ESSAYS ON CONTINUOUS-TIME BARGAINING

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Abstract

This thesis studies bargaining games in continuous time. In chapter 1, I construct a continuous time model to study the problem of a durable good monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. Time-varying costs modify the results on the Coase conjecture. When the distribution of consumer valuations is discrete, the monopolist is able to exercise market power and the outcome is inefficient. In contrast, with a continuous distribution the monopolist is unable to extract additional surplus from buyers with higher valuations. Moreover, the outcome is efficient in this setting.

Chapter 2 introduces a new continuous time bargaining model to investigate bilateral negotiations in settings in which the relative bargaining power of the parties might change over time. The model has a unique equilibrium, in which players reach an immediate agreement. The players’ payoffs are characterized by a system of ordinary differential equations. The equilibrium of the continuous time model corresponds to the limiting subgame perfect equilibrium of a discrete time bargaining game, when players can make offers arbitrarily frequently. The chapter also presents two applications of the baseline model featuring delays and inefficiencies.

Finally, chapter 3 studies legislative negotiations with supermajority requirements within the context of the continuous time bargaining model of chapter 2. The model has two key features. First, there is an exogenous diffusion process representing the parties’ relative political strength, whose realization determines the identity of the party making proposals. Second, the party responding to offers incurs a concession cost $c \geq 0$ whenever it accepts a proposal put forward by its opponent. If $c = 0$, the parties always come to an immediate agreement. On the other hand, if $c > 0$ the equilibrium involves a delay region and an agreement region. When the diffusion process is in the delay region gridlock emerges, and policies are only implemented when this process reaches the agreement region. The model delivers positive implications concerning when legislative inaction is most likely to emerge.
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Chapter 1

Durable Goods Monopoly with Stochastic Costs

1.1 Introduction

Consider the problem of a monopolist who produces a durable good and who cannot commit to a path of prices. For settings in which marginal costs do not change over time, Coase (1972) argued that such a producer would not be able to sell at the static monopoly price. After selling the initial quantity, the monopolist has the temptation to reduce prices to reach consumers with lower valuation. This temptation leads the monopolist to continue cutting prices after each sale. Forward looking consumers expect prices to fall, so they are unwilling to pay a high price. Coase conjectured that these forces would lead the monopolist to post an opening price arbitrarily close to marginal cost. The monopolist would then serve the entire market “in the twinkling of an eye”, and the outcome would be fully efficient. The classic papers on durable goods monopoly (Stokey, 1981 and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist’s opening price converges to the lowest consumer valuation. In the limit, all
consumers trade immediately and the monopolist earns the same profits she would get if all consumers had the lowest valuation (zero “excess profits”).

The purpose of this paper is to study the problem of a durable good monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. The assumption that marginal costs are subject to stochastic shocks is natural in many markets. For instance, costs may vary over time as a consequence of changes in input prices. Time-varying costs may also arise as a result of changes in productivity. An example of this is high-tech consumer goods, whose costs of production typically fall rapidly over time (e.g., Conlon, 2010). Finally, fluctuations in exchange rates will also lead to time-varying costs if the monopolist sells an imported good, or if she uses imported inputs.

Time-varying costs introduce an option value of delaying trade. The efficient outcome in this setting is that the monopolist serves consumers with valuation \( v > 0 \) the first time costs fall below a threshold \( z_v \). This threshold is decreasing in the valuation, so under the optimal outcome the monopolist serves consumers sequentially as costs decrease. Selling to all consumers immediately is therefore inconsistent with efficiency in this setting, so at least one of these features of Coase’s original conjecture cannot hold.

In this paper, I show that the Coase conjecture fails to hold in its entirety when costs are time-varying and the distribution of consumer valuations is discrete. With discrete valuations, the monopolist can truthfully commit to delay trade with low type consumers until costs decrease. This allows the monopolist to extract additional surplus from consumers with higher valuations, enabling her to obtain excess profits. Moreover, the outcome in this setting involves inefficiencies in the form of delayed trade. In contrast, a generalization of Coase’s theorem does hold when the distribution of valuations is continuous. In this case, the monopolist has an incentive to serve the next buyer arbitrarily soon after her last sale. This forces the monopolist’s profits down to what she would earn if all consumers had the lowest valuation (i.e., zero excess profits), and the outcome is fully efficient. Consumers with higher valuations trade earlier, and end up paying higher prices.
Coase’s original arguments illustrate the forces that prevent a monopolist producer of a durable good from exercising market power. The results in this paper show that these forces are more general than what Coase described. In particular, these forces do not rely on serving the entire market immediately, nor on serving every consumer at the same price. To attain efficiency and zero excess profits, it is enough that the monopolist cannot credibly commit to delay trade from one sale to the next.

The model is set up in continuous time and the monopolist’s marginal cost $x_t$ evolves as a diffusion process. Continuous time methods are especially suitable to perform the option value calculations that arise with time-varying costs, allowing me to obtain a tractable characterization of the equilibrium. I show that the monopolist’s profits solve an ordinary differential equation with appropriate boundary conditions. The model delivers simple expressions for the prices at which buyers are willing to trade as a function of costs, allowing the computation of profit margins as a function of costs and the level of market penetration.

To see how time-varying costs modify the results on the Coase conjecture, consider first a setting with two types of consumers: high types, who value the good at $v_H$, and low types, who value the good at $v_L < v_H$. Low type consumers buy when the price is weakly lower than $v_L$. After high types leave the market, the monopolist’s problem is to choose when to sell to the remaining low valuation consumers. When costs do not change over time, it is optimal for the monopolist to sell to low types immediately after selling to high types. This is the force behind the Coase conjecture: high valuation consumers are not willing to pay a high price, since they expect prices to fall rapidly after they buy. Time-varying costs give the monopolist the option value of delaying trade with low type buyers until costs decrease. In this case, the monopolist will only sell to low types when costs fall below a threshold $z_L < v_L$. High valuation consumers know that it will take a non-negligible amount of time for prices to drop to $v_L$ when costs are above $z_L$, so they are willing to pay a higher price. In a sense, time-varying costs provide commitment power to the monopolist.
The equilibrium dynamics with two types of buyers are as follows. If costs are initially larger than a threshold $x_0 > z_L$, the monopolist first sells to all high valuation consumers, and then sells to all low type buyers when costs fall below $z_L$. When costs are initially below a threshold $x_0 < z_L$, the monopolist sells immediately to high and low valuation consumers and the market closes. When costs initially lie between $x_0$ and $\overline{x}_0$, the monopolist gradually sells to high type buyers and continues to do so until costs either fall below $x_t < z_L$ or rise above $\overline{x}_t > z_L$. The cutoffs $x_t$ and $\overline{x}_t$ change over time as the level of market penetration increases. When costs fall below $x_t$, the monopolist sells to all remaining consumers (high and low types) and the market closes. When costs rise above $\overline{x}_t$, the monopolist sells to all remaining high type buyers, and then sells to low types when costs fall below $z_L$.

The intuition for the delayed trade when $x_0 \in (x_0, \overline{x}_0)$ is as follows. High type consumers expect prices to fall rapidly to $v_L$ after they have all left the market when $x_0$ lies in this region. Thus, the monopolist would not be able to charge a price significantly larger than $v_L$ if she were to sell to all high type consumers. However, the monopolist has the option value of waiting and obtaining a larger profit margin in the future. The cutoffs $x_0$ and $\overline{x}_0$ are such that the monopolist gets a larger payoff by waiting than by selling to all high type consumers immediately when $x_0 \in (x_0, \overline{x}_0)$. When $x_0$ lies within this region, the monopolist gradually sells to high type consumers at a price that leaves her indifferent between selling now or waiting and obtaining a larger margin in the future. High valuation consumers are willing to postpone their purchase since they expect prices to fall at a rate that compensates their cost of delay. The cutoffs $x_t$ and $\overline{x}_t$ change over time, since the gains from delaying trade change as the number of high type consumers in the market decreases.

The equilibrium outcome is inefficient when costs initially lie between $x_0$ and $\overline{x}_0$. The monopolist serves high type consumers sequentially in this case, but the first-best outcome is that all high valuation consumers trade immediately. In addition, the level of market penetration at each moment in time $s > 0$ depends upon the entire history of costs when $x_0$ lies in this region. Since prices are a function of costs and market penetration, the prices
that the monopolist charges also display history dependence. Finally, the monopolist is able to obtain excess profits, since time-varying costs allow her to extract additional surplus from high valuation consumers. These results generalize to settings in which the distribution of valuations takes any finite number of values.

I study markets in which the distribution of consumer valuations is continuous by analyzing a sequence of models with discrete valuations that approximate the desired continuous distribution. I show that the equilibrium outcome converges to the efficient outcome as the distribution becomes continuous. In the limit, the monopolist serves consumers sequentially as costs decrease, precisely at the point in time that maximizes total surplus. Moreover, the monopolist’s profits converge to what she would earn if all consumers had the lowest valuation (i.e., zero excess profits).

To see the intuition behind these results, suppose first that the distribution of valuations is discrete, taking values $v_1 < \ldots < v_n$. After consumers with valuation $v_k$ leave the market, the monopolist can truthfully commit to delaying trade with consumers with valuation $v_{k-1}$ until costs decrease. This allows the monopolist to charge $v_k$-consumers a price significantly larger than the price $v_{k-1}$-consumers are willing to pay. This commitment power disappears as the gap between valuations becomes vanishingly small, since now the monopolist will serve $v_{k-1}$-consumers arbitrarily soon after serving consumers with valuation $v_k$. The monopolist is therefore unable to obtain excess profits as the distribution becomes continuous, and the limiting equilibrium outcome is fully efficient.

### 1.1.1 Related literature

The literature on durable goods monopoly has identified different ways in which a monopolist can exercise market power. For instance, a durable good monopolist can ameliorate her lack of commitment by renting her good rather than selling it (Bulow, 1982), or by introducing best-price provisions (Butz, 1990). The Coase conjecture also fails when the monopolist faces capacity constraints (Kahn, 1986, and McAfee and Wiseman, 2008), or when consumers use
non-stationary strategies (Ausubel and Deneckere, 1989 and Sobel, 1991). The current paper identifies a new setting in which a dynamic monopolist can exercise market power. When marginal costs vary over time and the distribution of valuations is discrete, a monopolist producer of a durable good can commit to delaying trade with low valuation consumers until costs decrease. This allows the monopolist to extract more surplus from consumers with higher valuation, enabling her to obtain excess profits.¹

This paper also shares some features with models of bargaining with one-sided incomplete information and one-sided offers (Fudenberg, Levine and Tirole, 1985). Deneckere and Liang (2006) study a bargaining game in which the valuation of the buyer is correlated with the cost of the seller (see also Evans, 1989 and Vincent, 1989). They show that trade occurs via atoms in this setting, with short periods of high probability of agreement followed by long periods of inaction. In the current paper’s model, trade also occurs via atoms when the distribution of types is discrete. For instance, with two types of buyers the monopolist will first sell to all high types whenever costs are initially large, and will then sell to all low types when costs fall below $z_L$.

Fuchs and Skrzypacz (2010) study a one-sided incomplete information bargaining model in which a new trader may arrive according to a Poisson process. When a new trader arrives, the seller runs a second price auction between the two potential buyers. Fuchs and Skrzypacz (2010) show that a generalization of the Coase conjecture holds in this setting: the seller’s inability to commit to a path of offers drives her profits down to her outside option of waiting for the arrival of a new buyer. Moreover, the possibility of arrivals leads to inefficient delays, with the seller slowly screening out high type buyers. In the current paper, the monopolist is also unable to obtain excess profits when the distribution of valuations is discrete.

¹Other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment pricing path of a durable good monopolist when costs evolve deterministically over time. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Biehl (2001) studies a setting in which the buyers’ valuations are subject to shocks.
continuous. However, the equilibrium outcome is fully efficient, with the seller serving the
different consumers exactly at the point in time that maximizes total surplus.

Finally, this paper adds to the growing literature that uses continuous time methods to
analyze strategic interactions. The analysis of games in continuous time presents technical
difficulties. First, there are measurability problems related to the fact that players can con-
dition their actions on “instantaneous” events (e.g., Simon and Stinchcombe, 1989). Second,
subgame perfection has less bite when the monopolist can change her price in continuous
time, leading to a multiplicity of equilibria. The reason for this is that consumers do not
face a cost of delay after rejecting a price when the game is in continuous time, since they
can always accept a new price within the next instant. Following the recent literature on
continuous time games (e.g., Sannikov 2007, 2008), I deal with the first issue by imposing
measurability conditions on strategies that guarantee that outcomes and payoffs are well
defined. I deal with the second issue by imposing intuitive conditions on strategies that
resemble the conditions that would necessarily arise in a subgame perfect equilibrium of a
discrete time durable good monopoly game.

1.2 Model

A monopolist faces a unit measure of non-atomic consumers indexed by \( i \in [0, 1] \). Consumers
are in the market to buy one unit of the monopolist’s good. Time is continuous, and
consumers can make their purchase at any time \( t \in [0, \infty) \). The valuations of the consumers
are defined by the non-increasing and left-continuous function \( f : [0, 1] \rightarrow [\underline{v}, \overline{v}] \) with \( \overline{v} > \underline{v} > 0 \); consumer \( i \) has valuation \( f(i) \). Consumers and the monopolist are risk-neutral
expected utility maximizers and discount future payoffs at rate \( r > 0 \). I assume that \( f \) is
a step function taking \( n \) values \( v_1, \ldots, v_n \), with \( 0 < v_1 < v_2 \ldots < v_n \). For \( k = 1, \ldots, n \), let

\[ v_k = \begin{cases} v_1 & \text{if } k = 1 \\ \frac{v_{k+1} - v_k}{n-k} & \text{if } 2 \leq k < n \\ v_n & \text{if } k = n \end{cases} \]

For instance, continuous time methods have been used to study the provision of incentives
in dynamic settings (Sannikov 2007, 2008), political campaigns (Gul and Pesendorfer, 2011)
and dynamic markets for lemons (Daley and Green, 2011).
\[ \alpha_k = \max\{i \in [0, 1] : f(i) = v_k\} \] That is, \( \alpha_k \) is the highest indexed consumer with valuation \( v_k \). Section 6 considers the case in which \( f \) approximates a continuous function \( h \).

Let \( B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\} \) be a one-dimensional Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\).\(^3\) The Brownian motion \( B \) drives the monopolist’s marginal cost \( x_t \),

\[ dx_t = \mu x_t dt + \sigma x_t B_t, \tag{1.1} \]

with \( x_0 = x > 0, \sigma > 0 \) and \(|\mu| < r\). At time \( t \) the monopolist can produce any desired quantity at marginal cost \( x_t \). The assumption that \(|\mu| < r\) guarantees that the monopolist will always produce on demand: under this condition it is never optimal for the monopolist to produce when costs are low to sell in the future when costs are high.\(^4\) The constants \( \mu \) and \( \sigma \) measure the expected rate of change of \( x_t \) and the volatility of \( x_t \), respectively. The process \( x_t \) is publicly observable and its underlying structure is common knowledge: monopolist and consumers commonly know that \( x_t \) evolves as (1.1). The assumption that \( x_t \) evolves as (1.1) is for convenience. The main results in this paper continue to hold if \( x_t \) follows a more general diffusion process (see Section 7).

A (stationary) strategy for consumer \( i \in [0, 1] \) is a function \( P : \mathbb{R}_+ \to \mathbb{R}_+ \) that describes the maximum price that \( i \) is willing to pay for the good given any level of marginal costs. Suppose consumer \( i \) is still in the market at time \( t \). Then, under strategy \( P(\cdot, i) \) consumer \( i \) purchases the good at time \( t \) if and only if the price that the monopolist charges is weakly lower than \( P(x_t) \).

Let \( P = P(x,i) \) be a strategy profile for the consumers, with \( P(\cdot, i) \) denoting the strategy of consumer \( i \in [0, 1] \). In equilibrium, the strategy profile of the consumers must satisfy the skimming property: for all \( i < j, P(x,i) \geq P(x,j) \) for all \( x \). That is, consumers with higher valuation are willing to pay higher prices. The reason for this is that it is more costly for

\(^3\)The filtration \( \{\mathcal{F}^B_t : 0 \leq t < \infty\} \) is assumed to include all sets of measure zero, and is therefore right-continuous: for every \( t \geq 0, \mathcal{F}^B_t = \cap_{s \geq 0} \mathcal{F}^B_{t+s} \).

\(^4\)This assumption also guarantees that the stopping problems in equation (6) have a finite solution.
consumers with higher valuation to delay their purchase: if consumers with valuation \( v_k \) find it weakly optimal to purchase at some time \( t \) given a future path of prices, then consumers with valuation \( v_{k'} > v_k \) will find it strictly optimal to purchase at time \( t \). I will restrict attention to strategy profiles such that \( P(x,i) \) is left-continuous in \( i \) and continuous in \( x \).

The skimming property implies that at any time \( t \) there exists \( a_t \in [0,1] \) such that consumers \( i \leq a_t \) have already left the market, while consumers \( i > a_t \) are still in the market. The cutoff \( a_t \) describes the level of market penetration at time \( t \). At each time \( t \), the level of market penetration \( a_t \) and the monopolist’s marginal cost \( x_t \) describe the payoff relevant state of the game.

Given a strategy profile \( P \), the problem of the monopolist is to choose a path of prices to maximize her profits. Since \( P \) satisfies the skimming property, by setting a price \( p \) the monopolist effectively chooses the level of market penetration: if the monopolist sets price \( p \) at time \( t \), there will be an \( a \in [0,1] \) such that \( P(x_t,i) \geq p \) if and only if \( i \leq a \). Moreover, the monopolist will charge \( P(x_t,a) \) if consumer \( a \) is the marginal buyer at time \( t \). Thus, I can alternatively specify the monopolist’s problem as choosing a non-decreasing process \( \{a_t\} \) with \( a_0 = 0 \) and \( a_t \leq 1 \) for all \( t \), describing the level of market penetration at any time \( t \). With this specification, under strategy \( \{a_t\} \) the monopolist charges price \( P(x_t,a_t) \) at every time \( t \), and at this price all consumers \( i \leq a_t \) who are still in the market buy.

**Remark 1.1** With this specification, the process \( \{a_t\} \) must satisfy the following condition: suppose \( P(x,i) = p \) for all \( i \in [l,h] \subseteq [0,1] \) and the monopolist chooses a strategy \( \{a_t\} \) such that \( da_t > 0 \) when \( a_{t-} \in (l,h) \) and \( x_t = x \) (i.e., the monopolist makes some sales at state \((a,x)\) with \( a \in (l,h)\)). Then, in this case it must be that \( da_t \geq h - a_{t-} : \) in order to sell at time \( t \) with \( a_{t-} \in (l,h) \) and \( x_t = x \) the monopolist has to set a price of at most \( P(x,a_{t-}) \); and at this price all consumers \( i \in [a_{t-},h] \) will buy the good. Thus, in this case the level of market penetration \( \{a_t\} \) jumps at time \( t \).

**Monopolist’s problem:** Given a strategy profile \( P \) of the consumers, a strategy for the seller is an \( \mathcal{F}_t \)-progressively measurable process \( \{a_t\} \) satisfying the conditions in Remark 1 such
that \(a_0 = 0\), \(a_t\) is non-decreasing with \(a_t \leq 1\) for all \(t\), and \(\{a_t\}\) is right-continuous with left-hand limits.\(^5\) Let \(A^P\) denote the set of all such processes. Given a strategy profile \(P\) of the consumers and a strategy \(\{a_t\} \in A^P\), the monopolist’s discounted profits are\(^6\)

\[
\Pi = E \left[ \int_{[0,\infty]} e^{-rt} (P(x_t, a_t) - x_t) \, da_t \right]. \tag{1.2}
\]

Let \(\Pi(x, a)\) denote the monopolist’s future discounted profits conditional on the current state being \((x, a)\), and let \(A^P_{a,t}\) denote the set of processes in \(A^P\) such that \(a_t = a\). Then, the monopolist’s payoffs conditional on state \((x_t, a_t)\) are

\[
\Pi(x_t, a_t) = \sup_{\{a_t\} \in A^P_{a,t}} E \left[ \int_{(t, \infty)} e^{-r(s-t)} (P(x_s, a_s) - x_s) \, da_s \bigg| x_t, a_t \right]. \tag{1.3}
\]

Condition (1.3) is the requirement that the monopolist’s strategy \(\{a_t\}\) is subgame perfect (i.e., time-consistent), since \(\{a_t\}\) must be optimal at every state \((x_t, a_t)\).

_**Consumer’s problem:** Given a strategy of the monopolist \(\{a_t\}\) and a strategy profile \(P\) of the consumers, the path of prices is \(\{P(x_t, a_t)\}\). The strategy \(P(x, i)\) of each consumer \(i\) must be optimal given the path of prices \(\{P(x_t, a_t)\}\): the payoff that consumer \(i\) gets from buying at the time strategy \(P(x, i)\) tells her to buy must be weakly larger than what she would get from purchasing at any other point in time.

I impose two additional conditions on the consumers’ strategies. First,

\[
\forall i \text{ such that } f(i) = v_1, P(x; i) = v_1 \text{ for all } x. \tag{1.4}
\]

---

\(^5\)These requirements on \(\{a_t\}\) together with the continuity requirements on \(P(x, i)\) guarantee that the integrals in (2) and (3) are well-defined.

\(^6\)Given the discontinuities in \(\{a_t\}\), I use set notation in the integrals to avoid ambiguities: \(\int_{[s, T]} f(a_t) \, da_t\) denotes the integral between time \(s\) and \(T\), whereas \(\int_{[s, T]} f(a_t) \, da_t\) denotes the integral between \(s^-\) and \(T\).
In words, all consumers with the lowest valuation are willing to pay a price equal to their valuation. The second condition I impose is as follows. Fix a strategy profile \((P, \{a_t\})\).

Recall that for \(k = 1, \ldots, n\), \(\alpha_k\) is the highest indexed consumer with valuation \(v_k\). For \(k = 1, \ldots, n\), let \(\tau(k)\) denote the (possibly random) time at which the monopolist starts selling to consumers with valuation \(v_k\), i.e., \(\tau(k) = \inf\{t : a_t > \alpha_{k-1}\}\). Then, for \(k = 2, \ldots, n\),

\[
v_k - P(x_t, \alpha_k) = E\left[e^{-\tau(k-1)-t} \left(v_k - P(x_{\tau(k-1)}, a_{\tau(k-1)})\right)\right| x_t, \alpha_k].
\]

Equation (1.5) is an incentive compatibility condition stating that the price consumer \(\alpha_k\) is willing to pay must leave her indifferent between buying at that price or waiting and buying at the price at which consumers with valuation \(v_{k-1}\) start buying.

**Definition 1.1** A strategy profile \((P, \{a_t\})\) is an equilibrium if:

(i) \(\{a_t\}\) is optimal for all \((x_t, a_t)\) given \(P\),

(ii) For each \(i\), \(P(x, i)\) is optimal given \(\{a_t\}\) and \(P\), and

(iii) \(P\) satisfies conditions (1.4) and (1.5), given \(\{a_t\}\).

On games played in continuous time: The analysis of games in continuous time presents technical difficulties. First, there are measurability problems related to the fact that players can condition their actions on “instantaneous” events (e.g., Simon and Stinchcombe, 1989). In this paper, I deal with these issues by restricting consumers to use stationary strategies and by restricting the strategy \(\{a_t\}\) of the monopolist to be \(\mathcal{F}_t\)-progressively measurable. These restrictions guarantee that payoffs and outcomes are well-defined.

Second, in continuous time durable good monopoly games the notion of subgame perfection has less bite, leading to a multiplicity of equilibria. The reason for this is that consumers do not face a cost of delay after they reject a price when the monopolist can change prices in continuous time, since they can always accept a new price within the next instant. To see
this, suppose that the game I described so far was in discrete time. In that case, one can easily show that the following two conditions would hold in any SPE: (a) the monopolist would never charge a price below the lowest consumer valuation \( v_1 > 0 \) (so \( v_1 \)-consumers would always accept a price of \( v_1 \)), and (b) the price that the last buyer with valuation \( v_k \) is willing to pay leaves her indifferent between trading at that price or delaying trade until the purchase of the next buyer.

In this paper, I directly impose these conditions in the definition of equilibrium; see condition (iii) in Definition 1. When players can take actions in continuous time, there are equilibria that don’t satisfy these conditions. For example, in continuous time the strategy profile in which the monopolist always charges a price equal to marginal cost and in which consumers choose optimally the time at which to buy (given that prices will always be equal to \( x_t \)) satisfies conditions (i) and (ii) in Definition 1. Under this strategy profile, consumers always reject prices higher than \( x_t \) because they expect the monopolist to charge a lower price within the next instant. Against this strategy of the consumers, the monopolist can do no better than to charge a price equal to \( x_t \) at all times. Conditions (4) and (5) should be seen as a refinement, which rule out non-intuitive equilibria (like the one in which the monopolist always sets price equal to marginal cost) that violate them.

### 1.3 First-best outcome

This section computes the first-best outcome. Recall that the function \( f : [0, 1] \rightarrow [v, \bar{v}] \) describing the valuation of the consumers is a step function taking \( n \) values \( v_1 < v_2 < \ldots < v_n \). To compute the efficient outcome, consider first the problem of choosing the surplus maximizing time at which to serve a homogeneous group of consumers with valuation \( v_k \),

\[
V_k (x) = \sup_{\tau \in T} E \left[ e^{-\tau \tau} (v_k - x_\tau) \mid x_0 = x \right],
\]  
\[ (1.6) \]
where $T$ is the set of stopping times. Let $\lambda$ be the negative root of $\frac{1}{2}\sigma^2y(y-1)+\mu y = r$, and for $k = 1, ..., n$ let $z_k := \frac{-\lambda}{1-\lambda}v_k$.

**Lemma 1.1** The stopping time $\tau_k = \inf \{ t : x_t \leq z_k \}$ solves (1.6). Moreover,

$$
V_k(x) = \begin{cases} 
(v_k - z_k)(\frac{x}{z_k})^\lambda & x > z_k, \\
v_k - x & x \leq z_k.
\end{cases}
$$

**Proof:** See Appendix A.1.1.

Lemma 1.1 captures the option value that arises when the monopolist’s costs vary over time. The total surplus from serving consumers with valuation $v_k$ is maximized by waiting until costs fall below $z_k$. One can show that $\partial z_k/\partial \mu > 0$ and $\partial z_k/\partial \sigma < 0$, so that it is optimal to wait longer when costs fall faster or when they are more volatile. By Lemma 1.1, the first-best outcome is that the monopolist serves consumers with valuation $v_k$ at time $\tau_k$. When $x_0 > z_n$, under the optimal outcome the monopolist serves consumers with valuation $v_n$ the first time $x_t = z_n$. After that, the monopolists serves to consumers with valuation $v_{n-1}$ the first time $x_t = z_{n-1}$, and so on. On the other hand, when $x_0 < z_n$ the optimal outcome is that the monopolist sells immediately to all consumers whose valuation $v_k$ is such that $x_0 \leq z_k$. After this initial sale, the monopolist sells to the remaining groups of consumers sequentially as costs decrease.
1.4 Markets with two-types of consumers

1.4.1 Equilibrium

In this section, I characterize the equilibrium dynamics for markets with two types of consumers. That is, I consider the case in which

\[ f(i) = \begin{cases} 
  v_2 & i \in [0, \alpha], \\
  v_1 & i \in (\alpha, 1], 
\end{cases} \]

with \( v_2 > v_1 > 0 \) and \( \alpha \in (0, 1) \).

By equation (1.4), consumers with valuation \( v_1 \) will only buy when the price equals \( v_1 \). That is, \( \forall i \in (\alpha, 1], P(x,i) = v_1 \) for all \( x \). Let \( \Pi(x,\alpha) \) denote the monopolist’s profits when the only consumers left in the market are those with valuation \( v_1 \) (i.e., when the level of market penetration is \( \alpha \)). Since all consumers with valuation \( v_1 \) buy at the same instant, at state \( (x,\alpha) \) the problem of the monopolist is to optimally choose the time at which to sell to all consumers remaining in the market: \( \Pi(x,\alpha) = (1 - \alpha) \sup_x E[e^{-rt}(v_1 - x_r)|x_0 = x] \).

By Lemma 1.1, the solution to this problem is \( \tau_1 = \inf\{t : x_t \leq z_1\} \), and

\[ \Pi(x,\alpha) = \begin{cases} 
  (1 - \alpha) (v_1 - z_1) \left( \frac{x}{z_1} \right)^{\lambda} & x > z_1, \\
  (1 - \alpha) (v_1 - x) & x \leq z_1. 
\end{cases} \]  

(1.7)

For future reference, note that \( \Pi(x,\alpha) \in C^1 \) in \( x \).

Consider next the case in which the level of market penetration is \( a \in [0, \alpha) \), so there are \( \alpha - a \) high valuation consumers remaining in the market. To study equilibrium behavior at these states, I proceed in two steps. First, I establish a lower bound \( L(x,a) \) on the monopolist’s payoffs for states \( (x,a) \) with \( a \in [0, \alpha) \). Second, I show that in equilibrium the monopolist’s profits are exactly equal to this lower bound \( L(x,a) \).
Consider the strategy $P(x, \alpha)$ of consumer $\alpha$, the highest indexed consumer with valuation $v_2$. After consumer $\alpha$ buys and leaves the market, the monopolist faces only consumers with valuation $v_1$. After consumer $\alpha$ makes her purchase, the monopolist will sell to the remaining low valuation consumers when costs fall below the threshold $z_1$. Therefore, by equation (1.5), $P(x, \alpha)$ must satisfy

\[ P(x, \alpha) = v_2 - E\left[ e^{-r\tau_1} (v_2 - v_1) \mid x_0 = x \right] . \tag{1.8} \]

That is, for all $x > 0$ consumer $\alpha$ must be indifferent between buying at price $P(x, \alpha)$ or waiting until costs fall below $z_1$ and obtaining the good at price $v_1$. Equation (1.8) highlights the commitment power that time-varying costs provide to the monopolist. When $x_t > z_1$, consumer $\alpha$ knows that prices will not fall to $v_1$ until costs fall below $z_1$, so she is willing to pay a price strictly larger than $v_1$ (see Figure 1.1 for a plot of $P(x, \alpha)$).

**Lemma 1.2** $P(x, \alpha) - x > V_1(x)$ for all $x \in (z_1, z_2]$. Moreover,

\[ P(x, \alpha) = \begin{cases} v_2 - (v_2 - v_1) \left( \frac{x}{x_1} \right)^{\lambda} & x > z_1, \\ v_1 & x \leq z_1. \end{cases} \tag{1.9} \]
Proof: See Appendix A.1.1.

Since the strategy profile of consumers satisfies the skimming property, for all $i < \alpha$, $P(x, i) \geq P(x, \alpha)$ for all $x$. This implies that, at any time $t$, the monopolist can sell to all remaining high type buyers at price $P(x_t, \alpha)$. Therefore, for all states $(x, a)$ with $a \in [0, \alpha)$ the monopolist’s profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} E \left[ e^{-r \tau} \left[ (\alpha - a) (P(x_\tau, \alpha) - x_\tau) + \Pi(x_\tau, \alpha) \right] | x_0 = x \right], \quad (1.10)$$

where $P(x, \alpha)$ and $\Pi(x, \alpha)$ are given by (1.7) and (1.9), respectively. That is, at states $(x, a)$ with $a < \alpha$ the monopolist can choose optimally the time $\tau$ at which to sell to the remaining high valuation consumers at price $P(x_\tau, \alpha)$, obtaining profits of $(\alpha - a)(P(x_\tau, \alpha) - x_\tau)$ from these sales plus a continuation payoff of $\Pi(x_\tau, \alpha)$.

Lemma 1.3 For every $a \in [0, \alpha)$, there exists $\underline{x}(a) \in (0, z_1)$ and $\overline{x}(a) \in (z_1, z_2)$ such that $\tau(a) = \inf \{ t : x_t \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2] \}$ solves (1.10). Moreover, $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{a \to \alpha} \underline{x}(a) = \lim_{a \to \alpha} \overline{x}(a) = z_1$.

Proof: See Appendix A.1.2.

To gain intuition behind the solution to (1.10), let $g(x, a) := (\alpha - a)(P(x, \alpha) - x) + \Pi(x, \alpha)$. This implies that $L(x, a) = \sup_{\tau \in T} E[e^{-r \tau} g(x_\tau, a) | x_0 = x]$. Since $P(x, \alpha)$ has a convex kink at $z_1$ (see Figure 1.1) and $\Pi(x, \alpha) \in C^1$, $g(x, a)$ also has a convex kink at $z_1$. Therefore, when $x \in (\underline{x}(a), \overline{x}(a))$ the monopolist can obtain larger profits by delaying trade with high type consumers than by serving all of them at price $P(x, \alpha)$ (see Figure 1.2). The solution to (1.10) also involves delaying when costs are above $z_2$: serving high types is too expensive when $x > z_2$, so in this case it is optimal to wait for costs to fall.

For all $x \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]$, $L(x, a) = g(x, a)$. The proof of Lemma 1.3 shows that

$$r L(x, a) = \mu x L_x(x, a) + \frac{1}{2} \sigma^2 L_{xx}(x, a) \quad \text{for all } x \in (\underline{x}(a), \overline{x}(a)). \quad (1.11)$$
The general solution to (1.11) is $L(x,a) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2}\sigma^2y(y-1) + \mu y = r$, and $A$ and $B$ are constants. There are four unknowns: $A$ and $B$ and the thresholds $\underline{x}(a)$ and $\overline{x}(a)$. The four equations that determine these unknowns are

\begin{align}
L(\underline{x}(a),a) &= g(\underline{x}(a),a), \\
L(\overline{x}(a),a) &= g(\overline{x}(a),a), \\
L_x(\underline{x}(a),a) &= g_x(\underline{x}(a),a), \\
L_x(\overline{x}(a),a) &= g_x(\overline{x}(a),a).
\end{align}

The proof of Lemma 3 shows that there exists a unique solution to this system of equations, with $\underline{x}(a) < z_1 < \overline{x}(a) < z_2$.

The optimal stopping problem (1.10) is defined for all $a \in [0,\alpha)$. That is, for each $a \in [0,\alpha)$ there are cutoffs $\underline{x}(a)$ and $\overline{x}(a)$ such that the solution to (1.10) involves stopping the first time $x_t \in [0,\underline{x}(a)] \cup [\overline{x}(a),z_2]$. Lemma 1.3 shows that $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{a \to \alpha^-} \underline{x}(a) = \lim_{a \to \alpha^-} \overline{x}(a) = z_1$. In words, the delay region $(\underline{x}(a),\overline{x}(a))$ shrinks as $a$ increases, and in the limit as $a$ converges to $\alpha$ it becomes optimal to stop when $x_t \leq z_2$. The reason for this is that the kink that $g(x,a)$ has at $z_1$ gradually disappears as $a$ increases; and therefore delaying when $x$ is around $z_1$ becomes less profitable. Intuitively, the gains from delaying trade decrease when there are fewer high type consumers remaining in the market.
Let \((P, \{a_t\})\) be an equilibrium and let \(\Pi(x, a)\) denote the monopolist’s profit function. An equilibrium \((P, \{a_t\})\) is regular if \(\Pi(x, a)\) is piecewise \(C^{2,1}\).

**Theorem 1.1** There exists a unique regular equilibrium. In this equilibrium, at every state \((x, a)\) with \(a \in [0, \alpha)\) the monopolist’s profits are \(L(x, a)\). Moreover, for all \(t \geq 0\)

(i) if \(x_t > z_2\), the monopolist doesn’t sell (so \(d \alpha_t = 0\)),

(ii) if \(x_t \in [\overline{x}(a_t), z_2]\), the monopolist sells to all remaining high type consumers at price \(P(x_t, \alpha)\) (so \(d \alpha_t = \alpha - \alpha_t\)),

(iii) if \(x_t \leq \overline{x}(a_t)\), the monopolist sells to all remaining consumers (high and low type) at price \(v_1\) (so \(d \alpha_t = 1 - \alpha_t\)),

(iv) while \(x_t \in (\overline{x}(a_t), \overline{x}(a_t))\), the monopolist gradually sells to high type consumers at price \(P(x_t, \alpha_t) = x_t - L_a(x_t, \alpha_t)\) (so \(\alpha_t\) is continuously increasing).

**Proof:** See Appendix A.1.3.

Theorem 1.1 shows that the monopolist’s profits are equal to the lower bound \(L(x, a)\) for every state \((x, a)\) with \(a \in [0, \alpha)\). When \(x_t \in [\overline{x}(a_t), z_2]\), the monopolist sells to all remaining high type buyers at price \(P(x_t, \alpha)\), and then sells to low types when costs drop below \(z_1\). When \(x_t \leq \overline{x}(a_t)\), the monopolist sells to both low and high type consumers at price \(v_1\) and the market closes. When \(x_t > z_2\), the monopolist waits for costs to decrease.

When \(x_t \in (\overline{x}(a_t), \overline{x}(a_t))\), it is never optimal for the monopolist to sell to all remaining high type buyers immediately: by doing this the monopolist earns \(g(x_t, a_t) < L(x_t, a_t)\). On the other hand, it cannot be an equilibrium for the monopolist to wait until \(\tau(a_t) := \inf \{s > t : x_s \notin (\overline{x}(a_t), \overline{x}(a_t))\}\) and sell to all high types at that time. By doing this, the monopolist would earn \(E[e^{-r(\tau(a_t)-t)}(P(x_{\tau(a_t)}, \alpha) - x_{\tau(a_t)}) \mid x_t]\) on each high type consumer. In this case, for all \(x_t \in (\overline{x}(a_t), \overline{x}(a_t))\) the marginal buyer \(a_t^+\) would be willing to buy at a price

\[
P(x_t, a_t^+) = v_2 - E[e^{-r(\tau(a_t)-t)}(v_2 - P(x_{\tau(a_t)}, \alpha)) \mid x_t].
\]
That is, if the monopolist were to delay sales until \( \tau(a_t) \), the marginal buyer \( a_t^+ \) would be willing to pay a price \( P(x_t, a_t^+) \) that leaves her indifferent between buying at that price or waiting until time \( \tau(a_t) \) and buying at price \( P(x_{\tau(a_t)}, a) \). Note then that, for \( x_t \in (\underline{x}(a_t), \bar{x}(a_t)) \),

\[
P(x_t, a_t^+) - x_t - E \left[ e^{-r(\tau(a_t)-t)} \left( P(x_{\tau(a_t)}, a) - x_{\tau(a_t)} \right) \big| x_t \right] = v_2 - x_t - E \left[ e^{-r(\tau(a_t)-t)} (v_2 - x_{\tau(a_t)}) \big| x_t \right] > 0,
\]

where the inequality follows from the fact that \( v_2 - x = \sup_{\tau} E[ e^{-rt} (v_2 - x_{\tau}) | x_0 = x ] \) for all \( x_t \leq z_2 \) (Lemma 1.1). This implies that it cannot be optimal for the monopolist to delay sales until time \( \tau(a_t) \) when \( x_t \in (\underline{x}(a_t), \bar{x}(a_t)) \). Therefore, the monopolist must sell gradually to high types when \( x_t \in (\underline{x}(a_t), \bar{x}(a_t)) \).

I now show how to determine the price that the monopolist charges and the rate at which she sells when \( x_t \in (\underline{x}(a_t), \bar{x}(a_t)) \). Suppose \( x_t \in (\underline{x}(a_t), \bar{x}(a_t)) \) and let \( \tau = \inf \{ s > t : x_s \notin (\underline{x}(a_s), \bar{x}(a_s)) \} \). By Theorem 1.1, \( \{ a_s \} \) is continuously increasing in \( s \) for \( s \in [t, \tau) \), so \( da_s = a_s ds \). At \( t \) the monopolist’s discounted profits (which by Theorem 1.1 are \( L(x_t, a_t) \)) are

\[
L(x_t, a_t) = E \left[ \int_{[t, \tau]} e^{-r(s-t)} \left( P(x_s, a_s) - x_s \right) da_s ds + e^{-r(\tau-t)} L(x_{\tau}, a_{\tau}) \big| x_t, a_t \right].
\]

By the Law of Iterated Expectations, the process

\[
Y_t = \int_{[0, t]} e^{-rs} (P(x_s, a_s) - x_s) da_s + e^{-rt} L(x_t, a_t)
\]

\[
= E \left[ \int_{[0, \tau]} e^{-rs} (P(x_s, a_s) - x_s) da_s + e^{-rt} L(x_{\tau}, a_{\tau}) \big| \mathcal{F}_t \right],
\]

is a continuous martingale for all \( t < \tau \). By the Martingale Representation Theorem (Karatzas and Shreve, page 182), there exists a progressively measurable process \( \beta \in \mathcal{L}^* \)
such that \( dY_t = e^{-rt} \beta_t dB_t \). Differentiating the left-hand side of (1.12) with respect to \( t \) and using the fact that \( dY_t = e^{-rt} \beta_t dB_t \) gives

\[
dY_t = e^{-rt} (P(x_t, a_t) - x_t) \hat{a}_t dt - re^{-rt} L(x_t, a_t) dt + e^{-rt} dL(x_t, a_t) \Rightarrow
\]

\[
dL(x_t, a_t) = (rL(x_t, a_t) - (P(x_t, a_t) - x_t) \hat{a}_t) dt + \beta_t dB_t.
\]

Since \( L(x, a) \in C^{2,2} \) for all \( x \in (\bar{x}(a), \bar{\bar{x}}(a)) \) (Lemma A.1.5), by Ito’s Lemma

\[
dL(x_t, a_t) = \left( \mu x_t L_x(x_t, a_t) + \frac{1}{2} \sigma^2 x_t^2 L_{xx}(x_t, a_t) \right) dt + L_a(x_t, a_t) \hat{a}_t dt + \sigma x_t L_x(x, a) dB_t.
\]

Combining these two equations,

\[
 rL(x_t, a_t) = (P(x_t, a_t) - x_t) \hat{a}_t + L_a(x_t, a_t) \hat{a}_t + \mu x_t L_x(x_t, a_t) + \frac{1}{2} \sigma^2 x_t^2 L_{xx}(x_t, a_t). \tag{1.13}
\]

The left-hand side of (1.13) is the monopolist’s expected flow payoff at state \((x_t, a_t)\), while the right-hand side shows the sources of this flow payoff. The term \((P(x_t, a_t) - x_t)\hat{a}_t\) represents the flow payoff that the monopolist gets from her sales, while the term \(L_a(x_t, a_t)\hat{a}_t\) represents the drop in the monopolist’s continuation payoff due to the fact that consumers are leaving the market at rate \(\hat{a}_t\). Finally, the term \(\mu x_t L_x + \frac{1}{2} \sigma^2 x_t^2 L_{xx}\) gives the change in the monopolist’s continuation payoff due to changes in marginal cost.

Comparing equations (1.13) and (1.11), it follows that

\[
P(x_t, a_t) = x_t - L_a(x_t, a_t), \tag{1.14}
\]

for all \((x_t, a_t)\) such that \(x_t \in (\bar{x}(a_t), \bar{\bar{x}}(a_t))\). That is, the profit margin \(P(x_t, a_t) - x_t\) that the monopolist earns on each sale must be equal to the cost \(-L_a(x_t, a_t)\) that she incurs in terms of a lower continuation payoff. Equation (1.14) has the following interpretation. The

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7 A process \( \beta \) belongs to \( \mathcal{L}^* \) if \( E[\int_0^t \beta_s^2 ds] < \infty \) for all \( t \in [0, \infty) \).
monopolist sells at rate $\dot{a}_t > 0$ when $x_t \in (\underline{x}(a_t), \bar{x}(a_t))$. If $P(x_t, a_t) - x_t > -L_a(x_t, a_t)$, the monopolist could increase her profits by selling at a faster rate. Similarly, if $P(x_t, a_t) - x_t < -L_a(x_t, a_t)$ the monopolist would be better off not selling at all. Therefore, for $\dot{a}_t > 0$ to be optimal, equation (1.14) must hold for all $t$ such that $x_t \in (\underline{x}(a_t), \bar{x}(a_t))$. Since $L(x, a) = E[e^{-r\tau(a)}g(x_{\tau(a)}, a) | x_0 = x]$, it follows that $-L_a(x, a) = E[e^{-r\tau(a)} (P(x_{\tau(a)}, \alpha) - x_{\tau(a)}) | x_0 = x]$. Figure 1.3 plots the price $P(x, a) = x - L_a(x, a)$ that the monopolist charges when $x \in (\underline{x}(a), \bar{x}(a))$, for different values of $a$.

To close the equilibrium, I need to pin down the rate $\dot{a}_t$ at which the monopolist sells to high valuation consumers when $x_t \in (\underline{x}(a_t), \bar{x}(a_t))$. In equilibrium, all high valuation consumers must get the same payoff; otherwise, it would be profitable for a consumer who gets a lower payoff to mimic the strategy of one who is getting a larger payoff. Since the monopolist serves high type buyers sequentially while $x_t \in (\underline{x}(a_t), \bar{x}(a_t))$, prices must evolve in such a way that high type consumers are indifferent between purchasing at any time $s \in [t, \tau]$ (where $\tau = \inf\{s > t : x_s \notin (\underline{x}(a_s), \bar{x}(a_s))\}$). That is, for any $s, u \in [t, \tau]$,
s < u,

\[ v_2 - P(x_s, a_s) = E \left[ e^{-r(u-s)} (v_2 - P(x_u, a_u)) \right] | x_s, a_s ] \Rightarrow \]

\[ e^{-rs} (v_2 - P(x_s, a_s)) = E \left[ e^{-ru} (v_2 - P(x_u, a_u)) \right] | x_s, a_s ] . \quad (1.15) \]

By the Law of Iterated Expectations, \( M_s := E[e^{-ru} (v_2 - P(x_u, a_u)) | x_s, a_s] \) is a continuous martingale. By the Martingale Representation Theorem, there exists a progressively measurable process \( \gamma \in \mathcal{L}^s \) such that \( dM_s = e^{-rs} \gamma_s dB_s \). Differentiating (1.15) with respect to \( s \) and using \( dM_s = e^{-rs} \gamma_s dB_s \), gives

\[ d \left( e^{-rs} (v_2 - P(x_s, a_s)) \right) = -re^{-rs} (v_2 - P(x_s, a_s)) ds - e^{-rs} dP(x_s, a_s) \Rightarrow \]

\[ dP(x_s, a_s) = -r (v_2 - P(x_s, a_s)) ds - \gamma_s dB_s. \quad (1.16) \]

Equation (1.16) shows that (in expectation) prices must fall at rate \( -r(v_2 - P(x_s, a_s)) \) in order to maintain high valuation buyers indifferent. By equation (1.14), \( P(x_s, a_s) = x_s - L_a(x_s, a_s) \) for all \( s \in [t, \tau) \). The proof of Lemma 1.3 shows that \( L(x, a) \in C^{2,2} \) for all \( x \in (\underline{x}(a), \overline{x}(a)) \), so \( P(x, a) \in C^{2,1} \) for all \( x \in (\underline{x}(a), \overline{x}(a)) \). Itô’s Lemma then implies that for all \( s \in [t, \tau] \),

\[ dP(x_s, a_s) = \left( \mu x P_x(x_s, a_s) + \frac{1}{2} \sigma^2 x^2 P_{xx}(x_s, a_s) + a(x_s, a_s) \right) ds \]

\[ + P_x(x_s, a_s) \sigma x dB_s. \]

Combining these two expressions and rearranging gives

\[ \hat{a}_s = \frac{-r (v_2 - P(x_s, a_s)) - \mu x P_x(x_s, a_s) - \frac{1}{2} \sigma^2 x^2 P_{xx}(x_s, a_s)}{P_a(x_s, a_s)} . \]

Finally, the proof of Lemma 1.3 also shows that \( L_a(x, a) \) solves

\[ rL_a(x, a) = \mu x L_{ax}(x, a) + \frac{1}{2} \sigma^2 x^2 L_{axx}(x, a) \quad \text{for all} \quad x \in (\underline{x}(a), \overline{x}(a)). \]
Using this together with equation (1.14) gives

\[
\dot{a}_s = - \frac{r (v_2 - x_s) + \mu x_s}{P_a (x_s, a_s)} = \frac{r (v_2 - x_s) + \mu x_s}{L_{aa} (x_s, a_s)} > 0,
\]

where the inequality follows from the fact that \(L (x, a)\) is strictly convex in \(a\) for all \(x \in (x (a), \bar{x} (a))\) (Lemma A.1.6) and from the fact that \(r (v_2 - x) + \mu x > 0\) for all \(x < z_2\). Equation (1.17) gives the rate at which the monopolist sells while \(x_t \in (\bar{x} (a_t), \bar{x} (a_t))\).

### 1.4.2 Features of the equilibrium

**Failure of the Coase conjecture:** In his classic paper, Coase (1972) conjectured that a durable good monopolist would be post an initial price arbitrarily close to marginal cost. The monopolist would then serve the entire market “in the twinkling of an eye”, and the market outcome would be competitive. The classic papers on durable goods monopoly (Stokey, 1981 and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist’s opening price converges to the lowest consumer valuation. In the limit, all consumers trade immediately and the monopolist earns the same profits she would get if all consumers had the lowest valuation.

Time-varying costs introduce an option value of delaying trade. By Lemma 1.1, the efficient outcome in this setting is that the monopolist serves consumers with valuation \(v_k\) the first time costs fall below \(z_k\). This threshold is decreasing in the valuation, so under the optimal outcome the monopolist serves consumers sequentially as costs decrease. Selling to all consumers immediately is therefore inconsistent with efficiency in this setting, so at least one of these features of Coase’s original conjecture will not hold.

With time-varying costs, the profits a monopolist would earn if all consumers had the lowest valuation \(v_1\) are \(V_1 (x) = \sup_r E [e^{-r} (v_1 - x)] | x_0 = x\). Say that a monopolist producer of a durable good earns zero excess profits if her payoffs are exactly equal to \(V_1 (x)\).

---

\(^8\)One can check that \(rv_2 > z_2 (r - \mu)\) whenever \(|\mu| < r\). Thus, \(rv_2 - x(r - \mu) > 0\) for all \(x < z_2\).
A natural generalization of the Coase conjecture to this paper’s setting is that the monopolist earns zero excess profits, and the equilibrium outcome is fully efficient.

This generalized Coase conjecture fails to hold when there are two types of consumers in the market. First, the equilibrium is inefficient when \( x_0 \in (\underline{x}(0), \bar{x}(0)) \). The monopolist sells to high type consumers at a rate given by (1.17) when costs initially lie within this range, but the efficient outcome is to serve them immediately. When \( x_0 \in (\underline{x}(0), \bar{x}(0)) \), it is never optimal for the monopolist to sell to all remaining high types. Instead, the monopolist serves high type buyers gradually; and these buyers are willing to postpone their purchases since they expect prices to fall at a rate that compensates their cost of delay.

Second, time-varying costs allow the monopolist to obtain excess profits. By Lemma 1.2, \( P(x, \alpha) - x > V_1(x) \) for all \( x \in (z_1, z_2) \), so \( L(x, 0) \geq g(x, 0) > V_1(x) \) for all \( x \in (z_1, z_2) \). The intuition for why the monopolist is able to obtain excess profits is as follows. When marginal costs are fixed, a monopolist lacking commitment power will sell to low type consumers immediately after selling to those with high valuation. This limits the price high valuation buyers are willing to pay, since they expect prices to fall rapidly after they buy. With time-varying costs, the monopolist can truthfully commit to wait and serve low valuation consumers when costs fall below \( z_1 \). If \( x_t \) is large, high types know that it will take a non-negligible amount of time for prices to drop to \( v_1 \), so the monopolist is able to extract more surplus from them.

The monopolist’s profits under full commitment are

\[
\Pi^{FC}(x) = \sup_{\tau} E[e^{-\tau r} \alpha (v_2 - x_\tau) \mid x_0 = x]
\]

when \( \alpha v_2 > v_1 \). That is, a monopolist who can commit to a path of prices would find it optimal to sell only to high types (at a price of \( v_2 \)) when the share of high types is large. High type buyers would be willing to pay a price equal to \( v_2 \) in this case, since the monopolist can
Figure 1.4: Parameters: $\alpha = 0.7$, $v_1 = 0.5$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

commits to keep prices above $v_2$ after they purchase. Figure 1.4 shows that the monopolist obtains a large fraction of the full commitment profits when costs vary over time.

**History dependence:** Suppose $x_0 \in \mathcal{(x(0), \pi(0))}$ and let

$$\tau = \inf\{ t : x_t \notin \mathcal{(x(a_t), \pi(a_t))} \}.$$ 

For all $s \in [0, \tau)$, the rate $\dot{a}_s$ at which the monopolist sells to high valuation consumers at time $s$ depends on the current marginal cost $x_s$ and on the current level of market penetration $a_s$ (see equation 1.17). Therefore, for all $t \in [0, \tau]$ the level of market penetration $a_t = \int_0^t \dot{a}_s ds$ depends upon the entire path of costs from time zero to $t$. This implies that the price $P(x_t, a_t)$ that the monopolist charges at time $t$ depends upon the path of $x_s$ up to time $t$. In other words, the price that the monopolist charges at each instant in time $t \in [0, \tau]$ is not Markovian on $x_t$, but depends upon the entire history of costs.

**Upward sloping demand:** Suppose $x_0 \in \mathcal{(x(0), \pi(0))}$, and again let

$$\tau = \inf\{ t : x_t \notin \mathcal{(x(a_t), \pi(a_t))} \}.$$ 

\footnote{Clearly, this strategy of the monopolist is not time-consistent: after selling to high type buyers, it is in the monopolist’s best interest to sell to low types when costs fall below $z_1$.}
Consider histories in which \( x_\tau = \bar{x}(a_\tau) \). At such histories, at time \( \tau \) all high type buyers remaining in the market buy at a price \( P(\bar{x}(a_\tau), a_\tau) = P(\bar{x}(a_\tau), \alpha) \). Since \( P(x, a) \) is increasing in \( x \), under such histories a mass of consumers buys at a moment in which prices are actually increasing. If we plotted prices and quantities sold after such histories, we would observe that demand is (locally) upward sloping.

**Rate of price changes and costs:** The model in this section predicts that prices fall at a faster rate when costs are lower. For \( x_t \in [\bar{x}(a_t), \bar{z}_2] \), the monopolist charges a price \( P(x_t, \alpha) \). Applying Ito’s Lemma on equation (1.9) gives

\[
\frac{dP(x_t, \alpha)}{dt} = -r(v_2 - P(x_t, \alpha)) dt + \sigma x P_x(x_t, \alpha) dB_t, \tag{1.18}
\]

for all \( x \in [\bar{x}(a), \bar{z}_2] \). Similarly, by equation (1.16) the drift of \( P(x_s, a_s) \) is also \( -r(v_2 - P(x_s, a_s)) \) for all \( x_s \in (\bar{x}(a_s), \bar{z}(a_s)) \). Since \( P(x, a) \) is strictly increasing in \( x \) for all \( x \in (\bar{x}(a), \bar{x}(a)) \) and since \( P(x, \alpha) \) is strictly increasing in \( x \) for \( x \in [z_1, z_2] \), it follows that prices fall (on average) at a faster rate when marginal costs are lower. The intuition behind this is as follows. Prices must evolve in such a way that high type consumers are indifferent between purchasing at any time. High type consumers get a larger payoff from buying when prices are low (i.e., when costs are low). Therefore, when costs are low, prices need to fall faster to compensate high type consumers for their cost of delay.

**Gap vs. no gap:** The literature on the Coase conjecture distinguishes two cases: (i) the case in which there is a positive gap between the lowest consumer valuation and the monopolist’s marginal cost, and (ii) the case in which this gap is zero. With fixed costs and a positive gap, there is a unique equilibrium, which is stationary and satisfies the Coase conjecture (Gul, Sonnenschein and Wilson, 1986). In the no-gap case, there are also non-stationary equilibria in which the monopolist obtains excess profits (Ausubel and Deneckere, 1989).

With fixed costs and a positive gap, the price that the monopolist charges to high type consumers is increasing in the gap. In this paper’s setting, we can think of \( v_1 \) as measuring
the “gap”. Interestingly, with two types of consumers and time-varying costs the price that the monopolist can charge to high type buyers may be increasing or decreasing in the gap. By equation (1.9),

$$\frac{\partial P(x, \alpha)}{\partial v_1} = \frac{1}{v_1} (v_1 (1 - \lambda) + \lambda v_2) \left( \frac{x}{z_1} \right)^\lambda$$

for $x > z_1$.

Thus, $\partial P(x, \alpha)/\partial v_1 < 0$ if and only if $v_1 < \frac{-\lambda}{1-\lambda} v_2 = z_2$. That is, the price that high type buyers are willing to pay is decreasing in $v_1$ for low values of $v_1$, and its increasing in $v_1$ for high values of $v_1$. The price $P(x, \alpha)$ depends on two quantities: the time $\tau_1$ at which the monopolist starts selling to low type consumers, and the price $v_1$ that the monopolist charges at $\tau_1$. An increase in $v_1$ affects both of these quantities: it decreases the stopping time $\tau_1$ and it increases the price $v_1$ the monopolist charges at $\tau_1$. The second effect dominates when $v_1$ is large, so an increase in $v_1$ leads to an increase in $P(x, \alpha)$. In contrast, the first effect dominates when $v_1$ is low, so an increase in $v_1$ reduces $P(x, \alpha)$.

It follows from equation (1.9) that $\lim_{v_1 \to 0} P(x, \alpha) = v_2$ for all $x > 0$. The monopolist will wait an arbitrarily long time to sell to low type buyers when $v_1$ is arbitrarily small. In the limit as $v_1 \to 0$, it is as if the market was comprised only of high valuation consumers, so the monopolist can charge them a price of $v_2$. This implies that, as $v_1 \to 0$, the monopolist’s profits converge to the full commitment profits $\Pi^{FC} (x) = \sup_\tau E \left[ e^{-r\tau} \alpha (v_2 - x_\tau) \right] \mid x_0 = x$.

**Consumer heterogeneity and prices:** The model in this section gives predictions about how the degree of heterogeneity among consumers affects the evolution of prices. Let $m = \alpha v_2 + (1 - \alpha) v_1$ be the average valuation, so $v_1 = (m - \alpha v_2)/(1 - \alpha)$. By equation (1.9),

$$P(x, \alpha) = v_2 - \left( v_2 - \frac{m - \alpha v_2}{1 - \alpha} \right) \left( \frac{x}{\frac{-\lambda}{1-\lambda} \frac{m - \alpha v_2}{1-\alpha}} \right)^\lambda$$

for $x > z_1$. (1.19)

By varying $v_2$ in equation (1.19) we can trace how a mean-preserving change in the distribution of valuations affects prices. Suppose $x_0 \geq z_2$, so in equilibrium prices are $P(x_t, \alpha)$ for
all \( x_t \leq z_2 \). By equation (1.18), the drift of \( P(x_t, \alpha) \) is \(-r(v_2 - P(x_t, \alpha))\). Equation (1.19) then implies that the drift of \( P(x_t, \alpha) \) is decreasing in \( v_2 \) if and only if \( \frac{v_2}{v_1} < \frac{1-\alpha\lambda}{\alpha\lambda} \). Thus, prices will decrease at a faster rate in markets in which the degree of heterogeneity among consumers is larger, provided \( v_2/v_1 \) is small.

**Comparative statics with respect to \( \sigma \):** Equation (1.9) implies \( \partial P(x, \alpha)/\partial \sigma > 0 \) if and only if \( x > z_1 \exp(1/(1-\lambda)) \). In words, at high levels of costs the monopolist is able to charge a higher price when the volatility of \( x_t \) is larger. A change in \( \sigma \) has two opposing effects on \( P(x, \alpha) \). First, a higher \( \sigma \) lowers the threshold \( z_1 \) at which the monopolist starts selling to low valuation consumers. Second, a higher \( \sigma \) means that costs will (on average) reach \( z_1 \) faster. The second effect dominates when \( x \in (z_1, z_1 \exp(1/(1-\lambda))) \), while the first effect dominates when \( x > z_1 \exp(1/(1-\lambda)) \).

**Comparative statics with respect to \( \mu \):** By equation (1.9), \( \partial P(x, \alpha)/\partial \mu > 0 \) if and only if \( x > z_1 \exp(1/(1-\lambda)) \): at high levels of costs a monopolist charges higher prices in settings in which costs fall at a slower rate. Again, a change in \( \mu \) has two opposing effects on \( P(x, \alpha) \). First, a higher \( \mu \) raises the threshold \( z_1 \) at which the monopolist starts selling to low type consumers. Second, a higher \( \mu \) means that costs will (on average) take longer to fall to \( z_1 \). The first effect dominates when \( x \in (z_1, z_1 \exp(1/(1-\lambda))) \), while the second effect dominates when \( x > z_1 \exp(1/(1-\lambda)) \).

### 1.5 Markets with \( n \) types of consumers

In this section, I show how the results in Section 1.4 generalize to settings in which the function \( f : [0, 1] \rightarrow [\underline{v}, \overline{v}] \) describing the valuations of the consumers is a left-continuous, non-increasing step function taking a finite number of values \( v_1 < v_2 < \ldots < v_n \). For \( k = 1, \ldots, n \), let \( \alpha_k = \max\{i \in [0, 1] : f(i) = v_k\} \) denote the highest indexed consumer with valuation \( v_k \), so \( f(i) = v_k \) for all \( i \in (\alpha_{k+1}, \alpha_k] \). Let \( \alpha_{n+1} = 0 \).
As a first step towards analyzing this more general setting, note that at any state \((x,a)\) with \(a \geq \alpha_3\) there are either one or two types of consumers remaining in the market: consumers with valuation \(v_1\) and consumers with valuation \(v_2\). Thus, for any state \((x,a)\) with \(a \geq \alpha_3\) the equilibrium outcome is the one described in Section 1.4. At states \((x,a)\) with \(a \geq \alpha_2\), there are only consumers with valuation \(v_1\) in the market, so the monopolist’s profits are \((1 - \alpha_2) V_1(x)\). On the other hand, at states \((x,a)\) with \(a \in [\alpha_3, \alpha_2)\) there are \(\alpha_2 - a\) consumers with valuation \(v_2\) in the market. In this case, there are cutoffs \(\underline{x}(a)\) and \(\bar{x}(a)\) such that the monopolist sells to all remaining consumers when \(x \leq \underline{x}(a)\), and sells to all remaining consumers with valuation \(v_2\) when \(x \in [\bar{x}(a), z_2]\). When \(x \in (\underline{x}(a), \bar{x}(a))\), the monopolist sells gradually to consumers with valuation \(v_2\) at a rate given by equation (1.17), and when \(x_t > z_2\) the monopolist doesn’t sell. For states \((x,a)\) with \(a \geq \alpha_3\), let \(L(x,a)\) denote the (unique) equilibrium profits of the monopolist (derived in Section 1.4).

Consider next states \((x,a)\) with \(a \in [\alpha_4, \alpha_3)\), so that there are \(\alpha_3 - a\) consumers with valuation \(v_3\) still in the market. Let \(P_2(x) = \sup_{i \in [\alpha_3, \alpha_2]} P(x,i)\) be the highest price that a consumer with valuation \(v_2\) is willing to pay. The analysis in Section 1.4 implies that \(P_2(x) = P(x, \alpha_2)\) for all \(x \in [0, \underline{x}(\alpha_3)] \cup [\bar{x}(\alpha_3), \infty)\) (where \(P(x, \alpha_2) = v_2 - E[e^{-r \tau_1}(v_2 - v_1)|x_0 = x]\)), and \(P_2(x) = x - L_a(x, \alpha_3)\) for all \(x \in (\underline{x}(\alpha_3), \bar{x}(\alpha_3))\). By equation (1.5), the strategy \(P(x, \alpha_3)\) of consumer \(\alpha_3\) (the highest indexed consumer with valuation \(v_3\)) satisfies

\[
P(x, \alpha_3) = v_3 - E\left[e^{-r \tau_2}(v_3 - P_2(x_{\tau_2}))\right| x_0 = x],
\]

where \(\tau_2 = \inf\{t : x_t \leq z_2\}\) is the time at which the monopolist starts selling to consumers with valuation \(v_2\) when the level of market penetration is \(\alpha_3\) (i.e., when all consumers with valuation \(v_3\) have left the market).
By the skimming property, the monopolist can sell to all remaining $v_3$-consumers at price $P(x, \alpha_3)$. Therefore, at states $(x,a)$ with $a \in [\alpha_4, \alpha_3)$ her profits are bounded below by

$$L(x,a) = \sup_{\tau \in T} E \left[ e^{-\tau} ((\alpha_3 - a) (P(x_\tau, \alpha_3) - x_\tau) + e^{-\tau} L(x_\tau, \alpha_3)) \right] | x_0 = x].$$  

(1.20)

By arguments similar to those in Lemma 1.3, there exists thresholds $\overline{x}_1(a), \overline{x}_1(a), \overline{x}_2(a)$, $\overline{x}_2(a)$ with $\overline{x}_1(a) < z_1 < \overline{x}_1(a)$ and $\overline{x}_2(a) < z_2 < \overline{x}_2(a) < z_3$ such that

$$\tau(a) = \inf \{ t : x_t \in [0, \overline{x}_1(a)] \cup [\overline{x}_1(a), \overline{x}_2(a)] \cup [\overline{x}_2(a), z_3] \},$$

is a solution to the optimal stopping problem (1.20). That is, the solution to (1.20) is such that the monopolist sells immediately to all remaining consumers with valuation $v_3$ at price $P(x, \alpha_3)$ whenever $x_t \in [0, \overline{x}_1(a_t)] \cup [\overline{x}_1(a), \overline{x}_2(a)] \cup [\overline{x}_2(a), z_3]$. On the other hand, when $x_t \in (\overline{x}_1(a_t), \overline{x}_1(a_t)) \cup (\overline{x}_2(a_t), \overline{x}_2(a_t))$ it is optimal to delay. By arguments similar to those in Section 1.4, at states $(x_t, a_t)$ with $a_t \in [\alpha_4, \alpha_3)$ and $x_t \in (\overline{x}_1(a_t), \overline{x}_1(a_t)) \cup (\overline{x}_2(a_t), \overline{x}_2(a_t))$ the monopolist sells gradually to $v_3$-consumers at a price equal to $x_t - L_a(x_t, a_t)$. The rate at which the monopolist sells when costs are in this region can be derived following the steps leading to equation (1.17). Finally, for $x > z_3$ the monopolist doesn’t sell.

Next, consider states $(x,a)$ with $a \in (\alpha_5, \alpha_4)$. At such states there are $\alpha_4 - a$ consumers with valuation $v_4$ still in the market. Let $P_3(x) = \sup_{i \in (\alpha_4, \alpha_3]} P(x,i)$ be the highest price that a consumer with valuation $v_3$ is willing to pay. From the arguments above, $P_3(x) = P(x, \alpha_3)$ for $x \in [0, \overline{x}_1(a_t)] \cup [\overline{x}_1(a), \overline{x}_2(a_t)] \cup [\overline{x}_2(a), \infty)$, and $P_3(x) = x_t - L_a(x_t, \alpha_4)$ for all $x \in (\overline{x}_1(a_t), \overline{x}_1(a)) \cup (\overline{x}_2(a_t), \overline{x}_2(a_t))$. By equation (1.5), the strategy $P(x, \alpha_4)$ of consumer $\alpha_4$ (the highest indexed consumer with valuation $v_4$) satisfies

$$P(x, \alpha_4) = v_4 - E \left[ e^{-\tau_{\alpha_4}} (v_4 - P_3(x_{\tau_{\alpha_4}})) \right] | x_0 = x].$$
where $\tau_3 = \inf \{ t : x_t \leq z_3 \}$ is the time at which the monopolist starts selling to consumers with valuation $v_3$ after all consumers with valuation $v_4$ have left the market. The skimming property again implies that at states $(x, a)$ with $a \in (\alpha_5, \alpha_4]$ the monopolist can sell to all remaining consumers with valuation $v_4$ at price $P(x, \alpha_4)$. Therefore, at all such states the monopolist’s profits are bounded below by

$$L(x, a) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\tau \tau} \left( (\alpha_4 - a) (P(x_\tau, \alpha_4) - x_\tau) + e^{-\tau \tau} L (x_\tau, \alpha_4) \right) \bigg| x_0 = x \right].$$

Continuing in this way, I can extend the function $L(x, a)$ to all $a \in [0, 1]$ in such a way that, for $k = 1, \ldots, n$ and all $a \in [\alpha_{k+1}, \alpha_k)$,

$$L(x, a) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\tau \tau} \left( (\alpha_k - a) (P(x_\tau, \alpha_k) - x_\tau) + e^{-\tau \tau} L (x_\tau, \alpha_k) \right) \bigg| x_0 = x \right]. \quad (1.21)$$

**Theorem 1.2** In any regular equilibrium, the monopolist’s profits are $L(x, a)$ at every state $(x, a)$.

**Proof:** See Appendix A.1.4.

Theorem 1.2 shows that the results in Theorem 1.1 extend to the case in which $f : [0, 1] \rightarrow [\underline{v}, \overline{v}]$ takes any finite number of values: in this more general setting, the monopolist’s equilibrium profits are also equal to the lower bound $L(x, a)$.

At states $(x, a)$ with $a \in [\alpha_{k+1}, \alpha_k)$, consumers with valuation $v_{k+1}$ and higher have already left the market. For these states, the solution to the optimal stopping problem (1.21) involves delaying when $x$ is around $z_1$, $z_2$, ..., or $z_{k-1}$, and when $x > z_k$. If $x < z_k$ is in the delay region of the optimal stopping problem (1.21), the monopolist sells gradually to those consumers with valuation $v_k$ (the highest valuation remaining in the market). By arguments similar to those in Section 1.4, the rate at which the monopolist sells in this case is such that consumers with valuation $v_k$ are indifferent between buying at time $t$ or delaying their purchase; and the price that the monopolist charges at each instant is $P(x, a) = x - L_a(x, a)$. 31
If $x > z_k$, the monopolist does not sell until costs fall to $z_k$ (and at this point she sells to all $v_k$-consumers at price $P(z_k, \alpha_k)$). Finally, if $x$ lies in the stopping region of (1.21), the monopolist sells to all remaining consumers with valuation $v_k$ at price $P(x, \alpha_k)$, and the state moves to $(x, \alpha_k)$. At state $(x, \alpha_k)$, the solution to the the optimal stopping problem (1.21) involves delaying when $x$ is around $z_1, z_2, \ldots$, or $z_{k-2}$, and when $x > z_{k-1}$. Again, the monopolist sells gradually to consumers with valuation $v_{k-1}$ (the highest remaining buyers in the market) if $x < z_{k-1}$ lies inside the delay region of the optimal stopping problem (21). If $x > z_{k-1}$, the monopolist waits for costs to fall, while if $x$ lies in the stopping region of (1.21) the monopolist sells to all remaining consumers with valuation $v_{k-1}$ at price $P(x, \alpha_k)$, and the state moves to $(x, \alpha_{k-1})$.

The equilibrium of this more general model shares many of the same features of the two type case analyzed in Section 1.4. The generalized Coase conjecture also fails to hold in this setting. First, the monopolist is able to obtain excess profits. To see this, note that the monopolist can always sell to all consumers with valuation $v_2$ and higher at price $P(x, \alpha_2)$, obtaining a margin of $P(x, \alpha_2) - x$. By Lemma 1.2, $P(x, \alpha_2) - x > V_1(x)$ for all $x \in (z_1, z_2]$. Therefore, the monopolist’s profits $L(x, a)$ are strictly larger than $(1 - a)V_1(x)$ for all $x \in (z_1, z_2]$. More generally, arguments similar to those in Lemma 1.2 imply that for $k \geq 2$, $P(x, \alpha_k) - x > V_1(x)$ for all $x \in [z_{k-1}, z_k]$. Since the monopolist can sell to all consumers with valuation $v_k$ and higher at a price of $P(x, \alpha_k)$, this implies that $L(x, a) > (1 - a)V_1(x)$ for all $x \in [z_{k-1}, z_k]$. Second, the equilibrium outcome also involves inefficiencies in the form of delayed trade. Suppose $x_0 < z_n$ lies inside the delay region of (1.21). In this case, the efficient outcome is to serve all consumers with valuation $v_n$ immediately, but the monopolist sells to them gradually. In contrast, the outcome is fully efficient when $x_0 \geq z_n$: in this case, for $k = 1, \ldots, n$ the monopolist serves $v_k$-consumers at the surplus maximizing time $\tau_k$.

The equilibrium outcome also displays history dependence when costs initially lie inside the delay region of the optimal stopping problem (1.21), since the rate at which the monopo-
list sells at each instant depends on the current level of marginal costs. Finally, in this more
general model there will also be histories under which a positive mass of consumers buys
after an increase in prices. Suppose \( x_t \) lies within the delay region of the optimal stopping
problem (1.21). In this case, there will be a threshold \( \overline{a} (a_t) \) such that all consumers with
the highest valuation in the market buy if \( x_t \) increases above \( \overline{a} (a_t) \). Since the price that the
monopolist charges is increasing in \( x \), if costs raise rapidly above \( \overline{a} (a_t) \) all high valuation
consumers will buy at a moment in which prices are actually going up.

1.6 Continuous distributions and the generalized Coase
conjecture

In this section, I study markets in which the valuations of the consumers are described
by a continuous and strictly decreasing function \( h: [0, 1] \rightarrow [v, \overline{v}] \), with \( \overline{v} > v > 0 \). I
study this setting by considering a sequence of models with step functions \( f^n \) such that
\[ \sup_{i \in [0,1]} |f^n (i) - h(i)| \rightarrow 0 \] as \( n \rightarrow \infty \). For simplicity, I consider approximations \( f^n \) that
satisfy the following property: for \( n = 2, 3, \ldots, f^n \) is a left-continuous step function taking \( n \)
values \( v^n_1, \ldots, v^n_n \), with \( v^n_1 = v \) and for \( k = 2, \ldots, n, v^n_k = v^n_{k-1} + (\overline{v} - v) / (n - 1) \).

For \( n = 2, 3, \ldots \), let \( L^n(x, a) \) denote the monopolist’s profits at state \( (x, a) \) in an
environment in which the valuations of the consumers are described by \( f^n \). Recall that
\( V_1(x) = \sup_x E[e^{-r \tau} (\overline{v} - x_\tau) | x_0 = x] \) are the profits that the monopolist would earn if
all consumers in the market had the lowest valuation \( v > 0 \).

**Theorem 1.3** For all states \((x, a) \in \mathbb{R}_+ \times [0, 1], \lim_{n \rightarrow \infty} L^n (x, a) = (1 - a) V_1 (x).**

**Proof:** See Appendix A.1.5.

\( ^{10} \)For instance, we can construct such sequence \( \{f^n\} \) as follows. For \( k = 1, \ldots, n, \) let
\( \alpha^n_k = \max \{ i \in [0,1] : f^n (i) = v^n_k \} \) be the highest indexed consumer with valuation \( v^n_k \).
Thus, \( f^n (i) = v^n_k \) for all \( i \in (\alpha^n_{k+1}, \alpha^n_k] \). Let \( f^n \) be such that \( \alpha^n_1 = 1 \) and, for \( k = 2, \ldots, n, \)
\( \alpha^n_k = (h^{-1} (v^n_{k-1}) + h^{-1} (v^n_k)) / 2 \). One can check that \( \sup_i |f^n (i) - h(i)| \rightarrow 0 \) as \( n \rightarrow \infty \).
Theorem 1.3 shows that the monopolist’s profits at state \((x, a)\) converge to \((1 - a)V_1(x)\) in the limit as the distribution of consumer valuations becomes continuous. That is, a monopolist producer of a durable good earns zero excess profits when she faces a continuous distribution of consumer valuations.

To see the intuition behind Theorem 1.3, consider first a setting with two types of consumers: high types, with valuation \(v\), and low types, with valuation \(v\). After high types have left the market, the monopolist will sell to low types when costs fall below \(z = \frac{-\lambda}{1 - \lambda} v\). In this case, the monopolist can truthfully commit to maintain high prices until costs fall below \(z\). High type buyers know that prices will fall to \(v\) only when \(x_t \leq z\). Thus, when costs are above \(z\) they are willing to pay higher prices.

Consider next the case in which there are three types of consumers, with valuations \(v\), \((v + v)/2\) and \(v\). After consumers with valuation \(v\) have left the market, the monopolist can only commit to keep prices high until costs fall below \(\frac{-\lambda}{1 - \lambda} \frac{(v + v)}{2}\). At this point, it becomes optimal for the monopolist to sell to consumers with intermediate valuation \((v + v)/2\). This puts a limit to the price consumers with valuation \(v\) are willing to pay when \(x_t > \frac{-\lambda}{1 - \lambda} \frac{(v + v)}{2}\), since now they can wait until costs fall to \(\frac{-\lambda}{1 - \lambda} \frac{(v + v)}{2}\) and obtain a lower price.

More generally, the proof of Theorem 1.3 shows that the price consumers are willing to pay monotonically decreases as \(n \to \infty\). In the limit as the gap between valuations becomes vanishingly small, the monopolist’s profits fall to what she would earn if all consumers had the lowest valuation \(v\). In other words, the monopolist loses all commitment power when she faces a continuous distribution of valuations, since in this case she always has an incentive to serve the next buyer arbitrarily soon after her last sale (Figure 1.5 plots the prices consumers are willing to pay for \(n = 2, 3, 4\) and \(5\)).

**Corollary 1.1** In the limit as \(n \to \infty\), the monopolist sells at price \(P_t = x_t + V_1(x_t)\) and the equilibrium outcome is fully efficient.

Corollary 1.1 and Theorem 1.1 together imply that the generalized Coase conjecture holds when the distribution of valuations is continuous: the outcome is fully efficient in this
Figure 1.5: Parameters: $v = \frac{1}{2}$, $\bar{v} = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

case, and the monopolist is unable to obtain excess profits. To see why Corollary 1.1 must hold, note first that the monopolist can always guarantee herself a profit of $V_1(x_t)$ on every consumer by treating all of them as low types. This implies that the monopolist will never sell at a price below $x_t + V_1(x_t)$: selling at such a price would give her a profit lower than $V_1(x_t)$. On the other hand, the monopolist’s profits would be strictly larger than $V_1(x_t)$ if she could sell at prices strictly higher than $x_t + V_1(x_t)$, contradicting Theorem 3.

With a continuous distribution the path of prices is then given by $\{x_t + V_1(x_t)\}$ and the monopolist’s profit margin on each sale is $V_1(x_t)$. Given this path of prices, a consumer with valuation $v \in [v, \bar{v}]$ chooses optimally when to buy, solving

$$\sup_{\tau} E[e^{-rt}(v - x_t - V_1(x_t)) | x_0 = x].$$

The solution to this stopping problem is $\tau_v = \inf\{t : x_t \leq \frac{-\ln(\bar{v})}{\lambda}\}$: with a continuous distribution, a consumer with valuation $v$ buys at time $\tau_v$. By Lemma 1.1, $\tau_v$ is the surplus maximizing time at which to sell to a consumer with valuation $v$. Thus, the limiting outcome
is fully efficient: the monopolist serves consumers sequentially as cost decreases, precisely at the point in time that maximizes total surplus. Consumers with higher valuations trade earlier, and end up paying higher prices (since \( x + V_1(x) \) is strictly increasing in \( x \)).

Finally, in this setting we can think of the lowest valuation \( v \) as measuring the “gap”. Note that \( V_1(x) = (v - \bar{z}) (x/\bar{z})^\lambda \rightarrow 0 \) as \( v \rightarrow 0 \). Therefore, when the function describing the valuations of the consumers is continuous, the price \( P_t = V_1(x_t) + x_t \) at which the monopolist sells her good converges to marginal cost \( x_t \) as \( v \) goes to zero: in the no gap case, the equilibrium outcome is competitive and the monopolist earns zero profits.

1.7 Conclusion

This paper studies the effect time-varying costs have on the equilibrium dynamics of an otherwise standard durable goods monopoly model. When the distribution of consumer valuations is discrete, time-varying costs provide commitment power to the monopolist. This allows the monopolist to extract more surplus from consumers with higher valuations, modifying the entire equilibrium dynamics. This commitment power disappears when the distribution of valuations is continuous. The monopolist earns zero excess profits in this case, and the equilibrium outcome is fully efficient.

Continuous time methods lead to a tractable characterization of the equilibrium. The model delivers a variety of predictions about how prices and margins relate to the different features of the environment. For instance, the model with two types of buyers predicts that prices fall at a faster rate when the monopolist’s costs are lower, and that prices also fall faster in markets in which there is more heterogeneity among consumers. These and other predictions of the model could serve as a benchmark for future empirical work on durable goods pricing.

Throughout the paper, I assumed that costs follow a particular diffusion process. In applications, it might be important to have flexibility regarding the choice of the cost process,
especially if we have information about how these costs actually evolve. The main results of the paper continue to hold if costs follow a more general process of the form

\[ dx_t = \mu(x_t) \, dt + \sigma(x_t) \, dB_t. \]  

(1.22)

For instance, suppose there are two types of buyers. If costs evolve as (1.22), we could still compute the lower bound \( L(x, a) \) on the monopolist’s profits using the procedure of Section 1.4, and this lower bound would still characterize the monopolist’s equilibrium profits.

Finally, the paper assumes that the process driving marginal costs is exogenous. However, in many settings firms are able to influence how their costs evolve, for instance, through investments in R&D. One way to incorporate this feature into the model is to assume that the monopolist’s investment decisions affect the drift and/or volatility of the costs process. Although incorporating this feature to the model would make the analysis more complex, we could still use the methods put forward in this paper to study the dynamics of prices and sales under this environment.

\[ \text{11The coefficients } \mu : \mathbb{R} \to \mathbb{R} \text{ and } \sigma : \mathbb{R} \to \mathbb{R} \text{ in (1.22) must satisfy conditions for existence and uniqueness of a strong solution to this stochastic differential equation; see Theorem 5.2.9 in Karatzas and Shreve (1998).} \]
Chapter 2

A Continuous Time Model of Bilateral Bargaining

2.1 Introduction

In many real life negotiations the relative strength of the parties changes over time. While one side might start bargaining from a strong position, bargaining power might change hands as the negotiation proceeds. For instance, in wage negotiations the relative bargaining power of the firm and the workers depends on the unemployment rate, a variable that fluctuates along with the economic cycle. In legislative bargaining the ability of a political party to implement its preferred policy depends on the number of seats it controls in Congress, a quantity that moves over time together with the party’s political power and popularity. In mergers and acquisitions the way in which the gains from joint operations are divided among the firms may also depend on time-varying variables, such as market valuations and the general economic environment.

This paper introduces a continuous time model of bilateral bargaining that captures the effects that time-varying bargaining power has on the outcomes of negotiations. The model’s key variable is an exogenous and publicly observable diffusion process $x_t$, whose realization
determines the identity of the proposer at each moment in time: player 1 makes offers at time $t$ if $x_t > 0$ and player 2 makes offers at time $t$ if $x_t < 0$. The value that the process $x_t$ takes at any time $t$ is then a measure of the players’ relative bargaining power. This bargaining protocol, though particular, captures evolving bargaining power in the two applications of this model that I consider in this paper: legislative negotiations in two party systems and horizontal takeovers in duopolistic markets (see below for more details on these applications).

An outcome of this model is a pair $(A, \eta)$, where $A \subseteq \mathbb{R}$ is an agreement region and $\eta$ is a function from $A$ to the set of possible divisions of the surplus. The region $A$ determines the set of values of $x_t$ at which players reach an agreement: under outcome $(A, \eta)$ players reach an agreement the first time $x_t$ reaches $A$. On the other hand, the function $\eta$ determines the way in which the players split the surplus once an agreement is reached.

An outcome $(A, \eta)$ must satisfy two conditions to be an equilibrium. First, the responder’s acceptance threshold must equal her expected continuation value of waiting until she regains the right to make proposals. That is, in equilibrium the responder should always accept offers that give her a payoff equal to what she would get by waiting until the process $x_t$ reaches zero. Second, the agreement region $A$ must be such that the proposer finds it optimal to make an acceptable offer to her opponent when $x_t \in A$, and she finds it optimal to delay when $x_t \notin A$. Put differently, bargaining should come to an end only when this is optimal for the proposer to make an offer that will satisfy the expectations of her opponent.

I show that this model has a unique equilibrium. Players always reach an immediate agreement in equilibrium, and their payoffs are characterized by a system of ordinary differential equations. Closed form solutions to these equations are available when the process $x_t$ that determines relative bargaining power evolves as a Brownian motion with constant drift $\mu$ and constant volatility $\sigma > 0$. When $x_t$ follows a more general diffusion process the solution to the system of differential equations can be found using numerical methods.

I also study a discrete time version of this continuous time bargaining model. In this discrete time game the realization of the process $x_t$ determines the identity of the proposer
and responder in the same way as in the continuous time model, but players can only make offers at points on the grid \(\{0, \Delta, 2\Delta, \ldots\}\). I show that the equilibrium of the continuous time model corresponds to the limiting subgame perfect equilibrium of this discrete time game, when players can make offers arbitrarily frequently (i.e., when \(\Delta \to 0\)). That is, the continuous time bargaining model and its discrete time analog give the same unique prediction about the outcome of negotiations in settings in which bargaining power fluctuates over time.

The tractability of the model makes it amenable to a variety of extensions and applications. I present two such applications in this paper. First, I use the continuous time model to study legislative negotiations with supermajority requirements in two party systems. In this legislative bargaining framework the value of \(x_t\) represents the fraction of legislators supporting each party, and at each moment in time the leaders of the majority party have proposal power. I consider a setting in which the parties may learn verifiable information about the benefits of the different policies while bargaining. I show that parties may end up implementing a policy before the arrival of information, even in cases in which it is socially optimal to wait. Moreover, I show that inefficiencies tend to disappear as the supermajority requirement increases.\(^1\)

I also present an application of the continuous time bargaining model to horizontal takeovers in duopolistic markets. In this setting the process \(x_t\) represents the firms’ relative market shares. At each instant \(t\) the manager of the firm with a larger market share can make a takeover bid to its opponent. After a takeover takes place the remaining firm obtains monopoly profits. I show that the managers of the firms always come to an immediate agreement about the terms of the takeover when they have common beliefs about how the process \(x_t\) evolves. However, inefficient delays necessarily arise when managers are sufficiently optimistic about the prospects of their firms.

\(^1\)In Chapter 3 I use this paper’s continuous time model to analyze potential causes of gridlock in legislative negotiations.
It is well known that there are some technical issues that arise when modelling games in continuous time (Simon and Stinchcombe, 1989, and Bergin and MacLeod, 1993). The study of bargaining games in continuous time presents additional challenges besides these issues: when there are no restrictions on the timing of offers the notion of subgame perfection loses its bite, since it doesn’t refine the set of outcomes as it does when players can take actions in discrete time. The reason for this is that players do not face a fixed cost of delay after rejecting an offer when the game is in continuous time, since they can always accept a new offer within an arbitrarily short period of time. One of the contributions of this paper is to propose a new method for analyzing bargaining games in continuous time. Indeed, I show that we can avoid these complications by focusing on the outcomes of the bargaining model and on the payoffs that these outcomes induce. I also show that we can define an equilibrium directly in terms of outcomes and obtain a unique prediction. Importantly, this unique prediction corresponds to the (also unique) limiting subgame perfect of a discrete time bargaining game, when players can make offers arbitrarily frequently.

2.1.1 Related Literature

The present paper relates to the literature on bargaining games in continuous time. Perry and Reny (1993) construct a continuous time bilateral bargaining game in which players can choose to make offers at any point in time (see also Sakovics, 1993). There are two key assumptions in their model. First, after making an offer players have to wait a fixed time period (i.e., a waiting time) before they can make a new proposal. Second, players also have to wait a fixed time period (i.e., a reaction time) before replying to an offer. The authors show that all subgame perfect equilibria of this model are efficient when players can react arbitrarily quickly to their opponent’s offers (i.e., when reaction times are vanishingly small).

The fact that subgame perfection does not refine the set of outcomes in continuous time bargaining games is not new. Indeed, Bergin and MacLeod (1993) show that any division of the surplus can be supported as an equilibrium in a bargaining game in which players can take actions in continuous time.
Ambrus and Lu (2010) study a continuous time coalitional bargaining game with a fixed deadline in which players get random opportunities to make proposals through a Poisson process. The paper shows that this continuous time game has a unique Markov perfect equilibrium. The players’ equilibrium payoffs are fully characterized by the time left until the deadline and the arrival rates of proposal opportunities.

A common feature of these models is the presence of restrictions on the timing of offers and counteroffers. In Perry and Reny (1993) these restrictions appear in the form of waiting times and reaction times. In Ambrus and Lu (2010) the restrictions are at the heart of the model, since players can only make proposals when the Poisson process hits. These restrictions on the timing of offers allow these authors to sidestep the technical issues that arise when modelling games in continuous time, and bring the analysis closer to that of discrete time bargaining games. In contrast, the model I present in this paper features no restrictions on the timing of offers (besides the identity of the proposer), and therefore the standard discrete time methods cannot be directly applied.

The model I construct in this paper also relates to the stochastic bargaining games introduced by Merlo and Wilson (1995, 1998) (see also Cripps, 1998). These games feature an exogenous stochastic process whose value at each bargaining round determines the size of the surplus over which the players are bargaining and the identity of the proposer. One of the main results in these papers is that players will delay in reaching an agreement if they expect the surplus to grow fast enough in the future. The discrete time version of the model I present in this paper is a special case of Merlo and Wilson’s model: in my model the size of the surplus is fixed, and the exogenous stochastic process (i.e., the diffusion process $x_t$) only determines the identity of the proposer at each bargaining round.

Yildiz and Simsek (2009) also study a bilateral bargaining model in which the players’ bargaining power evolves stochastically over time. A key feature in their analysis is that

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3 Abreu and Gul (2000) study a continuous time bargaining game with exogenous irrational types. The paper shows that the player’s uncertainty about their opponent’s rationality can lead to inefficient delays, since rational players can build a reputation for being irrational by delaying an agreement.
players may have optimistic beliefs about their future bargaining power. The paper shows that this type of optimism might lead to inefficient delays if players expect bargaining power to become more durable at some future date. There are some important differences between their model and the one I present in this paper. First, Yildiz and Simsek (2009) use techniques that are different from the ones I use here: their model is setup in discrete time, and they don’t assume that bargaining power follows a diffusion process. In contrast, the continuous time methods I use in this paper allow me to obtain a very clean and tractable characterization of bargaining outcomes, which opens the door to a variety of extensions and applications.

More broadly, the present paper relates to the literature on continuous time games. Sannikov (2007, 2008) studies continuous time games in which the players’ unobservable actions affect the drift of a publicly observable signal which evolves as a diffusion process (see also DeMarzo and Sannikov, 2006). The main theme of these papers is to show how the realization of the public signal can be used to provide incentives in dynamic settings. Gul and Pesendorfer (2011) construct a continuous time model of campaigns in which parties provide costly information to voters. Daley and Green (2011) use continuous time methods to study a setting in which a seller has private information about the value of the asset she is selling, and in which the market gradually learns about the quality of the seller’s asset.

### 2.2 Model

I study a bilateral bargaining model under complete information. Two players, \( i = 1, 2 \), bargain over how to divide a perfectly divisible surplus of size 1. Time is continuous and players can reach an agreement at any time \( t \in [0, \infty) \), so there is no exogenous deadline to the negotiations.

The key variable of the model is relative bargaining power, which I denote by \( x_t \). Let \( B = \{B_t, \mathcal{F}_t^B : 0 \leq t < \infty \} \) be a one-dimensional Brownian motion on the probability space
$(\Omega, \mathcal{F}, \mathbf{P})$, where $\{\mathcal{F}_t^B : 0 \leq t < \infty\}$ is the completion of the filtration generated by the Brownian motion.\footnote{That is, the filtration $\{\mathcal{F}_t^B : 0 \leq t < \infty\}$ includes all sets of measure zero and is therefore right-continuous: for every $t \geq 0$, $\mathcal{F}_t^B = \cap_{s>0} \mathcal{F}_{t+s}^B$.} The Brownian motion $B$ drives the publicly observable process $x_t$. For most of the paper I will assume that $x_t$ evolves as a Brownian motion with constant drift $\mu$ and constant volatility $\sigma > 0$. That is, $x_t$ solves

$$dx_t = \mu dt + \sigma dB_t,$$

with $x_0 = x \in \mathbb{R}$ fixed. In this case I say that $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion. Section 4 shows how the results extend to the case in which $x_t$ follows a more general diffusion process.

Bargaining proceeds as follows: player 1 makes offers at time $t$ if $x_t > 0$, and player 2 makes offers at time $t$ if $x_t < 0$. That is, at each time $t \in [0, \infty)$ the value of $x_t$ determines the identity of the proposer and the responder. Suppose for instance that $x_0 > 0$.\footnote{If $x_0 = 0$, I assume that player 1 starts making offers. As it will become clear below, in this case the identity of the proposer at $t = 0$ will have no effect on the equilibrium outcome.} In this case player 1 will be the proposer from time 0 until time $\tau_1 := \inf\{t : x_t \leq 0\}$. At any moment until $\tau_1$, player 1 can make an offer $z \in \{y \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$ to player 2. There are no restrictions on the number of offers that player 1 can make between $t = 0$ and $\tau_1$. If player 2 accepts an offer before $\tau_1$ the game ends and each player collects her payoff. If player 2 does not accept any offer between time 0 and $\tau_1$, then player 2 becomes the proposer from $\tau_1$ until $\tau_2 := \inf\{t > \tau_1 : x_t \geq 0\}$. Bargaining continues this way, with players alternating in their right to make proposals according to the realization of the process $x_t$, until a player accepts an offer (Figure 2.1 plots a sample path of $x_t$).

Players are risk neutral expected utility maximizers and discount future payoffs at the common rate $r > 0$. Therefore, the payoff that a player gets from receiving a (possibly random) share $z \in [0, 1]$ of the surplus at some (possibly random) time $\tau$ is $E[e^{-\tau} z \mid x_0 = x]$. If players fail to reach an agreement in finite time they both get a payoff of zero.
An outcome of this bargaining model is a pair \((A, \eta)\), where \(A \in \mathcal{A} := \{A \subseteq \mathbb{R} : A \text{ is closed}\}\) is an agreement region and \(\eta\) a function mapping the agreement region \(A\) to the set of possible divisions of the surplus \(\{y \in \mathbb{R}_+^2 : y_1 + y_2 = 1\}\). The agreement region \(A\) determines the set of values of the state variable \(x_t\) at which players reach an agreement. Put differently, under an outcome \((A, \eta)\) players reach an agreement at time \(t\) if and only if \(x_t \in A\), so the agreement date is \(\tau(A) = \inf\{t \geq 0 : x_t \in A\}\). On the other hand, the function \(\eta : A \to \{y \in \mathbb{R}_+^2 : y_1 + y_2 = 1\}\) gives the share of the surplus that each player gets when they reach an agreement.

Let \(x_0 = x\) be the initial level of relative bargaining power. Then, the payoff that player \(i\) gets from outcome \((A, \eta)\) is

\[
V_i (x) = \begin{cases} 
\eta_i (x) & x \in A, \\
E \left[ e^{-r\tau(A)} \eta_i (x_{\tau(A)}) \big| x_0 = x \right] & x \notin A.
\end{cases}
\] (2.1)

Note that \(V_1 (x) + V_2 (x) = 1\) for all \(x \in A\). In what follows, I will denote an outcome \((A, \eta)\) as a triplet \((A, V_1, V_2)\), with \(A \in \mathcal{A}\) and with \(V_i\) satisfying (2.1) for \(i = 1, 2\).

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\(^6\)The restriction that the agreement region \(A\) must belong to the set \(\mathcal{A}\) is only to get rid of trivial multiplicities - see footnote 12.
Remark 2.1 The definition of an outcome has a stationarity assumption implicitly built-in: under outcome \((A, \eta)\) the agreement date \(\tau (A)\) depends only on the realization of \(x_t\), and the division of the surplus at time \(\tau (A)\) depends only on \(x_{\tau (A)}\). In Section 2.5, I show that the bargaining game that results from the discrete time version of this model has a unique subgame perfect equilibrium. Importantly, this unique equilibrium is supported by a profile of stationary strategies; and hence the outcome that the equilibrium induces is stationary. I also show that the equilibrium of the discrete time game converges to the unique equilibrium of the continuous time model as players can make offers arbitrarily frequently.\(^7\) Thus, there is a strong sense in which the restriction to stationary outcomes in this continuous time model is without loss of generality.

For any Borel set \(H \subseteq \mathbb{R}\) let \(\tau (H) = \inf \{t \geq 0 : x_t \in H\}.\(^8\) Let \(T_1\) denote the set of stopping times \(\tau (H)\) with \(H \subseteq [0, \infty)\). Similarly, let \(T_2\) denote the set of stopping times \(\tau (H)\) with \(H \subseteq (-\infty, 0]\).

**Definition 2.1** An outcome \((A, V_1, V_2)\) is an equilibrium if, for \(i, j = 1, 2, j \neq i\),

\[
V_i (x) = \sup_{\tau \in T_i} E \left[ e^{-r \tau} (1 - V_j (x_{\tau})) \mid x_0 = x \right],
\]

for all \(x \in \mathbb{R}\).

The idea behind this equilibrium notion is that the player responding to offers should always be willing to accept any proposal that gives her what she would get by waiting until she regains the right to make offers. Put differently, in an equilibrium the responder’s acceptance threshold must be equal to her expected continuation value. Note that a proposer will never offer the responder more than her acceptance threshold. Therefore, the responder’s payoff should always be equal to her expected continuation value. Definition 1 guarantees

\(^7\)I derive the unique equilibrium of this continuous time model in Section 2.3.

\(^8\)Since the filtration \(\{\mathcal{F}_t^B : 0 \leq t < \infty\}\) is right-continuous, then \(\tau (H) := \inf \{t \geq 0 : x_t \in H\}\) is a stopping time with respect to \(\{\mathcal{F}_t^B : 0 \leq t < \infty\}\) for every Borel set \(H\) - see Oksendal (2007), page 117.
that this will occur under any equilibrium outcome. Indeed, suppose $x_0 = x$ is such that player $i$ is responding to offers at $t = 0$ and let $\tau(0) := \inf\{t \geq 0 : x_t = 0\}$. Note that $\tau(0) \geq \tau$ for all $\tau \in T_i$ for such an $x_0$. Thus, equation (2.2) implies that

$$V_i(x) = E \left[ e^{-\tau(0)}V_i(0) \mid x_0 = x \right],$$

(2.3)

regardless of whether players reach an agreement when $x_t = x$ or not (i.e., regardless of whether $x$ belongs to the agreement region $A$ or not).

On the other hand, when player $i$ is proposer she takes player $j$’s acceptance threshold $V_j(x)$ as given. At each moment in time before she loses the right to make proposals player $i$ has to decide whether to make an acceptable offer of $V_j(x_t)$ to her opponent and end the bargaining (keeping $1 - V_j(x_t)$ for herself), or to delay the agreement until she can strike a better deal. That is, the proposer’s problem is to optimally choose the time at which to make an acceptable offer to her opponent. Definition 2.1 says that an outcome $(A, V_1, V_2)$ is an equilibrium if the proposer always finds it optimal to delay when $x_t \notin A$, and always finds it optimal to make an acceptable offer when $x_t \in A$.

In discrete time bargaining games a la Rubinstein (1982), the responder only accepts offers that give her a payoff at least as large as the value she expects to get by delaying an agreement for one period. That is, the responder’s acceptance threshold is always given by her continuation value. At each bargaining round the proposer decides whether to make an acceptable offer to her opponent and end the game or to delay the agreement for one period.9 Similarly, in an equilibrium of the continuous time model the responder’s acceptance threshold is given by her expected continuation value. Taking her opponent’s acceptance threshold as given, the proposer’s problem is to optimally choose when to make an acceptable offer to her opponent, knowing that she will lose the right to make offers when the process $x_t$ crosses 0.

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9Of course, in a subgame perfect equilibrium of a bargaining game a la Rubinstein the proposer always makes an acceptable offer at the beginning of the game, and the game ends in immediate agreement.
On games played in continuous time: It is well known that there are technical issues when modelling games in which players can take actions in continuous time. These issues arise due to the fact that in continuous time players can condition their actions on instantaneous events (e.g., Simon and Stinchcombe, 1989, and Bergin and MacLeod, 1993). The way in which the recent literature on continuous time game theory has addressed these technical issues is by imposing restrictions on the strategies that players can use. For instance, in Sannikov (2007, 2008) players are restricted to use strategies that are adapted to the filtration generated by the flow of public information.

The analysis of bargaining games in continuous time presents additional challenges on top of these issues. To see this, consider Rubinstein’s discrete time bargaining game with alternating offers. The following strategy profile is a Nash equilibrium in this game: player 2 only accepts offers that give her all of the surplus when she is responder and always offers to keep all of the surplus when she is proposer; player 1 always offers all of the surplus to her opponent when she is proposer and accepts any proposal when she is responder. Of course, this strategy profile is not subgame perfect: player 2 would be better-off by accepting an offer of $\delta + \varepsilon$ than by rejecting it (as rejection would give her a payoff of at most $\delta$). Indeed, this logic implies that in any subgame perfect equilibrium the responder’s payoff must be equal to her continuation value of waiting until she can make offers.

Note however that this will no longer be true if players can make offers in continuous time. Fix $\Delta > 0$ and consider the continuous time game in which player 1 is proposer at $t \in [0, \Delta)$, player 2 is proposer at $t \in [\Delta, 2\Delta)$, etc. In this setting, the outcome in which player 2 gets the entire surplus can be supported in a subgame perfect equilibrium: at each instant $t$ at which she is responder player 2 will reject any offer strictly lower than 1, in the expectation that player 1’s offer will increase up to 1 at a rate fast enough that compensates the cost of delay. Note that there is nothing player 1 can do against player 2’s strategy; hence, after such a deviation it is optimal for her to increase her offer rapidly up to 1 and confirm player
2’s expectations. Thus, in a continuous time bargaining game subgame perfection does not refine the set of outcomes in the way it does when the game is played in discrete time.

In this paper, I do two things in order to avoid these complications that arise when modelling bargaining games in continuous time. First, I focus directly on the outcomes and payoffs of the continuous time bargaining model, and don’t fully specify the set of strategies of the players. Second, I define an equilibrium to be an outcome that satisfies the optimality conditions in (2.2). As equation (2.3) shows, this definition of an equilibrium implies that the responder’s payoff is always equal to her expected continuation value, just as in a subgame perfect equilibrium of a discrete time bargaining game.\footnote{A different approach would be to impose restrictions on the strategies that players can use; for instance, to restrict strategies to be adapted to the filtration generated by public information. On top of this one would need to impose additional restrictions in order to recover subgame perfect-type behavior. Although one can get the same results by following this approach, in this paper I choose to focus directly on the outcomes of the model, as this makes the analysis simpler and more economic in terms of notation.}

2.3 Equilibrium

In this section I show that the continuous time bargaining model introduced in Section 2.2 has a unique equilibrium. That is, I show that there is a unique outcome \((A, V_1, V_2)\) such that \(V_1\) and \(V_2\) satisfy condition (2.2). As a first step, the following result establishes that every equilibrium outcome must involve immediate agreement.

**Proposition 2.1 (Immediate Agreement)** Let \((A, V_1, V_2)\) is an equilibrium outcome. Then, it must be that \(A = \mathbb{R}\).

**Proof:** Suppose \((A, V_1, V_2)\) is an equilibrium outcome and assume by contradiction that \(A\) is a strict subset of \(\mathbb{R}\). Since \(\mathbb{R}\setminus A\) is open (because \(A \in \mathcal{A}\)), there exists an open interval \((y, \bar{y})\) such that \((y, \bar{y}) \notin A\), so \(\tau (A) > 0\) whenever \(x_0 \in (y, \bar{y})\).\footnote{If I allow for agreement regions \(A\) that don’t belong to the set \(\mathcal{A}\), then there would be equilibrium agreement regions of the form \(\mathbb{R}\setminus Z\), where \(Z\) is a set of measure zero. The restriction to \(A \in \mathcal{A}\) rules out this (trivial) source of multiplicity.}
Define $W(x) := V_1(x) + V_2(x)$. Then, for all $x \in (y, \bar{y})$ it must be that

$$W(x) = E \left[ e^{-r\tau(A)} \big| x_0 = x \right] < 1,$$

where the inequality follows from the fact that $\tau(A) > 0$ whenever $x_0 \in (y, \bar{y})$. Thus, $V_1(x) + V_2(x) < 1$ for all $x \in (y, \bar{y})$. But this implies that, when $x_t \in (y, \bar{y})$, proposer $i$ is better off by offering $V_j(x_t)$ to her opponent (and obtaining a payoff of $1 - V_j(x_t) > V_i(x_t)$ for herself) than by delaying. Therefore, $(A, V_1, V_2)$ cannot be an equilibrium outcome.

Proposition 1 shows that players will always reach an immediate agreement. That is, any equilibrium outcome $(A, V_1, V_2)$ must have $A = A^* := \mathbb{R}$.\(^{12}\) Define $A_i^* = [0, +\infty)$ and $A_i^* = (-\infty, 0]$, and for $i = 1, 2$ let $\tau(A_i^*) = \inf\{t \geq 0 : x_t \in A_i^*\}$.

**Corollary 2.1** Let $(A^*, V_1, V_2)$ be an equilibrium outcome. Then, for $i = 1, 2$, $i \neq j$,

$$V_i(x) = \begin{cases} 
1 - V_j(x) & \text{if } x \in A_i^*, \\
E \left[ e^{-r\tau(A_i^*)} (1 - V_j(0)) \big| x_0 = x \right] & \text{if } x \notin A_i^*. 
\end{cases} \tag{2.4}$$

**Proof:** Let $(A^*, V_1, V_2)$ be an equilibrium outcome. Since $x_t$ has continuous sample paths, it follows that $\tau(A_i^*) = \tau(0)$ whenever $x_0 \notin A_i^*$. Equation (2.3) then implies that, for all $x \notin A_i^*$,

$$V_i(x) = E \left[ e^{-r\tau(A_i^*)} V_i(0) \big| x_0 = x \right] = E \left[ e^{-r\tau(A_i^*)} (1 - V_j(0)) \big| x_0 = x \right],$$

where the second equality follows from the fact that $V_1(x) + V_2(x) = 1$ for all $x$. Finally, this also implies that $V_i(x) = 1 - V_j(x)$ for all $x \in A_i^*$.

\(^{12}\)Thus, if $(A^*, V_1, V_2)$ is an equilibrium outcome, the quantity $V_i(x)$ is the share of the surplus that player 1 gets when $x_0 = x$.  

50
Corollary 2.1 provides a partial characterization of the payoffs that can arise in an equilibrium of this model: player $i$’s payoff when she is responder is given by the expected discounted value of waiting until $x_t$ reaches 0 and getting $1 - V_j(0)$ of the surplus at that point. On the other hand, when player $i$ is making proposals she immediately makes an acceptable offer to her opponent, thus receiving a payoff of $1 - V_j(x)$.

Let $(A^*, V_1, V_2)$ be an equilibrium outcome. By Corollary 2.1, for $i = 1, 2$ and for all $x \not\in A_i^*$ it must be that

$$V_i(x) = E \left[ e^{-rt(A_i^*)} (1 - V_j(0)) \right]_{x_0 = x}.$$

The fact that $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion therefore implies that $V_i(x)$ solves

$$rV_i(x) = V_i'(x) + \frac{1}{2} \sigma^2 V_i''(x) \text{ for all } x \not\in A_i^*,$$

(2.5)

with boundary conditions $V_i(0) = 1 - V_j(0)$ and $\lim_{x \to -\infty} V_i(x) = \lim_{x \to -\infty} V_2(x) = 0$ (see, for instance, Chapter 3 in Harrison, 1985). The general solution to equation (2.5) is

$$V_i(x) = C_i e^{-\alpha x} + D_i e^{\beta x},$$

where $\alpha = (\sqrt{\mu^2 + 2\sigma^2 r} + \mu)/\sigma^2$ and $\beta = (\sqrt{\mu^2 + 2\sigma^2 r} - \mu)/\sigma^2$. The boundary conditions in the limit as $|x| \to \infty$ imply that $C_1 = D_2 = 0$, so

$$V_1(x) = \begin{cases} D_1 e^{\beta x} & \text{if } x < 0, \\ 1 - C_2 e^{-\alpha x} & \text{if } x \geq 0, \end{cases} \quad V_2(x) = \begin{cases} 1 - D_1 e^{\beta x} & \text{if } x \leq 0, \\ C_2 e^{-\alpha x} & \text{if } x > 0. \end{cases}$$

(2.6)

The boundary condition at 0 (i.e., $V_1(0) + V_2(0) = 1$) implies that $D_1 = 1 - C_2$. Note next that in any equilibrium it must be that $V_i(x) \in [0, 1]$ for all $x \in \mathbb{R}$ and for $i = 1, 2$. That is, a player cannot get negative payoffs (as she can always guarantee herself zero by delaying an agreement forever), and cannot get a payoff larger than 1 (since a player cannot receive more than the total surplus). This implies that $C_2 \in [0, 1]$. Therefore, there is a one-dimensional family of functions $(V_1, V_2)$ satisfying (2.6) and the boundary conditions.
For any $c \in [0, 1]$, let $V^c = (V^c_1, V^c_2)$ be the functions in equation (2.6) with $C_2 = c$ and $D_1 = 1 - c$. Let $c^* = \beta / (\alpha + \beta)$. One can show that $c^*$ is the unique $c \in [0, 1]$ such that the payoff functions $V^c_1$ and $V^c_2$ are differentiable at 0. That is, $(V^c_1, V^c_2)$ satisfy the smooth pasting condition $(V^c_1)'(0^-) = -(V^c_2)'(0^+)$. For $i = 1, 2$, let $V^*_i = V^c_i$.

**Theorem 2.1** The outcome $(A^*, V^*_1, V^*_2)$ is the unique equilibrium.

**Proof:** See Appendix A.2.1.

Theorem 2.1 shows that the continuous time model yields a unique prediction regarding the outcome of negotiations in settings in which bargaining power fluctuates over time. The following examples illustrate how the share of the surplus that each player gets depends on the model’s parameters. The first example considers a situation in which bargaining power evolves as a Brownian motion with no drift and shows how the share of the surplus that player 1 gets changes with the volatility $\sigma$ of the diffusion process. The second example shows how player 1’s payoffs are affected by changes in the drift of the stochastic process.

**Example 1** Suppose that $x_t$ evolves as $(0, \sigma)$-Brownian motion. Figure 2.2 plots $V^*_1(x)$ for different values of $\sigma$. When volatility is low player 1 obtains a large fraction of the surplus even when her bargaining power is only relatively high, and she gets a small fraction of the surplus when her bargaining power is only relatively low. In other words, when $\sigma$ is low the function $V^*_1$ is very steep at 0. As $\sigma$ increases, $V^*_1$ becomes flatter around 0. The intuition behind this is as follows. When $\mu = 0$ the process $x_t$ is only driven by the Brownian motion $B$. In this case the expected waiting time until $x_t$ crosses 0 increases when the volatility $\sigma$ decreases. Therefore, when volatility is low the responder has to wait a long time in order to make offers, so she is willing to accept proposals that give her a low payoff. This implies that the proposer is able to obtain a large share of the surplus when $\sigma$ is low.

**Example 2** Suppose next that $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion with $\mu \geq 0$. Figure 2.3 plots $V^*_1(x)$ for different values of $\mu$, holding $\sigma = 0.5$. Player 1’s payoff increases for
all values of $x$ as $\mu$ increases. As $\mu$ increases, the expected waiting time until $x_t$ crosses the threshold 0 goes down if $x_0 < 0$, and goes up if $x_0 > 0$. Therefore, when $\mu$ is large player 1 can extract more surplus both when she is responder (since the expected waiting time until she can make offers is short) and when she is the proposer (since now player 2’s continuation value is low, as she has to wait a long time in order to make offers).
2.4 Other stochastic processes

In Sections 2.2 and 2.3 I analyzed the continuous time bargaining model when relative bargaining power evolves as a \((\mu, \sigma)\)-Brownian motion. In this section I show how the results in the previous sections extend to the case in which \(x_t\) follows a more general diffusion process.

Let \(B = \{B_t, \mathcal{F}_t^B : 0 \leq t < \infty\}\) be a one-dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with \(\mathcal{F}_t^B : 0 \leq t < \infty\) the completion of the filtration generated by the Brownian motion. Assume that the process \(x_t\) solves

\[
dx_t = \mu(x_t) \, dt + \sigma(x_t) \, dB_t, \tag{2.7}
\]

with \(x_0 \in \mathbb{R}\) and with \(\mu : \mathbb{R} \to \mathbb{R}\) and \(\sigma : \mathbb{R} \to \mathbb{R}\) satisfying conditions for existence and uniqueness of a strong solution to this stochastic differential equation (see, for instance, Theorem 5.2.9 in Karatzas and Shreve (1998)). Assume further \(\mu(\cdot)\) and \(\sigma(\cdot)\) are \(C^2\), and that \(x_t\) can take values in entire real line. The bargaining protocol is the same as in the previous sections: player 1 makes offers when \(x_t > 0\), and player 2 makes offers when \(x_t < 0\).

Note first that Proposition 2.1 continues to hold in this more general setup. That is, any equilibrium outcome \((A, V_1, V_2)\) must have \(A = A^*\), so players always reach an immediate agreement. Moreover, by Corollary 2.1 (which also continues to hold in this setting), for any equilibrium outcome \((A^*, V_1, V_2)\) the payoff functions \(V_1\) and \(V_2\) satisfy

\[
V_i(x) = \begin{cases} 
1 - V_j(x) & \text{if } x \in A_i^*, \\
E\left[ e^{-r(A^*)} (1 - V_j(0)) \right]_{x_0 = x} & \text{if } x \notin A_i^*. 
\end{cases} \tag{2.8}
\]

One can show that in this case the functions \(V_1(x)\) and \(V_2(x)\) satisfying (2.8) solve

\[
r V_i(x) = \mu(x) V_i'(x) + \frac{1}{2} \sigma^2(x) V_i''(x) \quad \text{for } x \notin A_i^*,
\]
with boundary conditions \( \lim_{x \to -\infty} V_1(x) = \lim_{x \to \infty} V_2(x) = 0 \) and \( V_1(0) + V_2(0) = 1 \).\(^{13}\)

Finally, using arguments similar to the ones in the proof of Theorem 1 one can show that any equilibrium outcome \((A^*, V_1, V_2)\) must satisfy the smooth pasting condition \((V_1)'(0^-) = -(V_2)'(0^+)\). These four boundary conditions pin down the functions \(V_1^*\) and \(V_2^*\).

I stress however that when \(x_t\) follows the more general diffusion process (2.7), the system of differential equations that characterizes equilibrium payoffs will typically not have a closed form solution. In these cases, one needs to resort to numerical methods to solve for the equilibrium payoffs \(V_1^*\) and \(V_2^*\).

**Example 3 (mean reverting bargaining power)** Suppose \(x_t\) evolves as

\[
dx_t = -\lambda x_t dt + \sigma dB_t,
\]

for some \(\lambda > 0, \sigma > 0\) and \(x_0 = x\). In this case relative bargaining power \(x_t\) always reverts to its long run mean of 0, with the parameter \(\lambda\) measuring the speed of mean reversion. Figure 2.4 plots player 1’s equilibrium payoff \(V_1^*\) for different values of \(\lambda\), holding \(\sigma = 0.5\). The figure shows that player 1’s payoff becomes flatter when \(\lambda\) is large. The reason for this is that the expected time the responder has to wait to become proposer decreases with \(\lambda\), since the process \(x_t\) will revert to its long-run mean of 0 at a faster rate. Therefore, when \(\lambda\) is large the responder is able to obtain a sizable share of the surplus even when \(x_t\) is far from zero.

**Remark 2.2** The limiting conditions \(\lim_{x \to -\infty} V_1^*(x) = \lim_{x \to \infty} V_2^*(x) = 0\) come from the assumption that \(x_t\) takes values on the whole real line. If \(x_t\) only took values on some interval, then the boundary conditions need to be modified. For instance, suppose that \(x_t\) evolves as in equation (2.7), but has absorbing boundaries at \(a < 0\) and \(b > 0\). That is, \(x_0 \in (a, b)\) and if \(x_t\) hits either \(a\) or \(b\) it stays there forever. In this case the equilibrium payoffs also solve the same ordinary differential equation, but with boundary conditions \(V_1^*(a) = 0, V_2^*(b) = 0, V_1^*(0) + V_2^*(0) = 1\) and \((V_1^*)'(0^-) = -(V_2^*)'(0^+)\).

\(^{13}\)One can show that this ordinary differential equation has a solution when \(\mu(\cdot)\) and \(\sigma(\cdot)\) are \(C^2\) and satisfy the conditions for existence and uniqueness of a solution to (2.11).
2.5 Discrete time bargaining games

In the previous sections I presented a continuous time bargaining model with time-varying bargaining power and characterized its unique equilibrium. In this section I study the discrete time version of the continuous time model. The main objective is to show that the equilibrium of the continuous time formulation corresponds to the limiting subgame perfect equilibrium of the discrete time bargaining game, when players can make offers arbitrarily frequently. In other words, the goal of this section is to establish that both the continuous time model and its discrete time counterpart provide the same unique prediction.

The discrete time game is as follows. Two players, $i = 1, 2$, bargain over how to divide a perfectly divisible surplus of size 1. Players can only make offers at points on the grid $T(\Delta) = \{0, \Delta, 2\Delta, \ldots\}$, where $\Delta > 0$ measures the time interval between consecutive offers. As in Sections 2 and 3, I assume that bargaining power $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion, for some constants $\mu$ and $\sigma > 0$ and with some initial value $x_0 \in \mathbb{R}$. That is, the process $x_t$ determining relative bargaining power evolves in continuous time, but players can only make offers at points on the grid $T(\Delta)$. The fact that $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion implies that $x_{t+\Delta} | x_t \sim N(x_t + \mu \Delta, \sigma^2 \Delta)$.$^{14}$

$^{14}$Appendix A.2.6 generalizes the results in this section to the more case in which $x_t$ solves the stochastic differential equation (2.7).
The bargaining protocol is as follows. At any bargaining round \( t \in T(\Delta) \) the realization of \( x_t \) determines the identity of the proposer: player 1 is proposer at time \( t \) if \( x_t \geq 0 \), and player 2 is proposer at time \( t \) if \( x_t < 0 \).\(^{15}\) The proposer makes an offer \( z \in \{ y \in \mathbb{R}_+^2 : y_1 + y_2 \leq 1 \} \). The other player, the responder, can either accept or reject the offer. If the responder accepts the offer, the game ends and players collect their payoffs. If the responder rejects the offer, the game moves to period \( t + \Delta \).

An outcome of this model is a pair \((u, \tau)\), where \( \tau \) is a stopping time taking values on \( T(\Delta) \) and \( u = (u_1, u_2) \) is a measurable random variable with \( u(x_\tau) \in \{ y \in \mathbb{R}_+^2 : y_1 + y_2 \leq 1 \} \) if \( \tau < \infty \) and \( u = 0 \) if \( \tau = \infty \). That is, an outcome \((u, \tau)\) is given by a stopping time \( \tau \) at which players reach an agreement and a random variable \( u \) which determines the share of the surplus (and therefore the utility) that each player gets at the agreement date. Note that an outcome \((u, \tau)\) need not be stationary, since the stopping time \( \tau \) may not be stationary and the random variable \( u \) may depend on things other than the value of \( x_\tau \); for instance, both \( \tau \) and \( u \) may depend on the history of offers made.

Players are risk neutral expected utility maximizers and that they share the same discount factor \( \delta (\Delta) := e^{-r \Delta} \) (with \( r > 0 \)). Therefore, player \( i \)'s payoff from outcome \((u, \tau)\) given an initial state \( x_0 = x \) is \( E[e^{-r \tau} u_i(x_\tau) | x_0 = x] \). In what follows I will denote by \( \Gamma_\Delta \) the discrete time bargaining game with interval between offers \( \Delta > 0 \).

For each player, a strategy specifies a feasible action at every history at which that player must act (either make an offer or respond to one). A strategy profile is a pair of strategies, one for each player. A strategy profile \( s = (s_1, s_2) \) induces an outcome \((u, \tau)\). A strategy profile \( s = (s_1, s_2) \) is a subgame perfect equilibrium (SPE) if, for \( i = 1, 2 \), \( s_i \) is a best response to \( s_{-i} \) at every history.

**Theorem 2.2** For any \( \Delta > 0 \), \( \Gamma_\Delta \) has a unique SPE. In the unique SPE players reach an immediate agreement. For \( i = 1, 2 \), let \( V^{\Delta}_i(x) \) denote player \( i \)'s SPE payoff when bargaining

\(^{15}\)Note that I’m assuming that player 1 makes offers at \( t \) when \( x_t = 0 \). As it will become clear below, in this setting the identity of the proposer when \( x_t = 0 \) has no effect on equilibrium payoffs.
power is equal to \( x \). These payoffs satisfy

\[
V_1^\Delta (x) = \begin{cases} 
   e^{-r\Delta} E \left[ V_1^\Delta (x_{t+\Delta}) \mid x_t = x \right] & \text{if } x < 0, \\
   1 - e^{-r\Delta} E \left[ V_2^\Delta (x_{t+\Delta}) \mid x_t = x \right] & \text{if } x \geq 0, 
\end{cases}
\]

\[ (2.9) \]

\[
V_2^\Delta (x) = \begin{cases} 
   1 - e^{-r\Delta} E \left[ V_1^\Delta (x_{t+\Delta}) \mid x_t = x \right] & \text{if } x < 0, \\
   e^{-r\Delta} E \left[ V_2^\Delta (x_{t+\Delta}) \mid x_t = x \right] & \text{if } x \geq 0. 
\end{cases}
\]

\[ (2.10) \]

**Proof:** See Appendix A.2.2.

The content of Theorem 2.2 can be described as follows. In a subgame perfect equilibrium the responder accepts any offer that gives her at least her continuation payoff (which depends on the current level of relative bargaining power \( x_t \)) and rejects any offer that gives her less than this quantity. Knowing this, the proposer always makes the lowest offer that the responder is willing to accept, and the game ends with an immediate agreement.

Merlo and Wilson (1998) study bargaining games with transferable utility in which the realization of an exogenous time homogeneous Markov process determines at each period both the size of the surplus and the identity of the proposer. The model in this section is a special case of the stochastic bargaining game in Merlo and Wilson (1998): in my model the size of the surplus is constant, and the realization of the Markov process (i.e., the diffusion process) only determines the identity of the proposer at reach bargaining round. One of the main insights in Merlo and Wilson (1998) is that players will delay reaching an agreement only if they expect the size of the surplus to grow fast enough in the near future. In the setup that I’m analyzing the size of surplus is constant, so players always reach an agreement in the first bargaining round.

In this section’s game the player responding to offers has to wait until \( x_t \) crosses 0 in order to gain the right to make proposals. Therefore, the offers that the responder is willing to accept at any time \( t \) depend on the value of relative bargaining power \( x_t \): when \( x_t \) is far from 0 the responder is willing to accept low proposals, as this implies that she will have
to wait for a long period to make offers. Indeed, the share of the surplus that a player gets in equilibrium is closely related to the probability with which she expects to make offers in future periods (and therefore to the current value $x_t$). To see this, suppose that $x_t$ is such that player $i$ is proposer at time $t$. By equations (2.9) and (2.10), player $i$'s payoff when bargaining power is $x_t$ is given by

$$V_i^\Delta(x) = 1 - e^{-r\Delta} E[V_j^\Delta(x_{t+\Delta})|x_t = x]$$

$$= 1 - e^{-r\Delta} + e^{-r\Delta} E[V_i^\Delta(x_{t+\Delta})|x_t = x],$$

where the second equality follows from the fact that $V_i^\Delta(x) + V_j^\Delta(x) = 1$ for all $x \in \mathbb{R}$. Player $i$ receives a payoff of $1 - e^{-r\Delta}$ plus her continuation value $e^{-r\Delta} E[V_i^\Delta(x_{t+\Delta})|x_t = x]$ when she is proposer. When player $i$ is the responder, she only gets her continuation value.

For any $s \geq 0$, let $P_i(t+s, x_t = x)$ denote the probability with which player $i$ is proposer in period $t+s$ given $x_t = x$. That is, $P_1(t+s, x_t = x) = \Pr(x_{t+s} \geq 0|x_t = x)$ and $P_2(t+s, x_t = x) = \Pr(x_{t+s} < 0|x_t = x)$. Since $x_t$ is a time-homogeneous Markov process, the probability $P_i(t+s, x_t = x)$ only depends only on $x$ and $s$; therefore, I can write $P_i(s, x)$. Note also that $P_i(0, x) = 1$ if $x \geq 0$ and $P_i(0, x) = 0$ if $x < 0$ (and $P_2(0, x) = 1 - P_1(0, x)$).

Using this notation and solving equation (2.11) forward yields

$$V_i^\Delta(x) = (1 - e^{-r\Delta}) P_i(0, x) + e^{-r\Delta} P_i(\Delta, x) (1 - e^{-r\Delta}) + ...$$

$$= (1 - e^{-r\Delta}) \sum_{k=0}^{\infty} e^{-rk\Delta} P_i(k\Delta, x) = \frac{(1 - e^{-r\Delta})}{\Delta} \sum_{k=0}^{\infty} e^{-rk\Delta} \Delta P_i(k\Delta, x).$$

For $i = 1, 2$, define $U_i(x) := r \int_0^\infty e^{-rs} P_i(s, x) \, ds$.

**Lemma 2.1** Fix a sequence $\{\Delta_n\} \to 0$. Then, for $i = 1, 2$, $V_i^{\Delta_n} \to U_i$ uniformly as $n \to \infty$.

**Proof:** Recall that $x_{t+s}|x_t \sim N(x_t + \mu s, \sigma^2 s)$. Thus, $P_2(s, x) = \Phi((x - \mu s)/\sigma \sqrt{s})$ and $P_1(s, x) = 1 - \Phi((x - \mu s)/\sigma \sqrt{s})$ and , where $\Phi(\cdot)$ is the cdf of the standard Normal distribution. Note that for all $x \in \mathbb{R}$ the function $e^{-rs} P_i(s, x)$ is Riemann integrable in $s$ for
$i = 1, 2$. Indeed, for all $x \neq 0$ both $e^{-rs}P_1(s, x)$ and $e^{-rs}P_2(s, x)$ are continuous in $s$ and hence Riemann integrable. At $x \neq 0$, both $P_1(s, 0)$ and $P_2(s, 0)$ have a single discontinuity point at $s = 0$.\footnote{Indeed, for $i = 1, 2$, $\lim_{s \to 0} e^{-rs}P_i(s, 0) = \frac{1}{2}$, but $P_1(0, 0) = 1$ and $P_2(0, 0) = 0.$} However, $e^{-rs}P_i(s, 0)$ is still Riemann integrable in $s$ for $i = 1, 2$, since any function with a finite number of discontinuity points is Riemann integrable.

Fix a sequence $\{\Delta_n\} \to 0$. Since $e^{-rs}P_i(s, x)$ is Riemann integrable in $s$, it follows that

$$V_i^\Delta_n(x) = \frac{1 - e^{-r\Delta_n}}{\Delta_n} \sum_{k=0}^{\infty} e^{-r k \Delta_n} \Delta_n P_i(k \Delta_n, x) \to r \int_0^\infty e^{-rs}P_i(s, x) \, ds.$$

This establishes that $V_i^\Delta_n \to U_i$ pointwise as $n \to \infty$. The proof that this convergence is uniform is in Appendix A.2.2.

Lemma 1 shows that $V_i^\Delta_n \to U_i$ as $n \to \infty$. The next step is to show that $U_i(x) = V_i^*(x)$ for $i = 1, 2$, where $V_1^*(x)$ and $V_2^*(x)$ are the equilibrium payoffs of the continuous time bargaining model. Let $p(y, x, s)$ denote the transition density of $x$; that is, for any $x, y \in \mathbb{R}$ and any $s \geq 0$, $p(y, x, s) = \Pr(x_{t+s} \in dy \mid x_t = x)$. It is well known that $p(y, x, s)$ satisfies Kolmogorov’s backward equation (i.e., Chapter 3.1 in Harrison, 1985),

$$\frac{\partial}{\partial s} p(y, x, s) = \mu \frac{\partial}{\partial x} p(y, x, s) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} p(y, x, s). \tag{2.12}$$

Since $P_1(s, x) = \int_0^\infty p(y, x, s) \, dy$, by Leibniz’s rule $P_1(s, x)$ also satisfies this backward equation. On the other hand, the rule of integration by parts implies that, for all $x \neq 0$,}

$$U_1(x) = r \int_0^\infty e^{-rs}P_1(s, x) \, ds = -e^{-rs}P_1(s, x) \bigg|_0^\infty + \int_0^\infty e^{-rs} \frac{\partial P_1(s, x)}{\partial s} \, ds.$$
Note that \(-e^{-rs}P_1(s,x)\big|_0^\infty = 0\) for all \(x < 0\). Therefore, for all \(x < 0\) it must be that

\[
U_1(x) = \int_0^\infty e^{-rs} \frac{\partial P_1(s,x)}{\partial s} \, ds
= \int_0^\infty e^{-rs} \left( \mu \frac{\partial}{\partial x} P_1(s,x) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} P_1(s,x) \right) \, ds,
\]

(2.13)

where the second equality follows from the fact that \(P_1\) satisfies (2.12). Since \(U'_1(x) = r \int_0^\infty e^{-rs} \frac{\partial P_1(s,x)}{\partial x} \, ds\) and \(U''_1(x) = r \int_0^\infty e^{-rs} \frac{\partial^2 P_1(s,x)}{\partial x^2} \, ds\), equation (2.13) yields

\[
rU_1(x) = \mu U'_1(x) + \frac{1}{2} \sigma^2 U''_1(x) \quad \text{for all } x < 0.
\]

A symmetric argument establishes that

\[
rU_2(x) = \mu U'_2(x) + \frac{1}{2} \sigma^2 U''_2(x) \quad \text{for all } x > 0.
\]

The general solutions to \(U_1\) and \(U_2\) are given by \(U_1(x) = E_1 e^{-\alpha x} + F_1 e^{\beta x}\) for \(x < 0\) and \(U_2(x) = E_2 e^{-\alpha x} + F_2 e^{\beta x}\) for \(x > 0\). To show that \(U_i = V_i^*\) for \(i = 1, 2\) it suffices to show that the four constants \((E_i, F_i)_{i=1,2}\) are the same constants in \(V_1^*\) and \(V_2^*\). Since \(\lim_{x \to -\infty} e^{-rs} P_1(s,x) = \lim_{x \to +\infty} e^{-rs} P_2(s,x) = 0\) for all \(s \geq 0\), it follows that \(\lim_{x \to -\infty} U_1(x) = \lim_{x \to +\infty} U_2(x) = 0\). This implies that \(E_1 = F_2 = 0\). The other two boundary conditions needed to solve for \(E_2\) and \(F_1\) follow from Lemma 2.2.

**Lemma 2.2** \(U_1(x)\) and \(U_2(x)\) are continuous in \(x\). Moreover, \(U_1(0) = \frac{\alpha}{\alpha + \beta}\) and \(U_2(0) = \frac{\beta}{\alpha + \beta}\).

**Proof:** See Appendix A.2.3.

By Lemma 2.2, \(U_1\) and \(U_2\) satisfy the boundary conditions \(U_1(0) = \alpha/ (\alpha + \beta)\) and \(U_2(0) = \beta/ (\alpha + \beta)\). These boundary conditions imply that \(U_1(x) = \frac{\alpha}{\alpha + \beta} e^{\beta x} = V_1^*\) for all \(x \leq 0\) and \(U_2(x) = \frac{\beta}{\alpha + \beta} e^{-\alpha x} = V_2^*\) for all \(x \geq 0\). It then follows that \(U_i = V_i^*\) for \(i = 1, 2\).
For any $\Delta > 0$, let $V^\Delta(x) = (V_1^\Delta(x), V_2^\Delta(x))$ be the SPE payoffs of $\Gamma_\Delta$ and let $V^*(x) = (V_1^*(x), V_2^*(x))$ be the players’ equilibrium payoffs of the continuous time bargaining model of Sections 2.2 and 2.3. The analysis above implies the following result:

**Theorem 2.3** Suppose bargaining power $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion. Then, $V^{\Delta_n}$ converges uniformly to $V^*$ as $n \to \infty$ for any sequence $\{\Delta_n\} \to 0$.

**Remark 2.3** The arguments leading to Theorem 3 rely on the fact that the transition density $p(y, x, s)$ satisfies Kolmogorov’s backward equation (2.16) when $x_t$ evolves as a $(\mu, \sigma)$-Brownian motion. The class of one-dimensional diffusion processes whose transition density satisfies the backward equation is very broad. Suppose $x_t$ solves (2.7), with $\mu(\cdot)$ and $\sigma(\cdot)$ both $C^2$ and satisfying conditions for existence and uniqueness of a solution to (2.7). Rogers (1985) showed that the transition density $p(y, x, s)$ of such a diffusion process satisfies the following version of Kolmogorov’s backward equation:

$$
\frac{\partial}{\partial s} p(y, x, s) = \mu(x) \frac{\partial}{\partial x} p(y, x, s) + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} p(y, x, s).
$$

When $x_t$ follows such a diffusion process, for every $\Delta$ the discrete time bargaining game $\Gamma_\Delta$ will also have a unique SPE, with equilibrium payoffs $(V_1^\Delta, V_2^\Delta)$ solving (2.9) and (2.10). In Appendix A.2.6, I show that in this setting, $V^{\Delta_n} \to V^*_i$ as $n \to \infty$ for any $\{\Delta_n\} \to 0$, where $V_1^*$ and $V_2^*$ solve

$$
r V_1^*(x) = \mu(x) (V_1^*)'(x) + \frac{1}{2} \sigma^2(x) (V_1^*)''(x) \text{ for all } x < 0,
$$

$$
r V_2^*(x) = \mu(x) (V_2^*)'(x) + \frac{1}{2} \sigma^2(x) (V_2^*)''(x) \text{ for all } x > 0,
$$

with boundary conditions $\lim_{x \to -\infty} V_1^*(x) = \lim_{x \to -\infty} V_2^*(x) = 0$, $V_1^*(0) + V_2^*(0) = 1$ and $(V_1^*)'(0^-) = -(V_2^*)'(0^+)$. Therefore, Theorem 2.3 extends to this more general case.
2.6 Applications

2.6.1 Legislative bargaining and efficient delays

Consider a legislative bargaining setting in which two parties \((i = 1, 2)\) have to decide which policy in \([0, 1]\) to implement. There are two possible states of the world, \(S = \{s_A, s_B\}\). The state determines the preferences of the parties over policies \([0, 1]\). Conditional on the state of the world, both parties have the same preferences over policies: the parties’ utility index over policies \(z \in [0, 1]\) at state \(s \in S\) is

\[
u(z; s) = \begin{cases} 
1 - z & \text{if } s = s_A, \\
z & \text{if } s = s_B.
\end{cases}
\]

That is, policy 0 (policy 1) is the parties’ preferred policy when the state is \(s_A\) (when the state is \(s_B\)). For an example, suppose the parties are deciding whether to open up their country to free trade or not, and the policy space \([0, 1]\) represents the degree of openness. In state \(s_A\) it is optimal to maintain the country closed, while in state \(s_B\) it is optimal to fully open the country’s economy to world trade.

Parties have different beliefs about the likelihood of each state. Let \(p_i\) be the probability that party \(i\) assigns to the state being \(s_A\). I assume that \(p_1 > 1/2 > p_2\), so party 1 believes that state \(s_A\) is more likely and party 2 believes that state \(s_B\) is more likely. This implies that, from an ex-ante perspective, party 1’s ideal policy is 0 and party 2’s ideal policy is 1. I assume that this difference in beliefs in common knowledge. To simplify the analysis, I further assume that \(p_1 = 1 - p_2 = p > 1/2\).

Parties are expected utility maximizers (with utility index \(u(s, z)\)) and discount future payoffs at rate \(r > 0\): party \(i\)’s payoff if (the possibly random) alternative \(v\) is implemented at (the possibly random) time \(\tau\) is

\[E[e^{-r\tau}(p_i u(s_A, v) + (1 - p_i) u(s_B, v))].\]

At each moment in time until agreement is reached, parties may receive a signal about the true state of the world with Poisson intensity \(\lambda\): if the signal has not arrived by time
For simplicity, I assume that the signal is fully informative, so that parties learn the true state of the world upon its arrival. This implies that if parties have not reached an agreement by the time information arrives, then at that time they will immediately implement the correct policy. 

Let $u_i(z)$ be party $i$’s expected utility if policy $z$ is implemented prior to the arrival of information; that is, $u_1(z) = p + z(1 - 2p)$ and $u_2(z) = 1 - p + z(2p - 1)$. Note that $u_1(z) + u_2(z) = 1$ for all $z \in [0, 1]$, with $u_i(z) \in [1 - p, p]$ for all $z \in [0, 1]$ and $i = 1, 2$. Thus, one can think of this model as one which parties bargain over a unit surplus, but in which there is a lower bound $1 - p$ on the fraction of the surplus that each party can get.

As in Sections 2.2 and 2.3, the process $x_t$ evolves as $(\mu, \sigma)$-Brownian motion, and its realization determines the identity of the party making offers at each instant: party 1 makes offers at $t$ if $x_t > 0$, while party 2 makes offers at $t$ if $x_t < 0$. In this legislative bargaining setting, one can interpret the process $x_t$ as measuring the fraction of legislators supporting each party at any moment in time: party 1 controls a majority of seats in Congress when $x_t > 0$, while party 2 controls a majority of seats when $x_t < 0$. The assumption that the number of legislators supporting each party fluctuates may reflect a setting in which the preferences of the median voter are changing over time. With this interpretation in mind, I will sometimes refer to the party making offers as the majority party and to the party responding to offers as the minority party.

A supermajority rule is a number $\phi > 0$ such that party 1 can unilaterally implement a policy if $x_t \geq \phi$ and party 2 can unilaterally implement a policy if $x_t \leq -\phi$. Under a supermajority rule a party with enough support in Congress has the ability to implement policies unilaterally, without the approval of the minority party. On the other hand, when neither party has enough legislators (i.e., when $x_t \in (-\phi, \phi)$), the majority party needs the support of the minority party in order to implement policies.

Let $\tilde{r}$ denote the time at which information arrives. Suppose parties wait until $\tilde{r}$ and then implement the correct policy. The payoff that party $i$ gets from this outcome is $E[e^{-r\tilde{r}}] = $
\[ \int_{0}^{\infty} e^{-r s} \lambda e^{-(r+\lambda)s} ds = \lambda / (r + \lambda), \] so the total payoff from this outcome (i.e., the sum of the parties’ payoffs) is \( 2\lambda / (r + \lambda) \). On the other hand, implementing some policy \( z \in [0, 1] \) at some time \( 0 \leq t < \hat{t} \) yields a total payoff of \( e^{-rt} \leq 1 \). Thus, from a utilitarian point of view delaying an agreement until time \( \hat{t} \) is socially optimal if and only if \( 2\lambda / (r + \lambda) \geq 1 \), or \( \lambda \geq r \).

If parties are to implement a policy before time \( \hat{t} \), the highest utility that party \( i \) can get is \( p \). In what follows, I will assume that \( \lambda \) is such that \( \lambda / (r + \lambda) < p \), or \( \lambda < \bar{\lambda} := r p / (1 - p) \). This assumption bounds the rate at which information arrives to the point that each party would rather implement its preferred policy at the outset of negotiations than to wait until the arrival of information. This implies that a party obtaining a supermajority prior to the arrival of the public signal will always choose to implement its ex-ante preferred policy immediately.

An outcome of this model is given by a pair \((A, \eta)\), where \( A \in A := \{ A \subseteq \mathbb{R} : A \) is closed\} is the agreement region and \( \eta : A \rightarrow [0, 1] \) a policy function. The region \( A \) describes the set of states at which parties reach an agreement prior to the arrival of public information, and \( \eta \) describes the policy that is implemented if parties reach an agreement before information arrives. If parties have not reached an agreement by the time information arrives, then they immediately implement the correct policy. Under supermajority rule \( \phi \), an equilibrium outcome \((A, \eta)\) must satisfy the following conditions: (i) \( A \supseteq (-\infty, -\phi] \cup [\phi, \infty) \), (ii) \( \eta (x) = 1 \) for all \( x \in [\phi, \infty) \), and (iii) \( \eta (x) = 0 \) for all \( x \in (-\infty, -\phi] \). Since \( \lambda < \bar{\lambda} \), under supermajority rule \( \phi \) the bargaining must end whenever a party has a supermajority, in which case that party will implement its preferred policy immediately; this is the content of (i)-(iii). Let \( \mathcal{O}^\phi \) denote the set of outcomes satisfying these conditions.

Party \( i \)'s payoff from outcome \((A, \eta) \in \mathcal{O}^\phi\) is given by

\[ V_i (x) = E \left[ \lambda \int_{0}^{\tau(A)} e^{-(r+\lambda)s} ds + e^{-r\tau(A)} u_i (\eta (x_{\tau(A)})) \bigg| x_0 = x \right], \quad (2.14) \]
where \( \tau(A) := \inf \{ t : x_t \in A \} \). Under outcome \((A, \eta)\), party \( i \) receives a payoff of \( u_i(\eta(x_{\tau(A)})) \) at the agreement date. Prior to the agreement date, information may arrive according to the Poisson process with arrival rate \( \lambda \); if information arrives at time \( s < \tau(A) \) each party gets a payoff of 1, since at this point the correct policy is implemented. As above, I will denote an outcome \((A, \eta)\) as a triplet \((A, V_1, V_2)\), with \( A \in \mathcal{A} \) and with \( V_i \) satisfying (2.14) for \( i = 1, 2 \). Let \( \tau^\phi = \inf\{ t : x_t \notin (-\phi, \phi) \} \).

**Definition 2.2** Under supermajority rule \( \phi \) an outcome \((A, V_1, V_2) \in \mathcal{O}^\phi \) is an equilibrium if, for \( i = 1, 2 \), \( j \neq i \),

\[
V_i(x) = \sup_{\tau \in T_i} E \left[ \lambda \int_0^{\tau \wedge \tau^\phi} e^{-(r+\lambda)s} ds + e^{-(r+\lambda)(\tau \wedge \tau^\phi)} (1 - V_j(x_{\tau})) \mid x_0 = x \right], 
\tag{2.15}
\]

for all \( x \in \mathbb{R} \).

Definition 2.2 adapts definition 2.1 to this subsection’s model: in an equilibrium the responder’s payoff should always be equal to her expected continuation value of waiting until she regains the right to make offers.\(^{17}\) On the other hand, the party making proposals optimally chooses the time at which to make an acceptable offer to its opponent. The only difference is that in the current setting the parties’ continuation value incorporate the possibility that information arrives before \( x_t \) reaches \( A \), in which case both parties get a payoff of 1.

**Proposition 2.2** There is a unique equilibrium \((A^*, V_1^*, V_2^*)\). If \( \lambda < r \) the unique equilibrium outcome has \( A^* = \mathbb{R} \). If \( \lambda \in (r, \lambda) \) the unique equilibrium outcome has \( A^* = (-\infty, -\phi] \cup [\phi, \infty) \).

**Proof:** See Appendix A.2.4.

\(^{17}\)Note that if the party responding to offers delays an agreement, her opponent might obtain the required supermajority level and implement its preferred policy before \( x_t \) crosses 0. This risk is taken into account in equation (2.15), since any outcome \((A, V_1, V_2) \in \mathcal{O}^\phi \) has \( V_1(-\phi) = 0 = V_2(\phi) \).
Recall from the discussion above that delaying an agreement until the arrival of the public signal is socially optimal if and only if $\lambda \geq r$. Proposition 2.2 then shows that the equilibrium outcome is efficient if $\lambda < r$: parties optimally reach an immediate agreement in this case, without waiting for the arrival of public information. Intuitively, delaying an agreement is not beneficial in this setting, since the gains from waiting for the public signal are outweighed by the cost of delay.

When $\lambda \in (r, \bar{\lambda})$ parties only delay an agreement while $x_t \in (-\phi, \phi)$, and they reach an immediate agreement when one party has a supermajority. Waiting for the arrival of the public signal is socially beneficial when $\lambda \in (r, \bar{\lambda})$, so parties delay an agreement when $x_t \in (-\phi, \phi)$. However, since $\lambda < \bar{\lambda}$ a party with a supermajority will always find it optimal to implement its ex-ante preferred policy immediately. The resulting equilibrium outcome is inefficient, since parties might end up implementing a policy too soon, even when its optimal to wait until the arrival of the public signal. Note that, in this setting, a more stringent supermajority requirement (a larger level of $\phi$) induces parties to wait longer. Therefore, when $\lambda \in (r, \bar{\lambda})$ a more stringent supermajority rule leads to more efficient outcomes.

### 2.6.2 Takeovers in duopolistic industries

Consider a market for a homogeneous product in which there are two firms ($i = 1, 2$) competing. Firms produce at zero marginal cost and there is a unit mass of consumers who make flow purchases of this good. Let $r \cdot a$ denote the price at which firms sell their product when they are both on the market and assume that $a \in (1/2, 1)$.

In this setting the stochastic process $x_t$ measures the firm’s relative market shares. Let $B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. The process $x_t$ evolves as

$$dx_t = \mu dt + \sigma dB_t,$$  \hfill (2.16)
with \( x_0 \in (0, 1) \) and with reflecting boundaries at 0 and 1. The process \( x_t \) represents firm 1’s market share. That is, \( x_t = 0 \) represents a situation in which firm 2 controls the entire market, while \( x_t = 1 \) represents a setting in which firm 1 controls the entire market. When \( x_t = 1/2 \) each firm has a market share of 50%. The flow payoffs of firm 1 and 2 are \( \pi_1 (x) = rax \) and \( \pi_2 (x) = ra \left( 1 - x \right) \), respectively. Thus, firm \( i \)’s discounted expected profits are

\[
\Pi_i^D (x) = E \left[ \int_0^\infty e^{-rs} \pi_i (x) \, ds \bigg| x_0 = x \right],
\]

and the entire industry profits are \( \Pi^D (x) = \Pi_1^D (x) + \Pi_2^D (x) = a \) for all \( x \). On the other hand, a monopolist in this market would charge a price \( r > ar \), thus obtaining discounted profits of \( \Pi^M = 1 > a \). This implies that there are incentives for the firms to merge.

Assume that at each instant \( t \) the manager of the firm with the higher market share considers an amicable takeover of its opponent: when \( x_t > 1/2 \) firm 1 makes takeover offers to firm 2, and when \( x_t < 1/2 \) firm 2 makes takeover offers to firm 1. That is, in this setting the process \( x_t \) determines the bargaining protocol in the same way as in the baseline model of Sections 2 and 3: firm 1 is proposer when \( x_t > 1/2 \) and firm 2 is proposer when \( x_t < 1/2 \). A takeover offer in this setting is a price \( p \) at which the larger firm intends to buy the smaller one. If the offer \( p \) is successful the remaining firm becomes the monopoly supplier of the good, obtaining a (net) payoff of \( 1 - p \). Thus, one can think of a takeover as a split of a unit surplus, with firms obtaining a payoff in \( \{ y \in \mathbb{R}_+^2 : y_1 + y_2 = 1 \} \).

An outcome of this model is again given by a pair \( (A, \eta) \), where \( A \in \mathcal{A} := \{ A \subseteq [0, 1] : A \text{ is closed} \} \) is the agreement region and \( \eta : A \to \{ y \in \mathbb{R}_+^2 : y_1 + y_2 = 1 \} \) a function giving the net payoff that firms get after a deal has been reached. Firm \( i \)’s payoff from outcome \( (A, \eta) \) is given by

\[
V_i (x) = E \left[ \int_0^{\tau (A)} e^{-rs} \pi_i (x) \, ds + e^{-r \tau (A)} \eta_1 (x_{\tau (A)}) \bigg| x_0 = x \right].
\] (2.17)
Under outcome \((A, \eta)\) firm \(i\) receives a payoff of \(\eta_i (x_{\tau(A)})\) at the agreement date. Prior to this date firm \(i\) collects its flows profits \(\pi_i (x)\). I will denote an outcome \((A, \eta)\) as a triplet \((A, V_1, V_2)\), with \(A \in \mathcal{A}'\) and with \(V_i\) satisfying (2.17) for \(i = 1, 2\).

Let \(T'_i\) be the set of stopping times \(\tau(B)\) with \(B \subseteq (\langle -\infty, 1/2\rangle\) and let \(T'_2\) be the set of stopping times \(\tau(B)\) with \(B \subseteq [1/2, \infty)\).

**Definition 2.3** An outcome \((A, V_1, V_2)\) is an equilibrium if, for \(i = 1, 2, j \neq i\),

\[
V_i (x) = \sup_{\tau \in T'_i} E \left[ \int_0^{\tau} e^{-\tau s} \pi_i (x_s) ds + e^{-\tau \tau} (1 - V_j (x_\tau)) \bigg| x_0 = x \right],
\]

for all \(x \in \mathbb{R}\).

Definition 2.3 again adapts definition 2.1 to this subsection’s setting: in an equilibrium the responder’s payoff should always be equal to her expected continuation value of waiting until she regains the right to make offers, and then optimally choosing when to make an acceptable offer to her opponent. The difference is that definition 2.3 incorporates the fact that in this setting the firms earn flow payoffs while bargaining.

**Proposition 2.3** There is a unique equilibrium outcome \((A^*, V_1^*, V_2^*)\). In equilibrium, \(A^* = [0, 1]\) and \((V_1^*, V_2^*)\) solve

\[
rV_1^* (x) = \mu (V_1^*)' (x) + \frac{1}{2} \sigma^2 (V_1^*)'' (x) + \pi_1 (x) \text{ for } x \in [0, 1/2],
\]

\[
rV_2^* (x) = \mu (V_2^*)' (x) + \frac{1}{2} \sigma^2 (V_2^*)'' (x) + \pi_2 (x) \text{ for } x \in [1/2, 1],
\]

with \((V_1^*)' (0) = (V_2^*)' (1) = 0, V_1^* (1/2) + V_2^* (1/2) = 1\) and \((V_1^*)' (1/2) = -(V_2^*)' (1/2)\).

**Proof:** See Appendix A.2.5.

Proposition 2.5 establishes that the firms will come to an immediate agreement, with the larger firm acquiring the smaller one at time \(t = 0\). The functions \(V_1^*\) and \(V_2^*\) characterize
the terms at which the takeover takes place: for \( x < 1/2 \), \( V_1^*(x) \) represents the price that firm 2 pays to acquire firm 1; and \( V_2^*(x) \) represents the price that firm 1 pays to acquire firm 2 for \( x > 1/2 \).

The model so far assumes that the firms hold the same beliefs about the evolution of their market shares. However, in real life market shares will usually depend on many factors, some of which are extremely difficult to predict. A natural consequence in such cases is that firms may end up holding different views about how their market shares will evolve. Thus, I now consider the case in which the firms’ managers have optimistic beliefs about their firms future market shares, and study the effect that these differences in beliefs have on the equilibrium outcome.

Assume that \( x_t \) evolves as in equation (2.16), with reflecting at 0 and 1. For \( i = 1, 2 \) let 
\[
B^i = \{B^i_t, F^i_t : 0 \leq t < \infty\}
\]
be a one-dimensional Brownian motion on the probability space \((\Omega, F^i, P^i)\). Firm \( i \) believes that the process \( x_t \) evolves as

\[
dx_t = \mu_i dt + \sigma dB^i_t,
\]

with reflecting boundaries at 0 and 1. To capture optimism I assume that \( \mu_1 > \mu > \mu_2 \), so that firm 1 (firm 2) believes that the drift of \( x_t \) is larger (smaller) than what it really is. For simplicity I consider the case with \( \mu_1 = d > 0 \) and \( \mu_2 = -d \). Note that in this case the parameter \( d \) measures the level of optimism.

An outcome is still given by a pair \((A, \eta)\). The difference is that, from firm \( i \)'s managers perspective, the payoff that firm \( i \) obtains from outcome \((A, \eta)\) is

\[
V_i(x) = E_i \left[ \int_0^{\tau(A)} e^{-rs} \pi_i(x) ds + e^{-r\tau(A)} \eta_1(x_{\tau(A)}) \right]_{x_0 = x},
\]

where \( E_i[\cdot] \) denotes the expectation relative to firm \( i \)'s beliefs.
**Definition 2.4** An outcome \((A, V_1, V_2)\) is an equilibrium if, for \(i = 1, 2, j \neq i\),

\[
V_i(x) = \sup_{\tau \in T_i^0} E_i \left[ \int_0^\tau e^{-r_s} \pi_i(x_s) \, ds + e^{-r_s} (1 - V_j(x_s)) \right| x_0 = x ,
\]

for all \(x \in \mathbb{R}\).

**Proposition 2.4** There exists \(\overline{d} > \underline{d} > 0\) such that:

(i) for \(d > \overline{d}\) the unique equilibrium \((A^*, V_1^*, V_2^*)\) has \(A^* = \emptyset\);

(ii) for \(d < \underline{d}\) the unique equilibrium \((A^*, V_1^*, V_2^*)\) has \(A^* = [0, 1]\);

(iii) for \(d \in (\underline{d}, \overline{d})\) the unique equilibrium \((A^*, V_1^*, V_2^*)\) has \(A^* = [0, \kappa] \cup [1 - \kappa, 1]\), with \(\kappa \in (0, 1/2)\).

**Proof:** See Appendix A.2.5.

Proposition 2.6 shows that inefficiencies may result when managers hold optimistic views about the prospects of their companies. Indeed, the efficient outcome would be for firms to merge at \(t = 0\). However, in the presence of optimism takeovers may only take place when one of the firms has a decisive advantage in terms of market power. Indeed, takeovers will only occur when \(x_t \notin [\kappa, 1 - \kappa]\) (where \(\kappa \in (0, 1/2)\)) when the level of optimism \(d\) moderate (i.e., \(d \in (\underline{d}, \overline{d})\)). Moreover, takeovers may never occur if the level of optimism is sufficiently large (i.e., if \(d > \overline{d}\)).

To understand the intuition behind Proposition 2.6, let

\[
\Pi_i^D(x; d) = E_i \left[ \int_0^\infty e^{-r_s} \pi_i(x_s) \, ds \right| x_0 = x ,
\]

be firm \(i\)'s expected discounted profits if a takeover never occurs (computed under firm \(i\)'s optimistic beliefs). Note that \(\Pi_i^D(x; d)\) is a lower bound to firm \(i\)'s equilibrium payoff, since firm \(i\) can always guarantee this much by indefinitely delaying a takeover. Let \(\Pi^D(x; d) := \Pi_1^D(x; d) + \Pi_2^D(x; d)\) be the total expected discounted profits if a takeover never occurs. In the absence of optimism (i.e., \(d = 0\)), \(\Pi^D(x; 0) = a\) for all \(x\). However, \(\Pi^D(x; d)\) is no longer
Figure 2.5: Plot of $\Pi^D(x)$. Parameters: $\sigma = 0.15$, $r = 0.05$, $a = 0.9$.

constant when firms hold optimistic views about $x_t$. Figure 2.5 plots $\Pi^D(x; d)$ for different values of $d$. The figure shows two things: (i) for a fix $d$, $\Pi^D(x; d)$ reaches a maximum at $x = 1/2$ and decreases as $x$ moves away from $1/2$; (ii) fixing $x$, $\Pi^D(x; d)$ is increasing in $d$.

The value $\overline{d}$ in Proposition 2.6 is the threshold such that $\Pi^D(x; d) > 1$ for all $x \in [0, 1]$ whenever $d > \overline{d}$. In this case there can never be a takeover, as there is no agreement that would satisfy the expectation of the two firms. On the other hand, when $d \in (\underline{d}, \overline{d})$ the firms only reach a deal when $x_t \notin [\kappa, 1 - \kappa]$, so that takeovers occur only when the firms’ relative market power is unbalanced. Importantly, one can show that there are values of $d \in (\underline{d}, \overline{d})$ such that $\Pi^D(x; d) < 1$ for all $x$. That is, delay may occur even in situations in which there gains from merger for all values of $x$, even taking into account the differences in beliefs. The intuition for why delay can occur even in this case is as follows. When $x$ is around $1/2$ the firms are effectively bargaining over a small surplus of size $1 - \Pi^D(x; d)$. Since $\Pi^D(x; d)$ attains a maximum at $1/2$, the effective size of the surplus increases as $x$ moves away from $1/2$. Suppose $x_0 > 1/2$ is close to $1/2$. In this case any offer that firm 2 will accept will give firm 1 a payoff only slightly above her outside option of $\Pi^D_1(x_0; d)$. On the other hand, by delaying an agreement until $x_t$ is above $1/2$ firm 1 can obtain a payoff that is significantly larger than $\Pi^D_1(x_t; d)$, since firm 2’s acceptance threshold decreases with $x_t$. The proof of Proposition 2.6 establishes that when the level of optimism is large enough
(i.e., when $d \in (d, \bar{d})$), the option of delaying an agreement becomes valuable for both firms in situations in which their market shares are balanced (i.e., when $x_t$ is close to $1/2$), and so they delay an agreement.

The results in Proposition 2.6 relate to the literature on bargaining with optimism. Yildiz (2003) studies a bilateral bargaining game a la Rubinstein (1982) in which players have optimistic beliefs about their bargaining power. The main result in his paper is that agreement will always be immediate whenever optimism is persistent and the number of bargaining rounds is sufficiently large. Since Yildiz’s work there has been several papers identifying conditions under which inefficient delays may arise when players are optimistic. The model in this section contributes to this literature by illustrating how optimism may lead to inefficiencies in settings in which players earn flow payoffs while bargaining.

## 2.7 Conclusion

This paper introduces a new continuous time model of bilateral bargaining in which the relative bargaining power of the players evolves over time as a diffusion process. The model has a unique equilibrium, in which players reach an immediate agreement. The player’s equilibrium payoffs solve a system of ordinary differential equations. Closed form solutions to these differential equations are available for the case in which bargaining power evolves as a $(\mu, \sigma)$-Brownian motion. For more general diffusion processes, the solution to these differential equations can be found using numerical methods.

The paper also studies a discrete time version of the bargaining model. This discrete time game has a unique subgame perfect equilibrium. Importantly, as offers are arbitrarily frequently the payoffs that the players get in this discrete time game converge to the payoffs they get in the continuous time model. Therefore, the equilibrium of the continuous time model

\[ \text{See Yildiz (2011) for a review of the literature on bargaining with optimism.} \]
formulation captures essentially the same behavior than the solution concept of subgame perfect equilibrium.

Finally, the paper presents two extensions of the baseline model: a model of legislative bargaining with supermajority requirements in which parties may learn verifiable information about the benefits of the different policies, and a model of takeovers in duopolistic industries. The tractability of the continuous-time model allows for a full characterization of the equilibrium outcome for these extensions.
Chapter 3

A Model of Legislative Gridlock

3.1 Introduction

Bargaining delays are a pervasive phenomenon in legislative negotiations. What are the main factors that drive gridlock? What features of the political environment are more likely to lead to legislative inaction? Which institutions are better in terms of promoting efficiency in the process of legislative policymaking? The model I present in this paper provides new insights to these and related questions.

I study a model of legislative negotiations with two political parties, $i = 1, 2$, who have distinct preferences over policies and must bargain over which policy to implement. The model is in continuous time and it’s key variable is an exogenous and publicly observable diffusion process $x_t$ taking values in $[0, 1]$. The value that this process takes at each moment in time measures the fraction of legislators supporting each party at that moment: a majority of legislators support party 1 when $x_t > 1/2$, while party 2 controls a majority of seats when $x_t < 1/2$. At each instant $t$ the majority party has proposal power.\footnote{The model in this paper is an adaptation of the continuous time bargaining model in Chapter 2.}

The assumption that the number of legislators supporting each party varies stochastically over time is natural in many settings. For instance, these fluctuations might reflect a situation...
in which the preferences of the median voter are changing over time. Also, time-varying political power might arise in a setting in which voters periodically receive new information about the quality of each parties’ preferred policies.

Under a supermajority rule a party that has enough political power can implement policies unilaterally, without the approval of the minority party. In this model a supermajority rule is characterized by a number \( \phi \in [0, 1/2) \) such that party 1 (party 2) can unilaterally implement policies at time \( t \) if \( x_t \geq 1 - \phi \) (if \( x_t \leq \phi \)). Under supermajority rule \( \phi \) the majority party needs the approval of its opponent in order to implement a policy whenever \( x_t \in (\phi, 1 - \phi) \). On the other hand, when \( x_t \notin (\phi, 1 - \phi) \) the majority party unilaterally implements its preferred policy and the negotiations end.

I assume that the minority party incurs a concession cost \( c \geq 0 \) whenever it accepts a proposal put forward by its opponent. The parameter \( c \) represents the political cost that legislators in the minority party incur when they give in to their opponents’ proposals. For example, a legislator in the minority party might believe that her reputation will suffer if she supports proposals made by the majority party, since the resulting policy would be far away from what she campaigned for. Also, a legislator who supports proposals made by the opposing party might have lower chances of getting reelected: by supporting such proposals she might loose the support of her party’s base, and would thus have lower chances of winning the primary in her own district. Importantly, the minority party only incurs the concession cost \( c \) when it accepts a proposal made by the majority party. A party with a supermajority implements policies unilaterally, without the support of the minority party. In this case the minority party does not incur any cost, since it is not conceding to its opponent’s proposal.

I show that this legislative bargaining model has a unique equilibrium. In the baseline model with no concession costs parties always come to an agreement at the beginning of the negotiations, regardless of the initial level of relative political power \( x_0 \). The model delivers closed form expressions for the parties’ payoffs, allowing for many comparative statics
exercises. For instance, when the process \( x_t \) has no drift a more stringent supermajority requirement always benefits the minority party.

When parties have a positive concession cost \( c > 0 \) the model’s unique equilibrium always involves a delay region and an agreement region. When \( x_t \) lies inside the delay region there is no agreement that satisfies both parties’ expectations. In this case gridlock emerges, and policies are only implemented when \( x_t \) reaches the agreement region. When the cost of conceding is larger than some threshold \( \bar{c} \), the delay region is given by \((\phi, 1 - \phi)\). In other words, if \( c \) is sufficiently large policies will only be implemented when a party obtains the support of a supermajority of legislators (i.e., when the public opinion definitely favours the policies if one of the parties).

When \( c \in (0, \bar{c}) \) the delay region is always of the form \((\phi, x] \cup [\bar{x}, 1 - \phi)\), for some \( x \) and \( \bar{x} \) such that \( \phi < x < \bar{x} < 1 - \phi \). That is, in this case gridlock only emerges in situations in which political power is unbalanced and one party is close to obtaining the support of a supermajority of legislators. The intuition for why delay emerges in these situations is as follows. When \( c > 0 \), the minority party will only accept proposals that compensate for the cost of concession. However, the majority party is not willing to make such offers when \( x_t \in (\phi, x] \cup [\bar{x}, 1 - \phi)\), since by delaying an agreement that party might obtain the support of a supermajority of legislators and would thus be able to implement its preferred policy. On the other hand, parties are able to reach an agreement when \( x_t \in [x, \bar{x}]\), since in this case the cost of delaying until a party obtains a supermajority outweighs the concession cost. I also show that the size of the delay region increases in \( c \) when \( c \in (0, \bar{c}) \), so a higher concession cost leads to more gridlock.

The model with intermediate costs of concession generates a variety of novel implications. First, the shape of the delay region implies that legislative gridlock is more likely to emerge when political power is unbalanced (i.e., when \( x_t \in (\phi, x] \cup [\bar{x}, 1 - \phi)\)). Second, the model predicts that a party having only a very weak majority might actually not be able to fully translate its political power into the policies that get implemented. The reason for this is that
the majority party always has to compensate the minority party for the cost of concession in order to pass any policy. Finally, the model also predicts that the likelihood of gridlock increases with the level of party polarization, as more polarization would lead to a higher concession costs.

The next subsection presents an overview the related literature. Section 2.2 introduces the continuous time bargaining model when there are no concession costs (i.e., when \( c = 0 \)) and characterizes the model’s unique equilibrium. Section 2.2 also briefly describes the connection between this paper’s continuous time bargaining model and discrete time bargaining games. Section 2.3 studies the model in which parties face a positive concession cost and shows how gridlock may arise in this setting. Section 2.3 also discusses some of the positive implications of the model. Section 2.4 presents concluding remarks. Some proofs appear in the appendix.

### 3.1.1 Related literature

Starting with the seminal paper by Baron and Ferejohn (1989), there is a large body of literature in political economy that uses non-cooperative game theory to analyze legislative bargaining.\(^2\) Banks and Duggan (2000, 2006) generalize the Baron-Ferejohn model by allowing players to bargain over a multidimensional policy space. A series of papers use these workhorse models to study the effect that different institutional arrangements have on legislative outcomes. Winter (1996) and McCarty (2000a) analyze models a la Baron-Ferejohn with the presence of veto players.\(^3\) Baron (1998) and Diermeier and Feddersen (1998) study legislatures with vote of confidence procedures. Diermeier and Myerson (1999), Ansolabehere et al. (2003) and Kalandrakis (2004) study legislative bargaining under bicameralism. Snyder et al. (2005) analyze the effects of weighted voting within the Baron-Ferejohn framework. Cardona and Ponsati (2011) analyze the effects of supermajority rules in legislative bargaining within the model of Banks and Duggan (2000).

\(^2\)Eraslan (2002) proved uniqueness of stationary subgame perfect equilibrium payoffs in the Baron-Ferejohn model.

\(^3\)Chari et al. (1997) and McCarty (2000b) study bargaining models with executive veto.
Although this literature has been very successful in characterizing legislative outcomes under different institutions, the models by Baron and Ferejohn (1989) and Banks and Duggan (2000, 2006) are not well suited to analyze the causes of bargaining delays and legislative gridlock. In the Baron-Ferejohn model the unique stationary subgame perfect equilibrium always involves immediate agreement; while Banks and Duggan (2000, 2006) (and, to the best of my knowledge, all the papers that apply their framework) focus their analysis on stationary subgame perfect equilibria that exhibit no delay. In order to study legislative inaction, in this paper I construct a new continuous time model of legislative bargaining with supermajority requirements in which the parties’ political strength fluctuates over time. In this model, legislative gridlock arises naturally when it is costly for the parties to concede to proposals put forward by their opponents.\footnote{Diermeier and Vlaicu (2010) construct a legislative bargaining model to study the differences between parliamentarism and presidentialism in terms of their legislative success rate (i.e., the frequency with which bills endorsed by the executive government are approved).}

Merlo and Wilson (1995, 1998) study complete information bargaining games in which the realization of an exogenous stochastic process determines at each bargaining round the size of the surplus to be divided among the players and the identity of the proposer (see also Cripps, 1998).\footnote{Merlo (1997) uses this framework to study government formation in parliamentary democracies. Eraslan and Merlo (2002) extend this stochastic bargaining model to settings in which agreement may require less than unanimity.} A key feature of this model is that players will delay in reaching an agreement if they expect the surplus to grow fast enough in the future. The model I present in this paper relates to Merlo and Wilson’s setup in two ways. First, the discrete time version of the model without concession costs (see Section 2.3 below) is a special case of Merlo and Wilson’s model: in the discrete time version of my model with no concession costs the size of the surplus is fixed, and the exogenous stochastic process (i.e., the diffusion process $x_t$) only determines the identity of the proposer at each bargaining round. Second, the delays that arise in the continuous time model with concession costs are closely related to the delays that arise in Merlo and Wilson’s framework. When parties incur a positive cost of conceding the effective size of the surplus jumps at the point at which a party obtains the support of
a supermajority of legislators, since at this point parties avoid paying the concession cost. In this setting parties delay in reaching an agreement when \( x_t \) is close to either \( \phi \) or \( 1 - \phi \), as there is a high probability that the effective size of the surplus will jump by a discrete amount in the near future.

The model I present in this paper also relates to Dixit et al. (2000), who study a model in which the parties’ political power evolves over time according to a Markov chain. The model in Dixit et al. (2000) features two political parties who interact repeatedly, with the party in power unilaterally deciding how to allocate a unit surplus. The paper characterizes efficient divisions of the surplus that are self-enforcing over time. In spite of the similarities, there are important differences between the model in Dixit et al. (2000) and the one I present in this paper. First, in this paper I study a canonical bargaining situation in which parties are negotiating over one single policy, whereas in their model parties divide a different surplus at each period. Moreover, their model is set up in discrete time, while the model I construct in this paper is in continuous time.

### 3.2 Model without concession costs

In this section I introduce the legislative bargaining model without concession costs. I show that parties always reach an immediate agreement, and that the parties’ payoffs are fully characterized by a system of differential equations. Section 3 presents the model in which parties incur a positive cost when they agree to proposals made by their opponents, and shows how this cost of concession can generate substantive delays in legislative negotiations.

#### 3.2.1 Framework

Let \([0, 1]\) be a set of alternatives or policies.\(^6\) Two political parties, \( i = 1, 2 \), bargain over which alternative in \([0, 1]\) to implement. Parties are expected utility maximizers and discount

\(^6\)In this paper, I use the terms alternatives and policies interchangeably.
future payoffs at the common rate \( r > 0 \). Therefore, the expected payoff that party \( i \) gets if some (possibly random) alternative \( v \) is implemented at some (possibly random) time \( \tau \) is 
\[ E[e^{-r\tau}u_i(v)], \]
where \( u_i: [0, 1] \to \mathbb{R} \) is party \( i \)'s utility index.

I assume that party \( i \)'s utility index is \( u_i(z) = 1 - |z - z_i| \), where \( z_i \in [0, 1] \) is party \( i \)'s ideal policy (with \( z_1 \neq z_2 \)). Throughout the paper, I will maintain the assumption that \( z_1 = 1 \) and \( z_2 = 0 \), so that \( u_1(z) = z \) and \( u_2(z) = 1 - z \) for all \( z \in [0, 1] \).

The model's key variable is an exogenous and publicly observable diffusion process \( \mathbf{x}_t \), which determines the identity of the proposer at each instant of time. Let \( B = \{B_t, \mathcal{F}_t^B : 0 \leq t < \infty \} \) be a one-dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}, P)\), where \( \{\mathcal{F}_t^B : 0 \leq t < \infty \} \) is the completion of the filtration generated by the Brownian motion.\(^8\)

The Brownian motion \( B \) drives the diffusion process \( \mathbf{x}_t \). I assume that \( \mathbf{x}_t \) evolves as Brownian motion with constant drift \( \mu \) and constant volatility \( \sigma > 0 \), with absorbing boundaries at 0 and 1; that is, \( \mathbf{x}_t \) evolves as

\[ dx_t = \mu dt + \sigma dB_t, \tag{3.1} \]

with \( x_0 \in [0, 1] \). When \( \mathbf{x}_t \) hits either 0 or 1, it stays there forever. As it will become clear below, the interaction between the parties will have ended by the time \( \mathbf{x}_t \) reaches either 0 or 1, so the absorption at the boundaries will play no role in what follows.

The process \( \mathbf{x}_t \) determines the bargaining protocol: party 1 makes offers at \( t \) if \( \mathbf{x}_t > 1/2 \), while party 2 makes offers at \( t \) if \( \mathbf{x}_t < 1/2 \). Since the ability to make offers is the source of bargaining power, the process \( \mathbf{x}_t \) can be thought of as measuring of the parties' relative

\(^7\)The assumption that the parties' ideal policies are at the extremes of the policy space is without loss of generality. Indeed, suppose that the policy space is \([a, b]\), with \( a < 0 \) and \( b > 1 \), and that \( z_1 = 1 \), \( z_2 = 0 \). In this case, alternatives in \([a, 0) \cup (1, b]\) are strictly suboptimal, so parties will never implement such an alternative. Thus, there is no loss of generality in restricting attention to this interval of alternatives.

\(^8\)That is, \( \{\mathcal{F}_t^B : 0 \leq t < \infty \} \) includes all sets of measure zero and is right-continuous (i.e., for any \( t \geq 0 \), \( \mathcal{F}_t^B = \bigcap_{s>0} \mathcal{F}_{t+s}^B \)).
political power. For instance, party 1’s political power is large when \( x_t \) is close to 1, as this implies that party 1 will (on average) be making offers for longer. Similarly, party 2’s political power is large when \( x_t \) is close to zero.

A more concrete interpretation is that the process \( x_t \) measures the number of legislators that supports each party at any moment in time: party 1 controls a majority of seats in Congress when \( x_t > 1/2 \), while party 2 controls a majority of seats when \( x_t < 1/2 \). With this interpretation in mind, I will sometimes refer to the party making offers as the majority party and to the party responding to offers as the minority party.

The underlying idea behind the assumption that the number of legislators that support each party fluctuates is that the preferences of the median voter might be changing over time. In principle, each legislator would choose to support her own party. However, a legislator may prefer to support the opposing party if her own party’s popularity decreases (i.e., if the median voter’s preferences move closer to those of the opposing party). For instance, this may occur if individual legislators are seeking reelection in their own district, and think they would benefit more by supporting the proposals that are closer to their constituency’s preferences. In this sense, this model assumes that individual legislators have some degree of party discipline, but that they might choose to support the opposing party whenever the benefit from doing this is large enough.

A supermajority rule is a number \( \phi \in [0, 1/2) \) such that party 1 can unilaterally implement a policy if \( x_t \geq 1 - \phi \), and party 2 can unilaterally implement a policy if \( x_t \leq \phi \). Therefore, under supermajority rule \( \phi \) the length of the bargaining is bounded above by \( \tau^\phi = \inf\{t : x_t \notin (\phi, 1-\phi)\} \), the first time either party obtains a supermajority. Indeed, if parties have not reached an agreement by time \( \tau^\phi \), then at that time the party with the supermajority implements its preferred alternative immediately: if \( x_{\tau^\phi} \geq 1 - \phi \), party 1 unilaterally implements policy 1, while party 2 unilaterally implements policy 0 if \( x_{\tau^\phi} \leq \phi \). When the rule is one of simple majority (i.e., \( \phi \approx 1/2 \)) the interaction between the parties ends immediately, with the majority party implementing its preferred policy. As the super-
majority rule becomes more stringent (i.e., as $\phi$ decreases), there is a larger range of values of $x_t$ at which parties have to bargain in order to reach an agreement: in this case, while $x_t \in (\phi, 1 - \phi)$ the party making proposals needs the approval of its opponent in order to implement a policy.

Suppose for instance that $x_0 > 1/2$. In this case, party 1 will be proposer from time 0 until time $\tau_1 = \inf\{t : x_t \leq 1/2\}$. At any moment until $\tau_1$, party 1 can make an offer $z \in [0,1]$. There are no restrictions on the number of offers that party 1 can make until $\tau_1$. If party 2 accepts an offer before $\tau_1$ or if party 1 obtains a supermajority before $\tau_1$, the bargaining ends and parties collect their payoffs. If neither of these two events occur, then party 2 becomes proposer between $\tau_1$ and time $\tau_2 = \inf\{t > \tau_1 : x_t \geq 1/2\}$. Bargaining continues this way, with parties alternating in their right to make proposals, until either a party accepts an offer or until a party obtains a supermajority (see Figure 3.1 for a plot of a sample path of $x_t$).

An outcome of this legislative bargaining model is a pair $(A, \eta)$, where $A \in \mathcal{A} := \{A \subseteq [0,1] : A \text{ is closed}\}$ is an agreement region and $\eta : A \to [0,1]$ is a policy function mapping the agreement region $A$ to the set of possible policies. The agreement region $A$ determines the set of values of the state variable $x_t$ at which parties reach an agreement. Put differently,
under an outcome \((A, \eta)\) parties reach an agreement at time \(t\) if and only if \(x_t \in A\), so the agreement date is \(\tau(A) := \inf\{t : x_t \in A\}\). On the other hand, the function \(\eta\) gives the policy that gets implemented when parties reach an agreement.

Under supermajority rule \(\phi\), an equilibrium outcome \((A, \eta)\) must satisfy the following conditions:

(i) \(A \supseteq [0, \phi] \cup [1 - \phi, 1]\),

(ii) \(\eta(x) = 1\) for all \(x \in [1 - \phi, 1]\), and

(iii) \(\eta(x) = 0\) for all \(x \in [0, \phi]\).

That is, under supermajority rule \(\phi\) the bargaining must end whenever a party has a super-majority, so \([0, \phi] \cup [1 - \phi, 1]\) must belong to the agreement region. Moreover, at these states the party with a supermajority will implement its preferred policy. I will denote by \(O^\phi\) the set of outcomes that satisfy conditions (i)-(iii).

The notion of an outcome has a stationarity assumption implicitly built-in. Indeed, under outcome \((A, \eta)\) the policy that gets implemented depends only on the value that \(x_t\) takes at the agreement date, and is independent of the history leading up to that point. Chapter 2 shows that the bargaining game that results from the discrete time version of this continuous time model has a unique subgame perfect equilibrium (see Section 2.3 for a description of the discrete time version of this model). Importantly, this unique equilibrium is supported by a profile of stationary strategies, so the outcome that the equilibrium induces is stationary. I also show that, in the limit when players can make offers arbitrarily frequently, the equilibrium of the discrete time game converges to the unique equilibrium of the continuous time model.\(^9\) Therefore, there is a strong sense in which this restriction to stationary outcomes is without loss of generality.

\(^9\)In Section 2.2, I derive the unique equilibrium of this continuous time model.
An outcome \((A, \eta) \in \mathcal{O}^\phi\) determines the parties’ payoffs as follows. Given \(x_0 = x\), party \(i\)'s payoff from outcome \((A, \eta)\) is

\[
V_i(x) = \begin{cases} 
  u_i(\eta(x)) & x \in A, \\
  E \left[ e^{-r\tau(A)}u_i(\eta(x)) \right] & x \notin A.
\end{cases}
\] (3.2)

In what follows, I will denote an outcome \((A, \eta)\) as a triplet \((A, V_1, V_2)\), with \(V_i\) satisfying (3.2) for \(i = 1, 2\).

### 3.2.2 Equilibrium

For any Borel set \(H \subseteq [0, 1]\), let \(\tau(H) = \inf\{t : x_t \in H\}\). Let \(T_1\) denote the set of stopping times \(\tau(H)\) with \(H \subseteq [1/2, 1]\). Similarly, let \(T_2\) denote the set of stopping times \(\tau(H)\) with \(H \subseteq [0, 1/2]\). Recall that \(\tau^\phi = \inf\{t : x_t \notin (\phi, 1 - \phi)\}\).

**Definition 3.1** Under supermajority rule \(\phi\) an outcome \((A, V_1, V_2) \in \mathcal{O}^\phi\) is an equilibrium if, for \(i = 1, 2\), \(j \neq i\),

\[
V_i(x) = \sup_{\tau \in T_i} E \left[ e^{-r\tau(\wedge)\tau^\phi} (1 - V_j(\tau(\wedge)\tau^\phi)) \right] x_0 = x \]

for all \(x \in [0, 1]\).

The idea behind an equilibrium is that the party responding to offers should always be willing to accept proposals that give the party a payoff equal to what it would get by waiting until it gains the right to make offers. In other words, in an equilibrium the acceptance threshold of a party responding to offers must be equal to that party’s expected continuation value.\(^{10}\)

Since the proposer will never offer the responder more than her acceptance threshold, this

\(^{10}\)Note that if the party responding to offers delays an agreement, her opponent might obtain the required supermajority level and implement its preferred policy before \(x_t\) crosses the threshold 1/2. This risk is taken into account in equation (3), since any outcome \((A, V_1, V_2) \in \mathcal{O}^\phi\) has \(V_1(\phi) = 0 = V_2(1 - \phi)\).
implies that the responder’s payoff should always be equal to her expected continuation value. Definition 1 guarantees that this will always be the case. To see this, suppose $x_0$ is such that party $i$ is responding to offers at the beginning of the negotiations, and define $\tau(1/2) = \inf\{t : x_t = 1/2\}$. Since $\tau(1/2) \geq \tau$ for all $\tau \in T_i$ when $x_0$ is such that party $i$ starts responding to offers, equation (3) implies that

$$V_i(x) = E \left[ e^{-\tau \phi \wedge \tau(1/2)} V_i \left( x_{\tau \phi \wedge \tau(1/2)} \right) \right]_{x_0 = x}, \quad (3.4)$$

regardless of whether parties reach an agreement when $x_t = x$ or not (i.e., regardless of whether $x$ belongs to the agreement region $A$ or not).

On the other hand, when making proposals party $i$ takes party $j$’s acceptance threshold $V_j(x)$ as given. At each moment in time before it loses the right to make offers, party $i$ has to decide whether to make an acceptable offer of $V_j(x_t)$ to her opponent and end the bargaining (obtaining a payoff of $1 - V_j(x_t)$), or to delay the agreement until it can strike a better deal. That is, the problem of the party making offers is to optimally choose a time at which to make an acceptable offer to her opponent. Definition 1 says that an outcome $(A, V_1, V_2)$ is an equilibrium if the proposer always finds it optimal to delay when $x_t \notin A$, and always finds it optimal to make an acceptable offer when $x_t \in A$.

In a discrete time bargaining games a la Rubinstein (1982), the responder only accepts offers that give her at least the value she expects to get by delaying an agreement for one period. That is, the responder’s acceptance threshold is always given by her expected continuation value. At each bargaining round, the proposer decides whether to make an acceptable offer to her opponent and end the game, or to delay the agreement for one period.\footnote{Of course, in a subgame perfect equilibrium of a discrete time bargaining game a la Rubinstein the proposer always makes an acceptable offer at the beginning of the game, and the game ends in immediate agreement.} Similarly, in an equilibrium of the continuous time model the responder’s acceptance threshold is given by her expected continuation value. Taking her opponent’s acceptance threshold as
given, the proposer’s problem is to choose an optimal time at which to make an acceptable offer, knowing that she will loose the right to make offers when the process \( x_t \) reaches \( 1/2 \).

The following Proposition shows that any equilibrium outcome \((A, V_1, V_2)\) of this bargaining model must have \( A = [0, 1] \), so that parties always come to an agreement at the beginning of the negotiations.

**Proposition 3.1** Let \((A, V_1, V_2) \in \mathcal{O}^\phi\) be an equilibrium outcome. Then, \( A = [0, 1] \).

**Proof:** Suppose \((A, V_1, V_2)\) is an equilibrium outcome and assume by contradiction that \( A \) is a strict subset of \([0, 1]\). Since \([0, 1] \setminus A\) is open (because \( A \in \mathcal{A} \)), there exists an interval \((a, b)\) with \( \phi \leq a < b \leq 1 - \phi \) such that \((a, b) \notin A\). This implies that \( \tau(A) > 0 \) whenever \( x_0 \in (a, b) \).

Define \( U(x) := V_1(x) + V_2(x) \). Then, for all \( x \in (a, b) \) it must be that

\[
U(x) = E[e^{-\tau(A)}|x_0 = x] < 1,
\]

where the inequality follows from the fact that \( V_1(x) + V_2(x) = 1 \) for all \( x \in A \) and from the fact that \( \tau(A) \geq 1 \) whenever \( x_0 \in (a, b) \). Thus, \( V_1(x) + V_2(x) < 1 \) for all \( x \in (a, b) \). But this implies that, when \( x_t \in (a, b) \), proposer \( i \) is better off by offering \( V_j(x_t) \) to her opponent (and obtaining a payoff of \( 1 - V_j(x_t) > V_i(x_t) \) for herself) than by delaying. Therefore, \((A, V_1, V_2)\) cannot be an equilibrium outcome.

Define \( A^* = [0, 1] \), so that any equilibrium outcome \((A, V_1, V_2)\) must have \( A = A^* \). Define also \( A^*_1 = [1/2, 1] \) and \( A^*_2 = [0, 1/2] \).

**Corollary 3.1** Let \((A^*, V_1, V_2) \in \mathcal{O}^\phi\) be an equilibrium outcome. Then, \( V_1 \) and \( V_2 \) are such that

\[
V_i(x) = \begin{cases} 
1 - V_j(x) & \text{if } x \in A^*_i, \\
E[ e^{-r\tau^\phi \land \tau(1/2)} (1 - V_j(x_{\tau^\phi \land \tau(1/2)}) ) \big| x_0 = x ] & \text{if } x \notin A^*_i.
\end{cases}
\] (3.5)
Proof: Let \((A^*, V_1, V_2)\) be an equilibrium outcome. Equation (3.4) then implies that, for all \(x \notin A_i^*\),

\[
V_i(x) = E \left[ e^{-\tau^\phi \wedge \tau(1/2)} V_i(x_{\tau^\phi \wedge \tau(1/2)}) \bigg| x_0 = x \right] \\
= E \left[ e^{-\tau^\phi \wedge \tau(1/2)} (1 - V_j(x_{\tau^\phi \wedge \tau(1/2)})) \bigg| x_0 = x \right],
\]

where the second equality follows from the fact that \(V_1(x) + V_2(x) = 1\) for all \(x \in [0, 1]\).

Finally, this also implies that \(V_i(x) = 1 - V_j(x)\) for all \(x \in A_i^*\).

Corollary 3.1 provides a partial characterization of the payoffs that can arise in an equilibrium of this model. Party \(i\)'s payoff when it is responding to offers is given by the expected discounted value of waiting until time \(\tau^\phi \wedge \tau(1/2)\) and getting a payoff of \(1 - V_j(x_{\tau^\phi \wedge \tau(1/2)})\) at that point. On the other hand, when making proposals, party \(i\) immediately makes an acceptable offer to party \(j \neq i\), thus receiving a payoff of \(1 - V_j(x)\).

The next Theorem shows that this bargaining model has a unique equilibrium, and shows that the parties' equilibrium payoffs solve a system ordinary differential equations.

**Theorem 3.1** For any \(\phi\), there is a unique equilibrium outcome \((A^*, V_1^\phi, V_2^\phi)\). Equilibrium payoffs \(V_1^\phi\) and \(V_2^\phi\) satisfy

\[
r V_1^\phi(x) = \mu(V_1^\phi)'(x) + \frac{1}{2} \sigma^2 (V_1^\phi)''(x) \text{ for } x \in (\phi, 1/2) , \quad (3.6) \\
r V_2^\phi(x) = \mu(V_2^\phi)'(x) + \frac{1}{2} \sigma^2 (V_2^\phi)''(x) \text{ for } x \in (1/2, 1 - \phi) , \quad (3.7)
\]

with boundary conditions \(V_1^\phi(\phi) = V_2^\phi(1 - \phi) = 0, V_1^\phi(1/2) + V_2^\phi(1/2) = 1\) and

\[
(V_1^\phi)'(1/2^-) = -(V_2^\phi)'(1/2^+) . \quad \text{(SP)}
\]

Proof: See Appendix A.3.1.
The proof of Theorem 3.1 is in Appendix A.3.1. Here, I present a sketch of its main arguments. Let \((A, V_1, V_2)\) be an equilibrium outcome. By Proposition 3.1 and Corollary 3.1, \(A = A^*\) and \((V_1, V_2)\) satisfy (3.5). Lemma A.3.1 in the appendix then implies that \(V_1\) and \(V_2\) must solve

\[
\begin{align*}
  rV_1(x) &= \mu V_1'(x) + \frac{1}{2} \sigma^2 V_1''(x) \quad \text{for } x \in (\phi, 1/2), \\
  rV_2(x) &= \mu V_2'(x) + \frac{1}{2} \sigma^2 V_2''(x) \quad \text{for } x \in (1/2, 1 - \phi),
\end{align*}
\]

with boundary conditions \(V_1(\phi) = V_2(1 - \phi) = 0\) and \(V_1(1/2) + V_2(1/2) = 1\).

In Appendix A.3.1, I show that there is a one-dimensional family of solutions to this system of differential equations. However, the equilibrium payoffs \(V_1^\phi\) and \(V_2^\phi\) are the unique solution that satisfies the smooth pasting condition (SP). This condition states that \(V_1^\phi\) and \(V_2^\phi\) are \(C^1\) in \([\phi, 1 - \phi]\). If this condition did not hold, then either \(V_1\) or \(V_2\) would have a convex kink at \(x = 1/2\). In Appendix A.3.1 I show that this cannot occur in an equilibrium, since the party whose payoff function has the convex kink at \(x = 1/2\) would find it optimal to delay an agreement when \(x\) is close 1/2, and this would contradict Proposition 3.1.

Theorem 3.1 shows that the continuous time bargaining model has a unique equilibrium outcome, and fully characterizes the payoff that each party gets in equilibrium as a function of relative political power. The ordinary differential equations (3.6) and (3.7) and the four boundary conditions pin down the parties’ equilibrium payoffs when they are responding to offers. This is enough to characterize parties equilibrium payoffs for all values of the state variable \(x\), since a party \(i\)’s payoff \(V_i^\phi(x)\) when making offers is given by \(1 - V_j^\phi(x)\). The solution to the system of differential equations (3.6) and (3.7) (with the boundary conditions) is given by

\[
\begin{align*}
  V_1^\phi(x) &= A_1 e^{-\alpha x} + B_1 e^{\beta x} \quad \text{for } x \in (1/2 - \phi, 1/2), \\
  V_2^\phi(x) &= A_2 e^{-\alpha x} + B_2 e^{\beta x} \quad \text{for } x \in (1/2, 1/2 + \phi),
\end{align*}
\]
Figure 3.2: Party 1’s payoff. Parameters: \( \mu = 0, \sigma = 0.05 \) and \( r = 0.05 \).

where \( \alpha = (\mu + \sqrt{\mu^2 + 2r^2})/\sigma^2 \), \( \beta = (-\mu + \sqrt{\mu^2 + 2r^2})/\sigma^2 \), and where \( (A_i, B_i)_{i=1,2} \) are constants such that the boundary conditions hold.\(^{12}\)

**Example 4** Suppose \( \mu = 0 \). Figure 3.2 plots \( V_1^\phi(x) \) for different supermajority levels, keeping \( \sigma \) and \( r \) fixed. The plot shows that when party 1 is the majority party (i.e., when \( x > 1/2 \)), its payoff decreases with an increase in the supermajority requirement. On the other hand, when party 1 is the minority party, its payoff increases as the supermajority requirement becomes more stringent. In other words, when \( \mu = 0 \) more stringent supermajority rules always protect the minority party.\(^{13}\) To understand the intuition behind this, recall that the equilibrium payoff of the minority party is given by its expected continuation value of waiting until \( \tau^\phi \wedge \tau(1/2) \). If \( \tau^\phi < \tau(1/2) \), then the minority party gets a payoff of zero; if \( \tau(1/2) < \tau^\phi \), it receives some positive payoff (which depends on the policy that gets implemented when \( x = 1/2 \)). As the supermajority rule becomes more stringent (i.e., as \( \phi \) decreases), the probability that \( \tau(1/2) < \tau^\phi \) increases, and therefore the payoff of the minority party goes up.

\(^{12}\)The full expressions for \( V_1^\phi \) and \( V_2^\phi \) are in Appendix A.1.

\(^{13}\)Using the expressions for \( V_1^\phi \) in the appendix, one can show that this result always holds; that is, when \( \mu = 0 \), \( \partial V_1^\phi(x)/\partial \phi > 0 \) \((< 0)\) for all \( x < 0 \) (for all \( x > 0 \)).
Remark 3.1  Throughout this paper I maintain the assumption that the process $x_t$ evolves as a diffusion with constant drift and constant volatility as in (3.1), with absorbing boundaries at 0 and 1. The results in this paper would continue to hold if $x_t$ followed a more general diffusion of the form

$$dx_t = \mu(x_t) \, dt + \sigma(x_t) \, dB_t, \quad x_0 \in [0,1]$$

with absorbing boundaries at 0 and 1 and with $\mu(\cdot), \sigma(\cdot) \in C^2$. One can show that in this more general case the equilibrium payoffs satisfy a system of differential equations similar to (3.6) and (3.7). However, when $x_t$ follows a more general diffusion process the system of differential equations that characterizes equilibrium payoffs will typically not have a closed form solution. In these cases, one can calculate the equilibrium payoffs using numerical methods.

3.2.3 Relation with discrete time bargaining games

In the previous subsection I presented a continuous time bargaining model and characterized its unique equilibrium. This model is an adaptation of the model in Chapter 2 to a setting of legislative negotiations. There, I show that the equilibrium of this continuous time model corresponds to the limiting SPE of a discrete time bargaining game, when players can make offers arbitrarily frequently.

The discrete time bargaining game that approximates the continuous time model is as follows. Two parties, $i = 1, 2$, bargain over which policy in $[0,1]$ to implement. Parties are expected utility maximizers and discount future payoffs at rate $r > 0$, with utility indices $u_1(z) = 1 - z$ and $u_2(z) = z$. Relative political power $x_t$ evolves in as (3.1) with absorbing boundaries at 0 and 1. However, parties can only make offers at points on the grid $T(\Delta) = \{0, \Delta, 2\Delta, \ldots\}$, where $\Delta > 0$ measures the time interval between consecutive offers. That is, $x_t$ evolves in continuous time, but parties can only make offers at $t \in T(\Delta)$. 

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The game's bargaining protocol is as follows. At any bargaining round $t \in T(\Delta)$, the realization of $x_t$ determines the identity of the proposer: if $x_t \geq 1/2$, then party 1 is the proposer at $t$; if $x_t < 0$, then party 2 is the proposer at $t$. If $x_t \notin (\phi, 1 - \phi)$, then the party with a supermajority can unilaterally implement any policy. If $x_t \in (\phi, 1 - \phi)$, the proposer makes an offer $z \in [0, 1]$. The other party, the responder, can either accept or reject the offer. If the responder accepts the offer, the game ends and players collect their payoffs. If the responder rejects the offer, the game moves to period $t + \Delta$. Since parties discount payoffs at rate $r$, the discount factor between consecutive bargaining rounds is $e^{-r\Delta}$.

This discrete time game is a special case of Merlo and Wilson's (1995, 1998) stochastic bargaining model. Merlo and Wilson (1995, 1998) study bargaining games in which the realization of an exogenous Markov process determines at each period both the size of the surplus and the identity of the proposer. In the model I described above, the size of the surplus is constant, and the realization of the Markov process (i.e., the diffusion process) only determines the identity of the proposer at each bargaining round.

Adapting arguments in Merlo and Wilson (1995), one can show that this game has a unique SPE. In equilibrium, parties reach an agreement in the first bargaining round. This unique SPE is supported by a profile of stationary strategies, so the outcome that it induces is also stationary. For any supermajority $\phi$, let $V_i^{\phi}(x; \Delta)$ denote party $i$'s equilibrium payoff when relative political power is $x$ and the interval between bargaining rounds is $\Delta > 0$. In Chapter 2 I show that for $i = 1, 2$, $V_i^{\phi}(x; \Delta_n) \to V_i^{\phi}(x)$ uniformly for any sequence $\{\Delta_n\} \to 0$. That is, the payoffs of the continuous time model correspond to the limiting SPE payoffs of this discrete time bargaining game, when parties can make offers arbitrarily frequently. Therefore, the continuous time model and its discrete time counterpart give the same unique prediction in this legislative bargaining setting in which the parties' political power fluctuates over time.
3.3 Cost of conceding and political gridlock

3.3.1 Costs of concession

In the model of the previous section, the minority party’s acceptance threshold is given by its expected continuation value. That is, the minority party is always willing to accept proposals that give a payoff equal to its expected continuation value, no matter how low this value is. In particular, when \( x_t \) is very close to either \( \phi \) or \( 1 - \phi \), the minority party accepts proposals that are essentially equal to its opponents’ preferred policy. In these situations, the minority party’s political power is so low that its views are hardly incorporated into the laws that get approved.

Although this may seem a reasonable approximation of reality, there are many settings in which legislators in the minority party might be unwilling to accept proposals that are so far away from their preferred policy. In other words, legislators may not be willing to concede to such proposals by their opponents, even when their own party’s political power is very low. For example, a legislator in the minority party might think that her reputation will suffer if she supports such proposals, since the resulting policy would be very far away from what she campaigned for. A legislator in the minority party might also think that supporting such a proposal could negatively affect her chances of reelection: by supporting such proposals, she might loose the support of her party’s base, and would thus have low chances of winning the primary in her own district.

To account for these situations, in this section I assume that the minority party incurs a concession cost of \( c \geq 0 \) if it accepts a proposal made by the other party. In other words, if the minority party \( i \) accepts a proposal by party \( j \neq i \) to implement policy \( z \), party \( i \)’s payoff is \( u_i(z) - c \) (where the utility indices \( u_1 \) and \( u_2 \) are the same as in the previous section). When a party obtains a supermajority, then that party implements its preferred policy unilaterally, without the support of the minority party. In this case the minority party does not incur any cost, since it is not conceding to the majority party’s proposal.
Indeed, a party will only obtain a supermajority in situations in which the public opinion is clearly favouring that party’s policies; and in such a situation, (at least some) legislators in the minority party could costlessly support proposals put forward by the majority party.

The parameter $c$ then captures the cost that the minority party incurs when it accepts a proposal by the majority party. One can think of this cost $c$ as measuring the level of polarization between the parties, or the degree of radicalization of the parties’ electoral base. The cost of concession $c$ might also capture how ideological the issue over which parties are bargaining is. Note that when $c = 0$, the model is identical to the one in Section 3.2.

When $c > 0$, the minority party will never accept a proposal $z \in [0, 1]$ that gives her a utility less than $c$ (i.e., a proposal $z$ with $u_i(z) < c$). Indeed, the minority party can always guarantee itself a non-negative payoff by rejecting all offers, so it will never find it optimal to accept such a proposal. In what follows, I will assume that party 1 pays the cost of conceding if agreement is reached when $x_t = 1/2$. I stress however that this assumption has no effect on the equilibrium outcome.

An outcome in this setting is again given by a pair $(A, \eta)$, where $A$ is the agreement region and $\eta : A \to [0, 1]$ gives the policy that gets implemented when parties reach an agreement. As in the previous section, I denote by $O^\phi$ the set of outcomes consistent with supermajority rule $\phi$; that is, $O^\phi$ is the set of outcomes satisfying conditions (i)-(iii) in Section 3.2.1.

For any supermajority rule $\phi$, let $c_i^\phi(x)$ denote party $i$’s cost of conceding when parties reach an agreement at $t$ with $x_t = x$. Then,

$$c_1^\phi(x) = \begin{cases} 
  c & x \in (\phi, 1/2], \\
  0 & x \notin (\phi, 1/2], 
\end{cases}$$

$$c_i^\phi(x) = \begin{cases} 
  c & x \in (1/2, 1 - \phi), \\
  0 & x \notin (1/2, 1 - \phi). 
\end{cases}$$
Let \( x_0 = x \) be the initial level of relative political power. Then, party \( i \)'s payoff from outcome \( (A, \eta) \in \mathcal{O}^\phi \) is

\[
V_i(x) = \begin{cases} 
  u_i(\eta(x)) - c_i^\phi(x) & x \in A, \\
  E\left[e^{-\tau(x)}\left(u_i\left(\eta\left(x_{\tau(A)}\right)\right)\right) \right] \big| x_0 = x & x \notin A,
\end{cases}
\]

That is, party \( i \)'s payoff at \( x \in A \) is \( u_i(\eta(x)) \) if party \( i \) is not conceding at \( x \), and it is \( u_i(\eta(x)) - c \) if party \( i \) is conceding. Let \( c^\phi(x) = c_1^\phi(x) + c_2^\phi(x) \), so that \( c^\phi(x) = c \) for \( x \in (\phi, 1 - \phi) \) and \( c^\phi(x) = 0 \) for \( x \notin (\phi, 1 - \phi) \). Note that for all \( x \in A \),

\[
V_1(x) + V_2(x) = 1 - c^\phi(x),
\]

since \( u_1(\eta(x)) + u_2(\eta(x)) = 1 \) for all \( x \in A \). In the rest of this section, I will denote an outcome \( (A, \eta) \in \mathcal{O}^\phi \) as a triplet \((A, V_1, V_2)\), with \( V_1 \) and \( V_2 \) satisfying (3.8).

### 3.3.2 Equilibrium

Suppose \( x_t = x \in (\phi, 1 - \phi) \) is such that party \( i \) is making proposals. If party \( i \) makes an offer \( z \in [0, 1] \) that gives a payoffs of \( V_j(x) \) to party \( j \), then it must be that

\[
V_j(x) = u_j(z) - c \iff u_j(z) = V_j(x) + c.
\]

Since \( u_i(z) + u_j(z) = 1 \) for all \( z \in [0, 1] \), it follows that such an offer would give party \( i \) a payoff of \( 1 - u_j(z) = 1 - V_j(x) - c \). In words, if party \( i \) wants to make an offer \( z \) that would give party \( j \) a payoff of \( V_j(x) \), then the offer must compensate party \( j \) for the cost of conceding \( c \).
Definition 3.2 Under supermajority rule \( \phi \), an outcome \( (A, V_1, V_2) \in \mathcal{O}^\phi \) is an equilibrium if, for \( i = 1, 2, j \neq i \)

\[
V_i(x) = \sup_{\tau \in T_i} E \left[ e^{-r \tau \wedge \tau^\phi} (1 - V_j(x_{\tau \wedge \tau^\phi}) - c^\phi(x_{\tau \wedge \tau^\phi})) \bigg| x_0 = x \right],
\]

for all \( x \in [0, 1] \).

This definition of equilibrium adapts definition 3.1 to the current setting. Indeed, definition 3.2 again implies that the acceptance threshold (and hence the payoff) of the party responding to offers is always equal to its expected continuation value: if \( x \) is such that party \( i \) is responding to offers, then by equation (3.9) it follows that

\[
V_i(x) = E \left[ e^{-r \tau \wedge \tau^\phi} V_i(x_{\tau \wedge \tau^\phi}) \bigg| x_0 = x \right],
\]

regardless of whether parties reach an agreement when \( x_t = x \) or not (i.e., regardless of whether \( x \) belongs to the agreement region \( A \) or not).

On the other hand, when party \( i \) is making proposals, it takes party \( j \)'s acceptance threshold \( V_j(x) \) as given. At each moment in time before it loses the right to make offers, party \( i \) has to decide whether to make an offer that gives party \( j \) a payoff of \( V_j(x_t) \) and end the bargaining, or to wait until it can strike a better deal. The difference in this case is that if parties reach an agreement while \( x_t \in (\phi, 1 - \phi) \), the payoff that the party making offers obtains is \( 1 - V_j(x_t) - c \), since in this case one of the parties will have to pay the concession cost \( c \). Definition 3.2 says that an outcome \( (A, V_1, V_2) \) is an equilibrium if the proposer always finds it optimal to delay when \( x_t \notin A \), and always finds it optimal to make an acceptable offer when \( x_t \in A \).

Suppose that the supermajority rule is \( \phi \). Note that, for \( i = 1, 2 \), there always exists \( \tau \in T_i \) such that \( \tau \geq \tau^\phi \).\(^{14}\) By choosing such a stopping time \( \tau \), party \( i = 1, 2 \) can guarantee

\(^{14}\)Indeed, for any \( H \subseteq [1 - \phi, 1] \), \( \tau(H) \in T_1 \) and \( \tau(H) \geq \tau^\phi \). Similarly, for any \( H \subseteq [0, \phi] \), \( \tau(H) \in T_2 \) and \( \tau(H) \geq \tau^\phi \).
itself a payoff of

\[
W_1^\phi(x) = E \left[ e^{-r^\phi} 1_{\{x_{r^\phi} \geq 1-\phi\}} \big| x_0 = x \right], \quad (3.11)
\]
\[
W_2^\phi(x) = E \left[ e^{-r^\phi} 1_{\{x_{r^\phi} \leq \phi\}} \big| x_0 = x \right]. \quad (3.12)
\]

Therefore, if \((A, V_1, V_2)\) is an equilibrium outcome, then \(V_i \geq W_i^\phi\) for \(i = 1, 2\). That is, in any equilibrium party \(i\)'s payoff must be bounded below by \(W_i^\phi(x)\). Note that, for \(i = 1, 2\), \(W_i^\phi(x) \geq 0\) for all \(x \in [0, 1]\). That is, parties can always guarantee themselves a non-negative payoff by delaying an agreement until time \(\tau^\phi\). Finally, note that \(W_1^\phi\) and \(W_2^\phi\) do not depend on the cost of conceding \(c\).

Define \(W^\phi(x) := W_1^\phi(x) + W_2^\phi(x)\), so that \(W^\phi(x)\) is a lower bound for the sum of the parties’ payoffs at any equilibrium. Combining (3.11) and (3.12), it follows that \(W^\phi(x) = E[e^{-r^\phi} \mid x_0 = x]\).

**Lemma 3.1** For all \(x \in (\phi, 1-\phi)\), \(W^\phi(x)\) solves the ordinary differential equation

\[
rv(x) = \mu v'(x) + \frac{1}{2}\sigma^2 v''(x), \quad (3.13)
\]

with boundary conditions \(W^\phi(\phi) = W^\phi(1-\phi) = 1\).

**Proof:** Follows from Lemma A.3.1 in the appendix.

If parties reach an agreement at time \(t\) with \(x_t \in (\phi, 1-\phi)\), then the sum of their payoffs at \(t\) is \(1-c\), as one party will have to pay the cost of conceding. Therefore, parties will never reach an agreement at any \(x \in (\phi, 1-\phi)\) such that \(W^\phi(x) > 1-c\): agreement at \(x\) would imply that the sum of the parties’ payoff is \(1-c < W^\phi(x)\), a contradiction to the fact that \(W_i^\phi(x)\) is a lower bound to party \(i\)'s payoff. In other words, if \(W^\phi(x) > 1-c\) for some \(x \in (\phi, 1-\phi)\), there is no policy that could be implemented when \(x_t = x\) that would satisfy both parties’ expectations, so delay is the only possible outcome.
In particular, if \( W^\phi (x) > 1 - c \) for all \( x \in (\phi, 1 - \phi) \), parties will delay until one of them has a supermajority (i.e., until time \( \tau^\phi \)). In this case, the unique equilibrium outcome is \((S^\phi, W_1^\phi, W_2^\phi)\), where \( S^\phi = [0, \phi] \cup [1 - \phi, 1] \) is the set of states at which either party has a supermajority.\(^{15}\)

The solution to the differential equation (3.13) and boundary conditions is given by

\[
W^\phi (x) = \begin{cases} 
1 & \text{for } x \notin (\phi, 1 - \phi), \\
\frac{e^{\beta x}(e^{-\alpha x} - e^{-\alpha(1-\phi)}) + e^{-\alpha x}(e^{\beta(1-\phi)} - e^{\beta \phi})}{e^{\beta(1-\phi)} e^{-\alpha x} - e^{\beta \phi} e^{-\alpha(1-\phi)}} & \text{for } x \in (\phi, 1 - \phi). 
\end{cases} 
\]  
(3.14)

Note that \( W^\phi (x) \) is strictly convex for all \( x \in (\phi, 1 - \phi) \), attaining a value of 1 at \( \phi \) and \( 1 - \phi \).\(^{16}\) Therefore, \( W^\phi (x) \) has a unique minimum \( w^\phi := \min_{x \in [0,1]} W^\phi (x) \).

**Lemma 3.2** Let \( \phi, \widetilde{\phi} \in [0,1/2] \), \( \phi > \widetilde{\phi} \). Then, \( W^\phi (x) \geq W^{\widetilde{\phi}} (x) \) for all \( x \in [0,1] \), with strict inequality for all \( x \in (\widetilde{\phi}, 1 - \widetilde{\phi}) \).

**Proof:** See Appendix A.3.2.

Since \( W^\phi (x) \) is increasing in \( \phi \) for all \( x \), it follows that \( w^\phi \) is also increasing in \( \phi \). Therefore, \( w^0 \leq W^\phi (x) \) for all \( \phi \in [0,1/2] \) and all \( x \in [0,1] \). Define \( \overline{c} := 1 - w^0 \).\(^{17}\)

**Proposition 3.2** If \( c > \overline{c} \), then the unique equilibrium outcome is \((S^\phi, W_1^\phi, W_2^\phi)\).

**Proof:** Fix \( \phi \in [0,1/2] \). Suppose \( c > \overline{c} \), and note that \( W^\phi (x) \geq w^0 = 1 - \overline{c} > 1 - c \) for all \( x \in (\phi, 1 - \phi) \). By the discussion above, there cannot be agreement at any \( x \in (\phi, 1 - \phi) \). Therefore, in this case the equilibrium agreement region is \( S^\phi \), and the parties’ equilibrium payoffs are \((W_1^\phi, W_2^\phi)\).

\(^{15}\)In this case, for \( i = 1,2 \) any stopping time \( \tau \in T_i \) with \( \tau \geq \tau^\phi \) is a solution to the optimal stopping problem (9).

\(^{16}\)Indeed, for all \( x \in (\phi, 1 - \phi) \),

\[
(W^\phi)^{''} (x) = \frac{\beta^2 e^{\beta x}(e^{-\alpha x} - e^{-\alpha(1-\phi)}) + \alpha^2 e^{-\alpha x}(e^{\beta(1-\phi)} - e^{\beta \phi})}{e^{\beta(1-\phi)} e^{-\alpha x} - e^{\beta \phi} e^{-\alpha(1-\phi)}} > 0.
\]

\(^{17}\)Since \( W^0 (x) > 0 \) for all \( x \in (0,1) \), it follows that \( w^0 > 0 \), and thus \( \overline{c} < 1 \).
Proposition 3.2 shows that if the cost of conceding is large enough (i.e., if \( c > c \)), then parties will never come to an agreement while \( x_t \in (\phi, 1 - \phi) \). In this case, policies will only be implemented when a party obtains a supermajority. Note that the agreement date is \( \tau^\phi \), so the amount of delay increases as \( \phi \) decreases (i.e., as the supermajority rule becomes more stringent): as \( \phi \) decreases, it will take longer for the process \( x_t \) to leave the interval \((\phi, 1 - \phi)\). In other words, stringent supermajority requirements lead to gridlock and political inaction when parties face a large cost of conceding. The sum of the parties’ payoffs is given by \( W^\phi \), which is increasing in \( \phi \) (Lemma 3.2). Therefore, when the cost of concession is larger than \( c \) an utilitarian social planner setting the supermajority rule would choose \( \phi \approx 1/2 \) (i.e., weak majority).

Next, I study the case with \( c \in (0, \bar{c}) \). Since \( W^\phi \) is continuous in \( \phi \), by Berge’s theorem of the maximum \( w^\phi \) is also continuous in \( \phi \).\(^{18}\) For any \( c \in (0, \bar{c}) \), define \( \bar{\phi}(c) := \inf\{\phi \in [0, 1/2) : w^\phi \geq 1 - c\} \).

**Lemma 3.3** For all \( c \in (0, \bar{c}) \), \( \bar{\phi}(c) \) is well-defined. Moreover, \( \bar{\phi}(c) \) is decreasing in \( c \) and \( w^{\bar{\phi}(c)} = 1 - c \).

**Proof:** See Appendix A.3.2.

**Proposition 3.3** For every \( c \in (0, \bar{c}) \), there exists a unique equilibrium outcome \((A, V_1^\phi, V_2^\phi)\).

(i) If \( \phi > \bar{\phi}(c) \), the unique equilibrium outcome has \( A = S^\phi \). (ii) If \( \phi < \bar{\phi}(c) \), the unique equilibrium outcome has \( A = S^\phi \cup [\underline{x}, \bar{\tau}] \), with \( \phi < \underline{x} < \bar{\tau} < 1 - \phi \).

**Proof:** See Appendix A.3.3.

Proposition 3.3 characterizes the equilibrium agreement region for intermediate costs of concession (i.e., for \( c \in (0, \bar{c}) \)). In this case there are two possibilities: (i) \( \phi > \bar{\phi}(c) \) and (ii) \( \phi < \bar{\phi}(c) \). To understand the importance of the threshold \( \bar{\phi}(c) \), recall that \( W^\phi \) is increasing.

\(^{18}\)See, for instance, pages 301-302 in de la Fuente (2000) for a statement and proof of Berge’s theorem of the maximum.
in \( \phi \) (Lemma 3.2). The threshold \( \bar{\phi}(c) \) is the smallest value of \( \phi \) such that \( W^\phi(x) \geq 1 - c \) for all \( x \). Thus, for \( \phi > \bar{\phi}(c) \) the sum of the party’s payoff from delaying until \( \tau^\phi \) is large relative to the cost of conceding \( c \) (i.e., \( W^\phi(x) > 1 - c \) for all \( x \in (\phi, 1 - \phi) \)). By the discussion above, in this case the equilibrium outcome is \((S^\phi, W_1^\phi, W_2^\phi)\), and policies are only implemented when a party has a supermajority (see dashed line in Figure 3.3).

Consider next the case in which \( \phi < \bar{\phi}(c) \), so that the supermajority rule is stringent relative to the concession cost. In this case there exists numbers \( a < b \) with \( a > \phi \) and \( b < 1 - \phi \) such that \( W^\phi(x) \leq 1 - c \) for \( x \in [a, b] \) and \( W^\phi(x) > 1 - c \) for \( x \notin [a, b] \) (see solid line in Figure 3.3). This implies that there will never be an agreement when a party is close to obtaining a supermajority (i.e., when \( x_t \in (\phi, a) \cup (b, 1 - \phi) \)), since there cannot be agreement when \( W^\phi(x) > 1 - c \). To understand the intuition behind this, suppose \( x_t \) is close to \( 1 - \phi \), so that party 1 is close to reaching the required supermajority. In this case, party 1’s value of delaying an agreement is large, as there is high probability that it will obtain a supermajority in a short time. This implies that party 1 would only make offers to party 2 that are close to its preferred policy (i.e., policies \( z \in [0, 1] \) close to 1). However, party 2 would never agree to such policies, since they do not compensate party 2 for the concession cost \( c \). Therefore, when \( x_t \) is close to \( 1 - \phi \) there are no policies that satisfy both parties’ expectations, so delay is the only possible outcome. A symmetric reasoning explains why there is delay when \( x_t \) is close to \( \phi \).

In Appendix A.3.3 I show that when \( \phi < \bar{\phi}(c) \) the equilibrium agreement region must be of the form \( A = S^\phi \cup [x, \bar{x}] \), with \( \phi < x < \bar{x} < 1 - \phi \). That is, in this case policies are only implemented either when one party has a supermajority, or when relative political power is not very unbalanced (i.e., when \( x_t \in [x, \bar{x}] \)). On the other hand, legislative gridlock emerges in situations in which one of the parties is close to obtaining a supermajority.

Suppose the agreement region \( A = S^\phi \cup [x, \bar{x}] \) is such that \( x < 1/2 < \bar{x} \), and let \((V_1^\phi, V_2^\phi)\) be the equilibrium payoffs.\(^{19}\) Since \( 1/2 \in A \), it follows that \( V_1^\phi(1/2) + V_2^\phi(1/2) = 1 - c \).

\(^{19}\)In Appendix A.3.3 I show that \( x < 1/2 < \bar{x} \) whenever \( |\mu| < k \) for some \( k > 0 \).
Figure 3.3: $W^\phi (x)$. Parameters: $\mu = 0$, $\sigma = 0.15$, $r = 0.05$ and $c = 0.2$.

Equation (3.10) then implies that

$$V_1^\phi (x) = E \left[ e^{-rt_\phi \wedge \tau (1/2)} \left( 1 - V_2^\phi (x_{t_\phi \wedge \tau (1/2)}) - c^\phi (x_{t_\phi \wedge \tau (1/2)}) \right) \right]_{x_0 = x},$$

for all $x \in (\phi, 1/2)$. Similarly,

$$V_2^\phi (x) = E \left[ e^{-rt_\phi \wedge \tau (1/2)} \left( 1 - V_1^\phi (x_{t_\phi \wedge \tau (1/2)}) - c^\phi (x_{t_\phi \wedge \tau (1/2)}) \right) \right]_{x_0 = x},$$

for all $x \in (1/2, 1 - \phi)$. By arguments similar to those used in Theorem 3.1, $V_1^\phi$ and $V_2^\phi$ solve equations (3.5) and (3.6) with boundary conditions $V_1^\phi (\phi) = V_2^\phi (1 - \phi) = 0$, $V_1^\phi (1/2) + V_2^\phi (1/2) = 1 - c$ and the smooth pasting condition $(V_1^\phi)' (1/2^-) = -(V_2^\phi)' (1/2^+).$ The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. These conditions pin down $V_1^\phi$ for all $x \in (\phi, 1/2)$ and $V_2^\phi$ for all $x \in (1/2, 1 - \phi)$. On the other hand, since $A = S^\phi \cup [\underline{x}, \overline{x}]$, it follows that $V_1^\phi (x) + V_2^\phi (x) = 1 - c$ for all $x \in [\underline{x}, \overline{x}]$. Therefore, $V_1^\phi (x) = 1 - V_2^\phi (x) - c$ for all $x \in [1/2, \overline{x}]$ and $V_2^\phi (x) = 1 - c - V_1^\phi (x)$ for all $x \in [\underline{x}, 1/2]$. These conditions pin down $V_1^\phi$ for all $x \in [\phi, \overline{x}]$, and $V_2^\phi$ for all $x \in [\underline{x}, 1 - \phi]$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2$. The reason why the smooth pasting condition has to hold in this setting is exactly the same as in Theorem 3.1: if this condition did not hold, then one of the parties would prefer to delay an agreement when $x$ is close to $1/2.
Note next that $V_1^\phi(x) = E[e^{-\tau(A)}u_1(\eta(x_{\tau(A)}))] \mid x_0 = x$ for all $x \in (\overline{x}, 1 - \phi)$. Indeed, $x_{\tau(A)}$ is either $1 - \phi$ or $\overline{x} > 1/2$ whenever $x_0 \in (\overline{x}, 1 - \phi)$, so in this case party 1 will never pay the concession cost. Note also that $u_1(\eta(1 - \phi)) = 1$ and $u_1(\eta(\overline{x})) = V_1(\overline{x}) = 1 - V_2^\phi(\overline{x}) - c$. Thus, for all $x \in (\overline{x}, 1 - \phi)$

$$V_1^\phi(x) = E\left[e^{-\tau(A)}g_1(x_{\tau(A)})\right] \mid x_0 = x,$$

where $g_1(x) = 1$ if $x \geq 1 - \phi$ and $g_1(x) = 1 - V_2^\phi(x) - c$ if $x \leq \overline{x}$. Lemma A.3.1 then implies that $V_1^\phi(x)$ solves (A.3.1) on $x \in (\overline{x}, 1 - \phi)$, with boundary conditions $V_1^\phi(\overline{x}) = 1 - V_2^\phi(\overline{x}) - c$ and $V_1^\phi(1 - \phi) = 1$. On top of these boundary conditions, $V_1^\phi$ must also satisfy the smooth pasting condition $(V_1^\phi)'(\overline{x}^+) = -(V_2^\phi)'(\overline{x})$; if this condition did not hold, party 1 would either have an incentive to delay when $x = \overline{x}$ (if $(V_1^\phi)'(\overline{x}^+) < -(V_2^\phi)'(\overline{x})$), or an incentive to agree at some $x > \overline{x}$ (if $(V_1^\phi)'(\overline{x}^+) > -(V_2^\phi)'(\overline{x})$); in either case, this would contradict the fact that the equilibrium agreement region is $A = S^\phi \cup [\underline{x}, \overline{x}]$. The boundary conditions and the smooth pasting condition pin down the value of $V_1^\phi$ for all $x \in (\overline{x}, 1 - \phi)$ and the value of $\overline{x}$. A symmetric argument shows that $V_2^\phi$ solves (A.3.1) for $x \in (\phi, \underline{x})$, with boundary conditions $V_2^\phi(\underline{x}) = 1 - V_1^\phi(\underline{x}) - c$ and $V_2^\phi(\phi) = 1$. Besides these conditions, $V_2^\phi$ must also satisfy the smooth pasting condition $(V_2^\phi)'(\underline{x}^-) = -(V_1^\phi)'(\underline{x})$. These three conditions pin down the value of $V_2^\phi$ for all $x \in (\phi, \underline{x})$ and the value of $\underline{x}$.

**Example 5** Figure 3.4 plots a situation with $c \in (0, \overline{x})$ and $\phi = 0 < \overline{\phi}$. The left panel plots party 1’s payoff as a function of relative political power $x$. The right panel plots the implemented policy and the delay region (the dotted line on the right panel is party 1’s payoff). For $x \leq 1/2$, party 1’s payoff is $V_1^\phi(x) = u_1(\eta(x)) - c = \eta(x) - c$, since for these values of $x$ party 1 pays the concession cost $c$ (here $\eta(x)$ denotes the policy that gets implemented when parties reach an agreement and $x_t = x$). Thus, $\eta(x) = V_1^\phi(x) + c$. On the other hand, for $x > 1/2$ party 1’s payoff is exactly equal to the implemented policy, as in this region it is party 2 who pays the concession cost. Therefore, the implemented policy is discontinuous in
Figure 3.4: Party 1’s payoff and Implemented policy. Parameters: $\mu = 0$, $\sigma = 0.1$, $r = 0.05$, $c = 0.1$ and $\phi = 0$.

the state variable $x$, with a downward jump at $x = 1/2$. The left panel also shows the delay regions at which parties fail to reach an agreement.

**Remark 3.2** Merlo and Wilson (1995, 1998) study complete information bargaining games in which the realization of an exogenous stochastic process determines at each bargaining round the size of the surplus to be divided among the players and the identity of the proposer. The authors show that, in this setting, players will delay in reaching an agreement if they expect the surplus to grow fast enough in the future. The delays that arise in this model with concession costs are closely related to the delays in Merlo and Wilson’s game. Indeed, in this section’s model the effective size of the surplus to be divided among the parties is $1 - c^\phi(x)$.

If $c > 0$ the effective size of the surplus jumps when $x_t$ reaches either $\phi$ or $1 - \phi$, since at this point no party has to pay the concession cost. Therefore, parties always delay whenever $x_t$ is close to either $\phi$ or $1 - \phi$, since there is high probability that the size of the surplus will jump in the near future.
Suppose \( \phi < \overline{\phi}(c) \), and let \((A, V_1^\phi, V_2^\phi)\) be the equilibrium outcome with \( A = S^\phi \cup [\underline{x}, \overline{x}] \). Define \( V^\phi(x) := V_1^\phi(x) + V_2^\phi(x) \). Then,

\[
V^\phi(x) = \begin{cases} 
1 & \text{if } x \in S^\phi; \\
1 - c & \text{if } x \in [\underline{x}, \overline{x}]; \\
E \left[ e^{-\tau(A)} \left( 1 - c^\phi(x_{\tau(A)}) \right) \right] x_0 = x & \text{if } x \in (\phi, \underline{x}) \cup (\overline{x}, 1 - \phi),
\end{cases}
\]

That is, the sum of the parties’ payoffs is equal to 1 when a party has a supermajority and equal to \( 1 - c \) when parties reach an agreement at \( x \in [\underline{x}, \overline{x}] \). When parties are delaying (i.e., when \( x \in (\phi, \underline{x}) \cup (\overline{x}, 1 - \phi) \)), the sum of the parties’ payoffs is equal to the expected value of waiting until \( x \) reaches \( A \), in which case parties will get a total payoff of either 1 (if \( x_{\tau(A)} \notin (\phi, 1 - \phi) \)) or \( 1 - c \) (if \( x_{\tau(A)} \in (\phi, 1 - \phi) \)).

By Lemma A.3.1, \( V^\phi \) solves ODE (3.13) for all \( x \in (\phi, \underline{x}) \cup (\overline{x}, 1 - \phi) \), with boundary conditions \( V^\phi(\phi) = V^\phi(1 - \phi) = 1 \) and \( V^\phi(\underline{x}) = V^\phi(\overline{x}) = 1 - c \). Moreover, \( V^\phi \in C^1 \) (since \( V_1^\phi \) and \( V_2^\phi \) are \( C^1 \), see Lemma A.3.6). Therefore, it must be that \( (V^\phi)'(\underline{x}) = (V^\phi)'(\overline{x}) = 0 \).

**Proposition 3.4** Fix \( c \in (0, \overline{c}) \) and let \( \phi_2 < \phi_1 < \overline{\phi}(c) \). For \( k = 1, 2 \), let \( A_k = S^{\phi_k} \cup [\underline{x}_k, \overline{x}_k] \) be the agreement region under supermajority \( \phi_k \). Then, \( \underline{x}_2 = \underline{x}_1 + \phi_2 - \phi_1 < \underline{x}_1 \), and \( \overline{x}_2 = \overline{x}_1 + \phi_1 - \phi_2 > \overline{x}_1 \).

**Proof:** I prove that \( \underline{x}_2 = \underline{x} + \phi_2 - \phi_1 \). The proof that \( \overline{x}_2 = \overline{x}_1 + \phi_1 - \phi_2 \) is symmetric and omitted. To see that this holds, for \( k = 1, 2 \) let \( V^{\phi_k}(x) = V_1^{\phi_k}(x) + V_2^{\phi_k}(x) \). By the discussion above, \( V^{\phi_k}(x) \) solves equation (3.13) for all \( x \in (\phi_k, \underline{x}_k) \), with \( V^{\phi_k}(\phi_k) = 1 \), \( V^{\phi_k}(\underline{x}_k) = 1 - c \) and \( (V^{\phi_k})'(\underline{x}_k, \phi_k) = 0 \). For all \( x \), let \( \widetilde{V}(x) = V^{\phi_1}(x + \phi_1 - \phi_2) \); that is, \( \widetilde{V}(x) \) is equal to \( V^{\phi_1}(x) \), but shifted to the left. Then, \( \widetilde{V}(x) \) solves (3.13) for all \( x \in (\phi_2, \underline{x}_1 + \phi_2 - \phi_1) \), with \( \widetilde{V}(\phi_2) = V^{\phi_1}(\phi_1) = 1 \), \( \widetilde{V}(\underline{x}_1 + \phi_2 - \phi_1) = V^{\phi_1}(\underline{x}_1) = 1 - c \) and \( \widetilde{V}'(\underline{x}_1 + \phi_2 - \phi_1) = (V^{\phi_1})'(\underline{x}_1) = 0 \). Therefore, since there is a unique solution to (3.13)
satisfying all these conditions (see proof of Lemma A.3.8 in the appendix), it must be that
\[ \tilde{V}(x) = V^{\phi_2}(x), \] and \( x_2 = x_1 + \phi_2 - \phi_1 < x_1. \)

Proposition 3.4 shows how the agreement region changes with the supermajority requirement in settings in which the cost of concession is moderately low (i.e., \( c < \tau \)) and the supermajority is stringent relative to the concession cost (i.e., \( \phi < \tilde{\phi}(c) \)). In these cases, the agreement region is \( A = S^\phi \cup [\underline{x}, \overline{x}] \). The result establishes that the range of intermediate values of the state variable \( x_t \) at which parties reach an agreement expands when the supermajority requirement becomes more stringent (i.e., when \( \phi \) decreases). That is, a stringent supermajority requirement is more effective in inducing parties to come to an understanding in settings in which their relative political power is not too unbalanced.

The next result shows the effect that an increase in the cost of concession has on the outcome of legislative negotiations.

**Proposition 3.5** Let \( c_1 < c_2 \), and let \( A_1 \) and \( A_2 \) be the agreement regions under \( c_1 \) and \( c_2 \) respectively. Then, \( A_1 \subseteq A_2 \).

**Proof:** Note first that \( \tilde{\phi}(c_1) > \tilde{\phi}(c_2) \) (since \( \tilde{\phi}(\cdot) \) is a decreasing function). There are three cases to consider: (a) \( \phi > \tilde{\phi}(c_1) > \tilde{\phi}(c_2) \), (b) \( \tilde{\phi}(c_1) > \phi > \tilde{\phi}(c_2) \) and (c) \( \tilde{\phi}(c_1) > \phi > \tilde{\phi}(c_2) \). In case (a), the equilibrium outcome remains unchanged when the concession cost increases, since under either \( c_1 \) or \( c_2 \) the parties will never come to an agreement while \( x_t \in (\phi, 1 - \phi) \): in this case \( A_1 = A_2 = S^\phi \). Consider next case (b). By Proposition 3(i), under cost \( c_2 \) the equilibrium agreement region is \( A_2 = S^\phi \). On the other hand, by Proposition 3(ii) the equilibrium agreement region under cost \( c_1 \) is \( A_1 = S^\phi \cup [\underline{x}_1, \overline{x}_1] \) (with \( \phi < \underline{x}_1 < \overline{x}_1 < 1 - \phi \)), so \( A_1 \subseteq A_2 \).

Finally, consider case (c). By Proposition 3.3(ii), for \( k = 1, 2 \), \( A_k = S^\phi \cup [\underline{x}_k, \overline{x}_k] \). Let \( V^\phi(x; c_k) \) be the sum of the parties’ payoffs when the concession cost is \( c_k \). I now show that \( x_2 > x_1 \). For \( k = 1, 2 \), \( V^\phi(x; c_k) \) solves (13) on \( (\phi, \underline{x}_k) \) with \( V^\phi(x; c_k) = 1 \), \( V^\phi(x; c_k) = 1 - c_k \) and \( (V^\phi)'(\underline{x}_k; c_k) = 0 \). Suppose by contradiction that \( x_2 \leq x_1 \), so
\( V^\phi(\bar{x}_2; c_2) = 1 - c_2 < 1 - c_1 \leq V^\phi(\bar{x}_1; c_1) \). Then, by Lemma A.3.4 in the appendix it must be that \((V^\phi)'(\phi; c_1) > (V^\phi)'(\phi; c_2)\), so \((V^\phi)'(x; c_1) > (V^\phi)'(x; c_2)\) for all \(x \in (\phi, \bar{x}_2)\). Moreover, since \(V^\phi(\cdot, c_k)\) is convex (see Lemma A.3.7) it follows that \((V^\phi)'(x; c_i) < 0\) for all \(x < \bar{x}_1\), with \((V^\phi)'(\bar{x}_1; c_1) = 0\). Thus, \(0 \geq (V^\phi)'(\bar{x}_2; c_1) > (V^\phi)'(\bar{x}_2; c_2)\), which contradicts the fact that \((V^\phi)'(\bar{x}_2; c_2) = 0\). Thus, it must be that \(\bar{x}_2 > \bar{x}_1\). A symmetric argument establishes that \(\bar{x}_2 < \bar{x}_1\). Thus, in this case \(A_1 = S^\phi \cup [\bar{x}_1, \bar{x}_2] \subset S^\phi \cup [\bar{x}_2, \bar{x}_2] = A_2\).

Proposition 3.5 shows that, when the cost of concession that the parties face increases, the region of the state space at which parties delay an agreement becomes larger. In other words, the model predicts that gridlock should increase when it is costlier for the parties to agree to proposals originating on the opposing side.

### 3.3.3 Some positive implications

The first positive implication of the model with concession costs relates to when legislative gridlock is more likely to emerge. Consider a case with intermediate concession costs \((c \in (0, \bar{c}))\) and with a stringent supermajority rule \((\phi < \bar{\phi}(c))\). By Proposition 3.3, in this setting a policy will be implemented either when there is a supermajority of legislators supporting one parties proposal (i.e., when \(x_t \in S^\phi\)), or when relative political power is not too unbalanced (i.e., when \(x_t \in [x, \bar{x}]\)). Put differently, this model predicts that gridlock is more likely to emerge when one of the parties has a relatively strong bargaining position vis-a-vis the other party. In these situations, the weak party will only agree to offers that compensate its cost of conceding. However, the strong party is not willing to make such offers, since there is a chance it might obtain a supermajority within a short time horizon.

A second implication of this model concerns the benefits for political parties of having a weak majority. Consider again a situation with intermediate cost of concession \((c \in (0, \bar{c}))\) and a stringent supermajority rule \((\phi < \bar{\phi}(c))\), so that the equilibrium agreement region is \(S^\phi \cup [\bar{x}, \bar{x}]\). Assume further that \(\mu\) is close to zero, so that \(\bar{x} < 1/2 < \bar{x}\). In this case, the policies that the parties end up agreeing upon when \(x_t \in [x, \bar{x}]\) are non-monotonic in \(x\).
(see right panel in Figure 3.4). That is, the implemented policy might be closer to party $i$’s preferred policy when party $i$ is in slight disadvantage relative to party $j$ than when party $i$ has only a weak majority. The reason for this is that the majority party always has to make generous offers to its opponent, to compensate for the cost of conceding $c$.

Finally, consider an increase in the cost of conceding $c$. Proposition 3.5 implies that such an increase in $c$ would lead to more political inaction, as this would decrease the size of the agreement region $A$. One should expect concession costs to increase with the level of party polarization. Also, an increase in the concession cost could reflect the fact that the issue over which parties are bargaining is more ideological. The model therefore predicts that the probability of gridlock will increase with the degree of polarization, and that gridlock will also be more likely when the issue at hand is highly ideological.

### 3.4 Conclusion

This paper presents a new legislative bargaining model in which the relative political power of the parties changes over time according to a diffusion process. The baseline model has a unique equilibrium, in which the parties always come to an immediate agreement. The policy that parties implement in equilibrium depends on the different parameters of the model: the drift and volatility of the process that drives political power, and on the supermajority requirement. The model is highly tractable, delivering closed-form form solutions to the parties’ equilibrium payoffs.

Departing from the baseline model, the paper shows how delays may emerge when it is costly for a party to concede to policies put forward by its opponent. When the cost of concession is sufficiently large, then there will never be a policy that satisfies both parties’ expectations. In this case, policies will only be implemented when a party has a supermajority. For intermediate costs of concession, gridlock arises in situations in which political power is unbalanced and one party is close to obtaining a supermajority. In these situations,
the weak party will only agree to offers that compensate its cost of conceding. However, the strong party is not willing to make such offers, since there is a chance it might obtain a supermajority within a short time horizon.

The model with concession costs provides some novel positive implications. First, the model predicts that legislative gridlock is more likely to emerge when one of the parties has a relatively strong bargaining position \textit{vis-a-vis} the other party. Also, a party having only a slight majority might not be very effective in securing policies that are close to its preferences. Indeed, in these situations the majority party still has to make generous offers in order to get the minority party to agree to them. Finally, the model also predicts that it will be harder to get laws approved by Congress when the level of partisanship is higher, and also when the issue at stake is more ideological.
Appendix A

Appendices

A.1 Appendix to Chapter 1

A.1.1 Proofs of Lemmas 1.1 and 1.2

Let $\tau_y = \inf \{ t : x_t \notin (y_1, y_2) \}$ for some $0 < y_1 < y_2$, and let $\tau_{y_1} = \inf \{ t : x_t \leq y_1 \}$.

**Lemma A.1.1** Let $g$ be a bounded function, and let $W$ be the solution to

$$rW(x) = \mu xW'(x) + \frac{1}{2} \sigma^2 x^2 W''(x),$$

with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. Then, $W(x) = E[e^{-\tau_{y_0} g(x_{\tau_y})} | x_0 = x]$ for all $x \in (y_1, y_2)$.

**Proof:** Let $W$ satisfy (A.1.1), with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. The general solution to (A.1.1) is $W(x) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2} \sigma^2 q(q - 1) + \mu q = r$, and where $A$ and $B$ are constants determined by $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$:

$$A = \frac{g(y_2) y_1^\kappa - g(y_1) y_2^\kappa}{y_1^\kappa y_2^\kappa - y_1^\lambda y_2^\lambda}, \quad B = -\frac{g(y_2) y_1^\lambda - g(y_1) y_2^\lambda}{y_1^\kappa y_2^\kappa - y_1^\lambda y_2^\lambda}$$

(A.1.2)
Let \( f(x, t) = e^{-rt} W(x) \). By Ito’s Lemma, for \( x_t \in (y_1, y_2) \)

\[
df(x_t, t) = e^{-rt} \left( -rw(x_t) + \mu x W'(x_t) + \frac{1}{2} \sigma^2 x^2 W''(x_t) \right) dt + e^{-rt} \sigma x W'(x_t) dB_t,
\]

where the second equality follows from the fact that \( W \) solves (A.1.1). Then,

\[
E[e^{-rt}g(x_{\tau_y})x_0 = x] = E[f(x_{\tau_y}, \tau_y)|x_0 = x] = f(x, 0) + E \left[ \int_0^\tau df(x_t, t) | x_0 = x \right] = W(x) + E \left[ \int_0^\tau e^{-rt} \sigma x W'(x_t) dB_t | x_0 = x \right] = W(x),
\]

since \( \int_0^\tau e^{-rt} \sigma x W'(x_t) dB_t \) is a martingale with expectation zero.

**Corollary A.1.1** Let \( g \) be a bounded function, and let \( w \) be a solution to (A.1.1) with \( w(y_1) = g(y_1) \) and \( \lim_{x \to \infty} w(x) = 0 \). Then, \( w(x) = E[e^{-rt}g(x_{\tau_y})x_0 = x] \) for all \( x > y_1 \). Moreover, \( w(x) = g(y_1) (x/y_1)^{\lambda} \) for all \( x > y_1 \).

**Proof:** Let \( \tau_y = \inf\{t : x_t \leq y_1\} \) and note that \( \tau_y = \inf\{t : x_t \notin (y_1, y_2)\} \to \tau_y \) as \( y_2 \to \infty \). By monotone convergence,

\[
W(x) = E[e^{-rt}g(x_{\tau_y})x_0 = x] \quad \to \quad E[e^{-rt}g(x_{\tau_y})x_0 = x] = w(x).
\]

By Lemma A.1.1, \( W(x) = Ax^\lambda + Bx^\kappa \) for \( x \in (y_1, y_2) \), with \( A \) and \( B \) satisfying (A.1.2).

Since \( \lim_{y_2 \to \infty} B = 0 \) and \( \lim_{y_2 \to \infty} A = g(y_1)/y_1^{\lambda} \), \( w(x) = \lim_{y_2 \to \infty} W(x) = g(y_1) (x/y_1)^{\lambda} \).

**Proof of Lemma 1.1:** Let \( V_k(\cdot) \) be as in the statement of the Lemma, and note that \( V_k(\cdot) \in C^1 \). One can show that \( V_k(x) > v_k - x \) for \( x > z_k \). Moreover, in this range \( V_k(\cdot) \) solves (A.1.1), with \( V_k(z_k) = v_k - z_k \) and \( \lim_{x \to \infty} V_k(x) = 0 \). By Corollary A.1.1,
\( V_k(x) = E[e^{-rt_k}(v_k - x_{r_k})|x_0 = x] \). One can also show that

\[
r_{j}(v_k - x) = rV_k(x) > \mu x V_k'(x) + \frac{1}{2}\sigma^2 x^2 V_k''(x) = -\mu x,
\]

for all \( x \leq z_k \). Therefore, by standard verification results \( V_k(\cdot) \) is the solution to (1.6) (e.g., Theorem 3.17 in Shiryaev, 2008).

**Remark A.1.1** Since \( V_k \) is a solution to the optimal stopping problem (1.6), then \( e^{-rt}V_k(x_t) \) is superharmonic; i.e., \( V_k(x) \geq E[e^{-rt}V_k(x_{\tau})|x_0 = x] \) for any stopping time \( \tau \) (e.g., Theorem 10.1.9 in Oksendal, 2008). I will use this property repeatedly in what follows.

**Proof of Lemma 1.2:** By equation (1.8) and Lemma 1.1,

\[
P(x, \alpha) - x - V_1(x) = v_2 - x - E[e^{-rt_1}(v_2 - x_{\tau_1})|x_0 = x] > 0
\]

for all \( x \in (z_1, z_2) \), since by Lemma 1.1, \( v_2 - x = V_2(x) > E[e^{-rt_1}(v_2 - x_{\tau_1})|x_0 = x] \).

**A.1.2 Proof of Lemma 1.3**

I divide the proof of Lemma 1.3 in a series of Lemmas. Lemmas A.1.2 and A.1.3 give properties of solutions to equation (A.1.1). Lemma A.1.4 characterizes the solution to the optimal stopping problem (1.10), while Lemmas A.1.5 and A.1.6 prove some properties of this solution.

**Lemma A.1.2** Let \( U \) be a solution to (A.1.1) with \( U(y) = (1-a)(v-y) \) and \( U'(y) = -(1-a) \) for some \( y \in (0, z_1) \) and \( a \in [0, \alpha) \). Then, \( U \) is strictly convex for all \( x > 0 \).

**Proof:** The general solution to (A.1.1) is \( U(x) = Ax^\lambda + Bx^\kappa \). Using the initial conditions,

\[
A = y^{-\lambda}(1-a) \frac{\kappa(v_1-y) + y}{\kappa - \lambda} > 0 \text{ and } B = y^{-\kappa}(1-a) \frac{-(v_1-y)\lambda - y}{\kappa - \lambda} > 0,
\]
where the second inequality follows from the fact that \( y < z_1 = -v_1\lambda/(1 - \lambda) \). Thus, \( U''(x) = \lambda(\lambda - 1)Ax^{\lambda^2} + \kappa(\kappa - 1)Bx^{\kappa^2} > 0 \) for all \( x > 0 \) (since \( \kappa > 1 \)).

**Lemma A.1.3** Let \( U \) and \( \tilde{U} \) be two solutions to \((A.1.1)\). If \( \tilde{U}(y) \geq U(y) \) and \( \tilde{U}'(y) > U'(y) \) for some \( y > 0 \), then \( \tilde{U}''(x) > U''(x) \) for all \( x > y \), and so \( \tilde{U}(x) > U(x) \) for all \( x > y \). Similarly, if \( \tilde{U}(y) \leq U(y) \) and \( \tilde{U}'(y) > U'(y) \) for some \( y > 0 \), then \( \tilde{U}''(x) > U''(x) \) for all \( x < y \), and so \( \tilde{U}(x) < U(x) \) for all \( x < y \).

**Proof:** I prove the first statement of the Lemma. The proof of the second statement is symmetric and omitted. Suppose the claim is not true, and let \( y_1 > y \) be the smallest point with \( U'(y_1) = \tilde{U}'(y_1) \). Therefore, \( \tilde{U}''(x) > U''(x) \) for all \( x \in [y, y_1) \), so \( \tilde{U}(y_1) > U(y_1) \). Since \( U \) and \( \tilde{U} \) solve \((A.1.1)\), then

\[
\tilde{U}''(y_1) = \frac{2(rU(y_1) - \mu y_1 U'(y_1))}{\sigma^2 y_1^2} > \frac{2(rU(y_1) - \mu y_1 U'(y_1))}{\sigma^2 y_1^2} = U''(y_1).
\]

But this implies that \( U'(y_1 - \varepsilon) > \tilde{U}'(y_1 - \varepsilon) \) for \( \varepsilon \) small enough, a contradiction.

For all \( a \in [0, \alpha) \) and all \( x > 0 \), let \( g(x, a) = (\alpha - a)(P(x, \alpha) - x) + \Pi(x, \alpha) \), where \( \Pi(x, \alpha) \) and \( P(x, \alpha) \) are given by equations \((1.7)\) and \((1.9)\). Let \( L(x, a) = \sup_x E[e^{-rt}g(x, a)] \) for \( x = x \).

**Lemma A.1.4** For all \( a \in [0, \alpha) \), there exists \( \underline{x}(a) \in (0, z_1) \) and \( \overline{x}(a) \in (z_1, z_2) \) such that

\[
\tau(a) = \inf\{t : x_t \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]\} \text{ solves } (1.10).
\]

Moreover,

(i) for all \( x \in (\underline{x}(a), \overline{x}(a)) \cup (z_2, \infty) \), \( L(x, a) \) solves \((A.1.1)\), with \( \lim_{x \to \infty} L(x, a) = 0 \).

(ii) for all \( x \leq \underline{x}(a) \) and \( x \in [\overline{x}(a), z_2] \), \( L(x, a) = g(x, a) \).

(iii) the cutoffs \( \underline{x}(a) \) and \( \overline{x}(a) \) are such that

\[
L(\underline{x}(a), a) = g(\underline{x}(a), a), \quad L(\overline{x}(a), a) = g(\overline{x}(a), a), \quad (VM)
\]

\[
L_x(\underline{x}(a), a) = g_x(\underline{x}(a), a), \quad L_x(\overline{x}(a), a) = g_x(\overline{x}(a), a). \quad (SP)
\]

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**Proof:** First I show that there exists a function $L(x, a)$ satisfying conditions (i)-(iii). I start by showing that there exists a function $L(x, a)$ and unique cutoffs $\bar{x}(a)$ and $\bar{\pi}(a)$ such that $L(x, a)$ solves (A.1.1) on $(\bar{x}(a), \bar{\pi}(a))$ and satisfies (iii). To see this, consider solutions $U$ to (A.1.1) with $U(y) = g(y, a) = (1 - a)(v_1 - y)$ and $U'(y) = g_x(y, a) = -(1 - a)$ for some $y < z_1$. By Lemma A.1.2, such solutions are strictly convex. Since solutions to (A.1.1) are continuous in initial conditions, then the solutions I’m considering are continuous in $y$. If $y$ is small enough, then $U(x)$ will remain above $g(x, a)$ for all $x > y$. On the other hand, if $y$ close to $z_1$ then $U$ will cross $g(x, a)$ at some $\tilde{x} > y$ (see solutions I-IV in Figure A.1.1). By Lemma A.1.3, the point $\tilde{x}$ moves to the right as $y$ decreases. Let $\bar{x}(a)$ be the smallest $y$ such that $U$ reaches $g(x, a)$ for some $\bar{\pi}(a) > y$. Since a solution with $y < \bar{x}(a)$ never reaches $g(x, a)$, it follows that $U(x) \geq g(x, a)$ for all $x$. Thus, $U$ is tangent to $g(x, a)$ at $\bar{\pi}(a)$, so $U'(\bar{\pi}(a)) = g_x(\bar{\pi}(a), a)$ (solution III in Figure A.1.1). Note that $U$ is the unique solution to (A.1.1) that satisfies (VM) and (SP). Hence, $L(x, a) = U(x)$ for $x \in [\bar{x}(a), \bar{\pi}(a)]$.

By (ii), $L(x, a) = g(x, a)$ for $x \leq \bar{x}(a)$ and $x \in [\bar{\pi}(a), z_2]$. By (i), $L(\cdot, a)$ solves (A.1.1) for $x > z_2$, with $\lim_{x \to \infty} L(x, a) = 0$ and $L(z_2, a) = g(z_2, a)$. Corollary A.1.1 then implies that

$$L(x, a) = E[e^{-\tau_2 g(x_{\tau_2}, a)} | x_0 = x] = g(z_2, a)(x/z_2)^\lambda$$

for all $x > z_2$.

For future reference, one can check that $L_x(z_2, a) = g_x(z_2, a)$; i.e., $L$ satisfies the smooth pasting condition at $z_2$. Also, one can check that, for all $x > z_2$, $g(x, a) < g(z_2, a)(x/z_2)^\lambda = E[e^{-\tau_2 g(x_{\tau_2}, a)} | x_0 = x]$.

Let $L(x, a)$ be the (unique) function satisfying conditions (i)-(iii). Then, $L(x, a)$ is twice differentiable in $x$, with a continuous first derivative. Moreover,

$$- rL(x, a) + \mu x L_x(x, a) + \frac{1}{2} \sigma^2 x^2 L_{xx}(x, a) \leq 0, \text{ with equality on } (\bar{x}(a), \bar{\pi}(a)) \cup (z_2, \infty).$$

(A.1.3)

Indeed, $L(x, a)$ satisfies (A.1.3) with equality on $(\bar{x}(a), \bar{\pi}(a)) \cup (z_2, \infty)$ since it solves (A.1.1) in this region. One can also check that $rL(x, a) > \mu x L_x(x, a) + \frac{1}{2} \sigma^2 x^2 L_{xx}(x, a)$
for all $x \in [0, \underline{x}(a)] \cup [\overline{x}(a), z_2]$. By standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008), $L(x, a) = \sup_x E[e^{-\tau x} g(x_\tau, a) | x_0 = x]$. By Lemma A.1.1 and Corollary A.1.1, $L(x, a) = E[e^{-\tau x} g(x_\tau, a) | x_0 = x]$, so $\tau(a)$ solves (1.10).

Finally, note that by construction it must be that $\underline{x}(a) < z_1$ and that $\overline{x}(a) > z_1$. I now show that $\underline{x}(a) < z_2$. Suppose by contradiction that $\overline{x}(a) > z_2$. In this case, $L(\overline{x}(a), a) = g(\overline{x}(a), a) < E[e^{-\tau x} g(z_2, a) | x_0 = \overline{x}(a)]$, contradicting the fact that $L(\overline{x}(a), a)$ solves the optimal stopping problem (1.10). Therefore, it must be that $\overline{x}(a) < z_2$.

**Lemma A.1.5** $L(x, a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$ and all $a \in [0, \alpha)$. Moreover, $\underline{x}(a)$ and $\overline{x}(a)$ are continuous in $a$, with $\lim_{a \to \alpha} \underline{x}(a) = \lim_{a \to \alpha} \overline{x}(a) = z_1$.

**Proof:** By Lemma A.1.4, $L(x, a) = A(a) x^\lambda + B(a) x^\kappa$ for all $x \in (\underline{x}(a), \overline{x}(a))$, where $A(a)$, $B(a)$, $\underline{x}(a)$ and $\overline{x}(a)$ are determined by the system of equations (VM) + (SP). Denote this system of equations by $F(\underline{x}(a), \overline{x}(a), A(a), B(a)) = 0$. Note that $F \in C^2$. Moreover, its Jacobian at $(\underline{x}(a), \overline{x}(a), A(a), B(a))$ has a non-zero determinant. By the Implicit Function Theorem, the functions $A(a)$, $B(a)$, $\underline{x}(a)$ and $\overline{x}(a)$ are all $C^2$ with respect to $a$ (e.g., de la Fuente, 2000, pages 210-211). Since $L(x, a) = A(a) x^\lambda + B(a) x^\kappa$ for all $x \in (\underline{x}(a), \overline{x}(a))$, this implies that $L(x, a) \in C^{2,2}$ for all $x \in (\underline{x}(a), \overline{x}(a))$. 

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Next, I show that \( \lim_{a \to \alpha} \varphi(a) = \lim_{a \to \alpha} \varphi(a) = z_1 \). Let \( \varphi := \lim_{a \to \alpha} \varphi(a) \) and \( \varphi := \lim_{a \to \alpha} \varphi(a) \). Since \( \varphi(a) < z_1 \) and \( \varphi(a) > z_1 \) for all \( a \) (Lemma A.1.4), it follows that \( \varphi \leq z_1 \leq \varphi \). Let \( \hat{\tau} := \inf \{ t : x_t \in [0, \varphi] \cup [\varphi, z_2] \} \), so \( \tau(a_n) \to \hat{\tau} \) for every sequence \( \{a_n\} \to \alpha \). Note that \( L(x, a) \geq g(x, a) \geq \Pi(x, \alpha) = (1 - \alpha) V_1(x) \) for all \( a \leq \alpha \), so \( \lim_{a \to \alpha} L(x, a) \geq (1 - \alpha) V_1(x) \).

Let \( \{a_n\} \to \alpha \). Since \( \lim_{a \to \alpha} g(x, a) = (1 - \alpha) V_1(x) \), by Dominated Convergence

\[
L(x, a_n) = E \left[ e^{-r\tau(a_n)} g(x_{\tau(a_n)}, a_n) \right] \xrightarrow{\text{as } n \to \infty} E \left[ e^{-r\tau(\alpha)} (1 - \alpha) V_1(x) \right] \xrightarrow{x_0 = x} ,
\]

Suppose by contradiction that \( \varphi < z_1 \). Then, for \( x \in (\varphi, \varphi) \),

\[
(1 - \alpha) E \left[ e^{-r\hat{\tau}} V_1(x) \right] \xrightarrow{x_0 = x} = (1 - \alpha) \Pr (x_{\hat{\tau}} = \varphi) E \left[ e^{-r\tau(\varphi)} (v_1 - \varphi) \right] \xrightarrow{x_0 = x} + (1 - \alpha) \Pr (x_{\hat{\tau}} = \varphi) E \left[ e^{-r\tau(\varphi)} V_1(\varphi) \right] \xrightarrow{x_0 = x} < (1 - \alpha) V_1(x) ,
\]

where the inequality follows from Remark A.1.1 and the fact that \( E[e^{-r\tau(\varphi)} (v_1 - \varphi)] \leq V_1(x) = E[e^{-r\tau_1} (v_1 - z_1)] \) (Lemma 1.1). This contradicts \( \lim_{a \to \alpha} L(x, a) \geq (1 - \alpha) V_1(x) \), so \( \varphi = z_1 \).

Suppose next that \( \varphi > z_1 \). Let \( \text{W} (x) = E[e^{-r\tau} (P(x_{\tau}, \alpha) - x_{\tau})] \xrightarrow{x_0 = x} \). Since \( \varphi = z_1 \), it follows that \( \hat{\tau} = \inf \{ t : x_t \in [0, z_1] \cup [\varphi, z_2] \} \). Let \( Y_t = e^{-rt} (P(x_t, \alpha) - x_t) \). By Ito’s Lemma,

\[
dY_t = e^{-rt} \left( -r (P(x_t, \alpha) - x_t) + \mu x_t (P_x (x_t, \alpha) - 1) + \frac{\sigma^2 x_t^2}{2} P_{xx} (x_t, \alpha) dt \right) + \sigma x_t P_x (x_t, \alpha) dB_t,
\]

for all \( x_t \in (z_1, \varphi) \). Equation (1.9) implies \( rP(x, \alpha) = rv_2 + \mu x P_x (x, \alpha) + \frac{\sigma^2 x^2}{2} P_{xx} (x, \alpha) \), so

\[
dY_t = e^{-rt} (-r (v_2 - x_t) - \mu x_t) dt + e^{-rt} \sigma x_t P_x (x_t, \alpha) dB_t.
\]
Therefore, for \( x \in (z_1, \overline{x}) \),

\[
W(x) = E[Y \mid x_0 = x] = Y_0 + E \left[ \int_0^{\overline{x}} e^{-rt} (-r(v_2 - x) - \mu x_t) dt \mid x_0 = x \right].
\]

One can check that \(-r(v_2 - x) < \mu x\) for all \( x < \overline{x} < z_2 \), so \( W(x) < Y_0 = P(x, \alpha) - x \).

For each \( a \in [0, \alpha) \), let \( W(x, a) = E[e^{-r\tau(a)}(P(x_\tau(a), \alpha) - x_\tau(a))] \mid x_0 = x \). Pick a sequence \( \{a_n\} \to \alpha \), and note that \( \tau(a_n) \to \overline{\tau} \) as \( n \to \infty \). By dominated Convergence, \( W(x, a_n) \to W(x) \) as \( n \to \infty \). Fix \( x \in (z_1, \overline{x}) \). Since \( W(x) < P(x, \alpha) - x \), there exists \( N \) such that \( W(x, a_n) < P(x, \alpha) - x \) for all \( n > N \). On the other hand,

\[
E[e^{-r\tau(a_n)}V_1(x_\tau(a_n))] \mid x_0 = x \leq V_1(x) \text{ for all } x \text{ (see Remark A.1.1).}
\]

Therefore, for \( n > N \)

\[
L(x, a_n) = E_x \left[ e^{-r\tau(a_n)} \left( (\alpha - a_n)(P(x_\tau(a_n), \alpha) - x_\tau(a_n)) + (1 - \alpha)V_1(x_\tau(a_n)) \right) \right]
\]

\[
< (\alpha - a_n)(P(x, \alpha) - x) + (1 - \alpha)V_1(x) = g(x, a_n),
\]

which contradicts the fact that \( L(x, a_n) = \sup_{\tau} E[e^{-r\tau}g(x_\tau, a_n) \mid x_0 = x] \). Thus, \( \overline{x} = z_1 \).

**Proof of Lemma 1.3:** Follows directly from Lemmas A.1.4 and A.1.5.

**Lemma A.1.6** \( L(x, a) \) is strictly convex in \( a \) for all \( x \in (\underline{x}(a), \overline{x}(a)) \).

**Proof:** I first show that \( \underline{x}'(a) > 0 \) and \( \overline{x}'(a) < 0 \). For \( a \in [0, \alpha) \), let

\[
W(x, a) = E[e^{-r\tau(a)}(P(x_\tau(a), \alpha) - x_\tau(a))] \mid x_0 = x,
\]

\[
U(x, a) = E[e^{-r\tau(a)}V_1(x_\tau(a))] \mid x_0 = x,
\]

so \( L(x, a) = (\alpha - a)W(x, a) + (1 - \alpha)U(x, a) \). By Lemma A.1.1, \( U(x, a) \) solves (A.1.1) with \( U(\underline{x}(a), a) = v_1 - \underline{x}(a) = V_1(\underline{x}(a)) \) and \( U(\overline{x}(a), a) = (v_1 - z_1)(\overline{x}(a)/z_1)^\lambda = V_1(\overline{x}(a)) \).

Note that \( U_x(\underline{x}(a), a) < -1 \) and \( U_x(\overline{x}(a), a) > V_1'(\overline{x}(a)) \). To see this, note that \( V_1(x) \) also solves (A.1.1) for \( x \geq z_1 \), with \( V_1(z_1) = v_1 - z_1 \) and \( V_1'(z_1) = -1 \). Suppose by contradiction
By Lemma A.1.2, \( U \) is strictly convex, so \( U'(x) > -1 \) and \( U(x) > v_1 - x \) for all \( x > \bar{x}(a) \). Lemma A.1.3 then implies that \( U(x, a) > V_1(x) \) for all \( x > \bar{x}(a) \), a contradiction to the fact that \( U(\bar{x}(a), a) = V_1(\bar{x}(a)) \). Hence, \( U_x(\bar{x}(a), a) < -1 \). Similarly, if \( U_x(\bar{x}(a), a) \leq V'_1(\bar{x}(a)) \) then by Lemma A.1.3 \( U(x, a) > V_1(x) \) for all \( x < \bar{x}(a) \), which contradicts \( U_x(\bar{x}(a), a) = V_1(\bar{x}(a)) \). Hence, \( U_x(\bar{x}(a), a) > V'_1(\bar{x}(a)) \). Since \( L_x(\bar{x}(a), a) = g_x(\bar{x}(a), a) = -(1 - a) \) and \( L_x(\bar{x}(a), a) = g_x(\bar{x}(a), a) = (\alpha - a)(P_x(\bar{x}(a), a) - 1) + (1 - \alpha)V'_1(\bar{x}(a)) \), it follows that \( W_x(\bar{x}(a), a) > -1 \) and \( W_x(\bar{x}(a), a) < P_x(\bar{x}(a), a) - 1 \).

Let \( a' < a \). The analysis above implies that,

\[
(\alpha - a') W_x(\bar{x}(a), a) + (1 - a) U_x(\bar{x}(a), a) > g_x(\bar{x}(a), a') \\
(\alpha - a') W_x(\bar{x}(a), a) + (1 - a) U_x(\bar{x}(a), a) < g_x(\bar{x}(a), a').
\]

Let \( F(x) = (\alpha - a') W(x, a) + (1 - a) U(x, a) \). Since \( W(x, a) \) and \( U(x, a) \) both solve (A.1.1) for all \( x \in (\bar{x}(a), \bar{x}(a)) \) (Lemma A.1.1), then so does \( F \). Moreover, \( F(\bar{x}(a)) = g(\bar{x}(a), a') \) and \( F(\bar{x}(a)) = g(\bar{x}(a), a') \). Thus, by Lemma A.1.1 \( F(x) = E[e^{-r_1(x)}g(x_{\tau(a)}, a') | x_0 = x] \).

Suppose by contradiction that \( \bar{x}(a') \geq \bar{x}(a) \). Let \( H \) be the solution to (A.1.1) with \( H(\bar{x}(a)) = g(\bar{x}(a), a') = (1 - a')(v_1 - \bar{x}(a)) \) and \( H'(\bar{x}(a)) = g_x(\bar{x}(a), a') = -(1 - a') \).

By Lemma A.1.2, \( H \) is strictly convex, so \( H'(x) > -(1 - a') \) for all \( x > \bar{x}(a) \). Since \( F \) solves (A.1.1) with \( F(\bar{x}(a)) = g(\bar{x}(a), a') \) and \( F'(\bar{x}(a)) = g_x(\bar{x}(a), a') \), it follows by Lemma A.1.3 that \( F'(x) > H'(x) = -(1 - a') \) for all \( x \geq \bar{x}(a) \) and \( F(x) > H(x) > g(x, a') \) for all \( x \in (\bar{x}(a), z_1) \). By Lemma A.1.4, \( L(x, a) \) solves (A.1.1) on \( (\bar{x}(a'), \bar{x}(a')) \), with \( L(\bar{x}(a'), a') = g(\bar{x}(a'), a') \) and \( L_x(\bar{x}(a'), a') = g_x(\bar{x}(a'), a') \). The arguments above together with Lemma A3 then imply that \( F(x) > L(x, a') \) for all \( x > \bar{x}(a') \), a contradiction to the fact that \( L(x, a') = \sup_x E[e^{-r_1(x)}g(x_{\tau(a')}, a') | x_0 = x] \). Thus, it must be that \( \bar{x}(a') < \bar{x}(a) \).

Similarly, suppose that \( \bar{x}(a') \leq \bar{x}(a) \). By a symmetric argument, one can show that \( L(\bar{x}(a'), a') = g(\bar{x}(a'), a') \leq F(\bar{x}(a')) \) and \( L_x(\bar{x}(a'), a') = g_x(\bar{x}(a'), a') > F_x(\bar{x}(a')) \).
Lemma A.1.3 then implies that $F(x) > L(x, a')$ for all $x < \pi(a')$, contradicting the fact that $L(x, a') = \sup_{\tau} E [e^{-\tau} g(x_\tau, a')] | x_0 = x$. Thus, it must be that $\pi(a') > \pi(a)$.

Finally, I show that $L(x, a)$ is strictly convex in $a$ for all $x \in (\pi(a), \pi(a))$. Take $a' < a < \alpha$, and let $a^\gamma = \gamma a + (1-\gamma)a'$ for some $\gamma \in (0, 1)$. Note that $g(x, a^\gamma) = \gamma g(x, a) + (1-\gamma)g(x, a')$. Moreover, $\pi(a') < \pi(a^\gamma) < \pi(a)$ and $\pi(a') > \pi(a^\gamma) > \pi(a)$. Therefore,

$$L(x, a^\gamma) = \gamma E_x [e^{-\tau(a^\gamma)} g(x_\tau(a^\gamma), a)] + (1-\gamma) E_x [e^{-\tau(a)} g(x_\tau(a), a')]$$

$$< \gamma L(x, a) + (1-\gamma) L(x, a'),$$

for all $x \in (\pi(a), \pi(a))$, so $L(x, a)$ is strictly convex in $a$ on $(\pi(a), \pi(a))$.

**A.1.3 Proof of Theorem 1.1**

The proof of Theorem 1.1 is organized as follows. Lemmas A.1.7 and A.1.8 establish conditions that hold in any regular equilibrium. Using these conditions, Lemma A.1.9 provides a partial characterization of the monopolist’s equilibrium payoff. Finally, Lemma A.1.10 establishes that in any regular equilibrium the monopolist’s payoff are equal to $L(x, a)$.

**Lemma A.1.7** Let $(\{a_t\}, P)$ be an equilibrium. Then,

(i) for all $t$ such that $x_t < z_2$ and $a_t < \alpha$, the monopolist always sells (i.e., $da_t > 0$),

(ii) for all $t$ such that $x_t > z_2$ and $a_t < \alpha$, the monopolist doesn’t sell (i.e., $da_t = 0$).

**Proof:** (i) Suppose that $a_t < \alpha$ and $da_t = 0$ while $x_t < z_2$. Let $\bar{\tau} = \inf\{s > t : da_s > 0\}$, so $\bar{\tau} > 0$. In this case, the price at which the marginal buyer $a_t^+$ is willing to buy satisfies

$$P(x_t, a_t^+) = v_2 - E \left[ e^{-(\bar{\tau}-t)} (v_2 - P(x_{\bar{\tau}}, a_{\bar{\tau}})) \right] | x_t, a_t.$$

That is, at time $t$ the marginal buyer $a_t^+$ is willing to pay a price that leaves her indifferent between buying at that price, or waiting until $\bar{\tau}$ and getting the good at price $P(x_{\bar{\tau}}, a_{\bar{\tau}})$. 

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The monopolist gets a profit of \( E[e^{-r(t-\tau)}(P(x_\tau,a_\tau) - x_\tau)|x_t,a_t]\) from selling to consumer \( a_t^+\) at time \( \tau \). The monopolist gets \( P(x_t,a_t^+) - x_t\) from selling to \( a_t^+\) at time \( t \). Note that,

\[
P(x_t,a_t^+) - x_t - E[e^{-r(\tau-t)}(P(x_\tau,a_\tau) - x_\tau)|x_t,a_t] = v_2 - x_t - E[e^{-r(\tau-t)}(v_2 - x_\tau)|x_t,a_t] > 0,
\]

where the last inequality follows from the fact that \( v_2 - x = \sup_\tau E[e^{-r\tau}(v_2 - x_\tau)|x_0 = x] \) for all \( x \leq z_2 \). Thus, the monopolist is better off selling to consumer \( a_t^+ \) at \( t \), a contradiction to the assumption that \( (\{a_t\}, P) \) is an equilibrium.

(ii) Suppose the monopolist sells while \( x_t > z_2 \). Let \( \tau(\alpha) \) denote the time at which consumer \( \alpha \) buys and recall that \( \tau_2 = \inf\{t : x_t \leq z_2\} \). Let \( \tau = \min\{\tau(\alpha), \tau_2\} \). Since all high valuation consumers must get the same payoff in equilibrium, the price the monopolist charges at any time \( s \in [t, \tau] \) must be such that

\[
P(x_s,a_s) = v_2 - E\left[e^{-r(\tau-s)}(v_2 - P(x_\tau,a_\tau))|x_s,a_s\right].
\]

If \( \tau(\alpha) < \tau_2 \), then \( a_\tau = \alpha \). After time \( \tau(\alpha) \), the monopolist sells to low type consumers when costs fall below \( z_1 \). Recall that \( \tau_1 = \inf\{t : x_t \leq z_1\} \). By equation (1.5), it follows that

\[
P(x_\tau,\alpha) = v_2 - E\left[e^{-r(\tau_1-\tau)}(v_2 - v_1)|x_\tau\right] = v_2 - E\left[e^{-r(\tau_2-\tau)}(v_2 - P(x_{\tau_2},\alpha))|x_\tau\right],
\]

since \( P(x_{\tau_2},\alpha) = v_2 - E\left[e^{-r(\tau_1-\tau_2)}(v_2 - v_1)|x_{\tau_2}\right] \). By the law of iterated expectations,

\[
P(x_s,a_s) = v_2 - E\left[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2},a_\tau))|x_s,a_s\right]
\]

\[
= v_2 - E\left[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2},a_{\tau_2}))|x_s,a_s\right],
\]

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for \( s < \tau \), since in this case \( a_{\tau_2} = \alpha \). On the other hand, if \( \tau (\alpha) \geq \tau_2 \), then

\[
P(x_s, a_s) = v_2 - E \left[ e^{-r(\tau_2-s)} (v_2 - P(x_{\tau_2}, a_{\tau_2})) \right] x_s, a_s.
\]

The profits that the monopolist gets from selling to high valuation consumers between time \( t \) and \( \tau \) are \( E_t[e^{-r(s-t)} \int_t^\tau (P(x_s, a_s) - x_s) da_s] \). If instead the monopolist waits until time \( \tau_2 \) and sells to all consumers \( i \in [a_t, a_{\tau_2}] \) at that instant, her profits are \( E_t[e^{-r(\tau_2-t)}(P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2})(a_{\tau_2} - a_t)] \). Note that for all \( s \in [t, \tau_2] \)

\[
P(x_s, a_s) - x_s - E_s \left[ e^{-r(\tau_2-s)} (P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2}) \right]
\]

\[
= v_2 - E_s \left[ e^{-r(\tau_2-s)} (v_2 - P(x_{\tau_2}, a_{\tau_2})) \right] - x_s - E_s \left[ e^{-r(\tau_2-s)} (P(x_{\tau_2}, a_{\tau_2}) - x_{\tau_2}) \right]
\]

\[
= v_2 - x_s - E_s \left[ e^{-r(\tau_2-s)} (v_2 - x_{\tau_2}) \right] < 0,
\]

since \( \tau_2 \) solves \( \sup_{\tau} E[e^{-r\tau} (v_2 - x_\tau)] \). Hence, the monopolist is better off by delaying sales until time \( \tau_2 \), contradiction the fact that \( (\{a_t\}, P) \) is an equilibrium.

**Lemma A.1.8** Let \( (\{a_t\}, P) \) be a regular equilibrium and let \( \Pi(x, a) \) denote the monopolist’s equilibrium profits. If \( a_s \) is continuously increasing in \( s \in [t, \tau] \) for some \( \tau > 0 \), then \( P(x_s, a_s) = x_s - \Pi_a(x_s, a_s) \) for all \( s \in [t, \tau] \).

**Proof:** Let \( (\{a_t\}, P) \) be a regular equilibrium. Suppose \( a_s \) is continuously increasing for \( s \in [t, \tau] \), with \( \dot{a}_s = da_s/ds \). Then,

\[
\Pi(x_t, a_t) = E_t \left[ \int_{[t,\tau]} e^{-r(s-t)} (P(x_s, a_s) - x_s) \dot{a}_s ds + e^{-r(\tau-t)} \Pi (x_\tau, a_\tau) \right].
\]

By the Law of iterated expectations, the process

\[
Y_t = \int_{[0,t]} e^{-rs} (P(x_s, a_s) - x_s) da_s + e^{-rt} \Pi (x_t, a_t)
\]

\[
= E \left[ \int_{[0,\tau]} e^{-rs} (P(x_s, a_s) - x_s) da_s + e^{-r\tau} \Pi (x_\tau, a_\tau) \right],
\]
is a martingale. The Martingale Representation Theorem implies that there exists a process 
\( \beta_t \in \mathcal{L}^* \) such that 
\[ dY_t = e^{-rt} \beta_t dB_t. \]
Differentiating (A.1.4) with respect to \( t \) yields
\[ dY_t = e^{-rt} \left( \frac{\partial}{\partial s} \Pi_t(s,a_s) - x_s \right) \frac{\partial}{\partial s} \Pi_t(s,a_s) dt - \frac{\partial}{\partial s} \Pi_t(s,a_s) \frac{\partial}{\partial x} \Pi_t(s,a_s) dt + e^{-rt} \frac{\partial}{\partial s} \Pi_t(s,a_s) \frac{\partial}{\partial x} \Pi_t(s,a_s) dB_t. \]

One the other hand, since \( \Pi \in C^{2,1} \) Ito’s Lemma implies that
\[ d\Pi_t(x_t,a_t) = \left( \mu x_t \Pi_t(x_t,a_t) + \frac{1}{2} \sigma^2 x_t^2 \Pi_{xx}(x_t,a_t) \right) dt + \Pi_a(x_t,a_t) \frac{\partial}{\partial x} \Pi_t(s,a_s) dB_t. \]
Combining these two equations gives
\[ r \Pi_t(x_t,a_t) = \mu x_t \Pi_t(x_t,a_t) + \frac{1}{2} \sigma^2 x_t^2 \Pi_{xx}(x_t,a_t) + \left( \frac{\partial}{\partial s} \Pi_t(s,a_s) - x_s \right) \frac{\partial}{\partial s} \Pi_t(s,a_s) dt + e^{-rt} \frac{\partial}{\partial s} \Pi_t(s,a_s) \frac{\partial}{\partial x} \Pi_t(s,a_s) dB_t. \]

Suppose that \( \Pi_t(x_t,a_t) \neq x_s - \Pi_a(x_s,a_s) \) on a set of positive measure in \( s \in [t, \tau] \), and let \( \{b_s\} \in \mathcal{A}_{a_t,t}^\pi \) be a process such that \( \hat{b}_s = \hat{a}_s \) for all \( s \) such that \( \Pi_t(x_t,a_t) = x_s - \Pi_a(x_s,a_s) \), and \( \hat{b}_s > \hat{a}_s \) (\( \hat{b}_s < \hat{a}_s \)) for all \( s \) such that \( \Pi_t(x_t,a_t) > x_s - \Pi_a(x_s,a_s) \) \((\Pi_t(x_t,a_t) < x_s - \Pi_a(x_s,a_s))\).

Let \( U_t \) denote the monopolist’s profits from following strategy \( \{b_s\} \) on \( s \in [t, \tau] \), so
\[ U_t = E_t \left[ \int_t^\tau e^{-r(s-t)} \left( \Pi_t(x_t,b_s) - x_s \right) \hat{b}_s ds + e^{-r(r-t)} \Pi_t(x_t,b_r) \right]. \]

By Ito’s Lemma, under process \( \{b_s\} \)
\[ de^{-r(s-t)} \Pi_t(x_t,b_s) = e^{-r(s-t)} \left( -r \Pi_t(x_t,b_s) + \mu x_t \Pi_t(x_t,b_s) + \frac{1}{2} \sigma^2 x_t^2 \Pi_{xx}(x_t,b_s) \right) ds + \Pi_a(x_t,b_s) \hat{b}_s ds + \sigma x_t \Pi_t(x_t,b_s) dB_s. \]
Therefore,

\[ E_t \left[ e^{-r(t-s)} \Pi(x_t, b_t) \right] = \Pi(x_t, a_t) + E_t \left[ \int_t^\tau e^{-r(s-t)} \left( -r \Pi(x_s, b_s) + \mu x_s \Pi_x(x_s, b_s) + \frac{1}{2} \sigma^2 x_s^2 \Pi_{xx}(x_s, b_s) + \Pi_a(x_s, b_s) \dot{b}_s \right) ds \right]. \]

Since \( P(x_s, a_s) \neq x_s - \Pi_a(x_s, a_s) \) on a set of positive measure in \( s \in [t, \tau] \), the equation above together with (A.1.6) gives

\[ U_t = E_t \left[ \Pi(x_t, a_t) + \int_t^\tau e^{-r(s-t)} \left( -r \Pi(x_s, b_s) + \mu x_s \Pi_x(x_s, b_s) + \frac{1}{2} \sigma^2 x_s^2 \Pi_{xx}(x_s, b_s) + \Pi_a(x_s, b_s) \dot{b}_s \right) ds \right] > \Pi(x_t, a_t), \]

a contradiction to the fact that \( \{a_t, P\} \) is an equilibrium. Thus, in equilibrium it must be that \( P(x_s, a_s) - x_s + \Pi_a(x_s, a_s) = 0 \) for all \( s \in [t, \tau] \).

**Corollary A.1.2** Let \( \{a_t, P\} \) be a regular equilibrium and let \( \Pi(x, a) \) be the monopolist’s profits. Then, \( \Pi(x, a) \) solves (A.1.1) at states \((x, a)\) with \( a < \alpha \) such that (i) \( x > z_2 \), or (ii) \( \{a_t\} \) is continuously increasing at time \( s \) when \((x_s, a_s) = (x, a)\).

**Proof:** (i) Let \((x, a)\) be such that \( a < \alpha \) and \( x > z_2 \). By Lemma A.1.7, at such a state the monopolist will not sell until costs fall below \( z_2 \), so \( \Pi(x, a) = E[e^{-r\tau} \Pi(z_2, a)|x_0 = x] \). Thus, by Corollary A.1.1 \( \Pi(x, a) \) solves (A.1.1).

(ii) Suppose \((x, a)\) is such that \( \{a_t\} \) is continuously increasing at time \( s \) when \((x_s, a_s) = (x, a)\). By the arguments in the proof of Lemma A.1.8, \( \Pi(x, a) \) solves (A.1.5). Moreover, by Lemma A.1.8 \( P(x, a) = x - \Pi_a(x, a) \), so \( \Pi(x, a) \) solves (A.1.1).

**Lemma A.1.9** Let \( \{a_t, P\} \) be a regular equilibrium and let \( \Pi(x, a) \) be the monopolist’s profits. Let \((x, a)\) with \( a < \alpha \) be such that \( \{a_t\} \) is continuously increasing at time \( s \) when \((x_s, a_s) = (x, a)\). Then, there exists \( x_s(a) < x < x^*(a) \) with either \( x^*(a) \leq z_2 \) or \( x^*(a) = \infty \).
such that \( \{a_t\} \) jumps at \( t \) if \( a_{t-} = a \) and \( x_t \in \{x_*(a), x^*(a)\} \). Moreover,

\[
\Pi(y, a) = E_y \left[ e^{-\tau^*(a)} \left( (P \left( x_{\tau^*(a)}, a_{\tau^*(a)} \right) - x_{\tau^*(a)}) da_{\tau^*(a)} + \Pi \left( x_{\tau^*(a)}, a + da_{\tau^*(a)} \right) \right) \right],
\]

(A.1.7)

for all \( y \in (x_*(a), x^*(a)) \), where \( \tau^*(a) = \inf \{ t : x_t \notin (x_*(a), x^*(a)) \} \) and where \( da_{\tau^*(a)} \) denotes the jump of \( \{a_t\} \) at state \( (x_{\tau^*(a)}, a) \).

**Proof:** Note first that for every such state \( (x, a) \) there must exist \( \overline{y}(a) < x \) such that \( a_t \) jumps when \( x_t = \overline{y}(a) \) and \( a_{t-} = a < \alpha \). To see this, suppose by contradiction that \( a_t \) is continuous at time \( s \) when \( (x_s, a_s) = (y, a) \) for every \( y < x \). By Corollary A.1.2 \( \Pi(y, a) \) solves (A.1.1) for all \( y \leq x \), so \( \Pi(y, a) = Ay^\lambda + By^\kappa \) for some constants \( A \) and \( B \). If \( A \neq 0 \) or \( B \neq 0 \), \( \Pi(\cdot, a) \) explodes as \( y \to 0 \) or as \( y \to \infty \), which cannot occur in equilibrium. Otherwise, \( A = B = 0 \) implies that \( \Pi(y, a) = 0 \) for all \( y \leq x \), which cannot occur either since \( \Pi(y, a) \geq L(y, a) > 0 \). Thus, there must exist \( \underline{y}(a) < x \) such that \( a_t \) jumps to some \( a' > a \) when \( x_t = \underline{y}(a) \) and \( a_{t-} = a \). Let \( x_*(a) \) denote the supremum over all such \( \underline{y}(a) \). If \( a_t \) is continuous for all \( y > x_*(a) \) whenever \( a_t = a \), then by Corollary A.1.2 \( \Pi(y, a) \) solves (A.1.1) for all \( y > x_*(a) \). Thus, \( \Pi(y, a) = Ay^\lambda + By^\kappa \) for all \( y > x_*(a) \). Since \( \kappa > 1 \), in this case it must be that \( B = 0 \); otherwise, \( \Pi(y, a) \) would explode as \( y \to \infty \). Since \( \{a_t\} \) jumps to \( a' \) when \( x_t = x_*(a) \) and \( a_{t-} = a \), it follows that \( \Pi(x_*(a), a) = (P(x_*(a), a') - x_*(a))(a' - a) + \Pi(x_*(a), a') \). Thus, Corollary A.1.1 implies that \( \Pi(y, a) \) satisfies (A.1.7) (with \( x^*(a) = \infty \)). Otherwise, there exists \( \overline{y}(a) > x \) such that \( \{a_t\} \) jumps to some \( \overline{a} > a \) when \( x_t = \overline{y}(a) \) and \( a_{t-} = a \). By Lemma A.1.7, \( \overline{y}(a) \leq z_2 \). Let \( x^*(a) \) be the infimum over all such \( \overline{y}(a) \), so \( x^*(a) \leq z_2 \). In this case, \( \Pi(y, a) \) solves (A.1.1) for all \( y \in (x_*(a), x^*(a)) \), with \( \Pi(y, a) = (P(y, a) - x)da_{\tau^*(a)} + \Pi(y, a + da_{\tau^*(a)}) \) whenever \( y \in \{x_*(a), x^*(a)\} \). Thus, by Lemma A.1.1 \( \Pi(y, a) \) satisfies (A.1.7) for all \( y \in (x_*(a), x^*(a)) \).

**Lemma A.1.10** Let \( \{a_t\} \) be a regular equilibrium, and let \( \Pi(x, a) \) denote the monopolist’s profits. Then, \( \Pi(x, a) = L(x, a) \) for all states \( (x, a) \) with \( a < \alpha \).
**Proof:** By the arguments in the main text, $\Pi(x, a) \geq L(x, a)$ for all states $(x, a)$ with $a < \alpha$. By Lemma A.1.9, for all $(x, a)$ such that $\{a_t\}$ is continuously increasing at time $s$ when $(x_s, a_s) = (x, a)$, there exists $x_*(a) < x < x^*(a)$ such that $da_t = a_t - a_{t-} > 0$ when $a_{t-} = a$ and $x_t \in \{x_*(a), x^*(a)\}$. Moreover, $\Pi(x, a)$ satisfies (A.1.7) for all $x \in (x_*(a), x^*(a))$. Suppose that $\{a_t\}$ jumps to $\alpha$ when $a_{t-} = a$ and $x_t \in \{x_*(a), x^*(a)\}$, so $\Pi(x, a) = (P(x, \alpha) - x)(\alpha - a) + \Pi(x, \alpha) = g(x, a)$ for $x \in \{x_*(a), x^*(a)\}$. Thus,

$$\Pi(x, a) = E[e^{-rt}(a)g(x_{r^*(a)}, a)| x_0 = x] \leq L(x, a) = \sup_T E[e^{-rt}g(x_r, a)| x_0 = x],$$

for all $x \in (x_*(a), x^*(a))$, so $\Pi(x, a) = L(x, a)$ for all such states $(x, a)$.

Suppose next that $da_t = \tilde{a} - a < \alpha - a$ when $a_{t-} = a$ and $x_t = x_*(a)$ or $x_t = x^*(a)$. Thus, $\Pi(x_t, a) = (P(x_t, \tilde{a}) - x_t)(\tilde{a} - a) + \Pi(x_t, \tilde{a})$ when $x_t \in \{x_*(a), x^*(a)\}$. By Lemma A.1.7, the monopolist must continue selling gradually after $a_t$ jumps (since $x_*(a) < x^*(a) \leq z_2$). By Lemma A.1.8, it must be that $P(x_t, \tilde{a}^+) = x_t - \Pi_a(x_t, \tilde{a})$. Note that prices cannot jump at time $t$. If prices jumped down at $t$, then those consumers who buy at $t^-$ would be strictly better off by delaying their purchase an instant, which cannot occur in equilibrium. Thus, it must be that $P(x_t, \tilde{a}) = P(x_t, \tilde{a}^+)$. By Lemma A.1.9, $\Pi(x_t, \tilde{a})$ satisfies (A.1.7), so

$$P(x_t, \tilde{a}) - x_t = -\Pi_a(x_t, \tilde{a}) = E[e^{-r(r^*(\tilde{a})-t)}(P(x_{r^*(\tilde{a})}, a_{r^*(\tilde{a})}) - x_{r^*(\tilde{a})})| x_t].$$

That is, the margin that the monopolist gets from selling to consumers $[a, \tilde{a}]$ at state $(x_t, a)$ with $x_t \in \{x_*(a), x^*(a)\}$ is the same as the expected discounted margin she gets at state $(x, \tilde{a})$ with $x \in \{x_*(\tilde{a}), x^*(\tilde{a})\}$. Since $\Pi(x_t, a) = (P(x_t, \tilde{a}) - x_t)(\tilde{a} - a) + \Pi(x_t, \tilde{a})$ and since $\Pi(x_t, \tilde{a})$ satisfies (A.1.7),

$$\Pi(x_t, a) = E \left[ e^{-r(r^*(\tilde{a})-t)} \left( \left( P(x_{r^*(\tilde{a})}, a_{r^*(\tilde{a})}) - x_{r^*(\tilde{a})} \right) \times \begin{pmatrix} da_{r^*(\tilde{a})} + \tilde{a} - a \\ \Pi(x_{r^*(\tilde{a})}, a_{r^*(\tilde{a})}) \end{pmatrix} \right) | x_t \right], \quad (A.1.8)$$

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where \( da_{r^*}(\tilde{a}) \) denotes the jump of \( \{a_t\} \) when \( a_{s-} = \tilde{a} \) and \( x_s \in \{x_{s}(\tilde{a}), x^*(\tilde{a})\} \). There are two possibilities: (i) \( a_{r^*}(\tilde{a}) = \alpha \) with probability 1, so \( da_{r^*}(\tilde{a}) = \alpha - \tilde{a} \); or (ii) \( a_{r^*}(\tilde{a}) = \tilde{a} < \alpha \), so \( da_{r^*}(\tilde{a}) = \tilde{a} - \tilde{a} < \alpha - \tilde{a} \). In the first case, \( da_{r^*}(\tilde{a}) + \tilde{a} - a = \alpha - a \), so \( P(x_{r^*}(\tilde{a}), \alpha) - x_{r^*}(\tilde{a}) + \tilde{a} - a + \Pi(x_{r^*}(\tilde{a}), a_{r^*}(\tilde{a})) = g(x_{r^*}(\tilde{a}), a) \). Using (A.1.8), this implies that

\[
\Pi(x_{r^*}(\tilde{a}), a_{r^*}(\tilde{a})) = E \left[ e^{-r(t^* - \tau^*)} (P(x_{r^*}(\tilde{a}), a_{r^*}(\tilde{a})) - x_{r^*}(\tilde{a})) \times \left( \left( da_{r^*}(\tilde{a}) + \tilde{a} - \tilde{a} \right) \right) \right]_{t^*}(\tilde{a}). \tag{A.1.9}
\]

Moreover, the same arguments as above also imply that

\[
P(x_{r^*}(\tilde{a}), a_{r^*}(\tilde{a})) - x_{r^*}(\tilde{a}) = E \left[ e^{-r(t^* - \tau^*)} (P(x_{r^*}(\tilde{a}), a_{r^*}(\tilde{a})) - x_{r^*}(\tilde{a})) \right]_{x_{r^*}(\tilde{a})}. \]

Using this equation and (A.1.9) in (A.1.8), it follows that

\[
\Pi(x_t, a) = E \left[ e^{-r(\tau - \tau)} \left( (P(x_{m}, a_{m}) - x_{m}) (da_{m} + \tilde{a} - a) \right) + \Pi(x_{m}, a_{m}) \right]_{x_t}, \]

for some stopping time \( \tau \). Again, there are two possibilities: (i) \( a_{\tau} = \alpha \) with probability 1 (so \( da_{\tau} = \alpha - \tilde{a} \)), or (ii) \( a_{\tau} < \alpha \). In case (i), \( P(x_{\tau}, a_{\tau}) - x_{\tau}) (da_{\tau} + \tilde{a} - a) + \Pi(x_{\tau}, a_{\tau}) = g(x_{\tau}, a) \), so \( \Pi(x_t, a) = E \left[ e^{-r(\tau - \tau)} g(x_{\tau}, a) \right]_{x_t} \leq L(x_t, a) \). Hence, \( \Pi(x_t, a) = L(x_t, a) \). In case (ii), we can again repeat the same argument. Eventually, we’ll get to a point at which \( a \) jumps to \( \alpha \), so \( \Pi(x_t, a) = E \left[ e^{-r(\tau - \tau)} g(x_{\tau}, a) \right]_{x_t} \) for some stopping time \( \tau \). Hence \( \Pi(x_t, a) = L(x_t, a) \).

**Proof of Theorem 1.1:** Lemma A.1.10 shows that in any regular equilibrium the monopolist’s profits are given by \( L(x, a) \) for all \( x \). Thus, the monopolist sells to all high type con-
sumers when \(x_t \in [0, \bar{x}(a_t)] \cup [\bar{x}(a_t), z_2]\) (and also to low type consumers when \(x_t \leq \bar{x}(a_t)\)). By Lemma A.1.7, the monopolist must sell at a positive rate while \(x_t \in (\bar{x}(a_t), \bar{x}(a_t))\). The arguments in the text pin down the rate at which the monopolist sells and the price she charges when \(x_t \in (\bar{x}(a_t), \bar{x}(a_t))\).

### A.1.4 Proof of Theorem 1.2

The proof of Theorem 1.2 is a generalization of the proof of Theorem 1.1. Suppose that there are \(n \geq 3\) types of consumers in the market. Here I provide a sketch of the arguments. Note that when \(a \geq \alpha_3\), the only consumers left in the market are those with valuations \(v_1\) and \(v_2\). By Theorem 1.1, in any regular equilibrium the monopolist’s profits are equal to \(L(x, a)\) for states \((x, a)\) with \(a \geq \alpha_3\).

Consider next states \((x, a)\) with \(a \in [\alpha_4, \alpha_3)\), so there are \(\alpha_3 - a\) consumers with valuation \(v_3\) in the market (if there are only three types of consumers in the market, let \(\alpha_4 = 0\)). Let \(P_2(x) = \sup_{i \in (\alpha_3, \alpha_2]} P(x, i)\) be the highest price a consumer with valuation \(v_2\) is willing to pay. By equation (1.5), the strategy \(P(x, \alpha_3)\) of consumer \(\alpha_3\) (the highest indexed consumer with valuation \(v_3\)) satisfies

\[
P(x, \alpha_3) = v_3 - E \left[ e^{-r\tau_2} (v_3 - P_2(x_{\tau_2})) \mid x_0 = x \right],
\]

where \(\tau_2 = \inf\{t : x_t \leq z_2\}\) is the time at which the monopolist starts selling to consumers with valuation \(v_2\) when the level of market penetration is \(\alpha_3\). By the skimming property, the monopolist can always sell to all consumers with valuation \(v_3\) at price \(P(x, \alpha_3)\). Therefore, at states \((x, a)\) with \(a \in [\alpha_4, \alpha_3)\) the monopolist’s profits are bounded below by

\[
L(x, a) = \sup_{\tau \in T} E \left[ e^{-r\tau} \left( (\alpha_3 - a) (P(x_{\tau}, \alpha_3) - x_{\tau}) + e^{-r\tau} L(x_{\tau}, \alpha_3) \right) \mid x_0 = x \right]. \tag{A.1.10}
\]

Let \(\bar{x}(\alpha_3)\) and \(\bar{\alpha}(\alpha_3)\) be the cutoffs that characterize the solution to the optimal stopping problem (1.10) when \(a = \alpha_3\) (i.e., when all consumers with valuation \(v_3\) have left the market).
The first thing to note is that the solution to (A.1.10) is such that it is optimal to continue when \( x_t \in (\underline{x}(\alpha_3), \overline{x}(\alpha_3)) \). The reason for this is that the expected payoff from delaying when \( x_t \in (\underline{x}(\alpha_3), \overline{x}(\alpha_3)) \) is larger when \( a < \alpha_3 \) than when \( a = \alpha_3 \), since in the former case there are more high valuation consumers to sell to.\(^1\)

Using arguments similar to those in Lemma A.1.4, one can show that the solution to (A.1.10) is of the form

\[
\tau(a) = \inf\{t : x_t \in [0, \underline{x}_1(a)] \cup [\overline{x}_1(a), \underline{x}_2(a)] \cup [\overline{x}_2(a), \underline{x}_3(a)]\},
\]

with \( \underline{x}_1(a), \overline{x}_1(a), \underline{x}_2(a), \overline{x}_2(a) \) such that \( \underline{x}_1(a) < z_1 < \overline{x}_1(a) \) and \( \underline{x}_2(a) < z_2 < \overline{x}_2(a) < z_3 \). That is, the solution to (A.1.10) involves delaying when \( x \) is around \( z_1 \) or \( z_2 \) and when \( x > z_3 \). Using arguments similar to those in Lemma A.1.5 the thresholds \( \underline{x}_1(a), \overline{x}_1(a), \underline{x}_2(a) \) and \( \overline{x}_2(a) \) are continuous in \( a \), with \( \lim_{a \to \alpha_3} \underline{x}_2(a) = \lim_{a \to \alpha_3} \overline{x}_2(a) = z_2 \), and \( \lim_{a \to \alpha_3} \underline{x}_1(a) = \underline{x}(\alpha_3) \) and \( \lim_{a \to \alpha_3} \overline{x}_1(a) = \overline{x}(\alpha_3) \) (where \( \underline{x}(\alpha_3) \) and \( \overline{x}(\alpha_3) \) are the cutoffs that characterize the solution to (1.10) when \( a = \alpha_3 \)). In addition, \( L(x,a) \in C^{2,2} \) for all \( x \in (\underline{x}_1(a), \overline{x}_1(a)) \cup (\underline{x}_2(a), \overline{x}_2(a)) \).

Next, by arguments similar to those in Lemma A.1.7 the monopolist will always sell to consumers with valuation \( v_3 \) at states \((x_t, a_t)\) with \( x_t \leq z_3 \) and \( a_t \in [\alpha_4, \alpha_3) \), and will never sell to them when \( x_t > z_3 \). Moreover, arguments similar to those in Lemma A.1.8 imply that in any regular equilibrium, \( P(x_s, a_s) = x_s - \Pi_a(x_s, a_s) \) whenever the monopolist is selling at a continuous rate (i.e., whenever \( da_t = \dot{a}_t dt \)). Finally, by arguments similar to those in Lemma A.1.10, in any regular equilibrium the monopolist’s profits must be equal to \( L(x,a) \) at all states \((x,a)\) with \( a \in [\alpha_4, \alpha_3) \).

At states \((x_t, a_t)\) with \( a_t \in [\alpha_4, \alpha_3) \) the equilibrium dynamics are as follows. If \( x_t > z_3 \), the monopolist doesn’t sell and waits for costs to decrease. When \( x_t \in [\overline{x}_2(a_t), z_3] \), the monopolist sells immediately to all remaining consumers with valuation \( v_3 \), and then

\(^1\)This can be proved formally using the arguments in the proof of Lemma A6, where I show that the cutoffs \( \underline{x}(a) \) and \( \overline{x}(a) \) of the solution to (1.10) satisfy \( \underline{x}'(a) > 0 \) and \( \overline{x}'(a) < 0 \).
equilibrium play continuous as in the case with two consumers. When \( x_t \in [\bar{\alpha}_1 (a_t), \bar{\alpha}_2 (a_t)] \),
the monopolist sells immediately to all remaining consumers with valuation \( v_3 \); however, since \( z_2 > \bar{\alpha}_3 (a_t) \) and \( \bar{\alpha}_1 (a_t) > \bar{\alpha}_3 \) (where \( \bar{\alpha}_4 \) is the cutoff that describes the solution to
the optimal stopping problem \( L (x, a) \) at state \((x, \alpha_3)\)), in this case the monopolist also sells
to all consumers with valuation consumers \( v_2 \). When \( x_t \leq \bar{\alpha}_1 (a_t) \) the monopolist sells to all
remaining consumers at price \( v_1 \) and the market closes. Finally, when \( x_t \in (\bar{\alpha}_1 (a_t), \bar{\alpha}_2 (a_t)) \cup
(\bar{\alpha}_3 (a_t), \bar{\alpha}_2 (a_t)) \), the monopolist sells gradually to consumers with valuation \( v_3 \) at a rate that
leaves them indifferent between purchasing at \( t \) or delaying their purchase. One can derive
this rate in a way similar to the derivation of equation (1.17) in the main text.

Consider next state \((x, a)\) with \( a \in [\alpha_5, \alpha_4) \), at which there are \( \alpha_4 - a \) consumers with
valuation \( v_4 \) in the market (if there are only four types of consumers in the market, let \( \alpha_5 = 0 \)). Let \( P_3 (x) = \sup_{i \in [\alpha_4, \alpha_3]} P (x, i) \), and let \( P (x, \alpha_4) \) be the strategy of consumer \( \alpha_4 \)
(i.e., the last consumer with valuation \( v_4 \)). By equation (1.5), it must be that

\[
P (x, \alpha_4) = v_4 - E \left[ e^{-r\tau_3} (v_4 - P_3 (x_{\tau_3})) \bigg| x_0 = x \right],
\]

where \( \tau_3 = \inf \{ t : x_t \leq z_3 \} \) is the time at which the monopolist starts selling to consumers
with valuation \( v_3 \) when \( a = \alpha_4 \). Since the monopolist can sell to all consumers with valuation
\( v_4 \) at price \( P (x, \alpha_4) \), at states \((x, a)\) with \( a \in [\alpha_5, \alpha_4) \) her profits are bounded below by

\[
L (x, a) = \sup_{\tau \in T} E \left[ e^{-r\tau} ((\alpha_4 - a) (P (x_{\tau}, \alpha_4) - x_{\tau}) + L (x_{\tau}, \alpha_4)) \bigg| x_0 = x \right].
\]

Repeating the same arguments as above, one can show that in any regular equilibrium the
monopolist’s profits must be given by \( L (x, a) \) at all states \((x, a)\) with \( a \in [\alpha_5, \alpha_4) \). More
generally, for \( k \geq 5 \) one can extend \( L (x, a) \) for all \( x \in [\alpha_{k+1}, \alpha_k) \) in a similar way, and show
that in any regular equilibrium the monopolist’s profits are \( L (x, a) \) for all \( a \geq [\alpha_{k+1}, \alpha_k) \).
A.1.5 Proof of Theorem 1.3

For each valuation \( v^n_k \), let \( z^n_k = \frac{\lambda}{1-\lambda} v^n_k \). For \( n = 2, 3, \ldots \), define the function \( P^n(x) \) as follows. For \( x \leq z^n_1 \), \( P^n(x) = v^n_1 = v \). For \( k = 2, \ldots, n \), and \( x \in (z^n_{k-1}, z^n_k] \), let \( P^n(x) = P(x, \alpha^n_k) \).

That is, for all \( x \in (z^n_{k-1}, z^n_k) \), \( P^n(x) \) is equal to the price at which consumer \( \alpha^n_k \) is willing to trade (where \( \alpha^n_k \) is the highest indexed consumer with valuation \( v^n_k \)). By equation (1.5), for \( k = 2, \ldots, n \) and \( x \in (z^n_{k-1}, z^n_k) \),

\[
P^n(x) = P(x, \alpha^n_k) = v^n_k - E[e^{-\tau^n_k} (v^n_k - P(z^n_{k-1}, \alpha^n_{k-1})) | x_0 = x]
= v^n_k - E[e^{-\tau^n_k} (v^n_k - P(z^n_{k-1})) | x_0 = x], \tag{A.1.12}
\]

where for \( k = 1, 2, \ldots, n \), \( \tau^n_k = \inf \{ t : x_t \leq z^n_k \} \) is the time at which the monopolist starts selling to buyers with valuation \( v^n_k \) when \( v^n_k \) is the highest valuation remaining in the market.

**Lemma A.1.11** For \( k = 2, \ldots, n \) and \( x \in (z^n_{k-1}, z^n_k) \),

\[
P^n(x) = v^n_k - \sum_{j=1}^{k-1} (v^n_{j+1} - v^n_j) \left( \frac{x}{z^n_j} \right)^\lambda. \tag{A.1.13}
\]

**Proof:** The proof is by induction. By equation (1.9), \( P^n(x) = v^n_2 - (v^n_2 - v^n_1) (x/z^n_2)^\lambda \) for \( x \in (z^n_1, z^n_2) \), so the statement is true for \( k = 2 \). Suppose the statement is true for \( l = 2, \ldots, k - 1 \). Equation (A.1.12), Corollary A.1.1 and the induction hypothesis then imply that

\[
P^n(x) = v^n_k - (v^n_k - P^n(z^n_{k-1})) \left( \frac{x}{z^n_{k-1}} \right)^\lambda = v^n_k - \sum_{j=1}^{k-1} (v^n_{j+1} - v^n_j) \left( \frac{x}{z^n_j} \right)^\lambda,
\]

for \( x \in (z^n_{k-1}, z^n_k) \).

Recall that \( v^n_1 = v \) and \( v^n_n = \bar{v} \) for all \( n \). Let \( \overline{z} := \frac{\lambda}{1-\lambda} v = z^n_n \) and \( \underline{z} := \frac{\lambda}{1-\lambda} \bar{v} = z^n_1 \).

**Lemma A.1.12** \( P^n(x) - x \to V_1(x) \) uniformly on \([0, \overline{z}]\) as \( n \to \infty \).

**Proof:** I first show that \( \lim_{n \to \infty} P^n(x) = V_1(x) + x \) for all \( x \in [0, \overline{z}] \). Note first that, for all \( n \), \( P^n(x) - x = v_1 - x = V_1(x) \) for all \( x \leq z_1 \). Next, fix \( x \in (z_1, \overline{z}] \) with \( x \in (z^n_{k-1}, z^n_k] \)
for some \( k \leq n \), and let \( v(x) = \frac{1-\lambda}{\lambda} x \). Recall that \( v^n_{j+1} - v^n_j = (\bar{v} - v) / (n - 1) \). Equation (A.1.13) and the fact that \( x / z^n_j = v(x) / v^n_j \) then imply that

\[
P^n(x) = v^n_k - \sum_{j=1}^{k-1} \frac{n - v}{n - 1} \left( \frac{v(x)}{v^n_j} \right)^\lambda.
\]

Note that \( z^n_k = \frac{-\lambda}{1-\lambda} v^n_k \to x \) as \( n \to \infty \), so \( v^n_k \to v(x) \). Since \((v(x) / v)^\lambda\) is Riemann integrable,

\[
\lim_{n \to \infty} P^n(x) = v(x) - \int_x^{v(x)} \left( \frac{v(x)}{v} \right)^\lambda dv = x + (v - x) \left( \frac{x}{\bar{z}} \right)^\lambda = x + V_1(x).
\]

Finally, since \( P^n(x) \) is increasing in \( x \) for all \( x \in [0, \bar{z}] \) and since \( \lim_{n \to \infty} P^n(x) = V_1(x) + x \) in this range, it follows that \( P^n(x) \) converges uniformly to \( V_1(x) + x \) as \( n \to \infty \). Thus, \( P^n(x) - x \) converges uniformly to \( V_1(x) \).

**Proof of Theorem 1.3:** I prove that, for all \( x \), \( L^n(x, 0) \to V_1(x) \) as \( n \to \infty \). The proof that \( L^n(x, a) \to (1 - a) V_1(x) \) as \( n \to \infty \) for \( a > 0 \) is symmetric and omitted. Note first that \( L^n(x, 0) \geq V_1(x) \) for all \( x \), since at any state \((x, 0)\) the monopolist can wait until time \( \tau_1 \) and sell to all consumers at price \( v_1 \), obtaining a profit of \( E[e^{-r\tau_1} (v_1 - x_n)] \big| x_0 = x] = V_1(x) \).

Consider next the case in which \( x_0 = x \geq \bar{z} \). In this case, in equilibrium the monopolist sells to consumers with valuation \( v^n_k \) at time \( \tau^n_k = \inf \{ t : x_t \leq z^n_k \} \) (for \( k = 1, \ldots, n \)), at a price \( P(z^n_k, \alpha^n_k) = P^n(z^n_k) \). Recall that for \( k = 1, \ldots, n \), \( \alpha^n_k = \max \{ i : f^n(i) = v^n_k \} \), and let \( \alpha^n_{k+1} = 0 \). Thus, the monopolist’s profits are

\[
L^n(x, 0) = E \left[ \sum_{k=1}^{n} e^{-r\tau^n_k} (P^n(z_k) - z_k) \left( \alpha^n_k - \alpha^n_{k+1} \right) \bigg| x_0 = x \right].
\]
Since \( P^n(x) - x \to V_1(x) \) uniformly on \([0, \bar{z}]\) as \( n \to \infty \), for every \( \eta > 0 \) there exists \( N \) such that \( P^n(x) - x - V_1(x) < \eta \) for all \( n > N \) and all \( x \in [0, \bar{z}] \). Thus, for \( n > N \),

\[
L^n(x, 0) < E \left[ \sum_{k=1}^{n} e^{-r x_k} V_1(z_k) \, d\alpha_k \right] x_0 = x + \eta = \sum_{k=1}^{n} d\alpha_k E \left[ e^{-r x_k} V_1(z_k) \right] x_0 = x + \eta,
\]

(A.1.14)

where \( d\alpha_k = \alpha_k - \alpha_{k+1} \) (so \( \sum_{k=1}^{n} d\alpha_k = 1 \)). Note further that for \( x \geq \bar{z} \) and \( k = 1, 2, \ldots, n \),

\[
E \left[ e^{-r x_k} V_1(z_k^n) \right] x_0 = x = E \left[ e^{-r x_k^n} E \left[ e^{-r (x_k^n - x_\tau^n)} (v_1 - x_\tau^n) \right] x_\tau^n \right] x_0 = x = V_1(x).
\]

Using this and the fact \( \sum_{k=1}^{n} d\alpha_k = 1 \) in (A.1.14) gives \( V_1(x) \leq L^n(x, 0) < V_1(x) + \eta \) for all \( n > N \). Therefore, \( \lim_{n \to \infty} L^n(x, 0) = V_1(x) \) for all \( x \geq \bar{z} \).

Consider next the case with \( x < \bar{z} \). Suppose by contradiction that there exists \( x < \bar{z} \) such that \( L^n(x, 0) \nrightarrow V_1(x) \) as \( n \to \infty \). Since \( L^n(x, 0) \geq V_1(x) \) for all \( n \), there exists \( N \) and \( \gamma > 0 \) such that \( L^n(x, 0) > V_1(x) + \gamma \) for all \( n > N \). Fix \( y \geq \bar{z} \) and let \( \tau_x := \inf\{ t : x_t \leq x \} \).

Since the monopolist can always delay trade until time \( \tau_x \), for all \( n > N \) it must be that

\[
L^n(y, 0) \geq E[e^{-r \tau_x} L(x, 0) \mid x_0 = y] > E[e^{-r \tau_x} V_1(x) \mid x_0 = y] + E[e^{-r \tau_x} \gamma \mid x_0 = y] = E[e^{-r \tau_x} E \left[ e^{-r (\tau_1 - \tau_x)} (v_1 - x_\tau) \right] x_\tau] x_0 = y + \left( \frac{y}{x} \right)^\lambda \gamma = V_1(y) + \left( \frac{y}{x} \right)^\lambda \gamma,
\]

a contradiction to \( \lim_{n \to \infty} L^n(y, 0) = V_1(y) \) (since \( y \geq \bar{z} \)); thus, \( \lim_{n \to \infty} L^n(x, 0) = V_1(x) \) for all \( x < \bar{z} \).
A.2 Appendix to Chapter 2

A.2.1 Proof of Theorem 2.1

To establish that \((A^*, V_1^*, V_2^*)\) is an equilibrium, I show that, for all \(x \in \mathbb{R}\), \(V_1^*(x)\) solves

\[
V_1^*(x) = \sup_{\tau \in T_1} E \left[ e^{-r\tau} (1 - V_2^*(x_\tau)) \right]_{x_0 = x}.
\]  

(A.2.1)

The proof that \(V_2^*\) also solves the optimal stopping problem is symmetric and omitted. To see that \(V_1^*\) solves (A.2.1), let \(F_2 : \mathbb{R} \to \mathbb{R}\) be the solution to the ODE (2.5) with boundary conditions \(F_2(0) = \frac{\beta}{\alpha + \beta}\) and \(\lim_{x \to -\infty} F_2(x) = 0\). That is, \(F_2(x) = \frac{\beta}{\alpha + \beta} e^{-\alpha x}\) for all \(x \in \mathbb{R}\). Let \(G_2(x) = \min \{1, F_2(x)\}\) and note that \(G_2(x) = V_2^*(x)\) for all \(x \geq 0\). Consider the optimal stopping problem

\[
G_1(x) = \sup_{\tau \in T} E \left[ e^{-r\tau} (1 - G_2(x_\tau)) \right]_{x_0 = x},
\]  

(A.2.2)

where \(T\) is the set of all stopping times. Since \(G_2(x) = V_2^*(x)\) for all \(x \geq 0\) and since \(T_1 \subseteq T\), it follows that \(G_1(x) \geq \sup_{\tau \in T_1} E \left[ e^{-r\tau} (1 - V_2^*(x_\tau)) \right]_{x_0 = x}\). Therefore, in order to show that \(V_1^*\) satisfies (A.2.1) it suffices to show that \(V_1^* = G_1\).

The function \(V_1^*\) is twice differentiable, with a continuous first derivative. One can check that \(V_1^*(x) > 1 - G_2(x)\) for all \(x \in (-\infty, 0)\), and \(V_1^*(x) = 1 - G_2(x)\) for all \(x \in [0, \infty)\). Finally, \(V_1^*(x)\) satisfies

\[
-r V_1^*(x) + \mu (V_1^*)' (x) + \frac{1}{2} \sigma^2 (V_1^*)'' \leq 0, \text{ with equality on } x \in (-\infty, 0).
\]  

(A.2.3)

To see this, note first that \(V_1^*\) satisfies (2.5) on \((-\infty, 0)\), so \(V_1^*\) satisfies (A.2.3) with equality on \((-\infty, 0)\). Moreover, \(F_2\) also satisfies (2.5), and \(V_1^*(x) = 1 - G_2(x) = 1 - F_2(x)\) for all
This implies that, for all \( x \in [0, \infty) \),

\[
    rV_1^* (x) = r - rF_2 (x) = r + \mu (V_1^*) (x) + \frac{1}{2} \sigma^2 (V_1^*)'' \Rightarrow \\
    0 > -r = -rV_1^* (x) + \mu (V_1^*) (x) + \frac{1}{2} \sigma^2 (V_1^*)''.
\]

Therefore, the function \( V_1^* \) is twice differentiable, with a continuous first derivative and satisfies (A.2.3). By standard verification theorems, it follows that \( V_1^* \) is a solution to the optimal stopping problem (A.2.2) (see, for instance, Theorem 3.17 in Shiryaev (2008)). Hence, \( V_1^* \) also solves (A.2.1).

Next, I show that \((A^*, V_1^*, V_2^*)\) is the unique equilibrium. By Proposition 2.1, any equilibrium outcome \((A, V_1, V_2)\) must have \( A = A^* = \mathbb{R} \). Let \((A^*, V_1, V_2)\) be an outcome different from \((A^*, V_1^*, V_2^*)\). The discussion prior to the statement of Theorem 2.1 implies that \( V_1' \) and \( V_2' \) are not continuous at 0; that is, \( V_1' (0^-) \neq -V_2' (0^+) \). Therefore, it must be that either (i) \( V_1' (0^-) < -V_2' (0^+) \), or (ii) \( V_1' (0^-) > -V_2' (0^+) \).

Consider case (i). Let \( Q_1 (x) = (1 - c) e^{\beta x} \), so that \( V_1 (x) = Q_1 (x) \) for all \( x < 0 \). Since \( V_1 (0) = 1 - V_2 (0) \) and since \( Q_1' (0) = V_1' (0^-) < -V_2' (0^+) \), there exists \( \varepsilon > 0 \) such that \( 1 - V_2 (x) > Q_1 (x) \) for all \( x \in (0, \varepsilon] \). Let \( \tau^\varepsilon = \inf \{ t \geq 0 : x_t \geq \varepsilon \} \) and note that \( \tau^\varepsilon \in T_1 \). Let \( V_1^\varepsilon (x) \) be given by

\[
    V_1^\varepsilon (x) = E \left[ e^{-r \tau^\varepsilon} (1 - V_2 (x_{\tau^\varepsilon})) \bigg| x_0 = x \right].
\]

Since the sample paths of \( x_t \) are continuous, \( x_{\tau^\varepsilon} = \varepsilon \) whenever \( x_0 < 0 \). Using the same arguments in the paragraphs prior to Proposition 2.2, one can show that \( V_1^\varepsilon (x) = De^{\beta x} \) for all \( x < \varepsilon \), where \( D \) is a constant such that \( De^{\beta \varepsilon} = 1 - V_2 (\varepsilon) > Q_1 (\varepsilon) = (1 - c) e^{\beta \varepsilon} \). This implies that \( D > 1 - c \), which in turn implies that \( V_1^\varepsilon (x) > V_1 (x) \) for all \( x < 0 \). Therefore, \( V_1 (x) \) cannot a solution to the optimal stopping problem (2.2) in the main text, so \((A^*, V_1, V_2)\) cannot be an equilibrium outcome.
In case (ii), a symmetric argument establishes that there exists $\xi > 0$ such that

$$V_2^\xi (x) = E \left[ e^{-r\xi} (1 - V_1 (x_\tau)) \middle| x_0 = x \right] > V_2 (x),$$

for all $x > 0$, where $\tau^\xi = \inf \{ t \geq 0 : x_t \leq -\xi \}$. Hence, in this case $(A^*, V_1, V_2)$ cannot be an equilibrium outcome either.

**A.2.2 Proof of Theorem 2.2**

Let $F^2$ be the set of bounded functions on $\mathbb{R}$ taking values in $\mathbb{R}^2$. Let $\| \cdot \|^2$ denote the sup norm on $\mathbb{R}^2$; i.e. for any $x \in \mathbb{R}^2$, $\| x \| = \max_{i \in \{1, 2\}} |x_i|$. For any $f \in F^2$, let $\| f \|^R = \sup_{z \in \mathbb{R}} \| f (z) \|$.

Fix $\Delta > 0, r > 0$, and let $\delta := e^{-r\Delta} < 1$. Define $\psi : F^2 \to F^2$ as follows: for any $f \in F^2$,

$$\psi_1 (f) (x) = \begin{cases} 
\delta E [f_1 (x_{t+\Delta}) | x_t = x] & \text{if } x < 0,

1 - \delta E [f_2 (x_{t+\Delta}) | x_t = x] & \text{if } x \geq 0,
\end{cases}$$

$$\psi_2 (f) (x) = \begin{cases} 
1 - \delta E [f_1 (x_{t+\Delta}) | x_t = x] & \text{if } x < 0,

\delta E [f_2 (x_{t+\Delta}) | x_t = x] & \text{if } x \geq 0.
\end{cases}$$

Let $B_1 := \{ x \in \mathbb{R} : x \geq 0 \}$ and $B_2 := \{ x \in \mathbb{R} : x < 0 \}$, so $B_i$ is the set of values of $x$ at which player $i$ is proposer.

**Lemma A.2.1** The operator $\psi : F^2 \to F^2$ is a contraction.

**Proof:** Let $f, g \in F^2$. Consider first any $x \notin B_i$, and note that

$$|\psi_i (f) (x) - \psi_i (g) (x)| = |\delta E [f_i (x_{t+\Delta}) - g_i (x_{t+\Delta}) | x_t = x]|$$

$$\leq \delta E [|f_i (x_{t+\Delta}) - g_i (x_{t+\Delta}) | | x_t = x] \leq \delta \| f - g \|^R.$$
Consider next \( x \in B_i \). In this case,

\[
|\psi_i(f)(x) - \psi_i(g)(x)| = |\delta E [g_i(x_{t+\Delta}) - f_i(x_{t+\Delta}) | x_t = x]| \\
\leq \delta E [||g_i(x_{t+\Delta}) - f_i(x_{t+\Delta})| | x_t = x] \leq \delta \|f - g\|^R.
\]

Therefore, \( \|\psi(f) - \psi(g)\|^R \leq \delta \|f - g\|^R \), so \( \psi \) is a contraction of modulus \( \delta \).

**Proof of Theorem 2.2:** In order to prove Theorem 2.2, I start out assuming that the set of SPE payoffs is not empty. At the end of the proof I show that the game indeed has a SPE.

Let \((u, \tau)\) be an SPE outcome and let \( f_i(x) = E[e^{-\tau t}u_i | x_0 = x] \) denote the payoff that player \( i \) gets from this SPE when the initial state is \( x \). Let \( \overline{M} = (\overline{M}_1, \overline{M}_2) \in F^2 \) and \( \underline{m} = (\underline{m}_1, \underline{m}_2) \in F^2 \) denote the supremum and infimum SPE payoffs at each state \( x \in \mathbb{R} \) for players 1 and 2 (so that \( \overline{M}_i(x) \geq f_i(x) \geq \underline{m}_i(x) \) for all \( x \in \mathbb{R}, i = 1, 2 \)).

Individual rationality implies that, for \( i = 1, 2 \), \( 1 \geq \overline{M}_i(x) \geq \underline{m}_i(x) \geq 0 \) for all \( x \in \mathbb{R} \). If \( x \in B_i \), player \( j \) who is responding to an offer can guarantee herself a payoff of at least \( \delta E[\underline{m}_j(x_{\Delta}) | x_0 = x] \) by rejecting the offer. Similarly, at state \( x \in B_j \) player \( j \) is the proposer so her payoff has to be at least as large as \( 1 - \delta E[\overline{M}_i(x_{\Delta}) | x_0 = x] \), since player \( i \) never rejects proposals that give her \( \delta E[\overline{M}_i(x_{\Delta}) | x_0 = x] \). In either case, \( f_1(x) \geq \psi_1(\overline{M}, \underline{m}) (x) \) and \( f_2(x) \geq \psi_2(\overline{M}, \underline{m}) (x) \). Define \( G(\overline{M}, \underline{m}) := (\psi_1(\overline{M}, \underline{m}), \psi_2(\overline{M}, \underline{m})) \).

Note next that \( f_i(x) \geq \underline{m}_i(x) \). Therefore, since \( f_j(x) + f_i(x) \leq 1 \) for all \( x \), it follows that \( f_j(x) \leq 1 - \underline{m}_i(x) \) whenever \( x \in B_j \). On the other hand, if \( x \in B_i \) (so player \( j \) is responding to an offer), then it must be that \( f_j(x) \leq \delta E[\overline{M}_j(x_{\Delta}) | x_0 = x] \), as player \( j \) will always accept any offer that gives her this much. Thus, it follows that \( f_1(x) \leq \psi_1(\overline{M}, \underline{m}) (x) \) and \( f_2(x) \leq \psi_2(\overline{M}, \underline{m}) (x) \). Define \( F(\overline{M}, \underline{m}) := (\psi_1(\overline{M}, \underline{m}), \psi_2(\overline{M}, \underline{m})) \).

Let \( M', M'' \) and \( m', m'' \) (all bounded functions on \( \mathbb{R} \) taking values on \( \mathbb{R}^2 \)) be such that \( M_i'(x) \geq M_i''(x) \) for all \( x \in \mathbb{R}, i = 1, 2 \) and \( m_i'(x) \leq m_i''(x) \) for all \( x \in \mathbb{R}, i = 1, 2 \). If this is the case, one can check that \( F_i(M', m')(x) \geq F_i(M'', m'')(x) \) for all \( x \in \mathbb{R}, i = 1, 2 \).
and $G_i(M', m') (x) \leq G_i(M'', m'') (x)$ for all $x \in \mathbb{R}, i = 1, 2$. In what follows, for any pair $f, g \in F^2$ I will write $f \geq g$ if $f \neq g$ and $f_i (x) \geq g_i (x)$ for all $x \in \mathbb{R}, i = 1, 2$.

Define the sequences $\{M^r\}$ and $\{m^r\}$ as follows: $(M^1, m^1) = (\bar{M}, \bar{m})$, and for all $r \geq 2$ let $(M^r, m^r) = (F(M^{r-1}, m^{r-1}), G(M^{r-1}, m^{r-1}))$. Note that $M^2 = F(M^1, m^1) \geq M^1$ and $m^2 = G(M^1, m^1) \leq m^1$. It follows then by induction and using the observation in the previous paragraph that $\{M^r\}$ is an increasing sequence and $\{m^r\}$ is a decreasing sequence. Moreover, it must be that $m^r \geq 0$ for all $r$ and $M^r \leq 1$ for all $r$. Thus, both $\{M^r\}$ and $\{m^r\}$ are bounded and monotonic sequences, so there exists $M^*$ and $m^*$ such that $\{M^r\} \to M^*$ and $\{m^r\} \to m^*$. Note further that $m^* \leq m^1 = \bar{m} \leq \bar{M} = M^1 \leq M^*$. Finally, since both $F$ and $G$ are continuous maps, it must be that $F(M^*, m^*) = M^*$ and $G(M^*, m^*) = m^*$.

Since $F(M^*, m^*) = M^*$, it follows from the definition of $F$ that $\psi_1 (M^*_1, m^*_2) = M^*_1$ and $\psi_2 (m^*_1, M^*_2) = M^*_2$. Similarly, $G(M^*, m^*) = m^*$ implies that $\psi_1 (m^*_1, M^*_2) = m^*_1$ and $\psi_2 (M^*_1, m^*_2) = m^*_2$. Therefore, both $(M^*_1, m^*_2)$ and $(m^*_1, M^*_2)$ are fixed points of $\psi$. Since $\psi$ has a unique fixed point (Lemma A.2.1), it must be that $M^*_1 = m^*_1$ for $i = 1, 2$. Finally, since $m^* \leq \bar{m} \leq \bar{M} \leq M^*$, it follows that $\bar{m} = \bar{M}$, so SPE payoffs are unique.

So far, I showed that if the set of SPE is not empty, then all SPE are payoff equivalent. I now show that the set of SPE is indeed non-empty. To see this, let $V^\Delta$ be such that $\psi (V^\Delta) = V^\Delta$ (i.e., $V^\Delta$ is the fix point of $\psi$). Note that $V^\Delta = (V^\Delta_1, V^\Delta_2)$ satisfies equations (2.9) and (2.10) in the main text. Let $(u, \tau)$ be the outcome with $\tau = 0$ and $u (x) = V^\Delta (x)$ for all $x \in \mathbb{R}$. Therefore, $V^\Delta$ is the players’ payoffs from outcome $(u, \tau)$.

To show that $(u, \tau)$ is the outcome induced by a SPE consider the following strategy profile. At any state $x \in B_i$, agent $i$ proposes division $u (x) = V^\Delta (x)$. Responder $j \neq i$ accepts any proposal which yields her a payoff of no less than $V^\Delta_j (x) = \delta E[V^\Delta_j (x)] \mid x_0 = x]$, and rejects any proposal that gives her a lower payoff. Note that this strategy profile induces outcome $(u, \tau)$. Moreover, it is easy to see that no player can gain by unilaterally deviating from her strategy at any $x \in \mathbb{R}$.
Proof of Lemma 2.1: The main text shows that $V_i^{\Delta_n} \to r \int_0^\infty e^{-rs} P_1(s, x) \, ds$. Here, I show that this convergence is uniform. Note first that for any $\Delta > 0$, the functions $V_i^{\Delta}(x)$ and $V_2^{\Delta}(x)$ are monotonic in $x$ (this because $P_1(s, x)$ and $P_2(s, x)$ are monotonic in $x$ for every $s \geq 0$). This implies that for any arbitrary closed interval $[a, b]$ and for $i = 1, 2$, $V_i^{\Delta_n} \to V_i^*$ uniformly on $[a, b]$ for any sequence $\{\Delta_n\} \to 0$. That is, for any closed interval $[a, b]$ and for every $\varepsilon > 0$ there exists an integer $N_{a,b}(\varepsilon)$ such that, for all $n > N_{a,b}(\varepsilon)$,

$$\max_{x \in [a,b]} |V_i^{\Delta_n}(x) - V_i^*(x)| < \varepsilon. \quad (2)$$

Moreover, for every $\varepsilon > 0$, $\Delta > 0$ there exists $c > 0$ s.t. $|V_i^{\Delta'}(x) - U_i(x)| < \varepsilon$ for all $x$ with $|x| > c$ and all $\Delta' \leq \Delta$. To see this note that for any $\Delta > 0$

$$\lim_{x \to \infty} \frac{1 - e^{-r\Delta}}{\Delta} \sum_{k=0}^\infty e^{-rk\Delta} \Delta P_1(k\Delta, x) = 1 = \lim_{x \to \infty} r \int_0^\infty e^{-rs} P_1(s, x) \, ds.$$

Therefore, for every $\varepsilon > 0$ there exists $c_+ > 0$ such that $|V_i^{\Delta}(x) - U_i(x)| < \varepsilon$ for all $x > c_+$.

A symmetric argument establishes that there exists $c_- < 0$ such that $|V_i^{\Delta}(x) - U_i(x)| < \varepsilon$ for all $x < c_-$. Let $c := \max\{|c_+|, |c_-|\}$, so that $|V_i^{\Delta}(x) - U_i(x)| < \varepsilon$ for all $x$ with $|x| > c$.

Finally, note that for any $\Delta' \leq \Delta$, $|V_i^{\Delta'}(x) - U_i(x)| \leq |V_i^{\Delta}(x) - U_i(x)| < \varepsilon$ for all $x$ with $|x| > c$.

Fix a sequence $\{\Delta_n\} \to 0$. Then, for every $\varepsilon$ there exists $c > 0$ such that $|V_i^{\Delta_n}(x) - U_i(x)| < \varepsilon$ for all $n$ and all $x$ with $|x| > c$. Moreover, there exists $N_{-c,c}(\varepsilon)$ such that $|V_i^{\Delta_n}(x) - U_i(x)| < \varepsilon$ for all $n > N_{-c,c}(\varepsilon)$ and all $x \in [-c, c]$. Thus, $|V_i^{\Delta_n}(x) - U_i(x)| < \varepsilon$ for all $n > N_{-c,c}(\varepsilon)$ and all $x \in \mathbb{R}$, so that $V_i^{\Delta_n} \to U_i(x)$ uniformly.

Proof of Lemma 2.2: In order to establish the first part of the Lemma, I show that $U_1(\cdot)$ is continuous. Since $U_2(x) = 1 - U_1(x)$, this will imply that $U_2(\cdot)$ is also continuous. Note

\[\int_0^\infty \int_0^{x} e^{-(s+t)} \, ds \, dt = e^{-x} \int_0^{\infty} e^{-ts} \, dt = e^{-x}.\]

This follows from a result in Boas (1996), page 113, which states: Let $f_n \to f$ pointwise on a compact interval $[a, b]$, with $f$ continuous and with $f_n$ monotonic for all $n$. Then, $f_n \to f$ uniformly on $[a, b]$.
first that the function \( P_1(s,x) \) is continuous in \( x \) for every \( s > 0 \) (indeed, this function is \( C^2 \) in \( x \) for all \( s > 0 \)). For every \( \varepsilon > 0 \), define \( U_1^\varepsilon(x) := r \int_0^\infty e^{-rs} P_1(s,x) \, ds \). Since \( P_1(s,x) \) is continuous in \( x \) for every \( s > 0 \), \( U_1^\varepsilon(\cdot) \) is also continuous for every \( \varepsilon > 0 \). Fix a decreasing sequence \( \{\varepsilon^n\} \to 0 \), so that \( U_1^{\varepsilon_n}(\cdot) \) is continuous for all \( n \). To show that \( U_1(\cdot) \) is continuous it suffices to show that \( \{U_1^{\varepsilon_n}\} \to U_1 \) uniformly. To see this, note that for any \( \varepsilon > 0 \) and \( x \in \mathbb{R} \),

\[
|U_1(x) - U_1^\varepsilon(x)| = r \int_0^\varepsilon e^{-rs} P_1(s,x) \, ds \leq r \int_0^\varepsilon e^{-rs} \, ds = 1 - e^{-r\varepsilon},
\]

where the inequality follows from the fact that \( P_1(s,x) \leq 1 \) for all \( s,x \). For every \( 1 > \eta > 0 \), there exists \( \varepsilon(\eta) > 0 \) such that \( 1 - e^{-r\varepsilon(\eta)} = \eta \) (i.e. \( \frac{-\ln(1-\eta)}{r} \)). Let \( N(\eta) = \inf \{ n : \varepsilon_n < \varepsilon(\eta) \} \), so that \( \varepsilon_n < \varepsilon(\eta) \) for all \( n > N(\eta) \). It then follows that, for every \( \eta > 0 \),

\[
|U_1(x) - U_1^{\varepsilon_n}(x)| \leq 1 - e^{-r\varepsilon_n} < \eta \text{ for all } n > N(\eta) \text{ and all } x \in \mathbb{R}, \text{ so } \{U_1^{\varepsilon_n}\} \to U_1 \text{ uniformly.}
\]

Therefore, \( U_1(\cdot) \) is continuous.

To show the second part of the Lemma, recall that \( P_1(s,x) = \Pr(x_{t+s} \geq 0 \mid x_t = x) = 1 - \Phi((-x-\mu s)/\sigma \sqrt{s}) \). Since \( \Phi((-x-\mu s)/\sigma \sqrt{s}) = \frac{1}{2}(1+\text{Erf}((-x-\mu s)/\sqrt{2s\sigma})) \) (where \( \text{Erf}(\cdot) \) denotes the error function), it follows that \( P_1(s,0) = \frac{1}{2}(1+\text{Erf}(\mu \sqrt{s}/\sqrt{2\sigma})) \) for all \( s > 0 \).

One can check that,

\[
r \int e^{-rs} P_1(s,0) \, ds = -\frac{e^{-rs}}{2} \left( 1 + \text{Erf} \left( \frac{\mu \sqrt{s}}{\sqrt{2\sigma}} \right) \right) + \frac{\mu \text{Erf} \left( \frac{\sqrt{s} \sqrt{\mu^2 + 2r \sigma^2}}{\sqrt{2 \sigma}} \right)}{2 \sqrt{\mu^2 + 2r \sigma^2}}. \tag{A.2.4}
\]

Equation (A.2.4), together with the fact that \( \text{Erf}(0) = 0 \) and \( \lim_{z \to \infty} \text{Erf}(z) = 1 \), yields \( U_1(0) = r \int_0^\infty e^{-rs} P_1(s,0) \, ds = \frac{\mu}{(2\sqrt{\mu^2 + 2r \sigma^2}) + \frac{1}{2} = \frac{\alpha}{\alpha + \beta}} \), and \( U_2(0) = 1 - U_1(0) = \frac{\beta}{\alpha + \beta} \).

### A.2.4 Proof of Proposition 2.2

Consider first the case with \( \lambda < r \). Let \((A,V_1,V_2)\) be an equilibrium outcome and assume by contradiction that \( A \neq \mathbb{R} \). Since \( A^c \) is open, there exists an interval \((\underline{x},\overline{x})\) with \( \underline{x},\overline{x} \in A \).
and \((x, \bar{x}) \not\in A\). Define \(W(x) = V_1(x) + V_2(x)\). Then, for all \(x \in (x, \bar{x})\)

\[
W(x) = E \left[ e^{-(r+\lambda)x} + 2\lambda \int_0^{\tau(A)} e^{-(r+\lambda)s} ds \right]_{x_0 = x}.
\]

For all \(x \in (x, \bar{x})\), \(W(x)\) solves

\[
rW(x) = \mu W'(x) + \frac{1}{2} \sigma^2 W''(x) + \lambda (2 - W(x)),
\]

(A.2.5)

with boundary conditions \(W(x) = W(\bar{x}) = 1\). The solution to this ODE is

\[
W(x) = \frac{2\lambda}{\lambda + r} + \frac{r - \lambda}{r + \lambda} \left( \frac{e^{\mu x} (e^{-\gamma x} - e^{-\gamma \bar{x}}) + e^{-\gamma x} (e^{\mu \bar{x}} - e^{\mu x})}{e^{\mu \bar{x}} e^{-\gamma x} - e^{\mu x} e^{-\gamma \bar{x}}} \right) < 1
\]

for all \(x \in (x, \bar{x})\), where \(\gamma = \frac{1}{\sigma^2} (\mu + \sqrt{\mu^2 + 2(r + \lambda) \sigma^2})\) and \(\nu = \gamma - \frac{2\mu}{\sigma^2}\), and where the inequality follows from \(\lambda < r\). Thus, \(V_1(x) + V_2(x) < 1\) for all \(x \in (x, \bar{x})\). But this implies that, when \(x_t \in (x, \bar{x})\), proposer \(i\) is better off by offering \(V_j(x_t)\) to her opponent (and obtaining a payoff of \(1 - V_j(x_t) > V_i(x_t)\) for herself) than by delaying. Therefore, \((A, V_1, V_2)\) cannot be an equilibrium outcome. Using arguments similar to those in the proof of Theorem 2.1, one can show that in this case there is unique equilibrium outcome \((A^*, V_1^*, V_2^*)\) with \(A^* = \mathbb{R}\) and with \((V_1^*, V_1^*)\) solving a system of differential equations.

Consider next the case with \(\lambda \in (r, \bar{\lambda})\). Recall that \(\tau^\phi = \inf\{t : x_t \notin (-\phi, \phi)\}\). Note that, in any equilibrium, party \(i\)'s payoffs are bounded below by

\[
\tilde{W}_1(x) = E \left[ e^{-(r+\lambda)\tau^\phi} 1_{\{x_{\tau^\phi} = \phi\}} + \lambda \int_0^{\tau^\phi} e^{-(r+\lambda)s} ds \right],
\]

\[
\tilde{W}_2(x) = E \left[ e^{-(r+\lambda)\tau^\phi} 1_{\{x_{\tau^\phi} = -\phi\}} + \lambda \int_0^{\tau^\phi} e^{-(r+\lambda)s} ds \right],
\]

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since a party can always unilaterally delay an agreement until time $\tau^\phi$. Let

$$
\tilde{W} (x) = \tilde{W}_1 (x) + \tilde{W}_2 (x) = E \left[ e^{-(r+\lambda)\tau^\phi} + 2\lambda \int_0^{\tau^\phi} e^{-(r+\lambda)s} ds \right].
$$

$\tilde{W}$ solves ODE (A.2.5), with boundary conditions $\tilde{W} (-\phi) = \tilde{W} (\phi) = 1$, so

$$
\tilde{W} (x) = \frac{2\lambda}{\lambda + r} + \frac{r - \lambda}{r + \lambda} \left( e^{\nu x} \left( e^{\gamma \phi} - e^{-\gamma \phi} \right) + e^{-\gamma x} \left( e^{\nu \phi} - e^{-\nu \phi} \right) \right).
$$

One can check that, for all $x \in (-\phi, \phi)$, the term in brackets is always strictly less than one. It then follows that $\tilde{W} (x) > \frac{2\lambda}{\lambda + r} + \frac{r - \lambda}{r + \lambda} = 1$ for all $x \in (-\phi, \phi)$. Since $\tilde{W} (x)$ is a lower bound to the sum of the parties’ payoffs in any equilibrium, there cannot be an agreement when $x_t \in (-\phi, \phi)$. Finally, when $\lambda \in (r, \bar{\lambda})$ a party with dictator power will choose to implement its preferred policy immediately, so in this case $A = (-\infty, -\phi] \cup [\phi, \infty)$.

### A.2.5 Proofs of Proposition 2.3 and 2.4

Let $\tau (a) = \inf \{ t : x_t = a \}$ with $a \in (0, 1)$. Suppose $x_t$ evolves as (2.16) with reflecting boundaries at 0 and 1 and let

$$
W (x) = E \left[ \int_0^{\tau (a)} e^{-rs} g (x_s) ds + e^{-r\tau (a)} G (x_{\tau (a)}) \right] \bigg| x_0 = x,
$$

where $g$ and $G$ are bounded functions. One can show then that $W$ solves

$$
r W (x) = \mu W' (x) + \frac{1}{2} \sigma^2 W'' (x) + g (x),
$$

(A.2.6)

for all $x \in (0, a)$, with boundary conditions $W (a) = G (a)$ and $W' (0) = 0$ (Harrison, 1982). Moreover, $W$ also solves (A.2.6) for all $x \in (a, 1)$, $W (a) = G (a)$ and $W' (1) = 0$. Finally, one can also show that $\tilde{W} (x) = E [\int_0^\infty e^{-rs} g (x_s) ds \mid x_0 = x]$ also solves (A.2.6), boundary conditions with $\tilde{W}' (0) = \tilde{W}' (1) = 0$. 

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Proof of Proposition 2.3: Let \((A, V_1, V_2)\) be an equilibrium outcome, and suppose by contradiction that \(A\) is a strict subset of \([0, 1]\). Then, there exists \(0 < y_1 < y_2 < 1\) such that \((y_1, y_2) \not\in A\) and \(y_1, y_2 \in A\). This implies that \(\tau(A) = \inf \{t : x_t \notin (y_1, y_2)\}\) > 0 whenever \(x_0 \in (y_1, y_2)\). Let \(W(x) := V_1(x) + V_2(x)\). Then, for all \(x \in (y_1, y_2)\) it must be that
\[
W(x) = E \left[ \int_0^{\tau(A)} e^{-rs} rads + e^{-r\tau(A)} \left| x_0 = x \right. \right] < 1,
\]
where the inequality follows from the fact that \(a < 1\). Thus, \(V_1(x) + V_2(x) < 1\) for all \(x \in (y_1, y_2)\). But this implies that, when \(x_t \in (y_1, y_2)\), proposer \(i\) is better off by offering \(V_j(x_t)\) to her opponent (and obtaining a payoff of \(1 - V_j(x_t) > V_i(x_t)\) for herself) than by delaying. Therefore, \((A, V_1, V_2)\) cannot be an equilibrium outcome. Thus, every equilibrium outcome \((A, V_1, V_2)\) must have \(A = [0, 1]\).

Let \(H_1 = [0, 1/2]\) and \(H_2 = [1/2, 1]\). Then, by an argument similar to that in Corollary 2.1, it must be that
\[
V_i(x) = \begin{cases} 1 - V_j(x) & x \notin H_i, \\ E \left[ \int_0^{\tau(1/2)} e^{-rs} \pi_i(x_s) ds + e^{-r\tau(1/2)}V_1(1/2) \left| x_0 = x \right. \right] & x \in H_i, \end{cases}
\]
where \(\tau(1/2) = \inf \{t : x_t = 1/2\}\). It then follows that
\[
r V_i(x) = \mu_i V_i'(x) + \frac{1}{2} \sigma^2 V_i''(x) + \pi_i(x) \text{ for all } x \in H_i, \tag{A.2.7}
\]
with boundary conditions \(V_1'(0) = V_2'(1) = 0\) and \(V_1(1/2) + V_2(1/2) = 1\). The general solution to the (A.2.7) is
\[
V_1(x) = ax + \frac{ad}{r} + A_1 e^{-\alpha x} + B_1 e^{\beta x}, \tag{A.2.8}
\]
\[
V_2(x) = a (1 - x) + \frac{ad}{r} + A_2 e^{-\alpha(1-x)} + B_2 e^{\beta(1-x)}, \tag{A.2.9}
\]

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where the constants \((A_i, B_i)_{i=1,2}\) solve appropriate boundary conditions. As in the standard model of Sections 2.2 and 2.3, one can show that there is a one-dimensional family of solutions to (A.2.8) and (A.2.9) satisfying the boundary conditions \(V_1'(0) = V_2'(1) = 0\) and \(V_1(1/2) + V_2(1/2) = 1\). Arguments similar to those used in the proofs of Propositions 2.2 and 2.3 imply that the unique equilibrium outcome \((A^*, V_1^*, V_2^*)\) is such that \(V_1^*\) and \(V_2^*\) also satisfy the smooth pasting condition \((V_1^*)' (1/2) = -(V_2^*)' (1/2)\).

From now on, let \(\alpha = (d + \sqrt{d^2 + 2r\sigma^2})/\sigma^2\) and \(\beta = (-d + \sqrt{d^2 + 2r\sigma^2})/\sigma^2\).

**Proof of Proposition 2.4(i):** Let \((A, V_1, V_2)\) be an equilibrium outcome. Then, for \(i = 1, 2\) it must be that \(V_i(x) \geq \Pi_i^D(x;d)\) for all \(x\), where \(\Pi_i^D(\cdot; d)\) is given by (2.18) in the main text. Therefore, \(\Pi_i^D(\cdot; d)\) solves

\[
\begin{align*}
r\Pi_i^D(x;d) &= \mu_i (\Pi_i^D)' (x;d) + \frac{1}{2} \alpha^2 (\Pi_i^D)'' (x;d) + \pi_i (x),
\end{align*}
\]

with \((\Pi_i^D)' (0;d) = (\Pi_i^D)' (1;d) = 0\). The solution to these equation are

\[
\begin{align*}
\Pi_1^D(x;d) &= a x + \frac{ad}{r} + \frac{a}{\alpha} e^{-\alpha x} + F \left( \frac{\beta}{\alpha} e^{-\alpha x} + e^{\beta x} \right), \\
\Pi_2^D(x;d) &= a (1-x) + \frac{ad}{r} + \frac{a}{\alpha} e^{-\alpha (1-x)} + F \left( \frac{\beta}{\alpha} e^{-\alpha (1-x)} + e^{\beta (1-x)} \right),
\end{align*}
\]

where \(F = -a(1 - e^{-\alpha})/(\beta e^\beta - \beta e^{-\alpha})\). Let \(\Pi^D(x;d) := \Pi_1^D(x;d) + \Pi_2^D(x;d)\), so

\[
\Pi^D(x;d) = a + \frac{2ad}{r} + \frac{a}{\alpha} \left( e^{-\alpha x} + e^{-\alpha (1-x)} \right) + F \left( \frac{\beta}{\alpha} \left( e^{-\alpha x} + e^{-\alpha (1-x)} \right) + e^{\beta x} + e^{\beta (1-x)} \right).
\]

One can check that \(\Pi^D(x;d) = a < 1\) when \(d = 0\) (i.e., when both managers have the same beliefs). Moreover, \(\Pi^D(x;d)\) is strictly increasing in \(d\) with \(\lim_{d \to -\infty} \Pi^D(x;d) = 2a\) for all \(d\). Hence, there exists \(\bar{d}\) such that \(\Pi^D(x;d) > 1\) for \(d > \bar{d}\) and for all \(x \in [0,1]\). Note then that any equilibrium outcome must have \(A^* = \emptyset\) when \(d > \bar{d}\), as in this case there is no agreement
that can satisfy both managers’ expectations. Therefore, the unique equilibrium outcome \((A^*, V_1^*, V_2^*)\) must have \(A^* = \emptyset\) and \(V_i^* (x) = \Pi_i^D (x; d)\) for \(i = 1, 2\).

From now on I focus on the case with \(d < \overline{d}\). Note that when \(d < \overline{d}\) any equilibrium outcome \((A, V_1, V_2)\) must have \(A \neq \emptyset\). Indeed, if \(A = \emptyset\) then \(V_i = \Pi_i^D\) for \(i = 1, 2\). Since \(d < \overline{d}\), there exists \(x\) such that \(V_1 (x) + V_2 (x) = \Pi^D (x) < 1\). But then the proposing firm \(i\) would find it optimal to make an offer of \(\overline{d}\) to her opponent and obtain a payoff of \(1 - V_2 (x) > V_1 (x)\). Thus, \(A \neq \emptyset\). Moreover, symmetry considerations imply that \(A_i \neq \emptyset\) for \(i = 1, 2\). Thus, for \(d < \overline{d}\) the values \(b_1 = \inf \{ x \in A_1 \}\) and \(b_2 = \sup \{ x \in A_2 \}\) are well defined for any equilibrium \((A, V_1, V_2)\).

**Lemma A.2.2** Let \((A, V_1, V_2)\) be an equilibrium and suppose that \(d < \overline{d}\). Then, must be that (i) \(V_1' (b_1) = -V_2' (b_1)\) and \(V_1'' (b_1) + V_2'' (b_1) \geq 0\); and (ii) \(V_1' (b_2) = -V_2' (b_2)\) and \(V_1'' (b_2) + V_2'' (b_2) \geq 0\).

**Proof:** Let \((A, V_1, V_2)\) be an equilibrium. The definition of equilibrium then implies that

\[
V_1 (x) = E_1 \left[ \int_0^{\tau(b_1)} e^{-rs} \pi_1 (x_s) \, ds + e^{-r \tau(b_1)} (1 - V_2 (b_1)) \right] \bigg| x_0 = x \quad \text{for } x < b_1. \tag{A.2.10}
\]

and

\[
V_2 (x) = E_2 \left[ \int_0^{\tau(b_2)} e^{-rs} \pi_2 (x_s) \, ds + e^{-r \tau(b_2)} (1 - V_1 (b_2)) \right] \bigg| x_0 = x \quad \text{for } x > b_2, \tag{A.2.11}
\]

where for \(i = 1, 2\), \(\tau (b_i) = \inf \{ t : x_t = b_i \}\). The argument that \(V_1' (b_1) = -V_2' (b_1)\) and \(V_1' (b_2) = -V_2' (b_2)\) is similar to the one in the proof of Proposition 2.3, and is therefore omitted.

I now establish that \(V_1'' (b_1) + V_2'' (b_1) \geq 0\) (the proof that \(V_1'' (b_2) + V_2'' (b_2) \geq 0\) is symmetric and omitted). The function \(V_1 (x)\) solves

\[
r V_1 (x) = d V_1' (x; d) + \frac{1}{2} \sigma^2 V_1'' (x) + \pi_1 (x), \tag{A.2.12}
\]

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with boundary conditions $V_1' (0) = 0$ and $V_1 (b_1) = 1 - V_2 (b_2)$. Let $f_1$ be the (unique) solution to (A.2.12) and boundary conditions (so that $f_1 (x) = V_1 (x)$ for all $x \leq b_1$). Towards a contradiction suppose that $V_1'' (b_1) + V_2'' (b_1) < 0$. Since $f_1' (b_1) = V_1' (b_1) = -V_2'' (b_1)$, there must exist $\varepsilon > b_1$ such that $f_1' (x) < -V_2' (x)$ for all $x \in (b_1, \varepsilon]$. Since $f_1 (b_1) = 1 - V_2 (b_1)$, it follows that $f_1 (\varepsilon) < 1 - V_2 (\varepsilon)$. Let

$$W^{\varepsilon}_1 (x) = E_1 \left[ \int_0^{\tau (\varepsilon)} e^{-r _s (x_s)} d s + e^{-r \tau (\varepsilon)} (1 - V_2 (\varepsilon)) \right] _{x_0 = x},$$

where $\tau (\varepsilon) = \inf \{ t : x_t \geq \varepsilon \}$. Thus, $W^{\varepsilon}_1 (x)$ solves (A.2.13) with boundary conditions $(W^{\varepsilon}_1)' (0) = 0$ and $W^{\varepsilon}_1 (\varepsilon) = 1 - V_2 (\varepsilon)$. Hence, $W^{\varepsilon}_1 (\varepsilon) > f_1 (\varepsilon)$. Since two different solutions to (A.2.13) with $(W^{\varepsilon}_1)' (0) = 0$ cannot cross, it must be that $W^{\varepsilon}_1 (x) > f_1 (x)$ for all $x < \varepsilon$. But since $\tau (\varepsilon) \in T_1$, this contradicts the assumption that $(A, V_1, V_2)$ is an equilibrium outcome. Therefore, if $(A, V_1, V_2)$ is an equilibrium outcome, it must be that $V_1'' (b_1) \geq V_2'' (b_1)$.

Let $(A, V_1, V_2)$ be an equilibrium. Since $V_1$ and $V_2$ satisfy (A.2.10) and (A.2.11), then

$$r V_1 (x) = d V_1' (x; d) + \frac{1}{2} \sigma ^2 V_1'' (x) + r a x \text{ for } x < b_1,$$

$$r V_2 (x) = -d V_2' (x; d) + \frac{1}{2} \sigma ^2 V_2'' (x) + r a (1 - x) \text{ for } x > b_2,$$

with boundary conditions $V_1' (0) = V_2' (1) = 0$ and $V_1 (a_i) + V_2 (a_i) = 1$ for $i = 1, 2$. The general solutions to these ODEs are $V_1 (x) = ax + \frac{ad}{r} + C_1 e^{-ax} + D_2 e^{\beta x}$ and $V_2 (x) = a (1 - x) + \frac{ad}{r} + C_2 e^{-\alpha (1 - x)} + D_2 e^{\beta (1 - x)}$ for some constants $(C_i, D_i)_{i=1,2}$. The boundary conditions $V_1' (0) = V_2' (1) = 0$ then imply that

$$V_1 (x) = a x + \frac{ad}{r} + \frac{a}{\alpha} e^{-ax} + D_1 \left( \frac{\beta e^{-ax} + e^{\beta x}}{\alpha} \right) \text{ for } x < b_1,$$

\footnote{Indeed, if $w_1$ and $w_2$ both solve (A.2.6) with $w_k' (0) = 0$ for $k = 1, 2$ and $w_1 (z) = w_2 (z)$ for some $z$, then it must be that $w_1 (x) = w_2 (x)$ for all $x$.}
and

$$V_2(x) = a (1 - x) + \frac{ad}{r} + \frac{a}{\alpha} e^{-\alpha(1-x)} + D_2 \left( \frac{\beta}{\alpha} e^{-\alpha(1-x)} + e^{\beta(1-x)} \right)$$ for $x > b_2$. \hspace{1cm} (A.2.16)

Since $V'_1(x) > 0$ for all $x \in (0, b_1)$ and $V'_2(x) < 0$ for all $x \in (b_2, 1)$, and since $V'_1(0) = V'_2(1) = 0$, it must be that $V''_1(0) > 0$ and $V''_2(1) > 0$. This implies that $D_i > \frac{-\alpha}{\beta(\beta + \alpha)}$ for $i = 1, 2$. If $D_i > 0$, then $V''_i(x) > 0$ for all $x$. Otherwise, $V''_i(x) < 0$ for all $x$ if $D_1 \in (-a\alpha/\beta(\beta + \alpha), 0)$ and $V''_2(x) > 0$ for all $x$ if $D_2 \in (-a\alpha/\beta(\beta + \alpha), 0)$. Since $V''_i$ is continuous for $i = 1, 2$, in this case then there exists $y_1 > 0$ and $y_2 < 1$ such that $V''_1(x) > 0$ iff $x < y_1$ and $V''_2(x) > 0$ iff $x > y_2$.

Let $(A, V_1, V_2)$ be an equilibrium outcome and suppose $d < \bar{d}$. There are two possibilities: (i) $b_1 = b_2 = 1/2$, or (ii) $b_1 > b_2$. By Lemma A.2.4 it must be that $V_1(b_1) + V_2(b_1) = 1$ and $V'_1(b_i) + V'_2(b_i) = 0$ for $i = 1, 2$. One can show that in case (i) these conditions imply that $D_1 = D_2$. On the other hand, one can also show that in case (ii) these conditions imply $D_1 = D_2$ and $b_2 = 1 - b_1$.

**Lemma A.2.3** Let $(A, V_1, V_2)$ be an equilibrium and suppose $d < \bar{d}$. Then it must be that $b_1 \geq y_2$ and $b_2 \leq y_1$.

**Proof:** Since $(A, V_1, V_2)$ is an equilibrium, it must be that $V_1$ and $V_2$ satisfy (A.2.15) and (A.2.16) with $D_1 = D_2$. Towards a contradiction suppose that $b_1 < y_2$, so that $V''_2(b_1) < 0$. Since $V''_1(b_1) + V''_2(b_1) > 0$, it must be that $V''_1(b_1) > 0$. This implies that $y_1 > b_1$, so $V''_1(x) > 0$ for all $x < b_1$. In particular, $0 < V''_1(1-b_1)$. But the fact that $D_1 = D_2$ implies that $V''_2(b_1) = V''_1(1-b_1) > 0$, a contradiction to the assumption that $V''_2(b_1) < 0$. Therefore, it must be that $b_1 \geq y_2$. A symmetric argument establishes that $b_2 \leq y_1$. 

\[4\] Indeed, since $V_i(x) \geq \Pi^D_i(x; d)$ it follows that $D_i > -a(1-e^{-\alpha})/(\beta e^\beta - \beta e^{-\alpha})$. One can show that $-a\alpha/\beta^2 - \beta < -a(1-e^{-\alpha})/(\beta e^\beta - \beta e^{-\alpha})$.

\[5\] Indeed, $V''_1(x) = \beta^2 D_1 e^{\beta x} - \alpha^2 e^{-\alpha x} (a + \beta D_1)$. If $D_1 < 0$ the first term is negative. Moreover, one can check that the second term is positive when $D_1 > -a\alpha/\beta(\beta + \alpha)$. Thus, $V''_1(x) < 0$ if $D_1 \in (-a\alpha/\beta(\beta + \alpha), 0)$. 

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Lemma A.2.4 Let \((A, V_1, V_2)\) be an equilibrium and suppose \(d < \bar{d}\). Then it must be that \(A_1 = [b_1, 1]\) and \(A_2 = [0, b_2]\) with \(b_1 \in [1/2, 1]\) and \(b_2 \in [0, 1/2]\).

Proof: Since \(d < \bar{d}\), it must be that \(A_1\) is not empty. Let \(b_1 = \inf\{x \in A_1\}\), so that \(b_1 \leq 1\). Moreover, since \(A_1\) is closed it must be that \(b_1 \in A_1\). The result follows when \(b_1 = 1\). Thus, suppose that \(b_1 < 1\). Let \(Y_t := \int_0^t e^{-rs} \pi_1(x_s) \, ds + e^{-rt}(1 - V_2^A(x_t))\). According to player 1’s beliefs, for \(x_t \in (b_1, 1)\) the process \(Y_t\) evolves as

\[
dY_t = e^{-rt} \left( \pi_1(x_t) - r + rV_2(x_t) - dV_2'(x_t) - \frac{1}{2} \sigma^2 V_2''(x_t) \right) \, dt + e^{-rt} \sigma V_2'(x_t) \, dB_t \\
= e^{-rt} \left[ (-r(1-a) - 2dV'_2(x_t)) \, dt + \sigma V_2'(x_t) \, dB_t \right],
\]

where the last equality follows from the fact that \(V_2\) solves \((A.2.14)\). The goal is to show that \(Y_t\) has a negative drift for all \(x_t > b_1\). By Lemma A.2.2,

\[
0 \leq V'_1(b_1) + V''_2(b_1) = \frac{1}{\sigma^2} \left( -a + V_2'(b_1) \right) - d \left( V'_1(b_1) - V'_2(b_1) \right) - ra \\
= \frac{1}{\sigma^2} \left( r(1-a) + 2dV'_2(b_1) \right),
\]

where the first equality follows from the fact that \(V_1\) and \(V_2\) satisfy \((A.2.13)\) and \((A.2.14)\) respectively, and the second from the fact that \(V'_1(b_1) + V_2'(b_1) = 1\) and \(V''_1(b_1) = -V''_2(b_1)\). Since \(V''_2(b_1) \geq 0\), it follows that \(V''_2(x) > 0\) for all \(x > b_1\), so that \(V'_2(x) > V'_2(b_1) > -r(1-a)/2d\). Hence, \(Y_t\) has negative drift for all \(x_t \geq b_1\).

I now show that firm 1 has a strict incentive to end the bargaining immediately whenever \(x_t > b_1\), so that \(A_1 = [b_1, 1]\). Towards a contradiction, suppose that there exists \((z_1, z_2) \subseteq [b_1, 1]\) such that \((z_1, z_2) \not\subseteq A_1\). For all \(x \in (z_1, z_2)\),

\[
V_1(x) = E_1 \left[ \int_0^{\tau(A)} e^{-rs} \pi_1(x_s) \, ds + e^{-rt}(1 - V_2(x_{\tau(A)})) \right| x_0 = x] \\
= E_1 \left[ Y_{\tau(A)} \right| x_0 = x] \\
= Y_0 + E \left[ \int_0^{\tau(A)} e^{-rt} (-r(1-a) - 2dV'_2(x_t)) \, dt \right] < Y_0 = 1 - V_2(x),
\]

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so firm 1 strictly prefers to make an acceptable proposal when \( x_t > b_1 \). Therefore, it must be that any equilibrium outcome \((A, V_1, V_2)\) has \( A_1 = [b_1, 1] \). A symmetric argument establishes that any equilibrium outcome must also have \( A_2 = [0, b_2] \).

**Proof of Proposition 2.4(ii):** Let \((A^*, V_1^*, V_2^*)\) be an equilibrium, with \( A^* = [0, 1] \). By the discussion above, in any equilibrium \( V_1^* \) and \( V_2^* \) must satisfy (A.2.15) and (A.2.16). By Lemma A.2.2, it must be that \( V_1^* (1/2) + V_2^* (1/2) = 1 \) and \( (V_1^*)' (1/2) = -(V_2^*)' (1/2) \). These conditions imply that

\[
D_1 = D_2 = D^* = \frac{\alpha \left( \frac{1-a}{2} - \frac{ad}{r} \right) - ae^{-\alpha/2}}{\beta e^{-\alpha/2} + \alpha e^{\beta/2}}.
\]

Lemma A.2.2 also implies that \((V_1^*)'' (1/2) + (V_2^*)'' (1/2) > 0\), and \( D_1 = D_2 = D^* \) implies that \((V_1^*)'' (1/2) = (V_2^*)'' (1/2) \). Thus, it must be that \((V_1^*)'' (1/2) > 0\), or that \( aae^{-\alpha/2} + \beta D^* (\beta e^{\beta/2} + \alpha e^{-\alpha/2}) > 0 \). One can check that there exists \( d \in (0, \overline{d}) \) such that this inequality holds iff \( d < \overline{d} \).

Since \( A^* = [0, 1] \), then \( V_1^* (x) = 1 - V_2^* (x) \) for \( x > 1/2 \). Therefore,

\[
-rV_1^* (x) + d (V_1^*)' (x) + \frac{1}{2} \sigma^2 (V_1^*)'' (x) + \pi_1 (x) = -r - d (V_2^*)' (x) - \frac{1}{2} \sigma^2 (V_2^*)'' (x) + \pi_2 (x) - d (V_2^*)' (x) + \frac{1}{2} \sigma^2 (V_2^*)'' (x) + \pi_1 (x) = -r (1 - a) - 2d (V_2^*)' (x) \leq 0, \tag{A.2.17}
\]

where the inequality was proved in Lemma A.2.4. Therefore, (A.2.17) implies that

\[
-rV_1 (x) + dV_1' (x) + \frac{1}{2} \sigma^2 V_1'' (x) + \pi_1 (x) \leq 0, \text{ with equality for } x \in [0, 1/2].
\]

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Moreover, $V_1^*$ is twice differentiable with a continuous first derivative. Thus, by standard verification theorems, $V_1^*$ solves the optimal stopping problem

$$ V_1^*(x) = \sup_\tau E_i \left[ \int_0^\tau e^{-rs} \pi_1(x_s) \, ds + e^{-r\tau} (1 - V_2^*(x_\tau)) \right]_{x_0 = x}. $$

A symmetric argument establishes that $V_2^*(x)$ also solves the optimal stopping problem. Hence, $([0,1], V_1^*, V_2^*)$ are an equilibrium outcome.

To establish uniqueness note that any equilibrium $(A, V_1, V_2)$, $V_1$ and $V_2$ must satisfy (A.2.14) and (A.2.15) for some $D_1 = D_2 = D$. If $D < D^*$ then $V_1(1/2) + V_2(1/2) < 1$, which cannot occur in equilibrium. On the other hand, $D > D^*$ implies $V_1(x) + V_2(x) > 1$ for all $x \in [0,1]$, which cannot be an equilibrium either since $d < \overline{d}$.

**Proof of Proposition 2.4(iii):** Let $(A^*, V_1^*, V_2^*)$ be an equilibrium outcome, and suppose $d \in (d, \overline{d})$. By the arguments above it must be that $A^* = [0, b_2] \cup [b_1, 1]$ with $0 < b_1 < b_2 < 1$. By Lemma A.2.2 it must also be that $(V_1^*)'(b_i) = -(V_2^*)'(b_i)$ and $(V_1^*)''(b_i) + (V_2^*)'(b_i) > 0$ for $i = 1, 2$ (and $V_1^*(b_i) + V_2^*(b_i) = 1$ for $i = 1, 2$). This is a system of four equations, with unknowns $b_1$, $b_2$, $D_1$ and $D_2$. One can show that this system of equations has a unique solution with $b_1 < 1/2$ and $b_2 > 1/2$. Moreover, $D_1 = D_2 = D^*$ and $b_1 = 1 - b_2 = \kappa$.

To show that $(A^*, V_1^*, V_2^*)$ is an equilibrium, I need to show that $V_1^*$ and $V_2^*$ are solutions to the optimal stopping problem. By arguments similar to those in the proof of Proposition 2.4(ii), one can show that

\begin{align*}
-rV_1^*(x) + d (V_1^*)'(x) + \frac{1}{2} \sigma^2 (V_1^*)''(x) + \pi_1(x) &\geq 0, \text{ with equality for } x \in [0, b_1], \\
rV_2^*(x) + d (V_2^*)'(x) + \frac{1}{2} \sigma^2 (V_2^*)''(x) + \pi_2(x) &\geq 0, \text{ with equality for } x \in [1, b_2].
\end{align*}

Therefore, by standard verification theorems, for $i = 1, 2$, $j \neq i$, $V_i^*$ solves

$$ V_i^*(x) = \sup_\tau E_i \left[ \int_0^\tau e^{-rs} \pi_i(x_s) \, ds + e^{-r\tau} (1 - V_j^*(x_\tau)) \right]_{x_0 = x}. $$

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Uniqueness then follows from the fact that, if $D < D^*$, then there exists $x$ such that $V_1(x) + V_2(x) < 1$ (which cannot occur in equilibrium). On the other hand, $D > D^*$ implies that $V_1(x) + V_2(x) > 1$ for all $x$, which cannot be since $d < d$.

### A.2.6 Convergence under general stochastic processes

In this Appendix, I study a discrete time bargaining game similar to the one I studied in Section 2.5. The only difference is that I allow relative bargaining power $x_t$ to solve

$$dx_t = \mu(x_t) dt + \sigma(x_t) dB_t, x_0 \in \mathbb{R},$$

with $\mu(\cdot), \sigma(\cdot) \in C^2$ and satisfying conditions for existence and uniqueness of a solution to this differential equation. The process $x_t$ evolves in continuous time, but players can only make offers at times $t \in T(\Delta) = \{0, \Delta, 2\Delta, \ldots\}$. The main goal is to establish the analog of Theorem 2.3 for this more general setup.

As in the main text, let $P_1(s,x) = \Pr(x_s \geq 0|x_0 = x)$, $P_2(s,x) = 1 - P_1(s,x)$. In this case, the transition density $p(y,x,s)$ of $x_t$ satisfies the following version of Kolmogorov’s backward equation (see Rogers, 1985)

$$\frac{\partial}{\partial s} p(y,x,s) = \mu(x) \frac{\partial}{\partial x} p(y,x,s) + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} p(y,x,s). \quad (A.2.18)$$

Since $P_1(s,x) = \int_0^\infty p(y,x,s) dy$ and $P_2(s,x) = \int_{-\infty}^0 p(y,x,s) dy$, by Leibniz’s rule $P_1(s,x)$ and $P_2(s,x)$ also satisfies (A.2.14).

Note first that Theorem 2.2 continuous to hold in this setting. Indeed, there is nothing in the proof of Theorem 2 that is specific to the process that characterizes the evolution of $x_t$. Let $(V_1^\Delta, V_2^\Delta)$ denote the players’ SPE payoffs of this discrete time game when the interval between offers is $\Delta$; $(V_1^\Delta, V_2^\Delta)$ satisfy equations (2.13) and (2.14) in the main text. Fix a sequence $\{\Delta_n\} \to 0$. By Lemma 2.1 (which continuous to hold in this setting), $V_i^\Delta(x)$ converges uniformly to $V_i^*(x) := r \int_0^\infty P_i(s,x) ds$ as $n \to \infty$. As in the main text, let
$A_1^* = [0, \infty)$ and $A_2^* = (-\infty, 0]$. Then, by the same arguments used in Section 2.5, $V_1^*(x)$ and $V_2^*(x)$ solve
\begin{equation}
rV_i^*(x) = (V_i^*)'(x) + \frac{1}{2} \sigma^2(x) (V_i^*)''(x) \quad \text{for all } x \notin A_i^*, \tag{A.2.19}
\end{equation}

$V_i^*(x) = 1 - V_j^*(x)$ for $x \in A_i^*$. Since $V_1^*$ and $V_2^*$ are continuous, one of the boundary conditions is $V_1^*(0) = 1 - V_2^*(0)$. Moreover, since $\lim_{x \to -\infty} P_1(s, x) = \lim_{x \to -\infty} P_2(s, x) = 0$ for all $s > 0$, it follows that $\lim_{x \to -\infty} V_1^*(x) = \lim_{x \to -\infty} V_2^*(x) = 0$. Therefore, in order to prove the analog of Theorem 2.3 for this setting, I need to show that $V_1^*(x)$ and $V_2^*(x)$ are differentiable at 0, so that $(V_1^*)'(0-) = -(V_2^*)'(0^+)$. The following Lemma establishes this result, and this completes the proof.

**Lemma A.2.5** The functions $V_1^*$ and $V_2^*$ are differentiable at 0.

**Proof:** Suppose by contradiction that they are not. Since $V_1^*(x) = 1 - V_2^*(x)$ for all $x \in \mathbb{R}$, it follows that either $V_1^*$ or $V_2^*$ has a convex kink at 0. Assume that $V_1^*$ has a convex kink. I will show that in this case, player 1 has a profitable deviation whenever $\Delta$ is small enough.

For $\kappa > 0$, define $\tau^\kappa := \inf \{t \geq 0 : x_t \geq \kappa\}$. Let
\[
U_1^\kappa(x) = E[e^{-rt\kappa} (1 - V_2^*(x_{\tau^\kappa}))| x_0 = x],
\]
so $U_1^\kappa(x) = V_1^*(x)$ for all $x \geq \kappa$. Moreover, $U_1^\kappa$ solves (A.2.15) for all $x < \kappa$, with boundary conditions $\lim_{x \to -\infty} U_1^\kappa(x) = 0$ and $U_1^\kappa(\kappa) = V_1^*(\kappa)$. Since $V_1^*$ has a convex kink at 0, there exists $\varepsilon > 0$ such that $U_1^\varepsilon(x) > V_1^*(x)$ for all $x < \varepsilon$.

Consider the following deviation for player 1. For all $x \geq \varepsilon$, offer $V_2^\Delta(x)$ to player 2 (an offer that player 2 accepts). For all $x < \varepsilon$, reject all offers when she is the responder, and offer $0 < V_2^\Delta(x)$ to player 2 when she is the proposer (so that these offers are rejected by player 2). For any $\Delta > 0$, define the stopping time $\tau^\varepsilon_\Delta := \inf \{t \in T(\Delta) : x_t \geq \varepsilon\}$. The payoff

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\footnote{Continuity of $V_1^*$ and $V_2^*$ follows from Lemma 2.2, which continuous to hold in this setting.}
player 1 gets from following this strategy (when player 2 follows her equilibrium strategy) is
\[ E \left[ e^{-rt_{\Delta}} \left( 1 - V_2^\Delta \left( x_{\tau_{\Delta}} \right) \right) \right] | x_0 = x ] . \] Moreover, \( \tau_\Delta^\varepsilon \geq \tau^\varepsilon \) for any \( \Delta > 0 \).

Let \( \Delta_n = 2^{-n} \). Since \( T(\Delta_n) \subseteq T(\Delta_n') \) for every \( n' > n \), it follows that \( \tau_{\Delta_n}^\varepsilon \) is a decreasing sequence, bounded below by \( \tau^\varepsilon \). Let \( \tau^* := \lim_{n \to \infty} \tau_{\Delta_n}^\varepsilon \); \( \tau^* \) is an stopping time, since \( \{ F_t : 0 \leq t < \infty \} \) is right-continuous. I will show that \( \tau^* = \tau^\varepsilon \) almost surely. To see this, let \( D := \{ \omega \in \Omega : \tau^*(\omega) > \tau^\varepsilon(\omega) \} \). That is, \( D \) is the set of sample paths of \( x_t \) such that \( \tau^* \) is larger than \( \tau^\varepsilon \). These sample paths are such that: (i) \( \tau^\varepsilon(\omega) \) does not belong to the dyadic rationals (so that \( \tau^\varepsilon(\omega) \notin T(\Delta_n) \) for all \( n \)), and (ii) there exists \( \rho > 0 \) such that \( x_{\tau^\varepsilon + s} < \varepsilon \) for all \( s \in [0, \rho] \). In words, the paths of \( x_t \) such that \( \tau^* > \tau^\varepsilon \) are such that \( x_t \) hits \( \varepsilon \) and then stays below \( \varepsilon \) for some positive amount of time. The set of all these paths has measure zero, so \( \tau^* = \tau^\varepsilon \) almost surely.

Let \( \xi \) be such that \( U^\xi_1(x) > V^{*}_1(x) + \xi \) for all \( x \in [a, b] \), with \( a < b < \varepsilon \). Since \( V_{i}^{\Delta_n}(x) \to V^{*}_i(x) \) uniformly, there exists \( N \) s.t. for all \( n > N \), \( |V^{\Delta_n}_i(x) - V^{*}_i(x)| < \xi/2 \) for all \( x \) and for \( i = 1, 2 \). Player 1’s payoff from this deviation is

\[ E \left[ e^{-rt_{\Delta_n}} \left( 1 - V_2^\Delta \left( x_{\tau_{\Delta_n}} \right) \right) \right] | x_0 = x ] > E \left[ e^{-rt_{\Delta_n}} \left( 1 - V_2^\Delta \left( x_{\tau_{\Delta_n}} \right) \right) \right] | x_0 = x ] - \frac{\xi}{2}, \]

for all \( n > N \). Since \( \tau_{\Delta_n}^\varepsilon \to \tau^\varepsilon \), Lebesgue’s Dominated Convergence Theorem implies

\[ E \left[ e^{-rt_{\Delta_n}} \left( 1 - V_2^\Delta \left( x_{\tau_{\Delta_n}} \right) \right) \right] | x_0 = x ] \to U^\varepsilon_1(x) - \frac{\xi}{2}. \]

Finally, since \( U^\xi_1(x) - \frac{\xi}{2} > V^{*}_1(x) + \frac{\xi}{2} \) for all \( x \in [a, b] \), and since \( V_{1}^{\Delta_n}(x) < V^{*}_1(x) + \frac{\xi}{2} \) for all \( n > N \), it follows that for \( n \) large enough,

\[ E \left[ e^{-rt_{\Delta_n}} \left( 1 - V_2^\Delta \left( x_{\tau_{\Delta_n}} \right) \right) \right] | x_0 = x ] > V_{1}^{\Delta_n}(x) \text{ for all } x \in [a, b]. \]
Therefore, if $V_1^*$ has a convex kink at 0, then player 1 has a profitable deviation whenever $\Delta_n$ is small enough. But this is a contradiction to the fact that $V_1^{\Delta_n} (x)$ is player 1’s SPE payoff for every, so $V_1^*$ cannot have a convex kink at 0.

A symmetric proof establishes that $V_2^*$ cannot have a convex kink at 0 either. Since $V_1^* (x) + V_2^* (x) = 1$ for all $x$, it follows that $V_1^*$ and $V_2^*$ are differentiable at 0.

A.3 Appendix to Chapter 3

A.3.1 Proof of Theorem 3.1

The proof of Theorem 3.1 involves two steps. The first step (Lemma A.3.2) is to show that the outcome $(A^*, V_1^\phi, V_2^\phi)$, with $V_1^\phi$ and $V_2^\phi$ satisfying equations (3.5) and (3.6) in the main text is indeed an equilibrium. The second step (Lemma A.3.3) is to show that this is the unique equilibrium outcome. Before turning to the first step of the proof, I present a Lemma that will be used repeatedly. Let $y_1, y_2 \in [0, 1], y_1 < y_2$ and define $\tau_y = \inf\{t : x_t \leq y_1 \text{ or } x_t \geq y_2\}$.

**Lemma A.3.1** Let $v(x) = E[e^{-r \tau_y} g (x_{\tau_y}) | x_0 = x]$ for some bounded function $g$. Then, for all $x \in (y_1, y_2)$, $v$ solves

$$rv(x) = \mu v'(x) + \frac{1}{2} \sigma^2 v''(x),$$

(A.3.1)

with boundary conditions $v(y_1) = g(y_1)$ and $v(y_2) = g(y_2)$.

**Proof:** Note first that solutions to (A.3.1) exist and are continuous in initial conditions. Indeed, (A.3.1) has a general solution of the form $v(x) = \eta e^{-\alpha x} + \lambda e^{\beta x}$, where $\alpha = (\mu + \sqrt{\mu^2 + 2r \sigma^2})/\sigma^2$, $\beta = (-\mu + \sqrt{\mu^2 + 2r \sigma^2})/\sigma^2$ and where $\eta$ and $\lambda$ are constants determined by the boundary conditions.
Let \( \tilde{v} \) be the solution to (A.3.1) with \( \tilde{v}(y_1) = g(y_1) \) and \( \tilde{v}(y_2) = g(y_2) \). Let \( f(x,t) = e^{-rt}\tilde{v}(x) \), so that \( f(x,t) = e^{-rt}g(x) \) for \( x \in \{y_1, y_2\} \). By Ito’s formula, for all \( x \in (y_1, y_2) \),

\[
df(x,t) = e^{-rt}\left(-r\tilde{v}(x_t) + \mu\tilde{v}'(x_t) + \frac{1}{2}\sigma^2\tilde{v}''(x_t)\right)dt + e^{-rt}\sigma\tilde{v}'(x_t)dB_t
\]

where the second equality follows from the fact that \( \tilde{v}(x) \) solves (A.3.1) for all \( x \in (y_1, y_2) \). Then,

\[
E\left[e^{-r\tau_y}g(x_{\tau_y})\mid x_0 = x\right] = E\left[f(x_{\tau_y}, \tau_y)\mid x_0 = x\right] = \tilde{v}(x) + E\left[\int_0^{\tau_y} df(x,t) \mid x_0 = x\right] = \tilde{v}(x) + E\left[\int_0^{\tau_y} e^{-rt}\sigma\tilde{v}'(x_t)dB_t \mid x_0 = x\right] = \tilde{v}(x),
\]

where the last equality follows from the fact that \( \int_0^{\tau_y} e^{-rt}\sigma\tilde{v}'(x_t)dB_t \) has zero expectation.\(^7\)

Therefore, \( \tilde{v}(x) = v(x) \) for all \( x \in (y_1, y_2) \), so \( v(x) \) solves (A.3.1) for all \( x \in (y_1, y_2) \) and satisfies the boundary conditions.

By Proposition 3.1 and Corollary 3.1, any equilibrium outcome \((A, V_1, V_2)\) has \( A = A^* = [0, 1] \) and \( V_1 \) and \( V_2 \) satisfying equation (3.4) in the main text. Since \( A = A^* \), it follows that \( V_1(x) + V_2(x) = 1 \) for all \( x \in [0, 1] \); in particular, \( V_1(1/2) + V_2(1/2) = 1 \). Therefore, by Lemma A.3.1, for any equilibrium outcome \((A, V_1, V_2)\), the payoffs \( V_1, V_2 \) solve

\[
\begin{align*}
    rV_1(x) &= \mu V_1'(x) + \frac{1}{2}\sigma^2 V_1''(x) \quad \text{for all } x \in (\phi, 1/2), \quad (A.3.2) \\
    rV_2(x) &= \mu V_2'(x) + \frac{1}{2}\sigma^2 V_2''(x) \quad \text{for all } x \in (1/2, 1 - \phi), \quad (A.3.3)
\end{align*}
\]

with boundary conditions \( V_1(\phi) = 0 = V_2(1 - \phi) \) and \( V_1(1/2) + V_2(1/2) = 1 \).

\(^7\)Let \( M_t = \int_0^t e^{-rs}\sigma\tilde{v}'(x_s)dB_s \), so that \( M_0 = 0 \). Since \( e^{-rs}\sigma\tilde{v}'(x_s) \) is bounded, it follows that \( E\left[\int_0^t (e^{-rs}\sigma\tilde{v}'(x_s))^2ds\right] < \infty \) for all \( t \). Therefore, by Proposition 3.2.10 in Karatzas and Shreve (1998), \( M_t \) is a Martingale, so \( E\left[M_t\right] = 0 \) for all \( t > 0 \).
The general solution to equations (A.3.2) and (A.3.3) is \( V_i(x) = D_i e^{-\alpha x} + E_i e^{\beta x} \). The boundary conditions \( V_1(\phi) = V_2(1 - \phi) = 0 \) imply that \( D_1 = -E_1 e^{(\alpha + \beta) \phi} \) and \( D_2 = -E_2 e^{(\alpha + \beta)(1 - \phi)} \). Therefore, for \( i, j = 1, 2, j \neq i \),

\[
V_i(x) = \begin{cases} 
E_i \left(-e^{(\alpha + \beta) \phi} e^{-\alpha x} + e^{\beta x}\right) & \text{if } x \in A_i^\phi, \\
1 - V_j(x) & \text{if } x \notin A_i^\phi.
\end{cases} \tag{A.3.4}
\]

The condition that \( V_1(1/2) + V_2(1/2) = 1 \) implies that

\[
E_1 \left(-e^{(\alpha + \beta) \phi} e^{-\frac{x}{2}} + e^{\frac{\beta}{2}}\right) = 1 - E_2 \left(-e^{(\alpha + \beta) \phi} e^{-\frac{x}{2}} + e^{\frac{\beta}{2}}\right). \tag{A.3.5}
\]

Finally, since \( V_i(x) \in [0, 1] \) for all \( x \), it must be that \( 0 \leq E_2 \leq 1 / (-e^{(\alpha + \beta) \phi} e^{-\alpha/2} + e^{\beta/2}) \).

Thus, there is a one dimensional family of functions \((V_1, V_2)\) satisfying (A.3.2), (A.3.3) and the boundary conditions \( V_1(\phi) = 0 = V_2(1 - \phi) \) and \( V_1(1/2) + V_2(1/2) = 1 \). Let \((V_1^\phi, V_2^\phi)\) be the unique element in this family that satisfies the smooth pasting condition (SP); one can show that

\[
V_1^\phi(x) = \begin{cases} 
\left(\beta e^{\alpha \frac{1}{2} - \frac{x}{2}} + \alpha e^{\beta \frac{1}{2} - \frac{1}{2} \phi}\right) \left( e^{\beta x} e^{-\alpha \phi - \alpha x} - e^{\alpha x} e^{\beta \phi} \right) & x \in \left(\phi, \frac{1}{2}\right), \\
1 - \left(\beta e^{\alpha \frac{1}{2} - \frac{x}{2}} + \alpha e^{\beta \frac{1}{2} - \frac{1}{2} \phi}\right) \left( e^{\beta x} e^{-\alpha \phi - \alpha x} - e^{\alpha x} e^{\beta \phi} \right) & x \in \left[\frac{1}{2}, 1 - \phi\right],
\end{cases}
\]

and with \( V_2^\phi(x) = 1 - V_1^\phi(x) \) for all \( x \in (\phi, 1 - \phi) \).

**Lemma A.3.2** The outcome \((A^*, V_1^\phi, V_2^\phi)\) is an equilibrium.

**Proof:** I show that, for all \( x \in (\phi, 1 - \phi) \), \( V_1^\phi(x) \) solves

\[
V_1^\phi(x) = \sup_{\tau \in T_i} E \left[ e^{\tau \land \tau^\phi} \left(1 - V_2^\phi(x_{\tau \land \tau^\phi})\right) \mid x_0 = x \right]. \tag{A.3.6}
\]

The proof that \( V_2^\phi \) also solves the optimal stopping problem is symmetric and omitted. To see that \( V_1^* \) satisfies (A.3.6), let \( F_2: \mathbb{R} \to \mathbb{R} \) be the solution to the (A.3.3), with boundary
conditions $F_2(1/2) = V_2^\phi(1/2)$ and $F_2(1 - \phi) = 0$. Let $G_2(x) := \min\{1, F_2(x)\}$ and note that $G_2(x) = V_2^\phi(x)$ for all $x \geq 1/2$. Consider the optimal stopping problem

$$G_1(x) = \sup_{\tau \in T} E\left[e^{-r\tau} \cdot \left(1 - G_2(x_{\tau\wedge \tau^\phi})\right) \big| x_0 = x\right],$$

(A.3.7)

where $T$ is the set of all stopping times. Since $G_2(x) = V_2^\phi(x)$ for all $x \geq 1/2$ and since $T_1 \subseteq T$, it follows that $G_1(x) \geq \sup_{\tau \in T_1} E[e^{-r\tau\wedge \tau^\phi}(1 - V_2^\phi(x_{\tau\wedge \tau^\phi})) | x_0 = x]$. Therefore, in order to show that $V_1^\phi$ satisfies (A.3.6) it suffices to show that $V_1^\phi = G_1$.

The function $V_1^\phi$ is twice differentiable on $(\phi, 1 - \phi)$, with a continuous first derivative. One can check that $V_1^\phi(x) > 1 - G_2(x)$ for all $x \in (\phi, 1/2)$, and $V_1^\phi(x) = 1 - G_2(x)$ for all $x \in [1/2, 1 - \phi)$. Finally, $V_1^\phi(x)$ satisfies

$$-r V_1^\phi(x) + \mu(V_1^\phi)'(x) + \frac{1}{2} \sigma^2(V_1^\phi)''(x) \leq 0, \text{ with equality on } x \in (\phi, 1/2).$$

(A.3.8)

To see this, note first that $V_1^\phi$ satisfies (A.3.2) on $(\phi, 1/2)$, so $V_1^\phi$ satisfies (A3.8) with equality on $(\phi, 1/2)$. Further, $F_2$ satisfies the differential equation (A.3.3), and $V_1^\phi(x) = 1 - G_2(x) = 1 - F_2(x)$ for all $x \in [1/2, 1 - \phi)$. Therefore, for all $x \in [1/2, 1 - \phi)$,

$$r V_1^\phi(x) = r - r F_2(x) = r + \mu(V_1^\phi)'(x) + \frac{1}{2} \sigma^2(V_1^\phi)''(x) \iff 0 > -r = -r V_1^\phi(x) + \mu(V_1^\phi)'(x) + \frac{1}{2} \sigma^2(V_1^\phi)''(x).$$

The function $V_1^\phi$ is twice differentiable, with a continuous first derivative and satisfies (A.3.8). By standard verification theorems (see, for instance, Theorem 3.17 in Shiryaev (2008)), it follows that $V_1^\phi$ is a solution to the optimal stopping problem (A.3.7). Therefore, $V_1^\phi$ also solves the optimal stopping problem (A.3.6).

**Lemma A.3.3** The outcome $(A^*, V_1^\phi, V_2^\phi)$ is the unique equilibrium.

**Proof:** Let $(A^*, V_1, V_2)$ be an equilibrium outcome different from $(A^*, V_1^\phi, V_2^\phi)$. By the discussion in the previous paragraphs, $(V_1, V_2)$ satisfy (A.3.4), with $E_1$ and $E_2$ satisfying
(A.3.5) and with \( 0 \leq E_2 \leq 1/(e^{(\alpha+\beta)\phi}e^{-\phi/2} + e^{\beta}) \). Since \((A^*, V_1, V_2)\) is different from \((A^*, V_1^\phi, V_2^\phi)\), then it must be that \( V_1' (1/2^-) \neq -V_2' (1/2^+) \). Therefore, it must be that either (i) \( V_1' (1/2^-) < -V_2' (1/2^+) \), or (ii) \( V_1' (1/2^-) > -V_2' (1/2^+) \).

Consider case (i). Let \( Q_1 (x) = \min \{0, E_1 (-e^{(\alpha+\beta)\phi}e^{-\alpha x} + e^{\beta x})\} \), so that \( V_1 (x) = Q_1 (x) \) for all \( x \in [0, 1/2] \). Since \( V_1 (1/2) = 1 - V_2 (1/2) \) and since \( Q_1' (1/2) = V_1' (1/2^-) < -V_2' (1/2^+) \), there exists \( \varepsilon > 0 \) such that \( 1 - V_2 (x) > Q_1 (x) \) for all \( x \in (1/2, 1/2 + \varepsilon] \). Consider the stopping time \( \tau^\varepsilon = \inf \{t : x_t \in [1/2 + \varepsilon, 1 - \phi]\} \), and note that \( \tau^\varepsilon \in T_1 \). Let \( \tilde{V}_1 (x) \) be given by

\[
\tilde{V}_1 (x) = E \left[ e^{-\tau^{\varepsilon \wedge \tau^\phi}} (1 - V_2 (x^{\tau^{\varepsilon \wedge \tau^\phi}})) \bigg| x_0 = x \right].
\]

One can show that \( \tilde{V}_1 (x) = F(-e^{(\alpha+\beta)\phi}e^{-\alpha x} + e^{\beta x}) \) for all \( x \in (\phi, 1/2 + \varepsilon) \), where \( F \) is a constant such that \( F(-e^{(\alpha+\beta)\phi}e^{-\alpha (1+\varepsilon)} + e^{\beta (1+\varepsilon)}) = 1 - V_2 (1/2 + \varepsilon) > Q_1 (1/2 + \varepsilon) = E_1 (-e^{(\alpha+\beta)\phi}e^{-\alpha (1/2+\varepsilon)} + e^{\beta (1/2+\varepsilon)}) \). This implies that \( F > E_1 \), which in turn implies that \( \tilde{V}_1 (x) > V_1 (x) \) for all \( x \in (\phi, 1/2 + \varepsilon) \). Therefore, \( V_1 (x) \) cannot be a solution to the optimal stopping problem (3.3), so \((A^*, V_1, V_2)\) cannot be an equilibrium outcome.

In case (ii), a symmetric argument establishes that there exists \( \xi > 0 \) such that

\[
\tilde{V}_2 (x) = E \left[ e^{-\tau^{\xi \wedge \tau^\phi}} (1 - V_1 (x^{\tau^{\xi \wedge \tau^\phi}})) \bigg| x_0 = x \right] > V_2 (x),
\]

for all \( x > 0 \), where \( \tau^{\xi} = \inf \{t : x_t \in S^{\phi} \cup [\phi, 1/2 - \xi]\} \). Hence, in this case \((A^*, V_1, V_2)\) cannot be an equilibrium outcome either.

**Proof of Theorem 3.1:** Follows immediately from Lemmas A.3.2 and A.3.3.
A.3.2 Proofs of Lemmas 3.2 and 3.3

Lemma A.3.4 Let \( U \) and \( \tilde{U} \) be two solutions to (A.3.1), with \( U (z) = \tilde{U} (z) \) and \( U' (y) > \tilde{U}' (y) \). Then, \( U' (x) > \tilde{U}' (x) \) for all \( x \). Therefore, \( U (x) > \tilde{U} (x) \) for all \( x > y \) and \( U (x) < \tilde{U} (x) \) for all \( x < y \).

Proof: I prove the claim for \( x > y \). The proof for \( x < y \) is symmetric and omitted. Suppose the claim in the Lemma is not true, and let \( y_1 > y \) be the smallest point with \( U' (y_1) = \tilde{U}' (y_1) \). Therefore, \( U' (x) > \tilde{U}' (x) \) for all \( x \in [z, y_1) \), so \( U (y_1) > \tilde{U} (y_1) \). Since \( U \) and \( \tilde{U} \) solve (A.3.1), then

\[
\tilde{U}'' (y_1) = 2 \left( r \tilde{U} (y_1) - \mu \tilde{U}' (y_1) \right) / \sigma^2 < 2 (r U (y_1) - \mu U' (y_1)) / \sigma^2 = U'' (y_1).
\]

But this implies that \( U' (y_1 - \varepsilon) < \tilde{U}' (y_1 - \varepsilon) \) for \( \varepsilon \) small, a contradiction.

Proof of Lemma 3.2: Note first that \( W^{\tilde{\phi}} (x) = 1 = W^{\phi} (x) \) for all \( x \in [0, \tilde{\phi}] \cup [1 - \tilde{\phi}, 1] \), and that \( W^{\tilde{\phi}} (x) < W^{\phi} (x) = 1 \) for all \( x \in (\tilde{\phi}, \phi] \cup [1 - \phi, 1 - \tilde{\phi}] \)-see equation (3.14). Thus, to prove the Lemma I need to show that \( W^{\tilde{\phi}} (x) < W^{\phi} (x) \) for all \( x \in (\phi, 1 - \phi) \). Since \( W^{\tilde{\phi}} \) and \( W^{\phi} \) are continuous, it must be that \( W^{\tilde{\phi}} (x) < W^{\phi} (x) \) for \( x \) close to \( \phi \) and for \( x \) close to \( 1 - \phi \). Suppose the claim in the Lemma is not true, so there exists \( y \in (\phi, 1 - \phi) \) with \( W^{\tilde{\phi}} (y) = W^{\phi} (y) \).

There are three possibilities: (i) \( (W^{\tilde{\phi}})' (y) > (W^{\phi})' (y) \), (ii) \( (W^{\tilde{\phi}})' (y) < (W^{\phi})' (y) \) and (iii) \( (W^{\tilde{\phi}})' (y) = (W^{\phi})' (y) \).

Cases (i) and (ii) cannot be. Indeed, by Lemma A.3.4 case (i) implies that \( W^{\tilde{\phi}} (x) > W^{\phi} (x) \) for all \( x > y \), while case (ii) implies \( W^{\tilde{\phi}} (x) > W^{\phi} (x) \) for all \( x < y \), a contradiction to the fact that \( W^{\tilde{\phi}} (x) < W^{\phi} (x) \) for \( x \) close to \( \phi \) and for \( x \) close to \( 1 - \phi \). Finally, case (iii) cannot be either, as it implies that \( W^{\tilde{\phi}} (x) = W^{\phi} (x) \) for all \( x \). Hence, it must be that \( W^{\tilde{\phi}} (x) < W^{\phi} (x) \) for all \( x \in (\phi, 1 - \phi) \).

Proof of Lemma 3.3: Note first that \( \{ \phi \in [0, 1/2] : w^{\phi} \geq 1 - c \} \) is non-empty, since \( w^{\phi} \to 1 > 1 - c \) as \( \phi \to 1/2 \); therefore, \( \overline{\phi} (c) \) is well-defined. Moreover, \( \overline{\phi} (c) \in (0, 1/2) \), since
$w^0 < 1 - c$ for all $c \in (0, \overline{c})$ and $w^\phi \to 1 > 1 - c$ as $\phi \to 1/2$. The fact that $w^\phi$ is continuous with respect to $\phi$ implies that $w^\phi(c) = 1 - c$. Finally, since $w^\phi$ is increasing in $\phi$, it follows that $\phi(c)$ is decreasing in $c$.

**A.3.3 Proof of Proposition 3.3**

**Proof of Proposition 3.3 (i):** Fix $c \in (0, \overline{c})$, and note that for $\phi > \overline{\phi}(c)$, $W^\phi(x) > 1 - c$ for all $x \in (\phi, 1 - \phi)$ (indeed, $W^\phi(x) \geq w^\phi > 1 - c$). Therefore, the outcome $(S^\phi, W^\phi_1, W^\phi_2)$ is the unique equilibrium whenever $\phi > \overline{\phi}(c)$.

Suppose next that $\phi < \overline{\phi}(c)$, so $w^\phi < 1 - c$. Since $W^\phi$ is continuous and strictly convex, there exists $a, b$ with $\phi < a < b < 1 - \phi$ such that $W^\phi(x) \leq 1 - c$ for all $x \in [a, b]$ (with strict inequality for all $x \in (a, b)$), and $W^\phi(x) > 1 - c$ for all $x \notin [0, a) \cup (b, 1]$. In this case, $(S^\phi, W^\phi_1, W^\phi_2)$ cannot be an equilibrium outcome. Indeed, if $(S^\phi, W^\phi_1, W^\phi_2)$ were an equilibrium, then for every $x \in (a, b)$ the proposing party $i$ would be better off offering a policy that gives party $j$ a payoff of $W^\phi_j$, since in this case party $i$ would obtain a payoff of $1 - c - W^\phi_j(x) > W^\phi_i(x)$. Therefore, for any equilibrium outcome $(A, V_1, V_2)$ it must be that $A \cap [a, b] \neq \emptyset$. On the other hand, any equilibrium outcome $(A, V_1, V_2)$ must be such that $(\phi, a) \cup (b, 1 - \phi) \not\subseteq A$. Indeed, if $x \in (\phi, a) \cup (b, 1 - \phi), x \in A$, then $V_1(x) + V_2(x) = 1 - c < W^\phi(x)$. But this cannot be, since $W^\phi(x)$ is a lower bound for the sum of the parties’ equilibrium payoffs.

**Lemma A.3.5** Suppose $\phi < \overline{\phi}(c)$ and let $(A, V_1, V_2)$ be an equilibrium outcome. Then, it must be that $A = S^\phi \cup [\underline{x}, \overline{x}]$ for some $a \leq \underline{x} < \overline{x} \leq b$.

**Proof:** From the discussion above, $A \cap [a, b] \neq \emptyset$. Let $\underline{x} = \min\{x \in A \cap [a, b]\}$ and $\overline{x} = \max\{x \in A \cap [a, b]\}$, so that $\underline{x} \leq \overline{x}$.

Consider first the case in which $\underline{x} < \overline{x}$, and suppose by contradiction that $A \cap [a, b] \neq [\underline{x}, \overline{x}]$. Since $A^c$ is open, there must exist $\underline{x} < y_1 < y_2 < \overline{x}$.

---

\footnote{Since $A$ is closed (because $A \subseteq A$), then $\min\{x \in A \cap [a, b]\} = \inf\{x \in A \cap [a, b]\}$ and $\max\{x \in A \cap [a, b]\} = \sup\{x \in A \cap [a, b]\}$.}
such that \((y_1, y_2) \notin A\). Let \(x \in (y_1, y_2)\), and note that

\[
V_1(x) + V_2(x) = E \left[ e^{-r \tau(A)} (1 - c) \middle| x_0 = x \right] < 1 - c,
\]

where the inequality follows from the fact that \(\tau(A) > 0\) whenever \(x_0 \in (y_1, y_2)\), and from the fact that one party has to pay the concession cost when they reach an agreement. But this contradicts the fact that \((A, V_1, V_2)\) is an equilibrium, since the party \(i\) proposing at \(x \in (y_1, y_2)\) is better off offering a policy which gives party \(j\) a total utility of \(V_j(x)\), obtaining a payoff \(1 - c - V_j(x) > V_i(x)\) for itself.

Finally, I prove that \(x < \overline{x}\). To see this, suppose by contradiction that \(x = \overline{x}\), and define \(V(x) = V_1(x) + V_2(x)\). Note that \(V(x) = E \left[ e^{-r \tau(A)} g(x_{\tau(A)}) \middle| x_0 = x \right]\), where \(g(x) = 1\) if \(x \notin (\phi, 1 - \phi)\) and \(g(x) = 1 - c\) if \(x \in (\phi, 1 - \phi)\). Since \((A, V_1, V_2)\) is an equilibrium, it must be that \(V(x) \geq 1 - c\) for all \(x \in (\phi, 1 - \phi)\); otherwise, if \(V(x) < 1 - c\) for some \(x \in (\phi, 1 - \phi)\) both parties would be better off by reaching an agreement at \(x\). By Lemma A.3.1, \(V\) solves (A.3.1) for all \(x \in (\phi, \overline{x}) \cup (\overline{x}, 1 - \phi)\), with boundary conditions \(V(\phi) = V(1 - \phi) = 1\) and \(V(\overline{x}) = 1 - c\). Thus,

\[
V(x) = \begin{cases} 
\eta_1 e^{-\alpha x} + \lambda_1 e^{\beta x} & \text{if } x \in (\phi, \overline{x}), \\
\eta_2 e^{-\alpha x} + \lambda_2 e^{\beta x} & \text{if } x \in (\overline{x}, 1 - \phi),
\end{cases}
\]

where \((\eta_i, \lambda_i)_{i=1,2}\) are constants such that the boundary conditions hold. The fact that \(V(x) \geq 1 - c\) for all \(x\) implies that \(V'(\overline{x}^-) \leq 0\) and \(V'(\overline{x}^+) \geq 0\), so \(V'(\overline{x}^-) \leq V'(\overline{x}^+)\). I now show that \(V'(\overline{x}^-) < V'(\overline{x}^+)\). Otherwise, \(V'(\overline{x}^-) = V'(\overline{x}^+) = 0\). But note that this implies that \(\eta_1 = \eta_2\) and \(\lambda_1 = \lambda_2\), since the conditions \(V(\overline{x}) = 1 - c\) and \(V'(\overline{x}) = 0\) would pin down both \((\eta_1, \lambda_1)\) and \((\eta_2, \lambda_2)\). Therefore, \(V(x) = \eta_1 e^{-\alpha x} + \lambda_1 e^{\beta x}\) for all \(x \in (\phi, 1 - \phi)\), with \(V(\phi) = W(1 - \phi) = 1\). But this implies that \(V(x) = W^\phi(x)\) for all \(x \in (\phi, 1 - \phi)\), with \(W^\phi(x) \geq 1 - c\), a contradiction to the assumption that \(\phi < \overline{\phi}(c)\). Hence, it must be that \(V'(\overline{x}^-) < V'(\overline{x}^+)\). 159
Since $V'(x^-) < V'(x^+)$, it follows that $\eta_1 e^{-\alpha x} + \lambda_1 e^{\beta x} < \eta_2 e^{-\alpha x} + \lambda_2 e^{\beta x}$ for all $x \in (x, 1 - \phi)$. In particular, $\eta_1 e^{-\alpha (1 - \phi)} + \lambda_1 e^{\beta (1 - \phi)} < 1$. Lemma A.3.4 then implies that $(W^\phi)'(\phi) > V'(\phi)$. A similar argument establishes that $(W^\phi)'(1 - \phi) < V'(1 - \phi)$. But, by Lemma A4, this implies that $W^\phi(x) > V(x) \geq 1 - c$ for all $x \in (\phi, 1 - \phi)$, which again contradictions the assumption that $\phi < \overline{\phi}(c)$. Therefore, it must be that $x < x$.

By Lemma A.3.4, any equilibrium outcome $(A, V_1, V_2)$ must have $A = S^\phi \cup [x, \overline{x}]$. There are three cases to consider: (a) $x \leq 1/2 \leq \overline{x}$ (with at least one inequality, since $x < \overline{x}$), (b) $x < \overline{x} < 1/2$ and (c) $1/2 < x < \overline{x}$.

Consider first (b). Note that

$$V_1(x) = E[e^{-r_1(A)} \left( \eta \left( x_{\tau(A)} \right) - c_1^\phi \left( x_{\tau(A)} \right) \right) \Bigg| x_0 = x] \text{ for all } x \notin A,$$

where, for all $x \in A$, $\eta(x)$ is the policy that parties agree upon. On the other hand, by equation (3.10)

$$V_1(x) = E \left[ e^{-r_1(A)} x_{\tau(A)} \right] \left| x_0 = x \right] \text{ for all } x \in (\phi, 1/2).$$

Applying the Law of Iterated Expectations, it follows that

$$V_1(x) = E \left[ e^{-r_1(A)} \eta(x_{\tau(A)}) \left| x_0 = x \right] \right].$$

Finally, since $(A, \eta) \in \mathcal{O}^\phi$, then $\eta(x) = 1$ for all $x \geq 1 - \phi$, and $\eta(x) = 0$ for all $x \leq \phi$. Using this in the equation above yields that $V_1(x) = W_1^\phi(x)$.

On the other hand,

$$V_2(x) = E \left[ e^{-r_1(A)} (1 - V_1(x_{\tau(A)}) - c_1^\phi(x_{\tau(A)})) \right] .$$
Therefore, $V_2(x) = 1 - V_1(x) - c$ for all $x \in [\underline{x}, \overline{x}]$; for all $x \in (\phi, \underline{x}) \cup (\overline{x}, 1 - \phi)$, $V_2$ solves (A.3.1), with boundary conditions $V_2(\phi) = 1$, $V_2(\underline{x}) = 1 - V_1(\underline{x}) - c$, $V_2(\overline{x}) = 1 - V_1(\overline{x}) - c$ and $V_2(1 - \phi) = 0$. Since $V_1(x) = W_1^\phi(x)$, these conditions completely characterize $V_2(x)$.

Consider next (c). This case is symmetric to (b), so by similar arguments $V_2(x) = W_2^\phi(x)$ for all $x$. On the other hand, $V_1(x) = 1 - V_2(x) - c$ for all $x \in [\underline{x}, \overline{x}]$; for all $x \in (\phi, \underline{x}) \cup (\overline{x}, 1 - \phi)$, $V_1$ solves (A.3.1), with boundary conditions $V_1(\phi) = 0$, $V_1(\underline{x}) = 1 - V_2(\underline{x}) - c$, $V_1(\overline{x}) = 1 - V_2(\overline{x}) - c$ and $V_1(1 - \phi) = 1$. Since $V_2(x) = W_2^\phi(x)$, these conditions completely characterize $V_1(x)$.

Finally, consider (a). Equation (3.10), together with the fact that $V_1(1/2) + V_2(1/2) = 1 - c$, implies that

\[
V_1(x) = E_x \left[ e^{-r^\phi \land \tau(1/2)} \begin{pmatrix}
1 - V_2(x_{\tau^\phi \land \tau(1/2)}) \\
-c(\phi(x_{\tau^\phi \land \tau(1/2)})
\end{pmatrix} \right] \quad \text{if } x \in (\phi, 1/2),
\]

\[
V_2(x) = E_x \left[ e^{-r^\phi \land \tau(1/2)} \begin{pmatrix}
1 - V_1(x_{\tau^\phi \land \tau(1/2)}) \\
-c(\phi(x_{\tau^\phi \land \tau(1/2)})
\end{pmatrix} \right] \quad \text{if } x \in (1/2, 1 - \phi).
\]

Thus, $V_1$ solves (A.3.1) on $(\phi, 1/2)$ and $V_2$ solves (A.3.1) on $(1/2, 1 - \phi)$, with boundary conditions $V_1(\phi) = 0$, $V_1(1/2) + V_2(1/2) = 1 - c$ and $V_2(1 - \phi) = 0$. On the other hand, $V_1(x) + V_2(x) = 1 - c$ for all $x \in [\underline{x}, \overline{x}]$. Finally, one can show that $V_1$ solves (A.3.1) on $(\overline{x}, 1 - \phi)$, with boundary conditions $V_1(\overline{x}) = 1 - V_2(\overline{x}) - c$ and $V_1(1 - \phi) = 1$. Similarly, $V_2$ solves (A1) on $(\phi, \underline{x})$, with boundary conditions $V_2(\phi) = 1 - V_1(\phi) - c$ and $V_2(1 - \overline{x}) = 1$.

**Lemma A.3.6** Suppose $\phi < \overline{\phi}(c)$ and let $(A, V_1, V_2)$ be an equilibrium outcome. Then, $V_i \in C^1$ in $(\phi, 1 - \phi)$ for $i = 1, 2$.

**Proof:** By Lemma A.3.4, it must be that $A \cap [a, b] = [\underline{x}, \overline{x}]$. There are three cases to consider:

(a) $\underline{x} \leq 1/2 \leq \overline{x}$; (b) $\underline{x} < \overline{x} < 1/2$ and (c) $1/2 < \underline{x} < \overline{x}$. Consider case (a). By equation (3.10) in the main text, $V_1 \in C^1$ for all $x \in (\phi, 1/2)$ and $V_2 \in C^1$ for all $x \in (1/2, 1 - \phi)$. Arguments similar to those in Lemma A.3.3 imply that, in this case, any equilibrium outcome
must satisfy the smooth pasting condition \( V'_i(1/2^-) = -V'_2(1/2^+) \). Indeed, if this condition did not hold, then either \( V_1 \) or \( V_2 \) would have a convex kink at \( x = 1/2 \); and the party whose value function has the convex kink would find it optimal to delay when \( x \) is around \( 1/2 \).

Note further that \( V_1(x) + V_2(x) = 1 - c \) for all \( x \in [\underline{x}, \overline{x}] \). Therefore, it follows that \( V_1 \in C^1 \) for all \( x \in (\phi, \overline{x}) \) and \( V_2 \in C^1 \) for all \( x \in (\underline{x}, 1 - \phi) \). Next, I show that \( V_2 \) is indeed \( C^1 \) on \((\phi, 1 - \phi)\) (not only on \((\underline{x}, 1 - \phi)\)). The proof that \( V_1 \in C^1 \) on \((\phi, 1 - \phi)\) is symmetric and omitted. Note that when \( x \in (\phi, \overline{x}) \), parties delay an agreement until \( \tau(A) \). Therefore, \( V_2(x) = E[e^{-\tau(A)}u_2(x_{\tau(A)}) | x_0 = x] \) for all \( x \in (\phi, \overline{x}) \), since either \( x_{\tau(A)} = \phi \) or \( x_{\tau(A)} = \overline{x} \) (so in either case party 2 doesn’t pay the concession cost). Moreover, at \( \overline{x} \) parties reach an agreement, so \( V_2(x) = 1 - V_1(x) - c \). Therefore, for all \( x \in (\phi, \overline{x}) \), \( V_2 \) solves \((A.3.1)\), with boundary conditions \( V_2(\phi) = 1 \) and \( V_2(\overline{x}) = 1 - V_1(\overline{x}) - c \); so \( V_2(x) \in C^1 \) for all \( x \in (\phi, \overline{x}) \).

The last step is to show that \( V'_2(\underline{x}^+) = V'_2(\overline{x}^-) = -V'_1(\overline{x}) \). Again, the argument that this condition has to hold follows from the fact that reaching an agreement at \( \overline{x} \) must be optimal for party 2: if this condition did not hold, then party 2 would be better off either by delaying until some \( x' > \underline{x} \) (if \( V'_2(\underline{x}) < -V'_1(\underline{x}) \)), or by reaching an agreement earlier at some \( x'' < \underline{x} \) (if \( V'_2(\underline{x}) > -V'_1(\underline{x}) \)).

Next I present the proof for case (b). The proof for case (c) is symmetric and omitted. In case (b), there is agreement only when party 2 is making offers. In this case, it must be that \( V_1(x) = E[e^{-\tau(A)}1_{x_{\tau(A)} \geq 1 - \phi} | x_0 = x] \), since there is always delay while party 1 is making offers. Thus, in this case \( V_1 \in C^1 \) for all \( x \in (\phi, 1 - \phi) \). Also, \( V_2 \in C^1 \) for all \( x \in (\phi, \overline{x}) \cup (\overline{x}, 1 - \phi) \), since \( V_2 \) satisfies \((A.3.1)\) in this region. On the other hand, \( V_2(x) = 1 - V_1(x) - c \) for all \( x \in (\underline{x}, \overline{x}) \), so \( V_2 \in C^1 \) for all \( x \in (\underline{x}, \overline{x}) \). Therefore, to show that \( V_2 \in C^1 \) for all \( x \in (\phi, 1 - \phi) \), it suffices to show that the following smooth pasting conditions hold:

\[
\begin{align*}
V'_2(\underline{x}^-) &= V'_2(\underline{x}^+) = -V'_2(\overline{x}^-), \\
V'_2(\overline{x}^+) &= V'_2(\overline{x}^-) = -V'_1(\overline{x}).
\end{align*}
\]
But this condition must hold in equilibrium, since otherwise party would have a strict incentive to change its behavior (either delay an agreement or reach an agreement earlier).

**Lemma A.3.7** Let $U$ be a solution to (A.3.1), with $U'(y) < 0$. Then, $U$ is either strictly convex for all $x$ or strictly concave for all $x$. In particular, if $U'(	ilde{y}) = 0$ for some $\tilde{y} > y$, then $U$ is strictly convex.

**Proof:** Since $U$ solves (A.3.1), then $U(x) = \eta e^{-\alpha x} + \lambda e^{\beta x}$ for some constants $\eta$, $\lambda$. The result is obvious if either $\eta = 0$ or $\lambda = 0$. Suppose next that $\eta, \lambda \neq 0$. Since $U'(y) < 0$, it follows that $-\alpha \eta e^{-\alpha y} + \beta \lambda e^{\beta y} < 0$, or $e^{(\alpha+\beta)y} < \frac{\eta}{\beta \lambda}$. Therefore, it must be that sign($\eta$) = sign($\lambda$), as $e^{(\alpha+\beta)y}, \frac{\eta}{\beta \lambda} > 0$. If $\eta$ and $\lambda$ are positive, then $U''(x) = \alpha^2 \eta e^{-\alpha x} + \beta^2 \lambda e^{\beta x} > 0$; if $\eta$ and $\lambda$ are negative, then $U''(x) = \alpha^2 \eta e^{-\alpha x} + \beta^2 \lambda e^{\beta x} < 0$.

**Lemma A.3.8** For any $\phi < \bar{\phi}(c)$, there exists a unique equilibrium outcome.

**Proof:** Let $(A, V_1, V_2)$ be an equilibrium outcome. By Lemma A.3.5, it must be that $A = S^\phi \cup [\underline{x}, \bar{x}]$, for some $\underline{x} < \bar{x}$. By Lemma A.3.6, $V_i \in C^1$ on $[\phi, 1 - \phi]$ for $i = 1, 2$. Let $V(x) = V_1(x) + V_2(x)$, so that $V \in C^1$ on $[\phi, 1 - \phi]$. Note that $V(x) = 1 - c$ for all $x \in [\underline{x}, \bar{x}]$ and $V(x) = 1$ for all $x \in S^\phi$. Moreover, $V(x) = E[e^{-r_T(A)} f(x_{r(T)})] | x_0 = x]$ for all $x \in (\phi, \underline{x}) \cup (\bar{x}, 1 - \phi)$, where $f(x) = 1$ if $x \notin (\phi, 1 - \phi)$ and $f(x) = 1 - c$ if $x \in (\phi, 1 - \phi)$. Therefore, by Lemma A.3.1 $V$ solves (A.3.1) for $x \in (\phi, \underline{x})$ and $x \in (\bar{x}, 1 - \phi)$, with boundary conditions $V(\phi) = V(1 - \phi) = 1$ and $V(\underline{x}) = V(\bar{x}) = 1 - c$. Finally, since $W \in C^1$ it must be that $V'(\underline{x}) = V'(\bar{x}) = 0$.

To show that the equilibrium is unique, I need to show that there exists unique $\underline{x}$ and $\bar{x}$ such that these conditions hold. I show that there is a unique $\underline{x}$ such that $V$ solves (A.3.1) with $V(\phi) = 1$, $V(\underline{x}) = 1 - c$ and $V'(\underline{x}) = 0$. The proof that $\bar{x}$ is unique is symmetric and omitted. Since $E[e^{-r_T(A)} f(x_{r(T)})] | x_0 = x] < 1$ for all $x \in (\phi, 1 - \phi)$, it must be that $V'(\phi) < 1$. Consider solutions $U$ to (A.3.1) with $U(\phi) = 1$, $U'(\phi) < 0$ and $U'(y) = 0$ for some $y > \phi$. By Lemma A.3.7, such $U$ must be strictly convex. Moreover, since solutions to
(A.3.1) are continuous in initial conditions, then the solutions I’m considering are continuous in $U'(\phi)$.

Note first that if $U'(\phi)$ is large enough, then $U(\phi)$ will remain above $1-c$ for all $x > \phi$. Indeed, $U'(\phi) = 0$ implies that $U(x) > 1$ for all $x > \phi$, since $U$ is convex. On the other hand, if $U'$ is small enough, then $U$ will cross $1-c$ at some $y > \phi$ (see solutions I, II and III in Figure A.3.1). Let $U'(\phi)$ be the largest slope such that $U$ reaches $1-c$ for some $y > \phi$. Since solutions with a higher slope never reach $1-c$, it follows that $U(x) \geq 1-c$ for all $x > \phi$; hence, $U$ is tangent to $1-c$ at $y$; that is, $U'(y) = 0$ (solution II in Figure A.3.1).

Note that $U$ is the unique solution to (A.3.1) that satisfies the three conditions $U(\phi) = 1$, $U(y) = 1-c$ and $U'(y) = 0$. Therefore, if $(A, V_1^\phi, V_2^\phi)$ with $A = S^\phi \cup [x, \pi]$ is an equilibrium outcome, it must be that $x = y$, so $U(x) = V(x)$ for all $x \in (\phi, x)$.

**Proof of Proposition 3.3 (ii):** Follows from Lemmas A.3.5, A.3.6 and A.3.8.
Bibliography


