] - \Lambda'[I_0 - I_B[1 - F[\hat{\theta}^{|SP|}_B(I_B, I_0)]]],
\]

\[
\propto |I_0 - I_B[1 - F(\theta_B)]| \Lambda''[I_0 - I_B[1 - F(\theta_B)]] \leq 0
\]

so it must be the case that \( \gamma > \xi_3[\hat{\theta}^{|SP|}_B(I_B, I_0)] \), which verifies \((**)>0\).

In light of this uniqueness result, we can conclude that the planner also obeys a pecking order when trading off between the intensive and extensive margins. In particular, he prefers to use the latter until \( \Xi^{SP}_B(\theta_B, I_0, I_0) = 0 \), at which point he switches to the former. To get more specific about the conditions under which this switch occurs, I’ll take an approach similar to that in my previous subsection:

**Sublemma 2.A.6.** If \( \hat{\theta}^{|SP|}_B(I_0) := \hat{\theta}^{|SP|}_B(I_0, I_0) \), then \( (\hat{\theta}^{|SP|}_B)'(I_0) < 0 \), with \( \hat{\theta}^{|SP|}_B(I_0) \leq \hat{\theta}^{|SP|}_B(I_0, I_0) \), \( \forall I_0 \in \mathbb{R}_+ \).

**Proof.** That \( \hat{\theta}^{|SP|}_B(I_0) \leq \hat{\theta}^{|SP|}_B(I_0, I_0) \) should be clear from (2.33). As for my claim that \( (\hat{\theta}^{|SP|}_B)'(I_0) < 0 \), I note that

\[
(\Xi^{SP}_B(I_B[I_0[\hat{\theta}^{|SP|}_B(I_0)], I_0, I_0] + (\Xi^{SP}_B(I_0[I_0[\hat{\theta}^{|SP|}_B(I_0), I_0, I_0])
\]

\[
> 0
\]

\[
\left[ \frac{1}{\xi_3[\hat{\theta}^{|SP|}_B(I_0)]} - \frac{1}{\gamma} \right] F[\hat{\theta}^{|SP|}_B(I_0)] \Lambda''[I_0 F[\hat{\theta}^{|SP|}_B(I_0)]]
\]

\[
< 0
\]

\[
\frac{F[\hat{\theta}^{|SP|}_B(I_0)][(\lambda^{\text{Rev}})^T[I_0 F[\hat{\theta}^{|SP|}_B(I_0)]]]}{\gamma \chi_B}
\]

\[
< 0.
\]

Since I’ve already shown that

\[
(\Xi^{SP}_B(I_B[I_0[\hat{\theta}^{|SP|}_B(I_0), I_0, I_0] < 0,
\]

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we can conclude that \((\hat{\theta}^{SP})'(I_0) < 0\), as desired. More specifically,

\[
(\hat{\theta}^{SP})'(I_0) = \frac{(-1)[(\Xi^{SP})_{I_B}[\hat{\theta}^{\Xi|SP}(I_0), I_0, I_0] + (\Xi^{SP})_{I_B}[\hat{\theta}^{\Xi|SP}(I_0), I_0, I_0]]}{(\Xi^{SP})_{\theta}[\hat{\theta}^{\Xi|SP}(I_0), I_0, I_0]}
\]

\[
\cdots \times \left[\frac{1}{\xi_3[\hat{\theta}^{\Xi|SP}(I_0)]} - \frac{1}{\gamma} \right] \frac{\Lambda''[I_0 F[\hat{\theta}^{\Xi|SP}(I_0)]]}{\chi_B} + \frac{(\lambda^{Rev})''[I_0 F[\hat{\theta}^{\Xi|SP}(I_0)]]}{\gamma \lambda_B}
\]

\[
\frac{-\xi_1[\hat{\theta}^{\Xi|SP}(I_0)]}{[\xi_3[\hat{\theta}^{\Xi|SP}(I_0)]]^2} \left[\frac{\rho + \Lambda'[I_0 F[\hat{\theta}^{\Xi|SP}(I_0)]]}{\chi_B} - \xi_1[\hat{\theta}^{\Xi|SP}(I_0)]\right]
\]

\[
+ I_0 f[\hat{\theta}^{\Xi|SP}(I_0)] \times \cdots
\]

\[
\cdots \times \left[\frac{1}{\xi_3[\hat{\theta}^{\Xi|SP}(I_0)]} - \frac{1}{\gamma} \right] \frac{\Lambda''[I_0 F[\hat{\theta}^{\Xi|SP}(I_0)]]}{\chi_B} + \frac{(\lambda^{Rev})''[I_0 F[\hat{\theta}^{\Xi|SP}(I_0)]]}{\gamma \lambda_B}
\]

\[\tag{2.40}\]

namely due to the derivations above, along with \((2.35)\). \hfill \blacksquare

**Sublemma 2.A.7.** \(\forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D - \Delta \mu(D) \geq 0, \exists! \tilde{T}_B^{ED|SP}(D) \in [\tilde{T}_B^{ED}(D), \infty) \text{ s.t.}\)

\[I_0 \succ \tilde{T}_B^{ED|SP}(D) \iff (E^b + D - I_0) + \Delta_B^{SP}[\hat{\theta}^{\Xi|SP}(I_0), I_0, I_0] \leq \Delta \mu(D), \forall I_0 \in \mathbb{R}_+.
\]

Moreover, this function has the property that \(\Xi^{SP}_{I_B}\) admits a unique solution, namely of the “extensive distortion” type, whenever \(I_0 \in (\tilde{T}_B^{LS}(D), \tilde{T}_B^{ED|SP}(D))\). It’s also strictly decreasing, with \(E^b + D - \Delta \mu(D) = 0 \implies \tilde{T}_B^{ED|SP}(D) = 0\).

**Proof.** Under a solution of the “extensive distortion” type, the planner sets \(I_B = I_0\), while \(\theta_B\) satisfies the following conditions:

\[
(E^b + D - I_0) + \Delta_B^{SP}(\theta_B, I_0, I_0) - \Delta \mu(D) = 0 \tag{2.41}
\]

\[
\theta_B \in [\hat{\theta}^{\Xi|SP}(I_0), \theta_B^{LS}(I_0)] \tag{2.42}
\]
Since $\hat{\theta}_B^{\Xi|SP}(I_0) > \theta_B^{\Delta|SP}(I_0, I_0)$, it should be clear that this system admits a unique solution when the following two conditions obtain:

\[(E^b + D - I_0) + \Delta_B^{SP}[\hat{\theta}_B^{\Xi|SP}(I_0), I_0, I_0] - \Delta \mu(D) \geq 0 \quad (2.43)\]

\[(E^b + D - I_0) + \Delta_B^{SP}[\theta_B^{LS}(I_0), I_0, I_0] - \Delta \mu(D) < 0 \quad (2.44)\]

On the other hand, if either of these conditions fails, then no solution exists.

Now, (2.43) clearly fails whenever $D$ is large enough that $E^b + D - \Delta \mu(D) < 0$, so I’ll restrict attention to choices on $D$ satisfying $E^b + D - \Delta \mu(D) \geq 0$. Under this restriction, (2.44) holds i.f.f. $I_0 \leq \bar{T}_B^{LS}(D)$. As for (2.43), it should be clear that its left-hand side is non-negative at $I_0 = \bar{T}_B^{ED}(D)$, namely due to my previous lemma, combined with the aforementioned fact that $\hat{\theta}_B^{\Xi|SP}(I_0) > \theta_B^{\Delta|SP}(I_0, I_0)$. It should also be clear that the left-hand side of (2.43) goes to $-\infty$ as $I_0 \nearrow \infty$ and is strictly decreasing in $D$. I further claim that it’s strictly decreasing in $I_0$ as well. The argument is as follows. First, recall that

\[
\Delta_B^{SP}[\hat{\theta}_B^{\Xi|SP}(I_0), I_0, I_0]
\]

\[
= I_0 \int_{\hat{\theta}_B^{\Xi|SP}(I_0)}^{1} (\theta \chi_B - \rho) dF(\theta) - I_0 \gamma \chi_B \mathbb{E} [\hat{\theta}_B^{\Xi|SP}(I_0)] + \lambda^{Rev}[I_0 F[\hat{\theta}_B^{\Xi|SP}(I_0)]],
\]

so, using $\cdot$ to suppress obvious arguments,

\[
\frac{d}{dI_0} \left[ (E^b + D - I_0) + \Delta_B^{SP}[\hat{\theta}_B^{\Xi|SP}(I_0), I_0, I_0] - \Delta \mu(D) \right]
\]

\[
= \int_{\hat{\theta}_B^{\Xi|SP}(I_0)}^{1} (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max(\cdot) + F(\cdot)(\lambda^{Rev})'(\cdot) - 1
\]

\[
+ \left[ (\hat{\theta}_B^{\Xi|SP})'(I_0) I_0 f(\cdot)[\rho - \chi_B \hat{\theta}_B^{\Xi|SP}(I_0) - \gamma \chi_B (F/f)(\cdot) + (\lambda^{Rev})'(\cdot)]
\]

\[
\times I_0 f(\cdot)[\chi_B \hat{\theta}_B^{\Xi|SP}(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{Rev})'(\cdot)]
\]

\[
- \left[ \frac{1}{(\hat{\theta}_B^{\Xi|SP})'(I_0)} \right] \times \cdots
\]
\[ I_0 f(\cdot) [\chi_B \hat{\varpi}^{\Xi|SP}(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \]

+ \left[ \frac{-\xi_1(\cdot)}{\xi_3(\cdot)^2} \left[ \frac{\rho + \lambda'(\cdot)}{\chi_B} - \xi_1(\cdot) \right] \right. \\
\left. + \frac{1}{\xi_3(\cdot)} \left[ \frac{1}{\gamma} \Lambda''(\cdot) + \frac{(\lambda^{\text{Rev}})''(\cdot)}{\gamma \chi_B} \right] \right] \frac{I_0 f(\cdot)}{F(\cdot)} \times \ldots

\]

\[ \ldots \times \left[ \int_{\theta_B^{|SP}(I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \max(\cdot) + F(\cdot)(\lambda^{\text{Rev}})'(\cdot) - 1 \right] \]

\[ \frac{I_0 f(\cdot)}{F(\cdot)} \times \ldots \]

\[ \ldots \times \left[ \int_{\theta_B^{|SP}(I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \max(\cdot) + F(\cdot)(\lambda^{\text{Rev}})'(\cdot) - 1 \right] \]

\[ < \int_{\theta_B^{|SP}(I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \max(\cdot) + F(\cdot) \Lambda'(\cdot) - 1 \]

\[ = \int_{\theta_B^{|SP}(I_0)}^1 [\theta \chi_B - \rho - \Lambda'(\cdot)] - \gamma \chi_B \max(\cdot) + \Lambda'(\cdot) - 1 \]

\[ < \int_{\theta_B^{|SP}(I_0)}^1 [\theta \chi_B - \rho - \Lambda'(\cdot)] - \gamma \chi_B \max(\cdot) \]

\[ = \Delta_B [\hat{\varpi}^{\Xi|SP}(I_0), \Lambda'(\cdot)] \]

\[ < 0 \]

\[ < I_0 f(\cdot) [\chi_B \hat{\varpi}^{\Xi|SP}(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \]

\[ + \left[ \frac{I_0 f(\cdot)}{F(\cdot)} \right] \times \ldots \]

\[ \ldots \times \left[ \int_{\theta_B^{|SP}(I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \max(\cdot) + F(\cdot)(\lambda^{\text{Rev}})'(\cdot) - 1 \right] \]
\[ \propto F(\cdot)[\chi_B \hat{\theta}_B^{\Xi|SP}(I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot)] \]

\[ + \int_{\hat{\theta}_B^{\Xi|SP}(I_0)}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] + F(\cdot)(\lambda^{\text{Rev}})'(\cdot) - 1 \]

\[ = (1 - \gamma) \chi_B \mathbb{E} \max(\cdot) + \gamma \chi_B F(\cdot)(F/f)(\cdot) - (1 + \rho) \]

\[ = [\chi_B \mathbb{E} \max(\cdot) + \gamma \chi_B [F(\cdot)(F/f)(\cdot) - \mathbb{E} \max(\cdot)]] - (1 + \rho). \]

The monotonicity being claimed would thus go through if we could show that the starred term is non-positive. Now, if \( F[\hat{\theta}_B^{\Xi|SP}(I_0)](F/f)[\hat{\theta}_B^{\Xi|SP}(I_0)] - \mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] \leq 0 \), then this follows immediately from subassumption 2.32, so it would suffice if we could rule out the case where \( F[\hat{\theta}_B^{\Xi|SP}(I_0)](F/f)[\hat{\theta}_B^{\Xi|SP}(I_0)] - \mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] > 0 \). Now, if this last inequality were to hold, we would have

\[ \Xi[\hat{\theta}_B^{\Xi|SP}(I_0)] = \hat{\theta}_B^{\Xi|SP}(I_0) + \frac{(F/f)[\hat{\theta}_B^{\Xi|SP}(I_0)](\mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] - \hat{\theta}_B^{\Xi|SP}(I_0))}{\mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] + (F/f)[\hat{\theta}_B^{\Xi|SP}(I_0)][1 - F[\hat{\theta}_B^{\Xi|SP}(I_0)]]} \]

\[ > \mathbb{E} \max[\hat{\theta}_B^{\Xi|SP}(I_0)] \]

\[ > \rho + \mathcal{N}[I_0 F[\hat{\theta}_B^{\Xi|SP}(I_0)]] \]

\[ \chi_B \]

where (2.45) follows from (2.20)-\( (2.20) \), while (2.47) follows from the fact that \( \hat{\theta}_B^{\Xi|SP}(I_0) > \hat{\theta}_B^{\Xi|SP}(I_0) \). However, it should be clear from (2.33) that this contradicts the fact that \( \Xi[\hat{\theta}_B^{\Xi|SP}(I_0), I_0, I_0] = 0 \).

We can finally conclude the following. \( \forall D \in \mathbb{R}_+ \) s.t. \( E^b + D - \Delta \mu(D) \geq 0, \exists \mathcal{T}_B^{ED|SP}(D) \in [\mathcal{T}_B^{ED}(D), \infty) \) s.t.

\[ I_0 \geq \mathcal{T}_B^{ED|SP}(D) \iff (E^b + D - I_0) + \Delta_B^{SP}[\hat{\theta}_B^{\Xi|SP}(I_0), I_0, I_0] - \Delta \mu(D) \leq 0, \forall I_0 \in \mathbb{R}_+, \]

with \( \mathcal{T}_B^{ED|SP}(D) \) strictly decreasing. The sublemma’s claims on \( (\mathcal{T}_B^{SP}) \), along with the claim that \( E^b + D - \Delta \mu(D) = 0 \implies \mathcal{T}_B^{ED|SP}(D) = 0 \), then follow immediately.
Sublemma 2.A.8. \( \forall D \in \mathbb{R}_+ \) s.t. \( E^b + D - \Delta \mu(D) \geq 0 \), \((P^{SP}_{B} - \text{rex})\) admits a unique solution, namely of the “dual distortion” type, whenever \( I_0 \in (T^{ED}_{B}SP(D), T^{DD}_{B}D(D)) \).

Proof. Under a solution of the “dual distortion” type, the planner sets \( \theta_B = \theta_B^{\Xi|SP}(I_B, I_0) \), where \( I_B \) satisfies the following conditions: 

\[
(E^b + D - I_0) + \Delta^B_{SP}[	heta_B^{\Xi|SP}(I_B, I_0), I_B, I_0] - \Delta \mu(D) = 0 \tag{2.48}
\]

\[
I_B \in [0, I_0] \tag{2.49}
\]

Now, I claim that the left-hand side of (2.48) is strictly decreasing in \( I_B \), though the argument is a bit involved. As a first step, note that 

\[
\Xi^{SP}_{B}[	heta_B^{\Xi|SP}(I_B, I_0), I_B, I_0] 
\]

\[
= \frac{1}{\xi_3[\theta_B^{\Xi|SP}(I_B, I_0)]} \times \ldots 
\]

\[
\ldots \times \left[ \rho + \Lambda'[I_0 - I_B[1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]] \right] - \Xi^{\Xi|SP}_{B}(I_B, I_0) 
\]

\[
+ (\Lambda^{\text{Rev}})'[I_0 - I_B[1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]] 
\]

\[
\gamma \chi_B 
\]

\[
- \Lambda'[I_0 - I_B[1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]] 
\]

\[
\gamma \chi_B 
\]

\[
= 0, 
\]

with 

\[
(\Xi^{SP}_{B})_{|I_B}[\theta_B^{\Xi|SP}(I_B, I_0), I_B, I_0] 
\]

\[
= (-1)[1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \times \ldots 
\]

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Since I’ve already shown in my proof of sublemma 2.A.5 that
\[ (\Xi_{SP}^B)^{\theta_{SP}^B(I_B, I_0)} < 0, \]
we can conclude that \((\theta_{SP}^B)^{I_B(I_B, I_0)} > 0\). Now, in the aforementioned proof, I also noted that \(\xi_3'(\theta_B) < 0\), and

\[ \frac{d}{d\theta_B} \left[ \frac{\Xi(\theta_B)}{\xi_3(\theta_B)} \right] = \xi_1(\theta_B) \frac{d}{d\theta_B} \left[ \frac{1}{\xi_3(\theta_B)} \right], \]

so

\[ \frac{(-1)(\theta_{SP}^B)^{I_B(I_B, I_0)}\xi_3'[\theta_{SP}^B(I_B, I_0)]}{[\xi_3[\theta_{SP}^B(I_B, I_0)]^2} \times \cdots \]
\[ \times \left[ \frac{\rho + \Lambda'[I_0 - I_B[1 - F[\theta_{SP}^B(I_B, I_0)]]]}{\chi_B} - \xi_1[\theta_{SP}^B(I_B, I_0)] \right] \]

\[ < 0, \text{namely } b/c \theta_{SP}^B(I_B, I_0) > \theta_{SP}^B(I_B, I_0) \]
\[
\frac{d}{dI_B} \left[ I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \right] = \frac{1}{\xi_3(\theta_B^{\Xi|SP}(I_B, I_0)) - \frac{1}{\gamma}} \times \ldots \cdot \Lambda''[I_0 - I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]] \chi_B \underbrace{\Lambda''[I_0 - I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]]}_{>0} + \underbrace{(\lambda_{\text{Rev}}^{\prime\prime})'[I_0 - I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]]}_{<0},
\]

which means that
\[
\frac{d}{dI_B} \left[ I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \right] > 0.
\]

This is useful because
\[
\Delta_B^{\text{SP}}[\theta_B^{\Xi|SP}(I_B, I_0), I_B, I_0]
\]
\[
= \lambda_{\text{Rev}}[I_0 - I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]] + I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \times \ldots \times [\chi_B \xi_1[\theta_B^{\Xi|SP}(I_B, I_0)] - \rho - \gamma \chi_B \xi_2[\theta_B^{\Xi|SP}(I_B, I_0)],
\]

so
\[
\frac{d}{dI_B} \left[ (E^b + D - I_0) + \Delta_B^{\text{SP}}[\theta_B^{\Xi|SP}(I_B, I_0), I_B, I_0] - \Delta \mu(D) \right]
\]
\[
= \left( \theta_B^{\Xi|SP} \right)_{I_B(I_B, I_0)} I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \chi_B \left[ \xi_1[\theta_B^{\Xi|SP}(I_B, I_0)] - \gamma \xi_2[\theta_B^{\Xi|SP}(I_B, I_0)] \right] \chi_B \left[ \xi_1[\theta_B^{\Xi|SP}(I_B, I_0)] - \gamma \xi_2[\theta_B^{\Xi|SP}(I_B, I_0)] \right]
\]
\[
- \frac{d}{dI_B} \left[ I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]] \right] (\lambda_{\text{Rev}}^{\prime\prime})'[I_0 - I_B [1 - F[\theta_B^{\Xi|SP}(I_B, I_0)]]]
\]

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\[ + \frac{d}{dI_B} \left[ I_B \left[ 1 - F[I_B^{SP}(I_B, I_0)] \right] \right] \cdot \left[ \chi_B \left[ I_B^{SP}(I_B, I_0) \right] - \rho - \gamma \chi_B \left[ I_B^{SP}(I_B, I_0) \right] \right]. \]

As desired.

We can now conclude that the system on lines 2.48 and 2.49 admits a unique solution if the following two conditions hold:

\[ (E^b + D - I_0) + \Delta^{SP}_B [\theta_B^{\Xi^{SP}}(0, I_0), 0, I_0] \]

\[ = (E^b + D - I_0) + I_0 \Lambda'(I_0) \geq \Delta \mu(D) \] (2.50)

\[ (E^b + D - I_0) + \Delta^{SP}_B [\theta_B^{\Xi^{SP}}(I_0, I_0), I_0, I_0] \]

\[ = (E^b + D - I_0) + \Delta^{SP}_B [\theta_B^{\Xi^{SP}}(I_0, I_0), I_0, I_0] \leq \Delta \mu(D) \] (2.51)

On the other hand, if either of these conditions fails, then no solution exists. Now, it should be clear that (2.50) fails whenever \( D \) is large enough that \( E^b + D < \Delta \mu(D) \), so I’ll restrict attention to choices on \( D \) satisfying \( E^b + D \geq \Delta \mu(D) \). Moreover, given any such choice on \( D \), it should be clear that (2.50) \( \iff \) \( I_0 > T^{E|D|SP}_B(D) \), and (2.51) \( \iff \) \( I_0 \leq T^{DD}_B(D) \). The sublemma’s claims then follow immediately.

**Sublemma 2.A.9.** If instead \( E^b + D - \Delta \mu(D) < 0 \), or \( E^b + D - \Delta \mu(D) \geq 0 \) with \( I_0 > T^{DD}_B(D) \), then \( (P^{SP}_{B}-\text{rex}) \) is insoluble.

Proof. This should now be obvious.
should be clear that it would then suffice for us to strengthen subassumption 2.3.3 as follows:

Subassumption 2.3.3': \[ \lim_{I_0 \to \infty} \{ \theta_B^{SP}(0, I_0) \} > \theta_B^{\Psi + \Delta}. \]

That this condition is more likely to obtain the lower is \( \chi_B \) can easily be verified.

2.A.3 Proof of proposition 2.1

Suppose that banks face some tax \( \tau_B \) on storage in the bad state, along with a lump sum transfer \( T_B \) at \( t = 2 \). In this case, a repeat of the arguments underlying lemma 1.3.1 will show that their expected payout in the bad state is given by

\[
(1 - \tau_B)[(E^b + D - I_0) + \ell_B I_0] + T_B + I_B \int_{\theta_B}^{1} [\theta \chi_B - (1 - \tau_B)(\rho + \ell_B)]dF(\theta)
- \Delta \mu(D),
\]

(2.52)

while their financial constraint reads as

\[
(1 - \tau_B)[(E^b + D - I_0) + \ell_B I_0] + T_B + I_B \left[ \Pi_B(\theta_B, \ell_B, \tau_B) - \gamma \chi_B \mathbb{E} \max(\theta_B) \right] \geq \Delta \mu(D),
\]

(2.53)

with the physical constraint unchanged. Let \( (\bar{\Pi}_B) \) denote the task of choosing the subcontract \((\theta_B, I_B) \in [0, 1] \times [0, I_0] \) so as to maximize (2.52), subject to (2.53) and the usual physical constraint, while \( (\bar{\Pi}_B\text{-}rex) \) denotes a relaxed version of this program from which we drop the physical constraint.

Now, if \( \tau_B \in [0, 1) \), it should be clear that the post-tax per-unit surplus \( \bar{\Pi}_B(\theta_B, \ell_B, \tau_B) \), viewed as a function of the choice on \( \theta_B \), peaks around

\[
(1 - \tau_B)(\rho + \ell_B)/\chi_B =: \bar{\theta}_B(\ell_B, \tau_B)
\]

and either exhibits single-crossing from below, namely at some \( \bar{\theta}_B^{\Pi}(\ell_B, \tau_B) < \bar{\theta}_B^{\Pi}(\ell_B, \tau_B) \), or otherwise is positive over all of \([0, 1]\), in which case I set \( \bar{\theta}_B^{\Pi}(\ell_B, \tau_B) = 0 \). It should also be clear that the post-tax per-unit pledgable surplus \( \bar{\Delta}_B(\theta_B, \ell_B, \tau_B) \) peaks around a lower type
\( \tilde{\theta}_B^\Delta(\ell_B, \tau_B) < \tilde{\theta}_B^\Pi(\ell_B, \tau_B) \) satisfying

\[
(1 - \tau_B)(\rho + \ell_B) = \chi_B \tilde{\theta}_B^\Delta(\ell_B, \tau_B) + \chi_B \gamma(F/f)[\tilde{\theta}_B^\Delta(\ell_B, \tau_B)].
\]

So, if the subcontract \((\theta_B, I_B) = [\tilde{\theta}_B^\Pi(\ell_B, \tau_B), I_0] \) satisfies \((2.53)\), then we know that this subcontract uniquely solves \((\tilde{P}_B\text{-rex})\). Otherwise, \((2.53)\) binds, and the rate of transformation along this constraint is given by

\[
(dI_B/d\theta_B)\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B) + (\tilde{\Delta}_B)_{\theta}(\theta_B, \ell_B, \tau_B) = 0
\]

\[
\iff dI_B/d\theta_B = (-1)(\tilde{\Delta}_B)_{\theta}(\theta_B, \ell_B, \tau_B)/\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B).
\]

Reliance on the extensive margin is thus worthwhile i.f.f.

\[
(dI_B/d\theta_B)\tilde{\Pi}_B(\theta_B, \ell_B, \tau_B) + (\tilde{\Pi}_B)_{\theta}(\theta_B, \ell_B, \tau_B) \leq 0
\]

\[
\iff (\tilde{\Pi}_B)_{\theta}(\theta_B, \ell_B, \tau_B) \leq \tilde{\Pi}_B(\theta_B, \ell_B, \tau_B)(\tilde{\Delta}_B)_{\theta}(\theta_B, \ell_B, \tau_B)/\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B).
\]

Now, assuming that \(\tau_B\) satisfies \(\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B) < 0, \forall \theta_B \in [0, 1]\), this last inequality can be re-written as

\[
(\tilde{\Pi}_B)_{\theta}(\theta_B, \ell_B, \tau_B)\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B) \geq \tilde{\Pi}_B(\theta_B, \ell_B, \tau_B)(\tilde{\Delta}_B)_{\theta}(\theta_B, \ell_B, \tau_B),
\]

which obviously fails at \(\theta_B = \max\{\tilde{\theta}_B^\Delta(\ell_B, \tau_B), \tilde{\theta}_B^\Pi(\ell_B, \tau_B)\}\) but holds with strict inequality at \(\theta_B = \tilde{\theta}_B^\Pi(\ell_B, \tau_B)\). Conclude that the interval between these two points must admit some critical type \(\tilde{\theta}_B^\Xi(\ell_B, \tau_B)\) at which \((2.54)\) holds with equality. In fact, \(\tilde{\theta}_B^\Xi(\ell_B, \tau_B)\) is unique, namely because the arguments in subsection 1.A.3 can be used to re-write \((2.54)\) as

\[
\chi_B \Xi(\theta_B) \geq (1 - \tau_B)(\rho + \ell_B),
\]

with \(\Xi'(\cdot) > 0\).

With these points in mind, it’s best to proceed as follows:

**Sublemma 2.A.10.** For any initial balance sheet \((D, I_0)\) satisfying

\[
I_0 \in (T_B^{LS}(D), T_B^{ED|SP}(D)),
\]
the ex-post intervention \((\tau_B, T_B)\) can be chosen s.t. the subcontract \((\theta_B, I_B) = [\theta_B^{ED}(D, I_0), I_0]\) solves \((\tilde{P}_B)\) when \(\ell_B = \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]\) — i.e., \((\tau_B, T_B)\) implements the solution for the planner’s subproblem in the bad state. Moreover, the appropriate intervention satisfies \(\tau_B > 0\).

Proof. Fix some such initial balance sheet \((D, I_0)\), and set \(\ell_B = \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]\). I’ll begin by arguing that the pair \((\tau_B, T_B)\) can be chosen s.t. the subcontract \((\theta_B, I_B) = [\theta_B^{ED}(D, I_0), I_0]\) solves \((\tilde{P}_B)-\text{rex})\). Now, based on the analysis above, it should be clear that it would suffice if we could set \((\tau_B, T_B)\) s.t.

\[
\chi_B \Xi[\theta_B^{ED}(D, I_0)] \geq (1 - \tau_B)(\rho + \ell_B) > \chi_B \theta_B^{ED}(D, I_0), \tag{2.56}
\]

\[
(1 - \tau_B)[(E^b + D - I_0) + \ell_B I_0] + T_B + I_0 \tilde{\Delta}_B[\theta_B^{ED}(D, I_0), \ell_B, \tau_B] = \Delta \mu(D), \tag{2.57}
\]

\[
\tau_B \in (0, 1), \tag{2.58}
\]

and

\[
\tilde{\Delta}(\theta_B, \ell_B, \tau_B) < 0, \quad \forall \theta_B \in [0, 1]. \tag{2.59}
\]

Of course, \(T_B\) can always be chosen to satisfy (2.57), so I’ll focus on choosing \(\tau_B\) in a manner consistent with the other conditions in question. In particular, I’ll argue that the interval

\[
\left\{ \max \left\{ 0, 1 - \frac{\chi_B \Xi[\theta_B^{ED}(D, I_0)]}{\rho + \ell_B} \right\} , \min \left\{ 1 - \frac{\chi_B \theta_B^{ED}(D, I_0)}{\rho + \ell_B} , \frac{\xi_3[\theta_B^{ED}(D, I_0)]}{\gamma} (-1)\delta[I_0 F[\theta_B^{ED}(D, I_0)]] \right\} \right\} \subset (0, 1) \tag{2.60}
\]

admits some tax rate \(\tau_B^{ED}(D, I_0)\) satisfying

\[
\frac{\chi_B \theta_B^{ED}(D, I_0) - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B)}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B)} = \frac{\chi_B \theta_B^{ED}(D, I_0) - \rho - \ell_B}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - \rho - \ell_B - \delta[I_0 F[\theta_B^{ED}(D, I_0)]]}, \tag{2.61}
\]

and that setting \(\tau_B = \tau_B^{ED}(D, I_0)\) will do the trick.
As a first step in this direction, I make the following two observations about the lower bound of the interval on line 2.60:

(i). If this lower bound is equal to zero, it must be the case that

\[ \rho + \ell_B < \chi_B \Xi[\theta_B^{ED}(D, I_0)] \]

\[ = \chi_B[\theta_B^{ED}(D, I_0) + \xi_3[\theta_B^{ED}(D, I_0)](F/f)[\theta_B^{ED}(D, I_0)]] \]

\[ < \chi_B[\theta_B^{ED}(D, I_0) + \gamma(F/F)[\theta_B^{ED}(D, I_0)]] \]

where the equality follows from (2.20)-(2.22), combined with (1.12), and the last inequality follows from the fact that \( \xi_3[\theta_B^{ED}(D, I_0)] < \gamma \) (as shown in my proof of sublemma 2.A.5), along with the fact that \( \xi_3(\cdot) < 0 \) (as argued in subsection 1.A.3). In turn, using \( \cdot \) to suppress obvious arguments,

\[ \frac{\chi_B \theta_B^{ED}(D, I_0) - (1 - \tau_B)(\rho + \ell_B)}{\chi_B[\theta_B^{ED}(D, I_0) + \gamma \chi_B(F/f)(\cdot) - (1 - \tau_B)(\rho + \ell_B)]}_{\tau_B = \max\left\{ 0, 1 - \frac{\chi_B \Xi(\cdot)}{\rho + \ell_B} \right\}} < 0, \quad b/c \theta_B^{ED}(D, I_0) < \theta_B^{LS}(I_0) \]

\[ = \frac{\chi_B \theta_B^{ED}(D, I_0) - \rho - \ell_B}{\chi_B[\theta_B^{ED}(D, I_0) + \gamma \chi_B(F/f)(\cdot) - \rho - \ell_B]} > 0 \]

\[ < \frac{\chi_B \theta_B^{ED}(D, I_0) - \rho - \ell_B}{\chi_B[\theta_B^{ED}(D, I_0) + \gamma \chi_B(F/f)(\cdot) - \rho - \ell_B - \delta(I_0 F)(\cdot)]}_{< 0} \]

(ii). What if the lower bound in question is instead positive? Well in this case, again using \( \cdot \) to suppress obvious arguments, I note that

\[ \frac{\chi_B \theta_B^{ED}(D, I_0) - (1 - \tau_B)(\rho + \ell_B)}{\chi_B[\theta_B^{ED}(D, I_0) + \gamma \chi_B(F/f)(\cdot) - (1 - \tau_B)(\rho + \ell_B)]}_{\tau_B = \max\left\{ 0, 1 - \frac{\chi_B \Xi(\cdot)}{\rho + \ell_B} \right\}} = \frac{\theta_B^{ED}(D, I_0) - \Xi(\cdot)}{\theta_B^{ED}(D, I_0) + \gamma(F/f)(\cdot) - \Xi(\cdot)} \]

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\[
\begin{align*}
&= (-1)\xi_3(\cdot) \\
&\quad \frac{\gamma - \xi_3(\cdot)}{\gamma - \xi_3(\cdot)} > 0, \text{ as argued in (i) above} \\
&\quad (2.62)
\end{align*}
\]

\[
< \frac{\chi_B\theta_B^{ED}(D, I_0) - \rho - \ell_B}{\chi_B\theta_B^{ED}(D, I_0) + \gamma\chi_B(F/f)(\cdot) - \rho - \ell_B - \delta[I_0F(\cdot)]} > 0, \text{ b/c } \theta_B^{ED}(D,I_0) > \hat{\theta}_B^{SP}(I_0) > \hat{\theta}_B^{ISP}(I_0,I_0)
\]

\[
\iff (-1)\xi_3(\cdot) \times \cdots
\]

\[
\cdots \times \left[\chi_B\theta_B^{ED}(D, I_0) + \gamma\chi_B(F/f)(\cdot) - \rho - \ell_B - \delta[I_0F(\cdot)]\right]
\]

\[
< [\gamma - \xi_3(\cdot)][\chi_B\theta_B^{ED}(D, I_0) - \rho - \ell_B]
\]

\[
\iff (-1)\xi_3(\cdot)[\gamma\chi_B(F/f)(\cdot) - \delta[I_0F(\cdot)]]
\]

\[
< \gamma[\chi_B[\theta_B^{ED}(D, I_0)] - \rho - \ell_B]
\]

\[
\iff \xi_3(\cdot)/\gamma[\delta[I_0F(\cdot)] + \rho + \ell_B
\]

\[
< \chi_B[\theta_B^{ED}(D, I_0) + (F/f)(\cdot)\xi_3(\cdot)]
\]

\[
= \chi_B\Xi(\cdot)
\]

\[
\iff \Xi^{SP}_B[\theta_B^{ED}(D, I_0), I_0, I_0] < 0
\]

(2.65)

where (2.62) and (2.64) follow from (2.20)-(2.22), combined with (1.12), while (2.65) follows from (2.34) and obviously holds, namely because we’re given that the initial balance sheet \((D, I_0)\) lies in a region where the planner prefers to rely only on the extensive margin.

Repeating for the upper bound of the interval on line (2.60)
(iii). Suppose first that this upper bound is equal to $1 - \chi_B \theta_B^{ED}(D, I_0) / (\rho + \ell_B)$. In this case, using $\cdot$ to suppress obvious arguments, I note that

$$
\frac{\chi_B \theta_B^{ED}(D, I_0) - (1 - \tau_B)(\rho + \ell_B)}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F / f)(\cdot) - (1 - \tau_B)(\rho + \ell_B)} \bigg|_{\tau_B = \min \{1 - \chi_B \theta_B^{ED}(D, I_0) / (\rho + \ell_B), \ldots\}}
$$

$$
= 0 > \underbrace{\chi_B \theta_B^{ED}(D, I_0) - \rho - \ell_B}_{<0, \ b/c \ \theta_B^{ED}(D, I_0) < \theta^{L^2}(I_0)}
- \underbrace{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F / f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]}_{>0, \ b/c \ \theta_B^{ED}(D, I_0) > \theta^{L^2}(I_0, I_0)};
$$

(iv). Otherwise, it’s useful to recall that

$$
\Xi_{SP}^{B}[\theta_B^{ED}(D, I_0), I_0, I_0] < 0 \iff \rho + \ell_B + [\xi_3[\theta_B^{ED}(D, I_0)] / \gamma] \delta[I_0 F[\theta_B^{ED}(D, I_0)]]
$$

$$
< \chi_B \Xi[\theta_B^{ED}(D, I_0)]
= \chi_B \theta_B^{ED}(D, I_0)
+ \chi_B \xi_3[\theta_B^{ED}(D, I_0)](F / f)[\theta_B^{ED}(D, I_0)],
$$

where the bicondition follows from (2.34), while the final equality follows from (2.20)-(2.22), combined with (1.12). What’s more, since I’ve already shown that \(\xi_3[\theta_B^{ED}(D, I_0)] < \gamma\) [see (i) above], we can further conclude that

$$
\rho + \ell_B + [\xi_3[\theta_B^{ED}(D, I_0)] / \gamma] \delta[I_0 F[\theta_B^{ED}(D, I_0)]]
$$

$$
< \chi_B \theta_B^{ED}(D, I_0) + \gamma(F / f)[\theta_B^{ED}(D, I_0)].
$$

This last inequality is useful for the following reasons. If we use $\cdot$ to suppress obvious arguments, then

$$
\frac{\chi_B \theta_B^{ED}(D, I_0) - (1 - \tau_B)(\rho + \ell_B)}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F / f)(\cdot) - (1 - \tau_B)(\rho + \ell_B)} \bigg|_{\tau_B = \min \{1 - \chi_B \theta_B^{ED}(D, I_0) / (\rho + \ell_B), \ldots\}}
$$

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\[
\frac{\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B - [\xi_3(\cdot)/\gamma]\delta[I_0 F(\cdot)]}{\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - [\xi_3(\cdot)/\gamma]\delta[I_0 F(\cdot)]} > 0
\]

\[
\frac{\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B}{\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]} > 0, \text{ b/c } \theta_B^{ED} (D, I_0) > \theta_B^{P} (I_0) > \theta_B^{SF} (I_0, I_0)
\]

\[
\iff [\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B - [\xi_3(\cdot)/\gamma]\delta[I_0 F(\cdot)]] \times \cdots
\]

\[
\cdots \times [\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]]
\]

\[
= [\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B] \times \cdots
\]

\[
\cdots \times [\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]]
\]

\[
- [\xi_3[\theta_B^{ED} (D, I_0)]/\gamma]\delta[I_0 F(\cdot)] \times \cdots
\]

\[
\cdots \times [\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]]
\]

\[
> [\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B] \times \cdots
\]

\[
\cdots \times \left[ \begin{array}{c}
\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B \\
- [\xi_3(\cdot)/\gamma]\delta[I_0 F(\cdot)]
\end{array} \right]
\]

\[
\iff \chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B + [\xi_3(\cdot)/\gamma] \times \cdots
\]

\[
\cdots \times [\chi_B \theta_B^{ED} (D, I_0) + \gamma \chi_B (F/f)(\cdot) - \rho - \ell_B - \delta[I_0 F(\cdot)]]
\]

\[
> [\chi_B \theta_B^{ED} (D, I_0) - \rho - \ell_B][\xi_3(\cdot)/\gamma]
\]

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\[ \iff \chi_B \theta_B^{ED}(D, I_0) + \chi_B \xi_3(\cdot)(F/f)(\cdot) \]

\[ = \chi_B \Xi(\cdot) \quad (2.66) \]

\[ > \rho + \ell_B + [\xi_3(\cdot)/\gamma]\delta[I_0 F(\cdot)], \]

\[ \iff \Xi^{SP}[\theta_B^{ED}(D, I_0), I_0, I_0] < 0 \quad (2.67) \]

where \((2.66)\) follows from \((2.20)-(2.22)\), combined with \((1.12)\), while \((2.65)\) follows from \((2.34)\) and obviously holds, namely because we’re given that the initial balance sheet \((D, I_0)\) lies in a region where the planner prefers to rely only on the extensive margin.

In light of these observations, combined with the fact that

\[
\frac{d}{d\tau_B} \left[ \frac{\chi_B \theta_B^{ED}(D, I_0) - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B)}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B)} \right] \\
= \frac{(\rho + \ell_B)\gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)]}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B)^2} > 0,
\]

we can conclude that the interval on line \((2.60)\) is non-empty and admits a unique tax rate \(\tau_B^{ED}(D, I_0)\) satisfying \((2.61)\), along with \((2.56)\) and \((2.58)\). As for \((2.59)\), I note that all \(\theta_B \in [0, 1]\) satisfy

\[
\tilde{\Delta}_B[\theta_B^{ED}(D, I_0), \ell_B, \tau_B^{ED}(D, I_0)] = \int_{\theta_B}^{1} \left[ \chi_B \theta - [1 - \tau_B^{ED}(D, I_0)](\rho + \ell_B) \right] dF(\theta) \\
- \gamma \chi_B \mathbb{E} \max(\theta_B) \quad (2.68)
\]
\(- \gamma \chi_B \mathbb{E} \max(\theta_B)\)

\(< \int_{\theta_B}^1 [\theta \chi_B - \rho - \ell_B - \delta [I_0 F[\theta^{ED}_B(D, I_0)]]] dF(\theta)\)

\(- \gamma \chi_B \mathbb{E} \max(\theta_B)\)

\(= \int_{\theta_B}^1 [\theta \chi_B - (\chi^\text{Rev})' I_0 F[\theta^{ED}_B(D, I_0)]] dF(\theta)\)

\(- \gamma \chi_B \mathbb{E} \max(\theta_B)\)

\(< \int_{\theta_B}^1 (\theta \chi_B - \rho) dF(\theta) - \gamma \chi_B \mathbb{E} \max(\theta_B)\)

\(= \Delta_B(\theta_B, 0)\)

\(< 0,\)

where (2.69) follows from the upper bound on line 2.60, while (2.74) follows from subassumption 2.3.1.

Having thus confirmed that the aforementioned choices on \((\tau_B, T_B)\) have the property that the subcontract \((\theta_B, I_B) = [\theta^{ED}_B(D, I_0), I_0]\) solves \((\tilde{P}_B\text{-rex})\), all that remains is to argue that this solution generalizes to \((\tilde{P}_B)\). Fortunately, this is immediate. In particular, it should be clear that the physical constraint facing banks coincides with that facing the planner when the pair \((\tau_B, T_B)\) has been chosen as prescribed above, so the desired generalization follows from the fact that the latter constraint is is lax, as shown at the end of my last subsection.

\textbf{Sublemma 2.A.11.} \textit{For any initial balance sheet \((D, I_0)\) satisfying}

\[I_0 \in (\tilde{T}_B^{ED,SP}(D), \tilde{T}_B^{DD}(D)),\]

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the ex-post intervention \((\tau_B, T_B)\) can be chosen s.t. the subcontract \((\theta_B, I_B) = [\theta_B^{DD|SP}(D, I_0), I_B^{DD|SP}(D, I_0)]\) solves \((\tilde{\mathbb{P}}_B)\) when
\[
\ell_B = \Lambda'[I_0 - I_B^{DD|SP}(D, I_0)[1 - F[\theta_B^{DD|SP}(D, I_0)]]]
\]
— i.e., \((\tau_B, T_B)\) implements the solution for the planner’s subproblem in the bad state. Moreover, the appropriate intervention satisfies \(\tau_B > 0\).

**Proof.** Fix some such initial balance sheet \((D, I_0)\), and set \(\ell_B = \Lambda'[I_0 - I_B^{DD|SP}(D, I_0)[1 - F[\theta_B^{DD|SP}(D, I_0)]]]\). I’ll begin by arguing that the pair \((\tau_B, T_B)\) can be chosen s.t. the subcontract \((\theta_B, I_B) = [\theta_B^{DD|SP}(D, I_0), I_B^{DD|SP}(D, I_0)]\) solves \((\tilde{\mathbb{P}}_B-\text{rex})\). Now, based on the analysis at the beginning of this subsection, it should be clear that it would suffice if we could set \((\tau_B, T_B)\) s.t.
\[
\chi_B \Xi[\theta_B^{ED}(D, I_0)] = (1 - \tau_B)(\rho + \ell_B) \tag{2.75}
\]
\[
(1 - \tau_B)[(E^b + D - I_0) + \ell_B I_0] + T_B + I_0 \tilde{\Delta}_B[\theta_B^{ED}(D, I_0), \ell_B, \tau_B] = \Delta \mu(D), \tag{2.76}
\]
\[
\tau_B \in (0, 1), \tag{2.77}
\]
and
\[
\tilde{\Delta}_B(\theta_B, \ell_B, \tau_B) < 0, \ \forall \theta_B \in [0, 1]. \tag{2.78}
\]

Of course, \(T_B\) can always be chosen to satisfy \([2.76]\), so I’ll focus on choosing \(\tau_B\) in a manner consistent with the other conditions in question. In particular, I note that
\[
\Xi_B^{SP}[\theta_B^{DD|SP}(D, I_0), I_B^{DD|SP}(D, I_0), I_0] = 0
\]
\[
\iff \chi_B \Xi[\theta_B^{DD|SP}(D, I_0)] = \rho + \ell_B + [\xi_3[\theta_B^{DD|SP}(D, I_0)]/\gamma] \times \cdots
\]
\[
\cdots \times \delta[I_0 I_B^{DD|SP}(D, I_0)[1 - F[\theta_B^{DD|SP}(D, I_0)]]], \tag{2.79}
\]
where the bicondition follows from \([2.34]\). Comparing \([2.75]\) and \([2.79]\), we can conclude that
\[
\tau_B = \frac{\xi_3[\theta_B^{DD|SP}(D, I_0)](-1)\delta[I_0 - I_B^{DD|SP}(D, I_0)[1 - F[\theta_B^{DD|SP}(D, I_0)]]]}{\gamma} \frac{\rho + \ell_B}{\rho + \ell_B}
\]
That this choice on $\tau_B$ satisfies the lower bound in (2.77) should be obvious, and a repeat of the arguments on lines 2.68 through 2.74 will show that it also satisfies (2.78), so all that remains is to verify the upper bound in (2.77). On this front, it suffices to recall from my previous subsection that $\theta_B^{DD|SP}(D,I_0) = \theta_B^{\Xi|SP} [I_B^{DD|SP}(D,I_0), I_0]$, with $\xi_3[\theta_B^{\Xi|SP} [I_B^{DD|SP}(D,I_0), I_0]] < \gamma$, and then note that

$$\frac{(-1)\delta[I_0 - I_B^{DD|SP}(D,I_0)](1 - F[\theta_B^{DD|SP}(D,I_0)])]}{\rho + \ell_B} < 1$$

$$\Leftrightarrow 0 < \rho + \ell_B + \delta[I_0 - I_B^{DD|SP}(D,I_0)](1 - F[\theta_B^{DD|SP}(D,I_0)]) = (\lambda^{rev})[I_0 - I_B^{DD|SP}(D,I_0)](1 - F[\theta_B^{DD|SP}(D,I_0)]) > 0 \text{ (see assumption 2.5)}.$$ (2.81)

All that remains is then to argue that our solution for $(\bar{P}_B\text{-rex})$ generalizes to $(\bar{P}_B)$. Fortunately, this should be immediate. In particular, it should be clear that the physical constraint facing banks coincides with that facing the planner when the pair $(\tau_B, T_B)$ has been chosen as prescribed above, so the desired generalization follows from the fact that the latter constraint is is lax, as shown at the end of my last subsection.

Remark 1. To summarize, for all initial balance sheets $(D, I_0) \in \mathbb{R}_+^2$ satisfying $E^b + D \geq \Delta \mu(D)$ and $I_0 \leq T_B^{DD}(D)$, the appropriate ex-post intervention is given by

$$\tau_B = \begin{cases} 
0 & \text{if } I_0 \in [0, T_B^{LS}(D)] \\
\tau_B^{ED}(D,I_0) & \text{if } I_0 \in (T_B^{LS}(D), T_B^{ED|SP}(D)] \\
\tau_B^{DD}(D,I_0) & \text{if } I_0 \in (T_B^{ED|SP}(D), T_B^{DD}(D)]
\end{cases}$$

$$=: \tau_B^*(D,I_0).$$ (2.82)
and

\[
T_B = \begin{cases} 
0 & \text{if } I_0 \in [0, T^{LS}_B(D, I_0)] \\
\Delta \mu(D) - [1 - \tau^{ED}_B(D, I_0)][(E^b + D - I_0) - I_0 \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]]] \\
- I_0 \tilde{\Delta}_B \left[ \theta^{ED}_B(D, I_0), \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)], \tau^{ED}_B(D, I_0) \right] & \text{if } I_0 \in (T^{LS}_B(D), T^{ED|SP}_B(D, I_0)] \\
\Delta \mu(D) - [1 - \tau^{DD}_B(D, I_0)] \times \cdots \\
\cdots \times \left[ (E^b + D - I_0) \\
+ I_0 \Lambda'[I_0 - I^{DD|SP}_B(D, I_0)[1 - F[\theta^{DD|SP}_B(D, I_0)]]] \right] \\
- I_0 \tilde{\Delta}_B \left[ \Lambda'[I_0 - I^{DD|SP}_B(D, I_0)[1 - F[\theta^{ED}_B(D, I_0)]]], \right. \\
\left. \tau^{DD}_B(D, I_0) \right] & \text{if } I_0 \in (T^{ED|SP}_B(D), T^{DD}_B(D, I_0)] \\
=: T^*_B(D, I_0), & \text{if } I_0 \in (T^{LS}_B(D), T^{ED}_B(D)] 
\end{cases}
\]

(2.83)

where \(\tau^{ED}_B(D, I_0)\) and \(\tau^{DD}_B(D, I_0)\) are defined on lines 2.61 and 2.80 respectively.

**Remark 2.** That \(\tau^*_B(D, I_0) > 0\) when \(I_0 \in (T^{LS}_B(D), T^{ED}_B(D)]\) may be surprising, since we know that there’s no need for ex-post intervention when initial balance sheets lie in this range. However, recall that policymakers target an inequality of the form on line 2.56 whenever initial balance sheets satisfy \(I_0 \in (T^{LS}_B(D), T^{ED|SP}_B(D)]\), rather than an equality of the form on line 2.75, which they target when \(I_0 \in (T^{ED|SP}_B(D), T^{DD}_B(D)]\). As a result, when \(I_0 \in (T^{LS}_B(D), T^{ED}_B(D)]\), multiple choices on \((\tau_B, T_B)\) will implement the solution for the planner’s ex-post problem, including \((\tau_B, T_B) = (0, 0)\). The particular choice on which
I’ve chosen to focus will prove especially tractable when we eventually turn our attention to
the task of introducing an ex-ante intervention into the overall policy mix.

2.A.4 Proof of lemmata 2.3.7 and 2.3.8

I begin by importing the following subassumptions from chapter [1] with adjustments ensuring
that they hold over the full range of liquidation values:

**Subassumption 2.3.4.** \( \min \{ \Delta_G[\theta_G^c[\Lambda'(0)], \Lambda'(0)], \rho - [\rho + \Lambda'(0)]F[\theta_G^c[\Lambda'(0)] \} > 0 \), where
\( \theta_G^c(\ell_G) \) is defined in subsection [1.A.5.5], though I’ve adjusted my notation to be explicit about
its dependence on the liquidation value \( \ell_G \).

**Subassumption 2.3.5.** \((1 - \gamma)\chi_B \max (\rho / \chi_B) > [\rho + \Lambda'(0)]F[\chi^{-1}_G[\rho + \Lambda'(0)]]\).

That these conditions are more likely to hold the greater is \( \chi_G \) can easily be verified.

To see why subassumption 2.3.4 is useful, fix some secondary-market price \( \ell_G \) and consider
banks’ behaviour when they take this price as given. Based on the analysis in subsection [1.A.5.5], we know that they will make initial balance-sheet choices which allow them to place
\( \theta_G \) somewhere in the interval \([\theta_\Pi_G(\ell_G), \theta_\zeta_G(\ell_G)]\). With \( \theta_G \) in this range, a sufficient condition
for \( \Delta_G(\theta_G, \ell_G) > 0 \) would be that \( \Delta_G[\theta_G^c(\ell_G), \ell_G] > 0 \), which subassumption 2.3.4 can easily
be shown to verify. So, in any monotonic equilibrium, it must be the case that

\[
(E^b + D - I_0) + \ell_G I_0 + I_G \Delta_G(\theta_G, \ell_G) \geq (E^b + D - I_0) + \ell_B I_0 + I_B \Delta_B(\theta_B, \ell_B),
\]

where the inequality follows from the fact that \( \ell_G \geq \ell_B \), combined with the fact that sub-
assumption 2.3.1 keeps \( \Delta_B(\theta_B, \ell_B) \) strictly negative, irrespective as to \((\theta_B, \ell_B)\). Conclude
that the financial constraint associated with the bad state must be tighter than the financial
constraint associated with the good state, so the latter must be lax.

Having thus confirmed that the physical constraint is all the physical constraint is all we
have to worry about in the good state, most of the claims made in lemma 2.3.7 can easily
be verified. The only potentially ambiguous claim is that \( T_G^{LS}(0) < T_B^{LS}(0) \). Fortunately, a
repeat of the arguments in subsection [1.A.4] should make it clear that it would suffice if we
could show that

\[
1 - \Lambda'[T_G^{LS}(0) F[\theta_G^{LS} T_G^{LS}(0)]] + \Psi_G[\theta_G^{LS} T_G^{LS}(0)], \Lambda'[T_G^{LS}(0) F[\theta_G^{LS} T_G^{LS}(0)]]
\]

\((*)\)
Now, using \( \cdot \) to suppress arguments, we have

\[
(\ast) = 1 - \Lambda'(\cdot) + \Psi_G \left[ \frac{\rho + \Lambda'(\cdot)}{\chi_G}, \Lambda'(\cdot) \right] = 1 + \rho - [\rho + \Lambda'(\cdot)]F \left[ \frac{\rho + \Lambda'(\cdot)}{\chi_G} \right],
\]

and

\[
(\ast\ast) = 1 - \Lambda'(\cdot) - \Delta_B \left[ \frac{\rho + \Lambda'(\cdot)}{\chi_B}, \Lambda'(\cdot) \right] = 1 + \rho - (1 - \gamma)\chi_B \mathbb{E} \max \left[ \frac{\rho + \Lambda'(\cdot)}{\chi_B} \right],
\]

where the final equality follows from integration by parts. That subassumption 2.3.5 suffices should then be obvious.

As for lemma 2.3.8, I’ll take a somewhat less direct approach. In particular, I let \((\mathbb{P}_G^{SP})\) denote the planner’s ex-post problem in the good state, and then let \((\mathbb{P}_G^{SP}-\text{rex})\) denote a relaxed version of this program from which I drop the financial constraint. Similarly, I let \((\mathbb{P}_G^{SP})\) denote the planner’s problem at \(t = 0\), and let \((\mathbb{P}_G^{SP}-\text{rex})\) denote a relaxed version of this program from which I dropped the physical constraint associated with the bad state, and the financial constraint associated with the good state, along with the non-negativity constraint \(I_0 \leq E^b + D\). That the solution described in 2.3.8 solves \((\mathbb{P}_G^{SP}-\text{rex})\) should then be obvious, so it would suffice if we could also show that solutions for \((\mathbb{P}_G^{SP}-\text{rex})\) generalize to \((\mathbb{P}_G^{SP})\), which I do in subsubsection 2.A.6.5.

\section*{2.A.5 Proof of lemmata 2.4.1 and 2.4.2}

\subsection*{2.A.5.1 Notation}

I begin by introducing some notation:

- first, I let \(\overline{D}_B^{LS}\) denote the unique point at which the functions \(\overline{T}_B^{LS}(D)\) and \(\overline{T}_G^{LS}(D)\) intersect, with \(\overline{D}_B^{ED}\) and \(\overline{D}_B^{DD}\) defined analogously;

- also, \(\forall (D, I_0) \in \mathbb{R}_+^2\) satisfying \(I_0 \leq \overline{T}_B^{DD}(D)\) and \(E^b + D - \Delta \mu(D) \geq 0\), I define \(v_{DD}^{\tau_B}(D, I_0)\)
\[
\frac{1 - \Delta \mu'(D)}{[1 - \Delta \mu'(D)] \times \ldots}
\]
\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]]}{(\Delta_B)[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]]} \right]
\]
\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - I_B^{DD}(D, I_0)][1 - F[\theta_B^{DD}(D, I_0)]]]}{(\Delta_B)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 F[\theta_B^{DD}(D, I_0)]]]} \right]
\]\n
\[
[1 - \Delta \mu'(D)] \times \ldots
\]
\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - I_B^{DD}(D, I_0)][1 - F[\theta_B^{DD}(D, I_0)]]]}{(\Delta_B)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 F[\theta_B^{DD}(D, I_0)]]]} \right]
\]\n
---

i.e., \( v_{BD}^r(D, I_0) \) gives banks’ return from the marginal deposit in the bad state. Similarly,

\[
v_{BI}^r(D, I_0)
\]

\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - I_B^{DD}(D, I_0)][1 - F[\theta_B^{DD}(D, I_0)]]]}{(\Delta_B)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 F[\theta_B^{DD}(D, I_0)]]]} \right]
\]\n
\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - I_B^{DD}(D, I_0)][1 - F[\theta_B^{DD}(D, I_0)]]]}{(\Delta_B)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 F[\theta_B^{DD}(D, I_0)]]]} \right]
\]\n
\[
[1 - \Delta \mu'(D)] \times \ldots
\]
\[
\ldots \times \left[ 1 - \frac{\Pi_B[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - I_B^{DD}(D, I_0)][1 - F[\theta_B^{DD}(D, I_0)]]]}{(\Delta_B)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 F[\theta_B^{DD}(D, I_0)]]]} \right]
\]\n
---

---

\[
v_{GD}^r(D, I_0)
\]
\[
\begin{align*}
1 - \Delta \mu'(D) & \quad \text{if } r_G = LS \\
1 - \Delta \mu'(D) + \frac{\left(\Pi_G\right)_0[\theta^L_G(D, I_0), \Lambda'[I_0 F[\theta^L_G(D, I_0)]]]}{\left(\Psi_G\right)_0[\theta^L_G(D, I_0), \Lambda'[I_0 F[\theta^L_G(D, I_0)]]]} & \quad \text{if } r_G = LR,
\end{align*}
\]

and

\[
v_{GI}^r(D, I_0) =
\begin{cases}
\Lambda'[I_0 F[\theta^L_G(I_0)]] + \Pi_G[\theta^L_G(I_0), \Lambda'[I_0 F[\theta^L_G(I_0)]]] - 1 & \quad \text{if } r_G = LS \\
\Lambda'[I_0 F[\theta^L_G(D, I_0)]] + \Pi_G[\theta^L_G(D, I_0), \Lambda'[I_0 F[\theta^L_G(D, I_0)]]] - 1 \\
-1 & \quad \text{if } r_G = LR;
\end{cases}
\]

finally, \( \forall (D, I_0) \in \mathbb{R}_+^2 \) satisfying \( I_0 \leq \bar{T}^{DD}(D) \) and \( E^b + D - \Delta \mu(D) \geq 0 \), I let \( v_D(D, I_0) \) denote banks' expected return from the marginal deposit, computed on an unconditional basis at \( t = 0 \), after taking lemmata 2.3.1 through 2.3.7 into account — e.g.,

\[
\begin{align*}
I_0 & \in (\bar{T}^{LS}_G(D), \bar{T}^{LS}_B(D)) \implies v_D(D, I_0) = \alpha_G v^L_G(D, I_0) + \alpha_B v^L_B(D, I_0).
\end{align*}
\]

Define \( v_I(D, I_0) \) analogously.

**Remark 1.** It can easily be verified that all of the marginal return functions defined above are continuous in both their arguments, even around the boundaries separating regimes. This is a consequence of the envelope theorem.

**Remark 2.** For clarity, figure 2.6 illustrates some of the notation used in this subsection.
2. A. 5. 2 Some preliminary results

As a first step, it will be useful to differentiate the marginal return functions defined above. I’ll proceed by taking cases on regimes and states:

Case one: $r_G = LR$. Under this case, $\theta^L_R(D, I_0)$ solves

$$(E^b + D - I_0) + I_0 \Lambda'[I_0 F[\theta^L_R(D, I_0)]] = I_0 \Psi_G[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]]],$$
or equivalently

$$(E^b + D - I_0) + \lambda^{Rev}[I_0 F[\theta^L_R(D, I_0)]] = \rho I_0[1 - F[\theta^L_R(D, I_0)].$$

Differentiating w.r.t. $I_0$ then yields

$$(\theta^L_R)_I(D, I_0) = \frac{1 + \rho - F[\theta^L_R(D, I_0)][\rho + (\lambda^{Rev})'[I_0 F[\theta^L_R(D, I_0)]]]}{I_0 f[\theta^L_R(D, I_0)][\rho + (\lambda^{Rev})'[I_0 F[\theta^L_R(D, I_0)]]] > 0, \quad (2.84)$$

with

$$\frac{d}{dI_0} \left[I_0 F[\theta^L_R(D, I_0)] \right] = \frac{1 + \rho}{\rho + (\lambda^{Rev})'[I_0 F[\theta^L_R(D, I_0)]]} > 0. \quad (2.85)$$

On the other hand, differentiating w.r.t. $D$ yields

$$(\theta^L_R)_D(D, I_0) = \frac{-1}{I_0 f[\theta^L_R(D, I_0)][\rho + (\lambda^{Rev})'[I_0 F[\theta^L_R(D, I_0)]]] < 0, \quad (2.86)$$

with

$$\frac{d}{dD} \left[I_0 F[\theta^L_R(D, I_0)] \right] = \frac{-1}{\rho + (\lambda^{Rev})'[I_0 F[\theta^L_R(D, I_0)]]} < 0. \quad (2.87)$$

Moreover,

$$v_{GL}^L(D, I_0) = \Lambda'[I_0 F[\theta^L_R(D, I_0)]] + \Pi_G[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]] - 1$$

$$- [1 - \Lambda'[I_0 F[\theta^L_R(D, I_0)]] + \Psi_G[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]]] \times \cdots$$

$$\cdots \times \frac{(\Pi_G)_0[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]]]}{(\Psi_G)_0[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]]]}$$

$$= \Lambda'[I_0 F[\theta^L_R(D, I_0)]] + \Pi_G[\theta^L_R(D, I_0), \Lambda'[I_0 F[\theta^L_R(D, I_0)]] - 1$$
\[
\begin{align*}
&- [1 - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] + \Psi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] \times \ldots \\
&\ldots \times \frac{\chi_G \theta_G^{LR}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\end{align*}
\]

\[
\pi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]
\]

\[
\begin{align*}
&+ \Psi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] \\
&- [1 - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] + \Psi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] \times \ldots \\
&\ldots \times \frac{\chi_G \theta_G^{LR}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\end{align*}
\]

\[
\begin{align*}
&= \int_{\theta_G^{LR}(D, I_0)}^{1} \theta \chi_G dF(\theta) \\
&- [1 + \rho - [\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] F[\theta_G^{LR}(D, I_0)] \times \ldots \\
&\ldots \times \frac{\chi_G \theta_G^{LR}(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\end{align*}
\]

\[
\begin{align*}
&= \int_{\theta_G^{LR}(D, I_0)}^{1} \theta \chi_G dF(\theta) - \frac{(1 + \rho) \chi_G \theta_G^{LR}(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]} + \chi_G \theta_G^{LR}(D, I_0) \\
&= \chi_G \left[ \mathbb{E} \max[\theta_G^{LR}(D, I_0)] - \frac{(1 + \rho) \theta_G^{LR}(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]} \right], \quad (2.88)
\end{align*}
\]

so, \( \forall x \in \{D, I_0\} \), we have

\[
(\psi^{LR}_{GI})_x(D, I_0) \propto (\theta_G^{LR})_x(D, I_0) \left[ F[\theta_G^{LR}(D, I_0)] - \frac{(1 + \rho)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]} \right]_{< 0}
\]
\[
- \frac{d}{dx} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \frac{<0}{\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)](1 + \rho)\theta_G^{LR}(D, I_0)} \frac{\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)](1 + \rho)\theta_G^{LR}(D, I_0)}{[\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]^2},
\]

which means that

\[
(v_{GL}^{LR})_I(D, I_0) < 0 < (v_{GD}^{LR})_D(D, I_0).
\]

Similarly,

\[
v_{GD}^{LR}(D, I_0) = 1 - \Delta \mu'(D) + \frac{(\Pi_G)_\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}{(\Psi_G)_\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\]

\[
= \frac{\chi_G \theta_G^{LR}(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]} - \Delta \mu'(D),
\]

so

\[
(v_{GD}^{LR})_I(D, I_0) = \frac{\chi_G(\theta_G^{LR})_I(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]
\]

\[
- \frac{d}{dI_0} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \frac{\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]\chi_G \theta_G^{LR}(D, I_0)}{[\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]^2}
\]

\[
> 0,
\]

and

\[
(v_{GD}^{LR})_D(D, I_0) = \frac{\chi_G(\theta_G^{LR})_D(D, I_0)}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]
\]

\[
- \frac{d}{dD} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \frac{\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]\chi_G \theta_G^{LR}(D, I_0)}{[\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]^2}
\]

\[
- \Delta \mu''(D)
\]

\[
< 0.
\]
Case two: $r_B = ED$. Under this case, the financial constraint can be written as

$$(E^b + D - I_0) + \lambda^{Rev}[I_0F[\theta^ED_B(D, I_0)]]$$

$$+ I_0 \left[ \int_{\theta^ED_B(D, I_0)}^{1} (\theta\chi_B - \rho)dF(\theta) - \gamma\chi_B E \max[\theta^ED_B(D, I_0)] \right] = \Delta \mu(D),$$

which differentiates as follows, where $\cdot$ suppresses obvious arguments:

$$\left( \theta^ED_B \right)_D(D, I_0) = \frac{1 - \Delta \mu'(D)}{I_0f(\cdot) \times \cdots \times \left[ \chi_B \theta^ED_B(D, I_0) + \gamma\chi_B (F/f)(\cdot) - \rho - (\lambda^{Rev})' \right]}$$

$$< 0 \quad (2.91)$$

$$\left( \theta^ED_B \right)_I(D, I_0) = \frac{1 - F(\cdot)(\lambda^{Rev})'(\cdot)}{\int_{\theta^ED_B(D, I_0)}^{1} (\theta\chi_B - \rho)dF(\theta) - \gamma\chi_B E \max(\cdot)}$$

$$- (\cdot) I_0 f(\cdot) \times \cdots \times \left[ \chi_B \theta^ED_B(D, I_0) + \gamma\chi_B (F/f)(\cdot) - \rho - (\lambda^{Rev})' \right]$$

$$< 0 \quad (2.92)$$

Conclude that

$$\frac{d}{dI_0} \left[ I_0F[\theta^ED_B(D, I_0)] \right]$$

$$= F[\theta^ED_B(D, I_0)] + I_0f[\theta^ED_B(D, I_0)][\theta^ED_B'(D, I_0)'](D, I_0)$$
by $E$ is indeed the case, note that assumption 2.3 keeps this difference strictly bounded from below

easily be shown to imply that

where I claim that $(\ast)$

Next, note that

Conclude that

Next, note that

$\nu_{B_1}^{ED}(D,I_0)$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$
\[- F[\theta_B^{ED}(D, I_0)] [\chi_B \theta_B^{ED}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] \]

\[+ [1 - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]] - \Delta_B[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] \times \cdots \]

\[\cdots \times \frac{\chi_B \theta_B^{ED}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - \rho - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]} \]

\[= \chi_B \mathbb{E} \max[\theta_B^{ED}(D, I_0)] - (1 + \rho) \]

\[+ [\rho + \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]] - \chi_B \theta_B^{ED}(D, I_0)] \times \cdots \]

\[\left[ (1 - \gamma) \chi_B \mathbb{E} \max[\theta_B^{ED}(D, I_0)] + \gamma \chi_B F[\theta_B^{ED}(D, I_0)](F/f)[\theta_B^{ED}(D, I_0)] \right] \]

\[\cdots \times \frac{-(1 + \rho)}{\chi_B \theta_B^{ED}(D, I_0) + \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)] - \rho - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]} \]

\[= : \chi_B \mathbb{E} \max[\theta_B^{ED}(D, I_0)] - (1 + \rho) \]

\[\left[ (1 - \gamma) \chi_B \mathbb{E} \max[\theta_B^{ED}(D, I_0)] + \gamma \chi_B F[\theta_B^{ED}(D, I_0)](F/f)[\theta_B^{ED}(D, I_0)] \right] \times \cdots \]

\[\cdots \times [(\Pi)_\theta/(\Delta)_\theta][\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]], \]

where all \(x \in \{D, I_0\}\) satisfy

\[\frac{d}{dx} \left[ [(\Pi)_\theta/(\Delta)_\theta][\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] \right] \]

\[\propto \chi_B (\theta_B^{ED})_x(D, I_0) - \frac{d}{dx} \left[ I_0 F[\theta_B^{ED}(D, I_0)] \right] \Lambda''[I_0 F[\theta_B^{ED}(D, I_0)]] \times \cdots \]
\[
\cdots \times [\chi^B \theta^E_D(D, I_0) + \gamma \chi^B(F/f)[\theta^E_D(D, I_0)] - \rho - \Lambda[I_0F[\theta^E_D(D, I_0)]]] \\
> 0
\]

\[
- \left[ \chi^B[\theta^E_D]_x(D, I_0)[1 + \gamma(F/f)[\theta^E_D(D, I_0)]] - \frac{d}{dx} [I_0F[\theta^E_D(D, I_0)]] \times \cdots \right] \times \cdots \\
\cdots \times \Lambda''[I_0F[\theta^E_D(D, I_0)]]
\]

\[
\cdots \times [\chi^B \theta^E_D(D, I_0) - \rho - \Lambda'I_0F[\theta^E_D(D, I_0)]] \\
< 0
\]

and thus

\[
(v^E_{BD})_x(D, I_0)
\]

\[
= \chi^B[\theta^E_D]_x(D, I_0)F[\theta^E_D(D, I_0)] \times \cdots \\
\cdots \times \left[ 1 - [1 + \gamma(F/f)[\theta^E_D(D, I_0)]] \times \cdots \right] \\
\cdots \times [(\Pi_B)_{|\theta|/\Delta_{B}}][\theta^E_D(D, I_0), \Lambda'I_0F[\theta^E_D(D, I_0)]]
\]

\[
< 0
\]

Similar arguments yield

\[
(v^E_{BD})_D(D, I_0)
\]

140
\[
= (-1)\Delta \mu''(D) \left[1 - \left[\frac{\Pi_B}{\Delta B}\right] \left[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]\right]\right] \\
< 0
\]

\[
- \left[1 - \Delta \mu'(D)\right] \frac{d}{dD} \left[\left[\frac{\Pi_B}{\Delta B}\right] \left[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]\right]\right] \\
< 0
\]

< 0,

and

\[
(v_{BD}^{ED})_t(D, I_0)
\]

\[
= (-1) \left[1 - \Delta \mu'(D)\right] \frac{d}{dI_0} \left[\left[\frac{\Pi_B}{\Delta B}\right] \left[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]\right]\right] \\
< 0
\]

Case three: \( r_B = DD \). If we let \( M_B^{DD}(D, I_0) := I_B^{DD}(D, I_0)[1 - F[\theta_B^{DD}(D, I_0)]] \) denote the total volume of projects being maintained, then the financial constraint can be re-written as

\[
(E_b + D - I_0) + \chi_{Rev}[I_0 - M_B^{DD}(D, I_0)]
\]

\[
+ M_B^{DD}(D, I_0)[\chi_B\xi_1[\theta_B^{DD}(D, I_0)] - \rho - \gamma \chi_B \xi_2[\theta_B^{DD}(D, I_0)]] = \Delta \mu(D),
\]

and the pair \([\theta_B^{DD}(D, I_0), M_B^{DD}(D, I_0)]\) is pinned down by this equation, along with the marginal condition

\[
\chi_B \Xi'[\theta_B^{DD}(D, I_0)] = \rho + \Lambda'[I_0 - M_B^{DD}(D, I_0)].
\]

Differentiating this system w.r.t. \( D \) then yields

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\[
(M_B^{DD})_D(D, I_0) \left[ \chi_B \xi_1 (\theta_B^{DD} (D, I_0)) - \rho - (\lambda^{Rev})' [I_0 - M_B^{DD} (D, I_0)] - \gamma \chi_B \xi_2 (\cdot) \right] < \chi_B \xi_1 (\cdot) - \rho - \gamma \chi_B \xi_2 (\cdot) \times \Delta_B [\theta_B^{DD} (D, I_0), 0] < 0
\]

\[
+ (\theta_B^{DD})_D(D, I_0) M_B^{DD} (D, I_0) \chi_B [\xi_1' (\cdot) - \xi_2' (\cdot)] = \Delta \mu' (D) - 1,
\]

and

\[
(M_B^{DD})_D(D, I_0) \Lambda'' (\cdot) + (\theta_B^{DD})_D(D, I_0) \chi_B \Xi' (\cdot) = 0,
\]

where \(\cdot\) suppresses obvious arguments. An application of Cramer’s rule then yields

\( (\theta_B^{DD})_D(D, I_0) < 0 \) and \( (M_B^{DD})_D(D, I_0) < 0 \), and similar arguments yield \( (\theta_B^{DD})_I(D, I_0) < 0 \) and \( 1 - (M_B^{DD})_I(D, I_0) > 0 \).

Now, we know that

\[
v_B^{DD}_D(D, I_0) = [1 - \Delta \mu' (D)] \left[ 1 - \frac{\Pi_B [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]]}{\Delta_B [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]]} \right],
\]

so

\[
(v_B^{DD})_D(D, I_0)
\]

\[
= \Delta \mu'' (D) (\Pi_B / \Delta_B) [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]]
\]

\[
+ [\Delta \mu' (D) - 1] (\Pi_B / \Delta_B)_\theta [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]] (\theta_B^{DD})_D(D, I_0)
\]

\[
+ [\Delta \mu' (D) - 1] (\Pi_B / \Delta_B)_\ell [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]] \times \cdots
\]

\[
\cdots \times \Lambda'' [I_0 - M_B^{DD} (D, I_0)] (-1) (M_B^{DD})_D(D, I_0)
\]

\[
= \Delta \mu'' (D) (\Pi_B / \Delta_B) [\theta_B^{DD} (D, I_0), \Lambda' [I_0 - M_B^{DD} (D, I_0)]]
\]

\[
< 0
\]

\[
> 0
\]

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\[
\mu'(D) - 1 \quad (\Pi_{B}/\Delta_{B})\, [\theta_B^{DD}(D, I_0), \Lambda'[I_0 - M_B^{DD}(D, I_0)]]
\]
\[
\times \cdots \times \Lambda''[I_0 - M_B^{DD}(D, I_0)](-1) \quad (M_B^{DD})_D(D, I_0)
\]
\[
< 0,
\]

and

\[
(v_{BD}^{DD})_I(D, I_0) = \mu'(D) - 1 \quad (\Pi_{B}/\Delta_{B})\, [\theta_B^{DD}(D, I_0), \Lambda'[I_0 - M_B^{DD}(D, I_0)]]
\]
\[
\times \cdots \times \Lambda''[I_0 - M_B^{DD}(D, I_0)](1 - (M_B^{DD})_I(D, I_0)]
\]
\[
< 0.
\]

Similar arguments yield

\[
(v_{BI}^{DD})_I(D, I_0) = \Lambda''[I_0 - M_B^{DD}(D, I_0)](1 - (M_B^{DD})_I(D, I_0)]
\]
\[
\times \cdots \times \Lambda''[I_0 - M_B^{DD}(D, I_0)](1 - (M_B^{DD})_I(D, I_0)]
\]
\[
< 0,
\]

and

\[
(v_{BI}^{DD})_I(D, I_0) = \Lambda''[I_0 - M_B^{DD}(D, I_0)](-1) \quad (M_B^{DD})_D(D, I_0)
\]
\[
	imes \cdots \times \Lambda''[I_0 - M_B^{DD}(D, I_0)](1 - (M_B^{DD})_I(D, I_0)]
\]
\[
< 0,
\]
\[
\begin{align*}
&\left[ 1 - \left( \frac{\Pi_B}{\Delta_B} \right)[\theta_B^{DD}(D, I_0), \Lambda'[I_0 - M_B^{DD}(D, I_0)]] \right] \\
&\cdots \times \\
&\left[ 1 - \Lambda'[I_0 - M_B^{DD}(D, I_0)] \right] \times \cdots \\
&\left[ 1 - \left( \frac{\Pi_B}{\Delta_B} \right)[\theta_B^{DD}(D, I_0), \Lambda'I_0 - M_B^{DD}(D, I_0)] \right]
\end{align*}
\]

< 0.

Case four: \( r_\omega = LS, (\omega \in \{B, G\}) \). Under this case, we know that \( \theta^{LS}_\omega(I_0) \) satisfies

\[
\chi_\omega \theta^{LS}_\omega(I_0) = \rho + \Lambda'[I_0 F[\theta^{LS}_\omega(I_0)]],
\]

so

\[
(\theta^{LS}_\omega)'(I_0) = \frac{F[\theta^{LS}_\omega(I_0)]\Lambda''[I_0 F[\theta^{LS}_\omega(I_0)]]}{\chi_B - I_0 F[\theta^{LS}_\omega(I_0)]\Lambda''[I_0 F[\theta^{LS}_\omega(I_0)]]} < 0,
\]

with

\[
\frac{d}{dI_0} \left[ I_0 F[\theta^{LS}_\omega(\theta)] \right] > 0.
\]

Next, note that

\[
v^{LS}_\omega(D, I_0) = \Lambda'[I_0 F[\theta^{LS}_\omega(I_0)]] + \Pi_\omega[\theta_B^{LS}(I_0), \Lambda'[I_0 F[\theta^{LS}_\omega(I_0)]]] - 1
\]

\[
= \Lambda'[I_0 F[\theta^{LS}_\omega(I_0)]] + \int_{\theta^{LS}_\omega(I_0)}^1 \left[ \theta \chi_\omega - \rho - \Lambda'[I_0 F[\theta^{LS}_\omega(I_0)]] \right] dF(\theta) - 1
\]

\[
= \chi_\omega \mathbb{E} \max[\theta^{LS}_\omega(I_0)] - F[\theta^{LS}_\omega(I_0)] \left[ \chi_\omega \theta^{LS}_\omega - \rho - \Lambda'[I_0 F[\theta_B^{LS}(I_0)]] \right]_{\theta = 0} - (1 + \rho)
\]

\[
= \chi_\omega \mathbb{E} \max[\theta^{LS}_\omega(I_0)] - (1 + \rho), \tag{2.93}
\]

so

\[
(v^{LS}_\omega)_{I}(D, I_0) = \chi_\omega F[\theta^{LS}_\omega(I_0)](\theta^{LS}_\omega)'(I_0) < 0 = (v^{LS}_\omega)_{D}(D, I_0). \tag{2.94}
\]
On the other hand,

\[ v^{LS}_{\omega D}(D, I_0) = 1 - \Delta \mu'(D) < 0, \tag{2.95} \]

with

\[ (v^{LS}_{\omega D})_D(D, I_0) = (-1)\Delta \mu''(D) < 0 = (v^{LS}_{\omega D})_I(D, I_0). \tag{2.96} \]

**2.A.5.3 Existence, uniqueness, and partial characterization of equilibrium**

We’ll now start searching for monotonic equilibria. As a first step in this direction, I note that assumption 2.1 ensures that the non-negativity constraint \( I_0 \geq 0 \) must be lax for banks. Similarly, assumption 2.4, combined with the fact that \( \mu'(E^h) = 1 \), ensures that both sides of the constraint \( D \in [0, E^h] \) are lax. Finally, based on the analysis in subsection 1.A.5.5, it should be clear that subassumption 2.3.4 suffices to ensure that the non-negativity constraint for storage, \( I_0 \leq E_b + D \), is also lax. As a result, monotonic equilibria must fall under one of two cases. The first would be an interior case under which banks find that the “no-default” constraint

\[ (E_b + D - I_0) + \ell_B I_0 \geq \Delta \mu(D) \]

is lax, and thus settle on initial balance sheets satisfying

\[ v_x(D, I_0) = 0, \quad \forall x \in \{D, I_0\}. \]

The alternative would be a corner case under which the aforementioned “no-default” constraint binds.

Now, based on the analysis in my previous subsubsection, it should be clear that the marginal return function \( v_I(D, I_0) \) is strictly decreasing in its first argument. At the same time, assumption 2.1 implies that \( v_I(D, 0) > 0 \) \( \forall D \in \mathbb{R}_+ \) s.t. \( E_b + D \geq \Delta \mu(D) \). As a result, for all values of \( D \) in this range, one of two possible cases obtains. If \( v_I[D, T^{DD}_B(D)] \leq 0 \), then the marginal return \( v_I(D, I_0) \), viewed as a function of the choice on \( I_0 \), exhibits single-crossing from above over the interval \([0, T^{DD}_B(D)]\), namely at some point \( I^*(D) > 0 \). If \( D \) instead satisfies \( v_I[D, T^{DD}_B(D)] > 0 \), then the marginal return \( v_I(D, I_0) \) is strictly positive over all of the aforementioned interval, and I adopt a convention that \( I^*(D) = T^{DD}_B(D) \).

So, when searching for monotonic equilibria, we can restrict attention to pairs of the form \((D, I_0) = [D, I^*(D)]\). Given any such pair, I’ll now check if banks have an incentive to deviate in their initial-balance sheet choices, namely by making some small adjustment to their choice on \( D \). Now, if the pair in question has has the property that the “no-default” constraint is lax, then banks’ return from a marginal increase in \( D \) is simply given by \( v_D[D, I^*(D)] \). If instead the “no-default” constraint binds, then increases in \( D \) must be offset by decreases in
$I_0$, and the relevant return reads as

$$v_D[D, I^*(D)] + \left[\frac{1 - \Delta \mu'(D)}{1 - N[I^*(D)]}\right] v_I[D, I^*(D)],$$

where the starred term gives the rate of transformation along the “no-default” constraint for banks taking secondary-market prices as given. So, $\forall D \in \mathbb{R}_+ \text{ s.t. } E^b + D \geq \Delta \mu(D)$, I define a function

$$h(D) := \begin{cases} 
  v_D[D, I^*(D)] & \text{if } v_I[D, \bar{T}_B^{DD}(D)] \leq 0 \\
  v_D[D, \bar{T}_B^{DD}(D)] + \left[\frac{1 - \Delta \mu'(D)}{1 - N[\bar{T}_B^{DD}(D)]}\right] v_I[D, \bar{T}_B^{DD}(D)] & \text{if } v_I[D, \bar{T}_B^{DD}(D)] > 0.
\end{cases}$$

Now, it should be clear that this function $h(\cdot)$ is continuous. It should also be clear that it’s strictly positive when $D = 0$, but strictly negative when $E^b + D = \Delta \mu(D)$, namely due to the fact that all initial balance sheets satisfying $r_G = LS$ have the property that the marginal deposit sits idle in the good state, and in the bad state either continues to sit idle or otherwise contributes to a tighter financial constraint. I further claim that function $h(\cdot)$ is strictly decreasing. To see that this, we’ll have to take cases on $D$:

**Case 1a.** Suppose first that $D$ has the property that the pair $[D, I^*(D)]$ satisfies $(r_B, r_G) = (LS, LR)$. In this case, we want to confirm that

$$h'(D) = (v_D)_D[D, I^*(D)] + (I^*)'(D)(v_D)_I[D, I^*(D)]$$

$$= (v_D)_D[D, I^*(D)] - \frac{(v_I)_D[D, I^*(D)](v_D)_I[D, I^*(D)]}{(v_I)_I[D, I^*(D)]} < 0,$$

or equivalently

$$\underbrace{(v_I)_I[D, I^*(D)](v_D)_D[D, I^*(D)] - (v_I)_D[D, I^*(D)](v_D)_I[D, I^*(D)]}_{(*)} > 0.$$
Now, if we use \( \cdot \) to suppress arguments, then the derivations in my previous subsubsection imply

\[
(*) = [\alpha_G(v_{GI}^{LR})I(\cdot) + \alpha_B(v_{BI}^{LS})I(\cdot)][\alpha_G(v_{GD}^{LR})D(\cdot) + \alpha_B(v_{BD}^{LS})D(\cdot)]
\]

\[
- [\alpha_G(v_{GI}^{LR})D(\cdot) + \alpha_B(v_{BI}^{LS})D(\cdot)][\alpha_G(v_{GD}^{LR})I(\cdot) + \alpha_B(v_{BD}^{LS})I(\cdot)]
\]

\[
= [\alpha_G(v_{GI}^{LR})I(\cdot) + \alpha_B(v_{BI}^{LS})I(\cdot)][\alpha_G(v_{GD}^{LR})D(\cdot) + \alpha_B(v_{BD}^{LS})D(\cdot)]
\]

\[
- \alpha_G^2(v_{GI}^{LR})D(\cdot)(v_{GD}^{LR})I(\cdot)
\]

\[
> \alpha_G^2(v_{GI}^{LR})I(\cdot) \frac{d}{dD} \left[ \frac{\chi_G^0 \theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right] - \alpha_G^2(v_{GI}^{LR})D(\cdot)(v_{GD}^{LR})I(\cdot)
\]

\[
\propto \left[ F(\cdot)(\theta_G^{LR})I(\cdot) - (1 + \rho) \frac{d}{dI_0} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right] \right] \frac{d}{dD} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right]
\]

\[
- \left[ F(\cdot)(\theta_G^{LR})D(\cdot) - (1 + \rho) \frac{d}{dD} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right] \right] \frac{d}{dI_0} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right]
\]

\[
\propto (\theta_G^{LR})I(\cdot) \frac{d}{dD} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right] - (\theta_G^{LR})D(\cdot) \frac{d}{dI_0} \left[ \frac{\theta_G^{LR}(\cdot)}{\rho + N(\cdot)} \right]
\]

\[
= (-1) \frac{d}{dD} \left[ I_0 F(\cdot) \right] \frac{(\theta_G^{LR})I(\cdot)(\theta_G^{LR})D(\cdot)\Lambda''(\cdot)}{[\rho + N(\cdot)]^2}
\]
\[ + \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \frac{\theta^R_G(\cdot) \theta^R_G(\cdot) \Lambda''(\cdot)}{[\rho + N'(\cdot)]^2} \]

\[ \propto (\theta^R_G(\cdot)) \frac{d}{dD} \left[ I_0 F(\cdot) \right] - (\theta^R_G(\cdot)) \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \]

\[ \propto (-1)[1 + \rho - F(\cdot)[\rho + (\lambda_{\text{Rev}})'(\cdot)] + (1 + \rho) \]

\[ = F(\cdot)[\rho + (\lambda_{\text{Rev}})'(\cdot)] > 0, \]

as desired.

Case 1b. Very similar arguments go through when \( D \) has the property that the pair \([D, I^*(D)]\) satisfies \((r_B, r_G) = (LS, LS)\).

Case 2a. Suppose next that \( D \) has the property that the pair \([D, I^*(D)]\) satisfies \((r_B, r_G) = (ED, LR)\). In this case, we still need to confirm that

\[ (v_I)_D[D, I^*(D)][(v_D)_D[D, I^*(D)] - (v_I)_D[D, I^*(D)](v_D)_I[D, I^*(D)] > 0. \]

Now, if we use \( \cdot \) to suppress arguments, then this inequality can be rewritten as

\[ [\alpha_G(v^R_{GI})_I(\cdot) + \alpha_B(v^E_{BI})_I(\cdot)][\alpha_G(v^R_{GD})_D(\cdot) + \alpha_B(v^E_{BD})_D(\cdot)] \]

\[ > [\alpha_G(v^R_{GI})_D(\cdot) + \alpha_B(v^E_{BI})_D(\cdot)][\alpha_G(v^R_{GD})_I(\cdot) + \alpha_B(v^E_{BD})_I(\cdot)], \]

or equivalently

\[ \alpha^2_G \left[ (v^R_{GI})_I(\cdot)(v^R_{GD})_D(\cdot) - (v^R_{GI})_D(\cdot)(v^R_{GD})_I(\cdot) \right] \]

\[ \quad + \alpha_G \alpha_B \left[ (v^R_{GI})_I(\cdot)(v^E_{BD})_D(\cdot) + (v^R_{GD})_D(\cdot)(v^E_{BI})_I(\cdot) - (v^R_{GI})_D(\cdot)(v^E_{BD})_I(\cdot) - \right. \]

\[ \left. (v^R_{GD})_I(\cdot)(v^E_{BI})_D(\cdot) \right] \]
\[
\frac{\partial}{\partial \theta} \left[ (v_{BI}^E)_{I}(\cdot)(v_{BD}^E)_{D}(\cdot) - (v_{BI}^E)_{D}(\cdot)(v_{BD}^E)_{I}(\cdot) \right] > 0.
\]

Now, in my analysis of case 1a, I showed that the single-starred term is strictly positive. Moreover, in light of the derivations in my previous subsubsection, we know that the double-starred term signs as

\[
\begin{align*}
&(v_{GI}^{LR})_{I}(\cdot)(v_{BD}^{ED})_{D}(\cdot) + (v_{GI}^{LR})_{D}(\cdot)(v_{BI}^{ED})_{I}(\cdot) < 0
\end{align*}
\]

\[
- (v_{GI}^{LR})_{D}(\cdot)(v_{BD}^{ED})_{I}(\cdot) - (v_{GD}^{LR})_{I}(\cdot)(v_{BI}^{ED})_{D}(\cdot) > 0,
\]

while the triple-starred term reads as

\[
(v_{BI}^{ED})_{I}(\cdot)(v_{BD}^{ED})_{D}(\cdot) - (v_{BI}^{ED})_{D}(\cdot)(v_{BD}^{ED})_{I}(\cdot)
\]

\[
= (v_{BI}^{ED})_{I}(\cdot) \left[ (-1)\Delta \mu''(D) \left[ 1 - \left[ (\Pi_B)_{\theta}/(\Delta_B)_{\theta}(\cdot) \right] \right] \right.
\]

\[
- (v_{BI}^{ED})_{D}(\cdot) \left[ (\Pi_B)_{\theta}/(\Delta_B)_{\theta}(\cdot) \right] \]

\[
= \chi_B(\theta_B^{ED})_{I}(\cdot) F(\cdot) \left[ 1 - [1 + \gamma(F/f)^{'}(\cdot)][(\Pi_B)_{\theta}/(\Delta_B)_{\theta}(\cdot)] \right] \times \cdots
\]

\[
\cdots \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dD} \left[ (\Pi_B)_{\theta}/(\Delta_B)_{\theta}(\cdot) \right] \]

\[
- \frac{d}{d\theta} \left[ (\Pi_B)_{\theta}/(\Delta_B)_{\theta}(\cdot) \right] \left[ (1 - \gamma)\chi_B \max(\cdot) + \gamma\chi_B F(\cdot)(F/f)(\cdot) - (1 + \rho) \right] \times \cdots
\]

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\[\begin{align*}
\cdots & \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dD} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
- \chi_B (\theta_B^{ED})_D (\cdot) F(\cdot) \left[ 1 - \left[ 1 + \gamma(F/f)'(\cdot) \right] \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \right] \times \cdots \\
\cdots & \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
+ \frac{d}{dD} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \left[ (1 - \gamma) \chi_B \mathbb{E} \max (\cdot) + \gamma \chi_B F(\cdot) (F/f)(\cdot) - (1 + \rho) \right] \times \cdots \\
\cdots & \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
= & \chi_B (\theta_B^{ED})_I (\cdot) F(\cdot) \left[ 1 - \left[ 1 + \gamma(F/f)'(\cdot) \right] \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \right] \times \cdots \\
\cdots & \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dD} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
- \chi_B (\theta_B^{ED})_D (\cdot) F(\cdot) \left[ 1 - \left[ 1 + \gamma(F/f)'(\cdot) \right] \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \right] \times \cdots \\
\cdots & \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
\propto & \left( \theta_B^{ED} \right)_I (\cdot) \frac{d}{dD} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] - \left( \theta_B^{ED} \right)_D (\cdot) \frac{d}{dI_0} \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
= & \left( \theta_B^{ED} \right)_I (\cdot) \left( \theta_B^{ED} \right)_D (\cdot) \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right] \\
+ & \left( \theta_B^{ED} \right)_I (\cdot) \frac{d}{dD} \left[ I_0 F(\cdot) \right] \Lambda''(\cdot) \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right]_{<0} \\
- & \left( \theta_B^{ED} \right)_I (\cdot) \left( \theta_B^{ED} \right)_D (\cdot) \left[ \left( \Pi_B \right)_{\theta/(\Delta_B)\theta} (\cdot) \right]
\end{align*}\]
\[-(\theta_B^{ED})_I(\cdot)\frac{d}{dI_0}[I_0F(\cdot)]\Lambda''(\cdot)[(\Pi_B)_{\theta}/(\Delta_B)_{\theta}]\epsilon(\cdot)\]

\[\propto (\theta_B^{ED})_I(\cdot)\frac{d}{dD}[I_0F(\cdot)] - (\theta_B^{ED})_D(\cdot)\frac{d}{dI_0}[I_0F(\cdot)] \]

\[\frac{1 - F(\cdot)(\lambda^{Rev})'(\cdot) - \left[ \int_{\theta_B^{ED}(D,I_0)}^{1} (\theta_{\chi_B} - \rho) dF(\theta) - \gamma_{\chi_B} E \max(\cdot) \right]}{(1 - \Delta\mu'(D))} \]

\[\propto (1 - \gamma)\chi_B \max E(\cdot) + \gamma_{\chi_B} (F/f)(\cdot) F(\cdot) - (1 + \rho) \]

\[+ \left[ 1 - F(\cdot)(\lambda^{Rev})'(\cdot) - \int_{\theta_B^{ED}(D,I_0)}^{1} (\theta_{\chi_B} - \rho) dF(\theta) - \gamma_{\chi_B} E \max(\cdot) \right] \]

\[= F(\cdot)[\chi_B \theta_B^{ED}(\cdot) + \gamma_{\chi_B} (F/f)(\cdot) - \rho - (\lambda^{Rev})'(\cdot)] \]

> 0,

as desired.

**Case 2b.** Very similar arguments go through when \(D\) has the property that the pair \([D, I^*(D)]\) satisfies \((r_B, r_G) = (ED, LS)\).

**Case 3a.** Suppose next that \(D\) has the property that the pair \([D, I^*(D)]\) satisfies \((r_B, r_G) = (DD, LR)\), with the “no-default” constraint lax. In this case, we still need to confirm that

\[(v_I)_I[D, I^*(D)](v_D)_D[D, I^*(D)] - (v_I)_D[D, I^*(D)](v_D)_I[D, I^*(D)] > 0.\]

Now, if we use \(\cdot\) to suppress arguments, then this inequality can be rewritten as
\[ \alpha_G(v_{GI}^{LR})_I(\cdot) + \alpha_B(v_{BI}^{DD})_I(\cdot) \leq [\alpha_G(v_{GI}^{LR})_D(\cdot) + \alpha_B(v_{BD}^{DD})_D(\cdot)] \]

\[ > [\alpha_G(v_{GI}^{LR})_D(\cdot) + \alpha_B(v_{BI}^{DD})_D(\cdot)] [\alpha_G(v_{GI}^{LR})_I(\cdot) + \alpha_B(v_{BD}^{DD})_I(\cdot)], \]

or equivalently

\[ \alpha_G^2 [(v_{GI}^{LR})_I(\cdot)(v_{GD}^{LR})_D(\cdot) - (v_{GI}^{LR})_D(\cdot)(v_{GD}^{LR})_I(\cdot)] \]
\[ \leq \alpha_G \alpha_B \left[ (v_{GI}^{LR})_I(\cdot)(v_{BD}^{DD})_D(\cdot) + (v_{GD}^{LR})_D(\cdot)(v_{BI}^{DD})_I(\cdot) - (v_{GI}^{LR})_D(\cdot)(v_{BB}^{DD})_I(\cdot) \right] \]
\[ - (v_{GD}^{LR})_I(\cdot)(v_{BI}^{DD})_D(\cdot) \]

\[ > \alpha_B^2 [(v_{BI}^{DD})_I(\cdot)(v_{BD}^{DD})_D(\cdot) - (v_{BI}^{DD})_D(\cdot)(v_{BD}^{DD})_I(\cdot)] > 0. \]

Now, in my analysis of case 1a, I showed that the single-starred term is strictly positive. Moreover, in light of the derivations in my previous subsubsection, we know that the double-starred term signs as

\[ (v_{GI}^{LR})_I(\cdot) < 0, \quad (v_{BD}^{DD})_D(\cdot) < 0, \quad (v_{BI}^{DD})_I(\cdot) < 0 \]

\[ - (v_{GI}^{LR})_D(\cdot) (v_{BD}^{DD})_I(\cdot) - (v_{GD}^{LR})_I(\cdot) (v_{BI}^{DD})_D(\cdot) > 0, \]

while the triple-starred term reads as

\[ (v_{BI}^{DD})_I(\cdot) [\Delta \mu''(D) \left( \frac{\Pi_B}{\Delta_B} \right) (\Delta) \cdot] - [\Delta \mu'(D) - 1] (\Pi_B/\Delta_B)_I(\cdot) \Lambda''(\cdot)(M_B^{DD})_D(\cdot)] \]
\[ - (v_{BI}^{DD})_D(\cdot)(v_{BD}^{DD})_I(\cdot) \]

\[ > (v_{BI}^{DD})_I(\cdot) [\Delta \mu'(D) - 1] (\Pi_B/\Delta_B)_I(\cdot) \Lambda''(\cdot)(-1)(M_B^{DD})_D(\cdot) - (v_{BI}^{DD})_D(\cdot)(v_{BD}^{DD})_I(\cdot) \]

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\[
= \Lambda''(\cdot)[1 - (M_B^{DD})_I(\cdot)][1 - (\Pi_B/\Delta_B)(\cdot) + [1 - \Lambda'(\cdot)](\Pi_B/\Delta_B)_I(\cdot)] \times \cdots
\]
\[
\cdots \times [\Delta \mu'(D) - 1](\Pi_B/\Delta_B)_I(\cdot)\Lambda''(\cdot)(-1)(M_B^{DD})_D(\cdot)
\]
\[
+ \Lambda''(\cdot)(M_B^{DD})_D(\cdot)[1 - (\Pi_B/\Delta_B)(\cdot) + [1 - \Lambda'(\cdot)](\Pi_B/\Delta_B)_I(\cdot)] \times \cdots
\]
\[
\cdots \times [\Delta \mu'(D) - 1](\Pi_B/\Delta_B)_I(\cdot)\Lambda''(\cdot)[1 - (M_B^{DD})_I(\cdot)]
\]
\[
= 0,
\]
as desired.

Case 3b. Very similar arguments go through when \(D\) has the property that the pair \([D, I^*(D)]\) satisfies \((r_B, r_G) = (LS, DD)\), with the “no-default” constraint lax.

Case 4a. Suppose next that \(D\) has the property that the “no-default” constraint binds, with \(r_G = LR\). In this case, we have

\[
h(D) = v_D[D, T_B^{DD}(D)] + \left[ \frac{1 - \Delta \mu'(D)}{1 - \Lambda'(T_B^{DD}(D))} \right] v_I[D, T_B^{DD}(D)]
\]
\[
\quad = \alpha_G \left[ v_{GD}^{LR}[D, T_B^{DD}(D)] + \left[ \frac{1 - \Delta \mu'(D)}{1 - \Lambda'(T_B^{DD}(D))} \right] v_{GI}^{LR}[D, T_B^{DD}(D)] \right]
\]
\[
\quad + \alpha_B \left[ v_{BD}^{DD}[D, T_B^{DD}(D)] + \left[ \frac{1 - \Delta \mu'(D)}{1 - \Lambda'(T_B^{DD}(D))} \right] v_{BI}^{DD}[D, T_B^{DD}(D)] \right],
\]
so

\[
h'(D) \propto (v_{GD}^{LR})_D[D, T_B^{DD}(D)] + (T_B^{DD})'_I(D)(v_{GD}^{LR})_I[D, T_B^{DD}(D)]<0 \quad \text{and} \quad >0
\]
\[ \frac{d}{dD} \left[ \frac{1 - \Delta \mu'(D)}{1 - N'[\tilde{T}^D_B(D)]} \right] v^L_{GI}[D, \tilde{T}^{DD}_B(D)] \]

\[ + \left[ \frac{1 - \Delta \mu'(D)}{1 - N'[\tilde{T}^D_B(D)]} \right] \left( v^L_{GI}(D, \tilde{T}^{DD}_B(D)) + (\tilde{T}^{DD}_B)'(D)(v^L_{GI})_1[D, \tilde{T}^{DD}_B(D)] \right) \]

which means that the desired monotonicity would go through if we could show that \( v^L_{GI}[D, \tilde{T}^{DD}_B(D)] > 0 \). To see that this is indeed the case, recall that case 4a only obtains if

\[ v_I[D, \tilde{T}^{DD}_B(D)] = \alpha_G v^L_{GI}[D, \tilde{T}^{DD}_B(D)] + \alpha_B v^D_{BI}[D, \tilde{T}^{DD}_B(D)] \]

\[ > 0. \]

Case 4b. Very similar arguments go through when the “no-default” constraint binds, with \( r_G = LS \).

We can finally conclude that the function \( h(D) \) reaches zero at exactly one point in its domain. This point represents our only candidate for a monotonic equilibrium. To confirm that this candidate actually constitutes a monotonic equilibrium, two additional steps are needed. First of all, while our candidate obviously has the property that banks have no incentive to make local deviations in their choice on \((D, I_0)\), we’ve not yet addressed the possibility of non-local deviations. Fortunately, I’ve already shown that first-order conditions suffice to pin down the solution for the optimal contracting problem facing banks in chapter 1, and the problem now facing banks is unchanged apart from the fact that liquidation values are now state-specific. That first-order conditions still suffice despite this change can easily be verified, though I’ve chosen to omit a formal proof since the relevant changes to the arguments given in chapter 1 are relatively cosmetic.

The second and more important step is then to confirm that our candidate is indeed monotonic. To see that this is the case, recall that our candidate has the property that the physical constraint binds in the good state but not in the bad state — that is,

\[ (E^b + D - I_0) + I_0 \lambda'[I_0 - I_G[1 - F(\theta_G)]] = I_G \int_{\theta_G}^{1} [\rho + \lambda'[I_0 - I_G[1 - F(\theta_G)]]] dF(\theta), \]

but

\[ (E^b + D - I_0) + I_0 \lambda'[I_0 - I_B[1 - F(\theta_B)]] \geq I_B \int_{\theta_B}^{1} [\rho + \lambda'[I_0 - I_B[1 - F(\theta_B)]]] dF(\theta). \]
With \( M_\omega = I_\omega[1 - F(\theta_\omega)] \) denoting the total volume of investments being maintained in state \( \omega \), we can then conclude that

\[
\rho M_G - \lambda \text{Rev}(I_0 - M_G) \geq \rho M_B - \lambda \text{Rev}(I_0 - M_B),
\]

which implies that \( M_G \geq M_B \) — i.e., more liquidations take place in the bad state, as desired.

Having thus established that a monotonic equilibrium exists and is unique, I’ll finally turn my attention to the claim in lemma 2.4.1 that this equilibrium also satisfies \( r_G = LR \). Fortunately, the usual arguments still go through on this front: if \( r_G = r_B = LS \), then assumption 2.1 leaves banks with a strict incentive to increase their choice on \( I_0 \); if instead \( r_G = LS \neq r_B \), then we again have a situation in which the marginal deposit sits idle in the good state but contributes to a tighter financial constraint in the bad state, leaving banks with a strict incentive to reduce their choice on \( D \).

Similar arguments can be used to rule out a non-monotonic equilibrium. In particular, it should now be clear that any such equilibrium would have

\[
\rho M_B - \lambda \text{Rev}(I_0 - M_B) > \rho M_G - \lambda \text{Rev}(I_0 - M_G),
\]

which ensures some slack in the physical constraint associated with the good state. As for the physical constraint associated with the bad state, its laxity was already established in subsection 2.3.1. As a result, we only have to worry about financial constraints, at which point the usual arguments kick in: if the financial constraints obtaining in both states are lax, then assumption 2.1 leaves banks with a strict incentive to increase their choice on \( I_0 \); otherwise, banks have a strict incentive to reduce \( D \), since the marginal deposit contributes to a tighter financial constraint in those states where the constraint binds, while sitting idle in any states where the constraint is lax.

2.A.5.4 Dependence on parameters

In this subsubsection, I show how the equilibrium allocation responds to changes in the model’s parameters. As explained in the main text, the particular parameters on which I focus are the probability on the good state, \( \alpha_G \), and the payout that successful projects generate in this state, \( \chi_G \). I thus let \( \beta := (\alpha_G, \chi_G) \) collect these parameters and will now include it as an explicit argument in any functions into which these parameters enter. It’s then best to proceed as follows:
Step one. In this first step, I check if the equilibrium identified in my previous subsubsection satisfies $r_B = LS$. In this case, the equilibrium value for the pair $(D, I_0)$ would lie above the locus

$$\{(D, I_0) \in [0, \overline{D}_B] \times \mathbb{R}_+ \text{ s.t. } I_0 = \overline{T}_G^LS (D)\}, \quad (2.97)$$

but below the locus

$$\{(D, I_0) \in [0, \overline{D}_B] \times \mathbb{R}_+ \text{ s.t. } I_0 = \bar{T}_B^LS (D)\}. \quad (2.98)$$

Now, in light of assumption 2.1, it should be clear that all points on the lower locus satisfy $v_I(D, I_0, \beta) > 0$. As for the upper locus, I note from the analysis in subsubsection 2.A.5.2 that all $D \in [0, \overline{D}_B]$ satisfy

$$v_I[D, T_B^LS (D), \beta] = \alpha_G v_{G-I}^LR[D, T_B^LS (D), \beta] + (1 - \alpha_G) v_{BI}^LS[D, T_B^LS (D)]$$

$$\propto \frac{(1/\chi_G)v_{G-I}^LR[D, T_B^LS (D)] - (1 + \rho)v_{G-I}^LR[D, T_B^LS (D)]}{\rho + \Lambda[I_0 F[\theta_B^R[D, T_B^LS (D)]]]}$$

$$+ \left(\frac{1 - \alpha_G}{\alpha_G \chi_G}\right) \left[\chi_B \mathbb{E} \max[\theta_B^LS[D, T_B^LS (D)]] - (1 + \rho)\right], \quad (2.99)$$

where the right-hand side of (2.99) admits the following derivative w.r.t. $D$:

$$\alpha_G (v_{G-I}^LR[D, T_B^LS (D), \beta] + \alpha_G (T_B^LS)'(D) (v_{G-I}^R)_{I[D, T_B^LS (D), \beta]}$$

$$+ (1 - \alpha_G) (v_{BI}^LS[D, T_B^LS (D), \beta] + (1 - \alpha_G) (T_B^LS)'(D) (v_{BI}^LS)_{I[D, T_B^LS (D), \beta]} > 0,$$

On the other hand, the right-hand side of (2.100) admits the following derivative w.r.t. $x \in \{\alpha_G, \chi_G\}$:

$$\frac{d}{dx} \left[\frac{1 - \alpha_G}{\alpha_G \chi_G}\right] \left[\chi_B \mathbb{E} \max[\theta_B^LS[D, T_B^LS (D)]] - (1 + \rho)\right] > 0.$$
We can then conclude that one of two cases must obtain, depending on parameters:

- the first would be that \( \beta \) is large enough that \( v_I[0, T_B^{LS}(0), \beta] > 0 \). In this case, all \( D \in [0, \bar{D}_B^{LS}] \) must satisfy \( v_I[D, T_B^{LS}(D), \beta] > 0 \). Since the analysis in subsubsection 2.A.5.2 implies that the function \( v_I(D, I_0, \beta) \) is decreasing in its second argument, we can conclude that this function is strictly positive at all points between the loci described by (2.97) and (2.98), which precludes \( r_B = LS \) as an equilibrium outcome;

- the alternate case would be that \( \beta \) is small enough that \( v_I[0, T_B^{LS}(0), \beta] \leq 0 \), in which case the composition \( v_I[D, T_B^{LS}(D), \beta] \) exhibits single-crossing from below over the interval \([0, \bar{D}_B^{LS}]\), namely at a point which I’ll denote \( \hat{D}_B^{LS}(\beta) \). In light of the analysis in my previous subsection, it should then be clear that \( r_B = LS \) obtains in equilibrium i.f.f. \( v_D[\hat{D}_B^{LS}(\beta), T_B^{LS}(\hat{D}_B^{LS}(\beta)), \beta] \leq 0 \). Now, I claim that this inequality is more likely to obtain the lower is \( \beta \). To show this, I first define \( \bar{I}_B^{LS}(\beta) := I^*[\hat{D}_B^{LS}(\beta)] \), then note that the pair \([\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta)]\) is pinned down by the following system:

\[
\frac{1}{\chi_G} v_{GI}^{LR}[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta), \beta] + [(1 - \alpha_G)/\alpha_G \chi_G] v_{BI}^{LS}[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta)] = 0
\]

\[
[E^b + \hat{D}_B^{LS}(\beta) - \hat{I}_B^{LS}(\beta)] + \bar{I}_B^{LS}(\beta) \Lambda'[\bar{I}_B^{LS}(\beta)] F[\theta_B^{LS}][\bar{I}_B^{LS}(\beta)]]
\]

\[
+ \bar{I}_B^{LS}(\beta) \Delta_B[\theta_B^{LS}[\bar{I}_B^{LS}(\beta)], \Lambda'[\bar{I}_B^{LS}(\beta)] F[\theta_B^{LS}][\bar{I}_B^{LS}(\beta)]] = \Delta \mu[\hat{D}_B^{LS}(\beta)]
\]

This system differentiates as follows w.r.t. \( x \in \{\alpha_G, \chi_G\} \):

\[
(\hat{D}_B^{LS})_x(\beta) \begin{cases}
(1/\chi_G) \left( v_{GI}^{LR} \right)_{D}[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta), \beta] > 0 \\
+ [(1 - \alpha_G)/\alpha_G \chi_G] \left( v_{BI}^{LS} \right)_{D}[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta)] = 0
\end{cases}
\]

\[
+ (\bar{I}_B^{LS})_x(\beta) \begin{cases}
(1/\chi_G) \left( v_{GI}^{LR} \right)_I[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta), \beta] < 0 \\
+ [(1 - \alpha_G)/\alpha_G \chi_G] \left( v_{BI}^{LS} \right)_I[\hat{D}_B^{LS}(\beta), \bar{I}_B^{LS}(\beta)] < 0
\end{cases}
\]
\begin{align*}
\frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] v_{BI}^{LS} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta)] &= 0 \\
(\hat{D}_B^{LS})_x(\beta) \left[ 1 - \frac{\Delta}{\chi_G} \right] < 0
\end{align*}

\begin{align*}
= (\hat{I}_B^{LS})_x(\beta) \frac{d}{dI_0} \left[ I_0 - I_0 \Lambda'[\theta_B^{LS}(I_0)] - I_0 \Delta_B[\theta_B^{LS}(I_0), \Lambda'[\theta_B^{LS}(I_0)]] \right]_{I_0=\hat{I}_B^{LS}(\beta)} > 0, \text{as shown in my proof of sublemma 2.A.1}
\end{align*}

An application of Cramer’s rule then yields $(\hat{I}_B^{LS})_x(\beta) > 0 > (\hat{D}_B^{LS})_x(\beta)$, so

\begin{align*}
\frac{d}{dx} \left[ (1/\chi_G) v_{GD}^{LR} \hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta), \beta \right] + [(1 - \alpha_G)/\alpha_G \chi_G] v_{BD}^{LS} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta)]
\end{align*}

\begin{align*}
= (\hat{D}_B^{LS})_x(\beta) [1/\chi_G] v_{GD}^{LR} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta), \beta]
\end{align*}

\begin{align*}
+ [(1 - \alpha_G)/\alpha_G \chi_G] \left[ v_{BD}^{LS} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta)] \right]
\end{align*}

\begin{align*}
+ (\hat{I}_B^{LS})_x(\beta) [1/\chi_G] v_{GD}^{LR} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta), \beta]
\end{align*}

\begin{align*}
+ [(1 - \alpha_G)/\alpha_G \chi_G] \left[ v_{BD}^{LS} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta)] \right] _= 0
\end{align*}

\begin{align*}
+ \frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] v_{BD}^{LS} [\hat{D}_B^{LS}(\beta), \hat{I}_B^{LS}(\beta)]
\end{align*}

\begin{align*}
> 0,
\end{align*}

where the inequalities follow from the analysis in subsubsection 2.A.5.2 and suffice to verify my claim.
We can finally conclude that \( R_B = LS \) is more likely to obtain in equilibrium the lesser is \( \beta \). So, let’s suppose for a moment that the only parameter being varied is the probability \( \alpha_G \), with \( \alpha_G \) denoting the value for this probability beneath which my parametric assumptions begin to fail. In light of the analysis above, we see that there are three cases to consider. The first is that all \( \alpha_G \in (\alpha_G, 1] \) have the property that \( R_B = LS \) in equilibrium, in which case I set \( \alpha_{LS}^G = 1 \). The second possibility is that all \( \alpha_G \in (\alpha_G, 1] \) instead satisfy \( R_B \neq LS \) in equilibrium, in which case I set \( \alpha_{LS}^G = \alpha_G \). The third possibility would be that the interval \( (\alpha_G, 1] \) admit a threshold such that \( R_B = LS \) holds as an equilibrium outcome i.f.f. \( \alpha_G \) lies below this threshold, in which case I set \( \alpha_{LS}^G \) equal to this threshold. \( \chi^L_S \) can be derived analogously.

**Step two.** In this next step, I check if the equilibrium identified in my previous subsubsection instead satisfies \( R_B = ED \). In this case, the equilibrium value for the pair \( (D, I_0) \) would lie above the locus

\[
\{(D, I_0) \in [0, \overline{D}^E_B] \times \mathbb{R}_+ \text{ s.t. } I_0 = \max \{T^L_B(D), T^L_G(D)\}\},
\]

but below the locus

\[
\{(D, I_0) \in [0, \overline{D}^E_B] \times \mathbb{R}_+ \text{ s.t. } I_0 = T^E_B(D)\}.
\]

Now, in light of the analysis in my previous step, it should be clear that a necessary condition for the equilibrium’s lying above the lower locus is that \( \alpha_G \geq \alpha_{LS}^G \) or \( \chi_G \geq \chi_{LS}^G \), depending on which parameter is being varied, so I’ll herein restrict attention to parameters satisfying this condition. As for the upper locus, I note from the analysis in subsubsection 2.A.5.2 that all \( D \in [0, \overline{D}^E_B] \) satisfy

\[
v_I[D, T^E_B(D), \beta] = \alpha_G v_{GI}^{LR}[D, T^E_B(D), \beta] + (1 - \alpha_G) v_{BI}^{DD}[D, T^E_B(D)]
\]

\[
\propto \frac{1}{\chi_G} v_{GI}^{LR}[D, T^E_B(D), \beta]
\]

\[
- \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \overline{\Lambda} [T^E_B(D) F[\hat{\xi} B[T^E_B(D)]]] \times \cdots
\]

\[
\cdots \times \left[ 1 - (\Pi_B/\Delta_B) \hat{\xi} B[T^E_B(D)] , \overline{\Lambda} [T^E_B(D) F[\hat{\xi} B[T^E_B(D)]]] \right],
\]

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where the left-hand side of this last line is strictly increasing in $x \in \{\alpha_G, \chi_G\}$ and furthermore satisfies

$$
\frac{d}{dD} \left[ (1/\chi_G)v_{GL}^{LR}[D, T_B^{ED}(D), \beta] \right. \\
- \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
= (1/\chi_G)\left[(v_{GL}^{LR})_D[D, T_B^{ED}(D), \beta] \right. \\
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
n + \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
= 0, \text{ where } \hat{\theta}_B^G[T_B^{ED}(D)]\text{ satisfies (II)_\theta[T_B^{ED}(D), \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \Delta_B()=(\Delta_B)_\theta()\Pi_B()}
$$

$$
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

$$
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \left[ 1 - \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \\
\left. \cdots \times \left[ 1 - (\Pi_B/\Delta_B)\theta [\hat{\theta}_B^G[T_B^{ED}(D)], \Lambda'[T_B^{ED}(D)F[\hat{\theta}_B^G[T_B^{ED}(D)]]] \times \cdots \right] \right]
$$

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that one of three cases must obtain, depending on parameters:

- the first would be that $\beta$ is large enough that $v_I[0, \overline{T_B^{ED}}(0), \beta] > 0$. In this case, it should be clear that all $D \in [0, \overline{T_B^{ED}}]$ satisfy $v_I[D, T_B^{ED}(D), \beta] > 0$, and in turn that $r_B = DD$ must obtain in equilibrium;

- if $\beta$ is instead small enough that $v_I[\overline{T_B^{ED}}, T_B^{ED}[\overline{T_B^{ED}}], \beta] < 0$, then it should be clear that all $D \in [0, \overline{T_B^{ED}}]$ satisfy $v_I[D, T_B^{ED}(D), \beta] < 0$, and in turn that $r_B = ED$ must obtain in equilibrium;

- finally, when $\beta$ takes an intermediate value, it should be clear that the composition $v_I[D, T_B^{ED}(D), \beta]$ exhibits single-crossing from below over the interval $[0, \overline{T_B^{ED}}]$, namely at a point which I’ll denote $\hat{D}_B^{ED}(\beta)$. In light of the analysis in my previous subsubsection, it should then be clear that $r_B = ED$ obtains in equilibrium i.f.f. $v_D[\hat{D}_B^{ED}(\beta), T_B^{ED}[\hat{D}_B^{ED}(\beta)], \beta] \leq 0$. Now, I claim that this inequality is more likely to
obtain the lower is $\beta$. To show this, I first define $\hat{I}_B^{ED}(\beta) = I^* [\hat{D}^{ED}_B(\beta)]$ and then note that the pair $[\hat{D}^{ED}_B(\beta), \hat{I}_B^{ED}(\beta)]$ is pinned down by the following system:

$$(1/\chi_G)\nu_{GI}^{LR}[\hat{D}^{ED}_B(\beta), \hat{I}_B^{ED}(\beta), \beta]$$

$$= [(1 - \alpha_G)/\alpha_G \chi_G] [1 - \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]] \times \cdots$$

$$\cdots \times [1 - (\Pi_B/\Delta_B) [\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)], \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]]]$$

$$[E^b + \hat{D}^{ED}_B(\beta) - \hat{I}_B^{ED}(\beta)] + \hat{I}_B^{ED}(\beta) \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]]$$

$$+ \hat{I}_B^{ED}(\beta) \Delta_B [\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)], \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]] = \Delta \mu[\hat{D}^{ED}_B(\beta)]$$

This system differentiates as follows w.r.t. $x \in \{\alpha_G, \chi_G\}$:

$$(\hat{D}^{ED}_B)_x(\beta)(1/\chi_G) \left( \nu_{GI}^{LR}[\hat{D}^{ED}_B(\beta), \hat{I}_B^{ED}(\beta), \beta] + (\hat{I}_B^{ED})_x(\beta) \times \cdots \right)$$

$$= \frac{d}{dx} \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] \frac{d}{dI_0} \left[ \Lambda'[I_0 F[\hat{\phi}_B^{\beta}(I_0)]] \right] \bigg|_{I_0 = \hat{I}_B^{ED}(\beta)} \times \cdots$$

$$\cdots \times \left[ 1 - \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]] \times \cdots \right]$$

$$\cdots \times (\Pi_B/\Delta_B) \delta[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)], \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]])$$

$$+ [1 - (\Pi_B/\Delta_B) [\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)], \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]]]$$

$$= \frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] [1 - \Lambda'[\hat{I}_B^{ED}(\beta) F[\hat{\phi}_B^{\beta}[\hat{I}_B^{ED}(\beta)]]]] \times \cdots$$

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An application of Cramer’s rule then yields \((\hat{I}^E_B)x(\beta) > 0 > (\hat{D}^E_B)x(\beta)\). In turn,

\[
\frac{d}{dx} \left[ (1/\chi_G)v^{LR}_{GD}[\hat{D}^E_B(D), \hat{I}^E_B(D), \beta] + [(1 - \alpha_G)/\alpha_G\chi_B]v^{DD}_{BB}[\hat{D}^E_B(D), \hat{I}^E_B(\beta)] \right]
\]
\[ + \frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] v_{BB}^{DD} \left[ \hat{D}^{ED}_B(D), \hat{I}^{ED}_B(\beta) \right] \left[ \frac{1}{\hat{I}^{ED}_B(\beta)} \right] \]

\[ =: (\ast). \]

It’s then useful to note that (2.101) can be re-written as

\[ \frac{1}{\chi_G} \left( \hat{D}^{ED}_B(\beta)(v_{GL}^{LR})_D[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta), \beta] \right) \]

\[ + (\hat{I}^{ED}_B(\beta)(v_{GI}^{LR})_I[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta), \beta] \]

\[ + \left[ (1 - \alpha_G) / \alpha_G \chi_G \right] \left( \hat{D}^{ED}_B(\beta)(v_{BI}^{DD})_D[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta)] \right) \]

\[ + (\hat{I}^{ED}_B(\beta)(v_{BI}^{DD})_I[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta)] \]

\[ + \frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] v_{BI}^{DD} \left[ \hat{D}^{ED}_B(D), \hat{I}^{ED}_B(\beta) \right] = 0, \]

or equivalently

\[ \frac{d}{dx} \left[ \frac{1 - \alpha_G}{\alpha_G \chi_G} \right] \left( \hat{I}^{ED}_B(\beta) \right) \]

\[ = \left[ \frac{1}{\chi_G} \left( \hat{D}^{ED}_B(\beta)(v_{GL}^{LR})_D[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta), \beta] \right) \right. \]

\[ + (\hat{I}^{ED}_B(\beta)(v_{GI}^{LR})_I[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta), \beta] \]

\[ + \left[ (1 - \alpha_G) / \alpha_G \chi_G \right] \left( \hat{D}^{ED}_B(\beta)(v_{BI}^{DD})_D[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta)] \right) \]

\[ + \left. (\hat{I}^{ED}_B(\beta)(v_{BI}^{DD})_I[\hat{D}^{ED}_B(\beta), \hat{I}^{ED}_B(\beta)] \right) \]

\[ = \frac{-1}{\left( -1 \right) v_{BI}^{DD} \left[ \hat{D}^{ED}_B(D), \hat{I}^{ED}_B(\beta) \right]. \]
Using this equality to eliminate \((\hat{I}_{B}^{ED})_{x}(\beta)\) on line \textcolor{red}{202} then yields

\[
(*) = (1/\chi_{G}) \left[ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{LB})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{<0} > 0
\]

\[+ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{GB})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{>0}
\]

\[+ (v_{GD})_{L}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]
\]

\[+ [(1 - \alpha_{G})/\alpha_{G}\chi_{B}] \left[ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{DD})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{<0} > 0
\]

\[+ (v_{DD})_{I}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta)]
\]

\[= [(v_{DD})_{I}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]/(v_{DD})_{B}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]] \times \ldots
\]

\[\ldots \times (1/\chi_{G}) \left[ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{LB})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{<0} > 0
\]

\[+ (v_{LD})_{L}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]
\]

\[= [(v_{DD})_{I}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]/(v_{DD})_{B}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]] \times \ldots
\]

\[\ldots \times [(1 - \alpha_{G})/\alpha_{G}\chi_{G}] \left[ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{DD})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{<0} > 0
\]

\[+ (v_{DD})_{I}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta)]
\]

\[> [(1 - \alpha_{G})/\alpha_{G}\chi_{G}] \left[ \frac{[(\hat{D}_{B}^{ED})_{x}(\beta)/(\hat{I}_{B}^{ED})_{x}(\beta)]}{(v_{DD})_{D}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta), \beta]} \right]^{<0} > 0
\]

\[+ (v_{DD})_{I}[\hat{D}_{B}^{ED}(\beta), \hat{I}_{B}^{ED}(\beta)]
\]

\[= [(v_{DD})_{I}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]/(v_{DD})_{B}[\hat{D}_{B}^{ED}(D), \hat{I}_{B}^{ED}(\beta)]] \times \ldots
\]

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\[
\cdots \times \frac{(1 - \alpha_G)}{\alpha_G \chi_G} \left[ \frac{(\hat{D}^{E_D} B_x(\beta)/\hat{I}_B^{E_D} x(\beta)) (v_{B1}^{E_D}) D [\hat{D}^{E_D} B(\beta), \hat{I}_B^{E_D} (\beta)]}{\alpha_G \chi_G} \right. \\
\left. + (v_{B1}^{E_D}) I [\hat{D}^{E_D} B(\beta), \hat{I}_B^{E_D} (\beta)] \right] =: \frac{(1 - \alpha_G)}{\alpha_G \chi_G} (**) \]

The derivations in case 3a of subsubsection 2.A.5.3 then yield the following, where I’ve used · to suppress arguments:

\[
(**) = \left[ \frac{(\hat{D}^{E_D} B_x(\beta)/\hat{I}_B^{E_D} x(\beta))}{(\hat{I}_B^{E_D} x(\beta)) \cdot \cdots < 0} \left[ \Delta \mu''(\cdot) (\Pi_B / \Delta_B)(\cdot) - [1 - \Delta \mu'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot) \Lambda''(\cdot) (-1)(M_B^{DD}) (\cdot) \right] \right. \\
\left. - [1 - \Delta \mu'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot) \Lambda''(\cdot) [1 - (M_B^{DD}) I (\cdot)] \right. \\
+ [[1 - \Delta \mu'(\cdot)]/[1 - \lambda'(\cdot)]] [\hat{D}^{E_D} B_x(\beta)/\hat{I}_B^{E_D} x(\beta)] \cdot \cdots \\
\cdots \times \Lambda''(\cdot) (-1)(M_B^{DD}) D(\cdot) [1 - (\Pi_B / \Delta_B)(\cdot) + [1 - \lambda'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot)] \\
+ [[1 - \Delta \mu'(\cdot)]/[1 - \lambda'(\cdot)] ] \Lambda''(\cdot) [1 - (M_B^{DD}) I (\cdot)] \times \cdots \\
\cdots \times [1 - (\Pi_B / \Delta_B)(\cdot) + [1 - \lambda'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot)] \\
\left. > (-1)[1 - \Delta \mu'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot) \Lambda''(\cdot) \times \cdots ight. \\
\left. \cdots \times [1 - (M_B^{DD}) I (\cdot) - [\hat{D}^{E_D} B_x(\beta)/\hat{I}_B^{E_D} x(\beta)] (M_B^{DD}) D(\cdot)] \right. \\
+ \left[ \frac{1 - \Delta \mu'(\cdot)}{1 - \lambda'(\cdot)} \right] [1 - (\Pi_B / \Delta_B)(\cdot) + [1 - \lambda'(\cdot)] (\Pi_B / \Delta_B) \ell(\cdot)] \Lambda''(\cdot) \times \cdots \\
\left. \cdots \times [1 - (M_B^{DD}) I (\cdot) - [\hat{D}^{E_D} B_x(\beta)/\hat{I}_B^{E_D} x(\beta)] (M_B^{DD}) D(\cdot)] \right]
\]

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\[
\begin{align*}
&= \left[1 - \Delta \mu'(\cdot)\right] / \left[1 - \Lambda'(\cdot)\right] \left[1 - (\Pi_B / \Delta B)(\cdot)\right] \Lambda''(\cdot) \times \cdots \\
&\cdots \times \left[1 - (M_B^{DD})_I(\cdot) - [(\hat{D}_B^{ED})_x(\beta)/(\hat{I}_B^{ED})_x(\beta)](M_B^{DD})_D(\cdot)\right] \\
&\propto 1 - (M_B^{DD})_I(\cdot) - [(\hat{D}_B^{ED})_x(\beta)/(\hat{I}_B^{ED})_x(\beta)](M_B^{DD})_D(\cdot).
\end{align*}
\]

So, it would finally suffice if we could show that the expression on this last line is positive. To see that this is indeed the case, note that the pair \([\hat{D}_B^{ED}(\beta), \hat{I}_B^{ED}(\beta)]\) satisfies

\[
\chi_B \Xi'[\hat{\theta}_B^{\Xi}[\hat{I}_B^{ED}(\beta)]] = \rho + \Lambda'[\hat{I}_B^{ED}(\beta) - M_B^{DD}[\hat{D}_B^{ED}(\beta), \hat{I}_B^{ED}(\beta)],
\]

and this condition differentiates as follows:

\[
\begin{align*}
&\chi_B \Xi'[\hat{\theta}_B^{\Xi}[\hat{I}_B^{ED}(\beta)]] \hat{\theta}_B^{\Xi}[\hat{I}_B^{ED}(\beta)] = \Lambda''[\hat{I}_B^{ED}(\beta) - M_B^{DD}[\hat{D}_B^{ED}(\beta), \hat{I}_B^{ED}(\beta)]] \times \cdots \\
&\cdots \times \left[1 - (M_B^{DD})_I(\beta, \hat{I}_B^{ED}(\beta)] - \left[(\hat{D}_B^{ED})_x(\beta)/(\hat{I}_B^{ED})_x(\beta)\right] (M_B^{DD})_D(\hat{D}_B^{ED}(\beta), \hat{I}_B^{ED}(\beta))\right].
\end{align*}
\]

We can now finally conclude that \(r_B = ED\) is more likely to obtain in equilibrium the lesser is \(\beta\). So, let’s suppose for a moment that the only parameter being varied is the probability \(\alpha_G\). In light of the analysis above, we see that there are three cases to consider. The first is that all \(\alpha_G \in (\alpha_G^{LS}, 1]\) have the property that \(r_B = ED\) in equilibrium, in which case I set \(\alpha_G^{ED} = 1\). The second possibility is that all \(\alpha_G \in (\alpha_G^{LS}, 1]\) instead satisfy \(r_B \neq ED\) in equilibrium, in which case I set \(\alpha_G^{ED} = \alpha_G^{LS}\). The third possibility would be that the interval \((\alpha_G^{LS}, 1]\) admit a threshold such that \(r_B = ED\) holds as an equilibrium outcome i.f.f. \(\alpha_G\) lies below this threshold, in which case I set \(\alpha_G^{ED}\) equal to this threshold. \(\chi_G^{ED}\) can be derived analogously.

**Step three.** Very similar arguments can be applied to the locus

\[
\{(D, I_0) \in [0, \overline{D}_B^{DD}] \times \mathbb{R}_+ \text{ s.t. } I_0 = T_B^{DD}(D)\}
\]

so as to derive \(\alpha_G^{DD}\) and \(\chi_G^{DD}\).
Figure 2.6: Visual aid for proving lemmata 2.4.1 and 2.4.2
INTERBANK MARKETS AND THEIR OPTIMAL REGULATION

VOLUME TWO

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A DISSERTATION PRESENTED TO THE FACULTY OF PRINCETON UNIVERSITY IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE BY THE DEPARTMENT OF ECONOMICS

ADVISER: NOBUHIRO KIYOTAKI

MAY 2016
2.A.6 Proof of lemma 2.4.3

2.A.6.1 Some preliminary results

I’ll begin by deriving the planner’s analogues for the marginal return functions appearing in my previous subsection. To this end, I take cases on states and regimes:

Case one: \( r^{sp}_G = LR \). Under this case, utilitarian welfare is given by

\[
(E^h + D - I_0) + \Lambda[I_0 F[\theta^{LR}_G (D, I_0)]] + I_0 \int_{\theta^{LR}_G (D, I_0)}^{1} (\theta \chi_G - \rho) dF(\theta) + \mu(E^h - D).
\]

If \( v^{LR|sp}_G (D, I_0) \) denotes this function’s first derivative in \( x \in \{D, I_0\} \), then

\[
v^{LR|sp}_G (D, I_0) = F[\theta^{LR}_G (D, I_0)] \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]] + \int_{\theta^{LR}_G (D, I_0)}^{1} (\theta \chi_G - \rho) dF(\theta) - 1
\]

\[
+ I_0(\theta^{LR}_G)'_1 (D, I_0) f[\theta^{LR}_G (D, I_0)][\rho + \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]]
\]

\[
- \chi_G \theta^{LR}_G (D, I_0)
\]

\[
= \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]] + \Pi_G[\theta^{LR}_G (D, I_0), \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]] - 1
\]

\[
+ I_0(\theta^{LR}_G)'_1 (D, I_0)(\Pi_G) \theta[\theta^{LR}_G (D, I_0), \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]]]
\]

\[
= \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]] + \Pi_G[\theta^{LR}_G (D, I_0), \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]] - 1
\]

\[
- \left[ 1 + \rho - F[\theta^{LR}_G (D, I_0)][\rho + (\lambda^{rev})'[I_0 F[\theta^{LR}_G (D, I_0)]]]
\]

\[
\times \left( -1 \right) f[\theta^{LR}_G (D, I_0)][\rho + (\lambda^{rev})'[I_0 F[\theta^{LR}_G (D, I_0)]]
\]

\[
\times \ldots
\]

\[
\text{see (2.84)}
\]

\[
\cdots \times (\Pi_G) \theta[\theta^{LR}_G (D, I_0), \Lambda'[I_0 F[\theta^{LR}_G (D, I_0)]]]
\]

(2.103)
\[ = \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] + \Pi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] - 1 \]

\[
\frac{1 + \rho - F[\theta_G^{LR}(D, I_0)][\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]]
- F[\theta_G^{LR}(D, I_0)]I_0 F[\theta_G^{LR}(D, I_0)]\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]]}{(-1)f[\theta_G^{LR}(D, I_0)][\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\times \ldots \]

\[
\cdots \times (\Pi_G)\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] \]

\[ = \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] + \Pi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] - 1 \]

\[
\frac{1 - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] + \Psi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]]
- F[\theta_G^{LR}(D, I_0)]I_0 F[\theta_G^{LR}(D, I_0)]\Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]]}{(\Psi_G)\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}
\times \ldots \]

\[
\cdots \times (\Pi_G)\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]],
\]

and

\[
v_G^{LR,SP}(D, I_0)
= 1 - \mu'(E^h - D) + I_0(\theta_G^{LR})_D(D, I_0)f[\theta_G^{LR}(D, I_0)] \times \ldots \]

\[
\cdots \times [\rho + \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]] - \chi_G\theta_G^{LR}(D, I_0)]
\]

\[= 1 - \Delta \mu'(D) + I_0(\theta_G^{LR})_D(D, I_0)(\Pi_G)\theta[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]] \]
\begin{align}
&= 1 - \Delta \mu'(D) + \frac{(\Pi_G)_{\theta}[\theta_{LR}^L(D, I_0), \Lambda'[I_0 F[\theta_{LR}^L(D, I_0)]]]}{(-1)f[\theta_{LR}^L(D, I_0)][\rho + (\lambda^{Rev})'[I_0 F[\theta_{LR}^L(D, I_0)]]]} \\
&\quad \text{see } 2.86
\end{align}
\[
\begin{align*}
&\left[1 - F[\theta_B^{ED}(D, I_0)](\Lambda^{\text{Rev}}')\left[I_0 F[\theta_B^{ED}(D, I_0)]\right]
+ \int_{\theta_B^{ED}(D, I_0)}^{1} (\theta \chi_B - \rho) dF(\theta) + \gamma \chi_B \mathbb{E}\max[\theta_B^{ED}(D, I_0)]
\right]
\times \ldots
\end{align*}
\]

\[
\left[\frac{\rho + (\Lambda^{\text{Rev}}')\left[I_0 F[\theta_B^{ED}(D, I_0)]\right]}{\rho + (\Lambda^{\text{Rev}}')\left[I_0 F[\theta_B^{ED}(D, I_0)]\right]}
\right]
\times \ldots
\]

\[\ldots \times (\Pi_B)_{\theta}[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] = \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]] + \Pi_B[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] - 1\]

\[
\begin{align*}
&\left[1 - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]
+ \int_{\theta_B^{ED}(D, I_0)}^{1} [\theta \chi_B - \rho - \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] dF(\theta)
+ \gamma \chi_B \mathbb{E}\max[\theta_B^{ED}(D, I_0)] - F[\theta_B^{ED}(D, I_0)] \times \ldots
\right]
\times \ldots
\end{align*}
\]

\[
\left[\frac{\rho + \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]] - \chi_B \theta_B^{ED}(D, I_0) - \gamma \chi_B (F/f)[\theta_B^{ED}(D, I_0)]}{\rho + \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]}
\right]
\times \ldots
\]

\[\ldots \times (\Pi_B)_{\theta}[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] = \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]] + \Pi_B[\theta_B^{ED}(D, I_0), \Lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] - 1\]
\[
\begin{align*}
&1 - \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]] \\
&- \Delta_B[\theta^{ED}_B(D, I_0), \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]]] - F[\theta^{ED}_B(D, I_0)] \times \cdots \\
&+ \left[ \cdots \times I_0 F[\theta^{ED}_B(D, I_0)] \Lambda''[I_0 F[\theta^{ED}_B(D, I_0)]] \right] \\
&\frac{(\Delta_B)_0[\theta^{ED}_B(D, I_0), \Lambda[I_0 F[\theta^{ED}_B(D, I_0)]]]}{\Lambda''[I_0 F[\theta^{ED}_B(D, I_0)]]} \\
&+ f[\theta^{ED}_B(D, I_0)] I_0 F[\theta^{ED}_B(D, I_0)] \Lambda''[I_0 F[\theta^{ED}_B(D, I_0)]] \\
&\times \cdots \times \cdots \times (\Pi_B)_0[\theta^{ED}_B(D, I_0), \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]]],
\end{align*}
\]

and

\[
u^{ED|SP}_{BD}(D, I_0)
\]

\[
= 1 - \mu'(E^h - D) + I_0(\theta^{ED}_B)_D(D, I_0) \times \cdots \\
\cdots \times f[\theta^{ED}_B(D, I_0)][\rho + \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]] - \chi_B \theta^{ED}_B(D, I_0)] \\
= 1 - \Delta \mu'(D) + I_0(\theta^{ED}_B)_D(D, I_0)(\Pi_B)_0[\theta^{ED}_B(D, I_0), \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]]] \\
= 1 - \Delta \mu'(D)
\]

\[
\frac{1 - \Delta \mu'(D)}{f[\theta^{ED}_B(D, I_0)]} \left[ \rho + (\lambda^{Rev})'[I_0 F[\theta^{ED}_B(D, I_0)]] - \chi_B \theta^{ED}_B(D, I_0) \right] \\
- \gamma \chi_B(F/f)[\theta^{ED}_B(D, I_0)]
\]

see (2.91)

\[
= 1 - \Delta \mu'(D)
\]
\[
\begin{align*}
&= 1 - \Delta \mu'(D)
\end{align*}
\]

Case three: \(r^SP_B = DD\). Under this case, utilitarian welfare can be written as

\[
(E^b + D - I_0) + \Pi^SP_B[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0] + \mu(E^b - D),
\]

where the pair \([\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0)]\) is pinned down by the financial constraint, which can be written as

\[
(E^b + D - I_0) + \Pi^SP_B[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0] = \Delta \mu(D),
\]

and the marginal condition

\[
(\Pi^SP_B)_{I_0}[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0] \times \cdots
\]

\[
\cdots \times (\Delta^SP_B)_{I_0}[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0]
\]

\[
= (\Delta^SP_B)_{I_0}[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0] \times \cdots
\]

\[
\cdots \times (\Pi^SP_B)_{I_0}[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0],
\]

or equivalently

\[
\Xi^SP_B[\theta^{DD|SP}_B(D, I_0), I^{DD|SP}_B(D, I_0), I_0] = 0.
\]
Now, if use \( \cdot \) to suppress arguments, then differentiating (2.107) and (2.109) w.r.t. \( D \) yields

\[
(\theta^{|DB|SP}_B)_D(\cdot) = \frac{\Delta \mu'(D) - 1}{(\Delta^{|SP}_B)_\theta(\cdot) - (\Delta^{|SP}_B)_I_B(\cdot)(\Xi^{|SP}_B)_\theta(\cdot)/(\Xi^{|SP}_B)_I_B(\cdot)},
\]

(2.110)

with

\[
(I^{|DB|SP}_B)_D(\cdot) = (-1)(\theta^{|DB|SP}_B)_D(\cdot)(\Xi^{|SP}_B)_\theta(\cdot)/(\Xi^{|SP}_B)_I_B(\cdot).
\]

(2.111)

So,

\[
v^{|DB|SP}_B(\cdot) = 1 - \mu'(E^h - D) + (\theta^{|DB|SP}_B)_D(\cdot)(\Pi^{|SP}_B)_\theta(\cdot) + (I^{|DB|SP}_B)_D(\cdot)(\Pi^{|SP}_B)_I_B(\cdot)
\]

\[
= 1 - \Delta \mu'(D) + (\theta^{|DB|SP}_B)_D(\cdot)(\Pi^{|SP}_B)_\theta(\cdot) + (I^{|DB|SP}_B)_D(\cdot)(\Pi^{|SP}_B)_I_B(\cdot)
\]

\[
= 1 - \Delta \mu'(D) + (\theta^{|DB|SP}_B)_D(\cdot) \times \ldots
\]

\[
\ldots \times [((\Pi^{|SP}_B)_\theta(\cdot) - (\Pi^{|SP}_B)_I_B(\cdot)(\Xi^{|SP}_B)_\theta(\cdot)/(\Xi^{|SP}_B)_I_B(\cdot)]]
\]

(2.112)

\[
= 1 - \Delta \mu'(D) + (\theta^{|DB|SP}_B)_D(\cdot)[((\Pi^{|SP}_B)_I_B(\cdot)/(\Delta^{|SP}_B)_I_B(\cdot)]] \times \ldots
\]

\[
\ldots \times [((\Delta^{|SP}_B)_\theta(\cdot) - (\Delta^{|SP}_B)_I_B(\cdot)(\Xi^{|SP}_B)_\theta(\cdot)/(\Xi^{|SP}_B)_I_B(\cdot)]]
\]

(2.113)

\[
= 1 - \Delta \mu'(D) - [1 - \Delta \mu'(D)][((\Pi^{|SP}_B)_I_B(\cdot)/(\Delta^{|SP}_B)_I_B(\cdot)]]
\]

(2.114)

\[
= [1 - \Delta \mu'(D)] \left[ 1 - \frac{\Pi_B(\cdot)}{\Delta_B(\cdot) - [1 - F(\cdot)][I_0 - I^{|DD|SP}_B(\cdot)][1 - F(\cdot)]}\Lambda''(\cdot) \right]
\]

where (2.112), (2.113), and (2.114) respectively follow from (2.111), the marginal condition above, and (2.110). Similar arguments yield

\[
v^{|DD|SP}_B(\cdot) = (-1) + \Lambda'(\cdot) + [1 - (\Lambda^{|Rev}'(\cdot))][((\Pi^{|SP}_B)_I_B(\cdot)/(\Delta^{|SP}_B)_I_B(\cdot)]]
\]

\[
= (-1) + \Lambda'(\cdot) + [1 - \Lambda'(\cdot) - [I_0 - I^{|DD|SP}_B(\cdot)][1 - F(\cdot)]\Lambda''(\cdot)] \times \ldots
\]

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\[
\cdot \times \frac{\Pi_B(\cdot)}{\Delta_B(\cdot) - [1 - F(\cdot)][I_0 - I^{DD|SP}_B(\cdot)][1 - F(\cdot)]\Lambda''(\cdot)}.
\] (2.115)

Case four: \(r^{SP}_\omega = LS (\omega \in \{B, G\})\). If we define \(v^{LS|SP}_{\omega x}(D, I_0)\) in analogy with my last three cases, then it should be clear that

\[
v^{LS|SP}_{\omega x}(D, I_0) = v^{LS}(D, I_0), \quad \forall \{D, I_0\}.
\] (2.116)

2.A.6.2 Notation

It’s now useful to introduce some notation:

- first, I let \(D^{ED|SP}_B\) denote the unique point at which the functions \(T^{ED|SP}_B(D)\) and \(T^{LS}_G(D)\), while retaining \(D^{LS}_B\) and \(D^{DD}_B\) from my previous subsection;

- also, I let \(v^{SP}_D(D, I_0)\) denote the planner’s expected return from the marginal deposit, computed on an unconditional basis at \(t = 0\), after taking the analysis in subsections 2.3.1 and 2.3.2 into account — e.g.,

\[
I_0 \in (T^{LS}_G(D), T^{LS}_B(D)) \implies v^{SP}_D(D, I_0) = \alpha_Gv^{LR|SP}_{GD}(D, I_0) + \alpha^{LS|SP}_{BD}(D, I_0).
\]

Define \(v^{SP}_I(D, I_0)\) analogously.

Remark 1. It can easily be verified that the functions \(v^{SP}_D(D, I_0)\) and \(v^{SP}_I(D, I_0)\) are continuous in both their arguments, even around the boundaries separating regimes. This is a consequence of the envelope theorem.

Remark 2. For clarity, figure 2.7 illustrates some of the notation used in this subsection.

2.A.6.3 Some more preliminary results

It will now be useful to differentiate the marginal return functions defined above. I’ll again proceed by taking cases on states and regimes:
Case one: \( r_{SP}^{LR} = LR \). Revisiting (2.103) yields

\[
v_{GI}^{LR_{SP}}(D, I_0) = \Lambda'[I_0F[\theta_G^{LR}(D, I_0)] + \Pi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0F[\theta_G^{LR}(D, I_0)]] - 1
\]

\[
- \left[ 1 + \rho - F[\theta_G^{LR}(D, I_0)]\left[ \rho + (\lambda^{Rev})' [I_0F[\theta_G^{LR}(D, I_0)]] \right] \right] \times \ldots
\]

\[
\ldots \times (\Pi_G)_{\theta}[\theta_G^{LR}(D, I_0), \Lambda'[I_0F[\theta_G^{LR}(D, I_0)]]
\]

\[
= \Lambda'[I_0F[\theta_G^{LR}(D, I_0)] + \Pi_G[\theta_G^{LR}(D, I_0), \Lambda'[I_0F[\theta_G^{LR}(D, I_0)]] - 1
\]

\[
+ \left[ \frac{1 + \rho}{\rho + (\lambda^{Rev})' [I_0F[\theta_G^{LR}(D, I_0)]]} - F[\theta_G^{LR}(D, I_0)] \right] \times \ldots
\]

\[
\ldots \times \left[ \rho + \Lambda'[I_0F[\theta_G^{LR}(D, I_0)] - \chi_G \theta_G^{LR}(D, I_0) \right]
\]

\[
= \chi_G \varepsilon \max[\theta_G^{LR}(D, I_0)] - (1 + \rho) + \left[ \frac{1 + \rho}{\rho + (\lambda^{Rev})' [I_0F[\theta_G^{LR}(D, I_0)]]} \right] \times \ldots
\]

\[
\ldots \times \left[ \rho + \Lambda'[I_0F[\theta_G^{LR}(D, I_0)] - \chi_G \theta_G^{LR}(D, I_0) \right],
\]

so, \( \forall x \in \{D, I_0\} \),

\[
v_{GIx}^{LR_{SP}}(D, I_0)
\]

\[
= \chi_G(\theta_G^{LR})_{x}(D, I_0) \left[ F[\theta_G^{LR}(D, I_0)] - \frac{1 + \rho}{\rho + (\lambda^{Rev})' [I_0F[\theta_G^{LR}(D, I_0)]]} \right]
\]

\[
- \frac{d}{dx} \left[ I_0F[\theta_G^{LR}(D, I_0)] \right] \times \ldots
\]

\[
\ldots \times (\lambda^{Rev})'' [I_0F[\theta_G^{LR}(D, I_0)]] \left( 1 + \rho \right) \left[ \rho + \Lambda[I_0F[\theta_G^{LR}(D, I_0)] - \chi_G \theta_G^{LR}(D, I_0) \right]
\]

\[
= \chi_G \varepsilon \max[\theta_G^{LR}(D, I_0)] - (1 + \rho) + \left[ \frac{1 + \rho}{\rho + (\lambda^{Rev})' [I_0F[\theta_G^{LR}(D, I_0)]]} \right] \times \ldots
\]

\[
\ldots \times \left[ \rho + \Lambda[I_0F[\theta_G^{LR}(D, I_0)] - \chi_G \theta_G^{LR}(D, I_0) \right],
\]

(2.117)
\[
+ \frac{d}{dx} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]] \frac{1 + \rho}{\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]} < 0
\]

in which case (2.84) through (2.87) together imply that

\[
v_{GII}^{LR|SP}(D, I_0) < 0 < v_{GID}^{LR|SP}(D, I_0).
\]  

Similarly, revisiting (2.104) yields

\[
v_{GD}^{LR|SP}(D, I_0) = 1 - \Delta \mu'(D) + \frac{(\Pi_G)_0[\theta_G^{LR}(D, I_0), \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]]}{(-1)f[\theta_G^{LR}(D, I_0)][\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]]}
\]

\[
= 1 - \Delta \mu'(D) + \frac{\chi G \theta_G^{LR}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}{\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]},
\]  

so

\[
v_{GDI}^{LR|SP}(D, I_0)
\]

\[
= (\theta_G^{LR|SP})_1(D, I_0) \frac{\chi G}{\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]} < 0
\]

\[
- \frac{d}{dI_0} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \Lambda''[I_0 F[\theta_G^{LR}(D, I_0)]] \frac{1}{\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]} < 0
\]

\[
- \frac{d}{dI_0} \left[ I_0 F[\theta_G^{LR}(D, I_0)] \right] \times \cdots
\]

\[
\cdots \times (\lambda^{\text{Rev}})''[I_0 F[\theta_G^{LR}(D, I_0)]] \frac{\chi G \theta_G^{LR}(D, I_0) - \rho - \Lambda'[I_0 F[\theta_G^{LR}(D, I_0)]]}{\rho + (\lambda^{\text{Rev}})'[I_0 F[\theta_G^{LR}(D, I_0)]]^2} < 0
\]

and

\[
v_{GDD}^{LR|SP}(D, I_0)
\]
\[
\begin{align*}
&= (\theta^{LR|SP}_G)_{D}(D, I_0) \frac{\chi_G}{\rho + (\lambda^{Rev})'\left[I_0 F[\theta^{LR}_G(D, I_0)]\right]} \\
&\quad - \frac{d}{dD} \left[I_0 F[\theta^{LR}_G(D, I_0)]\right] \left[\frac{\Lambda''[I_0 F[\theta^{LR}_G(D, I_0)]]}{\rho + (\lambda^{Rev})'[I_0 F[\theta^{LR}_G(D, I_0)]]} \right] \frac{1}{\rho + (\lambda^{Rev})'[I_0 F[\theta^{LR}_G(D, I_0)]]} \\
&\quad - \frac{d}{dD} \left[I_0 F[\theta^{LR}_G(D, I_0)]\right] \times \ldots \\
&\quad \ldots \times (\lambda^{Rev})''[I_0 F[\theta^{LR}_G(D, I_0)]] \frac{\chi_G \theta^{LR}_G(D, I_0) - \rho - \Lambda'[I_0 F[\theta^{LR}_G(D, I_0)]]}{\rho + (\lambda^{Rev})'[I_0 F[\theta^{LR}_G(D, I_0)]]}^2 \\
&\quad - \Delta \mu''(D),
\end{align*}
\]

in which case (2.84) through (2.87) now imply that

\[
v^{LR|SP}_{GDD}(D, I_0) < 0 < v^{LR|SP}_{GDI}(D, I_0).
\]

Case two: \(r^{SP}_B = ED\). Revisiting (2.105) yields

\[
v^{ED|SP}_{BI}(D, I_0)
\]

\[
= \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]] + \Pi_B[\theta^{ED}_B(D, I_0), \Lambda'[I_0 F[\theta^{ED}_B(D, I_0)]]] - 1
\]
\[
\begin{align*}
1 - \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]] \\
- \int_{\theta_B^{ED}(D, I_0)}^{1} [\theta_B - \Lambda'F[\theta_B^{ED}(D, I_0)]]dF(\theta) \\
+ \gamma_{\theta B}\mathbb{E}\max[\theta_B^{ED}(D, I_0)] - F[\theta_B^{ED}(D, I_0)] \times \cdots \\
+ \sum \cdots I_0F[\theta_B^{ED}(D, I_0)]^{\text{II}}[I_0F[\theta_B^{ED}(D, I_0)]] \times \cdots \\
f[\theta_B^{ED}(D, I_0)] \\
\left[\begin{array}{c}
\rho + \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]
\end{array}\right] \\
\left[\begin{array}{c}
-\chi_B\theta_B^{ED}(D, I_0) - \gamma_{\theta B}(F/f)[\theta_B^{ED}(D, I_0)] \\
+ I_0F[\theta_B^{ED}(D, I_0)]^{\text{II}}[I_0F[\theta_B^{ED}(D, I_0)]]
\end{array}\right] \\
\cdots \times (\Pi_B)[\theta_B^{ED}(D, I_0), \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]] \\
= \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]) + \Pi_B[\theta_B^{ED}(D, I_0), \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]] - 1 \\
\left[\begin{array}{c}
1 - F[\theta_B^{ED}(D, I_0)][(\Lambda^{\text{Rev}})'I_0F[\theta_B^{ED}(D, I_0)]]
\end{array}\right] \\
- \left[\begin{array}{c}
\int_{\theta_B^{ED}(D, I_0)}^{1} (\theta_B - \rho)dF(\theta) - \gamma_{\theta B}\mathbb{E}\max[\theta_B^{ED}(D, I_0)]
\end{array}\right] \\
\chi_B\theta_B^{ED}(D, I_0) + \gamma_{\theta B}(F/f)[\theta_B^{ED}(D, I_0)] - \rho - (\Lambda^{\text{Rev}})'[I_0F[\theta_B^{ED}(D, I_0)]] \\
\cdots \times [\chi_B\theta_B^{ED}(D, I_0) - \rho - \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]] \\
= \chi_B\mathbb{E}\max[\theta_B^{ED}(D, I_0)] - (1 + \rho) \\
- F[\theta_B^{ED}(D, I_0)][\chi_B\theta_B^{ED}(D, I_0) - \rho - \Lambda'[I_0F[\theta_B^{ED}(D, I_0)]]]
\end{align*}
\]
\[
1 - F[\theta^E_B(D, I_0)](\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]]
+ \left[ \int_{\theta^E_B(D, I_0)}^{1} (\theta \chi_B - \rho)dF(\theta) - \gamma \chi_B \mathbb{E}\max[\theta^E_B(D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]] \right] \chi_B \theta^E_B(D, I_0) + \gamma \chi_B(F/f)[\theta^E_B(D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]] 
\times \ldots
\]

\[
\ldots \times [\chi_B \theta^E_B(D, I_0) - \rho - \Lambda'[I_0F[\theta^E_B(D, I_0)]]]
\]

\[
= \chi_B \mathbb{E}\max[\theta^E_B(D, I_0)] - (1 + \rho) + \left[ \chi_B \theta^E_B(D, I_0) - \rho - \Lambda'[I_0F[\theta^E_B(D, I_0)]] \right] \times \ldots
\]

\[
\ldots \times \left[ (1 + \rho) - (1 - \gamma)\chi_B \mathbb{E}\max[\theta^E_B(D, I_0)] - \gamma \chi_B F[\theta^E_B(D, I_0)](F/f)[\theta^E_B(D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]] \right]
\]

\[
\times \chi_B \theta^E_B(D, I_0) + \gamma \chi_B(F/f)[\theta^E_B(D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]]
\]

\[
\times \ldots 
\]

where, \( \forall x \in \{D, I_0\} \), we have

\[
\frac{d}{dx} \left[ (\Pi^{SP}_B)^{(\Delta^{SP}_B)}/[\theta^E_B(D, I_0), I_0, I_0] \right]
\]

\[
\propto \chi_B \theta^E_B(D, I_0) \left[ \chi_B \theta^E_B(D, I_0) + \gamma \chi_B(F/f)[\theta^E_B(D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0F[\theta^E_B(D, I_0)]] \right] \times \ldots
\]

\[
\times \left( \Lambda''[I_0F[\theta^E_B(D, I_0)]] \right) \times \ldots
\]

\(<0\) (again, see subsection 2.A.5.2)
\[
\cdots \times [\chi B \theta ED_B (D, I_0) + \gamma \chi_B (F/f) [\theta ED_B (D, I_0)] - \rho - (\lambda^{\text{Rev}})'[I_0 F[\theta ED_B (D, I_0)]]] \\
- \chi_B (\theta ED_B)_x (D, I_0) [1 + \gamma (F/f)' [\theta ED_B (D, I_0)]] \times \cdots \\
\cdots \times \left[\frac{\chi B \theta ED_B (D, I_0) - \rho - \lambda'[I_0 F[\theta ED_B (D, I_0)]]}{<0}\right] \\
+ \frac{d}{dx} \left[I_0 F[\theta ED_B (D, I_0)]\right] (\lambda^{\text{Rev}})''[I_0 F[\theta ED_B (D, I_0)]] \times \cdots \\
\cdots \times [\chi B \theta ED_B (D, I_0) - \rho - \lambda'[I_0 F[\theta ED_B (D, I_0)]]] \\
< 0, \\
(2.121)
\]

and thus

\[
u_{BIx}^{ED|SP} (D, I_0) = \chi_B (\theta ED|SP)_x (D, I_0) F[\theta ED|SP_B (D, I_0)] \times \cdots \\
\cdots \times \left[1 - [1 + \gamma (F/f)' [\theta ED_B (D, I_0)]][(\Pi^{SP}_B)_{\theta}/(\Delta^{SP}_B)_{\theta}][\theta ED_B (D, I_0), I_0, I_0] \right] \\
+ \frac{d}{dx} \left[(\Pi^{SP}_B)_{\theta}/(\Delta^{SP}_B)_{\theta}][\theta ED_B (D, I_0), I_0, I_0] \right] \times \cdots \\
\cdots \times \left[(1 + \rho) - (1 - \gamma) \chi_B \max[\theta ED_B (D, I_0)] - \gamma \chi_B F(\cdot)(F/f)(\cdot)\right] \\
< 0,
\]

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where \( \cdot \) suppresses obvious arguments. Similar derivations yield

\[
v_{BDD}^{ED|SP}(D, I_0) = (-1) \Delta \mu''(D) \left[ 1 - \left[ (\Pi_B^{SP})_\theta / (\Delta_B^{SP})_\theta \right][\theta_B^{ED}(D, I_0), \lambda'[I_0 F[\theta_B^{ED}(D, I_0)]]] \right]_{> 0} < 0
\]

\[
- [1 - \Delta \mu'(D)] \frac{d}{dD} \left[ (\Pi_B^{SP})_\theta / (\Delta_B^{SP})_\theta \right][\theta_B^{ED}(D, I_0), I_0, I_0]_{< 0} < 0
\]

\[
< 0,
\]

and

\[
v_{BDI}^{ED|SP}(D, I_0) = (-1) [1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ (\Pi_B^{SP})_\theta / (\Delta_B^{SP})_\theta \right][\theta_B^{ED}(D, I_0), I_0, I_0]_{< 0} < 0
\]

Case three: \( r_B^{SP} = DD \). If we let \( M_{B}^{DD|SP}(D, I_0) := I_{B}^{DD|SP}(D, I_0)[1 - F[\theta_{B}^{DD|SP}(D, I_0)]] \) denote the total volume of investments being maintained, then the financial constraint can be re-written as

\[
(E^b + D - I_0) + \lambda^{rev}[I_0 - M_{B}^{DD|SP}(D, I_0)] + M_{B}^{DD|SP}(D, I_0) \times \ldots
\]

\[
\times [\chi_{B1} \xi_1[\theta_{B}^{DD|SP}(D, I_0)] - \rho - \gamma \chi_{B2}[\theta_{B}^{DD|SP}(D, I_0)]] = \Delta \mu(D), \quad (2.122)
\]

and the pair \( [M_{B}^{DD|SP}(D, I_0), \theta_{B}^{DD|SP}(D, I_0)] \) is pinned down by this constraint, along with the marginal condition

\[
(\Pi_B^{SP})_\theta[\theta_B^{DD|SP}(D, I_0), M_{B}^{DD|SP}(D, I_0)/[1 - F[\theta_{B}^{DD|SP}(D, I_0)]]], I_0] \times \ldots
\]

\[
\times (\Delta_B^{SP})_I[\theta_B^{DD|SP}(D, I_0), M_{B}^{DD|SP}(D, I_0)/[1 - F[\theta_{B}^{DD|SP}(D, I_0)]]], I_0]
\]

\[
= (\Delta_B^{SP})_\theta[\theta_B^{DD|SP}(D, I_0), M_{B}^{DD|SP}(D, I_0)/[1 - F[\theta_{B}^{DD|SP}(D, I_0)]]], I_0] \times \ldots
\]

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\[
\ldots \times (\Pi_B^{SP})_I [\theta^{DD|SP}_B (D, I_0), M^{DD|SP}_B (D, I_0)/[1 - F[\theta^{DD|SP}_B (D, I_0)], I_0],
\]
which can be re-written as
\[
\chi_B \Xi_\theta [\theta^{DD|SP}_B (D, I_0)] - \rho - N[I_0 - M^{DD|SP}_B (D, I_0)] = [\xi_3 [\theta^{DD|SP}_B (D, I_0)]/\gamma] \times \ldots
\]
\[
\ldots \times [(\lambda^{Rev})'[I_0 - M^{DD|SP}_B (D, I_0)] - N[I_0 - M^{DD|SP}_B (D, I_0)]. (2.123)
\]
Differentiating (2.122) and (2.123) w.r.t. \(D\) then yields
\[
(M^{DD|SP}_B)_{D(D, I_0)} \times \ldots
\]
\[
\ldots \times \left[ \chi_B \xi_1 [\theta^{DD|SP}_B (D, I_0)] - \rho - (\lambda^{Rev})'[I_0 - M^{DD|SP}_B (D, I_0)] - \gamma \chi_B \xi_2 (\cdot) \right] < 0
\]
\[
+ (\theta^{DD|SP}_B)_{D(D, I_0)} \chi_B [\xi_1 (\cdot) - \gamma \xi_2 (\cdot)] = \Delta \mu' (D) - 1,
\]
and
\[
0 = (M^{DD|SP}_B)_{D(D, I_0)} \times \ldots
\]
\[
\ldots \times \left[ (\lambda^{Rev})''[I_0 - M^{DD|SP}_B (D, I_0)] + [1 - (1/\gamma) \xi_3 (\cdot)] \Lambda'' (\cdot) \right] < 0
\]
\[
+ (\theta^{DD|SP}_B)_{D(D, I_0)} [\chi_B \Xi'_\theta (\cdot) - [\xi'_3 (\cdot)/\gamma][(\lambda^{Rev})'(\cdot) - N'(\cdot)],
\]
where \(\cdot\) suppresses obvious arguments. An application of Cramer’s rule will then confirm that \((\theta^{DD|SP}_B)_{D(D, I_0)} < 0\) and \((M^{DD|SP}_B)_{D(D, I_0)} < 0\), and similar arguments yield \((\theta^{DD|SP}_B)_{I(D, I_0)} < 0\) and \(1 - (M^{DD|SP}_B)_{I(D, I_0)} > 0\).

Now, recall from (2.114) that
\[
v^{DD|SP}_{BD} (D, I_0)
\]
\[= 1 - \Delta \mu'(D) - [1 - \Delta \mu'(D)] \times \ldots\]

\[
\ldots \times \left(\frac{\prod_{B}^{SP} I_{B} [\theta_{B}^{DD|SP} (D, I_{0}) \cdot M_{B}^{DD|SP} (D, I_{0})] / [1 - F[\theta_{B}^{DD|SP} (D, I_{0})]], I_{0}]}{\prod_{B}^{\Delta SP} I_{B} [\theta_{B}^{DD|SP} (D, I_{0}) \cdot M_{B}^{DD|SP} (D, I_{0})] / [1 - F[\theta_{B}^{DD|SP} (D, I_{0})]], I_{0}]}\right) \]

with

\[
\frac{d}{dD} \left[\left(\frac{\prod_{B}^{SP} I_{B} [\theta_{B}^{DD|SP} (D, I_{0}) \cdot M_{B}^{DD|SP} (D, I_{0})] / [1 - F[\theta_{B}^{DD|SP} (D, I_{0})]], I_{0}]}{\prod_{B}^{\Delta SP} I_{B} [\theta_{B}^{DD|SP} (D, I_{0}) \cdot M_{B}^{DD|SP} (D, I_{0})] / [1 - F[\theta_{B}^{DD|SP} (D, I_{0})]], I_{0}]}\right)\right]
\]
\[
\frac{\theta_B^{DD|SP}}{D(D,I_0)} f[\theta_B^{DD|SP}(D,I_0)] \times \ldots
\]
\[
\ldots \times \left[ \rho + \Lambda'[I_0 - M_B^{DD|SP}(D,I_0)] - \chi_B \theta_B^{DD|SP}(D,I_0) \right] \times \ldots
\]
\[
= \frac{1}{f(\cdot)}(\Pi_B^{SP})_0(\cdot)/|M_B^{DD|SP}(D,I_0)|/|1 - F(\theta_B^{DD|SP}(D,I_0))|
\]
\[
\ldots \times (\Delta_B^{SP})_B \left[ \theta_B^{DD|SP}(D,I_0), \frac{M_B^{DD|SP}(D,I_0)}{1 - F(\theta_B^{DD|SP}(D,I_0))}, I_0 \right]
\]
\[
+ (M_B^{DD|SP})_B(D,I_0) \Lambda''[I_0 - M_B^{DD|SP}(D,I_0)] \times \ldots
\]
\[
\ldots \times (\Delta_B^{SP})_B \left[ \theta_B^{DD|SP}(D,I_0), \frac{M_B^{DD|SP}(D,I_0)}{1 - F(\theta_B^{DD|SP}(D,I_0))}, I_0 \right]
\]
\[
- (\theta_B^{DD|SP})_B(D,I_0) f[\theta_B^{DD|SP}(D,I_0)] \times \ldots
\]
\[
\ldots \times \left[ \rho + (\lambda^{Rev})'[I_0 - M_B^{DD|SP}(D,I_0)] - \chi_B \theta_B^{DD|SP}(D,I_0) \right] \times \ldots
\]
\[
= \frac{1}{f(\cdot)}(\Delta_B^{SP})_0(\cdot)/|M_B^{DD|SP}(D,I_0)|/|1 - F(\theta_B^{DD|SP}(D,I_0))|
\]
\[
\ldots \times (\Pi_B^{SP})_B \left[ \theta_B^{DD|SP}(D,I_0), \frac{M_B^{DD|SP}(D,I_0)}{1 - F(\theta_B^{DD|SP}(D,I_0))}, I_0 \right]
\]
\[
- (M_B^{DD|SP})_B(D,I_0) (\lambda^{Rev})''[I_0 - M_B^{DD|SP}(D,I_0)] \times \ldots
\]
\[
= \left[ (\Delta_B^{SP})_B \left[ \theta_B^{DD|SP}(D,I_0), \frac{M_B^{DD|SP}(D,I_0)}{1 - F(\theta_B^{DD|SP}(D,I_0))}, I_0 \right] \right]^{2}
\]
\[
= (M_B^{DD|SP})_B(D,I_0) \times \ldots
\]
(2.124)
\[
\left( \Lambda''[I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots \right)_{<0}
\]

\[
\ldots \times (\Delta_B^{SP})_{IB} \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]_{<0}
\]

\[
- (\lambda^{Rev})^u[I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\ldots \times (\Pi_B^{SP})_{IB} \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]_{>0}
\]

\[
\left( (\Delta_B^{SP})_{IB} [\theta_B^{DD|SP}(D, I_0), M_B^{DD|SP}(D, I_0)/[1 - F[\theta_B^{DD|SP}(D, I_0)]], I_0] \right)^2
\]

\[
< 0,
\]

where the middle step follows from the marginal condition above. Similarly,

\[
\frac{d}{dI_0} \left[ (\Pi_B^{SP})_{IB} \right] \left[ \theta_B^{DD|SP}(D, I_0), M_B^{DD|SP}(D, I_0)/[1 - F[\theta_B^{DD|SP}(D, I_0)]], I_0 \right]
\]
\[
(\theta_B^{DD|SP})_I(D, I_0)f[\theta_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\ldots \times \left[ \rho + \Lambda'[I_0 - M_B^{DD|SP}(D, I_0)] - \chi_B^{DD|SP}(D, I_0) \right] \times \ldots
\]

\[
= [1/f(\cdot)][\Pi_B^{SP}(\cdot)/[M_B^{DD|SP}(D, I_0)/[1 - F[\theta_B^{DD|SP}(D, I_0)]]]
\]

\[
\ldots \times \left( \Delta^B_I \right)_B \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]
\]

\[
- [1 - (M_B^{DD|SP})_I(D, I_0)] \Lambda' [I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\ldots \times \left( \Delta^B_I \right)_B \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]
\]

\[
- (\theta_B^{DD|SP})_I(D, I_0)f[\theta_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\ldots \times \left[ \rho + (\lambda^{\text{Rev}})'[I_0 - M_B^{DD|SP}(D, I_0)] - \chi_B^{DD|SP}(D, I_0) \right]
\]

\[
\ldots \times \left( \Delta^B_I \right)_B \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]
\]

\[
+ [1 - (M_B^{DD|SP})_I(D, I_0)][(\lambda^{\text{Rev}})^\prime][I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\ldots \times \left( \Pi^B_I \right)_B \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right]
\]

\[
= [(\Delta^B_I)_B[\theta_B^{DD|SP}(D, I_0), M_B^{DD|SP}(D, I_0)/[1 - F[\theta_B^{DD|SP}(D, I_0)]], I_0]^2
\]

\[
= [1 - (M_B^{DD|SP})_I(D, I_0)] \times \ldots
\]

(2.125)
\[
(\lambda^\text{Rev})''[I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\cdots \times (\Pi_B^{SP})_{I_B} \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right] > 0
\]

\[
- \Lambda''[I_0 - M_B^{DD|SP}(D, I_0)] \times \ldots
\]

\[
\cdots \times (\Delta_B^{SP})_{I_B} \left[ \theta_B^{DD|SP}(D, I_0), \frac{M_B^{DD|SP}(D, I_0)}{1 - F[\theta_B^{DD|SP}(D, I_0)]}, I_0 \right] < 0
\]

\[
\left[ (\Delta_B^{SP})_{I_B} [v_B^{DD|SP}(D, I_0), M_B^{DD|SP}(D, I_0) / [1 - F[\theta_B^{DD|SP}(D, I_0)]], I_0] \right]^2 < 0.
\]

So, using \( \cdot \) to suppress arguments, we have

\[
v_B^{DD|SP} (\cdot) = (-1) \Delta \mu''(D) \left[ 1 - [(\Pi_B^{SP})_{I_B} / (\Delta_B^{SP})_{I_B}] (\cdot) \right]
\]

\[
- [1 - \Delta \mu'(D)] \frac{d}{dD} \left[ [(\Pi_B^{SP})_{I_B} / (\Delta_B^{SP})_{I_B}] (\cdot) \right] < 0,
\]

and

\[
v_B^{DD|SP} (\cdot) = (-1) [1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ [(\Pi_B^{SP})_{I_B} / (\Delta_B^{SP})_{I_B}] (\cdot) \right] < 0.
\]

Similar arguments yield

\[
v_B^{DD|SP} (\cdot) = \Lambda''(\cdot) \left[ 1 - (M_B^{DD|SP})_{I} (\cdot) \right]
\]
\[-(\lambda^{\text{rev}})^{''}(\cdot)[1 - (M_B^{\text{DD}\mid\text{SP}})I(\cdot)][(\Pi_B^{\text{SP}})I_B/(\Delta_B^{\text{SP}})I_B](\cdot)\]

\[+ [1 - (\lambda^{\text{rev}})'(\cdot)] d \frac{d}{dI_0} \left[ [(\Pi_B^{\text{SP}})I_B/(\Delta_B^{\text{SP}})I_B](\cdot) \right]\]

\[< 0,\]

and

\[v_{\text{DD}\mid\text{SP}}^{\text{DD}\mid\text{SP}}(\cdot) = \Lambda^{''}(\cdot)(-1) (M_B^{\text{DD}\mid\text{SP}})_D(\cdot) + (\lambda^{\text{rev}})^{''}(\cdot)(M_B^{\text{DD}\mid\text{SP}})_D(\cdot) [(\Pi_B^{\text{SP}})I_B/(\Delta_B^{\text{SP}})I_B](\cdot)\]

\[+ [1 - (\lambda^{\text{rev}})'(\cdot)] d \frac{d}{dD} \left[ [(\Pi_B^{\text{SP}})I_B/(\Delta_B^{\text{SP}})I_B](\cdot) \right]\]

\[< 0.\]

### 2.A.6.4 Solution for (P^{\text{SP}}\text{-rex}): existence, uniqueness, and partial characterization

At this point, we’re ready to solve (P^{\text{SP}}\text{-rex}). Now, (P^{\text{SP}}\text{-rex}) has been defined as a relaxed version of the planner’s ex-ante problem under which we drop the financial constraint associated with the good state and the physical constraint associated with the bad state, along with the non-negativity constraint $I_0 \leq E^b + D$. Moreover, it should be clear that assumption 2.1 ensures some slack in the lower bound $I_0 \geq 0$, while assumption 2.4, combined with the fact that $\mu'(E^b) = 1$, ensures slack in the constraint $D \in [0, E^b]$. As a result, solutions for (P^{\text{SP}}\text{-rex}) must fall under one of two cases. The first would be an interior case under which the “no-default” constraint

\[(E^b + D - I_0) + I_0\Lambda'(I_0) = (E^b + D - I_0) + \lambda^{\text{rev}}(I_0) \geq \Delta\mu(D)\]

is lax, leading the planner to settle on an initial balance sheet satisfying

\[v_x^{\text{SP}}(D, I_0) = 0, \forall x \in \{D, I_0\}.\]
The alternative would be a corner case under which the “no-default” constraint binds.

Now, based on the analysis in my previous subsubsection, it should be clear that the marginal return function \( v_{SP}^{SP}(D, I_0) \) is strictly decreasing in its first argument. At the same time, assumption 2.1 ensures that \( v_{SP}^{SP}(D, 0) > 0 \) \( \forall D \in \mathbb{R}_+ \) s.t. \( E^b + D \geq \Delta \mu(D) \). So, for choices on \( D \) in this range, we have two cases to consider. If \( v_{SP}^{SP}[D, T_B^{DD}(D)] \leq 0 \), then the marginal return \( v_{SP}^{SP}(D, I_0) \) exhibits single-crossing from above in its second argument, namely at some point \( I^{*|SP}(D) \in (0, T_B^{DD}(D)) \). If instead \( v_{SP}^{SP}[D, T_B^{DD}(D)] > 0 \), then \( v_{SP}^{SP}(D, I_0) \) is strictly positive over all of the interval \([0, T_B^{DD}(D)]\), and I set \( I^{*|SP}(D) = T_B^{DD}(D) \).

We can now conclude that solutions for \((P_{SP-rex})\) must take the form \((D, I_0) = [D, I^{*|SP}(D)]\). Given any such pair, I’ll now check if the planner has an incentive to deviate in his initial balance-sheet choices, namely making some small adjustment to the choice on \( D \). Now, if the pair in question has the property that the “no-default” constraint is lax, then the planner’s return from a marginal increase in \( D \) is simply given by \( v_{SP}^{SP}[D, I^{*|SP}(D)] \). If the “no-default” constraint instead binds, then increases in \( D \) must be offset by reductions in \( I_0 \), and the relevant return reads as

\[
 v^{SP}_D[D, T_B^{DD}(D)] + (T_B^{DD})'(D)v^{SP}_I[D, T_B^{DD}(D)].
\]

So, \( \forall D \in \mathbb{R}_+ \) s.t. \( E^b + D \geq \Delta \mu(D) \), I define a function

\[
 h^{SP}(D) := \begin{cases} 
 v^{SP}_D[D, I^{*|SP}(D)] & \text{if } v^{SP}_I[D, T_B^{DD}(D)] \leq 0 \\
 v^{SP}_D[D, T_B^{DD}(D)] + (T_B^{DD})'(D)v^{SP}_I[D, T_B^{DD}(D)] & \text{if } v^{SP}_I[D, T_B^{DD}(D)] > 0.
\end{cases}
\]

Now, it should be clear that this function is continuous. It should also be clear that it’s strictly positive when \( D = 0 \), but strictly negative when \( E^b + D = \Delta \mu(D) \). Existence and uniqueness would thus follow if we could show that the function \( h^{SP}(\cdot) \) is strictly decreasing. To see that this is indeed the case, we’ll have to take cases on \( D \):

**Case 1a.** Suppose first that \( D \) has the property that the pair \([D, I^{*|SP}(D)]\) satisfies \((r_B^{SP}, r_C^{SP}) = (LS, LR)\). In this case, we want to confirm that

\[
 (h^{SP})'(D) = v^{SP}_{DD}[D, I^{*|SP}(D)] + (I^{*|SP})'(D)v^{SP}_{DI}[D, I^{*|SP}(D)].
\]
or equivalently

\[
\frac{v^S_{II}[D, I^*[SP](D)]v^S_{DD}[D, I^*(D)] - v^S_{ID}[D, I^*[SP](D)]v^S_{DI}[D, I^*[SP](D)] > 0,}{(\ast)}
\]

which we recognize as the usual second-order condition. Now, if we use \cdot to suppress arguments, then the derivations in subsubsections 2.A.6.1 and 2.A.6.3 imply that

\[
(\ast) = [\alpha_G v^L_{GII}(\cdot) + \alpha_B v^L_{BH}(\cdot)][\alpha_G v^L_{GDD}(\cdot) + \alpha_B v^L_{BDD}(\cdot)]
\]

\[
- [\alpha_G v^L_{GID}(\cdot) + \alpha_B v^L_{BID}(\cdot)][\alpha_G v^L_{GDI}(\cdot) + \alpha_B v^L_{BDI}(\cdot)]
\]

\[
= [\alpha_G v^L_{GII}(\cdot) + \alpha_B v^L_{BII}(\cdot)] [\alpha_G v^L_{GDD}(\cdot) + \alpha_B v^L_{BDD}(\cdot)] < 0
\]

\[
- [\alpha_G v^L_{GID}(\cdot) + \alpha_B v^L_{BID}(\cdot)] [\alpha_G v^L_{GDI}(\cdot) + \alpha_B v^L_{BDI}(\cdot)] = 0
\]

\[
> \alpha_G v^L_{GII}(\cdot)[\alpha_G v^L_{GDD}(\cdot) + \alpha_B v^L_{BDD}(\cdot)] - 2 \alpha^2_G v^L_{GID}(\cdot)v^L_{GDI}(\cdot)
\]

\[
= \alpha_G v^L_{GII}(\cdot) \left[ \alpha_G \frac{d}{dD} \left[ \frac{\chi_G \theta^L_G(\cdot) - \rho - \lambda'(\cdot)}{\rho + (\lambda)^{\lambda'}(\cdot)} \right] - \Delta \mu''(\cdot) > 0 \right]
\]

\[
- 2 \alpha^2_G v^L_{GID}(\cdot)v^L_{GDI}(\cdot)
\]

\[
> \alpha^2_G v^L_{GII}(\cdot) \frac{d}{dD} \left[ \frac{\chi_G \theta^L_G(\cdot) - \rho - \lambda'(\cdot)}{\rho + (\lambda)^{\lambda'}(\cdot)} \right] - \alpha^2_G v^L_{GID}(\cdot)v^L_{GDI}(\cdot)
\]

\[
\propto v^L_{GII}(\cdot) \frac{d}{dD} \left[ \frac{\chi_G \theta^L_G(\cdot) - \rho - \lambda'(\cdot)}{\rho + (\lambda)^{\lambda'}(\cdot)} \right] - v^L_{GID}(\cdot)v^L_{GDI}(\cdot)
\]

\[
= \left[ \chi_G (\theta^L_G)(\cdot) F(\cdot) - (1 + \rho) \frac{d}{dI_0} \left[ \frac{\chi_G \theta^L_G(\cdot) - \rho - \lambda'(\cdot)}{\rho + (\lambda)^{\lambda'}(\cdot)} \right] \right] \times \ldots
\]

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\[ \ldots \times \frac{d}{dD} \left[ \frac{\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)}{\rho + (\lambda^\text{Rev})'(\cdot)} \right] \]

\[ - \left[ \chi_G(\theta_G^{LR})_D(\cdot)F(\cdot) - (1 + \rho) \frac{d}{dD} \left[ \frac{\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)}{\rho + (\lambda^\text{Rev})'(\cdot)} \right] \right] \times \ldots \]

\[ \ldots \times \frac{d}{dI_0} \left[ \frac{\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)}{\rho + (\lambda^\text{Rev})'(\cdot)} \right] \]

\[ \propto (\theta_G^{LR})_I(\cdot) \frac{d}{dD} \left[ \frac{\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)}{\rho + (\lambda^\text{Rev})'(\cdot)} \right] - (\theta_G^{LR})_D(\cdot) \frac{d}{dI_0} \left[ \frac{\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)}{\rho + (\lambda^\text{Rev})'(\cdot)} \right] \]

\[ = (\theta_G^{LR})_I(\cdot) \frac{d}{dD} \left[ I_0 F(\cdot) \right] \frac{(\lambda^\text{Rev})''(\cdot)[\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)]}{\rho + (\lambda^\text{Rev})'(\cdot)} \]

\[ - (\theta_G^{LR})_D(\cdot) \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \frac{(\lambda^\text{Rev})''(\cdot)[\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)]}{\rho + (\lambda^\text{Rev})'(\cdot)} \]

\[ + (\theta_G^{LR})_D(\cdot) \frac{d}{dD} \left[ I_0 F(\cdot) \right] \frac{(\lambda^\text{Rev})''(\cdot)[\chi_G \theta_G^{LR}(\cdot) - \rho - \Lambda'(\cdot)]}{\rho + (\lambda^\text{Rev})'(\cdot)} \]

\[ \propto (\theta_G^{LR})_I(\cdot) \frac{d}{dD} \left[ I_0 F(\cdot) \right] - (\theta_G^{LR})_D(\cdot) \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \]

\[ = (-1) \frac{1 + \rho - F(\cdot)[\rho + \lambda'(\cdot)]}{I_0 f(\cdot)\rho + (\lambda^\text{Rev})'(\cdot)[\rho + (\lambda^\text{Rev})'(\cdot)]^2} + \frac{1 + \rho}{I_0 f(\cdot)\rho + (\lambda^\text{Rev})'(\cdot)[\rho + (\lambda^\text{Rev})'(\cdot)]^2} \]

\[ > 0, \]

where the penultimate line follows from (2.84)-(2.87).

**Case 1b.** Very similar arguments go through when \( D \) has the property that \([D, I^*|_{SP}(D)]\) satisfies \((r_B^{SP}, r_C^{SP}) = (LS, LS)\).
Case 2a. Suppose next that $D$ has the property that $[D, I^*|SP(D)]$ satisfies $(v^SP_B, v^SP_G) = (ED, LR)$. In this case, we still need to confirm that

$$v^SP_H[D, I^*|SP(D)]v^SP-DD[D, I^*|SP(D)]v^SP-ID[D, I^*|SP(D)]v^SP-DI[D, I^*|SP(D)] > 0.$$ 

Now, if we use \( \cdot \) to suppress arguments, then this inequality can be rewritten as

$$[\alpha_G^L{LR}|SP_G(\cdot) + \alpha_B^E{ED}|SP_B(\cdot)][\alpha_G^L{LR}|SP_G(\cdot) + \alpha_B^E{ED}|SP_B(\cdot)]$$

$$> [\alpha_G^L{LR}|SP_G(\cdot) + \alpha_B^E{ED}|SP_B(\cdot)][\alpha_G^L{LR}|SP_G(\cdot) + \alpha_B^E{ED}|SP_B(\cdot)],$$

or equivalently

$$\alpha_G^2 [v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot) - v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)]$$

$$+ \alpha_G \alpha_B \left[ v^L{LR}|SP_G(\cdot)v^E{ED}|SP_B(\cdot) + v^E{ED}|SP_B(\cdot)v^L{LR}|SP_G(\cdot) 
- v^L{LR}|SP_G(\cdot)v^E{ED}|SP_B(\cdot) - v^E{ED}|SP_B(\cdot)v^L{LR}|SP_G(\cdot) \right]$$

$$+ \alpha_B^2 [v^E{ED}|SP_B(\cdot)v^E{ED}|SP_B(\cdot) - v^E{ED}|SP_B(\cdot)v^E{ED}|SP_B(\cdot)] > 0.$$ (***)

Now, in my analysis of case 1a, I showed that the single-starred term is strictly positive. Moreover, in light of the derivations in my previous subsubsection, we know that the double-starred term signs as

$$\frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} + \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} - \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} > 0,$$

$$\frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{>0} + \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} - \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} > 0,$$

$$\frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} + \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} - \frac{v^L{LR}|SP_G(\cdot)v^L{LR}|SP_G(\cdot)}{<0} > 0.$$
while the triple-starred term reads as

\[
\left( -1 \right) \frac{\Delta \mu''(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]

\[
\left[ (1 + \rho) - (1 - \gamma)\chi_{B}E \max(\cdot) - \gamma \chi_{B}F(\cdot)(F/f)(\cdot) \right] \times \ldots
\]

\[
\left( -1 \right) \frac{\Delta \mu'(D)}{\rho} \left[ 1 - \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{LR,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right] - v_{BII}^{ED,SP}(-1) \frac{d}{dD} \left[ \left( \frac{\Pi_{B}^{SP}}{\Delta_{B}^{SP}} \right) \theta \right]
\]
\[ \alpha (\theta^E_D)_{I}(\cdot) \]

\[ \propto (\theta^E_B)_{I}(\cdot) \]

\[ \alpha (\theta^E_D)_{I}(\cdot) \]

\[ \frac{d}{dD} \left[ I_0 F(\cdot) \right] \times \ldots \]

\[ \cdot \times \Lambda''(\cdot) \left[ \chi_B \theta^E_B(\cdot) + \gamma \chi_B (F/f)(\cdot) - \rho - (\lambda^{\text{Rev}})'(\cdot) \right] \]

\[ \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \times \ldots \]

\[ - \chi_B(\theta^E_B)_{D}(\cdot) \left[ 1 + \gamma (F/f)'(\cdot) \right] [\chi_B \theta^E_B(\cdot) - \rho - \Lambda'(\cdot)] \]

\[ \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \left( \lambda^{\text{Rev}}''(\cdot) \left[ \chi_B \theta^E_B(\cdot) - \rho - \Lambda'(\cdot) \right] \right) \]

\[ \alpha (\theta^E_B)_{I}(\cdot) \frac{d}{dD} \left[ I_0 F(\cdot) \right] - (\theta^E_D)_{D}(\cdot) \frac{d}{dI_0} \left[ I_0 F(\cdot) \right] \]

\[ > 0, \]  

(2.127)

where (2.126) follows from (2.121), while the final inequality was established in my analysis of case 2a in subsubsection 2.A.5.3.

Case 2b. Very similar arguments go through when \( D \) has the property that \([D, I^*_{SP}(D)]\) satisfies \((r^S_B, r^S_C) = (ED, LS)\).
Case 3a. Suppose next that \( D \) has the property that \( [D, I^*_{\text{SP}}(D)] \) satisfies \( (r^\text{SP}_B, r^\text{SP}_G) = (DD, LR) \), with the “no-default” constraint lax. In this case, we still need to confirm that \( v^\text{SP}_I[D, I^*_{\text{SP}}(D)]v^\text{SP}_{DD}[D, I^*(D)] - v^\text{SP}_{ID}[D, I^*_{\text{SP}}(D)]v^\text{SP}_{DI}[D, I^*_{\text{SP}}(D)] > 0 \).

Now, if we use \( \cdot \) to suppress arguments, then this inequality can be rewritten as

\[
[\alpha_G v^\text{LR}_{G\text{II}}(\cdot) + \alpha_B v^\text{DD}_{B\text{II}}(\cdot)][\alpha_G v^\text{LR}_{G\text{DD}}(\cdot) + \alpha_B v^\text{DD}_{B\text{DD}}(\cdot)]
\]

or equivalently

\[
\alpha^2_G \left[ v^\text{LR}_{G\text{II}}(\cdot)v^\text{LR}_{G\text{DD}}(\cdot) - v^\text{LR}_{G\text{ID}}(\cdot)v^\text{LR}_{G\text{DI}}(\cdot) \right]
\]

\[
+ \alpha_B \left[ v^\text{DD}_{B\text{II}}(\cdot)v^\text{DD}_{B\text{DD}}(\cdot) - v^\text{DD}_{B\text{ID}}(\cdot)v^\text{DD}_{B\text{DI}}(\cdot) \right]
\]

\[
\alpha^2_B \left[ v^\text{LR}_{B\text{II}}(\cdot)v^\text{LR}_{B\text{DD}}(\cdot) - v^\text{LR}_{B\text{ID}}(\cdot)v^\text{LR}_{B\text{DI}}(\cdot) \right] > 0.
\]

Now, in my analysis of case 1a, I showed that the single-starred term is strictly positive. Moreover, in light of the derivations in my previous subsubsection, we know that the double-starred term signs as

\[
\alpha^2_G \left[ v^\text{LR}_{G\text{II}}(\cdot)v^\text{LR}_{G\text{DD}}(\cdot) - v^\text{LR}_{G\text{ID}}(\cdot)v^\text{LR}_{G\text{DI}}(\cdot) \right] < 0
\]

\[
\alpha^2_B \left[ v^\text{DD}_{B\text{II}}(\cdot)v^\text{DD}_{B\text{DD}}(\cdot) - v^\text{DD}_{B\text{ID}}(\cdot)v^\text{DD}_{B\text{DI}}(\cdot) \right] < 0
\]
while the triple-starred term reads as

\[
\begin{aligned}
& v_{BBII}^{DD|SP} (<0) \left[ (-1) \frac{\Delta \mu''(D)}{>0} \left[ 1 - \left[ \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right] \frac{d}{dD} \right] \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \right] \\
& > v_{BBH}^{DD|SP} (-1) \left[ 1 - \Delta \mu'(D) \right] \frac{d}{dD} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \\
& - v_{BID}^{DD|SP} (-1) v_{BDI}^{DD|SP} \end{aligned}
\]

\[
\begin{aligned}
& \left[ \Lambda''(\cdot) \left[ 1 - (M^B_{BD|SP})_I(\cdot) \right] \frac{d}{dD} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \right] \\
& = \left[ (\lambda^{Rev})''(\cdot) \left[ 1 - (M^B_{BD|SP})_I(\cdot) \right] \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \times \ldots \\
& + \left[ 1 - (\lambda^{Rev})'(\cdot) \right] \frac{d}{dI_0} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \times \ldots \\
& \times (-1) \left[ 1 - \Delta \mu'(D) \right] \frac{d}{dD} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \\
& \left[ \Lambda''(-1)(M^B_{BD|SP})_D(\cdot) \right] \\
& = \left[ (\lambda^{Rev})''(\cdot)(M^B_{BD|SP})_D(\cdot) \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \times \ldots \\
& + \left[ 1 - (\lambda^{Rev})'(\cdot) \right] \frac{d}{dD} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \times \ldots \\
& \times (-1)[1 - \Delta \mu'(D)] \frac{d}{dI_0} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right] \\
& \propto (-1)(M^B_{BD|SP})_D(\cdot) \frac{d}{dI_0} \left[ \left( \frac{(\Pi^B)_{IB}/(\Delta^B)_{IB}}{>0} \right) \right]
\end{aligned}
\]
\[- \left[ 1 - (M_B^{DD|SP})_I(\cdot) \right] \frac{d}{dD} \left[ \left( (\Pi_B^{SP})_{I_B}/(\Delta_B^{SP})_{I_B} \right) (\cdot) \right] \]

= 0,

where the last line follows from comparing (2.124) and (2.125).

**Case 3b.** Very similar arguments go through when \( D \) has the property that \([D, I^*|SP(D)]\) satisfies \((r_{SP}^{SP}, r_{SP}^{G}) = (DD, LS),\) with the “no-default” constraint lax.

**Case 4a.** Suppose next that \( D \) has the property that the “no-default” constraint binds, with \( r_{SP}^{G} = LR.\) In this case, we have

\[ h^{SP}(D) = v_{D}^{SP}[D, T_B^{DD}(D)] + (T_B^{DD})'(D) v_{I}^{SP}[D, T_B^{DD}(D)] \]

\[ = \alpha_G [v_{GD}^{LR|SP}[D, T_B^{DD}(D)] + (T_B^{DD})'(D) v_{GI}^{LR|SP}[D, T_B^{DD}(D)]] \]

\[ + \alpha_B [v_{BD}^{DD|SP}[D, T_B^{DD}(D)] + (T_B^{DD})'(D) v_{BI}^{DD|SP}[D, T_B^{DD}(D)]] \]

where

\[ (T_B^{DD})'(D) = \frac{1 - \Delta \mu'(D)}{1 - (\lambda_{Rev})'[T_B^{DD}(D)]} < 0, \]

(2.128)

and, in turn,

\[ (T_B^{DD})''(D) \propto (-1) \Delta \mu''(D) + [(T_B^{DD})'(D)]^2 (\lambda_{Rev})''[T_B^{DD}(D)] < 0. \]

So,

\[ (h^{SP})'(D) = \alpha_G [v_{GD}^{LR|SP}[D, T_B^{DD}(D)] + \alpha_G (T_B^{DD})'(D) v_{GI}^{LR|SP}[D, T_B^{DD}(D)]] \]

\[ < 0 \text{ (see subsubsecn 2.A.6.3)} \]

\[ > 0 \text{ (ditto)} \]

\[ + \alpha_G (T_B^{DD})'(D) v_{GI}^{LR|SP}[D, T_B^{DD}(D)] + \alpha_G [(T_B^{DD})'(D)]^2 v_{GI}^{LR|SP}[D, T_B^{DD}(D)] \]

\[ < 0 \text{ (ditto)} \]

\[ + \alpha_B \frac{d}{dD} [v_{BD}^{DD|SP}[D, T_B^{DD}(D)]] + \alpha_B (T_B^{DD})'(D) \frac{d}{dD} [v_{BI}^{DD|SP}[D, T_B^{DD}(D)]] \]

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\[ + (T_B^{DD})''(D) v_I^{SP}(D, T_B^{DD}(D)) \]

> 0, since case 4a would not obtain otherwise

\[ > \alpha_B \left( \frac{d}{dD} [v_{BD}^{DD}(D, T_B^{DD}(D))] + (T_B^{DD})'(D) \frac{d}{dD} [v_{BI}^{DD}(D, T_B^{DD}(D))] \right). \]

Now, recall from subsubsection 2.6.3 that

\[ v_{BD}^{DD}(D, T_B^{DD}(D)) = [1 - \Delta \mu'(D)] \left[ 1 - \frac{(\Pi_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]}{(\Delta_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]} \right], \]

and

\[ v_{BI}^{DD}(D, T_B^{DD}(D)) = (-1)[1 - \Lambda[T_B^{DD}(D)] \]

\[ + [1 - (\lambda^{Rev})'[T_B^{DD}(D)]] \frac{(\Pi_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]}{(\Delta_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]}. \]

These two observations, combined with (2.128) above, imply that

\[ (*) = (-1) \Delta \mu''(D) \left( 1 - \frac{(\Pi_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]}{(\Delta_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]} \right) \]

\[ + [(T_B^{DD})'(D)]^2 \left[ \Lambda''[T_B^{DD}(D)] - (\lambda^{Rev})''[T_B^{DD}(D)] \times \cdots \right. \]

\[ \cdots \times \frac{(\Pi_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]}{(\Delta_B^{SP})_{IB} [\theta_B^{DD|SP}(D, T_B^{DD}(D)), 0, T_B^{DD}(D)]} \]

< 0,

as desired.
Case 4b. Very similar arguments go through when $D$ has the property that the “no-default” constraint binds, with $r^G_D = LS$.

Having thus established that $(\mathbb{P}^G_{SP}-\text{rex})$ admits a unique solution, I’ll now turn my attention to the claim made in lemma 2.4.3 that this solution satisfies $r^G_{SP} = LR$, which can be verified using the usual arguments. In particular, if instead $r^G_{SP} = r^B_{SP} = LS$, then assumption 2.1 leaves the planner with a strict incentive to increase his choice on $I_0$. On the other hand, if $r^G_{SP} = LS \neq r^B_{SP}$, then we’re back in a situation where the marginal deposit sits idle in the good state but contributes to a tighter financial constraint in the bad state, leaving the planner with a strict incentive to reduce his choice on $D$.

**Remark.** For later derivations, it will also be useful to note that the planner’s solution is monotonic in the usual sense that $\ell_G \geq \ell_B$. With the physical constraint lax in the bad state but tight in the good state, this can be shown using essentially the same arguments as in subsubsection 2.A.5.3

### 2.A.6.5 Generalization to $(\mathbb{P}^{SP})$

I’ll now argue that the solution for $(\mathbb{P}^{SP}-\text{rex})$ also solves $(\mathbb{P}^{SP})$. This will require that we verify three constraints: the physical constraint associated with the bad state, $(PC^B_{SP})$; the financial constraint associated with the good state, $(FC^G_{SP})$; and the non-negativity constraint on storage, $I_0 \leq E^b + D$. Fortunately, the first of these three constraints was already dispatched in subsection 2.A.2. As for the other two, it’s useful to recall that the planner’s choice on $(D, I_0)$ satisfies

$$\alpha^G_{DLR_{SP}}(D, I_0) + \alpha^B_{B_{SP}}(D, I_0) \geq 0,$$

where subassumption 2.3.2 ensures that $v^B_{\text{SP}}(D, I_0) < 0$, regardless as to the planner’s choice on $r^B_{SP}$. So,

$$v^G_{DLR_{SP}}(D, I_0) = \chi_G^E \max[\theta^G_{DLR}(D, I_0)] - (1 + \rho)$$

$$+ \left[\frac{1 + \rho}{\rho + N[I_0 F[\theta^G_{DLR}(D, I_0)]] + \delta[\theta^G_{DLR}(D, I_0, I_0, I_0)]} \right] \times \cdots$$
Figure 2.7: Visual aid for proving lemma 2.4.3
\[
\cdots \times \left[ \rho + \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]] - \chi_G \theta_G^{LR}(D,I_0) \right] ^< 0
\]

> 0,

where the equality follows from (2.117). Conclude that

\[
v_{GL}(D,I_0) = \chi_G \mathbb{E} \max[\theta_G^{LR}(D,I_0)] - (1 + \rho) + \left[ \frac{1 + \rho}{\rho + \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]} \right] \times \cdots \times \left[ \rho + \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]] - \chi_G \theta_G^{LR}(D,I_0) \right] ^> 0,
\]

in which case the analysis in subsection 1.A.5.5 yields

\[
\theta_G^{LR}(D,I_0) < \theta_G^C[\Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]]
\]

— or, more precisely, \( \theta_G^{LR}(D,I_0) \in (\theta_G^R[\Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]], \theta_G^C[\Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]]) \). (\( FC^{SP}_G \)) can then be verified using essentially the same approach as in subsection 2.A.4. In particular, with \( \theta_G^{LR}(D,I_0) \) in the aforementioned range, a sufficient condition for

\[
\Delta_G[\theta_G^{LR}(D,I_0), \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]] > 0
\]

would be that

\[
\Delta_G[\theta_G^C[\Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]], \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]] > 0,
\]

which subassumption 2.3.4 can easily be shown to ensure. In light of this inequality, combined with our previous finding that the planner’s solution is monotonic, we can conclude that the planner’s choices on \((\theta_B, I_B, \ell_B)\) satisfy

\[
(E^b + D - I_0) + I_0 \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]] + I_0 \Delta_G[\theta_G^{LR}(D,I_0), \Lambda'[I_0 F[\theta_G^{LR}(D,I_0)]]] \geq (E^b + D - I_0) + \ell_B I_0 + I_B \Delta_B(\theta_B, \ell_B),
\]
where the inequality follows from the fact that \( \Lambda'(I_0 F[\theta^L_0 G(D, I_0)]) \geq \ell_B \), as remarked at the end of my last subsubsection, along with subassumption 2.3.1, which guarantees that \( \Delta_B(\theta_B, \ell_B) < 0 \). Conclude that (FC\(_B^{SP}\)) must be tighter than (FC\(_G^{SP}\)), making the latter lax.

On the other hand, the inequality \( I_0 \leq E_b + D \) can be confirmed using essentially the same approach as in subsection 1.A.5.5. As explained therein, the binding physical constraint associated with the good state can be re-written as

\[
\frac{E_b + D - I_0}{I_0} = \rho - \left[ \rho + \Lambda'(I_0 F[\theta^L_0 G(D, I_0)]) F[\theta^L_0 G(D, I_0)] \right],
\]

so it would suffice if we could show that

\[
\rho - \left[ \rho + \Lambda'(I_0 F[\theta^L_0 G(D, I_0)]) F[\theta^L_0 G(D, I_0)] \right] \geq 0,
\]

which subassumption 2.3.4 can easily be shown to ensure.

### 2.A.7 Proof of lemma 2.4.4

In this subsection, I suppose that the solution for the planner’s ex-ante problem has the property that one of the distorted regimes occurs in the bad state. I’ll then argue that the economy’s ex-ante equilibrium must also have this property. As a first step in this direction, I note that all \( D \in [0, D_B^{LS}] \) satisfy

\[
v^I_{SP} = \alpha_G v^{LSP}_G [D, T^{LS}_B (D)] + \alpha_B v^{LSP}_B [D, T^{LS}_B (D)]
\]

\[= \alpha_G v^{LSP}_G [D, T^{LS}_B (D)] + \alpha_B v^{LSP}_B [D, T^{LS}_B (D)] \tag{2.129}\]

\[
= \alpha_G \left[ \chi_G \max[\theta^L_G [D, T^{LS}_B (D)] \right] - (1 + \rho) - (1 + \rho) \times \ldots \\
\]

\[
= \alpha_G \left[ \chi_G \theta^L_G [D, T^{LS}_B (D)] - \rho - \Lambda'(T^{LS}_B (D) F[\theta^L_0 G(D, T^{LS}_B (D)])] \right] \geq 0
\]

\[
\cdot \times \frac{\chi_G \theta^{LRS}_G [D, T^{LS}_B (D)] - \rho - \Lambda'(T^{LS}_B (D) F[\theta^L_0 G(D, T^{LS}_B (D)])]}{\rho + \Lambda'(T^{LS}_B (D) F[\theta^L_0 G(D, T^{LS}_B (D)])]} < 0
\]

\[\text{see (2.117)}\]

\[9 \text{ Recall from the main text that } \delta(L) := L \Lambda''(L), \forall L \in \mathbb{R}_+. \]

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\[ \alpha_B \left[ \chi_B \mathbb{E} \max \left\{ \theta_B^{L_B} \left[ T_B^{L_B} (D) \right] \right\} - (1 + \rho) \right] \] (2.130)

\[ \leq v_I [D, T_B^{L_B} (D)] \] (2.131)

\[ = \alpha_G v_{G1}^{LR} [D, T_B^{L_B} (D)] + \alpha_B v_{B1}^{LS} [D, T_B^{L_B} (D)] \] (2.132)

\[ = \alpha_G \left[ \chi_G \mathbb{E} \max \left\{ \theta_G^{L_G} [D, T_B^{L_B} (D)] \right\} \right] - (1 + \rho) - (1 + \rho) \times \cdots \]

\[ \cdots \times \chi_G \theta_G^{L_G} [D, T_B^{L_B} (D)] - \rho - \lambda'(T_B^{L_B} (D)) F\left[ \theta_G^{L_G} [D, T_B^{L_B} (D)] \right] \]

\[ \rho + \lambda'(T_B^{L_B} (D)) F\left[ \theta_G^{L_G} [D, T_B^{L_B} (D)] \right] \]

see (2.88)

\[ + \alpha_B \left[ \chi_B \mathbb{E} \max \left\{ \theta_B^{L_B} [T_B^{L_B} (D)] \right\} - (1 + \rho) \right], \] (2.133)

with

\[ \frac{d}{dD} \left[ v_I^{SP} [D, T_B^{L_B} (D)] \right] = \alpha_G \left[ v_{G1D}^{LRSP} [D, T_B^{L_B} (D)] + (T_B^{L_B})'(D) v_{G1I}^{LRSP} [D, T_B^{L_B} (D)] \right] > 0 \] (2.118)

\[ \frac{d}{dD} \left[ v_I^{SP} [D, T_B^{L_B} (D)] \right] = 0 \] (2.118)

\[ + \alpha_B \left[ v_{B1D}^{LS|SP} [D, T_B^{L_B} (D)] + (T_B^{L_B})'(D) v_{B1I}^{LS|SP} [D, T_B^{L_B} (D)] \right] < 0 \] (ditto)

\[ > 0, \] (2.134)

and

\[ \frac{d}{dD} \left[ v_I [D, T_B^{L_B} (D)] \right] = \alpha_G \left[ v_{G1D}^{LR} [D, T_B^{L_B} (D)] + (T_B^{L_B})'(D) v_{G1I}^{LR} [D, T_B^{L_B} (D)] \right] > 0 \] (2.89)

\[ \frac{d}{dD} \left[ v_I [D, T_B^{L_B} (D)] \right] = 0 \] (2.89)

\[ + \alpha_B \left[ v_{B1D}^{LS} [D, T_B^{L_B} (D)] + (T_B^{L_B})'(D) v_{B1I}^{LS} [D, T_B^{L_B} (D)] \right] < 0 \] (ditto)
Moreover, assumption 2.1 ensures that
\[
v_I^{SP}[D, \overline{T}^{LS}_B(D)]
\]
\[
= v_I[D, \overline{T}^{LS}_B(D)]
\]
\[
= \alpha_G \mathbb{E} \max[\theta^L_D[D, \overline{T}^{LS}_B(D)]] + \alpha_B \chi B \mathbb{E} \max[\theta^L_B[T^{LS}_B(D)]] - (1 + \rho)
\]
\[
> 0.
\]
(2.135)

though the signs on \(v_I^{SP}[0, \overline{T}^{LS}_B(0)]\) and \(v_I[0, \overline{T}^{LS}_B(0)]\) are ambiguous, so we’ll have to take cases.

Suppose first that \(v_I[0, \overline{T}^{LS}_B(0)] > 0\). In this case, (2.135) yields
\[
v_I[D, \overline{T}^{LS}_B(D)] > 0, \ \forall D \in [0, \overline{D}^{LS}_B].
\]
Moreover, since it was shown in subsubsection 2.A.5.2 that the function \(v_I(D, I_0)\) is always decreasing in its second argument, so we can further conclude that
\[
v_I(D, I_0) > 0, \ \forall (D, I_0) \in \mathbb{R}_+ \times [\overline{T}^{LS}_G(D), \overline{T}^{LS}_B].
\]
This last inequality clearly precludes an ex-ante equilibrium under which the bad state is associated with a liquidity surplus, precisely as desired.

So, let’s herein suppose that \(v_I[0, \overline{T}^{LS}_B(0)] \leq 0\) instead. In this case, (2.131) and (2.134)-(2.136) together imply that \(\exists [\hat{D}^{LS}_B, \hat{D}^{LS|SP}_B] \in [0, \overline{D}^{LS}_B]^2\) s.t.
\[
D \geq \hat{D}^{LS}_B \iff v_I[D, \overline{T}^{LS}_B(D)] \geq 0, \ \forall D \in [0, \overline{D}^{LS}_B],
\]
and
\[
D \geq \hat{D}^{LS|SP}_B \iff v_I^{SP}[D, \overline{T}^{LS}_B(D)] \geq 0, \ \forall D \in [0, \overline{D}^{LS}_B],
\]
with \(\hat{D}^{LS|SP}_B \geq \hat{D}^{LS}_B\). In light of the analysis in subsubsection 2.A.5.3, it should then be clear that it would suffice if we could show that
\[
v_D[\hat{D}^{LS}_B, \overline{T}^{LS}_B(\hat{D}^{LS}_B)] = \alpha_G v^{LR}_G[\hat{D}^{LS}_B, \overline{T}^{LS}_B(\hat{D}^{LS}_B)] + \alpha_B v^{LS}_B[\hat{D}^{LS}_B, \overline{T}^{LS}_B(\hat{D}^{LS}_B)]
\]
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\[
\begin{align*}
&= \alpha_G \begin{bmatrix}
1 - \Delta \mu'(\hat{D}_B^{LS}) \\
\chi_G \theta_{G}^{LR}[\hat{T}_B^{LS}, \hat{T}_B^{LS}] \\
- \rho - \Lambda'[\hat{T}_B^{LS}, \hat{T}_B^{LS}] F[\theta_{G}^{LR}[\hat{T}_B^{LS}, \hat{T}_B^{LS}]] \\
\rho + \Lambda'[\hat{T}_B^{LS}, \hat{T}_B^{LS}] F[\theta_{G}^{LR}[\hat{T}_B^{LS}, \hat{T}_B^{LS}]]
\end{bmatrix}
\end{align*}
\]

or equivalently

\[
\begin{align*}
&> 0, \\
&\text{see (2.90)}
\end{align*}
\]

Now, as a first step toward verifying (2.137), recall that we’ve been given that the solution
for the planner’s ex-ante problem associates the bad state with one of the distorted regimes.
In light of the analysis in subsubsection 2.A.6.4, this means that

\[
\begin{align*}
v_D[\hat{D}_B^{LS}, \hat{T}_B^{LS}(\hat{D}_B^{LS})] + \frac{v_I[\hat{D}_B^{LS}, \hat{T}_B^{LS}(\hat{D}_B^{LS})]}{1 + \rho} &= 0 \quad \text{see (2.95)} \\
&= \alpha_G \chi_{G} \mathbb{E} \max[\theta_{G}^{LR}[\hat{T}_B^{LS}, \hat{T}_B^{LS}]] + \alpha_B \chi_{B} \mathbb{E} \max[\theta_{B}^{LS}[\hat{T}_B^{LS}, \hat{T}_B^{LS}]] - \Delta \mu'(\hat{D}_B^{LS}) \\
&=: k[\hat{D}_B^{LS}, \hat{T}_B^{LS}(\hat{D}_B^{LS})] > 0. 
\end{align*}
\] (2.137)

Now, as a first step toward verifying (2.137), recall that we’ve been given that the solution
for the planner’s ex-ante problem associates the bad state with one of the distorted regimes.
In light of the analysis in subsubsection 2.A.6.4, this means that

\[
\begin{align*}
v_D^{SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})] &= \alpha_G \nu_{GD}^{LR|SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})] + \alpha_B \nu_{BD}^{LS|SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]
\end{align*}
\]
\[
\begin{aligned}
\alpha_G & \left[ 1 - \Delta \mu'\left( \hat{D}_B^{LS|SP} \right) \right] \\
&= \alpha_G \left[ 1 - \Delta \mu'\left( \hat{D}_B^{LS|SP} \right) \right] \\
&\quad + \left[ \chi_G \theta_G^{LR}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})] \\
&\qquad - \rho - \Lambda'[\hat{T}_B^{LS}(\hat{D}_B^{LS|SP})F[\theta_G^{LR}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]]] \\
&\quad \left[ \rho + \Lambda'[\hat{T}_B^{LS}(\hat{D}_B^{LS|SP})F[\theta_G^{LR}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]]] \\
&\quad \left[ \Delta \mu'\left( \hat{D}_B^{LS|SP} \right) \right] \\
&\quad \right. \\
&\quad \left. \text{see } (2.119) \right]
\end{aligned}
\]

or equivalently

\[
v_D^{SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})] + \frac{\nu_I^{SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]}{1 + \rho} = 0
\]

\[
= \frac{\alpha_G \chi_G \varepsilon \max[\theta_G^{LR}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]]}{1 + \rho} + \frac{\alpha_B \chi_B \varepsilon \max[\theta_B^{LS}[\hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]]}{1 + \rho} - \Delta \mu'\left( \right)
\]

\[
= k[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]
\]

or equivalently

\[
v_D^{SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})] + \frac{\nu_I^{SP}[\hat{D}_B^{LS|SP}, \hat{T}_B^{LS}(\hat{D}_B^{LS|SP})]}{1 + \rho} = 0
\]

In light of this last inequality, combined with the fact that \( \hat{D}_B^{LS|SP} \geq \hat{D}_B^{LS} \), we see that a sufficient condition for \( (2.137) \) would be that the composition \( k[D, \hat{T}_B^{LS}(D)] \) decrease weakly. On this front, I note that

\[
\frac{d}{dD}[k[D, \hat{T}_B^{LS}(D)]] \propto \alpha_G \chi_G F[\theta_G^{LR}[D, \hat{T}_B^{LS}(D)]] \times \ldots
\]
\[
\ldots \times \left( \frac{(\theta^L_G)_D[D, \bar{T}_B^{\text{LS}}(D)] + (\bar{T}_B^{\text{LS}})'(D)(\theta^L_G)_I[D, \bar{T}_B^{\text{LS}}(D)]}{1 + \rho} \right)
\]

\[+ \frac{\alpha_B \chi_B F[\theta^L_B \bar{T}_B^{\text{LS}}(D)][(\bar{T}_B^{\text{LS}})'(D)(\theta^L_B)'[\bar{T}_B^{\text{LS}}(D)]}{1 + \rho} \]

\[- \Delta \mu''(D) \]

\[< \langle \bar{T}_B^{\text{LS}} \rangle' \langle D \rangle \]

\[< \left[ \alpha_G \chi_G F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\theta^L_G]'[D, \bar{T}_B^{\text{LS}}(D)] \right] \]

\[+ \alpha_B \chi_B F[\theta^L_B \bar{T}_B^{\text{LS}}(D)][(\theta^L_B)'[\bar{T}_B^{\text{LS}}(D)] > 0 \]

It would thus suffice if we could show that

\[\alpha_G \chi_G F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\theta^L_G]'[D, \bar{T}_B^{\text{LS}}(D)] + \alpha_B \chi_B F[\theta^L_B \bar{T}_B^{\text{LS}}(D)][(\theta^L_B)'[\bar{T}_B^{\text{LS}}(D)] > 0 \]

or equivalently

\[\chi_G F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\theta^L_G]'[D, \bar{T}_B^{\text{LS}}(D)] > (1)(\frac{\alpha_B}{\alpha_G}) \chi_B F[\theta^L_B \bar{T}_B^{\text{LS}}(D)][(\theta^L_B)'[\bar{T}_B^{\text{LS}}(D)]]. \tag{2.138}\]

Now, w.r.t. the left-hand side of this last inequality, I note from (2.84) that

\[\chi_G F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\theta^L_G]'[D, \bar{T}_B^{\text{LS}}(D)] = \chi_G F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\theta^L_G]'[D, \bar{T}_B^{\text{LS}}(D)] \times \ldots \]

\[> 0, \text{namely due to subassumption 2.3.4} \]

\[\ldots \times 1 + \rho - F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\rho + (\lambda^{\text{Rev}})F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]]] \]

\[\bar{T}_B^{\text{LS}}(D)F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]][\rho + (\lambda^{\text{Rev}})F[\theta^L_G [D, \bar{T}_B^{\text{LS}}(D)]]] \]

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\[
\chi_G(F/f)[\theta^L_{LR}[D, T^L_B (D)]] / T^L_B (D)|\rho + (\lambda^{Rev}|[T^L_B (D) F[\theta^L_{LR}[D, T^L_B (D)]]]] = \chi_G(F/f)[\theta^L_{LR}[D, T^L_B (D)]] / T^L_B (D)|\rho + N[T^L_B (D) F[\theta^L_{LR}[D, T^L_B (D)]]]] + \delta[T^L_B (D) F[\theta^L_{LR}[D, T^L_B (D)]]]]) < 0
\]

\[
> \chi_G(F/f)[\theta^L_{LR}[D, T^L_B (D)]] / T^L_B (D)|\rho + N[T^L_B (D) F[\theta^L_{LR}[D, T^L_B (D)]]]]
\]

\[
= \chi_G(F/f)[\theta^L_{LR}[D, T^L_B (D)]] / T^L_B (D)|\rho + N[T^L_B (D) F[\theta^L_{LR}[D, T^L_B (D)]]]]
\]

\[
> \frac{1}{T^L_B (D)}
\]

where (2.139) follows from assumption 2.6 while (2.187) follows from the fact that \(\theta^L_{LR}[D, T^L_B (D)] > \theta^L_{G}[T^L_B (D)]\). So, a sufficient condition for (2.138) would be that

\[
1 > (-1)T^L_B (D)(\alpha_B/\alpha_G)\chi_B F[\theta^L_B [T^L_B (D)]][\theta^L_B (T^L_B (D))]
\]

On this front, I finally from (2.2) that

\[
(-1)T^L_B (D)(\alpha_B/\alpha_G)\chi_B F[\theta^L_B [T^L_B (D)]][\theta^L_B (T^L_B (D))]
\]

\[
\geq \frac{(\alpha_B/\alpha_G)\chi_B(-1)T^L_B (D) F[\theta^L_B [T^L_B (D)]]}{\chi_B - T^L_B (D) F[\theta^L_B [T^L_B (D)]]}< 0
\]

\[
< (\alpha_B/\alpha_G)\lambda[T^L_B (D) F[\theta^L_B [T^L_B (D)]]]< 0
\]

\[
< (\alpha_B/\alpha_G)\lambda[T^L_B (D) F[\theta^L_B [T^L_B (D)]]]
\]

(2.142)
where (2.142) and (2.143) respectively follow from (2.5) and the fact that \( \alpha_G > 1/2 \).  

\[ \text{(2.143)} \]

\[ \text{(2.142)} \]

### 2.A.8 Proof of propositions 2.2 and 2.3

#### 2.A.8.1 Notation

- Let \((D^{SP}, I_{0}^{SP})\) denote the initial balance sheet preferred by the planner, with \( s^{SP} := (E_b + D^{SP} - I_{0}^{SP})/I_{0}^{SP} \);
- Let \( \ell^{SP}_{\omega} \) denote the price at which the planner would like the secondary market to clear in state \( \omega \);
- Let \((\tau^{SP}_{B}, T^{SP}_{B}) := [\tau^{*}_{B}(D^{SP}, I_{0}^{SP}), T^{*}_{B}(D^{SP}, I_{0}^{SP})] \), where \( \tau^{*}_{B}(\cdot) \) and \( T^{*}_{B}(\cdot) \) are defined on lines 2.82 and 2.83 respectively;
- Let \((\tilde{P}_0)\) denote the task of choosing \((D, I_{0}, \theta_G, I_G, \theta_B, I_B) \in [0, D^{SP}] \times [0, (E + D)/(1 + \rho s^{SP})] \times [0, 1] \times [0, I_{0}] \times [0, 1] \times [0, I_{0}]\) so as to maximize

\[
\alpha_G [(E^b + D - I_{0}) + \ell^{SP}_{G} I_{0} + I_G \Pi_G(\theta_G, \ell^{SP}_{G}) - \Delta\mu(D)]
\]

\[
+ \alpha_B [(1 - \tau^{SP}_{B})(E^b + D - I_{0}) + \ell^{SP}_{B} I_{0}] + T^{SP}_{B} + I_B \tilde{\Pi}_B(\theta_B, \ell^{SP}_{B}, \tau^{SP}_{B}) - \Delta\mu(D)]
\]

subject to

\[
(E^b + D - I_{0}) + \ell^{SP}_{G} I_{0} + I_G \Delta_G(\theta_G, \ell^{SP}_{G}) \geq \Delta\mu(D), \quad \text{\( (\tilde{F}C_G) \)}
\]

\[
(1 - \tau^{SP}_{B})(E^b + D - I_{0}) + \ell^{SP}_{B} I_{0}] + T^{SP}_{B} + I_B \tilde{\Delta}_B(\theta_B, \ell^{SP}_{B}, \tau^{SP}_{B}) \geq \Delta\mu(D), \quad \text{\( (\tilde{F}C_B) \)}
\]

\[
(E^b + D - I_{0}) + \ell^{SP}_{G} I_{0} \geq I_G \Psi_G(\theta_G, \ell^{SP}_{G}), \quad \text{\( (\tilde{P}C_G) \)}
\]

To be clear w.r.t. (2.142), assumption 2.5 \( \Rightarrow \) \( (\Lambda_{\text{Rev}})'(L) = \Lambda'(L) + L\Lambda''(L) > 0, \forall L \in \mathbb{R}_+ \) \( \iff \) \( \Lambda'(L) > -1L\Lambda''(L), \forall L \in \mathbb{R}_+ \).
and
\[(E^b + D - I_0) + \ell^S_B I_0 \geq I_B \Psi_B(\theta_B, \ell^S_B), \quad (\tilde{PC}_B)\]
where \(\tilde{\Pi}_B(\cdot)\) and \(\tilde{\Delta}_B(\cdot)\) are defined on lines 2.52 and 2.53, respectively;

- let \((\tilde{\mathcal{P}}_0\text{-}rex)\) denote a relaxed version of \((\tilde{\mathcal{P}}_0)\) from which we drop \((\tilde{FC}_G)\) and \((\tilde{PC}_B)\);

- as in subsection 2.A.3, let \((\tilde{\mathcal{P}}_B\text{-}rex)\) denote the task of choosing the subcontract \((\theta_B, I_B) \in [0, 1] \times [0, I_0]\) so as to maximize
\[
(1 - \tau^{SP}_B)[(E^b + D - I_0) + \ell^{SP}_B I_0] + T^{SP}_B + I_B \tilde{\Pi}_B(\theta_B, \ell^{SP}_B, \tau^{SP}_B) - \Delta \mu(D),
\]
subject only to \((\tilde{FC}_B)\), taking some initial balance sheet \((D, I_0)\) as given;

- similarly, let \((\tilde{\mathcal{P}}_G\text{-}rex)\) denote the task of choosing the subcontract \((\theta_G, I_G) \in [0, 1] \times [0, I_0]\) so as to maximize
\[
(E^b + D - I_0) + \ell^{SP}_G I_0 + I_G \Pi_G(\theta_G, \ell^{SP}_G) - \Delta \mu(D),
\]
subject only to \((\tilde{PC}_G)\), again taking some initial balance sheet \((D, I_0)\) as given.

2.A.8.2 Details on \((\tilde{\mathcal{P}}_B\text{-}rex)\)

**Sublemma 2.A.12.** \(\forall D \in [0, D^{SP}], \exists [\tilde{I}^{LS}_B(D), \tilde{I}^{ED}_B(D), \tilde{I}^{DD}_B(D)] \in \mathbb{R}^3_{++}, \text{ s.t.}\)

\[
I_0 \gtrless \tilde{I}^{LS}_B(D) \iff (1 - \tau^{SP}_B)[(E^b + D - I_0) + \ell^{SP}_B I_0] + T^{SP}_B
\]
\[
+ I_0 \tilde{\Delta}_B[\tilde{\omega}_B(\ell^{SP}_B, \tau^{SP}_B), \ell^{SP}_B, \tau^{SP}_B] \lesssim \Delta \mu(D), \quad (2.144)
\]

\[
I_0 \gtrless \tilde{I}^{ED}_B(D) \iff (1 - \tau^{SP}_B)[(E^b + D - I_0) + \ell^{SP}_B I_0] + T^{SP}_B
\]
\[
+ I_0 \tilde{\Delta}_B[\tilde{\omega}_B(\ell^{SP}_B, \tau^{SP}_B), \ell^{SP}_B, \tau^{SP}_B] \gtrsim \Delta \mu(D), \quad (2.145)
\]

and

\[
I_0 \gtrless \tilde{I}^{DD}_B(D) \iff (1 - \tau^{SP}_B)[(E^b + D - I_0) + \ell^{SP}_B I_0] + T^{SP}_B \lesssim \Delta \mu(D), \quad (2.146)
\]
∀\(I_0 \in \mathbb{R}_+\), where \(\tilde{\theta}_B^\Pi(\cdot)\) and \(\tilde{\theta}_B^\Xi(\cdot)\) were defined at the beginning of subsection 2.A.3. Moreover, these functions are strictly decreasing and satisfy

\[
\tilde{I}_B^{LS}(D) < \tilde{I}_B^{ED}(D) < \tilde{I}_B^{DD}(D), \quad \forall D \in [0, D^{SP}].
\]

Proof. \(\forall D \in [0, D^{SP}]\), it should be clear that \(\exists [\tilde{I}_B^{LS}(D), \tilde{I}_B^{ED}(D), \tilde{I}_B^{DD}(D)] \in \mathbb{R}_+^3\) s.t. all \(I_0 \in \mathbb{R}_+\) satisfy (2.144), (2.145), and (2.146). It should also be clear that these functions are strictly decreasing. So, it would suffice if we could show that these functions furthermore satisfy

\[
0 < \tilde{I}_B^{LS}(D) < \tilde{I}_B^{ED}(D) < \tilde{I}_B^{DD}(D), \quad \forall D \in [0, D^{SP}].
\]

To this end, I’ll proceed from left to right in this chain:

• if the planner’s solution exhibits a liquidity surplus in the bad state, then it should be clear that

\[
(1 - \tau_B^{SP})[(E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP}] + T^{SP}_B + I_0^{SP} \tilde{\Delta}_B[\theta_B^{LS}(I_0^{SP}), \ell_B^{SP}, \tau_B^{SP}] < 0 \quad \text{(see subsection 2.A.3)}
\]

\[
= (E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP} + I_0^{SP} \Delta_B[\theta_B^{LS}(I_0^{SP}), \ell_B^{SP}]
\]

\[
\geq \Delta \mu(D^{SP}).
\]

If the planner’s solution instead exhibits an extensive distortion in the bad state, then it should be clear that

\[
(1 - \tau_B^{SP})[(E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP}] + T^{SP}_B + I_0^{SP} \tilde{\Delta}_B[\theta_B^{ED}(D^{SP}, I_0^{SP}), \ell_B^{SP}, \tau_B^{SP}] < 0 \quad \text{(see subsection 2.A.3)}
\]

\[
= (E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP} + I_0^{SP} \Delta_B[\theta_B^{ED}(D^{SP}, I_0^{SP}), \ell_B^{SP}]
\]

\[
= \Delta \mu(D^{SP}).
\]

On the other hand, if the planner’s solution exhibits a dual distortion in the bad state, then it should be clear that

\[
(1 - \tau_B^{SP})[(E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP}] + T^{SP}_B + I_B^{DD|SP}(D^{SP}, I_0^{SP}) \times \ldots
\]

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\[= (E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP} + I_B^{DD|SP} (D^{SP}, I_0^{SP}) \Delta_B [\theta_B^{DD|SP} (D^{SP}, I_0^{SP}), \ell_B^{SP}] \]

\[= \Delta \mu (D^{SP}). \]

In all three cases, we have

\[ (1 - \tau_B^{SP}) [(E^b + D^{SP} - I_0^{SP}) + \ell_B^{SP} I_0^{SP}] + T_B^{SP} > \Delta \mu (D^{SP}), \]

and, in turn,

\[ (1 - \tau_B^{SP}) (E^b + D) + T_B^{SP} > \Delta \mu (D), \ \forall D \in [0, D^{SP}]. \]

Combining this inequality with (2.144) then yields \( \tilde{I}_B^{LS} (D) > 0, \ \forall D \in [0, D^{SP}]; \)

- next, \( \forall D \in [0, D^{SP}], \) I note that

\[ (1 - \tau_B^{SP}) [E^b + D - \tilde{I}_B^{LS} (D)] + T_B^{SP} + \tilde{I}_B^{LS} (D) \Delta_B [\tilde{\theta}_B^{LS} (\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]

\[> (1 - \tau_B^{SP}) [E^b + D - (1 - \ell_B^{SP}) \tilde{I}_B^{LS} (D)] + T_B^{SP} + \tilde{I}_B^{LS} (D) \times \ldots \]

\[\ldots \times \Delta_B [\tilde{\theta}_B^{LS} (\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]

\[= \Delta \mu (D), \]  \hspace{1cm} (2.147)

where the inequality follows from the fact that

\[ \tilde{\theta}_B^{LS} (\ell_B^{SP}, \tau_B^{SP}) > \tilde{\theta}_B^{DD} (\ell_B^{SP}, \tau_B^{SP}) > \tilde{\theta}_B^{E} (\ell_B^{SP}, \tau_B^{SP}), \]

as shown in subsection 2.A.3  Combining (2.147) with (2.145) then yields \( \tilde{I}_B^{ED} (D) > \tilde{I}_B^{LS} (D), \ \forall D \in [0, D^{SP}]; \)

- finally, \( \forall D \in [0, D^{SP}], \) I note that

\[ (1 - \tau_B^{SP}) [E^b + D - \tilde{I}_B^{ED} (D)] + \ell_B^{SP} \tilde{I}_B^{ED} (D)] + T_B^{SP} \]
\[ > (1 - \tau_B^{SP})[E^b + D - \hat{I}_B^{ED}(D) + \ell_B^{SP} \hat{I}_B^{ED}(D)] + T_B^{SP} + \hat{I}_B^{ED}(D) \Delta_B(\hat{\theta}^\Xi_B, \ell_B^{SP}, \tau_B^{SP}) \]

\[ = \Delta \mu(D). \]

Combining with (2.146) then yields \( \hat{I}_B^{DD}(D) > \hat{I}_B^{ED}(D), \forall D \in [0, D^{SP}]. \)

**Sublemma 2.A.13.** The three functions derived above have the property that the subcontract \((\theta_B, I_B)\) exhibits the following dependence on initial balance sheets under \((\bar{P}_B\text{-}rex)\):

- if \( I_0 \in [0, I_B^{LS}(D)] \), then \((\theta_B, I_B) = [\hat{\theta}_B^{\Pi}(\ell_B^{SP}, \tau_B^{SP}), I_0] \) — i.e., banks opt for a liquidity surplus;
- if \( I_0 \in (I_B^{LS}(D), \hat{I}_B^{ED}(D)) \), then \( I_B = I_0 \), with \( \theta_B \) set s.t. \( \bar{FC}_B \) holds with equality — i.e., banks opt for an extensive distortion;
- if \( I_0 \in (\hat{I}_B^{ED}(D), I_B^{DD}(D)) \), then \( \theta_B = \hat{\theta}_B^{\Xi}(\ell_B^{SP}, \tau_B^{SP}) \), with \( I_B \) set s.t. \( \bar{FC}_B \) holds with equality — i.e., banks opt for a dual distortion;
- otherwise, the subcontract \((\theta_B, I_B)\) cannot be chosen to satisfy \( \bar{FC}_B \) — i.e., \((\bar{P}_B\text{-}rex)\) is insoluble.

See figure 2.8 for an illustration.

**Proof.** This is just a corollary of my previous sublemma, combined with the analysis in subsection 2.A.3.

**Remark.** For future derivations, it will be useful to note the following:

- in the case of a liquidity surplus, banks’ expected payout in the bad state is given by

\[ \hat{v}_B^{LS}(D, I_0) := (1 - \tau_B^{SP})[(E^b + D - I_0) + \ell_B^{SP} I_0] + T_B^{SP} + I_0 \hat{\Pi}_B[\hat{\theta}_B^{\Pi}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]

\[ - \Delta \mu(D), \]
with
\[
\tilde{v}^{LS}_{BD}(D, I_0) = (1 - \tau^S_P) - \Delta \mu'(D),
\]
\[
\tilde{v}^{LS}_{BI}(D, I_0) = (1 - \tau^S_P)(\ell^S_P - 1) + \tilde{\Pi}_B[\tilde{\theta}^S_B(\ell^S_P, \tau^S_P), \ell^S_P, \tau^S_P],
\]
and
\[
\tilde{v}^{LS}_{BDD}(D, I_0) = -(1)\Delta \mu''(D) < 0
\]
\[
= \tilde{v}^{LS}_{BDI}(D, I_0) = \tilde{v}^{LS}_{BID}(D, I_0) = \tilde{v}^{LS}_{BII}(D, I_0); \tag{2.151}
\]
• in the case of an extensive distortion, banks’ expected payout in the bad state is given by
\[
\tilde{v}^{ED}_B(D, I_0)
\]
\[
:= (1 - \tau^S_P)[(E^b + D - I_0) + \ell^S_P I_0] + T^S_P + I_0 \tilde{\Pi}_B[\tilde{\theta}^{ED}_B(D, I_0), \ell^S_P, \tau^S_P]
\]
\[- \Delta \mu(D),
\]
where \(\tilde{\theta}^{ED}_B(D, I_0)\) solves
\[
(1 - \tau^S_P)[(E^b + D - I_0) + \ell^S_P I_0] + T^S_P + I_0 \tilde{\Delta}_B[\tilde{\theta}^{ED}_B(D, I_0), \ell^S_P, \tau^S_P] = \Delta \mu(D).
\]
So,
\[
(\tilde{\theta}^{ED}_B)_D(D, I_0) = \frac{\Delta \mu'(D) - (1 - \tau^S_P)}{I_0 \tilde{\Delta}_B[\tilde{\theta}^{ED}_B(D, I_0), \ell^S_P, \tau^S_P]} < 0,
\]
<0, b/c \(\tilde{\theta}^{ED}(D, I_0) > \tilde{\Delta}(\ell^S_P, \tau^S_P)\), as shown in subsection 2.A.3
and
\[
(\tilde{\theta}^{ED}_B)_I(D, I_0) = \frac{(1 - \tau^S_P)(1 - \ell^S_P) - \Delta \tilde{\mu}_B[\tilde{\theta}^{ED}_B(D, I_0), \ell^S_P, \tau^S_P]}{I_0 \tilde{\Delta}_B[\tilde{\theta}^{ED}_B(D, I_0), \ell^S_P, \tau^S_P]} < 0.
\]
Figure 2.8: Illustration of sublemma 2.A.13
In turn, 

\[ \tilde{v}_{BD}(D, I_0) = (1 - \tau_B^{SP}) - \Delta \mu'(D) + I_0(\tilde{\theta}_B^{ED})_D(D, I_0)(\tilde{\Pi}_B)_\theta[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] \]

\[ = [(1 - \tau_B^{SP}) - \Delta \mu'(D)] \left[ 1 - \frac{(\tilde{\Delta}_B)_\theta[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]}{(\tilde{\Pi}_B)_\theta[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]} \right] \]

\[ =: [(1 - \tau_B^{SP}) - \Delta \mu'(D)] \times \cdots \]

\[ \cdots \times [1 - [(\tilde{\Pi}_B)_\theta/(\tilde{\Delta}_B)_\theta][\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]] \quad (2.152) \]

and

\[ \tilde{v}_{BI}(D, I_0) = (1 - \tau_B^{SP})(\ell_B^{SP} - 1) + \tilde{\Pi}_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] \]

\[ + I_0(\tilde{\theta}_B^{ED})_D(D, I_0)(\tilde{\Pi}_B)_\theta[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] \]

\[ = (1 - \tau_B^{SP})(\ell_B^{SP} - 1) + \tilde{\Pi}_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] \]

\[ + [(1 - \tau_B^{SP})(1 - \ell_B^{SP}) - \tilde{\Delta}_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]] \times \cdots \]

\[ \cdots \times [(\tilde{\Pi}_B)_\theta/(\tilde{\Delta}_B)_\theta][\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}], \quad (2.153) \]

where

\[ [(\tilde{\Pi}_B)_\theta/(\tilde{\Delta}_B)_\theta][\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] \]

\[ < \text{b/c } \tilde{\theta}_B^{ED}(D, I_0) < \tilde{\theta}_B^{ED}(\ell_B^{SP}, \tau_B^{SP}), \text{ as shown in subsection 2.A.3} \]

\[ = \frac{\chi_B\tilde{\theta}_B^{ED}(D, I_0) - (1 - \tau_B^{SP})(\rho + \ell_B^{SP})}{\chi_B\tilde{\theta}_B^{ED}(D, I_0) + \gamma \chi_B[F/f][\tilde{\theta}_B^{ED}(D, I_0)][(1 - \tau_B^{SP})(\rho + \ell_B^{SP})]} \]

\[ < \text{b/c } \tilde{\theta}_B^{ED}(D, I_0) > \tilde{\theta}_B^{ED}(\ell_B^{SP}, \tau_B^{SP}), \text{ as shown in subsection 2.A.3} \]
\[< 0,\]

so

\[
[\overline{\Pi}_B\theta/\overline{(\Delta_B)\theta}][\theta_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]
\]

\[\propto \chi_B \theta_B^{ED}(D, I_0) + \chi_B(F/f)[\theta_B^{ED}(D, I_0)] - (1 - \tau_B^{SP})(\rho + \ell_B^{SP})
\]

\[- [1 + \gamma(F/f)][\theta_B^{ED}(D, I_0)][\chi_B \theta_B^{ED}(D, I_0) - (1 + \tau_B^{SP})(\rho + \ell_B^{SP})]
\]

\[> 0.\]

Therefore,

\[
\tilde{v}_{BDDD}^{ED}(D, I_0)
\]

\[= (-1)\overline{\Delta^{''}}(D)[1 - [\overline{(\Pi}_B\theta/\overline{(\Delta_B)\theta}][\theta_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]]
\]

\[< 0,
\]

\[< 0, (2.154)
\]

\[
\tilde{v}_{BDI}^{ED}(D, I_0)
\]

\[= (-1)[(1 - \tau_B^{SP}) - \Delta^{'}(D)](\theta_B^{ED})_{D}(D, I_0) [\overline{(\Pi}_B\theta/\overline{(\Delta_B)\theta}][\theta_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]
\]

\[< 0, (2.155)
\]
and

\[
\tilde{v}_{BX}^{ED}(D, I_0) = \left[ (1 - \tau_B^{SP})(1 - \ell_B^{SP}) - \frac{\Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]}{\tilde{\theta}_B^{ED}(D, I_0)} \right] \times \ldots
\]

\[
\ldots \times \frac{\Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]}{\tilde{\theta}_B^{ED}(D, I_0)} \leq 0.
\]

\[
< 0, \ \forall x \in \{D, I\}; \quad (2.156)
\]

• in the case of a dual distortion, banks’ expected payout in the bad state is given by

\[
\tilde{v}_B^{DD}(D, I_0) := (1 - \tau_B^{SP})[(E^b + D - I_0) + \ell_B^{SP}I_0] + T_B^{SP} + \tilde{I}_B^{DD}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0) \Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] = \Delta_B(D),
\]

where \(\tilde{I}_B^{DD}(D, I_0)\) solves

\[
(1 - \tau_B^{SP})[(E^b + D - I_0) + \ell_B^{SP}I_0] + T_B^{SP} + \tilde{I}_B^{DD}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0)\tilde{\theta}_B^{ED}(D, I_0) \Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}] = \Delta_B(D).
\]

So,

\[
\tilde{I}_B^{DD}(D, I_0) = \underbrace{\Delta_B'(D) - (1 - \tau_B^{SP})}_{> 0} \left[ \frac{\Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]}{\tilde{\theta}_B^{ED}(D, I_0)} \right] < 0,
\]

and

\[
\tilde{I}_B^{DD}(D, I_0) = \frac{(1 - \tau_B^{SP})(1 - \ell_B^{SP})}{\Delta_B[\tilde{\theta}_B^{ED}(D, I_0), \ell_B^{SP}, \tau_B^{SP}]} < 0.
\]
In turn,
\[ \tilde{v}_{BB}^{DD}(D, I_0) = (1 - \tau_B^{SP}) - \Delta \mu'(D) + (\tilde{I}_{BB}^{DD})_D(D, I_0) \tilde{\Pi}_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]
\[ = [(1 - \tau_B^{SP}) - \Delta \mu'(D)] \left[ 1 - \tilde{\Pi}_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \right] \Delta_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}], \tag{2.157} \]

and
\[ \tilde{v}_{BI}^{DD}(D, I_0) = (1 - \tau_B^{SP})(\ell_B^{SP} - 1) + (\tilde{I}_{BI}^{DD})_D(D, I_0) \tilde{\Pi}_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]
\[ = (1 - \tau_B^{SP})(\ell_B^{SP} - 1) \left[ 1 - \tilde{\Pi}_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \right] \Delta_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}], \tag{2.158} \]

so
\[ \tilde{v}_{BDD}^{DD}(D, I_0) = (-1)\Delta \mu''(D) \left[ \tilde{\Pi}_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \right] \Delta_B[\tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \ell_B^{SP}, \tau_B^{SP}] \]
\[ = 0 > 0, b/c \tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}) > \tilde{\theta}_B^{\mu}(\ell_B^{SP}, \tau_B^{SP}), \text{ as shown in subsection 2.A.3} \]
\[ < 0 \text{ (see subsection 2.A.3)} \]
\[ \tilde{v}_{BDD}^{DD}(D, I_0) = \tilde{v}_{BID}^{DD}(D, I_0) = \tilde{v}_{BII}^{DD}(D, I_0). \tag{2.159} \]

2.A.8.3 Details on \((\tilde{P}_G\text{-rex})\)

Sublemma 2.A.14. All \(D \in [0, D^{SP}]\) admit some \(\tilde{I}_G^{LS}(D) \in (0, (E^b + D)(1 + \rho_\Sigma^{SP}))\) s.t.
\[ I_0 \geq \tilde{I}_G^{LS}(D) \iff (E^b + D - I_0) + \ell_G^{SP}I_0 \leq I_0\Psi_G[(\rho + \ell_G^{SP})/\mu_G, \ell_G^{SP}], \forall I_0 \in \mathbb{R}_+. \tag{2.160} \]

Moreover, this function is strictly increasing.

Proof. \(\forall D \in [0, D^{SP}]\), it should be clear that \(\exists \tilde{I}_G^{LS}(D) \in \mathbb{R}_{++}\) satisfying (2.160). It should also be clear that this function is strictly increasing. So, it would suffice if we could show that this function furthermore satisfies \(\tilde{I}_G^{LS}(D) < (E^b + D)/(1 + \rho_\Sigma^{SP})\), \(\forall D \in [0, D^{SP}]\) — i.e., the liquidity coverage ratio is lax when initial balance sheets satisfy \(I_0 = \tilde{I}_G^{LS}(D)\). To see that this is indeed the case, note that
\[ (E^b + D^{SP} - I_0^{SP}) + \ell_G^{SP}I_0^{SP} = I_0^{SP}\Psi_G[\ell_G^{LR}(D^{SP}, I_0^{SP}), \ell_G^{SP}] \]
\[ \frac{E^b + D^{SP} - I_0^{SP}}{I_0^{SP}} + \ell_G^{SP} = \rho \bar{z}^{SP} + \ell_G^{SP} = \Psi_G[\vartheta_G^{LR}(D^{SP}, I_0^{SP}), \ell_G^{SP}], \quad (2.161) \]

whereas

\[ [E^b + D - \bar{I}_G^{LS}(D)] + \ell_G^{SP} \bar{I}_G^{LS}(D) = \Psi_G[(\rho + \ell_G^{SP})/\chi_G, \ell_G^{SP}] \]

\[ \iff \frac{E^b + D - \bar{I}_G^{LS}(D)}{\bar{I}_G^{LS}(D)} + \ell_G^{SP} = \Psi_G[(\rho + \ell_G^{SP})/\chi_G, \ell_G^{SP}], \quad \forall D \in [0, D^{SP}], \quad (2.162) \]

with

\[ \Psi_G[(\rho + \ell_G^{SP})/\chi_G, \ell_G^{SP}] > \Psi_G[\vartheta_G^{LR}(D^{SP}, I_0^{SP}), \ell_G^{SP}]. \quad (2.163) \]

Comparing (2.161) and (2.162) thus yields

\[ \frac{E^b + D - \bar{I}_G^{LS}(D)}{\bar{I}_G^{LS}(D)} > \rho \bar{z}^{SP}, \quad \forall D \in [0, D^{SP}], \]

as desired. \(\blacksquare\)

**Sublemma 2.A.15.** The function derived above has the property that the subcontract \((\vartheta_G, I_G)\) exhibits the following dependence on initial balance sheets under \((\bar{P}_G, \text{rex})\):

- if \(I_0 \in [0, \bar{I}_G^{LS}(D)]\), then \((\vartheta_G, I_G) = [(\rho + \ell_G^{SP})/\chi_G, I_0]\) — i.e., banks opt for a liquidity surplus;

- if instead \(I_0 \in (\bar{I}_G^{LS}(D), (E^b + D)/(1 + \rho \bar{z}^{SP})]\), then \(I_G = I_0\), with \(\vartheta_G\) set s.t. \((P_{CG})\) holds with equality — i.e., banks settle for liquidity rationing.

See figure 2.9 for an illustration.

**Proof.** This should be obvious. \(\blacksquare\)

**Remark.** For future derivations, it will be useful to note the following:

- in the case of a liquidity surplus, banks’ expected payout in the good state is given by

\[ \bar{v}_G^{LS}(D, I_0) = (E^b + D - I_0) + \ell_G^{SP} I_0 + I_0 \Pi_G[(\rho + \ell_G^{SP})/\chi_G, \ell_G^{SP}] - \Delta \mu(D), \]
so

\[ \tilde{v}^{LS}_{GD}(D, I_0) = 1 - \Delta \mu'(D) \]

\[ \tilde{v}^{LS}_{GI}(D, I_0) = \ell^G + \Pi_G[\rho + \ell^G] - 1, \]

and

\[ \tilde{v}^{LS}_{GDD}(D, I_0) = (-1)\Delta \mu''(D) < 0 \]

\[ = \tilde{v}^{LS}_{GDI}(D, I_0) = \tilde{v}^{LS}_{GI}(D, I_0) = \tilde{v}^{LS}_{GII}(D, I_0). \quad (2.164) \]

- in the case of liquidity rationing, banks’ expected payout in the good state is given by

\[ \tilde{v}^{LR}_{G}(D, I_0) = (E^b + D - I_0) + \ell^G I_0 + I_0 \Psi_G[\tilde{\theta}^{LR}_{G}(D, I_0), \ell^G] - \Delta \mu(D), \]

where \( \tilde{\theta}^{LR}_{G}(D, I_0) \) solves

\[ (E^b + D - I_0) + \ell^G I_0 = I_0 \Psi_G[\tilde{\theta}^{LR}_{G}(D, I_0), \ell^G] \]

\[ = I_0(\rho + \ell^G)[1 - F[\tilde{\theta}^{LR}_{G}(D, I_0)]], \quad (2.165) \]

or equivalently

\[ E^b + D = I_0[1 + \rho - (\rho + \ell^G) F[\tilde{\theta}^{LR}_{G}(D, I_0)]]. \]

So,

\[ (\tilde{\theta}^{LR}_{G})_D(D, I_0) = \frac{-1}{I_0 f[\tilde{\theta}^{LR}_{G}(D, I_0)](\rho + \ell^G)} < 0, \]

and

\[ (\tilde{\theta}^{LR}_{G})_I(D, I_0) = \frac{1 + \rho - (\rho + \ell^G) F[\tilde{\theta}^{LR}_{G}(D, I_0)]}{I_0 f[\tilde{\theta}^{LR}_{G}(D, I_0)](\rho + \ell^G)} > 0. \]
Figure 2.9: Illustration of sublemma 2.A.15
In turn,
\[
\tilde{v}^{LR}_{GD}(D, I_0) = 1 - \Delta \mu'(D) + I_0(\tilde{\theta}^{LR}_G)_D(D, I_0)(\Pi_G)_0[\tilde{\theta}^{LR}_G(D, I_0), \ell^{SP}_G]
\]
\[= 1 - \Delta \mu'(D) + \frac{\chi_G \tilde{\theta}^{LR}_G(D, I_0) - \rho - \ell^{SP}_G}{\rho + \ell^{SP}_G}, \tag{2.166}
\]
and
\[
\tilde{v}^{LR}_{GI}(D, I_0)
\]
\[= \ell^{SP}_G + \Pi_G[\tilde{\theta}^{LR}_G(D, I_0), \ell^{SP}_G] - 1 + I_0(\tilde{\theta}^{LR}_G)_I(D, I_0)(\Pi_G)_0[\tilde{\theta}^{LR}_G(D, I_0), \ell^{SP}_G]
\]
\[= \ell^{SP}_G + \Pi_G[\tilde{\theta}^{LR}_G(D, I_0), \ell^{SP}_G] - 1 - [1 + \rho - (\rho + \ell^{SP}_G)F[\tilde{\theta}^{LR}_G(D, I_0)]] \times \cdots \]
\[\cdots \times \frac{\chi_G \tilde{\theta}^{LR}_G(D, I_0) - \rho - \ell^{SP}_G}{\rho + \ell^{SP}_G}. \tag{2.167}
\]
Moreover,
\[
\tilde{v}^{LR}_{GDD}(D, I_0) = (-1)\Delta \mu''(D) + \frac{\chi_G(\tilde{\theta}^{LR}_G)_D(D, I_0)}{\rho + \ell^{SP}_G} < 0,
\]
\[
\tilde{v}^{LR}_{GDI}(D, I_0) = \frac{\chi_G(\tilde{\theta}^{LR}_G)_I(D, I_0)}{\rho + \ell^{SP}_G} > 0,
\]
\[
\tilde{v}^{LR}_{GID}(D, I_0) = (\tilde{\theta}^{LR}_G)_D(D, I_0)(\Pi_G)_0[\tilde{\theta}^{LR}_G(D, I_0), \ell^{SP}_G]
\]
\[+ f[\tilde{\theta}^{LR}_G(D, I_0)](\tilde{\theta}^{LR}_G)_D(D, I_0)\chi_G \tilde{\theta}^{LR}_G(D, I_0) - \rho - \ell^{SP}_G]
\[= \frac{[1 + \rho - (\rho + \ell^{SP}_G)F[\tilde{\theta}^{LR}_G(D, I_0)]] \cdot \chi_G(\tilde{\theta}^{LR}_G)_D(D, I_0)}{\rho + \ell^{SP}_G}
\]
\[= \frac{(-1)[1 + \rho - (\rho + \ell^{SP}_G)F[\tilde{\theta}^{LR}_G(D, I_0)]] \cdot \chi_G(\tilde{\theta}^{LR}_G)_D(D, I_0)}{\rho + \ell^{SP}_G}
\]
> 0,

and

$$\tilde{u}_{\text{GII}}^{LR}(D, I_0) = \frac{(-1)[1 + \rho - (\rho + \ell_{G}^{SP})F[\tilde{\theta}_{G}^{LR}(D, I_0)]]}{\rho + \ell_{G}^{SP}} \cdot \chi_{G}(\tilde{\theta}_{G}^{LR}(D, I_0)) < 0.$$  

(2.168)

2.A.8.4 A bit more notation

- let $\tilde{v}_{\omega D}(D, I_0)$ denote banks’ expected return from the marginal deposit in state $\omega$ — e.g.,

$$I_0 \in (\tilde{I}_{B}^{LS}(D), \tilde{I}_{B}^{EP}(D)) \implies \tilde{v}_{BD}(D, I_0) = \tilde{v}_{BD}^{EP}(D, I_0).$$

Define $\tilde{v}_{\omega I}(D, I_0)$ analogously;

- let $\tilde{v}_{D}(D, I_0)$ denote banks’ expected return from the marginal deposit, computed on an unconditional basis at $t = 0$ — i.e.,

$$\tilde{v}_{D}(D, I_0) = \alpha_{G}\tilde{v}_{GD}(D, I_0) + \alpha_{B}\tilde{v}_{BD}(D, I_0).$$

Define $\tilde{v}_{I}(D, I_0)$ analogously.

Remark. It can easily be verified that all of the marginal return functions defined above are continuous in both their arguments, even around the boundaries separating regimes. This is a consequence of the envelope theorem.

2.A.8.5 Details on ($\tilde{\mathbb{F}}_{0}$-rex)

Sublemma 2.A.16. All $D \in [0, D^{SP}]$ satisfy

$$\tilde{\theta}_{G}^{LR}[D, (E + D)/(1 + \rho_{2}^{SP})] = \theta_{G}^{LR}(D^{SP}, I_0^{SP}).$$

Proof. That

$$\tilde{\theta}_{G}^{LR}[D^{SP}, (E + D^{SP})/(1 + \rho_{2}^{SP})] = \theta_{G}^{LR}(D^{SP}, I_0^{SP}) = \theta_{G}^{LR}(D^{SP}, I_0^{SP})$$
should be obvious. To see that $\tilde{\theta}_G^{LR}(D, (E + D)/(1 + \rho_2^{SP})) = \theta_G^{LR}(D^{SP}, I_0^{SP}) \forall D \in [0, D^{SP})$ as well, note from (2.165) that

$$\frac{E^b + D - I_0}{I_0} + \ell_G^{SP} = \Psi_G[\tilde{\theta}_G^{LR}(D, I_0), \ell_G^{SP}],$$

$$\forall (D, I_0) \in [0, D^{SP}] \times \left(\tilde{I}_G^{LS}(D), (E^b + D)/(1 + \rho_2^{SP})\right),$$

so the ratio $(E^b + D - I_0)/I_0$ pins down the threshold $\tilde{\theta}_G^{LR}(D, I_0)$.

**Sublemma 2.A.17.** $\forall (D, I_0) \in [0, D^{SP}] \times [0, (E^b + D)/(1 + \rho_2^{SP})]$, the marginal return $\tilde{v}_I(D, I_0)$ is weakly decreasing in its first argument.

**Proof.** See (2.151), (2.156), (2.159), (2.164), and (2.168). □

**Sublemma 2.A.18.** $\forall D \in [0, D^{SP}]$, the composition $\tilde{v}_I[D, (E^b + D)/(1 + \rho_2^{SP})]$ is weakly decreasing.

**Proof.** Fix some $D \in [0, D^{SP}]$ and note from sublemmata 2.A.15 and 2.A.16 along with (2.167), that the state-specific return

$$\tilde{v}_GI[D, (E^b + D)/(1 + \rho_2^{SP})] = \tilde{v}_G^{LR}[D, (E^b + D)/(1 + \rho_2^{SP})]$$

$$= \ell_G^{SP} + \Pi_G[\theta_G^{LR}(D^{SP}, I_0^{SP}), \ell_G^{SP}] - 1$$

$$- [1 + \varrho - (\varrho + \ell_G^{SP})F[\theta_G^{LR}(D^{SP}, I_0^{SP})]] \times \cdots$$

$$\cdots \times \frac{\chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \varrho - \ell_G^{SP}}{\varrho + \ell_G^{SP}}$$

is constant. As for the complementary return $\tilde{v}_B[I[D, (E^b + D)/(1 + \rho_2^{SP})]]$, it suffices to note from (2.151), (2.156), and (2.159) that the function $\tilde{v}_B[I, \cdot]$ is weakly decreasing in both its arguments. □

**Sublemma 2.A.19.** $\tilde{v}_I(D^{SP}, I_0^{SP}) > 0$.

**Proof.** It’s best to take cases as follows:
Case one: $I_{0}^{SP} \leq T_{B}^{LS}(D^{SP})$. Under this case, the planner’s expected return on the marginal investment is given by

$$
\alpha G \tilde{v}_{GI}^{LR}(D^{SP}, I_{0}^{SP}) + \alpha B \tilde{v}_{BI}^{LS}(D^{SP}, I_{0}^{SP})
$$

$$
= \alpha_{G} \left[ \chi_{G} \mathbb{E} \max[\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})] - (1 + \rho) \right] - \left[ \frac{1 + \rho}{\rho + \ell_{G}^{SP} + \delta_{G}^{LR}(D^{SP}, I_{0}^{SP})} \right] \left[ \chi_{G} \theta_{G}^{LR}(D^{SP}, I_{0}^{SP}) - \rho - \ell_{G}^{SP} \right] > 0
$$

where

$$
\tilde{v}_{I}(D^{SP}, I_{0}^{SP}) = \alpha_{G} \tilde{v}_{GI}^{LR}(D^{SP}, I_{0}^{SP}) + \alpha_{B} \tilde{v}_{BI}^{LS}(D^{SP}, I_{0}^{SP}),
$$

with $\tau_{B}^{SP} = 0$. Now, revisiting (2.167) yields

$$
\tilde{v}_{GI}^{LR}(D^{SP}, I_{0}^{SP})
$$

$$
= \ell_{G}^{SP} + \Pi_{G}[\theta_{G}^{LR}(D^{SP}, I_{0}^{SP}), \ell_{G}^{SP}] - 1 - [1 + \rho - (\rho + \ell_{G}^{SP})F[\theta_{G}^{LR}(D, I_{0})]] \times \cdots
$$

$$
\cdots \times \frac{\chi_{G} \theta_{G}^{LR}(D^{SP}, I_{0}^{SP}) - \rho - \ell_{G}^{SP}}{\rho + \ell_{G}^{SP}}
$$

$$
= \ell_{G}^{SP} + \int_{\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})}^{1} (\theta \chi_{G} - \rho - \ell_{G}^{SP})dF(\theta) - 1
$$

$$
- [1 + \rho - (\rho + \ell_{G}^{SP})F[\theta_{G}^{LR}(D, I_{0})]] \times \cdots
$$

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\[ \chi_G \theta_G^{LR}(D_{SP}, I_0^{SP}) - \rho - \ell_G^{SP} \]

\[ = \ell_G^{SP} + \chi_G \mathbb{E} \max[\theta_G^{LR}(D_{SP}, I_0^{SP})] - \theta_G^{LR}(D_{SP}, I_0^{SP}) F[\theta_G^{LR}(D_{SP}, I_0^{SP})] \\
- (\rho + \ell_{SP}^G)[1 - F[\theta_G^{LR}(D_{SP}, I_0^{SP})]] - 1 \\
- [1 + \rho - (\rho + \ell_{SP}^G) F[\theta_G^{LR}(D, I_0)]] \times \cdots \]

\[ \cdots \times \frac{\chi_G \theta_G^{LR}(D_{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP}} \] \hspace{1cm} (2.170)

\[ = \chi_G \mathbb{E} \max[\theta_G^{LR}(D_{SP}, I_0^{SP})] - (1 + \rho) - \left( (1 + \rho)/(\rho + \ell_G^{SP}) \right) \times \cdots \]

\[ \cdots \times [\chi_G \theta_G^{LR}(D_{SP}, I_0^{SP}) - \rho - \ell_G^{SP}], \] \hspace{1cm} (2.171)

where (2.170) follows from integration by parts. At the same time, revisiting (2.149) yields

\[ \tilde{v}_{LS}(D_{SP}, I_0^{SP}) \]

\[ = \ell_B^{SP} + \tilde{\Pi}_B[\tilde{\Pi}_B(\ell_B^{SP}, 0), \ell_B^{SP}, 0] - 1 \]

\[ = \ell_B^{SP} + \int_0^1 \left( \theta \chi_B - \rho - \ell_B^{SP} \right)dF(\theta) - 1 \]

\[ = \ell_B^{SP} + \chi_B \left[ \mathbb{E} \max \left( \frac{\rho + \ell_B^{SP}}{\chi_B} \right) - \left( \frac{\rho + \ell_B^{SP}}{\chi_B} \right) F\left( \frac{\rho + \ell_B^{SP}}{\chi_B} \right) \right] \\
- (\rho + \ell_B^{SP}) \left[ 1 - F\left( \frac{\rho + \ell_B^{SP}}{\chi_B} \right) \right] - 1, \] \hspace{1cm} (2.173)

\[ = \chi_B \mathbb{E} \max \left[ (\rho + \ell_B^{SP})/\chi_B \right] - (1 + \rho) \]
where (2.176) again follows from integration by parts. So,

\[ \tilde{v}_I(D^{SP}, I_0^{SP}) \]

\[
= \alpha_G \left[ \chi_G \mathbb{E} \max[\theta_{G}^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho) - \left( \frac{1 + \rho}{\rho + \ell_G^{SP}} \right) \times \cdots \right] \\
\cdots \times \left[ \chi_G \theta_{G}^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP} \right] \\
+ \alpha_B \chi_B \mathbb{E} \max \left[ (\rho + \ell_B^{SP})/\chi_B \right] - (1 + \rho) \]

(2.174)

> 0,

where the inequality follows from comparison with (2.169).

Case two: \( I_0^{SP} \in (T_B^{LS}(D^{SP}), T_B^{ED}(D^{SP})) \). Under this case, the planner’s expected return on the marginal investment is given by

\[ \alpha_G v_L^{SP}(D^{SP}, I_0^{SP}) + \alpha_B v_B^{SP}(D^{SP}, I_0^{SP}) \]

\[
= \alpha_G \left[ \chi_G \mathbb{E} \max[\theta_{G}^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho) \right] - \frac{1 + \rho}{\rho + \ell_G^{SP} + \delta[I_0 F[\theta_{G}^{LR}(D^{SP}, I_0^{SP})]]} \left[ \chi_G \theta_{G}^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP} \right] \\
\text{see } (2.117)
\]
\[
\chi_B \mathbb{E} \max[\theta_B^{ED}(D^{SP}, I_0^{SP})] - (1 + \rho) \\
+ \alpha_B \left[ (1 + \rho) - (1 - \gamma) \chi_B \max[\theta_B^{ED}(D^{SP}, I_0^{SP})] \right. \\
\left. - \gamma \chi_B F[\theta_B^{ED}(D^{SP}, I_0^{SP})](F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})] \right] \times \cdots \\
\cdots \times \left[ \frac{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) - \rho - \ell_B^{SP}}{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})]} \right. \\
\left. - \rho - \ell_B^{SP} - \delta[I_0^{SP} F[\theta_B^{ED}(D^{SP}, I_0^{SP})]] \right] \times \cdots 
\]

\[= 0, \quad \text{(2.175)}\]

whereas

\[
\tilde{v}_I(D^{SP}, I_0^{SP}) \\
= \alpha_G \tilde{v}_{GI}^{LR}(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_{BI}^{ED}(D^{SP}, I_0^{SP}) \\
= \alpha_G \left[ \chi_G \mathbb{E} \max[\theta_G^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho) - \left( \frac{1 + \rho}{\rho + \ell_G^{SP}} \right) \times \cdots \right. \\
\left. \cdots \times [\chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}] \right. \\
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\[
\ldots \times \frac{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) - (1 - \tau_B)(\rho + \ell_B^{SP})}{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED} (D^{SP}, I_0^{SP})]}
\]

\[
= (1 - \tau_B^{SP})(\ell_B^{SP} - 1) + \int_0^1 \theta_B^{ED} (D^{SP}, I_0^{SP}) \big[ \theta_B - (1 - \tau_B^{SP})(\rho + \ell_B^{SP}) \big] dF(\theta)
\]

\[
+ [(1 - \tau_B^{SP})(1 - \ell_B^{SP}) - \tilde{\Delta}_B[\theta_B^{ED} (D^{SP}, I_0^{SP}), \ell_B^{SP}, \tau_B^{SP}]] \times \ldots
\]

\[
\ldots \times \frac{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) - (1 - \tau_B)(\rho + \ell_B^{SP})}{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED} (D^{SP}, I_0^{SP})]}
\]

\[
= (1 - \tau_B^{SP})(\ell_B^{SP} - 1)
\]

\[
+ \chi_B \big[ \mathbb{E} \max[\theta_B^{ED} (D^{SP}, I_0^{SP})] - \theta_B^{ED} (D^{SP}, I_0^{SP})F[\theta_B^{ED} (D^{SP}, I_0^{SP})] \big]
\]

\[
- (1 - \tau_B^{SP})(\rho + \ell_B^{SP})[1 - F[\theta_B^{ED} (D^{SP}, I_0^{SP})]]
\]

\[
+ [(1 - \tau_B^{SP})(1 - \ell_B^{SP}) - \tilde{\Delta}_B[\theta_B^{ED} (D^{SP}, I_0^{SP}), \ell_B^{SP}, \tau_B^{SP}]] \times \ldots
\]

\[
\ldots \times \frac{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) - (1 - \tau_B)(\rho + \ell_B^{SP})}{\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED} (D^{SP}, I_0^{SP})]}
\]

\[
= \chi_B \big[ \mathbb{E} \max[\theta_B^{ED} (D^{SP}, I_0^{SP})] - (1 - \tau_B^{SP})(1 + \rho)
\]

\[
- F[\theta_B^{ED} (D^{SP}, I_0^{SP})][\chi_B \theta_B^{ED} (D^{SP}, I_0^{SP}) - (1 - \tau_B^{SP})(\rho + \ell_B^{SP})] \big]
\]
\[
\begin{align*}
+ [(1 - \tau_{SP}^B)(1 - \ell_{SP}^B) - \tilde{\Delta}_B[\theta_{SP}^{ED}(D^{SP}, I^{SP}_0), \ell_{SP}^B, \tau_{SP}^B]] \times \ldots
\end{align*}
\]

\[
\begin{align*}
\cdots \times \frac{\chi_B\theta_{SP}^{ED}(D^{SP}, I^{SP}_0) - (1 - \tau_{SP}^B)(\rho + \ell_{SP}^B)}{\chi_B\theta_{SP}^{ED}(D^{SP}, I^{SP}_0) + \gamma\chi_B(F/f)[\theta_{SP}^{ED}(D^{SP}, I^{SP}_0)] - (1 - \tau_{SP}^B)(\rho + \ell_{SP}^B)}
\end{align*}
\]

\[
= \chi_B\max[\theta_{SP}^{ED}(D^{SP}, I^{SP}_0)] - (1 - \tau_{SP}^B)(1 + \rho)
\begin{align*}
\begin{bmatrix}
\chi_B\theta_{SP}^{ED}(D^{SP}, I^{SP}_0) - (1 - \tau_{SP}^B)(\rho + \ell_{SP}^B) \\
\chi_B\theta_{SP}^{ED}(D^{SP}, I^{SP}_0) + \gamma\chi_B(F/f)[\theta_{SP}^{ED}(D^{SP}, I^{SP}_0)] - (1 - \tau_{SP}^B)(\rho + \ell_{SP}^B)
\end{bmatrix}
\end{align*}
\times \ldots
\]
\[
\begin{aligned}
&= (1 - \tau_B^{SP})(1 - \ell_B^{SP}) - \Delta_B[\theta_B^{ED}(D^{SP}, I_0^{SP}), \ell_B^{SP}, \tau_B^{SP}] \\
&\quad - F[\theta_B^{ED}(D^{SP}, I_0^{SP})] \times \cdots \\
&\quad \cdots \times \left[ \chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})] \right] \\
&\quad \cdots \times \\
&\quad \cdots \times F[\theta_B^{ED}(D^{SP}, I_0^{SP})] \\
&= (1 - \tau_B^{SP})(1 - \ell_B^{SP}) \\
&\quad - \int_{\theta_B^{ED}(D^{SP}, I_0^{SP})}^{1} \left[ \theta_B - (1 - \tau_B^{SP})(\rho + \ell_B^{SP}) \right] dF(\theta) \\
&\quad + \gamma \chi_B \mathbb{E} \max[\theta_B^{ED}(D^{SP}, I_0^{SP})] - F[\theta_B^{ED}(D^{SP}, I_0^{SP})] \times \cdots \\
&\quad \cdots \times \left[ \chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})] \right] \\
&\quad \cdots \times \\
&\quad \cdots \times F[\theta_B^{ED}(D^{SP}, I_0^{SP})] \\
&= (1 - \tau_B^{SP})(1 + \rho) - (1 - \gamma) \chi_B \mathbb{E} \max[\theta_B^{ED}(D^{SP}, I_0^{SP})] \\
&\quad - \gamma \chi_B F[\theta_B^{ED}(D^{SP}, I_0^{SP})](F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})]
\end{aligned}
\]
\[
\chi_E \max[\theta^E_D(D^{SP}, I_0^{SP})] \cdot (1 - \tau^E_B)(1 + \rho) + \\
\left[ (1 - \tau^E_B)(1 + \rho) - (1 - \gamma)\chi_E \max[\theta^E_D(D^{SP}, I_0^{SP})] - \\
\gamma \chi_E F[\theta^E_D(D^{SP}, I_0^{SP})](F/f)[\theta^E_D(D^{SP}, I_0^{SP})] \right] \times \ldots
\]

\[
\ldots \times \frac{\chi_E \theta^E_D(D^{SP}, I_0^{SP}) - (1 - \tau^E_B)(\rho + \ell^E_B)}{\chi_E \theta^E_D(D^{SP}, I_0^{SP}) + \gamma \chi_E (F/f)[\theta^E_D(D^{SP}, I_0^{SP})]} - (1 - \tau^E_B)(\rho + \ell^E_B)
\]

\[
\chi_E \max[\theta^E_D(D^{SP}, I_0^{SP})] \cdot (1 - \tau^E_B)(1 + \rho) + \\
\left[ (1 - \tau^E_B)(1 + \rho) - (1 - \gamma)\chi_E \max[\theta^E_D(D^{SP}, I_0^{SP})] - \\
\gamma \chi_E F[\theta^E_D(D^{SP}, I_0^{SP})](F/f)[\theta^E_D(D^{SP}, I_0^{SP})] \right] \times \ldots
\]

\[
\ldots \times \frac{\chi_E \theta^E_D(D^{SP}, I_0^{SP}) - \rho - \ell^E_B}{\chi_E \theta^E_D(D^{SP}, I_0^{SP}) + \gamma \chi_E (F/f)[\theta^E_D(D^{SP}, I_0^{SP})]} - \rho - \ell^E_B
\]

\[
\ldots - \delta[I_0^{SP} F[\theta^E_D(D^{SP}, I_0^{SP})]]
\]

where (2.176) follows from integration by parts, while (2.177) follows from the way that I’ve constructed the function \(\tau^E_B(\cdot, \cdot)\) — see (2.61) in particular. So,

\[
\tilde{v}_I(D^{SP}, I_0^{SP})
\]

\[
\alpha_G \left[ \chi_G \max[\theta^L_R(D^{SP}, I_0^{SP})] - (1 + \rho) - \left( \frac{1 + \rho}{\rho + \ell^G_S} \right) [\chi_G \theta^L_R(D^{SP}, I_0^{SP}) - \rho - \ell^G_S] \right]
\]

(2.178)
\[ \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \tau_{SP}^{B})(1 + \rho) \]

\[ + \left[ \begin{array}{c}
(1 - \tau_{SP}^{B})(1 + \rho) - (1 - \gamma) \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \\
- \gamma \chi_B F[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \times \ldots \\
\ldots \times (F/f)[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]
\end{array} \right] \times \ldots \]

\[ + \alpha_B \left[ \begin{array}{c}
\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 + \rho) \\
\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \gamma) \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \\
\ldots \times (F/f)[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]
\end{array} \right] \times \ldots \]

\[ + \alpha_B \left[ \begin{array}{c}
\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 + \rho) \\
\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \gamma) \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \\
\ldots \times (F/f)[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]
\end{array} \right] \times \ldots \]

\[ \ldots \times \frac{\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 + \rho)}{\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \gamma) \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]} \times \ldots \]

\[ \ldots \times \frac{\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 + \rho)}{\chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \gamma) \chi_B \mathbb{E} \max [\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]} \times \ldots \]

\[ > 0, \]

where the inequality follows from comparison with (2.175).

Case three: \( I_{0}^{SP} \in (T_{B}^{ED}, T_{B}^{DD}) \). Under this case, the planner’s expected return on the marginal investment is given by

\[ \alpha_{G}^{v_{G}^{LR|SP}}(D^{SP}, I_{0}^{SP}) + \alpha_{B}^{v_{B}^{DD|SP}}(D^{SP}, I_{0}^{SP}) \] (2.179)

\[ = \alpha_{G} \left[ \begin{array}{c}
\chi_G \mathbb{E} \max [\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})] - (1 + \rho) \\
\chi_G \mathbb{E} \max [\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})] - (1 - \gamma) \chi_G \mathbb{E} \max [\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})] \\
\ldots \times (F/f)[\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})]
\end{array} \right] \times \ldots \]

\[ \ldots \times \left[ \frac{1 + \rho}{\rho + \ell_{G}^{SP} + \delta[I_{0} F[\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})]]} \right] \times \ldots \]

\[ \ldots \times \left[ \frac{1 + \rho}{\rho + \ell_{G}^{SP} + \delta[I_{0} F[\theta_{G}^{LR}(D^{SP}, I_{0}^{SP})]]} \right] \times \ldots \]

see (2.177)

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\[
\begin{align*}
- \alpha_B & \left[ (1 - \ell_{SP}^{I_0}) \right. \\
& - \left. [1 - \ell_{SP}^{I_0} - \delta[I_0^{SP} - I_B^{DD|SP}(D^{SP}, I_0^{SP})[1 - F[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]]]] \times \ldots \\
& \ldots \times \frac{(\Pi_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]}{(\Delta_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]} \right] \\
\geq 0,
\end{align*}
\]

where the inequality allows for the possibility that the planner’s solution entails an interbank collapse. At the same time,

\[
\tilde{v}_I(D^{SP}, I_0^{SP}) = \alpha_G \tilde{v}_{G}^{LR}(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_{B}^{DD}(D^{SP}, I_0^{SP})
\]

\[
= \alpha_G \left[ \chi_G \mathbb{E} \max[\theta_G^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho) - \left( \frac{1 + \rho}{\rho + \ell_{SP}^{I_0}} \right) \times \ldots \\
\ldots \times \left[ \chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_{SP}^{I_0} \right] \right]
\]

\[
- \alpha_B (1 - \tau_{SP}^{I_0})(1 - \ell_{SP}^{I_0}) \left[ 1 - \frac{\Pi_B[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), \ell_{SP}^{I_0}, \tau_{SP}^{I_0}]}{\Delta_B[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), \ell_{SP}^{I_0}, \tau_{SP}^{I_0}]} \right],
\]

with \( \tau_{SP}^{I_0} = \tau_{SP}^{DD}(D^{SP}, I_0^{SP}) > 0 \). Now, based on the planner’s marginal condition in the bad state (line 2.108), we know that

\[
\frac{(\Pi_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]}{(\Delta_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]} = \frac{(\Pi_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]}{(\Delta_B^{SP})_0[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), I_B^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP}]}.
\]

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\[
\chi_B\theta_B^{DD|SP}(D^{SP}I_0^{SP}) - \rho - \ell_B^{SP} \\
\chi_B\theta_B^{DD|SP}(D^{SP}, I_0^{SP}) + \gamma\chi_B(F/f)[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})] - \rho - \ell_B^{SP} \\
- \delta[I_0^{SP} - I_B^{DD|SP}(D^{SP}, I_0^{SP})[1 - F[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]]] \\
\]

Moreover, using (2.109), combined with (2.34), (2.20)-(2.22), and (1.12), this marginal condition can be re-written as

\[
\chi_B[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}) + \xi_3[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})](F/f)[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]] \\
= \rho + \ell_B + \left[\xi_3[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]/\gamma\right][\delta[I_0^{SP} - I_B^{DD|SP}(D^{SP}, I_0^{SP})[1 - F[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]]] \\
- \chi_B(F/f)[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})] \\
- \chi_B(F/f)[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})] \\
+ \gamma\chi_B(F/f)[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})] - \rho - \ell_B^{SP} \\
- \delta[I_0^{SP} - I_B^{DD|SP}(D^{SP}, I_0^{SP})[1 - F[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]]] \\
\]

\[
= \frac{(-1)[\xi_3[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]]}{\gamma - \xi_3[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]} \\
< 0,
\]

where last line follows from the fact that $\gamma > \xi_3[\theta_B^{DD|SP}(D^{SP}, I_0^{SP})]$, as shown in my proof of sublemma 2.1.5. Conclude that

\[
\alpha_{G^{v_{GI}}}(D^{SP}, I_0^{SP}) + \alpha_{B^{v_{BI}}}(D^{SP}, I_0^{SP})
\]

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\[
\alpha \left[ \chi G \mathbb{E} \max [\theta^L (D^{SP}, I_0^{SP})] - (1 + \rho) \right] \\
- \left[ \frac{1 + \rho}{\rho + \ell^{SP} + \delta[I_0 F[\theta^L G (D^{SP}, I_0^{SP})]]} \right] \times \ldots \]
\[
\ldots \times \left[ \chi G \theta^L G (D^{SP}, I_0^{SP}) - \rho - \ell^{SP} \right] \]
\[
\geq 0. \quad (2.180)
\]

Since banks obey a very similar marginal condition in the bad state [see (2.54) and (2.55)], analogous arguments yield
\[
\tilde{v}_I (D^{SP}, I_0^{SP}) = \alpha_G \left[ \chi G \mathbb{E} \max [\theta^L (D^{SP}, I_0^{SP})] - (1 + \rho) - \left( \frac{1 + \rho}{\rho + \ell^{SP}} \right) \times \ldots \right] \\
\ldots \times \left[ \chi G \theta^L G (D^{SP}, I_0^{SP}) - \rho - \ell^{SP} \right] \\
- \alpha_B (1 - \tau^B) (1 - \ell^B) \left[ 1 + \frac{\xi^3 [\theta^D (D^{SP}, I_0^{SP})]}{\gamma - \xi^3 [\theta^D (D^{SP}, I_0^{SP})]} \right] \quad (2.181)
\]
\[
> 0,
\]
where the inequality follows from comparison with (2.180).
Sublemma 2.A.20. \( \forall D \in [0, D^{SP}] \), the composition

\[
\tilde{v}_D[D, (E + D)/(1 + \rho s^{SP})] + (1 + \rho s^{SP})^{-1}\tilde{v}_I[D, (E + D)/(1 + \rho s^{SP})]
\]

is strictly decreasing.

Proof. That \( \tilde{v}_I[D, (E + D)/(1 + \rho s^{SP})] \) is weakly decreasing has already been shown (sublemma 2.A.17). As for \( \tilde{v}_D[D, (E + D)/(1 + \rho s^{SP})] \), I first note from sublemmata 2.A.15 and 2.A.16, along with (2.166), that the state-specific return

\[
\tilde{v}_{GD}[D, (E^b + D)/(1 + \rho s^{SP})] = \tilde{v}_{LRGD}[D, (E^b + D)/(1 + \rho s^{SP})] = 1 - \Delta \mu'(D) + \chi_G \theta_{LRG}(D^{SP}, I_0^{SP}) - \rho - \ell^{SP}_{G}
\]

is strictly decreasing in \( D \). As for the complementary marginal return \( \tilde{v}_{BD}[D, (E^b + D)/(1 + \rho s^{SP})] \), it suffices to note from (2.150), (2.154), (2.155), and (2.159) that the function \( \tilde{v}_{BD}(\cdot, \cdot) \) is weakly decreasing in both its arguments. \[\Box\]

Sublemma 2.A.21. \( \tilde{v}_D(D^{SP}, I_0^{SP}) + (1 + \rho s^{SP})^{-1}\tilde{v}_I(D^{SP}, I_0^{SP}) > 0 \).

Proof. It’s best to take cases as follows:

Case one: \( I_0^{SP} \leq T^{LS}_B(D^{SP}) \). Under this case, the planner’s expected return on the marginal deposit is given by

\[
\alpha_G \tilde{v}_{GD}^L(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_{BD}^L(D^{SP}, I_0^{SP})
\]

\[
= \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \theta_{LRG}(D^{SP}, I_0^{SP}) - \rho - \ell^{SP}_{G}}{\rho + \ell^{SP}_{G} + \delta[I_0^{SP} F[\theta_{LRG}(D^{SP}, I_0^{SP})]]} \right] + \alpha_B \left[ 1 - \Delta \mu'(D^{SP}) \right]
\]

\[
= 0
\]

Combining with (2.169) then yields

\[
\alpha_G \left[ \tilde{v}_{GD}^L(D^{SP}, I_0^{SP}) + \frac{\tilde{v}_{GI}^L(D^{SP}, I_0^{SP})}{1 + \rho s^{SP}} \right]
\]

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\[ + \alpha_B \left[ v_{BD}\overset{LS}{\leftarrow}^{SP}(D^{SP}, I_0^{SP}) + \frac{v_{BI}\overset{LS}{\leftarrow}^{SP}(D^{SP}, I_0^{SP})}{1 + \rho_s^{SP}} \right] \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi G E\overline{\max}[\theta_G^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho)}{1 + \rho_s^{SP}} \right] \]

\[ + \sum_{\rho_s^{SP}} \left[ \frac{\chi G \theta_G^{LR}(D^{SP}, I_0^{SP})}{\rho + \ell_G^{SP}} + \delta[I_0^{SP} F[\theta_G^{LR}(D, I_0)]] \right] \]

\[ = \alpha_B \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi B E\overline{\max}[(\rho + \ell_B^{SP})/\chi_B] - (1 + \rho)}{1 + \rho_s^{SP}} \right] \]

\[ = 0, \quad (2.182) \]

where my claim that \( 1 < (1 + \rho)/(1 + \rho_s^{SP}) \) follows from the fact that

\[(E^b + D^{SP} - I_0^{SP}) + \ell_G^{SP} = I_0^{SP} = \sum_{\rho_s^{SP}} \left[ \frac{\chi G \theta_G^{LR}(D^{SP}, I_0^{SP})}{\rho + \ell_G^{SP}} + \delta[I_0^{SP} F[\theta_G^{LR}(D, I_0)]] \right] \]

\[ \iff \frac{E^b + D^{SP} - I_0^{SP}}{I_0^{SP}} + \ell_G^{SP} = (\rho + \ell_G^{SP})[1 - F[\theta_G^{LR}(D^{SP}, I_0^{SP})]] \]

\[ \iff \rho_s^{SP} = \rho - (\rho + \ell_G^{SP})F[\theta_G^{LR}(D^{SP}, I_0^{SP})], \]

which means that \( \rho_s^{SP} < 1 \). At the same time, we know that \( \tau_B^{SP} = 0 \), and also that

\[ \tilde{v}_D(D^{SP}, I_0^{SP}) \]

\[ = \alpha_G \tilde{v}_{GD}(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_{BD}(D^{SP}, I_0^{SP}) \]
Combining with (2.174) then yields

\[
\tilde{v}_D(D^{SP}, I_0^{SP}) + \tilde{v}_I(D^{SP}, I_0^{SP}) = \alpha_G \left[ 1 - \Delta\mu'(D^{SP}) + \frac{\chi G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP} + \delta(I_0^{SP} F[\theta_G^{LR}(D^{SP}, I_0^{SP})])} \right] \\
= \alpha_G \left[ 1 - \Delta\mu'(D^{SP}) \right] + \frac{\chi G \max[\theta_G^{LR}(D^{SP}, I_0^{SP})] - (1 + \rho)}{1 + \rho S^{SP}} + \left( 1 - \frac{1 + \rho}{1 + \rho S^{SP}} \right) \times \ldots \\
\ldots \times \frac{\chi G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP}} \\
\alpha_B \left[ 1 - \Delta\mu'(D^{SP}) \right] + \frac{\chi B \max[(\rho + \ell_B^{SP})/\chi_B] - (1 + \rho)}{1 + \rho S^{SP}} > 0,
\]

where the inequality follows from comparison with (2.182).

Case two: \( I_0^{SP} \in (T_B^{LS}(D^{SP}), T_B^{ED}(D^{SP})) \). Under this case, the planner’s expected return on the marginal deposit is given by

\[
\alpha_G \mathcal{U}^{LR|SP}_{GD}(D^{SP}, I_0^{SP}) + \alpha_B \mathcal{U}^{ED|SP}_{BD}(D^{SP}, I_0^{SP}) \\
= \alpha_G \left[ 1 - \Delta\mu'(D^{SP}) + \frac{\chi G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP} + \delta(I_0^{SP} F[\theta_G^{LR}(D^{SP}, I_0^{SP})])} \right] \\
\text{see } (2.119) \\
+ \alpha_B \left[ 1 - \Delta\mu'(D^{SP}) \right].
\]
\[ + \alpha_B \left[ 1 - \Delta \mu'(D^{SP}) \right] \left[ 1 - \frac{\chi_B \dot{\theta}_B^{ED}(D^{SP}, I_0^{SP}) - \rho - \ell_B^{SP}}{\chi_B \dot{\theta}_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B(F/f)\dot{\theta}_B^{ED}(D^{SP}, I_0^{SP})} \right] = 0. \]

Combining with (2.175) then yields

\[ \alpha_{Gv_GD}^{LR|SP}(D^{SP}, I_0^{SP}) \]

\[ + \alpha_{B^{UD|SP}}(D^{SP}, I_0^{SP}) + \alpha_{Gv_{GI}}^{LR|SP}(D^{SP}, I_0^{SP}) + \alpha_{B^{UD|SP}}(D^{SP}, I_0^{SP}) \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{1 + \rho \ell_G^{SP}} \right] + \left( 1 - \frac{1 + \rho}{1 + \rho \ell_G^{SP}} \right) \left( \frac{\chi_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP} + \delta[I_0^{SP} F[\theta_G^{LR}(D^{SP}, I_0^{SP})]]} \right) \]

\[ < 0 \]
\[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_B \mathbb{E} \max[\theta_B^E(D^{SP}, I_0^{SP})] - (1 + \rho)}{1 + \rho_s^{SP}} \]

\[ + \alpha_B \]

\[ 1 - \Delta \mu'(D^{SP}) - \frac{\chi_B \theta_B^E(D^{SP}, I_0^{SP}) - \rho - \ell^{SP}}{\chi_B \theta_B^E(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^E(D^{SP}, I_0^{SP})]} \]

\[ \times \ldots \]

\[ \ldots \times \left( \frac{\chi_B \theta_B^E(D^{SP}, I_0^{SP}) - \rho - \ell^{SP}}{\chi_B \theta_B^E(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^E(D^{SP}, I_0^{SP})]} \right) \]

\[ = \alpha_G \tilde{v}_{GD}^{LR}(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_{BD}^{ED}(D^{SP}, I_0^{SP}) \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell^{SP}}{\rho + \ell^{SP}} \right] \]

\[ + \alpha_B [(1 - \tau_B^{SP}) - \Delta \mu'(D^{SP})] \times \ldots \]

\[ = 0. \] \hspace{1cm} (2.183)
\[
\cdots \times \left[1 - \frac{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) - (1 - \tau_B^{SP})(\rho + \ell_B^{SP})}{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f) [\theta_B^{ED}(D^{SP}, I_0^{SP})]} \right] \\
\cdots \times \left[1 - \frac{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) - \rho - \ell_B^{SP}}{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f) [\theta_B^{ED}(D^{SP}, I_0^{SP})]} \right] \\
= \alpha_G \left[1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP}} \right] \\
+ \alpha_B \left[(1 - \tau_B^{SP}) - \Delta \mu'(D^{SP})\right] \times \cdots
\]

where the last line follows from the way that I've constructed the function \(\tau_B^{ED}(\cdot, \cdot)\) — see (2.61) in particular. Combining with (2.178) then yields

\[
\tilde{v}_D(D^{SP}, I_0^{SP}) + \frac{\tilde{v}_I(D^{SP}, I_0^{SP})}{1 + \rho_{s}^{SP}}
\]

\[
= \alpha_G \left[1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - (1 + \rho)}{1 + \rho_{s}^{SP}} \right] \\
+ \left(1 - \frac{1 + \rho}{1 + \rho_{s}^{SP}} \right) \frac{\chi_G \theta_G^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_G^{SP}}{\rho + \ell_G^{SP}}
\]

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\[
\begin{align*}
(1 - \tau_{SP}^B) - \Delta \mu'(D^{SP}) \\
\quad + \chi_B \mathbb{E} \max[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] - (1 - \tau_{SP}^B)(1 + \rho) \\
\quad \times \frac{1 + \rho_{SP}}{1 + \rho_{SP}} \\
\quad + \alpha_B \\
\quad - (1 - \gamma) \chi_B \mathbb{E} \max[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \\
\quad - \gamma \chi_B F[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})] \times \ldots \\
\quad \times \frac{\chi_B \theta_{B}^{ED}(D^{SP}, I_{0}^{SP}) - \rho - \ell_{SP}^B}{\chi_B \theta_{B}^{ED}(D^{SP}, I_{0}^{SP}) + \gamma \chi_B (F/f) \theta_{B}^{ED}(D^{SP}, I_{0}^{SP})} \\
\quad - \rho - \ell_{B} \\
\quad - \delta[I_{0}^{SP} F[\theta_{B}^{ED}(D^{SP}, I_{0}^{SP})]] \\
\quad \ldots \times (F/f) \frac{\chi_G \theta_{G}^{LR}(D^{SP}, I_{0}^{SP})}{1 + \rho_{SP}} \\
\quad + \left(1 - \frac{1 + \rho}{1 + \rho_{SP}}\right) \frac{\chi_G \theta_{G}^{LR}(D^{SP}, I_{0}^{SP}) - \rho - \ell_{G}^{SP}}{\rho + \ell_{G}^{SP}} \\
\end{align*}
\]

\[= \alpha_G \]
\[
1 - \Delta \mu'(D^{SP}) + \frac{\chi_B \bar{E}_\text{max} \left[ \theta_B^{ED}(D^{SP}, I_0^{SP}) \right]}{1 + \rho^{SP}} - (1 + \rho) \\
- \left[ \frac{1 + \rho}{1 + \rho^{SP}} \right] \times \ldots \\
\left[ \frac{(1 - \gamma)\chi_B \bar{E}_\text{max} \left[ \theta_B^{ED}(D^{SP}, I_0^{SP}) \right]}{1 + \rho^{SP}} \right] \times \ldots \\
\left[ \frac{\gamma\chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})]}{1 + \rho^{SP}} \right] - \rho - \ell_B \\
- \delta[I_0^{SP} F[\theta_B^{ED}(D^{SP}, I_0^{SP})]]
\]

\[
- \alpha_B \tau_B^{SP} \left( 1 - \frac{1 + \rho}{1 + \rho^{SP}} \right) \\
+ \alpha_B \tau_B^{SP} \left( 1 - \frac{1 + \rho}{1 + \rho^{SP}} \right) \times \ldots
\]

\[
\ldots \times \frac{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) - \rho - \ell_B^{SP}}{\chi_B \theta_B^{ED}(D^{SP}, I_0^{SP}) + \gamma \chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})]} \\
\left[ \frac{\gamma\chi_B (F/f)[\theta_B^{ED}(D^{SP}, I_0^{SP})]}{1 + \rho^{SP}} \right] - \rho - \ell_B \\
- \delta[I_0^{SP} F[\theta_B^{ED}(D^{SP}, I_0^{SP})]]
\]

\[
> 0,
\]

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where the inequality follows from comparison with (2.183).

Case three: \( I_0^{SP} \in (I_B^{ED|SP}(D^{SP}), I_B^{DD}(D^{SP})) \). Under this case, the planner’s expected return on the marginal deposit is given by

\[
\alpha_{G} v_{GD}^{LR|SP}(D^{SP}, I_0^{SP}) + \alpha_{B} v_{BD}^{DD}(D^{SP}, I_0^{SP})
\]

\[
= \alpha_{G} \left[ 1 - \Delta^e(D^{SP}) + \frac{\chi_{G} \theta_{G}^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_{G}^{SP}}{\rho + \ell_{G}^{SP} + \delta I_0^{SP} F[\theta_{G}^{LR}(D^{SP}, I_0^{SP})]} \right]
\]

\[
+ \alpha_{B} \left[ 1 - \Delta^e(D^{SP}) \right] \left[ 1 - \frac{(\Pi_{B}^{SP}) I_{B}^{DD|SP}(D^{SP}, I_0^{SP}), I_{B}^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP})}{(\Delta_{B}) I_{B}^{DD|SP}(D^{SP}, I_0^{SP}), I_{B}^{DD|SP}(D^{SP}, I_0^{SP}), I_0^{SP})} \right]
\]

\[
= \alpha_{G} \left[ 1 - \Delta^e(D^{SP}) + \frac{\chi_{G} \theta_{G}^{LR}(D^{SP}, I_0^{SP}) - \rho - \ell_{G}^{SP}}{\rho + \ell_{G}^{SP} + \delta I_0^{SP} F[\theta_{G}^{LR}(D^{SP}, I_0^{SP})]} \right]
\]

\[
+ \alpha_{B} \left[ 1 - \Delta^e(D^{SP}) \right] \left[ 1 + \frac{\xi_{3} \theta_{B}^{DD|SP}(D^{SP}, I_0^{SP})}{\gamma - \xi_{3} \theta_{B}^{DD|SP}(D^{SP}, I_0^{SP})} \right] \quad (2.184)
\]

\[
\geq 0, \quad (2.185)
\]

where the middle equality follows from the analysis of the present case given in my proof of sublemma 2.A.19 while the final inequality allows for the possibility that the planner’s solution entails an interbank collapse. Combining with (2.180) then yields

\[
\alpha_{G} v_{GD}^{LR|SP}(D^{SP}, I_0^{SP}) + \alpha_{B} v_{BD}^{DD}(D^{SP}, I_0^{SP})
\]

\[
+ \frac{\alpha_{G} v_{GI}^{LR|SP}(D^{SP}, I_0^{SP}) + \alpha_{B} v_{BI}^{ED|SP}(D^{SP}, I_0^{SP})}{1 + \rho_{SP}} \quad (2.186)
\]
\[
1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \max[\theta^{LR}_G(D^{SP}, I_0^{SP})] - (1 + \rho)}{1 + \rho \bar{\mu}^{SP}}
\]
\[
= \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \max[\theta^{LR}_G(D^{SP}, I_0^{SP})] - (1 + \rho)}{1 + \rho \bar{\mu}^{SP}} \right]
\]
\[
\left( 1 - \frac{1 + \rho}{1 + \rho \bar{\mu}^{SP}} \right) \frac{\chi_G \theta^{LR}_G(D^{SP}, I_0^{SP}) - \rho - \ell^G}{\rho + \ell^G + \delta[I_0^{SP} F[\theta^{LR}_G(D^{SP}, I_0^{SP})]]} \right]
\]
\[
\left[ 1 - \Delta \mu'(D^{SP}) - \frac{1 - \ell^B}{1 + \rho \bar{\mu}^{SP}} \right] \left[ 1 + \frac{\xi_3[\theta^{DD|SP}_B(D^{SP}, I_0^{SP})]}{\gamma - \xi_3[\theta^{DD|SP}_B(D^{SP}, I_0^{SP})]} \right]
\]
\[
\left[ \frac{\delta[I_0^{SP} - I_B^{DD|SP}(D^{SP}, I_0^{SP})][1 - F[\theta^{DD|SP}_B(D^{SP}, I_0^{SP})]]}{1 + \rho \bar{\mu}^{SP}} \right] \times \ldots
\]
\[
\ldots \times \frac{\xi_3[\theta^{DD|SP}_B(D^{SP}, I_0^{SP})]}{\gamma - \xi_3[\theta^{DD|SP}_B(D^{SP}, I_0^{SP})]} \right)
\]
\[
\geq 0. \quad (2.187)
\]

At the same time, we know that \( \tau_B^{SP} = \tau_B^{DD}(D^{SP}, I_0^{SP}) > 0 \), and also that

\[
\tilde{v}_D(D^{SP}, I_0^{SP}) = \alpha_G \tilde{v}_G^{LR}(D^{SP}, I_0^{SP}) + \alpha_B \tilde{v}_B^{DD}(D^{SP}, I_0^{SP})
\]
\[
= \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \theta^{LR}_G(D^{SP}, I_0^{SP}) - \rho - \ell^G}{\rho + \ell^G} \right]
\]
\[
\text{see (2.166)}
\]
\[
+ \alpha_B \left[ (1 - \tau_B^{SP}) - \Delta \mu'(D^{SP}) \right] \left[ 1 - \frac{\tilde{\Pi}_B[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), \ell^{SP}, \tau_B^{SP}]}{\Delta_B[\theta_B^{DD|SP}(D^{SP}, I_0^{SP}), \ell^{SP}, \tau_B^{SP}]} \right]
\]
\[
\text{see (2.157)}
\]
\[
= \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \theta^{LR}_G(D^{SP}, I_0^{SP}) - \rho - \ell^G}{\rho + \ell^G} \right]
\]
\[
\text{254}
\]
\[ + \alpha_B [(1 - \tau^S_{B}) - \Delta \mu'(D^{SP})] \times \ldots \]

\[ \cdots \times \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right], \] (2.188)

where (2.188) can be verified using arguments very similar to those underlying (2.184).

Combining with (2.181) then yields

\[ \tilde{v}_{D}(D^{SP}, I^{SP}_0) + \frac{\tilde{v}_{I}(D^{SP}, I^{SP}_0)}{1 + \rho_{2}^{SP}} \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \text{E} \max[\theta^L_{G}(D^{SP}, I^{SP}_0)] - (1 + \rho)}{1 + \rho_{2}^{SP}} \right] \]

\[ + \alpha_B \left[ (1 - \tau^S_{B}) - \Delta \mu'(D^{SP}) - \frac{(1 - \tau^S_{B})(1 - \ell^S_{B})}{1 + \rho_{2}^{SP}} \right] \times \ldots \]

\[ \cdots \times \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right] \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \text{E} \max[\theta^L_{G}(D^{SP}, I^{SP}_0)] - (1 + \rho)}{1 + \rho_{2}^{SP}} \right] \]

\[ + \alpha_B \left[ 1 - \Delta \mu'(D^{SP}) - \frac{1 - \ell^S_{B}}{1 + \rho_{2}^{SP}} \right] \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right] \]

\[ - \alpha_B \tau^S_{B} \left( 1 - \frac{1 + \rho}{1 + \rho_{2}^{SP}} \right)^{<0} \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right]^{>0} \]

\[ = \alpha_G \left[ 1 - \Delta \mu'(D^{SP}) + \frac{\chi_G \text{E} \max[\theta^L_{G}(D^{SP}, I^{SP}_0)] - (1 + \rho)}{1 + \rho_{2}^{SP}} \right] \]

\[ + \alpha_B \left[ 1 - \Delta \mu'(D^{SP}) - \frac{1 - \ell^S_{B}}{1 + \rho_{2}^{SP}} \right] \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right] \]

\[ - \alpha_B \tau^S_{B} \left( 1 - \frac{1 + \rho}{1 + \rho_{2}^{SP}} \right)^{<0} \left[ 1 + \frac{\xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]}{\gamma - \xi_3[\theta^D_{B}(D^{SP}, I^{SP}_0)]} \right]^{>0} \]
where the inequality follows from comparison with (2.187).

**Sublemma 2.A.22.** Under $(\bar{P}_0)$, it’s strictly optimal to set $(D, I_0) = (D^{SP}, I_0^{SP})$.

*Proof.* I’ll proceed in steps:

**Step one.** I’ll first argue that the liquidity coverage requirement, $I_0 \leq (E^b + D)/(1 + \rho_s^{SP})$, must bind under $(\bar{P}_0$-rex). To see this, note that sublemmata 2.A.18 and 2.A.19 together imply that $\bar{v}_I[D, (E^b + D)/(1 + \rho_s^{SP})] > 0 \forall D \in [0, D^{SP}]$, in which case sublemma 2.A.17 implies that $\bar{v}_I(D, I_0) > 0 \forall (D, I_0) \in [0, D^{SP}] \times [0, (E^b + D)/(1 + \rho_s^{SP})]$.

**Step two.** I’ll next argue that it’s strictly optimal to set $(D, I_0) = (D^{SP}, I_0^{SP})$ under $(\bar{P}_0$-rex). To see this, note that a marginal increase in $D$ yields return $v_D[D, (E^b + D)/(1 + \rho_s^{SP})] + (1 + \rho_s^{SP})^{-1}v_I[D, (E^b + D)/(1 + \rho_s^{SP})]$ when the liquidity coverage requirement binds, and sublemmata 2.A.20 and 2.A.21 together imply that this object is strictly positive $\forall D \in [0, D^{SP}]$.

**Step three.** I’ll finally argue that solutions for $(\bar{P}_0$-rex) generalize to $(\bar{P}_0)$, which requires that we verify $(\bar{FC}_G)$ and $(\bar{PC}_B)$. Fortunately, when $(D, I_0) = (D^{SP}, I_0^{SP})$, it should be clear that these two constraints coincide with the analogous constraints facing the planner, and the laxity of these analogues has already been established, namely in subsection 2.A.6.

$\square$
Chapter 3

A model for spillovers between the deposit and interbank markets

3.1 Introduction

In this last chapter, I extend the baseline model from chapter 1 in an alternate direction, namely by considering an environment in which pecuniary externalities arise in the market for deposits, rather than the secondary market. More specifically, I restore the assumption that liquidation values are exogenous, but abandon the island structure assumed in chapter 1 which matched particular banks with particular depositors. Instead, all agents now have access to a common deposit market, which clears competitively. Banks’ failure to internalize their price impact in this market then creates a potential role for policy quite distinct from that explored in chapter 2 since the interest rate on deposits influences the tightness of banks’ financial constraints and, by extension, the severity of any interbank distortions arising at the interim date $t = 1$.

The main motivation for this exercise is that it provides an especially stark example of the importance of ex-ante policy intervention, reinforcing one of the key themes in chapter 2. More specifically, with liquidation values exogenous and the interest rate on deposits locked in at the initial date $t = 0$, the model admits no useful role for ex-post interventions, despite the fact that the economy’s inefficiencies ultimately manifest themselves in the form of (overly severe) interbank distortions at $t = 1$. Instead, to the extent that policy’s needed, it must take the form of an ex-ante intervention. In particular, I show that a leverage limit targeting banks’ deposit-raising at $t = 0$ will suffice to restore constrained efficiency, if appropriately specified (proposition 3.1).
A second motivation is that the contrast between this chapter’s findings and those in chapter 1 offers some insight into the potential dangers associated with competition in the market for deposits, an issue which has attracted attention both from academics and from policymakers. This is because my only point of departure from the framework in chapter 1 is that banks now compete for deposits, and this change in the economic environment suffices to overturn the “no-go” result highlighted in chapter 1. This complements theoretical work by Keeley (1990), Hellmann et al. (2000), Matutes and Vives (2000), Cordella and Yeyati (2002), and Repullo (2004) elucidating potential links between competition in the deposit market and overall financial stability. However, they focus on the possibility that an erosion in franchise values during periods of heightened deposit competition may lead banks to shift risk on to the deposit insurance system while I place my emphasis on a novel channel involving pecuniary externalities and spillovers into the interbank market. My findings in this area also complement empirical work by Demirguc-Kunt and Detragiache (1998, 2002), Beck et al. (2005, 2006), and Craig and Dinger (2013) linking lower competition among banks with greater financial stability in international panels.

This chapter’s findings also admit connections with the debates on the merits of countercyclical capital regulation and regulatory forbearance. The first of these connections arises because the parameter space includes regions in which no policy interventions are needed. This happens when banks’ initial balance-sheet choices at \( t = 0 \) are so liquid and unlevered that financial constraints are sure to stay lax at \( t = 1 \), so the aforementioned pecuniary externalities remain dormant. Since banks are only willing to adopt such conservative balance sheets when expected productivity at \( t = 0 \) is relatively low, the overall need for the leverage limit described above is procyclical (proposition 3.2). Similar arguments show that intervention may not be needed when banks are sufficiently capitalized and, by extension, that shocks to banks’ net worth may force the economy into a part of the parameter space where policy action suddenly becomes necessary (also proposition 3.2). This last finding crucially suggests that regulatory forbearance, when extended to banks that have recently suffered

---

1 On the policy side, I note that this issue historically motivated regulation Q and its analogues throughout the developed world. It also figures in China’s on-going regulation of the interest rate on commercial bank deposits.

2 The literature in this area is too vast to do it much justice here. See Carletti and Hartmann (2002) for a review, and Boyd and de Nicolo (2005) for a countervailing framework in which the link between bank competition and financial stability can be weakened or even reversed. Their insight is that banks may also face competition in the loan market and that the borrowers in this market may exert some unobservable influence on the riskiness of the projects in which funds are ultimately invested. In this case, low competition in the loan market may drive interest rates up to a point where borrowers have an incentive to engage in moral hazard. The overall link between competition and stability then depends on the relative strength of this channel versus the aforementioned deposit-market channel. See Caminal and Matutes (2002) for a model with similar flavour.
large losses, may entail costs distinct from the moral-hazard concerns usually highlighted in
this area.

The remainder of the chapter is organized as follows. In section 3.2 I present the model. In sections 3.3 and 3.4 I then solve the model and explore its implications for policy. Section 3.5 concludes.

## 3.2 Model

### 3.2.1 Changes to the economic environment

The structure of the economy is mostly unchanged relative to the baseline model presented in chapter 1. As explained in my introduction, the only exception is that I now abandon the island structure assumed in chapter 1 which matched particular banks with particular households from whom they could collect deposits. Instead, I grant all banks and households access to a common deposit market. Assuming that this market clears on a competitive basis, the interest rate on deposits is then pinned down by the first-order condition

$$
R \begin{cases} 
\leq 1 & \text{if } D = 0 \\
= \mu'(E^h - D) & \text{if } D \in (0, E^h) \\
\geq \mu'(0) & \text{if } D = E^h.
\end{cases} \tag{3.1}
$$

### 3.2.2 Definitions

In this new environment, an equilibrium can be defined as follows:

**Definition.** An *equilibrium* is an interbank contract

$$
\mathcal{C} := \left[ D, I_0, \{ S_\omega(\theta), I_\omega(\theta), T_{\omega f}(\theta) \Delta T_\omega(\theta) \}_{(\theta, \omega) \in [0,1] \times \{B,G\}} \right]
$$

and an interest rate $R$ on deposits such that the following two conditions obtain:

- the interbank contract maximizes banks’ expected payout,

$$
\sum_{\omega \in \{B,G\}} \alpha_\omega \left[ \int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) - RD \right],
$$

With respect to the case where $D = 0$, recall that households’ outside option satisfies $\mu'(E^h) = 1$ — i.e., households lack a good use to which they can put the last unit of their endowment.
taking $R$ as given, subject to the truth-telling constraint

\[
S_\omega(\theta) - T_{\omega f}(\theta) - RD + \theta [\chi_\omega I_\omega(\theta) - \Delta T_\omega(\theta)] \\
\geq S_\omega(\theta') - T_{\omega f}(\theta') - RD + \theta [\chi_\omega I_\omega(\theta') - \Delta T_\omega(\theta')] ,
\]

\[\forall (\theta, \theta', \omega) \in [0, 1]^2 \times \{B, G\}; \quad \text{(TT)}\]

the limited-pledgability constraint

\[\Delta T_\omega(\theta) \leq (1 - \gamma) \chi_\omega I_\omega(\theta), \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\}; \quad \text{(LP)}\]

and the following feasibility constraints:

\[\begin{align*}
(D, I_0) &\in \mathbb{R}_+ \times [0, E^b + D] \quad \text{(F0)} \\
(E^b + D - I_0) + \ell I_0 &= \int_0^1 [S_\omega(\theta) + (\rho + \ell) I_\omega(\theta)] dF(\theta), \quad \forall \omega \in \{B, G\} \quad \text{(F1a)} \\
S_\omega(\theta) &\geq 0, \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\} \quad \text{(F1b)} \\
0 &= \int_0^1 [T_{\omega f}(\theta) + \theta \Delta T_\omega(\theta)] dF(\theta), \quad \forall \omega \in \{B, G\} \quad \text{(F2a)} \\
S_\omega(\theta) &\geq T_{\omega f}(\theta) + RD, \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\} \quad \text{(F2b)} \\
S_\omega(\theta) + \chi_\omega I_\omega(\theta) &\geq T_{\omega f}(\theta) + \Delta T_\omega(\theta) + RD, \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\} \quad \text{(F2c)} \\
I_\omega(\theta) &\in [0, I_0], \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\} \quad \text{(F1c)} \\
I_\omega(\theta) &\implies \Delta T_\omega(\theta) = 0, \quad \forall (\theta, \omega) \in [0, 1] \times \{B, G\}; \quad \text{(F2d)}
\end{align*}\]

• the deposit market clears — i.e., (3.1) holds, with $D \leq E^h$. 

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Now, when evaluating the economy’s efficiency in equilibrium, the benchmark on which I focus is one under which the social planner internalizes the price-setting process in the deposit market but otherwise faces the same information and pledgability frictions as do banks. More formally:

**Definition.** The *social planner’s problem* involves choosing an interbank contract

\[
\left[ D, I_0, \{S_\omega(\theta), I_\omega(\theta), T_\omega(\theta)\Delta T_\omega(\theta)\}_{(\theta,\omega)\in[0,1] \times \{B,G\}} \right]
\]

and an interest rate \( R \) on deposits so as to maximize utilitarian welfare,

\[
\sum_{\omega \in \{B,G\}} \alpha_\omega \left[ \int_0^1 [S_\omega(\theta) + \theta \chi_\omega I_\omega(\theta)] dF(\theta) + \mu(E^h - D) \right],
\]

subject to the same constraints facing banks in equilibrium, along with (3.1) and a requirement that \( D \leq E^h \).

### 3.2.3 Parametric assumptions

I’ll now close this section with my parametric assumptions, most of which I import from chapter 1:

**Assumption 3.1.** Investment at \( t = 0 \) is profitable — i.e.,

\[
\sum_{\omega \in \{B,G\}} \alpha_\omega \mathbb{E} [\max \{\ell, \theta \chi_\omega - \rho\}] > 1.
\]

**Assumption 3.2.** Continuation at \( t = 1 \) is always profitable for some types — \( \chi_B > \rho + \ell \).

**Assumption 3.3.** The cumulative function \( F \) is strictly log-concave.

**Assumption 3.4.** The bad state is “sufficiently bad”, namely in the sense that the payout \( \chi_B \) satisfies an upper bound given in the appendix. Conversely, the good state is “sufficiently good”, namely in the sense that the payout \( \chi_G \) satisfies a lower bound given in the appendix.

**Assumption 3.5.** The household endowment \( E^h \) is relatively large in comparison with the endowment \( E^b \) received by banks — specifically, \( E^b + E^h - E^h \mu'(0) < 0 \).

Apart from these assumptions, it will also be useful to impose a mild regularity on the household utility function:

**Assumption 3.6.** \( \frac{d^2}{dD^2} [D \mu'(E^h - D)] > 0, \forall D \in (0, E^h].\)
In light of the market-clearing condition for deposits, (3.1), we here recognize $D\mu'(E^h - D)$ as the total interest and principal promised to households at $t = 2$ as a function of the deposit volume $D$ collected at $t = 0$, so assumption 3.6 simply requires that this function be convex. This holds for a wide range of utility functions, including the exponential, isoleastic, and logarithmic cases, along with any cases for which absolute risk aversion is decreasing.\footnote{To see this, note that}

\[
\frac{d^2}{dD^2} \left[ D\mu'(E^h - D) \right] = D\mu'''(E^h - D) - 2\mu''(E^h - D),
\]

so a sufficient condition would be that $\mu'''(\cdot) \geq 0$. That this inequality holds for all of the aforementioned examples is well known.

### 3.3 Solution and optimal policy at $t = 1$

I’ll now begin solving the model and exploring its implications for policy. More specifically, the present section focuses on the economy’s behaviour at $t = 1$, taking banks’ initial balance-sheet choices as given, while section TBC will focus on $t = 0$.

Now, since the interest rate $R$ on deposits is locked in at $t = 0$, lemma 1.3.1 and proposition 1.1 still respectively hold as descriptions of banks’ behaviour when $t = 1$ in general and $\omega = B$ in particular. More specifically:

**Lemma 3.3.1.** Given some state $\omega$ and some initial balance sheet $(D, I_0, R)$, banks’ optimal choice on the state-specific subcontract

\[
C_\omega := \{S_\omega(\theta), I_\omega(\theta), T_\omega f(\theta)\Delta T_\omega(\theta)\}_{\theta \in [0,1]}
\]

can be summarized by a pair $(\theta_\omega, I_\omega) \in [0,1] \times [0, I_0]$, with the usual interpretation that all types in $[0, \theta_\omega)$ liquidate completely, whereas (almost) all others keep operating at scale $I_\omega$. More specifically, banks choose this pair to maximize the state-specific payout

\[
I_\omega \int_{\theta_\omega}^1 (\theta \chi_\omega - \rho - \ell) \, dF(\theta),
\]

subject to the usual physical and financial constraints:

\[
(E_b^b + D - I_0) + \ell I_0 \geq I_\omega \int_{\theta_\omega}^1 (\rho + \ell) \, dF(\theta)
\]

(3.2)
\[(E^b + D - I_0) + \ell I_0 \geq RD - I_0 \Delta_B(\theta_B^\Pi),\]

\[= \Delta_\omega(\theta_\omega) \]

Moreover, if \(\omega = B\), then this program either admits a unique solution or is insoluble and in the former case has the property that the physical constraint is lax. More specifically, the situation is as follows:

- if initial balance sheets exhibit low leverage and high liquidity, namely in the sense that
  \[(E^b + D - I_0) + \ell I_0 \geq RD - I_0 \Delta_B(\theta_B^\Pi),\]
  with \(\theta_B^\Pi := (\rho + \ell_B)/\chi_B\), then banks opt for a liquidity surplus \((r_B = LS)\) — i.e., \((\theta_B, I_B) = \theta_B^\Pi, I_0\);

- if initial balance sheets exhibit moderate leverage and moderate liquidity, namely in the sense that
  \[(E^b + D - I_0) + \ell I_0 \in [RD - I_0 \Delta_B(\theta_B^\Xi), RD - I_0 \Delta_B(\theta_B^\Pi)],\]
  with \(\theta_B^\Xi\) defined in subsection 1.3.2, then banks opt for an extensive distortion \((r_B = ED)\) — i.e., \((\theta_B, I_B) = [\theta_B^{ED}(D, I_0, R), I_0]\), where \(\theta_B^{ED}(D, I_0, R)\) solves
  \[(E^b + D - I_0) + \ell I_0 = RD - I_0 \Delta_B(\theta_B^{ED}(D, I_0, R))]; \quad (3.3)\]

- if initial balance sheets exhibit high leverage and low liquidity, namely in the sense that
  \[(E^b + D - I_0) + \ell I_0 \in [RD, RD - I_0 \Delta_B(\theta_B^\Xi)],\]
  then banks opt for a dual distortion \((r_B = DD)\) — i.e., \((\theta_B, I_B) = [\theta_B^\Xi, I_B^{DD}(D, I_0, R)]\), where \(I_B^{DD}(D, I_0, R)\) solves
  \[(E^b + D - I_0) + \ell I_0 = RD - I_B^{DD}(D, I_0, R) \Delta_B(\theta_B^\Xi)]; \quad (3.4)\]

- finally, if initial balance sheets exhibit very high leverage and very low liquidity, namely in the sense that
  \[(E^b + D - I_0) + \ell I_0 < RD,\]
  then the program above is insoluble, namely because the financial constraint fails under all candidate solutions.
At the same time, the lower bound imposed on the “good” payout $\chi_G$ (assumption $\text{3.4}$) has the usual upshot that the constraint banks have to worry about in the good state is physical, rather than financial, the intuition being that this state is one in which many banks draw fundamentals which justify maintenance, leaving relatively few who must be bribed to admit that their fundamentals warrant liquidation. As a result, banks’ behaviour in the good state still follows the relatively mechanical pattern established in subsection $\text{1.3.3}$.

**Lemma 3.3.2.** If $\omega = G$ and the initial balance sheet $(D, I_0, R)$ is one onto which banks would be willing to select at $t = 0$, then the program described in lemma $\text{3.3.1}$ admits a unique solution, namely under which the financial constraint is lax. More specifically, the situation is as follows:

- if initial balance sheets are relatively liquid, namely in the sense that
  
  $$(E^b + D - I_0) + \ell I_0 \geq I_0 \Psi_G (\theta^\Pi_G),$$

  then banks opt for a liquidity surplus ($r_G = LS$) — i.e., $(\theta_G, I_G) = (\theta^\Pi_G, I_0)$;

- otherwise, banks settle for liquidity rationing ($r_G = LR$) — i.e., $(\theta_G, I_G) = [\theta^LR_G(D, I_0), I_0]$, where $\theta^LR_G(D, I_0)$ solves

  $$(E^b + D - I_0) + \ell I_0 = I_0 \Psi_G[\theta^LR_G(D, I_0)].$$

See figure $\text{3.1}$ for an illustration and subsection $\text{3.A.2}$ in the appendix for details on this figure’s construction.

**Efficiency and optimal policy.** Now, since we’ve returned to the baseline case where the liquidation value $\ell$ is exogenous, we should expect that the planner has no reason to prefer a different allocation at $t = 1$ relative to that described above, taking as given some initial balance $(D, I_0, R)$. Indeed:

**Lemma 3.3.3.** For any initial balance sheet $(D, I_0, R)$ on to which the planner would be willing to select at $t = 0$, the social planner’s optimal choice on the subcontract $C_B (C_G)$ takes the form described in lemma $\text{3.3.1}$ ($\text{3.3.2}$).

As a result, to the extent that the economy admits a role for policy, that role must take the form of an ex-ante intervention at $t = 0$, as discussed in my introduction. The details on this intervention are covered in my next subsection.

---

5 When constructing figure $\text{3.1}$ I’ve used the market-clearing condition for the deposit market, $\text{(3.1)}$, to eliminate the interest rate $R$, so the pair $(D, I_0)$ suffices as a summary of banks’ initial balance-sheet choices.
3.4 Solution and optimal policy at $t = 0$

If we now shift our attention to the initial balance sheet $(D, I_0, R)$ that banks adopt at $t = 0$, it can be shown that several of the insights established in chapter 1 still go through in this new environment, including the general form of the solution on which banks settle, along with the procyclicality in the risk of interbank distortions:

**Lemma 3.4.1.** An equilibrium exists, is unique, and has the property that banks experience liquidity rationing in the good state. As for their behaviour in the bad state, one of four cases must obtain:

- **the first case has a liquidity surplus occurring in the bad state, with banks’ choices on $D$ and $I_0$ respectively pinned down by the first-order conditions**

$$1 - \mu'(E^h - D) + \alpha_G \frac{\Pi_G[\theta^L_R(D, I_0)]}{\Psi_G(\cdot)} = 0, \quad (3.6)$$

and

$$\alpha_G \left[ \ell + \Pi_G(\cdot) - 1 - \left[ 1 - \ell + \Psi_G(\cdot) \right] \frac{\Psi'_G(\cdot)}{\Pi'_G(\cdot)} \right] + \alpha_B \left[ \ell + \Pi_B[\theta^H_B] - 1 \right] = 0, \quad (3.7)$$

where I use $\cdot$ to suppress obvious arguments;

- **the second case has an extensive distortion occurring in the bad state, so banks’ first-order conditions read as**

$$\alpha_G \left[ 1 - \mu'(E^h - D) + \frac{\Pi'_G[\theta^L_R(D, I_0)]}{\Psi'_G(\cdot)} \right]$$

$$+ \alpha_B [1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi'_B[\theta^E_D(D, I_0, \mu'(E^h - D))]}{\Delta_B(\cdot)} \right] = 0,$$

and

$$\alpha_G \left[ \ell + \Pi_G(\cdot) - 1 - \left[ 1 - \ell + \Psi_G(\cdot) \right] \frac{\Psi'_G(\cdot)}{\Pi'_G(\cdot)} \right]$$

$$+ \alpha_B \left[ \ell + \Pi_B(\cdot) - 1 + \left[ 1 - \ell - \Delta_B(\cdot) \right] \frac{\Pi'_B(\cdot)}{\Delta'_B(\cdot)} \right] = 0; \quad (3.8)$$
Figure 3.1: Banks’ behaviour at $t = 1$ as a function of their initial balance-sheet choices
the third case has a dual distortion occurring in the bad state, but no interbank collapse, so banks’ first-order conditions read as
\[
\alpha_G \left[ 1 - \mu'(E^h - D) + \frac{\Pi'_G[\theta'_G(D, I_0)]}{\Psi'_G(\cdot)} \right] + \alpha_B[1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi_B(\theta^e_B)}{\Delta_B(\theta^e_B)} \right] = 0,
\]
and
\[
\alpha_G \left[ \ell + \Pi_G(\cdot) - 1 - \left[ 1 - \mu'(E^h - D) \right] \frac{\Psi'_G(\cdot)}{\Pi'_G(\cdot)} \right] = \alpha_B(1 - \ell) \left[ 1 - \frac{\Pi_B(\theta^e_B)}{\Delta_B(\theta^e_B)} \right]; \quad (3.9)
\]
the final case has an interbank collapse occurring in the bad state, so banks’ initial balance-sheet choices are pinned down by the “no-default” constraint
\[
(E^b + D - I_0) + \ell I_0 = D\mu'(E^h - D) \quad (3.10)
\]
and a first-order condition given in the appendix.

**Lemma 3.4.2.** Fix all parameters save for the payout that successful banks generate in the good state, \(\chi_G\), and let \(\chi_G\) denote the lower bound on this payout at which my parametric assumptions begin to fail. The range of potential values for this payout then admits a partition
\[
\chi_G \leq \chi_G^{LS} \leq \chi_G^{ED} \leq \chi_G^{DD} \leq \infty
\]
with the property that banks behave as follows in equilibrium:

- if \(\chi_G \in (\chi_G^{LS}, \chi_G^{LS}]\), then banks experience a liquidity surplus in the bad state;
- if \(\chi_G \in (\chi_G^{LS}, \chi_G^{ED}]\), then banks experience an extensive distortion in the bad state;
- if \(\chi_G \in (\chi_G^{ED}, \chi_G^{DD}]\), then banks experience a dual distortion in the bad state, but the “no-default” constraint remains lax;
- if \(\chi_G \in (\chi_G^{DD}, \infty)\), then an interbank collapse occurs in the bad state.

Similar results obtain if the parameter being varied is instead the probability on the good state, \(\alpha_G\).

Moreover, derivations similar to those underlying lemma 3.4.2 will show that distortions are more likely to occur when banks are poorly capitalized, as in chapter 1.

**Lemma 3.4.3.** Fix all parameters save for the endowment \(E^b\) received by banks, and let \(\overline{E}\) denote the upper bound on this endowment implied by the requirement that households be deep-pocketed (assumption 3.5). The interval \([0, \overline{E}]\) then admits a partition
\[
0 \leq E^{DD} \leq E^{ED} \leq E^{LS} \leq \overline{E}
\]
with the property that banks behave as follows in equilibrium:

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• if \( E_b \in [E^{LS}, E] \), then banks experience a liquidity surplus in the bad state;
• if \( E_b \in [E^{ED}, E^{LS}) \), then banks experience an extensive distortion in the bad state;
• if \( E_b \in [E^{DD}, E^{ED}) \), then banks experience a dual distortion in the bad state, but the “no-default” constraint remains lax;
• if \( E_b \in (0, E^{DD}) \), then an interbank collapse occurs in the bad state.

Efficiency and optimal policy. As for the planner’s solution, we should expect that his internalization of the market-clearing condition for deposits gives rise to wedges whenever the financial constraint associated with the bad state binds, and also that these wedges make the planner’s preferred initial balance sheet more “conservative” in some sense. Indeed:

**Lemma 3.4.4.** The planner’s solution admits a unique solution, and this solution has the property that liquidity rationing occurs in the good state. As for the bad state, one of four cases must obtain:

• the first case has a liquidity surplus occurring in the bad state \( (r^{SP}_B = LS) \), with planner’ choices on \( D \) and \( I_0 \) respectively pinned down by the first-order conditions on \( D \) and \( I_0 \) respectively pinned down by \((3.6)\) and \((3.7)\):

\[
\alpha_G \left[ 1 - \mu'(E^h - D) \right] + \frac{\Pi'_G [\theta^L_G(D, I_0)]}{\Psi'_G [\theta^R_G(D, I_0)]} + \alpha_B [1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi'_B [\theta^{ED}_B(D, I_0, \mu'(E^h - D))]}{\Delta'_B [\theta^{ED}_B(D, I_0, \mu'(E^h - D))]} \right] - \alpha_B D \mu''(E^h - D) \frac{\Pi'_B [\theta^{ED}_B(D, I_0, \mu'(E^h - D))]}{\Delta'_B [\theta^{ED}_B(D, I_0, \mu'(E^h - D))]} > 0
\]

while the choice on \( I_0 \) is pinned down by \((3.8)\);
• the third case has a dual distortion occurring in the bad state \( \tau_B^{SP} = DD \), but no interbank collapse, with the planner’s choice on \( D \) pinned down by the first-order condition

\[
\alpha_G \left[ 1 - \mu'(E^h - D) + \frac{\Psi_G'(\theta^L_B(D, I_0))}{\Psi_G'(\theta^L_B(D, I_0))} \right] + \alpha_B[1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi_B(\theta^L_B)}{\Delta_B(\theta^L_B)} \right] - \alpha_B D \mu''(E^h - D) \frac{\Pi_B(\theta^L_B)}{\Delta_B(\theta^L_B)} > 0
\]

while the choice on \( I_0 \) is pinned down by (3.9);

• the final case has an interbank collapse occurring in the bad state, so the planner’s initial balance-sheet choices are pinned down by the “no-default” constraint (3.10), along with a first-order condition given in the appendix.

**Lemma 3.4.5.** The planner’s solution has the property that one of the distorted regimes obtains in the bad state only if this is also true in equilibrium. It also has the property that a dual distortion (interbank collapse) occurs in the bad state only if this is also true in equilibrium.

Another way to formalize the notion that the planner prefers a more “conservative” initial balance sheet is as follows:

**Lemma 3.4.6.** Fix all parameters save for the “good” payout \( \chi_G \). The range of potential values for this parameter admits a partition \( \chi_G \leq \chi^L_{SP} \leq \chi^E_{SP} \leq \chi^D_{SP} \leq \infty \) with the property that the planner’s solution connects the bad state with a liquidity surplus (extensive distortion, dual distortion, interbank collapse) when \( \chi_G \in [\chi^L_{SP}, \chi^E_{SP}], \chi^D_{SP} \in (\chi^L_{SP}, \chi^E_{SP}], \chi^D_{SP} \in (\chi^E_{SP}, \chi^D_{SP}], \chi^D_{SP} \in (\chi^D_{SP}, \infty)] \), and this partition satisfies \( \chi^x_{SP} \geq \chi^x_G \forall x \in \{LS, ED, DD\} \). Similar results obtain if the parameter being varied is instead the probability on the good state, \( \alpha_G \).

**Lemma 3.4.7.** If the parameter being varied is instead the endowment \( E^b \) received by banks, then the range of potential values for this parameter admits a partition \( 0 \leq E^{DD}_{SP} \leq E^{ED}_{SP} \leq E^{LS}_{SP} \leq E \) with the property that the planner’s solution connects the bad state with a liquidity surplus (extensive distortion, dual distortion, interbank collapse) when \( E^b \in [E^{LS}_{SP}, E] [E^b \in [E^{ED}_{SP}, E^{LS}_{SP}], E^b \in [E^{DD}_{SP}, E^{ED}_{SP}], E^b \in (0, E^{DD}_{SP}) \), and this partition satisfies \( E^x_{SP} \leq E^x \forall x \in \{LS, ED, DD\} \).

See figure 3.2 for an illustration of these last two lemmata.
Figure 3.2: Illustration of lemmata 3.4.6 and 3.4.7

NB: $\alpha_G$ denotes the lower bound on $\alpha_G$ implied by my parametric assumptions.
These last few lemmata have several important implications for policy. First of all, the wedges identified in lemma \[3.4.4\] suggest that interventions, if needed, should target banks’ deposit-raising at \( t = 0 \). As in chapter \[2\] a natural candidate would be leverage limit of the form

\[
\frac{D}{E^b} \leq \overline{d}.
\]  

(3.11)

Indeed:

**Proposition 3.1.** If policymakers are able to impose the aforementioned leverage limit at \( t = 0 \), then \( \overline{d} \) can be chosen to implement the planner’s solution as an equilibrium, with no additional intervention at \( t = 0 \) or \( t = 1 \).

That an ex-ante intervention in the deposit market thus suffices to restore constrained efficiency, despite the fact that the economy’s inefficiencies ultimately manifest themselves ex-post in the interbank market, provides an especially emphatic example of the importance of ex-ante intervention and more generally highlights the potential for spillovers between these two markets.

That said, lemmata \[3.4.1\] through \[3.4.3\] imply that the parameter space includes regions in which the aforementioned intervention may not be necessary. In particular, if the bad state ends up being associated with a liquidity surplus in equilibrium, then banks’ financial constraints would stay lax in both states, leaving dormant the pecuniary externality on deposits and thus eliminating the need for policy:

**Proposition 3.2.** If parameters are such that the first case in lemma \[3.4.1\] obtains — e.g., \( \chi_G \leq \chi_G^{LS} \), \( E^b \geq E^{LS} \) — then the allocation obtaining in equilibrium also solves the social planner’s solution, and the leverage limit on line \[3.11\] is lax.

Since this means that intervention is only needed when expected productivity at \( t = 0 \) is relatively high (e.g., \( \chi_G > \chi_G^{LS} \)), the model provides a potential rationale for countercyclical capital regulation of the form now being implemented under Basel III (see, e.g., Basel Committee on Banking Supervision \[2010\]). At the same time, proposition \[3.2\] also implies that intervention may not be needed when banks are well capitalized and thus that shocks to banks’ net worth may force the economy into a part of the parameter space where intervention is suddenly needed. As mentioned in my introduction, this is especially important because regulators would likely face some temptation to extend forbearance following shocks of this sort.
3.5 Conclusion

I’ll now close with a brief description of this chapter’s findings. In this chapter, I considered an alternate extension of the baseline model from chapter \[1\], namely in which pecuniary externalities arise in the market for deposits, rather than the secondary market. I then characterized the role for policy to which this externality gives rise. In particular, I showed that a leverage limit imposed at the initial date \(t = 0\) can be used to implement the planner’s solution, without any subsequent ex-post intervention at the interim date \(t = 1\). This provides an especially stark example of the importance of ex-ante prudential policies, reinforcing a key theme from chapter \[2\]. However, I also showed that this policy may be unnecessary if expected productivity at \(t = 0\) is relatively low, making the overall need for intervention procyclical. In this way, the model provides a potential rationale for countercyclical capital regulation. It also provides some insight into the potential dangers associated with competition in the market for deposits, since its policy implications differ dramatically from those of the baseline model in which banks faced no competition for deposits.

In terms of future work, the most natural avenue for extending the work done in this dissertation would be quantitative. More specifically, it would be useful to get a precise sense of the size and cyclical of the optimal policies emerging from my analysis, along with the welfare gains associated with their implementation. I am currently working on embedding this framework inside a larger macro model which can then be calibrated to precisely this end.
3.A Appendix

3.A.1 Proof of lemma 3.3.1

I begin by importing the following subassumptions from chapter 1:

Subassumption 3.4.1. \( \Delta_B \left( \theta^\Delta_B \right) < 0. \)

Subassumption 3.4.2. \( \Theta^\Xi_B > \theta^\Psi_B + \Delta \), where \( \theta^\Psi_B + \Delta \) is defined in subsection 1.A.5.5.

Subassumption 3.4.3. \( \ell + \Pi_B(\theta^\Pi_B) < 1. \)

That these conditions are more likely to hold the lesser is \( \chi_B \) can easily be verified. Moreover, a repeat of the arguments made in subsection 1.A.3 and the first part of subsection 1.A.5.5 will show that they together suffice to ensure that banks find the physical constraint lax in the bad state and obey the usual pecking order viz-á-viz the financial constraint. All of the lemma’s claims then follow immediately.

3.A.2 Proof of lemma 3.3.2 and construction of figure 3.1

I begin by importing the following subassumptions from chapter 1:

Subassumption 3.4.4. \( \min\{ \Delta_G(\theta^\xi_G), \rho - (\rho + \ell) F(\theta^\xi_G) \} > 0 \), where \( \theta^\xi_G \) is defined in subsection 1.A.5.5.

Subassumption 3.4.5. \( \Delta_B(\theta^\Pi_B) > (-1) \Psi_G(\theta^\Pi_G). \)

That these conditions are more likely to hold the greater is \( \chi_G \) can easily be verified. Moreover, a repeat of the arguments in the last part of subsection 1.A.5.5 will show that subassumption 3.4.4 ensures that the financial constraint in the good state is lax for any initial balance sheet onto which banks would be willing to select at \( t = 0 \), in which case the claims made in lemma 3.3.2. As for figure 3.1, its only potentially ambiguous property is the relative placement of the loci

\[ \{(D, I_0) \in \mathbb{R}^2_+ \text{ s.t. } (E^b + D - I_0) + \ell I_0 = I_0 \Psi_G(\theta^\Pi_G)\} \]

and

\[ \{(D, I_0) \in \mathbb{R}^2_+ \text{ s.t. } (E^b + D - I_0) + \ell I_0 + I_0 \Delta_B(\theta^\Xi_B) = D \rho'(E^h - D)\}. \]
specifically w.r.t. to their vertical intercepts. Fortunately, a repeat of the arguments in subsection 1.6 will confirm that the latter admits a higher intercept than the former, as shown in figure 3.1, namely due to subassumption 3.4.5.

3.A.3 Proof of lemma 3.3.3

This should be obvious.

3.A.4 Proof of lemmata 3.4.1 through 3.4.3

3.A.4.1 Some preliminary results

I begin by deriving banks’ marginal returns on \( D \) and \( I_0 \), which will require that we take cases on regimes and states:

**Case one: \( r_G = LR \).** Under this case, banks’ expected payout is given by

\[
(E^b + D - I_0) + \ell I_0 + I_0 \Pi_G[\theta^{LR}_G(D, I_0)] - RD =: v^{LR}_G(D, I_0, R),
\]

with

\[
(3.5) \implies \theta^{LR}_{GD}(D, I_0) = \frac{-1}{I_0 \Psi_{G}'[\theta^{LR}_G(D, I_0)](\rho + \ell)} < 0,
\]

so

\[
v^{LR}_{GD}(D, I_0, R) = 1 - R + I_0 \theta^{LR}_{GD}(D, I_0) \Pi_{G}'[\theta^{LR}_G(D, I_0)] = 1 - R + \frac{\Pi_G[\theta^{LR}_G(D, I_0)]}{\Psi_{G}'[\theta^{LR}_G(D, I_0)]}.
\]

Similarly,

\[
(3.5) \implies \theta^{LR}_{GI}(D, I_0) = \frac{(-1)[1 - \ell + \Psi_{G}[\theta^{LR}_G(D, I_0)]]}{I_0 \Psi_{G}'[\theta^{LR}_G(D, I_0)]}
\]

\[
= \frac{1 - \ell + \Psi_{G}[\theta^{LR}_G(D, I_0)]}{I_0 \Psi_{G}'[\theta^{LR}_G(D, I_0)](\rho + \ell)} > 0,
\]

so

\[
v^{LR}_{GI}(D, I_0, R) = \ell + \Pi_G[\theta^{LR}_G(D, I_0)] - 1 + I_0 \theta^{LR}_{GI}(D, I_0) \Psi_{G}'[\theta^{LR}_G(D, I_0)]
\]

\[
= \ell + \Pi_G[\theta^{LR}_G(D, I_0)] - 1 - [1 - \ell + \Psi_{G}[\theta^{LR}_G(D, I_0)]] \times \cdots
\]
\[
\ldots \times \frac{\Pi'_G[\theta^{LR}_{G}(D,I_0)]}{\Psi'_G[\theta^{LR}_{G}(D,I_0)]},
\]

(3.16)

Case two: \(r_B = ED\). Under this case, banks’ expected payout is given by

\[
(E^b + D - I_0) + \ell I_0 + I_0 \Pi_B[\theta^{ED}_{B}(D,I_0,R)] - RD =: v^{ED}_{B}(D,I_0,R),
\]

(3.17)

with

\[
\implies \theta^{ED}_{BD}(D,I_0,R) = (R - 1)/I_0 \Delta'_{BD}[\theta^{ED}_{B}(D,I_0,R)],
\]

(3.18)

so

\[
v^{ED}_{BD}(D,I_0,R) = 1 - R + I_0 \theta^{ED}_{BD}(D,I_0,R) \Pi'_B[\theta^{ED}_{B}(D,I_0,R)]
\]

\[
= (1 - R)[1 - \frac{\Pi'_B[\theta^{ED}_{B}(D,I_0,R)]}{\Delta_B[\theta^{ED}_{B}(D,I_0,R)]}],
\]

(3.19)

Similarly,

\[
\implies \theta^{ED}_{BI}(D,I_0,R) = [1 - \ell - \Delta_B[\theta^{ED}_{B}(D,I_0,R)]]/I_0 \Delta'_{BD}[\theta^{ED}_{B}(D,I_0,R)]
\]

\[
< 0,
\]

(3.20)

so

\[
v^{ED}_{BI}(D,I_0,R) = \ell + \Pi_B[\theta^{ED}_{B}(D,I_0,R)] - 1 + I_0 \theta^{ED}_{BI}(D,I_0,R) \Pi'_B[\theta^{ED}_{B}(D,I_0,R)]
\]

\[
= \ell + \Pi_B[\theta^{ED}_{B}(D,I_0,R)] - 1 + [1 - \ell - \Delta_B[\theta^{ED}_{B}(D,I_0,R)]] \times \ldots
\]

\[
\ldots \times \frac{\Pi'_B[\theta^{ED}_{B}(D,I_0,R)]}{\Delta_B[\theta^{ED}_{B}(D,I_0,R)]}
\]

(3.21)

Case three: \(r_B = DD\). Under this case, banks’ expected payout is given by

\[
(E^b + D - I_0) + \ell I_0 + I^{DD}_{B}(D,I_0,R) \Pi_B(\theta^{ED}_{B}) - RD =: v^{DD}_{B}(D,I_0,R),
\]

(3.22)
with
\[(3.4) \implies I_{BD}^{DD}(D, I_0, R) = (R - 1)/\Delta_B(\theta_B^\Xi),\]
so
\[
v_{BD}^{DD}(D, I_0, R) = 1 - R + (I_{BD}^{DD})_D(D, I_0, R)\Pi_B(\theta_B^\Xi) = (1 - R) \left[ 1 - \frac{\Pi_B(\theta_B^\Xi)}{\Delta_B(\theta_B^\Xi)} \right] \leq 0. \tag{3.23}
\]
Similarly,
\[
(3.4) \implies I_{BI}^{DD}(D, I_0, R) = (1 - \ell)/\Delta_B(\theta_B^\Xi),
\]
so
\[
v_{BI}^{DD}(D, I_0, R) = \ell - 1 + I_{BI}^{DD}(D, I_0, R)\Pi_B(\theta_B^\Xi) = (\ell - 1) \left[ 1 - \frac{\Pi_B(\theta_B^\Xi)}{\Delta_B(\theta_B^\Xi)} \right]. \tag{3.24}
\]

Case four: \(r_\omega = LS \ (\omega \in \{B, G\})\). Under this case, banks’ expected payout is given by
\[
(E^b + D - I_0) + \ell I_0 + I_0\Pi_B(\theta_B^\Pi) - RD =: v_{\omega}^{LS}(D, I_0, R), \tag{3.25}
\]
so
\[
v_{\omega D}^{LS}(D, I_0, R) = 1 - R. \tag{3.26}
\]
Similarly,
\[
v_{\omega I}^{LS}(D, I_0) = \ell + \Pi_B(\theta_\omega^\Pi) - 1. \tag{3.27}
\]

3.A.4.2 Notation

I’ll now introduce some notation:

- \(\forall I_0 \in [0, E^b/[1 - \ell - \Delta_B(\theta_B^\Pi)]]\), let \(D_B^\Pi(I_0) \in \mathbb{R}_+\) denote the level of deposits at which the financial constraint associated with the bad state begins to bind after taking the market-clearing condition (3.1) into account — i.e.,
\[
[E^b + D_B^\Pi(I_0) - I_0] + \ell I_0 + I_0\Delta_B(\theta_B^\Pi) = D_B^\Pi(I_0)\mu'[E^b - D_B^\Pi(I_0)];
\]

- \(\forall I_0 \in [0, E^b/[1 - \ell - \Delta_B(\theta_B^\Xi)]]\), let \(D_B^\Xi(I_0) \in \mathbb{R}_+\) denote the level of deposits at which banks transition from the extensive distortion regime to the dual distortion regime after taking (3.1) into account — i.e.,
\[
[E^b + D_B^\Xi(I_0) - I_0] + \ell I_0 + I_0\Delta_B(\theta_B^\Xi) = D_B^\Xi(I_0)\mu'[E^b - D_B^\Xi(I_0)];
\]
• $\forall I_0 \in [0, E_b/(1-\ell)]$, let $\overline{D}_B(I_0) \in \mathbb{R}_+$ denote the level of deposits at which the “no-default” constraint binds after taking (3.1) into account — i.e.,

$$[E_b + \overline{D}_B(I_0) - I_0] + \ell I_0 = \overline{D}_B(I_0) \mu'[E^h - \overline{D}_B(I_0)];$$

• $\forall I_0 \in \mathbb{R}$, let $D^\Pi_G(I_0)$ denote the level of deposits at which the physical state associated with the good state binds after taking (3.1) into account — i.e.,

$$[E_b + D^\Pi_G(I_0) - I_0] + \ell I_0 = I_0 \Psi_G(\theta^\Pi_G);$$

• finally, $\forall (D, I_0) \in \mathbb{R}^2_+$ satisfying $(E^b + D - I_0) + \ell I_0 \geq D \mu'(E^h - D)$, let $v_D(D, I_0)$ give banks’ expected return on the marginal deposit, computed on an unconditional basis at $t = 0$ after taking account of (3.1) and the analysis in section 3.3 — e.g.,

$$D \in \min\{D^\Pi_G(I_0), D^\Pi_B(I_0)\}$$

$$\implies v_D(D, I_0) = \alpha_G v^LR_G[D, I_0, \mu'(E^h - D)] + \alpha_B v^LS_B[D, I_0, \mu'(E^h - D)].$$

Define $v_I(D, I_0)$ analogously.

**Remark 1.** It can easily be verified that the marginal return functions $v_D(D, I_0)$ and $v_I(D, I_0)$ are continuous in both their arguments, even around the boundaries separating regimes. This is a consequence of the envelope condition.

**Remark 2.** For clarity, figure 3.3 illustrates some of the notation used in this subsection.

### 3.A.4.3 Some more preliminary results

**Sublemma 3.A.1.** The function $v_D(D, I_0)$ is strictly decreasing in its first argument.

**Proof.** I’ll again take cases on states and regimes.

**Case one: $r_B = ED$.** Under this case, we know that

$$v^{ED}_{BD}[D, I_0, \mu'(E^h - D)] = [1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi'_B[D, I_0, \mu'(E^h - D)]}{\Delta'_B[D, I_0, \mu'(E^h - D)]} \right],$$

$$=:(W'_B/\Delta'_B)[\theta^ED_B[D, I_0, \mu'(E^h - D)] < 0$$
\[ \frac{d}{dD} \left[ \theta^E_B[D, I_0, \mu'(E^h - D)] \right] < 0, \]

and

\[ (\Pi'_B / \Delta'_B)'[\theta^E_B[D, I_0, \mu'(E^h - D)]] > 0, \]

namely due to (3.19), (3.52), and (1.15), respectively. So,

\[ \frac{d}{dD} \left[ v^E_B[D, I_0, \mu'(E^h - D)] \right] = \mu''(E^h - D) \left[ 1 - (\Pi'_B / \Delta'_B)[\theta^E_B[D, I_0, \mu'(E^h - D)]] \right] \]

\[ - [1 - \mu'(E^h - D)] \frac{d}{dD} \left[ \theta^E_B[D, I_0, \mu'(E^h - D)] \right] \times \cdots \]

\[ \cdots \times (\Pi'_B / \Delta'_B)'[\theta^E_B[D, I_0, \mu'(E^h - D)]] \]

\[ < 0. \]

Case two: \( r_B = DD \). Under this case, we know that

\[ v^{DD}_{BB}[D, I_0, \mu'(E^h - D)] = [1 - \mu'(E^h - D)] \left[ 1 - \frac{\Pi_B(\theta^E_B)}{\Delta_B(\theta^E_B)} \right] \]

\[ =: (\Pi_B / \Delta_B)(\theta^E_B < 0) \]

namely due to (3.23), so

\[ \frac{d}{dD} \left[ v^{DD}_{BB}[D, I_0, \mu'(E^h - D)] \right] = \mu''(E^h - D) [1 - (\Pi_B / \Delta_B)(\theta^E_B)] < 0. \]

Case three: \( r_G = LR \). Under this case, we know that

\[ v^{LR}_{GD}[D, I_0, \mu'(E^h - D)] = 1 - \mu'(E^h - D) + \frac{\Pi'_G[\theta^LR_G(D, I_0)]}{\Psi'_G[\theta^LR_G(D, I_0)]}, \]

\[ =: (\Pi'_G / \Psi'_G)[\theta^LR_G(D, I_0)] \]

\[ \theta^LR_{GD}(D, I_0) < 0, \]

and

\[ (\Pi'_G / \Psi'_G)'[\theta^LR_G(D, I_0)] > 0, \]

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namely due to (3.14), (2.86), and (1.17), respectively. So,

\[
\frac{d}{dD}\left[v_{GD}^{LR}[D, I_0, \mu'(E^h - D)]\right] = \mu''(E^h - D) + (\theta_G^L)'(D, I_0)(\Pi_G'/\Psi_G')[\theta_G^L(D, I_0)]
\]

\[
< 0.
\]

Case four: \( r_\omega = LS \), with \( \omega \in \{B, G\} \). Under this case, we know that

\[
\frac{d}{dD}\left[v_{BD}^{LS}[D, I_0, \mu'(E^h - D)]\right] = \mu''(E^h - D) < 0,
\]

namely due to (3.26).

**Sublemma 3.A.2.** The composition \( v_D[D_B(I_0), I_0] \) exhibits single-crossing from below over the interval \([0, E_b/(1 - \ell)]\), namely at some interior point \( I_B \) satisfying \( D_B(I_B) < D^H_G(I_B) \).

**Proof.** It should be clear that the function \( D_B(I_0) \) intersects the function \( D^H_G(I_0) \) once from above over the interval \([0, E_b/(1 - \ell)]\), namely at some interior point which I’ll denote \( \tilde{I} \). Moreover, \( \forall I_0 \in [0, \tilde{I}] \), it should also be clear that

\[
v_D[D_B(I_0), I_0]
\]

\[
= \alpha_G v_{GD}^{LS}[D_B(I_0), I_0, \mu'(E^h - D_B(I_0))] + \alpha_B v_{BD}^{DD}[D_B(I_0), I_0, \mu'(E^h - D_B(I_0))]
\]

\[
= \alpha_G [1 - \mu'[E^h - D_B(I_0)]] + \alpha_B \left[ 1 - \mu'[E^h - D_B(I_0)] \right] \left[ 1 - (\Pi_B/\Delta_B)(\theta_B^L) \right]
\]

\[
< 0,
\]

whereas

\[
v_D[D_B[E^b/(1 - \ell)], E^b/(1 - \ell)]
\]

\[
= v_D[0, E^b/(1 - \ell)]
\]

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\[ \alpha_G v_{GD}^{LR}[0, E^b/(1 - \ell), \mu'(E^h)] + \alpha_B v_{BD}^{DD}[0, E^b/(1 - \ell), \mu'(E^h)] \]
\[ = \alpha_G \left[ 1 - \mu'(E^h) + (\Pi'_G/\Psi'_G)[\theta_G^{LR}[0, E^b/(1 - \ell)]] \right] \]
\[ + \alpha_B \left[ 1 - \mu'(E^h) \right][1 - (\Pi_B/\Delta_B)(\theta_B^s)] \]
\[ = \alpha_G (\Pi'_G/\Psi'_G)[\theta_G^{LR}[0, E^b/(1 - \ell)]] \]
\[ > 0, \]

so it would suffice if we could show that the composition
\[ v_{D}[\overline{D}_B(I_0), I_0] \]
\[ = \alpha_G v_{GD}^{LR}[\overline{D}_B(I_0), I_0, \mu'[E^h - \overline{D}_B(I_0)]] + \alpha_B v_{BD}^{DD}[\overline{D}_B(I_0), I_0, \mu'[E^h - \overline{D}_B(I_0)]] \]
is strictly increasing over the interval \((\tilde{I}, E^b/(1 - \ell))\). Fortunately, my previous sublemma, combined with the fact that the function \(\overline{D}_B(\cdot)\) is strictly decreasing, implies that it would finally suffice to note that
\[ v_{GD}^{LR}[\overline{D}_B(I_0), I_0, \mu'[E^h - \overline{D}_B(I_0)]] = 0, \]
with
\[ v_{BD}^{DD}[\overline{D}_B(I_0), I_0, \mu'[E^h - \overline{D}_B(I_0)]] = 0, \]
where the last equality follows from (3.23).

**Sublemma 3.A.3.** \(\forall x \in \{\Pi, \Xi\}, \) the composition \(v_D[D^\pi_B(I_0), I_0]\) exhibits single-crossing from below over the interval \([0, E^b/[1 - \ell - \Delta_B(\theta_B^s)]\)], namely at some interior point \(I^*_B\) satisfying \(D^\pi_B(I^*_B) < D^\Pi_G(I^*_B)\).

**Proof.** This can be verified using essentially the same argument as for my previous sublemma.
3.A.4.4 Existence and uniqueness

I’ll now start searching for equilibria. As a first step in this direction, I note that assumption 3.1 ensures that the non-negativity constraint $I_0 \geq 0$ must be lax for banks. Similarly, the fact that $\mu'(E^h) = 1$ ensures some slack in the constraint $D \geq 0$. Moreover, a repeat of the argument made in the last part of subsection 1.A.5.5 will show that the non-negativity constraint for storage, $I_0 \leq E^h + D$, is also lax, namely due to subassumption 3.4.4. As a result, equilibria must fall under one of two cases. The first would be an interior case under which banks find that the “no-default” constraint

$$(E^h + D - I_0) + \ell I_0 \geq D \mu'(E^h - D)$$

is lax, and thus settle on initial balance sheets satisfying

$$v_x(D, I_0) = 0, \ \forall x \in \{D, I_0\}.$$ 

The alternative would be a corner case under which the aforementioned “no-default” constraint binds.

Now, irrespective as to which of these two cases obtains, I claim that equilibria must satisfy $D < D^\Pi_B(I_0)$. To see this, first note that all initial balance sheets satisfying $D \in (\max\{D_B^\Pi(I_0), D_G^\Pi(I_0)\}, D_B(I_0))$ also satisfy

$$v_D(D, I_0) = \alpha_G v_G^L[D, I_0, \mu'(E^h - D)] + \alpha_B v_B^D[D, I_0, \mu'(E^h - D)]$$

$$\overset{\text{see (3.26)}}{=} \alpha_G \left[1 - \mu'(E^h - D)\right] + \alpha_B \left[1 - \mu'(E^h - D)\right] \left[1 - \left(\Pi_B/\Delta_B\right)(\theta_B^2)\right]$$

$$< 0.$$ 

Similarly, if $D \in (\max\{D_B^\Pi(I_0), D_G^\Pi(I_0)\}, D_B^\Xi(I_0))$, then

$$v_D(D, I_0)$$

$$\overset{\text{see (3.26)}}{=} \alpha_G v_G^L[D, I_0, \mu'(E^h - D)] + \alpha_B v_B^E[D, I_0, \mu'(E^h - D)]$$

$$\overset{\text{see (3.26)}}{=} \alpha_G \left[1 - \mu'(E^h - D)\right] + \alpha_B \left[1 - \mu'(E^h - D)\right] \left[1 - \left(\Pi_B/\Delta_B\right)(\theta_B^2)\right]$$

$$< 0.$$ 

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\[= \alpha_G[1 - \mu'(E^h - D)] + \alpha_B \left[ 1 - \mu'(E^h - D) \right] \left[ 1 - \left( \Pi_B' / \Delta_B' \right) \theta_B^{ED} [D, I_0, \mu'(E^h - D)] \right] < 0. \]

Finally, if \( D \in [\max\{0, D^H_G(I_0)\}, D^H_B(I_0)] \), then

\[
v_I(D, I_0) = \alpha_G v_G^{LS} [D, I_0, \mu'(E^h - D)] + \alpha_B v_B^{LS} [D, I_0, \mu'(E^h - D)] \\
= \ell + \mathbb{E} [\Pi_{\omega}(\theta^H)] - 1 \tag{3.28}
\]

\[
> 0, \tag{3.29}
\]

where (3.28) and (3.29) respectively follow from (3.27) and assumption 3.1. Conclude that banks have a strict incentive to adjust their choice on \( D \) or \( I_0 \) whenever \( D \geq D^H_G(I_0) \).

So, we can now restrict attention to candidate equilibria satisfying \( D < D^H_G(I_0) \) and, by extension, \( I_0 \in (E^b/[1 - \ell - \Delta_B(\theta^H)], E^b/(1 - \ell)] \). Now, given any choice on \( I_0 \) in this range, sublemma 3.A.1 implies that one of two cases must obtain. The first would be that the marginal return \( v_D(D, I_0) \), viewed as a function of the choice on \( D \), exhibits single-crossing from above over the interval \([0, \min\{D^H_G(I_0), \bar{D}_B(I_0)\}]\), namely at a point which I’ll denote \( D^*(I_0) \). Otherwise, it must be the case that the marginal return \( v_D(D, I_0) \) is strictly positive over all of the aforementioned interval, in which case I adopt a convention that \( D^*(I_0) = \min\{D^H_G(I_0), \bar{D}_B(I_0)\} \). Moreover, sublemma 3.A.2 gives us a clean way to separate these two cases. In particular, it implies that \( D^*(I_0) = \bar{D}_B(I_0) \) whenever \( I_0 \in [\bar{T}_B, E^b/(1 - \ell)] \), and otherwise \( D^*(I_0) < \min\{D^H_G(I_0), \bar{D}_B(I_0)\} \).

With \( D^*(I_0) \) defined in this way, we can now further restrict attention to candidate equilibria satisfying \( D = D^*(I_0) \). Given any such candidate, I’ll now check if banks have an incentive to deviate in their initial balance-sheet choices, namely by adjusting their choice on \( I_0 \). Now, if \( I_0 < \bar{T}_B \), then the candidate in question has the property that the “no-default” constraint is lax, and banks’ return from a marginal increase in \( I_0 \) is given by \( v_I[D^*(I_0), I_0] \). Otherwise, the “no-default” constraint binds, and increases in \( I_0 \) must be offset by decreases
in $D$, so the relevant return reads as

$$v_I[D^*(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'(E^h - D^*(I_0))} \right] v_D[D^*(I_0), I_0],$$

where the starred term gives the rate of transformation along the “no-default” constraint for banks taking the interest rate on deposits as given. So, $\forall I_0 \in [E^b/[1 - \ell + \Psi_G(\theta^G)], E^b/(1 - \ell)]$, I define a function

$$h(I_0) := \begin{cases} v_I[D^*(I_0), I_0] & \text{if } I_0 \leq \overline{I}_B \\ v_I[D^*(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'(E^h - D^*(I_0))} \right] v_D[D^*(I_0), I_0] & \text{if } I_0 > \overline{I}_B. \end{cases}$$

Now, it should be clear that this function (i) is continuous; (ii) satisfies

$$h[E^b/(1 - \ell + \Psi_G(\theta^G))] > 0,$$

namely due to assumption 3.1 and (iii) satisfies

$$\lim_{I_0 \to E^b/(1 - \ell)} \{h(I_0)\} = -\infty,$$

namely due to the fact that $\mu'(E^h) = 1$. I furthermore claim that it’s strictly decreasing, though the proof requires that we take cases:

**Case one.** Suppose first that $I_0$ has the property that the pair $[D^*(I_0), I_0]$ lies in the region where $r_B = LS$. In this case, we want to confirm that

$$h'(I_0) = v_{II}[D^*(I_0), I_0] + (D^*)'(I_0)v_{ID}[D^*(I_0), I_0]$$

$$= v_{II}[D^*(I_0), I_0] - \frac{v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0]}{v_{DD}[D^*(I_0), I_0]} < 0,$$

or equivalently

$$v_{II}[D^*(I_0), I_0]v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0.$$
Now, in the good state, we know that

\[
v_{GI}^{LR}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] = \ell + \Pi_G[\theta_G^{LR}[D^*(I_0), I_0]] - 1
\
- \left[1 - \ell + \Psi_G[\theta_G^{LR}[D^*(I_0), I_0]]\right] \times \cdots
\
\cdots \times (\Pi_G'/\Psi_G')[\theta_G^{LR}[D^*(I_0), I_0]],
\]

\[
v_{GD}^{LR}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] = 1 - \mu'[E^h - D^*(I_0)] + (\Pi_G'/\Psi_G')[\theta_G^{LR}[D^*(I_0), I_0]],
\]

\[
\theta_{GI}^{LR}[D^*(I_0), I_0] > 0 > \theta_{GD}^{LR}[D^*(I_0), I_0],
\]

and

\[
(\Pi_G'/\Psi_G')[\theta_G^{LR}[D^*(I_0), I_0]] > 0,
\]

namely due to (3.16), (3.14), (3.15), (3.13), and (1.17), respectively. So,

\[
v_{GI}^{LR}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] = (-1)[1 - \ell + \Psi_G[\theta_G^{LR}[D^*(I_0), I_0]]] \times \cdots
\
\cdots \times (\theta_{GI}^{LR}[D^*(I_0), I_0] (\Pi_G'/\Psi_G') [\theta_G^{LR}[D^*(I_0), I_0]])
\]

\[
< 0, \tag{3.30}
\]

\[
\frac{d}{dD} \left[ v_{GI}^{LR}[D, I_0, \mu'(E^h - D)] \right] \bigg|_{D=D^*(I_0)}
\]

\[
= v_{GID}^{LR}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]
\]

\[
= (-1)[1 - \ell + \Psi_G[\theta_G^{LR}[D^*(I_0), I_0]]] \times \cdots
\]

\[
\cdots \times (\theta_{G}^{LR})_D[D^*(I_0), I_0] (\Pi_G'/\Psi_G') [\theta_G^{LR}[D^*(I_0), I_0]]
\]

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\[
\frac{d}{dD} \left[ v^{LR}_{GD} (D, I_0, \mu' (E^h - D)) \right]_{D=D^*(I_0)} = \mu'' [E^h - D^*(I_0)] \\
\quad + (\theta^{LR}_{G})_D [D^*(I_0), I_0] (\Pi'_G / \Psi'_G)' [\theta^{LR}_{G} [D^*(I_0), I_0]] < 0,
\]

On the other hand, in the bad state, we have
\[
\frac{d}{dD} \left[ v^{LS}_{BD} (D, I_0, \mu' (E^h - D)) \right]_{D=D^*(I_0)} = \mu'' [E^h - D^*(I_0)]
\]

\[
< 0
\]

and
\[
v^{LR}_{GD1} [D^*(I_0), I_0] = \theta^{LR}_{GI} [D^*(I_0), I_0] (\Pi'_G / \Psi'_G)' [\theta^{LR}_{G} [D^*(I_0), I_0]] > 0.
\]

namely due to (3.26) and (3.27). So, using \cdot to suppress arguments, we have
\[
v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]

\[
= v_{II}(\cdot) [\mu''(\cdot) + \alpha_G(\theta^{LR}_{G})_D(\cdot)(\Pi'_G / \Psi'_G)'(\cdot)] - v_{DI}(\cdot)v_{ID}(\cdot)
\]

(3.35)
\[ v_{II}(\cdot)\alpha_G(\theta^L_{G})_{D}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot) \]  
(3.36)

\[ \propto v_{GII}^L(\cdot)(\theta^L_{G})_{D}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot) - v_{GDI}^L(\cdot)v_{GID}^L(\cdot) \]  
(3.37)

\[ = (-1)[1 - \ell + \Psi_{G}(\cdot)]\theta^L_{GI}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot)(\theta^L_{G})_{D}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot) \]

\[ + \theta^L_{GI}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot)[1 - \ell + \Psi_{G}(\cdot)](\theta^L_{G})_{D}(\cdot)(\Pi'_{G}/\Psi'_G)'(\cdot) \]  
(3.38)

\[ = 0, \]  
(3.39)

as desired.

Case two. Now suppose that \( I_0 \) has the property that the pair \([D^*(I_0), I_0]\) lies in the region where \( r_B = ED \). In this case, we still want to confirm that

\[ v_{II}[D^*(I_0), I_0]v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0, \]

and banks’ behaviour in the bad state is unchanged relative to my previous case. As for their behaviour in the bad state, we know that

\[ v_{BD}^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \]

\[ = \ell + \Pi_B[\theta^E_{B}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]] - 1 \]

\[ + [1 - \ell - \Delta_B[\theta^E_{B}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]]] \times \cdots \]

\[ \cdots \times (\Pi'_{B}/\Delta'_B)[\theta^E_{B}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]], \]

\[ v_{BD}^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \]

\[ = [1 - \mu'[E^h - D^*(I_0)]] \times \cdots \]
\[
\cdots \times [1 - (\Pi'_B/\Delta'_B)[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]],
\]

\[
\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] < 0,
\]

\[
\left. \frac{d}{dD} \left[ \theta^{ED}_B[D, I_0, \mu'(E^h - D)] \right] \right|_{D = D^*(I_0)} < 0,
\]

and

\[
(\Pi'_B/\Delta'_B)[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]] > 0,
\]

namely due to (3.21), (3.19), (3.20), (3.52), and (1.15), respectively. So,

\[
v^{ED}_{BI}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] = [1 - \ell - \Delta_B[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]]] \times \cdots
\]

\[
\cdots \times \theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \times \cdots
\]

\[
\cdots \times (\Pi'_B/\Delta'_B)[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]]
\]

\[
< 0, \quad (3.40)
\]

\[
\frac{d}{dD} \left[ v^{ED}_{BI}[D, I_0, \mu'(E^h - D)] \right] \bigg|_{D = D^*(I_0)}
\]

\[
= [1 - \ell - \Delta_B[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]]] \times \cdots
\]

\[
\cdots \times \frac{d}{dD} \left[ \theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \right] \times \cdots
\]

\[
\cdots \times (\Pi'_B/\Delta'_B)[\theta^{ED}_B[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]]
\]

\[
< 0,
\]

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\[
\begin{align*}
\frac{d}{dD} \left[ v_{BD}^{ED}[D, I_0, \mu'(E^h - D)] \right]_{D=D^*(I_0)} &= \mu''[E^h - D^*(I_0)] \times \cdots \\
\cdots \times [1 - (\Pi'_B / \Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]] \\
- [1 - \mu'[E^h - D^*(I_0)]] \times \cdots \\
\cdots \times \frac{d}{dD} \left[ \theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \right] \times \cdots \\
\cdots \times (\Pi'_B / \Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]] \\
< 0,
\end{align*}
\]

and
\[
\begin{align*}
v_{BDl}^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] &= (-1)[1 - \mu'[E^h - D^*(I_0)]] \times \cdots \\
\cdots \times \theta_B^{EI}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \times \cdots \\
\cdots \times (\Pi'_B / \Delta'_B)[\theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]]] \\
< 0.
\end{align*}
\]

Moreover, using \( \cdot \) to suppress arguments, we also have
\[
\begin{align*}
v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
= [\alpha_Gv_{GII}^{LR}(\cdot) + \alpha_Bv_{BII}^{ED}(\cdot)] \left[ \alpha_G \frac{d}{dD} \left[ v_{GD}^{LR}(\cdot) \right] \right]_{D=D^*(I_0)} \\
+ \alpha_B \frac{d}{dD} \left[ v_{BD}^{ED}(\cdot) \right]_{D=D^*(I_0)}
\end{align*}
\]
\[- \left[ \alpha_G v_{GDI}^{LR}(\cdot) + \alpha_B v_{BDI}^{ED}(\cdot) \right] \left[ \alpha_G v_{GID}^{LR}(\cdot) + \alpha_B \frac{d}{dD} \left[ v_{BD}^{ED}(\cdot) \right] \right]_{D=D^*(I_0)} \]

\[= \alpha_G^2 \left[ v_{GII}^{LR}(\cdot) \frac{d}{dD} \left[ v_{GD}^{DR}(\cdot) \right] \right]_{D=D^*(I_0)} - v_{GDI}^{LR}(\cdot) v_{GID}^{LR}(\cdot) \]

\[+ \alpha_B^2 \left[ v_{BII}^{ED}(\cdot) \frac{d}{dD} \left[ v_{BD}^{ED}(\cdot) \right] \right]_{D=D^*(I_0)} - v_{BDI}^{ED}(\cdot) v_{GID}^{LR}(\cdot) \]

\[+ \alpha_G \alpha_B \left[ \frac{d}{dD} \left[ v_{GII}^{LR}(\cdot) \right] \right]_{D=D^*(I_0)} v_{BII}^{ED}(\cdot) - \frac{d}{dD} \left[ v_{BII}^{ED}(\cdot) \right]_{D=D^*(I_0)} v_{GII}^{LR}(\cdot) \]

so it would suffice if we could show that the starred terms are all positive. To see that this is indeed the case, first note that

\[v_{GII}^{LR}(\cdot) \frac{d}{dD} \left[ v_{GII}^{LR}(\cdot) \right] \right]_{D=D^*(I_0)} - v_{GDI}^{LR}(\cdot) v_{GID}^{LR}(\cdot) \]

\[= v_{GII}^{LR}(\cdot) \left[ \mu''(\cdot) + (\theta_G^{LR})_D(\cdot) (\Pi_G'/\Psi_G')'(\cdot) \right] - v_{GDI}^{LR}(\cdot) v_{GID}^{LR}(\cdot) \]

\[> v_{GII}^{LR}(\cdot) (\theta_G^{LR})_D(\cdot) (\Pi_G'/\Psi_G')'(\cdot) - v_{GDI}^{LR}(\cdot) v_{GID}^{LR}(\cdot) \]

\[= 0, \]  

where the last equality was established on lines (3.37) through (3.39). Similarly,
\[= \nu^{ED}_{BII}(\cdot)\mu''(\cdot)\left[1 - (\Pi'_B/\Delta'_B)(\cdot)\right] - \nu^{ED}_{BII}(\cdot)\left[1 - \mu'(\cdot)\right] \frac{d}{dD}\left[\theta^{ED}_B(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot)\]

\[= \nu^{ED}_{BII}(\cdot)\frac{d}{dD}\left[\nu^{ED}_{BI}(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot)\]

\[> (-1)\nu^{ED}_{BII}(\cdot)\left[1 - \mu'(\cdot)\right] \frac{d}{dD}\left[\theta^{ED}_B(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot)\]

\[= (-1)[1 - \ell - \Delta_B(\cdot)]\theta^{ED}_B(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)[1 - \mu'(\cdot)] \times \cdots\]

\[\cdots \times \frac{d}{dD}\left[\theta^{ED}_B(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot)\]

\[+ [1 - \mu'(\cdot)]\theta^{ED}_B(\cdot)(\Pi'_B/\Delta'_B)'(\cdot)[1 - \ell - \Delta_B(\cdot)] \times \cdots\]

\[\cdots \times \frac{d}{dD}\left[\theta^{ED}_B(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot)\]

\[= 0.\]

Moreover,
\[\left< 0 \right> \nu^{LR}_{GII}(\cdot) \frac{d}{dD}\left[\nu^{ED}_{BDI}(\cdot)\right]_{\Pi'_B/\Delta'_B}(\cdot) \left< 0 \right> - \nu^{ED}_{BDI}(\cdot)\nu^{LR}_{GID}(\cdot) > 0.\]
Finally,
\[
\frac{d}{dD} \left[ v_{GD}^L(\cdot) \right]_{D=D^*(I_0)} < 0 \quad \frac{d}{dD} \left[ v_{BI}^E(\cdot) \right]_{D=D^*(I_0)} < 0 \quad \frac{d}{dD} \left[ v_{BD}^L(\cdot) \right]_{D=D^*(I_0)} > 0.
\]

**Case three.** Suppose next that \( I_0 \) has the property that the pair \([D^*(I_0), I_0]\) lies in the region where \( r_B = DD \), but the “no-default” constraint is lax. In this case, we still want to confirm that
\[
v_I[D^*(I_0), I_0]v_{DD}[D^*(I_0), I_0] - v_{DI}[D^*(I_0), I_0]v_{ID}[D^*(I_0), I_0] > 0,
\]
and banks’ behaviour in the good state is unchanged relative to case one. As for their behaviour in the bad state, we know that
\[
\frac{d}{dD} \left[ v_{BD}^{DD}(D, I_0, \mu'(E^h - D)) \right]_{D=D^*(I_0)} = \mu''[E^h - D^*(I_0)][1 - (\Pi_B/\Delta_B)(\theta_B^2)]
\]
\[
< 0
\]
\[
= v_{BDI}^{DD}[D^*(I_0), I_0, \mu'(E^h - D^*(I_0))]
\]
\[
= \frac{d}{dD} \left[ v_{BI}^{DD}(D, I_0, \mu'(E^h - D)) \right]_{D=D^*(I_0)}
\]
\[
= v_{BDI}^{DD}[D^*(I_0), I_0, \mu'(E^h - D^*(I_0))], \quad (3.45)
\]

namely due to \((3.23)\) and \((3.24)\). So, using \( \cdot \) to suppress arguments, we have
\[
v_{II}(\cdot)v_{DD}(\cdot) - v_{DI}(\cdot)v_{ID}(\cdot)
\]
\[
= v_I(\cdot)\left[ \alpha_G \frac{d}{dD} \left[ v_{GD}^L(\cdot) \right]_{D=D^*(I_0)} + \alpha_B v_{BD}^{DD}(\cdot) \right] - v_{DI}(\cdot)v_{ID}(\cdot)
\]
\[
> v_I(\cdot)\alpha_G \frac{d}{dD} \left[ v_{GD}^L(\cdot) \right]_{D=D^*(I_0)} - v_{DI}(\cdot)v_{ID}(\cdot)
\]

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\[ v_{\text{GII}}^{LR}(\cdot) \frac{d}{dD} \left[ v_{\text{GD}}^{LR}(\cdot) \right] \bigg|_{D=D^{*}(I_0)} - v_{\text{GID}}^{LR}(\cdot) v_{\text{GDI}}^{LR}(\cdot) \]

\[ = 0, \]

where the last equality was established on lines 3.41 through 3.44.

**Case four.** Suppose finally that \( I_0 \) has the property that the “no-default” constraint binds at the point \( [D^*(I_0), I_0] \) — i.e., \( I_0 \in (I)_B, E^b/(1 - \ell) \). In this case, we know that

\[ h(I_0) \]

\[ = v_I[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h - D_B(I_0)]} \right] v_D[D_B(I_0), I_0] \]

\[ = \alpha_G v_{\text{GI}}^{LR}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]] + \alpha_B v_{\text{BD}}^{DD}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]] \]

\[ + (1 - \ell) \times \cdots \]

\[ \cdots \times \left[ \alpha_G v_{\text{GI}}^{LR}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]] + \alpha_B v_{\text{BD}}^{DD}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]] \right] \]

\[ = \alpha_G v_{\text{GI}}^{LR}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]] + \frac{(1 - \ell)\alpha_G v_{\text{GI}}^{LR}[D_B(I_0), I_0, \mu'[E^h - D_B(I_0)]]}{1 - \mu'[E^h - D_B(I_0)]}, \]

where the last equality follows from the fact that banks’ payout in the bad state is zero whenever the “no-default” constraint binds, since all resources go to depositors. So, using \( \cdot \) to suppress arguments, we have

\[ h'(I_0) \propto v_{\text{GII}}^{LR}(\cdot) + \underbrace{D_B(\cdot)}_{<0} v_{\text{GID}}^{LR}(\cdot) \]

\[ - \frac{\underbrace{D_B(\cdot)\mu''(\cdot)(1 - \ell)}_{<0} v_{\text{GDI}}^{LR}(\cdot)}{[1 - \mu'(\cdot)]^2} \]

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\[
\begin{align*}
&+ \frac{(1 - \ell) v_{G_D}^{LR}()} {1 - \mu'()}^{>0} \\
&+ \left[ \frac{(1 - \ell \bar{D}_B()}{1 - \mu'()} \right] \frac{d}{d\bar{D}} \left[ v_{G_D}^{LR}() \right] \bigg|_{\bar{D} = \bar{D}_B()} < 0,
\end{align*}
\]

where the final inequality holds because case four only obtains when

\[
v_D[\bar{D}_B(I_0), I_0] = \alpha_G v_{G_D}^{LR}[\bar{D}_B(I_0), I_0, \mu'(E^h - D)] + \alpha_B v_{BD}^{DD}[\bar{D}_B(I_0), I_0, \mu'(E^h - D)] > 0,
\]

and we’ve already seen that

\[
v_{BD}^{DD}[\bar{D}_B(I_0), I_0, \mu'(E^h - D)] < 0,
\]

namely in (3.23).

Having dispatched this last case, we can finally conclude that the function \( h(I_0) \) reaches zero at exactly one point in its domain. This point represents our only candidate for equilibrium. I furthermore claim that it indeed constitutes an equilibrium. In principle, this would require that we rule out banks’ having some incentive to engage in non-local deviations, since the candidate has only been constructed in a way which precludes their having some incentive to engage in local deviations. Fortunately, this can easily be verified, though I’ve chosen to omit a formal proof for brevity.

3.A.4.5 Dependence on parameters

In this last subsubsection, I show how the equilibrium identified above responds to changes in the model’s parameters. I thus let \( \beta = (\chi_G, \alpha_G, E^h) \) collect all the parameters on which we’ll focus and will include \( \beta \) as an explicit argument in any functions into which these parameters enter. Now, based on the analysis in my previous subsubsection, it should be clear that

\[
h[I_B^\Pi(\beta), \beta] > h[I_B^\Xi(\beta), \beta] > h[I_B(\beta), \beta],
\]

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and also that the regime associated with the bad state in equilibrium depends on where zero lies in this chain of inequalities — e.g., a liquidity surplus occurs i.f.f. $h[I_B^H(\beta), \beta] \leq 0$. So, this subsubsection will focus on determining how parameters influence the signs on $h[I_B^H(\beta), \beta]$, $h[I_B^L(\beta), \beta]$, and $h[I_B(\beta), \beta]$.

Let’s begin with $h[I_B^H(\beta), \beta]$. In particular, let’s define $D_B^H(\beta) := D^*[I_B^H(\beta), \beta]$, so that the pair $[D_B^H(\beta), I_B^H(\beta)]$ solves

$$[E^b + D_B^H(\beta) - I_B^H(\beta)] + \ell I_B^H(\beta) + I_B^H(\beta)\Delta_B(\theta_B^\beta) = D_B^H(\beta)\mu'[E^h - D_B^H(\beta)]$$

$$\iff E^b + D_B^H(\beta) - D_B^H(\beta)\mu'[E^h - D_B^H(\beta)] = I_B^H(\beta)[1 - \ell - \Delta_B(\theta_B^\beta)], \quad (3.46)$$

and

$$\alpha_G v_{GD}^L[D_B^H(\beta), I_B^H(\beta), \beta] + \alpha_B v_{BD}^L[D_B^H(\beta), I_B^H(\beta), \beta] = 0$$

$$\iff 1 - \mu'[E^h - D_B^H(\beta)] + \alpha_G(\Pi'_G/\Psi_G)[\theta_{LR}^L[G_D^H(\beta), I_B^H(\beta), \beta], \beta] = 0, \quad (3.47)$$

with

$$[E^b + D_B^H(\beta) - I_B^H(\beta)] + \ell I_B^H(\beta) = I_B^H(\beta)\Psi_G[\theta_{LR}^L[D_B^H(\beta), I_B^H(\beta), \beta]]$$

$$\iff E^b + D_B^H(\beta) = I_B^H(\beta)[1 - \ell + \Psi_G[\theta_{LR}^L[D_B^H(\beta), I_B^H(\beta), \beta]]]. \quad (3.48)$$

Now, dividing (3.46) by (3.48) yields

$$\left[1 - \frac{D_B^H(\beta)\mu'[E^h - D_B^H(\beta)]}{E^b + D_B^H(\beta)}\right] \times \ldots$$

$$\times \left[1 - \ell + \Psi_G[\theta_{LR}^L[D_B^H(\beta), I_B^H(\beta), \beta]]\right] = 1 - \ell - \Delta_B(\theta_B^H). \quad (3.49)$$

Differentiating (3.47) and (3.49) w.r.t. $\chi_G$ then yields the following system, where $\cdot$ suppresses arguments:

$$\frac{d}{d\chi_G}\left[\theta_{LR}^G(\cdot)\right](\Pi'_G/\Psi_G)_{\theta}(\cdot) + (\Pi'_G/\Psi_G)_{\chi_G}(\cdot) + (D_B^H)_{\chi_G}(\beta)(1/\alpha_G)\mu''(\cdot) = 0$$
Figure 3.3: Visual aid for proving lemmata \(3.4.1\) through \(3.4.3\)
\[
\frac{d}{d\chi_G} \left[ \theta_{LR}^G(\cdot) \right] < 0,
\]

An application of Cramer’s rule then yields

\[
\frac{d}{d\alpha_G} \left[ \theta_{LR}^G(\cdot) \right] < 0 < \frac{d}{dE_b} \left[ \theta_{LR}^G(\cdot) \right].
\]

Moreover, a repeat of the arguments on lines 1.22 through 1.27 yields

\[
\tilde{h}[I_B^H(\beta), \beta] \propto \int_{\theta_{LR}^G[D_B^H(\beta), I_B^H(\beta), \beta]}^{1} \theta dF(\theta)
\]

\[
- \left[ 1 - \ell + \int_{\theta_{LR}^G[D_B^H(\beta), I_B^H(\beta), \beta]}^{1} (\rho + \ell) dF(\theta) \right] \left[ \frac{\theta_{LR}^G[D_B^H(\beta), I_B^H(\beta), \beta]}{\rho + \ell} \right]
\]

\[
+ \left[ (1 - \alpha_G)/\alpha_G \chi_G \right] [\ell + \Pi_B(\theta_B^H) - 1]
\]

\[= : \tilde{h}(\beta), \]

with

\[
\tilde{h}_{\chi G}(\beta) = \frac{d}{d\chi_G} \left[ \theta_{LR}^G[D_B^H(\beta), I_B^H(\beta), \beta] \right] \left( \frac{-1}{\rho + \ell} \right) \times \ldots
\]

\[
\ldots \times \left[ 1 - \ell + \int_{\theta_{LR}^G[D_B^H(\beta), I_B^H(\beta), \beta]}^{1} (\rho + \ell) dF(\theta) \right]
\]

\[
- \left[ (1 - \alpha_G)/\alpha_G \chi_G^2 \right] [\ell + \Pi_B(\theta_B^H) - 1]
\]

\[> 0,\]
where the last inequality follows from subassumption 3.4.3. Repeating for $\alpha_G$ and $E^b$ then yields

$$h_{\alpha_G}(\beta) > 0 > h_{E^b}(\beta),$$

so $h[I_B^\Pi(\beta), \beta]$ is more likely to be positive the greater is $(\chi_G, \alpha_G)$ and the lesser is $E^b$. That these tendencies also hold for $h[I_B^\Omega(\beta), \beta]$ and $h[I_B(\beta), \beta]$ can be shown using similar arguments.

At this point, the partitions described in lemmata 3.4.2 and 3.4.3 can be constructed in the usual way. In particular, we can conclude that the function $h[I_B^\Pi(\beta), \beta]$, viewed as a function of the parameter $\chi_G$, is either single-signed over the interval $(\chi_G, \infty)$ or otherwise exhibits single-crossing from below over this interval. In the latter case, I set $\chi_G^{LS}$ equal to the crossing point. Otherwise, I set $\chi_G^{LS} = \chi_G$ ($\chi_G^{LS} = \infty$) in the case of uniform positivity (negativity). The rest of the partition can be constructed on a mutatis mutandis basis, and similar constructions go through w.r.t. $\alpha_G$ and $E^b$.

### 3.A.5 Proof of lemmata 3.4.4 through 3.4.7

#### 3.A.5.1 Some preliminary results

I begin by deriving the planner’s marginal returns on $D$ and $I_0$, which will require that we take cases on regimes and states:

**Case one: $r_G = LR$.** Under this case, utilitarian welfare is given by

$$v_G^{LR}[D, I_0, \mu'(E^h - D)] + D\mu'(E^h - D) + \mu(E^h - D) =: v_G^{LR|SP}(D, I_0),$$

in which case it should be clear that

$$v_G^{LR|SP}(D, I_0) = v_G^{LR|SP}(D, I_0, \mu'(E^h - D)), \quad \forall x \in \{D, I_0\}.$$  \hspace{1cm} (3.50)

**Case two: $r_B = ED$.** Under this case, utilitarian welfare is given by

$$v_B^{ED}[D, I_0, \mu'(E^h - D)] + D\mu'(E^h - D) + \mu(E^h - D) =: v_B^{ED|SP}(D, I_0),$$

in which case it should be clear that

$$v_B^{ED|SP}(D, I_0) = v_B^{ED|SP}(D, I_0, \mu'(E^h - D)).$$  \hspace{1cm} (3.51)
Moreover,

\[\frac{d}{dD} \left[ \theta_B^{ED} [D, I_0, \mu'(E^h - D)] \right] = \frac{\mu'(E^h - D) - 1 - D\mu''(E^h - D)}{I_0 \Delta_B [\theta_B^{ED} [D, I_0, \mu'(E^h - D)]]} < 0, \tag{3.52}\]

so

\[v_{BD}^{SP} (D, I_0) = 1 - \mu'(E^h - D) + I_0 \frac{d}{dD} \left[ \theta_B^{ED} [D, I_0, \mu'(E^h - D)] \right] \times \ldots \times \Pi_B' [\theta_B^{ED} [D, I_0, \mu'(E^h - D)]]\]

\[= 1 - \mu'(E^h - D) - [1 - \mu'(E^h - D) + D\mu''(E^h - D)] \times \ldots \times \frac{\Pi_B' [\theta_B^{ED} [D, I_0, \mu'(E^h - D)]]}{\Delta_B [\theta_B^{ED} [D, I_0, \mu'(E^h - D)]]}. \tag{3.53}\]

Case three: \(r_B = DD\). Under this case, utilitarian welfare is given by

\[v_B^{DD} [D, I_0, \mu'(E^h - D)] + D\mu'(E^h - D) + \mu(E^h - D) =: v_B^{DD|SP} (D, I_0),\]

in which case it should be clear that

\[v_B^{DD|SP} (D, I_0) = v_B^{DD} [D, I_0, \mu'(E^h - D)]. \tag{3.54}\]

Moreover,

\[\frac{d}{dD} \left[ I_B^{DD} [D, I_0, \mu'(E^h - D)] \right] = \frac{\mu'(E^h - D) - 1 - D\mu''(E^h - D)}{\Delta_B (\theta_B^B)} ,\]

so

\[v_B^{DD|SP} (D, I_0) = 1 - \mu'(E^h - D) + \frac{d}{dD} \left[ I_B^{DD} [D, I_0, \mu'(E^h - D)] \right] \Pi_B (\theta_B^B) \]

\[= 1 - \mu'(E^h - D) - [1 - \mu'(E^h - D) + D\mu''(E^h - D)] \times \ldots \underbrace{< 0}_{< 0} \]
\[
\cdots \times \frac{\Pi_B(\theta_B^c)}{\Delta B(\theta_B^c)} < 0.
\]

(3.55)

Case four: \( r_\omega = LS \) (\( \omega \in \{B, G\} \)). Under this case, utilitarian welfare is given by

\[
v^{LS}_\omega [D, i_0, \mu'(E^h - D)] + D\mu'(E^h - D) + \mu(E^h - D) =: v^{LS|SP}_\omega (D, i_0),
\]

in which case it should be clear that

\[
v^{LS|SP}_\omega (D, i_0) = v^{LS}_\omega [D, i_0, \mu'(E^h - D)] \forall x \in \{D, i_0\}. \tag{3.56}
\]

3.A.5.2 Notation

It will be useful to supplement the notation established in subsubsection 3.A.4.2 as follows. Let \( v^{SP}_D(D, i_0) \) denote the planner’s expected return on the marginal deposit, computed on an unconditional basis at \( t = 0 \) after taking account of the analysis in section 3.3 — e.g.,

\[
D \leq \min\{D^H_G(i_0), D^H_B(i_0)\} \implies v^{SP}_D(D, i_0) = \alpha_G v^{LR|SP}_{GD}(D, i_0) + \alpha_B v^{LS|SP}_{BD}(D, i_0).
\]

Define \( v^{SP}_I(D, i_0) \) analogously.

Remark 1. It can easily be verified that the marginal return functions \( v^{SP}_D(D, i_0) \) and \( v^{SP}_I(D, i_0) \) are continuous in both their arguments, even around the boundaries separating regimes. This is a consequence of the envelope condition.

Remark 2. For clarity, figure 3.4 illustrates some of the notation used in this subsection.

3.A.5.3 Some more preliminary results

Sublemma 3.A.4. The function \( v^{SP}_D(D, i_0) \) is strictly decreasing in its first argument.

Proof. I’ll take cases on regimes and states:
Case one: $r^{SP}_B = ED$. Under this case, we know that
\[
v^{ED|SP}_{BD}(D, I_0) = 1 - \mu'(E^h - D) - [1 - \mu'(E^h - D) + D\mu''(E^h - D)]\times\ldots
\]
\[
\ldots \times (\Pi'_B/\Delta'_{B})[\theta^{ED}_{B}[D, I_0, \mu'(E^h - D)]],
\]
\[
\frac{d}{dD} \left[ \theta^{ED}_{B}[D, I_0, \mu'(E^h - D)] \right] < 0,
\]
and
\[
(\Pi'_B/\Delta'_{B})'[\theta^{ED}_{B}[D, I_0, \mu'(E^h - D)] > 0,
\]
namely due to (3.53), (3.52), and (1.15). So,
\[
v^{ED|SP}_{BDD}(D, I_0) = \mu''(E^h - D)
\]
\[
< 0 \quad \text{— see assumption 3.6}
\]
\[
- [2\mu''(E^h - D) - D\mu'''(E^h - D)]\times\ldots
\]
\[
\ldots \times (\Pi'_B/\Delta'_{B})[\theta^{ED}_{B}[D, I_0, \mu'(E^h - D)]]
\]
\[
< 0
\]
\[
- [1 - \mu'(E^h - D) + D\mu''(E^h - D)]\times\ldots
\]
\[
\ldots \times \frac{d}{dD} \left[ \theta^{ED}_{B}[D, I_0, \mu'(E^h - D)] \right] (\Pi'_B/\Delta'_{B})'[\theta^{ED}_{B}[D, I_0, \mu'(E^h - D)]]
\]
\[
< 0.
\]

Case two: $r^SP_B = DD$. Under this case, we know that
\[
v^{DD|SP}_{BD}(D, I_0) = 1 - \mu'(E^h - D) - [1 - \mu'(E^h - D) + D\mu''(E^h - D)](\Pi_B/\Delta_B)(\theta^E_{B}),
\]
namely due to (3.55), so

\[ v_{BDD}^{DD|SP}(D, I_0) = \mu''(E^h - D) - [2\mu''(E^h - D) - D\mu'''(E^h - D)] \left( \Pi_B / \Delta_B \right)(\theta_B^2) < 0. \]

Case three: \( v_G^{SP} = LR \). Under this case, we know that

\[ v_{GD}^{LR|SP}(D, I_0) = v_{GD}^{LR}(D, I_0), \]

where the function on the right-hand side has already been shown to decrease strictly in its first argument (see case three in sublemma 3.A.1 in particular).

Case four: \( r_\omega = LS \), with \( \omega \in \{B, G\} \). Under this case, we know that

\[ v_{\omega D}^{LS|SP}(D, I_0) = v_{\omega D}^{LR}(D, I_0), \]

where the function on the right-hand side has already been shown to decrease strictly in its first arguments (see case four in sublemma 3.A.1 in particular).

Sublemma 3.A.5. The composition \( v_{D}^{SP}[\overline{D}_B(I_0), I_0] \) exhibits single-crossing from below over the interval \([0, E^b/(1 - \ell)]\), namely at some interior point \( \overline{T}_B^{SP} \) satisfying \( \overline{D}_B(\overline{T}_B^{SP}) < D_B^{H}(\overline{T}_B^{SP}) \).

Proof. It should be clear that the function \( \overline{D}_B(I_0) \) intersects the function \( D_B^{H}(I_0) \) once from above over the interval \([0, E^b/(1 - \ell)]\), namely at some interior point which I’ll denote \( \tilde{I} \). Moreover, \( \forall I_0 \in [0, \tilde{I}] \), it should also be clear that

\[ v_D^{SP}[\overline{D}_B(I_0), I_0] = \alpha_G v_{GD}^{LS|SP}[\overline{D}_B(I_0), I_0] + \alpha_B v_{BD}^{DD|SP}[\overline{D}_B(I_0), I_0] \]

\[ = \alpha_G [1 - \mu' [E^h - \overline{D}_B(I_0)]], \]

see (3.50) and (3.26).
\[ \begin{align*}
&1 - \mu'[E^h - \overline{D}_B(I_0)] \\
&\quad + \alpha_B \left[ -[1 - \mu'[E^h - \overline{D}_B(I_0)] + \overline{D}_B(I_0) \mu''[E^h - \overline{D}_B(I_0)] \right] \times \cdots < 0 \\
&\quad \cdots \times \left( \Pi_B/\Delta_B \right) (\theta_{\overline{B}}^\overline{G}) < 0
\end{align*} \]

whereas
\[
v^{SP}_D[\overline{D}_B(E^b/(1 - \ell), E^b/(1 - \ell)] = v^{SP}_D[0, E^b/(1 - \ell)]
\]
\[
= \alpha_Gv^{LR|SP}_G[0, E^b/(1 - \ell)] + \alpha_Bv^{DD|SP}_{BD}[0, E^b/(1 - \ell)] \\
= \alpha_G[1 - \mu'(E^h) + (\Pi'_G/\Psi'_G)[\theta^{LR}_G[0, E^b/(1 - \ell)]]] \\
+ \alpha_B[1 - \mu'(E^h)][1 - (\Pi_B/\Delta_B)(\theta_{\overline{B}}^\overline{G})] \\
= \alpha_G(\Pi'_G/\Psi'_G)[\theta^{LR}_G[0, E^b/(1 - \ell)]]
\]
\[
> 0,
\]
so it would suffice if we could show that the composition
\[
v^{SP}_D[\overline{D}_B(I_0), I_0] = \alpha_Gv^{LR|SP}_G[\overline{D}_B(I_0), I_0] + \alpha_Bv^{DD|SP}_{BD}[\overline{D}_B(I_0), I_0]
\]
is strictly increasing over the interval \((I, E^b/(1 - \ell))\). Fortunately, my previous sublemma, combined with the fact that the function \(\overline{D}_B(\cdot)\) is strictly decreasing, implies that it’s sufficient to note that
\[
v^{LR|SP}_{GD}[\overline{D}_B(I_0), I_0] = \frac{\theta^{LR}_G[\overline{D}_B(I_0), I_0], (\Pi'_G/\Psi'_G)[\theta^{LR}_G[\overline{D}_B(I_0), I_0]]}{\overline{D}_B(I_0), I_0} > 0 \quad \text{— see (3.50) and (1.14)}
\]
\[
v^{LR|SP}_{GD}[\overline{D}_B(I_0), I_0] > 0 \quad \text{— see (3.15)}
\]

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with
\[ v^{DD|SP}_{BDI}(D_B(I_0), I_0) = 0, \]
where the last equality follows from (3.55).

**Sublemma 3.A.6.** \( \forall x \in \{\Pi, \Xi\}, \) the composition \( v^{SP}_{D}(D_B(I_0), I_0) \) exhibits single-crossing from below over the interval \([0, E^b/[1 - \ell - \Delta_B(\theta^*_B)])\), namely at some interior point \( I^{|SP}_B \) satisfying \( D^r_B(I^{|SP}_B) < D^H_G(I^{|SP}_B). \)

**Proof.** This can be verified using essentially the same argument as for my previous sublemma.

**3.A.5.4 Existence and uniqueness**

I’ll now argue that the planner’s problem admits a unique solution. As a first step in this direction, I note that assumption 3.1 ensures that the non-negativity constraint \( I_0 \geq 0 \) is lax. Similarly, the fact that \( \mu'(E^h) = 1 \) ensures some slack in the constraint \( D \geq 0. \) Moreover, arguments very similar to those made in the last part of subsection 1.A.5.5 will show that the non-negativity constraint for storage, \( I_0 \leq E^b + D, \) must also be lax, namely due to subassumption 3.4.4. As a result, solutions for the planner’s problem must full under one of two cases. The first would be an interior case under which the planner finds that the “no-default” constraint
\[(E^b + D - I_0) + \ell I_0 \geq D \mu'(E^h - D)\]
is lax, and thus settle on initial balance sheets satisfying
\[ v^{SP}_x(D, I_0) = 0, \quad \forall x \in \{D, I_0\}. \]

The alternative would be a corner case under which the aforementioned “no-default” constraint binds.

Now, irrespective as to which of these two cases obtains, I claim that solutions for the planner’s problem must satisfy \( D < D^H_G(I_0). \) To see this, first note that all initial balance sheets satisfying \( D \in (\max\{D^{\Xi}_B(I_0), D^{H}_G(I_0)\}, \overline{D}_B(I_0)] \) also satisfy
\[ v^{SP}_D(D, I_0) = \alpha_G v^{LS|SP}_{GD}(D, I_0) + \alpha_B v^{DD|SP}_{BD}(D, I_0) \]
\[
\alpha_G \left[ 1 - \mu'(E^h - D) \right] \alpha_B \times \ldots
\]

\[
\ldots \times \left[ 1 - \mu'(E^h - D) - \left[ 1 - \mu'(E^h - D) \right] + D \left[ \mu''(E^h - D) \right] \left( \Pi_B / \Delta_B \right) \left( \theta_B^\Omega \right) \right] \ldots
\]

\[
< 0.
\]

Similarly, if \( D \in (\max\{D_B^\Pi(I_0), D_G^\Pi(I_0)\}, D_B^\Xi(I_0)) \), then
\[
v_D^{SP}(D, I_0) = \alpha_G v_{GD}^{LS|SP}(D, I_0) + \alpha_B v_{BD}^{ED|SP}(D, I_0)
\]
\[
= \alpha_G[1 - \mu'(E^h - D)]
\]
\[
+ \alpha_B \left[ \begin{array}{c}
1 - \mu'(E^h - D) \\
\left[ 1 - \mu'(E^h - D) + D \mu''(E^h - D) \right] < 0 \\
\ldots \times \left( \Pi_B' / \Delta_B' \right) [\theta_B^{ED}(D, I_0)] < 0
\end{array} \right] \ldots
\]

\[
< 0.
\]

Finally, if \( D \in \max\{0, D_G^\Pi(I_0)\}, D_B^\Pi(I_0) \), then
\[
v_I^{SP}(D, I_0) = \alpha_G v_{GI}^{LS|SP}(D, I_0) + \alpha_B v_{BI}^{LS|SP}(D, I_0)
\]
\[
= \alpha_G v_{GI}^{LS}(D, I_0) + \alpha_B v_{BI}^{LS}(D, I_0) \quad (3.57)
\]
\[
= \ell + E[\Pi_\omega(\theta_\omega^\Pi)] - 1 \quad (3.58)
\]
where (3.57), (3.58), and (3.59) respectively follow from (3.56), (3.27), and assumption 3.1. Conclude that the planner must have a strict incentive to adjust \( D \) or \( I_0 \) whenever \( D \geq D_G^H(I_0) \).

So, we can now restrict attention to candidate solutions satisfying \( D < D_G^H(I_0) \) and, by extension, \( I_0 \in (E^b/[1 - \ell - \Delta B(\theta_B)], E^b/(1 - \ell)] \). Now, given any choice on \( I_0 \) in this range, sublemma 3.A.4 implies that one of two cases must obtain. The first would be that the marginal return \( v_B^{SP}(D, I_0) \), viewed as a function of the choice on \( D \), exhibits single-crossing from above over the interval \([0, \min\{D_G^H(I_0), \mathcal{D}_B(I_0)\}]\), namely at a point which I’ll denote \( D^*(I_0) \). Otherwise, it must be the case that the marginal return \( v_D^{SP}(D, I_0) \) is strictly positive over all of the aforementioned interval, in which case I adopt a convention that \( D^*(I_0) = \mathcal{D}_B(I_0) \). Moreover, sublemma 3.A.5 gives us a clean way to separate these two cases. In particular, it implies that \( D^*(I_0) = \mathcal{D}_B(I_0) \) whenever \( I_0 \in [\mathcal{T}_B^{SP}, E^b/(1 - \ell)] \), and otherwise \( D^*(I_0) < \min\{D_G^H(I_0), \mathcal{D}_B(I_0)\} \).

With \( D_B^{*|SP}(I_0) \) defined in this way, we can now further restrict attention to candidate solutions satisfying \( D = D^*(I_0) \). Given any such candidate, I’ll now check if the planner has an incentive to deviate in his initial balance-sheet choices, namely by adjusting his choice on \( I_0 \). Now, if \( I_0 < \mathcal{T}_B^{SP} \), then the “no-default” constraint is lax, and the planner’s return from a marginal increase in \( I_0 \) is given by \( v_I^{SP}[D^*(I_0), I_0] \). Otherwise, the “no-default” constraint binds, and increases in \( I_0 \) must be offset by decreases in \( D \), and the relevant return reads as

\[
v_I^{SP}[D^*(I_0), I_0] + \frac{1 - \ell}{1 - \mu'[E^h - D^*(I_0)]} v_D^{SP}[D^*(I_0), I_0]
\]

where the starred term gives the planner’s rate of transformation along the “no-default” constraint. So, \( \forall I_0 \in [E^b/[1 - \ell + \Psi_G(\theta_H^G)], E^b/(1 - \ell)] \), I define a function

\[
h^{SP}(I_0) := \begin{cases} 
  v_I^{SP}[D^*(I_0), I_0] & \text{if } I_0 \leq \mathcal{T}_B^{SP} \\
  v_I^{SP}[D^*(I_0), I_0] + \frac{(1 - \ell)v_D^{SP}[D^*(I_0), I_0]}{1 - \mu'[E^h - D^*(I_0)]} & \text{if } I_0 > \mathcal{T}_B^{SP} 
\end{cases}
\]
Now, it should be clear that this function (i) is continuous; (ii) satisfies
\[ h^{SP}[E^h/[1 - \ell + \Psi_G(\theta^*_G)]] > 0, \]

namely due to assumption $3.1$ and (iii) satisfies
\[ \lim_{I_0 \uparrow E^b/(1-\ell)} \{ h^{SP}(I_0) \} = -\infty, \]

namely due to the fact that $\mu'(E^h) = 1$. Existence and uniqueness of the planner’s solution would then follow if we could show that the function $h^{SP}(\cdot)$ is strictly decreasing. To see that this is indeed the case, we’ll have to take cases on $I_0$:

**Case one.** Suppose first that $I_0$ has the property that the pair $[D^{|SP}\{I_0\}, I_0]$ lies in the region where $r_B^{SP} = LS$. In this case, we want to confirm that
\[
(h^{SP})'(I_0) = v^{SP}_I[D^{|SP}\{I_0\}, I_0] + (D^{|SP})'(I_0)v^{SP}_I[D^{|SP}\{I_0\}, I_0] < 0,
\]

or equivalently
\[
v^{SP}_I[D^{|SP}\{I_0\}, I_0]v^{SP}_DD[D^{|SP}\{I_0\}, I_0] - v^{SP}_DI[D^{|SP}\{I_0\}, I_0]v^{SP}_ID[D^{|SP}\{I_0\}, I_0] > 0.
\]

Fortunately, since the present case has the property that liquidity rationing (surplus) occurs in the good (bad) state, $3.50$ and $3.56$ together imply that
\[
v^{SP}_I[D^{|SP}\{I_0\}, I_0]v^{SP}_DD[D^{|SP}\{I_0\}, I_0] - v^{SP}_DI[D^{|SP}\{I_0\}, I_0]v^{SP}_ID[D^{|SP}\{I_0\}, I_0] > 0.
\]

where a repeat of the arguments on lines $3.35$ through $3.39$ will show that the right-hand side of this last equality is strictly positive.
Case two. Now suppose that $I_0$ has the property that the pair $[D^*|SP(I_0), I_0]$ lies in the region where $r_{SP} = ED$. In this case, we still want to confirm that

$$v_H^{SP}[D^{*|SP}(I_0), I_0]v_{DD}^{SP}[D^{*|SP}(I_0), I_0] - v_{DI}^{SP}[D^{*|SP}(I_0), I_0]v_{ID}^{SP}[D^{*|SP}(I_0), I_0] > 0.$$ 

Now, in the good state, we know that

$$v_{G_{x'y}}^{LR[SP][D^{*|SP}(I_0), I_0]} = v_{G_{x'y}}^{LR}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]],$$

$$v_{G_{DD}}^{LR[SP][D^{*|SP}(I_0), I_0]} = \frac{d}{dI_{DD}}[v_{G_{x'y}}^{LR}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]]] < 0,$$

namely due to (3.50), combined with (3.30) through (3.33). On the other hand, in the bad state, we know that

$$v_{B_{x'y}}^{ED[SP][D^{*|SP}(I_0), I_0]} = \ell + \Pi_B[\theta_B^{ED}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]]] - 1$$

$$+ [1 - \ell - \Delta_B[\theta_B^{ED}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]]] \times \cdots$$

$$\cdots \times (\Pi_B' / \Delta_B')[\theta_B^{ED}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]]],$$

$$v_B^{ED[SP][D^{*|SP}(I_0), I_0]}$$

$$= 1 - \mu'[E^h - D^{*|SP}(I_0)]$$

$$- [1 - \mu'[E^h - D^{*|SP}(I_0)] + D^{*|SP}(I_0)\mu''[E^h - D^{*|SP}(I_0)]] \times \cdots$$

$$\cdots \times (\Pi_B' / \Delta_B')[\theta_B^{ED}[D^{*|SP}(I_0), I_0, \mu'[E^h - D^{*|SP}(I_0)]]],$$

$$\theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] < 0,$$
\[
\frac{d}{dD} \left[ \theta_B^{ED}[D, I_0, \mu'(E^h - D)] \right] \bigg|_{D=D^*(I_0)} < 0,
\]

and

\[
\left( \Pi'_B/\Delta'_B \right) \left[ \theta_B^{ED}[D^*(I_0), I_0, \mu'[E^h - D^*(I_0)]] \right] > 0,
\]

respectively due to (3.51) and (3.21); (3.53); (3.20); (3.52); and (1.15). So,

\[
v_{BDI}^{ED|SP}[D^*|SP(I_0), I_0] = \left[ 1 - \ell - \Delta_B \left[ \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \right] \right] \times \cdots
\]

\[
\cdots \times \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \times \cdots
\]

\[
\cdots \times \left( \Pi'_B/\Delta'_B \right) \left[ \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \right] > 0,
\]

\[
v_{BDI}^{ED|SP}[D^*|SP(I_0), I_0] = \left[ 1 - \ell - \Delta_B \left[ \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \right] \right] \times \cdots
\]

\[
\cdots \times \frac{d}{dD} \left[ \theta_B^{ED}[D, I_0, \mu'(E^h - D)] \right] \bigg|_{D=D^*|SP(I_0)} \times \cdots
\]

\[
\cdots \times \left( \Pi'_B/\Delta'_B \right) \left[ \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \right] > 0,
\]

\[
v_{BDI}^{ED|SP}[D^*|SP(I_0), I_0] = (-1)[1 - \mu'[E^h - D^*|SP(I_0)] + D^*|SP(I_0)\mu''[E^h - D^*|SP(I_0)]] \times \cdots
\]

\[
\cdots \times \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \times \cdots
\]

\[
\cdots \times \left( \Pi'_B/\Delta'_B \right) \left[ \theta_B^{ED}[D^*|SP(I_0), I_0, \mu'[E^h - D^*|SP(I_0)]] \right] > 0.
\]
\( < 0 \),

and

\[
v_{BDD}^{ED|SP} [D^*|SP (I_0), I_0]
= \mu'' [E^h - D^*|SP (I_0)]
- \frac{2\mu'' [E^h - D^*|SP (I_0)] - D^*|SP (I_0)\mu'' [E^h - D^*|SP (I_0)]]}{\mu'' [E^h - D^*|SP (I_0)]} \times \ldots
\]

\(< 0 \quad \text{— see assumption 3.6}\)

\[
\ldots \times \left( \frac{\Pi_B' / \Delta_B'}{\theta_B'^{ED}[D^*|SP (I_0), I_0], \mu'[E^h - D^*|SP (I_0)]} \right)
- \left[ 1 - \mu'[E^h - D^*|SP (I_0) + D^*|SP (I_0)\mu'' [E^h - D^*|SP (I_0)]] \right] \times \ldots
\]

\[
\ldots \times \frac{d}{dD} \left. [\theta_B^{ED}[D, I_0, \mu'(E^h - D)] \right|_{D=D^* (I_0)} \times \ldots
\]

\[
\ldots \times \left( \frac{\Pi_B' / \Delta_B'}{\theta_B'^{ED}[D^*|SP (I_0), I_0], \mu'[E^h - D^*|SP (I_0)]} \right)
\]

\(< 0 \).

At the same time, using \( \cdot \) to suppress arguments, we also know that

\[
v_{II}^{SP} (\cdot) v_{DD}^{SP} (\cdot) - v_{DI}^{SP} (\cdot) v_{ID}^{SP} (\cdot)
= \left[ \alpha_G v_{GII}^{LR|SP} (\cdot) + \alpha_B v_{BII}^{ED|SP} (\cdot) \right] \left[ \alpha_G v_{GDD}^{LR|SP} (\cdot) + \alpha_B v_{BDD}^{ED|SP} (\cdot) \right]
- \left[ \alpha_G v_{GDI}^{LR|SP} (\cdot) + \alpha_B v_{BDI}^{ED|SP} (\cdot) \right] \left[ \alpha_G v_{GID}^{LR|SP} (\cdot) + \alpha_B v_{BID}^{ED|SP} (\cdot) \right]
= \alpha_G^2 [v_{GII}^{LR|SP} (\cdot) v_{GDD}^{LR|SP} (\cdot) - v_{GDI}^{LR|SP} (\cdot) v_{GID}^{LR|SP} (\cdot)]
\]

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\[ + \alpha_B^2 v_{BII}^{ED|SP} (\cdot) v_{BDD}^{ED|SP} (\cdot) - v_{BII}^{ED|SP} (\cdot) v_{BID}^{ED|SP} (\cdot) \]

\[ - \alpha_G \alpha_B [v_{GI}^{LR|SP} (\cdot) v_{BDD}^{ED|SP} (\cdot) - v_{BII}^{ED|SP} (\cdot) v_{GID}^{LR|SP} (\cdot)] \]

\[ - \alpha_G \alpha_B [v_{GI}^{LR|SP} (\cdot) v_{BII}^{ED|SP} (\cdot) - v_{BII}^{ED|SP} (\cdot) v_{GDI}^{LR|SP} (\cdot)], \]

so it would suffice if we could show that all of the terms in square brackets after the last equality are positive. To see that this is indeed the case, first note that

\[ v_{GI}^{LR|SP} (\cdot) v_{GDD}^{LR|SP} (\cdot) - v_{GDI}^{LR|SP} (\cdot) v_{GID}^{LR|SP} (\cdot) \]

\[ = v_{GI}^{LR|SP} (\cdot) \frac{d}{dD} \left[ v_{GDD}^{LR|SP} (\cdot) \right]_{D = D^{*|SP}(\cdot)} - v_{GDI}^{LR|SP} (\cdot) v_{GID}^{LR|SP} (\cdot), \]

where a repeat of the arguments on lines 3.41 through 3.44 will show that the right-hand side is strictly positive. Similarly,

\[ v_{BII}^{ED|SP} (\cdot) v_{BDD}^{ED|SP} (\cdot) - v_{BII}^{ED|SP} (\cdot) v_{BID}^{ED|SP} (\cdot) \]

\[ = v_{BII}^{ED|SP} (\cdot) \left[ \begin{array}{c} \mu'' (\cdot) - \left[ 2 \mu'' (\cdot) - D^{*|SP}(\cdot) \mu'' (\cdot) \right] (\Pi'_B/\Delta'_B) (\cdot) \\ < 0 \\
- [1 - \mu' (\cdot) + D^{*|SP}(\cdot) \mu'' (\cdot)] \frac{d}{dD} \left[ \theta^{ED}_B (\cdot) \right]_{D = D^{*|SP}(\cdot)} (\Pi'_B/\Delta'_B)' (\cdot) \\
< 0 
\end{array} \right] \]

\[ - v_{BII}^{ED|SP} (\cdot) v_{BID}^{ED|SP} (\cdot) \]

\[ > v_{BII}^{ED|SP} (\cdot) (-1) [1 - \mu' (\cdot) + D^{*|SP}(\cdot) \mu'' (\cdot)] \frac{d}{dD} \left[ \theta^{ED}_B (\cdot) \right]_{D = D^{*|SP}(\cdot)} (\Pi'_B/\Delta'_B)' (\cdot) \]

\[ - v_{BII}^{ED|SP} (\cdot) v_{BID}^{ED|SP} (\cdot) \]

\[ = [1 - \ell - \Delta_B (\cdot)] \theta^{ED}_B (\cdot) (\Pi'_B/\Delta'_B)' (\cdot) \times \cdots \]
\[
\cdots \times (-1)[1 - \mu'(\cdot) + D^*|SP(\cdot)\mu''(\cdot)] \frac{d}{dD} \left[ \theta_E^{ID}(\cdot) \right]_{D = D^*|SP(\cdot)} (\Pi'_B/\Delta'_B)'(\cdot)
\]
\[
+ [1 - \mu'(\cdot) + D^*|SP(\cdot)\mu''(\cdot)]\theta_E^{ED}(\cdot)(\Pi'_B/\Delta'_B)'(\cdot) \times \cdots
\]
\[
\cdots \times [1 - \ell - \Delta_B(\cdot)] \frac{d}{dD} \left[ \theta_E^{ID}(\cdot) \right]_{D = D^*|SP(\cdot)} (\Pi'_B/\Delta'_B)'(\cdot)
\]
\[= 0.
\]
Moreover,
\[
<0 \underbrace{v_{GII}|SLP(\cdot)} <0 \underbrace{v_{BDI}|SLP(\cdot)} - \underbrace{v_{GID}|SLP(\cdot)}_{\geq 0} > 0.
\]
Finally,
\[
<0 \underbrace{v_{GDD}|SLP(\cdot)} <0 \underbrace{v_{BDI}|SLP(\cdot)} - \underbrace{v_{GDI}|SLP(\cdot)}_{\geq 0} > 0.
\]

**Case three.** Suppose next that \(I_0\) has the property that the pair \([D^*|SP(I_0), I_0]\) lies in the region where \(r_{SP} B = DD\), but the “no-default” constraint is lax. In this case, we still want to confirm that
\[
v_{II}|SLP[D^*|SP(I_0), I_0]v_{DD}|SLP[D^*|SP(I_0), I_0] - v_{DI}|SLP[D^*|SP(I_0), I_0]v_{ID}|SLP[D^*|SP(I_0), I_0] > 0,
\]
and banks’ behaviour in the good state is unchanged relative to my previous case. As for their behaviour in the bad state, we know that
\[
v_{BD|DD}[D^*|SP(I_0), I_0]
\]
\[= \mu''[E^h - D^*|SP(I_0)]
\]
\[< 0 \text{ — see assumption 3.6}
\]
\[-\underbrace{2\mu''[E^h - D^*|SP(I_0)] - D^*|SP(I_0)\mu''[E^h - D^*|SP(I_0)]}_{\geq 0} (\Pi_B/\Delta_B)(\theta_B^2)
\]

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\[ \langle 0 \]

\[ = v_{BD}^{SP} [D^{1/3} (I_0), I_0] \]

\[ = v_{BI}^{DD} [D^{1/3} (I_0), I_0], \quad \forall x \in \{ D, I_0 \} \]

namely due to (3.55), (3.54), and (3.24). So, using \( \cdot \) to suppress arguments, we have

\[ v^\text{SP}_I (\cdot) v^\text{SP}_D (\cdot) v^\text{SP}_I (\cdot) v^\text{SP}_I (\cdot) = v^\text{SP}_I (\cdot) [\alpha_G v^\text{LR}_D (\cdot) + \alpha_B v^\text{DD}_D (\cdot)] - v^\text{SP}_I (\cdot) v^\text{SP}_I (\cdot) \]

\[ > v^\text{SP}_I (\cdot) \alpha_G v^\text{LR}_D (\cdot) - v^\text{SP}_I (\cdot) v^\text{SP}_I (\cdot) \]

\[ \propto v^\text{LR}_I (\cdot) v^\text{LR}_D (\cdot) - v^\text{LR}_I (\cdot) v^\text{LR}_I (\cdot) \]

\[ = v^\text{LR}_I (\cdot) \frac{d}{dD} \left[ v^\text{LR}_D (\cdot) \right] \bigg|_{D = D^{1/3} (\cdot)} - v^\text{LR}_I (\cdot) v^\text{LR}_I (\cdot) \]

where a repeat of the arguments on lines 3.41 through 3.44 will show that the right-hand side of the last equality is strictly positive.

\textbf{Case four.} Suppose finally that \( I_0 \) has the property that the “no-default” constraint binds at \( [D^{1/3} (I_0), I_0] \). In this case, we have

\[ h^{SP} (I_0) = v^\text{SP}_I [\overline{D}_B (I_0), I_0] + \left( \frac{1 - \ell}{1 - \mu' [E^h - \overline{D}_B (I_0)]} \right) \left[ v^\text{SP}_D [\overline{D}_B (I_0), I_0] \right] \]

\[ = \alpha_G \left[ v^\text{LR}_G I [\overline{D}_B (I_0), I_0] + \overline{D}_B (I_0) v^\text{LR}_G D [\overline{D}_B (I_0), I_0] \right] \]

\[ + \alpha_B \left[ v^\text{DD}_B [\overline{D}_B (I_0), I_0] + \overline{D}_B (I_0) v^\text{DD}_B [\overline{D}_B (I_0), I_0] \right], \]

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with
\[
\bar{D}_B''(I_0) = \frac{(-1) \bar{D}_B' (I_0) \left[ 2 \mu'' [E^h - \bar{D}_B(I_0)] - \bar{D}_B(I_0) \mu''' [E^h - \bar{D}_B(I_0)] \right]}{[1 - \mu'' [E^h - \bar{D}_B(I_0)] + \bar{D}_B(I_0) \mu'' [E^h - \bar{D}_B(I_0)]^2]} < 0.
\]

So,
\[
(h^{SP})'(I_0) = \alpha_{G} \left( \bar{D}_B''(I_0), I_0 \right) + \alpha_{G} \bar{D}_B'(I_0) \left( \bar{D}_B''(I_0), I_0 \right)
\]
\[
+ \alpha_{G} \bar{D}_B'(I_0) \left( \bar{D}_B''(I_0), I_0 \right) + \alpha_{G} \left[ \bar{D}_B'(I_0)^2 \left( \bar{D}_B''(I_0), I_0 \right) \right]
\]
\[
+ \alpha_B \left[ \bar{D}_B''(I_0)^2 \left( \bar{D}_B''(I_0), I_0 \right) \right] + \bar{D}_B''(I_0) \left( \bar{D}_B''(I_0), I_0 \right)
\]
\[
< 0,
\]
as desired.

3.A.5.5 Dependence on parameters

In this next subsubsection, I show how the solution identified above responds to changes in the model’s parameters. I thus let \( \beta = (\chi_G, \alpha_G, E^h) \) collect all the parameters on which we’ll focus and will include \( \beta \) as an explicit argument in any functions into which these parameters enter. Now, based on the analysis in my previous subsubsection, it should be clear that
\[
h^{SP}[I_B^{I|SP}(\beta), \beta] > h^{SP}[I_B^{I|SP}(\beta), \beta] > h^{SP}[I_B^{I|SP}(\beta), \beta],
\]
and also that the regime associated with the bad state under the planner’s solution depends on where zero lies in this chain of inequalities — e.g., a liquidity surplus occurs i.f.f. \( h^{SP}[I_B^{I|SP}(\beta), \beta] \leq 0 \). So, this subsubsection will focus on determining how parameters influence the signs on \( h^{SP}[I_B^{I|SP}(\beta), \beta], h^{SP}[I_B^{I|SP}(\beta), \beta], \) and \( h^{SP}[I_B^{I|SP}(\beta), \beta] \).
For expositional purposes, it’s best to begin with $h^{SP}[I_B^{\Xi|SP}(\beta), \beta]$. In particular, let’s define $D_B^{\Xi|SP}(\beta) := D^{*|SP}[I_B^{\Xi|SP}(\beta), \beta]$, so that the pair $[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta)]$ solves

$$[E^h + D_B^{\Xi|SP}(\beta) - I_B^{\Xi|SP}(\beta)] + \ell I_B^{\Xi|SP}(\beta) + I_B^{\Xi|SP}(\beta) \Delta_B(\theta^\Xi_B) = D_B^{\Xi|SP}(\beta) \times \cdots$$

$$\cdots \times \mu'[E^h - D_B^{\Xi|SP}(\beta)]$$

$$\iff \quad E^h + D_B^{\Xi|SP}(\beta)[1 - \mu'[E^h - D_B^{\Xi|SP}(\beta)] = I_B^{\Xi|SP}(\beta)[1 - \ell - \Delta_B(\theta^\Xi_B)], \quad (3.60)$$

and

$$\alpha_G v^{LR|SP}_G[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta), \beta] + (1 - \alpha_G) v^{DD|SP}_B[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta), \beta] = 0$$

$$\iff 1 - \mu'[E^h - D_B^{\Xi}(\beta)] + \alpha_G(\Pi_G/\Psi_G)[\theta_G^{\Xi}[D_B^{\Xi}(\beta), I_B^{\Xi}(\beta), \beta] - (1 - \alpha_G)[1 - \mu'[E^h - D_B^{\Xi}(\beta)] + D_B^{\Xi}(\beta)\mu'[E^h - D_B^{\Xi}(\beta)])(\Pi_B/\Delta_B)(\theta^\Xi_B) = 0, \quad (3.61)$$

with

$$[E^h + D_B^{\Xi|SP}(\beta) - I_B^{\Xi|SP}(\beta)] + \ell I_B^{\Xi|SP}(\beta) + I_B^{\Xi|SP}(\beta)\Psi_G[\theta_G^{\Xi}[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta), \beta]$$

$$\iff E^h + D_B^{\Xi|SP}(\beta) = I_B^{\Xi|SP}(\beta)[1 - \ell + \Psi_G[\theta_G^{\Xi}[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta), \beta]]. \quad (3.62)$$

Now, dividing (3.60) by (3.62) yields

$$\left[1 - \frac{D_B^{\Xi|SP}(\beta)\mu'[E^h - D_B^{\Xi|SP}(\beta)]}{E^h + D_B^{\Xi|SP}(\beta)} \right] \times \cdots$$

$$\cdots \times [1 - \ell + \Psi_G[\theta_G^{\Xi}[D_B^{\Xi|SP}(\beta), I_B^{\Xi|SP}(\beta), \beta]] = 1 - \ell - \Delta_B(\theta^\Xi_B), \quad (3.63)$$

with $g^{SP}_D[D_B^{\Xi|SP}(\beta), \beta] > 0$. Differentiating (3.61) and (3.63) w.r.t. $\chi_G$ then yields the following system, where I’ve used $\cdot$ to suppress arguments:
Figure 3.4: Visual aid for proving lemmata 3.4.4 through 3.4.7
\[ \frac{d}{d\chi_G} \left[ \theta_G^{LR}(\cdot) \right] \alpha_G(\Pi'_G/\Psi'_G)\theta(\cdot) + \alpha_G(\Pi'_G/\Psi'_G)\chi_G(\cdot) > 0 \]

\[ = (D_{\beta}^{\Xi|SP})\chi_G(\cdot) \left[ (1 - \alpha_G) [2\mu''(\cdot) - D_{\beta}^{\Xi|SP}(\cdot)\mu''(\cdot)] (\Pi_B/\Delta_B)(\cdot) - \mu''(\cdot) \right] > 0 \]

\[ \frac{d}{d\chi_G} \left[ \theta_G^{LR}(\cdot) \right] [1 - g^{SP}(\cdot)]\Psi'_G(\cdot) = (D_{\beta}^{\Xi|SP})\chi_G(\cdot) g^{SP}_{\beta}(\cdot)[1 - \ell + \Psi_G(\cdot)] \]

An application of Cramer’s rule then yields

\[ \frac{d}{d\chi_G} \left[ \theta_G^{LR}(\cdot) \right] < 0, \]

and similar arguments yield

\[ \frac{d}{d\alpha_G} \left[ \theta_G^{LR}(\cdot) \right] < 0 \quad \frac{d}{d\varepsilon^b} \left[ \theta_G^{LR}(\cdot) \right]. \]

Moreover, a repeat of the arguments on lines 1.22 through 1.27 yields

\[ h^{SP}[I_{\beta}^{\Xi|SP}(\beta, \beta)] \propto \int_{\theta_G^{LR}[D_{\beta}^{\Xi|SP}(\beta), I_{\beta}^{\Xi|SP}(\beta, \beta)]}^{1} \theta dF(\theta) \]

\[ - \left[ 1 - \ell + \int_{\theta_G^{LR}[D_{\beta}^{\Xi|SP}(\beta), I_{\beta}^{\Xi|SP}(\beta, \beta)]}^{1} (\rho + \ell) dF(\theta) \right] \times \]

\[ \cdots \times \left[ \frac{\theta_G^{LR}[D_{\beta}^{\Xi|SP}(\beta), I_{\beta}^{\Xi|SP}(\beta, \beta)]}{\rho + \ell} \right] \]

\[ + [(1 - \alpha_G)/\alpha_G\chi_G](\ell - 1)[1 - (\Pi_B/\Delta_B)(\theta_{\beta}^\Xi)] \]

\[ =: \tilde{h}^{SP}(\beta) \]
with

\[ \hat{h}_{\chi G}^{SP}(\beta) = \frac{d}{d\chi G} \left[ \theta_{G}^{LR} [D_{B}^{\Xi|SP}(\beta), I_{B}^{\Xi|SP}(\beta), \beta] \left( \frac{-1}{\rho + \ell} \right) \times \cdots \right. \]

\[ \left. \cdots \times \left[ 1 - \ell + \int_{\theta_{G}^{LR}|D_{B}^{\Xi|SP}(\beta), I_{B}^{\Xi|SP}(\beta), \beta}^{1} (\rho + \ell) dF(\theta) \right] \right. \]

\[ \left. - \left[ (1 - \alpha_{G}) / \alpha_{G} \chi_{G}^{2} \right] (\ell - 1) [1 - (\Pi_{B} / \Delta_{B})(\theta_{B}^{\Xi})] \right] > 0. \]

Repeating for \( \alpha_{G} \) and \( E^{b} \) yields

\[ \hat{h}_{\alpha G}^{SP}(\beta) > 0 > \hat{h}_{E^{b}}^{SP}(\beta), \]

so \( h^{SP}[I_{B}^{\Xi|SP}(\beta), \beta] \) is more likely to be positive the greater is \( (\chi_{G}, \alpha_{G}) \) and the lesser is \( E^{b} \). That these tendencies also hold for \( h^{SP}[I_{B}^{\Pi|SP}(\beta), \beta] \) and \( h^{SP}[I_{B}^{SP}(\beta), \beta] \) can be shown using similar arguments.

The usual partitions of the parameter space can then be constructed. In particular, we can conclude that the function \( h^{SP}[I_{B}^{\Xi|SP}(\beta), \beta] \), viewed as a function of the parameter \( \chi_{G} \), is either single-signed over the interval \( (\chi_{G}, \infty) \) or otherwise exhibits single-crossing from below over this interval. In the latter case, I set \( \chi_{G}^{ED|SP} \) equal to the crossing point. Otherwise, I set \( \chi_{G}^{ED|SP} = \chi_{G} \) \( (\chi_{G}^{ED|SP} = \infty) \) in the case of uniform positivity (negativity). The rest of the partition can be constructed on a mutatis mutandis basis, and similar constructions go through w.r.t. \( \alpha_{G} \) and \( E^{b} \).

3.A.5.6 Comparison with the equilibrium allocation (part one)

In this subsubsection, I focus on proving lemma \[3.4.5\]. For expositional purposes, I’ll begin with the claim made therein that the planner’s solution exhibits a dual distortion in the bad state only if the economy also exhibits this property in equilibrium. In light of the analysis in subsections \[3.4.5\] and \[3.5.5\] it should be clear that a sufficient condition for this claim would be that \( h(I_{B}^{\Xi}) > h^{SP}(I_{B}^{\Xi|SP}) \). As a first step toward verifying this inequality, recall lemma \[3.4.3\] that the compositions

\[ v_{D}[D_{B}^{\Xi}(I_{0}), I_{0}] \]
\[ v^G_G \] LR \ D \ \Xi_B \left( I_0 \right), I_0, \mu' \left[ E^h - D^G_B \left( I_0 \right) \right] + \alpha_B v^DD_B \left[ D^G_B \left( I_0 \right), I_0, \mu' \left[ E^h - D^G_B \left( I_0 \right) \right] \right] \\

and

\[ v^D_D \left[ D^B_B \left( I_0 \right), I_0 \right] = \alpha_G v^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] + \alpha_B v^{DD}_B \left[ D^B_B \left( I_0 \right), I_0 \right] \]

both exhibit single-crossing from below over the interval

\[ \{ I_0 \in \mathbb{R}_+ \text{ s.t. } D^B_B \left( I_0 \right) \leq D^H_H \left( I_0 \right) \}, \] (3.64)

namely at interior points \( I^B_B \) and \( I^B_B | SP \), respectively. Moreover, it should be clear that

\[ v^D_D \left[ D^B_B \left( I^B_B \right), I^B_B \right] < v^D_D \left[ D^B_B \left( I^B_B \right), I^B_B \right], \]

namely due to (3.23), (3.50), and (3.55), so we can conclude that \( I^B_B < I^B_B | SP \). Next, note that all points in the interval on line (3.64) satisfy

\[ h^SP \left( I_0 \right) = \alpha_G v^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] + \alpha_B v^{DD}_B \left[ D^B_B \left( I_0 \right), I_0 \right] \]

\[ = \alpha_G v^{LR}_G \left[ D^B_B \left( I_0 \right), I_0, \mu' \left[ E^h - D^B_B \left( I_0 \right) \right] \right] \]

\[ + \alpha_B v^{DD}_B \left[ D^B_B \left( I_0 \right), I_0, \mu' \left[ E^h - D^B_B \left( I_0 \right) \right] \right] \] (3.65)

\[ = \alpha_G \left[ \ell + \Pi_G \left[ \theta^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] \right] - \left[ 1 - \ell \right] + \Psi_G \left[ \theta^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] \right] \right] \]

\[ \times \cdots \times \left( \Pi'_G / \Psi'_G \right) \left[ \theta^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] \right] \]

\[ = \alpha_G \left( 1 - \ell \right) \left( \Pi_B / \Delta_B \right) \left( \theta^B_B \right), \] (3.66)

where (3.65) follows from (3.54) and (3.50), while (3.66) follows from (3.24) and (3.16). In turn,

\[ h' \left( I_0 \right) = \alpha_G \left( -1 \right) \left[ 1 - \ell + \Psi_G \left[ \theta^{LR}_G \left[ D^B_B \left( I_0 \right), I_0 \right] \right] \right] \times \cdots \]

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Conclude that
\[ h(I_B^{\Xi}) = h^{SP}(I_B^{\Xi}) > h^{SP}(I_B^{\Xi|^{SP}}), \]
as desired. Similar arguments yield
\[ h(I_B^{\Pi}) = h^{SP}(I_B^{\Pi|^{SP}}), \]
with
\[ h(I_B) > h^{SP}(I_B^{SP}), \]
and it should be clear that these results respectively imply that (i) the planner’s solution has the property that one of the distorted regimes obtains in the bad state i.f.f. this is also true in equilibrium; and (ii) the planner’s solution has the property that an interbank collapse occurs in the bad state only if the economy also exhibits this property in equilibrium.

3.A.5.7 Comparison with the equilibrium allocation (part two)

At this point, all that remains is to verify lemmata 3.4.6 and 3.4.7. Fortunately, it should be clear that they both follow immediately from lemma 3.4.5, combined with the analysis in subsubsections 3.A.4.5 and 3.A.5.5.

3.A.6 Proof of proposition 3.1

3.A.6.1 Notation

I’ll begin by supplementing the notation established in subsubsections (3.A.4.2) and (3.A.5.2) as follows:

- let \( (D^{SP}, I_0^{SP}) \) denote the initial balance sheet preferred by the planner;
- let \( \hat{I}_B \) solve
  \[ \overline{D}_B(\hat{I}_B) = D^{SP}. \]
3.A.6.2 Some preliminary results

Sublemma 3.A.7. All initial balance sheets satisfying $D \in [D^H_G(I_0), \overline{D}_B(I_0)]$ have the following three properties: (i) the function $v_I(D, I_0)$ is strictly decreasing in its second argument; (ii) $v_D(D, I_0) \geq v^{SP}_D(D, I_0)$; (iii) $v_I(D, I_0) = v^{SP}_I(D, I_0)$;

Proof. I’ll proceed claim-by-claim:

- with respect to (i), I note that all initial balance sheets satisfying $D \in [D^H_G(I_0), \overline{D}_B(I_0)]$ also satisfy

$$v_I(D, I_0) = \alpha_G v^{LR}_G[D, I_0, \mu'(E^h - D)]$$

$$+ \alpha_B \begin{cases} v^{LS}_B[D, I_0, \mu'(E^h - D)] & \text{if } D \in [0, \min\{D^H_G(I_0), D^H_B(I_0)\}] \\ v^{ED}_B[D, I_0, \mu'(E^h - D)] & \text{if } D \in (D^H_B(I_0), \min\{D^H_G(I_0), D^B_B(I_0)\}] \\ v^{DP}_B[D, I_0, \mu'(E^h - D)] & \text{if } D \in (D^B_B(I_0), \min\{D^H_G(I_0), \overline{D}_B(I_0)\}] \end{cases},$$

in which case it suffices to note that the functions $v^{LR}_G[D, I_0, \mu'(E^h - D)]$ and $v^{LS}_B[D, I_0, \mu'(E^h - D)]$ are strictly decreasing in their second arguments, while the functions $v^{LS}_B[D, I_0, \mu'(E^h - D)]$ and $v^{DP}_B[D, I_0, \mu'(E^h - D)]$ are constant in their second arguments — see (3.30), (3.40), (3.34), and (3.45), respectively;

- with respect to (ii), I note that all initial balance sheets satisfying $D \in [D^H_G(I_0), \overline{D}_B(I_0)]$ also satisfy

$$v_D(D, I_0) = \alpha_G v^{LR}_G[D, I_0, \mu'(E^h - D)]$$
This should be obvious.

with respect to (iii), see (3.50), (3.56), (3.51), and (3.54).

Sublemma 3.A.8. The planner’s solution satisfies $I_{0}^{SP} \leq \hat{T}_{B}$.

Proof. This should be obvious.
Sublemma 3.A.9. The composition
\[ v_I[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h - \overline{D}_B(I_0)]} \right] v_D[D_B(I_0), I_0] \]
is strictly decreasing over the interval \([\overline{I}_B, E^h/(1 - \ell)]\).

Proof. This has already been established — see case four in subsubsection 3.A.4.4 in particular.

\[ \blacksquare \]

Sublemma 3.A.10. All \( I_0 \in (T^{SP}_B, E^h/(1 - \ell)) \) satisfy
\[ v_I^{SP}[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h + \overline{D}_B(I_0)] + \overline{D}_B(I_0)\mu''[E^h - \overline{D}_B(I_0)]} \right] v_D^{SP}[D_B(I_0), I_0] \]
\[ > v_I[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h - \overline{D}_B(I_0)]} \right] v_D[D_B(I_0), I_0]. \]

Proof. Fix some \( I_0 \in (T^{SP}_B, E^h/(1 - \ell)) \) and note that
\[ v_D[D_B(I_0), I_0] \geq v_D^{SP}[D_B(I_0), I_0] > 0, \]
where the first inequality follows from item (ii) in sublemma 3.A.7 while the second follows from sublemma 3.A.5. So,
\[ v_I^{SP}[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h + \overline{D}_B(I_0)] + \overline{D}_B(I_0)\mu''[E^h - \overline{D}_B(I_0)]} \right] v_D^{SP}[D_B(I_0), I_0] \]
\[ > v_I^{SP}[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h - \overline{D}_B(I_0)]} \right] v_D[D_B(I_0), I_0] \]
\[ > v_I^{SP}[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h + \overline{D}_B(I_0)]} \right] v_D[D_B(I_0), I_0] \]
\[ = v_I[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h + \overline{D}_B(I_0)]} \right] v_D[D_B(I_0), I_0]. \]
where the last line follows from item (iii) in sublemma 3.A.7.

Sublemma 3.A.11. It must be the case that $\bar{T}_B^{SB} \geq T_B$.

Proof. This is just a corollary of sublemmata (3.A.2) and (3.A.5), combined with item (ii) in sublemma 3.A.7.

3.A.6.3 Main result

In this last subsubsection, I argue that policymakers can ensure a unique equilibrium, namely under which banks set $(D, I_0) = (D^{SB}, I_0^{SB})$, by imposing the leverage constraint proposed in proposition 3.1 with $d = D^{SP}/E^b$. Now, even when facing a regulation of this form, the usual arguments will confirm that banks still find some slack in the non-negativity constraints $I_0 \geq 0$, $D \geq 0$, and $I_0 < E^b + D$. As a result, equilibria must fall under one of four distinct cases. The first would be an interior case under which the leverage and “no-default” constraints are both lax, so

$$v_x(D, I_0) = 0, \quad \forall x \in \{D, I_0\}.$$ 

The other three cases would be corner cases under which at least one of the leverage and “no-default” constraints binds.

As a first step toward narrowing down these cases, I note that the usual arguments preclude banks’ settling on any initial balance sheet satisfying $D \geq D^H(I_0)$, so we can restrict attention to initial balance sheets satisfying $I_0 \in [E^b/[1 - \ell + \Psi G(\theta^H_G)], E^b/(1 - \ell)]$. Moreover, for any choice on $I_0$ in this range, sublemma 3.A.1 implies that one of two scenarios must obtain. The first would be that the marginal return $v_D(D, I_0)$, viewed as a function of the choice on $D$, exhibits single-crossing from above over the interval $[0, \min\{D^{SP}, D^H_G(I_0), \bar{D}_B(I_0)\}]$, namely at a point which I’ll denote $\hat{D}^*(I_0)$. The alternative scenario has the marginal return $v_D(D, I_0)$ strictly positive over all of the interval $[0, \min\{D^{SP}, D^H_G(I_0), \bar{D}_B(I_0)\}]$, in which case I adopt a convention that $\hat{D}^*(I_0) = \min\{D^{SP}, D^H_G(I_0), \bar{D}_B(I_0)\}$.

Based on this analysis, we can further restrict attention to initial balance sheets satisfying $D = \hat{D}^*(I_0)$. Given any initial balance sheet of this form, I’ll now check if banks have an incentive to deviate in their initial balance-sheet choices, namely by making some small adjustment to their choice on $I_0$. Of course, adjusting $I_0$ may require an offsetting adjustment in $D$ if the “no-default” constraint binds. On this front, I note from sublemma 3.A.2 that $I_0 < \max\{I_B, \hat{I}_B\} \implies \hat{D}^*(I_0) < \min\{D^H_G(I_0), \bar{D}_B(I_0)\}$, and otherwise $\hat{D}^*(I_0) = \bar{D}_B(I_0)$. So, if $I_0 < \max\{I_B, \hat{I}_B\}$, then banks’ return from a marginal increase in $I_0$ is given by $v_I[\hat{D}^*(I_0), I_0]$. Otherwise, the “no-default” constraint binds, and increases in $I_0$ must be
offset by decreases in \( D \), so the relevant return reads as
\[
v_I \tilde{D}^*(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^b - \tilde{D}^*(I_0)]} \right] v_D \tilde{D}^*(I_0), I_0],
\]
where the starred terms gives the rate of transformation along the “no-default” constraints for banks taking the interest rate on deposits as given.

With these points in mind, I suggest that we now take cases as follows:

**Case one.** Suppose first that the planner’s solution has the property that the “no-default” constraint is lax, with \( I_B > \hat{I}_B \), so the situation is as given in figure 3.5. In this case, it should be clear that the function
\[
\hat{h}(I_0) := \begin{cases} v_I \tilde{D}^*(I_0), I_0] & \text{if } I_0 < \hat{I}_B \\
v_I \tilde{D}^*(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^b - \tilde{D}^*(I_0)]} \right] v_D \tilde{D}^*(I_0), I_0] & \text{if } I_0 \geq \hat{I}_B
\end{cases}
\]
is continuous over the interval \([E^b/(1 - \ell + \Psi G(\theta \Pi G)), E^b/(1 - \ell))\), with
\[
\hat{h}[E^b/[1 - \ell + \Psi G(\theta \Pi G))] > 0,
\]
and
\[
\lim_{I_0 \uparrow E^b/(1 - \ell)} \{\hat{h}(I_0)\} = -\infty,
\]
namely due to assumption 3.1 along with the fact that \( \mu'(E^b) = 1 \). I furthermore claim that this function is strictly decreasing over the interval \([E^b/[1 - \ell + \Psi G(\theta \Pi G)), E^b/(1 - \ell))\). That this must be true whenever \( I_0 \geq \hat{I}_B \) — or \( I_0 < \hat{I}_B \) with \( \hat{D}^*(I_0) < D^{SP} \) — should be clear from the analysis in subsubsection 3.A.4.4 while the complementary case under which \( I_0 < \hat{I}_B \) with \( \hat{D}^*(I_0) = D^{SP} \) is dispatched by item (i) in sublemma 3.A.7.

We can now conclude that the function \( \hat{h}(I_0) \) exhibits single-crossing from above over the interval \([E^b/[1 - \ell + \Psi G(\theta \Pi G)), E^b/(1 - \ell))\), namely at an interior point which I’ll denote \( \hat{I}_0^* \). The point \( (D, I_0) = [\hat{D}^*(\hat{I}_0^*), \hat{I}_0^*] \) thus constitutes our only candidate for equilibrium under the present case. I furthermore claim that this candidate is indeed an equilibrium. In principle, this would require that we rule out banks’ having some incentive to engage in non-local deviations, since the candidate has only been constructed in a way which precludes their...
having some incentive to engage in local deviations. Fortunately, this can easily be verified, though I’ve chosen to omit a formal proof for brevity.

At this point, all that remains for case one is to check that \( [\hat{D}^*(\hat{I}_0^*), \hat{I}_0^*] = (D^{SP}, I_0^{SP}) \). As a first step in this direction, I note from item (ii) in sublemma 3.A.7 that

\[
v_D(D^{SP}, I_0^{SP}) \geq v_D^*(D^{SP}, I_0^{SP}) = 0,
\]

where the equality follows from the fact that case one only obtains when the planner’s solution is interior. Combining this result with sublemma 3.A.8 then yields \( \hat{D}^*(I_0^{SP}) = D^{SP} \).

It then suffices to note that

\[
\hat{h}(I_0^{SP}) = v_I[\hat{D}^*(I_0^{SP}), I_0^{SP}] = v_I(D^{SP}, I_0^{SP}) = v_I^*(D^{SP}, I_0^{SP}) = 0,
\]

where the penultimate equality follows from item (iii) in sublemma 3.A.7, while the final equality again follows from interiority of the planner’s solution.

**Case two.** Suppose next that the planner’s solution still has the property that the “no-default” constraint is lax, but now \( I_B \leq \hat{I}_B \), so the situation is as given in figure 3.6. In this case,

\[
v_I(D^{SP}, \hat{I}_B) \leq v_I(D^{SP}, I_0^{SP}) = 0,
\]

where the inequality follows from sublemma 3.A.8 combined with item (i) in 3.A.7, while the equality follows from the fact that case two only obtains when the planner’s solution is interior. At the same time,

\[
v_D(D^{SP}, \hat{I}_B) \geq 0,
\]

namely due to sublemma 3.A.2 combined with the fact that \( I_B \leq \hat{I}_B \). Conclude that

\[
v_I(D^{SP}, \hat{I}_B) + \left[ \frac{1 - \ell}{1 - \mu'[E^h - D^{SP}]} \right] v_D(D^{SP}, \hat{I}_B) \leq 0,
\]

in which case sublemma 3.A.9 combined with the fact that \( I_B \leq \hat{I}_B \), yields

\[
v_I(\overline{D}_B(I_0), I_0) + \left[ \frac{1 - \ell}{1 - \mu'[E^h - \overline{D}_B(I_0)]} \right] v_D(\overline{D}_B(I_0), I_0) < 0, \quad \forall I_0 \in (\hat{I}_B, E^h/\ell) .
\]

This means that we can restrict attention to candidate equilibria under which \( I_0 \in [E^h/[1 - \ell + \Psi_G(\theta)]) , \hat{I}_B \).
Of course, for any such candidate, banks’ return from a small increase in $I_0$ is given by

$$v_I[\hat{D}^*(I_0), I_0].$$

(3.69)

Now, it should be clear that this function (i) is continuous over the interval $[E_b/[1 - \ell + \Psi_G(\theta^H_G)], \hat{I}_B]$; (ii) satisfies

$$\hat{h}[E_b/[1 - \ell + \Psi_G(\theta^H_G)]] > 0,$$

namely due to assumption 3.1; and (iii) satisfies

$$v_I[\hat{D}^*(\hat{I}_B), \hat{I}_B] = v_I(D^{SP}, \hat{I}_B) \leq 0,$$

where the equality follows from (3.68), while the inequality has already been established — see (3.67) in particular. The usual arguments will further show that the function on line 3.69 is also strictly decreasing over the interval $[E_b/[1 - \ell + \Psi_G(\theta^H_G)], \hat{I}_B]$, so we can conclude that this interval admits a (potentially non-interior) point $\hat{I}_0^*$ around which we have single-crossing from above. The point $(D, I_0) = [\hat{D}^*(I_0^*), I_0^*]$ thus constitutes our only candidate for equilibrium. In fact, I claim that this candidate is indeed an equilibrium. As in my previous case, this would, in principle, require that we rule out banks’ having some incentive to engage in non-local deviations, since the candidate has only been constructed in a way which precludes their having some incentive to engage in local deviations. Fortunately, this can easily be verified, though I’ve chosen to omit a formal proof for brevity.

At this point, all that remains for case two is to check that $[\hat{D}^*(I_0^*), I_0^*] = (D^{SP}, I_0^{SP})$. On this front, the relevant arguments from case one still go through. In particular, item (ii) in sublemma 3.A.7 yields

$$v_D(D^{SP}, I_0^{SP}) \geq v_D^{SP}(D^{SP}, I_0^{SP}) = 0,$$

where the equality follows from the fact that case two only obtains when the planner’s solution is interior. Combining this result with 3.A.8 then yields $\hat{D}^*(I_0^{SP}) = D^{SP}$. It then suffices to note that

$$v_I[\hat{D}^*(I_0^{SP}), I_0^{SP}] = v_I(D^{SP}, I_0^{SP}) = v_I^{SP}(D^{SP}, I_0^{SP}) = 0,$$

where the penultimate equality follows from item (iii) in sublemma 3.A.7 while the final equality again follows from interiority of the planner’s solution.

**Case three.** Suppose finally that the planner’s solution has the property that the “no-default” constraint binds. In this case, it should be clear that $(D^{SP}, I_0^{SP}) = [\overline{D}_B(\hat{I}_B), \hat{I}_B]$, with
Figure 3.5: Illustration of cases one and three in my proof of proposition 3.1
\( \hat{I}_B > I^\text{SP}_B \geq I_B \) where the last inequality follows from sublemma [3.A.11]. The situation is thus again as depicted in figure [3.5]. It should also be clear that

\[
v^\text{SP}_D[D_B(\hat{I}_B), \hat{I}_B] > 0, \tag{3.70}
\]

with

\[
v^\text{SP}_I[D_B(\hat{I}_B), \hat{I}_B] = 0. \tag{3.71}
\]

Combining (3.71) with sublemmata 3.A.10 and 3.A.9 then yields

\[
v[I[D_B(I_0), I_0] + \left[ \frac{1 - \ell}{1 - \mu'[E^h - D_B(I_0)] + D_B(I_0)\mu''[E^h - D_B(I_0)]} \right] v^\text{SP}_D[D_B(I_0), I_0] < 0 \tag{3.72}
\]

for all \( I_0 \in (\hat{I}_B, E^b/(1 - \ell)) \supset (\hat{I}_B, E^b/(1 - \ell)) \), so we can restrict attention to candidate equilibria under which \( I_0 \in [E^b/[1 - \ell + \Psi_G(\theta_G^\Pi)], \hat{I}_B] \).

Of course, for any such candidate, banks’ return from a small increase in \( I_0 \) is given by

\[
v[I[D^*(I_0), I_0],
\]

and the usual arguments will confirm that this function is strictly decreasing over the interval \([E^b/[1 - \ell + \Psi_G(\theta_G^\Pi)], \hat{I}_B] \). I furthermore note that it satisfies

\[
v[I[D^*(\hat{I}_B), \hat{I}_B] = v^\text{SP}_I[D^*(\hat{I}_B), \hat{I}_B] > 0, \tag{3.73}
\]

where the equality follows from item (iii) in sublemma [3.A.7] while the inequality follows from (3.70) and (3.71). This makes \( I^\text{SP}_\text{0} \) our only candidate equilibrium. In fact, I claim that this candidate is indeed an equilibrium. As in my previous case, this would, in principle, require that we rule out banks’ having some incentive to engage in non-local deviations, since the candidate has only been constructed in a way which precludes their having some incentive to engage in local deviations. Fortunately, this can easily be verified, though I’ve chosen to omit a formal proof for brevity. All that remains is to check that \( D^*(I^\text{SP}_\text{0}) = D^\text{SP} \), but this follows immediately from (3.70), combined with item (ii) in sublemma [3.A.7].
Figure 3.6: Illustration of case two in my proof of proposition 3.1
3.A.7 Proof of proposition 3.2

This should now be obvious.
Bibliography


