EXAMINATION OF THE RATE-STATE FRICTION
EQUATIONS UNDER LARGE PERTURBATIONS FROM
STEADY SLIDING: A THEORETICAL AND
EXPERIMENTAL STUDY.

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Abstract

The laboratory derived rate-state friction (RSF) relationships are the most widely used constitutive equations for fault friction in numerical models of fault mechanics. But even after more than three decades of these being first proposed, we are far from certain about the identity of the ‘proper’ set of these equations which describe all laboratory friction data. In fact, the two most popular choices of the ‘state’ evolution component of RSF represent two end-member physical pictures of how frictional strength evolves – with time even without slip (Aging law) or only with slip (Slip law). Yet both these viewpoints have traditionally been inferred to be independently supported by different classes of friction experiments which (sometimes) access similar portions of the RSF parameter space. We present a set of comprehensive studies which establish, both theoretically and with inversion of laboratory data, that in fact all the widely used experimental protocols provide evidence that friction dominantly evolves with slip even when the interface is sliding at the lowest slip rates accessed by these experiments.

We examined these state evolution laws under a diverse range of sliding conditions – up to 3.5 orders of velocity steps on both initially bare rock and gouge, up to $3 \times 10^4$ s long holds on initially bare rock performed using machine stiffnesses differing by 1.5 orders of magnitude and 5% normal stress steps on initially bare rock carried out at an order of magnitude different sliding rates. For all of these experimental regimes, the widely used Aging law generally performed worse than the Slip law, even in those parts of the parameter space where conventional RSF wisdom would have predicted it to find strong support. Additionally, across all these experiments, more recent prescriptions of state evolution were generally found to fit the data only as well as the Slip law even with the freedom of extra parameters. We argue that these findings contradict the traditional view that the state variable is a proxy for the ‘quantity’ of true contact area alone, it is likely that some measure of the ‘quality’ of contacts contributes significantly to state evolution as well.
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To my parents, Nupur and Pranab, and Jaya
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solution for long holds. Velocity neutral and strengthening solutions show
continual relaxation of stress but, importantly, the rate of stress relaxation
is stiffness independent for long holds. Note that the experiments of Beeler
\textit{et al.} (1994) span the range from \(0.3 - 1.0 \lesssim V_s/r_{hold}/D_c \lesssim 3 \times 10^3 - 10^4\)
between the high and normal stiffness experiments.

4.5 Stress relaxation under the Slip law during a long hold \((\sim 10^5 \text{ s})\) for different
values of the normalized stiffness, \(kD_c/a = 1\) (blue), 10 (ochre) and 100
(green), and different values of \(b/a\). (A) Velocity weakening, \(b/a = 2\), (B)
velocity neutral, \(b/a = 1\) and (C) velocity strengthening, \(b/a = 0.5\). The
solid color lines are the numerically integrated values of \(\Delta \mu/a\). Corresponding
dashed lines denote the relevant analytical approximations derived in
Appendix D.3.2. In the small stiffness limit (dashed blue lines), the velocity
weakening Slip law predicts a \(\log(\log(t_{hold}))\) trajectory while the velocity
strengthening trajectories are linear in \(\log(t_{hold})\). The large stiffness limit
for all trajectories is \(\Delta \mu/a = \ln(V/V_s/r)\) (dashed red lines).

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4.6 Aging and Slip law fits to the time evolution of $\Delta \mu_{\text{peak}}$ in initially bare granite from Beeler et al. (1994). The red squares and circles are the low stiffness data, the corresponding blue symbols are the high stiffness data. (A) Aging law fit (solid line) with $a - b$ fixed at -0.0027. (B) Aging (solid line) and Slip law (dashed line) fits without any constraint on $a - b$. Note that the Slip law fit to the time evolution of $\Delta \mu_{\text{peak}}$ is as good as the Aging law with the healing rate being stiffness independent. However, for both the Aging and Slip laws, the parameter choices which fit the peaks very well completely fail to match the corresponding values of $\Delta \mu_{\text{hold}}$.

4.7 The evolution of $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ with hold time for the Slip law fits to peak stresses alone. (A) The hatched and filled regions depict an ensemble of simulated time series for $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ respectively drawn from the posterior generated in obtaining the Slip law fit in Figure 4.6b. The density of color represents number density of samples from the posterior. This ensemble represents $\sim$ 400 samples (10% of the total number accepted) drawn randomly from the steady-state posterior realized by running a Markov chain 50,000 samples long. Squares are the Beeler et al. (1994) values of $\Delta \mu_{\text{peak}}$ scaled by $b$. The $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ values (solid lines) were read from the full time series from which the Slip law fits in Figure 4.6b were derived. Dashed lines show the coarsely sampled actual fit to peak stresses in Figure 4.6b. (B), (C) and (D) show the actual posterior distributions of $a$, $b$ and $D_c$. The yellow squares show the location of the fit shown in Figure 4.6(b), this is statistically equivalent to the maximum posterior solution. The thick red lines show the maximum likelihood Gaussian fits to the posteriors.
4.8 Aging and Slip law fits to the time evolution of $\Delta \mu_{\text{hold}}$ in initially bare granite from [Beeler et al. (1994)]. (A) Aging (solid line) and Slip (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Fits without any constraint on $a - b$. The two Aging law fits are the maximum a posteriori (solid line) and the minimum misfit solutions (dot-dashed line). These two solutions are separated by orders of magnitude of parameter values for this inversion, likely due to numerical noise around the minimum misfit solution which assumes unrealistic parameter values. The Slip law fit is the dashed line.

4.9 Detailed comparison between the stress relaxation during the 3162 s and 10,000 s holds with the stress relaxation predicted by the Slip and Aging law fits (maximum a posteriori) to the stress minima from Figure 4.8b. The abscissa is the hold duration scaled to unity to help with visualization. (A) Slip and (B) Aging law predictions of stress relaxation from the fits to the low stiffness $\Delta \mu_{\text{hold}}$ data. (C) Slip and (D) Aging law predictions of stress relaxation from the fits to the high stiffness $\Delta \mu_{\text{hold}}$ data. Insets in (A) and (C) show the details of stress evolution during the first 5% of the hold. In all the plots, blue is data, red is Slip law and green is Aging law. Note that the best Slip law fit explains the full stress-relaxation time series very well.

4.10 Aging and Slip law fits to the time evolution of both $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ in initially bare granite from [Beeler et al. (1994)]. (A) Aging (solid line) and Slip (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Fits without any constraint on $a - b$; the solid lines show Aging and the dashed lines Slip law fits.
4.11 Kato and Nagata law fits to the $\Delta \mu_{\text{hold}}$ time series for the initially bare granite data from Beeler et al. (1994). (A) Nagata (dashed line) and Kato (dash-dotted line) law fits with $a - b$ fixed at -0.0027. The corresponding Slip law fit (solid line) from Figure 4.10(A) is plotted for reference; the fit to the stress minima by all three laws are identical. (B) Fits without any constraint on $a - b$; the dashed lines show the Nagata law fits. Note that the stress minima are again identical to the corresponding Slip law fit (solid line, from Figure 4.10(B)). Dotted lines show the best Nagata law fit for $c = 10$. We did not attempt the corresponding fit with the Kato law, since the Slip law already provided an excellent fit to the $\Delta \mu_{\text{hold}}$ data alone.

4.12 Kato and Nagata law fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ time series from the initially bare granite data of Beeler et al. (1994). (A) Kato (solid line) and Nagata (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Nagata law fit (dashed line) to stress minima without any constraint on $a - b$; the solid lines show the corresponding Slip law fit. Note that the Kato and Nagata law fits in (B) are nearly identical, and also identical to the corresponding Slip law fit in Figure 4.10(B). The dotted line shows the best Nagata law fit when $c = 10$ is held constant.
4.13 Summary of the important features of the quartzite data from Beeler et al. (1994). (A)-(C) Fits to a sequence of 1 order steps with the servo controlled stiffer setup. \( a - b = -0.0062, a \sim 0.012 \). (D) Evolution of \( \Delta \mu_{\text{peak}} \) and \( \Delta \mu_{\text{hold}} \) time series. Note that the healing rate is close to \( a - b \). Solid line shows Aging law fit to peak stresses alone. Dashed lines shows the Slip law fit to both \( \Delta \mu_{\text{peak}} \) and \( \Delta \mu_{\text{hold}} \), note that this fit explains the peaks and the stress minima equally well. (E) Fits to \( \Delta \mu_{\text{hold}} \) with the Aging and Slip laws. Solid lines-Aging, dashed lines-Slip law without any constraint on \( a - b \) and dash-dotted lines - Slip law with \( a - b = -0.0061 \). Note that, just like the granite data, the stiffness dependence of the stress relaxation is well modeled by the Slip law when \( a - b \) is not constrained.

5.1 Large velocity step decreases and increases on initially bare granite. Most of the step decreases at cumulative slip larger than 120 mm are plotted. The legend shows the velocity step order (jump denotes \( \log_{10}(V_{\text{final}}/V_{\text{initial}}) \)) and total cumulative slip at the beginning of the step. The slip is set to zero at the stress minimum (or maximum for the step increases). The data are smoothed over 0.2 \( \mu \)m for the step downs and over 0.4 \( \mu \)m for the step ups. The main panel shows the friction normalized by its change between 0 and 4 \( \mu \)m of slip. The slip evolution of stress is nearly identical over this length scale for all the velocity steps in the plot. Upper inset shows friction normalized by its change between 0 and 25 \( \mu \)m of slip for comparison. That some of the velocity steps exhibit a second, slower evolution of friction beginning at slip distances of a few microns becomes more evident at this scale. But the evolution of friction with slip still plot on the top of each other for most of the velocity steps.
5.2 Finite stiffness simulations of large rate step increases and decreases, with normalized stiffness \( k = 0.065 \, \mu m^{-1} \), generated at 50 Hz sampling to mimic the time history represented by the experimental data. The modeled time series is then smoothed identically to the data in Figure 5.1 (A) and (C) show the evolution of friction for 0.5-3.5 orders of magnitude step increases and decreases (order in the legend denotes \( \log_{10}(V_{\text{final}}/V_{\text{initial}}) \)) for the Aging and Slip laws respectively, such that the changes in friction are referenced to the pre-step value \( \mu_{\text{prior}} \). The parameters used are \( a = 0.013 \), \( a - b = -0.003 \) and \( D_c = 2 \mu m \); these values are derived from fits to the data (see Appendices E.1 and E.2). (B) and (D) are rescaled versions of (A) and (C) respectively such that changes in friction are measured from its value at 4\( \mu m \) and are normalized by the maximum amplitude of this change, as in the main panel of Figure 5.1. The slip is set to zero at the stress minimum (or maximum) for all panels.

5.3 (A) Same set of velocity steps as in Figure 5.1 but plotted as non-normalized friction versus slip. Slip is set to zero at minimum stress and shear stress is set to zero at the pre-step level. The data are smoothed as in Figure 5.1. Inset shows the evolution of the stress minimum (\( \Delta \text{Friction}_{\text{min}} \)) with log step size. (B) Evolution of the stress minima following a large velocity step decrease with step-size from the (smoothed) finite stiffness simulations in Figure 5.1 (A) for the Aging law. (C) Same as (B) but for the Slip law. Note how the Slip law predicts linear evolution of the stress minima with log step-size. The Aging law, on the other hand, predicts that the stress minima deviate to significantly shallower values from the initial linear trend as step size increases. The black dashed lines in (B) and (C) have a slope of \( a = 0.013 \). The trend of the stress minima from the data is linear, as predicted by the Slip law.
5.4 (A) Evolution of log velocity and (B) and log state with slip between the onset of the velocity step and the stress minimum for the numerical simulations of the velocity step decreases in Figure 5.2. Solid lines – Aging law; dashed lines – Slip law; zero of slip is set at minimum stress. Note how the rate of Aging law state evolution with slip increases dramatically with slip as the size of the velocity step increases, unlike the modest changes observed for the corresponding Slip law evolution. (C) Evolution of friction with log slip rate under the Aging law for the numerical simulations of the velocity step decreases in Figure 5.2. The dashed trajectories show the stress relaxation trajectories of a 5000s hold with the same $a$ and $D_c$ as the velocity steps, but the blue curve has the same $a - b$ as the steps while the red curve has $a - b$ of the same value but of opposite sign. (D) Same as (C) but for the Slip law. Note how most of the stress evolution between the onset of the step and the stress minimum follows the corresponding stress relaxation trajectory for the hold. The vertical dashed lines in panels (C) and (D) show the hold durations (for the trajectories with $a - b < 0$) corresponding to the lowest slip rate accessed during the largest velocity step decrease.
5.5 5-6% Normal stress steps at two different constant loading rates \( (V_{lp} = 0.3162 \, \mu m s^{-1} \) and \( V_{lp} = 0.03162 \, \mu m s^{-1} \) \) from the same experimental run as the velocity steps in Figure 5.1. The normal stress, shear stress and closure (fault normal displacement) data are smoothed over 0.075 \( \mu m \) and 0.0375 \( \mu m \) for the data at \( V_{lp} = 0.3162 \, \mu m s^{-1} \) (solid lines) and \( V_{lp} = 0.03162 \, \mu m s^{-1} \) (dashed) respectively. The normal stress, shear stress and closure data (in \( \mu m MPa^{-1} \)) are jointly normalized by the size of the normal stress step. Top panels: Normal stress step increases. (A) - Normal stress in red, shear stress in blue. (B) - Normal stress in red, closure in blue. Bottom panels: (C) and (D) correspond to (A) and (B) respectively, but for normal stress step decreases. There are two instances of each normal stress step at each sliding speed, showing that these experiments are highly reproducible.

When plotted against slip, the shear stress evolution in response to the step is nearly identical for the two sliding speeds. There is no increase in shear stress at zero slip in response to the normal stress steps even though there is significant near-instantaneous fault compaction/dilation.
5.6  (A) Evolution of normalized shear stress with slip. Unlike Figure 5.5, the shear stress is normalized here by its total change between 0 and 10 \( \mu m \) of slip. The negative shear stress responses to step decreases have been flipped to plot on the top of the responses to step increases to aid visual comparison. The slip evolution of shear stress in response to normal stress step increases (solid lines) and decreases (dashed lines) track each other irrespective of the nominal sliding rate. The legend shows the nominal slip rate \( V \) in \( \mu m s^{-1} \) and the sign of the step ( +1 for normal stress increase and -1 for decrease) (B) Evolution of normalized \( \Delta \) Closure-\( \Delta \) Normal Stress versus slip. Normal stress and closure are independently normalized by their total change between 0 and 10 \( \mu m \) of slip. The difference between normalized closure and normal stress reveals the slower evolution of closure even after normal stress has reached a uniform, constant value. (C) and (D) show the slip rate excursion in response to the normal stress step decreases/increases at \( V_{lp} = 0.3162 \mu m s^{-1} \) and \( V_{lp} = 0.03162 \mu m s^{-1} \) respectively. The slip rate excursions are instantaneous and of opposite sign to that of the normal stress step.
A.1 The different types of transition between linear and exponential slip weakening shown by the analytical stress evolution solution for the Nagata law velocity step response and its limiting approximations. We used \( a - b = 0.0026 \) and \( V_{\text{max}}/V_b = 10^{15} \). The scaling relations in Eq. (2.24) were used to obtain values for \( a, b \) and \( D_c \) for values of \( c \) different from 2.0 – (A) For \( c = 2.0 \), the linear slip weakening approximation (equation (2.19)) is appropriate to calculate fracture energy; (B) for \( c = 10.0 \) neither the linear slip weakening nor the exponential weakening approximation (Eq. (2.23), \( c \gg 1 \)) can fully account for the fracture energy. The decay of stress at large slips is well approximated by Eq. (A.1a). (C) For \( c = 100.0 \) the exponential weakening can account for most of the fracture energy; (D) for \( c = 1000.0 \) the exponential solution is essentially exact.

A.2 Evolution of \( cab^{-1}D_cV(0)/V(0)^2 \) with increasing slip speed at the center of the localized nucleation zone for \( a = 0.05, D_c = 0.33 \mu \text{m}, a/b = 0.60 \) for \( c = 2.0 \). The fault was initially everywhere below steady state with a locally peaked load. The scaling relations in Eqs. (2.24a), (2.24b) and (2.24c) were used to obtain the parameters for \( c = 5.0, 10.0, 20.0 \) and 100.0.
A.3 Numerical simulation for $a - b = -0.033$ and $c = 100$. This value of $a - b$ corresponds to $a/b = 0.6$ (given $a = 0.05$) for the $c = 2.0$ case. For $c = 100$, the scaling relations in Eqs. (2.24) imply $a/b \approx 0.98$. (A) Slip rate and (B) $\Omega$ profiles when the initial condition was a locally peaked load at the center of the patch. (C) Slip rate profile when the initial condition was spatially randomized $\Omega$ between 0 and 1. The length scale bounded by the red, dashed, vertical lines in (A) is $1.3774L_b$ which is the Aging law type fixed length solution. The curves outlined by blue diamonds signify a length scale $1.3774L_b/\ln(\Omega)$ with $\Omega$ evaluated at the center of the nucleation zone. (D) shows plots of $caD_c b^{-1}V(0)/V(0)^2$, $\Omega(0)$ and $k_s(0)/k_b$ with $V(0)$. Our approximation in Eq. (??) is supported by the fact that $k_s(0)/k_b \ll 1$ and $\Omega(0) \approx 1$ up to slip rates of $\approx 1 \text{ ms}^{-1}$. Eventual logarithmic convergence of the nucleation length scale to $\sim 1.3774L_b$ is suggested by $caD_c b^{-1}V(0)/V(0)^2 \sim 40$ in accordance with Eq. (??) and a continually increasing $\Omega(0)$ as the nucleating patch is driven to slip rates of $\approx 10^{80} \text{ ms}^{-1}$.

A.4 Evolution of $\Omega(0)$ with $V_{\text{max}}$ on the fault for the expanding crack like solutions. The colors represent different values of $a/b$. The dashed lines represent the analytical approximation in Eq. (A.24). All simulations were carried out with $a = 0.05$ and $D_c = 0.33 \mu m$. The fault was initially everywhere below steady state with a locally peaked load.
A.5 Normalized stress drop versus normalized slip from the Nagata law simulations for \( c = 10.0 \) and \( a - b = -0.0026 \). The values of \( a \) and \( D_c \) were determined assuming \( a = 0.05 \) and \( D_c = 0.33 \mu m \) for \( c = 2.0 \) using the scaling relations in Eqs. (2.24). The decay in stress is neither purely exponential nor linear. There is a transition between the two types between jump sizes and with slip distance. Inset: we plot the value of \( \alpha(V_{\text{max}}/V_{\text{bg}})/2 \) over the values of the jump sizes encountered in the simulations.

A.6 The scaling of fracture energy with \( V_{\text{max}}/V_{\text{bg}} \) for the different values of \( c \) used in the simulations and \( c = 20.0 \). We used \( a - b = 0.0026 \). The scaling relations in Eqs. (2.24) were used to obtain values for \( a, b \) and \( D_c \) for values of \( c \) different from 2.0. Scaled form means \( G_c \) is given by Eq. (A.25). For all other curves, the fracture energy is assumed to be \( G_c = \int_0^{\infty} \Delta \tau(\delta)d\delta \). Full integral means Eq. (2.15), linear means Eq. (2.19) and exponential means Eq. (2.23) for \( \Delta \tau(\delta) \). (A) For \( c = 2.0 \), the linear slip weakening approximation is appropriate to calculate fracture energy; (B) for \( c = 5.0 \) the linear approximation can no longer account for the fracture energy for the smaller jumps. For larger jumps, there is a consistent underestimation. (C) For \( c = 10.0 \) the exponential weakening can account for jumps up to 4 orders of magnitude. (D) For \( c = 20.0 \) we see a similar result as for \( c = 10.0 \). The scaled form always does an excellent job of matching the full integral implying that, if the area under the curves in Figure A.1 is a good approximation for \( G_c \), then the scaled form in Eq. (A.25) is a good measure of \( G_c \).
A.7 The scaling of fracture energy with $V_{\text{max}}/V_{bg}$ for the different values of $c$. We used $a - b = 0.0026$. All fracture energies are corrected for $\Omega_{bg} \neq 1$ according to Eq. (A.33). The scaling relations in Eqs. (2.24) were used to obtain values for $a$, $b$ and $D_c$ for values of $c$ different from 2.0. Scaled form means $G_c(\Omega_{bg} = 1)$ is given by Eq. (A.25). For all other curves, we use $G_c(\Omega_{bg} = 1) = \int_0^\infty \Delta \tau(\delta) d\delta$. Full integral means Eq. (2.15), linear means Eq. (2.19) and exponential means Eq. (2.23) for $\Delta \tau(\delta)$. (A), (B) and (C) show the $G_c$ vs. $V_{\text{max}}/V_{bg}$ curves with $\Omega_{bg} = 0.2$ for $c = 0, 10$ and 100. The full integral with $\Omega_{bg} \neq 1$ agrees reasonably well with the scaled form from Eq. (A.25) corrected for $\Omega_{bg} \neq 1$ by using Eq. (A.33). For smaller jumps, $G_c < 0$ implying that the expanding nucleation zone is not a physically valid solution when the region ahead of an early stage propagating front is below steady state. (D), (E) and (F) show the $G_c$ vs. $V_{\text{max}}/V_{bg}$ curves with $\Omega_{bg} = 5$ for $c = 0, 10$ and 100. Again, the full integral agrees well with the corrected scaled form from Eq. (A.25).
B.1 Estimate of standard error of the data from the steady state portions of
the experimental data (dataset p1060). We assume that shear stress during
steady state sliding following a velocity step should be constant. Therefore
the consecutive measurements of shear stress during steady state sliding
could be considered as repeated measurements of the same stress state.
This means that the distribution of shear stress fluctuations about such a
constant value could give us some estimate, most likely the lower bound,
of the data error. The histogram of errors for the sections A, B, C and D of
the data (labeled in the left hand panel showing the stress data) are plotted
in parts (A), (B), (C) and (D). Maximum-likelihood estimates of Gaussian
parameters for these error distributions are overlain on these histograms
with the corresponding fits shown as red solid lines and their confidence
intervals as red dashed lines.

B.2 A typical converged Markov Chain for the Nagata law. This is the same chain
shown in Figures 3 and 7 in the main text for dataset p1060. (A)–(C) show
the posteriors for \(a\), \(D_c\) and \(c\), (D)–(F) show the auto correlation function
(ACF) along the chain for each parameter along with 5% confidence level
shown as horizontal blue lines, (G)–(I) show the trace plots along the chain
for each parameter. The ACF estimates show that the samples are mostly
uncorrelated at the 5% level. The trace plots show that the sequence is well
mixed above a lower bound in \(c\).

C.1 Stiffness estimation for the different datasets. The initial portion of the
excursion of \(\Delta \mu\) following a large velocity increase or reslide following a
hold (if performed during the same experimental run) is used to estimate the
stiffness. The slope of a linear fit to the \(\Delta \mu\) vs. \(\delta_{lp}\) plot yields the stiffness
if \(V_f/V_i \gg 1\), with \(V\) here referring to the load point velocity.
C.2 Fault displacement vs. ram displacement for the largest velocity increases in p1060, p1169 and p1180. When compared to Figure A1, it is clear that over the range load point displacements used to estimate the stiffness $k$, the fault slips minimally with respect to the ram.

C.3 (A)–(C) The Nagata law posteriors for $a$, $D_c$ and $c$ for dataset p1060 with $S_e = 0.00045$ (the maximum standard deviation from the steady state error distributions constructed in Figure B.1 in the Appendix B). The posteriors are well bounded Gaussians as shown in the inset by overlaying maximum likelihood estimates and the corresponding 5% error bounds as red solid and dashed lines respectively. The mean of these Gaussians coincide exactly with the parameter values representing the global minimum of the Nagata law RMSE for the chain in Figure 4.3 which was run with $S_e = 0.004$. This helps us identify the strongly peaked regions in the posteriors in Figure 4.3 with these bounded Gaussians whose precise structure emerges at this stricter constraint on data error. (D)–(F) The Nagata law posteriors for $a$, $D_c$ and $c$ for fits to the best Slip law fit to dataset p1060 at the same standard error as parts (A)–(C). The posteriors reveal a quasi-uniform distribution with a well defined lower bound on $c$. This lets us recognize the quasi-uniform tails in the posteriors in Figure 4.3 as Nagata law fits which are identical to the best Slip law fit.
C.4 The Nagata law posteriors for $a$, $D_c$, $c$ for datasets (A)–(C) p1060, (D)–(F) p1169 and (G)–(I) p1180. The posterior distributions in blue in the background are the same as those in Figure 4.7 i.e. they represent the MCMC samples drawn for the whole dataset. The semi-transparent red posteriors in the foreground represent fits to only the 1 order of magnitude steps in the respective datasets at the same level of standard error as the blue posteriors. It is clear that the lower bound on $c$ for producing Nagata law fits at least as good as the best Slip law fit to the 1 order of magnitude steps only is smaller than the corresponding bound for the fits which also include the 2 order steps. Additionally, the lower bound on $c$ to produce Nagata law fits identical to the best Slip law fits is also smaller when fitting only 1 order of magnitude steps. This lower bound on $c$ is marked approximately by the onset of the quasi-uniform region of the posterior distribution of $c$. 

C.5 Comparing the Aging law posteriors for $a$ and $D_c$ ($a - b = -0.0002$ fixed) between fits with the stiffness constrained at $k = 0.0011 \mu m^{-1}$ ((A) and (B), ochre posteriors) and allowed to vary as a free parameter ((C), (D) and (E), green posteriors). The constrained stiffness value was obtained from fitting the initial portions of the velocity steps. Note in (E) how the fixed stiffness value is well separated from the stiffness posterior obtained by running the Markov chain at this level of data error; the same as for the fits in the main text. Also note the feedback between $a$, $D_c$ and $k$ in the bottom panel as evidenced by the fit requiring a larger $a$ to accommodate the larger $D_c$ and $k$. 

C.6 Same as Figure C.5 but for the Slip law. Note that the constrained stiffness value is nearly equal to the mean of the quasi-Gaussian posterior derived by treating stiffness as a free parameter in the inversions (in (E)). The two sets of posteriors derived for $a$ and $D_c$ are identical for all practical purposes.
C.7 (A) The Aging law fits to the data corresponding to the mode of the posteriors in Figure C.5; ochre - fixed stiffness, green - varying stiffness. The loading portions of (B) the one order and (C) the two order velocity steps from p1060. The dashed lines show the reference slopes corresponding to $k = 0.0011 \mu \text{ms}^{-1}$ (pink) and $k = 0.00126 \mu \text{ms}^{-1}$ (green), the solid green lines show the fit. Note that the fit to the larger velocity step is more sensitive to the 15% difference between the two stiffnesses.

D.1 (A) Plots of $\ln(V_{s/r}/\theta/D_c)$ at the end of the hold (solid lines) and at peak stress (dashed lines) for the set of Aging law numerical simulations from Figure 4.3. Red colors represent the low stiffness simulations, blue high stiffness. $a = 0.009, b = 0.01$ and $D_c = 3.0 \mu m$. (B) The same as (A) but now the dashed lines show $\ln(V_{s/r}/\theta/D_c) - \Delta \ln(V_{s/r}/\theta/D_c)$ where $\Delta \ln(V_{s/r}/\theta/D_c)$ is the analytical estimate of the (negative) change in state between beginning of reslide and peak stress from Eq. (D.8). The approximation becomes better for longer hold times.

D.2 Plots of $\ln(V_{s/r}/\theta/D_c)$ at the end of the hold (solid lines) and at peak stress (dashed lines) for the set of Slip law numerical simulations from Figure 4.3. Red colors represent the low stiffness simulations, blue high stiffness. $a = 0.009, b = 0.01$ and $D_c = 3.0 \mu m$. Note that the peak friction for each individual stiffness is quasi-linear with log of hold time with a stiffness dependent healing rate. However, the healing rates for the Slip law are less than $\ln \theta_{\text{hold}}$ which is shown for comparison. Also, at least for the lower stiffness simulations, $\partial / \partial \theta_{\text{hold}} \left[ \ln(\theta_{\text{hold}}) \right]$ is clearly different from $\partial / \partial \theta_{\text{peak}} \left[ \ln(\theta_{\text{peak}}) \right]$.
D.3 Plots of total change in closure measured (in $\mu$m) since the beginning of the hold (red in (A), blue in (B)) and corresponding shear stress changes (in green) during the reslide following a $10^4$ s holds for (A) the natural stiffness apparatus (B) the stiffer apparatus. The vertical black dashed lines are plotted as a visual aid to locate peak stress. (C) Evolution of the change in closure during the hold (circles) with $t_{\text{hold}}$ (positive changes denote compaction). Also shown is the evolution of the dilation between the end of the hold and peak stress (squares) with $t_{\text{hold}}$. (D) The amount of slip occurring between the end of the hold and peak stress ($\Delta \delta_{\text{peak}}$) plotted as a function of $\ln t_{\text{hold}}$. Red colors represent the low and blue the high stiffness data for all plots. Note that the change in closure between the onset of the reslide and peak stress is a significant fraction of the total change during the hold. Also, $\Delta \delta_n$ clearly evolves with $\ln t_{\text{hold}}$ for both stiffnesses.

D.4 (A) The argument of the Lambert W function in Eq. (D.24) for velocity weakening (red) and strengthening (blue) parameter combinations and $K = 1$ (dashed lines) and $K = 10$ (solid lines). The black dashed line shows $-e^{-1}$. (B) The branches of Lambert W, principal in blue ($W_0$) and auxiliary in red ($W_{-1}$), for real valued arguments, the branch point is $-e^{-1}$. Note that as $\phi$ decreases during holds, $-\beta v^{-\beta}$ increases from closer to $-e^{-1}$ to closer to 0. The velocity weakening trajectories for Slip law holds follow the auxiliary branch $W_{-1}$ while the velocity strengthening trajectories follow the principal branch $W_0$. 

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D.5 Evolution of frictional strength under the Sliplaw during a long hold (∼ 10^5 s) for different values of the normalized stiffness, \( K = 1 \) (blue), 10 (ochre) and 100 (green), and different values of \( \beta \). (A) Velocity weakening, \( \beta = 2 \); (B) velocity neutral, \( \beta = 1 \); and (C) velocity strengthening, \( \beta = 0.5 \).

The solid color lines are the numerically integrated values of \( \psi \). The black dashed lines are the exact analytical solutions for \( \psi \) derived using the numerical predictions of \( \phi \) from the simulations. For \( \beta \neq 1 \) the solutions are represented by the branches of the Lambert W function (Eq. (D.24)) and for \( \beta = 1 \) these are the parabolic trajectories given in Eq. (D.33).

D.6 As in Figure D.5, but now the black dashed lines are the analytical approximations for \( \psi \) from Eqs. (D.24) (for \( \beta \neq 1 \)) and (D.33) (for \( \beta = 1 \)), using Eq. (D.26) to estimate \( \phi \) as a function of hold time.

D.7 Evolution of frictional strength under the Nagata law during a long hold (∼ 10^5 s) for different values of the normalized stiffness, \( \langle K \rangle = 1 \) (blue), 10 (ochre) and 100 (green), and different values of \( \langle \beta \rangle r \). Top panel is for velocity weakening \( \langle \beta \rangle r = 2 \), (A) \( c = 1 \), (B) \( c = 10 \), (C) \( c = 100 \). Bottom panel is for velocity strengthening \( \langle \beta \rangle r = 0.5 \), (D) \( c = 1 \), (E) \( c = 10 \), (F) \( c = 100 \). The solid color lines are the numerically integrated values of \( \langle \psi \rangle \).

The black dashed lines in (C) and (F) are the approximate, large \( c \), analytical solutions for \( \langle \psi \rangle \) (Eq. (D.46)) calculated using the numerical predictions of \( \phi \) from the simulations.
D.8 Estimations of stiffness from the [Beeler et al. (1994)](1994) holds on granite. Change in friction ($\Delta \mu$) versus load point displacement ($\delta_{lp}$) since reload for (A) lower stiffness ($k_n$) and (B) higher stiffness data ($k_s$). The colors represent holds of different durations: blue – 300 s; green – 1000 s; red – 3000 s and cyan – 10000 s. The slope of the initial portion of this loading curve is the stiffness as long as no appreciable slip has commenced. Insets show the time windows used to obtain this slope. Note that we find $k_s \sim 0.055 \mu m^{-1}$ whereas [Beeler et al. (1994)](1994) reported $k_s \sim 0.074 \mu m^{-1}$. D.9 Fitting the velocity steps on initially bare granite from [Beeler et al. (1994)](1994). The steps are all half order, the numbers in blue are sequence of load point velocities used in $\mu m{s}^{-1}$. The velocity steps on the left panels ((A) and (C)) were imposed with the natural machine stiffness while those on the right ((B) and (D)) were imposed with the higher stiffness set up. (A) and (C) show fits where all the four steps (both stiffnesses together) were inverted simultaneously such that the misfits between onset and peak stress were proportionally weighted for all the steps. (C) and (D) show fits where the step in (D) was weighted preferentially and only the pre-step steady-state sliding was weighted in (C). Red curves show Slip and ochre Aging law fits. The light, dotted curves show the respective weight distributions. Both sets of weights constrain $a - b \sim -0.0025$, we choose $-0.0027$ as our preferred value. The values of $a$ and $D_c$ change within a factor of 2 between the top and bottom panels.
D.10 Fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ time series on granite from Beeler et al. (1994) with the strain rate dependence in $a$ specified in Eq. (D.49). (A) The actual fit to the data; solid lines denote Aging and dashed Slip laws. The color scheme specifying stiffness is identical to figures in the main text. (B) and (C) show the posteriors for $\gamma$ and $V^*$ (in $\mu\text{m}^{-1}$) respectively. The blue area plots denote Slip law posteriors, the ochre area plots denote Aging law posteriors. (D) Actual variations in $a$ and $a - b$ during the holds and reslides. Pink curves denote $a$, blue $b$. Solid curves denote Aging law; dashed Slip law.

D.11 Fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data on granite from Beeler et al. (1994) with the second-order corrected friction law specified in Eq. (D.50). (A) The actual fit to the data, solid lines denote Aging and dashed Slip laws. The color denoting stiffness is the same as figures in the main text. (B) and (C) show the posteriors for $\mu_0$ and $V_*$ respectively. The blue area plots denote Slip law posteriors, the ochre area plots denote Aging law posteriors. (D) Actual variations in $\mu_{\text{ss}}$ with $V$ and $V_0$ which effectively denotes a strain rate dependence in $a - b$ (Eq. (D.51)). Solid curves denote Aging law, dashed Slip law.

E.1 Estimation of stiffness for the servo-controlled, artificially stiffened, rotary shear apparatus. (A) Change in friction ($\Delta \mu$) since reslide versus load point displacement ($\delta_{\text{lp}}$) since reslide for reslides following holds of $\sim 3000$ s and $\sim 10000$ s at three reslide rates each – squares at $1 \mu\text{m}^{-1}$, crosses at $0.3162 \mu\text{m}^{-1}$ and dots at $0.03162 \mu\text{m}^{-1}$. The initial reloading rate fixes the stiffness; we used $1/7$th of the total number of points between the onset of reslide and eventual peak strength for the fits. (B) A zoomed in version of the fits in (A). We use a stiffness of $0.065 \mu\text{m}^{-1}$ in our analyses.
E.2 Estimation of $a - b$ from a subset of the sequence of the velocity steps.

(A)-(F) Velocity step increases and decreases of 1 and 1.5 order showing steady state velocity weakening beyond total accumulated slip (the numbers in blue in each panel) of 120 mm. The 1 order velocity step decrease in (F) was imposed between the two sets of normal stress steps at $0.316 \mu m s^{-1}$ and $0.0316 \mu m s^{-1}$. (G) $\Delta \mu_{ss}$ versus $\ln(V_{final}/V_{initial})$ from a sequence of velocity steps including those shown in (A)-(F). Mean $a - b \sim -0.003$. 

E.3 Fitting a sequence of large velocity step decreases with the Slip law jointly with the same set of parameters. We modeled only the first 3 $\mu$m of slip following the onset of the velocity step to avoid potential problems with the long-term stress transients in some of the steps. (A)-(E) Fits to the stress data, (F)-(J) predictions of slip rate from the corresponding fits to the stress data. Blue – data; red – Slip law fit. The data were weighted to ensure that the misfits for all the velocity steps got equal weights. Note that $a - b$ was not pre-constrained, the inversion by itself comes up with a value which agrees well with our independent, a priori estimate of $a - b$. 

...
E.4 Frictional response of an interface sliding at constant rate when subjected to a 5% normal stress step. State evolution was computed by coupling Eq. (5.10) with the Aging (A and C) and Slip (B and D) laws. The simulations were carried out at two different sliding rates – $V_{lp} = 0.3162 \, \mu m s^{-1}$ (solid lines) and $V_{lp} = 0.03162 \, \mu m s^{-1}$ (dashed lines). The simulated data points were generated at 50 Hz sampling rate and then smoothed identically to the data in Figure 5.5. Top panels show shear stress response, bottom panels the resultant excursions in slip rate. The lines with dot markers (which appear as bold lines where dots are dense) are the response to step decreases while those without the dots are step increases. For all the simulations, $\eta_{up} = \eta_{down} = 0$. The values of $a$, $b$ and $D_{c}$ are borrowed from the Slip law fits to velocity steps. Unlike the data in Figure 5.5, there is significant asymmetry between the normal stress increases and decreases, and the velocity excursion is larger – large enough to give the appearance of a discontinuity in shear stress at zero slip for the step increases.
E.5 Markov Chain Monte Carlo fits to the shear stress evolution following the normal stress steps. The state evolution response to the normal stress step is assumed to be Eq. (E.18) with the instantaneous response \( \eta = 1 - n_\Delta / n_0 \).

The state evolution with slip is modeled with the Aging and Slip laws.

We invert for \( a, D_c, \eta_{up}, \eta_{down} \) and \( \mu^* \). \( \eta_{up} \) is the strength dilution for up-steps and \( \eta_{down} \) for down steps. Blue-data; ochre-Aging Law fits; red-Slip law fits. The blue dotted lines in the background are the relative weights used to design the residual; the weights decay exponentially with slip from the onset of the normal stress step. (A)-(D) \( V_{lp} = 0.3162 \mu m s^{-1} \), (E)-(H) \( V_{lp} = 0.03162 \mu m s^{-1} \). Note that while the fits are only shown to the first 10\( \mu m \) of slip, there is a long term evolution in the data which can’t be explained by these laws with the values of \( D_c \) obtained by fitting the velocity steps.

E.6 Posteriors for the parameters from the Markov Chain Monte Carlo chains which yielded the fits in Figure E.5 as maximum a-posteriori solutions. The distributions with red outlines used the Slip law, the ochre posteriors used the Aging. The solid squares of the same colors as the distributions show the maximum a posteriori solutions used for the fits in Figure E.4 (A) \( P(a|\text{data}) \), (B) \( P(D_c|\text{data}) \), (C) \( P(\eta|\text{data}) \) with the solid outlines representing \( \eta_{up} \) and the dashed outlines \( \eta_{down} \) and (D) \( P(\mu^*|\text{data}) \). In panels (A) and (B), the dashed, vertical black line shows the values of the corresponding parameters derived from the velocity steps. Note that the values of \( a \) and \( D_c \) yielded by both the Aging and Slip laws do not agree well with those derived from the velocity steps. But the Slip law derived values are marginally better since the values derived from the velocity steps are within the tail of the corresponding marginal posteriors.
Chapter 1

Introduction

Fault friction is an important factor in determining the diverse and rich phenomenology of the earthquake cycle on natural faults; being particularly influential during its aseismic portions. For example, the recurrence periods of repeating earthquakes are thought to be controlled by frictional healing during the interseismic period such that the magnitude of healing determines the amount of high frequency shaking caused by the eventual earthquake (Marone, 1998a; McLaskey et al., 2012). On shorter time scales, the occurrence of episodic tremor and slip phenomena along many major plate boundaries suggests that the friction-controlled aseismic transients which drive these phenomena might play a critical role in modulating the mechanical conditions, and the related seismic hazard, in the prominent seismogenic regions of the earth (Obara, 2002; Dragert et al., 2001; Peng and Gomberg, 2010; Schwartz and Rokosky, 2007; Uchida et al., 2016). Additionally, recent analysis of inter-plate mainshocks in the North Pacific suggest that, like tremor, foreshocks may also be driven by extended-duration aseismic slip (Bouchon et al., 2013), an observation which has parallels in laboratory nucleation experiments on meter-scale synthetic faults (McLaskey and Kilgore, 2013). At the other end of the fault-size spectrum, waste-water injection induced seismicity has been observed to be dominantly mediated by aseismic creep on much smaller (~ 500 m long) but well-instrumented natural faults (Guglielmi, 2010).
et al., 2015). Together, these observations suggest that realistic numerical modeling of the friction controlled portions of the earthquake cycle could prove a substantial step towards a physics-based approach to earthquake hazard assessment and early warning.

For our numerical simulations of the aseismic portions of the earthquake cycle to be realistic, we need to prescribe the ‘proper’ constitutive equations for fault friction in our models. In the absence of in situ constraints on natural faults, the development of such constitutive equations has almost exclusively been guided by laboratory friction experiments. Rate and state friction (RSF) is the most widely used representative of such laboratory-derived constitutive equations for fault friction [Dieterich (1972, 1979, 1981); Ruina (1980, 1983); Tullis and Weeks (1986); Rice and Ruina (1983)].

Besides accounting for a logarithmic dependence of friction on slip rate, the specification of frictional strength under RSF also requires the description of the time evolution of a ‘state’ variable, usually interpreted as a proxy for the ‘true’ area of contact formed by asperities bridging the sliding interface [Bowden and Tabor, 1964; Dieterich and Kilgore, 1994; Beeler and Tullis, 1997; Baumberger and Caroli, 2006]. While the rate dependence of friction can be probed using relatively straightforward experimental protocols, the obvious difficulties in tracking the evolution of a multi-contact interface at the micron scale has meant that direct observational constraints on state evolution are very limited. In the absence of direct observational constraints, the development of a first-principles theoretical basis for state evolution has proven difficult. As a result, prescriptions of state evolution under RSF have remained largely empirical and non-unique.

The different versions of these state evolution laws differ mainly in their choice of the physical variables which dominantly control state evolution in particular portions of the parameter space [Dieterich, 1978; Ruina, 1983; Beeler et al., 1994; Nakatani, 2001; Nagata et al., 2012]. The most popular state evolution laws represent two end-member views with regards to this choice: the Slip law state evolution [Dieterich, 1979], which predicts that slip is necessary for state to evolve, and Aging law state evolution [Ruina, 1983], which predicts
that state can evolve with time even without slip. More recently, Nagata et al. (2012) used continuous measurements of ultrasonic P-wave transmissivity across laboratory granite faults as an indirect ‘measure’ for state evolution to propose that, along with time and slip, the shear stress acting on the slip interface can also contribute to changes in the state variable. However, none of the widely used state evolution laws can explain the full range of laboratory data, let alone being relevant to realistic simulations of natural faults. Thus, even after more than three decades since RSF was first proposed, the identity of the ‘proper’ state evolution law remains unknown.

The first step towards the design of the ‘proper’ state evolution law is to ascertain the relative successes and failures of the existing formulations vis-à-vis laboratory data. Secondly, in view of our ultimate goal of reproducing realistic fault behavior in numerical simulations, it is essential to examine these laws, both theoretically and experimentally, in those parts of the parameter space which determine the gross aseismic behavior of numerical models. Therefore, these state evolution laws need to be tested against laboratory experiments which probe those parts of the parameter space where – (a) The responses of the different candidate formulations are the most disparate (b) The response of the RSF equations are most relevant to the modeling of the aseismic portions of the earthquake cycle. Fortunately, the same portion of the parameter space satisfies both (a) and (b): far from steady-state. In fact, under small perturbations from steady sliding (at some constant velocity) the most widely used state evolution laws all produce identical mechanical response; their differences become apparent only when pushed far from steady state. And these differences in the mechanical response of the friction law to large perturbations in slip rate from steady sliding can lead to markedly different styles of earthquake nucleation (under large velocity increases) or fault healing (under large velocity decreases) in numerical models (Beeler et al., 1994; Marone, 1998b; Ampuero and Rubin, 2008; Bhattacharya and Rubin, 2014).
Unfortunately, the testing of different RSF formulations with laboratory velocity steps has generally been limited to imposition of small velocity steps. This has led to a lot of ambiguity with regards to the existing experimental support for the different versions of state evolution. In particular, while rate stepping experiments have been cited as showing support for the Slip law \cite{Ruina1983, Nakatani2001, Bhattacharya2015}, slide-hold-slide (SHS) experiments (designed to examine fault healing at low slip rates) have been traditionally interpreted as being more consistent with the Aging law \cite{Beeler1994, Marone1998}. This thesis attempts to address these issues by analytically and numerically examining the widely used state evolution formulations against the largest known laboratory velocity steps, long holds and normal stress steps at orders of magnitude different slip rates. Using this approach reveals that some widely held ideas about state evolution need to be reassessed, in particular we show that both large velocity steps and slide-hold-slides support slip dependent state evolution. In doing so, we also raise some important questions about the physical content of the state variable and outline arguments which suggest that it is very likely that state is more than just the total area of ‘true’ contact. In the rest of this introduction, we outline the scheme in which these findings are laid out in this thesis on a chapter-by-chapter basis.

\subsection{Organization of the thesis}

One of the initial goals of my thesis was to rigorously examine the new shear-stress dependent state evolution law proposed by \cite{Nagata2012} with our entire suite of experimental data. The first step towards this goal was to analyze the theoretical response of the Nagata law in different portions of its parameter space in order to recognize what type of data would be most effective in its experimental assessment. Chapter 2 of this thesis details the theoretical treatment of the Nagata law response to large velocity steps and its implications for earthquake nucleation on an elastically deformable fault \cite{Bhattacharya2014}.  

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Specifically, we show that the Nagata law response to large velocity increases transitions from Aging law type linear slip weakening to Slip law type exponential weakening as the shear stress dependence becomes increasingly dominant. In our numerical simulations of earthquake nucleation, this Aging law – Slip law transition manifests itself as a switch from expanding crack type (characteristic of Aging law) to propagating pulse type (typical for Slip law) nucleation as the shear-stress effect on state evolution becomes more dominant.

In Chapter 3, I present the experimental evaluation of the Aging, Slip and Nagata laws by attempting to fit 1–3 orders of magnitude velocity steps on simulated fault gouge ([Bhattacharya et al.], 2015). These velocity steps were carried out using specially designed experimental protocols to artificially stiffen the experimental apparatus such that large perturbations from steady state sliding could be imposed on the slip interfaces. Our inversions confirm the results from smaller velocity step tests that the Aging law cannot explain the observed response and that the Slip law produces much better fits to the data. The stress-dependent Nagata law produced fits identical to, and sometimes slightly better than, those produced by the Slip law using a sufficiently large value of the additional free parameter \( c \) that controls the stress dependence of state evolution. We further show that a Monte Carlo search of the parameter space confirms the analytical results that velocity step data that are well represented by the Slip law can only impose a lower bound on the acceptable values of \( c \) and that this lower bound increases with the size of the velocity step being fit. We find that our 1–3 orders of magnitude velocity steps on synthetic gouge impose this lower bound on \( c \) to be 10–100, significantly larger than the value of 2 obtained by [Nagata et al. (2012)] based on experiments on initially bare rock surfaces with generally smaller departures from steady state.

Chapter 4 deals with the experimental evaluation of the Aging, Slip, Nagata and a fourth, Aging-Slip hybrid state evolution law with the slide-hold-slide experiments of [Beeler et al. (1994)] on initially bare granite. While rate stepping experiments show support for the Slip law, laboratory observed frictional behavior near zero slip rates has traditionally been used
to interpreted experimental support for Aging law style time-dependent healing, in particular from the slide-hold-slide experiments of \cite{Beeler94}. Using a combination of new analytical results and numerical inversion, we show instead that the slide-hold-slide data of \cite{Beeler94} favor slip-dependent state evolution during holds. We also show that both the Aging-Slip hybrid and Nagata laws reproduce the best Slip law fits to the slide-hold-slide data even with the freedom of an extra parameter which constitutes independent evidence against Aging law type time-dependent healing. However, we find that none of the state evolution laws used in this study can explain all the features of the \cite{Beeler94} data satisfactorily when their parameters are also constrained by prior velocity steps.

Chapter 5 of the thesis builds on the surprising result of chapter 4, that frictional healing far below steady state appears to be slip-dependent rather than time-dependent, by studying laboratory friction data derived from up to 3.5 orders of magnitude velocity step decreases on initially bare granite. These extreme velocity step decreases were complemented by up to 2.5 orders velocity step increases and 5\% normal stress steps at order of magnitude different sliding rates during the same experimental run. We find that, across these diverse sliding conditions, the data are much more consistent with the Slip law version of slip-dependence than the time-dependence formulated in the Aging law. The shear stress response to normal stress steps is also consistently better explained by the Slip law when paired with a Linker-Dieterich type response to normal stress perturbations. Finally, high quality measurements of interface compaction from the normal-stress steps suggest that the instantaneous changes in state and contact area are opposite in sign, indicating that state evolution is most likely not controlled by the quantity of ‘true’ contact area alone.
1.2 List of publications from this thesis

1.2.1 Papers

Under Preparation


Refereed Publications


1.2.2 Conference Presentations


Chapter 2

Frictional response to velocity steps and 1-D fault nucleation under a state evolution law with stressing-rate dependence

2.1 Introduction

The fast slip during earthquakes is preceded by a phase of accelerating slow slip on the fault during which the elastic forces are very nearly balanced by the frictional forces. This is called the nucleation phase of the rupture. Theoretical models suggest that, under appropriate conditions, this nucleation phase can culminate in a seismogenic instability when the slip speeds become high enough to make inertial effects dominate frictional resistance.

It has been suggested that, in general, foreshock occurrence and mainshock initiation might be controlled by aseismic nucleation processes along plate boundaries – particularly in subduction zones and along both continental margin and oceanic transform faults (McGuire [9])
et al., 2005; Kato et al., 2012; Bouchon et al., 2011, 2013). There is some observational evidence that the dimensions of these aseismic nucleation zones (as inferred from the spatial extent of foreshock activity) scale with the seismic moments of the eventual earthquakes (Dodge et al., 1996). In addition, combined geodetic and seismological observations have established the occurrence of a class of episodic aseismic transients along plate boundaries (downdip from the seismogenic zone) collectively referred to as slow slip and non volcanic tremor (Dragert et al., 2001; Obara, 2002; Nadeau and Dolenc, 2005; Shelly et al., 2007; Obara, 2010; Peng and Gomberg, 2010). The apparent ubiquity of these aseismic slip processes and their inferred influences on faulting suggests that the proper detection of these phases might have implications for both earthquake hazard estimation and early warning. But it is difficult to observe the details of earthquake nucleation phases from seismological or geodetic data (Beroza and Ellsworth, 1996; Bouchon et al., 2011). To aid in the interpretation of such data it would be useful to simulate rupture nucleation using reliable numerical models to gain a better understanding of the properties of the nucleation processes. Under such friction dominated slow slip conditions it is important to prescribe the ‘proper’ friction laws on fault surfaces to be able to accurately predict fault behaviour. However, the physics of friction is not well understood and this is especially true on natural faults due to lack of observational constraints. This makes the ‘proper’ constitutive law for fault friction an elusive mathematical formulation.

The most widely adopted formulation of the frictional constitutive law for natural faults comes from a laboratory derived framework called ‘Rate- and State-dependent Friction’ (RSF), wherein the frictional stress at a point is determined by the ‘rate’ at which the fault is slipping, and the physical/mechanical ‘state’ of the micro-contacts bridging the two interfaces (Dieterich, 1978; Ruina, 1983). The precise mathematical form of the evolution of the ‘state’ variable remains an unresolved problem as none of the existent formulations agree fully with experimental data (Beeler et al., 1994; Marone et al., 1995; Bayart et al., 2006). Under the assumption that these experiments apply to (at least slow) sliding phenomena,
it is of interest to keep searching for formulations which describe laboratory data better. The importance of finding the ‘proper’ state evolution law is amplified by the fact that different state evolution laws can lead to very different earthquake nucleation styles in numerical models (Ampuero and Rubin, 2008). Nagata et al. (2012) have recently proposed a modified state evolution law that, they claim, does a better job of matching laboratory rock friction data than the state evolution laws commonly used in numerical modelling. In what follows, we report our analytical and numerical exploration of the frictional behaviour that this new evolution law imposes on single-degree-of-freedom systems. We also analyse the type of earthquake nucleation this law produces on a 1-D fault in an elastic continuum.

2.2 Background

It is widely believed that the first order increase/decrease in frictional strength of an un-lubricated sliding interface is controlled by the increase/decrease in the area of adhesive contacts between the sliding surfaces (Bowden and Tabor, 1964). This is also the basis for the physical explanation of Amonton’s law which gives the frictional strength $\tau$ as

$$\tau = \mu \sigma$$  \hspace{1cm} (2.1)

where $\mu$ is the friction coefficient and $\sigma$ is the normal stress acting on the interface. There are, however, experimentally observed time and slip dependent variations in $\mu$. These include increases in strength with time for interfaces at rest or slipping at very small slip speeds away from steady-state (Dieterich, 1972; Dieterich and Kilgore, 1994; Beeler et al., 1994), and weakening with slip (Dieterich, 1978, 1979; Ruina, 1980, 1983; Rice and Ruina, 1983), which are essential features in describing fault friction over seismic cycles. RSF attempts to capture these variations, usually, in terms of two, coupled, first order ODE’s in two variables: 1) The relative slip rate across the contact interface, denoted by $V$, and 2) a measure of the strength of the asperities bridging the sliding surface at a reference slip.
speed, often interpreted as the true area of contact and called the state variable, denoted by \( \theta \). The first of these equations describes the relationship of the frictional strength with \( V \) and \( \theta \). This we shall call the friction law,

\[
\frac{\tau}{\sigma} = \mu^* + a \log \frac{V}{V^*} + b \log \frac{\theta}{\theta^*},
\]

(2.2)

where \( a \) is the ‘direct effect’ coefficient and accounts for the variations in frictional strength due to changes in slip rate, and \( b \) is the ‘evolution effect’ coefficient which determines the change in friction due to evolution of state. In general, at not very high temperatures, \( a \) and \( b \) are constants of the order of 0.01, but they can vary by as much as an order of magnitude with varying temperature and moisture content (Blanpied et al., 1998). The other parameters \( \mu^* \), \( V^* \) and \( \theta^* \) are the values of the friction coefficient, slip rate and state at some reference steady state, the choice of which does not affect the dynamics of the system. Throughout this article we consider only dry rocks and hence will not make a distinction between normal stress and effective normal stress. The mathematical framework of the problem is complete with the statement of an evolution equation for \( \theta \) in terms of the variables of interest,

\[
\dot{\theta} = F(\theta, V, \dot{\tau}, \Gamma),
\]

(2.3)

where \( \Gamma \) stands for any generalized family of frictional parameters. In the form represented in Eq. (2.3), \( \theta \) is generally expressed in units of time, which can be interpreted as a measure of contact age relative to some arbitrarily defined reference value. Eq. (2.3) is generally
referred to as the state evolution law and, relative to the friction law, has been a matter of much debate. The two most widely used forms are

\[
\text{Dieterich (Aging) Law : } \dot{\theta} = 1 - \frac{V\theta}{D_c} \quad (2.4a)
\]

\[
\text{Ruina (Slip) Law : } \dot{\theta} = -\frac{V\theta}{D_c} \ln \frac{V\theta}{D_c} \quad (2.4b)
\]

where \( D_c \) is some characteristic slip weakening length scale ([Dieterich, 1978; Ruina, 1983]). Eq. (2.4a) is often referred to as the Aging law because state increases linearly with time when \( V = 0 \). Eq. (2.4b) is referred to as the Slip law in that no state evolution occurs unless \( V \neq 0 \). Though the responses of the two laws differ far from steady state (steady state being \( \dot{\theta} = 0 \)), in the vicinity of steady state sliding the two laws are asymptotically identical. Given that for both laws steady state sliding occurs when \( V\theta/D_c = 1 \), one can readily deduce from Eq. (2.2) that the change in frictional strength between two consecutive steady states interrupted by a velocity perturbation is

\[
\frac{\Delta \tau}{\sigma} = (a - b) \log \frac{V_2}{V_1}, \quad (2.5)
\]

where \( V_2 \) and \( V_1 \) are the two steady state velocities. For \( (a - b) < 0 \) the surface is steady state velocity weakening and can experience velocity instabilities when the sliding is infinitesimally perturbed about the steady state. For \( (a - b) > 0 \), such instabilities are not possible and hence nucleation is not possible on a velocity strengthening fault patch.

It has become clear over the years that neither the Aging law nor the Slip law accurately describes all robust features of laboratory friction experiments ([Beeler et al., 1994; Marone, 1998a; Kato and Tullis, 2001]). [Beeler et al., 1994] demonstrated through ‘slide-hold-reslide’ experiments that the amount of healing of frictional interfaces in granite and quartzite increased as the logarithm of hold time (of the load point) and not with ever decreasing slip rates for large hold times. This feature is built-in in the Aging law due to
the healing term in equation (2.4a) (the first term on the right). The Slip law is at odds with this observation as it predicts no healing as \( V \rightarrow 0 \).

On the other hand, laboratory experiments also show that the stress change following a step up or down in load point velocity is such that the length scale of slip over which the stress evolves back to the new steady state value is independent of the magnitude or sign of the step (Ruina, 1980, 1983; Tullis and Weeks, 1986; Marone, 1998a; Blanpied et al., 1998; Bayart et al., 2006). For a velocity step, this evolution in stress change with slip can be solved for analytically for both the Aging and the Slip laws (e.g. Rice, 1993; Nakatani, 2001). These solutions and numerical solutions for finite stiffness machines show that the transient response of the Aging law is asymmetric with respect to the sign of the jump with much more evolution occurring for step downs (for amounts of slip \( \delta \ll D_c \)) than for step ups in load point velocities (see Figure 2.1) (Linker and Dieterich, 1992; Marone, 1998a; Kato and Tullis, 2001; Nakatani, 2001). Moreover, for velocity increases the approach to steady state occurs over a length scale which increases with the size of the jump (Ruina, 1980; Nakatani, 2001; Rubin and Ampuero, 2005). In particular, the decay of stress with slip is linear with slope \( b \sigma / D_c \) when the velocity step up pushes the sliding surface far above steady state. The Slip law leads to a symmetric response to velocity increases and decreases with the slip weakening length independent of the size or sign of the step in velocity (Figure 2.1) (Ampuero and Rubin, 2008).

It has been argued that the ‘proper’ state evolution law for simulating earthquake nucleation needs to predict the correct amount of slip over which stress evolves back to steady state following rapid velocity increases in order to evaluate the fracture energy of an expanding nucleation zone (Rubin, 2008). Given its similarity to laboratory velocity step up test results, the Slip law is a better candidate for nucleation simulations (Bayart et al., 2006). But the Aging law, due to its time dependent healing property, has been claimed to be better suited to describe fault healing during interseismic periods (Beeler et al., 1994; Marone et al., 1995; Marone, 1998a). To avoid this dichotomy of evolution laws for the
two portions of the seismic cycle where sliding friction is the dominant controlling factor, one would ideally hope for an evolution law similar to the Aging law in that state increases linearly with time at near zero slip speeds and simultaneously similar to the Slip law in that the stress evolves to steady state over a fixed length scale in response to step changes in sliding velocity (Kato and Tullis 2001).

Recently, Nagata et al. (2012) carried out a series of ‘stress step’ experiments by servo controlling a particular shear stress applied at the sliding interface. Further, following Nagata et al. (2008), they measured the variation in acoustic transmissivity across the interface as a measure of the fluctuations in true contact area and thus as a proxy for variations in state. Using this information and estimating the velocity of sliding after the stress step, they estimated the ‘direct effect’ term $a$ without recourse to the details of a particular state evolution law. They obtained values of $a \approx 0.05$ which were much larger than the usual estimates of $a < 0.01$. Both the Slip and Aging laws fared poorly in attempting to fit their data with such a large value of $a$. To fit their data, they modified the Aging law by including a dependence upon stressing rate:

Nagata Law: 

\[
\dot{\theta} = 1 - \frac{V\theta}{D_c} - \frac{c}{b} \frac{\dot{\tau}}{\sigma}
\]

(2.6)

where $c$ is another parameter and all other symbols carry over their respective meanings (but not necessarily values) from Eqs. (2.2) and (2.4a). For $c = 0$, this law reduces to the Aging law. It is worth pointing out that this revision also required a $D_c$ much smaller the usual estimates of a few microns to match the lab data. Additionally, the requirement of the same amount of steady state velocity weakening/strengthening means $a - b$ is identical between the Aging, Slip and Nagata law. Given the much larger value of $a$ this further requires that $b$ also be larger (Nagata et al., 2012). Nagata et al. (2012) found $c = 2.0$ to be an appropriate value for their experiments. In the remainder of this paper, the stand-alone symbols $a, b$ and $D_c$ are used solely to denote appropriate values for the Nagata law. Whenever we use
the corresponding values for either the Aging or Slip laws, we make that explicit using appropriate subscripts.

Surprisingly, in spite of the close mathematical similarity to the Aging law, the authors claimed with numerical simulations that the addition of the stressing rate term led to nearly symmetric responses to velocity step tests \cite{Nagata2012}. This promise of a single evolution law with time dependent healing and the proper response to velocity step tests was the primary motivation behind us taking up the Nagata law for further study. The Nagata law has been recently studied (mostly numerically) in the context of earthquake cycle simulations with spring-sliders and aftershock triggering \cite{Kame2013a, Kame2013b}. Here we focus instead on formulating a rigorous mathematical description of the slip-weakening behavior under Nagata law RSF in order to describe nucleation (in a 1D elastic continuum) in full detail. In what follows, we first present a detailed analytical exploration of the transient response of the Nagata law to velocity step ups and downs. We then give a complete analytical description of the various limiting forms of the Nagata law velocity step response with particular emphasis on transitions from Aging to Slip law type behavior. Finally, we apply the Nagata law to an elastic continuum to study nucleation on 1D faults, and use the results from the velocity steps to understand the resultant process analytically and numerically.

### 2.3 Single degree of freedom systems

In this section, we analyze the frictional response to velocity steps under Nagata law RSF to find an analytical expression for the variation of shear stress with slip. The particular functional form of the shear stress variation with slip due to a step velocity increase is of central importance in understanding the mechanics of nucleation under a given state evolution law. Before we begin, we note that for single degree of freedom systems there exists a critical spring stiffness above which a small perturbation to steady state sliding cannot grow into an instability. For the Aging and Slip laws, this critical stiffness $k_c$ is
given as \( k_c = (b - a)\sigma / D_c \) \(\text{[Ruina, 1983]}\). Noting that steady state sliding for the Nagata law implies \( V \theta / D_c = 1 \), it can be shown that \(\text{[Kame et al., 2013a]}\)

\[
k_c = \frac{(b - a)\sigma}{D_c(1 + c)}.
\] (2.7)

This shows that the critical stiffness for the Nagata law has the same mathematical form as that for the Slip and Aging laws, but the slip-weakening distance is scaled by the factor \( (1 + c) \) \(\text{[Ruina, 1983]}\). For \( c = 0 \) it reduces to Ruina’s result for the Aging law as expected.

### 2.3.1 Response to velocity steps

The Aging and the Slip laws predict no state evolution across a velocity step. For both these laws, with the slider velocity constant, one can solve for the evolution of state following the velocity step using this simple initial condition. But, for the Nagata law, one needs to estimate the state change across the step due to the third term in Eq. (2.6) \(\text{[Nagata et al., 2012]}\). It is this instantaneous evolution of state that gives the Nagata law its distinctive features.

In what follows, we shall use the subscript \( i \) for \( t \to 0^- \) and the subscript \( f \) for the values of the variables of interest at \( t \to 0^+ \) where \( t = 0 \) marks the time of the step. To find the change in state across the step we use Eq. (2.6) and the fact that this change \( \delta \theta \) happens in a vanishingly small time due to a corresponding change in shear stress \( \delta \tau \) (for \( \Delta t \to 0 \), the other two terms are negligible):

\[
\frac{\delta \theta}{\theta} = -\frac{c}{b} \frac{\delta \tau}{\sigma}.
\] (2.8)

In terms of the values of the state variable just before and after the jump we can then write

\[
\ln \frac{\theta_f}{\theta_i} = -\frac{c}{b} \frac{\Delta \tau_{f-i}}{\sigma},
\] (2.9)
where $\Delta \tau_{f-i}$ denotes the stress change across the step.

At this point we also need to evaluate the change in stress across the step using the friction law as that gives us the second equation needed to evaluate $\theta_f$ and $\Delta \tau_{f-i}$. From the friction law we have

$$\frac{\tau_f}{\sigma} = \mu_* + a \log \frac{V_f}{V_*} + b \log \frac{\theta_f}{\theta_*},$$

(2.10a)

$$\frac{\tau_i}{\sigma} = \mu_* + a \log \frac{V_i}{V_*} + b \log \frac{\theta_i}{\theta_*}.$$  

(2.10b)

Using Eqs. (2.9), (2.10a) and (2.10b) one can easily verify that

$$\Delta \tau_{f-i} = a \sigma \frac{1}{1 + c} \ln \left( \frac{V_f}{V_i} \right),$$

(2.11a)

$$\theta_f = \theta_i \left( \frac{V_f}{V_i} \right)^{-\frac{\mu_* + a}{c+1}}.$$  

(2.11b)

$$\frac{\dot{\tau}}{\sigma} = b \frac{\dot{\theta}}{\theta}, \quad t > 0 + .$$

(2.12)

Substituting Eq. (2.12) into (2.6), for a constant sliding velocity $V_f$, the Nagata evolution law takes the form

$$\dot{\theta} = \frac{1}{1 + c} - \frac{V_f \theta}{D_c (1 + c)}, \quad t > 0 + .$$

(2.13)

This equation can be solved for $\theta$ using the initial condition derived in Eq. (2.11b):

$$\theta(\delta) = \frac{D_c}{V_f} \left[ 1 - \left( 1 - \frac{V_i \theta_i}{D_c} \left( \frac{V_f}{V_i} \right)^{1 - \frac{\mu_* + a}{c+1}} \right) \exp \left( -\frac{\delta}{D_c (1 + c)} \right) \right], \quad t > 0 + .$$

(2.14)
We can combine this with Eq. (2.2) to find the stress change with respect to the steady-state stress level achieved over long slip distances after the step. This choice of the zero level is useful in estimating the peak to residual stress drop when a slip front passes through a region of a fault that previously was slipping slowly, which will be used when we examine nucleation on a 1D fault. The exact expression for the variation of the stress change with slip is

$$\Delta \tau(\delta) = b \sigma \ln \left[ 1 - \left( 1 - \frac{V_f}{V_i} \left( \frac{V_f}{V_i} \right)^{1 - \Omega_i} \right) \exp \left( -\frac{\delta D_c}{(1 + c)} \right) \right], \quad t > 0+, \quad (2.15)$$

where, for the sake of notational brevity, we have introduced a new variable $\Omega$ for the combination $V\theta/D_c$. In particular, we have

$$\Omega_i = \frac{V_i \theta_i}{D_c}. \quad (2.16)$$

Note that the value of $\Omega$ relative to 1 denotes how far the sliding surface is from steady state. In particular, $\Omega = 1$ denotes steady state while $\Omega \gg 1$ denotes far above and $\Omega \ll 1$ far below steady state.

It is of interest to determine how the size of the velocity jump affects the response to the velocity step. Before proceeding with this analysis in Section 3.2, it is important to note that typical velocity step up/down experiments impose velocity steps from at or very near steady state. Therefore, for the practical purposes of comparing our analytical results with lab data, we can assume $\Omega_i \approx 1$. But in general, most of the approximations given below are mathematically sound for the less stringent requirement $\Omega_i = O(1)$. This is our assumption about the magnitude of $\Omega_i$ unless made explicit otherwise.
2.3.2 Aging law - Slip Law transition

Assuming $\Omega_i = O(1)$, we have the following important limiting forms of Eq. (2.15) for large velocity jumps of either sign:

\[
\lim_{\left(\frac{V_f}{V_i}\right) \gg 1} \Delta \tau(\delta) \approx b\sigma \ln \left[ 1 + \Omega_i \left(\frac{V_f}{V_i}\right)^r \exp \left( -\frac{\delta}{D_c(1+c)} \right) \right], \quad \infty > \delta \geq 0 \tag{2.17a}
\]

\[
\lim_{\left(\frac{V_f}{V_i}\right) \ll 1} \Delta \tau(\delta) \approx b\sigma \ln \left[ 1 - \exp \left( -\frac{\delta}{D_c(1+c)} \right) \right], \quad \infty > \delta > 0 \tag{2.17b}
\]

where

\[
r = 1 - \frac{c}{c + 1} \frac{a}{b} \tag{2.18}
\]

For every pair of $V_f/V_i$ and $c$ that satisfies the limit in one of Eqs. (2.17), the pair $V_i/V_f$ and $c$ must satisfy the limit in the other. It is crucial to note here that both of Eqs. (2.17) are not only dependent on the size and sign of the velocity jump but also on the specified value of $c$. In particular, we show in Section 3.3 that the exponent $r$ tends to zero for $c \gg 1$ (the ratio $a/b \to 1$ for $c \gg 1$ when $a-b$ is held constant). In this limit the approximations in Eqs. (2.17a) and Eq. (2.17b) are valid only for velocity jumps that lie outside the range accessible during earthquake nucleation; e.g. for $c = 100$, $(V_f/V_i)^r \gg 1$ requires $\log_{10}(V_f/V_i) \sim 100$. It is worth noting that Eq. (2.17b) is mathematically analogous to the Aging law response for large velocity step downs derived by Ampuero and Rubin (2008). For both Eqs. (2.17a) and (2.17b), the Aging law-Nagata law correspondence is exact for $c = 0$.

Given that Eq. (2.17a) is the appropriate limiting form for a specified $V_f/V_i$ and $c$, it automatically follows that the second term within the logarithm in Eq. (2.17a) is much larger than 1 over a range of finite values of $\delta$. This leads to linear slip-weakening similar to the Aging law. In this linear slip-weakening limit, the stress variation with slip has the form
\[
\Delta \tau(\delta) \approx b\sigma \left[ \ln \Omega_i + \left( 1 - \frac{c}{c + 1} b \right) \ln \left( \frac{V_f}{V_i} \right) \right] - \frac{b\sigma}{D_c(1 + c)} \delta, \quad \delta < ND_c(c + 1), \quad (2.19)
\]

where \( ND_c(c + 1) \) is the slip distance over which linear slip weakening behavior is observed. An estimate of this critical value of \( \delta \), given the values of \( c \) and \( V_f/V_i \), can be obtained by requiring that

\[
\left( \frac{V_f}{V_i} \right)^{1 - \frac{c}{c + 1} b} \exp(-N) \gtrsim O(10). \quad (2.20)
\]

Note that as we have assumed \( \Omega_i = O(1) \), it does not feature in the order of magnitude calculations here. Therefore, for a given finite value of \( c \), it follows from Eqs. (2.17a) and (2.20) that there always exists a value of \( V_f/V_i \) above which the Nagata law response will show linear slip weakening behavior over finite slip distances. The size of this critical velocity jump increases with the value of \( c \). The approximate equality in Eq. (2.20) provides the relationship which gives the lower bound on the step size \( V_f/V_i \) (given \( c \)) or equivalently the upper bound on \( c \) (given \( V_f/V_i \)) to achieve linear slip weakening behavior. Numerically, we have seen that a value of about 20 for the \( O(10) \) term on the right hand side of Eq. (2.20) works pretty well. From the second term on the right in Eq. (2.19) the rate of linear slip-weakening is \( b\sigma/D_c(1 + c) \), whereas the corresponding Aging law slip weakening rate is \( b\sigma/D_c \).

Even for values of \((V_f/V_i, c)\) which satisfy Eq. (2.17a), the Nagata law exhibits a transition from linear to exponential slip weakening for values of slip much larger than \( ND_c(c + 1) \), where \( N \) is given by Eq. (2.20) (Appendix A.1). Figure 2.2b clearly shows this behavior for jumps of \( \gtrsim \pm 6 \) orders of magnitude. Such a slip dependent transition in the style of slip-weakening also occurs for the Aging law. But for the Nagata law, this transition from linear to exponential slip weakening occurs at slip distances small enough to affect the area under the shear stress vs. slip curve (see Figure 2.2b), and hence the fracture energy of an
expanding nucleation zone. We discuss this issue and its relevance to nucleation in greater detail in Appendices A and E.

### 2.3.3 Pure Slip law behavior

We mentioned in Section 3.2 that for \( c \gg 1 \), velocity jumps needed to satisfy the limiting condition in Eq. (2.17a) are not physically realizable. For such values of \( c \), another useful approximation exists. To show this, in this section, we will use the simplification that \( \Omega_i = 1 \). As mentioned before, this is not problematic for laboratory velocity step experiments. With this simplifying assumption, we will show that for \( c \gg 1 \), the Nagata law response is exactly equivalent to the Slip law response for all plausible values of \( V_f/V_i \) and all slip distances \( \delta \).

In order to derive the desired limiting form, we will use the identity

\[
\lim_{r \to 0} x^r = 1 + r \ln x, \quad x > 0.
\]  

Therefore, in the limit \( \frac{c}{c+1} \frac{a}{b} \to 1 \) (which we show later in this section to be equivalent to \( c \gg 1 \)), we have

\[
\left( \frac{V_f}{V_i} \right)^{1 - \frac{c}{c+1} \frac{a}{b}} \approx 1 - \left( \frac{c}{c + 1} \frac{a}{b} - 1 \right) \ln \left( \frac{V_f}{V_i} \right),
\]

where \( x \equiv V_f/V_i \) and \( r \) is substituted by the right hand of side Eq. (2.18). This reduces Eq. (2.15) to

\[
\Delta \tau(\delta) = b \sigma \left( 1 - \frac{c}{c + 1} \frac{a}{b} \right) \ln \left( \frac{V_f}{V_i} \right) \exp \left( -\frac{\delta}{D_c(1 + c)} \right)
\]

for any value of the slip. This is exactly the Slip law response given certain scaling relationships between the Nagata and Slip law parameters we tabulate below.

The preceding paragraph, along with the discussion in Section 3.2, presents a clear picture of the role of the stressing rate parameter in modifying the response of the Aging
law. Without any contribution of the stressing rate term \((c = 0)\), we have exactly the Aging law. When there is some contribution, we observe a transition from the Aging to Slip law depending on the value of \(c\) and how far above steady state we are. When the stressing rate contribution dominates the other terms in Eq. (2.6), the response looks exactly like the Slip law. This behavior is shown clearly in Figures 2.2 and 2.3. One more point needs to be made with respect to the identity in Eq. (2.21). It is strictly valid for all values of \(x > 0\) only when \(r \to 0\). But for \(0 < r \ll 1\), Eq. (2.21) is still a good approximation as long as \(x\) lies within some logarithmic neighborhood of 1, the size of the neighborhood increasing with decreasing but positive values of \(r\). For example, if we choose \(c = 10\) and \(a = b\) we get \(r = 1 - ca/(c + 1)b = 0.09\). For \(x = 10^2\) (\(x\) can be thought of as a jump size) the right hand side of Eq. (2.21) is 1.5136 while the left hand side is 1.4145. This explains the near exact correspondence between the Slip law and the Nagata law responses for \(V_f/V_i \lesssim 10^{\pm 2}\) with \(c = 10\) in Figure 2.2. A corollary to the above statement is that whenever \(0 < r \ll 1\), there always exist values of \(x\) for which Eq. (2.21) is no longer a good approximation. This again leads us to the conclusion that for a given finite value of \(c\) there always exists a value of \(V_f/V_i\) above which the Nagata law response will show the linear slip weakening behaviour over non-zero slip distances.

To complete the discussion of the response of the Nagata law to velocity step tests, we tabulate three scaling relationships that describe the correspondence of the Slip law parameters and the Nagata law parameters in terms of \(c\). That is, for a sufficiently large \(c\) and sufficiently small velocity jump \(V_f/V_i\) (such that the Nagata law response is identical to the Slip law), the following scaling relationships show how the other Nagata law parameters must be chosen to match the Slip law response given the value of \(c\):

23
\[ D_c(1 + c) \equiv D_c|_{\text{Slip}}, \quad (2.24a) \]
\[ \frac{a}{1 + c} \equiv a|_{\text{Slip}}, \quad (2.24b) \]
\[ (a - b) \equiv (a - b)|_{\text{Slip}}. \quad (2.24c) \]

Eq. (2.24a) follows from equating the slip weakening rates of the Slip and Nagata laws. This explains the small value of \( D_c \) needed by Nagata et al. (2012) to fit their data. Eq. (2.24b) follows from Eq. (2.11a) and is imposed by the requirement of having the same peak stress for the Nagata and Aging/Slip laws after a given velocity step up/down. This indicates why the larger \( a \) values observed in experiments carried out by Nagata et al. (2012) could be accommodated by introducing the stressing rate dependent term with a value \( c \approx 2 \). As mentioned before, Eq. (2.24c) follows from the requirement to have the same amount of steady state velocity weakening/strengthening independent of the state evolution law. These scaling relations imply that given lab data with Slip law like responses to each of a series of increasingly large velocity jumps, no upper bound on the value of \( c \) exists. In other words, if one considers \( c \) to be a material property like \( a, b \) and \( D_c \), then it is endowed with this strange property that any velocity step test data that is well modeled by the Slip law cannot constrain \( c \) to any better than a lower bound. Finally, we have the following expression for \( a/b \) given a constant value of \( b - a = \Delta \):

\[ \frac{a}{b} = \frac{a|_{\text{Slip/Aging}}(1 + c)}{a|_{\text{Slip/Aging}}(1 + c) + \Delta}. \quad (2.25) \]

Therefore, for a given value of \( \Delta \), \( a/b \rightarrow 1 \) for \( c \gg 1 \). Given this, we have the result that \( \frac{c}{c+1} \frac{a}{b} \rightarrow 1 \) for \( c \gg 1 \) as claimed previously. These scaling relationships are observed to hold quite well when fitting the same velocity step up/down data with both the Nagata and the Slip laws as we discuss in Section 3.2.
A set of scaling relationships similar to Eqs. (2.24a), (2.24b) and (2.24c) hold for the Aging-Nagata law correspondence as well. These are

\[
D_c \left\{ 1 - c \frac{(a - b)}{b} \right\} \equiv D_c \mid_{\text{Aging}}, \tag{2.26a}
\]

\[
\frac{a}{1 + c} \equiv a \mid_{\text{Aging}}, \tag{2.26b}
\]

\[
(a - b) \equiv (a - b) \mid_{\text{Aging}}. \tag{2.26c}
\]

Eq. (2.26a) follows from the requirement of equality of the slip weakening rates of the Aging and Nagata laws (equating \(b \sigma / D_c(c + 1)\) from Eq. (2.19) with the linear slip-weakening rate \(b \sigma / D_c\) for the Aging law). There is an important distinction between the scaling relations in Eqs. (2.24) and (2.26). For a given value of \(c\), Eqs. (2.24) define an exact (asymptotic) correspondence between the Nagata and Slip laws given a small enough velocity jump (up or down). But for the same value of \(c\), the correspondence between the Nagata and Aging laws defined by the relations in Eqs. (2.26) applies only for (sufficiently large) velocity increases, and only for slip distances \(\delta\) such that the linear slip weakening approximation holds (\(\delta \lesssim ND_c(c + 1)\)). The Aging-Nagata scaling relation for \(D_c\) for velocity step downs is, in fact, the same as Eq. (2.24a). This is because the Nagata law velocity step down response in Eq. (2.17b) is exactly analogous to its Aging law counterpart with the Aging law \(D_c\) replaced by the Nagata law \(D_c(c + 1)\). But the \(D_c\) scaling for step downs is not relevant to understanding 1D fault nucleation. Eqs. (2.26b) and (2.26c) are identical to their Slip law counterparts and, actually, hold both for velocity step ups and downs. In practice, however, existing velocity step tests of up to 2 orders of magnitude are known to agree well with the Slip law (Bayart et al., 2006), and therefore Eq. (2.24a) is probably the more useful scaling relationship for \(D_c\) in understanding Nagata law fits to laboratory velocity step tests.
2.3.4 How large must $c$ be?

A crucial question to ask at this point is how large does $c$ need to be for the Nagata law to explain existing large magnitude velocity step up/down data as well as the Slip law. As discussed above, the real test for the existence of a characteristic length scale for the decay/rise of the transient shear stress response to a velocity step (for a single degree of freedom system) comes from pushing the system far from steady state. The two order of magnitude velocity step up/down data of Bayart et al. (2006) thus provide a good test. A double-direct shear geometry apparatus was used to shear 3 mm thick layers of granular quartz gouge between rough surfaces held at 25 MPa normal stress. The surface was very nearly velocity-neutral ($a - b \approx 0$). Bayart et al. (2006) have shown that these data are well fit by the Slip law but not the Aging law. Given that these experiments drive the sliding interface further from steady state than the experiments by Nagata et al. (2012), our analytical results suggest that values of $c$ larger than what they found might be required to fit the data as well as the Slip law. To confirm this, we inverted the dataset for the relevant Nagata law parameters using a Bayesian algorithm (Bhattacharya et al., 2015). We found $c \approx 10.0$ to be an appropriate lower bound on $c$ to fit these larger velocity step up/down data as closely as the Slip law. This is considerably larger than the $c = 2.0$ adopted by Nagata et al. (2012). The corresponding (estimated) Nagata law parameters follow the scaling prescriptions in Eqs. (2.24a), (2.24c) and (2.24b) quite well, indicating the validity of these relationships in this case.

2.4 Nucleation on 1-D faults

Although the results for the single-degree of freedom system are instructive, the details of Nagata law nucleation on elastically deformable faults can be revealed only by numerical simulation. We use a wavenumber domain, parallelized, boundary element code to run simulations for a 1D fault embedded in a 2D elastic continuum. In what follows, we
introduce the model, discuss the numerical features of the Nagata law nucleation and then describe, through a series of approximate analytical solutions, the salient features of the numerical simulations.

2.4.1 Mathematical formulation

We start by describing the full suite of equations that govern the physics of our problem. From elasticity, we write down the shear stress on the 1D fault as:

\[
\tau(x, t) = \tau^\infty(x, t) + \tau^{el}(x, t) = \tau^\infty(x, t) + \frac{\mu'}{2} \mathcal{H}[\delta^\prime](x, t),
\]

(2.27)

where \(\tau^\infty(x)\) is the remote tectonic loading, \(\tau^{el}\) is the elastic stress produced by non-uniformity of slip and \(\mathcal{H}\) denotes the real Hilbert transform. Differentiating Eq. (2.27) we can equate the shear stressing rate with the rate of change of frictional strength from Eq. (2.2) to write:

\[
a \frac{\dot{V}}{V} + b \frac{\dot{\theta}}{\theta} = \frac{\tau^\infty}{\sigma} + \frac{\mu'}{2\sigma} \mathcal{H}[V'],
\]

(2.28)

We assume \(\sigma, a, b\) and \(\dot{\tau}^\infty\) to be spatially uniform and solve Eq. (2.28) coupled with Eq. (2.6).

In keeping with typical values for natural faults, we prescribe \(\dot{\tau}^\infty = 10^{-2}\ \text{Pa/s}\) and an initial, spatially uniform velocity \(10^{-9}\ \text{m/s}\). Nucleation is started in our simulations by either prescribing a local peak in velocity with a Gaussian profile of width \(2L_b\) (defined below) or by adding a spatially random perturbation to the steady state \(\Omega = V\theta/D_c\) field to vary it between 0 and 1.

[Rubin and Ampuero (2005)] showed that Aging law nucleation shows a clear dichotomy in nucleation styles depending on the value of \(a/b\). For \(a/b < 0.3781\) a fixed length nucleation patch was observed with the fault pushed so far above steady state everywhere within the patch that the healing term in Eq. (2.4a) could be dropped. This was called the ‘no-healing’ limit. For larger values of \(a/b\), an expanding crack-like nucleation patch
was observed where only the two propagating edges of the patch were far above steady state, while the center was close to steady state. \cite{Rubin and Ampuero 2005} called this the ‘constant-weakening’ regime as the value of the weakening term \((V\theta/D_c)\) in Eq. \eqref{eq:2.4a} turned out to be quasi-constant in time over the nucleating patch.

Slip law nucleation shows a similar \(\Omega \gg 1\) limit for small values of \(a/b\) \((\lesssim 0.6 \sim 0.7)\) in the simulations of \cite{Ampuero and Rubin 2008}, but these values seem more sensitive to boundary and initial conditions than for the Aging law, but the length scale of localization is inversely proportional to \(\sim b\sigma \ln(V\theta/D_c)\) rather than \(b\sigma\). This makes the \(\Omega \gg 1\) regime nucleation zone gradually shrink with increasing slip speeds \cite{Ampuero and Rubin 2008}. However, for larger values of \(a/b\), the \(\Omega \gg 1\) limit is not applicable and a propagating nucleation front travels along the fault as a unidirectional slip pulse. The analysis of the Nagata law velocity step response gives clear indications that tuning the value of \(c\) from \(c = 0\) to \(c \gg 1\) can take us from Aging law behavior to Slip law behavior. Given this, we expect to see a similar transition in nucleation style by varying the value of \(c\). In the rest of this paper, we discuss the details of Nagata law nucleation under variations in both \(a/b\) and \(c\).

Before concluding this section, we will note a few details about the length scales of relevance to the 1D fault nucleation problem. The critical stiffness \(k_c\) (Eq. \eqref{eq:2.7}) arises from analyzing the stability of sliding of a spring block slider. To study the stability of sliding on a deformable fault one needs an analogue of \(k_c\). Following \cite{Rice 1993} and \cite{Rubin and Ampuero 2005}, we derive a dimensional estimate using the fact that the effective stiffness at the center of a 2-D elastic crack (the stress drop for a given amount of slip) is \(k_* = \Delta \tau /\delta = \mu' /2L\). Here \(2L\) is the crack length and \(\mu'\) is the shear modulus \(\mu\) for anti-plane strain and \(\mu/(1 - \nu)\) for plane strain, where \(\nu\) is Poisson’s ratio. Demanding that \(k_*\) at the center of the fault be equal to \(k_c\) leads to the following estimate for the minimum half-length of the nucleation zone required for instability, termed \(L_{b-a}\) because it is inversely proportional to \((b - a)\),
\[ L_{b-a} = \frac{\mu' D_c (1 + c)}{\sigma (b - a)}. \]  
\hspace{2cm} (2.29)

To within a multiplicative constant, this is analogous to the \( h^* \) parameter introduced by \cite{Rice1993} who considered a sinusoidally varying load instead of a crack.

Far above steady state \cite{Dieterich1992}, the smallest length scale that one needs to resolve with the Aging law to observe the relevant features of nucleation is smaller than this length scale by \((b - a)/b\). For the Nagata law, we define this length scale to be (motivated by the linear slip weakening length scale in Eq. (2.19)):

\[ L_b = \frac{\mu' D_c (1 + c)}{\sigma b}. \]  
\hspace{2cm} (2.30)

The length scales in both Eqs. (2.29) and (2.30) are equivalent to the definitions of \( L_b \) and \( L_{b-a} \) from previous numerical studies with the Aging and Slip laws \cite{Dieterich1992, Rubin2005, Ampuero2008} given the scaling relationships in Section 3.4. For nucleation simulations, it is important to choose a grid size small enough to resolve all the spatial heterogeneity in slip or stress. For the expanding crack type nucleation with the Aging law, this length scale is the length scale over which stress drops occur at the edges of the nucleating patch, approximately \( L_b \) \cite{Ampuero2008}. For the Slip law, the relevant length scale is \( L_b \ln(V\theta/D_c)^{-1} \) and, with growing slip speeds, can shrink to much smaller than \( L_b \) \cite{Ampuero2008}. This led \cite{Ampuero2008} to choose a grid resolution of 100 per \( L_b \) for the Slip law. We choose 40-1000 points per \( L_b \).

### 2.4.2 General features of nucleation for \( c = 2.0 \)

Our first suite of simulations were run with \( c = 2.0, D_c = 0.33 \mu\text{m} \) and \( a = 0.05 \). These values were reported by \cite{Nagata2012} as providing the best fit to their lab data. Figure 2.4 shows the results of these simulations. For values of \( a/b < 0.6 - 0.7 \) we observe a clear localization of the nucleation patch just like the ‘no-healing’ regime of the Aging law.
(Figure 2.4a, c and d). This is characterized by the nucleation patch being far above steady state ($\Omega \gg 1$) everywhere (Figure 2.4c). To determine the suitable form of the Nagata state evolution law for $\Omega = V\theta/D_c \gg 1$, it is useful to start by using the quasi-static force balance to replace the stressing rate in Eq. (2.6) with the left hand side of Eq. (2.28) to write:

$$\dot{\theta} = \frac{1}{c + 1} - \frac{V\theta}{D_c(c + 1)} - \frac{c}{c + 1} \frac{a}{b} \frac{\dot{V}}{V}. \quad (2.31)$$

For $\Omega \gg 1$ we can drop the first term in Eq. (2.31), leading to the ‘no-healing’ limit of the Nagata law:

$$\dot{\theta} \approx -\frac{V\theta}{D_c(c + 1)} - \frac{c}{c + 1} \frac{a}{b} \frac{\dot{V}}{V}. \quad (2.32)$$

Using Eq. (2.28) to express $a\dot{V}/V$ in terms of the stressing rate, Eq. (2.32) becomes

$$\frac{\dot{\theta}}{\theta} \approx -\frac{V}{D_c} - \frac{c}{b\sigma} \tau. \quad (2.33)$$

For $a/b > 0.7$, expanding crack type nucleation is observed with the Nagata law (Figure 2.4b, d and f). Figure 2.4f shows that $\Omega$ is quasi-constant in time, and of the order 1, at the centre of the expanding patch in this regime. This is very similar to the ‘constant-weakening’ limit observed in Aging law nucleations (for $a/b > 0.4$). In the subsequent sections we show, following Rubin and Ampuero (2005) and Ampuero and Rubin (2008), that these regimes of Nagata law nucleation are amenable to some quantitative understanding through a series of semi-analytical approximations.

2.4.2.1 Localized nucleation

The numerical results (Figure 2.4e) suggest that localized slipping for Nagata law nucleation with $c = 2$ requires $\Omega \gg 1$ everywhere within the slipping patch. The slip rate profile has a clear spatial structure that is maintained while the velocity increases in time (Figure 2.4a).
Following the analysis of Rubin and Ampuero (2005) for the Aging law, we search for a variable separable solution of the form:

\[ V(x,t) = V_t(t)V(\xi), \quad \xi = \frac{x}{L} \leq 1; \]  
\[ V(x,t) = 0, \quad \xi > 1, \]  
(2.34a)  
(2.34b)

where \( V_t(t) \) and \( V(\xi) \) are the independent time and space solutions such that \( V(0) = 1, \) \( L \) is the fixed length of the localized patch and \( \xi \) is dimensionless position. The stressing rate transforms as

\[ \dot{\tau}^{el}(x,t) = V_t(t)\frac{\mu'}{2L} \hat{T}(\xi), \]  
(2.35)

where \( \hat{T}(\xi) \) is the spatial distribution of the stressing rate. For slip speeds large enough that the remote loading rate is much smaller than the elastic loading rate (\( |\dot{\tau}^{el}| \gg \dot{\tau}^{\infty} \)) we can drop the background loading term altogether and, using Eq. (2.35), write

\[ \frac{\dot{V}_t}{V_t^2} = \frac{b}{aD_c} \left[ V(\xi) + \frac{L_b}{2L} \hat{T}(\xi) \right], \]  
(2.37)

identical to Eq. (14) of Rubin and Ampuero (2005) with \( L_b \) generalized for non-zero \( c \) as in Eq. (2.30). The evolution of \( V_t(t) \) can be obtained by evaluating Eq. (2.37) at \( \xi = 0, \)

\[ \frac{\dot{V}_t}{V_t^2} = \frac{b}{aD_c} \left[ 1 + \frac{L_b}{2L} \hat{T}(0) \right] \equiv \frac{C_{\Omega \gg 1}}{D_c}, \]  
(2.38)
which has the solution:

\[ V_t(t) = \frac{D_c}{C_{\Omega \gg 1}}(t^* - t)^{-1}. \] \hspace{1cm} (2.39)

Here \( t^* \) is the so-called ‘time of instability’ for \( C_{\Omega \gg 1} > 0 \) (as \( \lim_{t \to t^*} V_t(t) \to \infty \)) and is given by

\[ t^* = \frac{D_c}{C_{\Omega \gg 1} V_t(0)}. \] \hspace{1cm} (2.40)

To obtain a semi-analytical solution for \( \mathcal{V}(\xi) \), we follow \( \text{Rubin and Ampuero} \) (2005) and use Eqs. (2.37) and (2.38) to write

\[ \mathcal{V}(\xi) = 1 - \frac{L_b}{2L} [\hat{T}(\xi) - \hat{T}(0)]. \] \hspace{1cm} (2.41)

This gives us a solution for \( \mathcal{V}(\xi) \) given the particular stressing rate distribution imposed by elasticity. \( \text{Rubin and Ampuero} \) (2005) outlined a method to solve for \( \mathcal{V}(\xi) \) semi-analytically for all \( \left| \xi \equiv x/L \right| \leq 1 \) as an eigenvalue problem for a given value of \( L/L_b \). Their approach uses known elastostatic analytical solutions for \( \mathcal{V}(\xi) \) given a piecewise linear and continuous stressing rate distribution \( \hat{T}(\xi) \). Using the fact that the slipping zone is localized, they further imposed that the stress intensity factor at the tips of the nucleation zone be zero. This fixes both \( L/L_b \) and \( C_{\Omega \gg 1} \). As our Eq. (2.41) is exactly analogous to their Eq. (18) (where \( L_b \) in our case is given by Eq. (2.30)), we can use their results directly to fix

\[ L \approx 1.3774L_b, \] \hspace{1cm} (2.42a)

\[ \hat{T}(0) \approx -1.7132. \] \hspace{1cm} (2.42b)
For the no-healing regime, we estimate the patch length from the numerical simulations by defining it to be half of the distance separating the two stressing rate peaks.

Finally, using Eqs. (2.33) and (2.41), we have the following expression for the evolution of state:

\[
\frac{\dot{\Omega}}{\dot{\theta}} = -\frac{V(t)}{D_c} \left[ \mathcal{V}(\xi) + \frac{c}{c+1} \frac{L_b}{2L} \dot{T}(\xi) \right].
\]  

(2.43)

Integrating both sides of Eq. (2.43) and using the expression for \( V(t) \) from Eq. (2.39), it can be shown that the analytical solution for \( \Omega \) at a given space-time co-ordinate is:

\[
\frac{\Omega(\xi, t)}{\Omega(\xi, 0)} = \left( \frac{V(t)}{V(0)} \right)^{\frac{1}{\frac{1}{C_{\Omega>>1}} - 1}} \left[ \mathcal{V}(\xi) + \frac{c}{c+1} \frac{L_b}{2L} \dot{T}(\xi) \right].
\]  

(2.44)

The no-healing regime is characterized by \( \Omega \gg 1 \) everywhere within the nucleation zone as the patch approaches instability. This requires \( \Omega(\xi, t) \) to be a monotonically increasing function over the nucleating patch or equivalently

\[
\frac{1}{C_{\Omega>>1}} \left[ \mathcal{V}(\xi) + \frac{c}{c+1} \frac{L_b}{2L} \dot{T}(\xi) \right] \leq 1.
\]  

(2.45)

In Appendix A.2, we show that the condition for monotonically increasing \( \Omega \) is violated first (if at all) at the center of the nucleation zone (\( \xi = 0; \mathcal{V}(0) = 1 \)). This leads to the following condition for the no-healing limit:

\[
1 + \frac{c}{c+1} \frac{L_b}{2L} \dot{T}(0) \leq C_{\Omega>>1}.
\]  

(2.46)

Eq. (2.46) can be rewritten to provide a range of values of \( a/b \) for which a localized nucleation patch would be numerically observed with the Nagata law:

\[
\frac{a}{b} \leq \frac{1 + \frac{L_b}{2L} \dot{T}(0)}{1 + \frac{c}{c+1} \frac{L_b}{2L} \dot{T}(0)}.
\]  

(2.47)
where $L/L_b$ and $\dot{T}(0)$ are given by Eqs. (2.42a) and (2.42b) respectively and are both independent of $c$. Using numerical values of $L/L_b$ and $\dot{T}(0)$, we can rewrite Eq. (2.47) as

$$\frac{a}{b} \leq \frac{0.3781(1 + c)}{1 + 0.3781c},$$  \hspace{1cm} (2.48)

For $c = 0$, we have the Aging law limit obtained by [Rubin and Ampuero (2005)]:

$$\frac{a}{b} \leq 1 + \frac{L_b}{2L} \dot{T}(0) = 0.3781,$$  \hspace{1cm} (2.49)

where, for $c = 0$, $L_b$ is the same for the Nagata and Aging laws. Furthermore, if one writes $a/b$ in (48) in terms of $a_{\text{Aging}}$ and $b_{\text{Aging}}$ using the scaling relations in (26), one obtains $a_{\text{Aging}}/b_{\text{Aging}} \leq 0.3781$, as for the Aging law. In other words, for values of $a - b$ and the stress change across a velocity step determined by laboratory data, the condition for localized nucleation is independent of the choice of $c$, given the above scaling relations. In our case, for $c = 2$, Eq. (2.47) leads to the condition $\frac{a}{b} < 0.6459$. This is consistent with our observation that the transition from the localized nucleation patch to expanding crack-type nucleation occurs for $a/b \approx 0.6 - 0.7$.

### 2.4.2.2 Expanding crack-like nucleation

Nucleation on a homogeneous deformable fault can be viewed as a competition between two processes: elasticity and slip/velocity weakening friction. Elasticity works to smooth slip/velocity gradients whereas slip/velocity weakening works to keep regions of high slip rates localized. For $a/b$ smaller than a critical, $c$ dependent, value (or equivalently $\Omega \gg 1$), these two tendencies are balanced and we have a localized nucleating patch which keeps accelerating, as in Section 4.2.1. On the other hand, keeping $c$ fixed, as one decreases the amount of velocity weakening by increasing the value of $a/b$, the nucleating patch expands. For $c = 2.0$, this expansion is crack like.
As mentioned before, the constant weakening regime gets its name from the final, quasi-constant, nearly steady state value of $\Omega$ everywhere in the interior of the nucleating patch. The only exception occurs at the propagating tips where the fault is still well above steady state (Figure 2.4f). We derive approximate expressions for this value of $\Omega$ in Appendix A.4. The shear stress drop within the nucleating patch is spatially quasi-uniform but increases with time. Also, nucleating patches with larger values of $a/b$ expand more in approaching the same final maximum velocity (see Figure 2.5). This behavior is analogous to Aging law nucleation for values of $(a/b)_{\text{Aging}} > 0.3781$.

Following Rubin and Ampuero (2005), we can also obtain an estimate of the asymptotic length of the expanding nucleating patch by equating the mechanical energy release rate with the fracture energy. The velocity jump imposed at the tip of the nucleation zone is sufficiently abrupt that the fracture energy can be estimated by making use of the velocity step solutions derived in Section 3.1. In particular, as $c = 2.0 \gg 1$ and the jump in velocity at the tip of the nucleating zone becomes very large, we can make use of the linear slip weakening solution from Eq. (2.19) to estimate the critical amount of slip over which the peak stress is relaxed to the residual, (nearly) steady state, level in the interior of the nucleation zone. Therefore, for a peak to residual stress change $\Delta \tau_{p-r}$, the critical slip weakening distance is given by

$$
\delta_c = D_c (1 + c) \frac{\Delta \tau_{p-r}}{b \sigma}. \tag{2.50}
$$

Eq. (2.19) assumes a ‘true’ velocity step. Therefore, Eq. (2.50) only strictly applies if the peak slip rate imparted to a given point by the passing nucleation front is maintained at that point over all subsequent times. Figure 2.4b shows that this is clearly not the case in our numerical results. However, as the perturbation in velocity after the passage of the stressing rate peak through a point is much smaller than the perturbation due to the stressing rate peak arriving at that point, we find that Eq. (2.19) is a useful approximation. Figure 2.6 provides support to this approximation by showing that our linear stress weakening approximation is
a good proxy for the observed stress weakening in the simulations. Figure 2.6 also shows that the slip-weakening rate in the center of the nucleating zone tends to a constant value depending on $a$, $b$ and $c$.

Further, making use of the initial conditions for a step change in velocity, we can calculate the peak to residual stress drop. To make the derivation easier we first rewrite Eq. (2.2) as

$$\frac{\tau}{\sigma} = \mu^* + (a - b) \log \frac{V}{V_e} + b \log \Omega,$$

(2.51)

We use Eqs. (2.51) and (2.11b) to obtain

$$\Delta \tau_{p-r} = b \sigma \left[ \ln \left( \frac{\Omega_{bg}}{\Omega_{int}} \left( \frac{V_{max}}{V_{bg}} \right)^{r} \right) \right],$$

(2.52)

where $\Omega_{int}$ is the future (near) steady state value of $\Omega$ in the interior of the nucleation zone, $V_{bg}$ is the slip rate well ahead of the front, $\Omega_{bg}$ is the non-steady state value of $\Omega$ well ahead of the propagating front, $V_{max}$ is the maximum value of $V$ near the tip and $r$ is given by Eq. (2.18). It is important to note that Eq. (2.52) assumes a step change in velocity from $V_{bg}$ to the maximum sliding velocity $V_{max}$. Putting the expressions of the critical slip-weakening distance and the peak to residual stress change together we can write the following expression for the critical fracture energy $G_c$ following Lawn (1993):

$$G_c = \frac{1}{2} \Delta \tau_{p-r} \delta_c = \frac{b \sigma}{2} D_c (1 + c) \left[ \ln \left( \frac{\Omega_{bg}}{\Omega_{int}} \left( \frac{V_{max}}{V_{bg}} \right)^{r} \right) \right]^2.$$

(2.53)

For an equilibrium crack $G_c$ is balanced by the reduction in mechanical energy per increment of crack length $G$. In general, for a quasi-uniform stress drop in the interior of the nucleating zone (as in Figure 2.5), the mechanical energy release rate $G$ is given as Lawn (1993):

36
\[ G = \frac{\pi L}{2 \mu'} \Delta \tau_0^2, \] (2.54)

where \( L \) is the equilibrium crack half length and \( \Delta \tau_0 \) is the magnitude of the ambient to residual stress drop. Using the friction law (Eq. (2.51)), one can express \( \Delta \tau_0 \) as:

\[ \Delta \tau_0 = \left[ \sigma (b-a) \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right) - b \sigma \ln \left( \frac{\Omega_{\text{int}}}{\Omega_{bg}} \right) \right], \] (2.55)

This leads to the following expression for \( G \):

\[ G = \frac{\pi L}{2 \mu'} \left[ \sigma (b-a) \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right) - b \sigma \ln \left( \frac{\Omega_{\text{int}}}{\Omega_{bg}} \right) \right]^2, \] (2.56)

Using the condition \( G = G_c \) for an equilibrium crack (setting Eq. (2.53) equal to Eq. (2.56)),

\[ L = L_b \left( \frac{\Delta \tau_{p-r}}{\Delta \tau_0} \right)^2 = \frac{L_b}{\pi} \left[ \ln \left( \frac{\Omega_{bg}}{\Omega_{\text{int}}} \right) + r \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right) \right]^2, \] (2.57)

As the interior of the nucleating zone is always near steady state, \( \Omega_{\text{int}} \sim 1 \) is a good approximation for the constant weakening case. Additionally, we can make the assumption that \( \Omega_{bg} \approx O(1) \) as seen in Figure 2.4f. These assumptions reduce Eq. (2.57) to the following form:

\[ \frac{L}{L_b} \approx \frac{1}{\pi} \left[ \frac{\ln(O(1)) + r \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right)}{\ln(O(1)) + \frac{b-a}{b} \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right)} \right]^2. \] (2.58)

We note that for values of \( a/b \) increasing up to the upper limit 1 (for a given value of \( c \)), \( r \) tends to \( 1/(c + 1) \). Also, the coefficient of the \( \ln(V_{\text{max}}/V_{bg}) \) term in the denominator goes to 0 as \( a/b \rightarrow 1 \). Therefore, with increasing values of \( c \) and \( a/b \), one can drop the first term within the square brackets simultaneously in the numerator and denominator only at
progressively higher order values of $V_{\text{max}}$. Following this argument, for a value of $V_{\text{max}}/V_{bg}$ large enough (given the value of $c$ and $a/b$), we can reduce Eq. (2.57) to

$$\frac{L_\infty}{L_b} = \frac{1}{\pi} \left( \frac{b}{b-a} \right)^2 \left( 1 - \frac{c}{c+1/b} \right)^2, \quad (2.59)$$

where $L_\infty$ is the crack half-length in the limit $V_{\text{max}} \to \infty$. The corresponding estimate for the Aging law can be obtained by putting $c = 0$ in Eq. (2.59) leading to

$$\left( \frac{L_\infty}{L_b} \right)_{c=0} = \frac{1}{\pi} \left( \frac{b}{b-a} \right)^2, \quad (2.60)$$

where the values of $b$ and $b-a$ are equal to the corresponding Aging law parameters. It is important to note that Eqs. (2.57) to (2.59) only hold for combinations of $c$ and $V_{\text{max}}/V_{bg}$ for which the linear slip weakening assumption holds. The values of $V_{\text{max}}/V_{bg}$ for which linear slip weakening is valid with $c \gg 1$ are physically impossible to attain. So one cannot use Eq. (2.59) in the limit $c \gg 1$ for geologically plausible slip rates.

The scaling relationships in Eqs. (2.26b) and (2.26c) independently lead to an implicit relation between the Nagata law $b$ and the corresponding Aging law $b$:

$$\frac{b}{b_{\text{Aging}}} = \left( 1 - \frac{c}{c+1/b} \right)^{-1}, \quad (2.61)$$

This scaling tells us that a set of values of $a, b, D_c, c$ and $a_{\text{Aging}}, b_{\text{Aging}}, D_{c, \text{Aging}}$ (suppose from fitting laboratory friction data equally well with either law) which scale in accordance to the relationships in Eqs. (2.26a), (2.26b) and (2.26c) will predict the same limiting size for the expanding crack like nucleation patch. $L_\infty$ is the minimum length of a velocity-weakening fault patch to allow an expanding crack-type nucleation zone to reach instability. Our results further imply that velocity-weakening fault patches which are large enough to host an Aging law seismic instability would, following the scaling realtions in Eqs. (26), produce the same behaviour with the Nagata law as long as $c$ was small enough to lead to linear slip-weakening at the rupture tip during (most of) the nucleation process.
In a purely quasi-static model (as ours), there is no imposed upper limit on \( V_{\text{max}} \). In order to check if numerical simulations agree with Eq. (2.59) in the expanding crack regime, we let the nucleation zone achieve slip speeds which are not expected on natural faults (at least not for aseismic slip). We need to drop the \( O(1) \) terms from both the numerator and denominator in Eq. (2.58) to arrive at Eq. (2.59). This means that the choice of this maximum slip-speed, \( V_{\text{lim}} \), is dictated by:

\[
\ln \left( \frac{V_{\text{lim}}}{V_{bg}} \right) \gg \max \left[ \frac{\ln(O(1))}{(1 - \frac{a}{b})}, \frac{\ln(O(1))}{(1 - \frac{a}{b^c})} \right].
\]

But in practice, we have always observed from our simulations that the expanding crack attains the value of \( L_{\infty}/L_b \) predicted by Eq. (2.59) if our choice of \( V_{\text{lim}} \) satisfies the following less restrictive condition:

\[
\ln \left( \frac{V_{\text{lim}}}{V_{bg}} \right) \gg \ln(O(1)) \frac{1}{(1 - \frac{a}{b^c})}.
\]

For example, if we choose 5 as an appropriate value for the \( \frac{\Omega_{bg}}{\Omega_{\text{int}}} \approx O(1) \) in Eq. (2.58), we get \( \ln \left( \frac{V_{\text{lim}}}{V_{bg}} \right) \gg 4.02 \) for \( a/b = 0.9, c = 2.0 \). Assuming \( \ln \left( \frac{V_{\text{lim}}}{V_{bg}} \right) \approx 40 \) and as \( V_{bg} \approx 10^{-13} \text{ ms}^{-1} \), one has \( V_{\text{lim}} \approx 10^4 \text{ ms}^{-1} \). The plot of \( L_{\infty}/L_b \) vs. \( a/b \) is shown in Figure 2.7. We have also plotted the \( L/L_b \) values at \( V_{\text{max}} = 0.1 \text{ms}^{-1} \) for comparison. The actual values of \( L/L_b \) arising from the simulations for our chosen values of \( V_{\text{lim}} \) are in reasonable agreement with our analytical estimates except at the very highest value of \( a/b \) (0.99). These deviations may be due to two reasons – (1) For values of \( a/b \) really close to 1, maybe our less restrictive condition in Eq. (2.63) fails and we need to use Eq. (2.62) to compute \( V_{\text{lim}} \). (2) Our conservative choice of \( \ln \left( \frac{V_{\text{lim}}}{V_{bg}} \right) \) as being only one order of magnitude larger than the right hand side of Eq. (2.62). Attempting to rectify either of these problems would lead to remarkably large values of \( V_{\text{lim}} \) and we did not drive our simulations to such limits.
2.4.3 Increasing $c$ (Aging-Slip transition)

Our analysis of the velocity step up/down tests for single degree of freedom systems suggest that if we systematically increase the value of $c$ keeping $a - b$ constant, the response of the Nagata law should become progressively like the Slip law over an increasing range of velocity jumps, at least in the expanding crack-like nucleation regime. On an elastically deformable fault we can recast this statement to ask the following question: Given the range of velocity jumps consistent with earthquake nucleation on a 1D fault, how large does $c$ need to be observe Slip law style nucleation? For this purpose, we use $c$ up to 10 as an appropriate upper limit (the minimum bound on $c$ obtained from fitting 2-3 orders of magnitude velocity steps on simulated gouge in (Bhattacharya et al., 2015)).

2.4.3.1 Localized nucleation

The no-healing regime for $c = 2$ extended up to $a/b < 0.6459$. For $a \approx 0.05$, this corresponds to a range of values of $a - b (\lesssim -0.0274)$ over which we observed a localized nucleation patch. Values of $a - b$ this large are not observed experimentally, consequently the $\Omega \gg 1$ regime is not consistent with laboratory experiments, as for the Aging law. We used these same values of $a - b$ to study the effect of varying $c$ in this regime. Keeping $a - b$ fixed, the scaling relations in Eqs. (2.24a), (2.24b) and (2.24c) were used to specify the new values of $a$, $b$ and $D_c$. For these values of $a - b$ we still observe the localized nucleation patch for $c$ as high as 10 (Figure 2.8). This behavior is exactly what is expected from Eq. (2.48).

In general, however, increasing the value of $c$ also makes it increasingly difficult to push the fault far above steady state at geological slip rates. This is illustrated by the profiles of $\Omega$ in Figures 2.8b and d. For up to $c \approx 10.0$, the length scale over which slip localizes is $1.3774L_b$ as expected from the variable separable solution required to explain the $c = 2.0$ simulations. This is Aging law type behavior (see Figure 2.8a) and is unlike the Slip law ‘no-healing’ regime where the localized nucleating patch is observed to shrink over time.
as \((\ln \Omega)^{-1}\) (Ampuero and Rubin, 2008). However, we show in Appendix A.3 that for an order of magnitude larger value of \(c \approx 100\), the length scale of localization does seem to scale approximately as \(L_b (\ln \Omega)^{-1}\) for geological slip rates with \(\Omega \sim 1\) at the center of the nucleating patch. This shrinking of the localized nucleation patches (Figure A.3) is analogous to Slip law type behavior. However, as \(\ln \Omega\) is smaller than 1, the length scale of localization is actually larger than the linear slip weakening case. In Appendix A.3, we suggest a possible explanation for this behavior by showing that for \(c \sim 100\) and slip rates \(\sim 1\, \text{ms}^{-1}\), the Nagata law state evolution can be approximated by the Slip law at the center of the nucleating zone. We pushed the simulations to higher (non-physical) slip rates to verify the suggestion of an anonymous reviewer that, even for \(c \gg 1\), eventually \(\Omega \gg 1\) and the nucleating patch logarithmically localizes to the Aging law length scale \(1.3774L_b\). Our simulations support this prediction and indicate that, for slip rates approaching \(10^{80}\, \text{ms}^{-1}\), the appropriate approximate form of the Nagata law state evolution is indeed Eq. (2.33) (Appendix A.3), leading again to the limiting length scale of \(L = 1.3774L_b\) (as in Section 4.2.1). Therefore, whether the Nagata law localized nucleation behavior looks like the Slip or the Aging law is determined by the combination of the value of \(c\) and how fast the patch is slipping.

### 2.4.3.2 Non-localized solution

For typical laboratory observed values of \(a - b\), consistent with \(a/b\) considerably larger than is prescribed in Eq. (2.48), the balance between velocity weakening and elasticity is tipped in favor of elasticity and the stiffness of the developing nucleation zone is large enough to either drive it down towards steady state or never let it get far above steady state. For values of \(c \geq 4.0\), in this regime, Nagata law nucleation clearly shows uni-directional slip pulses analogous to Slip law behaviour as in Figure 2.9. The direction in which the pulses finally propagate is presumably controlled by the heterogeneous initial conditions. Figure 2.10 shows that increasing the value of \(c\) makes it harder to push the interior of the nucleation
zone far above steady state. For \( c \gtrsim 4.0 \), the tips are the only locations where the nucleation zone is pushed more than an order of magnitude above steady state by elasticity.

The physical reason behind the slip pulse type nucleation for the Slip law was explained by [Ampuero and Rubin (2008)] from energy balance arguments. These are similar in spirit to the ones presented in the case of the expanding crack type solution observed for \( c = 2 \). Assuming that the ‘true’ velocity step solution for the exponential decay of stress with slip (Eq. (2.23)) holds with \( c \gtrsim 4.0 \) for the range of velocity jumps experienced by the expanding front of the nucleation zone, one can write:

\[
G_c \approx b \sigma \left( 1 - \frac{c}{c + 1} a \right) \int_0^\infty \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right) \exp \left( -\frac{\delta}{D_c(1 + c)} \right) d\delta
\]

\[
= b \sigma \left( 1 - \frac{c}{c + 1} a \right) D_c(1 + c) \ln \left( \frac{V_{\text{max}}}{V_{bg}} \right),
\]

(2.64)

where \( V_{\text{max}} \) and \( V_{bg} \) have the same meaning as in Eqs. (2.52). For mathematical convenience, we have further assumed that the propagating front is always entering a region at steady state \( (\Omega_{bg} = 1) \). This clearly shows that \( G_c \) grows as \( \ln (V_{\text{max}}) \). But Eq. (2.56) shows that \( G \) for a uniform stress-drop crack \( G \) grows as \( \ln^2 (V_{\text{max}}) \). This makes crack-like expansion of the nucleation zone impossible. We show in Appendix A.6 that, for the more general case of \( \Omega_{bg} \neq 1 \) and values of \( r = 1 - \frac{c}{c + 1} a \) which lead to slip pulses as above, \( G_c \) remains a weaker function of \( V_{\text{max}} \) than \( \ln^2 (V_{\text{max}}) \). So, our conclusion from the \( \Omega_{bg} = 1 \) case is rather general for large values of \( c \) (\( \gtrsim 4 \)).

A final point needs to be made with respect to the expression for \( G_c \) in Eq. (2.64). In general, given \( \Omega_{bg} = 1 \), the approximations leading to this expression (the logarithmic approximation in Eq. (2.22)) hold only for \( c \gg 1 \). This requirement is not satisfied by even the largest \( c \) (= 10.0) used in the simulations here. In reality, as we discuss in Appendix A.5, for values of \( c \) that satisfy Eq. (2.17a) (given a particular \( V_{\text{max}}/V_{bg} \)) the fracture energy (for \( \Omega_{bg} = 1 \)) scales as:
\[ G_c \approx \frac{C \ln(V_{\text{max}}/V_{bg})^2}{\alpha(V_{\text{max}}/V_{bg})}, \]  

(2.65)

where \( C \) is a combination of \( a, b, c, D_c \) and \( \sigma \) and \( \alpha(V_{\text{max}}/V_{bg}) \) is a function of \( V_{\text{max}}/V_{bg} \).

We give an analytical approximation of \( \alpha(V_{\text{max}}/V_{bg}) \) in Appendix A.5. But even without the exact analytical form being available to us here, we can make some general comments about the Nagata law slip pulse with the knowledge of the limiting values of \( \alpha(V_{\text{max}}/V_{bg}) \).

If large velocity jumps are imposed at the tip of the nucleation zone, \( \alpha(V_{\text{max}}/V_{bg}) \approx 2.0 \) (this is the linear slip weakening limit). This leads to an expanding equilibrium crack-like solution. The magnitude of the jump that leads to this limiting value of \( \alpha(V_{\text{max}}/V_{bg}) \) is itself an increasing function of \( c \). For smaller jumps, \( \alpha(V_{\text{max}}/V_{bg}) \propto \ln(V_{\text{max}}/V_{bg}) \) (similar to the exponential slip weakening limit). For such values we have a Slip law like propagating pulse. However, for the same nucleation simulation, the rupture front might go through both regimes of velocity jumps. As a consequence the nucleation zone could transition from a pulse-like regime to a crack-type regime with increasing maximum slip rate. This also implies that the size of the nucleation zone for the Nagata law slip pulse is larger than that of a Slip law slip pulse. This is due to the fact that the fracture energy for the Slip law is a weaker function of \( \ln(V_{\text{max}}/V_{bg}) \) than for the Nagata law.

### 2.5 Comments on aseismic slip transients

Slow slip events on subducting plates might be thought of nucleation events that failed to reach instability. One mechanism to obtain slow slip events with standard RSF is to have a velocity weakening region that is large enough to let slip accelerate but not large enough to let the nucleation events reach instability ([Kato, 2003](#), [Liu and Rice, 2005](#), [2007](#), [Rubin](#), 2008). For the Aging law expanding crack-type nucleation, the range of fault lengths which allow such aseismic transients varies over an order of magnitude of length scales for values of \( a/b \approx 0.95 \) ([Rubin, 2008](#)). The underlying reason for this large range is linear slip
weakening of stress with slip following a large velocity step up ($G_c \propto \ln^2(V_{\text{max}})$) which has no experimental support. The Slip law (which explains the relevant lab data well), on the other hand, produces slip pulses which approach instability over a smaller nucleation zone size. This produces slow slip events over a range of fault lengths that seem too small to explain the prevalence of these phenomena in subduction zones (Rubin, 2008). The Nagata law, however, explains up to two orders of magnitude velocity step data as closely as the Slip law with large values of $c \gtrsim 10.0$ and, for such values of $c$, leads to a unidirectional slip pulse-like nucleation over slip rates of interest. Given that the fracture energy for the Nagata law is a stronger function of $\ln(V_{\text{max}}/V_bg)$ for $c \approx 10$ than is the case for the Slip law, these pulses are expected to require a larger region to approach instability than the Slip law slip pulse. This could lead to slow slip events over a larger range of fault lengths than the Slip law and perhaps allow their global prevalence. But again, just like the Aging law, this facility is a result of transition to a linear slip weakening regime (over at least modest slip distances) for very large velocity jumps, which has not been observed experimentally.

2.6 Summary and Conclusions

We examined the response of the Nagata RSF law when applied to single degree of freedom systems. We showed analytically that the Nagata law behaves similarly to the Aging law when pushed far above steady state in velocity step tests. After a large enough velocity step up, the shear stress decays linearly with slip over a distance that increases with increasing size of the jump, and at a rate that is independent of the size of that jump. The size of the jump required to observe this linear slip weakening increases with $c$. For jumps smaller than this $c$-dependent critical size, the Nagata law shows exponential slip weakening like the Slip law. Our analytical solutions also suggest scaling relations between the $a$, $b$ and $D_c$ values for the Nagata law and the corresponding values for the Slip law for a given value of $c$. Similar scaling relations were derived for the Aging law, but one of them (Eq. (2.26a))
applies only in the limit of large velocity increases, while the other two are the same as for the Slip law. Notably, the scaling laws imply that increasing the value of $c$ also increases the value of $a/b$ while keeping $a - b$ constant. These scalings are consistent with the high value of $a$ and low value of $D_c$ observed by [Nagata et al. (2012)] in their experiments.

Given these results from the analysis of single degree of freedom systems, we applied the Nagata law to study nucleation on a 1D fault. For the laboratory derived values of $a$, $c$ (= 2.0) and $D_c$ reported by [Nagata et al. (2012)], Nagata law nucleation is qualitatively similar to Aging law nucleation. For $c = 2.0$, we observed a localised nucleation patch for $a/b \leq 0.6459$ (as governed by Eq. (2.48)) for which the nucleation patch remains far above steady state everywhere in its interior ($\Omega \gg 1$) while approaching instability. For larger values of $a/b$, with $c$ fixed at 2.0, an expanding crack like nucleation zone is observed. We also analytically estimated the limiting size of the nucleation zone for values of $c$ which lead to expanding crack like nucleation given small enough values of $a - b$. The size of this expanding nucleation zone at (or near) instability is the same as for the corresponding Aging law simulations (with values of $D_c$, $a$ and $a - b$ scaled according to Eqs. (2.26a), (2.26b) and (2.26c)). The analytical estimate of the value of $a/b$ (given by Eq. (2.48)) at which the transition from localized to expanding nucleation zone occurs is an increasing function of $c$ in keeping with the aforementioned scaling relationships.

We also carried out a suite of simulations by varying $c$ up to 10.0. This upper limit of the value of $c$ is in keeping with our minimum estimate from parameter inversions on large magnitude velocity step up/down data in simulated gouge (Bhattacharya et al. [2013], manuscript under preparation). Keeping $a - b$ fixed, an increase in the value of $c$ leads to a dichotomy of behaviour in the nucleation style. Values of $a/b$ satisfying Eq. (48) still lead to $\Omega \gg 1$ and a localized nucleation patch with a half-length of $\sim 1.3774L_b$, with the value of $L_b$ suitably scaled, for $c$ up to $\approx 10$. But for an order of magnitude larger value of $c$, we observed $\Omega \sim 1$ and continually shrinking nucleation patches over geologically plausible slip rates. These patches seem to localize to length scales approximately proportional to

45
$L_b/\ln \Omega(0)$, similar to Slip law behaviour. Our simulations suggest that such shrinking nucleation patches ultimately converge to the length scale $1.3774L_b$ with $\Omega \gg 1$ in the limit of non-physically high slip rates ($> 10^{80} \text{ ms}^{-1}$). Values of $a/b$ much closer to 1 than those prescribed in Eq. (2.48) lead to uni-directional slip pulses for $c \gtrsim 4$ in analogy with the Slip law. We explain this based on a fracture energy argument wherein the effective fracture energy increases as a weaker function of slip velocity than would the mechanical energy release rate for a growing crack.

Given that the largest velocity jumps yet achieved in the laboratory seem very consistent with Slip law behavior, to really test the proposed Nagata law will require jumps larger than two orders of magnitude to check whether or not there is indeed a transition to linear slip weakening. Given such a dataset, if the largest velocity increases are still well described by the Slip law, then the Nagata law would need ever increasing values of $c$ to explain them. In the limit of $c \gg 1$ the Nagata law will mimic the Slip law and thus lose time dependent healing. In this case this state evolution law would not solve the problem it was intended to solve.
Figure 2.1 Analytical solutions for the evolution of stress with slip for the (A) Aging law and (B) Slip law for velocity step ups/downs ($V_f/V_i$) of $10^{\pm 2-8}$ with $a = b = 0.007$, $D_c = 10 \mu m$. The Aging law shows linear slip weakening when pushed far above steady state and the characteristic slip weakening distance increases with the size of the velocity jump. The Slip law shows a characteristic slip-weakening distance independent of the size and sign of the jump similar to laboratory velocity step up/down experiments (although these are limited to $\sim \pm 2$ orders of magnitude). Colors correspond to the size of the jumps.
Figure 2.2 Analytical behaviour of the Aging law (open triangles), Slip law (solid diamonds) and the Nagata law (solid curves) for velocity step ups/downs of magnitudes $10^{±2−8}$ with different values of $c$: (A) $c = 1$, (B) $c = 10$. For the Aging and Slip laws $a = b = 0.007, D_c = 10 \mu m$. For each value of $c$, the scaling relations in Eqs. (2.24a), (2.24b) and (2.24c) were used to obtain $a, b$ and $D_c$ for the Nagata Law. For non-zero $c$ one observes Slip law-like behaviour for small velocity jumps but Aging law-like linear slip weakening response for larger jumps. The magenta lines superposed on the Nagata law curves in (A) are the shear stress estimates from the linear slip weakening approximation in Eq. 31. The magnitude of the velocity jump required to see the Nagata law transition between Aging and Slip behaviour increases as $c$ increases. The vertical dashed lines in (A) are estimates of the slip distance over which linear slip weakening occurs from Eq. (2.20) using the equality with a value of 20 for the $O(10)$ term. The corresponding vertical lines in (B) are almost at zero slip for the range of velocity steps used. But the Nagata law response deviates from the Slip law response for velocity jumps $\gtrsim 10^{±6}$. Colors correspond to the size of the jumps.
Figure 2.3 Analytical behaviour of the Aging law (open triangles), Slip law (solid diamonds) and the Nagata law (solid curves) for velocity step ups/downs of magnitudes $10^{\pm 2-8}$ with $c = 100$. For the Aging and Slip laws $a = b = 0.007$, $D_c = 10 \mu m$. The scaling relations in Eqs. (2.24a), (2.24b) and (2.24c) were used to obtain $a$, $b$ and $D_c$ for the Nagata Law. The Nagata law response and the Slip law response are identical for $c = 100$ over the range of velocity jumps used. Colors correspond to the size of the jumps. Symbols for the Aging law velocity decreases (open triangles) plot on the top of one another.
Figure 2.4 Numerical simulation of nucleation on 1D fault using the Nagata law for $a = 0.05$, $D_c = 0.33 \mu m$ and $c = 2.0$. The initial condition was a randomized value of $\Omega$ between 0 and 1. Profiles of (A) slip rate, (C) normalized slip and (E) $\Omega$ for $a/b = 0.6$. A localized nucleation patch is observed. The profile of $\Omega$ shows the ‘no-healing’ regime wherein the nucleating patch is far above steady state everywhere and $\Omega$ remains a monotonically increasing function of time within the patch. Profiles of (B) normalized slip rate, (D) normalized slip and (F) $\Omega$ for $a/b = 0.95$ showing the expanding crack-like nucleation patch. The profile of $\Omega$ shows the ‘constant-weakening’ regime where the nucleating patch is far above steady state only at its edges. In the interior $\Omega$ is quasi-constant in time.
Figure 2.5 Shear stress change profile across the nucleating patch for (A) $a/b = 0.90$ and (B) $a/b = 0.95$ with $a = 0.05$, $D_c = 0.33$ µm and $c = 2.0$. Both the simulations were initiated with a randomized value of $\Omega$ between 0 and 1. Both simulations were terminated when the maximum velocity reached was 1 ms$^{-1}$. 
Figure 2.6 Normalized shear stress change vs. slip from the tip of the crack to the center, for snapshots at three separate times, showing that for the given value of $c = 2.0$ and $a/b = 0.95$ the Nagata law shows quasi-linear slip weakening for the velocity jumps experienced by the fault interior to the expanding nucleation zone. Our analytical estimate for the rate of slip-weakening (blue bold line, Eq. (2.19)) is a good approximation. The peak stress for the linear slip weakening curve is calculated using Eq. (2.11a) with $V_{\text{max}}/V_{\text{bg}}$ as the size of the velocity jump. The vertical extent of the linear slip weakening curve is the peak to residual stress drop (from Eq. (2.52)). The zero stress level is the ambient stress.
Figure 2.7 Plot of $L_{\text{lim}}$ (the crack half-length at $V_{\text{lim}}$) normalized by $L_b$ vs $a/b$ for $c = 2.0$ and two different initial conditions (blue triangles: Locally peaked velocity, uniform state; red squares: Random initial $\Omega$ between 0 and 1, uniform velocity). For progressively higher values of $a/b$, progressively higher values of $V_{\text{lim}}$ were chosen according to Eq. (2.62). The open symbols represent the $L/L_b$ values for the same initial conditions as the corresponding solid symbols but when $V_{\text{max}} = 0.1 \text{ms}^{-1}$. The transition from the fixed length solution (black dash-dotted line is our analytical estimate) to the expanding crack solution (green dashed line is our analytical estimate) is observed around $a/b \approx 0.7$. 
Figure 2.8 Numerical simulation for nucleation on 1D fault using the Nagata law for $a - b = -0.0333$ and $c = 5$ and 10. This value of $a - b$ corresponds to $a/b = 0.6$ in the $c = 2.0$ case. The simulations were initiated with a randomized value of $\Omega$ between 0 and 1. (A) Slip rate and (B) $\Omega$ profiles for $c = 5$ or equivalently $a/b \approx 0.75$. A localized nucleation patch is observed for this set of parameter values. (C) Normalized slip rate and (D) $\Omega$ profiles for $c = 10$ or equivalently $a/b \approx 0.85$ also showing localized nucleation patch. Note that these values of $a/b$ for both $c = 5$ and 10 are smaller than the corresponding right hand sides of Eq. (2.48). The length scale bounded by the red, dashed, vertical lines is $1.3774L_b$ which is the Aging law type fixed length solution. The curves outlined by blue diamonds signify a length scale smaller by $\ln(\Omega)$ with $\Omega$ evaluated at the center of the nucleation zone. This is the length scale for the size of the localized nucleation zone for the Slip law.
Figure 2.9 Numerical simulation for nucleation on 1D fault using the Nagata law for \( a - b = -0.0026 \) and \( c = 5.0 \) and 10.0. This value of \( a - b \) corresponds to \( a/b = 0.95 \) in the \( c = 2.0 \) case. The initial condition was a randomized value of \( \Omega \) between 0 and 1. (A) Slip rate and (B) normalized slip profiles for \( c = 5.0 \) or equivalently \( a/b \approx 0.98 \). These values of \( a/b \) for both \( c = 5 \) and 10 are considerably larger than the corresponding right hand sides of Eq. (2.48). A uni-directional slip pulse is observed for this set of parameter values. (C) Normalized slip rate and (D) normalized slip profiles for \( c = 10.0 \) or equivalently \( a/b \approx 0.99 \) also showing the uni-directional slip pulse.
Figure 2.10 Results from the same sets of simulations as in Figure 2.9. Profile of $\Omega$ for (A) $c = 5.0$ or equivalently $a/b \approx 0.98$ and (C) $c = 10.0$ or equivalently $a/b \approx 0.99$. Normalized stress drop profile for (B) $c = 5.0$ and (D) $c = 10.0$. 
Chapter 3

Critical evaluation of state evolution laws in rate and state friction: Fitting large velocity steps in simulated fault gouge with time-, slip-, and stress-dependent constitutive laws

3.1 Introduction

The proper description of the evolution of the ‘state’ variable in rate and state friction (RSF) remains an unresolved problem as none of the existent mathematical formulations agree with the full range of experimental data (Ruina [1983], Beeler et al. [1994], Marone [1998a, Karner and Marone [2001], Bayart et al. [2006]). This makes the formulation of predictive numerical models of earthquake nucleation very difficult, as different ‘state’ evolution laws can lead to very different nucleation styles (Ampuero and Rubin [2008]). These different nucleation styles arise from the differences in the response of the sliding surface to large
and rapid velocity increases, as the nucleation zone expands into regions that previously had been slipping very slowly. Thus the propagating edge of a nucleation zone undergoes a velocity history similar to that imposed during laboratory velocity step experiments, making such experiments very relevant to studies of earthquake nucleation (Ampuero and Rubin, 2008; Bhattacharya and Rubin, 2014). However, with a few exceptions (Ruina, 1983; Tullis and Weeks, 1986; Bayart et al., 2006), laboratory velocity step increases have been typically limited to at most a single order of magnitude, in part because of the inherent difficulty in stabilizing larger velocity excursions. These velocity steps are far smaller than the multiple orders of magnitude excursions encountered in numerical simulations. Such datasets are therefore ill-equipped to tell us which law to use for realistic simulations of earthquake nucleation.

In this paper, we present some of the largest velocity step laboratory data published to date, consisting of 1-3 orders of magnitude steps on simulated fault gouge. Simulated gouge is particularly suited for our experimental goals because it generally has a larger slip-weakening distance than bare rock (a larger value of the parameter $D_c$ introduced in Section 2), making it easier to stabilize large velocity steps. We carry out non-linear and Bayesian parameter inversions on this dataset using the most common formulations of state evolution, the Dieterich (Aging) and the Ruina (Slip) laws, as well as a recently proposed state evolution law which accounts for an apparent dependence of state upon shear stress (Nagata et al., 2012).

### 3.2 Background

RSF describes frictional resistance as a function of slip rate ($V$) and state ($\theta$). The relationship of the frictional resistance to the rate and state variables is given by the friction law:
\[
\frac{\tau}{\sigma} = \mu(V, \theta) = \mu_* + a \log \frac{V}{V_*} + b \log \frac{\theta}{\theta_*},
\]
(3.1)

where \(\tau\) is frictional strength, \(\sigma\) is normal stress, \(\mu\) is the ‘rate and state dependent’ friction coefficient, \(a\) is the ‘direct effect’ parameter accounting for the variations in frictional strength due to changes in slip rate, and \(b\) is the ‘evolution effect’ parameter which determines the change in friction due to evolution of state. The state variable is a measure of a small fraction of the total strength of the asperities bridging the sliding surface at a reference slip speed (most of the strength is considered contained in \(\mu_*\)), expressed either as average contact lifetime using units of time (Dieterich, 1979, 1981) or in terms of contact strength using units of stress (Nakatani, 2001). In general, at not very high temperatures \(a\) and \(b\) are constants of the order of 0.01 (Blanpied et al., 1998; Marone, 1998a). The other parameters \(\mu_*, V_*\) and \(\theta_* = D_c/V_*\) are the values of friction coefficient, slip rate and state at some reference steady state.

The mathematical framework of the problem is complete with an evolution equation for \(\theta\). Generally, in the RSF literature, evolution of the state variable has been physically interpreted as changes in one or more of: A) the real area of surface contact, B) the strength of contact junctions at a reference strain rate, or C) in the case of granular materials the granular packing and bulk porosity (e.g., Dieterich and Kilgore (1994); Marone (1998a); Beeler (2007); Li et al. (2011)). The two most widely used mathematical descriptions of state evolution assume that these quantities, and hence state, can evolve only with time and/or slip

\[
\text{Dieterich (Aging) Law : } \dot{\theta} = 1 - \frac{V\theta}{D_c}
\]
(3.2a)

\[
\text{Ruina (Slip) Law : } \dot{\theta} = -\frac{V\theta}{D_c} \ln \frac{V\theta}{D_c}
\]
(3.2b)
where $D_c$ is some characteristic slip weakening length scale (Dieterich, 1978; Ruina, 1983). Equation (3.2a) is often referred to as the Aging law, because state increases linearly with time for stationary contacts due to the presence of the ‘healing term’ (the 1 on the right). This property of the Aging law is usually referred to as ‘time-dependent-healing’ (Dieterich, 1972; Ruina, 1983; Beeler et al., 1994). Equation (3.2b) is referred to as the Slip law as state evolution occurs only for slipping contacts $\lim_{V \to 0} \hat{\theta} = 0$. At steady state sliding ($\hat{\theta} = 0$), both laws yield $V\theta/D_c = 1$; near steady state the two laws are asymptotically identical. Everywhere in this paper, ‘above’ and ‘below’ steady state implies $V\theta/D_c$ greater than and less than 1 respectively.

For sufficiently stiff experimental systems, the frictional response of both bare rock and gouge to a step change in load point velocity show that stress evolves to steady state over a slip distance that is independent of the magnitude or sign of the step (Ruina, 1980, 1983; Tullis and Weeks, 1986; Marone, 1998a; Blanpied et al., 1998; Bayart et al., 2006; Rathbun and Marone, 2013). Consistent with these observations, the Slip law response to a velocity step shows exponential slip-weakening behavior over a length scale that is independent of the size or sign of the step (Figure 3.1) (Rice, 1993; Nakatani, 2001; Ampuero and Rubin, 2008; Rathbun and Marone, 2013). In contrast, for large velocity step increases, the Aging law shows linear slip weakening with slope $b\sigma/D_c$. Given that the stress increase upon a step increase in velocity is an increasing function of the step size, this linear slip-weakening with a constant slope implies that the length scale for the evolution of the frictional strength to approach steady state increases with the size of the step (Figure 3.1) (Ruina, 1980; Nakatani, 2001; Ampuero and Rubin, 2008). For large velocity step decreases, on the other hand, instantaneously $V\theta/D_c \ll 1$ so $\hat{\theta} \approx 1$. This allows the Aging law $\theta$ to evolve substantially over slip distances $\delta \ll D_c$ and, ultimately, reach steady state over much smaller slip distances than for velocity step increases of the same size (Figure 3.1). Therefore, unlike the available laboratory data, large velocity step increases and decreases show very different characteristic length scales for state evolution under the Aging law.
In contrast to velocity step tests, slide-hold-slide tests are designed to measure the rate of frictional strengthening of the slip interface during periods of at or near zero slip (Dieterich [1972]). Such experimentally observed frictional strengthening has been widely documented as being consistent with the time-dependent-healing of the frictional interface (Beeler et al. [1994]; Nakatani and Mochizuki [1996]; Berthoud et al. [1999]). These experimental observations of time-dependent healing have been traditionally interpreted as providing experimental support to the Aging law (Beeler et al. [1994]). In particular, slide-hold-slide experiments on quartzite and granite show that the peak stresses following reslides after long holds evolve linearly with log[hold time] with a slope that is independent of machine stiffness. This feature is shared by the Aging law, where during long holds \( \dot{\theta} \sim 1 \) independent of the machine stiffness, but not the Slip law (Beeler et al. [1994]). For the Slip law, a higher stiffness leads to smaller increases in state during the load-point hold (the slip speed more rapidly approaches zero), so that the peak stress increases more slowly with log[hold time] than for a lower stiffness machine. However, this apparent experimental support for time-dependent healing as embodied by the Aging law may be negated by the fact that, again because of the healing term, the Aging law predicts stiffness independent evolution of the stress minima at the end of long holds, which is not observed in the data. The observed stiffness dependence in the stress minima are in much closer agreement with predictions of the Slip law. We leave a discussion of slide-hold-slide experiments to a separate paper (Bhattacharya et al. [2014a]).

Recently, Nagata et al. (2012) carried out a series of ‘shear stress step’ experiments on initially bare rock which were designed to produce very little state evolution across the step in order to estimate the ‘direct effect’ term \( a \) without recourse to a particular state evolution law. They further complemented their mechanical data by measuring the variations in acoustic transmissivity across the slipping interface as a proxy for variations in the state variable. Combining their independent estimate of ‘state’ with the mechanical data from their shear stress steps, they estimated \( a \) to be \( \sim 0.05 \), much larger than the usual estimate.
of $a \lesssim 0.01$. Both the Slip and Aging laws failed to fit their laboratory data with such a large value of $a$. To address this problem, [Nagata et al., 2012] modified the Aging law by including a dependence upon stressing rate:

$$
\dot{\theta} = 1 - \frac{V \theta}{D_c} - \frac{c}{b} \frac{\dot{\tau}}{\sigma},
$$

Here, $c$ is another material parameter and all other symbols carry over their respective meanings (but not necessarily values) from Eqs. (3.1) and (3.2a). The physical origin of such a stressing-rate term in the state evolution equation is unclear, but its mathematical form implies instantaneous state evolution across a velocity step ([Nagata et al., 2012], [Bhattacharya and Rubin, 2014]). In terms of a physical picture, it is conceivable that such rapid changes in state could be related to an instantaneous change in contact area brought about by elastic deformation of contact asperities under rapid stressing ([Nagata et al., 2012]) (assuming no contact area change, the stress change on the asperities is larger than the applied stress change by the ratio of the nominal to true contact area, raising the possibility of finite strains). Under steady state sliding, this stressing-rate term plays no role and the Nagata law predicts $V \theta / D_c = 1$, similar to the Aging (and Slip) law. The stressing-rate term vanishes for $c = 0$ as well, leading to purely Aging law state evolution. [Nagata et al., 2012] found $c = 2$ fit their data reasonably well.

The authors concluded, with support from numerical simulations, that the addition of the stressing-rate term led to a quasi-symmetric response to velocity step tests ([Nagata et al., 2012]), similar to the Slip law. When viewed alongside the fact that Eq. (3.3) retains the healing term from the Aging law, this suggests that the Nagata law might be able to explain the full range of robust observations from rock friction experiments better than either the Aging or the Slip laws. In comparison to the Aging and Slip laws, the stress dependent formulation of state evolution is based more directly on micro-mechanical observations of the sliding interface ([Nagata et al., 2012, 2014]). This makes it important to ascertain to
what extent the Nagata law explains large velocity step data and, therefore, decide whether
this new, 'physical' insight into friction can be utilized in earthquake nucleation simulations.

3.3 Response of the Nagata law to velocity steps

As mentioned in section 3.2, when a fault surface is subjected to a step change in sliding
velocity, both the Aging and Slip laws predict no instantaneous change in state. But under the
Nagata law, the stressing-rate term does produce an instantaneous change in state, opposite
in sign to the velocity change, given any $c \neq 0$. In this section, we evaluate the implications
of this shear stress dependence on state evolution by analyzing the Nagata law response to
velocity steps. Under the Nagata law, the exact expression for the variation of stress with
slip following a step in sliding speed (imposed at steady state) is given by Eq. (2.15) as

$$
\Delta \tau(\delta) = b \sigma \ln \left[ 1 - \left( 1 - \frac{V_f \theta_i}{D_c} \frac{V_f}{V_i} \right)^r \exp \left( -\frac{\delta}{D_c (1 + c)} \right) \right],
$$

where $\delta$ is slip since the velocity step, $V_f$ and $V_i$ are the final and initial slip rates respectively,
$\Delta \tau$ is measured with respect to the future steady state stress corresponding to the sliding
rate $V_f$, and $\theta_i$ is the value of the state variable immediately prior to the step. The exponent
$r$ is

$$
r = 1 - \frac{c}{c + 1} \frac{a}{b}.
$$

For the Aging law ($c = 0$), $r = 1$. For $c \gg 1$, $r \to 0$ (for a fixed velocity dependence of
steady state friction $a - b$, we justify below that $a/b \to 1$ as $c \to \infty$).

In laboratory velocity step experiments, it is generally the case that the velocity changes
are imposed on a surface previously sliding at steady state ($V_i / \theta_i / D_c = 1$). With this
simplification [Bhattacharya and Rubin (2014)] found the following two useful limiting
forms of Eq. (3.4), depending upon the value of $(V_f / V_i)^r$:
\( (V_f/V_i)^r \gg 1: \)

\[
\Delta \tau \approx b\sigma \left( 1 - \frac{c}{c + 1} \right) \ln \left( \frac{V_f}{V_i} \right) - \frac{b\sigma}{D_c(1 + c)} \delta, \quad \delta < N[D_c(c + 1)]; \quad (3.6)
\]

\( (V_f/V_i)^r \sim 1: \)

\[
\Delta \tau(\delta) \approx b\sigma \left( 1 - \frac{c}{c + 1} \right) \ln \left( \frac{V_f}{V_i} \right) \exp \left( -\frac{\delta}{D_c(1 + c)} \right). \quad (3.7)
\]

These equations show how the response of the Nagata law to velocity steps can mimic either the Aging law or the Slip law, depending upon \( c \) and the amplitude of the velocity step. Equation (3.6) exhibits linear slip-weakening behavior, similar to the Aging law, up to slip distances of \( N \) times \( D_c(c + 1) \) during the approach to steady state, where \( N \) is given roughly by

\[
 \left( \frac{V_f}{V_i} \right)^r \exp(-N) \approx O(10). \quad (3.8)
\]

The slip weakening rate, given by the second term on the right, is \( b\sigma/[D_c(1 + c)] \), compared to \( b\sigma/D_c \) for the Aging law. Considering the requirement for the applicability of Eq. (3.6), \( (V_f/V_i)^r \gg 1 \), Bhattacharya and Rubin (2014) showed that for any given value of \( c \), there exists a critical value of \( V_f/V_i \) above which the Nagata law response shows linear slip weakening behavior, similar to the Aging law, and that the size of this critical velocity jump increases with the value of \( c \). For velocity decreases as large as the increases that satisfy Eq. (3.6), the Nagata law shows effects of time-dependent healing with substantial state evolution occurring over \( \delta \ll D_c(c + 1) \). This leads to steady state being attained over much smaller slip distances than velocity increases of the same size, similar to the Aging law response (Figure 3.1).

For velocity jumps of either sign much smaller than this critical size, \( (V_f/V_i)^r \sim 1 \) and Eq. (3.7) applies. This equation exhibits exponential decay of stress over a characteristic
slip distance independent of the sign or size of the jump, just as for the Slip law. In fact, this
limiting form of the Nagata law is identical to the response of the Slip law to velocity steps,
given a set of scaling relationships between the Nagata and Slip law parameters described
in Eqs. 3.9a-c below. Furthermore, although mathematically for any finite \( c \) there is some
value of \( V_f/V_i \) above which the Nagata law response shows linear slip weakening behavior,
for \( c \gg 1 \) such velocity jumps are not realizable in nature (e.g., with \( c = 10 \) and \( a = b \),
\( (V_f/V_i)^r \sim 10 \) requires \( \log_{10}(V_f/V_i) \sim 25 \)). Thus for \( c = 0 \) the Nagata law is identical to the
Aging law, while for \( c \gg 1 \) the response of the Nagata law to physically plausible velocity
steps is asymptotically identical to that of the Slip law.

For velocity excursions imposed from steady state, there is a simple physical interpre-
tation of \( (V_f/V_i)^r \): It equals \( V_f \theta_f/D_c \), the measure of how far from steady state the surface
is brought immediately following the step (we define \( \theta_f \) as the value of state immediately
following the velocity step). One can think of the stressing rate term in Eq. (3.3) as adjusting
how far from steady state the slip surface is pushed by a given velocity jump, by controlling
the amount of state evolution across the jump. Viewed in this way, the linear slip-weakening
response occurs when \( c \) is small enough that the imposed velocity step can push the slip
surface far above steady state, while the exponential slip-weakening response occurs when
\( c \) is large enough to prevent this. These transitions are shown in Figure 3.1. Because the
Nagata law was developed by adding a stressing rate term to the Aging law, it might be
helpful to note that the exponential slip-weakening response when the fault is close to steady
state can be viewed as the near-steady-state response of the Aging law, since the Slip and
Aging laws are asymptotically identical near steady state (Appendix C in Bhattacharya and
Rubin (2014)).

To complete the discussion of the response of the Nagata law to velocity step tests, we
tabulate three scaling relationships that describe the exact correspondence between the Slip
and Nagata law parameters when Eq. (3.7) holds:
Equation (3.9a) follows from Eq. (3.7) because the characteristic slip weakening distance, which is a property of the frictional interface, is $D_c(1 + c)$ for the Nagata law instead of $D_c$ for the Slip law. Equation (3.9b) follows from the requirement of having the same peak stress value for the Nagata and Slip laws following a given velocity step up/down, and indicates why the larger $a$ values observed in experiments carried out by Nagata et al. (2012) could be accommodated by introducing the stressing rate term. Equation (3.9c) follows from the requirement to have the same amount of steady state velocity weakening/strengthening independent of the state evolution law. These scaling relations along with Eq. (3.7) tell us that if a velocity step dataset is closely fit by the Slip law, then we can always find a value of $c$ large enough for the Nagata law to match the data equally well. But, importantly, all values of $c$ larger than this critical value will also fit this dataset equally well but with $a$, $b$ and $D_c$ now scaled according to Eqs. (3.9)a-c. Therefore, velocity step data well explained by the Slip law cannot fix the value of $c$ to any better than a lower bound.

Finally, we have the following expression for $a/b$ given a constant value of $b - a = \Delta$:

$$\frac{a}{b} \bigg|_{\text{Nagata}} = \frac{a|_{\text{Slip}}(1 + c)}{a|_{\text{Slip}}(1 + c) + \Delta}.$$  \hfill (3.10)

Therefore, for a given value of $\Delta$, $a/b \to 1$ for $c \gg 1$. The scaling relationships in Eqs. (3.9) and (3.10) are observed to hold quite well when fitting the same set of velocity step up/down data with both the Nagata and the Slip laws (Section 4.2 and Appendix C.2).

We note that this discussion reinforces the importance of obtaining laboratory velocity step data with velocity excursions as far from steady state as possible. Existing experiments
are consistent with exponential slip weakening over a length scale independent of the magnitude of the velocity step, consistent with the Slip law. But the Nagata law, motivated by an indirect measure of the state variable, predicts that for any given \( c \) the response of the fault surface will transition from Slip-law-like to Aging-law-like as the velocity jump increases. If this transition occurs at less than elastodynamic speeds, as it would for the value \( c = 2 \) inferred by \[\text{Nagata et al.} (2012)\], it will influence the style of earthquake nucleation (\[\text{Bhattacharya and Rubin} \ (2014)\]). Nonetheless, such a Slip law - Aging law transition has not been observed in the laboratory. The experiments of \[\text{Nagata et al.} \ (2012)\] are limited to modest excursions from steady state; those we describe below include excursions up to nearly 3 orders of magnitude.

### 3.4 Fitting laboratory data

On the basis of the discussion in Section 3, it is reasonable to expect that a velocity step up/down dataset which is well fit by the Slip law can be equally well fit by the Nagata law with sufficiently large values of \( c \). In parameter inversions with such a dataset, we can only expect to find the lower bound on acceptable values of \( c \). We further expect that this lower bound on \( c \) will increase with the size of the velocity jump, i.e. how far the sliding surface is pushed from steady state. In this section, we make use of these analytical results while performing parameter inversions on laboratory data. For all our inversions our forward model is comprised of two coupled differential equations – force balance between elasticity and friction and the relevant state evolution law. Differentiating the friction law (Eq. (3.1)) with respect to time and equating this with the time-derivative of the spring force, the force balance for a single degree of freedom system is:

\[
\frac{dV}{V} + \frac{d\theta}{\theta} = k(V_{l_p} - V), \quad (3.11)
\]
where $V_{lp}$ is the load point velocity and $k$ is stiffness normalized by the normal stress (so $k$ has units of length$^{-1}$). The time-varying load point velocity is imposed by specifying the desired load point displacements as a function of time. The actual load point velocity achieved is determined from measured displacements of the load point, and the stiffness is that of an equivalent spring connecting the sliding interface to the point at which the load point displacement is measured. This stiffness is estimated from measurements made at the start of the largest velocity increases (Appendix C.1). We solve this forward model for a particular choice of the model parameters in order to obtain numerical values of the shear stress at all times at which the experimental shear stress was recorded. We use a weighted sum of squares ($l_2$ norm) of the difference between the numerical and experimental time-series to quantify the misfit. The weighted root mean square error (RMSE) is also used to construct the likelihood function for Bayesian inference on the data (see Appendix B).

### 3.4.1 Experimental setup

We used velocity step data from the experiments of Bayart et al. (2006) carried out in the Penn State Rock and Sediment Mechanics Lab. The experimental setup is a biaxial double-direct-shear configuration designed to measure the frictional and mechanical properties of granular materials or bare rock (Dieterich, 1972; Savage and Marone, 2007). For all the experiments discussed here, the sample consisted of simulated granular quartz gouge. In this case, the double-direct-shear configuration consists of two layers of sample material, each 3 mm thick, sandwiched between three steel forcing blocks. The quartz particles were initially sub-angular and have initial grain size of 50–150 µm. Grooves cut perpendicular (0.8 mm deep and 1 mm wide) to the sliding direction force shearing to occur within the layer rather than at the boundaries. Fast acting servo-hydraulic controllers are used to maintain specified normal stress on the sliding surfaces and displacement rate on the ram. Forces are measured with load cells mounted to the end of each ram. Displacements are recorded with direct current displacement transducers mounted both at the load point and on the sample,
straddling the slip surface. The nominal frictional contact area for our experiments was 10 cm x 10 cm. All experiments were carried out at 25 MPa normal stress under ambient room humidity (43% to 57%) and temperature (23.7°C to 26.7°C). The sliding velocities accessed in these experiments ranged from 0.5 µms⁻¹ – 500 µms⁻¹.

All experiments showed an initial transient regime where the ‘steady-state’ friction gradually increased up to slip distances of about 10 mm, followed by steadier behavior where the friction coefficient under steady sliding was more nearly constant. All the results that we discuss here are from experiments carried out in the latter regime, which from previous experiments in the Penn State lab corresponds to the case where well-developed shear bands have been formed within the gouge. The slip is thought to occur along a well developed sliding interface within the shear band (e.g. Rathbun and Marone (2013)). In this larger slip range the surface was very nearly velocity neutral (\(a \sim b\)).

### 3.4.2 Parameter inversion

As emphasized earlier, we need our velocity step experiments to push the slip surface as far above steady state as possible in order to fully explore the part of the parameter space most relevant to earthquake nucleation. However, owing to the finite stiffness of the experimental apparatus, a 3 orders of magnitude velocity step imposed at the load-point does not translate to a 3 orders velocity step on the sliding surface and, therefore, does not drive the surface as far from steady state as desired. Additionally, to qualitatively assess the successes or failures of the various laws it turns out to be useful to compare the data and the fits to the analytical expectations that arise from ‘true’ velocity steps (e.g. do the data support linear slip-weakening with a fixed slope, or exponential slip weakening over a fixed length scale, in both cases independent of the size of the jump?).
3.4.2.1 Velocity steps of up to 2 orders of magnitude

One way to bypass the limitations of finite stiffness machines and produce more accurate velocity steps is to servo-control the load point off a displacement transducer straddling the slip surface (Ruina, 1980, 1983; Linker and Dieterich, 1992; Bayart et al., 2006). In practice this means that the ram displacement is continually adjusted to negate any difference between the displacement measured close to the fault and the desired displacement history on the slip surface. Using this approach, we were able to stabilize 1-2 orders of magnitude velocity step up/downs (dataset p1060) and produced a close to ideal dataset to compare with analytical results (Bayart et al., 2006). Bayart et al. showed that the Slip law does a very good job of fitting the shear stress evolution for this dataset (Figure 3.2). On the other hand, besides producing a fit worse than the Slip law to the velocity step increases, the best fitting Aging law does a particularly poor job with the large velocity step downs. This is because the Aging law, with its time-dependent healing, predicts too much state evolution over $\delta \ll D_c$ following the velocity decreases. Therefore, given large perturbations to steady state sliding, the Slip law explains laboratory velocity step data much better than the Aging law. Furthermore, since these experiments drive the sliding interface further from steady state than the experiments of Nagata et al. (2012), our analytical results suggest that fitting the data as closely as the Slip law might require values of $c$ larger than the value ($c = 2$) that they found.

For preliminary inversions, we employed a downhill simplex scheme (Press et al., 1996) initiated with parameter values dictated by various trends in the data. We used time varying weights along the misfit time-series in order to calculate the RMSE values. The strategies adopted in designing such schemes are described in Appendix B. For most of our inversions we fix $a - b \approx -0.0002$ based on the behavior at steady state, and stiffness (between the sliding surface and the loading ram) $k = 0.0011 \mu m^{-1}$ based on the short-term stress response to the largest load point velocity increase (Figure C.1 in Appendix C.1). The fits for the Aging, Slip and Nagata laws are shown in Figure 3.2(A). Even though the misfit was
calculated between the observed and modeled stress time series, we plot the fits and the stress
data against fault slip, rather than time, to facilitate comparisons with the analytical results
for velocity steps. The data are plotted against fault slip calculated as \( \delta = \delta_{lp} - \Delta \mu/k \), where
\( \delta_{lp} \) and \( \Delta \mu \) are the observed load point displacement and friction coefficient (shear stress
divided by normal stress) respectively. The modeled friction coefficient is plotted against
the fault slip as determined by the forward model. Figure 3.2(B) compares the slip velocity
obtained by differentiating the fault slip calculated from the data (\( V = d\delta/dt \)) with the slip
velocities predicted by the respective state evolution law fits shown in Figure 3.2(A). The
best fitting Slip and Nagata laws evidently also explain the slip velocity data most closely.
For the Nagata law the downhill simplex algorithm gets trapped in many local minima in
the parameter space depending on the starting values of \( c \). For \( c = 2 \), the Nagata law does a
much worse job than the Slip law. Fits similar to those produced by the Slip law were found
only for \( c \gtrsim 10 \) by experimenting with various initial parameter values. The existence of
many local minima for \( c \) above the approximate lower bound of 10 is expected from the
scaling relationships in Eqs. (3.9a), (3.9b) and (3.9c). We also observed local minima for
\( c \lessapprox 10 \); all of these produced fits worse than the Slip law but better than the Aging law
(\( c = 0 \)).

To investigate the details of the full posterior distributions of the Nagata RSF parameters,
we designed a small world Markov Chain Monte Carlo global search with adaptive proposal
distribution (Rosenthal, 2011; Bai, 2009a,b; Guan et al., 2006). The details of this inversion
scheme are given in Appendix B. The results from a typical, long Markov chain (50,000
iterations with about an initial 25% being thrown out as burn-in) derived from dataset p1060
are shown in Figure 3.3. The most striking feature of the posterior distributions of \( a \), \( D_c \)
and \( c \) (Figures 3.3(A), (B) and (C) is their remarkable co-variance. The inter-relationships
of the shapes of the three posteriors are consistent with the trends predicted in the scaling
relations in Eqs. (3.9a), (3.9b) and (3.9c). The posterior distribution of \( c \) shows the presence
of a strongly peaked region corresponding to \( c \sim 10 - 100 \) and a quasi-uniform tail which
continues to orders of magnitude larger values of $c$. This clearly shows the existence of many possible Nagata law fits over orders of magnitude variations in $c$, all of which are statistically acceptable given the specified level of data error. Our chains were truncated at some pre-decided upper bound of $c$ by imposing constraints through the prior distribution. For the chain in Figure 3.3 this upper bound was at $c \sim 10^5$, leading to the slow fall-off of the quasi-uniform tail beyond $c \gtrsim 10^4$ in Figure 3.3 (C). Upon closer inspection (Figures 3.3 (D) and (E)) the fits to the data represented by the peaked region of the posterior appear not to be visually identical to the best Slip law fit. In particular, Figure 3.3 (D) shows that the fit to the 2 order step down with $c \sim 20$ (which produces the global minimum of the RMSE for the chain and is typical of the fits represented by the peaked region) looks slightly better than those produced by the Slip law. The Nagata law fits that are indeed visually identical to the Slip law are found only for $c \gtrsim 100$ and are represented by the quasi-uniform tail. Therefore, for experiment p1060, there are two critical values of $c$ that define the family of optimal Nagata law fits: (1) $c \sim 10$ represents the smallest value of $c$ with which one can produce a Nagata law fit to p1060 that is at least as good as the best fitting Slip law. (2) $c \sim 100$ represents the lower bound on $c$ to produce fits identical to the best fitting Slip law. We treat the statistical distinctions between these families in greater detail in Appendix C.2.

The absence of accepted fits to the stress data for $c \lesssim 10$ shows that in order to fit this dataset at least as closely as the Slip law, the lower bound on $c$ is $\sim 10$. This value is considerably larger than the $c = 2$ adopted by Nagata et al. (2012), who fit generally smaller departures from steady state. This supports our analytical expectation that given ever increasing laboratory velocity steps all well modeled by the Slip law, one needs to systematically increase $c$ to produce Nagata law fits similar to those produced by the Slip law. However, the fact that the Nagata et al. (2012) experiments were performed on bare rock also raises the possibility that this variation in $c$ reflects a difference between rock and gouge, an issue that we cannot address with gouge data alone. But if successively larger velocity steps on the same gouge material are all well modeled by the Slip law, this raises
the question of whether introducing the extra parameter $c$ in the Nagata law is justified for gouge. In the next section, we attempt to address this question by analyzing velocity step data containing $> 2$ orders of magnitude steps on the same gouge material as used in p1060.

### 3.4.2.2 Velocity steps larger than 2 orders of magnitude

The strategy of servo controlling the loading ram off the transducer straddling the fault was unable to stabilize velocity step increases larger than 2 orders of magnitude. To attempt larger increases, we designed a specific velocity history for the load point such that, given an estimate of the stiffness of the apparatus and the parameters of a particular state evolution law, it would produce a nearly ‘true’ velocity step on the sliding surface by adjusting the load point appropriately via servo control. In our case we assumed a set of Slip law parameters which fit previous experiments well. In practice, we used force balance and determined the change in shear stress by using the response of the Slip law to a step velocity increase (Eq. (7) in [Ampuero and Rubin (2008)]) to write the required load-point displacement history as

$$
\delta_{lp}' = \frac{a}{k} \ln \left( \frac{V_f}{V_i} \right) \left[ 1 + \frac{b}{a} \left\{ \exp \left( -\frac{V_f t}{D_c} \right) - 1 \right\} \right] + V_f t, \quad \delta_{lp}', t \geq 0
$$

where $\delta_{lp}'$ and $t$ are the ram displacement and time elapsed since the step respectively. An example of such a ram history is shown in Figure 3.4 designed to impose a 3 orders of magnitude velocity increase on the sliding surface. The design includes an instantaneous change in load point displacement, to produce an instantaneous change in stress, followed by a small reversal of the load point velocity (according to Equation 12) necessary to avoid velocity overshoot due to strong frictional weakening. The actual displacement history of the ram did not strictly follow the designed history due inherent limitations of servo-control. This meant that a ‘true’ velocity step could not be imposed on the fault (blue solid line in Figure 3.4). But even with perfect servo-control, modest variations of the Slip law
parameters from experiment to experiment means that the sliding histories would still not have approximated true velocity steps to any great degree of accuracy. In spite of these shortcomings, the main advantage of such designed velocity steps is that we can reasonably expect these to push the sliding surface further from steady state than ‘traditional’ load point velocity steps of the same order of magnitude. The velocity step decreases, however, were imposed as ‘traditional’ load point steps for these experimental runs.

Using such ‘designed’ load point displacement histories, we were able to generate two other sequences of large velocity steps on simulated gouge, each of which contained attempted 3 orders of magnitude velocity step increases (Figures 3.5(A), (D) and 3.6(A), (B)). Given that we did not impose a true velocity step at the sliding surface, we estimated the actual departure from steady state using a forward model that fits the data reasonably well. Assuming that the best fitting Slip law was a good approximation to the stress data, the maximum value of $V_\theta/D_c$ predicted by this forward model was used as a proxy for the maximum departure from steady state following a velocity increase. Defined in this way, the largest velocity step for the dataset p1169 in Figures 3.5(C) and (D) pushed the slip surface nearly 3 orders of magnitude from steady state (peak modeled Slip law $V_\theta/D_c \sim 750$). For comparison, the peak $V_\theta/D_c$ from the Slip law fit to the 2 order step in p1060 is around 80, but we trust this estimate more because the data look very much like an ‘ideal’ velocity step and the Slip law fit is excellent. The largest velocity increase in dataset p1180 (Figures 3.6(A)-(B)) pushed the Slip law $V_\theta/D_c$ to $\sim 300$, equivalent to a velocity step of $\sim 2.5$ orders of magnitude. The parameter inversions were constrained to a fixed stiffness ($k = 0.0012 \mu m^{-1}$ for both p1169 and p1180) based on the short-term stress response to the reslide following a load-point velocity hold performed later during both experimental runs. All the fits were also constrained at a fixed $a-b$ value ($\approx 0.00025$ for p1169 and $\approx 0.00001$ for p1180) based on the steady state response from the data.

We again used the downhill simplex algorithm for the preliminary fits. The Aging law again does the worst job of all the laws considered here (Figures 3.5(A) and 3.6(A)). The
Slip law produces a much better fit than the Aging law but a less impressive fit than for the 1-2 orders of magnitude steps in p1060. Interestingly, even though these experiments were carried out on the same starting material, the value of $D_c|_{\text{Slip}}$ obtained for the simulated gouge in p1169 is more than twice the value inferred for either p1060 or p1180. Recent laboratory experiments suggest that such variations in $D_c$ might be due to differences in the extent of shear localization and fabric development within the gouge sample (Rathbun and Marone, 2013). Also, our choice of stressing rate dependent weights (refer to Appendix B) led to the Aging law fitting the velocity step decreases better in p1169 and the step increases better for p1180. But the asymmetry in the Aging law response clearly prohibits it from fitting both the step increases and decreases as well as the Slip law. The Aging and Slip law solutions correspond to global minima for both p1169 and p1180. This was verified by choosing various initial parameter sets for the downhill simplex algorithm and, also, independently running the aforementioned MCMC code.

The MCMC posterior search revealed that the Nagata law fits accepted at the specified level of data error spanned many orders of magnitudes of values of $c$ for both p1169 and p1180 (Figures 3.7(A)-(I)). For p1169 (max Slip law $V/\theta/D_c \sim 750$), the posterior for $c$ showed a strongly peaked region between $c \sim 5 - 60$ and a quasi-uniform tail for $c \gtrsim 70$ extending over many orders of magnitude (Figure 3.7(C)). When compared to the quasi-uniform region, the Nagata law fits represented by the peaked region represented smaller RMSE values with $c \approx 10$ leading to the global minimum RMSE. This is similar to the posterior structure observed for p1060 in Figure 3.3(C). The Nagata law fit for $c = 10$ is visually not identical to the Slip law fit, as can be clearly seen in Figure 3.5(A). When compared to the Slip law fit, the Nagata law fit for $c = 10$ seems to have done a worse job of fitting the 2 orders of magnitude velocity increase but a better job for the $\sim 3$ orders of magnitude increase. The lower bound on $c$ to produce fits identical to the Slip law, which we recognize as the onset of the quasi-uniform tail region in the posterior distribution, is around 70. This value is smaller than $c \sim 100$, the corresponding lower bound for
 Additionally, the best Nagata law fit needs $c \sim 10$ whereas this value was $\sim 20$ for p1060 (Figures 3.7(C) and (F)). Finally, the global lower bound on $c$ for fits similar to the Slip law is also smaller for p1169 ($\sim 5$) than it is for p1060 ($\sim 10$). This appears to contradict our analytical expectations given the apparently larger departure from steady state in p1169 than p1060. However, it is worth pointing out that the stress data in p1169 show rather complicated behavior near peak stress for both the attempted 2 and 3 orders velocity increases (Figures 3.5(B), (C)). Our analytical expectations might be too simplistic to explain the features of the numerical fits to such datasets that deviate this strongly from ideal velocity steps.

For p1180 (maximum Slip law $V\theta/D_c \sim 300$), the posteriors for $a$, $D_c$, and $c$ were purely quasi-uniform beyond $c \sim 100$ and showed no prominent peaked regions around $c \sim 10$ like the ones seen for p1060 and p1169. In particular, the Nagata law with $c \approx 10$ led to worse fits than the Slip law, the most pronounced differences occurring between the fits to the 1 and 2 orders load point velocity step decreases (Figures 3.8(A), (B)). There seems to be no region of the parameter space that allows the Nagata law to produce fits that have a lower RMSE than the best Slip law fit, at least when $a - b$ is constrained by the observed steady state behavior. The lower bound on $c$ for generating fits identical to the Slip law appears to be $\sim 100$. This lower bound is the same as that found for p1060 (with a smaller apparent maximum excursion from steady state) and larger than $\sim 70$ as seen for p1169 (with a larger apparent maximum excursion from steady state).

### 3.4.2.3 Comparing analytical and inversion results

The analysis in sections 3.4.2.1 and 3.4.2.2 shows that: (1) The Slip law fits the $\geq 2$ order velocity step data much better than the Aging law; (2) Given the best Slip law fit to a particular dataset, we can always find lower bounds on $c$ such that the Nagata law fits these data either slightly better or identically well. It is important to recognize that we have no analytical expectations for those Nagata law fits which match the stress data better than
the Slip law. We only note from our inversion results that when this class of Nagata law fits do exist (for datasets p1060 and p1169), they appear as a strongly peaked region of the posterior distribution representing a well constrained minimum RMSE solution (see Appendix C.2). For both p1060 and p1169, this maximum aposteriori (MAP) value of $c$ is considerably smaller than the minimum value required to replicate the best Slip law fit with the Nagata law. However, given that the frictional properties of simulated gouge have been known to vary systematically with the extent of shear-displacement (Marone, 1998b; Rathbun and Marone, 2013; Marone and Saffer, 2015), we cannot rule out the possibility that the slightly worse (than the MAP Nagata law fit) Slip law fits might only be an artifact of drift of the material properties across the experimental run.

On the other hand, we do have some analytical expectations relevant to the Nagata law fits which exactly reproduce the best Slip law fits to a given velocity step dataset. These expectations, however, come with the caveat that they were derived from analytical results on ‘true’ velocity steps (section 3.3), a condition only approximately satisfied in our experiments. With this caveat, the main analytical predictions regarding these Slip law like fits are: (1) The values of $c$ which allow such fits can only be constrained to a lower bound; (2) this lower bound on $c$ is an increasing function of the maximum velocity step size that we choose to fit. In Appendix C.2, we show that the quasi-uniform tail of the marginal posteriors for $c$ in Figures 3.3 and 3.7 represent this family of Slip law like fits and that these quasi-uniform posteriors are only bounded by a minimum value of $c$. This is in agreement with expectation (1) above. But, contrary to expectation (2), the lower bounds on $c$ imposed by the onset of the quasi-uniform regions in the posteriors do not show any robust positive correlation with the size of the largest velocity step being fit. In particular, the dataset with the largest excursion from the steady state led to the smallest value for this lower bound ($c \sim 70$ for p1169). As pointed out above, it is possible that this particular analytical expectation, derived for ‘true’ velocity steps, is not relevant to ‘approximate’ laboratory velocity steps in general. On the other hand, inherent differences between the
various experimental runs make comparison of trends difficult across the different datasets and could possibly mask any correlations. Therefore, to be sure whether expectation (2) was consistent with our inversions or not, we compared the lower bounds on $c$ (for Slip law like fits) across different subsets of velocity steps from the same experimental run. We show in Appendix C.3 that the lower bound on $c$ (for Slip law like fits) derived from fitting only 1 order steps in each dataset is always smaller than the corresponding estimate for the whole dataset. This shows that the lower bound on $c$ to produce Slip law like fits increases with the size of the velocity step being fit. That is, our inversion results on approximate velocity steps in the laboratory agree with our major analytical expectations based on ‘ideal’ velocity steps.

3.5 Can there be too large a value of $c$?

Nagata et al. (2012) made micro-mechanical observations of the frictional surface but proposed a shear stress dependence of state which has no well-established micro-mechanical origin. We examined the implications of this additional factor in state evolution across a wide range of excursions from steady state sliding. The most striking of these implications is that velocity step data which is fit well by the Slip law is at least equally as well fit by the Nagata law over orders of magnitude variations in $c$ larger than a minimum. Therefore, it is important to ask if there exists a physical constraint on the maximum, physically reasonable, value for $c$. One approach could be to find a proxy for $c$ in the quantities which scale with $c$ but have a clearer physical meaning and hence might be easier to constrain, e.g. the direct effect $a$. Because $a$ scales linearly with $c$ (Eq. (3.9b)), the value of $a$ needed to fit the data increases to very large values as the value of $c$ increases to $c \gg 1$ (e.g. for $c = 100$, the appropriate value of the Nagata law $a$ for p1180 is $\sim 0.8$). Unlike $c$, $a$ has an independent physical interpretation based on only the friction law in Eq. (3.1): It determines the stress increase required to slip faster given a hypothetical fixed state (physical or chemical) of
the slip surface. Assuming this direct velocity effect to be an activated Arrhenius rate process, the traditional room-temperature Aging or Slip law estimate $a \sim 0.01$ corresponds to activation volumes of the order of the molecular volumes of tectosilicates ($\sim 0.1 \text{nm}^3$), provided contact stresses are assumed to be $\sim 1/10$ the shear modulus (Rice et al., 2001; Boettcher et al., 2007). The Nagata et al. (2012) value of $a \sim 0.05$ on bare rock seems close enough to the order of magnitude theoretical value to be plausible. In fact, Nagata et al. (2012) have argued that $a \sim 0.05$ implies activation volumes which correspond to the typical Si-Si or Si-O bond lengths in tectosilicates (5 times smaller than the volumes corresponding to $a \sim 0.01$). However, it is not immediately clear if activation volumes of atomic order are physically preferable to those of molecular order, especially since the ‘preferred’ value depends on the ‘preferred’ physical mechanism behind the instantaneous rate effect. But atomic volumes seem to be a reasonable physical lower bound and values of $a$ larger than $\sim 0.05$ might already be pushing this boundary. Therefore, the value of $a \gtrsim 0.8$ appropriate to fit p1180 with $c \gtrsim 100$ is probably unreasonably large and raises the possibility that such large values of $c$ could be falsified on this basis.

It should also be possible to test whether values of $c$ as large as 10 (the minimum value required to fit the data in Figure 3.2 as closely as the Slip law) are plausible using acoustic monitoring of large velocity steps. For example, in Figure 3.2(A) the excursions in friction following the two-orders-of-magnitude velocity steps are $\sim 0.03$. Given that the steady state friction coefficient in these experiments is $\sim 0.6$, this represents a roughly 5% change in frictional strength. From Figure 3.2(B) it appears that these velocity steps are sufficiently close to ideal that one can reasonably approximate the approach to steady state following the maximum stress excursion as occurring at constant slip speed. This implies that the stress change following the maximum excursion is due entirely to state evolution. If for concreteness we interpret state as true contact area, this implies a 5% change in contact area between the maximum stress excursion and the subsequent steady state.
Because this interpretation depends only upon the friction Eq. (3.1), it is independent of any state evolution law. However, we have already seen that the Nagata law implies an instantaneous change in state (contact area) across a stress step, and we can compare the magnitude of this change to the subsequent change with slip. The stress change with slip can be determined by substituting $\delta = 0$ into either of the approximations (6) or (7), yielding (in agreement with the exact Eq. (14) in Bhattacharya and Rubin (2014))

$$\Delta \tau_{\text{slip}} = b\sigma \left(1 - \frac{c}{c+1} \frac{a}{b}\right) \ln \left(\frac{V_f}{V_i}\right).$$  \hspace{1cm} (3.13)

The stress change associated with the instantaneous change in state can be determined from Eq. (9) in Bhattacharya and Rubin (2014):

$$\Delta \tau_{\Delta \theta_{\text{inst}}} = a\sigma \frac{c}{c+1} \ln \left(\frac{V_f}{V_i}\right).$$  \hspace{1cm} (3.14)

The ratio of the instantaneous area change to the subsequent area change with slip is then equal to $\Delta \tau_{\Delta \theta_{\text{inst}}}/\Delta \tau_{\text{slip}}$, or

$$\frac{\Delta A_{\text{inst}}}{\Delta A_{\text{slip}}} = \frac{c}{c+1} \frac{a}{b} \frac{1}{1 - \frac{c}{c+1} \frac{a}{b}}.$$  \hspace{1cm} (3.15)

This shows that for $a \sim b$ the instantaneous area change is larger than the subsequent evolution for all $c \gg 1$. For the $c = 10$ required to fit the friction data with the Nagata law, the instantaneous area change should be 10 times the subsequent evolution. Given that ultrasonic transmissivity was shown to detect apparent changes in contact area associated with different steady-state sliding velocities in the bare rock experiments of Nagata et al. (2012, 2014), such a large change should be easily visible.
3.6 Summary and conclusions

Numerical simulations show that the feature of a state evolution law most relevant to nucleation of earthquakes is its response to large velocity increases (Ampuero and Rubin, 2008; Bhattacharya and Rubin, 2014). To find the state evolution law most suitable for earthquake nucleation simulations, our study examines the extent to which the Aging, Slip and Nagata laws can fit laboratory data comprising velocity steps of up to nearly 3 orders of magnitude on simulated gouge. Interpretation of earlier experimental work on generally smaller velocity steps suggest that the Slip law does a much better job of matching velocity step data than the linear slip-weakening predicted by the Aging law far above steady state (Ruina, 1980; Tullis and Weeks, 1986). On the other hand, for a given value of \(c\) the Nagata law predicts a transition from Slip to Aging law behavior with increasing velocity step size (Bhattacharya and Rubin, 2014). Since our experiments on simulated gouge pushed the sliding surface farther above steady state than prior laboratory experiments, this dataset provided us a unique opportunity to examine the extent to which each of these laws explain rock friction data far above steady state. Additionally, given that our experiments pushed the sliding surface further from steady state than those of Nagata et al. (2012), we wanted to see if this led to the appropriate value of \(c\) for our experiments being different from the Nagata et al. (2012) value of 2.

Our datasets contained velocity steps of 1, 2 and nearly 3 orders of magnitude, all designed to produce close to ‘ideal’ velocity steps in order to facilitate comparisons with theoretical predictions. We observed that the Slip law fits all of these datasets well, and much more closely than the Aging law. The Nagata law parameters could always be suitably tuned to produce fits either slightly better than or equivalent to the best Slip law fits. But such Nagata law fits could only be generated for \(c\) in the range 10–100 which is significantly larger than the \(c \sim 2\) value reported by Nagata et al. (2012) for initially bare rock surfaces. For all our datasets, the inversion algorithm could only constrain the minimum bound on \(c\) such that the Nagata law fit was at least as good as the Slip law. This and the fact that these
lower bounds were also observed to be increasing functions of the size of the largest velocity step being fit, consistent with surfaces well described by the Slip law, raise questions about whether $c$ is a physically based material parameter.

We conclude that both the Slip law and Nagata constitutive laws adequately describe velocity step tests. However, given a parsimonious view of free parameters, we prefer the Slip law. In detail, the Nagata law can fit velocity step data as well as or slightly better than the Slip law with the value of $c$ larger than a minimum bound according to the size of the largest step size being fit. In fact, if one were to accept the Nagata law fit to p1180 for $c = 10$ as ‘close enough’ to that for $c = 100$, our data cannot rule out the possibility that the Nagata law with $c \sim 10$ and $a \sim 0.07 - 0.08$ is appropriate for gouge. But we find it significant that given the additional parameter $c$ introduced by the Nagata law, for the two experiments fit most closely by the models (p1060 and p1180), the inversion chooses to make that additional parameter large enough that the Nagata law appears nearly indistinguishable from the Slip law.

It is important to note that our datasets were designed specifically to study the response of the sliding surface to large velocity steps. This style of experiments has long been known to favor the Slip law, albeit with generally smaller velocity steps. More investigation of experimental data that also contain sequences of slide-hold-reslides is required to find which of the prevalent state evolution laws best explains all the robust features of laboratory friction. More experiments should also be carried out to fully understand the stress dependence of state, and the large values of $a$, found by Nagata et al. (2012) for bare rocks.
Figure 3.1 Analytical behavior of the Aging (open triangles in (A)), Slip (solid diamonds in (B)) and Nagata laws (solid curves) for velocity steps of magnitudes $10^{±2-6}$. For the Nagata law, values of $c$ are 1 (A) and 10 (B). For the Aging and Slip laws $a = b = 0.007, D_c = 10 \mu m$. The scaling relations in Eqs. (3.9a), (3.9b) and (3.9c) were used to obtain $a, b$ and $D_c$ for the Nagata Law with specified $c$. Note the linear slip-weakening response of the Aging law above steady state and evolution to steady state over $\delta \ll D_c$ below steady state for velocity steps $\geq 10^{±2}$. The Slip law shows exponential slip weakening over the full range of velocity steps. For non-zero $c$ one observes Slip law-like behavior for small velocity jumps but Aging law-like linear slip weakening response for larger jumps. The magenta lines superposed on the Nagata law curves in (A) are the slope estimates from the linear slip weakening approximation from Eq. (3.6). The magnitude of the velocity jump required to see the Nagata law transition between Slip and Aging behaviour increases as $c$ increases. The vertical dashed lines in (A) are estimates of the slip distance over which linear slip weakening occurs from Eq. (3.8). The corresponding vertical lines in (B) are at almost zero slip for the range of velocity steps used. But the Nagata law response deviates from the Slip law response for velocity jumps $\geq 10^{±4}$ (colors correspond to the size of the jumps).
Figure 3.2 Results from a velocity step up/down experiment (dataset p1060) carried out by servo-controlling off a displacement transducer mounted directly on the sample, hugging the sliding surface. (A) The fit to $\Delta \mu$. (B) The corresponding slip rates ($V$) predicted by the forward model for the best fitting $\Delta \mu$ time series with the stiffness estimated as described in text. Blue: data; ochre: Aging law; red: Slip law; green: Nagata law with $c = 2$; purple: Nagata law with $c \approx 10$. The numbers in blue in (B) denote measured load point velocities in $\mu$m s$^{-1}$. Plots scaled and shifted to improve visibility. The Slip law does a good job of fitting the data due to its characteristic exponential slip-weakening over a fixed length scale for step ups/downs of all magnitudes. The Nagata law does a slightly better job with $c \approx 10$. 
Figure 3.3 The marginal posteriors for the Nagata law Markov chain for the dataset in Figure 3.2 for (A) $a$, (B) $D_c$ and (C) $c$. The first 2500 accepted samples were discarded to account for a burn-in period. The Markov sampling was truncated at $c = 10^5$. The posteriors are dominated by a strongly peaked region (representing fits which look slightly better than the Slip law) and a quasi-uniform tail (representing exactly Slip law like fits) showing that there exist infinitely many solutions at this level of data error (see Appendix C.2 for details on estimation of data error). The green star denotes the initial point and the yellow square the least RMSE value. (D) The fit to $\Delta \mu$ values and (E) The corresponding slip rates ($V$) predicted by the forward models which fit the stress data. Blue: data; ochre: the best fitting Slip law; red: Nagata law with $c \approx 19.4$ which leads to the least RMSE in the Markov chain; pink: Nagata law with $c = 100$ fits the data almost exactly like the Slip law. Exact values of $a$ and $D_c$ for this value of $c$ were searched by redoing a downhill simplex with the values of $c$ and $a - b$ fixed.
Figure 3.4 Slip history of the loading ram when servo controlled off the ram transducer, with a displacement history corresponding to a desired three orders of magnitude velocity step increase on the sliding surface (from p1180). Red dashed line: intended ram displacement derived from the analytical solution for velocity steps under the Slip law; black solid line: actual ram displacement; blue solid line: resultant fault displacement time history; green dashed line: desired fault displacement. The assumed parameters are listed in red.
Figure 3.5 Fits to a sequence of large velocity steps (dataset p1169) on simulated gouge. From left to right in (A) and (D): 1 and 2 orders step increases/decreases and a ~ 3 orders velocity step increase. (A) The fit to $\Delta \mu$ values. (B), (C) Observed $\Delta \mu$ values across the attempted 2 and 3 orders velocity increases respectively. The complicated responses near peak stress can be seen. (D) The corresponding slip rates ($V$) predicted by the forward model for the fits to the $\Delta \mu$ time series in (A). Blue: data; ochre: Aging law; red: Slip law; purple: Nagata law with $c = 10$; cyan: Nagata law with $c = 25$; green: Nagata law with $c = 70$. (E), (F) Slip rates calculated from the data for the attempted 2 and 3 orders velocity increases respectively. The numbers in blue in (D) denote load point velocities in $\mu$m$s^{-1}$. 

$A$: $a = 0.0081$, $D_c = 40.47 \mu$m
$S$: $a = 0.0059$, $D_c = 32.25 \mu$m
$N$: $a = 0.0657$, $D_c = 3.17 \mu$m, $c = 10.0$
$N$: $a = 0.1546$, $D_c = 1.28 \mu$m, $c = 25.0$
$N$: $a = 0.4217$, $D_c = 0.46 \mu$m, $c = 70.0$

$\text{Slip (}\delta\text{) [}\mu\text{m]}$

$\Delta \mu$

$V_{l\beta} \sim 50 \mu$m$s^{-1}$
$V_{l\beta} \sim 0.5 \mu$m$s^{-1}$
$V_{l\beta} \sim 500 \mu$m$s^{-1}$
$V_{l\beta} \sim 0.5 \mu$m$s^{-1}$

$a-b = 0.00025$, $k = 0.0012 \mu$m$^{-1}$
Figure 3.6 Fits to a sequence of large velocity steps on simulated gouge (dataset p1180). From left to right in (A) and (B): one and two order step increases and decreases with an attempted three order velocity step up at the end that pushed $V\theta/D_c \sim 300$ for the best fitting Slip law. (A) The fit to $\Delta \mu$ values. (B) The corresponding slip rates ($V$) predicted by the forward model for the best fitting $\Delta \mu$ time series. Blue: data; ochre: Aging law; red: Slip law; purple: Nagata fit with $c = 10$; cyan: Nagata fit with $c = 100$. The numbers in blue in (B) denote load point velocities in $\mu$m s$^{-1}$. For the Aging law, the data are plotted over the numerical results to facilitate its viewing.
Figure 3.7 Nagata law posteriors for $a$, $D_c$, $c$ for datasets (A)–(C) p1060, (D)–(F) p1169 and (G)–(I) p1180. The green stars show the initiation point for the chain and the yellow squares show the parameter values corresponding to the global minimum RMSE fit. The value of $c$ was constrained to be less than some different upper-bound a-priori for each chain. This leads to the truncation of the quasi-uniform tails of the posteriors of $c$ (the effects of this truncation is also seen in the posteriors of $a$ and $D_c$). Comparing the posteriors of $c$ across the three datasets shows that the presence of a strongly peaked region is not a universal feature whereas the presence of a quasi-uniform region representing fits identical to the best fitting Slip law (see Appendix C.2) is universal.
Figure 3.8 The fits to the (A) one and (b) two orders step decreases in load point velocity from dataset p1180. Blue: data; red: Slip law; brown: Nagata fit with $c = 10$; cyan: Nagata fit with $c = 100$. The Slip law fit is identical to the Nagata fit with $c = 100$ and, hence, these fits lie on the top of each other. This plot makes clear that the Slip law fits the data better than the Nagata law with $c = 10$. One needs $c \gtrsim 100$ in order to produce Nagata law fits identical to those produced by Slip law.
Chapter 4

Is fault restrengthening in laboratory rock friction experiments really time-dependent?

4.1 Introduction

For a fault to fail repeatedly during the seismic cycle, it is necessary for it to restrengthen (heal) during the inter-seismic period. Friction experiments on both bare rock and gouge have shown that the slip interface heals during periods of little or no sliding called holds (Dieterich, 1972; Beeler et al., 1994; Dieterich and Kilgore, 1994; Nakatani and Mochizuki, 1996; Marone, 1998b; Berthoud et al., 1999; Bureau et al., 2002; Marone and Saffer, 2015). This restrengthening is evidenced by the fact that the peak static friction upon resliding is an increasing function of the duration of the preceding hold (Beeler et al., 1994; Marone, 1998b; Berthoud et al., 1999). On initially bare rock surfaces, e.g. granite and quartzite, such frictional healing has two robust characteristics: (1) The static friction peaks increase linearly with the logarithm of the hold time for holds longer than a threshold time (Dieterich, 1972; Beeler et al., 1994; Marone 1998b; Berthoud et al., 1999); (2) This constant rate of
healing/strengthening is independent of the stiffness of the testing apparatus (Beeler et al., 1994) (here, and elsewhere in this manuscript, the rate of healing/strengthening implies the rate of increase in static friction with log hold time, as evidenced by the peak stress following a reslide).

Observation (1) is not limited to bare rock surfaces; such log-linear healing with hold time has been reported for a wide range of materials, e.g., simulated gouge (Karner and Marone, 1998, 2001), steel (Dokos, 1946), Poly Methyl MethAcrylate (Berthoud et al., 1999) and paper (Heslot et al., 1994). It is noteworthy that the rates of healing across these materials are remarkably similar, \( \sim 10^{-2} \) per decade of hold duration in seconds. This points to a robust, (perhaps) material independent, physical mechanism governing frictional healing (Berthoud et al., 1999; Bureau et al., 2002). Since different stiffnesses lead to different amounts of slip during the holds, observation (2) was used by Beeler et al. (1994) to infer that this mechanism is dominantly time-dependent, i.e. frictional interfaces heal even at rest as the logarithm of the hold time. Such an inference is consistent with observations of time dependent growth of the size of micro-contacts bridging stationary interfaces revealed by direct optical measurements in Lucite acrylic and soda glass (Dieterich and Kilgore, 1994). Furthermore, observations of continued increase in static friction peaks with hold duration even at near zero shear stresses (thus ensuring near zero slip) for a variety of materials, including granite, also lends support to the suggestion that slip might not be necessary for frictional healing (Nakatani and Mochizuki, 1996; Bureau et al., 2002). All of these lines of evidence seem to suggest that time-dependent healing is a desirable property in constitutive relations of fault friction.

The most widely used constitutive relations for fault friction are the laboratory derived rate-and-state friction (RSF) equations. Within the RSF framework there are two end-member views of how frictional strength evolves – (a) the Slip law (Ruina, 1983), which allows frictional strength to evolve only with slip, and (b) the Aging law (Dieterich, 1978; Ruina, 1983), which allows frictional strength to evolve even without slip, purely as a
function of time. Beeler et al. (1994) used observation (2) above, and numerical simulations, to conclude that their data supported Aging law style time-dependent healing. Since typical laboratory holds subject the interface to rates of sliding many orders of magnitude smaller than the steady-state sliding speed prior to the hold, a corollary of this conclusion is that the Aging law is the appropriate friction constitutive description at such small slip rates.

However, it has long been recognized that velocity-step experiments are consistently better explained by the Slip law than by the Aging law. Bhattacharya et al. (2015, 2014b) recently extended these results to sequences of 2-3 order velocity step increases and decreases on bare rock and simulated gouge, which rapidly impose slip rates orders of magnitude larger or smaller than the preceding steady-state rate on the sliding interface. In fact, as we show later in this article, the Slip law outperforms the Aging law even when constrained to fit the large velocity step decreases alone. However, given that large velocity step decreases also access sliding regimes which promote frictional healing (brought about by the rapid deceleration from steady-state sliding), it seems inconsistent to simultaneously claim that (1) fault healing in rock is time dependent and (2) large velocity decreases are well modeled by the Slip law and not the Aging law.

In this paper, we investigate this inconsistency by reanalyzing the data of Beeler et al. (1994). We focus not only on the static friction peaks but also on the stress relaxation that occurs during long holds. We carry out detailed nonlinear inversions on the initially bare rock data from Beeler et al. (1994) to examine the Aging and Slip law fits to the stress relaxation during holds both in isolation and in conjunction with the evolution of static friction peaks with hold time. Additionally, we use two other laws - a Slip/Aging hybrid evolution law and a recently proposed shear stress dependent evolution law - both of which can be tuned to transition between Aging and Slip law behaviors to check if the data are better explained by a (particular) combination of Aging and Slip rather than Aging or Slip alone. We compare the properties of these fits with analytical predictions of the frictional response to long holds under the different formulations of RSF considered here.
Our results reveal that stiffness independence of the healing rate is not sufficient to rule out the Slip law; in fact it is possible to find Slip law parameters which fit the peak stress data as well as the Aging law. Additionally, we point out that, vis-à-vis the Aging vs. Slip argument, the more diagnostic (and robust) feature of the data is the strongly stiffness dependent rate of stress relaxation during holds, provided we consider the RSF parameters to be constant. Using both analytical and inversion results, we show that such data are consistent with the Slip law and, importantly, are sufficient to rule out the Aging law with constant RSF parameters. To relax this constraint, some of our inversions also introduced velocity dependent RSF parameters designed to add stiffness sensitivity to the otherwise stiffness independent rates of stress relaxation under the Aging law. All our inversions, including the ones with non-constant RSF parameters, show that the Slip law fits the slide-hold-slide data consistently better than Aging law, but the best fits generally required values of $a - b$ different from those derived from prior velocity steps. Additionally, that the alternative state evolution laws choose to tune their extra parameters to replicate the best Slip law fits to the data provides additional support for the Slip law. All of these findings provide evidence that frictional healing in slide-hold-slide experiments is dominantly slip-dependent, consistent with the unambiguous support for the Slip law from the largest laboratory velocity step decreases.

4.2 Rate and state background

RSF describes the frictional strength of an interface as a function of two variables: 1) Rate ($V$) – the relative slip rate across the contact interface, and 2) State ($\theta$) – a proxy (in units of time for the state evolution formulations we have chosen) for the strength of the asperities in contact across the sliding interface at a reference slip speed, often considered to scale with the true area of contact. These variables are related by two coupled equations. The first of these, called the friction law, describes the rate and state dependence of frictional strength:
\[
\frac{\tau}{\sigma} = \mu(V, \theta) = \mu_0 + a \ln \frac{V}{V_0} + b \ln \frac{\theta}{\theta_0},
\]  
(4.1)

where \(\tau\) is frictional strength, \(\sigma\) is normal stress, \(\mu\) is the ‘rate and state dependent’ friction coefficient, \(a\) is the ‘direct effect’ parameter accounting for the variations in frictional strength due to changes in slip rate, and \(b\) is the ‘evolution effect’ parameter which determines the change in friction due to evolution of state. In general, at not very high temperatures \(a\) and \(b\) are constants of the order of 0.01, but they can vary by as much as an order of magnitude with varying temperature and moisture content \((\text{Blanpied et al., 1998})\). The other parameters \(\mu_0, V_0\) and \(\theta_0\) are the values of friction coefficient, slip rate and state at an arbitrary reference steady-state. The system of equations is closed with an evolution equation for \(\theta\). The two most widely used forms are

Aging (Dieterich) Law: \(\dot{\theta} = 1 - \frac{V \theta}{D_c}\)  
(4.2a)

Slip (Ruina) Law: \(\dot{\theta} = -\frac{V \theta}{D_c} \ln \frac{V \theta}{D_c}\),  
(4.2b)

where \(D_c\) is some characteristic slip weakening length scale \((\text{Dieterich, 1978}; \text{Ruina, 1983})\).

Eq. (4.2a) is often referred to as the Aging law because state increases linearly with time for stationary contacts. Eq. (4.2b) is referred to as the Slip law as state evolution occurs only for slipping contacts (\(\lim_{V \to 0} \dot{\theta} = 0\)). At steady-state sliding (\(\dot{\theta} = 0\)), both the laws yield \(V \theta / D_c = 1\). We refer to \(V \theta / D_c > 1\) and \(V \theta / D_c < 1\) as being ‘above’ and ‘below’ steady-state respectively; we use the phrase ‘far from steady-state’ to imply \(V \theta / D_c\) significantly different from 1.

Given that at steady-state \(V \theta / D_c = 1\), Eq. (4.1) leads to the following expression for the change in frictional strength between two steady-states at velocities \(V_2\) and \(V_1\):

\[
\frac{\Delta \tau}{\sigma} = (a - b) \ln \frac{V_2}{V_1}.
\]  
(4.3)
For \((a - b) < 0\) the sliding surface is steady-state velocity weakening and can undergo velocity instabilities when the sliding is perturbed about the steady-state. For \((a - b) > 0\) (steady-state velocity strengthening) such instabilities are not possible.

While the mathematical form of the velocity dependence of friction in Eq. (4.1) is widely accepted to have a physical basis, the prescription for how state \((\theta)\) evolves with time and/or slip remains a matter of debate. Neither of the two formulations mentioned above, Aging or Slip, explains the full range of friction experiments \((Beeler et al., 1994; Marone, 1998b)\). In the rest of this section, we explore the characteristics of frictional sliding under large velocity perturbations and slide-hold-reslides to investigate the extent to which such laboratory friction data can constrain the time- or slip-dependence of healing.

### 4.2.1 Friction far from steady-state

The two end-member views of state evolution, Aging and Slip, though asymptotically equivalent near steady-state, produce very different responses when the sliding surface experiences large velocity perturbations from steady-state sliding \((Ruina, 1983; Ampuero and Rubin, 2008; Marone, 1998b; Bhattacharya et al., 2015)\). Therefore, if the goal is to achieve the clearest distinction between Aging and Slip law responses, one should design experiments which probe frictional sliding far from steady-state. Two experimental protocols are commonly used to study rock friction phenomenology in the laboratory - velocity steps and slide-hold-reslides. In the rest of this section, we will explore the features of friction far from steady-state revealed by these two experimental protocols. We also present new theoretical results for slide-hold-slides, and argue that data from these experimental protocols can be used in new ways to constrain RSF models.

#### 4.2.1.1 Observations from velocity step tests

An interface undergoing steady-state sliding in the laboratory can be driven far from steady-state by imposing rapid, orders of magnitude, changes in slip rate. On a sufficiently stiff
testing machine this produces a very rapid strength change of the same sign as the velocity step (from the first term in Eq. (4.1)) followed by a slower evolution of the frictional resistance to steady state. One robust observation from such large velocity steps (∼ 1-3 orders of magnitude changes) on both initially bare rock and synthetic gouge is that the evolution of frictional strength following the rapid extremum occurs over a quasi-constant length scale independent of the magnitude or sign of the step \( \text{[Ruina, 1980, 1983, Tullis and Weeks, 1986, Bhattacharya et al., 2014b, 2015]} \). Consistent with such data, the Slip law predicts exponential slip-weakening (or strengthening) over a characteristic length scale, \( D_c \), following a velocity step of arbitrary size or sign (\text{Rice, 1993, Nakatani, 2001, Ampuero and Rubin, 2008}).

In contrast, following a large velocity step the evolution of frictional strength under the Aging law occurs over slip scales which are functions of both the magnitude and sign of the velocity jump. Such asymmetry in the Aging law response is fundamentally tied to the relative amplitude of the two terms on the right hand side of Eq. (4.2a). For a large and sudden velocity increase, the surface is far above steady-state (\( V\theta/D_c \gg 1 \)) and one can neglect the “1” in Eq. (4.2a). Integrating the resulting equations under the assumption of constant slip speed and plugging the result into (1) leads to linear slip weakening with slope \( b\sigma/D_c \). That the rate of slip-weakening is independent of the size of the velocity step implies that the evolution of frictional strength to steady-state occurs over length scales which increase with the size of the jump (\text{Ruina, 1980, Nakatani, 2001, Rubin and Ampuero, 2005}). On the other hand, for a velocity step decrease large and rapid enough to instantaneously satisfy \( V\theta/D_c \ll 1 \), Aging law state evolution predicts \( \dot{\theta} \sim 1 \), i.e. there is no slip-scale for state evolution. In this limit, the post-step increase in Aging law state is just time elapsed since the velocity step and significant state evolution occurs over slip distance \( \delta \ll D_c \text{[Ampuero and Rubin, 2008]} \). As mentioned above, such asymmetry in the frictional response between large velocity increases and decreases is not supported by observations from velocity step experiments.
Large velocity step decreases and long holds are intimately connected in that they both access the portion of the parameter space where $V\theta/D_c \ll 1$, even though the slip rates at the end of long holds are much lower. Therefore, it seems inconsistent that one of these types of experiments would provide evidence for slip-dependent healing while the other for time-dependence. It is then reasonable to ask if the Aging law is capable of providing a decent fit to large velocity decrease data alone. For ideal velocity step decreases, the theoretical expectation is that the Aging law slip-strengthening length scale would decrease with increasing step size, while for the Slip law, one would expect slip to evolve over the same characteristic length scale for all step sizes (Figure 4.1a). Bhattacharya et al. (2015) reported some near ideal 1-2 order velocity steps on gouge (dataset p1060 therein), which clearly show that friction evolves over a constant length scale following these large step decreases in slip rate, consistent with the Slip law (Figure 4.1b). In fact, the Aging law clearly performs worse than the Slip law when constrained to fit the 1- and 2-order step decreases alone (Figure 4.1c). It is also noteworthy that the Slip law parameters adopted to fit the step decreases alone also fit the step increases very well. Therefore, the Aging law, and its prediction that $\dot{\theta} \sim 1$ when $V\theta/D_c \ll 1$, are not supported by laboratory velocity step data which access sliding regimes far below steady-state. It is particularly important to recognize that the Aging law’s apparent success in explaining the peak stress upon reslides following laboratory holds originates from the very ingredient that leads to its failure in fitting the velocity step decrease data: time-dependent healing. These observations provide the motivation for our re-examination of the slide-hold-slide dataset of Beeler et al. (1994).

4.2.1.2 Observations from slide hold slides: The data of Beeler et al. (1994)

Slide-hold-slide (SHS) experiments have traditionally been used in tribology to study healing of frictional interfaces undergoing little or no sliding. In a typical SHS test, the shear stress on an interface undergoing (usually steady) sliding at a rate $V_s/f_r$ is relaxed by bringing the load point abruptly to rest. After being held at rest for some duration $t_{\text{hold}}$ the
load point is re-driven, usually at the pre-hold steady-state speed $V_{s/r}$. This increases the stress until the slider is slipping as fast as the load point. Beyond this peak friction, stress decays back to steady-state with continued sliding. The difference between peak stress and the future steady-state ($\Delta \mu_{peak}$) has traditionally been used as a measure of frictional healing/strengthening during the preceding hold.

Beeler et al. (1994) studied the evolution of $\Delta \mu_{peak}$ with $t_{hold}$ in initially bare granite and quartzite by carrying out a sequence of SHS tests with holds from 0.4 to $\sim 3 \times 10^4$ s. This sequence of holds was repeated under two different setups of the testing apparatus which differed in effective stiffness by a factor of 30. The higher stiffness was achieved artificially by servo controlling the load point displacement off a near-fault transducer (Figure 4.2a). Their data show that $\Delta \mu_{peak}$ increases as a linear function of the logarithm of hold time such that the slope (which we call the healing rate) is independent of the stiffness of the testing apparatus (Figure 4.2c). Beeler et al. (1994) argued, based on this observation and numerical simulations with specified parameter values, that the stiffness independence of the time evolution of $\Delta \mu_{peak}$ supported continued strengthening of nearly stationary interfaces with time as formulated by the Aging law. In contrast, the authors further argued, the Slip law predicts stiffness dependent healing rates since the amount of slip accrued during the hold, and hence the concomitant state evolution, is stiffness dependent. In the rest of this section, we examine these conclusions and argue that some of them are suspect. Moreover, we will demonstrate that there are other properties of the data which provide more reliable diagnostic constraints on the class of state evolution laws we are considering as long as $a$, $b$ and $D_c$ are constants across the range of velocities accessed in the experiment.

The central motivation for using the variation of peak stress with hold duration as a proxy for state evolution during holds stems from the recognition that $\Delta \mu_{peak}$ has a simple relationship with the value of state at peak stress. To see this, one can begin by writing the expression for $\Delta \mu_{peak}$ using Eq. (4.1):
\[ \Delta \mu_{\text{peak}} = a \ln \left( \frac{V_{\text{peak}}}{V_{s/r}} \right) + b \ln \left( \frac{V_{s/r} \theta_{\text{peak}}}{D_c} \right), \]  

(4.4)

where \( \theta_{\text{peak}} \) and \( V_{\text{peak}} \) are the values of \( \theta \) and \( V \) at peak stress. If the sample is assumed to be a rigid slider block coupled to the machine via a spring of known stiffness, the shear stressing rate on the sample is specified by the difference between the slip rate of the block and the displacement rate imposed by the machine. This implies that \( V_{\text{peak}} = V_{s/r} \) at peak stress given the stressing rate is instantaneously zero. Owing to the finite stiffness of the setup, some slip is expected between the beginning of the reslide and peak stress. Denoting \( \Delta \theta \) as the resultant change in state between the beginning of the reslide and peak stress, we have:

\[ \Delta \mu_{\text{peak}} = b \ln \left( \frac{V_{s/r} (\Delta \theta + \theta_{\text{hold}})}{D_c} \right), \]

\[ = b \left[ \ln \left( \frac{V_{s/r} \theta_{\text{hold}}}{D_c} \right) + \ln \left( 1 + \frac{\Delta \theta}{\theta_{\text{hold}}} \right) \right], \]

\[ = b \left[ \ln \left( \frac{V_{s/r} \theta_{\text{hold}}}{D_c} \right) + \Delta \ln(\theta) \right], \]

(4.5)

where \( \theta_{\text{hold}} \) is the value of the state variable at the end of the hold and \( \Delta \ln(\theta) \) is the change in log state corresponding to \( \Delta \theta \). One issue with the use of peak stress as proxy for \( \theta_{\text{hold}} \) is immediately clear from Eq. (4.5) - how do we constrain \( \Delta \ln(\theta) \), the extent to which log state has evolved between the end of the hold and peak stress? [Beeler et al.](1994) recognized that a significant amount of state might be lost during the reslide. However, they noted that under the Aging law, the decrease in log state between the beginning of the reslide and peak stress is independent of hold duration. Mathematically, this statement implies:

\[ \frac{\partial \Delta \mu_{\text{peak}}}{\partial \ln \theta_{\text{hold}}} = b \frac{\partial \ln (\theta_{\text{hold}})}{\partial \ln \theta_{\text{hold}}}, \]

(4.6)
Beeler et al. (1994) suggested that Eq. (4.6) holds generally for the Aging law. We derive this result under a fairly non-restrictive set of assumptions in Appendix D.1. For holds long enough that \( V \theta / D_c \ll 1 \), from Eq. (4.2a), we have \( \theta_{\text{hold}} = t_{\text{hold}} + t_c \) (follows from \( \dot{\theta} = 1 \)) where \( t_c \) is a constant of integration. Therefore, given Eq. (4.6), the following is true for long holds governed by Aging law state evolution:

\[
\frac{\partial \Delta \mu_{\text{peak}}}{\partial \ln t_{\text{hold}}} = b. \tag{4.7}
\]

Therefore, purely time-dependent healing far below steady-state leads to \( \partial \Delta \mu_{\text{peak}} / \partial \ln t_{\text{hold}} \) being independent of stiffness, with healing rate \( b \). Figure 4.3 shows numerical simulations of \( \Delta \mu_{\text{peak}} \) using the Aging law (red curves) for typical laboratory derived rate-state parameters (the value of \( a - b \) appropriate for these experiments is known from velocity steps on the same sample) and two different machine stiffnesses (the same combination of stiffness and \( V_s / t \) values as in the Beeler et al. (1994) experiments). The peak stress upon reslide is indeed independent of the apparatus stiffness and in visual agreement with the data.

For the Slip law, however, the state at the end of the hold is expected to be a function of the total amount of slip accrued during the hold. Therefore, the time evolution \( \theta_{\text{hold}} \) is stiffness dependent. Beeler et al. (1994) argued that this would necessarily lead to a stiffness dependent time evolution of \( \Delta \mu_{\text{peak}} \). They verified this assertion with numerical simulations similar to the ones we show in Figure 4.3. Given the same parameter combinations chosen for the Aging law, the Slip law indeed leads to a stiffness dependent increase in \( \Delta \mu_{\text{peak}} \) with hold duration. However, since Eq. (4.6) is not generally true for the Slip law (see Figure D.2), the time evolution of peak stress is not necessarily a good proxy for the time evolution of state at the end of the hold. For example, it is unclear whether there might exist parameter combinations for the Slip law under which the stiffness dependence of \( \theta_{\text{hold}} \) and \( \Delta \theta_{\text{peak}} \) cancel each other to produce an effectively stiffness independent evolution of peak stress upon reslide. Therefore, the stiffness independence of the time evolution of \( \Delta \mu_{\text{peak}} \)
alone may not be a good diagnostic tool when choosing between competing state evolution laws which do not necessarily satisfy Eq. (4.6).

Furthermore, we show in Appendix D.1 that the Aging law satisfies Eq. (4.7) only due to the linear slip weakening form it assumes far above steady state (that is, $-V \theta / D_c \gg 1$ in Eq. (4.2a)). It is important to remind ourselves that such linear slip weakening far above steady state has zero support from large velocity step increase data (see Section 2.1.1). Therefore, the very ingredient in the Aging law that allows it to satisfy Eq. (4.6), which in turn allowed Beeler et al. (1994) to tie $\partial \Delta \mu_{\text{peak}} / \partial \ln t_{\text{hold}}$ to Aging law style time-dependent healing, is experimentally suspect.

We explore the state evolution across the reslide further in Appendix D.2 by gathering some qualitative idea about the evolution of log state with hold time from the granite data of Beeler et al. (1994). To do this, we follow Beeler and Tullis (1997) by using fault normal displacement (increase signifies compaction, decrease dilation), continuously measured during the experiments of Beeler et al. (1994), as a proxy for log state. This connection between closure and log state is derived from the expectation that variations in closure are dominated by anelastic deformations of the sliding interface which, in turn, are related to the changes in actual asperity contact area (Beeler and Tullis, 1997). We found that the amount of fault dilation between the onset of the reslide and peak stress is neither a negligible fraction of the compaction during the hold nor is it independent of hold duration. Interestingly, direct optical observations of the loss in real contact area between the onset of reslide and peak stress from the see-through experiments of Dieterich and Kilgore (1994) (on lucite and soda glass) also show similar, non-negligible, hold duration dependence for holds ranging from $10^{-10^4}$ s (their Figure 7). Therefore, if contact area is linearly related to log state, the property of the Aging law that allowed Beeler et al. (1994) to interpret their peak stress data as supporting time-dependent healing (as formulated within the Aging law) seems unlikely to be satisfied in typical laboratory slide-hold-slide experiments.
4.2.1.3 Reinterpreting SHS data: What other constraints are required to choose between competing state evolution descriptions?

To avoid the ambiguous assumptions required to interpret $\Delta \mu_{\text{peak}}$ in terms of $\theta_{\text{hold}}$ (state at the end of the hold), we fit the slide-hold-slide data of Beeler et al. (1994) with numerical models rather than rely on restrictive analytical expectations like Eq. (4.6). We also seek additional constraints on the numerical models by including both the stress minima at the end of the holds ($\Delta \mu_{\text{hold}}$) and peak stresses in our misfit calculations within a mathematical inversion framework. There are two distinct advantages to this approach: (1) The additional observational constraint imposed by $\Delta \mu_{\text{hold}}$ is more directly tied to frictional healing far below steady-state, in particular $\theta_{\text{hold}}$ (at least for constant $a$, $b$ and $D_c$; see Appendix D.3). (2) As alluded to in Section 4.2.1.2 understanding the holds is crucial to reconciling the apparently contradictory inferences from large velocity step decreases and SHS tests.

In Appendix D.3, we derive analytical approximations for the stress relaxation for both the Aging and Slip laws. These analyses show that, under the Aging law, the rate of stress relaxation is stiffness independent irrespective of the choice of $a$, $b$ and $D_c$ in the limit of hold times much longer than $D_c/V_{s/r}$ (see Appendix D.3.1 for details). Given $D_c$ around a few microns, and even the smallest $V_{s/r} = 0.316 \mu m s^{-1}$, $D_c/V_{s/r} \sim 10$ s and the longest hold durations accessed in the experiments of Beeler et al. (1994) $(10^3 - 10^4$ s) satisfy this criterion. In the limit of such long holds, velocity weakening ($a - b < 0$) Aging law stress-relaxation trajectories asymptote to a stiffness-dependent constant value (Figure 4.4a). On the other hand, when $a - b > 0$, friction decreases linearly with hold time with an asymptotic slope that is independent of stiffness (Figure 4.4c). Note that this is consistent with the requirement that, in order for a velocity-strengthening interface to remain below steady state during holds, shear stress needs to be a monotonically decreasing function of slip rate. Finally, for velocity neutral trajectories ($a - b = 0$), friction decreases as $\ln[\ln(t_{\text{hold}})]$ such that the rate of relaxation is asymptotically stiffness independent (Figure 4.4b).
The Slip law, on the other hand, always leads to stiffness dependent stress-relaxation during long holds such that both the actual value of $\Delta \mu_{\text{hold}}$ and the rate of decrease in friction ($\partial \Delta \mu_{\text{hold}}/\partial \ln t$) are stiffness dependent (see Appendix D.3.2 for details). In fact, for all of velocity weakening (Figure 4.5a), neutral (Figure 4.5b) and strengthening (Figure 4.5c) cases, $\partial \Delta \mu_{\text{hold}}/\partial \ln t$ increases with stiffness until, in the limit of infinite stiffness, $\partial \Delta \mu_{\text{hold}}/\partial \ln t = \partial \ln (V)/\partial \ln t$. Typical laboratory setups operate with stiffnesses far below this limit, e.g. even the higher stiffness in the [Beeler et al.] (1994) data are at least 10 times smaller than the corresponding infinite stiffness limit (Appendix D.3.2). Therefore, the Slip law prediction of the evolution of $\Delta \mu_{\text{hold}}$ with hold time is expected to retain its stiffness sensitivity for these data.

If one looks at the evolution of $\Delta \mu_{\text{hold}}$ in the [Beeler et al.] (1994) data (Figure 4.2a), it is immediately clear that $\partial \Delta \mu_{\text{hold}}/\partial \ln t$ is strongly stiffness dependent and continues to remain so even for the longest holds. If one further accounts for the fact that we know from preceding velocity steps during the same experimental run that $a - b < 0$, then the Aging law is even more clearly contradicted as it would predict asymptotically constant stresses for the longest holds. Therefore, even without formal inversion it is apparent that the Aging law, with constant parameter values, cannot explain the evolution of $\Delta \mu_{\text{hold}}$ in the data of [Beeler et al.] (1994). On the other hand, these properties of the data are clearly consistent with the Slip law formalism and give us hope that the apparent discrepancy between large velocity step decreases and long holds can be resolved. To fully address this issue, we report results from detailed Bayesian inference on both the SHS $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data from [Beeler et al.] (1994) in Section 3. While doing so, we also consider two other state evolution laws which can be tuned to switch between Aging and Slip law responses in different parts of their parameter space. We briefly introduce these alternative formulations of state evolution in the next two sections.
4.2.2 Stress dependent state evolution

In Section 2.1.2, we pointed out that one of the disadvantages in using $\Delta \mu_{\text{peak}}$ as a proxy for $\theta_{\text{hold}}$ for data from a finite stiffness testing machine was that both the Aging and Slip laws predict that there will be some state evolution between the start of the reslide and peak stress. We also pointed out that, in numerical modeling of finite stiffness reslides, this change in state is easily accounted for as long as we have an accurate estimate of the machine stiffness. However, thus far we have not considered the possibility that state may evolve between the start of the reslide and peak stress in a way that is not captured by these laws. Any such effect cannot be accounted for by numerical models without knowing the state evolution law which appropriately describes such phenomenology. That some such mechanism may be operating was suggested recently by Nagata et al. (2012) on the basis of ultrasonic monitoring of frictional sliding in initially bare granite. They carried out a series of shear stress step experiments and measured the variation in acoustic transmissivity across the slipping interface as a proxy for variations in real contact area and, by assumption, state. To fit their data, Nagata et al. (2012) proposed a modification of the Aging law by including a dependence upon stressing rate:

$$\dot{\theta} = 1 - \frac{V \theta}{D_c} - \frac{c}{b} \frac{\dot{\tau}}{\sigma}. \quad (4.8)$$

Here, $c$ is another parameter and all other symbols carry over their respective meanings (but not necessarily values) from Eq. (4.1). For $c = 0$, this law reduces to the Aging law state evolution. Nagata et al. (2012) found $c = 2.0$ to be an appropriate value for their experiments. Because the Nagata law predicts an instantaneous change in state in response to any rapid change in shear stress for non-zero $c$, this implies a considerable and rapid state evolution across the reslide. From Figures D.3a and D.3b (Appendix D.2), it is interesting to note that the slip interface in the experiments of Beeler et al. (1994) does seem to instantaneously dilate at the onset of the reslides after long holds, more pronouncedly so for
the higher stiffness setup. If fault normal closure and log state were indeed linearly related (Beeler and Tullis [1997]), both the Aging and Slip laws would have predicted continued compaction during the initial portion of the reslide for as long as the interface remained below steady state (leading to $\dot{\theta} > 0$ for both Aging and Slip laws).

Bhattacharya and Rubin [2014] showed that the Nagata law response to velocity steps transitions smoothly from the Aging law to the Slip law as $c$ increases. For $c > 0$, one observes Slip law behavior for small velocity jumps but Aging law behavior for larger jumps. The magnitude of the velocity jump required to see this transition increases as $c$ increases, and in the limit $c \gg 1$ the response mimics purely Slip law behavior for all physically plausible velocity jumps. In other words, the further the interface is pushed from steady-state, the larger the value of $c$ required for the Nagata law to reproduce Slip law behavior. A corollary is that velocity step experiments well fit by the Slip law cannot falsify the Nagata law; they can only impose a lower bound on $c$, a lower bound that increases as the size of the velocity step increases. Examining experiments that imposed velocity steps of 2–3 orders of magnitude on simulated gouge, with stress data that were fit either extremely well or reasonably well by the Slip law, Bhattacharya et al. [2015] found minimum values of $c$ required to fit the data as well as the Slip law to be $10^1$–$10^2$. This is far larger than the value $c \sim 2$ found by Nagata et al. for smaller departures from steady-state.

In Appendix D.3.3, we analyze stress-relaxation trajectories under Nagata law holds in detail. The conclusions are very similar to those for velocity steps: When $c = 0$, the Nagata law is the Aging law. As $c$ increases, the Nagata law trajectories start to diverge from their Aging law counterparts and tend more closely to the corresponding Slip law trajectories. As with velocity steps, for values of $c$ larger than a critical value (determined by a combination of $a$, $b$, $D_c$, stiffness and the length of the hold), holds under the Nagata law exactly reproduce the corresponding Slip law trajectories (see Appendix D.3.3 and Figure D.7 for details). And, just like velocity steps, the further the sliding surface is pushed below steady-state during the hold, the larger the value of $c$ required to reproduce Slip law trajectories with the
Nagata law. Therefore any stress relaxation trajectory that is well explained by the Slip law can only constrain $c$ to a lower bound; for all values of $c$ greater than this critical value, there exists a Nagata law fit equivalent to the Slip law fit with their parameters related according to Eqs. (D.34).

For typical laboratory parameter values and hold times, the evolution of $\Delta \mu_{\text{peak}}$ under the Nagata law also transitions from nearly stiffness independent for small values of $c$ to nearly as stiffness dependent as the Slip law for $c \sim 100$ (Figure 4.3). However, it is important to remember that the numerical predictions in Figure 4.3 are contingent upon our somewhat arbitrary choice of parameters. Our Nagata law inversions are motivated by the hope that there exists a sweet spot in the combination of $a, b, D_c$ and $c$ which might capture the stiffness dependent evolution of $\Delta \mu_{\text{hold}}$ observed in the data while simultaneously explaining the stiffness independent evolution of $\Delta \mu_{\text{peak}}$.

4.2.3 Aging-Slip hybrid state evolution

Kato and Tullis (2001) suggested a modification to the Slip law to introduce purely time dependent healing when the slip rate decreased below a critical value. Their motivation was that the Slip law models velocity steps well, while long holds access velocities far lower than can be reached on controlled velocity steps ($\sim 10^{-3} \mu m/s$). They proposed the rate of change of the state variable to be

$$
\text{Kato Law} : \dot{\theta} = \exp \left( -\frac{V}{V_c} \right) - \frac{V \theta}{D_c} \ln \frac{V \theta}{D_c}.
$$

(4.9)

The first term on the right (the modification to the Slip law) approaches unity at slip speeds much smaller than $V_c$. When applied to numerical simulations of SHS, the Kato law produces time dependent healing for long hold times as long as $V_c$ is at least a couple of orders of magnitude larger than the smallest velocities reached during the longest holds (Kato and Tullis 2001). In numerical simulations, the Kato law shows stiffness independent
evolution of $\Delta \mu_{\text{peak}}$ for long hold times but has a crossover behavior dependent on $V_c$ and the stiffness of the system for small hold times (Figure 4.3). This crossover behavior is not seen in typical SHS test data. However, it is not clear whether there exist combinations of $V_c$ and the conventional RSF parameters that might allow the Kato law to predict both stiffness-dependent evolution of $\Delta \mu_{\text{hold}}$ and stiffness-independent evolution of $\Delta \mu_{\text{peak}}$. Given that our Bayesian inference algorithm performs a global search, we can search for such a solution in the relevant parameter space.

### 4.3 Fitting the Beeler et al. (1994) slide-hold-slide data

The experiments of Beeler et al. (1994) were unique in their replication of a sequence of long duration holds at two different stiffnesses. While this protocol was originally designed to exploit the stiffness independence of the healing rate in distinguishing between Aging and Slip law responses, we have argued in Section 2.1.3 that the stiffness dependence of the stress relaxation during holds is also useful for this purpose. In this section, we carry out formal inversion of the Beeler et al. (1994) dataset. Unless otherwise specified, all the inversions are on the granite data; we will only briefly discuss the characteristics of their quartzite data in Section 4.

For our inversions we made use of two a-priori constraints on the relevant parameters, the stiffness of the testing apparatus and $a - b$. We estimated the natural stiffness of the machine to be $k_n = 0.0019 \, \mu m^{-1}$ (scaled by normal stress which was 25 MPa for all experiments) and the servo-controlled artificial stiffness to be $k_s = 0.055 \, \mu m^{-1}$ (Appendix D.4). The initially bare granite lab specimens appear to be moderately velocity weakening ($a - b = -0.0027$) from a sequence of velocity steps performed with both stiffnesses preceding the holds (Appendix D.4). The driving rate for the high stiffness holds was $V_{s/r} = 1 \, \mu m s^{-1}$, while the lower stiffness holds were carried out at $V_{s/r} = 0.316 \, \mu m s^{-1}$. 
We modeled the frictional sliding of the ‘block’ (in reality a hollow granite cylinder being torsionally rotated) as a single degree of freedom system wherein the net shear stress on the block is represented in terms of the imposed load point displacement history as

\[ \Delta \tau = k(\Delta \delta_{lp} - \Delta \delta), \] (4.10)

where \( k \) is the stiffness normalized by normal stress, and \( \delta \) and \( \delta_{lp} \) are fault and load point displacements respectively. This equation was solved under quasi-static force balance with frictional strength as specified in Eq. (4.1) and a particular choice of the state evolution law. We used a hybrid 4th-5th order Runge Kutta adaptive time stepper (Press et al., 1996) for numerical integration. The forward model uses adaptive time stepping between each time sample in the data, therefore the reported fits are down-sampled from the underlying full integrated time series.

We ran inversions involving misfits across the full time series, but do not report them here as our conclusions are generally insensitive to this choice. For our inversions, we used a small world Markov Chain Monte Carlo global search with adaptive proposal distribution (Rosenthal, 2011; Bai, 2009a,b; Guan et al., 2006), the details of which can be found in the supplementary material accompanying Bhattacharya et al. (2015). For all our inversions, we scale the \( \ell^2 \) norm of the misfit by the scalar data error covariance to obtain the argument for the likelihood function. However, as our misfits are dominated by modeling error, the convergence of the chain is only weakly sensitive to the specific value of the standard error. When reporting the parameters, we refine the fit to a (locally) minimum misfit solution by running a local downhill simplex (Press et al., 1996) initiated with random samples from the posterior. This minimum misfit solution is statistically indistinguishable from the maximum a posteriori fit for most of our inversions.

To fully exploit the dependence of the data on the differences in machine stiffnesses we inverted the data corresponding to both stiffnesses simultaneously. For the fits reported
here, the misfit between the data and forward model was evaluated only at the times of stress peaks and minima. The data peaks suffered from finite sampling frequency issues, e.g., the recorded peak stresses were randomly offset from the times at which \( V = V_{lp} \) (the theoretical requirement for stress maximum from Eq. (4.10)) following the reslide. However, as seen later in Fig. 4.7, the disagreements between the data peaks, numerical fits to the data peaks and the numerical estimate of the ‘true maximum’ (read from the full integrated time series at the times of \( V = V_{lp} \)) are all within the uncertainty represented by the family of statistically equivalent fits to the data.

4.3.1 Fitting only \( \Delta \mu_{\text{peak}} \)

We made the assertion in Section 2.1.2 that the stiffness independent evolution of \( \Delta \mu_{\text{peak}} \), while inherently consistent with the Aging law, was not sufficient to rule out the Slip law. One way to validate this assertion is to find a Slip law fit to the peak stresses that is as good as the Aging law fit. With \( a - b \) constrained to be -0.0027, the Slip law posteriors contained no such fit. The Aging law, not surprisingly, matches the stiffness independent healing rate, which is equal to \( b = 0.01 \) (see Eq. (4.7)), independent of \( a - b \) (Figure 4.6a). However, the actual values of \( \Delta \mu_{\text{peak}} \) are not very well modeled by this Aging law fit; the offset between \( \Delta \mu_{\text{peak}} \) for the two stiffnesses is nearly zero for the fits, while the data show an obvious and consistent offset between the two stiffnesses. Also, from fitting the velocity steps (Appendix D.4), we know that the lower bound on \( a \) is around 0.0011. Therefore, if we constrain \( a - b = -0.0027 \), then the Aging law requirement that \( b \sim 0.01 \) (Eq. (4.7)) implies \( a \sim 0.007 \), smaller than the minimum bound. We show in Section 4 that this deficiency of the Aging law is present to a much greater degree for the quartzite data.

When we relax the constraint on \( a - b \), the Aging law does a good job of matching the offset in \( \Delta \mu_{\text{peak}} \) between the two stiffnesses (Figure 4.6b). Figure 4.6b also shows that the best Slip law fit to the peaks is nearly visually indistinguishable from its Aging law counterpart when \( a - b \) is not constrained. Figure 7a shows an ensemble of numerical
predictions of the state evolution between the beginning of the reslide and peak stress drawn from the posterior derived from the Slip law inversion in Figure 4.6b. This confirms our expectation that the best Slip law fits to the peak stresses represent parameter combinations for which the stiffness dependence of state evolution between the beginning of the reslide and peak stress nearly cancels out the stiffness dependence of state evolution during the hold.

Before moving on to other fits, it is worth pointing out that the values of $D_c$ for both the Aging and Slip law fits in Figure 4.6B) appear too small for the slip weakening observed in the velocity steps. As a consequence, these numerical models show marginal instability (ringing and large overshoots) for both the Aging and Slip law fits, for the smaller stiffness apparatus.

### 4.3.2 Fitting only $\Delta \mu_{\text{hold}}$

The fits in Figure 4.6 support our first assertion, that stiffness independence of the healing rate might not be a sufficient condition to rule the Slip law out as incapable of explaining laboratory SHS data. We turn now to our second assertion, that the Aging law is incompatible with a stiffness-dependent rate of stress relaxation for long holds. Figure 4.8a shows the fits to $\Delta \mu_{\text{hold}}$ with $a - b$ constrained to -0.0027. As analytically predicted, the velocity weakening Aging law fit shallows out to a quasi-constant value for the longest hold times. The Slip law, on the other hand, captures the observed stiffness dependence of the stress minima reasonably well, the rate of evolution of $\Delta \mu_{\text{hold}}$ being better matched than actual values.

With the constraint on $a - b$ removed, the Aging law maximum a posteriori (the most sampled solution in the posterior) solution is velocity strengthening (Figure 4.8b). According to the theoretical treatment developed in Appendix D.3.1, this is consistent with the need to model the continually decreasing shear stresses, especially for the higher stiffness holds. However, these velocity strengthening solutions suffer from the stiffness independence of
the stress relaxation for the longest holds, consequently the quality of the fit worsens with increasing hold duration. It is also noteworthy that this fit severely overestimates the peak stresses. We found other, qualitatively similar but statistically significantly different, Aging law fits to $\Delta\mu_{\text{hold}}$ with smaller RMSE values than the maximum a posteriori solution reported here. These fits adopted unrealistically large $a$ and $b$ (0.4) and small $D_c$ ($\sim 0.03\mu\text{m}$) and we do not discuss them here.

The corresponding Slip law fit, on the other hand, matches the $\Delta\mu_{\text{hold}}$ very well with (very weakly) velocity weakening parameters. The peaks are much better matched than the corresponding Aging law fit but the healing rate is clearly stiffness dependent. Figure 4.9 shows how these Aging and Slip law fits compare with the full time history of stress relaxation during the longest holds (3162 s and $10^4$ s). This reveals that the Aging law velocity strengthening fits (panels b and d) overestimate the stress relaxation for the lower stiffness and underestimate the stress relaxation for the higher stiffness holds. This is consistent with the analytically predicted stiffness independence of the relaxation trajectories. But the Slip law fit matches the details of the stress history very faithfully for both stiffnesses and equally well.

### 4.3.3 Simultaneously fitting $\Delta\mu_{\text{peak}}$ and $\Delta\mu_{\text{hold}}$

In Section 3.2 we showed that, in keeping with our theoretical expectations, the stiffness dependence of stress relaxation during long holds in the initially bare granite data of [Beeler et al. (1994)] shows strong support for the Slip law. Additionally, our fits to peak stresses alone (Section 3.1) showed that there exist parameter combinations for the Slip law which could explain the observed peak stress evolution with hold duration nearly as well as the Aging law. In this section we explore whether one can find solutions with either the Aging or the Slip laws that explain both the stress peaks and minima equally well.

When we constrain $a-b=-0.0027$ (Figure 4.10(A)), neither the Aging nor the Slip law does a good job of fitting both $\Delta\mu_{\text{peak}}$ and $\Delta\mu_{\text{hold}}$. The Aging law fits the peaks marginally
better than the stress minima but gets the healing rate wrong. In contrast, the Slip law fits the high stiffness $\Delta \mu_{\text{hold}}$ and the low stiffness $\Delta \mu_{\text{peak}}$ data well but does a poor job of fitting the rest of the data.

When we relax the constraint on $a - b$, the Slip law chooses a velocity strengthening solution which fits the lower stiffness stress minima well (Figure 4.10b). The fit to the higher stiffness peak stresses is also better than that obtained with $a - b$ constrained. However, the stiffness dependence of peak stresses worsen the fit $t_{\text{hold}} \gtrsim 300$ s. The corresponding Aging law fit is also velocity strengthening. However, this Aging law fit overestimates the low stiffness stress minima and underestimates the high stiffness ones and, unlike the data but consistent with our theory, predicts the stress relaxation rate to be stiffness independent for the longest holds ($t_{\text{hold}} \gtrsim 1000$ s). The healing rate is also too high to fit the peaks well. Overall, although the Slip law does fit the peaks slightly worse than the stress minima, it generally explains the joint $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data much better than the Aging law when no constraint is imposed on $a - b$.

4.3.4 Fitting the data with Kato and Nagata laws

Given that neither the Aging nor the Slip laws can explain both the peak stresses and stress minima well, at least with $a - b$ constrained from the velocity-step data, it is important to consider whether alternative formulations of state evolution can achieve this goal. In Sections 2.2 and 2.3 we mentioned two such candidate state evolution formulations, the Nagata and the Kato laws. We carried out inversions on the granite data from Beeler et al. (1994) with these laws by fitting the $\Delta \mu_{\text{hold}}$ data both alone and jointly with $\Delta \mu_{\text{peak}}$.

4.3.4.1 Fitting only $\Delta \mu_{\text{hold}}$

Before we describe the results of the fit to the data with the Kato law, it is important to point out that the steady-state condition for the Kato law is the following transcendental equation
in $V$ and $\theta$:

$$\exp\left(-\frac{V}{V_c}\right) = -\frac{V\theta}{D_c}\ln\left(\frac{V\theta}{D_c}\right).$$  \hfill (4.11)

If $V_c \ll V_{s/r}$, then the steady-state condition before the beginning of a hold is the familiar $V\theta/D_c = 1$, which considerably simplifies the specification of initial conditions in our model. To avail this simplification we constrain $V_c \leq 0.01\mu\text{ms}^{-1}$ a priori for all our Kato law inversions. Note that [Kato and Tullis (2001)] recommended a constant $V_c = 0.01\mu\text{ms}^{-1}$ for the granite data being considered in this study. This choice not only simplifies the math but, given that $\partial\Delta\mu_{ss}/\partial V_{ss}$ is known to be fairly slip rate independent in the range of $V_{s/r}$ used in this study (Figure 6 in [Kato and Tullis (2001)]), also makes physical sense.

Given that the Slip law produced an excellent fit to the $\Delta\mu_{\text{hold}}$ data when $a - b$ was unconstrained, we only attempted the $a - b = -0.0027$ version when fitting the stress minima alone with the Kato law. With $a - b$ constrained, the minimum misfit Kato law adopted $V_c \sim 7 \times 10^{-6}\mu\text{ms}^{-1}$. This reproduced the corresponding Slip law fit to the stress minima (Figure 4.11a). The peak stress predictions for the Kato and Slip law fits were also identical except for the longest high stiffness hold where the peak stress deviated wildly from the corresponding Slip law prediction. Since the lowest slip rates accessed by the numerical models were $\sim 10^{-6}\mu\text{ms}^{-1}$ (reached at the end of the longest high stiffness hold), the reslide following the longest high stiffness hold was most sensitive to the $\exp(-V/V_c)$ term (the time dependent healing term) in the Kato law. For the other holds $V$ remained considerably larger than the $V_c$ chosen by the fit. However, this leads to a sudden break in the trend of the numerically predicted peak stresses for long hold durations which is unlike anything seen in the data.

The Nagata law also reproduced the corresponding Slip law fits both with $a - b$ constrained (Figure 4.11b) and unconstrained (Figure 4.11b). As predicted in Section 2.2, given that the $\Delta\mu_{\text{hold}}$ data are well explained by the Slip law, they cannot constrain $c$ to any better than a lower bound. This lower bound on $c$, given the combination of stiffness and hold duration from the experiments, is around 5 both when $a - b$ is constrained and
when it is unconstrained. The minimum misfit solutions adopt values of $c$ which are more than an order of magnitude larger. Figures 4.10a and b show that the quality of fits to the stress minima at these much larger values of $c$ is nearly indistinguishable from the best fits obtained in inversions where we constrained $c = 10$. These fits are shown as dotted lines in Figures 4.11a and b. It is noteworthy that the parameters of these Nagata law fits are related to the parameters of the corresponding Slip law fit in accordance with the scaling relationships in Eqs. (D.34). Therefore, these numerical inversions validate our theoretical expectations regarding the behavior of the Nagata law far below steady state.

4.3.5 Jointly fitting $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$

When fitting the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data jointly, the Kato law still chooses $V_c$ values considerably smaller than the imposed upper bound constraint $0.01 \mu \text{ms}^{-1}$. For example, in Figure 4.12a, the Kato law fit with $a - b$ constrained finds a value of $V_c \sim 3 \times 10^{-5} \mu \text{ms}^{-1}$. However, $V$ in the numerical models does drop to values comparable to and smaller than $V_c$ for some of the long high stiffness holds. This produces a fit different from the corresponding Slip law fit due to sensitivity to the $\exp(-V/V_c)$ term: less so for the stress minima but more so for the peaks. Similar to the Kato law fit to the stress minima alone, this results in non-smooth trends in the numerically predicted stress peaks and minima which are inconsistent with the data.

When the peak stresses are included in the misfit calculations, the Nagata law $c$ becomes constrained by a tightly bound Gaussian posterior when $a - b$ is fixed in the inversion. The minimum misfit and maximum posterior solutions both agree on $c \sim 2.7$. With $a - b$ fixed; this Nagata law fit outperforms the corresponding fits for all the other laws tested here (Figure 4.12a), and it does so with a value of $c$ that is not far from the value of 2.0 obtained by Nagata et al. (2012) in their granite experiments. However, the peak stress evolution exhibits significant stiffness dependence and the stiffness-peak stress magnitude relationship is flipped with respect to the data.
When the constraint on $a - b$ is eased, the Nagata and Kato law best fits are identical both to each other and to the corresponding Slip law fit. The Kato law chooses $V_c \sim 10^{-10} \mu m s^{-1}$, far below the lowest computed slip speeds in the fits to the data, and the posterior for $V_c$ is unbounded towards even smaller values. This is expected if the best Kato law fit represents no improvement upon the corresponding Slip law fit. The Nagata law posterior for $c$ again becomes unconstrained as was the case for fitting the values of $\Delta \mu_{\text{hold}}$ alone. The lower bound on $c$ for reproducing the corresponding Slip law fit is around 10. We found fits practically identical to this for $c$ as large as 1000 by fixing $c$ and varying $a$, $b$ and $D_c$. Fits for $c = 10$ (dotted curves) and $c = 450$ (the minimum misfit solution) are shown in Figure 4.12b.

### 4.4 Quartzite data from Beeler et al. (1994)

In addition to the granite data discussed in the previous sections, the slide-hold-slide dataset of Beeler et al. (1994) contained quartzite data generated using the same experimental protocols. The only difference was that the higher stiffness $k_s$ was slightly higher than its granite counterpart, around $0.08 \mu m^{-1}$. Before concluding this study, we briefly discuss how the features of the quartzite data compare with our findings on granite reported in Section 3.

The quartzite samples were found to be more rate weakening than their granite counterpart. We found $a - b = -0.0061$ from fitting 1 order velocity steps performed during the same experimental run (Figure 4.13a-c). Fitting the velocity steps with the Slip law also suggest that $a$ is no smaller than 0.012; in the absence of any fit or choice of state evolution law, the transient excursions in $\Delta \mu$ in Figure 4.13a-c require $a \gtrsim 0.009$. The $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data for the initially bare quartzite are shown in Figure 4.13d. The hold durations range from $3 \times 10^4$ s. Over this range of hold durations, the healing rate actually does not appear entirely stiffness independent, especially for the longest holds. However,
the peak stresses are noisier than for the granite data, and Beeler et al. (1994) inferred slightly different peaks than in this study. If one ignores these deviations as data noise, the healing rate is around 0.006.

For the Aging law predicted peak stresses to match these data, \( b \) needs to be within a few percent of 0.006 (Eq. (4.7)). But given that the velocity steps constrain \( a - b \approx -0.006 \approx -b \), this implies the non-physical condition that \( a \approx 0 \), far lower than the absolute minimum bound of \( a \approx 0.009 \). A similar, but less severe, problem was pointed out with the Aging law fit to \( \Delta \mu_{\text{peak}} \) values from the granite data (Section 3.1 and Figure 4.6).

When the constraint on \( a - b \) is relaxed, the best Aging law fit to the peak stresses is velocity strengthening with \( a \approx 0.009 \) and \( b \approx 0.007 \) (Figure 4.13d). The best Slip law fit to the peak stresses, on the other hand, is (nearly) identical to the best Slip law fit obtained by jointly fitting \( \Delta \mu_{\text{peak}} \) and \( \Delta \mu_{\text{hold}} \). Interestingly, just as for the granite data, this Slip law fit to the peak stresses is very similar to its Aging law counterpart, and the predicted evolution of peak stresses with hold time is nearly stiffness independent.

Finally, Figure 4.13e shows the fits to the \( \Delta \mu_{\text{hold}} \) data alone from the quartzite experiments. The stress relaxation during the holds is strongly stiffness dependent just as it was for the granite data. And as with the corresponding inversions on granite, the Aging law minimum misfit solution chose \( a/b \approx 1 \) with a too small \( D_c \) to exploit the weak stiffness dependence of the velocity neutral Aging law trajectories. But even with these seemingly unreasonable parameters, the stress relaxation rates in the fits had a demonstrably weaker stiffness dependence than the data. On the other hand, the observed stiffness dependence of the stress relaxation was very well captured by the Slip law. However, the Slip law fit which produced an excellent fit to the \( \Delta \mu_{\text{hold}} \) data did not do as good a job of fitting the full time series of stress relaxation during long holds as the corresponding Slip law fit to the granite data. In particular, the higher stiffness stress relaxation time history was well fit while the lower stiffness was not.
4.5 The case for additional physics at low slip rates: Non-constant RSF parameters

One enduring appeal of Aging law state evolution is that time-dependent healing has a clear, experimentally supported, physical picture: The growth of stationary contacts with time (Dieterich and Kilgore, 1994). However, in view of the discussions above, the systematic lack of support for the Aging law from long laboratory holds, and its clear refutation by large velocity step decreases, seems an equally compelling argument against Aging law style time-dependent healing. From Appendix D.3.1, it is clear that the Aging law cannot explain the properties of friction evolution observed during long laboratory holds with constant RSF parameters. Therefore, one way to reconcile Aging law style time-dependent healing with laboratory hold data is to consider physical mechanisms by which RSF parameters could vary at the low slip rates accessed at the end of long holds. As a first approximation, we restrict our discussion to only those physical mechanisms which could be modeled through rate dependencies of $a$ and/or $b$ (Boettcher et al., 2007; Rice et al., 2001).

Such an idea is not entirely without experimental motivation. For example, Marone and Saffer (2015) have recently pointed out that $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data from simulated gouge show systematic dependencies on $V_s/r$ which are not consistent with conventional RSF (with constant parameters) from the point of view of dimensional analysis. One way to explain such systematic loading rate dependence is to add a second velocity scale to traditional RSF models, a particular example of such a modification being rate dependence of $a$ and/or $b$. In Appendix D.5, we explore two such choices of rate dependence, both of which were motivated by the micro-mechanics of contacting asperities and, a priori, appeared helpful to the cause of the Aging law. In particular, both these modifications allowed the Aging law to predict (1) continual weakening of the interface at progressively smaller slip rates even when the interface was velocity-weakening at the reference slip rate $V_{s/r}$ and (2) stiffness-dependent rates of stress-relaxation during long holds.
However, in actual inversions, neither of these modifications qualitatively improved the Aging law fits to $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$. Surprisingly, at least one of these formulations (with $a$ increasing logarithmically with decreasing slip rate) actually improved the Slip law fit to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data from the granite experiments. Additionally, this Slip law fit adopted physically reasonable values for the extra parameters introduced due to the rate dependence of $a$ (Figure D.10). However, since our choice of these modifications are clearly not exhaustive, this exercise does not rule out the possibility that some other formulation of rate-dependent RSF parameters could address the lack of experimental support for the Aging law. Discovering that appropriate strain rate dependence, if it exists, requires additional research.

## 4.6 Discussion and conclusions

Over the last two decades, the observation that $\Delta \mu_{\text{peak}}$ increases linearly with log hold time, with a slope that is independent of the apparatus stiffness, has been cited as a major experimental support for the Aging law. This support is inferred based on two, fairly general, properties of the Aging law expected to be important for long holds – (1) the rate of state evolution far below steady state is independent of stiffness (time dominates slip), and (2) the change in state between the start of the reslide and peak stress is independent of both stiffness and hold duration (Beeler et al., 1994). However, it is important to recognize that the Slip law too can produce log-linear and stiffness-independent increase in peak stress for a more limited range of parameters. Such stiffness-independent healing rates ($\partial \Delta \mu_{\text{peak}} / \partial t_{\text{hold}}$) with the Slip law occur when the increased healing associated with the lower stiffness apparatus is offset by the greater state evolution, for that lower stiffness, between the start of the reslide and peak stress.

Therefore, to better test the two laws using $\Delta \mu_{\text{peak}}$ data alone, one must fit the actual data and not just the healing rate for large hold times. As additional constraints, one
can also ask how consistent the parameter values are with inferences from the velocity step tests performed during the same experimental run, $a - b$ being the most reliable and reproducible estimate for this purpose. With this independent constraint on $a - b$, the Aging law healing rate, $b$, required to match the $\Delta \mu_{\text{peak}}$ data of [Beeler et al. (1994)] was found to be systematically too small to accommodate realistic values of $a$. In particular, for the quartzite data, we found $b - a \sim b \sim 0.006$ which implied $a \sim 0$, whereas the absolute minimum lower bound from velocity steps was $a \sim 0.009$. When the constraint on $a - b$ was relaxed, we were able to obtain comparably good Aging and Slip law fits to $\Delta \mu_{\text{peak}}$ for both the granite and quartzite data. For the granite data, these best fits adopted slightly velocity weakening parameters ($a - b = -0.0001$ for Slip, $-0.0007$ for Aging) which are an order of magnitude smaller than the $a - b \sim -0.0027$ derived from the velocity steps. For the quartzite data, these best fits yield values of $a - b \sim 0.002$ (velocity strengthening) for both laws which is inconsistent with the results of velocity steps (velocity weakening; $a - b \sim -0.006$). The values of $D_c$ adopted by both laws for these fits were also an order of magnitude smaller than those estimated from velocity steps. However, the values of $a$ adopted by the best Slip law fits to $\Delta \mu_{\text{peak}}$ data were always consistent with those obtained from the fits to the velocity steps, while the corresponding Aging law fits systematically underestimated $a$. On this basis, we conclude that the $\Delta \mu_{\text{peak}}$ data of [Beeler et al. (1994)] are fit marginally better by the Slip law than by the Aging law, and that there is no good evidence from their peak stress data alone that fault healing in rock is time-dependent.

We also note that there are fundamental problems with the arguments that tie $\partial \Delta \mu_{\text{peak}} / \partial t_{\text{hold}}$ to state evolution during the hold. In particular, the linear relationship between $\partial \Delta \mu_{\text{peak}} / \partial t_{\text{hold}}$ and $\partial \theta_{\text{hold}} / \partial t_{\text{hold}}$ expressed in Eq. (4.6) is contingent upon assuming either negligible or hold-duration-independent log state evolution across the reslide, prior to peak stress. Optical observations of real contact area change across reslides ([Dieterich and Kilgore, 1994]) and fault normal closure (a linear proxy for log state) data collected during the experiments of [Beeler et al. (1994)] (Appendix D.2) both independently suggest that
these assumptions are likely to be not satisfied in laboratory slide-hold-slides. Additionally, the property of the Aging law that allows it to satisfy Eq. (4.6) is linear slip weakening far above steady state, behavior that is violated by large velocity step increase experiments.

Given this uncertainty regarding the extent of state evolution across the reslide, prior to peak stress, using $\Delta \mu_{\text{peak}}$ as a proxy for state at the end of the hold is problematic. A more direct way of constraining state evolution during the holds is to fit the stress data from the holds themselves. To do this in a straightforward way requires assuming that the RSF parameters remain constant even at slip speeds orders of magnitude below those typically accessed during velocity step tests, and this proposition is difficult to test. But given this assumption, it is clear the Slip law provides a far superior fit to the hold data. The most prominent features of the $\Delta \mu_{\text{hold}}$ values are that (1) they decay quasi-linearly with log hold time out to the longest holds, and (2) the rate of decay depends strongly on stiffness. Both features are consistent with our analytical results for the Slip law. In contrast, under the Aging law we have shown analytically that the rate of decay of stress with log hold time is independent of stiffness, and moreover for velocity-weakening surfaces (which we know the granite and quartzite to be, based on velocity step tests on the same samples) the stress does not decay with log hold time but asymptotes to a constant value (a direct result of the continued healing even at low slip speeds). Furthermore, for granite, the Slip law can fit the $\Delta \mu_{\text{hold}}$ data reasonably well even when $a - b$ is constrained to the value estimated from the velocity-step tests (Figure 4.8a). For quartzite this is not the case, but the $\Delta \mu_{\text{hold}}$ data can be well modeled by a velocity-neutral surface (Figure 4.13e). This issue of velocity steps and slide-hold-slides requiring significantly different $a - b$ values to be fit has also been reported by [Marone and Saffer, 2015] for slide-hold-slide experiments on simulated gouge. However, given this freedom on $a - b$, they too find the Slip law to be more consistent with their data than the Aging law ([Marone and Saffer, 2015]).

When the $\Delta \mu_{\text{hold}}$ and $\Delta \mu_{\text{peak}}$ values for granite are fit with no constraint from the velocity-step tests, the Slip law does far better than the Aging law, primarily because the
Aging law cannot satisfy the stiffness-dependent rate of stress relaxation during the holds (Figure 4.10b). The Slip law fit is quite good even though the rate of strengthening ($\Delta \mu_{peak}$) is not strictly linear with log hold time. The Slip law does well enough that the best Kato and Nagata fits, even with their extra parameter ($V_c$ or $c$, respectively), choose values for that parameter that make their fits indistinguishable from that of the Slip law (Figure 4.12b). The unconstrained Slip law fit to the quartzite data is also very good. However, these Slip law fits to both the granite and quartzite data are velocity-strengthening. When forced to use the velocity-weakening $a - b$ from the velocity steps, the Slip law does not do very well (Figure 4.10a for granite; 4.13e for quartzite). The best-fitting law under these conditions is (for granite) the Nagata law with $c \sim 2.7$, not far from the value advocated by Nagata et al. (2012), although this fit is still not very good.

One way to reconcile Aging law style time-dependent healing with the strongly stiffness-dependent rates of stress relaxation is to allow the RSF parameters to vary as functions of the variables of interest, e.g., slip rate, state, shear stress etc. It is very difficult to test this hypothesis within an inversion framework, primarily because it is challenging to come up with an exhaustive list of such modifications to RSF parameters. We, somewhat arbitrarily, chose two such constitutive relations: (1) A strain-rate dependence of $a$ (constant $b$) derived from the micro-mechanics of contact creep; (2) An effective rate dependence of $a - b$ brought about by the inclusion of (conventionally neglected) second order terms in RSF. However, these modifications did not improve the Aging law fits to the peaks and stress minima from the granite data (Appendix D.5). Instead, the model with strain-rate dependent $a$ improved the Slip law fit to $\Delta \mu_{peak}$ and $\Delta \mu_{hold}$ with reasonable values of the extra parameters. But, the strain-rate dependence on $a - b$ derived from this Slip law fit failed to match the values of $a - b$ inferred from the velocity steps near $V_{s/t}$. Whether some other formulation of variations in $a, b, D_c$ (with slip rate or any other relevant variable) can indeed reconcile Aging law style time-dependent healing with laboratory friction data remains an open question.
Given the above, we conclude that the slide-hold-slide data of Beeler et al. (1994) provide better support for the Slip law than for the Aging law. Given that large velocity step decreases show unambiguous support for slip-dependent rather than time-dependent healing (Figure 4.1), this study takes an important step toward alleviating the apparent contradiction between the inferences from velocity step decreases and slide-hold-slide experiments. In doing so, this study also presents a strong case in support of dominantly slip-dependent state evolution far below steady-state, at least at the slip rates accessed by the longest holds of Beeler et al. (1994).

However, it is important to remember that time-dependent healing, in particular the logarithmic in time growth of contact area, is supported by direct optical observations from the see-through experiments of Dieterich and Kilgore (1994). These observations on transparent plastic also seem consistent with the stiffness independent, log time, increase in across-fault compaction seen in Figure D.3c. But, provided $a$, $b$ and $D_c$ are held constant, such logarithmic time-dependent healing is clearly incompatible with the stiffness dependent relaxation rates observed far below steady state in the SHS experiments of Beeler et al. (1994). We have discussed the possibility of non-constant RSF parameters in Section 5; even though the constitutive relations we tried did not help the Aging law explain the data, there might be other formulations which achieve this goal. An alternative approach to resolving the conundrum that fault healing appears not to be strictly time-dependent while growth of contact area is, is to postulate that state is more than just the ‘quantity’ of contact area; that the proper description of state evolution needs to take into account the ‘quality’ of the contact. Such a hypothesis immediately rules out any a priori expectation, based on friction data alone, for the state variable to represent log time healing in its constitutive equation. Contact ‘quality’ could be related to the strength of bonds (Li et al., 2011; Liu and Szlufarska, 2012) or some other aspect of contact rheology (Bureau et al., 2002). Since state evolution seems to be dominantly slip-dependent across the slip rates accessed by traditional velocity step tests and SHS experiments, it seems reasonable to imagine
that contact ‘quality’ is a function of slip, possibly along with time. However, given that frictional strengthening has been observed to grow as the logarithm of hold duration even at (near) zero shear stresses (albeit at a reduced rate relative to normal holds) on rock \cite{Nakatani and Mochizuki, 1996} and glass \cite{Bureau et al., 2002}, it seems reasonable to imagine that contact ‘quality’ either switches to a time-dependent regime or contact ‘quantity’ becomes the dominant contributor to frictional strength at these low slip rates. Such phenomenology might have other independent experimental support – using holds of duration as long as two weeks \cite{Walsh and Goldsby, 2008} found that the rate of frictional strengthening in initially bare rock progressively decreases but remains non-zero for holds longer than $10^5$ s.

More experiments need to be carried out to explore these issues, especially by making use of optical and ultrasonic techniques to ascertain the extent to which contact area correlates with the evolution of frictional strength.
Table 4.1 The best fitting parameter values from the various inversions of the granite data from [Beeler et al.] (1994) reported in the paper. Capital letters denote the various state evolution laws - Aging (A), Slip (S), Nagata (N) and Kato (K) laws respectively.

<table>
<thead>
<tr>
<th></th>
<th>Law</th>
<th>$a$</th>
<th>$b$</th>
<th>$D_c$ [$\mu$m]</th>
<th>$c$</th>
<th>$V_c$ [$\mu$m s$^{-1}$]</th>
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<tbody>
<tr>
<td>Velocity steps $^a$</td>
<td>A</td>
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<td>0.0148</td>
<td>3.10</td>
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<td>S</td>
<td>0.0117</td>
<td>0.0144</td>
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<td>-</td>
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<tr>
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<td>0.0269</td>
<td>2.23</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>S</td>
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<td>0.0253</td>
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<td>-</td>
<td>-</td>
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<td>-</td>
</tr>
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<td></td>
<td>S</td>
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<td>-</td>
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<tr>
<td></td>
<td></td>
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<td>0.0162</td>
<td>0.34</td>
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</tr>
<tr>
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<td>-</td>
<td>3.18</td>
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<td>-</td>
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<td>5.69</td>
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</tr>
<tr>
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<td>-</td>
</tr>
<tr>
<td></td>
<td>K</td>
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<td>-</td>
<td>5.69</td>
<td>-</td>
<td>$7 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\Delta \mu_{hold}^d$</td>
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<td>0.0210</td>
<td>1.04</td>
<td>-</td>
<td>$1 \times 10^{-10}$</td>
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</tbody>
</table>

$^a$Low stiffness data better fit
$^b$High stiffness data better fit
$^c$ $a - b = -0.0027$
$^d$ $a - b$ unconstrained
Figure 4.1 (A) Numerical solution of the response of Aging (dashed lines) and Slip (solid lines) laws to velocity step decreases of 1 (red), 2 (green) and 3 (blue) orders of magnitude. The simulations use the best fitting Slip law parameters from panel (C) and the appropriate stiffness 0.008 $\mu$ms$^{-1}$. Friction values are normalized by minimum-to-residual stress change. Following the stress minimum, the Slip law curves for the different orders of magnitude plot on the top of each other. (B) Change in friction as a function of slip for load point velocity decreases of 1 and 2 orders of magnitude in simulated gouge (Bhattacharya et al., 2015). The data are scaled to the minimum-to-residual friction range as in (A). These large velocity steps carry the surface far below steady-state, yet note that the data for the 1- and 2-order of magnitude steps strengthen over the same slip distance, as predicted by the Slip law in (A). (C) Aging and Slip law fits to the velocity step data shown in (B). Only the velocity step decreases were fit with the 1- and 2-order decreases equally weighted. We constrained $a - b = -0.0002$. Blue: Data; Ochre: Aging law; Red: Slip law. Numbers in black denote load point displacement rate in $\mu$ms$^{-1}$.
Figure 4.2 (A) Shear stress evolution from the slide-hold-slides of [Beeler et al. (1994)]. The X-axis is time, scaled such that each phase of hold and slide/reslide has unit duration to aid in visualization. The holds span $0.5 \times 10^4$ s in $t_{\text{hold}}$. The red curve shows a slide-hold-slide sequence for the lower, natural stiffness setup ($k_n = 0.0019 \, \mu m^{-1}, V_{r/s} = 1.0 \, \mu m s^{-1}$), the blue curve shows a sequence for the stiffer apparatus ($k_s = 0.055 \, \mu m^{-1}, V_{r/s} = 0.32 \, \mu m s^{-1}$). (B) A 1000 s hold with the stiffer apparatus. The $t_{\text{hold}}, \Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ notation wherever used in the text is as defined in this figure. Numbers in blue represent load point velocities in $\mu m s^{-1}$. (C) Evolution of $\Delta \mu_{\text{peak}}$ (squares) and $\Delta \mu_{\text{hold}}$ (circles) with $t_{\text{hold}}$ for two sets of slide-hold-slide sequences with the low (red) and high (blue) stiffness apparatus setups. The time evolution of these quantities is remarkably reproducible from repeated experiments during the same experimental run.
Figure 4.3 Numerical behavior of the different evolution laws under SHS tests of varying hold durations $t_{\text{hold}}$. The solid curves were generated using $k_n = 0.0019 \mu m^{-1}$ and $V_{s/r} = 1.0 \mu ms^{-1}$. The dashed curves use $k_s = 0.055 \mu m^{-1}$ and $V_{s/r} = 0.316 \mu ms^{-1}$. Colors represent different state evolution laws- red: Aging; blue: Slip; cyan: Nagata with $c = 1$; and magenta: Nagata with $c = 10$. We used $a = 0.009$, $b = 0.01$, $D_c = 3 \mu m$ for the Aging and Slip laws. The scaling relations for the Nagata Law were used to find the corresponding appropriate $a$, $b$ and $D_c$. For the Kato law, $V_c = 10^{-2} \mu ms^{-1}$. Even for small values of $c$, the Nagata law shows stiffness dependence in $\Delta \mu_{\text{peak}}$. For $c = 100$, the Nagata law is nearly indistinguishable from the Slip law.
Figure 4.4 Evolution of frictional strength under the Aging law during a long hold (∼ $3 \times 10^4$ s) for different values of normalized stiffness, $kD_c/a = 1$ (blue), 10 (ochre) and 100 (green), and different values of $b/a$. (A) Velocity weakening, $b/a = 2$; (B) Velocity neutral, $b/a = 1$; (C) Velocity strengthening, $b/a = 0.5$. The solid lines show numerical solutions, corresponding dashed lines denote the analytical approximations to $\Delta \mu /a$ derived in Appendix D.3.1. A velocity weakening Aging law predicts a constant stress solution for long holds. Velocity neutral and strengthening solutions show continual relaxation of stress but, importantly, the rate of stress relaxation is stiffness independent for long holds. Note that the experiments of Beeler et al. (1994) span the range from $0.3 - 1.0 \lesssim V_s/t_{\text{hold}}/D_c \lesssim 3 \times 10^3 - 10^4$ between the high and normal stiffness experiments.
Figure 4.5 Stress relaxation under the Slip law during a long hold ($\sim 10^5$ s) for different values of the normalized stiffness, $kD_c/a = 1$ (blue), 10 (ochre) and 100 (green), and different values of $b/a$. (A) Velocity weakening, $b/a = 2$, (B) velocity neutral, $b/a = 1$ and (C) velocity strengthening, $b/a = 0.5$. The solid color lines are the numerically integrated values of $\Delta \mu/a$. Corresponding dashed lines denote the relevant analytical approximations derived in Appendix D.3.2. In the small stiffness limit (dashed blue lines), the velocity weakening Slip law predicts a log(log($t_{\text{hold}}))$ trajectory while the velocity strengthening trajectories are linear in log($t_{\text{hold}}$). The large stiffness limit for all trajectories is $\Delta \mu/a = \ln(V/V_{s/r})$ (dashed red lines).
Figure 4.6 Aging and Slip law fits to the time evolution of $\Delta \mu_{\text{peak}}$ in initially bare granite from [Beeler et al. (1994)]. The red squares and circles are the low stiffness data, the corresponding blue symbols are the high stiffness data. (A) Aging law fit (solid line) with $a - b$ fixed at -0.0027. (B) Aging (solid line) and Slip law (dashed line) fits without any constraint on $a - b$. Note that the Slip law fit to the time evolution of $\Delta \mu_{\text{peak}}$ is as good as the Aging law with the healing rate being stiffness independent. However, for both the Aging and Slip laws, the parameter choices which fit the peaks very well completely fail to match the corresponding values of $\Delta \mu_{\text{hold}}$. 

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**Figure 4.6: Aging and Slip Law Fits**

The red squares and circles represent the low stiffness data, while the corresponding blue symbols represent the high stiffness data. **(A)** Aging law fit with $a - b$ fixed at -0.0027. **(B)** Aging (solid line) and Slip law (dashed line) fits without any constraint on $a - b$. The Slip law fit is as good as the Aging law, with the healing rate being stiffness independent. However, the parameter choices which fit the peaks very well completely fail to match the corresponding values of $\Delta \mu_{\text{hold}}$. 

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Figure 4.7 The evolution of $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ with hold time for the Slip law fits to peak stresses alone. (A) The hatched and filled regions depict an ensemble of simulated time series for $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ respectively drawn from the posterior generated in obtaining the Slip law fit in Figure 4.6b. The density of color represents number density of samples from the posterior. This ensemble represents ~ 400 samples (10% of the total number accepted) drawn randomly from the steady-state posterior realized by running a Markov chain 50,000 samples long. Squares are the Beeler et al. (1994) values of $\Delta\mu_{\text{peak}}/b$. The $\theta_{\text{hold}}$ and $\theta_{\text{peak}}$ values (solid lines) were read from the full time series from which the Slip law fits in Figure 4.6b were derived. Dashed lines show the coarsely sampled actual fit to peak stresses in Figure 4.6b. (B), (C) and (D) show the actual posterior distributions of $a$, $b$ and $D_c$. The yellow squares show the location of the fit shown in Figure 4.6(B), this is statistically equivalent to the maximum posterior solution. The thick red lines show the maximum likelihood Gaussian fits to the posteriors.
Figure 4.8 Aging and Slip law fits to the time evolution of $\Delta \mu_{\text{hold}}$ in initially bare granite from Beeler et al. (1994). (A) Aging (solid line) and Slip (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Fits without any constraint on $a - b$. The two Aging law fits are the maximum a posteriori (solid line) and the minimum misfit solutions (dot-dashed line). These two solutions are separated by orders of magnitude of parameter values for this inversion, likely due to numerical noise around the minimum misfit solution which assumes unrealistic parameter values. The Slip law fit is the dashed line.
Figure 4.9 Detailed comparison between the stress relaxation during the 3162 s and 10,000 s holds with the stress relaxation predicted by the Slip and Aging law fits (maximum a posteriori) to the stress minima from Figure 4.8b. The abscissa is the hold duration scaled to unity to help with visualization. (A) Slip and (B) Aging law predictions of stress relaxation from the fits to the low stiffness $\Delta \mu_{\text{hold}}$ data. (C) Slip and (D) Aging law predictions of stress relaxation from the fits to the high stiffness $\Delta \mu_{\text{hold}}$ data. Insets in (A) and (C) show the details of stress evolution during the first 5% of the hold. In all the plots, blue is data, red is Slip law and green is Aging law. Note that the best Slip law fit explains the full stress-relaxation time series very well.

Slip: $a = 0.0261$, $b = 0.0264$, $D_c = 1.16 \mu m$

Aging: $a = 0.0435$, $b = 0.0402$, $D_c = 0.69 \mu m$
Figure 4.10 Aging and Slip law fits to the time evolution of both $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ in initially bare granite from [Beeler et al., 1994]. (A) Aging (solid line) and Slip (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Fits without any constraint on $a - b$; the solid lines show Aging and the dashed lines Slip law fits.
Figure 4.11 Kato and Nagata law fits to the $\Delta \mu_{\text{hold}}$ time series for the initially bare granite data from [Beeler et al. (1994)]. (A) Nagata (dashed line) and Kato (dash-dotted line) law fits with $a - b$ fixed at -0.0027. The corresponding Slip law fit (solid line) from Figure 4.10(A) is plotted for reference; the fit to the stress minima by all three laws are identical. (B) Fits without any constraint on $a - b$; the dashed lines show the Nagata law fits. Note that the stress minima are again identical to the corresponding Slip law fit (solid line, from Figure 4.10(B)). Dotted lines show the best Nagata law fit for $c = 10$. We did not attempt the corresponding fit with the Kato law, since the Slip law already provided an excellent fit to the $\Delta \mu_{\text{hold}}$ data alone.
Figure 4.12 Kato and Nagata law fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ time series from the initially bare granite data of Beeler et al. (1994). (A) Kato (solid line) and Nagata (dashed line) law fits with $a - b$ fixed at -0.0027. (B) Nagata law fit (dashed line) to stress minima without any constraint on $a - b$; the solid lines show the corresponding Slip law fit. Note that the Kato and Nagata law fits in (B) are nearly identical, and also identical to the corresponding Slip law fit in Figure 4.10(B). The dotted line shows the best Nagata law fit when $c = 10$ is held constant.
Figure 4.13 Summary of the important features of the quartzite data from Beeler et al. (1994). (A)-(C) Fits to a sequence of 1 order steps with the servo controlled stiffer setup. $a - b = -0.0062$, $a \sim 0.012$. (D) Evolution of $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ time series. Note that the healing rate is close to $a - b$. Solid line shows Aging law fit to peak stresses alone. Dashed lines show the Slip law fit to both $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$, note that this fit explains the peaks and the stress minima equally well. (E) Fits to $\Delta \mu_{\text{hold}}$ with the Aging and Slip laws. Solid lines-Aging, dashed lines-Slip law without any constraint on $a - b$ and dash-dotted lines - Slip law with $a - b = -0.0061$. Note that, just like the granite data, the stiffness dependence of the stress relaxation is well modeled by the Slip law when $a - b$ is not constrained.
Chapter 5

Where did the time go? Friction evolves with slip following large velocity and normal stress steps

5.1 Introduction

Rate-and-state friction (RSF) has been the constitutive framework of choice for modeling laboratory friction data over the last three decades (Dieterich, 1972; Ruina, 1983; Tullis and Weeks, 1986; Marone, 1998b; Marone and Saffer, 2015). Under the RSF framework, the evolution of friction is described in terms of the slip rate ($V$) of the interface, and a physically more nebulous, ‘state’ variable ($\theta$) which is traditionally viewed as the actual area of contact represented by the asperities bridging the nominally flat, but microscopically rough, interface (Bureau et al., 2002; Baumberger and Caroli, 2006). While a logarithmic rate dependence of friction is well established both from rate stepping experiments and first-principles physics (Rice et al., 2001; Nakatani, 2001; Boettcher et al., 2007), the ‘proper’ description of state evolution remains elusive. The two most popular versions of the state evolution formulation represent opposite end-member views of the physical quantity which
dominantly controls state evolution: slip alone (as with Slip law state evolution) or time even in the absence of slip (as with Aging law state evolution). Laboratory evidence for or against either of these views has remained equivocal – large laboratory velocity step increases and decreases show clear support for the slip dependent Slip law, while slide-hold-slide experiments, in which the slip interface is relaxed to very low stresses during periods of little or no sliding called holds, have been traditionally interpreted as showing support for the time-dependence inherent in the Aging law (Marone, 1998b; Nakatani, 2001; Marone and Saffer, 2015; Bhattacharya et al., 2015).

At the core of the differences between the slip-dependent and time-dependent views are their disagreements on whether frictional interfaces require time or slip to heal, where healing is defined as any increase in state, or equivalently increase in frictional strength at some reference sliding speed. Both the Aging and Slip laws allow frictional healing through an increase in the state variable in response to rapid and large reductions in sliding rate, but whereas the Aging law state grows purely as a function of time under such conditions, the Slip law predicts growth of state only with slip. Both the largest velocity step decreases and long holds access this ‘far below steady state’ regime of sliding (which we shall precisely quantify in Section 5.2), and it is therefore somewhat paradoxical that the former supports the Slip law while the latter has traditionally been inferred to support Aging law style time-dependent healing.

Bhattacharya et al. (2016) have recently re-interpreted the slide-hold-slide data of Beeler et al. (1994), considered the gold standard evidence for Aging law style time-dependent healing from friction experiments, to show that, in fact, the data provide only equivocal support for Aging law type time-dependent healing and are better explained by the Slip law. In particular, the stress-relaxation during long holds is well explained by the Slip law, much like the largest velocity step decreases are, and this seems consistent with the interpretation that both these experimental protocols reveal slip-dependent frictional healing far below steady state. However, most laboratory velocity step experiments do not push the sliding
surface very far below steady state and it is possible that the parameter space has just not been probed sufficiently to observe evidence of time-dependent healing (Beeler et al., 1994). This limitation can be overcome by examining these state evolution laws under larger and more rapid laboratory velocity steps. This presents an experimental challenge – the fidelity with which a rate step imposed at the load point is translated to the actual sliding interface is limited by the stiffness of the testing apparatus, and specially designed experimental protocols are usually required to maximize the size of the excursions from steady state (Bhattacharya et al., 2015). In this paper we address this issue by carrying out velocity step experiments on an artificially stiffened apparatus which allowed us to impose stable and reproducible velocity step increases and decreases of up to 3.5 orders of magnitude on initially bare Westerly granite samples.

If the state variable represents true contact area, any argument against time-dependent healing also has to address the direct observations of log time growth in size of truly stationary micro-contacts bridging acrylic and soda glass interfaces in the see-through experiments of Dieterich and Kilgore (1994). One way out of this conundrum is to argue that the RSF state variable is not contact area alone, and that the growth of contacts does not necessarily translate into additional frictional resistance. This line of thinking is not without experimental support – Kilgore et al. (2012) and Nagata et al. (2014) have recently shown that the evolution of shear resistance in response to normal stress step increases imposed on a steadily sliding sample (both Westerly granite and transparent plastic) occurs much more gradually than the change in real contact area ‘tracked’ by simultaneous measurements of fault dilation/compaction, across-fault ultrasonic transmissivity and, in the case of the plastic sample, direct optical measurements (like in the experiments of Dieterich and Kilgore (1994)). This observation, Kilgore et al. (2012) argue, might suggest that the regions of the interface brought rapidly into contact by the increase in normal stress are initially too weak to support any shear stress. It is likely that the authors inferred such extreme contrasts in strength between old and new contacts (that fresh contacts have exactly zero inherent
strength) only because they ignored the significant decreases in slip rate which occurred across these normal stress steps \cite{Beeler2016}; nevertheless, these observations suggest that differences in ‘quality’ across the population of contacts can play a significant role in determining friction phenomenology. To investigate whether the state variable physically represents a combination of contact ‘quality’ and ‘quantity’, we complemented our large velocity steps with 5% normal stress steps during the same experimental run. These normal stress steps were imposed at two different slip rates to explore the time- versus slip-dependence of the evolution of friction. These experiments were also accompanied by continuous measurements of fault normal displacement which further allowed us to probe the inter-relationship between the evolutions of friction and contact area following the arguments outlined in \cite{Kilgore2012}.

We show in this paper that our velocity step data provide unequivocal support for the Slip law. In particular, the large velocity step decreases show that shear resistance evolves over a characteristic slip distance irrespective of the size of the velocity step, a property inherent in the Slip law but at odds with Aging law style time-dependent healing. In addition, the normal stress step data show that shear resistance evolves over a characteristic slip distance but much slower than the rapid change in contact area suggested by the fault normal dilation/compaction measurements. However, the data also reveal an instantaneous excursion in slip rate across the normal stress steps and the evolution of $\tau$ over characteristic slip is in some part due to the subsequent evolution of the slip rate back to its pre-step value. But these slip rate excursions are reproducibly smaller in magnitude than would be expected if the new contact area had the same inherent strength as the existing steady-state contact area. Given this implied contrast in strength between old and new contacts, we proceed to assume a simple model of a multi-contact interface whose strength (or state) is determined by a combination contact ‘quality’ and ‘quantity’ to derive a state evolution response to normal stress variations without slip, with the contrast in contact ‘quality’ built-in as a model parameter. With either the Slip or the Aging law describing post-step changes in state with
accumulating slip, this new state evolution formulation explains our normal stress step data well by adopting significant contact ‘quality’ contrast between old and new contacts. That the Slip law might be the favored formulation for state evolution at constant normal stress is indicated by the fact that the relevant RSF parameters derived from fitting the normal stress steps turn out be more similar to those derived from the velocity steps with the Slip law rather than the Aging law.

5.2 Why large velocity steps are relevant to frictional healing

Mathematically, friction as a function of rate \( V \) and state \( \theta \) is expressed as:

\[
\tau_{fr} = \mu \sigma = \sigma \left[ \mu_s + a \ln \left( \frac{V}{V_s} \right) + b \ln \left( \frac{V \theta}{D_c} \right) \right],
\]  

(5.1)

where \( \tau_{fr} \) is shear resistance to sliding expressed as the product of the friction coefficient \( \mu \) (we refer to \( \mu \) as friction in the rest of the paper) with normal stress \( \sigma \), \( a \) is the ‘direct effect’ parameter accounting for the variations in friction due to changes in slip rate, and \( b \) is the ‘evolution effect’ parameter which determines the change in friction due to evolution of state. The state variable is traditionally considered a proxy (in units of time for the state evolution formulations we have chosen) for the strength of the asperities in contact across the sliding interface at a reference slip speed (most of the interfacial strength is considered as contained in the constant \( \mu_s \)), with log state generally assumed to scale with the true area of contact (Bowden and Tabor, 1964; Linker and Dieterich, 1992; Baumberger and Caroli, 2006). The constants \( a \) and \( b \) are of the order 0.01 at moderate temperatures (Blanpied et al., 1998). The other parameters \( \mu_s \), \( V_s \) and \( \theta_s \) are the values of friction, slip rate and
state at an arbitrary reference steady-state. To close the system, we need an equation for the
time evolution of $\theta$. As mentioned in Section 5.1, the two most popular choices are:

\begin{align}
\text{Aging (Dieterich) Law} & : \dot{\theta} = 1 - \frac{V \theta}{D_c} \\
\text{Slip (Ruina) Law} & : \dot{\theta} = -\frac{V \theta}{D_c} \ln \left( \frac{V \theta}{D_c} \right)
\end{align}

where $D_c$ is some characteristic slip weakening length scale (Dieterich, 1978; Ruina, 1983).

Eq. (5.2a), the Aging law, has the property that state increases linearly with time for
stationary contacts. The Slip law (Eq. (5.2b)), on the other hand, allows state to evolve
only when slip occurs. At steady-state sliding ($\dot{\theta} = 0$), both the laws yield $V \theta / D_c = 1$. For
the rest of this paper, we define $V \theta / D_c > 1$ and $V \theta / D_c < 1$ as being ‘above’ and ‘below’
steady-state respectively; consequently ‘far from steady-state’ implies $V \theta / D_c$ significantly
different from 1.

If a large decrease in sliding rate is imposed (from steady state) rapidly enough that
the change in state is negligible, then $V \theta / D_c$ drops to values much smaller than 1. Both
the Aging and Slip laws predict that state increases in response to such large, rapid and
negative perturbations to steady state. This increase in state leads to an increase in friction
following a large velocity step decrease, this is frictional healing. Under these conditions,
the Aging law in Eq. (5.2a) reduces to $\dot{\theta} \sim 1$, i.e. state increases as time, a phenomenon
usually referred to as time-dependent healing. On the other hand, the rate of state evolution
under the Slip law decreases to progressively smaller values as the size of the velocity step
increases with no state evolution occurring in the case that $V \rightarrow 0$, this is slip-dependent
healing.

However, the most widely cited support for time-dependent healing is inferred from a
different class of experiments called slide-hold-slide tests wherein the driving on the sample
is stopped for a finite duration called a hold and the shear stress on the interface is allowed
to relax to small values. As a result, at the end of the longest laboratory holds (typically about $10^5$ s), the slip rate can reduce to smaller than $10^{-6} \mu m s^{-1}$. Therefore, long holds push the frictional interface far below steady state just as large velocity step decreases do, albeit at much lower slip rates, thus allowing the interface to undergo frictional healing. At the end of the hold, the sample is usually reslid at the pre-hold slip rate and friction reaches a peak value. This peak friction upon reslide has been observed to increase linearly with the logarithm of hold time for holds longer than a threshold time \cite{Dieterich1972,Beeler1994,Marone1998,Berthoud1999}. Additionally, \cite{Beeler1994} performed a series of slide-hold-slides at two different machine stiffnesses to show that the rate of this increase in peak friction is independent of machine stiffness. Since the pre-hold and reslide rates are the same, the increase in peak friction with hold duration is usually ascribed to the state accrued during the hold \cite{Beeler1994}. Such an interpretation implies that the log time growth in peak friction originates from a linear in time growth in state far below steady state, a property inherent in the Aging law. And, if peak friction is only a function of the state accrued during the hold, then the data of \cite{Beeler1994} show that the very different amounts of slip accrued during the holds for the two different stiffnesses led to the same rate of increase in state. Based on numerical simulations, \cite{Beeler1994} argued that this ruled out slip-dependent healing.

However, it has been recently shown that the Slip law can fit the \cite{Beeler1994} peak stress data, simultaneously for the two stiffnesses, nearly as well as the Aging law, albeit for a narrow range of parameter values \cite{Bhattacharya2016}. Furthermore, \cite{Bhattacharya2016} pointed out that the rates of stress relaxation during the holds were strongly stiffness dependent. On the basis of analytical results and fits to the data, they showed that this feature of the data was incompatible with Aging law style time-dependent healing but naturally captured by the Slip law. In particular, with constant RSF parameters, their Slip law fits explain the stress relaxation during the holds well and systematically much better than the Aging law. The fact that, in slide-hold-slide experiments, the strongest argument
against Aging law style time-dependent healing and for slip-dependent healing comes from the portion of the parameter space where time-dependent healing has for long been assumed to be clearly superior raises an important question – Is frictional healing really dominantly slip-dependent far below steady state?

One way to address this question is to investigate very large velocity step decreases. Even though the largest laboratory velocity step decreases do not push the interface to slip rates as low as even moderately long holds do, as long as the interface is pushed far below steady state \( \frac{V\theta}{D_c} \ll 1 \), signatures of time-dependent healing should be evident from the data if the Aging law is the appropriate state evolution law. Additionally, since the post-step driving is held constant, most of the restrengthening post the friction minimum is a direct manifestation of state evolution far below steady state. As we shall see in Section 5.2.1, this allows us to make compelling diagnostic distinctions between time- and slip-dependent healing based on the friction data alone, even without carrying out full inversions.

Therefore the key to studying frictional healing from rate steps is to impose very large velocity step decreases. Traditionally, most experimental studies of velocity steps concentrate on small excursions from steady state (Beeler et al., 1994). This is partly due to experimental challenge. The key ingredient here is the stiffness of the testing apparatus, which controls how faithfully a velocity step imposed at the load point is translated to the sliding surface. Only a stiff machine can ensure a nearly ‘true’ velocity step on the sample, thus ensuring little state evolution across the rate step. This in turn ensures that a given large load point rate step actually leads to the desirable large departure from steady state. Additionally, for velocity step increases, a large stiffness helps keep the system stable even following large perturbations. The rotary shear apparatus at Brown University can be servo-controlled off a near fault transducer which artificially stiffens the machine to around 30 times its natural stiffness (depending on the type of rock sample, e.g. around 30 times for granite but 40 times for quartzite) (Beeler et al., 1994). This stiff testing apparatus can
be used vary slip rates between 3 and $0.001 \mu m s^{-1}$ allowing us rate steps of up to 3.5 orders of magnitude.

5.2.1 Large velocity steps on a stiff apparatus

The 0.5 to 3.5 orders of magnitude velocity steps were carried out at 25 MPa normal stress on a hollow, cylindrical sample of Westerly granite with outer and inner radii of 54 mm and 44 mm respectively. The sample was initially bare and polished, but we report experimental results only after around 120 mm of total slip. At these large values of total slip, the steady-state rate dependence of the sample reached a stable, quasi-constant, velocity weakening $(a - b \sim -0.003)$ value. Previous studies on the same apparatus have shown that, during the accumulation of $\sim 100$ mm of slip, a 70-100 $\mu m$ thick layer of gouge develops on initially bare Westerly granite samples under similar experimental conditions, with the total shear being accommodated in a narrow, quasi-planar, shear zone within this gouge layer (Beeler et al., 1996).

Data were collected at a sampling rate of 50 Hz throughout the experiments. All the velocity steps and normal stress steps reported in this study were performed during the same experimental run. Besides the rate and normal stress steps, the experimental run also consisted of a series of holds carried out from three different pre-hold sliding rates. We leave the discussion of the hold data for a separate paper, but use the initial portion of the reslides following six of our longest holds to estimate the stiffness of the testing apparatus (Appendix E.1). We found the stiffness (normalized by normal stress) to be $k = 0.065 \mu m^{-1}$. Knowing the stiffness of the apparatus allows us to use Eq. (5.3) and the known shear stress and load point history to estimate the actual slip on the frictional interface.

Figure 5.1 shows friction data from a large number of velocity step decreases and increases. In the main panel, the friction values have been normalized by the amplitude of its total change from the peak to the residual measured at 4 $\mu m$ of slip. When normalized in this fashion, the velocity decreases clearly show that the post-minimum frictional strength
evolves over a length scale that is independent of step size. For comparison, we have plotted numerical simulations of velocity step decreases with both the Aging and the Slip laws. These simulations were run with the appropriate stiffness (estimated in Appendix E.1) and $a$, $b$ and $D_c$ values (derived by fitting the velocity step data in Appendix E.2) for our experiments. Figure 5.2B shows the Aging law simulations, clearly the Aging law predicts that following a large velocity step decrease the interface strengthens over different length scales depending on the size of the velocity step. In particular, the larger the velocity step, the smaller the slip scale over which the interface strengthens back to steady state. For velocity steps which push the interface far below steady state ($V\theta/D_c \ll 1$), this observation can be rationalized by recognizing that the slip scale over which friction increases back to steady state is a decreasing function of the size of the velocity step. To show this, under conditions of time-dependent healing ($\dot{\theta} \sim 1$), we write the Aging law evolution of friction after its minimum as (assuming the slip rate to be quasi-constant after the velocity step)

$$\frac{1}{b\sigma} \frac{d\mu}{d\delta'} \approx \frac{d}{d\delta'} \log(\theta) \approx \frac{D_c}{V\theta},$$  \hspace{1cm} (5.3)

where $\delta'$ is the amount of slip experienced by the block normalized by $D_c$. For a ‘true’ velocity step, Aging law allows no state evolution across the velocity step, therefore at minimum friction $\theta$ is its pre-step steady-state value $D_c/V_0$ and consequently $V\theta/D_c = V_f/V_0$ where $V_0$ and $V_f$ are pre- and post-step steady state slip rates respectively. It follows, that for the finite stiffness case, $V\theta/D_c$ at minimum friction can be considered a proxy for the step size $V_f/V_0$ and, therefore from Eq. (5.3), we can make the prediction that (Aging law style) time-dependent healing implies that the rate of restrengthening with slip increases with increasing step size. Under the Slip law, on the other hand, the shear stress response to the large velocity step decreases evolves to steady state over a characteristic slip scale similar to the data (Figure 5.2D). This is the well known exponential slip-weakening/strengthening inherent in the Slip law (Linker and Dieterich 1992; Rice 1993; Nakatani 2001; Ampuero and Rubin 2008).
Figure 5.3A once again shows the friction versus slip plots for the velocity step data from Figure 5.1 but friction is now not normalized and the changes in friction are referenced to the pre-step steady-state stress level. These plots are visually comparable to the finite stiffness simulations in Figures 5.2A and C. The amplitude of the stress minimum increases linearly with the logarithm of the size of the velocity step for the data (inset in Figure 5.3A) with a slope of around $-0.01$. Using parameters derived from fits to all these velocity step data, Slip law simulations also show a linear growth in the amplitude of the friction minimum with log step size with a slope $\sim -0.009$ (Figure 5.3C). This linear increase in the amplitude of the stress minimum with log step size couples with the quasi-constant $d\mu/d\delta$ between the onset of the velocity step and the stress minimum to make the pre-minimum slip a nearly linear function of step size (Figure 5.2C) (in this interval, for a large velocity step decrease, $d\mu/d\delta \sim -k$ for as long as the block continues to slip at a rate much larger than the post-step driving rate). The slip accumulated between the onset of the velocity step and the friction minimum increases with step size for the data as well, more clearly so for the larger velocity steps (Figure 5.3A).

The Aging law, on the other hand, shows that the rate of increase in the amplitude of the friction minimum decreases with increasing log step size (Figure 5.3B). In general, any increase in state (healing) between the onset of the velocity step and the friction minimum reduces the eventual amplitude of the friction minimum for a given size of the velocity step. Under the Aging law, for velocity step decreases large enough that $V\theta/D_c \ll 1$ before the friction minimum, the rate of change of log state with slip increases linearly as the inverse of $V\theta/D_c$ thus producing progressively larger amounts of state evolution across the velocity step with increasing step size (the second equality in Eq. (5.3), see also solid lines in Figure 5.4B). As a result, with increasing step size, $\Delta \log(\theta)$ across the velocity step quickly becomes a significant fraction of the negative contribution of the direct effect (which is linear with log step size: $a \log(V_f/V_0)$) to the amplitude of the friction minimum. This leads to a considerable amount of shallowing of the amplitude of the friction minimum.
when compared to its ‘no state evolution’ value (which is just the direct effect). This shallowing effect is particularly pronounced when \( a < b \) as the contribution of \( \Delta \log(\theta) \) to the amplitude of the friction minimum has a larger weight attached to it. Bhattacharya et al. (2016) have shown that, because of continued healing, stress relaxation trajectories for the Aging law actually asymptote to a constant shear stress in the limit of long holds (or equivalently in the limit of \( \log(V/V_0)) \rightarrow -\infty \) when \( a - b < 0 \). Since load point holds can be thought of extreme velocity step decreases, this result is related to the pronounced shallowing of the friction minimum seen in Figure 5.3B. In fact, Figure 5.4C shows how, for successively larger velocity step decreases under the Aging law, most of the stress evolution between the onset of the velocity step and the stress minimum follows the stress relaxation trajectories of load point holds simulated with the same set of parameters. The velocity step trajectories depart from their corresponding hold trajectory when the slip rate has decreased enough to be comparable to the driving rate.

In comparison, when recast in terms of the slip derivative, the Slip law predicts that the rate of change of log state with slip only increases as the logarithm of the velocity perturbation:

\[
\frac{d}{d\delta'} \log(\theta) = - \log(V\theta/D_c). \tag{5.4}
\]

This leads to an almost linear growth in \( \Delta \log(\theta) \) with log step size across the step (dashed lines in Figure 5.4B). This combines with the log step size contribution of the direct effect to produce a quasi-linear increase in the amplitude of the friction minimum with step size, but with a slope which is smaller than \( a \).

The discussions above show not only that our velocity step decreases are much more consistent with the Slip law, but that the failings of the Aging law far below steady state are due to its time-dependent healing property. We also showed that some of these shortcomings of the Aging law vis-a-vis large velocity step decreases are closely related to similar shortcomings noted by Bhattacharya et al. (2016) on the basis of the slide-hold-slide data.
of [Beeler et al. (1994), e.g. the prediction of much shallower values of the stress minimum with increasing velocity step size as compared to the linear increase seen in the data. In particular, the friction minimum for the largest velocity step decrease for the Aging law corresponds to the slip rates accessed by a \( a - b < 0 \), see Figure 5.4C). Note that by 100 s the stress relaxation trajectory for \( a - b < 0 \) has already started to saturate to quasi-constant values. [Bhattacharya et al. (2016)] have pointed out that this shallowing of the stress relaxation trajectories, which is a direct consequence of time-dependent healing, leads to the interface being much too strong under the Aging law (for \( a - b < 0 \)) when compared to the data derived from the longest holds of [Beeler et al. (1994)]. This gives us confidence that these velocity steps are large enough to probe the portions of the parameter space where the Aging law predicts that effects of time-dependent healing should kick in, and similar to the hold data of [Beeler et al. (1994)], refute its implications on the evolution of friction.

Additionally, the velocity step increases shown in the main panel of Figure 5.1 show that the evolution of friction with slip happens over a characteristic length scale independent of the size or sign of the velocity step. Such data are consistent with the Slip law prediction of exponential slip-weakening over a characteristic length scale \( D_c \) (Figure 5.2D) [Linker and Dieterich 1992, Rice 1993, Nakatani 2001, Ampuero and Rubin 2008]. The Aging law, in contrast, predicts linear slip weakening for velocity step increases which push the sliding surface far above steady state (\( V\theta / D_c \gg 1 \)). This is because, in this limit, one can drop the ‘1’ in Eq. (5.2a) to write the following analog of Eq. (5.3)

\[
\frac{1}{b\sigma} \frac{d\mu}{d\delta'} \approx \frac{d}{d\delta'} \log(\theta) \approx -1. \tag{5.5}
\]

One can already see linear slip weakening kicking in for the 2 order step increase in Fig. 5.2B. The data show no such features. Further, since the amplitude of the friction peak increases with step size, such constant rate of weakening means that stress approaches steady state over increasingly large slip distances as the step size increases. Therefore normalizing the stress
by its total change between peak and $2D_c$, as done in the data and in Figures 5.2B and D, leads to very large overshoots below the zero level for the large steps as seen in Figure 5.2B. It is well established that such asymmetry in the frictional response between large velocity increases and decreases is not supported by velocity step experiments (Ruina, 1980, 1983; Tullis and Weeks, 1986; Linker and Dieterich, 1992; Nakatani, 2001; Bhattacharya et al., 2015).

Finally, it is important to point out that the Slip law does not explain all the features of the data. Some of the largest and smallest velocity decreases and most of the velocity increases show a long term transient evolution in stress over slip distances much larger than the value of $D_c$ derived by fitting the first few microns of the shear stress response. The panel inset in Figure 5.1, wherein friction is normalized by its change between its minimum (or peak) and the value at 25$\mu$m, makes these transients more apparent. For experiments on initially bare Westerly granite samples on the same rotary shear apparatus, Tullis and Weeks (1986) have discussed similar transients in response to up to 2 orders of magnitude velocity step increases and decreases (their Figure 7). They suggest fitting such transients with the inclusion of a second state variable with its own $b$ and $D_c$. For their inversions, this second $D_c$ comes out to be in the range 30 – 100$\mu$m while the second $b \sim 0.003 – 0.004$. This is the main reason why we avoid fitting more than the first few microns of slip evolution with our one state variable model. Note that, in obtaining the value $a - b = -0.003$ in Appendix E.1 we use the steady state values at more than 50$\mu$m of slip following our velocity steps (Figure E.2). This value of $a - b$ probably corresponds better to this two state variable picture. But, on the other hand, $a - b = -0.003$ seems also appropriate for the one state variable fits in Figure E.3. Nevertheless, the fact that the data from far below steady state is so consistent with the Slip law over the first few microns of slip indicates that slip-dependent healing is the dominant mechanism behind most of the frictional restrengthening observed in these experiments.
5.3 How to reconcile time-dependent contact area growth with slip-dependent healing?

In Section 5.2.1 we argued that the large velocity step decreases unequivocally support slip-dependent healing over time-dependent healing. The evidence for time-dependent healing, however, extends beyond just friction data. For example, assuming that log state is equivalent to contact area, time-dependent healing is consistent with time dependent growth of micro-contacts bridging stationary interfaces revealed by direct optical measurements in Lucite acrylic and soda glass (Dieterich and Kilgore 1994). Furthermore, the optically observed contact area increased linearly with the logarithm of contact duration, consistent with Aging law style time-dependent healing (Dieterich and Kilgore 1994). Given these observations, it may seem paradoxical to assert that contact area grows with log time even without slip, but that state cannot increase without slip. One possible solution to this conundrum is to argue that state is more than just contact area.

An important constraint on the inter-relationship between log state and contact area comes from laboratory normal stress step tests. One of the earliest detailed experimental study was carried out by Linker and Dieterich (1992). Based on 1-40% normal stress step tests on initially bare granite, they noted that the shear stress response to a normal stress step increase at a (nominally) constant sliding rate consists of two parts – an initial increase in shear stress over nearly zero slip (the shear stress increases along the elastic loading curve) and a slower transient evolution of shear stress with slip to the future steady state. Independently, Richardson and Marone (1999) and Hong and Marone (2005) confirmed this two stage response on initially bare rock and simulated quartz and quartz+clay gouge across slip rates spanning a few $\mu$ms$^{-1}$ to mms$^{-1}$. Based on the total amplitude of the (post-step) gradual change in friction with slip, which Linker and Dieterich (1992) found to scale linearly with the log step size with a scaling factor $\alpha$, they formulated the following expression for state evolution in response to a normal stress step:
\[ \Delta \Phi = -\alpha \frac{\Delta \sigma}{\sigma}, \]  
(5.6)

where \( \Phi = b \log(V_\theta/D_c) \) and \( \sigma \) is the pre-step normal stress. For \( \alpha > 0 \), Eq. (5.6) predicts that \( \Phi \) decreases in response to a normal stress step increase; it then recovers by the same amount because, at a given slip rate \( V \), \( \Phi \) is independent of normal stress. Based on numerical simulations, [Linker and Dieterich (1992)] argued that the features of the observed shear stress response could not be explained with \( \alpha = 0 \) when the post-step evolution of friction with slip was modeled either with the Aging or Slip laws.

The interpretation of the fault constitutive response to variable normal stress is made more complicated by the fact that not all experimental studies reproduce the two stage response in the evolution of shear resistance observed by [Linker and Dieterich (1992)] and [Hong and Marone (2005)]. In particular, normal stress steps imposed on hard metal by oblique shock impact at high sliding rates (of order 10 m/s) and normal stress (a few GPa) show a complete absence of the zero-slip, rapid increase in friction seen by [Linker and Dieterich (1992)] (Prakash, 1998). More recent experiments on initially bare granite and PMMA under identical experimental conditions as [Linker and Dieterich (1992)], but with data recording at much higher time resolutions, confirm that there is no time when the surface locks up and the stress increases along the elastic loading curve, in response to rapid normal stress changes (Kilgore et al., 2012; Nagata et al., 2014; Beeler et al., 2016).

The friction data of Kilgore et al. (2012) and Beeler et al. (2016) was complemented by simultaneous recording of fault normal displacement and ultrasonic transmissivity, both quantities considered as proxies for fractional contact area \( \Sigma_r \), providing an additional constraint on hypotheses regarding state evolution. The experiments of Nagata et al. (2014) were carried out on transparent PMMA samples which made it also possible to optically track contact area, similar to the experiments of Dieterich and Kilgore (1994). All of these proxies for contact area showed instantaneous increase in response to the step increases in normal stress. To explain these observations, Kilgore et al. (2012) hypothesized that the
contact area added in response to a normal stress increase has no inherent strength, and only gains strength with the passage of either time/slip thus leading to zero instantaneous change in shear stress. However, an alternative explanation is that the newly added contact area has the same strength as the existing contacts, the slip rate changes across the normal stress step instead and the shear stress increases continuously in accord with the finite machine stiffness. In that case, the absence of a rapid, zero-slip, Linker-Dieterich type response implies that the change in slip rate is not large enough to lock the interface and, therefore, the shear stress increase does not ride along the elastic loading curve.

In this context, it is important to spell out the details of the expected behavior of shear stress in response to a normal stress step on a finite stiffness machine. To start with, we assume that fractional changes in normal stress ($\Delta \sigma / \sigma$) are equal to fractional changes in contact area $\Delta A_c / A_c$ (since force balance implies $\sigma A = \sigma_c A_c$, where the contact level local normal stresses $\sigma_c$ are usually assumed to be a material property under small perturbations (Baumberger and Caroli, 2006)). For a normal stress increase under these conditions, there is no instantaneous change in slip rate if the new contact area being added has zero strength. Therefore, the entire post-step increase in shear stress must arise from changes in state $\Phi$. As the slip rate does not change, the shear stress grows continuously when plotted against slip. When the newly added contact area has some strength, the slip rate must go down instantaneously due to the added shear resistance. Yet, shear stress evolves continuously in time/load point displacement due to the finite stiffness and constant loading rate $V_{lp}$. If this decrease in $V$ is such that $V \ll V_{lp}$, then the interface effectively locks up and shear stress increases along the elastic loading curve; when plotted against interface slip, this increase appears discontinuous, i.e., there is a shear stress change at zero-slip. Finally, if the newly added contact area has the same strength as the existing contact area, then the decreases in slip rate is maximum for the specified normal stress change. In Appendix E.3, we show that under such conditions, $\Phi$ does not change across the normal stress step. Therefore, given a normal stress step increase, whether there is a discontinuous increase in shear stress when
plotted against slip (like in the experiments of [Linker and Dieterich, 1992] and [Hong and Marone, 2005]) depends on the magnitude of the accompanying slip rate excursion which, in turn, is determined by the magnitude of the strength or ‘quality’ contrast between old and new contacts.

The evidence above suggests that normal stress steps could provide important constraints on the inter-relationship between state, contact ‘quality’ and ‘quantity’ (or the real contact area). To this end we ran a series of 5% normal stress increases and decreases during the same experimental run as our velocity steps. We chose the smallest size of normal stress perturbations which were observed to produce clear signals in the experiments of Kilgore et al. (2012). This gave us the best chance of imposing stable and reproducible normal stress step decreases with our high stiffness setup. One problem with imposing normal stress steps in the biaxial geometry is that any probable rapid change in shear stress due to the constitutive response of the fault may be overprinted by machine response either due to Poisson effects accompanying the normal stress step or slight misalignment of the sample and loading frames (Linker and Dieterich, 1992; Hong and Marone, 2005). The geometrical configuration of the rotary shear apparatus naturally ensures that the shear stress data is relatively uncontaminated by such machine effects, thus, the shear stress data can be treated directly as the fault constitutive response without any additional correction. To study the effect of slip rate on the fault constitutive response, we repeated these normal stress steps at two orders of magnitude different sliding rates – 0.3162μms⁻¹ and 0.03162μms⁻¹. The low slip rates ensure better time resolution per unit slip. Our friction data were complemented by measurements of fault dilation/compaction to give independent (but indirect) constraints on contact area evolution.

5.3.1 Observations from normal stress step experiments

The data from our normal stress step experiments confirm that the shear stress increase (decrease) in response to a normal stress step increase (decrease) occurs continuously with
slip (no zero-slip change in shear stress) (Figures 5.5A and C), implying that the slip rate has not decreased by an order of magnitude or more. But contact area, as indirectly interpreted from rapid changes in fault dilation/compaction, does change instantaneously without slip in response to a normal stress step (Figures 5.5B and D). When plotted against slip, the shear stress evolves to steady state over the same slip scale in response to both the normal stress increases and decreases, irrespective of the sliding rate. It is relevant here to mention that what appears to be steady state shear stress at $10\mu m$ of slip in the panels of Figure 5.5 really keeps on slowly increasing to around $\mu \sim 0.7$ at around $40\mu m$ of slip in our experiments. As with the velocity steps, we ignore this longer-duration transient in our analyses. Figure 5.6A shows that the slip scale over which shear stress evolves to steady state is also independent of whether the normal stress increases or decreases. Figure 5.6B shows a plot of the difference between normalized (by the amplitudes of their respective total change) fault dilation/compaction and normal stress. This difference helps remove the initial large abrupt change in fault normal displacement to reveal a second, much smaller and slower evolution in fault normal displacement with continued slip. This gradual change in fault normal displacement represents compaction for a step increase and dilation for a decrease, but again occurs over the same slip scale irrespective of the sign of the normal stress step or the sliding rate. If the rapid response in contact area is elastic, then note that subtracting the normalized normal stress from normalized closure is similar to removing the elastic component of contact normal displacement (say for a Hertzian contact model) in response to the normal stress change provided the elastic contact area change is small (Eq. 8 in Kendall and Tabor (1971)). In this sense, the normalized $\Delta$Closure-$\Delta$Normal Stress might be thought of as a proxy for the inelastic deformation of the multi contact interface as long as the contacts are sparse enough to undergo elastic deformation independent of each other.

In all our normal stress step experiments, the slip rate varied immediately in response to the normal stress change and the displacement and stress data had high enough signal
to noise ratio for us to derive a time series for slip rate by differentiating the displacement (Figure 5.6C and D). Following this rapid change, the slip rate gradually evolves back to its pre-step value. Such slip rate variations complicate the interpretation of the friction data in terms of state. In particular, the observation that shear stress evolves to steady state over a characteristic slip scale irrespective of the sign of the normal stress step or the sliding rate cannot be interpreted as indicating slip-dependent state evolution. Through the direct effect term, the quasi-exponential slip rate variation is also a part of this remarkably reproducible shear stress variation and we can only separate the effects of ‘rate’ and ‘state’ with full RSF modeling. However, the fact that slip rate undergoes rapid change at (nearly) zero slip while the shear resistance does not implies the following constraint on state evolution across the normal stress step (see Appendix E.3):

$$
\mu_0(\sigma_0/\sigma_f - 1) = a \log(V_f/V_0) + \Delta\Phi,
$$

(5.7)

where the subscript ‘0’ denotes pre-step values, ‘f’ the corresponding value immediately post-step and $\Delta\Phi$ is the state change (if any) over zero slip. Now, if one assumes that $\Delta\Phi = 0$, then reading off the value of $V_f/V_0$ from Figures 5.6C and D and using the appropriate normal stress step sizes, would lead to a value of $a = 0.07 - 0.074$ between normal stress increases and decreases. But from fitting the velocity steps we know that $a \sim 0.013$ for the Slip law and $a \sim 0.03$ for Aging. Therefore, a priori, this value of $a$ seems too large to be consistent with the velocity step data. In fact, with the constraints imposed by Eq. (5.7), the only way to ‘normalize’ the value of $a$ is to have $\Delta\Phi < 0$ for a normal stress step increase (and $\Delta\Phi > 0$ for a normal stress decrease). In other words, given that the change in shear stress is continuous with slip and that slip-rate instantaneously changes across the normal stress steps, state must also evolve instantaneously (and in the opposite sense to normal stress) in order for us to reconcile the magnitude of the slip rate excursions with that of the normal stress steps provided the independent constraint on $a$ from the velocity step data. This is in qualitative agreement with the Linker and Dieterich
In their Appendix B, Linker and Dieterich (1992) identify $\Phi$ as the fractional contact area that takes part in time-dependent creep, a view that borrows from the classical Bowden and Tabor representation the shear resistance to sliding ($\tau_{fr} = \mu \tau$) as a product of some average contact strength times fractional contact area (Bowden and Tabor, 1964; Linker and Dieterich, 1992; Baumberger et al., 1999; Baumberger and Caroli, 2006):

$$\tau_{fr} = \tau_c \Sigma_c,$$

(5.8)

where $\tau_c$ is the average strength of individual contacts and $\Sigma_c = A_c / A$ is the total area represented by contacting asperities $A_c$ normalized by the nominal area $A$. The Bowden-Tabor picture, when borrowed to traditional RSF as in Linker and Dieterich (1992), is fine at constant normal stresses but can lead to inconsistencies when normal stress changes. To see this, first note that when the Bowden-Tabor picture is used to explain the RSF formulation, traditionally $\tau_c$ is treated as the source of the rate dependence of friction. $\Phi$ is identified as contact area (Linker and Dieterich (1992) qualify this as the portion of the contact area undergoing time-dependent creep) but all contact area is considered to be inherently of the same quality. By quality we mean some measure of contact strength independent of the rate dependent $\tau_c$. Also, as mentioned before, it is reasonable to expect normal stress to be proportional to the contact area fraction under small perturbations. Under these assumptions, any normal stress step would lead to zero state evolution across the step as long as the newly added (or removed) contacts are of the same quality as the pre-step old contacts (see Appendix E.3). Therefore, given that our normal stress step data dictate an abrupt change in state, we need to modify the Bowden-Tabor product form in favor of treating $\Phi$ as more than just contact quantity. Hong and Marone (2005) argue that the similar instantaneous state change predicted in Eq. (5.8) can be rationalized by considering $\theta$ as average contact age – the introduction of new contact area in response to a normal
stress step decreases the average age of contacts thus bringing state down. On the other hand, a normal stress decrease leads to a loss of preferentially younger (hence weaker) contacts thus increasing average contact age. In this context, it is important to note that the simultaneous physical interpretation of $\theta$ as contact age and $\Phi$ as contact area assumes the view that the only contribution of contact age to interfacial strength is the growth in contact area through anelastic creep. But if we think of the [Hong and Marone (2005)] interpretation, the core of the hypothesized decrease in average contact age in response to a normal stress decrease is the preferential loss of weaker contacts which are weaker due to their younger age. This view pre-supposes that age also equates to some measure of how strong the contacts are. It seems simpler to argue that contact age (or $\theta$) is really a measure of the combination of contact ‘quantity’ (area of contact) and some estimate of contact ‘quality’ or how inherently strong the contacts are. In the next section we formalize this view into a constitutive equation for state evolution in response to a normal stress step.

5.3.2 From contact rheology to a state evolution law

That contact age could be a proxy for not only contact area but also some measure of the contact ‘quality’ has been recognized previously in nano-tribology ([Li et al., 2011] [Liu and Szlufarska, 2012]). In experiments involving frictional interactions of nanometric single asperity contacts, [Li et al., 2011] observed logarithmic in time strengthening of quasi-stationary contacts without any plastic deformation at the tips. They argued that, in their experiments, contacts develop an increasing number of siloxane bonds across the interface with progressing age. In molecular dynamics simulations of siloxane bonding across such aging contacts, [Liu and Szlufarska, 2012] have shown that the concentration of bonds grows linearly with log age under a wide range of distributions of the energy barriers required to break these bonds. We, under the unproven assumption that these nano-metric observations hold for the micro-metric contacts which concern the present study, borrow this line of argument to revise the Bowden and Tabor product decomposition of frictional strength:
\[ \tau_{fr} = \tau_c n \Sigma_c, \]  
(5.9)

where \( \tau_c \) is reinterpreted as the average strength of individual bond, \( \Sigma_c \) is again the fractional contact area (actual/nominal) and \( n \) is an area averaged bond density which could be a function of contact age or contact slip among other things. The form of the Bowden-Tabor product in Eq. (5.9) is essentially a mean field approximation of the partition function prescribed by [Hatano (2015)] where he expresses the frictional strength as an explicit sum over the distribution of strength of bonds across the whole population of contacts. [Hatano (2015)] argues that by connecting the local shear rate on each bond to the force acting on it, one can derive a logarithmic rate dependence for \( \tau_c \) of the form consistent with the classical direct rate effect. Therefore, by expressing shear resistance to sliding in the form of Eq. (5.9), we identify \( \Phi \) as the average number of bonds bridging a unit fractional area of contact – \( n \Sigma_c \). In Appendix E.4 we use Eq. (5.9), and the assumption that \( \Delta \sigma / \sigma = \Delta \Sigma_c / \Sigma_c \), to derive an explicit constitutive equation for the evolution of \( \Phi \) in response to a normal stress step:

\[
\Delta \Phi = -\mu_0 \frac{\sigma_0}{\sigma_f} \frac{\Delta n}{n_0} \frac{\Delta \sigma}{\sigma_0}, \\
\Delta \Phi = -\mu_0 \frac{\sigma_0}{\sigma_f} \eta \frac{\Delta \sigma}{\sigma_0}, \tag{5.10}
\]

where \( n_0 \) is the average density of bonds on the pre-step population of contacts, \( n_\Delta \) is the density of bonds on the fraction of contact area lost or gained and, in that sense, \( \eta = 1 - n_\Delta / n_0 \) can be imagined as a ‘dilution of quality’ factor. For a normal stress step increase, when the newly added contact area has the same strength as the old contact area there is no dilution of contact quality and state does not evolve instantaneously. In Figure E.4, we show the numerical shear stress response to 5% normal stress steps with state evolution prescribed by Eq. (5.10) coupled with the Slip and Aging laws. In these simulations, \( \eta = 0 \) (the quality...
of the old and new contacts are identical) and this results in very asymmetric shear stress response to normal stress step increases and decreases. This is clearly inconsistent with the remarkably symmetric response seen in the data. Also, the velocity excursions in response to the normal stress steps are much larger when compared to the data in Figures 5.6C and 5.6D. This shows that, with Eq. (5.10) describing the zero-slip state evolution in response to a normal stress excursion, the symmetry of the shear stress response between up and down steps requires $\Phi$ to change across the step without slip. To further confirm this, we also explicitly fit the shear stress data from the normal stress step experiments with $a, D_c, \eta_{up}, \eta_{down}$ and $\mu_*$ as free parameters and $a - b = -0.003$ (Appendix E.4). We used the Markov Chain Monte Carlo algorithm used for fitting the velocity steps. We allowed the inversions the freedom of two different values of $\eta$ for the up and down steps, $\eta_{up}$ and $\eta_{down}$, in recognition of the possibility that the bond density on the contacts being lost and those being added could be different. The fits match the data well irrespective of whether the Aging or Slip law was used to model state evolution with slip (Figure E.5). However, while the velocity-step derived values of $a, a - b$ and $D_c$ were within the tails of the corresponding posteriors derived by fitting the normal stress data, the Aging law derived values were considerably different (Figure E.6). But, given the widths of these posteriors are sensitive to the poorly known level of data error, the tails of the posteriors are likely to be only lower bounds. Finally, both the Aging and Slip law inversions predict $\eta_{up} \gtrsim \eta_{down}$ with the Aging law fits requiring nearly a factor of 2 different values. This has the implication that $n_\Delta$ for the newly added contacts in response to a normal stress decrease is smaller than for the older contacts which are lost in response to a normal stress decrease; however for the Slip law this difference is only 10%.
5.4 Discussion

The central motivation behind our normal stress step experiments was to ascertain whether it was reasonable to assume that state, as in $\Phi$, could be considered equivalent to only contact area. An important factor in the argument that this is not the case is an estimate of a ‘reasonable’ value for $a$ for our experiments. It is important to note that what constitutes a reasonable value of $a$ in some cases might be a function of the state evolution law in use, e.g. the shear stress dependent Nagata law ([Nagata et al.], 2012) requires much larger values of $a$ to explain laboratory the same laboratory velocity step data as well as the Slip law ([Bhattacharya et al.], 2015). It might be worthwhile to couple the constitutive law in Eq. (5.10) with the Nagata law to verify whether it too would require $\eta \neq 0$ to explain the data. Further, the inversions reveal that the Slip law requires $n_\Delta \sim 0.3 - 0.4n_0$ while the Aging law fits the data with $n_\Delta \sim 0.5 - 0.7n_0$. It is clear that both these laws do not require as extreme a quality contrast between the old and new contacts as was hypothesized by [Kilgore et al.], (2012), who proposed $n_\Delta = 0$ for the up steps. That shear stress did not increase along the elastic loading curves in their normal step increases does not imply that the newly formed contacts have zero strength. [Beeler et al.], (2016) have re-analyzed the [Kilgore et al.], (2012) data to show that all of their normal stress steps show modest to large slip rate excursions. The [Kilgore et al.], (2012) prediction of $n_\Delta = 0$ probably stems from not taking into account the direct rate effect.

It is noteworthy that Eq. (5.10) has a similar mathematical form to the [Linker and Dieterich], (1992) constitutive law in Eq. (5.6) while retaining some important distinctions. First, the scaling between the change in $\Phi$ and $\sigma$ in Eq. (5.10) is through a combination of physical variables and not an arbitrary constant $\alpha$. The [Linker and Dieterich], (1992) definition of $\alpha$ comes from equating the total change in shear stress with slip between the value at the end of its initial abrupt change (at zero slip) and future steady state to the fractional change in normal stress. However, in deriving their constitutive relationship, [Linker and Dieterich], (1992) simply equate (their Eqs. 8 and 10):
\[ \Delta \tau = \alpha \sigma_f \log \left( \frac{\sigma_0}{\sigma_f} \right) = b \sigma_f \log \left( \frac{\theta}{\theta_{ss}} \right), \] 

(5.11)

where \( \theta \) is the value immediately following the normal stress step, but prior to any sliding, and \( \theta_{ss} \) corresponds to the future steady state. This derivation ignores any excursion in slip rate across the normal stress step in spite of the fact that slip rate excursions did occur in their experiments \citep{LinkerDieterich1992}. For experimental data which do contain slip rate excursions, \( \alpha \) read off a shear stress response plot (like suggested in Figure 5 of \citep{LinkerDieterich1992}) does not seem to be the appropriate value for use in the state evolution equation. For our dataset, which shows no zero-slip change in shear stress, reading \( \alpha \) off a plot of shear stress change versus slip would lead to \( \alpha = \mu_{ss} \) at the reference sliding rate \citep{Perfettini2001,HongMarone2005}. But due to the issues outlined above, it is difficult for us to compare \( \eta \) to \( \alpha \) in a meaningful way. The constitutive equation in Eq. (5.10) has the added advantage that \( \eta \) has a clear meaning and, upon being derived from data fitting, provides information about a micro-mechanical property of the interface. However, as with any constitutive equation, the real test of Eq. (5.10) is in fitting larger normal stress step data. Given that the slip rate excursions in response to the normal stress steps increase with the size of the step, and the fact that Aging and Slip law responses are also most distinct from each other far from steady state, larger normal stress steps might also provide a clearer choice in terms of the preferred state evolution law for coupling with Eq. (5.10). This also has an important implication in terms of the inferred properties of the interface – since the Aging law predicts a much greater contrast in quality between the newly added contacts and newly lost contacts than the Slip law, a reliable estimate of the contrast in quality between old and new contacts is also most likely to come from larger normal stress steps.

It is important to recognize that there exist alternative constitutive relations which are specially designed to explain the continuous increase in shear stress with slip seen in our normal stress step data. \citep{Prakash1998}, whose experimental data shared this feature,
proposed a constitutive relation which allows state to only evolve with slip even in response to a normal stress step. This was based on the assumption the contact area too increases continuously with slip in response to a normal stress step. This is clearly unlikely for our experiments and, for that matter, the Kilgore et al. (2012) experiments, given the fault closure and acoustic transmissivity data. Instead, in order to explain the Kilgore et al. (2012) data, Beeler et al. (2016) have proposed a constitutive equation outside the framework of RSF. Their constitutive equation assumes that the rate of change in shear resistance is equal to the difference between the instantaneous value of shear resistance and its pre-step value. This naturally leads to an exponential evolution of shear stress with slip over a characteristic slip distance as seen in the data. However, they ignore the direct effect term in their formulation.

Finally, our original motivation for doing the normal stress step experiments was to verify whether log state was more than just (normal stress normalized) contact area. This was in the hope that, since contact area shows log time growth for stationary contacts similar to Aging law style time-dependent healing, ascribing a measure of contact quality to state (along with contact area) would provide us the wiggle room to propose some alternative slip controlled mechanism to explain the unequivocal support for slip-dependent healing from the velocity step decreases. Though the normal stress step data established that state needed to be a combination of contact ‘quality’ and ‘quantity’, our arbitrary choice of average bond density as contact ‘quality’ only solves the original problem if contact ‘quality’ at least dominantly evolves with slip across the range of slip rates accessed in these experiments. In molecular dynamics (MD) simulations of sliding nanometric contacts, Mo et al. (2009) and Mo and Szlufarska (2010) have shown that friction at the nano-scale is determined by a Bowden-Tabor product of similar to Eq. (5.9), i.e., shear resistance = average contact strength × average number of interacting atomic contacts; this seems to suggest that our choice of area averaged bond density as contact ‘quality’ is not necessarily poor. On the other hand, average bond density $n$ is observed to grow logarithmically with contact age in MD simulations of stationary nano-contacts Liu and Szlufarska (2012) and
the experiments of Li et al. (2011) and might not be the desired micro-mechanical basis for slip-dependent healing. However, it is difficult to ascertain how relevant these numerical and experimental results derived for stationary, nanometric contacts are to our problem of sliding, micrometric contacts. But, equally, it might be possible to explain slip-dependent healing through some other choices of contact ‘quality’ which evolve dominantly with slip rather than time, e.g. contact ‘quality’ could be an increasing function of contact purity controlled by the adsorption-desorption of contaminants with slip.

5.5 Summary

We performed a set of large velocity step decreases (0.5-3.5 orders of magnitude) on initially bare granite to examine whether frictional healing far below steady state is time- or slip-dependent. Through a combination of numerical modeling and analytical results, we find that our velocity step data unequivocally support the Slip law, which predicts slip-dependent healing, over Aging law style time-dependent healing. However, given that the traditional view of log state as contact area and the observation of time dependent growth in size of stationary contacts in the experiments of Dieterich and Kilgore (1994) are most naturally connected by time-dependent healing, it seemed reasonable to hypothesize that the observed slip-dependent healing suggested that state was more than just contact area. In order to explore this hypothesis, we carried out 5% normal stress steps at two different slip rates (0.3162 µms\(^{-1}\) and 0.03162 µms\(^{-1}\)) during the same experimental run as the rate steps.

The shear stress response to the normal stress steps revealed that state must change instantaneously across the step which, can only happen if the new contact area has a strength or quality contrast with respect to the old, pre-step, contact population. We chose the average bond density across the population of contacts as the measure of this contact quality and derived a constitutive equation for state evolution in response to normal stress step. Our state, instead of scaling with contact area alone, is the product of contact quality
and quantity. We found that, when paired with the Aging and Slip laws, this constitutive equation fits the normal stress data well. However, the values of $a$ and $D_c$ derived from these fits were reasonably similar to their velocity-step derived counterparts only with the Slip law. Whether the constitutive equation for state evolution under variable normal stress proposed here can explain larger normal stress perturbations when coupled with the conventional state evolution laws remains to be tested in the laboratory.
Figure 5.1 Large velocity step decreases and increases on initially bare granite. Most of the step decreases at cumulative slip larger than 120 mm are plotted. The legend shows the velocity step order (jump denotes $\log_{10}(V_{\text{final}}/V_{\text{initial}})$) and total cumulative slip at the beginning of the step. The slip is set to zero at the stress minimum (or maximum for the step increases). The data are smoothed over 0.2 $\mu$m for the step downs and over 0.4 $\mu$m for the step ups. The main panel shows the friction normalized by its change between 0 and 4 $\mu$m of slip. The slip evolution of stress is nearly identical over this length scale for all the velocity steps in the plot. Upper inset shows friction normalized by its change between 0 and 25 $\mu$m of slip for comparison. That some of the velocity steps exhibit a second, slower evolution of friction beginning at slip distances of a few microns becomes more evident at this scale. But the evolution of friction with slip still plot on the top of each other for most of the velocity steps.
Figure 5.2 Finite stiffness simulations of large rate step increases and decreases, with normalized stiffness $k = 0.065 \, \mu m^{-1}$, generated at 50 Hz sampling to mimic the time history represented by the experimental data. The modeled time series is then smoothed identically to the data in Figure 5.1. (A) and (C) show the evolution of friction for 0.5-3.5 orders of magnitude step increases and decreases (order in the legend denotes $\log_{10}(V_{\text{final}}/V_{\text{initial}})$) for the Aging and Slip laws respectively, such that the changes in friction are referenced to the pre-step value $\mu_{\text{prior}}$. The parameters used are $a = 0.013$, $a - b = -0.003$ and $D_c = 2\mu m$; these values are derived from fits to the data (see Appendices E.1 and E.2). (B) and (D) are rescaled versions of (A) and (C) respectively such that changes in friction are measured from its value at $4\mu m$ and are normalized by the maximum amplitude of this change, as in the main panel of Figure 5.1. The slip is set to zero at the stress minimum (or maximum) for all panels.
Figure 5.3 (A) Same set of velocity steps as in Figure 5.1 but plotted as non-normalized friction versus slip. Slip is set to zero at minimum stress and shear stress is set to zero at the pre-step level. The data are smoothed as in Figure 5.1. Inset shows the evolution of the stress minimum (ΔFriction\textsubscript{min}) with log step size. (B) Evolution of the stress minima following a large velocity step decrease with step-size from the (smoothed) finite stiffness simulations in Figure 5.1(A) for the Aging law. (C) Same as (B) but for the Slip law. Note how the Slip law predicts linear evolution of the stress minima with log step-size. The Aging law, on the other hand, predicts that the stress minima deviate to significantly shallower values from the initial linear trend as step size increases. The black dashed lines in (B) and (C) have a slope of $a = 0.013$. The trend of the stress minima from the data is linear, as predicted by the Slip law.
Figure 5.4 (A) Evolution of log velocity and (B) and log state with slip between the onset of the velocity step and the stress minimum for the numerical simulations of the velocity step decreases in Figure 5.2. Solid lines – Aging law; dashed lines – Slip law; zero of slip is set at minimum stress. Note how the rate of Aging law state evolution with slip increases dramatically with slip as the size of the velocity step increases, unlike the modest changes observed for the corresponding Slip law evolution. (C) Evolution of friction with log slip rate under the Aging law for the numerical simulations of the velocity step decreases in Figure 5.2. The dashed trajectories show the stress relaxation trajectories of a 5000s hold with the same $a$ and $D_c$ as the velocity steps, but the blue curve has the same $a - b$ as the steps while the red curve has $a - b$ of the same value but of opposite sign. (D) Same as (C) but for the Slip law. Note how most of the stress evolution between the onset of the step and the stress minimum follows the corresponding stress relaxation trajectory for the hold. The vertical dashed lines in panels (C) and (D) show the hold durations (for the trajectories with $a - b < 0$) corresponding to the lowest slip rate accessed during the largest velocity step decrease.
Figure 5.5 5-6% Normal stress steps at two different constant loading rates ($V_{lp} = 0.3162 \mu m s^{-1}$ and $V_{lp} = 0.03162 \mu m s^{-1}$) from the same experimental run as the velocity steps in Figure 5.1. The normal stress, shear stress and closure (fault normal displacement) data are smoothed over 0.075 $\mu m$ and 0.0375 $\mu m$ for the data at $V_{lp} = 0.3162 \mu m s^{-1}$ (solid lines) and $V_{lp} = 0.03162 \mu m s^{-1}$ (dashed) respectively. The normal stress, shear stress and closure data (in $\mu m MPa^{-1}$) are jointly normalized by the size of the normal stress step. Top panels: Normal stress step increases. (A) - Normal stress in red, shear stress in blue. (B) - Normal stress in red, closure in blue. Bottom panels: (C) and (D) correspond to (A) and (B) respectively, but for normal stress step decreases. There are two instances of each normal stress step at each sliding speed, showing that these experiments are highly reproducible. When plotted against slip, the shear stress evolution in response to the step is nearly identical for the two sliding speeds. There is no increase in shear stress at zero slip in response to the normal stress steps even though there is significant near-instantaneous fault compaction/dilation.
Figure 5.6 (A) Evolution of normalized shear stress with slip. Unlike Figure 5.5, the shear stress is normalized here by its total change between 0 and 10 \( \mu m \) of slip. The negative shear stress responses to step decreases have been flipped to plot on the top of the responses to step increases to aid visual comparison. The slip evolution of shear stress in response to normal stress step increases (solid lines) and decreases (dashed lines) track each other irrespective of the nominal sliding rate. The legend shows the nominal slip rate \( V \) in \( \mu ms^{-1} \) and the sign of the step (+1 for normal stress increase and -1 for decrease) (B) Evolution of normalized \( \Delta \text{Closure-\Delta Normal Stress} \) versus slip. Normal stress and closure are independently normalized by their total change between 0 and 10 \( \mu m \) of slip. The difference between normalized closure and normal stress reveals the slower evolution of closure even after normal stress has reached a uniform, constant value. (C) and (D) show the slip rate excursion in response to the normal stress step decreases/increases at \( V_{lp} = 0.3162 \mu ms^{-1} \) and \( V_{lp} = 0.03162 \mu ms^{-1} \) respectively. The slip rate excursions are instantaneous and of opposite sign to that of the normal stress step.
Chapter 6

Future Work and Conclusions

The core scientific content of this thesis was to provide experimental constraints on the state evolution component of the widely used rate-state friction (RSF) laws. Through the scientific work contained in this thesis we have adopted a two pronged strategy to address this core issue – First, we develop an analytical understanding of the behavior of the given state evolution law across its parameter space and identify the portions which, if experimentally probed, would have the maximum diagnostic potential. Based on this analytical understanding, we use numerical inversion techniques to fit an experimental dataset that actually probes these relevant portions of the parameter space. This joint analytical and numerical approach has the advantage of not only informing us how well these state evolution laws perform in actually explaining the laboratory data but also gives us a detailed understanding of what are the aspects of the mathematical formulation of a given state evolution law that leads to its failures. Given that these mathematical features are tied to some physical picture of how state evolves, this approach lends us the ability to constrain the physics of state evolution using laboratory data. Since the connection between the physics of state evolution contained in a certain state evolution law and the resultant frictional behavior is through a set of coupled, non-linear, differential equations, the joint
analytical-numerical approach is crucial to verify whether the physics matches up to the data.

The most important consequence of this approach, contained mostly in the last two chapters of this thesis, is the realization that even at the lowest sliding rates accessed in conventional laboratory experiments, state seems to dominantly evolve with slip and not with time. The widely held expectation that state should evolve with time at low slip rates comes from two sources. First, the classical micro-mechanical picture of the origins of frictional strength based on the Bowden-Tabor decomposition of frictional strength inherently ties the state variable to contact area (Bowden and Tabor [1964]; Linker and Dieterich [1992]; Baumberger et al. [1999]; Baumberger and Caroli [2006]). Secondly, at the high stresses assumed to being exerted at the micrometric contacts, the actual contact area is expected to grow via plastic creep (Brechet and Estrin [1994]; Baumberger and Caroli [2006]) – an expectation corroborated by experimentally observed contact growth for stationery interfaces in the see-through experiments of Dieterich and Kilgore [1994]. Therefore, if state is contact area, it can be reasonably expected to grow with time even in the absence of slip. The finding that Aging law style time dependent healing, which is based on the premise of such time-dependent contact area creep, finds little experimental support even at the lowest slip rates accessed in the laboratory brings into question the validity of this physical picture for state evolution.

The normal stress step data in Chapter 5 further reinforces our belief that state is more than just contact area. In particular, we find that our experimental data requires an instantaneous change in state in response to a normal stress step such that state and contact area need to change in the opposite directions. In order to address this, we hypothesize that encoded in the RSF state variable is some measure of contact quality along with contact quantity, a view which is supported by observations of increase in contact strength with contact age in nano-scale asperities which undergo no demonstrable plastic deformation (Li et al. [2011]). We express this contact quality in terms of the average density of bonds
bridging a contact. While a state evolution equation for response to normal stress step derived on the basis of this physical picture explains the normal stress step data pretty well, the true test of our state evolution equation would comprise of larger normal stress perturbations than our 5% normal stress increases. These experiments would be a part of a larger study towards characterizing the proper fault constitutive response under variable normal stress.

Since none of the state evolution laws studied in this thesis seem to explain the full range of laboratory data satisfactorily, it is very likely that we do not know identity of the ‘proper’ state evolution law for explaining laboratory friction data. But based on the evidence above, the answer to this problem might lie in our ability to experimentally constrain the evolution of contact quality and quantity with time and slip. Fortunately, there have been recent developments in tracking contact quantity during friction experiments by using ultrasonic transmissivity across the sliding interface as proxy for real contact area which promise unprecedented experimental constraints on contact area evolution (Nagata et al., 2008, 2012, 2014). Simultaneous use of P- and S-wave amplitude variations could prove particularly useful in this regard since they induce displacements perpendicular and parallel, respectively, to the sliding direction thus providing more information concerning the state of the interface (Baik and Thompson, 1984; Pyrak-Nolte et al., 1990; Nagy, 1992). In addition to a different elastodynamic response of the interface to P and S waves, differences may arise in the amount of inelastic deformation (dissipation) during wave transmission (e.g., sliding at the contact points is geometrically permissible; interpenetration of those contacts is not). Therefore, besides providing constraints on contact area evolution, ultrasonic monitoring with P- and S-waves could provide information about the rheology of the multi-contact interface. We have active collaborations with the Penn State Rock mechanics laboratory where we are focussing on perfecting the best strategy for acquiring multi-component ultrasonic data to answer these questions. However, in experiments on see-through plastic where ultrasonic transmissivity was measured in conjunction with optically observed contact
Nagata et al. (2014) have observed that these two quantities do not always track each other. Therefore much work needs to be done on fully understanding the relationship between ultrasonic transmissivity and contact area.

If contact quality is truly the density of bonds on contacts, it seems challenging to evaluate changes in it with the passage of time and/or slip directly. However, with simultaneous measurements of friction and contact area (from a proxy like ultrasonic transmissivity), it might be possible to separate the effect of contact quality and quantity by tracking discrepancies between the inferred behavior of state and contact area. Such new experimental strategies might prove the crucial observational constraints for designing the ‘proper’ state evolution law for laboratory friction.

The major shortcoming of the RSF framework is that it is largely empirical. This has meant that the physical processes underlying state evolution can only be speculated upon but difficult to prove. As mentioned before, we have outlined how a first principles approach can be adopted to derive a state evolution law (for state evolution in response to variable normal stress) based on a clear physical description of contact rheology. Since the relevant parameters of such a state evolution law involves parameters with a clear physical interpretation, fitting laboratory data with such models actually allows us to infer properties of the contact interface. The ultimate goal in this regard is to do the same for state evolution with slip. The route to the ‘proper’ state evolution law for laboratory experiments is very likely through a better understanding of the micro-mechanics of the frictional interface. One hopes that new data such as ultrasonic transmissivity would provide the requisite observational constraints for this purpose.
Appendix A

Appendices to Chapter 2

A.1 Aging law - Slip law transition with slip

For values of $c$ that satisfy Eq. (2.17a) (given a particular value of $V_f/V_i > 1$) we can make the claim that, for large enough $\delta$, as $\ln(1 \pm x) = \pm x$ for small $x$, we have:

$$\lim_{V_f/V_i \gg 1} \Delta \tau \approx b \sigma \ln \left[ 1 + \Omega_i \left( \frac{V_f}{V_i} \right)^{1 - \frac{c}{c+1}} \exp \left( -\frac{\delta}{D_c(1+c)} \right) \right]$$

$$\approx b \sigma \Omega_i \left( \frac{V_f}{V_i} \right)^{1 - \frac{c}{c+1}} \exp \left( -\frac{\delta}{D_c(1+c)} \right), \quad \delta/D_c(c+1) \gg N,$$

(A.1a)

$$\lim_{V_f/V_i \ll 1} \Delta \tau \approx b \sigma \ln \left[ 1 - \exp \left( -\frac{\delta}{D_c(1+c)} \right) \right] \approx -b \sigma \exp \left( -\frac{\delta}{D_c(1+c)} \right).$$

(A.1b)

where $N$ is the cutoff slip distance for linear slip weakening given by Eq. (2.20). For the Slip law we have \cite{Ampuero and Rubin 2008}:

$$\Delta \tau(\delta) = b \sigma \left[ \ln \Omega_i + \ln \left( \frac{V_f}{V_i} \right) \right] \exp \left( -\frac{\delta}{D_c} \right),$$

(A.2)

which has a similar mathematical form to Eq. (A.1a). Between Eqs. eqrefeq19naac and (A.1a), we see that there is a transition from linear slip weakening to exponential decay
with increasing slip. From Figure 2.2, it could be argued that over the largest velocity step ups/downs achieved in the laboratory (of about two orders of magnitude), the Nagata law is nearly equivalent to the Slip law for $c \gtrsim 10$. But for jumps of $\gtrsim \pm 6$ orders of magnitude, the linear slip weakening to exponential slip weakening transitions happen at such values of slip that this would affect the fracture energy calculations. This highlights the fact that there are in fact two types of Aging law - Slip law transitions in the response of the Nagata law to step velocity increases. One results from increasing the value of $c$, the other with increasing amounts of slip. For a given value of $c$, we have a critical value of the velocity jump that would lead to a linear slip-weakening response for modest slips (as in Eq. (2.19)), but this would evolve to the exponential decay given by Eq. (A.1a) for larger slips. The critical jump size is itself an increasing function of $c$. But the critical slip distance at which the transition from linear slip weakening to exponential slip weakening takes place is a decreasing function of $c$. A lower bound for this transitional slip distance is given by Eq. (2.20). But for $c \gg 1$, the exponential decay or Slip law like response is observed over all slip distances for any size of the jump. In other words, in this limit, the critical jump size goes to infinity while the transitional slip distance tends to zero. The full range of this behaviour is shown in Figure A.1 where we choose a very large jump in velocity (representative of the maximum jump experienced by the tip of the nucleation zone during our simulations) and vary the value of $c$ to show these transitions.

### A.2 Existence of a global minimum of $\Omega$ in the no-healing limit

As outlined in Section 4.2.1, in order to maintain $\Omega \gg 1$ everywhere in the nucleation zone during nucleation, $\Omega$ needs to be a monotonically increasing function of time. In particular, given the expression for $\Omega(t, \xi)$ in Eq. (2.44), this requirement can be mathematically stated as follows:
We define

\[ F(\xi) = V(\xi) + \frac{c}{c + 1} \frac{L_b}{2L} \tilde{T}(\xi). \]  

(A.4)

If the global maximum of \( F(\xi) \) (within the nucleation zone i.e. \(|\xi| < 1\)) is at \( \xi_{\text{max}} \), then in order to satisfy Eq. (A.3) everywhere on the fault, one has to show that the following is true:

\[ F(\xi_{\text{max}}) \leq \frac{1}{C_{\Omega \gg 1}}. \]  

(A.5)

Therefore, we need to find \( \xi_{\text{max}} \) and use Eq. (A.5) to find the condition under which \( \Omega(\xi, t) \) is a monotonically increasing function everywhere in the nucleation zone. The condition for existence of a maximum of \( F(\xi) \) at \( \xi_{\text{max}} \) is

\[ \frac{\partial F(\xi)}{\partial (\xi)} \bigg|_{\xi_{\text{max}}} = 0, \]  

(A.6a)

\[ \frac{\partial^2 F(\xi)}{\partial (\xi)^2} \bigg|_{\xi_{\text{max}}} < 0. \]  

(A.6b)

Using Eq. (2.41) to eliminate \( \tilde{T}(\xi) \) from Eq. (A.4), one can write

\[ \frac{\partial F(\xi)}{\partial (\xi)} = \frac{1}{c + 1} \frac{\partial V(\xi)}{\partial \xi}, \]  

(A.7a)

\[ \frac{\partial^2 F(\xi)}{\partial (\xi)^2} = \frac{1}{c + 1} \frac{\partial^2 V(\xi)}{\partial \xi^2}. \]  

(A.7b)

Eqs. (A.7a) and (A.7b) indicate that any isolated maximum of \( F(\xi) \) is an isolated maximum of \( V(\xi) \). The global maximum of \( V(\xi) \) is at \( \xi = 0 \). In view of Eqs. (A.7a) and (A.7b), this means
\[ \xi_{\text{max}} = 0. \]  
\[ (A.8) \]

Therefore, from Eqs. (A.5) and (A.8), the condition under which \( \Omega(\xi, t) \) is a monotonically increasing function everywhere in the nucleation zone is

\[ F(0) \leq C_{\Omega \gg 1}. \]  
\[ (A.9) \]

### A.3 Appropriate approximations for the Nagata state evolution law

Because the Nagata state evolution law has 3 terms different approximations might be appropriate in different portions of the parameter space. To explain this, we start by rewriting Eq. (2.31) as:

\[ \dot{\theta} = \frac{1}{c + 1} \left[ 1 - \Omega \left( 1 + c \frac{a}{b} D_c \frac{\dot{V}}{V^2} \right) \right]. \]  
\[ (A.10) \]

To obtain the analytical approximations for the no-healing regime, we dropped the 1 inside the square brackets. As mentioned in the main text, \( \dot{V}/V^2 \) is a positive constant at the center of the ‘no-healing’ (or localized) nucleation patch, and the lowest value of \( \Omega \) occurs in the center of the nucleation zone. Therefore, the form of Eq. (A.10) shows that \( \Omega \gg 1 \) is a sufficient condition to drop the 1 within the square brackets. But for \( \Omega \gg 1 \) to be the necessary condition to drop that 1, the second term in the curly brackets needs to be of the order of unity. That this is true for modest values of \( c \) is clear from Figure A.2. As \( \Omega(0) \gg 1 \) for \( c \leq 10.0 \) (see Figures 2.4c and 2.8), \( \Omega(0)(1 + cd/aD_c b^{-1}\dot{V}/V^2) \) is therefore much larger than unity at the center of the patch.

In general, for \( c \leq 10 \), Eq. (2.48) gives us the condition for guaranteeing a localized nucleation zone of length \( 1.3774 L_b \) with the Nagata law. But increasing \( c \) to values much
higher than $\sim 10$ increasingly drives $\Omega(0)$ closer to 1 over geologically relevant slip rates. In particular, for $c \approx 100$, $\Omega(0) \sim 1$ and $caD_e b^{-1}V^2 / V \sim 10$ (Figures A.2 and A.3b) for slip rates up to $\sim 1 \text{ ms}^{-1}$. In this regime, it is no longer accurate to drop the 1 in the square brackets and the variable separable solution is no more applicable. For values of $c \approx 100$ (and $a/b$ values obeying Eq. (2.48)) we observe a shrinking, localized nucleation patch (Figures A.3a, b). This behavior is similar to Slip law nucleation for $(a/b)_{\text{Slip}} < 0.6 - 0.7$ ([Ampuero and Rubin, 2008]). To understand this behavior, we rewrite Eq. (2.6) as

$$\dot{\theta} = 1 - \Omega \left(1 + \frac{cD_e \tau}{b\sigma V}\right).$$

(A.11)

Recognizing $-\tau(0)/V(0)$ as a proxy for the stiffness $k_s(0)$ at the center of the nucleation patch we further simplify Eq. (A.11) as

$$\dot{\theta}(0) = 1 - \Omega(0) \left(1 - \frac{c}{c + 1} \frac{k_s(0)}{k_b}\right) \approx 1 - \Omega(0),$$

(A.12)

where $k_b$ is $\mu'/L_b$. The approximation on the right of Eq. (A.12) follows from Figure A.3d which shows that $k_s(0)/k_b \ll 1$ for slip rates up to $\sim 1 \text{ ms}^{-1}$. This approximation is only accurate (for $\Omega(0) \sim 1$ and $c \gg 1$) if

$$\Omega(0) - 1 \gg \frac{k_s(0)}{k_b}.$$

(A.13)

Our numerical simulations show that the left hand side of the inequality in Eq. (A.13) is about 3.5-8 times the right hand side for slip rates up to $\sim 1 \text{ ms}^{-1}$. For values of $\Omega(0) \approx 1$ we can rewrite Eq. (A.13) as

$$\dot{\theta}(0) \approx -\Omega(0) \ln[\Omega(0)];$$

(A.14)

i.e., $\dot{\theta}$ is well approximated by the Slip law. [Ampuero and Rubin, 2008] argued that the Slip law length scale of localization is $L_b/\ln[\Omega(0)]$. Figure A.3a shows that such a length scale of localization indeed seems appropriate for the Nagata law. Therefore there is a
Nagata law-Slip law correspondence for values of $a/b$ obeying Eq. (2.48). However, we must point out that the approximation in Eq. (??) only works well near the center of the nucleating patch and not at the tips (corresponding to the maxima in $\Omega$ in Figure A.3b). It is therefore possible that the estimate of $L_b/\ln[\Omega(0)]$ might not provide as good a fit to the numerical simulations given different initial conditions. We verified, however, that when the simulations were initiated with either a locally peaked load at the center of the patch or a spatially randomized $\Omega$ between 0 and 1, the length scale of localization was in close agreement with the $L_b/\ln[\Omega(0)]$ estimate (Figure A.3c). The agreement was even better for $c = 500$, where for slip speeds up to 1 m/s the maximum value of $\Omega$ at the peaks was only $\sim 1.12$.

An anonymous reviewer of this paper suggested the possibility of the logarithmic convergence of the nucleation zone width to $L = 1.3774L_b$ (Eq. (2.42a)) for large values of $c$ when the nucleation patch is driven to much higher (physically non-plausible) slip rates. When the localized nucleation patch for $c \approx 100$ is driven to $V(0) \sim 10^{80}$ ms$^{-1}$, our simulations show that $\Omega(0)$ goes up to $\sim 6$ (Figures A.2 and A.3b). This suggests that at such slip rates it is again appropriate to drop the 1 in the square brackets in Eq. (A.10) leading us again to the regime described in Section 4.2.1 where the localization length scale was indeed found to be $L = 1.3774L_b$. Additionally, Figure A.3d also suggests that $caDcb^{-1}\dot{V}(0)/V(0)^2$ logarithmically converges to a constant value $\gg 1$. In the limit of the localized variable separable slip rate profiles of Section 4.2.1, one can make use of Eq. (2.38) to estimate that this limiting value is

$$caDcb^{-1}\dot{V}(0)/V(0)^2 = c \left[ 1 + \frac{L_b}{2L} \dot{T}(0) \right] = c * 0.3781. \quad (A.15)$$

For $c = 100$ this value is in close agreement with our numerical results (Figure A.3d) and further reinforces the expectation that the shrinking nucleation patch does indeed converge logarithmically to the $1.3774L_b$ length scale for slip rates $> 10^{80}$ ms$^{-1}$. 

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Finally, we want to point out that Eq. eqrefeq48na assumes an Aging law like variable separable solution for the Nagata law localized nucleation patch. It guarantees that for any value of $a/b$ satisfying Eq. eqrefeq48na, nucleation ultimately localizes to the length scale $1.3774L_b$. However, Eq. eqrefeq48na might not have any relevance to the shrinking nucleation patch we see with the Slip law or with the Nagata law for $c \approx 100$ and physical slip rates. Therefore it does not simultaneously guarantee that localized solutions are not seen for values of $a/b$ modestly in excess of that given by Eq. eqrefeq48na. \cite{AmpueroRubin2008} reported (initial condition dependent) transitional values of $a/b$ separating localizing from propagating slip pulse nucleation styles for the Slip law to be in the range of $0.6 - 0.7$; conceivably these values are appropriate for the Nagata law for $c \gg 1$ as well.

### A.4 Effective stiffness at the center of the expanding crack like nucleation

To evaluate the stiffness at the center of the expanding crack, we start with the observation that $\Omega$ is quasi-constant there. This leads to

$$\frac{\dot{V}}{V} = -\frac{\dot{\theta}}{\theta}.$$  \hspace{1cm} (A.16)

Using this and the expression for $\dot{\theta}$ from Eq. (A.10), we can write:

$$\frac{\dot{V}}{V^2} = \frac{1 - \Omega^{-1}}{\left[1 - \frac{c}{c+1} \frac{g}{k}\right] D_c(c+1)}.$$ \hspace{1cm} (A.17)

Additionally, using Eq. (A.11) and the force balance in Eq. (2.28), we can write that

$$\frac{\dot{\tau}}{\sigma} = (a - b) \frac{\dot{V}}{V}.$$ \hspace{1cm} (A.18)
If one introduces an effective stiffness at the center of the crack, \( k_s(0) \), defined as \( \dot{\tau}(0) = -k_s(0)V(0) \), then using Eqs. (A.12) and (A.13), one can write

\[
k_s(0) = \frac{1 - \Omega^{-1}}{1 - \frac{c}{c+1} \frac{a}{b}} \left( 1 - \frac{a}{b} \right).
\]  
(A.19)

For a constant stress drop crack, we have the relationship

\[
\frac{\Delta \tau_0}{\delta} = \frac{\mu'}{2L}.
\]  
(A.20)

We take the time derivative of Eq. (A.15) to account for the increase in \( L \) with time and express the stiffness at the center of the expanding crack as

\[
k_s(0) = \frac{\Delta \tau_0(0)}{V(0)} = \frac{\mu'}{2L_\infty} \left[ \frac{L}{L_\infty} + \frac{\Delta \tau_0}{\Delta \tau_0} \frac{\dot{L}}{L_\infty} \right]^{-1}.
\]  
(A.21)

Following [Ampuero and Rubin (2008)], we can rewrite Eq. (A.16) as:

\[
k_s(0) = \frac{\mu'}{2L} \left[ 1 + \frac{\Delta \tau_0}{\Delta \tau_0} \frac{\dot{L}}{L} \right]^{-1}.
\]  
(A.22)

Further, using Eq. (2.57), we also have the following identity:

\[
\frac{\Delta \tau_0 \dot{L}}{\Delta \tau_0 L} = 2 \frac{\dot{\tau}_{p-r} \Delta \tau_0}{\Delta \tau_{p-r} \Delta \tau_0} - 2.
\]  
(A.23)

We can further use Eqs. (2.52), (2.55) and (2.59) to write the following expression for \( \dot{\tau}_{p-r}/\Delta \tau_0 \):

\[
\frac{\dot{\tau}_{p-r}}{\Delta \tau_0} = r \frac{b}{b - a} = \sqrt{\frac{\pi L_\infty}{L_b}},
\]  
(A.24)

where \( r = 1 - \frac{c}{c+1} \frac{a}{b} \). Substituting for \( \dot{\tau}_{p-r}/\Delta \tau_0 \) from Eq. (2.57) and using Eq. (A.19), we rewrite Eq. (A.18) as:
\[ \frac{\Delta \tau_0 \dot{L}}{\Delta \tau_0 L} = 2r \frac{b}{b-a} \frac{\Delta \tau_0}{\Delta \tau_{p-r}} - 2 = 2 \sqrt{\frac{L_\infty}{L}} - 2. \] (A.25)

Note that Eq. (A.20) is exactly analogous to equation (A4) in [Amuero and Rubin (2008)] where the authors derived the same quantity for the Aging law expanding crack-type nucleation. This leads to the same effective stiffness at the center of the Nagata law expanding crack as for the Aging law, given the generalized definition of \( L_\infty \) in Eq. (2.59):

\[ k^*_s(0) = \mu' \left( 2 \sqrt{\frac{L_\infty}{L}} - 1 \right)^{-1}. \] (A.26)

Eq. (A.21) can be further simplified to:

\[ \frac{k^*_s(0)}{\mu'/2L_\infty} = \Phi \left( \frac{L}{L_\infty} \right) = \left[ \frac{L_\infty}{L} \left( 2 \sqrt{\frac{L_\infty}{L}} - 1 \right) \right]^{-1}. \] (A.27)

If we express \( L = \Delta L + L_\infty \) where \( \Delta L/L_\infty \ll 1 \), then we have the following expression for \( \Phi \):

\[ \frac{k^*_s(0)}{\mu'/2L_\infty} = \Phi \left( \frac{L}{L_\infty} \right) = \left[ 1 - \left( 1 - \frac{L}{L_\infty} \right)^2 \right]^{-1}. \] (A.28)

This shows that \( \Phi \) has a broad minimum around \( L = L_\infty \) and, as long as the half length of the crack satisfies the condition \( |L - L_\infty| \ll L_\infty \), the effective stiffness at the center of the crack is \( \mu'/L_\infty \) to a good approximation. Eqs. (A.14) and (A.23) also lead to the following expression for \( \Omega(0) \):

\[ \Omega(0) = \left( 1 - \frac{1}{2} \sqrt{\frac{\pi L_p}{L_\infty}} \right)^{-1}. \] (A.29)

Figure [A.4b] shows that this estimate of \( \Omega(0) \) is close what is observed in the simulations for all values of \( a/b \).
A.5 Approximate analytical expression for the Nagata law slip pulse fracture energy

The style of nucleation for expanding nucleation zones is predominantly controlled by the velocity jumps, $V_{\text{max}}/V_{bg}$, experienced at the propagating front. For the Aging law, large values of the jump lead to linear slip weakening over all slip distances. This leads to expanding equilibrium cracks as the favored style of nucleation as both $G_c$ and $G$ increase as $\ln(V_{\text{max}}/V_{bg})^2$. For the Slip law, exponential slip weakening occurs for all jump sizes and slip distances. As a result, the nucleation zone approaches instability in the form of unilaterally travelling slip pulses as $G_c$ is a weaker function of $\ln(V_{\text{max}}/V_{bg})$ than would be $G$ for an expanding crack. We will now show that the Nagata law, for the largest values of $c$ used in the main text, behaves in between these two extremes.

In Section 4.3.2 we treated the energy balance for Nagata law slip pulses only in the case where $c \gg 1$, and hence assumed pure Slip law like response to a step jump in velocity of arbitrary size over all slip distances. In reality, even for the largest values of $c$ used here for expanding nucleation simulations ($c = 10.0$) this is not a valid assumption for velocity jumps of $\gtrsim \pm 6$ orders of magnitude (see Figure 2.2). For $c = 10.0$, the Nagata law stress vs. slip curves clearly show an initial linear slip weakening and subsequent exponential slip weakening for the largest velocity jumps encountered during nucleation (Figure A.1). As we approximate $G_c$ as the area under the curves in Figure A.1 these transitions significantly affect our calculations and make the related integrals particularly hard to solve for arbitrary $c$. We therefore resort to an asymptotic heuristic to estimate the fracture energy for any $c$. Additionally, Figure A.1 also clearly shows that using either the purely linear slip weakening or the purely exponential slip weakening extremes for calculating $G_c$ would lead to large errors. Figure 2.6 further suggests that this error is larger than that associated with the assumption of an instantaneous velocity jump at the edge of the expanding nucleation zone.
This makes an approximate analytical form of $G_c$, that is valid for arbitrary values of $c$ and $V_{\text{max}}/V_{\text{bg}}$, particularly useful.

In the purely linear slip weakening limit (for typical values of the velocity jump at the tip of the nucleation zone, $0 \leq c \lesssim 2$), the fracture energy $G_c$ scales as $\ln(V_{\text{max}}/V_{\text{bg}})^2$. For the purely exponential slip weakening limit, $G_c$ scales as $\ln(V_{\text{max}}/V_{\text{bg}})$. This suggests that to explain the simulations at any arbitrary value of $c$ and $V_{\text{max}}/V_{\text{bg}}$, the expression for $G_c$ for nearly velocity neutral values of $a - b$ can be written as:

$$G_c \approx C \frac{\ln(V_{\text{max}}/V_{\text{bg}})^2}{\alpha(V_{\text{max}}/V_{\text{bg}})}, \quad (A.30)$$

where $C$ is a combination of $a, b, c, D_c$ and $\sigma$ in keeping with Eqs. (2.53) and (2.64) and $\alpha(V_{\text{max}}/V_{\text{bg}})$ is a function of $V_{\text{max}}/V_{\text{bg}}$. The Aging law-Slip law transition in nucleation style dictates that $\alpha(V_{\text{max}}/V_{\text{bg}})$ be 2 for small values of $c$ and vary as $\ln(V_{\text{max}}/V_{\text{bg}})$ for large values of $c$. In other words, a proper choice of $\alpha(V_{\text{max}}/V_{\text{bg}})$ can be guessed by assuming Eq. (A.25) to switch between Eq. (2.53) for small $c$ and Eq. (2.64) for $c \gg 1$ for the largest jumps experienced by the nucleating patch. Using Eq. (2.21) as the tool for this switch in behaviour, we guess that $\alpha(V_{\text{max}}/V_{\text{bg}})$ must have the form:

$$\alpha(V_{\text{max}}/V_{\text{bg}}) \approx 2 \left[ 1 - \left( \frac{V_{\text{bg}}}{V_{\text{max}}} \right)^{4 \left( 1 - \frac{c - \frac{a}{b}}{c + \frac{a}{b}} \right)} \right], \quad (A.31)$$

which, given Eq. (2.22), has the right limits. This functional form of $\alpha(V_{\text{max}}/V_{\text{bg}})$ also accounts for the difference in the constant coefficients of the $\ln(V_{\text{max}}/V_{\text{bg}})$ terms between Eqs. (2.53) and (2.64). We can estimate $\alpha(V_{\text{max}}/V_{\text{bg}})$ for the range of velocity jumps encountered by the tip of the nucleation zone during the simulations and the inset of Figure A.5 shows that $\alpha(V_{\text{max}}/V_{\text{bg}})$ is a logarithmic function for the smallest jumps. This suggests exponential slip weakening for these jumps, a fact corroborated by the adjoining stress vs. slip plot in Figure A.5. Further, if the area under the curves in Figure A.1 is a good approximation for $G_c$, then Figure A.6 shows that $G_c$ as given by Eq. (A.12) (with
\( \alpha \left( \frac{V_{\text{max}}}{V_{bg}} \right) \) given by Eq. (A.26) is a good approximation for numerically calculated values of \( \int_{0}^{\infty} \Delta \tau(\delta) d\delta \) with \( \Delta \tau(\delta) \) given by Eq. (2.15).

### A.6 Analytical expression for \( G_{c}(\Omega_{bg} \neq 1) \) in terms of \( G_{c}(\Omega_{bg} = 1) \)

We start with the expression for \( \Delta \tau(\delta) \) from Eq. (2.15) and writing

\[
\Delta \tau(\delta) = b \sigma \ln \left[ 1 - \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\} \exp \left( -\frac{\delta}{D_c(1 + c)} \right) \right].
\]

Therefore for any value of \( \frac{V_{\text{max}}}{V_{bg}} \), \( \Omega_{bg} \) and \( r \), we have the following

\[
G_{c}' = \frac{G_{c}}{b \sigma D_c (c + 1)} = \int_{0}^{\infty} \ln \left[ 1 - \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\} \exp \left( -\frac{\delta'}{D_c(1 + c)} \right) \right] d\delta',
\]

where \( \delta' = \delta / D_c(c + 1) \). To solve this integral, let us differentiate \( G_{c} \) with respect to \( \Omega_{bg} \):

\[
\frac{\partial G_{c}'}{\partial \Omega_{bg}} = \int_{0}^{\infty} \frac{\left( \frac{V_{max}}{V_{bg}} \right)^r \exp(-\delta')}{1 - \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\} \exp(-\delta')} d\delta'.
\]

Some simple substitutions lead to the following:

\[
\frac{\partial G_{c}'}{\partial \Omega_{bg}} = - \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \ln \left\{ \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\} \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\}^{-1}.
\]
\[ G'(\Omega_{bg}) = G'(\Omega_{bg} = 1) + \int_{1}^{\Omega_{bg}} \frac{\partial G'_c}{\partial \Omega_{bg}'} d\Omega_{bg}'. \] (A.36)

Additionally, using the integral definition of the dilogarithm function,

\[ \text{Li}_2(z) = \int_{0}^{z} \frac{\ln(1 - z')}{z'} dz'. \]

some algebraic manipulation leads to

\[ \int_{1}^{\Omega_{bg}} \frac{\partial G'_c}{\partial \Omega_{bg}'} d\Omega_{bg}' = \text{Li}_2 \left\{ 1 - \left( \frac{V_{\text{max}}}{V_{bg}} \right)^{r} \right\} - \text{Li}_2 \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^{r} \right\} \] (A.37)

Finally, using Eqs. (6) and (7), one can write

\[ G'_c(\Omega_{bg}) = G'_c(\Omega_{bg} = 1) + \text{Li}_2 \left\{ 1 - \left( \frac{V_{\text{max}}}{V_{bg}} \right)^{r} \right\} - \text{Li}_2 \left\{ 1 - \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^{r} \right\} \] (A.38)

which holds for all values of \( \Omega_{bg} \) and \( V_{\text{max}}/V_{bg} \). The mathematical form of Eq. (A.33) makes it easy to use our approximate scaling form for \( G_c(\Omega_{bg} = 1) \) from Eq. (A.25) to obtain an approximate analytical form for \( G_c \) for any value of \( c, V_{\text{max}}/V_{bg} \) and \( \Omega_{bg} \). Figure A.7 shows the \( G_c \) vs. \( V_{\text{max}}/V_{bg} \) plot for different values of \( c \) and \( \Omega_{bg} \). The full numerical integral with \( \Delta \tau(\delta) \) from Eq. (2.15) agrees reasonably well with the scaled form from Eq. (A.12) corrected for \( \Omega_{bg} \neq 1 \) by using Eq. (A.33). Note that when \( \Omega_{bg} < 1, G_c < 0 \) for smaller jumps, implying that the expanding nucleation zone is not a physically valid solution when the region ahead of an aseismically propagating nucleation front is below steady state.

For \( V_{\text{max}}/V_{bg} \gg 1 \), we should have the Aging law approximation for the fracture energy. To show that this is the case we use the relations
\begin{align*}
\text{Li}_2 \left( \frac{1}{z} \right) &= -\text{Li}_2 (z) - \frac{1}{2} \ln^2 (-z) - \frac{\pi^2}{6}, \quad z \neq 0 \quad (A.39) \\
\text{Li}_2 \left( \frac{1}{z} \right) &= \sum_{k=1}^{\infty} \frac{1}{z^k k^2}, \quad |z| > 1. \quad (A.40)
\end{align*}

Both the above are valid on the complex plane outside the unit circle (|z| > 1). On the real line, these only hold for z < 0 and |z| > 1. Equations (10) and (11) also mean that for real |z| \gg 1 and z < 0, we have the identity

\begin{equation}
\text{Li}_2 (z) = -\frac{1}{2} \ln^2 (-z) - \frac{\pi^2}{6}, \quad z < 0, \ |z| \gg 1. \quad (A.41)
\end{equation}

Therefore in the linear slip weakening limit we have,

\begin{equation}
G'_c (\Omega_{bg}) = G'_c (\Omega_{bg} = 1) - \frac{1}{2} \ln^2 \left\{ \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\} + \frac{1}{2} \ln^2 \left\{ \left( \frac{\Omega_{bg} V_{\text{max}}}{V_{bg}} \right)^r \right\}. \quad (A.42)
\end{equation}

Also, we have, in the linear slip weakening limit,

\begin{equation}
G'_c (\Omega_{bg} = 1) = \frac{1}{2} \ln^2 \left( \frac{V_{\text{max}}}{V_{bg}} \right). \quad (A.43)
\end{equation}

Therefore, we have the result

\begin{equation}
G'_c (\Omega_{bg}) = \frac{1}{2} \ln^2 \left\{ \Omega_{bg} \left( \frac{V_{\text{max}}}{V_{bg}} \right)^r \right\}. \quad (A.44)
\end{equation}

which is exactly the linear slip weakening approximation. For r = 1, we have the exact Aging law fracture energy,

\begin{equation}
G'_c (\Omega_{bg}) = \frac{1}{2} \ln^2 \left( \frac{V_{\text{max}} \theta_{bg}}{D_c} \right). \quad (A.45)
\end{equation}
While deriving the fracture energy expression for the Slip law, we mentioned that Eq. (2.23) could only be derived for $\Omega_{bg} = 1$. Here we shall generalize the expression for $\Omega_{bg} \neq 1$. Before generalizing, however, it is useful to point out the reasons that make it necessary to use the strict condition $\Omega_{bg} = 1$ to obtain Eq. (2.23). One crucial step in deriving Eq. (2.23) involves using the condition $1 - (V_f/V_{bg})^{1-c/\pi} \ll 1$ (valid for $c \gg 1$) to rewrite the logarithm in Eq. (2.15) as a first degree approximation. But without assuming $\Omega_{bg} = 1$, we would have an expression of the form $1 - \Omega_{bg} (V_f/V_{bg})^{1-c/\pi} \ll 1$ within the curly brackets in Eq. (2.15) which, for $\Omega_{bg} = O(1)$, does not lead us to the simple, elegant form of Eq. (2.23). Importantly, the assumption of $\Omega_{bg} = 1$ in deriving Eq. (2.23), besides leading us to the simple Slip law type form, is also practically useful for understanding laboratory velocity step experiments. Additionally, as we shall show now, Eq. (2.23) is also a useful result for understanding nucleation simulation results even when strictly $\Omega_{bg} \neq 1$. This is because for the values of $r$ for which Eq. (2.22) and hence Eq. (2.64) are valid, the asymptotic form of the fracture energy $G_c$ is

$$G_c(\Omega_{bg}) \approx b\sigma D_c(1 + c) \left[ \left( 1 - \frac{c}{c + 1} \frac{a}{b} \right) \ln \left( \frac{V_{max}}{V_{bg}} \right) + \frac{1}{2} \ln^2(\Omega_{bg} - 1) + \frac{\pi^2}{6} + O \left( \frac{1}{\Omega_{bg} - 1} \right) \right],$$

$$|\Omega_{bg} - 1| > 1,$$

(A.46)

where we have implicitly assumed $\Omega_{bg} \gtrsim 1$ as in the simulations. This shows that $G_c(\Omega_{bg} \neq 1)$ is a weaker function of $\ln(V_{max})$ than the $\ln^2(V_{max})$ growth of $G$ for an expanding, uniform stress-drop crack. This generalizes the result obtained in Section 4.3.2 to the more general (and relevant to the simulations) case of $\Omega_{bg} \neq 1$ and shows that, as long as Eq. (2.22) is valid, Nagata law nucleation cannot proceed as an expanding crack.
Figure A.1 The different types of transition between linear and exponential slip weakening shown by the analytical stress evolution solution for the Nagata law velocity step response and its limiting approximations. We used $a - b = 0.0026$ and $V_{\text{max}}/V_{\text{bg}} = 10^{15}$. The scaling relations in Eq. (2.24) were used to obtain values for $a$, $b$ and $D_c$ for values of $c$ different from 2.0 – (A) For $c = 2.0$, the linear slip weakening approximation (equation (2.19)) is appropriate to calculate fracture energy; (B) for $c = 10.0$ neither the linear slip weakening nor the exponential weakening approximation (Eq. (2.23), $c \gg 1$) can fully account for the fracture energy. The decay of stress at large slips is well approximated by Eq. (A.1a). (C) For $c = 100.0$ the exponential weakening can account for most of the fracture energy; (D) for $c = 1000.0$ the exponential solution is essentially exact.
Figure A.2 Evolution of $cab^{-1}DcV(0)/V(0)^2$ with increasing slip speed at the center of the localized nucleation zone for $a = 0.05$, $Dc = 0.33 \mu m$, $a/b = 0.60$ for $c = 2.0$. The fault was initially everywhere below steady state with a locally peaked load. The scaling relations in Eqs. (2.24a), (2.24b) and (2.24c) were used to obtain the parameters for $c = 5.0, 10.0, 20.0$ and 100.0.
Figure A.3 Numerical simulation for $a - b = -0.033$ and $c = 100$. This value of $a - b$ corresponds to $a/b = 0.6$ (given $a = 0.05$) for the $c = 2.0$ case. For $c = 100$, the scaling relations in Eqs. (2.24) imply $a/b \approx 0.98$. (A) Slip rate and (B) $\Omega$ profiles when the initial condition was a locally peaked load at the center of the patch. (C) Slip rate profile when the initial condition was spatially randomized $\Omega$ between 0 and 1. The length scale bounded by the red, dashed, vertical lines in (A) is $1.3774L_b$ which is the Aging law type fixed length solution. The curves outlined by blue diamonds signify a length scale $1.3774L_b/\ln(\Omega)$ with $\Omega$ evaluated at the center of the nucleation zone. (D) shows plots of $caD_c b^{-1}\dot{V}(0)/V(0)^2$, $\Omega(0)$ and $k_\ast(0)/k_b$ with $V(0)$. Our approximation in Eq. (??) is supported by the fact that $k_\ast(0)/k_b \ll 1$ and $\Omega(0) \approx 1$ up to slip rates of $\approx 1 \text{ ms}^{-1}$. Eventual logarithmic convergence of the nucleation length scale to $\sim 1.3774L_b$ is suggested by $caD_c b^{-1}\dot{V}(0)/V(0)^2 \sim 40$ in accordance with Eq. (??) and a continually increasing $\Omega(0)$ as the nucleating patch is driven to slip rates of $\approx 10^{80} \text{ ms}^{-1}$. 
Figure A.4 Evolution of $\Omega(0)$ with $V_{\text{max}}$ on the fault for the expanding crack like solutions. The colors represent different values of $a/b$. The dashed lines represent the analytical approximation in Eq. (A.24). All simulations were carried out with $a = 0.05$ and $D_c = 0.33 \, \mu\text{m}$. The fault was initially everywhere below steady state with a locally peaked load.
Figure A.5 Normalized stress drop versus normalized slip from the Nagata law simulations for $c = 10.0$ and $a - b = -0.0026$. The values of $a$ and $D_c$ were determined assuming $a = 0.05$ and $D_c = 0.33 \, \mu m$ for $c = 2.0$ using the scaling relations in Eqs. (2.24). The decay in stress is neither purely exponential nor linear. There is a transition between the two types between jump sizes and with slip distance. Inset: we plot the value of $\alpha(V_{\text{max}}/V_{bg})/2$ over the values of the jump sizes encountered in the simulations.
Figure A.6 The scaling of fracture energy with $V_{max}/V_{bg}$ for the different values of $c$ used in the simulations and $c = 20.0$. We used $a - b = 0.0026$. The scaling relations in Eqs. (2.24) were used to obtain values for $a$, $b$ and $D_c$ for values of $c$ different from 2.0. Scaled form means $G_c$ is given by Eq. (A.25). For all other curves, the fracture energy is assumed to be $G_c = \int_0^\infty \Delta \tau(\delta)d\delta$. Full integral means Eq. (2.15), linear means Eq. (2.19) and exponential means Eq. (2.23) for $\Delta \tau(\delta)$. (A) For $c = 2.0$, the linear slip weakening approximation is appropriate to calculate fracture energy; (B) for $c = 5.0$ the linear approximation can no longer account for the fracture energy for the smaller jumps. For larger jumps, there is a consistent underestimation. (C) For $c = 10.0$ the exponential weakening can account for jumps up to 4 orders of magnitude, (D) For $c = 20.0$ we see a similar result as for $c = 10.0$. The scaled form always does an excellent job of matching the full integral implying that, if the area under the curves in Figure A.1 is a good approximation for $G_c$, then the scaled form in Eq. (A.25) is a good measure of $G_c$. 
Figure A.7 The scaling of fracture energy with $V_{\text{max}}/V_{bg}$ for the different values of $c$. We used $a-b = 0.0026$. All fracture energies are corrected for $\Omega_{bg} \neq 1$ according to Eq. (A.35). The scaling relations in Eqs. (2.24) were used to obtain values for $a$, $b$ and $D_c$ for values of $c$ different from 2.0. Scaled form means $G_c(\Omega_{bg} = 1)$ is given by Eq. (A.25). For all other curves, we use $G_c(\Omega_{bg} = 1) = \int_0^\infty \Delta \tau(\delta) d\delta$. Full integral means Eq. (2.15), linear means Eq. (2.19) and exponential means Eq. (2.23) for $\Delta \tau(\delta)$. (A), (B) and (C) show the $G_c$ vs. $V_{\text{max}}/V_{bg}$ curves with $\Omega_{bg} = 0.2$ for $c = 0, 10$ and 100. The full integral with $\Omega_{bg} \neq 1$ agrees reasonably well with the scaled form from Eq. (A.25) corrected for $\Omega_{bg} \neq 1$ by using Eq. (A.33). For smaller jumps, $G_c < 0$ implying that the expanding nucleation zone is not a physically valid solution when the region ahead of an early stage propagating front is below steady state. (D), (E) and (F) show the $G_c$ vs. $V_{\text{max}}/V_{bg}$ curves with $\Omega_{bg} = 5$ for $c = 0, 10$ and 100. Again, the full integral agrees well with the corrected scaled form from Eq. (A.25).
Appendix B

Inverting friction data: Adaptive, self-learning Monte Carlo in a correlated multidimensional space

Introduction
This section describes the details of the Markov Chain Monte Carlo (MCMC) algorithm used in the accompanying article. We describe below the non-standard modifications we needed to introduce to the classical Metropolis-Hastings approach to ensure fast converging and well-mixed chains. In addition, we also outline our philosophy and methodology for treating data-errors in our experiments, and how this information was used to construct a properly weighted likelihood function.

Details of our MCMC algorithm
The MCMC algorithm used here was based on the Metropolis-Hastings (MH) algorithm (Metropolis et al., 1953; Hastings, 1970) and makes use of the strategy of sampling the posterior using Bayes’ theorem. It is worthwhile to spell out here our philosophy behind attempting these detailed parameter estimations. We believe that the laboratory derived rate and state friction laws are likely inaccurate. We want to study how closely the best fitting
parameters for each of the state evolution laws explain the various experimental datasets. Our repeated experiments with the downhill simplex algorithm lead us to believe that the posteriors are multi-modal for some of these inverse problems. Therefore our central focus is on the existence and positions of these modes/mode. The most general approach to characterizing these modes, is to attempt to construct the full marginal posteriors for each parameter, a problem well-suited to Bayesian inference.

Before introducing the core structure of the MH variant used in this study, we need to list some background information pertaining and specific to the inversion of typical laboratory friction data. At the outset, it is important to recognize that the models used here do not allow for negative values of any of the parameters of interest for a realistic experimental dataset, with the possible but rare exception of $b$, which can be negative in some cases (Carpenter et al., 2011). This permits us to transform the parameter space to logarithmic scales, allowing us to restrict sampling only to the positive real axis. For setting up the sampling procedure, we make a few more simplifications. Firstly, we choose our priors to be uniform distributions in the logarithmic domain over many orders of magnitudes. The upper and lower bounds of this uniform distribution are not needed in practice due to their appearance in both the numerator and denominator terms in the acceptance ratio formulated from Bayes’ theorem. We further make the assumption that the priors of the different parameters are independent of each other. This is definitely not true in the case of the Nagata law. This choice is dictated by the fact that the covariances characterizing the joint prior are unknown to us. Finally, the time sampling of the stress measurements during our experiments is such that the number of data points recorded during sliding at a particular slip rate is a function of slip rate. Typically, slow sliding contributes many more data points than fast sliding. This a-priori bias is relaxed by using weights that are scaled according to the number of points within the particular phase of sliding such that all phases (each velocity increase and decrease) have equal contribution to the misfit. Further, these weights are scaled by numerical factors linearly proportional to $d\mu/d\delta$ ($= V^{-1}d\mu/dt$).
Within a particular phase of sliding, when \( V^{-1} d\mu/dt \) falls below a critical absolute value at a particular time, an exponential taper is applied for all subsequent times for as long as steady state sliding lasts (or until the end of the phase). This style of weight distribution is designed to promote better fits to non-steady state sliding and is typical of all the fits described in the text. Note that the steady state part of the data is independently used to constrain the steady state velocity dependence \( a - b \) to a fixed value in most of our inversions.

With this background set, we are now in a position to list the elements that are used to construct the MH acceptance ratio \( \alpha \) for rejection sampling in an \( n \)-dimensional parameter space given a \( N \)-tuple data vector:

1. Prior: In the logarithmic domain the prior is represented by a uniform distribution spanning many orders of magnitude about the initial guess. The initial guess comes from the Nelder-Mead inversions:

   \[
   P(x_a) = \prod_i \frac{1}{x_{\text{max}}^i - x_{\text{min}}^i},
   \]

   where \( x_{\text{max}}^i \) and \( x_{\text{min}}^i \) are the upper and lower bounds on the \( i^{th} \) parameter in the logarithmic domain.

2. Proposal: The proposal function is a center shifting, symmetric Gaussian about the present point in the Markov chain. Again, all parameters were deemed to be independent with regards to the proposal. The mathematical form of the distribution is:

   \[
   P(x_p|x_q) = \frac{1}{2\pi^{n/2}|\Sigma|^{-1}} \exp \left[ -\frac{1}{2}(x_p - x_q)^T \Sigma^{-1} (x_p - x_q) \right],
   \]

   where \( x_p \) is the proposed vector state and \( x_q \) is the last accepted one. For typical MH algorithms, the parameter covariance matrix \( \Sigma \) is assumed to be diagonal in the absence of a-priori knowledge of its full structure, i.e. \( \Sigma \sim \sigma I_n \) where \( \sigma \) is the vector
of (non-degenerate) standard deviations fixed by experimentation. For our typical Markov chains, especially with the strongly correlated parameters of the Nagata law, such diagonal parameter covariances led to many wasted samples and, therefore, to very slow mixing and convergence. To get around this problem, we ran long trial chains to ‘learn’ the proposal covariance matrix specific to a particular problem. In such a run, we initiated the chain with a diagonal parameter covariance but, beyond a large prescribed number of initial samples, calculated the proposal covariance from the parameter values accepted in the chain. For the remainder of the trial chain, we continued adaptively updating the proposal covariance on the run to end up with a relatively stationary estimate. This proposal was then scaled for optimal convergence following the prescriptions in Bai (2009a,b); Rosenthal (2011) and used to construct the posterior across various different chain initiations. In some cases, this approach led to more than an order of magnitude faster convergence.

3. Likelihood: The standard Gaussian likelihood was used under the assumption of point wise independence of the data set. Additionally, a scalar standard deviation, $S_e$, is used in the order of magnitude sense to represent data error. It is worthwhile to remind ourselves that the main source of the misfit in these models is not data uncertainty but model inadequacy. Given this, and the inherent lack of repeatability of laboratory friction experiments, we decided not to be more rigorous in characterizing data errors. However, even the value of $S_e$ is not known a-priori and, due to lack of many repeated friction experiments, is not trivial to estimate. One can, however, use the fact that steady state friction is assumed to be a constant frictional strength and treat consecutive measurements of the shear stress during steady state sliding as repeated measurements of the same physical state. Therefore any fluctuations in shear stress about this constant value can be used to construct a distribution of data errors (Figure B.1). This likely leads to a lower bound on data errors because typical repeated experiments on the same material can lead to much larger fluctuations in the
data values than the standard deviation of the resultant error distribution. Additionally, especially for gouge, there are often linear trends superposed on this constant steady state friction value which probably make our assertion regarding steady state and repeated measurements less tenable. We have checked that the distribution of errors about a linear trend is not very different from the distribution of errors about a constant steady state value. But we treat this estimate of the standard error strictly as a lower bound on the data error and typically use about 10-40 times this estimate as the value of $S_e$ in our inversions. We explain the rationale behind this scaling in greater detail in Appendix C. We typically run chains with different values of $S_e$ from this range, and we choose the minimum value above which the posterior structure does not change conspicuously.

Once we have fixed $S_e$, the likelihood can be calculated as:

$$P(d|x_p) = \frac{1}{\sqrt{2\pi} S_e} \exp \left[ -\frac{1}{2} \chi^2 \right], \quad (B.3)$$

where $\chi$ is the weighted RMSE scaled by $S_e$. With this formulation, the acceptance ratio is defined as

$$\alpha = \min \left[ 1, \frac{P(d|x_p)}{P(d|x_q)} \right] \quad (B.4)$$

When trying to construct the full posterior, one cannot rule out the possibility of multi-modality of the posterior, especially with the Nagata law which admits more than one local minimum.

Finally, it is worth mentioning that for well separated modes in the posterior, even an adaptive proposal MH algorithm might lead to a chain that is not well-mixed, e.g. the ‘learned’ parameter covariance might be deficient if the trial chains are not well mixed. One possible way to deal with this problem was suggested by Guan.
et al. (2006). They proposed a hybrid local-global proposal distribution which could populate well separated modes by using the form:

\[ Q(x_p|x_q) = (1 - p)P(x_p|x_q) + \frac{p}{|S|}, \]  

(B.5)

where the local proposal from \( P(x_p|x_q) \) is occasionally superseded by a wild, uniform distribution proposal over a given space of measure \( |S| \) with a small probability \( p \). Guan et al. (2006) prescribe an empirical estimate of the parameter \( p \) as

\[ p \approx \frac{1}{\sqrt{Mr}}, \]  

(B.6)

where \( M \) is the length of the Markov chain and \( r \) is the ratio of the area under the posterior contributed by the modes to the total area. This strategy has been observed to lead to rapid stabilization of the acceptance ratio of the chain for posteriors with well separated modes (Guan et al., 2006). Generally, the burn-in period (the initial portion of the chain not used for statistical inference) was decided by the amount of time it takes the code to saturate its acceptance rate to a stationary value. A typical converged Nagata law Markov Chain and its diagnostics are shown in Figure B.2.
Figure B.1 Estimate of standard error of the data from the steady state portions of the experimental data (dataset p1060). We assume that shear stress during steady state sliding following a velocity step should be constant. Therefore the consecutive measurements of shear stress during steady state sliding could be considered as repeated measurements of the same stress state. This means that the distribution of shear stress fluctuations about such a constant value could give us some estimate, most likely the lower bound, of the data error. The histogram of errors for the sections A, B, C and D of the data (labeled in the left hand panel showing the stress data) are plotted in parts (A), (B), (C) and (D). Maximum-likelihood estimates of Gaussian parameters for these error distributions are overlain on these histograms with the corresponding fits shown as red solid lines and their confidence intervals as red dashed lines.
Figure B.2 A typical converged Markov Chain for the Nagata law. This is the same chain shown in Figures 3 and 7 in the main text for dataset p1060. (A)–(C) show the posteriors for $a$, $D_c$ and $c$, (D)–(F) show the auto correlation function (ACF) along the chain for each parameter along with 5% confidence level shown as horizontal blue lines, (G)–(I) show the trace plots along the chain for each parameter. The ACF estimates show that the samples are mostly uncorrelated at the 5% level. The trace plots show that the sequence is well mixed above a lower bound in $c$. 
Appendix C

Appendices to Chapter 3

C.1 Estimating stiffness from data

In friction experiments, the effective stiffness of the experimental configuration depends upon the testing machine, the loading assembly, and the sample. Thus, the effective stiffness can vary depending on the material being used (Leeman et al., 2015). This makes it necessary to estimate the stiffness ($k$ in Eq. (3.11)) individually for each dataset, because stiffness may vary systematically with shear strain, sample damage, or other factors (Berthoud and Baumberger, 1998; Leeman et al., 2015). However, for our current work we neglect any such effect and treat $k$ as constant during our experiments. In order to estimate this constant stiffness, we note that immediately following a large load-point velocity increase one can write

$$\Delta \mu = k(\delta_{lp} - \delta) = k \delta_{lp} \left( 1 - \frac{\delta}{\delta_{lp}} \right) \approx k \delta_{lp}, \quad (C.1)$$

where $\delta$ and $\delta_{lp}$ are surface and load point displacements since the velocity increase respectively, and we assume that, instantaneously, $\delta / \delta_{lp} \ll 1$ for a sufficiently large velocity increase. A linear fit to the $\Delta \mu$ vs. $\delta_{lp}$ plot over the first few data points gives $k$ as the slope (Figure C.1). In Figure C.2 we show plots of fault vs. ram displacement for the largest
velocity steps for each of our datasets over the first few tens of microns of ram displacement.
This shows that there is clearly a time window over which \( \delta \ll \delta_{lp} \) (till about the ram
displaces around 30 microns); this is the time window over which the stiffness is estimated.

**C.2 The structure of the Nagata law posterior for p1060**

In section 3.4.2.1 and Figure 4.3, we have seen how the posterior distribution of \( c \) for p1060
shows a strongly peaked region and a quasi-uniform region. In this section we will try to
understand the nature of the fits that represent the two regimes. We already have some a
priori notion of the structure of the posterior by exploring the parameter space with the
downhill simplex algorithm over orders of magnitude variations in \( c \). To carry out this
analysis we typically fix \( a - b \) and \( c \) and vary \( a \) and \( D_c \) with the downhill simplex to obtain
the best Nagata law fit for a given value of \( c \). Two main features were observed: 1) For
\( c \sim 10 - 50 \) the Nagata law fits can be tuned to look slightly better than the best Slip law fit.
2) For \( c \gtrsim 100 \), the Nagata law parameters can be tuned to exactly reproduce the best Slip
law fit. Therefore it is reasonable to expect that the posterior of \( c \) is likely unbounded along
\( c \) with each value of \( c \gtrsim 10 \) corresponding to a fit with varying levels of support from the
data.

In order to quantify the visual qualities of the two regimes of the Nagata law fits from
a Bayesian perspective we first need to specify a value of \( S_e \) in the framework described in
Appendix B. We choose the maximum of the standard deviations obtained by analyzing the
fluctuations in shear stress about steady state frictional strength (Figure S1) therein. This
value, \( S_e = 0.00045 \), is a lower bound on the data error. Running the Markov chain at this
level data error leads to tightly bounded Gaussian posteriors for \( a \), \( D_c \) and \( c \) in Figure C.3.
The parameter values from this chain all lead to fits which ‘look’ better than the best Slip
law fit. The smallest value of the \( S_e \) at which the Nagata posteriors started showing the
unbounded structures was around 0.004, i.e. about 10 times the value obtained from the
error analysis of the steady state portions of the shear stress curves. It is noteworthy that
the gross features of the posterior stayed persistent even when we used a value of $S_e$ a few
times this value. We used this value of $S_e$ for the Markov chain showed in Figure 4.3.
At this value, the Markov chain shows a strongly peaked region and a quasi-uniform tail
for the posteriors. Given that the $a$, $D_c$ and $c$ values representing the peak are coincident
with the means of the Gaussians in Figure C.3(A)–(C), we identify this peaked region as
representing the global minimum of the Nagata law RMSE. These belong to the family of
Nagata law fits to p1060 which ‘look’ slightly better than the Slip law.

The next task is to describe the family of Nagata law fits to p1060 which are exactly
identical to the best Slip law fit. The easiest way to do this was to fit the best Slip law fit
to p1060 with the Nagata law. In other words, we generated the shear stress profile for the
best Slip law fit to p1060 and added Gaussian white noise with standard deviation 0.00045.
This ‘synthetic’ dataset was then analyzed using Markov chains with $S_e = 0.00045$. This
clearly reveals a quasi-uniform distribution for $c \gtrsim 100$ (Figure C.3(D)–(F)). This shows
that the quasi-uniform tail of the Markov chain in Figure 4.3 represents the Nagata law fits
which are exactly identical to the best Slip law fit to p1060.

Before concluding this appendix it is important to recognize that it is our choice of $S_e$
that allows the unbounded structure of the Nagata law posteriors in Figure 4.3 to be revealed.
Choosing $S_e = 0.00045$ would have led to well bounded Gaussian posteriors. One reason
to choose larger values of $S_e$ is that the friction values across small numbers of repeated
experiments typically show much larger fluctuations than those characterized by our analysis
of the steady-state portions, with $S_e$ easily up to an order of magnitude larger for common
granular materials (see Figures 4 and 5 in Rathbun and Marone (2013)). Additionally, the
choice of a scalar standard error is anyways a gross over-simplification as consecutive data
points sampled under non-steady sliding are strongly correlated. Most importantly, the
magnitude of modeling errors are much larger than the random fluctuation of data under
steady state sliding. In other words, we assume that these friction constitutive equations are
fundamentally deficient in describing the detailed features of laboratory experiments and, therefore, modeling error is the dominant source of misfit. In this sense, the estimate of $S_e$ from the analysis of steady state sliding also leads to a lower bound on the estimate of modeling error as the model also predicts constant steady state frictional strength. Then assuming data error to be negligible when compared to modeling error, one should naturally choose larger values of $S_e$ for poorly fit data. But as a practical strategy, we ran a number of chains with $S_e$ between 10-40 times the value estimated from steady state sliding (perhaps by chance, this is the range typically represented by the variations in frictional strength across the repeated experiments of Rathbun and Marone (2013)). We then chose the minimum of these trial values of $S_e$ such that the posteriors did not change their gross shapes even when higher values from this range were used as $S_e$. These values of $S_e$ always led to posteriors which were consistent with the features of the model parameter space revealed by probing the distribution of local minima over many orders of magnitude variations in $c$ using the downhill simplex. For the Markov chains corresponding to p1169 (Figure 4.7(D)–(F)), the estimate of $S_e$ from the steady state analysis was around 0.00025 while the value used was 0.008. Similarly for p1180, the steady state estimate of $S_e$ was $\sim 0.0003$ while we used the value 0.003 to construct the Markov chains in Figure 4.7(G)–(I). It is interesting that this pragmatic approach also leads to the largest $S_e$ being chosen for the worst fit dataset, p1169.

C.3 Does the minimum bound on $c$ for producing Slip law like fits increase with the size of velocity steps?

As described in sections 3.4.2.1, 3.4.2.3 our inversions of the laboratory velocity step data show that the lower bound on the value of $c$, above which the Nagata law can replicate the best Slip law fit to each dataset, shows no correlation with the size of the largest velocity step being fit. In contrast, our theoretical results predict that this lower bound should increase with the size of the velocity step being fit (see section 3.3). This disagreement between
theory and inversion could be due to the fact that the experiments represent velocity steps only approximately and, hence, the analytical results might not be relevant. On the other hand, the expectation that a positive correlation between the minimum bound on $c$ (for Slip law like fits) and the size of the velocity steps being fit would be apparent across the experimental runs implicitly assumes that these datasets differ only in the size of velocity steps they contain. But there do exist other differences, e.g. the Slip law parameters obtained from the three experimental runs do not agree with each other, particularly p1169 leading to estimates of $D_c$ almost twice as large as the estimates from p1060 and p1180. Therefore, it might be possible that even though our analytical expectations are relevant to the approximate experimental velocity steps, the predicted correlations across these datasets might be suppressed by the inherent differences between them. In the rest of this section we show that this is indeed the case.

If the analytical expectations from ideal velocity steps indeed apply to laboratory data, comparing the lower bounds on $c$ (for Slip law like fits) across different subsets of velocity steps from the same experimental run should reveal larger lower bounds for larger velocity steps. To show this, we ran the Nagata law MCMC posterior search on only the 1 order steps from each of the three datasets. Comparing these posteriors to those derived from the corresponding whole datasets, we observed that for all three experimental runs, the lower bounds on $c$ (to produce fits identical to the Slip law) were smaller for the 1 order of magnitude steps ($c \sim 1$ for p1060, $c \lesssim 7$ for p1169 and $c \sim 10$ for p1180) than for the whole datasets (Figures C.4(C), (F) and (I)). Note that in Appendix C.2 we have established that the quasi-uniform region of the posterior for $c$ represents Nagata law fits identical to the best Slip law fits to the respective datasets, and the stated lower bounds represent the onset of quasi-uniformity. These results show that, for given velocity step data, the lower bound on $c$ for producing Nagata law fits identical to the best fitting Slip law indeed increases with the size of the largest step being fit, a conclusion that agrees with our analytical prediction.
Note, for p1169, the posterior for $c$ obtained by fitting only 1 order steps shows a weakly peaked region while those for $a$ and $D_c$ show very clear, well-constrained, peaked regions (Figures C.4D–(F)). The lower bound of the weakly peaked region in the posterior for $c$ goes to values of $c \ll 1$. This regime of fits is equivalent to the best Aging law fit to the data. In fact, for all $c \ll 1$, the Nagata law reproduces the best Aging law fit to the data for a particular combination of $a$ and $D_c$ which show up as the peaks in the respective posteriors. Therefore, there is no strict lower bound on $c$ when one exclusively fits the 1 order steps in p1169 (at our specified level of data error). In this case, we imposed an artificial lower bound in our inversions through the choice of the prior.

### C.4 Problems with estimating stiffness as a free parameter in inversions

As pointed out before, the stiffness ($k$) is not treated as an independent parameter in our Markov chain runs. As described in Appendix C.1, we used the loading portion for the largest velocity increases to estimate this stiffness. However, [Noda and Shimamoto (2009)] have argued in favor of explicitly inverting for stiffness while fitting laboratory friction data with rate-state equations. In particular, [Noda and Shimamoto (2009)] argue that estimating the stiffness separately from the inversion procedure is risky when the amount of slip prior to peak stress following a velocity step increase (or prior to the stress minimum for a step decrease) is comparable to or larger than $D_c$. For such compliant systems, they suggest that stiffness should be treated as an independent parameter to avoid introducing errors to the estimates of the other rate-state parameters through correlations. Instead, we show here that it is the joint inversion for stiffness and other rate-state parameters that is generally affected by trade-off due to inter-parameter correlations.

The trade-off between stiffness and the other rate-state parameters can be understood in terms of the non-dimensional equations representing rate-state friction. The system stiffness
is relevant only in the following dimensionless recasting of the force-balance in Eq. (3.11):

\[ K(V_{lp} - V) = \frac{dV}{dT} + \frac{b}{a} \frac{d\theta}{dT}, \]  

(C.2)

where dimensionless stiffness \( K = kD_c/a \) and dimensionless \( T = V_{lp}t/D_c \). Therefore, stiffness has an inherent trade-off with \( a \) and \( D_c \) which can be utilized to improve certain aspects of a poor quality fit.

To show this, we compare the posteriors for the Aging (a poor quality fit) and Slip law fits (an excellent fit) to p1060 with the corresponding posteriors derived when stiffness is allowed to vary as a free parameter in the Markov chains. To invert for stiffness one could choose to start with some a-priori value and some width of variation. From the Bayesian perspective, this is akin to starting the Markov chain with a Gaussian prior for stiffness. Instead, to remove any a priori bias, we construct the chain with an unbounded uniform prior i.e. start without any prior knowledge of stiffness. In our inversions, we assume that stiffness does not vary with slip, damage etc. and assume that a constant value of stiffness is sufficient to describe the data. Leeman et al. (2015) have recently explored this issue of evolution of measured apparatus stiffness with increasing shear displacement across a series of stick-slip events on baked flour. They found that, following an initial period of strain localization, the measured apparatus stiffness tends to saturate to a quasi-constant level. Given that our experiments were carried out only after steady-state deformation had been reached, the assumption of constant stiffness seems reasonable. With this setup, we explore the range of values of stiffness for which a fit to the data is accepted at the specified data error level.

In Figures C.5 and C.6, the orange posteriors are for chains where stiffness was fixed at the value obtained from fitting the initial portions of the velocity steps, the green posteriors were obtained by inverting jointly for the stiffness and the other rate-state parameters. For both the Aging and Slip laws, the stiffness comes out to be very well constrained values.
For the worse fit to the data, the Aging law, the stiffness changes by about 15% from the fixed value used for the 2 parameter inversions and this change is translated into modest variations in $a$ and $D_c$. But from Figure C.7 it appears that these modest variations lead to nearly the same Aging law fit to the data as obtained when stiffness was fixed at the value obtained from the loading portions of the 1 and 2 order steps. However, these slight differences between the two Aging law fits are more marked for the larger step. Also, in parts (B) and (C), you’ll see that this 15% difference in stiffness does a visibly worse job of explaining the stress vs. load point displacement data immediately after the velocity step increases, more so for the largest velocity step. Therefore, at this level of data error, this 15% larger stiffness can be rejected based on the initial loading portion. This indicates that the optimum strategy for inverting for stiffness is to weight the initial loading following the velocity step increases more irrespective of how well the particular choice of the state-evolution law works. The extreme example of such a weighting scheme is to fit the initial linear loading to find the stiffness which is the strategy that we adopt for the fits with the stiffness fixed a priori. But this cannot be the optimum strategy for estimating the rest of the rate-state parameters since these are best constrained by other portions of the data. This is what makes the joint inversion of stiffness and rate-state parameters risky if the numerical fit to the data is poor.

For the Slip law, which fits the data very well, the stiffness estimated without any prior knowledge or bounds is within 0.5% difference of the stiffness obtained from the initial loading for velocity step increases and shows an approximately 5% variation. The posteriors of other rate-state parameters also seem quite insensitive to the treatment of stiffness as a free parameter.

Therefore, in principle, one could invert for stiffness as a free parameter only when the data is well fit by the numerical model. Even then, the primary check of the health of the fit should be with an a-priori fixed stiffness. But if the joint inversion of stiffness is a requirement, a Bayesian approach is favorable since it naturally predicts the range of
variations in this ‘free’ stiffness permitted by a specified level of data error. So the user can determine if such variations are physical or not given the initial loading phase following the velocity step increase, a post-facto check of the validity of such an inversion.

Figure C.1 Stiffness estimation for the different datasets. The initial portion of the excursion of $\Delta \mu$ following a large velocity increase or reslide following a hold (if performed during the same experimental run) is used to estimate the stiffness. The slope of a linear fit to the $\Delta \mu$ vs. $\delta_{lp}$ plot yields the stiffness if $V_f / V_i \gg 1$, with $V$ here referring to the load point velocity.
Figure C.2 Fault displacement vs. ram displacement for the largest velocity increases in p1060, p1169 and p1180. When compared to Figure A1, it is clear that over the range load point displacements used to estimate the stiffness $k$, the fault slips minimally with respect to the ram.
Figure C.3 (A)–(C) The Nagata law posteriors for $a$, $D_c$ and $c$ for dataset p1060 with $S_e = 0.00045$ (the maximum standard deviation from the steady state error distributions constructed in Figure B.1 in the Appendix B). The posteriors are well bounded Gaussians as shown in the inset by overlaying maximum likelihood estimates and the corresponding 5% error bounds as red solid and dashed lines respectively. The mean of these Gaussians coincide exactly with the parameter values representing the global minimum of the Nagata law RMSE for the chain in Figure 4.3 which was run with $S_e = 0.004$. This helps us identify the strongly peaked regions in the posteriors in Figure 4.3 with these bounded Gaussians whose precise structure emerges at this stricter constraint on data error. (D)–(F) The Nagata law posteriors for $a$, $D_c$ and $c$ for fits to the best Slip law fit to dataset p1060 at the same standard error as parts (A)–(C). The posteriors reveal a quasi-uniform distribution with a well defined lower bound on $c$. This lets us recognize the quasi-uniform tails in the posteriors in Figure 4.3 as Nagata law fits which are identical to the best Slip law fit.
Figure C.4 The Nagata law posteriors for $a$, $D_c$, $c$ for datasets (A)–(C) p1060, (D)–(F) p1169 and (G)–(I) p1180. The posterior distributions in blue in the background are the same as those in Figure 4.7 i.e. they represent the MCMC samples drawn for the whole dataset. The semi-transparent red posteriors in the foreground represent fits to only the 1 order of magnitude steps in the respective datasets at the same level of standard error as the blue posteriors. It is clear that the lower bound on $c$ for producing Nagata law fits at least as good as the best Slip law fit to the 1 order of magnitude steps only is smaller than the corresponding bound for the fits which also include the 2 order steps. Additionally, the lower bound on $c$ to produce Nagata law fits identical to the best Slip law fits is also smaller when fitting only 1 order of magnitude steps. This lower bound on $c$ is marked approximately by the onset of the quasi-uniform region of the posterior distribution of $c$. 
Figure C.5 Comparing the Aging law posteriors for $a$ and $D_c$ ($a - b = -0.0002$ fixed) between fits with the stiffness constrained at $k = 0.0011 \mu m^{-1}$ ((A) and (B), ochre posteriors) and allowed to vary as a free parameter ((C), (D) and (E), green posteriors). The constrained stiffness value was obtained from fitting the initial portions of the velocity steps. Note in (E) how the fixed stiffness value is well separated from the stiffness posterior obtained by running the Markov chain at this level of data error; the same as for the fits in the main text. Also note the feedback between $a, D_c$ and $k$ in the bottom panel as evidenced by the fit requiring a larger $a$ to accommodate the larger $D_c$ and $k$. 
Figure C.6 Same as Figure C.5, but for the Slip law. Note that the constrained stiffness value is nearly equal to the mean of the quasi-Gaussian posterior derived by treating stiffness as a free parameter in the inversions (in (E)). The two sets of posteriors derived for $a$ and $D_c$ are identical for all practical purposes.
Figure C.7 (A) The Aging law fits to the data corresponding to the mode of the posteriors in Figure C.5: ochre - fixed stiffness, green - varying stiffness. The loading portions of (B) the one order and (C) the two order velocity steps from p1060. The dashed lines show the reference slopes corresponding to $k = 0.0011 \, \mu\text{ms}^{-1}$ (pink) and $k = 0.00126 \, \mu\text{ms}^{-1}$ (green), the solid green lines show the fit. Note that the fit to the larger velocity step is more sensitive to the 15% difference between the two stiffnesses.
Appendix D

Appendices to Chapter 4

D.1 Evolution of Aging law state between beginning of reslide and peak stress

In Section 2.1.1 we pointed out that the relationship between $\Delta \mu_{\text{peak}}$ and $\theta_{\text{hold}}$ represented by Eq. (4.6) requires the assumption that the magnitude of $\Delta \ln(\theta)$ is independent of hold duration. Mathematically, this assumption implies:

$$\frac{\partial \Delta \ln \theta}{\partial \ln t_{\text{hold}}} = 0,$$

(D.1)

where $\theta = \frac{V_s}{\tau} \frac{\theta}{D_c}$ is defined as dimensionless state and $\Delta \ln \theta$ is the change in the logarithm of dimensionless state between the beginning of the reslide and peak stress. We will show in this section that Eq. (D.1) is approximately true for the Aging law under fairly non-restrictive conditions.

To begin we note that, during the initial portion of the reslide after a long hold, the sliding velocity increases and state continues to increase under the Aging law until $V\theta/D_c = 1$. At $V\theta/D_c = 1$, state reaches its peak value and begins to decrease as $V\theta/D_c$ increases further. For any net decrease in state to occur between the end of the hold and peak stress upon
reslide, the decrease in state that commences with $V\theta/D_c > 1$ must first erase the increase in state during the initial portions of the reslide. Therefore, only those portions of the reslide that satisfy $V\theta/D_c > 1$ (in the Aging law formulation) can contribute to $\Delta \ln(\vartheta) \leq 0$. Our principal assumption is that most of this net decrease in $\ln \vartheta$ occurs when the interface is sliding far above steady-state, i.e. $\dot{\vartheta} \approx -V\theta/D_c$.

Under these conditions, the non-dimensional form for the evolution of $\ln \vartheta$ is

$$\frac{d \ln(\vartheta)}{dT} = -\mathcal{V}, \quad (D.2)$$

where $T = tV_{s/r}/D_c$ and $\mathcal{V} = V/V_{s/r}$. Combining Eq. (D.2) with the time derivative of Eq. (4.10) in its non-dimensional form leads to the following non-linear ODE for $\mathcal{V}$:

$$\dot{\mathcal{V}} - K\mathcal{V} = (\beta - K)\mathcal{V}^2, \quad (D.3)$$

where $\dot{\cdot}$ signifies derivatives with respect to $T$, $K = kD_c/a$ and $\beta = b/a$. This is a Bernoulli non-linear ODE which can be solved by standard methods. The solution is

$$\mathcal{V} = \left[ (1 - \beta/K) - C \exp(-KT) \right]^{-1}, \quad (D.4)$$

where the constant of integration $C$ is of the form:

$$C = \mathcal{V}_i^{-1} + (\beta/K - 1), \quad (D.5)$$

with $\mathcal{V}_i$ being the initial value of $\mathcal{V}$. We argue later that the precise functional form of $\mathcal{V}_i$ is not important in evaluating $\Delta \ln(\vartheta)$ as long as we can assume certain properties for it.

Using Eqs. (D.2) and (D.5), we can derive the following expression for $\Delta \ln(\vartheta)$
\[
\Delta \ln(\vartheta) = - \int_0^{T_{\text{peak}}} \frac{\mathcal{V}_i \exp(KT)}{(1 - \beta/K)\mathcal{V}_i[\exp(KT) - 1] + 1} \, dT,
\]

\[
= - \frac{1}{K - \beta} \ln\{(1 - \beta/K)\mathcal{V}_i[\exp(KT_{\text{peak}}) - 1] + 1\},
\]

(D.6)

where \(T_{\text{peak}}\) is non-dimensional time at peak stress. Noting that \(\mathcal{V} = 1\) at \(T_{\text{peak}}\), we can use Eq. (D.4) to derive the following expression for \(\exp(KT_{\text{peak}})\):

\[
\exp(KT_{\text{peak}}) - 1 = \frac{K}{\beta} \left[\mathcal{V}_i^{-1} - 1\right].
\]

(D.7)

If we assume that \(\mathcal{V}_i \ll 1\) (i.e. \(V\theta/D_c \gg 1\) first occurs at slip rates orders of magnitude below the load point velocity), then using Eqs. (D.6) and (D.7) we have:

\[
\Delta \ln(\vartheta) = \frac{\ln(K/\beta)}{\beta - K}.
\]

(D.8)

This equation represents the formal proof of the conjecture in Eq. (D.1) under the assumptions stated above. Note that \(\Delta \ln(\vartheta)\) as expressed by Eq. (D.8) is always negative. Also, for \(\beta \to K\), the expression for \(\Delta \ln(\vartheta)\) simplifies to:

\[
\lim_{\beta \to K} \Delta \ln(\vartheta) = -K^{-1} \approx -\beta^{-1}.
\]

(D.9)

In order to show that our analytical results can indeed explain our numerical results, we use our numerical estimates of \(\theta_{\text{hold}}\) and \(\theta_{\text{peak}}\) from the set of simulations in Figure 4.3. Figure 4.1 shows that, when the analytical estimate of the change in state between the end of the hold and peak stress following reslide (Eq. (D.8)) is subtracted from the numerically predicted curves for \(\theta_{\text{peak}}\), the curves for \(\theta_{\text{hold}}\) (solid lines) and the adjusted \(\theta_{\text{peak}}\) (dashed lines) line up with each other for sufficiently long holds. The approximate analytical solution in Eq. (D.8) begins to agree with numerical estimates of \(\Delta \ln(V_{s/r}\theta/D_c)\) at smaller hold times for the stiffer apparatus than for the lower stiffness case. For comparison, we
show the corresponding numerical results for the Slip law in Figure D.2. With the Slip law, the simulations with the lower stiffness spring clearly show that $\Delta \ln (V_s/r \theta / D_c)$ is not independent of hold time, i.e. Eq. (D.1) is not generally true for the Slip law even at long hold times. Therefore, while the rate of peak stress evolution with respect to $t_{\text{hold}}$ can be used as a proxy for the rate of log state evolution at the end of the hold for the Aging law, the same is not true for the Slip law.

It is also important to note that the theoretically predicted hold duration independence of $\Delta \ln (\theta)$ under the Aging law is a direct consequence of its linear slip weakening behavior far above steady state (Eq. (D.2)). As pointed out in Section 2.1.1, linear slip weakening has zero support from large velocity step increase experiments on initially bare rock and simulated gouge. Therefore, the property of the Aging law that allows us to use simple analytical expectations like Eq. (4.6) to interpret SHS data seem unlikely to withstand experimental scrutiny.

**D.2 Is state evolution between the beginning of the reslide and peak stress independent of hold duration?**

In Section 2.1.2, we argued that the stiffness independence of $\partial \Delta \mu_{\text{peak}} / \partial t_{\text{hold}}$ observed in the data of Beeler et al. (1994) could be unambiguously tied to (Aging law style) time-dependent healing during the holds only if either of the following two conditions were satisfied: 1) The magnitude of state evolution between the onset of the reslide and peak stress (denoted by $\Delta \ln (\theta)$ from hereon) is a small fraction of the state accrued during the hold, 2) If, instead, $\Delta \ln (\theta)$ is non-negligible, then it must be nearly independent of hold duration for long holds.

In Appendix D.1, we showed that assumption (2) is theoretically true for the Aging law under fairly general circumstances. In this appendix, we use continuous measurements of normal displacement as a proxy for (log) state to qualitatively ascertain whether these
assumptions were likely to be satisfied during the slide-hold-slide experiments of Beeler et al. (1994).

Fault normal displacement (closure) measurements, if relatively uncontaminated by bulk strains, are expected to track deformations occurring within the frictional interface (Beeler and Tullis, 1997; Nagata et al., 2014). Imagining these interface deformations as due to the ploughing of an anelastic substrate by rigid asperities, Beeler and Tullis (1997) suggested that closure scales linearly with contact area (e.g., for asperities with spherical tips, contact area = \( \pi \times \text{tip radius} \times \text{closure} \)) and hence log state. In the granite experiments of Beeler et al. (1994), closure (expressed as positive for compaction here) is observed to increase during velocity decreases and holds and decrease across reslides or velocity increases mimicking the expected behavior of state (Beeler and Tullis, 1997) and Figure D.3a-b). This lends some credence to the proposed closure-log state relationship.

In Figure D.3c, we plot the changes in closure, \( \Delta \delta_n \), during holds and across reslides from the granite experiments. During holds, the closure data shows stiffness independent linear growth with log hold time similar to the observed stiffness-independence of the peak stress evolution. Beeler and Tullis (1997) noted that this was consistent with the expected behavior of state under Aging law style time-dependent healing.

Figure D.3c further shows data for the change in closure between the beginning of the reslide and peak stress. It is clear that this change in closure is a considerable fraction of the increase in closure during the preceding holds. Additionally, Figure D.3d shows that this change in closure is accompanied by at least a couple microns of slip independent of stiffness which is clearly significant relative to \( D_c \) of a few microns. Therefore, it seems unlikely that \( \Delta \ln(\vartheta) \) is a negligible fraction of the increase in log state during the preceding holds.

Finally, Figure D.3c also shows that the change in closure across the reslide is a function of hold duration, this is inconsistent with the predictions of Aging law state evolution if closure is indeed a linear proxy for log state. Notably, Dieterich and Kilgore (1994) have
shown (their Figure 7) that (optically observed) contact area loss across reslides following long holds on transparent acrylic and glass samples is also non-negligible and hold duration dependent. These observations bring into question the validity of assumption (2) in relation to laboratory friction data.

Therefore, we conclude that the assumptions under which the stiffness independence of the healing rate can be interpreted as support for Aging law style time-dependent healing during holds are most likely not satisfied during typical laboratory experiments. However, if closure is indeed linearly related to log state, then Figures D.3c and d jointly show that $\Delta \ln(\theta)$ is larger, even though the corresponding slip is consistently smaller, for the higher stiffness spring. Also, the rate of increase in $\Delta \delta_n$ across the reslide with hold duration seems to be nominally stiffness-independent. These observations suggest a lack of correlation between $\Delta \ln(\theta)$ and slip which is not consistent with the Slip law either. In fact, assuming that the stiffness-independent rate of increase in $\Delta \delta_n$ across the reslide to be a linear proxy for $\Delta \ln(\theta)$, one could still tie the observed stiffness independence of healing rates to time dependent healing. But, none of the state evolution laws considered here share this property of the closure data.

D.3 Analytical behavior of Aging, Slip and Nagata law state evolution during long holds

As explained in the main text, frictional healing during laboratory slide-hold-slides has traditionally been understood in terms of the peak friction after the reload. To use $\Delta \mu_{\text{hold}}$ as an additional constraint to differentiate between state evolution laws, it is also important to understand the time evolution of the stress minima. With reference to the [Beeler et al. (1994)] data, of particular importance is understanding the stiffness dependence of the evolution of friction during long holds. In this appendix we develop a detailed analytical framework to address this issue for the Aging, Slip and Nagata laws.
D.3.1 Aging law long holds

Ranjith and Rice (1998) (RR98) derived analytical solutions for stress as a function of slip rate during Aging law holds (the solutions for \( V_{lp} = 0 \) in Section 4 of their paper). These can be used to gain insight into the behavior of the stress minima at the end of holds. The approach adopted by RR98 is to eliminate the state variable and time from the system of equations and solve for phase plane trajectories defined by non-dimensional changes in stress \( \psi \) \((= (\mu - \mu_0)/a)\) and slip rate \( \phi \) \((= \ln(V/V_0))\), where the subscript 0 denotes an arbitrary reference state. In what follows, we choose the reference state to be steady-state at the reslide rate \( V_{s/r} \). RR98 derived these trajectories for two regimes of machine stiffnesses,

\[(1) \ K \neq K_{cr} :\]
\[
\exp \left[ \phi \left(1 - \frac{1}{\beta} \right) \right] = \left[ \frac{U}{K\beta} \exp \left\{ -\frac{\psi}{K} (\beta - 1 - k) \right\} + \frac{1}{\beta - 1 - K} \right] (\beta - 1) \exp \left( -\frac{\psi}{\beta} \right), \tag{D.10a}
\]

\[(2) \ K = K_{cr} :\]
\[
\exp \left[ \phi \left(1 - \frac{1}{\beta} \right) \right] = \frac{1}{\beta} (U + \psi \beta) \exp \left( -\frac{\psi}{\beta} \right), \tag{D.10b}
\]

where \( K = kD_c/a, \beta = b/a, \) \( U \) is a parameter specifying a particular trajectory fixed by initial conditions, and \( K_{cr} = \beta - 1 \) is the scaled critical stiffness for instability (Ranjith and Rice 1998). It follows from equations (D.10) that, for any load point hold commencing from steady state \((\psi = \phi = 0)\), we have

\[U = \frac{\beta K^2}{(K + 1 - \beta)(\beta - 1)}, \quad K \neq K_{cr} \quad \text{and} \quad \tag{D.11a}\]

\[U = \beta, \quad K = K_{cr}. \tag{D.11b}\]
These are the appropriate values of $U$ for all the holds being analyzed in this study.

RR98 claimed that in the limit of long hold times ($\phi = \ln(V/V_0) \rightarrow -\infty$), all the trajectories in equations (D.10) predict that the dimensionless friction $\psi \rightarrow -\infty$ (their Eq. (32)), but this is the case only for velocity strengthening surfaces. We show here that for velocity weakening materials ($\beta = b/a > 1$), $\psi$ approaches a constant value as $\phi \rightarrow -\infty$. To do this, we first point out that for $\phi \rightarrow -\infty$ and $\beta > 1$, the left hand side of equation (D.10a) approaches zero. This implies

$$
\lim_{\phi \rightarrow -\infty} \left[ \frac{U}{K\beta} \exp\left\{ -\frac{\psi}{K} (\beta - 1 - K) \right\} + \frac{1}{\beta - 1 - K} \right] (\beta - 1) \exp\left\{ -\frac{\psi}{\beta} \right\} = 0, \quad \beta > 1 \quad (D.12)
$$

For a finite value of $\psi$, equation (D.12) shows that for any $U$ and $\beta > 1$

$$
\lim_{\phi \rightarrow -\infty} \psi = \left[ \frac{K}{K + 1 - \beta} \ln \left( \frac{\beta}{U \left( \frac{K}{K + 1 - \beta} \right)} \right) \right], \quad K + 1 - \beta \leq 0, U \geq 0. \quad (D.13)
$$

For load point holds from steady-state, using the expression for $U$ from equation (D.11a), this implies

$$
\lim_{\phi \rightarrow -\infty} \psi = \left[ \frac{K}{K + 1 - \beta} \ln \left( \frac{\beta - 1}{K} \right) \right], \quad \beta > 1, \quad (D.14)
$$

which is a negative constant for all values of $K + 1 - \beta$. This constant value of friction depends on the parameter $\beta$ and the normalized stiffness $K$.

For $\beta < 1$, unlike Eq. (D.12), the L.H.S. of Eq. (D.10a) diverges exponentially for long hold times ($\phi \rightarrow -\infty$). Since $\beta - 1$ and $\beta - K - 1$ are both negative, the only asymptotic form of Eq. (D.10a) that is consistent with both the L.H.S. going to $+\infty$ and $\psi < 0$ is the following:
\[
\lim_{\phi \to -\infty} \exp \left[ \phi \left( 1 - \frac{1}{\beta} \right) \right] = \frac{\beta - 1}{\beta - 1 - K} \exp \left( -\frac{\psi}{\beta} \right), \quad \beta < 1. \tag{D.15}
\]

Upon rearrangement, we have for long hold times

\[
\lim_{\phi \to -\infty} \psi = \beta \ln \left( \frac{\beta - 1}{\beta - 1 - K} \right) - \phi (\beta - 1), \quad \beta < 1. \tag{D.16}
\]

Therefore, for velocity strengthening parameters, the Aging law predicts a stiffness-independent linear decrease in shear stress with \(\ln(V)\). We can use Eq. (D.16) to express \(\psi\) as a function of hold time making use of the quasi-static force balance and the fact that at the end of long holds the Aging law state variable \(\theta \approx t_{\text{hold}}\):

\[
\lim_{\phi \to -\infty} \psi \approx \ln \left( \frac{\beta - 1}{\beta - 1 - K} \right) - (\beta - 1) \ln \left( \frac{V_{s/r} t_{\text{hold}}}{D_c} \right), \quad \beta < 1, \tag{D.17}
\]

where we have used the Aging law identity \(\psi = \phi + \beta \ln(V_{s/r} t_{\text{hold}}/D_c)\) for long holds. Eqs. (D.14) and (D.17) were used to produce the dashed lines in Figure 4.4a and c.

RR98 did not solve for trajectories for a velocity neutral Aging law. However, their integrating function (Eq. 21 in RR98) can be modified (i.e. substitute \(\beta = 1\)) to solve for the phase trajectories corresponding to \(\beta = 1\) for a stationary load point:

\[
U = K\phi + \psi (1 - K) + \exp(-\psi), \quad K_{cr} = 0 \neq K \tag{D.18}
\]

where \(U = 1\) for holds from steady-state. Given the form of Eq. (D.18), we derive an asymptotic expression by recognizing that consistent solutions for \(\phi \to -\infty\) require that \(\psi\) acquire very large -ve values, for which the exponential term would dominate the linear term:

\[
\psi \approx -\ln \left[ 1 + \beta K \ln \left( \frac{V_{s/r} t_{\text{hold}}}{D_c} \right) \right], \quad t_{\text{hold}} > \frac{D_c}{V_{s/r}} \exp \left( -\frac{1}{\beta K} \right). \tag{D.19}
\]
Here, we have again assumed the time dependent healing limit for the Aging law ($\theta \approx t_{\text{hold}}$) similar to Eq. (D.17). Eq. (D.19) was used to plot the dashed lines in Figure 4.4b.

What is noteworthy from both Eqs. (D.14) and (D.17) is that the Aging law predicts stiffness independent rate of stress-relaxation for both velocity weakening and strengthening parameter combinations (Figures 4.4a, c). For the velocity neutral case, the stress relaxation has a weak logarithmic dependence on stiffness, which becomes progressively weaker as the hold duration increases (Figure 4.4b).

### D.3.2 Slip law long holds

Our analysis in this section borrows the phase plane trajectories for Slip law holds derived by [Gu et al.] (1984). The Slip law trajectory for a stationary load point is

$$
(\beta - 1)\phi = U \exp \left( -\frac{\beta - 1}{K} \psi \right) - \psi + \frac{K\beta}{\beta - 1},
$$

where all symbols are identical to Section D.3.1 and we choose the reference state to be steady-state at the reslide rate $V_{s/r}$. Eq. (D.20) implies that for any hold starting from steady-state ($\psi = \phi = 0$),

$$
U = -\frac{K\beta}{\beta - 1}.
$$

To express $\psi$ as an explicit function of $\phi$, we need to solve the exponential-polynomial equation in Eq. (D.20). To do this, recognize that Eq. (D.20) represents a special class of exponential-polynomial equation which can be re-written in the following standard form:

$$
-p\beta \exp(-p\beta) = -\beta \nu^{-\beta},
$$

where
Exponential-polynomial equations of the form of Eq. (D.22) can be converted to an expression for $\psi$ as follows:

\[ p = \exp \left( -\frac{\beta - 1}{K} \psi \right), \tag{D.23a} \]
\[ \nu = \exp \left[ 1 - \frac{(\beta - 1)^2}{K\beta} \phi \right]. \tag{D.23b} \]

where $W_i$ are the real branches of the Lambert-W function (Lambert 1758; Euler 1783; Corless et al. 1996). The Lambert-W function is analytic for real values in the interval $[-e^{-1}, \infty)$. Using Eq. (D.23), one can show that $-\beta\nu^{-\beta} \in (-e^{-1}, 0)$ for $\phi < 0$ (see Figure D.4(A)) and increases in value as $\phi$ decreases to larger negative values. $W$ has two real branches in the interval $[-e^{-1}, 0]$ with the branch point at $e^{-1}$, the increasing principal branch, $W_0$, and the decreasing auxiliary branch, $W_{-1}$ (see Figure D.4(B)) (Corless et al. 1996). Therefore, the physical requirement of $\psi < 0$ is satisfied only if the velocity-weakening solutions lie along $W_{-1}$ while the velocity-strengthening solutions lie along $W_0$. It is also worth pointing out that these closed form solutions for $\psi$ in terms of the Lambert-W are exact and standard numerical implementations of the Lambert-W (e.g. the lambertw package in MATLAB) reproduce numerically derived results (see Figures D.5(A) and (C)).

One limitation of expressing $\psi$ as a function of $\phi$ is that $\phi$ is itself an unknown function of time during holds. Therefore, it seems reasonable to ask if we can develop at least an approximate relationship between $\phi$ and our independent variable $t_{\text{hold}}$ in the limit of long hold durations as we did for the Aging law. To derive such an approximation, we note that when $\phi$ assumes large negative values (for long holds), it is reasonable to approximate Slip...
law state as evolving much slower than slip rate. This is because, denoting \( \Phi = \ln(V_{s/r}, \theta / D_c) \), the following expression is approximately true far below steady-state i.e. \( |\Phi| \ll |\phi| \):

\[
\lim_{|\Phi| \ll |\phi|} \Phi = \lim_{|\Phi| \ll |\phi|} -\exp(\phi) \left[ \phi + \Phi \right] = -\exp(\phi) \phi,
\]

(D.25)

where \( \dot{\Phi} \) represents derivatives with respect to \( V_{s/r}, t / D_c \). Making use of the quasi-static force balance equation,

\[
\psi = \phi + \beta \Phi,
\]

one can use Eq. (D.25) to solve for \( \phi \) in terms of the exponential integral function \( E_i \). A simpler expression for \( \phi \) can be derived under the more restrictive assumption \( \beta \Phi \ll \dot{\phi} \) (appropriate for large negative values of \( \phi \) encountered during long holds)

\[
\phi = -\ln \left[ K \left( \frac{V_{s/r}}{D_c} (t_{\text{hold}} - t_i) \right) \right] + \phi_i, \quad t_{\text{hold}} > t_i,
\]

\[
\approx -\ln \left[ K \left( \frac{V_{s/r} t_{\text{hold}}}{D_c} \right) \right], \quad t_{\text{hold}} \gg t_i, \phi \gg \phi_i,
\]

(D.26)

where \( \phi_i \) is the value of \( \phi \) at the beginning of the integration interval \( t_i \). In arriving at the approximation in Eq. (D.26), we have assumed the hold time is long enough to drop \( t_i \) and \( \phi \) large enough to drop \( \phi_i \). This approximation increases in precision with increasing stiffness and decreasing \( \beta \). Additionally, because the constants being dropped are additive, the rate of evolution of \( \phi \) as given by Eq. (D.26), when valid, is always a better approximation than its value for the same combination of relevant parameters.

Eq. (D.26) shows that it is possible to re-frame the evolution of \( \psi \) in Eq. (D.24) as a function of hold time. With \( \nu \) expressed as the function of \( \ln(t_{\text{hold}}) \) in Eq. (D.26), the evolution of \( \psi \) expressed in Eq. (D.24) is approximate with the evolution rate being better approximated than the actual value of \( \psi \) for small stiffnesses and finite holds due to reasons
outlined in the previous paragraph (see Figs D.6(A) and (C)). In particular, for long enough hold durations (around $10^3$ s for the parameters chosen in Figure D.6), the analytically predicted evolution of $\psi$ with hold time is in excellent agreement with $\psi$ and in exact agreement with $\dot{\psi}$ obtained numerically.

When the argument $-\beta \nu^{-\beta}$ is small, there exist simpler limiting expressions that can be derived from the exact expression in Eq. (D.24) making use of analytical expansions of the branches of the Lambert-W about 0. For typical durations of the longest laboratory holds, this could be considered a low stiffness limit ($K \lesssim 1$) given $\beta$ significantly different from 1. For $\beta > 1$, we make use of the following series expansion of the $W_{-1}$ branch which is convergent only for small -ve arguments (Corless et al., 1996):

$$W_{-1}(-z) = \ln(-z) - \ln(-\ln(-z)) + O\left(\frac{\ln(-\ln(-z))}{\ln(-z)}\right), \quad z \in [-e^{-1}, 0], z \to 0 - . \quad (D.27)$$

Dropping the terms of the order of $\ln(\ln())/\ln()$ in the limit $-\beta \nu^{-\beta} \to 0 -$, we have the following expression for $\psi$:

$$\lim_{-\beta \nu^{-\beta} \to 0 -, \beta > 1} \psi = -\frac{K}{\beta - 1} \ln \left[1 - \frac{\ln \beta}{\beta} - \frac{(\beta - 1)^2}{K\beta} \phi\right]. \quad (D.28)$$

Combining Eqs. (D.26) and (D.28), we have

$$\lim_{-\beta \nu^{-\beta} \to 0 -, \beta > 1} \psi = -\frac{K}{\beta - 1} \ln \left[1 - \frac{\ln \beta}{\beta} + \frac{(\beta - 1)^2}{K\beta} \ln \left(\frac{V_s r_{\text{hold}}}{D_c}\right)\right]. \quad (D.29)$$

In Figure 4.5a, we show that Eq. (D.29) is a good approximation for $\psi$ for holds longer than around 1000 s when $K = 1$ and $\beta = 2$. Gu et al. (1984) noted the existence of such weaker than logarithmic trajectories in their solutions (Eq. 29 therein and their following sentence).
For $\beta < 1$, the corresponding limiting trajectory ($-\beta v^{-\beta} \to 0$) is logarithmic. To show this, we use the Taylor series for $W_0$ about $z = 0$:

$$W_0(z) = \sum_{n \geq 1} \frac{(-n)^{n-1}}{n!} z^n, \quad z \in [-e^{-1}, e^{-1}].$$  \hspace{1cm} (D.30)

[Corless et al. (1996)] point out that this series is convergent in $[-e^{-1}, e^{-1}]$. In the limit $-\beta v^{-\beta} \to 0$, we only retain the first order term in the series in Eq. (D.30) to obtain

$$\lim_{-\beta v^{-\beta} \to 0, \beta > 1} \psi = \frac{K\beta}{\beta - 1} - (\beta - 1)\phi,$$

$$\approx \frac{K\beta}{\beta - 1} + (\beta - 1) \ln \left\{ K \left( \frac{V_s/r_{\text{hold}}}{D_c} \right) \right\},$$  \hspace{1cm} (D.31)

where we have made use of the approximate relationship between $\phi$ and $t$ from Eq. (D.26).

Figure 4.5c shows that Eq. (D.31) is a good approximation for $\psi$ for holds longer than around 1000s when $K = 1$ and $\beta = 0.5$.

It is clear from the numerical simulations in Figures D.5(A)-(C) that under the Slip law, the rate of stress relaxation during a hold, i.e. $\partial \psi / \partial t$, increases with increasing stiffness. It is worthwhile to investigate if this relaxation rate keeps on increasing with stiffness or does it reach a limiting value. To do this, we solve for the trajectory in Eq. (D.20) assuming that $(\beta - 1)/K$ small enough that the exponential term is well approximated as

$$\exp \left( -\frac{\beta - 1}{K} \psi \right) \approx 1 - \frac{\beta - 1}{K} \psi.$$  \hspace{1cm} (D.32)

For $\beta$ sufficiently different from 1, this is the large stiffness limit for long holds. Under this approximation, the resultant trajectory is simply

$$\psi = \phi = -\ln \left[ K \left( \frac{V_s/r_{\text{hold}}}{D_c} \right) \right].$$  \hspace{1cm} (D.33)
In fact, this trajectory is the limiting solution for both velocity weakening and strengthening regimes. Additionally, this solution physically implies that the spring stiffness is large enough to completely stop state evolution as soon as the load point is made stationary; in that case, $\dot{\psi} = \dot{\phi}$ at $t \to 0^+$. In our simulations, with $\beta$ significantly different from 1, Eq. (??) seems a good approximation for $K \gtrsim 1000$ and exact for $K \gtrsim 10000$ (see Figure 4.5). For comparison with laboratory experiments, for the granite data from Beeler et al. (1994), we have $a - b = -0.0027$ (Figure D.9) and the two stiffnesses are $k_n = 0.0019\mu m^{-1}$ and $k_s = 0.0019\mu m^{-1}$. Assuming $a \sim 0.02$ and $D_c \sim 2\mu m^{-1}$ these stiffnesses translate to $K_n = 0.2$ and $K_s = 6.0$ and $\beta - 1 \sim 0.27$. Given $|\psi| \sim 1.5$ and 7.5 respectively for the longest low and high stiffness holds (read off $\Delta \mu_{\text{hold}}$ values for the longest holds in Figure 4.2(C)), even the higher stiffness setup seems at least an order of magnitude (possibly more) smaller than the infinite stiffness limit $K \gg (\beta - 1)\psi$. Therefore, the infinite stiffness limit is likely not accessed in typical laboratory experiments, i.e. Slip law state likely evolves appreciably even at the end of the longest holds.

Finally, we consider the case of the velocity neutral Slip law ($\beta = 1$). Gu et al. (1984) did not seek the stationary load point trajectory for this case. But one can find the appropriate integrating factor by substituting $\beta = 1$ in their general integrating factor for a stationary load point (inline equation in the sentence above Eq. 25 in Gu et al. (1984)). The trajectory comes out to be a parabola in $(\psi, \phi)$,

$$\frac{\psi^2}{2} - K(\psi - \phi) = U = 0,$$

where the trajectory constant $U$ is zero for holds from steady-state ($\psi = \phi = 0$). It is then straightforward to write the expression for $\psi$:

$$\psi = K - \sqrt{K^2 - 2K \phi} \approx K - \sqrt{K^2 + 2K \ln \left[ K \left( \frac{V_s/r_{\text{hold}}}{D_c} \right) \right]},$$

(D.35)
where the approximation is, as before, only appropriate for large negative values of $\phi$. The $\ln(t_{\text{hold}})$ approximation is appropriate for all holds larger than about $10^3$ s (Figure 4.5). It is noteworthy that, similar to the behavior for $\beta \neq 1$, the stress relaxation rate (and actual values of $\psi$) is stiffness sensitive. Also, the large stiffness limit ($K \gg \psi \sim \phi$) is identical to Eq. (??) (a binomial approximation of the square root reveals this) consistent with the fact that, being the no state evolution limit, this approximation is insensitive to $\beta$.

Therefore, unlike the Aging law, the rate of stress relaxation ($\partial \psi / \partial t$) is always stiffness dependent under the Slip law as long as the stiffness is significantly smaller than the infinite stiffness limit. As pointed out above, typical laboratory experiments satisfy this constraint. Therefore, hold portions of slide-hold-slide experiments carried out at sufficiently different stiffnesses can be used to decide whether the data supports Aging or Slip law formulations.

### D.3.3 Nagata law holds

Before analyzing Nagata law holds, it is important to note that Bhattacharya and Rubin (2014) pointed out that meaningful comparisons between the Nagata law and laboratory velocity step data (that are well modeled by the Slip law) require that $a$, $D_c$ and $a - b$ obey the following scaling relations:

\[
\langle a \rangle = \frac{a}{c + 1}, \tag{D.36a}
\]
\[
\langle D_c \rangle = D_c(c + 1), \tag{D.36b}
\]
\[
\langle a - b \rangle = a - b, \tag{D.36c}
\]

where angular brackets represent Slip law values and those without brackets represent Nagata law values. Eqs. (D.34)(a) and (b) follow from the recognition that the effective direct rate dependence and slip-weakening distance under the Nagata law are $\langle a \rangle$ and $\langle D_c \rangle$ respectively (Bhattacharya and Rubin, 2014; Bhattacharya et al., 2015), both assumed material dependent constants under conventional RSF. Eq. (D.34)(c) satisfies the additional
requirement that the steady-state rate dependence should be a constant independent of \( c \).

Using these relations, we write the following set of non-dimensional equations:

\[
\langle \psi \rangle = (c + 1)\phi + \langle \beta \rangle \Phi, \tag{D.37a}
\]

\[
\frac{d\langle \psi \rangle}{dT} = (c + 1)\frac{d\phi}{dT} + \langle \beta \rangle \frac{d\Phi}{dT}, \tag{D.37b}
\]

\[
\frac{d\Phi}{dT} = (c + 1)e^{-\Phi} - \alpha e^\phi, \tag{D.37c}
\]

\[
\frac{d\langle \psi \rangle}{dT} = -\langle K \rangle e^\phi, \tag{D.37d}
\]

where

\[
\langle \psi \rangle = \frac{\tau - \tau_0}{\langle a \rangle \sigma}, \tag{D.38a}
\]

\[
T = \frac{V_{0t}}{\langle D_c \rangle}, \tag{D.38b}
\]

\[
\langle K \rangle = \frac{k\langle D_c \rangle}{\langle a \rangle}, \tag{D.38c}
\]

\[
\langle \beta \rangle = \frac{b}{\langle a \rangle} = \beta(c + 1), \tag{D.38d}
\]

\[
r = 1 - \frac{c}{\langle \beta \rangle}, \tag{D.38e}
\]

\[
\alpha = (c + 1) - (1 - r)\langle K \rangle, \tag{D.38f}
\]

and the definitions of \( \phi \) and \( \Phi \) are identical to previous sections. Note that \( \langle \beta \rangle \) is not the Slip law value of \( b/a \). This choice of the non-dimensional form ensures that the resultant trajectories would relate physical variables insensitive to the implicit scaling of the rate and state parameters with \( c \).

The Nagata law trajectories for a stationary load point can be derived following the steps outlined in [Ranjith and Rice (1998)] and [Gu et al. (1984)]. To do this, we first eliminate \( \Phi \) and \( T \) using Eqs. (D.35)(a)-(d) and obtain a differential equation of the form:
\[ P(\phi, \langle \psi \rangle) d\langle \psi \rangle + Q(\phi, \langle \psi \rangle) d\phi = 0, \quad \text{(D.39)} \]

where

\[
\frac{d\phi}{dT} = P(\phi, \langle \psi \rangle) = (\langle \beta \rangle \alpha - \langle K \rangle) e^{\phi} - (c + 1) \langle \beta \rangle e^{\frac{(c+1)\phi - \langle \psi \rangle}{\langle \psi \rangle}}, \quad \text{(D.40a)}
\]

\[
\frac{d\langle \psi \rangle}{dT} = Q(\phi, \langle \psi \rangle) = -\langle K \rangle e^{\phi}. \quad \text{(D.40b)}
\]

To solve Eq. (D.36), we seek an integrating factor \( e^{q(\phi, \langle \psi \rangle)} \) such that

\[
e^{q(\phi, \langle \psi \rangle)} \left[ P(\phi, \langle \psi \rangle) \langle \psi \rangle + Q(\phi, \langle \psi \rangle) d\phi \right] = dU = 0, \quad \text{(D.41)}
\]

where \( U \) is the trajectory constant. Inspecting the form of Eqs. (D.38)(a) and (b), we find that the integrating function is

\[
q(\phi, \langle \psi \rangle) = \langle \beta - 1 \rangle \left( \langle \beta \rangle \alpha - \langle K \rangle \right) \langle \psi \rangle \langle \beta \rangle - (c + 1) \frac{\phi}{\langle \beta \rangle}, \quad \text{(D.42)}
\]

where \( \langle \beta - 1 \rangle \) is the Nagata law critical stiffness, \( k_{cr} = (b - a)/\langle D \rangle \) (Kame et al. 2013a; Bhattacharya and Rubin, 2014), scaled in keeping with the definition of \( \langle K \rangle \). Solving Eq. (D.39), we obtain the following trajectory for the Nagata law when \( \langle K \rangle r = \langle K \rangle_{cr} = \langle \beta - 1 \rangle \),

\[
e^{\phi \left( \frac{\langle \beta - 1 \rangle}{\langle \beta \rangle} \right)} = \langle \beta - 1 \rangle \left[ \frac{U}{\langle K \rangle \langle \beta \rangle} e^{-\frac{(c+1)\phi - \langle \psi \rangle}{\langle \psi \rangle}} + \frac{1}{\langle \beta - 1 \rangle - \langle K \rangle r} \right] e^{\frac{\phi}{\langle \psi \rangle}}, \quad \langle K \rangle r \neq \langle K \rangle_{cr} \quad \text{(D.43)}
\]

where, for holds from steady-state sliding at \( V_{s,t} (\phi = \langle \psi \rangle = 0) \), \( U \) is given by

\[
U = \left( 1 - \frac{\langle \beta - 1 \rangle}{\langle \beta - 1 \rangle - \langle K \rangle r} \right) \frac{\langle \beta \rangle \langle K \rangle}{\langle \beta - 1 \rangle}. \quad \text{(D.44)}
\]
The \( c \) dependence of Eq. (D.41) is captured entirely by the two constants \( \langle \beta \rangle \) and \( r \). For \( c = 0 \), it is trivial to show that Eq. (D.41) is identical to the Aging law trajectories in Eq. (D.10)(a). However, for non-zero \( c \), the long hold duration (large, negative \( \phi \)) limiting behaviors of the Nagata law trajectories (Section D.3.1) begin to diverge from their Aging law counterparts. For example, for \( c = 1 \), while the velocity weakening trajectories converge to a stiffness dependent constant shear stress (Figure D.7(A)) the velocity strengthening trajectories exhibit stiffness dependent stress relaxation (Figure D.7(D)). This stiffness dependence of the velocity strengthening trajectories originates from the increasing relative importance of the second term \( (\alpha e^\phi) \) in Eq. (D.35)(c) with respect to the time dependent healing term \( ((c + 1)e^{-\Phi}) \) as \( \alpha \) increases, even for long holds. Since \( \alpha \) increases as a result of competition between \( c \) and \( \langle K \rangle \), the trajectories are sensitive to both \( c \) and stiffness.

As \( c \) increases in value, the nature of the velocity weakening trajectories changes markedly. For \( c = 10 \), the numerical solutions show that the velocity weakening trajectories do not asymptote to a constant stress anymore, instead relaxing to ever-decreasing stresses with increasing hold time (Figure D.7(B)). The advantage of our particular formulation of the trajectories is that we can understand this \( c \)-dependence in terms of the parameters \( \langle \beta \rangle \) and \( r \). The scaling relations in Eq. (D.34)(a) and (c) imply that when \( c \gg 1 \), \( b/a = \beta \) asymptotes to 1 which in turn implies that when \( c \gg 1 \), \( \langle \beta \rangle \rightarrow c \) and \( r \rightarrow 0 \). Therefore, for velocity weakening trajectories, we can no longer assume that the left hand side of Eq. (D.41) goes to zero, the crucial ingredient for the constant shear stress solution derived in Eq. (D.12) for the Aging law. The velocity strengthening trajectories retain their stiffness dependence and predict generally greater stress relaxation with increasing \( c \) in this intermediate range.

When \( c \gg 1 \), one can rewrite Eqs. (D.41) and (D.42) in the following form as long as \( \langle \psi \rangle / \langle \beta \rangle \ll 1 \) and \( \phi \left( \frac{\langle \beta - 1 \rangle}{\langle \beta \rangle} \right) \ll 1 \) (from Eq. (D.35)(a), these conditions are simultaneously true if \( |\phi + \Phi| \ll 1 \)):

\[
1 - \eta \phi = \exp (-\gamma \langle \psi \rangle) + \zeta \langle \psi \rangle, \quad |\phi + \Phi| \ll 1 \quad (D.45)
\]
where

\[
\zeta = \left< \beta - 1 \right> \frac{1}{\left< K \right> \left< \beta \right> r} - \frac{1}{\left< \beta \right>}, \quad \text{(D.46a)}
\]

\[
\tilde{\gamma} = \frac{\left< \beta - 1 \right>}{\left< K \right>} - r, \quad \text{(D.46b)}
\]

\[
\eta = \frac{\left< \beta - 1 \right>^2}{\left< K \right> \left< \beta \right> r} \left( 1 - \frac{\left< K \right> r}{\left< \beta - 1 \right>} \right), \quad \text{(D.46c)}
\]

\[
\text{(D.46d)}
\]

Eq. (D.43) has a Lambert W solution similar to the Slip law trajectories in Section D.3.2:

\[
\left< \psi \right> = -\frac{1}{\tilde{\gamma}} \ln \left[ -\frac{W_i(-\zeta^{-1} \tilde{\gamma}^{-1} \nu)}{\zeta^{-1} \tilde{\gamma}} \right], \quad i = -1, 0 \text{ for } \left< \beta - 1 \right> \gtrless 0. \quad \text{(D.47)}
\]

For values of \( c \) large enough that \( \left< \beta - 1 \right> \gg \left< K \right> r \), we have \( \eta \approx \left< \beta - 1 \right>^2 / \left< K \right> \left< \beta \right> r \). In this limit, we can write:

\[
\left< \psi \right> = -\frac{\left< K \right>}{\left< \beta - 1 \right>} \ln \left[ -\frac{W_i(-\left< \beta \right> r \nu^{-\left< \beta \right> r})}{-\left< \beta \right> r} \right], \quad i = -1, 0 \text{ for } \left< \beta - 1 \right> \gtrless 0. \quad \text{(D.48)}
\]

Given the equivalence

\[
\beta_{\text{Slip}} = \left< \beta \right> r, \quad \text{(D.49)}
\]

Eq. (D.46) exactly reproduces the Slip law trajectories. In practice, for velocity strengthening trajectories (\( \left< \beta - 1 \right> < 0 \)), this approximation is accurate even when \( \left< K \right> \sim \left< \beta - 1 \right> (c = 100) \) as shown in Figure [D.7 F]. However, under the same parameter regime, the approximation is worse for the velocity weakening trajectories and gets worse with increasing hold times (Figure [D.7 C]). For the ranges of hold times and stiffnesses in Figure [D.7] we have verified that for \( c \sim 1000 \), the numerically derived trajectories are in exact agreement with the analytical trajectories in Eq. (D.46) for both velocity weakening and strengthening regimes. 

*Bhattacharya and Rubin* (2014) had previously shown for velocity steps that the Nagata law response could be tuned to match the Slip law response for a given excursion from
steady-state by making $c$ larger than a critical value. This critical value was found to be an increasing function of the magnitude of the velocity step. Our analysis in this section shows that similar conclusions can be reached regarding holds under the Nagata law. When $c = 0$, the Nagata law is the Aging law. When $c$ assumes non-zero values, the Nagata law trajectories start to diverge from their Aging law counterparts and tend more closely to the corresponding Slip law trajectories. For values of $c$ large enough that (1) $\langle \beta - 1 \rangle \gg \langle K \rangle r$ and (2) $D_c$ is small enough that even far below steady-state $V \theta / D_c \sim 1$, the holds under the Nagata law exactly reproduce the corresponding Slip law trajectories. Conditions (1) and (2) both show that, just like velocity steps, the further the sliding surface is pushed below steady-state the larger the critical value of $c$ one requires to reproduce Slip law trajectories with the Nagata law. Because larger stiffnesses and/or longer hold times drive the sliding surface further below steady-state, larger values of $c$ are needed to satisfy conditions (1) and (2).

It is important to note that Eq. (D.47) is equivalent to the Nagata law parameters simultaneously satisfying Eqs. (D.34)(A) and (C) with the corresponding Slip law parameters as reference values. Along with the scaling in Eq. (D.34)(B), which is built in to all our dimensionless equations and is needed to satisfy condition (2) above, these scaling relationships represent Nagata law solutions which reproduce the corresponding Slip law stress relaxation trajectory when $c$ is larger than its critical value given the particular combination of hold length and stiffness. Interestingly, these scaling relations are identical to the Slip law - Nagata law correspondence derived for velocity steps by Bhattacharya and Rubin (2014). It follows, therefore, that stress relaxation data for holds which are well explained by the Slip law can only constrain $c$ to a lower bound. For any value of $c$ larger than this lower bound we can use the scaling relations in Eqs. (D.34) to find parameters which reproduce the corresponding Slip law trajectory. An equivalent result was established for data from laboratory velocity steps by Bhattacharya et al. (2015).
D.4 Estimating stiffness and $a - b$ from independent constraints

There are two important constraints on the Beeler et al. (1994) data which can be evaluated without formally inverting the slide-hold-reslide data: (1) the stiffness of the testing apparatus; (2) the value of $a - b$. These constraints reduce the dimensionality of our inverse problem. Additionally, these constraints allow us to resolve trade-offs between the parameters being inverted. For example, stiffness enters the dimensionless rate-state equations only in the combination $kD_c/a$ (see Eq. (D.3)), and could show strong trade-offs with both $a$ and $D_c$ while minimizing misfit in an inversion exercise. Such trade-offs are best resolved by independent a-priori constraints.

To obtain an estimate of $k$, we note that at the end of the longest holds, the fault is sliding orders of magnitude slower than $V_s/r$. Therefore, after the reload, for as long as the slip rate $V \ll V_{s/r}$, we have the following

$$
\Delta \mu = k(\delta_{lp} - \delta) = k\delta_{lp} \left(1 - \frac{\delta}{\delta_{lp}}\right) \approx k\delta_{lp},
$$

(D.50)

where $\delta$ and $\delta_{lp}$ are surface and load point displacements since the reload, respectively, and instantaneously $\delta/\delta_{lp} \ll 1$ for reslides following long holds. A linear fit to the $\Delta \mu$ vs. $\delta_{lp}$ plot over the first few data points following the reslide (after a long hold) gives $k$ as the slope (Figure D.8). Note that, while our estimate of the natural machine stiffness $k_n = 0.0019 \mu m^{-1}$ is identical to that of Beeler et al. (1994), we found the servo controlled effective stiffness to be $k_s = 0.055 \mu m^{-1}$ instead of $k_s = 0.075 \mu m^{-1}$ reported in Beeler et al. (1994).

The experimental runs of Beeler et al. (1994) contained sequences of velocity steps preceding the holds which allow us to constrain the value of $a - b$. Instead of using the steady-state friction values to constrain $a - b$, we ran full inversions on these sequence of
half order velocity steps for both stiffnesses. The misfits were weighted carefully to best constrain $a$ and $a - b$ (Figure D.9). Based on these inversions, we used $a - b = -0.0027$ for all our fits where $a - b$ was fixed a priori. The value of $a$ recovered from these inversions were different by a factor of 2 depending whether we did a good job of fitting the low stiffness steps (0.012 with the Slip law, Figure D.9a) or the high stiffness steps (0.023 with Slip law, Figure D.9d). Based on these results, we concluded that $a$ is at least unlikely to be smaller than 0.012. If instead, one attributes the stress minimum following the velocity step down entirely to the direct rate effect, the lower bound on decreases to $a \sim 0.009$ for the lower, and $a \sim 0.015$ for the higher stiffness steps respectively.

D.5 Can strain rate dependence of RSF parameters help the Aging law fit $\Delta \mu_{\text{hold}}$?

For a considerable portion of the main text, we have espoused the idea that stress relaxation during long holds is better explained by the Slip than the Aging law. In particular, based on inversions with the [Beeler et al., 1994] data, we established that an important argument against the Aging law is the lack of stiffness dependence of the stress relaxation during long holds. This is a direct consequence of time dependent healing and is a robust feature in the data. It is, therefore, worth asking if we can mathematically modify the RSF equations such that the Aging law stress relaxation can end up matching the observed stiffness dependence in the data. Equivalently, this amounts to asking if the slip rates accessed at the end of long holds are too low for the usual assumptions of RSF, e.g., constant $a, b, D_c$ etc., to remain valid.

D.5.1 Strain rate dependence of $a$

First principles derivations of the friction law (Eq. (4.1)) seem to suggest that $a$ might be expected to have both rate ([Boettcher et al., 2007]) and state dependence ([Rice et al., 2001]).
In this subsection, we formulate the rate dependence of $a$ by assuming that the rheology of asperity contacts has an inherent strain-rate sensitivity at the GPa level stresses expected at these length scales. We first express $a$ in terms of the parameters of an Arrhenius rate dependence \cite{Nakatani2001, Rice2001, Baumberger2006}:

$$a = \frac{k_B T}{\sigma_c \Omega},$$  

(D.51)

where $k_B$ is the Boltzmann constant, $\Omega$ the so-called activation volume such that the contact normal stress $\sigma_c$ times $\Omega$ gives the minimum work needed to activate a dislocation depinning \cite{Baumberger2006, Boettcher2007}. If the asperity contacts are in a state of plastic flow, the contact stress can be equated to yield stress \cite{Brechet1994, Baumberger2006}. \cite{Boettcher2007} used a low temperature plasticity flow-law for olivine to calculate these contact yield stresses:

$$\sigma_c = \sigma_p \left[ 1 - \left( -\frac{RT}{H} \ln \left( \frac{\dot{\varepsilon}}{B} \right) \right)^{1/q} \right],$$  

(D.52)

where $\dot{\varepsilon}$ is bulk strain-rate within the asperity, $\sigma_p = 8500$ MPa is Peierl’s stress, $R = 8.314 \text{ J mol}^{-1} \text{K}^{-1}$ is the universal gas constant, $H = 5.4 \times 10^5$ J mol$^{-1}$ is the activation enthalpy, $B = 5.7 \times 10^{11}$ s$^{-1}$ is an empirical constant, and the exponent $q = 2$. The strain-rate dependence of contact stresses expressed in Eq. (D.50) leads to a rate dependence in $a$.

To derive this, we write:

$$a_0 = \frac{k_B T}{\sigma_p \Omega},$$  

(D.53a)

$$\gamma = \frac{1}{q} \left( \frac{RT}{H} \right)^{1/q},$$  

(D.53b)

to get:
\[ a = a_0 \left[ 1 - \gamma \ln \left( \frac{V^*}{V} \right) \right]^{-1}, \quad (D.54) \]

where \( V^* \) is a velocity scale derived by multiplying \( B \) with a characteristic length scale \( l_{\text{def}} \) for strain localization. Assuming that \( a \) does not change much from \( a_0 \) even at the lowest slip rates accessed during long holds (i.e. \( \gamma \ln(V^*/V) \ll 1 \)), and regularizing the logarithm with \( \sinh^{-1} \) to accommodate small values of \( V^* \) (since we are imagining slip on a planar surface, \( l_{\text{def}} \to 0 \)), we express the parametrized rate dependence for \( a \) as:

\[ a = a_0 \left[ 1 + \gamma \sinh^{-1} \left( \frac{V^*}{V} \right) \right]. \quad (D.55) \]

Note that, since \( da/dV < 0 \) (\( a \) increases with decreasing slip rate), for the same hold duration, this velocity dependence of \( a \) would drop the Aging law \( \Delta \mu_{\text{hold}} \) to larger negative values such that the magnitude of this decrease is stiffness dependent. It is worth noting that the \( \Delta \mu_{\text{peak}} \) and \( \Delta \mu_{\text{hold}} \) data collected by [Marone and Saffer (2015)] on simulated gouge over a range of \( V_s/r \) shows that the magnitude of \( \Delta \mu_{\text{hold}} \) increases with decreasing \( V_s/r \); such a trend is consistent with \( da/dV < 0 \). Additionally, the form for rate dependence of \( a \) in Eq. (D.53) also allows the Aging law interface to continually weaken during long holds even for velocity-weakening interfaces. At \( V \gg V^* \), Eq. (D.53) implies \( a = a_0 \). Therefore, it is possible to choose a \( V^* \) that is small enough to allow \( a - b = -0.0027 \) at the values of \( V_s/r \) accessed in this study. In our inversions, we allow all of \( a_0, \gamma \) and \( V^* \) to vary since the olivine crystal values used by [Boettcher et al. (2007)] might not be appropriate for tectosilicates.

In actual inversions, this added strain rate dependence does not help the Aging law fit the \( \Delta \mu_{\text{hold}} \) data as closely as the Slip law (Figure [D.10(A)]). The fit is improved when compared to the Aging law fit in Figure [4.10(B)] but, unlike the data, the stress relaxation rate becomes steeper with increasing hold time. Remarkably, this strain rate dependence helps the Slip law fit instead. The peak stress evolution with hold duration becomes truly stiffness independent for the longest holds while maintaining the quality of the fits to the
stress minima. In fact, this version of the Slip law fit is the best joint fit to $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ of all the laws tested in this study. For room temperature studies, Eqs. (D.51)a and (D.51)b would suggest that $a_0$ would be of the order of typical values of $a$ while $\gamma \sim 0.035$; the corresponding values adopted by the Slip law fit are similar. The variations in $a - b$ introduced by these parameters are plotted in Figure D.10.

D.5.2 Second order corrections to RSF

As an alternative to the strain rate dependence of $a$ considered in Section D.5.1, the Bowden-Tabor decomposition of friction into a product of real contact area and contact strength can be used to come up with a velocity dependence of $a - b$ (Baumberger et al., 1999; Baumberger and Caroli, 2006). This rate-sensitivity arises naturally out of a second order dependence of friction on the product $\ln(V) \ln(\theta)$ which is a consequence of the product decomposition of Bowden-Tabor (Baumberger et al., 1999). The form recommended by Baumberger et al. (1999) is as follows (Eqs. 4, 5 and 6 therein):

$$\mu = \mu_0 + a \ln \left( \frac{V}{V_0} \right) + b \ln \left( \frac{V_0 \theta}{D_c} \right) + \frac{ab}{\mu_0} \ln \left( \frac{V}{V_0} \right) \ln \left( \frac{V_0 \theta}{D_c} \right),$$  \hspace{1cm} (D.56)

where we have used $\mu_0 = \sigma_{s0} \Sigma_0 / W$, $a = \alpha \sigma_{s0} / W$ and $b = m \Sigma_0 / W$ (the symbols $\sigma_{s0}$, $\Sigma_0$, $\alpha$ and $m$ are defined in Baumberger et al. (1999); $W$ is normal load) to convert the Baumberger et al. (1999) equations to the form used in this paper. Note that the reference slip rate $V_0$ here is the same as in Eq. (4.1). This second order correction to conventional RSF naturally gives rise to the following velocity dependence to steady-state friction $\mu_{ss}$:

$$\frac{\partial \mu_{ss}}{\partial \ln(V)} = (a - b) - \frac{ab}{\mu_0} \ln \left( \frac{V_{ss}}{V_0} \right),$$  \hspace{1cm} (D.57)

which effectively introduces a velocity dependence to $a - b$. For $a, b > 0$ and $V < V_0$, the correction to $a - b$ during long holds would be positive thus making the model increasingly velocity neutral or strengthening with decreasing slip rate in our case. Notably, this is the
same sense in which the strain rate dependence introduced in Section D.5.1 modifies $a - b$. Given that the Aging law needs a velocity strengthening solution to produce non-zero stress relaxation rate during the longest holds, we a priori constrained $V_0 > 10^{-5} \mu\text{ms}^{-1}$. This ensures that $V$ would always be smaller than $V_0$, and hence more velocity strengthening, at the lowest slip rates ($10^{-6} \mu\text{ms}^{-1}$) accessed by the numerical models. We also inverted for $\mu_0$ as a parameter of this model and constrained this in the range [0.3,0.8].

Figure [D.11(A)] shows the fits to the $\Delta\mu_{\text{peak}}$ and $\Delta\mu_{\text{hold}}$ data with the second-order corrected RSF equation. The general features of the fits are similar to the ones obtained with the strain-rate dependence of $a$. The Aging law fits are not drastically improved and this formulation also suffers from the steepening of the stress-relaxation rate at long hold times. The Slip law fits to $\Delta\mu_{\text{hold}}$ are worsened for the low stiffness data. However, the evolution of Slip law peak stresses becomes stiffness independent similar to the strain rate dependent fit in Figure [D.10(A)].
Figure D.1 (A) Plots of $\ln(V_s/\theta/D_c)$ at the end of the hold (solid lines) and at peak stress (dashed lines) for the set of Aging law numerical simulations from Figure 4.3. Red colors represent the low stiffness simulations, blue high stiffness. $a = 0.009$, $b = 0.01$ and $D_c = 3.0 \mu m$. (B) The same as (A) but now the dashed lines show $\ln(V_s/\theta_{\text{peak}}/D_c) - \Delta \ln(V_s/\theta/D_c)$ where $\Delta \ln(V_s/\theta/D_c)$ is the analytical estimate of the (negative) change in state between beginning of reslide and peak stress from Eq. (D.8). The approximation becomes better for longer hold times.
Figure D.2 Plots of $\ln(V_{s/r} \theta / D_c)$ at the end of the hold (solid lines) and at peak stress (dashed lines) for the set of Slip law numerical simulations from Figure 4.3. Red colors represent the low stiffness simulations, blue high stiffness. $a = 0.009$, $b = 0.01$ and $D_c = 3.0 \mu m$. Note that the peak friction for each individual stiffness is quasi-linear with log of hold time with a stiffness dependent healing rate. However, the healing rates for the Slip law are less than $\ln t_{\text{hold}}$ which is shown for comparison. Also, at least for the lower stiffness simulations, $\partial / \partial t_{\text{hold}} \left[ \ln(\theta_{\text{hold}}) \right]$ is clearly different from $\partial / \partial t_{\text{hold}} \left[ \ln(\theta_{\text{peak}}) \right]$. 

$k_n = 0.0019 \, \mu m^{-1}$, $V_{s/r} = 1.0 \, \mu m s^{-1}$

$k_s = 0.055 \, \mu m^{-1}$, $V_{s/r} = 0.316 \, \mu m s^{-1}$
Figure D.3 Plots of total change in closure measured (in $\mu$m) since the beginning of the hold (red in (A), blue in (B)) and corresponding shear stress changes (in green) during the reslide following a $10^4$ s holds for (A) the natural stiffness apparatus (B) the stiffer apparatus. The vertical black dashed lines are plotted as a visual aid to locate peak stress. (C) Evolution of the change in closure during the hold (circles) with $t_{\text{hold}}$ (positive changes denote compaction). Also shown is the evolution of the dilation between the end of the hold and peak stress (squares) with $t_{\text{hold}}$. (D) The amount of slip occurring between the end of the hold and peak stress ($\Delta\delta_{\text{peak}}$) plotted as a function of $\ln t_{\text{hold}}$. Red colors represent the low and blue the high stiffness data for all plots. Note that the change in closure between the onset of the reslide and peak stress is a significant fraction of the total change during the hold. Also, $\Delta\delta_n$ clearly evolves with $\ln t_{\text{hold}}$ for both stiffnesses.
Figure D.4 (A) The argument of the Lambert W function in Eq. (D.24) for velocity weakening (red) and strengthening (blue) parameter combinations and $K = 1$ (dashed lines) and $K = 10$ (solid lines). The black dashed line shows $-\frac{1}{e}$. (B) The branches of Lambert W, principal in blue ($W_0$) and auxiliary in red ($W_{-1}$), for real valued arguments, the branch point is $-\frac{1}{e}$. Note that as $\phi$ decreases during holds, $-\beta \nu^{-\beta}$ increases from closer to $-\frac{1}{e}$ to closer to 0. The velocity weakening trajectories for Slip law holds follow the auxiliary branch $W_{-1}$ while the velocity strengthening trajectories follow the principal branch $W_0$. 

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Figure D.5 Evolution of frictional strength under the Slip law during a long hold (∼ 10^5 s) for different values of the normalized stiffness, $K = 1$ (blue), 10 (ochre) and 100 (green), and different values of $\beta$. (A) Velocity weakening, $\beta = 2$; (B) velocity neutral, $\beta = 1$; and (C) velocity strengthening, $\beta = 0.5$. The solid color lines are the numerically integrated values of $\psi$. The black dashed lines are the exact analytical solutions for $\psi$ derived using the numerical predictions of $\phi$ from the simulations. For $\beta \neq 1$ the solutions are represented by the branches of the Lambert W function (Eq (D.24)) and for $\beta = 1$ these are the parabolic trajectories given in Eq. (D.33).
Figure D.6 As in Figure D.5, but now the black dashed lines are the analytical approximations for $\psi$ from Eqs. (D.24) (for $\beta \neq 1$) and (D.33) (for $\beta = 1$), using Eq. (D.26) to estimate $\phi$ as a function of hold time.
Figure D.7 Evolution of frictional strength under the Nagata law during a long hold (∼ $10^5$ s) for different values of the normalized stiffness, $⟨K⟩ = 1$ (blue), 10 (ochre) and 100 (green), and different values of $⟨β⟩_r$. Top panel is for velocity weakening $⟨β⟩_r = 2$, (A) $c = 1$, (B) $c = 10$, (C) $c = 100$. Bottom panel is for velocity strengthening $⟨β⟩_r = 0.5$, (D) $c = 1$, (E) $c = 10$, (F) $c = 100$. The solid color lines are the numerically integrated values of $⟨ψ⟩$. The black dashed lines in (C) and (F) are the approximate, large $c$, analytical solutions for $⟨ψ⟩$ (Eq. (D.46)) calculated using the numerical predictions of $φ$ from the simulations.
Figure D.8 Estimations of stiffness from the Beeler et al. (1994) holds on granite. Change in friction ($\Delta \mu$) versus load point displacement ($\delta_{lp}$) since reload for (A) lower stiffness ($k_n$) and (B) higher stiffness data ($k_s$). The colors represent holds of different durations: blue – 300 s; green – 1000 s; red – 3000 s and cyan – 10000 s. The slope of the initial portion of this loading curve is the stiffness as long as no appreciable slip has commenced. Insets show the time windows used to obtain this slope. Note that we find $k_s \sim 0.055 \, \mu m^{-1}$ whereas Beeler et al. (1994) reported $k_s \sim 0.074 \, \mu m^{-1}$. 
Figure D.9 Fitting the velocity steps on initially bare granite from Beeler et al. (1994). The steps are all half order, the numbers in blue are sequence of load point velocities used in \( \mu m s^{-1} \). The velocity steps on the left panels ((A) and (C)) were imposed with the natural machine stiffness while those on the right ((B) and (D)) were imposed with the higher stiffness set up. (A) and (C) show fits where all the four steps (both stiffnesses together) were inverted simultaneously such that the misfits between onset and peak stress were proportionally weighted for all the steps. (C) and (D) show fits where the step in (D) was weighted preferentially and only the pre-step steady-state sliding was weighted in (C). Red curves show Slip and ochre Aging law fits. The light, dotted curves show the respective weight distributions. Both sets of weights constrain \( a - b \sim -0.0025 \), we choose \(-0.0027\) as our preferred value. The values of \( a \) and \( D_c \) change within a factor of 2 between the top and bottom panels.
Figure D.10 Fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ time series on granite from Beeler et al. (1994) with the strain rate dependence in $a$ specified in Eq. (D.49). (A) The actual fit to the data; solid lines denote Aging and dashed Slip laws. The color scheme specifying stiffness is identical to figures in the main text. (B) and (C) show the posteriors for $\gamma$ and $V^*$ (in $\mu\text{ms}^{-1}$) respectively. The blue area plots denote Slip law posteriors, the ochre area plots denote Aging law posteriors. (D) Actual variations in $a$ and $a - b$ during the holds and reslides. Pink curves denote $a$, blue $b$. Solid curves denote Aging law; dashed Slip law.
Figure D.11 Fits to the $\Delta \mu_{\text{peak}}$ and $\Delta \mu_{\text{hold}}$ data on granite from Beeler et al. (1994) with the second-order corrected friction law specified in Eq. (D.50). (A) The actual fit to the data, solid lines denote Aging and dashed Slip laws. The color denoting stiffness is the same as figures in the main text. (B) and (C) show the posteriors for $\mu_0$ and $V_*$ respectively. The blue area plots denote Slip law posteriors, the ochre area plots denote Aging law posteriors. (D) Actual variations in $\mu_{ss}$ with $V$ and $V_0$ which effectively denotes a strain rate dependence in $a - b$ (Eq. (D.51)). Solid curves denote Aging law, dashed Slip law.
Appendix E

Appendices to Chapter 5

E.1 Estimating stiffness and $a - b$ from independent constraints

Our experiments were carried out on the artificially stiffened rotary shear apparatus at Brown. In order to obtain an estimate of this higher stiffness, we used the initial loading curve of the reslides following a sequence of long holds carried out during the same experimental run. At the end of long holds, the block is sliding at rates orders of magnitude smaller than the pre-hold steady sliding rate $V_{s/r}$. Therefore, following the reslide, also at the rate $V_{s/r}$, there is an initial time window over which the slip rate of the block continues to satisfy $V \ll V_{s/r}$. During this initial portion of the reload, assuming quasi-static force balance between the driving shear stress and friction, we have

$$
\Delta \mu = k(\delta_{lp} - \delta) = k\delta_{lp} \left(1 - \frac{\delta}{\delta_{lp}}\right) \approx k\delta_{lp},
$$

(E.1)

where $\delta$ and $\delta_{lp}$ are surface and load point displacements since the reslide respectively, and $\delta/\delta_{lp} \ll 1$ for reslides following long holds. Note that $k$ is the stiffness normalized by normal stress and $\Delta \mu$ is the change in friction. A linear fit to the $\Delta \mu$ vs. $\delta_{lp}$ plot over this
initial portion following the reslide (after a long hold) gives \( k \) as the slope (Figure E.1). We use the reslides following a sequence of 3000 s and 10000 s holds carried out at three different values of \( V_{s/r} \) spanning more than an order of magnitude – 1, 0.3162 and 0.03162 \( \mu \text{ms}^{-1} \). For the linear fit, we chose a seventh of the total number of points between the onset of the reslide and eventual peak strength to evaluate \( k \). This number was chosen based on trial-and-error such that the spread in the estimated value of \( k \) was the least between our six chosen reslides. We found \( k \sim 0.065 \) to be the mean stiffness from our fits.

In Figure E.2 we show the estimation of \( a - b \) from a sequence of velocity steps at different amounts of slip. All the data considered in this study was collected after the total accumulated slip exceeded 120 mm. Beeler et al. (1996) have shown, that for an identical Westerly granite sample at 25 MPa, a stable, narrow, quasi-planar shear fabric develops in the gouge layer accumulated between the initially bare granite surfaces at such large total slip. Most of the continued shear is accommodated within this shear zone. Beyond 120 mm of slip, the sample was velocity weakening with \( a - b \) fluctuating between \( \sim -0.0023 \) and \( \sim -0.0035 \). We estimated an average \( a - b \sim -0.003 \).

### E.2 Estimating \( a \) and \( D_c \) by fitting the velocity step decreases

Comparing the dependence of the stress minima of the velocity step decreases on the size of the velocity steps with the corresponding Slip law prediction gives us a lower bound on \( a \) for our sample. In order to get better constraints on \( a \) and \( D_c \), we also fit a representative size range (3.5-1 orders of magnitude) of the velocity step decreases with the Slip law. The forward model is simulated by simultaneously solving the time derivative of the quasi-static force balance equation in Eq. (E.1) (where \( \Delta \mu \) is calculated using Eq. (5.1)) along with the Slip law state evolution from Eq. (5.2b). We fix \( k = 0.0065 \mu \text{m}^{-1} \). We invert for \( a, b \) and \( D_c \) simultaneously and jointly fit all the velocity step decreases shown in Figures E.3(A)-(E).
For the inversion, we use an adaptive proposal, small-world, Markov chain Monte Carlo code. The algorithm, and the general inversion procedure, is described in the Supplementary materials accompanying Bhattacharya et al. (2015). We minimize the weighted square root misfit between the modeled time series and the data at every time sample in the data. We only model the evolution of friction over the first 3 µm of slip since the onset of the velocity step. This saves computational time and also avoids fitting secondary long term transients in the evolution of friction present in the data. The data are sampled uniformly at 50 Hz in our experiments. This implies that the portions of the data which access lower sliding rates have many more samples per unit slip. Therefore, we weight the misfit for each velocity step inversely by the total number of data samples in the fitting window to ensure that all the velocity steps contribute to the aggregate misfit equally.

The fits are shown in Figures E.3(A)-(E). The Slip law does a reasonable job of fitting all the velocity steps. However, the \( \sim 3.5 \) and \( \sim 3 \) orders of magnitude velocity steps are worse fit than the smaller velocity steps. The value of \( a \) comes out to be around 0.013 and \( D_c \sim 2 \) µm. Figures E.3(F)-(G) show the slip rate evolution predicted by the corresponding fits to the friction data. All the fits capture the slip rate excursions reasonably well. Also, note that the value of \( a - b \) derived from the fits is \( \sim -0.0031 \) which is in good agreement with our a priori estimate from Appendix E.1.

**E.3  Relating instantaneous state evolution to zero instantaneous friction change in response to a normal stress step**

In this section we will show that in the Bowden-Tabor picture, where all contact area is considered of the same quality and where changes in contact strength come only from rate perturbations, a fractional change in normal stress that leads to an identical fractional change
in contact area leads to zero change in state. To do so, let us consider an instantaneous change in normal stress \( \Delta \sigma \) imposed at \( t = 0 \). As a result, the change in shear resistance across the step is:

\[
\Delta \tau_{fr} = \mu_0 \Delta \sigma + \sigma_0 \Delta \mu + \Delta \sigma \Delta \mu,
\]

where \( \Delta \mu \) is the resultant change in friction and the subscript 0 denotes pre-step values. Now, following Bowden and Tabor (1964), Linker and Dieterich (1992) and Baumberger et al. (1999), we write the interfacial shear resistance as a product of the purely rate dependent contact strength \( \tau_c \) and the real contact area \( A_c \):

\[
\tau_{fr} A = \tau_c A_c,
\]

where \( A \) is the nominal area of the interface. As mentioned above, in this decomposition, changes in the contact strength term \( \tau_c \) are usually equated to the direct rate effect by assuming an underlying Arrhenius rate rheology, while changes in the fractional contact area are considered to be proportional to log state. From, Eq. (E.3), we can express the change in shear resistance \( \Delta \tau_{fr} \) as:

\[
\Delta \tau_{fr} = \tau_{c0} \Delta A_c + A_{c0} \Delta \tau_{c0} + \Delta A_c \Delta \tau_{c0}.
\]

Equating Eqs. (E.2) and (E.4) and normalizing by the reference shear resistance (\( \sigma_0 \mu_0 = \tau_{c0} A_{c0} \)) on both sides yields:

\[
\frac{\Delta \sigma}{\sigma_0} + \frac{\sigma_f \Delta \mu}{\sigma_0 \mu_0} = \frac{\Delta A_c}{A_{c0}} + \frac{A_{c_f} \Delta \tau_{c0}}{A_{c0} \tau_{c0}},
\]

where the subscript \( f \) denotes values immediately following the normal stress step. Eq. (E.5) can be rearranged to write
On the right hand side of Eq. (E.6), as long as the nominal normal stress is proportional to real contact area (since force balance implies $\sigma A = \sigma_c A_c$, this is expected to hold as long as the contact normal stresses $\sigma_c$ hold constant), the term in the brackets is equal to 0 and the term in the parentheses is equal to 1. This gives us

$$\Delta \mu = \mu_0 \frac{\Delta \tau_c}{\tau_c^0}. \quad (E.7)$$

As the fractional change in $\tau_c$ is the contribution of the direct effect, Eq. (E.7) implies $\Delta \Phi = 0$.

Now, we will show that our data does imply an instantaneous change in state, and therefore we must modify the Bowden-Tabor picture to add some inherent quality to contact area. First, our data shows that at zero slip the change in shear resistance is zero in response to normal stress step. Therefore, at zero slip, we can write from Eq. (E.2):

$$\mu_0(\sigma_0/\sigma_f - 1) = \Delta \mu. \quad (E.8)$$

From RSF $\Delta \mu$, the change in friction is:

$$\Delta \mu = \mu_f - \mu_0 = a \log(V_f/V_0) + \Delta \Phi, \quad (E.9)$$

where the first term is the direct rate effect for slip rate change from $V_0$ to $V_f$ over zero slip in response to the normal stress step. The second term is the change in state at zero slip where our usual state variable $\theta$ is related to $\Phi$ as follows:

$$\Phi = b \log(V_s \theta/D_c). \quad (E.10)$$
From Eqs. (E.8) and (E.9), it follows that

$$\mu_0(\sigma_0/\sigma_f - 1) = a \log(V_f/V_0) + \Delta \Phi. \quad (E.11)$$

Therefore, any state evolution at zero slip that obeys Eq. (E.11) leads to zero increase in shear resistance at zero slip. However, if we set the zero slip change in state to be zero, we obtain the following estimate of $a$ from Eq. (E.11):

$$\hat{a} = \frac{\mu_0(\sigma_0/\sigma_f - 1)}{\log(V_f/V_0)}. \quad (E.12)$$

For our experiments, all the quantities on the right hand side are known, e.g. for the normal stress increases $\sigma_0/\sigma_f \approx 0.95$, $V_f/V_0 \approx 0.6$ (from Figures 5.6C and D) and $\mu_0 \approx 0.7$. These values put $\hat{a} \approx 0.07$ which is more than 5 times the value derived by fitting the velocity steps. For the normal stress step decreases, using $V_f/V_0 \approx 1.6$, Eq. (E.12) fixes $\hat{a} \approx 0.074$.

One way to get $\hat{a}$ down to ‘reasonable’ values is to reintroduce state change at zero slip. Then Eq. (E.12) would be revised as:

$$\hat{a} = \frac{\mu_0(\sigma_0/\sigma_f - 1) - \Delta \Phi}{\log(V_f/V_0)}. \quad (E.13)$$

Note that $\mu_0(\sigma_0/\sigma_f - 1) < 0$ for a normal stress increase. Therefore the only way we obtain ‘reasonable’ values of $\hat{a}$ is if $\Delta \Phi < 0$. Therefore, state must decrease at zero slip in response to the normal stress steps in our data. Note, that the constitutive relation of Linker and Dieterich (1992) also predicts an instantaneous decrease in state following a normal stress decrease and therefore has the correct sign.

However, this result also implies that the Bowden-Tabor picture has to be modified to admit this zero slip change in state. In Appendix E.4 we explicitly introduce a contact quality term in the Bowden-Tabor product decomposition to derive our own state evolution formulation in response to normal stress change.
E.4 Derivation of the state evolution response to variable normal stress from the rheology of a multi-contact interface

We showed in Appendix E.3 that our normal stress step data indicate that, given the observed magnitude of the slip rate excursions, we need to have state evolve without slip in order to explain the zero increase in shear resistance at zero slip with reasonable values of the direct effect \(a\). Though this is in spirit similar to the Linker and Dieterich (1992) formulation of state change in response to normal stress change, we do not share their physical interpretation that log state (\(\Phi\) in Appendix E.3) is the portion of the contact area that contributes to creep. We, instead, want to view state as a combination of some measure of contact quality and quantity. To give this physical picture a concrete mathematical form, we arbitrarily choose area averaged bond density \(n\) as contact quality which could be a function of contact age or contact slip among other things and is might be different between new and old contacts (Liu and Szlufarska, 2012). Contact quantity is the total microscopic contact area of the multi-contact interface \(A_c\) normalized by the nominal area \(A\). With this picture in mind, following Hatano (2015), we modify the Bowden and Tabor (1964) decomposition in Eq. (E.3) to express shear resistance as:

\[
\tau_{fr} = \tau_c n \Sigma_c, \quad (E.14)
\]

where \(\tau_c\) is reinterpreted as the average strength of individual bonds, \(n\) is the area averaged bond density and \(\Sigma_c\) is \(A_c/A\). Hatano (2015) argue that by connecting the local shear rate on each bond to the force acting on it, one can derive a logarithmic rate dependence for \(\tau_c\) of the form consistent with the classical direct rate effect form:

\[
\tau_c = \frac{E}{l_v} + \frac{k_BT}{l_v} \log \left( \frac{V}{V_s} \right), \quad (E.15)
\]
where $E$ is the average activation energy for bond dissociation, $l$ is some length constant representing some displacement required to break bonds, $\nu$ is a characteristic area and $V^*$ is a reference slip rate. We need to derive the change in state in response to normal stress change alone, so we are going to hold $\tau_c$ constant (i.e. no change in slip rate). With this assumption, the normalized change in shear resistance in response to a normal stress step can be written as:

$$\frac{\Delta \sigma}{\sigma_0} + \frac{\sigma_f \Delta \mu}{\sigma_0 \mu_0} = \frac{\Delta \Sigma_c}{\Sigma_{c0}} + \frac{\Delta n}{n_0} + \frac{\Delta n \Delta \Sigma_c}{n_0 \Sigma_{c0}},$$

(E.16)

where the subscript ‘0’ denotes pre-step values and ‘f’ the corresponding value immediately following the step. Now, the average bond density can change instantaneously due to a normal stress step if the change in contact area is accompanied by a change in $n$, e.g. if the new area brought into contact by a normal stress step increase has a different density of bonds than what was already existing on the contact. Then $\Delta n/n_0$ is intrinsically connected to $\Delta \Sigma_c/\Sigma_{c0}$ as follows:

$$\frac{\Delta n}{n_0} = \frac{1}{n_0} \left[ n_f - n_0 \right],$$

$$= \frac{1}{n_0} \left[ \frac{n_\Delta \Delta \Sigma_c + n_0 \Sigma_{c0}}{\Delta \Sigma_c + \Sigma_{c0}} - n_0 \right],$$

(E.17)

where $n_0$ is the average density of bonds on the pre-step population of contacts, $n_f$ is the average density of bonds on the post-step population of contacts at zero slip and $n_\Delta$ is either the average bond density of the area brought into contact by a normal stress increase or the area separated from contact by a normal stress decrease at zero slip. As in Appendix E.3 we assume that normal stress is proportional to the fractional real contact area and use Eqs. (E.14) and (E.17) to write:
where at constant slip rate $\Delta \mu = \Delta \Phi$ and $\eta = 1 - n_\Delta/n_0$. Note that both for the loss or gain of contact area, $\Delta \Phi$ is opposite in sign to $\Delta \sigma$ as long as $n_\Delta/n_0 < 1$. This seems a priori reasonable since, if $n$ is indeed a function of contact age; For a normal stress increase, the contact area that is added might be expected to have a smaller average density of bonds due to its younger age, while for a normal stress decrease, the area of the contact which loses contact might be expected to be at the rims of the existing contact and thus younger on average than the contact interior. If on the other hand all contacts, old and new, had the same strength (like the classical Bowden-Tabor picture), $n_\Delta = n_0$. Under these conditions, Eq. (E.18) predicts no instantaneous state evolution in response to a normal stress step as expected from our discussion in Appendix E.3. When there is a quality contrast between old and new contacts, the term $1 - n_\Delta/n_0$ is in a sense a ‘dilution factor’ for contact quality: for normal stress increases it decreases the average quality of the contacts while doing the opposite for normal stress increases. The discrete form of Eq. (E.18) can be transformed to a continuous form for modeling purposes

\[
\frac{d\theta}{d\sigma} = -\mu_\theta \frac{\sigma}{b\sigma_f} \eta \frac{1}{\sigma_f},
\]  

(E.19)

where $\sigma$, $\mu$ and $\theta$ are the values of the variables at the beginning and $\sigma_f$ the value of normal stress at the end of the time interval over which the differential is being evaluated. Following the suggestions of Linker and Dieterich (1992), we couple Eq. (E.19) with prescriptions of state evolution with slip at constant normal stress to describe the full,
post-step, evolution of shear resistance back to steady state:

\[
d\theta = \left( \frac{\partial \theta}{\partial \delta} \right)_\sigma \ d\delta + \left( \frac{\partial \theta}{\partial \sigma} \right)_\delta \ d\sigma,
\]

\[
= \left( \frac{\partial \theta}{\partial t} \right)_\sigma \ dt - \mu \theta \ \frac{\sigma}{b \sigma_f} \eta \frac{d\sigma}{\sigma}.
\]  

(E.20)

The functional form of $\partial \theta / \partial t$ can be any of the widely used state evolution laws. We use the Aging and Slip laws for some rudimentary inversions described below.

The full test of a constitutive equation such as Eq. (E.18) will require additional experimental data to test its validity under a wider range of normal stress perturbations. But at present, we will first point out that, with the constitutive equation in Eq. (E.18) representing the instantaneous change in state in response to normal stress change and either the Aging or Slip laws prescribing the subsequent state evolution with slip, the remarkable symmetry observed in the shear stress evolution with slip between the normal stress increases and decreases cannot be explained with $\eta = 0$ (Figure E.4A and B). The corresponding slip rate variations are also much larger when compared to the data in Figures 5.6C and D. Second, as a proof of concept, we fit the shear stress response to our normal stress steps (Figure E.5) using the combination of Eq. (E.18) and Aging and Slip laws. We use the same adaptive proposal, small world Markov Chain Monte Carlo that we used for fitting the velocity steps in Appendix E.2. Both the combinations fit the data well with parameter values which predict instantaneous change in state in response to the normal stress perturbations. We allowed the inversions the freedom of two different values of $\eta$ for the up and down steps; $\eta_{up}$ and $\eta_{down}$, in recognition of the possibility that the bond density on the contacts being lost and those being added might be different. Interestingly, the Aging law chooses $\eta_{up}$ about twice as large as $\eta_{down}$ while the Slip law fit chooses $\eta_{up}$ only 10% larger than $\eta_{down}$. It is probably intuitive that $\eta_{up}$ would be larger than $\eta_{down}$ given the expectation that the density of bonds might be larger on the older contacts being lost in a down step than on those being freshly added in an up step.
The full Slip and Aging law posteriors for the fits in Figure E.5 are shown in Figure E.6.

Note that the values of $a$ and $D_c$ derived from fitting the normal stress steps do not match those derived from the velocity steps. The Slip law does slightly better in this regard since the velocity step derived parameters are within the tail of the corresponding normal stress step derived posteriors. But, given the widths of these posteriors are sensitive to the poorly known level of data error, we are not very confident about the tails.
Figure E.1 Estimation of stiffness for the servo-controlled, artificially stiffened, rotary shear apparatus. (A) Change in friction ($\Delta \mu$) since reslide versus load point displacement ($\delta_{lp}$) since reslide for reslides following holds of $\sim 3000$ s and $\sim 10000$ s at three reslide rates each – squares at $1 \mu\text{ms}^{-1}$, crosses at $0.3162 \mu\text{ms}^{-1}$ and dots at $0.03162 \mu\text{ms}^{-1}$. The initial reloading rate fixes the stiffness; we used $1/7$th of the total number of points between the onset of reslide and eventual peak strength for the fits. (B) A zoomed in version of the fits in (A). We use a stiffness of $0.065 \mu\text{m}^{-1}$ in our analyses.
Figure E.2 Estimation of $a - b$ from a subset of the sequence of the velocity steps. (A)-(F) Velocity step increases and decreases of 1 and 1.5 order showing steady state velocity weakening beyond total accumulated slip (the numbers in blue in each panel) of 120 mm. The 1 order velocity step decrease in (F) was imposed between the two sets of normal stress steps at $0.316 \mu \text{ms}^{-1}$ and $0.0316 \mu \text{ms}^{-1}$. (G) $\Delta \mu_{ss}$ versus $\ln(V_{\text{final}}/V_{\text{initial}})$ from a sequence of velocity steps including those shown in (A)-(F). Mean $a - b \sim -0.003$. 

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Figure E.3 Fitting a sequence of large velocity step decreases with the Slip law jointly with the same set of parameters. We modeled only the first 3 μm of slip following the onset of the velocity step to avoid potential problems with the long-term stress transients in some of the steps. (A)-(E) Fits to the stress data, (F)-(J) predictions of slip rate from the corresponding fits to the stress data. Blue – data; red – Slip law fit. The data were weighted to ensure that the misfits for all the velocity steps got equal weights. Note that \( a - b \) was not pre-constrained, the inversion by itself comes up with a value which agrees well with our independent, a priori estimate of \( a - b \).
Figure E.4  Frictional response of an interface sliding at constant rate when subjected to a 5% normal stress step. State evolution was computed by coupling Eq. (5.10) with the Aging (A and C) and Slip (B and D) laws. The simulations were carried out at two different sliding rates – $V_{lp} = 0.3162 \, \mu m s^{-1}$ (solid lines) and $V_{lp} = 0.03162 \, \mu m s^{-1}$ (dashed lines). The simulated data points were generated at 50 Hz sampling rate and then smoothed identically to the data in Figure 5.5. Top panels show shear stress response, bottom panels the resultant excursions in slip rate. The lines with dot markers (which appear as bold lines where dots are dense) are the response to step decreases while those without the dots are step increases. For all the simulations, $\eta_{up} = \eta_{down} = 0$. The values of $a$, $b$ and $D_c$ are borrowed from the Slip law fits to velocity steps. Unlike the data in Figure 5.5 there is significant asymmetry between the normal stress increases and decreases, and the velocity excursion is larger – large enough to give the appearance of a discontinuity in shear stress at zero slip for the step increases.
Figure E.5 Markov Chain Monte Carlo fits to the shear stress evolution following the normal stress steps. The state evolution response to the normal stress step is assumed to be Eq. (E.18) with the instantaneous response $\eta = 1 - n\Delta/n_0$. The state evolution with slip is modeled with the Aging and Slip laws. We invert for $a, D_c, \eta_{up}, \eta_{down}$, and $\mu_s$. $\eta_{up}$ is the strength dilution for up-steps and $\eta_{down}$ for down steps. Blue-data; ochre-Aging Law fits; red-Slip law fits. The blue dotted lines in the background are the relative weights used to design the residual; the weights decay exponentially with slip from the onset of the normal stress step. (A)-(D) $V_{lp} = 0.3162\mu\text{ms}^{-1}$, (E)-(H) $V_{lp} = 0.03162\mu\text{ms}^{-1}$. Note that while the fits are only shown to the first $10\mu\text{m}$ of slip, there is a long term evolution in the data which can’t be explained by these laws with the values of $D_c$ obtained by fitting the velocity steps.
Figure E.6 Posterior distributions for the parameters from the Markov Chain Monte Carlo chains which yielded the fits in Figure E.5 as maximum a-posteriori solutions. The distributions with red outlines used the Slip law, the ochre posteriors used the Aging. The solid squares of the same colors as the distributions show the maximum a posteriori solutions used for the fits in Figure E.4. (A) $P(a|data)$, (B) $P(D_c|data)$, (C) $P(\eta|data)$ with the solid outlines representing $\eta_{up}$ and the dashed outlines $\eta_{down}$ and (D) $P(\mu^*|data)$. In panels (A) and (B), the dashed, vertical black line shows the values of the corresponding parameters derived from the velocity steps. Note that the values of $a$ and $D_c$ yielded by both the Aging and Slip laws do not agree well with those derived from the velocity steps. But the Slip law derived values are marginally better since the values derived from the velocity steps are within the tail of the corresponding marginal posteriors.
Bibliography


Bhattacharya, P., A. M. Rubin, M. M. Scuderi, J. Leeman, K. Ryan, and C. Marone (2014b), The role of stressing rate in state evolution under rate-state friction, in *American Geophysical Union Fall Meeting*.


Hatano, T. (2015), Rate and state friction law as derived from atomistic processes at asperities.


Walsh, J. B., and D. L. Goldsby (2008), Modeling the mechanics of rate and state friction with linear viscoelasticity, *J. Geophys. Res.: Solid Earth*, 113(B9), n/a–n/a, doi:10.1029/2007JB005160, b09408.