Kähler-Einstein metrics, Bergman metrics, and higher alpha-invariants

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Abstract

The question of the existence of Kähler-Einstein metrics on a Kähler manifold $M$ has been a subject of study for decades. The Kähler manifolds on which this question may be studied divide naturally into three types. For two of these types the question was long ago settled by Yau and Aubin. For the third type, Fano manifolds, the question is (despite great recent progress) open for many individual manifolds.

In the first part of this thesis we define algebraic invariants $B_{m,k}(M)$ of a Fano manifold $M$, which codify certain properties of $M$’s Bergman metrics. We prove a criterion (Theorem 1.1.1) in terms of these invariants $B_{m,k}(M)$ for the existence of a Kähler-Einstein metric on $M$. The proof of Theorem 1.1.1 relies on Székelyhidi’s deep recent partial $C^0$-estimate, and on a new family of estimates for Fano manifolds.

We furthermore introduce a very general hypothesis on Bergman metrics, Conjecture 6.1.2, offering some partial results (Section 6.3) in evidence. Modulo this conjecture, we prove a variation of Theorem 1.1.1, which gives a criterion for the existence of a Kähler-Einstein metric on $M$ in terms of the well-known alpha-invariants, $\alpha_{m,k}(M)$. This result extends a theorem of Tian.

The second part of this thesis concerns Riemannian manifolds more generally. We give a characterization (Theorem 1.2.1) of conformal classes realizing a compact manifold’s Yamabe invariant. This characterization is the analogue of an observation of Nadirashvili for metrics realizing the maximal first eigenvalue, and of Fraser and Schoen for metrics realizing the maximal first Steklov eigenvalue.
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Chapter 1

Introduction

This thesis has two parts. Chapters 2-6 concern Kähler-Einstein metrics, and incorporate the preprint [Mac14b]. Chapters 7-9 concern the Yamabe invariant, and incorporate the preprint [Mac14a].

1.1 Kähler-Einstein metrics

An old idea, originating perhaps with the Uniformization Theorem and with Hodge theory, is that the solutions to natural partial differential equations on a manifold should reflect that manifold’s topological and algebro-geometric invariants. One partial differential equation long studied from this point of view is the equation characterizing those Riemannian metrics on a complex manifold which are Kähler-Einstein.

A rough necessary algebro-geometric criterion for the existence of a solution to this equation was quickly established [Cal57]: the manifold should be either Calabi-Yau; or an algebraic variety with ample canonical bundle; or an algebraic variety with ample anticanonical bundle. In the first two cases no further hypotheses are required; Yau and Aubin [Aub76b, Yau77, Aub78, Yau78] proved the existence of Kähler-Einstein metrics on these manifolds, developing foundational analytic techniques in the process. In the third case, the case of Fano manifolds, further hypotheses are required; a precise (both necessary and sufficient) and purely algebro-geometric criterion, K-stability, was developed conjecturally over many decades, and very recently proved [CDS15a, CDS15b, CDS15c, Tia].

K-stability, however, is in practice very difficult to verify, and so it is still a challenging question to determine whether a given Fano manifold admits a Kähler-Einstein metric. For example, the existence of a Kähler-Einstein metric is conjectured but not proven for many degree-\((n+1)\) hypersurfaces in \(\mathbb{CP}^{n+1}\) [CPW14],
for certain deformations of the Mukai-Umemura manifold [Don08, p45], or for various moduli spaces of semistable bundles on Riemann surfaces [Hwa00, Iye11]. For this reason, it is natural to work on developing simpler and more explicit (though less general) algebro-geometric criteria for the existence of Kähler-Einstein metrics on a Fano manifold $M$.

In Chapters 2-6 of this thesis we develop one such criterion. We define (in Section 4.4) algebraic invariants $B_{m,k}(M)$ of a Fano manifold $M$, which codify certain properties of the Bergman metrics (defined in Chapter 3) of $M$. We then prove the following theorem:

**Theorem 1.1.1.** Let $M$ be a Fano manifold of dimension $n$. Let $k$ be a natural number, with $2 \leq k \leq n$. Then there exists a natural number $m$, and a (explicitly computable) real number $\epsilon = \epsilon(n, k, B_{m,k}(M))$, such that if

$$B_{m,k}(M) > \frac{n}{n+1},$$

$$B_{m,1}(M) > \frac{n}{n+1} - \epsilon,$$

then $M$ admits a Kähler-Einstein metric.

Theorem 1.1.1 in fact follows from a slightly more general existence theorem (Theorem 4.5.1) by Székelyhidi's deep recent partial $C^0$-estimate Theorem 3.3.2.

The inspiration for Theorem 1.1.1 is a theorem of Tian for $k = 2$ [Tia91], involving the alpha-invariants $\alpha_{m,k}(M)$ (described in Chapter 5), which was used by him in proving the existence of Kähler-Einstein metrics on the last few dimension-2 manifolds for which that question had been open [Tia90a]:

**Theorem 1.1.2** ([Tia90a, Tia91], combined with the partial $C^0$-estimate of [Sze]). Let $M$ be a Fano manifold of dimension $n \geq 2$. There exists a natural number $m$, and a (explicitly computable) real number $\epsilon = \epsilon(n, \alpha_{m,2}(M))$, such that if

$$\alpha_{m,2}(M) > \frac{n}{n+1},$$

$$\alpha_{m,1}(M) > \frac{n}{n+1} - \epsilon,$$

then $M$ admits a Kähler-Einstein metric.

As we outline in Chapter 6, $\alpha_{m,1}(M) = B_{m,1}(M)$ (Proposition 6.2.2), and $B_{m,2}(\mathcal{L}) \geq \alpha_{m,2}(\mathcal{L})$ (Propositions 6.2.2 and 6.3.1), so this theorem may be deduced from Theorem 1.1.1. More generally, we introduce a
conjectural very general property of Bergman metrics (Conjecture 6.1.2), given which it follows (Proposition 6.2.2) that for all \( k \), \( \mathcal{B}_{m,k}(L) \geq \alpha_{m,k}(L) \). We therefore have:

**Theorem 1.1.3.** Suppose that Conjecture 6.1.2 holds. Let \( M \) be a Fano manifold of dimension \( n \). Let \( k \) be a natural number, with \( 2 \leq k \leq n \). Then there exists a natural number \( m \), and a (explicitly computable) real number \( \epsilon = \epsilon(n,k,\alpha_{m,k}(M)) \), such that if

\[
\alpha_{m,k}(M) > \frac{n}{n+1}, \quad \alpha_{m,1}(M) > \frac{n}{n+1} - \epsilon,
\]

then \( M \) admits a Kähler-Einstein metric.

Conjecture 6.1.2 is simple, natural, and of considerable independent interest, and its proof in general would be expected to combine analytic and algebraic ideas. We hope to address it in future work.

Another natural avenue for future work is the question of finding a Fano manifold \( M \) (or many such \( M \)) which satisfy the alpha-invariants criterion of Theorem 1.1.3, and which had not previously been known to admit Kähler-Einstein metrics. Since Tian’s Theorem 1.1.2 was enough to solve the Kähler-Einstein problem for surfaces, such a manifold \( M \) would be of complex dimension at least 3. Recent work in this direction includes computations by Cheltsov, Kosta, Shi and others (e.g. [Che09, Shi10, CK14]), of \( \alpha_{m,1}(M) \) for some Fano 3-folds \( M \) and \( \alpha_{m,2}(M) \) for all Fano 2-folds \( M \). Perhaps some of their methods can be adapted for higher alpha-invariants.

The organization of Chapters 2-6 is as follows. Chapters 2, 3 and 5 collect prerequisite facts about, respectively, Kähler-Einstein metrics, Bergman metrics, and alpha-invariants. With the possible exception of Proposition 2.9.1, this material is standard.

In Chapter 4, the invariants \( \mathcal{B}_{m,k}(M) \) are defined and Theorem 1.1.1 is proved. Chapter 6 is devoted to the statement, consequences, and evidence of Conjecture 6.1.2.

### 1.2 Conformal classes realizing the Yamabe invariant

Three decades ago, Schoen’s ground-breaking solution [Sch84] of the Yamabe problem established that, within any conformal equivalence class \( c \) of Riemannian metrics on a compact smooth \( n \)-manifold \( M \), the
total scalar curvature functional

\[ g \mapsto \frac{\int_M R(g) \, d\text{Vol}_g}{(\int_M d\text{Vol}_g)^{1-2/n}} \]

attains its infimum. The quantity

\[ I(c) := \min_{g \in c} \frac{\int_M R(g) \, d\text{Vol}_g}{(\int_M d\text{Vol}_g)^{1-2/n}}, \]

which is therefore well-defined, is known as the the \textit{Yamabe constant} of \( c \), and metrics \( g \) attaining this minimum are referred to as \textit{Yamabe metrics}. Computing the Euler-Lagrange equation of the restriction of the total scalar curvature functional to \( c \) shows that Yamabe metrics have constant scalar curvature.

One can study the properties of \( I(c) \) as it varies over all conformal classes \( c \) on a smooth manifold. In particular the \textit{Yamabe invariant} of \( M \) is defined to be the minimax expression

\[ Y(M) := \sup_c I(c) = \sup_c \min_{g \in c} \frac{\int_M R(g) \, d\text{Vol}_g}{(\int_M d\text{Vol}_g)^{1-2/n}}; \]

This quantity is finite, as follows from the observation of Aubin [Aub76a] that \( I(c) \leq n(n-1)\omega_n^{2/n} \) for all \( c \) (it is a corollary of the solution of the Yamabe problem that equality holds if and only if \( c \) is the conformal class of the round sphere).

In Chapters 7-9 of this thesis we study manifolds \( M \) whose Yamabe invariant \( Y(M) \) is attained. Our result is the following algebraic constraint on the set of Yamabe metrics in a conformal class attaining the Yamabe invariant:

\textbf{Theorem 1.2.1.} \textit{Let} \( M \) \textit{be a compact smooth} \( n \)-\textit{manifold,} \( c \) \textit{a conformal class attaining} \( Y(M) \), \textit{and} \( \mathcal{F} \) \textit{the set of unit-volume Yamabe metrics in} \( c \). \textit{There exists a (positive) measure} \( \mu \) \textit{on} \( \mathcal{F} \), \textit{such that}

\[ 0 = \int_{\mathcal{F}} (\text{Ric}(g) - \frac{1}{n} R(g)g) \, |d\text{Vol}_g|^{1-2/n} d\mu(g). \]

\textbf{Remarks.} 1. Here \( |d\text{Vol}_g| \) is the density associated to the volume form \( d\text{Vol}_g \); this density is a positive section of the density bundle \( |\Lambda^n M| \), an oriented line bundle, and so \( |d\text{Vol}_g|^{1-2/n} \) is a well-defined section of \( |\Lambda^n M|^{1-2/n} \).

2. The integral is in the sense of Pettis; see Section 9.4.

3. More explicitly: let \( g \) be a representative of \( c \); and \( \mathcal{U} \) the set of positive smooth functions \( u \) such that
$(u^2g)$ is a unit-volume Yamabe metric in $c$; then there exists a measure $\mu$ on $\mathcal{U}$, such that

$$0 = \int_{\mathcal{U}} u^{n-2} \left[ \text{Ric}(u^2g) - \frac{1}{n} R(u^2g)u^2g \right] d\mu(u).$$

**Corollary 1.2.2.** Let $M$ be a compact smooth $n$-manifold, and $c$ a conformal class attaining $Y(M)$, and suppose that there are, up to rescaling, only finitely many Yamabe metrics in $c$. Then there exist $N \in \mathbb{N}$, and Yamabe metrics $(g_i)_{1 \leq i \leq N}$ (not necessarily unit-volume) in $c$, such that

$$0 = \sum_{i=1}^{N} (\text{Ric}(g_i) - \frac{1}{n} R(g_i)g_i) |d\text{Vol}_{g_i}|^{1-2/n}.$$

**Proof.** Write $\mathcal{F}$ explicitly as $(\bar{g}_j)$. By Theorem 1.2.1 there exist nonnegative reals $(a_j)$ such that $0 = \sum_j a_j (\text{Ric}(\bar{g}_j) - \frac{1}{n} R(\bar{g}_j)\bar{g}_j) |d\text{Vol}_{\bar{g}_j}|^{1-2/n}$. Drop those $\bar{g}_j$ for which $a_j = 0$, and rescale the rest. 

Theorem 1.2.1 generalizes two previous results. In the case when the class $c$ contains exactly one unit-volume Yamabe metric (i.e., $|\mathcal{F}| = 1$), Theorem 1.2.1 is equivalent to the following well-known result:

**Lemma 1.2.3** (see for example [Sch89, discussion preceding Lemma 1.2]). Suppose that the conformal class $c$ attains $Y(M)$, and suppose that there is, up to rescaling, exactly one Yamabe metric $g$ in $c$. Then $g$ is Einstein.

**Proof of equivalence.** The tensor $\text{Ric}(g) - \frac{1}{n} R(g)g$ vanishes precisely when $g$ is Einstein. 

In the case when the class $c$ contains, up to rescaling, exactly two Yamabe metrics (i.e., $|\mathcal{F}| = 2$), we may write them explicitly as $g$ and $u^2g$; then Theorem 1.2.1 reduces (possibly after rescaling) to the following relationship, previously derived by Anderson [And05, equation 2.37]:

$$u^{-1}(1 + u^{n-2})[\text{Ric}(g) - \frac{1}{n} R(g)g] = -(n - 2)[\text{Hess}_g(u^{-1}) - \frac{1}{n} \Delta_g(u^{-1})].$$

The organization of Chapters 7-9 is as follows. In Chapter 7 are proved versions of various standard inequalities (the Sobolev and $L^p$ estimates) on Riemannian manifolds, in which the dependence of the constants on the metric is explicit. Similarly, in Chapter 8 are proved versions of Aubin’s estimates for constant scalar curvature metrics, in which the dependence of the constants on the conformal class is explicit. To our knowledge most theorems in these two chapters does not appear in the literature, although they are straightforward extensions of theorems which do.
Chapter 9 contains a proof, and some discussion, of Theorem 1.2.1. The proof of Theorem 1.2.1 appears in Section 9.5. In Section 9.6 we describe an interpretation of Theorem 1.2.1, as a nonlinear analogue of theorems of Nadirashvili and Fraser-Schoen on Laplace and Steklov eigenfunctions. In Section 9.7 we discuss a possible application to the existence of Einstein metrics.
Chapter 2

Classical estimates

Throughout Chapters 2-6 of this thesis $M$ will be a compact Kähler manifold of dimension $n$. When there is no ambiguity as to the Kähler class $a$ being considered, we write $V$ for the volume $\int_M a^n$, and $f_M$ for the averaged integration operator $V^{-1} \int_M$.

When a reference Kähler metric $\omega$ is fixed, we describe other Kähler metrics in the class by means of their potential functions $\varphi \in C^\infty(M, \mathbb{R})$, with the notation

$$\omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi.$$

2.1 Preliminaries

In this section, except where otherwise specified, we work on a compact Riemannian manifold $(M, g)$.

**Theorem 2.1.1** (Croke [Cro80], Li [Li80]). Let $\lambda$ and $V$ be positive reals.

(i) There exists a constant $C_0 = C_0(n, V, \lambda)$,

(ii) There exist constants $C_1 = C_1(n, V, \lambda)$, $C_2 = C_2(n, V, \lambda)$,

such that if $\int_M d\text{Vol}_g = V$ and $\text{Ric}(g) \geq \lambda g$, then

(i) for all functions $f$ on $M$ such that $\int_M f d\text{Vol}_g = 0$,

$$\left( \int_M |f|^{2n/(n-2)} d\text{Vol}_g \right)^{\frac{n-2}{n}} \leq C_0 \int_M |df|^2 d\text{Vol}_g.$$
(ii) For all functions \( f \) on \( M \),

\[
\left( \int_M |f|^{2n/(n-2)} \, d\text{Vol}_g \right)^{n-2} \leq C_1 \int_M |df|^2 \, d\text{Vol}_g + C_2 \int_M |f|^2 \, d\text{Vol}_g .
\]

Denote by \( G(x, y) \) the Green’s function of \((M, g)\). Our convention is to normalize \( G(x, y) \) so that for all \( x \in M \),

\[
\Delta_g G(x, \cdot) = \delta_x - \frac{1}{\text{Vol}(g)}
\]

\[
\int_{y \in M} G(x, y) \, d\text{Vol}_g(y) = 0.
\]

**Corollary 2.1.2.** Let \( \lambda \) and \( V \) be positive reals. There exists a constant \( C = C(n, V, \lambda) \), such that if \( \int_M d\text{Vol}_g = V \) and \( \text{Ric}(g) \geq \lambda g \), then for all \( x, y \in M \), \( G(x, y) \leq C \).

**Proof.** Follows from Theorem 2.1.1 (i) by Hölder’s inequality.

**Proposition 2.1.3.** Let \( \lambda \) be a positive real. There exists a constant \( C = C(n, V, \lambda) \), such that if \( \omega \) is a Kähler metric with \( \int_M \omega^n = V \) and \( \text{Ric}(\omega) \geq \lambda \omega \), then for all potentials \( \varphi \in C^\infty(M, \mathbb{R}) \) such that the \((1, 1)\)-form \( \omega_\varphi \) is Kähler,

\[
\sup_M \varphi \leq \int_M \varphi \omega^n + C.
\]

**Proof.** Since \( \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) is Kähler,

\[
0 < \text{tr}_\omega(\omega + \sqrt{-1} \partial \bar{\partial} \varphi) = n + \Delta_\omega \varphi.
\]

By Corollary 2.1.2, there exists a constant \( C = C(n, V, \lambda) \) such that for each point \( x \in M \), the function \( G(x, \cdot) - C \) is nonpositive. Thus for each point \( x \in M \), using the properties of \( G(x, y) \),

\[
0 \geq \int_{y \in M} [G(x, y) - C] [n + (\Delta_\omega \varphi)(y)] \, d\text{Vol}_g(y)
\]

\[
= n \int_{y \in M} [G(x, y) - C] \, d\text{Vol}_g(y) + \int_{y \in M} [G(x, y) - C] (\Delta_\omega \varphi)(y) \, d\text{Vol}_g(y)
\]

\[
= n \int_{y \in M} G(x, y) \, d\text{Vol}_g(y) - nC \int_{y \in M} \, d\text{Vol}_g(y) + \int_{y \in M} [\Delta_\omega G(x, \cdot)](y) \varphi(y) \, d\text{Vol}_g(y)
\]

\[
= 0 - nCV + \left( \varphi(x) - \frac{1}{V} \int_{y \in M} \varphi(y) \, d\text{Vol}_g(y) \right).
\]

Rearranging, we have that for all \( x \in M \), \( \int_M \varphi \omega^n + nCV \geq \varphi(x) \).
**Theorem 2.1.4** (Lichnerowicz, [Lic58]). Let $\lambda$ be a positive real, and suppose that $\text{Ric}(g) \leq \lambda g$. Then the first nonzero eigenvalue of the Laplace-Beltrami operator $\Delta_g$ is at least $\lambda$.

### 2.2 Continuity methods

In this chapter we study the continuity methods of [Yau77, Yau78] for two classical partial differential equations in Kähler geometry.

1. $\text{Ric}(\omega_t) = t\chi + (1 - t)\text{Ric}(\omega)$. This interpolates between (at $t = 0$) the tautological equation and (at $t = 1$) the **Calabi problem**.

2. $\text{Ric}(\omega_t) = t\omega_t + (1 - t)\omega$. This interpolates between (at $t = 0$) a special case of the Calabi problem and (at $t = 1$) the **Kähler-Einstein equation** on a Fano manifold.

In this section we set up these continuity methods in more detail.

**The Calabi problem**

Let $a \in H^{1,1}(M, \mathbb{R})$ be a Kähler cohomology class, and let $\chi$ be a $(1, 1)$-form in $2\pi c_1(M)$. The **Calabi problem** is to find a Kähler metric $\omega \in a$ such that $\text{Ric}(\omega) = \chi$.

To seek solutions to this problem, we select a reference Kähler metric $\omega \in a$, and study the one-parameter family of equations

$$\text{Ric}(\omega_t) = t\chi + (1 - t)\text{Ric}(\omega).$$

At $t = 0$ this is the tautological equation $\text{Ric}(\omega_t) = \text{Ric}(\omega)$, solved by $\omega_t = \omega$. At $t = 1$ this is the Calabi problem.

We may convert this equation to an equivalent system of equations on the Kähler potentials. Let $V := \int_M a^n$. Choose a function $f$ such that $\text{Ric}(\omega) - \chi = \frac{1}{n} \partial \bar{\partial} f$ and $\int_M e^f \omega^n = V$. For each $t \in [0, 1]$, let $c_t$ be the constant $-\log (\int_M e^f \omega^n)$, so that $\int_M e^{f + c_t} \omega^n = V$. (By Jensen’s inequality $c_t \leq -t \log (\int_M e^f \omega^n) = -t \log 1 = 0$.) Then the equation is equivalent to the equation

\[
\begin{align*}
\omega_{\varphi_t} &= e^{f + c_t} \omega^n, \\
\int_M \varphi_t \omega^n &= V,
\end{align*}
\]

(*$_t$)
Kähler-Einstein metrics on Fano manifolds

Let $M$ be a Fano manifold. The Kähler-Einstein problem on $M$ is to find a Kähler metric $\omega \in 2\pi c_1(M)$ such that $\text{Ric}(\omega) = \omega$.

To seek solutions to this problem, we select a reference Kähler metric $\omega \in 2\pi c_1(M)$, and study the one-parameter family of equations

$$\text{Ric}(\omega_t) = t\omega + (1-t)\omega.$$ 

At $t = 0$ this is the equation $\text{Ric}(\omega_t) = \omega$, the special case $a = 2\pi c_1(M)$, $\chi = \omega$ of the Calabi problem. At $t = 1$ this is the Kähler-Einstein problem.

We may convert this equation to an equivalent system of equations on the Kähler potentials. Let $\varphi$ be a solution at $t = 0$.

$$\omega_{\varphi_t}^n = e^{f - t\varphi} \omega^n. \quad (\ast_t)$$

We will also have use for an alternative equation on potentials:

$$\omega_{\varphi_t}^n = e^{f - t\varphi - \beta(f_0 \psi t \omega^n)} \omega^n. \quad (\ast'_t)$$

(Here $\beta$ is a fixed nonzero real, and $\varphi_0$ a solution at $t = 0$.)

2.3 Implicit function theorem

The Calabi problem

Let $C_0^\gamma(M, \mathbb{R})$, $C_0^{2, \gamma}(M, \mathbb{R})$ be the subspaces of (respectively) the Hölder spaces $C^\gamma(M, \mathbb{R})$, $C^{2, \gamma}(M, \mathbb{R})$ containing those functions $\psi$ such that $\int_M \psi \omega^n = 0$. Since $\psi \mapsto \int_M \psi \omega^n$ is bounded as a functional on $C^\gamma(M, \mathbb{R})$ or $C^{2, \gamma}(M, \mathbb{R})$, the subspaces $C_0^\gamma(M, \mathbb{R})$, $C_0^{2, \gamma}(M, \mathbb{R})$ are closed, and therefore Banach with respect to the restriction norm.

Consider the differential operator $F : C_0^{2, \gamma}(M, \mathbb{R}) \oplus \mathbb{R} \to C_0^\gamma(M, \mathbb{R})$,

$$F_*(\varphi, t) := \frac{\omega_{\varphi_t}^n}{\omega^n} - e^{tf_0}.$$ 

(The operator has range $C_0^\gamma(M, \mathbb{R})$ since for all $\varphi$, $\int_M \frac{\omega_{\varphi_t}^n}{\omega^n} = \int_M \omega^n = \int_M e^{tf_0} \omega^n$. ) The kernel of $F_*$ is
precisely the set of solutions to the continuity method \((\star_1)\).

The differential of \(F_\ast\) with respect to the first factor is \((DF_{\ast,t})|_\varphi : C^2_{0}(\gamma_\ast, \mathbb{R}) \to C^0_0(\gamma_\ast, \mathbb{R})\),

\[
(DF_{\ast,t})|_\varphi(\psi) = \frac{\partial}{\partial s} \big|_{s=0} F_\ast(\varphi + s\psi, t) = \frac{\omega_{\varphi}^{n}}{\omega^n} \Delta_{\omega_\varphi} \psi.
\]

By standard linear theory \(\Delta_{\omega_\varphi}\) is bounded and bijective as an operator from \(C^2_{0}(\gamma_\ast, \mathbb{R})\) to \(C^0_0(\gamma_\ast, \mathbb{R})\), and hence, by the Open Mapping Theorem, invertible. So \(DF_{\ast,t}\) is also invertible.

**Lemma 2.3.1.** The set of \(t \in [0, 1]\) for which there exists a solution to \((\star_1)\) is open.

**Proof.** Since at each solution \((\varphi, t)\) to \((\star_1)\) the first-factor differential \(DF_{\ast,t}\) is invertible, we may apply the Implicit Function Theorem. \(\square\)

### Kähler-Einstein metrics on Fano manifolds

Consider the differential operators \(F_\ast\) and \(F'_\ast\), both from \(C^2_{\gamma}(\mathbb{M}, \mathbb{R}) \oplus \mathbb{R}\) into \(C^\gamma(\mathbb{M}, \mathbb{R})\):

\[
F_\ast(\varphi, t) := \omega_{\varphi}^{n} - e^{f(t, \varphi)}
\]

\[
F'_\ast(\varphi, t) := \omega_{\varphi}^{n} - e^{f(t, \varphi) - \beta(f_{M} \varphi + \omega_{n}^{\varphi})},
\]

The kernels of \(F_\ast, F'_\ast\) are precisely the sets of solutions to the continuity methods \((\star_1), (\star_2)\).

The differentials of \(F_\ast, F'_\ast\) with respect to the first factor are the maps \((DF_{\ast,t})|_\varphi, (DF'_{\ast,t})|_\varphi\), both from \(C^2_{\gamma}(\mathbb{M}, \mathbb{R})\) to \(C^\gamma(\mathbb{M}, \mathbb{R})\),

\[
(DF_{\ast,t})|_\varphi(\psi) = \frac{\partial}{\partial s} \big|_{s=0} F_\ast(\varphi + s\psi, t) = \frac{\omega_{\varphi}^{n}}{\omega^n} \Delta_{\omega_\varphi} \psi + e^{f(t, \varphi)} t\psi,
\]

\[
(DF'_{\ast,t})|_\varphi(\psi) = \frac{\partial}{\partial s} \big|_{s=0} F'_\ast(\varphi + s\psi, t) = \frac{\omega_{\varphi}^{n}}{\omega^n} \Delta_{\omega_\varphi} \psi + e^{f(t, \varphi) - \beta(f_{M} \varphi + \omega_{n}^{\varphi})} (t\psi + \beta \int \psi \omega_{\varphi_0}^{n})
\]

In particular, at points \((\varphi, t)\) where (respectively) \(F_\ast(\varphi, t) = 0\) or \(F'_\ast(\varphi, t) = 0\), we have (respectively) \(e^{f(t, \varphi)} = \omega_{\varphi}^{n}/\omega^n\) or \(e^{f(t, \varphi) - \beta(f_{M} \varphi + \omega_{n}^{\varphi})} = \omega_{\varphi}^{n}/\omega^n\), and so

\[
(DF_{\ast,t})|_\varphi(\psi) = \frac{\omega_{\varphi}^{n}}{\omega^n} \left[ \Delta_{\omega_\varphi} + t \right] \psi
\]

\[
(DF'_{\ast,t})|_\varphi(\psi) = \frac{\omega_{\varphi}^{n}}{\omega^n} \left[ \Delta_{\omega_\varphi} + t + \beta \int \omega_{\varphi_0}^{n} \right] \psi.
\]

For \(0 < t < 1\), since \(\text{Ric}(\omega_{\varphi}) \geq \omega_{\varphi} + (1-t)\omega \geq (t + \epsilon)\omega_{\varphi}\), we have by Theorem 2.1.4 that the first nonzero eigenvalue of \(\Delta_{\omega_\varphi}\) is at least \(t + \epsilon\). Therefore \([\Delta_{\omega_\varphi} + t]\) is invertible, and hence \((DF_{\ast,t})|_\varphi\) is too.
For $t = 0$, by standard linear theory $\Delta \omega + \beta f \cdot \omega \varphi_0$ is bounded and bijective as an operator from $C^{2, \gamma}(M, \mathbb{R})$ to $C^{\gamma}(M, \mathbb{R})$, and hence, by the Open Mapping Theorem, invertible. So $DF_{*,t_0}'$ is also invertible.

Lemma 2.3.2 ([Aub84], see also [Aub98]). Let $t_0 \in [0, 1)$, and let $\varphi$ either

1. (if $0 < t_0 < 1$) be a solution to $(*)_{t_0}$; or,

2. (if $t_0 = 0$) be a solution to $(*)_0$ and moreover satisfy $\int_M \varphi \omega \varphi_0 = 0$.

Then there exists a neighbourhood $\mathcal{J} \subseteq [0, 1)$ of $t_0$, open in $[0, 1)$, and a 1-parameter family $(\varphi_t)_{t \in \mathcal{J}}$, continuous on $\mathcal{J}$ and differentiable on $\mathcal{J} \cap (0, 1)$, with $\varphi_{t_0} = \varphi$ and with each $\varphi_t$ a solution to the corresponding $(*)_t$.

Proof. We first study the case $0 < t_0 < 1$. In this case we have observed above that that the first-factor differential, $(DF_{*,t_0}')|_{\varphi}$ is invertible, so we may apply the Implicit Function Theorem to $F_{*}'$. The result follows.

We now turn to the case $t_0 = 0$. Since $\int_M \varphi \omega \varphi_0 = 0,$

$$\omega \varphi^n = e^{f - t \varphi_0} \omega^n = e^{f - t \varphi - \beta(f_M \varphi \omega \varphi_0^n)} \omega^n,$$

so $\varphi$ is also a solution to $(*)'_0$. We have observed above that $(DF_{*,0}')|_{\varphi}$ is invertible, so we may apply the Implicit Function Theorem to $F_{*}'$ at $\varphi$. We obtain that there exists an interval $\mathcal{J} := [0, \epsilon)$, and a 1-parameter family $(\tilde{\varphi}_t)_{t \in \mathcal{J}}$, differentiable in $t$, with $\tilde{\varphi}_0 = \varphi$ and with each $\tilde{\varphi}_t$ a solution to the corresponding $(*)'_t$.

Define a 1-parameter family $(\varphi_t)_{t \in \mathcal{J}}$ by,

$$\varphi_t = \begin{cases} 
\tilde{\varphi}_t + t^{-1} \beta \int_M \tilde{\varphi}_t \omega \varphi_0^n, & t > 0 \\
\varphi, & t = 0.
\end{cases}$$

The family is differentiable for $t > 0$, and each $\varphi_t$ is a solution to $(*)_t$ since

$$e^{f - t \varphi} + t^{-1} \beta \int_M \tilde{\varphi}_t \omega \varphi_0^n \omega^n = e^{f - t \varphi - \beta(f_M \varphi_0 \omega \varphi_0^n)} = \omega \varphi_0^n = \omega \varphi_t^n.$$

So it remains to be checked only that the family is continuous at $t = 0$. Indeed, implicitly differentiating $(*)'_t$, we see that at $t = 0$,

$$\beta \int_M \tilde{\varphi}_0 \omega \varphi_0^n = -\varphi - \Delta \omega \varphi \tilde{\varphi}_0.$$
Integrating this equation with respect to $\omega^n$,

$$
\beta \int_M \tilde{\varphi}_0 \omega^n = - \int_M \varphi \omega^n - \int_M \Delta \tilde{\varphi}_0 \omega^n = 0.
$$

Therefore

$$
\lim_{t \to 0^+} \varphi_t = \lim_{t \to 0^+} \left[ \tilde{\varphi}_t + t^{-1} \beta \int_M \tilde{\varphi}_t \omega^n \right] = \lim_{t \to 0^+} \tilde{\varphi}_t + \beta \int_M \tilde{\varphi}_t \omega^n = 0 + 0 = \varphi_0,
$$

so the family $\varphi_t$ is continuous.

**Remark.** If $M$ has no holomorphic vector fields, then the linearization $DF_t$ is invertible also at $t = 1$, and so Lemma 2.3.2 applies also at $t_0 = 1$. However, we will not use this result.

### 2.4 The $C^{2,\alpha}$ estimates of Evans-Krylov

Interior estimates for the real Monge-Ampère equation are due to Evans and to Krylov, and their arguments were applied by Siu [Siu87, Section 2.4] to give interior estimates for the complex Monge-Ampère equation. (Yau’s original work on the Calabi problem relied on somewhat different results [Yau77, Yau78], based on a $C^3$ estimate of Calabi [Cal58].)

We cite a version of the basic interior estimate due to Blocki.

**Theorem 2.4.1** ([Blo00, Theorem 3.1], [Blo12, Theorem 5.15]). Let $U$ be a domain in $\mathbb{C}^n$, let $\psi$ be a $C^4$ plurisubharmonic function on $U$, and let the function $F$ be such that $e^F = (\sqrt{-1} \partial \bar{\partial} \psi)^n$. For each subdomain $U_0$ compactly contained in $U$, there exists a positive constant $C$ and a real number $\alpha \in (0, 1)$, both dependent only on $n$, $U$, $U_0$ and on upper bounds for $||\psi||_{0,1,U}$, $\sup_U \Delta \psi$, $||F||_{0,1,U}$, such that

$$
||\psi||_{2,\alpha,U_0} \leq C.
$$

We will also require a simple lemma controlling first derivatives.

**Lemma 2.4.2** ([GT01, Theorem 3.9]). Let $U$ be a domain in $\mathbb{R}^n$, and let $\psi$ be a $C^2$ function on $U$. For each subdomain $U'$ compactly contained in $U$, there exists a positive constant $C$ dependent only on $n$, $U$, $U'$, such that

$$
\sup_{U'} |D\psi| \leq C \left[ \sup_U |\psi| + \sup_U |\Delta \psi| \right].
$$
These facts yield a $C^{2,\alpha}$ estimate for Kähler metrics.

**Corollary 2.4.3.** Let $\varphi = \varphi_t$ be a function arising in

1. the continuity method $(\star_t)$ for the Calabi problem; or,
2. the continuity method $(\star_t)$ for the Kähler-Einstein problem.

There exists a constant $C$ and a real number $\alpha \in (0, 1)$, both dependent only on $n$, $\omega$, $f$, $\sup |\varphi|$, and $n + \sup \Delta \omega \varphi$, such that $||\varphi||_{2,\alpha} \leq C$.

**Proof.** We will prove the result locally; the global version follows by compactness. Let $U$ be a contractible open set of $M$, identified with a domain in $\mathbb{C}^n$, and $U_0$ a domain compactly contained in $U$. Choose $\psi_0$ on $U$ such that $\omega = \sqrt{-1} \partial \overline{\partial} \psi_0$. We will apply Theorem 2.4.1 with $\psi = \psi_0 + \varphi$ and with $F := tf + c_t$ (in the Calabi case), or $F := f - t\varphi$ (in the Kähler-Einstein case). To do this we must verify the hypotheses, namely that the following quantities be controlled:

$$||\psi_0 + \varphi||_{0,1,U}, \quad \sup_U \Delta(\psi_0 + \varphi), \quad ||F||_{0,1,U}.$$ 

First note that

$$||\psi_0 + \varphi||_{0,1,U} \leq ||\psi_0||_{0,1,U} + ||\varphi||_{0,1,U},$$

$$||tf + c_t||_{0,1,U} \leq t||f||_{0,1,U} + |c_t| \leq 2||f||_{0,1,U},$$

$$||f - t\varphi||_{0,1,U} \leq ||f||_{0,1,U} + t||\varphi||_{0,1,U} \leq ||f||_{0,1,U} + ||\varphi||_{0,1,U},$$

so the first and last ($||\psi_0 + \varphi||_{0,1,U}$ and $||F||_{0,1,U}$) are controlled in terms of $f$, $\omega$ and $||\varphi||_{0,1,U}$.

Write $\omega_0$ for the Euclidean Kähler metric on $U$ (under its identification with a domain in $\mathbb{C}^n$). We have that (pointwise) $\omega \leq \lambda_{\text{max}}(\omega) \omega_0$, where $\lambda_{\text{max}}(\omega) := \sup_{|v|=1} \omega(v, v)$.

$$0 < \Delta(\psi_0 + \varphi) = \text{tr}_\omega \omega_0 \psi_0 \leq \lambda_{\text{max}}(\omega) \text{tr}_\omega \omega_0 \psi_0 \leq \lambda_{\text{max}}(\omega) [n + \Delta_\omega \varphi].$$

Thus also $|\Delta \varphi| \leq \max (-\Delta \psi_0, -\Delta \psi_0 + \lambda_{\text{max}}(\omega) [n + \Delta_\omega \varphi])$. So, shrinking $U$ slightly, to $U'$, we have by Lemma 2.4.2 that $\sup_{U'} |D\varphi|$ is bounded in terms of $\omega$, $\sup_U |\varphi|$ and $\sup_U (n + \Delta_\omega \varphi)$. This concludes the verification of the hypotheses of Theorem 2.4.1.
It follows that
\[ \|\varphi\|_{2,\alpha,U_0} \leq C + \|\psi_0\|_{2,\alpha,U_0} \leq C. \]

\subsection{2.5 $C^2$ estimates}

In this section we present the $C^2$ estimate of Yau [Yau77,Yau78], following [Aub98, Section 7.10].

\begin{proposition}
Let $f$ be a smooth function, let $\lambda_0$ be a positive constant, and let $\omega$ and $\omega_\varphi$ be Kähler metrics, with $\omega_\varphi^n = e^{F+\lambda\varphi}\omega^n$ for some $|\lambda| \leq \lambda_0$. There exist constants $C = C(\omega,F,\lambda_0)$, $K = K(\omega, \sup F, \sup(-\Delta_\omega F), \lambda_0)$, such that
\[ 0 < n + \Delta_\omega \varphi \leq Ce^{K\sup|\varphi|}. \]
\end{proposition}

In proving Proposition 2.5.1 we rely on the following curvature computation:

\begin{proposition}[Aub98, Section 7.10] Let $\omega$ and $\omega_\varphi$ be Kähler metrics, with $\omega_\varphi^n = e^F\omega^n$. There exists a constant $C = C(\omega)$, such that
\[ \Delta_\omega \log(n + \Delta_\omega \varphi) \geq -C \text{tr}_\omega \omega + (n + \Delta_\omega \varphi)^{-1}\Delta_\omega F. \]
\end{proposition}

\begin{proof}[Proof of Proposition 2.5.1]
Let $k$ be a real constant which we will fix later. Since $\text{tr}_\omega \omega - n = -\Delta_\omega \varphi$, and
\[ \Delta_\omega [F + \lambda \varphi] = -(n\lambda - \Delta_\omega F) + \lambda(n + \Delta_\omega \varphi), \]
we have by Proposition 2.5.2 that for some $C_0 = C_0(\omega)$,
\begin{equation}
\Delta_\omega [\log(n + \Delta_\omega \varphi) - k\varphi] \geq (k - C_0) \text{tr}_\omega \omega - nk + \lambda - (n + \Delta_\omega \varphi)^{-1}(n\lambda - \Delta_\omega F). \tag{2.1}
\end{equation}

We now apply the Maximum Principle. We select a point $x \in M$ at which $\log(n + \Delta_\omega \varphi) - k\varphi$ is maximal. At that point the left-hand side of (2.1) is positive, and so
\begin{equation}
k - \lambda \geq (k - C_0) \text{tr}_\omega \omega - (n + \Delta_\omega \varphi)^{-1}(n\lambda - \Delta_\omega F). \tag{2.2}
\end{equation}

\end{proof}
On the other hand, by the AM-GM inequality

$$(n + \Delta \omega \varphi)(\text{tr}_\omega \omega) = (\text{tr}_\omega \omega \varphi)(\text{tr}_\omega \omega) \geq n^2;$$

inserting this in (2.2) gives, at $x$,

$$nk - \lambda \geq [((k - C_0) - n^{-2}(n\lambda - \Delta \omega F)) \text{tr}_\omega \omega].$$

Choosing $k = k(C, n, \lambda_0, \sup(-\Delta \omega F))$ such that the right-hand side coefficient is necessarily at least 1, we obtain that for some $C = C(C_0, n, \lambda_0, \sup(-\Delta \omega F)) := nk - \lambda$,

$$C \geq (\text{tr}_\omega \omega)_{x}. \quad (2.3)$$

Now, again by the AM-GM inequality,

$$\log(n + \Delta \omega \varphi) = \log \text{tr}_\omega \omega \varphi \leq \log \left(n \left[\frac{\text{tr}_\omega \omega}{n-1}\right]^{n-1} \frac{\omega^n}{\omega^n} \right) = \log \left(n \left[\frac{\text{tr}_\omega \omega}{n-1}\right]^{n-1} \right) + \lambda \varphi + F.$$

Inserting (2.3), we obtain that, at the point $x$, for some (new) $C = C(C_0, n, \lambda_0, \sup(-\Delta \omega F), \sup F)$,

$$\log(n + \Delta \omega \varphi)|_{x} \leq C + \lambda \varphi(x) + F(x)$$

$$\left|\log(n + \Delta \omega \varphi) - k \varphi\right|_{x} \leq -(k - \lambda) \varphi + (C + F)$$

$$\leq -(k - \lambda) \inf \varphi + C.$$

Since $x$ was chosen to maximize $\log(n + \Delta \omega \varphi) - k \varphi$, we thus have that throughout the manifold,

$$\left|\log(n + \Delta \omega \varphi) - k \varphi\right| \leq -(k - \lambda) \inf \varphi + C$$

$$\log(n + \Delta \omega \varphi) \leq k \sup \varphi - (k - \lambda) \inf \varphi + C$$

$$\leq (2k + \lambda_0) \sup |\varphi| + C.$$

This is the desired result. \qed
Thus we may remove the dependence on \( n + \Delta_\omega \varphi \) in Corollary 2.4.3:

**Corollary 2.5.3.** Let \( \varphi = \varphi_t \) be a function arising in

1. the continuity method \((\star_t)\) for the Calabi problem; or,

2. the continuity method \((\star_t)\) for the Kähler-Einstein problem.

There exists a constant \( C \) and a real number \( \alpha \in (0, 1) \), both dependent only on \( n, \omega, f \) and \( \sup |\varphi| \), such that 

\[
||| \varphi |||_{2, \alpha} \leq C.
\]

**Proof.** In both cases we have a uniform (in \( t \)) bound 

\[
0 < n + \Delta_\omega \varphi \leq C(\omega, f, \sup |\varphi|).
\]

For the Calabi problem this comes from applying Proposition 2.5.1 with \( F = tf + c_t, \lambda = 0 \), and observing that the parameters have uniform (in \( t \)) bounds

\[
\sup(-\Delta_\omega F) = t \sup(-\Delta_\omega f) \leq C(f),
\]

\[
\sup F = t \sup f + c_t \leq \sup f + 0 \leq C(f).
\]

For the Kähler-Einstein problem this comes from applying Proposition 2.5.1 with \( F = f, \lambda = -t \), and observing that the parameters have the uniform (in \( t \)) bound \( |-t| \leq 1 \). Now apply Corollary 2.4.3.  

\( \square \)

### 2.6 Yau’s \( C^0 \) estimate

In this section we prove the fundamental \( C^0 \) estimate of Yau [Yau77, Yau78], as subsequently simplified in [Bou79] and [Siu87].

**Lemma 2.6.1.** Let \( \omega \) and \( \omega_\phi \) be Kähler metrics, with \( \omega_\phi = e^F \omega^n \). There exists a constant \( C = C(n, \sup F) \), such that for each real number \( a \geq 0 \),

\[
C \int_M |\varphi|^a \omega^n \geq \frac{a + 1}{(a + 2)^2} \int_M d \left( \varphi |\varphi|^{a/2} \right) \omega^n.
\]

**Proof.** For each real number \( a \geq 0 \), the function \( \varphi |\varphi|^a \) is differentiable. So

\[
\partial (\varphi |\varphi|^a) \wedge \partial \varphi = (a + 1) |\varphi|^a \partial (\varphi) \wedge \partial \varphi
\]
\[
= \frac{4(a + 1)}{(a + 2)^2} \left( \frac{a+2}{2} |\varphi|^{a/2} \partial \varphi \right) \wedge \left( \frac{a+2}{2} |\varphi|^{a/2} \partial \varphi \right)
\]
\[
= \frac{4(a + 1)}{(a + 2)^2} \partial \left( |\varphi|^{a/2} \right) \wedge \partial \left( |\varphi|^{a/2} \right).
\]
and therefore (using Stokes’ theorem, in the first line, and the fact that all \((1,1)\)-forms involved are positive, in the second-last line)

\[
\int_M \varphi|\varphi|^a (\omega^n - \omega_\varphi^n) = -\sum_{r=0}^{n-1} \int_M \varphi|\varphi|^a \sqrt{-1} \partial \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1}
= \sum_{r=0}^{n-1} \int_M \sqrt{-1} \partial (\varphi|\varphi|^a) \wedge \overline{\partial} \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1}
= \frac{4(a+1)}{(a+2)^2} \sum_{r=0}^{n-1} \int_M \sqrt{-1} \partial \left( \varphi|\varphi|^{a/2} \right) \wedge \overline{\partial} \left( \varphi|\varphi|^{a/2} \right) \wedge \omega_\varphi^r \wedge \omega^{n-r-1}
= \frac{4(a+1)}{(a+2)^2} \int_M \sqrt{-1} \partial \left( \varphi|\varphi|^{a/2} \right) \wedge \overline{\partial} \left( \varphi|\varphi|^{a/2} \right) \wedge \omega_\varphi^{n-1}
= \frac{4(a+1)}{n(a+2)^2} \int_M \left| \partial \left( \varphi|\varphi|^{a/2} \right) \right|^2 \omega^n.
\]

Now we observe that

\[
\int_M \varphi|\varphi|^a (\omega^n - \omega_\varphi^n) = \int_M \varphi|\varphi|^a (1 - e^F) \omega^n \leq \max(1, \sup F) \int_M |\varphi|^{a+1} \omega^n.
\]

\[\square\]

**Proposition 2.6.2.** Let \(\omega\) and \(\omega_\varphi\) be Kähler metrics, with \(\omega_\varphi^n = e^F \omega^n\). Suppose moreover that \(\int_M \varphi \omega^n = 0\). There exists a constant \(C = C(n, \omega, \sup F)\), such that \(\sup |\varphi| \leq C\).

**Remark.** In the general case (that is, \(\int_M \varphi \omega^n\) not necessarily 0), inserting \(\varphi - \int_M \varphi \omega^n\) in Proposition 2.6.2 as \(\varphi\) shows that there exists a constant \(C = C(n, \omega, \sup F)\), such that \(\sup \varphi - \inf \varphi \leq C\).

**Proof.** We argue by Moser iteration. By Lemma 2.6.1, there exists a constant \(C = C(n, \sup F)\), such that for each real number \(a \geq 0\),

\[
\frac{1}{a+2} \int_M \left| \partial \left( \varphi|\varphi|^{a/2} \right) \right|^2 \omega^n \leq \frac{2(a+1)}{(a+2)^2} \int_M \left| \partial \left( \varphi|\varphi|^{a/2} \right) \right|^2 \omega^n \leq C \int_M |\varphi|^{a+1} \omega^n
\]

\[
\leq C \int_M [1 + |\varphi|^{a+2}] \omega^n 
\leq CV + C \int_M |\varphi|^{a+2} \omega^n.
\]
So, applying Theorem 2.1.1 (ii) to the function \( f = \varphi |\varphi|^{\alpha/2} \) (so \( |f| = |\varphi|^{(\alpha+2)/2} \)),

\[
\left( \int_M |\varphi|^{(\alpha+2)\omega/(n-2)} \omega^n \right)^{\frac{n-2}{n}} \leq C_0 \int_M |d(\varphi |\varphi|^{\alpha/2})|^2 \omega^n + C_1 \int_M |\varphi|^{\alpha+2} \omega^n \\
\leq (a + 2)C_0CV + ((a + 2)C_0C + C_1) \int_M |\varphi|^{\alpha+2} \omega^n \\
\leq \max \left\{ 2C_0CV(a + 2), (2C_0C(a + 2) + C_1) \int_M |\varphi|^{\alpha+2} \omega^n \right\}.
\]

Thus, letting \( T_p := \log \left( \int_M |\varphi|^{p\omega^n} \right)^{1/p} \) (and setting \( b := a + 2 \)), we have that for all real numbers \( b \geq 2 \),

\[
T_{bn/(n-2)} \leq \max \left\{ b^{-1} \log (2C_0CVb), b^{-1} \log (2C_0Cb + C_1) + T_b \right\}.
\]

In particular, for all natural numbers \( r \),

\[
T_{2\left(\frac{n}{n-2}\right)^r} \leq \max \left\{ \frac{1}{2} \left( \frac{n-2}{n} \right)^r \log \left( 2C_0CV \cdot 2 \left( \frac{n}{n-2} \right)^r \right), \frac{1}{2} \left( \frac{n-2}{n} \right)^r \log \left( 2C_0C \cdot 2 \left( \frac{n}{n-2} \right)^r + C_1 \right) + T_{2\left(\frac{n}{n-2}\right)^r} \right\}
\]

for appropriate constants \( c_0, c_1, c_2 \) dependent only on \( n, \omega, \sup F \). By induction, for all \( k \),

\[
T_{2\left(\frac{n}{n-2}\right)^k} \leq \left[ c_1 \left( \sum_{r=1}^{k-1} \left( \frac{n-2}{n} \right)^r \right) + c_2 \left( \sum_{r=1}^{k-1} r \left( \frac{n-2}{n} \right)^r \right) \right] + \max \left( c_0, T_{2\left(\frac{n}{n-2}\right)^{k-1}} \right),
\]

and since these integrals converge, we have that for constants \( C = C(n, \omega, \sup F) \),

\[
\sup_M |\varphi| = \lim_{k \to \infty} \exp \left( T_{2\left(\frac{n}{n-2}\right)^k} \right) \leq C \exp \max \left( c_0, T_{2\left(\frac{n}{n-2}\right)^k} \right) = \max \left( C, \left( \int_M |\varphi|^{2n/(n-2)} \omega^n \right)^{\frac{n-2}{2n}} \right).
\]

Now we use the normalization \( \int_M \varphi \omega^n = 0 \). By the Poincaré inequality Theorem 2.1.1 (i), there exists a constant \( C = C(n, \omega) \) such that

\[
\left( \int_M |\varphi|^{2n/(n-2)} \omega^n \right)^{\frac{n-2}{n}} \leq C \int_M |d\varphi|^2 \omega^n.
\]
Thus by Lemma 2.6.1 there exists a constant $C = C(n, \omega, \sup F)$ such that

$$\left( \int_M |\varphi|^{2n/(n-2)} \omega^n \right)^{\frac{n-2}{n}} \leq C \int_M |\varphi| \omega^n.$$ 

By Hölder’s inequality this implies an absolute bound

$$\left( \int_M |\varphi|^{2n/(n-2)} \omega^n \right)^{\frac{n-2}{2n}} \leq C;$$

inserting this in (2.4) finishes the proof.

**Corollary 2.6.3.** Let $\varphi = \varphi_t$ be a function arising in the continuity method $(*)_t$ for the Calabi problem. There exists a constant $C$ and a real number $\alpha \in (0, 1)$, both dependent only on $n, \omega, f$, such that $|||\varphi|||_{2, \alpha} \leq C$.

**Proof.** The continuity method $(*)_t$ implies that $\int_M \varphi_t \omega^n = V$ and that $\omega^n = e^{f+c} \omega^n$, for some $t \in [0, 1]$. Since $\sup (tf + ct) \leq \sup f$ is bounded independently of $t$, we have by Proposition 2.6.2 that for some $C = C(n, \omega, \sup f)$, $\sup |\varphi| \leq C$. We now apply Corollary 2.5.3.

### 2.7 Carrying out the method of continuity

#### The Calabi problem

**Theorem 2.7.1** ([Yau77, Yau78]). Let $a \in H^{1,1}(M, \mathbb{R})$ be a Kähler cohomology class, and let $\chi$ be a $(1, 1)$-form in $2\pi c_1(M)$. Then there exists a Kähler metric $\omega \in a$ such that $\text{Ric}(\omega) = \chi$.

**Proof.** We study the continuity method $(*)_t$ in the Hölder spaces $C^{2,\alpha/2}(M, \mathbb{R})$, where $\alpha$ is as in Corollary 2.6.3. Let $\mathcal{I}$ be the subset of $t \in [0, 1]$ for which $(*)_t$ has a solution.

By Lemma 2.3.1, $\mathcal{I}$ is open in $[0, 1]$. Since for $\varphi = 0$ we have that $\omega^n = e^{\alpha} \omega^n$, the set $\mathcal{I}$ contains 0. By Corollary 2.6.3 the set of solutions $\{ \varphi : \varphi \text{ solves } (*)_t, t \in \mathcal{I} \}$ is compact in $C^{2,\alpha/2}(M, \mathbb{R})$, and so $\mathcal{I}$ is closed in $[0, 1]$. It follows that $\mathcal{I} = [0, 1]$.

**Kähler-Einstein metrics on Fano manifolds**

**Theorem 2.7.2** ([Aub84]). Let $M$ be a Fano manifold. Fix a Kähler metric $\omega$ in $2\pi c_1(M)$. There exist

1. a connected open subset $\mathcal{I}_0$ of $[0, 1)$, with $0 \in \mathcal{I}_0$,
2. a $1$-parameter family $(\varphi_t)_{t \in I}$, continuous on $I_0$ and differentiable on $I_0 \setminus \{0\}$, with each $\varphi_t$ a solution to the corresponding $(*)_t$ (in the continuity method with respect to $\omega$);

such that if $(\varphi_t)_{t \in I_0 \cap [\delta,1)}$ is bounded in $C^0(M,\mathbb{R})$, for some $\delta \in I_0$, then $M$ admits a Kähler-Einstein metric.

Proof. We study the continuity method $(*)_t$ in the Hölder spaces $C^{2,\alpha/2}(M,\mathbb{R})$, where $\alpha$ is as in Corollary 2.5.3. By Theorem 2.7.1 there exists a Kähler metric $\omega_0 \in 2\pi c_1(M)$ such that $\text{Ric}(\omega_0) = \omega$. Let $\varphi_0$ be the potential such that $\omega_{\varphi_0} = \omega_0$ and $\int_M \varphi_0 \omega_0^n = 0$. This is a solution to the continuity method $(*)_0$ at $t = 0$.

By Lemma 2.3.2 (2) there exists an open set $J \subseteq [0,1)$ on which there exists a $1$-parameter family $(\varphi_t)_{t \in J}$, continuous on $J$ and differentiable on $J \setminus \{0\}$, with each $\varphi_t$ a solution to the corresponding $(*)_t$.

Let $[0,T) = I_0 \subseteq [0,1)$ be a maximal such subset, and $(\varphi_t)_{t \in I_0}$ the corresponding $1$-parameter family.

If $(\varphi_t)_{t \in I_0 \cap [\delta,1)}$ is bounded in $C^0(M,\mathbb{R})$, for some $\delta \in I_0$, then by Corollary 2.5.3 $(\varphi_t)_{t \in I_0 \cap [\delta,1)}$ is pre-compact in $C^{2,\alpha/2}(M,\mathbb{R})$. So there exists a sequence $(t_k)_{k \in \mathbb{N}} \subseteq I_0$ tending to the right endpoint $T$ of $I_0$, such that $\varphi_T := \lim_{k \to \infty} \varphi_{t_k}$ exists and is a solution to $(*)_T$. By Lemma 2.3.2 (1) and the maximality of $I_0$, it is impossible that $T < 1$. So $T = 1$. It follows that $I = [0,1)$ and that $\omega_{\varphi_T}$ is Kähler-Einstein. \qed

2.8 Two standard functionals

We recall the functionals $I$ and $J$ introduced by Aubin [Aub84]:

\[ I(\varphi) := \sum_{r=0}^{n-1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1}; \]
\[ J(\varphi) := \sum_{r=0}^{n-1} \frac{n-r}{n+1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1}. \]

We also introduce our own:

\[ I_k(\varphi) := \sum_{r=0}^{k-2} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1}. \]

On occasion, when we wish to emphasize the dependence on the reference metric, we will use the notation $I_k(\varphi,\omega)$.

By construction $I(\varphi) = I_{n+1}(\varphi)$, and for any integer $k$ such that $2 \leq k \leq n$,

\[ (n+1)J(\varphi) - I(\varphi) = \sum_{r=0}^{n-2} (n-r-1) \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_\varphi^r \wedge \omega^{n-r-1} \geq (n-k+1)I_k(\varphi). \]
Also by construction, for any Kähler potential \( \varphi \),
\[
I(\varphi) - J(\varphi) = \sum_{r=0}^{n-1} \frac{r + 1}{n + 1} \int_M \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^r \wedge \omega^{n-r-1} \geq 0,
\]
(2.6)
and by Stokes’ theorem
\[
I_k(\varphi) = -\sum_{r=0}^{k-2} \int_M \varphi \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^r \wedge \omega^{n-r-1} = \int_M \varphi [\omega^{k-1} - \omega^k] \wedge \omega^{n-k+1}.
\]
(2.7)

### 2.9 Estimates for the Fano Kähler-Einstein problem

Throughout this section we let \( I_0 \) and \((\varphi_t)\) be as in Theorem 2.7.2.

**Proposition 2.9.1.** For all \( t \in I_0 \), for any integer \( k \) such that \( 2 \leq k \leq n \),
\[
\int_M (-\varphi_t) \omega_{\varphi_t}^n + (n - k + 1)I_k(\varphi_t) \leq n \sup_M \varphi_t.
\]

The proof of Proposition 2.9.1 uses the following identity for the family \((\varphi_t)\), which essentially appears in the proof of [Tia87, Proposition 2.3]; see also [Tia96, Proposition 4.3].

**Proposition 2.9.2.** For all \( t \in I_0 \setminus \{0\} \),
\[
-\frac{1}{t} \int_0^t [I(\varphi_s) - J(\varphi_s)]ds = J(\varphi_t) - \int_M \varphi_t \omega^n.
\]

**Proof of Proposition 2.9.1.** By (2.6), \( I(\varphi_s) - J(\varphi_s) \geq 0 \) for all \( s \), so
\[
0 \geq J(\varphi_t) - \int_M \varphi_t \omega^n
= J(\varphi_t) - \frac{n}{n + 1} \int_M \varphi_t \omega^n - \frac{1}{n + 1} \left[ I(\varphi_t) + \int_M \varphi_t \omega_{\varphi_t}^n \right].
\]
(The second line is an application of (2.7).) Multiplying through by \( n + 1 \) and rearranging,
\[
n \int_M \varphi_t \omega^n \geq [(n + 1)J(\varphi_t) - I(\varphi_t)] + \int_M (-\varphi_t) \omega_{\varphi_t}^n.
\]
Now apply (2.5) (on the right-hand side) and the estimate \( \sup_M \varphi_t \geq \int_M \varphi_t \omega^n \) (on the left-hand side). \( \square \)
Corollary 2.9.3. Let $\delta > 0$. There exists a constant $C = C(n, V, \delta)$, such that for all $t \in \mathcal{I}_0$,

$$\sup_M |\varphi_t| \leq n \sup_M \varphi_t + C.$$ 

Proof. By Proposition 2.9.1,

$$\int_M (-\varphi_t) \omega_{\varphi_t}^n \leq n \sup_M \varphi_t,$$

and by Proposition 2.1.3, since $\text{Ric}(\omega_t) \geq t \omega_t \geq \delta \omega$, there exists a constant $C = C(n, V, \delta)$ such that

$$\sup_M (-\varphi_t) \leq \int_M (-\varphi_t) \omega_0^n + C.$$ 

Theorem 2.9.4. Let $M$ be a Fano manifold. Fix a Kähler metric $\omega$ in $2\pi c_1(M)$. There exist

1. a connected open subset $\mathcal{I}_0$ of $[0, 1)$, with $0 \in \mathcal{I}_0$;

2. a 1-parameter family $(\varphi_t)_{t \in \mathcal{I}_0}$, continuous on $\mathcal{I}_0$ and differentiable on $\mathcal{I}_0 \setminus \{0\}$, with each $\varphi_t$ a solution to the corresponding $(\ast_t)$ (in the continuity method with respect to $\omega$);

such that if for some $\delta \in \mathcal{I}_0$, the suprema $(\sup \varphi_t)_{t \in \mathcal{I}_0 \cap [\delta, 1)}$ are uniformly bounded, then $M$ admits a Kähler-Einstein metric.

Proof. By Corollary 2.9.3, since $(\sup \varphi_t)_{t \in \mathcal{I}_0 \cap [\delta, 1)}$ are uniformly bounded, $(\sup |\varphi_t|)_{t \in \mathcal{I}_0 \cap [\delta, 1)}$ in $C^0(M, \mathbb{R})$, are also uniformly bounded. We now apply Theorem 2.7.2.

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Chapter 3

Bergman metrics and the Bergman kernel

In this chapter we review certain constructions of Tian [Tia90a, Tia90b, Tia91]. The objects of study are the powers $\mathcal{L}^m$ of a line bundle $\mathcal{L}$ over a complex variety $M$. In some sections we restrict to the case of $M$ a Fano variety and $\mathcal{L}$ the anticanonical bundle $K_M^{-1}$.

3.1 Bergman metrics

Definition. The base locus of $\mathcal{L}^m$ is the subvariety of $M$ on which all sections of $\mathcal{L}^m$ vanish. That is,

$$\text{Bs}(\mathcal{L}^m) := \bigcap_{s \in H^0(M, \mathcal{L}^m)} s^{-1}(0) = \{ x \in M : \forall s \in H^0(M, \mathcal{L}^m), s_x = 0 \}.$$ 

There is a natural smooth map $\iota_{\mathcal{L}^m}$ of $M \setminus \text{Bs}(\mathcal{L}^m)$ into a complex projective space (specifically, the projectivization of $V := H^0(M, \mathcal{L}^m)^*$). This map is defined as follows. First observe that there is a natural “evaluation” section $\text{ev}$ of the holomorphic vector bundle $V \otimes \mathcal{L}^m \cong \text{Hom}(H^0(M, \mathcal{L}^m), \mathcal{L}^m)$: explicitly, $\text{ev}(x) = (s \mapsto s_x)$. The zero locus of $\text{ev}$ is precisely $\text{Bs}(\mathcal{L}^m)$. The map $\iota_{\mathcal{L}^m}$ is the projectivization of $\text{ev}$:

$$\iota_{\mathcal{L}^m} := [\text{ev}] : M \setminus \text{Bs}(\mathcal{L}^m) \to \mathbb{C}P(V).$$

Definition. 1. $\mathcal{L}$ is very ample, if $\text{Bs}(\mathcal{L})$ is empty and $\iota_{\mathcal{L}}$ is an embedding.
2. \( \mathcal{L} \) is ample, if some positive power \( \mathcal{L}^m \) of \( \mathcal{L} \) is very ample.

**Example 3.1.1.** For each positive \( m \), the line bundle \( \mathcal{O}(m) = \mathcal{O}(1)^m \) over the variety \( \mathbb{C}P^1 \) is very ample, and the embedding \( \iota_{\mathcal{O}(m)} \) may be identified with the rational normal curve \( \nu_m : \mathbb{C}P^1 \to \mathbb{C}P^m \),

\[
[s : t] \mapsto [s^m : s^{m-1}t : \cdots : t^m].
\]

**Theorem 3.1.2** (Kodaira [Kod54]). The line bundle \( \mathcal{L} \) is ample if and only if the cohomology class \( c_1(\mathcal{L}) \) is Kähler.

When the bundle \( \mathcal{L}^m \) is very ample, the embedding \( \iota_{\mathcal{L}^m} \) induces a natural, noncompact finite-dimensional family \( \mathcal{M}_{\mathcal{L}^m} \) of Kähler metrics on \( M \), parametrized by the homogeneous space of inner products on \( V \). (This homogeneous space is isomorphic to \( SL_C(V)/SU(V) \).) The correspondence between inner products and these Kähler metrics goes as follows: for an inner product \( a \) on \( V \), there is an induced Fubini-Study metric \( \omega_{FS}(a) \) on \( \mathbb{C}P(V) \), and pulling back \( \omega_{FS}(a) \) under the embedding yields a Kähler metric \( \frac{1}{m}(\iota_{\mathcal{L}^m})^* \omega_{FS}(a) \) on \( M \), in the Kähler class \( 2\pi c_1(\mathcal{L}) \).

**Definition.** The \((m-)\)Bergman metrics of \( \mathcal{L} \) are the Kähler metrics in this family \( \mathcal{M}_{\mathcal{L}^m} \).

**Lemma 3.1.3.** Fix a reference Kähler metric \( \omega \in 2\pi c_1(M) \). Let \( h \) be a hermitian metric on \( \mathcal{L} \) of curvature \( \omega \). For a potential \( \psi \in C^\infty(M, \mathbb{R}) \), the following are equivalent:

1. \( \psi \) is the Kähler potential of some \( m \)-Bergman metric, with respect to \( \omega \). (That is, the \((1,1)\)-form \( \omega_\psi := \omega + \sqrt{-1}\partial\bar{\partial}\psi \) is an \( m \)-Bergman metric.)
2. \( \psi = \frac{1}{m} \log \left( \sum_{i=1}^{N} |s_i|^2 h_{\mathcal{L}^m} \right) \), for some basis \( (s_1, \ldots, s_N) \) of \( H^0(M, \mathcal{L}^m) \).
3. \( \psi = \frac{1}{m} \log \left( \sum_{i=1}^{N} \mu_i |s_i|^2 h_{\mathcal{L}^m} \right) \), for some real numbers \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N > 0 \) and some basis \( (s_1, \ldots, s_N) \) of \( H^0(M, \mathcal{L}^m) \) which is orthonormal with respect to a fixed inner product on \( H^0(M, \mathcal{L}^m) \).

**Proof.** (3) and (2) are equivalent by diagonalization. For the equivalence of (1) and (2), note that for any nonvanishing local holomorphic section \( s \) of \( \mathcal{L} \),

\[
\frac{1}{m} \sqrt{-1}\partial\bar{\partial} \log \left( \sum_{i=1}^{N} |s_i|^2 h_{\mathcal{L}^m} \right) = \frac{1}{m} \sqrt{-1}\partial\bar{\partial} \log \left( \sum_{i=1}^{N} \frac{|s_i|^2}{|s|^2} \right) + \sqrt{-1}\partial\bar{\partial} \log |s|^2.
\]

The first term is the pullback under \( \iota_{\mathcal{L}^m} = [s_1 : s_2 : \cdots : s_N] \) of the Fubini-Study metric \( \sqrt{-1}\partial\bar{\partial} \left( \sum_{i=1}^{N} |z_i|^2 \right) \). The negative of the second term is the curvature of \( h \); that is, \( \omega \). \( \square \)
3.2 The Bergman kernel

Let \( \omega \) be a Kähler metric in \( 2\pi c_1(\mathcal{L}) \), and let \( h \) be a hermitian metric on \( \mathcal{L} \) whose curvature is \( \omega \). These induce an inner product \( || \cdot || \) on \( H^0(M, \mathcal{L}^m) \):

\[
||s||^2 = \int_M |s|^2_{h,m} \omega^m.
\]

**Definition.** The \((m\text{-th})\) Bergman kernel of \( \omega \) is the function \( \rho_{\omega,m} : M \to \mathbb{R}^+ \), given by \( \rho_{\omega,m} := \sum_{i=1}^N |s_i|^2_{h,m} \) for some \( || \cdot ||\)-orthonormal basis \((s_1, \ldots, s_N)\) of \( H^0(M, \mathcal{L}^m) \).

**Remarks.**
1. Changing the choice of orthonormal basis does not change the Bergman kernel.
2. By Lemma 3.1.3, the \((1,1)\)-form \( \omega_{\omega,m} := \omega + \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \rho_{\omega,m} \) is an \( m\)-Bergman metric. In fact it is the \( m\)-Bergman metric associated to the inner product \( || \cdot || \) on \( H^0(M, \mathcal{L}^m) \).

The rest of this section is devoted to proving the following upper bound for the Bergman kernel, due to [Tia90a] (see also the exposition [Tos12]).

**Proposition 3.2.1.** Let \( \lambda \) be a positive real. There exists a constant \( C = C(n, m, \dim H^0(M, K_M^{-m}), c_1(M)^n, \lambda) \), such that if \( \omega \) is a Kähler metric in \( 2\pi c_1(M) \), with \( \text{Ric}(\omega) \geq \lambda \omega \), then \( \rho_{\omega,m} \leq C \).

We start by proving some preliminary results.

**Lemma 3.2.2.** Let \( \mathcal{L} \) be a positive holomorphic line bundle, let \( \omega \in 2\pi c_1(\mathcal{L}) \) be a Kähler metric, and let \( h \) be a hermitian metric on \( \mathcal{L} \) of curvature \( \omega \). For any positive integer \( m \), for any section \( s \) of \( \mathcal{L}^m \), and for any real number \( \alpha \),

\[
\Delta_\omega |s|^2_{h,m} = \frac{1}{2} \alpha^2 |s|^2_{h,m} - \nabla^{h,m} |s|^2_{h,m} - \alpha m |s|^2_{h,m}.
\]

In particular,

\[
\Delta_\omega |s|^2_{h,m} \geq -\alpha m |s|^2_{h,m}.
\]

**Proof.** We first claim that for any real function \( H \), holomorphic function \( f \) and real number \( \alpha \),

\[
\Delta_\omega [e^H |f|^{2\alpha}] = e^H |f|^{2(\alpha-1)} \left[ 2 |a\partial f + f\partial H|^2 + |f|^2 \Delta_\omega H \right].
\] (3.1)
Indeed, since $\partial \bar{f} = 0$, $\bar{\partial} f = 0$ and $\partial \ast \partial f = \bar{\partial} \ast \partial f = 0$,
\[
e^{-H} \partial \ast \partial \left[ e^H |f|^{2a} \right] = e^{-H} \partial \ast \partial \left( e^H |f|^{2(a-1)} \partial f + f \partial H \right)
= \partial \left( e^H |f|^{2(a-1)} \partial f + f \partial H \right) + e^H |f|^{2(a-1)} \partial \ast \partial \left( a \partial f + f \partial H \right)
= |f|^{2(a-1)} \left( a \partial f + f \partial H \right) + e^H |f|^{2(a-1)} \partial \ast \partial \left( a \partial f + f \partial H \right).
\]
\[
\Delta_{\partial,\omega} \left[ e^H |f|^{2a} \right] = e^H |f|^{2(a-1)} \left( a \partial f + f \partial H \right) + e^H |f|^{2(a-1)} \partial \ast \partial \left( a \partial f + f \partial H \right).
\]
Since $\Delta_{\partial,\omega} = \frac{1}{2} \Delta_{\omega}$, the claim follows.

Now let $\eta$ be a nonvanishing local section of $\mathcal{L}$, let $f = s / \eta^m$, let $a = \alpha / 2$, and let $H = \alpha m \log h(\eta, \eta)$.
Then
\[
|s|_{h}^{2a} = |f \eta^{m/2} |_{h}^{2a}
= e^{H} |f|^{2a}
\]
\[
\nabla^{h_{m}} s = (\partial f + f \partial [h^{m}(\eta^{m}, \eta^{m})]) \eta^{m}
= (\partial f + \frac{1}{2} f \partial H) \eta^{m}
\]
\[
\frac{1}{2} \alpha^{2} |s|_{h}^{2a-2} |\nabla^{h_{m}} s|_{h}^{2} = 2a^{2} |f|^{\alpha-2} |\eta^{m}|_{h}^{\alpha-2} \cdot |\partial f + \frac{1}{2} f \partial H|^{2} |\eta^{m}|_{h}^{2}
= 2 e^{H} |f|^{2(a-1)} |a \partial f + f \partial H|^{2}.
\]
Also, since the curvature of $h$ is $\omega$,
\[
-\alpha m = \frac{1}{2} \alpha m \Delta_{\omega} \log h(\eta, \eta) = \Delta_{\omega} H.
\]
Thus, inserting these values of $f$, $a$ and $H$ into (3.1), the result follows.

In the rest of this chapter we restrict the case of $M$ a Fano manifold, and $\mathcal{L}$ the anticanonical bundle $K^{-1}_M$.
We will denote by $V$ the quantity $(2\pi)^n c_1(M)^n$, so that for any Kähler metric $\omega \in 2\pi c_1(M)$, $V = \int_M \omega^n$.

**Proposition 3.2.3.** Let $\lambda$ be a positive real. There exists a constant $C = C(n, m, V, \lambda)$, such that if $\omega$ is a Kähler metric in $2\pi c_1(M)$, with $\text{Ric}(\omega) \geq \lambda \omega$, then for all holomorphic sections $s$ of $K^{-m}_M$,
\[
\sup_M |s|_{h}^{2} \leq C \int_M |s|_{h}^{2} \omega^n.
\]

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(where $h$ is any hermitian metric on $K_M^{-1}$ of curvature $\omega$).

**Proof.** We argue by Moser iteration. By Lemma 3.2.2, for each real number $\alpha$,

$$\alpha m \int_M |s|^{2n}_{h_0^n} \omega^n \geq - \int_M |s|^{2n}_{h_0^n} \Delta_\omega |s|^{2n}_{h_0^n} \omega^n = \int_M |d|s|^{2n}_{h_0^n} |^2 \omega^n.$$

So, applying Theorem 2.1.1 (ii) to the function $f = |s|^{2n}_{h_0^n}$,

$$\left( \int_M |s|^{2n/(n-2)}_{h_0^n} \omega^n \right)^{n-2} \leq C_0 \int_M |d|s|^{2n}_{h_0^n} |^2 \omega^n + C_1 \int_M |s|^{2n}_{h_0^n} \omega^n \leq (\alpha m C_0 + C_1) \int_M |s|^{2n}_{h_0^n} \omega^n.$$

Thus, letting $T_p := \log \left( \int_M |s|^{2n}_{h_0^n} \omega^n \right)^{1/p}$, we have that for all $\alpha$,

$$T_{2\alpha n/(n-2)} \leq \frac{1}{2\alpha} \log (\alpha m C_0 + C_1) + T_{2\alpha}.$$

By induction, for all $k$,

$$T_{2 \left( \frac{n}{n-2} \right)^k} \leq \frac{1}{2} \left( \sum_{r=0}^{k-1} \left( \frac{n-2}{n} \right)^r \log \left( \left( \frac{n}{n-2} \right)^r m C_0 + C_1 \right) \right) + T_2.$$

For all $r$ sufficiently large, $\left( \frac{n}{n-2} \right)^r m C_0 \geq C_1$; thus for all $k$,

$$\frac{1}{2} \left( \sum_{r=0}^{k-1} \left( \frac{n-2}{n} \right)^r \log \left( \left( \frac{n}{n-2} \right)^r m C_0 + C_1 \right) \right) \leq C(n, m, C_0, C_1) \left[ 1 + \left( \sum_{r=N(n, m, C_0, C_1)}^{k-1} \left( \frac{n-2}{n} \right)^r \right) + \left( \sum_{r=N(n, m, C_0, C_1)}^{k-1} r \left( \frac{n-2}{n} \right)^r \right) \right] = C(n, m, C_0, C_1) < \infty.$$

We conclude that for some $C = C(n, m, C_0, C_1)$,

$$\sup_M |s|^{2n}_{h_0^n} = \lim_{k \to \infty} \exp \left( 2T_{2 \left( \frac{n}{n-2} \right)^k} \right) \leq C \exp(2T_2) = C \int_M |s|^{2n}_{h_0^n} \omega^n.$$
Proof of Proposition 3.2.1. Let \( h \) be a hermitian metric on \( K_M^{-1} \) of curvature \( \omega \). Select a basis \( (s_1, s_2, \ldots, s_N) \) for \( H^0(M, K_M^{-m}) \) which is orthonormal with respect to the inner product \( ||s||^2 := \int_M |s|_h^2 \omega^m \). Then

\[
\sup_M \rho_{\omega,m} = \sup_M \sum_{i=1}^N |s_i|_h^2 m \leq \sum_{i=1}^N \sup_M |s_i|_h^2 m \\
\leq C \sum_{i=1}^N \int_M |s_i|_h^2 m \omega^n = NC.
\]

3.3 Approximation by Bergman metrics

For a class \( \mathcal{A} \) of Kähler metrics, an estimate of the form

\[
\inf_{\omega \in \mathcal{A}} \rho_{\omega,m} \geq a > 0
\]

is called a partial \( C^0 \)-estimate. According to longstanding conjectures of Tian (e.g. [Tia91]), such estimates should hold uniformly for quite general classes of metric.

They are in general proved using convergence theory for classes of manifolds whose metrics satisfy some analytic constraint. Tian’s work on complex surfaces [Tia90a] included a partial \( C^0 \)-estimate for Kähler-Einstein surfaces, proved using the orbifold convergence of Kähler-Einstein 4-manifolds:

**Theorem 3.3.1** ([Tia90a]). Let \( J_r \), for some \( r \) such that \( 5 \leq r \leq 8 \), be a moduli space of del Pezzo surfaces,

\[
J_r := \{ \text{Bl}_\Sigma \mathbb{C}P^2 : |\Sigma| = r \}.
\]

Let \( K \) be a compact subset of \( J_r \), and let

\[
\mathcal{A} := \{ (\omega, M) : M \in K, \omega \in 2\pi c_1(M), \omega \text{ Kähler-Einstein} \}.
\]

For each \( m \) sufficiently large and divisible by 2 (if \( r = 7, 8 \)) or by 6 (if \( r = 5, 6 \)), there exists a constant \( a > 0 \), such that for all \( (\omega, M) \in \mathcal{A} \), \( \rho_{\omega,m} \geq a \).
Deep, very recent work [CDS15a, CDS15b, CDS15c, DS14, Sze, Tia, Tia13] has produced partial \( C^0 \)-estimates for various classes of metrics in arbitrary dimension, proved using Cheeger-Colding theory.

In this thesis we will use one of these, due to Székelyhidi, applying to the class of metrics appearing while solving the Aubin continuity method (\( \ast_t \)) of Section 2.2.

**Theorem 3.3.2** ([Sze]). Let \( T \leq 1 \) be a positive real. Let \( (\omega_t) \), for \( t \in (0, T) \), be Kähler metrics, such that \( \text{Ric}(\omega_t) = t\omega_t + (1 - t)\omega \). Then there exist a natural number \( m = m(M, \omega) \) and a constant \( a = a(M, \omega) > 0 \), such that the family \( (\omega_t) \) satisfies a partial \( C^0 \)-estimate: for all \( t \in (0, T) \), \( \rho_{\omega_t, m} \geq a \).

A first consequence of a partial \( C^0 \) estimate (and the source of the appellation) is the following result on approximation by potentials of Bergman metrics:

**Proposition 3.3.3** ([Tia90a]). Let \( a \) and \( \lambda \) be positive reals, and \( \omega \) a fixed Kähler metric in \( 2\pi c_1(M) \).

There exists a constant \( C = C(n, m, N, V, a, \lambda, \omega) \), such that for any Kähler metric \( \omega \in 2\pi c_1(M) \), if

- \( \text{Ric}(\omega) \geq \lambda \),
- \( \text{partial } C^0 \)-estimate \( \rho_{\omega, m} \geq a \),

then there exists an \( m \)-Bergman metric \( \omega \in 2\pi c_1(M) \) such that

\[
\sup_M \left| (\varphi - \sup_M \varphi) - (\psi - \sup_M \psi) \right| \leq C.
\]

**Proof.** Let \( \psi := \varphi + \frac{1}{m} \log \rho_{\omega, m} \). The \((1, 1)\)-form \( \omega_\psi = \omega + \frac{1}{m} \sqrt{-1} \partial \bar{\partial} \log \rho_{\omega, m} \) is an \( m \)-Bergman metric, and there exists a constant \( C = C(n, m, N, V, a, \lambda, \omega) \) such that

\[
-C \leq \psi - \varphi \leq C,
\]

where the lower bound is by the partial \( C^0 \)-estimate and the upper bound is by Proposition 3.2.1. Now apply Lemma 3.3.4, below.

**Lemma 3.3.4.**

\[
\sup_M \left| (\varphi - \sup_M \varphi) - (\psi - \sup_M \psi) \right| \leq 2 \sup_M |\varphi - \psi|.
\]

**Proof.**

\[
\begin{align*}
\sup |\varphi - \psi| + \psi & \geq \varphi \geq -\sup |\varphi - \psi| + \psi \\
\sup |\varphi - \psi| + \sup \psi & \geq \sup \varphi \geq -\sup |\varphi - \psi| + \sup \psi \\
\sup |\varphi - \psi| & \geq \sup \varphi - \sup \psi \geq -\sup |\varphi - \psi|
\end{align*}
\]
so $\sup \varphi - \sup \psi \leq \sup |\varphi - \psi|$, so

$$\sup_M \left| (\varphi - \sup_M \varphi) - (\psi - \sup_M \psi) \right| \leq \sup |\varphi - \psi| + |\sup \varphi - \sup \psi| \leq 2 \sup |\varphi - \psi|.$$
Chapter 4

Existence theorem

In this chapter we study the validity of the following two classes of hypothesis:

(i) Hypothesis on $\varphi$, $\alpha$, $C$:

$$\log \int_M e^{\alpha (\sup_M \varphi - \varphi)} \omega^n \leq C.$$  \hfill ($*_{1,\alpha,C}(\varphi)$)

(ii) Hypothesis on $k \geq 2$, $\varphi$, $\alpha$, $\Lambda$, $C$:

$$\log \int_M e^{\alpha (\sup_M \varphi - \varphi)} \omega^n \leq \alpha \Lambda I_k(\varphi) + C.$$  \hfill ($*_{k,\alpha,\Lambda,C}(\varphi)$)

In these hypotheses $\omega$ is a representative of a Kähler class (sometimes, but not always, taken to be $2\pi c_1(M)$ on a Fano manifold $M$), and $\varphi$ is a Kähler potential with respect to $\omega$. The functionals $I_k$ were introduced in Section 2.8.

The main results are Theorems 4.5.1 and 4.5.2.

4.1 A basic estimate for Einstein potentials

In this section and the next, $M$ is a Fano manifold, and we consider estimates which can be deduced from the hypotheses ($*_{k,\alpha,\Lambda,C}(\varphi)$), ($*_{1,\alpha,C}(\varphi)$) with respect to a metric $\omega \in 2\pi c_1(M)$. These estimates are generalizations of those in [Tia87, Tia90a, Tia91] (for which a good exposition is available in [Tos12]).

**Proposition 4.1.1.** (i) Let $\alpha$, $\delta$, $C_0$ be positive reals. There exists a constant $C = C(n, m, \omega, \alpha, \delta, C_0)$, such that if $\varphi$ is a Kähler potential, which solves the Aubin continuity-method equation ($*_{1}$) for some
real number $t$ with $\delta \leq t \leq 1$, and if $(\ast_{1,\alpha,C_0}(\varphi))$ holds, then

$$\sup_M \varphi \leq \frac{1-\alpha}{\alpha} \int_M (-\varphi) \omega^n + C.$$ 

(ii) Let $k$ be a natural number, with $2 \leq k \leq n$. Let $\alpha, \delta, \Lambda, C_0$ be positive reals. There exists a constant $C = C(n, m, k, \omega, \alpha, \delta, C_0)$, such that if $\varphi$ is a Kähler potential, which solves the Aubin continuity-method equation $(\ast_t)$ for some real number $t$ with $\delta \leq t \leq 1$, and if $(\ast_{k,\alpha,\Lambda,C_0}(\varphi))$ holds, then

$$\sup_M \varphi \leq \frac{1-\alpha}{\alpha} \int_M (-\varphi) \omega^n + \Lambda I_k(\varphi) + C.$$ 

Proof. By Jensen’s inequality,

$$\alpha t \sup_M \varphi + (1-\alpha) \int_M t\varphi \omega^n \leq \log \left[ \int_M e^{\alpha t \sup_M \varphi + (1-\alpha)t\varphi} \omega^n \right]. \tag{4.1}$$

Let $f$ be the real function on $M$ such that

\begin{align*}
\{ &\sqrt{-1}\partial \bar{\partial} f = \text{Ric}(\omega) - \omega, \\
&\int_M e^f \omega^n = V.
\}
\end{align*}

By the Aubin equation $(\ast_t)$ on $\varphi$, and by construction of $f$, $e^{\varphi} \omega^n = e^f \omega^n$, so

$$\log \left[ \int_M e^{\alpha t \sup_M \varphi + (1-\alpha)t\varphi} \omega^n \right] = \log \left[ \int_M e^{\alpha t (\sup_M \varphi - \varphi)} + f \omega^n \right] \leq C + \log \left[ \int_M e^{\alpha t (\sup_M \varphi - \varphi)} \omega^n \right]. \tag{4.2}$$

By Hölder’s inequality,

$$\log \left[ \int_M e^{\alpha t (\sup_M \varphi)} \omega^n \right] \leq t \log \left[ \int_M e^{\alpha (\sup_M \varphi)} \omega^n \right]. \tag{4.3}$$

Since $(\ast_{k,\alpha,\Lambda,C_0}(\psi))$ or $(\ast_{1,\alpha,C_0}(\varphi))$ holds,

$$\log \int_M e^{\alpha (\sup_M \varphi)} \omega^n \leq \alpha \Lambda I_k(\varphi) + C_0. \tag{4.4}$$
Combining equations (4.1), (4.2), (4.3) and (4.4),
\[ \alpha t \sup_{\varphi} + (1 - \alpha) \int_M t \varphi \omega \varphi^n \leq C + tC + \alpha t \Lambda k(\varphi). \]

Rearranging, and using that \( t \geq \delta \),
\[ \sup_{\varphi} \leq \frac{1 - \alpha}{\alpha} \int_M (-\varphi) \omega \varphi^n + \Lambda k(\varphi) + C. \]

4.2 \( C^0 \) estimate

Throughout this section we let \( \mathcal{I}_0 \) and \( (\varphi_t) \) be as in Theorem 2.7.2. Thus the functions \( (\varphi_t) \) are, in particular, solutions to the Aubin continuity method \( *(\dot{\gamma}) \) on the interval \( \mathcal{I}_0 \).

**Proposition 4.2.1.** Let \( k \) be a natural number, with \( 2 \leq k \leq n \). Let \( \alpha_1, \alpha_k, \Lambda, \delta \) be positive reals, with \( \alpha_1 \leq 1 \), with \( \Lambda > 1 \) and with \( \delta > 0 \) small. Suppose that
\[
\frac{n}{n+1} < \alpha_k,
\]
\[
(n[\Lambda - 1] + k - 1) \left[ \frac{1}{\alpha_1} - \frac{n+1}{n} \right] < (n-k+1) \left[ \frac{n+1}{n} - \frac{1}{\alpha_k} \right],
\]
and that there exist constants \( C \) such that, for all \( t \in \mathcal{I}_0 \cap [\delta, 1) \), \( *(k, \alpha_k, \Lambda, C(\psi)) \) and \( *(1, \alpha_1, C(\psi)) \) hold. Then there exists a constant \( C \) such that, for all \( t \in \mathcal{I}_0 \cap [\delta, 1) \),
\[ \sup_{\varphi} \leq C. \]

**Proof.** Throughout this proof write \( \varphi \) for a potential \( \varphi_t \) satisfying the hypotheses of the proposition. By Proposition 2.9.1,
\[
\int_M (-\varphi) \omega \varphi^n + (n - k + 1)I_k(\varphi) \leq n \sup_{\varphi} \varphi. \quad (4.5)
\]

Let \( \mu_1 := (1 - \alpha_1)/\alpha_1 \) and \( \mu_k := (1 - \alpha_k)/\alpha_k \). We have that \( \mu_1 \geq 0 \) (since \( \alpha_1 \leq 1 \)) and \( \mu_k > -1 \), and the inequalities on \( \alpha_1, \alpha_k \) yield the constraints \( \mu_k < 1/n \),
\[
(n[\Lambda - 1] + k - 1) \left[ \mu_1 - \frac{1}{n} \right] < (n-k+1) \left[ \frac{1}{n} - \mu_k \right]. \quad (4.6)
\]
Since \((\ast_{1,\alpha_1,C}(\psi))\) holds, we obtain from Proposition 4.1.1 (i) that
\[
\sup_M \varphi \leq \mu_1 \int_M (-\varphi) \omega \varphi^n + C; \tag{4.7}
\]
since \((\ast_{k,\alpha_k,A,C}(\psi))\) holds, we obtain from Proposition 4.1.1 (ii) that
\[
\sup_M \varphi \leq \mu_k \int_M (-\varphi) \omega \varphi^n + \Lambda I_k(\varphi) + C. \tag{4.8}
\]

Consider the following nonnegative linear combination of the inequalities (4.5), (4.7), (4.8):
\[
\mu_1 \Lambda (4.5) + [\Lambda - (n - k + 1)\mu_k] (4.7) + (n - k + 1)\mu_1 (4.8). \tag{4.9}
\]
(Since \(\Lambda > 1 > \frac{n-k+1}{n}\) and \(\frac{1}{n} > \mu_k\) and \(\mu_1 \geq 0\), the coefficients of (4.5), (4.7), (4.8) are indeed nonnegative.)

After moving everything to the left-hand side, the coefficient of \(I_k(\varphi)\) in inequality (4.9) is:
\[
\mu_1 \Lambda \cdot (n - k + 1) + (n - k + 1)\mu_1 \cdot -\Lambda = 0.
\]
The coefficient of \(\int_M (-\varphi) \omega \varphi^n\) in inequality (4.9) is:
\[
\mu_1 \Lambda \cdot 1 + [\Lambda - (n - k + 1)\mu_k] \cdot -\mu_1 + (n - k + 1)\mu_1 \cdot -\mu_k = 0.
\]
The coefficient of \(\sup_M \varphi\) in inequality (4.9) is:
\[
\mu_1 \Lambda \cdot -n + [\Lambda - (n - k + 1)\mu_k] \cdot 1 + (n - k + 1)\mu_1 \cdot 1
\]
\[= (n\Lambda - n + k - 1) \left(\frac{1}{n} - \mu_1\right) + (n - k + 1) \left(\frac{1}{n} - \mu_k\right).
\]
which by (4.6) is positive. So the inequality (4.9) simplifies to the uniform bound
\[
\sup_M \varphi \leq \left\{ (n - k + 1) \left[\frac{1}{n} - \mu_k\right] - (n\Lambda - 1 + k - 1) \left[\mu_1 - \frac{1}{n}\right] \right\}^{-1} C.
\]
4.3 Approximation of Kähler potentials

In this section we study some technical properties of the functionals $I_k$.

Let $\omega$ be a Kähler metric.

**Lemma 4.3.1.** Let $\varphi$ and $\psi$ be Kähler potentials with respect to $\omega$. Let $r$ be an integer, $1 \leq r \leq n$. Then

$$\hat{M} (\varphi \omega^r - \psi \omega^r) \wedge \omega^{n-r} = \hat{M} \left[ (\varphi - \psi) \left( \sum_{i=0}^{r-1} \omega^i \wedge \omega^{r-i-1} + \sum_{i=0}^{k-1} \omega^i \wedge \omega^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega^i \wedge \omega^{k-i-2} \wedge \omega^{n-k+2} \right) \right].$$

**Proof.**

$$\varphi \omega^r - \psi \omega^r = (\varphi \omega^r - \omega^r) + (\varphi - \psi) \omega^r$$

$$= \varphi \sqrt{-1} \partial \bar{\partial} (\varphi - \psi) \left( \sum_{i=0}^{r-1} \omega^i \wedge \omega^{r-i-1} + \sum_{i=0}^{k-1} \omega^i \wedge \omega^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega^i \wedge \omega^{k-i-2} \wedge \omega^{n-k+2} \right).$$

Wedge with $\omega^{n-r}$, integrate, and apply Stokes’ theorem:

$$\hat{M} (\varphi \omega^r - \psi \omega^r) \wedge \omega^{n-r} = \hat{M} \left[ (\varphi - \psi) \left( \sum_{i=0}^{r-1} \omega^i \wedge \omega^{r-i-1} + \sum_{i=0}^{k-1} \omega^i \wedge \omega^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega^i \wedge \omega^{k-i-2} \wedge \omega^{n-k+2} \right) \right].$$

**Corollary 4.3.2.** Let $\varphi$ and $\psi$ be Kähler potentials. Let $k$ be an integer, $2 \leq k \leq n$. Then

$$I_k(\varphi) - I_k(\psi) = \hat{M} \left[ (\varphi - \psi) \left( \sum_{i=0}^{k-1} \omega^i \wedge \omega^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega^i \wedge \omega^{k-i-2} \wedge \omega^{n-k+2} \right) \right].$$

**Proposition 4.3.3.** Let $c > 0$. Let $k$ be an integer, $2 \leq k \leq n$. There exists $C = C(k, c)$, such that if $\varphi$ and
ψ are Kähler potentials with

\[ \sup_M |\varphi - \psi| \leq c, \]

then

\[ |I_k(\varphi) - I_k(\psi)| \leq C. \]

Proof. By Corollary 4.3.2, for any real number \( a \),

\[ I_k(\varphi) - I_k(\psi) = \int_M (\varphi - \psi + a) \left[ \omega^n - \sum_{i=0}^{k-1} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega_\varphi^i \wedge \omega_\psi^{k-i-2} \wedge \omega^{n-r+2} \right]. \]

(The constant \( a \) can be added since

\[ \int_M \left[ \omega^n - \sum_{i=0}^{k-1} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-k+1} + \sum_{i=0}^{k-2} \omega_\varphi^i \wedge \omega_\psi^{k-i-2} \wedge \omega^{n-r+2} \right] = 1 - (k-1) + (k-2) = 0. \]

Since \( \sup_M |\varphi - \psi| \leq c \),

\[ 0 \leq \varphi - \psi + c \leq 2c. \]

Also the forms

\[ \omega^n + \sum_{i=0}^{k-2} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-r+2}, \quad \sum_{i=0}^{k-1} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-k+1} \]

are positive. Hence we have the pointwise inequalities

\[ 0 \leq (\varphi - \psi + c) \left[ \omega^n + \sum_{i=0}^{k-2} \omega_\varphi^i \wedge \omega_\psi^{k-i-2} \wedge \omega^{n-r+2} \right] \leq 2c \left[ \omega^n + \sum_{i=0}^{k-2} \omega_\varphi^i \wedge \omega_\psi^{k-i-2} \wedge \omega^{n-r+2} \right], \]

\[ -2c \left[ \sum_{i=0}^{k-1} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-k+1} \right] \leq -(\varphi - \psi - c) \left[ \sum_{i=0}^{k-1} \omega_\varphi^i \wedge \omega_\psi^{k-i-1} \wedge \omega^{n-k+1} \right] \leq 0. \]

Summing, integrating and averaging,

\[ -2(k-1)c \leq I_k(\varphi) - I_k(\psi) \leq 2(k-1)c. \]

\[ \square \]

Proposition 4.3.4 (Change of reference metric). Let \( c > 0 \). Let \( k \) be an integer, \( 2 \leq k \leq n \). There exists
\[ C = C(k, c), \text{ such that if } \omega \text{ and } \omega_\psi \]
\[ \sup_M |\psi| \leq c, \]
then for each potential \( \varphi \) which is Kähler with respect to \( \omega \) (so that \( \omega_\varphi = \omega_\psi + \sqrt{-1} \partial \overline{\partial} (\varphi - \psi) = (\omega_\psi)_{\varphi - \psi}, \) and hence \( \varphi - \psi \) is Kähler with respect to \( \omega_\psi \)),
\[ |I_k(\varphi, \omega) - I_k(\varphi - \psi, \omega_\psi)| \leq C. \]

**Proof.** We observe that
\[ I_k(\varphi, \omega) = \int_M \varphi (\omega^n - \omega_\varphi^k \wedge \omega^{n-k}) \]
\[ = \int_M (-\varphi)(\omega_\varphi^n - \omega^n) - \int_M (-\varphi) (\omega_\varphi^n - \omega^{n-k} \wedge \omega_\varphi^k) \]
\[ = I_n(-\varphi, \omega_\varphi) - I_n(-\varphi, \omega_\varphi). \]

So \[ |I_k(\varphi, \omega) - I_k(\varphi - \psi, \omega_\psi)| \leq |I_n(-\varphi, \omega_\varphi) - I_n(\psi - \varphi, \omega_\varphi)| + |I_n(-\varphi, \omega_\varphi) - I_n(\psi - \varphi, \omega_\varphi)|. \] By Proposition 4.3.3 each of the right-hand side terms is bounded independently of \( \psi, \varphi. \) \( \square \)

### 4.4 The invariants \( \mathcal{B}_{m,k}(\mathcal{L}) \)

In this section, we introduce some invariants of a line bundle \( \mathcal{L} \), which characterize the validity of the inequalities \((*_k, \alpha, \Lambda, C(\psi))\) and \((*_1, \alpha, C(\psi))\) for \( \mathcal{L}'s \) \( m \)-Bergman metrics.

**Proposition 4.4.1** (Change of reference metric). (i) If there exists \( C_0 \), such that for all \( m \)-Bergman metrics \( \omega_0 + \sqrt{-1} \partial \overline{\partial} \psi_0 \), \((*_1, \alpha, C_0(\psi_0))\) holds, then there exists \( C \), such that for all \( m \)-Bergman metrics \( \omega + \sqrt{-1} \partial \overline{\partial} \psi \), \((*_1, \alpha, C(\psi))\) holds.

(ii) For \( k \) such that 2 \( \leq k \leq n \), if there exists \( C_0 \), such that for all \( m \)-Bergman metrics \( \omega_0 + \sqrt{-1} \partial \overline{\partial} \psi_0 \), \((*_k, \alpha, \Lambda, C_0(\psi_0))\) holds, then there exists \( C \), such that for all \( m \)-Bergman metrics \( \omega + \sqrt{-1} \partial \overline{\partial} \psi \), \((*_k, \alpha, \Lambda, C(\psi))\) holds.

**Proof.** We prove (ii); (i) is similar. Let \( \omega = \omega_0 + \sqrt{-1} \partial \overline{\partial} \rho \), and let \( c = \sup_M |\rho| \). Any \( m \)-Bergman metrics \( \omega + \sqrt{-1} \partial \overline{\partial} \psi \) may also be expressed as \( \omega_0 + \sqrt{-1} \partial \overline{\partial} \psi_0 \), with \( \psi_0 = \psi + \rho \), so (by Lemma 3.3.4)
\[ \sup_M \left| \left( \psi_0 - \sup_M \psi_0 \right) - \left( \psi - \sup_M \psi \right) \right| \leq 2 \sup_M |\psi_0 - \psi| = 2 \sup_M |\rho| = c. \]

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\[
\log \int_M e^{\alpha (\sup_M \psi - \psi)} \omega^n \leq C + \log \int_M e^{\alpha (\sup_M \psi_0 - \psi_0)} \omega^n.
\]

As for the right-hand side, by Proposition 4.3.4, \( I_k(\psi, \omega_0) \leq I_k(\psi, \omega) + C. \)

It follows that for any representatives \( \omega_0, \omega \) of the cohomological class \( 2\pi c_1(M) \), and for any \( 2 \leq k \leq n \),

\[
\sup \left\{ \alpha > 0 : \text{all } m\text{-Bergman metrics } \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_0, \right. \left. (*_{k,\alpha,C_0}(\psi_0)) \text{ holds} \right\}
= \sup \left\{ \alpha > 0 : \text{all } m\text{-Bergman metrics } \omega + \sqrt{-1} \partial \bar{\partial} \psi, \right. \left. (*_{k,\alpha,1+C}(\psi)) \text{ holds} \right\}
\]

We define \( B_{m,k}(\mathcal{L}) \) to be this supremum, an invariant only of \( m, k \) and \( \mathcal{L} \). Similarly,

\[
B'_{m,k}(\mathcal{L}) := \sup \left\{ \alpha > 0 : \text{there exist } \Lambda \text{ and } C, \text{ such that for all } m\text{-Bergman metrics } \omega + \sqrt{-1} \partial \bar{\partial} \psi, \right. \left. (*_{k,\alpha,C}(\psi)) \text{ holds} \right\}
\]

is well-defined and dependent only on \( m, k \) and \( \mathcal{L} \), and

\[
B_{m,1}(\mathcal{L}) := \sup \left\{ \alpha > 0 : \text{there exists } C, \text{ such that for all } m\text{-Bergman metrics } \omega + \sqrt{-1} \partial \bar{\partial} \psi, \right. \left. (*_{1,\alpha,C}(\psi)) \text{ holds} \right\}
\]

is well-defined and dependent only on \( m \) and \( \mathcal{L} \). By construction \( B'_{m,k}(\mathcal{L}) \geq B_{m,k}(\mathcal{L}) \).

In particular: on a Fano manifold \( M \), we define

\[
B_{m,k}(M) := B_{m,k}(K_M^{-1}), \quad B'_{m,k}(M) := B'_{m,k}(K_M^{-1}).
\]

**Lemma 4.4.2.** (i) For each \( \alpha < B_{m,1}(\mathcal{L}) \), there exists \( C \), such that for all \( m\)-Bergman metrics \( \omega + \sqrt{-1} \partial \bar{\partial} \psi, (*_{1,\alpha,C}(\psi)) \) holds.

(ii) For \( k \) such that \( 2 \leq k \leq n \), for each \( \alpha < B_{m,k}(\mathcal{L}) \), for each \( \epsilon > 0 \) there exists \( C \), such that for all
m-Bergman metrics $\omega + \sqrt{-1} \partial \overline{\partial} \psi$, $(\ast_{k,\alpha,1+\epsilon,C}(\psi))$ holds.

(iii) For $k$ such that $2 \leq k \leq n$, for each $\alpha < B'_{m,k}(L)$, there exist $\Lambda$ and $C$, such that for all $m$-Bergman metrics $\omega + \sqrt{-1} \partial \overline{\partial} \psi$, $(\ast_{k,\alpha,\Lambda,C}(\psi))$ holds.

Proof. We show that if $(\ast_{k,\alpha,0,\Lambda_0,1+\epsilon,C_0}(\psi))$ holds and $\alpha < \alpha_0$, then for some $C$, $(\ast_{k,\alpha,\Lambda,C}(\psi))$ holds. Indeed, by Hölder’s inequality,

$$\log \int_M e^{\alpha (\sup_M \varphi - \psi)} \omega^n \leq \frac{\alpha}{\alpha_0} \log \int_M e^{\alpha (\sup_M \varphi - \psi)} \omega^n + \left( 1 - \frac{\alpha}{\alpha_0} \right) \log \int_M \omega^n.$$ 

\[\Box\]

Proposition 4.4.3.  

(i) Suppose $B_{m,1}(L) > \alpha$. Then for each $a > 0$ there exists a constant $C$, such that if (partial $C^0$-estimate) $\varphi$ is a Kähler potential with $\rho_{\omega,\varphi,m} \geq a$, then the hypothesis $(\ast_{1,a,C}(\varphi))$ holds.

(ii) Let $k$ be such that $2 \leq k \leq n$, and suppose $B_{m,k}(L) > \alpha$. Then for each $a > 0$ and $\epsilon > 0$ there exists a constant $C$, such that if (partial $C^0$-estimate) $\varphi$ is a Kähler potential with $\rho_{\omega,\varphi,m} \geq a$, then the hypothesis $(\ast_{k,\alpha,1+\epsilon,C}(\varphi))$ holds.

(iii) Let $k$ be such that $2 \leq k \leq n$, and suppose $B'_{m,k}(L) > \alpha$. Then for each $a > 0$ there exist constants $\Lambda$ and $C$, such that if (partial $C^0$-estimate) $\varphi$ is a Kähler potential with $\rho_{\omega,\varphi,m} \geq a$, then the hypothesis $(\ast_{k,\alpha,\Lambda,C}(\varphi))$ holds.

Proof. We show (ii); (i) and (iii) are similar. First, by Proposition 3.3.3, there exists a constant $c$, such that for all Kähler potentials $\varphi$ with $\rho_{\omega,\varphi,m} \geq a$, there exists an $m$-Bergman metric $\omega + \partial \overline{\partial} \psi$ such that

$$\sup_M \left| (\varphi - \sup_M \varphi) - (\psi - \sup_M \psi) \right| \leq c. \quad (4.10)$$

Next, by Lemma 4.4.2, for each $\epsilon > 0$ there exists $C_0$, such that for all $m$-Bergman metrics $\omega + \sqrt{-1} \partial \overline{\partial} \psi$, $(\ast_{k,\alpha,1+\epsilon,C_0}(\psi))$ holds.

Now, by the assumption (4.10),

$$\log \int_M e^{\alpha (\sup_M \varphi - \psi)} \omega^n \leq C + \log \int_M e^{\alpha (\sup_M \psi - \psi)} \omega^n.$$

By hypothesis $(\ast_{k,\alpha,1+\epsilon,C_0}(\psi))$ holds.

1. If $k = 1$, then $I_k \equiv 0$ and we are done.
2. If \( k \geq 2 \), then by (4.10) and Proposition 4.3.3, \( I_k(\psi) \leq C + I_k(\varphi) \). This concludes the proof.

\[ \square \]

### 4.5 Existence of Kähler-Einstein metrics

**Theorem 4.5.1.** Let \( k \) be a natural number, with \( 2 \leq k \leq n \). Suppose that

\[
\frac{n}{n+1} < \mathcal{B}_{m,k}(M),
\]

\[
\frac{k-1}{\mathcal{B}_{m,1}(M)} + \frac{n-k+1}{\mathcal{B}_{m,k}(M)} < n+1,
\]

and that we have the following partial \( C^0 \) estimate: there exists \( a > 0 \), such that for all \( t \in \mathcal{I}_0 \cap [\delta, 1) \), \( \rho_{\omega_t, \psi} \geq a \). Then \( M \) admits a Kähler-Einstein metric.

**Proof.** The second part of the criterion may be rearranged as

\[
(k-1) \left[ \frac{1}{\mathcal{B}_{m,1}(M)} - \frac{n+1}{n} \right] < (n-k+1) \left[ \frac{n+1}{n} - \frac{1}{\mathcal{B}_{m,k}(M)} \right],
\]

where (by the first part of the criterion) \( \mathcal{B}_{m,k}(M) > n/(n+1) \), and so the right-hand side is positive.

Therefore there exist \( \epsilon > 0 \), \( \alpha_1 < \mathcal{B}_{m,1}(M) \) and \( \alpha_k \) with \( n/(n+1) < \alpha_k < \mathcal{B}_{m,k}(M) \), such that

\[
(\epsilon n + k - 1) \left[ \frac{1}{\alpha_1} - \frac{n+1}{n} \right] < (n-k+1) \left[ \frac{n+1}{n} - \frac{1}{\alpha_k} \right].
\]

Since we have a partial \( C^0 \) estimate, we may apply Proposition 4.4.3, and so there exist \( C \) such that for all \( t \in \mathcal{I}_0 \cap [\delta, 1) \), \((*_{k, \alpha_1, 1+\epsilon, C(\varphi_t)})\) and \((*_{1, \alpha_1, C(\varphi_t)})\) hold. By Proposition 4.2.1, this gives us the bound

\[
\sup_M \varphi_t \leq C.
\]

Now apply Theorem 2.9.4. \( \square \)

**Theorem 4.5.2.** Let \( k \) be a natural number, with \( 2 \leq k \leq n \). Suppose that

\[
\frac{n}{n+1} < \mathcal{B}'_{m,k}(M),
\]

\[
\frac{n}{n+1} = \mathcal{B}_{m,1}(M),
\]

\[ \]
and that we have the following partial $C^0$ estimate: there exists $a > 0$, such that for all $t \in I_0 \cap [\delta, 1)$, $\rho_{\omega_m, m} \geq a$. Then $M$ admits a Kähler-Einstein metric.

Proof. Let $\alpha_k$ be such that $n/(n + 1) < \alpha_k < B^\prime_{m,k}(M)$. Since we have a partial $C^0$ estimate, we may apply Proposition 4.4.3, and so there exist $\Lambda$ and $C$ such that for all $t \in I_0 \cap [\delta, 1)$, $(\ast_{k, \alpha_k, \Lambda, C}(\varphi_t))$ holds.

Therefore there exists $\alpha_1 < n/(n + 1)$ such that

$$(n[\Lambda - 1] + k - 1) \left[ \frac{1}{\alpha_1} - \frac{n + 1}{n} \right] < (n - k + 1) \left[ \frac{n + 1}{n} - \frac{1}{\alpha_k} \right].$$

Again, since we have a partial $C^0$ estimate, and (since $n/(n + 1) = B_{m,1}(M)$) we may apply Proposition 4.4.3, and so there exists $C$ such that for all $t \in I_0 \cap [\delta, 1)$, $(\ast_{1, \alpha_1, C}(\varphi_t))$ holds.

By Proposition 4.2.1, this gives us the bound

$$\sup_M \varphi_t \leq C.$$

Now apply Theorem 2.9.4. \qed
Chapter 5

Log-canonical thresholds and alpha-invariants

5.1 Local theory

As is conventional [Hör94, Dem12], our plurisubharmonic functions are locally-$L^1$ but need not be continuous.

Definition. The critical exponent of a plurisubharmonic function germ $\varphi$ at a point $x$ is

$$c_x(\varphi) := \sup \left\{ \alpha > 0 : \text{there exists a neighbourhood } U \ni x, \text{ such that } \int_U e^{-2\alpha \varphi} < \infty \right\}.$$

Remark. By the Openness Conjecture of Demailly-Kollár [DK01], recently proved by Berndtsson [Ber13], the set

$$\left\{ \alpha > 0 : \text{there exists a neighbourhood } U \ni x, \text{ such that } \int_U e^{-2\alpha \varphi} < \infty \right\}.$$

is an open interval $(0, c)$. In particular the supremum is never attained.

Theorem 5.1.1 (Demailly-Kollár [DK01]). Let $\varphi$ be plurisubharmonic, with $\alpha < c_x(\varphi)$. If $\varphi_i \to \varphi$ locally in $L^1$, then there exists a neighbourhood $U$ of $x$ on which $e^{-2\alpha \varphi_i} \to e^{-2\alpha \varphi}$ in $L^1$.

Definition. 1. The log-canonical threshold of a holomorphic function germ $f$ at a point $x$, $\text{lc}_x(f)$, is the critical exponent $c_x(\log |f|)$.

2. Let $\mathcal{J}$ be a coherent ideal sheaf, generated on some neighbourhood of a point $x$ by the holomor-
phic functions \((f_1, \ldots, f_k)\). The log-canonical threshold of \(J\) at \(x\), \(\text{lct}_x(J)\), is the critical exponent \(c_x(\log|f_1| + \cdots + |f_k|)\).

**Remark.** This definition is easily seen to be independent of the choice of generators of \(J\).

**Lemma 5.1.2.** If \(J_1 \subseteq J_2\), then \(\text{lct}_x(J_1) \leq \text{lct}_x(J_2)\) at each point \(x\).

**Example 5.1.3.** If \(f\) is nonzero at \(x\), then \(\text{lct}_x(f) = \infty\).

**Example 5.1.4.** If \(x\) is a nonsingular point of the hypersurface \(f^{-1}(0)\) (that is, \(f\) is zero at \(x\) but \(df|_x \neq 0\)), then \(\text{lct}_x(f) = 1\). For instance, \(\text{lct}_0(z) = 1\).

**Example 5.1.5.** If \(f = g^k\) (so that the hypersurface \(f^{-1}(0)\) is nonreduced, a multiplicity-\(k\) copy of a hypersurface \(g^{-1}(0)\)), and \(g^{-1}(0)\) contains and is nonsingular at \(x\), then \(\text{lct}_x(f) = 1/k\). For instance, \(\text{lct}_0(z^k) = 1/k\).

**Example 5.1.6.** Let \(f(z_1, z_2) = z_1^k - z_2^2\), for \(k \geq 3\), so that \(f\) defines a cusp singularity at 0. We may divide the polydisc \(P := \{(z_1, z_2) : |z_1| < r^{1/k}, |z_2| < r^{1/2}\} \subseteq \mathbb{C}^2\) as the union (modulo a set of measure 0) of

\[
R_1 := \{(z_1, z_2) : |z_2|^2 < |z_1|^k < r\}, \quad R_2 := \{(z_1, z_2) : |z_1|^k < |z_2|^2 < r\}.
\]

Also define punctured polydiscs

\[
B_1 := \{(z_1, w) : 0 < |z_1| < r^{1/k}, |w| < 1\}, \quad B_2 := \{(w, z_2) : |w| < 1, 0 < |z_2| < r^{1/2}\},
\]

and define maps \(h_i : R_i \to B_i\),

\[
h_1(z_1, z_2) := (z_1, z_2^2/z_1^k), \quad h_2(z_1, z_2) := (z_1^k/z_2^2, z_1).
\]

The maps \(h_1, h_2\) are local diffeomorphisms, covering maps of index 2, \(k\) respectively, with Jacobians \(|w|^{-1}|z_1|^k\) and \(|z_2|^{k-2}|w|^{-2(k-1)/k}\), and \(f\) factors through \(h_1, h_2\). Therefore

\[
\int_{(z_1, z_2) \in P} |z_1^k - z_2^2|^{-2\alpha} \sqrt{-1}dz_1 \wedge d\overline{z}_1 \wedge \sqrt{-1}dz_2 \wedge d\overline{z}_2
\]
is finite if and only if

\[
I_1 := \int_{(z_1, w) \in B_1} \left| (w - 1) z_1 \right|^{-2\alpha} \cdot |w|^{-1} |z_1|^k \sqrt{-1} dz_1 \wedge d\overline{z_1} \wedge \sqrt{-1} dw \wedge d\overline{w}
\]

\[
= \left( \int_{|z_1| < \frac{1}{r/k}} |z_1|^{k(1 - 2\alpha)} \sqrt{-1} dz_1 \wedge d\overline{z_1} \right) \left( \int_{|w| < 1} |w - 1|^{-2\alpha} |w|^{-1} \sqrt{-1} dw \wedge d\overline{w} \right),
\]

\[
I_2 := \int_{(w, z_2) \in B_2} \left| (1 - w) z_2 \right|^{-2\alpha} \cdot |z_2|^{4/k} |w|^{-2(k-1)/k} \sqrt{-1} dw \wedge d\overline{w} \wedge \sqrt{-1} dz_2 \wedge d\overline{z_2}
\]

\[
= \left( \int_{|w| < 1} |1 - w|^{-2\alpha} |w|^{-2(k-1)/k} \sqrt{-1} dw \wedge d\overline{w} \right) \left( \int_{|z_2| < \frac{1}{r/k}} |z_2|^{4((1/k) - \alpha)} \sqrt{-1} dz_2 \wedge d\overline{z_2} \right),
\]

are finite. This happens precisely for \( \alpha \) such that

\[4[(1/k) - \alpha] > -2, \quad k(1 - 2\alpha) > -2;\]

that is, for \( \alpha < (k + 2)/2k \). Therefore \( \text{lct}_0(f) = (k + 2)/2k \).

5.2 Global theory

Let \( M \) be a complex manifold.

**Definition.** 1. Let \( \mathcal{L} \) be a line bundle over \( M \), and \( \sigma \) a section of \( \mathcal{L} \). The log-canonical threshold of \( \sigma \) is the infimum

\[\text{lct}(\sigma) := \inf_{x \in M} \text{lct}_x(\sigma).\]

For \( D \) a divisor of \( M \), with associated line bundle \( \mathcal{O}(D) \) and defining section \( \sigma_D \), we also write \( \text{lct}(D) = \text{lct}(\sigma_D) \).

2. Let \( \mathcal{J} \) be a coherent ideal sheaf of \( M \). The log-canonical threshold of \( \mathcal{J} \) is the infimum

\[\text{lct}(\mathcal{J}) := \inf_{x \in M} \text{lct}_x(\mathcal{J}).\]

Let us calculate a few examples of log-canonical thresholds. We recall a useful enumerative theorem:

**Example 5.2.1.** We consider the complex manifold \( \mathbb{C}P^2 \).

1. Each hyperplane has log-canonical threshold 1.
2. A quadric curve is either nonsingular or the product of two hyperplanes (which may either coincide or not). A section of $\mathcal{O}(2)$ therefore has log-canonical threshold $\frac{1}{2}$, if it is the square $\sigma^2$ of a single section $\sigma$ of $\mathcal{O}(1)$, or 1, otherwise.

**Definition.** Let $\mathcal{L}$ be a line bundle over $M$. Let $G_k$ be the Grassmannian of $k$-dimensional vector subspaces of $H^0(M, \mathcal{L}^m)$. For a vector subspace $V \in G_k$, let $\mathcal{J}_V$ be the coherent ideal sheaf locally generated by the sections in $V$. The $(m,k)$-th alpha-invariant of $\mathcal{L}$ is

$$\alpha_{m,k}(\mathcal{L}) := m \inf_{V \in G_k} \operatorname{lct}(\mathcal{J}_V).$$

In particular, for a Fano manifold $M$, the $(m,k)$-th alpha-invariant of $M$ is $\alpha_{m,k}(M) := \alpha_{m,k}(K_M^{-1})$.

**Lemma 5.2.2.** $\alpha_{m,1}(\mathcal{L}) \leq \alpha_{m,2}(\mathcal{L}) \leq \alpha_{m,3}(\mathcal{L}) \leq \cdots \leq \alpha_{m,\dim H^0(M, \mathcal{L}^m) + 1}(\mathcal{L}) = \infty$.

**Proof.** Lemma 5.1.2.

**Example 5.2.3.** By Example 5.2.1, $\alpha_{1,2}(\mathbb{CP}^2) = 2$.

**Example 5.2.4** ([Tia90a]). We consider the complex manifold $M = \text{Bl}_\Sigma \mathbb{CP}^2$, where $|\Sigma| = 6$. $\alpha_{1,1}(M)$ is $\frac{2}{3}$ if $M$ has an Eckardt point, and $\frac{3}{4}$ if not.

**Example 5.2.5** ([Che01]). Let $M$ be a smooth hypersurface of degree $n+1$ in $\mathbb{CP}^{n+1}$. Then $M$ is Fano, and $\alpha_{m,1}(M) = \frac{n}{n+1}$ for all $m$, with equality if and only if $M$ contains a cone of dimension $n-1$.

**Proposition 5.2.6.** Suppose $\alpha < \alpha_{m,k}(\mathcal{L})$. Let $h$ be a hermitian metric on $\mathcal{L}$ and $\Omega$ a volume form on $M$, and equip $H^0(M, \mathcal{L}^m)$ with the inner product $\langle \sigma, \sigma' \rangle = \int_M h(\sigma, \sigma') \Omega$. Then there exists a positive constant $C$, such that for all orthonormal sets $(\sigma_1, \ldots, \sigma_k)$ in $H^0(M, \mathcal{L}^m)$,

$$\int_M \left( \sum_{i=1}^k |\sigma_i|^2 h^m \right)^{-\alpha/m} \Omega \leq C.$$

**Remark.** This characterization yields an equivalent definition of the alpha-invariants $\alpha_{m,k}(M)$, which was the definition originally given by Tian [Tia90a, Tia91].

**Proof.** Suppose for the sake of contradiction that there exists no such constant $C$. Then by the compactness of the space of orthonormal subsets $(\sigma_1, \ldots, \sigma_k)$ in $H^0(M, \mathcal{L}^m)$, there must therefore exist a convergent
sequence of such sets, \((\sigma_1^r, \ldots, \sigma_k^r)_{r \in \mathbb{N}}\), such that \((\sigma_1, \ldots, \sigma_k) := \lim_{r \to \infty} (\sigma_1^r, \ldots, \sigma_k^r)\) exists and such that

\[
\lim_{r \to \infty} \int_M \left( \sum_{i=1}^k |\sigma_i^r|^2_{h_m} \right)^{-\alpha/m} \Omega = \infty.
\]

Let \(x \in M\), and let \(s\) be a nonzero holomorphic section of \(L\) on a neighborhood of \(x\). On the neighbourhood of definition of \(s\), consider the plurisubharmonic functions

\[
\varphi_r := \frac{1}{2} \log \left( \sum_{i=1}^k \frac{|\sigma_i^r|^2}{s^m} \right),
\]

\[
\varphi := \frac{1}{2} \log \left( \sum_{i=1}^k \frac{|\sigma_i|^2}{s^m} \right).
\]

Since the functions \(e^{\varphi_r}\) are smooth and converge smoothly to \(e^{\varphi}\), there exists (possibly after shrinking slightly the neighborhood of definition) a uniform upper bound

\[
\varphi_r \leq C, \quad \varphi \leq C.
\]

By the standard compactness theory of subharmonic functions (e.g. [Hör94, Theorem 3.2.12]), there therefore exists a subsequence \(\varphi_{r_s}\) which converges to \(\varphi\) in \(L^1\) on some neighborhood of \(x\). Now, since \(\alpha < \alpha_{m,k}(L)\), we have that \(\alpha < mC_x(\varphi)\). Therefore, by the semicontinuity result Theorem 5.1.1, there exists a neighborhood of \(x\) on which \(e^{-2\alpha \varphi_{r_s}/m} \to e^{-2\alpha \varphi/m}\) in \(L^1\).

By the compactness of \(M\), therefore, \(e^{-2\alpha \varphi_{r_s}/m}\) converges in \(L^1\) to \(e^{-2\alpha \varphi/m}\) on the whole manifold \(M\). So

\[
\lim_{s \to \infty} \int_M \left( \sum_{i=1}^k |\sigma_i^r|^2_{h_m} \right)^{-\alpha/m} \Omega = \int_M \left( \sum_{i=1}^k |\sigma_i|^2_{h_m} \right)^{-\alpha/m} \Omega < \infty,
\]

a contradiction. \(\square\)
Chapter 6

Control on Bergman metrics

6.1 A conjecture about on Bergman metrics

In this section we study a very ample line bundle $\mathcal{L}$ over a complex variety $M$. Recall from Section 3.1 that there is a natural, noncompact finite-dimensional family $\mathcal{M}_\mathcal{L}$ of Kähler metrics on $M$, the 1-Bergman metrics (in this section just called Bergman metrics), parametrized by the homogeneous space of inner products on $H^0(M, \mathcal{L})$.

Fix an inner product $a_0$ on $H^0(M, \mathcal{L})$. Given any other inner product $a$ on $H^0(M, \mathcal{L})$, we may simultaneously diagonalize $a$ and $a_0$, producing a basis $(s_1, \ldots, s_N)$ of $H^0(M, \mathcal{L})$ and positive reals $\mu_1(a) \geq \mu_2(a) \geq \cdots \geq \mu_N(a) > 0$ such that

- $(s_1, \ldots, s_N)$ is orthonormal with respect to $a_0$;
- $(\mu_1(a)^{1/2}s_1, \ldots, \mu_N(a)^{1/2}s_N)$ is orthonormal with respect to $a$.

When we wish to emphasize the dependence of $\mu_1(a) \geq \mu_2(a) \geq \cdots \geq \mu_N(a) > 0$ on $a_0$, we write them as $\mu_1(a, a_0) \geq \mu_2(a, a_0) \geq \cdots \geq \mu_N(a, a_0) > 0$.

Lemma 6.1.1 (Change of reference inner product). For any two inner products $a_0, a'_0$ on $H^0(M, \mathcal{L})$, there exists a constant $C = C(a_0, a'_0)$, such that for each inner product $a$ on $H^0(M, \mathcal{L})$ and each $k$,

$$C^{-1}\mu_k(a, a_0) \leq \mu_k(a, a'_0) \leq C\mu_k(a, a_0).$$

Proof. Let $G_k$ be the Grassmannian of $k$-dimensional vector subspaces of $H^0(M, \mathcal{L})$. Let $C$ be such that
Then, for each inner product $a$ on $H^0(M,\mathcal{L})$, each $k$, and each $s \in H^0(M,\mathcal{L}) \setminus \{0\}$,

$$C^{-1} \max_{V \in G_k} \min_{V \notin V \setminus \{0\}} \frac{a_0(s,s)}{a(s,s)} \leq \max_{V \in G_k} \min_{V \notin V \setminus \{0\}} \frac{a_0'(s,s)}{a(s,s)} \leq C \max_{V \in G_k} \min_{V \notin V \setminus \{0\}} \frac{a_0(s,s)}{a(s,s)}.$$  

The result follows by the Courant minimax principle.

Fix a Kähler metric $\omega \in 2\pi c_1(\mathcal{L})$. Given an inner product $a$ on $H^0(M,\mathcal{L})$, we write $\psi_a$ for the Kähler potential of the Bergman metric associated to $a$. By Proposition 4.4.1, if $\omega'$ is any other Kähler metric in $2\pi c_1(\mathcal{L})$, then there exists a constant $C = C(\omega,\omega_0)$ such that for each Bergman metric $\omega_{\psi_a}$, $\omega'_{\psi_{a'}}$,

$$|I_k(\psi_a,\omega) - I_k(\psi_{a'},\omega')| \leq C.$$  

It follows that the following two definitions are independent of the choice of reference inner product $a_0$ and reference Kähler metric $\omega$:

**Definition.** Let $2 \leq k \leq n$.

1. Let $D$ be a subset of $\mathcal{M}_\mathcal{L}$. The $k$-th eigenvalue of $\mathcal{L}$ is controlled on $D$, if for each $\epsilon > 0$, there exists a constant $C$, such that for all inner products $a$ in $D$, with associated Bergman metrics $\omega_{\psi_a}$,

$$\log \left[ \frac{\mu_1(a)}{\mu_k(a)} \right] \leq (1 + \epsilon)I_k(\psi_a) + C.$$  

2. The $k$-th eigenvalue of $\mathcal{L}$ is controlled, if it is controlled throughout the full set $\mathcal{M}_\mathcal{L}$.

**Definition.** Let $2 \leq k \leq n$.

1. Let $D$ be a subset of $\mathcal{M}_\mathcal{L}$. The $k$-th eigenvalue of $\mathcal{L}$ is weakly controlled on $D$, if there exist constants $C$ and $\Lambda$, such that for all inner products $a$ in $D$, with associated Bergman metrics $\omega_{\psi_a}$,

$$\log \left[ \frac{\mu_1(a)}{\mu_k(a)} \right] \leq \Lambda I_k(\psi_a) + C.$$  

2. The $k$-th eigenvalue of $\mathcal{L}$ is weakly controlled, if it is weakly controlled throughout the full set $\mathcal{M}_\mathcal{L}$.

The $k$-th eigenvalue of $\mathcal{L}$ is obviously controlled on any compact subset $D$ of $\mathcal{M}_\mathcal{L}$; what is not obvious is whether it is controlled on the full, noncompact, $\mathcal{M}_\mathcal{L}$. This chapter and the next are devoted to some consequences of, and some evidence for, the following conjecture:

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Conjecture 6.1.2. Let $\mathcal{L}$ be very ample. Then for each $k$ with $2 \leq k \leq n$, the $k$-th eigenvalue of $\mathcal{L}$ is controlled.

6.2 Consequences of Conjecture 6.1.2

Let $a_0$ be an inner product on $H^0(M, \mathcal{L}^m)$, and let $\omega \in 2\pi c_1(\mathcal{L})$ be a Kähler metric. For an inner product $a$ on $H^0(M, \mathcal{L}^m)$, write $\psi_a$ for a Kähler potential such that $\omega \psi_a := \omega + \sqrt{-1} \partial \bar{\partial} \psi_a$ is the Bergman metric associated to $a$, and write $\mu_1(a) \geq \mu_2(a) \geq \cdots \geq \mu_N(a) > 0$ for the eigenvalues of $a$ with respect to $a_0$ (as in Section 6.1).

Lemma 6.2.1. If $\alpha < \alpha_{m,k}(\mathcal{L})$, then there exists a constant $C$ such that for all inner products $a$,

$$\log \int_M e^{\alpha (\sup_M \psi_a - \psi_a)} \omega^n \leq \frac{\alpha}{m} \log \left[ \frac{\mu_1(a)}{\mu_k(a)} \right] + C.$$

Remark. In particular, if $\alpha < \alpha_{m,1}(\mathcal{L})$, then there exists a constant $C$ such that for all inner products $a$,

$$\log \int_M e^{\alpha (\sup_M \psi_a - \psi_a)} \omega^n \leq C.$$

Proof. Let $h$ be hermitian metric on $\mathcal{L}$ of curvature $\omega$. By Lemma 3.1.3, there exists a basis $(s_1, \ldots, s_N)$ of $H^0(M, \mathcal{L}^m)$, orthonormal with respect to $a_0$, such that, after adding a constant if necessary,

$$\psi_a = \frac{1}{m} \log \left( \sum_{i=1}^N \mu_i(a) |s_i|^2_{h^m} \right).$$

We therefore calculate

$$\frac{1}{m} \log \mu_k(a) + \frac{1}{m} \log \left( \sum_{i=1}^k |s_i|^2_{h^m} \right) \leq \frac{1}{m} \log \left( \sum_{i=1}^N \mu_i(a) |s_i|^2_{h^m} \right) = \psi_a,$$

$$\sup_M \psi_a \leq \frac{1}{m} \sup_M \log \left( \sum_{i=1}^N \mu_i(a) |s_i|^2_{h^m} \omega^n \right) \leq \frac{1}{m} \log \mu_1(a) + C,$$

where the last line is since the function $\sum_{i=1}^N |s_i|^2_{h^m} \omega^n$ is independent of the choice of $a_0$-orthonormal basis.
Adding (6.2) and (6.2),
\[ \sup_M \psi_a - \psi_a \leq \frac{1}{m} \log \left( \frac{\mu_1(a)}{\mu_k(a)} \right) - \frac{1}{m} \log \left( \sum_{i=1}^k |s_i|_{h_m}^2 \right) + C. \]

Exponentiating and integrating,
\[ \int_M e^{\alpha (\sup_M \psi_a - \psi_a)} \omega^n \leq C \left( \left( \frac{\mu_1(a)}{\mu_k(a)} \right)^{\frac{\alpha}{m}} \left\{ \int_M \left( \sum_{i=1}^k |s_i|_{h_m}^2 \right)^{-\frac{\alpha}{m}} \omega^n \right\} \right). \]

Now observe that by Proposition 5.2.6, since \( \alpha < \alpha_{m,k}(\mathcal{L}) \),
\[ \int_M \left( \sum_{i=1}^k |s_i|_{h_m}^2 \right)^{-\frac{\alpha}{m}} \omega^n \leq C. \]

\[ \square \]

**Proposition 6.2.2.** (i) \( B_{m,1}(\mathcal{L}) = \alpha_{m,1}(\mathcal{L}) \).

(ii) For \( k \) such that \( 2 \leq k \leq n \), if the \( k \)-th eigenvalue of \( \mathcal{L}^m \) is controlled, then \( B_{m,k}(\mathcal{L}) \geq \alpha_{m,k}(\mathcal{L}) \).

(iii) For \( k \) such that \( 2 \leq k \leq n \), if the \( k \)-th eigenvalue of \( \mathcal{L}^m \) is weakly controlled, then \( B'_{m,k}(\mathcal{L}) \geq \alpha_{m,k}(\mathcal{L}) \).

**Proof.** We prove (ii); the other two are similar. We must show that if \( \alpha < \alpha_{m,k}(\mathcal{L}) \), then for each \( \epsilon > 0 \) there exists \( C \), such that for all \( m \)-Bergman metrics \( \omega + \sqrt{-1} \partial \bar{\partial} \psi \),
\[ \log \int_M e^{\alpha (\sup_M \psi - \psi)} \omega^n \leq \alpha (1 + \epsilon) I_k(\psi) + C. \] \hspace{1cm} (6.3)

Indeed, if \( a \) is an inner product on \( H^0(M, \mathcal{L}^m) \) associated to the \( m \)-Bergman metric \( \omega + \sqrt{-1} \partial \bar{\partial} \psi \), then by Lemma 6.2.1,
\[ \log \int_M e^{\alpha (\sup_M \psi - \psi)} \omega^n \leq \frac{\alpha}{m} \log \left( \frac{\mu_1(a)}{\mu_k(a)} \right) + C. \]

On the other hand, note that since \( \omega + \sqrt{-1} \partial \bar{\partial} \psi \) is an \( m \)-Bergman metric for \( \mathcal{L} \), the Kähler metric \( m\omega + \sqrt{-1} \partial \bar{\partial} m\psi \) is a \( (1) \)-Bergman metric for \( \mathcal{L}^m \). Since the \( k \)-th eigenvalue of \( \mathcal{L}^m \) is controlled, by definition, for each \( \epsilon > 0 \) there exists a constant \( C \), such that for all inner products \( a \) in \( D \),
\[ \log \left( \frac{\mu_1(a)}{\mu_k(a)} \right) \leq (1 + \epsilon) I_k(m\psi, m\omega) = (1 + \epsilon) mI_k(\psi, \omega) + C. \]
Combining these two gives (6.3).

**Theorem 6.2.3.** Suppose Conjecture 6.1.2 holds. Let $k$ be a natural number, with $2 \leq k \leq n$. Suppose that

\[
\frac{k - 1}{\alpha_{m,1}(M)} + \frac{n}{n + 1} < \frac{n}{n + 1} < n + 1,
\]

and that we have the following partial $C^0$ estimate: there exists $a > 0$, such that for all $t \in I_0 \cap [\delta, 1)$, $\rho_{\omega_{g,m}} \geq a$. Then $M$ admits a Kähler-Einstein metric.

**Theorem 6.2.4.** Let $k$ be a natural number, with $2 \leq k \leq n$. Suppose that the $k$-th eigenvalue of $L^m$ is weakly controlled, that

\[
\frac{n}{n + 1} < \alpha_{m,k}(M),
\]

\[
\frac{n}{n + 1} = \alpha_{m,1}(M),
\]

and that we have the following partial $C^0$ estimate: there exists $a > 0$, such that for all $t \in I_0 \cap [\delta, 1)$, $\rho_{\omega_{g,m}} \geq a$. Then $M$ admits a Kähler-Einstein metric.

**Proofs.** Theorems 4.5.1/4.5.2 together with Proposition 6.2.2. □

### 6.3 An approach to Conjecture 6.1.2

Let $G_{k-1}$ be the Grassmannian of $(k - 1)$-dimensional vector subspaces of $H^0(M, L)$. For $A \subseteq G_{k-1}$, let $D_A$ be the subset of $\mathcal{M}_L$ consisting of those inner products $\alpha$, such that some span $V := \langle s_1, \ldots, s_{k-1} \rangle$ of linearly independent first $(k - 1)$ eigenvectors of $\alpha$, ordered by eigenvalue, belongs to $A$. (Thus for each $1 \leq i \leq k - 1$, the section $s_i$ is a $\mu_i(\alpha)$-eigenvector.) It is natural to work on Conjecture 6.1.2 first on sets $D_A$, where $A$ is a neighbourhood of a single $(k - 1)$-dimensional subspace $V$ (and then conclude by appealing to the compactness of $G_{k-1}$).

To demonstrate this this method, we present a proofs of two partial results. The first is a result of Tian, which, combined with the arguments of Theorems 3.3.1 and 6.2.3, allowed him to prove the existence of Kähler-Einstein metrics on the last few dimension-2 manifolds for which that question had been open [Tia90a]:

\[\text{[Tia90a]:} \]
Proposition 6.3.1 ([Tia91]). The second eigenvalue of $L$ is controlled.

The second is a simple special case:

Lemma 6.3.2. For each $n$, the third eigenvalue of the line bundle $O(1)$ over $\mathbb{CP}^n$ is controlled.

Both will follow from the following theorem, which in turn follows from Proposition 6.4.4, proved in the next section:

Theorem 6.3.3. For each natural number $k$ such that $2 \leq k \leq n$, for each $(k-1)$-dimensional subspace $V \in G_{k-1}$ such that the subvariety $\cap_{s \in V} s^{-1}(0)$ of $M$ has pure codimension $k-1$, there exists a neighbourhood $A$ of $V$ in $G_{k-1}$, such that the $k$-th eigenvalue of $L$ is controlled on $D_A$.

Proof of Proposition 6.3.1. 1-dimensional subspaces of $H^0(M, L)$ are lines, and therefore determine divisors, which are always of codimension 1. So by Theorem 6.3.3 each line in $G_1$ has a neighbourhood $A$ such that the second eigenvalue of $L$ is controlled on $D_A$. By the compactness of $G_1$, this cover has a finite subcover. The second eigenvalue of $L$ is controlled on $M_L$ by the maximum of the constants of control over this finite subcover. □

Proof of Lemma 6.3.2. Any pair of hyperplanes in $\mathbb{CP}^n$ intersect in a subspace of codimension 2. So by Theorem 6.3.3 each plane in $G_2$ has a neighbourhood $A$ such that the third eigenvalue of $L$ is controlled on $D_A$. Now, again, by the compactness of $G_2$, this cover has a finite subcover, and the third eigenvalue of $L$ is controlled on $M_L$ by the maximum of the constants of control over this finite subcover. □

The obstacle to proving Conjecture 6.1.2 when $k \geq 3$ is precisely the algebro-geometric technicality hinted at by Theorem 6.3.3: the subvariety defined by the vanishing of all sections $s \in V$ may have not have pure codimension $k-1$. For example, the divisors defined by two linearly independent sections in $V$ could have a common component.

6.4 Estimates for $I_k$

Proposition 6.4.1. For each nonvanishing holomorphic section $s$ of $L$ over the closure of some open set $U$, for each $\epsilon > 0$, there exists a positive constant $C$, such that for each $\omega$-Kähler potential $\varphi$, the plurisubharmonic functions $\Phi := \varphi|_U - \log |s|^2_h$ satisfy,

$$\sum_{i=0}^{k-2} \sqrt{-1} \partial \Phi \wedge \bar{\partial} \Phi \wedge (\sqrt{-1} \partial \bar{\partial} \Phi)^i \wedge \omega^{n-i-1} \leq (1 + \epsilon) \sum_{i=0}^{k-2} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-i-1} + C \sum_{i=0}^{k-2} \omega^n.$$
Proof. By construction
\[ \sqrt{-1} \partial \Phi = \sqrt{-1} \partial \varphi - \sqrt{-1} \partial \log |s|^2_h = \omega \varphi. \]

Since \( \log |s|^2_h \) extends continuously without vanishing to the closure \( \overline{U} \), there exists a constant \( C \) such that
\[ \sqrt{-1} \partial \log |s|^2_h \wedge \partial (\log |s|^2_h) = C \omega. \]
Now by AM-GM
\[ \sqrt{-1} \partial \Phi \wedge \partial \Phi \leq (1 + \epsilon) \sqrt{-1} \partial \varphi \wedge \partial \varphi + C \sqrt{-1} \partial \log |s|^2_h \wedge \partial \log |s|^2_h \leq (1 + \epsilon) \sqrt{-1} \partial \varphi \wedge \partial \varphi + C \omega. \]

\[ \Box \]

**Proposition 6.4.2.** For each \( \epsilon > 0 \) there exists a constant \( C = C(\epsilon, k) \), such that for all \( \mu > 0 \), for all nonnegative functions \( F, \tilde{F} \) with
\[ \tilde{F} \leq \mu, \quad \sqrt{-1} \partial \tilde{F} \wedge \partial \tilde{F} \leq \mu^2 \omega, \]
and for which \( F, \tilde{F} \) and \( \Phi := \log(F + \tilde{F}) \) are plurisubharmonic,
\[ \sum_{i=0}^{k-2} \frac{F^{2i}}{(F + C\mu)^2(i+1)} \sqrt{-1} \partial F \wedge \partial \Phi \wedge (\sqrt{-1} \partial \log F)^i \wedge \omega^{n-i-1} \]
\[ \leq (1 + \epsilon) \sum_{i=0}^{k-2} \sqrt{-1} \partial \Phi \wedge \partial \Phi \wedge (\sqrt{-1} \partial \Phi)^i \wedge \omega^{n-i} + \epsilon \sum_{i=0}^{k-2} (\sqrt{-1} \partial \Phi)^i \wedge \omega^{n-i}. \]

Proof. Let \( \delta > 0 \), to be determined later. Let \( F := F + \tilde{F} \), so that \( \Phi = \log F \). By AM-GM, there exist constants \( C = C(\delta) \), such that
\[ \sqrt{-1} \partial F \wedge \partial F \leq (1 + \delta) \sqrt{-1} \partial F \wedge \partial F + C \sqrt{-1} \partial \tilde{F} \wedge \partial \tilde{F} \]
\[ = (1 + \delta) F^2 \sqrt{-1} \partial \Phi \wedge \partial \Phi + C \mu \omega \]
\[ \leq (1 + \delta) (F + \mu)^2 \sqrt{-1} \partial \Phi \wedge \partial \Phi + C \mu \omega \]
\[ \leq (F + C\mu)^2 \left[ (1 + \delta) \sqrt{-1} \partial \Phi \wedge \partial \Phi + \delta \omega \right]. \quad (6.4) \]
Similarly, by AM-GM there exist constants $C = C(\delta)$, such that

$$-\sqrt{-1} \partial F \wedge \bar{\partial} F \leq (1 + \delta)\sqrt{-1} \partial F \wedge \bar{\partial} F + C \sqrt{-1} \partial \tilde{F} \wedge \bar{\partial} \tilde{F}$$
$$\leq (1 + \delta)\sqrt{-1} \partial \Phi \wedge \bar{\partial} \Phi + C \mu \omega; \quad (6.5)$$

since $F$ and $\tilde{F}$ are nonnegative plurisubharmonic functions, $0 \leq \sqrt{-1} \partial \partial F \leq \sqrt{-1} \partial \bar{\partial} F$ and $0 \leq \tilde{F} \leq F$, so

$$0 \leq \tilde{F} \sqrt{-1} \partial \bar{\partial} F \leq F \sqrt{-1} \partial \bar{\partial} F; \quad (6.6)$$

adding (6.5) and (6.6),

$$F^2 \sqrt{-1} \partial \bar{\partial} \log F = F \sqrt{-1} \partial \bar{\partial} F - \sqrt{-1} \partial F \wedge \bar{\partial} F$$
$$\leq F \sqrt{-1} \partial \bar{\partial} F + (1 + \delta)\sqrt{-1} \partial \bar{\partial} F + C \mu \omega$$
$$= F^2 \sqrt{-1} \partial \bar{\partial} \Phi + \delta \sqrt{-1} F^3 \partial \Phi \wedge \bar{\partial} \Phi + C \mu \omega$$
$$\leq (\bar{F} + \mu)^2 [\sqrt{-1} \partial \bar{\partial} \Phi + \delta \sqrt{-1} \partial \bar{\partial} \Phi] + C \mu \omega$$
$$\leq (\bar{F} + C \mu)^2 [\sqrt{-1} \partial \bar{\partial} \Phi + \delta \sqrt{-1} \partial \bar{\partial} \Phi + \delta \omega]. \quad (6.7)$$

In the second-last line we used the plurisubharmonicity of $\Phi$.

Considering the expression $(6.4) \wedge (6.7)^i$,

$$\frac{F^{2i}}{(\bar{F} + C \mu)^{2(i+1)}} \sqrt{-1} \partial \bar{\partial} F \wedge \bar{\partial} F \wedge (\sqrt{-1} \partial \bar{\partial} \log F)^i \wedge \omega^{n-i-1}$$
$$\leq [(1 + \delta)\sqrt{-1 \partial \Phi \wedge \bar{\partial} \Phi + \delta \omega}]^i \wedge [\sqrt{-1 \partial \bar{\partial} \Phi + \delta \sqrt{-1 \partial \bar{\partial} \Phi \wedge \bar{\partial} \Phi + \delta \omega}]^i$$
$$= (1 + \delta)\sqrt{-1 \partial \Phi \wedge \bar{\partial} \Phi \wedge [\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega}]^i$$
$$+ \delta \omega \wedge \{[\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega]^i + i \delta \sqrt{-1 \partial \bar{\partial} \Phi \wedge \bar{\partial} \Phi \wedge [\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega}]^{i-1}} \}$$
$$= \sqrt{-1 \partial \Phi \wedge \bar{\partial} \Phi \{ (1 + \delta)\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega] + i \delta \omega \} \wedge [\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega]}^{i-1}$$
$$+ \delta \omega \wedge [\sqrt{-1 \partial \bar{\partial} \Phi + \delta \omega}. \quad (6.7)$$

Thus, summing and choosing $\delta = \delta(\epsilon, k)$ sufficiently small, the result follows. \[\square\]
Lemma 6.4.3. There exists a constant $C = C(k)$, such that for all $r_2 > r_1 > 0$ and all $\eta > 0$,

$$
\int_{r_1 \leq |z| \leq r_2} \frac{|z|^{2(k-2)} \sqrt{\overline{\partial} \partial |z|^2 \wedge \overline{\partial} |z|^2}}{(|z|^2 + \eta)^{k-1}} \geq 2\pi \log \left( \frac{r_2^2 + \eta}{r_1^2 + \eta} \right) - C.
$$

In particular, there exists a constant $C = C(k)$, such that for all $\eta > 0$

$$
\int_{|z| \leq r} \frac{|z|^{2(k-2)} \sqrt{\overline{\partial} \partial |z|^2 \wedge \overline{\partial} |z|^2}}{(|z|^2 + \eta)^{k-1}} > 2\pi \log \left( \frac{r^2}{\eta} \right) - C.
$$

Proof.

$$
\int_{r_1 \leq |z| \leq r_2} \frac{|z|^{2(k-2)} \sqrt{\overline{\partial} \partial |z|^2 \wedge \overline{\partial} |z|^2}}{(|z|^2 + \eta)^{k-1}} = 2\pi \int_{(r_1)^2}^{(r_2)^2} \frac{R^{k-2}dR}{(R + \eta)^{k-1}}
= 2\pi \int_{r_1^2 + \eta}^{r_2^2 + \eta} \sum_{i=0}^{k-2} \binom{k-2}{i} S^{-i-1}(-\eta)^i dS
= 2\pi \left[ \log S + \sum_{i=1}^{k-2} \binom{k-2}{i} \left( -\frac{\eta}{S} \right) \right] \frac{S^2 + \eta}{r_1^2 + \eta}
\geq 2\pi \log \left( \frac{r_2^2 + \eta}{r_1^2 + \eta} \right) - C.
$$

\[\square\]

For a subspace $V \in \mathcal{G}_{k-1}$, we will write $V^\perp$ for the orthogonal complement of $V$ in $H^0(M, \mathcal{L})$ with respect to the inner product $a_0$. Given inner products $\pi$ on $V$ and $\tilde{a}$ on $V^\perp$, we write $\pi \oplus \tilde{a}$ for the inner product on $H^0(M, \mathcal{L})$ which restricts to $\pi$ on $V$ and $\tilde{a}$ on $V^\perp$, and with respect to which the subspaces $V$ and $V^\perp$ are also orthogonal.

Proposition 6.4.4. Let $V_0 \in \mathcal{G}_{k-1}$ be such that the subvariety $\cap_{s \in V_0} s^{-1}(0)$ of $M$ has pure codimension $k - 1$. For each $\epsilon > 0$ there exist a constant $C$ and a neighbourhood $\Omega \subseteq \mathcal{G}_{k-1}$ of $V_0$, such that for all $V \in \Omega$, all positive reals $\pi$ and $\tilde{\mu}$, and all inner products $\pi \leq \pi^{-1}a_0|_V$ on $V$ and $\tilde{\mu} \geq \tilde{\mu}^{-1}a_0|_{V^\perp}$ on $V^\perp$,

$$
\log \left( \frac{\pi}{\tilde{\mu}} \right) \leq (1 + \epsilon)I_k(\psi_{\pi \oplus \tilde{a}}) + C.
$$

Proof. For each $V \in \mathcal{G}_{k-1}$, denote by $Z_V$ the subvariety $\{ x \in M : \forall s \in V, s|_x = 0 \} = \cap_{s \in V} s^{-1}(0)$ of $M$. By assumption $Z_{V_0}$ is of pure codimension $k - 1$.  

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Let $\delta > 0$, to be determined later. Choose nonvanishing holomorphic sections $(s_{\gamma})_{\gamma \in \Gamma}$ of $\mathcal{L}$ on the closures of finitely many disjoint open sets $(U_{\gamma})_{\gamma \in \Gamma}$ in $M$, in such a way that

$$
\sum_{\gamma \in \Gamma} \int_{U_{\gamma} \cap Z_{V_0}} \omega^{n-k+1} \geq (1 + \delta)^{-1} \int_{Z_{V_0}} \omega^{n-k+1} = (1 + \delta)^{-1}(2\pi)^{n-k+1}c_1(\mathcal{L})^n.
$$

For inner products $\pi, \tilde{a}$ satisfying the conditions of the Proposition, we define functions as follows:

$$
F_\gamma := \left| \frac{\text{ev}_{\gamma}}{s_{\gamma}} \right|^2_\pi, \quad F_{\gamma} := \left| \frac{\text{ev}_{\gamma+1}}{s_{\gamma}} \right|^2_\tilde{a},
$$

$$
\Psi_\gamma := \log(F_{\gamma} + \tilde{F}_{\gamma}) = \log \left| \frac{\text{ev}_{\gamma}}{s_{\gamma}} \right|^2_{\pi \oplus \tilde{a}},
$$

so that $\psi_{\pi \oplus \tilde{a}}$, the Kähler potential of the Bergman metric associated to $\pi \oplus \tilde{a}$, satisfies,

$$
\Psi_\gamma = \psi_{\pi \oplus \tilde{a}} - \log |s_{\gamma}|^2_{\tilde{h}}.
$$

Propositions 6.4.1 and 6.4.2 are applicable, and so

$$
\sum_{\gamma \in \Gamma} \int_{U_{\gamma}} \frac{F_{\gamma}}{(F_{\gamma} + C\hat{\mu})^{2(k-1)}} \sqrt{-1} \partial F_{\gamma} \wedge \bar{\partial} F_{\gamma} \wedge (\sqrt{-1} \partial \bar{\partial} \log F_{\gamma})^{k-2} \wedge \omega^{n-k+1} \\
\leq \sum_{\gamma \in \Gamma} \sum_{i=0}^{k-2} \int_{U_{\gamma}} \frac{\tilde{F}_{\gamma}}{(F_{\gamma} + C\hat{\mu})^{2(i+1)}} \sqrt{-1} \partial F_{\gamma} \wedge \bar{\partial} F_{\gamma} \wedge (\sqrt{-1} \partial \bar{\partial} \log F_{\gamma})^{i} \wedge \omega^{n-i-1} \\
\leq (1 + \delta)^2 \left( \sum_{\gamma \in \Gamma} \sum_{i=0}^{k-2} \int_{U_{\gamma}} \sqrt{-1} \partial \psi_{\pi \oplus \tilde{a}} \wedge \bar{\partial} \psi_{\pi \oplus \tilde{a}} \wedge (\sqrt{-1} \partial \bar{\partial} \psi_{\pi \oplus \tilde{a}})^i \wedge \omega^{n-i-1} + C \sum_{\gamma \in \Gamma} \sum_{i=0}^{k-2} \int_{U_{\gamma}} (\sqrt{-1} \partial \bar{\partial} \psi_{\pi \oplus \tilde{a}})^i \wedge \omega^{n-i} \right) \\
\leq (1 + \delta)^2 \left( \sum_{i=0}^{k-2} \int_{M} \sqrt{-1} \partial \psi_{\pi \oplus \tilde{a}} \wedge \bar{\partial} \psi_{\pi \oplus \tilde{a}} \wedge (\sqrt{-1} \partial \bar{\partial} \psi_{\pi \oplus \tilde{a}})^i \wedge \omega^{n-i-1} + C \sum_{i=0}^{k-2} \int_{M} (\sqrt{-1} \partial \bar{\partial} \psi_{\pi \oplus \tilde{a}})^i \wedge \omega^{n-i} \right) \\
= (1 + \delta)^2 [2\pi c_1(\mathcal{L})]^n I_k(\psi_{\pi \oplus \tilde{a}}) + C.
$$

There is a tautological vector bundle over $\mathcal{G}_{k-1}$, specifically $\mathcal{F}_{k-1} := \bigsqcup_{V \in \mathcal{G}_k} V^*$. For $\gamma \in \Gamma$ and $(V, z) \in \mathcal{F}_{k-1}$, there is an analytic subset

$$
Z_{V, z} := \left( \frac{\text{ev}_V}{s_{\gamma}} \right)^{-1} (z) = \{ x \in M : \forall s \in V, s|_x = z(s) s_{\gamma} \}
$$

of $U_{\gamma}$. In particular $Z_{V, 0, \gamma} = U_{\gamma} \cap Z_V$.

Since the subvariety $Z_{V_0, 0, \gamma} = U_{\gamma} \cap Z_{V_0}$ has pure dimension $k - 1$, the same is true of the analytic
subsets $Z_{V,z,\gamma}$, for all $(V,z)$ in some neighbourhood of $(V_0,0)$ in $\mathcal{F}_{k-1}$. On such a neighbourhood the function

$$(V,z) \mapsto \int_{Z_{V,z,\gamma}} \omega^{n-k+1}$$

is well-defined and continuous. (More precisely, we consider the function

$$(V,z) \mapsto \int_{U_\gamma} T_{V,z,\gamma} \wedge \omega^{n-k+1},$$

where $T_{V,z,\gamma}$ is the distributional $(k-1,k-1)$-form

$$\left(\sqrt{-1} \partial \bar{\partial} \left| \text{ev}_V - z \otimes s_\gamma \right|^2 \right)^{k-1}$$
on $U_\gamma$, which is well-defined precisely where $Z_{V,z,\gamma}$ has pure codimension $k-1$.) There therefore exists a neighbourhood $\Omega$ of $V_0$ in $\mathcal{G}_{k-1}$, and a radius $r > 0$, such that for all $\gamma \in \Gamma$, for $V \in \mathcal{G}_{k-1}$ and $z$ in the ball $B_{r,a_0}(V^*)$ of radius $r$ (with respect to the inner product $a_0$) in $V^*$,

$$\int_{Z_{V,z,\gamma}} \omega^{n-k+1} \geq (1 + \delta)^{-1} \int_{U_\gamma \cap Z_{V_0}} \omega^{n-k+1}.$$

We obtain that

$$\int_{U_\gamma} \frac{\overline{F}_\gamma}{(\overline{F}_\gamma + C\mu)^{2(k-1)}} \sqrt{-1} \partial \overline{\partial} \left| \text{ev}_V - z \otimes s_\gamma \right|^2 \wedge \omega^{n-k+1}$$

$$= \int_{z \in V^*} \frac{|z|^4}{(|z|^4 + C\mu)^{2(k-1)}} \sqrt{-1} \partial |z|^2 \wedge \overline{\partial} |z|^2 \wedge (\sqrt{-1} \partial \overline{\partial} \log |z|^2)^k \int_{Z_{V,z,\gamma}} \omega^{n-k+1}$$

$$\geq (1 + \delta)^{-1} \int_{z \in B_{r,a_0}(V^*)} \frac{|z|^4}{(|z|^4 + C\mu)^{2(k-1)}} \sqrt{-1} \partial |z|^2 \wedge \overline{\partial} |z|^2 \wedge (\sqrt{-1} \partial \overline{\partial} \log |z|^2)^k \int_{U_\gamma \cap Z_{V_0}} \omega^{n-k+1}. $$
Since \( \pi \leq p_{a_0}|_{V} \), we have that \( B_{r,a_0}(V^*) \supseteq B_{\pi/2,r,a}(V^*) \). Therefore by Lemma 6.4.3

\[
\int_{z \in B_{r,a_0}(V^*)} \frac{|z|^4 (k-2)}{|z|^2 + C\mu^2} \left( |z|^2 \right) \left( \sqrt{-1} \partial |z|^2 \right) \wedge \left( \sqrt{-1} \partial \log |z|^2 \right)^{k-2} \geq \int_{z \in B_{\pi/2,r,a}(V^*)} \frac{|z|^4 (k-2)}{|z|^2 + C\mu^2} \left( |z|^2 \right) \left( \sqrt{-1} \partial |z|^2 \right) \wedge \left( \sqrt{-1} \partial \log |z|^2 \right)^{k-2} = \int_{\gamma \in B_{\pi/2,r,a}(\mathbb{C}^n)} \frac{|\gamma|^4 (k-2)}{|\gamma|^2 + C\mu^2} \left( |\gamma|^2 \right) \left( \sqrt{-1} \partial |\gamma|^2 \right) \wedge \left( \sqrt{-1} \partial \log |\gamma|^2 \right)^{k-2} = (2\pi)^{k-2} \int_{\gamma \in B_{\pi/2,r,a}(\mathbb{C})} \frac{|\gamma|^4 (k-2)}{|\gamma|^2 + C\mu^2} \left( |\gamma|^2 \right) \left( \sqrt{-1} \partial |\gamma|^2 \right) \wedge \left( \sqrt{-1} \partial \log |\gamma|^2 \right)^{k-2} \geq (2\pi)^{k-1} \log \left( \frac{\mu}{\tilde{\mu}} \right) - C.
\]

Combining these three computations,

\[
(1 + \delta)^2 \left[ 2\pi c_1(L) \right] I_k(\psi_{\mathbb{P}^{n-1}}) + C \geq (1 + \delta)^{-2} \sum_{\gamma \in \Gamma} \left[ (2\pi)^{k-1} \log \left( \frac{\mu}{\tilde{\mu}} \right) - C \right] \int_{U_{\gamma} \cap Z_{\nu}} \omega^{n-k+1} \geq (1 + \delta)^{-2} \int_{Z_{\nu}} \omega^{n-k+1} \left[ (2\pi)^{k-1} \log \left( \frac{\mu}{\tilde{\mu}} \right) - C \right] \geq (1 + \delta)^{-2} (2\pi)^{n-k+1} c_1(L)^n \left[ (2\pi)^{k-1} \log \left( \frac{\mu}{\tilde{\mu}} \right) - C \right] .
\]

Thus \((1 + \delta)^4 I_k(\psi_{\mathbb{P}^{n-1}}) + C \geq \log \left( \frac{\mu}{\tilde{\mu}} \right) \).
Chapter 7

Explicit constants for Riemannian inequalities

In this chapter we collect versions of various standard inequalities in which the dependence of the constant on the metric is explicit. The main results are Propositions 7.2.4, 7.2.3, 7.3.3 and 7.4.2. Though perhaps known in principle to experts, the latter three have not, to our knowledge, appeared elsewhere in the literature; we give detailed proofs.

7.1 Technical results

Definition. Let $(M, g)$ be a smooth Riemannian $n$-manifold and let $x \in M$. Given $Q > 1$, $k \in \mathbb{N}$, and $p > n$, the $(Q, k, p)$-harmonic radius at $x$, $r_H(Q, k, p)(x)$, is the supremum of reals $r$ such that, on the geodesic ball $B_x(r)$ of center $x$ and radius $r$, there is a harmonic co-ordinate chart such that if $g_{ij}$ are the components of $g$ in these co-ordinates, then

1. $Q^{-1} \delta_{ij} \leq g_{ij} \leq Q \delta_{ij}$ as bilinear forms;
2. $\sum_{1 \leq |\beta| \leq k} r^{|\beta| - n/p} \|\partial_\beta g_{ij}\|_{L^p} \leq Q - 1$.

The $(Q, k, p)$-harmonic radius of $M$ is

$$r_H(Q, k, p)(M) := \inf_{x \in M} r_H(Q, k, p)(x).$$
Theorem 7.1.1 ([HH97], Theorem 11). Let \( n \in \mathbb{N}, Q > 1, p > n, i > 0 \). Suppose \( (M, g) \) is a Riemannian \( n \)-manifold with \( \text{injrad}(M, g) \geq i \).

1. Let \( \lambda \in \mathbb{R} \). There exists \( C = C(n, Q, p, i, \lambda) \), such that if \( \text{Ric} \geq \lambda g \), then the harmonic radius \( r_H(1, p)(M) \) is \( \geq C \).

2. Let \( k \geq 2 \), and let \( (C(j))_{0 \leq j \leq k-2} \) be positive constants. There exists \( C = C(n, Q, p, i, (C(j))_{0 \leq j \leq k-2}) \), such that if for each \( 0 \leq j \leq k-2 \) we have

\[ |\nabla^j \text{Ric}| \leq C(j), \]

then the harmonic radius \( r_H(Q, k, p)(M) \) is \( \geq C \).

Lemma 7.1.2 ([Heb96b], Lemma 1.6). Let \( n \in \mathbb{N}, \lambda \in \mathbb{R}, \) and let \( r \geq \rho > 0 \). Let \( (M, g) \) be a complete Riemannian \( n \)-manifold with \( \text{Ric} \geq \lambda g \). Then there exist \( N = N(n, \lambda, \rho, r) \), and an (at most) countable set \((x_i)\) of points in \( M \), such that

1. the family \((B(x_i, \rho))\) covers \( M \);
2. each point in \( M \) is contained in at most \( N(n, \lambda, \rho, r) \) balls of the family \((B(x_i, r))\).

Let \( U \subseteq \mathbb{R}^n \) be an open set, \( g \) a Riemannian metric on \( u \), \( \nabla \) its Levi-Civita connection, and \( \Gamma \) the Christoffel symbols of \( g \) on the co-ordinate patch \( U \). We write \( D \) for the (Euclidean) derivative on \( U \).

In the following we denote by \( S \) a multilinear map

\[ S : T_b^{a_1}(\mathbb{R}^n) \times T_{b_2}^{a_2}(\mathbb{R}^n) \times \ldots \times T_{b_c}^{a_c}(\mathbb{R}^n) \rightarrow T_b^a(\mathbb{R}^n), \]

composed of sums of traces (so \( a - b = \sum a_i - \sum b_i \)). It is to be understood that the particular map \( S \), and the \( a_i \)'s, \( b_i \)'s, \( a \) and \( b \) determining its domain and range, may vary from use to use and from line to line, but are independent of the choice of \( g \), and of the choice of \( A \) (in (1)) or of \( u \) (in (2)).

Lemma 7.1.3. 1. Let \( k \in \mathbb{N} \). For all covariant tensors \( A : U \rightarrow T^k(\mathbb{R}^n) \),

\[ \nabla A = DA + S(\Gamma, A). \]
2. Let \( m \in \mathbb{N} \). For all functions \( u : U \to \mathbb{R} \),
\[
(\nabla)^m u = D^m u + \sum_{k=1}^{m-1} \sum_{r=1}^{m-k} \sum_{a_1 \geq \cdots \geq a_r \geq 0, \atop a_1 + \cdots + a_r = m-k-r} S(D^{a_1} \Gamma, \ldots, D^{a_r} \Gamma, D^k u).
\]

Proof. 1. For any covariant tensor \( A \),
\[
(\nabla A)(\partial_j, \partial_{i_1}, \ldots, \partial_{i_k}) = \partial_j (A(\partial_{i_1}, \ldots, \partial_{i_k})) - \sum_r \Gamma^a_{ji} A(\partial_{i_1}, \ldots, \partial_{a}, \ldots, \partial_{i_k})
\]
2. For \( m = 0, 1 \) these are the identities
\[
u = u, \quad \nabla u = du.
\]

Thenceforth we proceed by induction. Suppose this is known for some \( m \). Then
\[
(\nabla)^{m+1} u = \nabla (D^m u) + \sum_{k=1}^{m-1} \sum_{r=1}^{m-k} \sum_{a_1 \geq \cdots \geq a_r \geq 0, \atop a_1 + \cdots + a_r = m-k-r} \nabla \left( S(D^{a_1} \Gamma, \ldots, D^{a_r} \Gamma, D^k u) \right),
\]
and we may calculate
\[
\nabla (D^m u) = D^{m+1} u + S(\Gamma, D^m u),
\]
\[
\nabla \left( S(D^{a_1} \Gamma, \ldots, D^{a_r} \Gamma, D^k u) \right) = \sum_r S(D^{a_1} \Gamma, \ldots, D^{a_i+1} \Gamma, \ldots, D^{a_r} \Gamma, D^k u)
\]
\[
+ S(D^{a_1} \Gamma, \ldots, D^{a_r} \Gamma, D^{k+1} u)
\]
\[
+ S(D^{a_1} \Gamma, \ldots, D^{a_r} \Gamma, \Gamma, D^k u).
\]

The result follows.

\[\square\]

7.2 Sobolev estimates

Theorem 7.2.1 (Sobolev estimate, Gilbarg-Trudinger [GT01] 7.10 & 7.25). For each \( n, p \neq n \), and bounded domain \( V \) with \( C^1 \) boundary, there exists \( C = C(n, p, V) \), such that for all functions \( u \in W^{1,p}(V) \), we have

1. if \( p < n \),
\[
\|u\|_{W^{1,p}(V)} \leq C\|u\|_{W^{1,1}(V)}.
\]
2. if $p > n$, 
\[
\sup_{V} |u| \leq C \|u\|_{W^{1,p},V}
\]

**Lemma 7.2.2** (Local Sobolev estimate). Let $n$, $m$, $i$, $\lambda$ be given.

1. Let $q < p < n$ be given. Then there exists $r = r(n, p, q, m, i, \lambda) < i$ and $C = C(n, p, q, m, i, \lambda)$, such that for each
   - complete Riemannian $n$-manifold $(M, g)$ with injectivity radius at least $i$ and $\text{Ric} \geq \lambda g$
   - point $x \in M$
   - smooth covariant $m$-tensor $A$ on $B(x, r)$,
we have
   
   1. (if $p < n$)
   \[
   \|A\|_{p,g,B(x,r)}^{p} \leq C \left[ \|\nabla A\|_{p,g,B(x,r)}^{p} + \|A\|_{p,g,B(x,r)}^{p} \right].
   \]
   2. (if $p > n$)
   \[
   \left( \sup_{B(x,r)} |A|_{g} \right)^{p} \leq C \left[ \|\nabla A\|_{p,g,B(x,r)}^{p} + \|A\|_{p,g,B(x,r)}^{p} \right].
   \]

Proof. If $p > n$, let $q := \frac{1}{2}(n + p)$. Then, either way, let
\[
s = \frac{pq}{p - q},
\]
and choose $r$ to be less than $i$ and less than the $C(n, Q := 1, s, i, \lambda)$ of Theorem 7.1.1 (1), so that the harmonic radius $r_{H}(Q := 1, 1, s)$ is greater than $r$. We therefore have uniform bounds in terms of $n$, $i$, $\lambda$, $p$, and (if $p < n$) $q$ on the co-ordinate norms $\|g\|_{s,B(x,r)}$, $\|g^{-1}\|_{s,B(x,r)}$, $\|\Gamma\|_{s,B(x,r)}$.

Write $B$ for $B(x, r)$ throughout. Let $u$ be a function on $B$, with $u \in W^{m+1,p}_{\text{loc}}(B) \cap L^{p}(B)$.

By our bounds on the components of the tensor $g$, the metrics $g$ and $g_{\text{eucl}}$ are comparable, so the norms $\|\cdot\|_{p,g,B}$ and $\|\cdot\|_{p,B}$ are comparable, the norms $\|\cdot\|_{p,g,B}$ and $\|\cdot\|_{p,g,B}$ are comparable, and the pointwise tensor norms $|\cdot|_{g}$ and $|\cdot|$ are comparable. It therefore it suffices to prove the inequalities with the latter norms.
Applying the Sobolev inequality Theorem 7.2.1 co-ordinatewise and combining, we have $C = C(n, m, q)$, such that

1. (if $p < n$)
   $$\|A\|_{q, B}^q \leq C \left(\|DA\|_{q, B}^q + \|A\|_{q, B}^q\right).$$

2. (if $p > n$)
   $$\left(\sup_B |A|\right)^q \leq C \left(\|DA\|_{q, B}^q + \|A\|_{q, B}^q\right).$$

By Lemma 7.1.3 (1) and the power means inequalities,

$$\left[\|DA\|_{q, B}^q + \|A\|_{q, B}^q\right]^\frac{p}{q} \leq 2^{\frac{p}{q} - 1} \left[\|DA\|_{p, B}^p + \|A\|_{p, B}^p\right]$$

$$\leq 2^{\frac{p}{q} - 1} \left(2^{p-1} \left(\|\nabla A\|_{p, B}^p + \|S(\Gamma, A)\|_{p, B}^p\right) + \|A\|_{p, B}^p\right).$$

By Hölder’s inequality, for $C = C(n, m)$,

$$\|\nabla A\|_{q, B} \leq C \|1\|_{s, B} \|\nabla A\|_{p, B}$$

$$\|S(\Gamma, A)\|_{q, B} \leq C \|\Gamma\|_{s, B} \|A\|_{p, B}$$

$$\|A\|_{q, B} \leq C \|1\|_{s, B} \|A\|_{p, B}$$

The terms other than $A, \nabla A$ in these right-hand sides are controlled by construction. The result follows.

\[\square\]

**Proposition 7.2.3** (Sobolev inequalities). Let $n, m, i, \lambda$ be given.

1. Let $q < p < n$ be given. Then there exists $C = C(n, p, q, m, i, \lambda)$,

2. Let $p > n$ be given. Then there exists $C = C(n, p, m, i, \lambda)$,

such that for each

- complete Riemannian $n$-manifold $(M, g)$ with injectivity radius at least $i$ and $\text{Ric} \geq \lambda g$

- smooth function $u$ on $M$,

we have

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1. (if \( p < n \))
\[
\|u\|_{W^{m, \frac{nq}{n-q}, g}} \leq C\|u\|_{W^{m+1, p, g}}.
\]

2. (if \( p > n \))
\[
\|u\|_{C^{m, g}} \leq C\|u\|_{W^{m+1, p, g}}.
\]

**Proof.** Choose \( r \) and \( C \) from the local Sobolev estimate Lemma 7.2.2.

By Lemma 7.1.2, there exist \( N = N(n, A, r, r) \) and an (at most) countable set \( (x_\alpha) \) of points in \( M \), such that each point in \( M \) is contained in at least one and at most \( N \) balls of the family \( (B(x_\alpha, r)) \). Let \( \chi_\alpha \) be the characteristic function of \( B(x_\alpha, r) \).

By Minkowski’s inequality (that is, the triangle inequality), we have,
\[
\|A\|^{\frac{p}{n-q}, B_\alpha} \leq \sum_\alpha \|\chi_\alpha|A|^p\|^{\frac{nq}{n-q}, B_\alpha}.
\]

By the local Sobolev inequality Lemma 7.2.2,
\[
\|A\|_{B_\alpha}^{\frac{p}{n-q}} \leq C \left[ \int_{B_\alpha} |\nabla A|^p + \int_{B_\alpha} |A|^p \right]
\]

So, since each point is in at most \( N \) of the \( B_\alpha \)’s,
\[
\|A\|_{B_\alpha}^{\frac{p}{n-q}} \leq C \sum_\alpha \left[ \int_{B_\alpha} |\nabla A|^p + \int_{B_\alpha} |A|^p \right] \leq NC \left[ \int_M |\nabla A|^p + \int_M |A|^p \right].
\]
Similarly, if \( p > n \), by the local Sobolev inequality Lemma 7.2.2,

\[
\left( \sup_M |A|_g \right)^p \leq C \max_{\alpha} \left[ \int_{B_{\alpha}} |\nabla A|^p + \int_{B_{\alpha}} |A|^p \right] \leq C \left[ \int_M |\nabla A|^p + \int_M |A|^p \right].
\]

Now, applying these inequalities simultaneously to the covariant tensors \( A = \nabla^i u \), for each \( 0 \leq i \leq m \), and summing, we obtain, as required,

1. (if \( p < n \))

\[
\sum_{i=0}^{m} \left\| \nabla^i u \right\|_{\frac{nq}{nq-p}} g \leq C \left( \sum_{i=0}^{m} \left\| \nabla^i u \right\|_{\frac{nq}{nq-p}} g \right)^{\frac{nq}{p(n-q)}} \leq C \left( \sum_{i=0}^{m} \left\| \nabla^i u \right\|_{p,g} \right)^{\frac{nq}{p(n-q)}}.
\]

2. (if \( p > n \))

\[
\sum_{i=0}^{m} \sup_M |\nabla^i u|_g \leq C \left( \sum_{i=0}^{m} \left( \sup_M |\nabla^i u|_g \right)^{p} \right)^{\frac{1}{p}} \leq C \left( \sum_{i=0}^{m+1} \left\| \nabla^i u \right\|_{p,g} \right)^{1/p}.
\]

Finally, we recall a variant of the above Sobolev inequalities, in which one constant is made more precise.

**Proposition 7.2.4** (Aubin’s sharp Sobolev inequality, [Aub76c, Heb96a], see exposition in [Heb96b, Theorem 4.6]). Let \( n, p < n, \epsilon, i, \lambda \) be given. Then there exists \( C = C(n, p, \epsilon, i, \lambda) \), such that if \((M, g)\) is a complete Riemannian \( n \)-manifold with injectivity radius at least \( i \) and \( \text{Ric} \geq \lambda g \), and \( u \) a smooth function on \( M \), then

\[
\left( \int_M u^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}} \leq \frac{1 + \epsilon}{\Lambda(n, p)} \int_M |du|^p + C \int_M u^p.
\]
7.3 Elliptic estimates

Theorem 7.3.1 \((L^p \text{ estimates, Gilbarg-Trudinger } [GT01] \text{ 9.11, modified})\). For each \(n, p, m, U, V \subset U\), \(\lambda, \Lambda, \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) increasing, there exists \(C = C(n, p, U, V, \lambda, \Lambda, \mu)\), such that if

\[
Lu := a^{ij} \partial_i \partial_j u + b^i \partial_i u
\]

satisfies

- For all \(x \in U\) and \(\xi \in \mathbb{R}^n\), \(a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2\);
- \(||a^{ij}||_{m, \infty}, ||b^i||_{m, \infty} \leq \Lambda;\)
- For all \(x, y \in U\), \(a^{ij}(x) - a^{ij}(y) \leq \mu(|x - y|)\)

then for all functions \(u \in W^{m+2, p}_{\text{loc}}(U) \cap L^p(U)\),

\[
||u||_{m+2, p; V} \leq C(||Lu||_{m, p; U} + ||u||_{p; U}).
\]

Lemma 7.3.2 (Local elliptic estimate). Let \(n, p, m, i, A\) be given. Then there exists \(r = r(n, p, m, i, A) < i \) and \(C = C(n, p, m, i, A)\), such that for each

- complete Riemannian \(n\)-manifold \((M, g)\) with injectivity radius at least \(i\) and \(||\text{Ric}||_{C^i, g} \leq A,\)
- point \(x \in M,\)
- smooth function \(u\) on \(B(x, r),\)

we have

\[
\sum_{i=0}^{m+2} ||(\nabla)^i u||^p_{p,g,B(x, r/2)} \leq C \left[ \left( \sum_{i=0}^{m} ||(\nabla)^i \Delta g u||^p_{p,g,B(x,r)} \right) + ||u||^p_{p,g,B(x,r)} \right]
\]

Proof. Let \(q > n\) be arbitrary. Choose \(r\) to be less than or equal to the \(C(n, Q := 1, q, i, (A)_{0 \leq j \leq m})\) of Theorem 7.1.1 (2), so that the harmonic radius \(r_H(Q := 1, m + 2, q)\) is at least \(r\). By the Sobolev estimate Theorem 7.2.1 (2), we therefore have uniform bounds in terms of \(n, m, i, A\) on \(||g||_{\infty, B(x,r)}, ||g^{-1}||_{\infty, B(x,r)}, \)

\(||D(g^{-1})|||_{\infty, B(x,r)}, ||\Gamma||_{m, \infty, B(x,r)}\).

For shorthand we write \(B_1\) for \(B(x, r/2)\) and \(B_2\) for \(B(x, r)\). Let \(u\) be a function on \(B_2\), with \(u \in W^{m+2, p}_{\text{loc}}(B_2) \cap L^p(B_2)\).
Since (by our bounds on the components of the tensor $g$) the metrics $g$ and $g_{\text{eucl}}$ are comparable, the norms $\| \cdot \|_{p,g,B_1}$ and $\| \cdot \|_{p,B_1}$ are comparable and the norms $\| \cdot \|_{p,g,B_2}$ and $\| \cdot \|_{p,B_2}$ are comparable. It therefore suffices to prove the inequality with the latter norms.

By Lemma 7.1.3 (2),

$$\sum_{i=0}^{m+2} \| (\nabla)^i u \|^p_{p,B_1} \leq C \left( \sum_{i=0}^m \| D^i \Gamma \|_{\infty,B_1} \right) \sum_{i=0}^{m+2} \| D^i u \|^p_{p,B_1}$$

$$\sum_{i=0}^m \| D^i \Delta_g u \|^p_{p,B_2} \leq C \left( \sum_{i=0}^{m-2} \| D^i \Gamma \|_{\infty,B_2} \right) \sum_{i=0}^m \| (\nabla)^i \Delta_g u \|^p_{p,B_2}$$

(where if $m - 2 < 0$ the second inequality is simply an identity).

Also, applying the $L^p$ estimate of Theorem 7.3.1 with the operator

$$L u = \Delta_g u = g^{ij} \partial_i \partial_j u + g^{ij} \Gamma^k_{ij} \partial_k u$$

shows: there exists $C = C(n, p, m, r, \| g \|_{m, \infty, B_2}, \| g^{-1} \|_{m, \infty, B_2}, \| \Gamma \|_{m, \infty, B_2})$ such that

$$\sum_{i=0}^{m+2} \| D^i u \|^p_{p,B_1} \leq C \left[ \left( \sum_{i=0}^m \| D^i \Delta_g u \|^p_{p,B_2} \right)^{\frac{1}{p}} + \| u \|_{p,B_2} \right]^p \leq 2^{p-1} C \left( \sum_{i=0}^m \| D^i \Delta_g u \|^p_{p,B_2} + \| u \|^p_{p,B_2} \right).$$

Combining these three inequalities gives the result. \qed

**Proposition 7.3.3** ($L^p$ estimate). Let $n$, $p$, $m$, $i$, $A$ be given. Then there exists $C = C(n, p, m, i, A)$, such that if $(M, g)$ is a complete Riemannian $n$-manifold with injectivity radius at least $i$, $\| \text{Ric} \|_{C^m,g} \leq A$, and $u$ a smooth function on $M$, then

$$\| u \|_{W^{m+2,p,g}} \leq C \| \Delta_g u \|_{W^{m,p,g}} + \| u \|_{p,g}.$$  

**Proof.** Choose $r$ and $C$ (dependent on $n, p, m, i, A$) as in the local elliptic estimate Lemma 7.3.2.

By Lemma 7.1.2 (setting $\rho = \frac{1}{2} r$), there exists $N = N(n, A, \frac{1}{2} r, r)$ and an (at most) countable set $(x_\alpha)$ of points in $M$, such that

1. the family $(B(x_\alpha, \frac{1}{2} r))$ covers $M$;
2. each point in $M$ is contained in at most $N$ balls of the family $(B(x_i, r))$.

So, by the local elliptic estimate Lemma 7.3.2,

$$\|u\|_{W^{m+2,p,g}} \leq \sum_{i=0}^{m+2} \sum_{\alpha} \|\nabla^i u\|_{p,g,B(x_{a\alpha},r/2)}^p \leq C \sum_{\alpha} \left( \sum_{i=0}^{m} \|\nabla^i \Delta g u\|_{p,g,B(x_{a\alpha},r)}^p + \|u\|_{p,g,B(x_{a\alpha},r)}^p \right) \leq NC \|\Delta g u\|_{W^{m,p,g}} + \|u\|_{p,g} \leq 2NC \|\Delta g u\|_{W^{m,p,g}} + \|u\|_{p,g}^p.$$  

\[ \square \]

### 7.4 Moser’s Harnack inequality

**Theorem 7.4.1** (Harnack inequality, Gilbarg-Trudinger [GT01] 8.21). For each $n, r, \lambda, \Lambda$, there exists $C = C(n, r, \lambda, \Lambda)$, such that if the operator $L$ on $W^{1,2}(B_{4r})$,

$$Lu := \partial_i (a^{ij} \partial_j u) + cu,$$

satisfies

- For all $x \in U$ and $\xi \in \mathbb{R}^n$, $a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2$;
- $\|a^{ij}\|_{\infty}, |c|_{\infty} \leq \Lambda$;

then for all functions $u \in W^{1,2}(B_{4r})$ with $u \geq 0$ and $Lu = 0$,

$$\sup_{B_r} u \leq C \inf_{B_r} u.$$  

**Proposition 7.4.2** (Harnack inequality). Let $n, i, \lambda, B, D$ be given. Then there exists $C = C(n, i, \lambda, B, D)$, such that for each

- compact Riemannian $n$-manifold $(M, g)$ with injectivity radius at least $i$, Ricci curvature $\text{Ric} \geq \lambda g$, and diameter at most $D$, 

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• smooth function $u$ on $M$ with $u \geq 0$ and $|\Delta_g u| \leq B|u|

we have

$$\sup_M u \leq C \inf_M u.$$ 

Proof. Let $p > n$ be arbitrary; choose $4r$ to be < $i$ and less than or equal to the $C(n, Q := 1, p, i, \lambda)$ of Theorem 7.1.1 (1), so that the harmonic radius $r_H(Q := 1, 1, p)$ is at least $4r$. By the Sobolev estimate Theorem 7.2.1 (2), we therefore have uniform bounds in terms of $n, i, \lambda$ on, for each $x \in M$, the co-ordinate norms $||g||_{\infty, B(x, 4r)}, ||g^{-1}||_{\infty, B(x, 4r)}$.

Define a measurable function $c$ on $M$ by,

$$c(x) = \begin{cases} 0, & \text{if } u(x) = 0 \\ -\frac{\sqrt{|g|} (\Delta_g u)(x)}{|u(x)|}, & \text{if } u(x) \neq 0. \end{cases}$$

This function satisfies the bound $||c||_{\infty} \leq \sqrt{n}||g||^{n/2}_{\infty, B(x, 4r)} B < \infty$. By construction $\sqrt{|g|} \Delta_g u + cu = 0$.

Applying the Harnack estimate of Theorem 7.4.1 with the operator

$$Lu = (\sqrt{|g|} \Delta_g + c)u = \partial_i (\sqrt{|g|} g^{ij} \partial_j u) + cu,$$

we deduce that for $C = C(n, i, \lambda, B)$, for each $x \in M$,

$$\sup_{B(x, r)} u \leq C \inf_{B(x, r)} u.$$

Applying Lemma 7.1.2 with $(\rho, r) = (r, D)$, we obtain an integer $N = N(n, \lambda, r, D)$ such that $M$ may be covered by a set of at most $N$ radius-$r$ balls. Let $(B_\alpha)_{\alpha \in A}$ be such a covering.

For any two balls $B_\alpha, B_\beta$ in the set, there exists a sequence $\alpha_0 := \alpha, \alpha_1, \ldots, \alpha_l := \beta$, with $l \leq |A| - 1$, such that each pair $B_{\alpha_i}, B_{\alpha_{i+1}}$ of adjacent balls in the sequence intersects. Thus, for each $0 \leq i \leq l - 1$,

$$\inf_{B_{\alpha_i}} u \leq \inf_{B_{\alpha_i} \cap B_{\alpha_{i+1}}} u \leq \sup_{B_{\alpha_{i+1}}} u.$$

Therefore, by induction,

$$\sup_{B_{\alpha}} u \leq C^N \inf_{B_{\beta}} u.$$
Since this holds for all $\alpha, \beta \in \mathcal{A}$, and $(B_\alpha)_{\alpha \in \mathcal{A}}$ cover $M$, we conclude

$$\sup_M u \leq C_N \inf_M u.$$
Chapter 8

A compactness theorem for Yamabe metrics

A well-known corollary of Trudinger’s and Aubin’s work on the Yamabe problem [Tru68, Aub76c, Aub76a] is the fact that, in a conformal class other than the conformal class of the round sphere, the set of Yamabe metrics is compact.

In this chapter we give an exposition of a slight generalization: in a compact set of conformal classes not containing the conformal class of the round sphere, the set of Yamabe metrics is compact. The point to be checked is the dependence, in some standard a priori estimates, of the constant $C$ on the background Riemannian metric $g$. Again, though known in principle to experts (see for instance [And05, discussion following Equation 2.1], [KMS09, Lemma 10.1]), this material has not, to our knowledge, appeared elsewhere in the literature.

8.1 $L^r$ bound on constant-scalar-curvature metrics

We now prove versions of standard bounds on metrics with (small) constant scalar curvature, in which the dependence of the bounds on the metric is made explicit. For the first, an $L^r$ bound, we adapt the presentation of [LP87] Theorem 4.4.

First observe that if $p = 2$ then the constant $S(n, p)$ of the sharp Sobolev inequality Proposition 7.2.4
has,
\[ S(n, 2) = n(n - 2) \left( \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n}{2} + 1)}{\Gamma(n + 1)} \right)^{2/n}. \]

By the Gamma duplication identity \( \Gamma\left( \frac{n}{2} + 1 \right) \Gamma\left( \frac{n}{2} \right) = 2^{-n} \sqrt{\pi} \Gamma(n + 1) \), and the sphere volume recursion formula \( \Gamma\left( \frac{n+1}{2} \right) \omega_n = \sqrt{\pi} \Gamma\left( \frac{n}{2} \right) \omega_{n-1} \) (here \( \omega_n \) is the volume of the \( n \)-sphere), we obtain
\[ S(n, 2) = \frac{n(n-2)}{4} \omega_n^2. \]

By convention we define
\[ \Lambda := 4 \frac{n-1}{n-2} S(n, 2) = 4n(n-1) \left( \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n}{2} + 1)}{\Gamma(n + 1)} \right)^{2/n} = \frac{n(n-1)\omega_n^2}{4}. \]

**Proposition 8.1.1.** Suppose given \( i, A, B, \eta \). There exists \( r > \frac{2n}{n-2}, C > 0 \) dependent only on \( n, i, A, B, \eta \), such that for each

- smooth metric \( g \) on \( M \) such that \( \text{Vol}(M, g) \leq A, \text{injrad}(M, g) \geq i, \text{Ric}(g) \geq -Bg, |R(g)| \leq nB \),
- smooth positive function \( \varphi \) on \( M \) such that \( \varphi^{\frac{4}{n+2}} g \) has volume 1 and constant scalar curvature \( \lambda \leq \Lambda - \eta \),

we have the uniform bound
\[ ||\varphi||_{L_r, g} \leq C. \]

**Proof.** Let \( g \) and \( \varphi \) be as given. Thus the function \( \varphi \) satisfies the scalar curvature equation
\[ -4 \frac{n-1}{n-2} \Delta_g \varphi + R(g)\varphi = \lambda \varphi^{\frac{n+2}{n-2}}. \]

Multiplying by \( \varphi^{1+\delta} \) and integrating (implicitly with respect to \( d\text{Vol}_g \)), we obtain, by Stokes’ theorem, the identity
\[ 4 \left( \frac{n-1}{n-2} \right) \left( \frac{1+\delta}{1+\delta^2} \right) \int_M |d(\varphi^{1+\delta})|^2_g = -4 \frac{n-1}{n-2} \int_M \varphi^{1+2\delta} \Delta_g \varphi \]
\[ = I([g]) \int_M \varphi^{\frac{2n}{n-2}+2\delta} - \int_M R(g)\varphi^{2+2\delta}. \]

Thus, by our bound on \( R(g) \), for some \( C(\delta) \)
\[ 4 \left( \frac{n-1}{n-2} \right) \int_M |d(\varphi^{1+\delta})|^2_g \leq \lambda \frac{(1+\delta)^2}{1+2\delta} \int_M \varphi^{\frac{2n}{n-2}+2\delta} + C \int_M \varphi^{2(1+\delta)}. \quad (8.1) \]
By the sharp Sobolev inequality Proposition 7.2.4 applied to the function $\varphi^{1+\delta}$, since we have bounds on $\text{injrad}(M, g)$ and $\inf_M \text{Ric}(g)$, there exists $C = C(n, \epsilon)$ such that

$$
\left( \int_M \varphi^{\frac{n}{n-2}(1+\delta)} \right)^\frac{n-2}{n} \leq 4(1+\epsilon) \left( \frac{n-1}{n-2} \right) \Lambda^{-1} \int_M \left| d(\varphi^{1+\delta}) \right|^2 + C \int_M \varphi^{2(1+\delta)}. \quad (8.2)
$$

By Hölder’s inequality, we have

$$
\int_M \varphi^{\frac{n}{n-2} + 2\delta} \leq \left( \int_M \varphi^{\frac{n}{n-2}(1+\delta)} \right)^\frac{n-2}{n} \left( \int_M \varphi^{\frac{n}{n-2}} \right)^\frac{2}{n},
$$

(8.3)

$$
\int_M \varphi^{2(1+\delta)} \leq \left( \int_M 1 \right)^{1-(1+\delta)\frac{n-2}{n}} \left( \int_M \varphi^{\frac{n}{n-2}} \right)^{(1+\delta)\frac{n-2}{n}},
$$

(8.4)

where in each case the second identity is from the condition

$$
\int_M \varphi^{\frac{n}{n-2}} = \text{Vol}(\varphi^{\frac{n}{n-2}} g) = 1,
$$

and where the second inequality also uses our bound on $\text{Vol}(M, g)$.

Combining the inequalities (8.1), (8.2), (8.3), (8.4), we see that for some $C = C(n, \epsilon, \delta)$,

$$
\left( \int_M \varphi^{\frac{n}{n-2}(1+\delta)} \right)^\frac{n-2}{n} \leq (1+\epsilon) \left( 1 + \frac{(1+\delta)^2}{1+2\delta} \right) \left( \frac{\lambda}{\Lambda} \right) \left( \int_M \varphi^{\frac{n}{n-2}(1+\delta)} \right)^\frac{n-2}{n} + C.
$$

Rearranging and using our bound $\Lambda - \lambda \geq \eta$,

$$
\left[ 1 - (1+\epsilon) \frac{(1+\delta)^2}{1+2\delta} \left( 1 - \frac{\eta}{\Lambda} \right) \right] \left( \int_M \varphi^{\frac{n}{n-2}(1+\delta)} \right)^\frac{n-2}{n} \leq C.
$$

There exist $\epsilon > 0$, $\delta > 0$, sufficiently small that

$$
(1+\epsilon) \frac{(1+\delta)^2}{1+2\delta} \left( 1 - \frac{\eta}{\Lambda} \right) < 1.
$$

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Therefore for this $\epsilon$, $\delta$ and for $r := \frac{2n}{n-2}(1 + \delta) > \frac{2n}{n-2}$,

\[
\left( \int_M \varphi^r \, d\text{Vol}_g \right)^{\frac{1}{r}} \leq \left[ 1 - (1 + \epsilon) \frac{(1 + \delta)^2}{1 + 2\delta} \left( 1 - \frac{\eta}{A} \right) \right]^{-1} C \leq C.
\]

\[\square\]

## 8.2 Higher-order bounds on constant-scalar-curvature metrics

In this section we follow the arguments of [LP87] Lemma 4.1, with slight modifications to yield sharper bounds.

**Lemma 8.2.1.** For each

- smooth metric $g$ on $M$
- smooth positive function $\varphi$ on $M$ such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda$,

we have, $\lambda \geq -\max_M [-R(g)] \text{Vol}(M, g)^{2/n}$.

**Proof.** Multiplying the condition on the scalar curvature of $\varphi^{\frac{4}{n-2}} g$,

\[-4 \frac{n-1}{n-2} \Delta_g \varphi + R(g) \varphi = \lambda \varphi^{\frac{n+2}{n-2}},\]

by $\varphi$, integrating, and using Stokes' theorem, we obtain,

\[4 \frac{n-1}{n-2} \int_M |d\varphi|^2_g + \int_M R(g) \varphi^2 = \lambda \int_M \varphi^{\frac{2n}{n-2}}.\]

Rearrange and apply Hölder’s inequality:

\[(-\lambda) \int_M \varphi^{\frac{2n}{n-2}} \leq \max_M [-R(g)] \int_M \varphi^2 \leq \max_M [-R(g)] \text{Vol}(M, g)^{2/n} \left( \int_M \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}.\]

Since $\varphi^{\frac{4}{n-2}} g$ has volume 1, $\int_M \varphi^{\frac{2n}{n-2}} = 1$. The result follows. \[\square\]

**Proposition 8.2.2.** Suppose given $p > 1$, $i$, $A$, $B$, $\eta$. There exists $C > 0$ dependent only on $n$, $p$, $i$, $A$, $B$, $\eta$, such that for each

- smooth metric $g$ on $M$ such that $\text{Vol}(M, g) \leq A$, $\text{injrad}(M, g) \geq i$, $|\text{Ric}(g)| \leq B$,
• smooth positive function $\varphi$ on $M$ such that $\varphi \rightarrow g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta,$ we have the uniform bound

$$||\varphi||_{W^{2,p},g} \leq C.$$

**Proof.** We prove this by an induction-type argument on $p$. In this proof $C$ denotes a constant dependent on $n, i, A, B, \eta; \text{ and } C([\text{other quantities}])$ denotes a constant dependent on those quantities and on the other quantities specified. We will show that for each $p > 1$,

$$||\varphi||_{W^{2,p},g} \leq C(p).$$

Let $S = \max(\Lambda - \eta, nBA^{2/n}).$ By Lemma 8.2.1, $|\lambda| \leq S.$

For each $p \geq 1,$ by our bounds on Vol$(M,g)$ and Ric$(g)$ and by Hölder's inequality, the scalar curvature equation

$$-\frac{4}{n-2} \Delta_g \varphi + R(g)\varphi = \lambda \varphi \frac{n+2}{n-2},$$

implies the uniform bounds

$$||\varphi||_{p,g} \leq \text{Vol}(M,g)^{\frac{1}{n+2}}||\varphi||_{\frac{n+2}{n-2},p,g}$$

$$\leq C||\varphi||_{\frac{n+2}{n-2},p,g};$$

$$||\Delta_g \varphi||_{p,g} \leq \frac{n-2}{4(n-1)} \left[||R(g)||_{\infty,g}||\varphi||_{p,g} + |\lambda||\varphi||_{\frac{n+2}{n-2},p,g}\right]$$

$$\leq \frac{n-2}{4(n-1)} \left[||R(g)||_{\infty,g} \text{Vol}(M,g)^{\frac{4}{n+2}}||\varphi||_{\frac{n+2}{n-2},p,g} + S||\varphi||_{\frac{n+2}{n-2},p,g}\right]$$

$$\leq C \left(||\varphi||_{\frac{n+2}{n-2},p,g}\right).$$

Therefore for each $p > 1,$ by the $L^p$ elliptic estimate Proposition 7.3.3 (which is available by our bounds on injrad$(M,g)$ and $|\text{Ric}(g)|$),

$$||\varphi||_{W^{2,p},g} \leq C(p) \left[||\Delta_g \varphi||_{p,g} + ||\varphi||_{p,g}\right] \leq C \left(p, ||\varphi||_{\frac{n+2}{n-2},p,g}\right).$$

By Proposition 8.1.1 there exist $r > \frac{2n}{n-2}$ and $C$, both with the appropriate dependence, such that

$$||\varphi||_{r,g} \leq C.$$
So, as a base case,

$$||\varphi||_{W^{2, \frac{n-2}{n+2}, g}} \leq C \left( \frac{n-2}{n+2} r \right).$$

Moreover, by the Sobolev inequality Proposition 7.2.3 (which is available by our bounds on injrad$(M, g)$ and inf$_M \text{Ric}(g)$), we have,

1. if $q < n/2$, then for each $p$ satisfying

$$1 < p < \left( \frac{n-2}{n+2} \right) \left( \frac{nq}{n-2q} \right),$$

we have

$$||\varphi||_{W^{2, p, g}} \leq C \left( p, ||\varphi||_{W^{2, \frac{n-2}{n+2} p, g}} \right) \leq C \left( p, ||\varphi||_{W^{2, q, g}} \right);$$

2. if $q > n/2$, then for each $p > 1$ (arbitrarily large), we have

$$||\varphi||_{W^{2, p, g}} \leq C \left( p, ||\varphi||_{W^{2, q, g}} \right) \leq C \left( p, ||\varphi||_{W^{2, q, g}} \right);$$

Thus it suffices to observe that if $r > \frac{2n}{n-2}$ then the sequence

$$\frac{n-2}{n+2} r, \left( \frac{n-2}{n+2} \right) \left( \frac{n}{n-2} \left( \frac{n-2}{n+2} r \right) \right), \ldots$$

1. stays greater than 1; and,

2. eventually becomes greater than $n/2$.

\[ \square \]

**Lemma 8.2.3.** Suppose given $i$, $D$, $B$, $\eta$. There exists $C > 0$ dependent only on $n$, $i$, $D$, $B$, $\eta$, such that for each

- smooth metric $g$ on $M$ such that $\text{diam}(M, g) \leq D$, $\text{injrad}(M, g) \geq i$, $|\text{Ric}(g)| \leq B$,

- smooth positive function $\varphi$ on $M$ such that $\varphi^{1/4} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,

we have the uniform bounds

$$||\varphi||_{C^0} \leq C, \quad ||\varphi^{-1}||_{C^0} \leq C.$$
Proof. In this proof

- $C$ denotes constants dependent on $n$, $i$, $D$, $B$, $\eta$,

- $g$ is a smooth metric on $M$ such that $\text{diam}(M,g) \leq D$, $\text{injrad}(M,g) \geq i$, $|\text{Ric}(g)| \leq B$,

- $\varphi$ is a smooth positive function on $M$ such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$.

Diameter and Ricci bounds imply a volume bound, so the hypotheses of Proposition 8.2.2 are satisfied. By the same argument as in Proposition 8.2.2, $|\lambda| \leq C$.

For the first inequality, choose $p > \frac{n}{2}$. By the Sobolev inequality Proposition 7.2.3 and by Proposition 8.2.2,

$$||\varphi||_{C^0} \leq C||\varphi||_{W^{2,p},g} \leq C(p).$$

This moreover shows

$$||\frac{n-2}{4(n-1)} \left( R(g) - \lambda \varphi^{\frac{n+2}{n-2}} \right) ||_{C^0} \leq C,$$

so by the Yamabe equation

$$-4\frac{n-1}{n-2} \Delta_g \varphi + R(g) \varphi = \lambda \varphi^{\frac{n+2}{n-2}}.$$

we obtain $C$ with only the given dependence such that $|\Delta_g \varphi| \leq C|\varphi|$. Thus the Harnack inequality Proposition 7.4.2 (applicable due to our bounds on $\text{diam}(M,g)$, $\text{injrad}(M,g)$ and $\text{Ric}(g)$) implies

$$||\varphi^{-1}||_{C^0} \leq C||\varphi||_{C^0}^{-1}.$$

We conclude by using Hölder’s inequality and the volume condition $||\varphi||_{\frac{2n}{n+2},g} = 1$ to bound $||\varphi||_{C^0}$ from below:

$$||\varphi||_{C^0}^{-1} \leq \text{Vol}(M,g)^{\frac{n-2}{n-1}} ||\varphi||_{C^0}^{-1} \leq C.$$

\[ \square \]

**Theorem 8.2.4.** Suppose given $m \geq 2$, $p > n$, $i$, $D$, $B$, $\eta$. There exists $C > 0$ dependent only on $n$, $m$, $p$, $i$, $D$, $B$, $\eta$, such that for each

- smooth metric $g$ on $M$ such that $\text{diam}(M,g) \leq D$, $\text{injrad}(M,g) \geq i$, $|\text{Ric}(g)|_{C^{m-2},g} \leq B$,

- smooth positive function $\varphi$ on $M$ such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$.
we have the uniform bound

$$||\varphi||_{W^{m,p},g} \leq C.$$ 

**Proof.** We prove this by induction on $m$. In this proof

- $C$ denotes a constant dependent on $n$, $i$, $D$, $B$, $\eta$,
- $C([\text{other quantities}])$ denotes a constant dependent on those quantities and on the [other quantities] specified,
- $g$ denotes a smooth metric on $M$ such that $\text{diam}(M, g) \leq D$, $\text{injrad}(M, g) \geq i$,
- $\varphi$ is a smooth positive function on $M$ such that $\varphi^{\frac{4}{n-2}} g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$.

By the same argument as in Proposition 8.2.2, $|\lambda| \leq C$. We will show that for each $m \geq 2$ and $p > n$, if $||\text{Ric}(g)||_{C^{m-2}} \leq B$ then

$$||\varphi||_{W^{m,p},g} \leq C(m, p).$$

The $m = 2$ case is Proposition 8.2.2. (Bounds on diameter and Ricci curvature imply a bound on volume, so the assumptions of Proposition 8.2.2 are strictly stronger than our assumptions here.)

Suppose the result is known for $m - 1$.

If now $||\text{Ric}(g)||_{C^{m-2},g} \leq B$, then by our bounds on $\text{Vol}(M, g)$ and $||R(g)||_{C^{m-2},g}$ and by Hölder’s inequality, the scalar curvature equation

$$-4\frac{n-1}{n-2} \Delta_g \varphi + R(g) \varphi = \lambda \varphi^{\frac{n+2}{n-2}}.$$
implies the uniform bound

\[ \|\Delta g \varphi\|_{W^{m-2,p,g}} \leq \frac{n-2}{4(n-1)} C(m) \left[ \|R(g)\|_{C^{m-2,g}} \|\varphi\|_{W^{m-2,p,g}} + |\lambda| \|\varphi^{\frac{n+2}{n-2}}\|_{W^{m-2,p,g}} \right] \]

\[ \leq C(m) \|\varphi\|_{W^{m-2,p,g}} \]

\[ + C(m) \sum_{i=0}^{m-2} \sum_{a_1 + \cdots + a_r = i} \|\varphi^{\frac{n+2}{n-2} - r} \nabla^{a_1} \varphi \cdots \nabla^{a_r} \varphi\|_{p,g} \]

\[ \leq C(m) \|\varphi\|_{W^{m-2,p,g}} \]

\[ + C(m) \operatorname{Vol}(M,g)^{\frac{1}{m}} \sum_{i=0}^{m-2} \sum_{a_1 + \cdots + a_r = i} \|\varphi^{\frac{n+2}{n-2} - r} \nabla^{a_1} \varphi \cdots \nabla^{a_r} \varphi\|_{C^0,g} \]

\[ \leq C(m) \|\varphi\|_{W^{m-2,p,g}} \]

\[ + C(m) \sum_{i=0}^{m-2} \sum_{a_1 + \cdots + a_r = i} \|\varphi^{\frac{n+2}{n-2} - r} \|_{C^0} \|\nabla^{a_1} \varphi\|_{C^0,g} \cdots \|\nabla^{a_r} \varphi\|_{C^0,g}. \]

Since, for each \( r \),

\[ \|\varphi^{\frac{n+2}{n-2} - r} \|_{C^0} \leq \max \left( \|\varphi\|_{C^0}^{\frac{n+2}{n-2} - r}, |\varphi^{-1}|_{C^0}^{\frac{n+2}{n-2}} \right), \]

this yields,

\[ \|\Delta g \varphi\|_{W^{m-2,p,g}} \leq C \left( m, \|\varphi\|_{W^{m-2,p,g}}, \|\varphi\|_{C^m,g}, |\varphi^{-1}|_{C^0} \right) \leq C(m,p), \]

where the final inequality comes from combining

- Lemma 8.2.3 that \( |\varphi^{-1}|_{C^0} \leq C \);

- the Sobolev inequality Proposition 7.2.3 that \( \|\varphi\|_{C^m,g} \leq \|\varphi\|_{W^{m-1,p,g}} \);

- the inductive hypothesis \( \|\varphi\|_{W^{m-1,p,g}} \leq C(m-1,p) \).

So, by the \( L^p \) elliptic estimate Proposition 7.3.3 (applicable by our bounds on \( \operatorname{injrad}(M,g) \) and \( \|\operatorname{Ric}(g)\|_{C^{m-2,g}} \)),

\[ \|\varphi\|_{W^{m,p,g}} \leq C \left[ \|\Delta g \varphi\|_{W^{m-2,p,g}} + \|\varphi\|_{p,g} \right] \leq C. \]

\[ \square \]
8.3 Compactness

Lemma 8.3.1. \( I \) is upper semi-continuous.

Proof. It is the infimum of a continuous functional.

In fact \( I \) is continuous [BB83], but we will not need that here.

Proposition 8.3.2. Let \( m \geq 3 \). Let \((M,c)\) be a conformal manifold with \( I(c) < \Lambda \). Let \((c_k)\) be a sequence of smooth conformal classes on \( M \) which \( C^m\)-converges to \( c \), and let \((g_k)\) be volume-1 Yamabe metrics for the classes \((c_k)\).

Then there exists a subsequence \((g_{k_i})\) which \( C^{m-1}\)-converges to a volume-1 Yamabe metric for \( c \).

Proof. Choose \( p > n \), and choose representatives \((\bar{g}_k)\), \( \bar{g} \) of the classes \((c_k)\), \( c \) with \( \bar{g}_k \to \bar{g} \) in the \( C^m \) topology. For sufficiently large \( k \),

1. \( I(c_k) \leq \frac{1}{2}[I(c) + \Lambda] \) (by Lemma 8.3.1);

2. there are uniform bound on the expressions

\[
\text{diam}(M,\bar{g}_k) \quad \text{injrad}(M,\bar{g}_k) \quad ||\text{Ric}(\bar{g}_k)||_{C^{m-2},\bar{g}_k};
\]

3. there exists a uniform \( C \) such that for \( k \) sufficiently large,

\[
C^{-1} || \cdot ||_{W^{m+1,p},\bar{g}_k} \leq || \cdot ||_{W^{m+1,p},\bar{g}} \leq C || \cdot ||_{W^{m+1,p},\bar{g}_k}.
\]

Therefore, by Theorem 8.2.4, if \( \varphi_k \) are smooth positive functions on \( M \) such that \( \varphi_k^{4/(n-2)}\bar{g}_k \) are volume-1 Yamabe metrics for \( c_k \), then we have the uniform bound \( ||\varphi_k||_{W^{m,p},\bar{g}} \leq C \), hence by Morrey’s inequality the uniform bound \( ||\varphi_k||_{C^{m-1,1-\frac{p}{n}},\bar{g}} \leq C \), and so there exists a subsequence \((k_i)\) such that \( \varphi_{k_i} \) \( C^{m-1}\)-converges.

It remains to be checked that the limit, \( \varphi \), makes \( \varphi^{4/(n-2)}\bar{g} \) a Yamabe metric for \( c \). Indeed, \( \varphi_k^{4/(n-2)}\bar{g}_k \) all have volume 1 and constant scalar curvature \( I(c_k) \). So their \( C^{m-1}\)-limit \( g := \varphi^{4/(n-2)}\bar{g} \) has volume 1, and constant scalar curvature \( \lim_{k \to \infty} I(c_k) \), which is \( \leq I(c) \) by Lemma 8.3.1; this is a contradiction unless equality holds and \( \bar{g} \) is a Yamabe metric.

\( \square \)
Chapter 9

Conformal classes realizing the Yamabe invariant

9.1 Notation

As sketched in the introduction, we denote by $|\Lambda^n M|$ the bundle of densities on $M$; it is a line bundle, equipped with a natural positive orientation, and hence all real powers $|\Lambda^n M|^\alpha$ are well-defined. We write $|\Omega|$ for the density associated to an $n$-form $\Omega$.

For each $\alpha$ there is a well-defined “determinant” map of bundle sections,

$$\det : C^\infty(M, \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^\alpha) \to C^\infty(M, |\Lambda^n M|^{2+n\alpha})$$

For $\alpha = 0$ this is the square of the volume form, $g \mapsto (d\text{Vol}_g)^{\otimes 2}$. The case $\alpha = -2/n$ (that is, the bundle $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$) has the special feature that its determinant has range $C^\infty(M, \mathbb{R})$.

Denote by $\mathcal{M}$ the space of smooth Riemannian metrics on $M$, and by $\mathcal{V}$ the space of smooth positive densities. The map from $\mathcal{M}$ to $\mathcal{V} \times C^\infty(M, \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n})$ given by

$$g \mapsto (|d\text{Vol}_g|, g \otimes |d\text{Vol}_g|^{-2/n})$$

is injective with image $\mathcal{V} \times \mathcal{C}$, where $\mathcal{C}$ denotes the set of smooth positive-definite sections of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$ with determinant 1. The inverse, an isomorphism from $\mathcal{V} \times \mathcal{C}$ to $\mathcal{M}$, is given by $(\Omega, c) \mapsto \Omega^{2/n} c$. 

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For any element $c \in C$, the set of Riemannian metrics in $M$ corresponding to $V \times \{c\}$ under this isomorphism is precisely a conformal equivalence class of Riemannian metrics. We therefore identify $c$ with that conformal equivalence class, and $C$ with the space of smooth conformal classes. The tangent space $T_c C$ to $C$ at $c$ is the kernel of the trace

$$\text{tr}_c : C^\infty(M, \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}) \to C^\infty(M, \mathbb{R}).$$

### 9.2 Continuity properties of the total scalar curvature functional

We introduce the notation $Q : M \to \mathbb{R}$ for the total scalar curvature functional,

$$Q(g) = \frac{\int_M R(g)|dVol_g|}{(\int_M dVol_g)^{1-2/n}},$$

and, according to the isomorphism described in the previous section, have an equivalent functional $Q : V \times C \to \mathbb{R},$

$$Q_\Omega(c) = Q(\Omega^{2/n} c).$$

We have, for a conformal class $c$,

$$I(c) = \inf_{g \in c} Q(g) = \inf_{\Omega \in V} Q_\Omega(c).$$

**Lemma 9.2.1.** Let $\Omega$ be a positive density. The functional $Q_\Omega : C \to \mathbb{R}$ is $C^1$, and its derivative $D_c(Q_\Omega) : T_c C \to \mathbb{R}$ at $c \in C$ is

$$D_c(Q_\Omega)(w) = \text{Vol}(\Omega)^{\frac{2}{n} - 1} \int_M \langle w, - \text{Ric}(\Omega^{2/n} c) \Omega^{1-2/n} c \rangle.$$

**Proof.** It is well-known (e.g. [Bes87, Proposition 4.17]) that for a Riemannian metric $g$ and symmetric 2-tensor $h$,

$$D_g(Q)(h) = \text{Vol}(g)^{2/n - 1} \int_M \left\langle h, - \text{Ric}(g) + \frac{1}{2} \left[ R(g) - \frac{\int_M R(g) dVol_g}{\text{Vol}(g)} \right] g \right\rangle |dVol_g|.$$

Recall from the previous section that $T_c C$ is the set of $c$-tracefree sections of $C^\infty(M, \text{Sym}^2(T^*M))$. If $w$ is such a section, the tangent vector $(0, w)$ to $V \times C$ at $(\Omega, c)$ corresponds to the tangent vector $\Omega^{2/n} w$ to $M$ at $\Omega^{2/n} c$, and since $\text{tr}_c w = 0$, the term $(\Omega^{2/n} w, \Omega^{2/n} c)_{\Omega^{2/n} c}$ vanishes. \qed
Proposition 9.2.2 (Modulus of continuity of $I$). Let $(M, c_0)$ be a conformal manifold. Let $v$ be a smooth section of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$, sufficiently small that $\det(c_0 + tv)$ is nonvanishing for $t \in [0, 1]$. Write

$$c_t := \frac{c_0 + tv}{\det(c_0 + tv)^{\frac{1}{n}}},$$

so that $(c_t)$ is a 1-parameter family of conformal classes starting at $c_0$. Let $\Omega$ be a density such that $g_1 = \Omega^{2/n} c_1$ is a Yamabe metric for $c_1$, and write $g_t = \Omega^{2/n} c_t$. Then

$$\text{Vol}(\Omega)^{1 - \frac{2}{n}} [I(c_1) - I(c_0)]$$

$$\geq \int_0^1 \int_M \left\langle v, \left( -\text{Ric}(g_t) + \frac{1}{n} R(g_t) g_t \right) \det(c_0 + tv)^{-1/n} \Omega^{1 - 2/n} \right\rangle_{c_t} dt.$$

**Proof.** Since $g$ is a Yamabe metric for $c_1$,

$$Q|_{dVol_g}(c_1) = Q(g) = I(c_1),$$

so

$$I(c_1) - I(c_0) = Q|_{dVol_g}(c_1) - \left( \inf_{\Omega \in V} Q_{\Omega}(c_0) \right)$$

$$\geq Q|_{dVol_g}(c_1) - Q|_{dVol_g}(c_0).$$

We calculate

$$\left. \frac{dc_t}{dt} \right|_{t=t} = \left. \frac{dc_0}{dt} \right|_{t=t} \left[ \frac{c_0 + tv}{\det(c_0 + tv)^{\frac{1}{n}}} \right]$$

$$= \frac{v \cdot \det(c_0 + tv)^{\frac{1}{n}} - (c_0 + tv) \cdot \frac{1}{n} \det(c_0 + tv)^{\frac{1}{n} - 1} \cdot \text{tr}_{c_0 + tv} v \det(c_0 + tv)}{\det(c_0 + tv)^{\frac{1}{n}}}$$

$$= \det(c_0 + tv)^{-\frac{1}{n}} \left[ v - \frac{1}{n} (\text{tr}_{c_t} v)c_t \right].$$

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so by Lemma 9.2.1,

\[
\begin{align*}
\text{Vol}(\Omega)^{1-\frac{2}{n}} \frac{d}{dt} \Big|_{t=t} \left[ Q \left( t \Omega^{\frac{1}{n}} \right) \right] \\
= & \int_M \left( v - \frac{1}{n} (\text{tr}_c v) c_t, - \text{Ric}(g_t) \det(c_0 + tv)^{-\frac{n}{2}} \Omega^{1-2/n} \right) c_t \\
= & \int_M \left( v, - \text{Ric}(g_t) + \frac{1}{n} R(g_t) g_t \right) \det(c_0 + tv)^{-\frac{n}{2}} \Omega^{1-2/n} c_t.
\end{align*}
\]

The result follows by the Fundamental Theorem of Calculus.

**9.3 An “Euler-Lagrange inequality”**

Let \( M \) be a compact connected smooth manifold, and suppose the conformal class \( c \) on \( M \) attains the Yamabe invariant of the manifold \( M \).

**Proposition 9.3.1.** For each distributional section \( v \) of \( \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n} \), there exists a Yamabe metric \( g \) for \( c \), such that

\[
\hat{M} \left\langle v, (\text{Ric}(g) - \frac{1}{n} R(g) g) \right| \text{dVol}_g \right|^{1-2/n} \rangle_c \geq 0.
\]

**Proof.** If \((M, c)\) is the conformally round sphere, then \( c \) contains an Einstein metric \( g \), the round metric; for this \( g \), the tensor \( \text{Ric}(g) - \frac{1}{n} R(g) g \) vanishes. The result follows.

If \((M, c)\) is not the conformally round sphere, by the solution of the Yamabe problem [Aub76a, Sch84], \( I(c) < \Lambda \), so the compactness result Proposition 8.3.2 applies. We now give a proof in these cases using it.

Let \((v_k)\) be a sequence of smooth \( c \)-tracefree sections of \( \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n} \) which converge distributionally to \( v \), and let \((t_k)\) be a sequence of positive reals such that for all \( m \) we have \( t_k \|v_k\|_{C^m, g} \to 0 \). (For instance, one might choose \( t_k := \frac{1}{k} \|v_k\|_{C^m, g}^{-1} \).) Thus the sequence

\[
c_k := \frac{c + t_k v_k}{\det(c + t_k v_k)^{\frac{1}{n}}}.
\]

of smooth conformal classes \( C^\infty \)-converges to \( c \). Let \((\Omega_k)\) be densities such that the metrics \( \Omega_k^{2/n} c_k \) are volume-1 Yamabe metrics for the classes \((c_k)\).

Since \( c \) attains the Yamabe invariant, we have that for each \( k \),

\[
0 \geq I(c_k) - I(c).
\]
Thus, applying Proposition 9.2.2 to $c$ and $c_k$,

$$0 \geq \frac{I(c_k) - I(c)}{t_k} \geq \int_0^1 \int_M \langle v_k, (-\text{Ric}(g_k,\tau) + \frac{1}{n} R(g_k,\tau)g_k,\tau) \det(c_0 + \tau t_k v) \Omega^{1-2/n} \rangle_{c_k,\tau} d\tau,$$

where we define the 1-parameter families of conformal classes $c_{k,\tau}$ by

$$c_{k,\tau} := \frac{c + \tau t_k v_k}{\det(c + \tau t_k v_k)^{\frac{1}{2}}}$$

and metrics $g_{k,\tau}$ by $\Omega^{2/n} c_{k,\tau}$.

By the compactness result Proposition 8.3.2 and a diagonal argument, we may choose a subsequence $(k_i)$ such that $g_{k_i}$ converges in $C^\infty$ to a volume-1 Yamabe metric, $g$ say, for $c$, with $\Omega_{k_i}$ converging in $C^\infty$ to $|d\text{Vol}_g|$. Therefore also the sequence of fields on $[0, 1] \times M$,

$$(x, \tau) \mapsto \langle \cdot, (-\text{Ric}(g_{k_i,\tau}) + \frac{1}{n} R(g_{k_i,\tau})g_{k_i,\tau}) \det(c_0 + \tau t_{k_i} v)^{-1/n} \Omega_{k_i}^{1-2/n} \rangle_{c_{k_i,\tau}},$$

$C^\infty$-converges to the constant-in-$\tau$ field

$$(x, \tau) \mapsto \langle \cdot, (-\text{Ric}(g) + \frac{1}{n} R(g)g) |d\text{Vol}_g|^{1-2/n} \rangle_c.$$

By construction, the sequence $(v_{k_i})$ of constant-in-$\tau$ fields on $[0, 1] \times M$ converges distributionally to the constant-in-$\tau$ field $v$. Pairing, it follows that

$$\int_M \langle v, (-\text{Ric}(g) + \frac{1}{n} R(g)g) |d\text{Vol}_g|^{1-2/n} \rangle_c = \lim_{i \to \infty} \int_0^1 \int_M \langle v_{k_i}, (-\text{Ric}(g_{k_i,\tau}) + \frac{1}{n} R(g_{k_i,\tau})g_{k_i,\tau}) \det(c_0 + \tau t_{k_i} v)^{-1/n} \Omega_{k_i}^{1-2/n} \rangle_{c_{k_i,\tau}} d\tau \leq 0.$$
9.4 Preliminaries from functional analysis

Theorem 9.4.1 (Hahn-Banach separation theorem [Rud91, Theorem 3.4], [Edw95, Corollary 2.2.3]). Let $X$ be a locally convex topological vector space over $\mathbb{R}$, and $A$ and $B$ nonempty closed convex subsets of $X$, with $B$ compact. If for all functionals $\varphi \in X^*$ we have

$$\sup_{a \in A} \varphi(a) \geq \inf_{b \in B} \varphi(b),$$

then $A \cap B \neq \emptyset$.

Theorem 9.4.2 ([Phe01, Proposition 1.2], [Edw95, Corollary 8.13.3]). Let $Y$ be a Hausdorff locally convex topological vector space over $\mathbb{R}$, and $K \subseteq Y$ a nonempty compact subset. If a point $y_0 \in Y$ is in the closed convex hull of $K$, then there exists a Borel probability measure $\mu$ on $K$, such that

$$y_0 = \int_K y \, d\mu(y).$$

Remarks. (i) The integral is to be taken as a Pettis integral ([AB06, Section 11.10], [Edw95, Section 8.14]). That is, for all functionals $\psi \in Y^*$, $\psi(y_0) = \int_K \psi(y) \, d\mu(y)$.

(ii) By a straightforward argument, the converse is also true.

Proof. Since the closed convex hull of $K$ contains $y_0$, there exist sequences $N_k$ of natural numbers, $(a^k_{\gamma})_{1 \leq \gamma \leq N_k}$ of finite sets of positive reals, and $(y^k_{\gamma})_{1 \leq \gamma \leq N_k}$ of finite sets of elements of $K$, such that $1 = \sum_{\gamma=1}^{N_k} a^k_{\gamma} y^k_{\gamma}$ for all $k$ and $y_0 = \lim_{k \to \infty} \sum_{\gamma=1}^{N_k} a^k_{\gamma} y^k_{\gamma}$. Define a sequence of discretely-supported probability measures $(\mu_k)$ on $K$ by $\mu_k := \sum_{\gamma=1}^{N_k} a^k_{\gamma} \delta_{y^k_{\gamma}}$. Thus for all functionals $\psi \in Y^*$, $\psi(y_0) = \lim_{k \to \infty} \int_K \psi(y) d\mu_k(y)$.

Applying the Banach-Alaoglu theorem to the set of Radon measures on $K$, there exists a subsequence $(\mu_{k_i})$ which weak-*-converges to some Borel probability measure $\mu$ on $K$. For all functionals $\psi \in Y^*$, $\psi(y_0) = \lim_{i \to \infty} \int_K \psi(y) d\mu_{k_i}(y) = \int_K \psi(y) \, d\mu(y)$. 

\[\Box\]

9.5 Proof of Theorem 1.2.1

Let $M$ be a compact smooth $n$-manifold, and $c$ a conformal equivalence class on $M$.

Lemma 9.5.1. The conformal equivalence class $c$ induces canonical pairings

$$\langle \cdot, \cdot \rangle : \Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^*) \times \mathcal{C}^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-4/n-n}) \to \mathbb{R},$$

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(Γ denoting distributional sections) such that the induced maps

\[ \Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^s) \to (C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-4/n-s}))^* \]

are isomorphisms of topological vector spaces.

**Proof.** Pick a metric \( g \in c \). The induced pairing \( \langle v, w \rangle = \int g(v, w) |dVol_g|^{4/n} \) is easily checked to be independent of the choice of \( g \in c \). \( \square \)

Suppose henceforth that the conformal class \( c \) attains \( Y(M) \). Denote by \( F \subseteq C^\infty(\text{Sym}^2(T^*M)) \) the set of volume-1 Yamabe metrics in the conformal class \( c \).

Define a function

\[ Q : c \to C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n}) \]

on the set of metrics in the conformal class \( c \), by

\[ Q(g) := \left( \text{Ric}(g) - \frac{1}{n} \text{R}(g) g \right) |dVol_g|^{1-2/n}. \]

**Proposition 9.5.2.** The closed convex hull of the set \( Q(F) \) contains \( 0 \).

**Proof.** Let \( A \) be the closed convex hull of the set \( Q(F) \). The topological vector space of sections

\[ C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n}) \]

is Fréchet and therefore locally convex. Thus it suffices to verify the hypothesis of Theorem 9.4.1. Indeed, let \( v \) be in

\[ \Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}) \cong (C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n}))^* \]

(the isomorphism is by Lemma 9.5.1). By Proposition 9.3.1, there exists (rescaling if necessary) a metric \( g \in F \) such that \( \langle v, Q(g) \rangle \geq 0 \). Since \( Q(g) \in A \), it follows that indeed \( \sup_{a \in A} \langle v, a \rangle \geq 0 = \langle v, 0 \rangle \). \( \square \)

The set \( F \), the set of volume-1 Yamabe metrics in the conformal class \( c \), is compact by Proposition 8.3.2. So \( Q(F) \) is also compact.

**Proposition 9.5.3.** There exists a Borel probability measure \( \mu \) on \( Q(F) \), such that \( 0 = \int_{Q(F)} a \, d\mu(a) \).

**Proof.** Combine Proposition 9.5.2 with Theorem 9.4.2. \( \square \)
Theorem 1.2.1 now follows by choosing an arbitrary Borel measure $\mu$ on $F$ such that $Q_* \mu = \pi$.

9.6 Connections with other geometric minimax problems

Theorem 1.2.1 was inspired by two closely analogous results in other geometric minimax contexts. Nadirashvili [Nad96] studies Riemannian metrics $g$ on a compact smooth manifold $M$ which maximize the weighted first Dirichlet eigenvalue $\lambda_1(g) \text{Vol}(M, g)^{2/n}$, where

$$\lambda_1(g) = \inf_{\{u \in C^\infty(M) : \int_M u^2 d\text{Vol}_g = 1\}} \frac{\int_M |\nabla u|^2 d\text{Vol}_g}{\int_M u^2 d\text{Vol}_g}.$$ 

Among other things, he observes (Theorem 5) that such metrics are $\lambda_1$-minimal; that is, that there exist a set of first Dirichlet eigenfunctions for $g$, such that the map of $M$ into Euclidean space defined by those eigenfunctions is an isometric embedding as a minimal submanifold of the unit sphere. The key point of the proof is to find a finite set $(u_i)$ of first eigenfunctions, such that

$$g = \sum_i \left[ (du_i)^{\otimes 2} + \frac{1}{4} \Delta_g(u_i^2) g \right].$$

The central identity is a finite sum, rather than (as in Theorem 1.2.1) an integral, because of a polarization argument that works for linear equations (as here) but not for nonlinear equations (such as the Yamabe problem).

Fraser and Schoen [FS12] study Riemannian metrics $g$ on a compact smooth manifold with boundary $M$ which maximize the weighted first Steklov eigenvalue $\sigma_1(g) \text{Vol}(\partial M, g|_{\partial M})^{1/(n-1)}$, where

$$\sigma_1(g) = \inf_{\{u \in C^0(M) : \int_{\partial M} u^2 d\text{Vol}_g|_{\partial M} = 1\}} \frac{\int_M |\nabla u|^2 d\text{Vol}_g}{\int_{\partial M} u^2 d\text{Vol}_g|_{\partial M}}.$$ 

Among other things, they observe (Proposition 5.2) that such a metric $g$ has a set of first Steklov eigenfunctions, such that the map of $M$ into Euclidean space defined by those eigenfunctions is a conformal embedding as a minimal submanifold of the unit ball, isometric on $\partial M$. The key point of the proof is to find a finite
set \( \{u_i\} \) of first eigenfunctions, such that

\[
0 = \sum_i \left[ (du_i)^{\otimes 2} - \frac{1}{2} |du_i|^2 g \right], \quad \text{on } M,
\]

\[
1 = \sum_i u_i^2, \quad \text{on } \partial M.
\]

Our argument, particularly in Sections 9.3 and 9.5, closely follows Fraser and Schoen’s.

### 9.7 Connection to the problem of existence of Einstein metrics

The interest of Theorem 1.2.1 and Corollary 1.2.2 lies in their connection to a subtle and very appealing open problem:

**Question 9.7.1.** *Do all conformal classes attaining the Yamabe invariant contain an Einstein metric?*

This question is motivated by the examples of manifolds for which the Yamabe invariant has to date been computed, which fall into two classes:

- **Einstein manifolds** \((M, g)\) for which it has been proved that \([g]\), the conformal class of the Einstein metric, attains \(Y(M)\); these include
  - \(S^n\) ([Aub76c]; see [LP87, Section 3] for an exposition);
  - manifolds admitting flat metrics ([SY79, GL83]; see [Sch89, Proposition 1.3] for an exposition);
  - general-type and Calabi-Yau complex surfaces [LeB96, LeB99];
  - \(\mathbb{CP}^2\) [LeB97];
  - \(\mathbb{RP}^3\) [BN04];

- **manifolds** \(M\) for which the existence of a conformal class attaining \(Y(M)\) is unclear, or for which it has been proved that no conformal class attains \(Y(M)\); these include
  - \(S^1 \times S^{n-1}\) and connect sums thereof [Kob87];
  - \(M \# k \mathbb{CP}^2\), where \(M\) is a complex surface of general type [LeB96];
  - \(M \# |S^1 \times S^3|\), where \(M\) is a 4-manifold with \(Y(M) \leq 0\) [Pet98];
  - \(\mathbb{RP}^3 \# k(S^1 \times \mathbb{RP}^2)\) [BN04];
as well as by the observation that the Yamabe invariant is a minimax quantity associated with the total scalar curvature functional
\[ g \mapsto \frac{\int_M R(g) \, dVol_g}{(\int_M dVol_g)^{1-2/n}}, \]
of which Einstein metrics are the critical points. If Question 9.7.1 were answered in the affirmative, it could perhaps be possible to find Einstein metrics on new manifolds by direct variational methods, by maximizing the functional \( I \).

It is our hope that Theorem 1.2.1 may eventually be used to answer Question 9.7.1 in the affirmative, at least when \( \mathcal{F} \) is finite. In the special cases \(|\mathcal{F}| = 1\) and \(|\mathcal{F}| = 2\), such arguments already exist. The case \(|\mathcal{F}| = 1\) was discussed in Chapter 1. The case \(|\mathcal{F}| = 2\) is due to Anderson, and we present here an reformulated version of Anderson’s argument, which we hope may admit a generalization to the case \( |\mathcal{F}| > 2 \).

**Theorem 9.7.2** ([And05, Theorem 1.2]). *Suppose that the conformal class \( c \) attains \( Y(M) \), and suppose that there are at most two (modulo rescaling) Yamabe metrics in \( c \). Then in fact there is exactly one Yamabe metric, and that metric is Einstein.*

**Proof.** The crux of the proof is the following Bochner-type identity for a pair \( \{v_1^{-2}g, v_2^{-2}g\} \) of conformally equivalent metrics:
\[ \text{div}_g(\Theta_g(v_1, v_2)) = \langle R_g(v_1, v_2), T_g(v_1, v_2) \rangle - f(v_1, v_2) \left[ \sum_i \| \text{Ric}^0(v_i^{-2}g) \|_g^2 \right]. \] (9.1)

Here we use the following notation:

- superscript-0 for “tracefree part” of a symmetric 2-tensor

- \( \Theta_g(v_1, v_2) \) for the symmetrization of
  \[ (n - 2)\langle [v_1 \nabla_g v_2 - v_2 \nabla_g v_1], v_1^{2-n} \text{Ric}^0(v_1^{-2}g) \rangle \]
  (an \( (n - 1) \)-form);

- \( R_g(v_1, v_2) \) for the symmetrization of \( v_1^{2-n} \text{Ric}^0(v_1^{-2}g) \) (a symmetric 2-covector);

- \( T_g(v_1, v_2) \) for the symmetrization of \( \langle \cdot, v_1 v_2 \text{Ric}^0(v_1^{-2}g) \rangle_g \) (a symmetric 2-vector);

- \( f(v_1, v_2) := v_1^{3-n} v_2 + v_1 v_2^{3-n} \) (a function);
Suppose that the conformal class \([g]\) maximizes the Yamabe invariant, and that there are, up to rescaling, only two Yamabe metrics in \([g]\): let us say, \(\{v_1^{-2}g, v_2^{-2}g\}\). Applying Theorem 1.2.1 and rescaling the metrics if necessary, we have that \(0 = \sum_{i=1}^{2} v_i^{2-n} \text{Ric}^0(v_i^{-2}g) = R_g(v_1, v_2)\). Inserting this in (9.1) and applying Stokes’ theorem,

\[
0 = \int_M f(v_1, v_2) \left[ \sum_i \| \text{Ric}^0(v_i^{-2}g) \|_{g_i}^2 \right] \, d\text{Vol}_g.
\]

So, for each \(i\), \(\text{Ric}^0(v_i^{-2}g) \equiv 0\). This is the desired result.

It remains to prove the identity (9.1). We recall the formula for the change in Ricci curvature under a conformal change in metric:

\[
v^{2-n} \text{Ric}^0(v^{-2}g) = v^{2-n} \text{Ric}^0(g) + (n-2)v^{-1} \text{Hess}^0_g(v).
\]

Let \(\Omega_g\) be the section of \((TM)^{\otimes 3} \otimes \Lambda^{n-1} T^* M\) given by \((\Omega_g)^{ijk}_l = g^{ik} g^{jm} (d\text{Vol}_g)_m l\); it varies tensorially with \(g\), and \(\Omega_{v^{-2}g} = v^{4-n} \Omega_g\). We have, by the Bianchi identity for \(v_1^{-2}g\),

\[
v_1^{4-n} \langle \text{Hess}^0_{v_1^{-2}g}(v_2/v_1), \text{Ric}^0(v_1^{-2}g) \rangle \, d\text{Vol}_g
\]

\[
= \langle \text{Hess}_{v_1^{-2}g}(v_2/v_1), \text{Ric}^0(v_1^{-2}g) \rangle \, d\text{Vol}_{v_1^{-2}g}
\]

\[
= \text{div}_{v_1^{-2}g}(d(v_2/v_1), \text{Ric}^0(v_1^{-2}g)) \, d\text{Vol}_{v_1^{-2}g}
\]

\[
= d \left[ \langle d(v_2/v_1) \otimes \text{Ric}^0(v_1^{-2}g), \Omega_{v_1^{-2}g} \rangle \right]
\]

\[
= d \left[ (v_1^{-2}[v_1 dv_2 - v_2 dv_1] \otimes \text{Ric}^0(v_1^{-2}g), v_1^{4-n} \Omega_g) \right]
\]

\[
= \text{div}_g \left[ ([v_1 \nabla_g v_2 - v_2 \nabla_g v_1], v_1^{2-n} \text{Ric}^0(v_1^{-2}g)) \right] \, d\text{Vol}_g.
\]

Also

\[
(n-2) \text{Hess}^0_{v_1^{-2}g}(v_2/v_1) = (v_2/v_1)^{3-n} [\text{Ric}^0(v_2^{-2}g) - \text{Ric}^0(v_1^{-2}g)]
\]

\[
v_1^{n-3} v_2 [v_2^{2-n} \text{Ric}^0(v_2^{-2}g)] - (v_2/v_1)^{3-n} \text{Ric}^0(v_1^{-2}g)
\]

\[
v_1^{n-3} v_2 [v_1^{2-n} \text{Ric}^0(v_1^{-2}g)] + v_2^{2-n} \text{Ric}^0(v_2^{-2}g)
\]

\[
- [(v_2/v_1) + (v_2/v_1)^{3-n}] \text{Ric}^0(v_1^{-2}g).
\]
Combining,

\[
(n - 2) \text{div}_g \left[ \left[ (v_1 \nabla_g v_2 - v_2 \nabla_g v_1), v_1^{2-n} \text{Ric}^0(v_1^{-2}g) \right] \right]
\]

\[
= v_1^{4-n} \langle (n - 2) \text{Hess}_{v_1^{-2}g}(v_2/v_1), \text{Ric}^0(v_1^{-2}g) \rangle_g
\]

\[
= v_1^{4-n} v_1^{n-3} v_2 \left[ v_1^{2-n} \text{Ric}^0(v_1^{-2}g) + v_2^{2-n} \text{Ric}^0(v_2^{-2}g) \right], \text{Ric}^0(v_1^{-2}g) \rangle_g
\]

\[
- v_1^{4-n} [(v_2/v_1) + (v_2/v_1)^3 - n] \langle \text{Ric}^0(v_1^{-2}g), \text{Ric}^0(v_1^{-2}g) \rangle_g
\]

\[
= \langle [v_1^{2-n} \text{Ric}^0(v_1^{-2}g) + v_2^{2-n} \text{Ric}^0(v_2^{-2}g)], v_1 v_2 \text{Ric}^0(v_1^{-2}g) \rangle_g
\]

\[
- \langle [v_1^{3-n} v_2 + v_1 v_2^{3-n}] \text{Ric}^0(v_1^{-2}g) \rangle_g^2
\]

However, let us note a different point of view, from which the available evidence regarding Question 9.7.1 is more equivocal. There is a bidirectional version of Lemma 1.2.3 which rephrases Question 9.7.1 as a question of uniqueness:

**Proposition 9.7.3** ([Oba72], see also [Sch89, Proposition 1.4]). *Let $M$ be a compact manifold other than the sphere, and suppose the conformal class $c$ attains $Y(M)$. Then there is an Einstein metric in $c$ if and only there is a unique (modulo rescaling) Yamabe metric in $c$; the Einstein metric is the Yamabe metric if so.*

In a general (that is, not necessarily maximizing) conformal class, the question of uniqueness (modulo rescaling) of Yamabe metrics is well-studied. Let $\Omega$ be the subset of conformal classes containing a unique Yamabe metric. $\Omega$ is large (for instance it contains all conformal classes $c$ such that $I(c) \leq 0$ [Aub70], and it is open and dense [And05]) but its complement need not actually be empty; the simplest counterexample is the round sphere, and many others have been found [Sch89, BP13].

Perhaps, therefore, it may in fact be possible to construct a set of metrics satisfying the algebraic criterion of Theorem 1.2.1, thus answering Question 9.7.1 in the negative.
Bibliography


