MEASUREMENT AND CHARACTERIZATION OF FAST ELECTRON CREATION, TRAPPING, AND ACCELERATION IN AN RF-COUPLED HIGH-MIRROR-RATIO MAGNETIC MIRROR

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Abstract

The PFRC-II in seed plasma mode is a tandem magnetic mirror. In one end cell is a low-power double-saddle antenna which produces cold (5 eV), tenuous (10^{11}/cm^3) plasma. Using x-ray pulse-height detectors to probe a previously unmeasured energy range of electrons, I measure a component with temperatures up to 3 keV. I characterize their life cycle, including a Fermi-Ulam-like acceleration process which allows them to attain energies in excess of 30 keV.

The fast electrons are born at 300 - 600 eV temperature in one end via secondary electron emission through an RF sheath. These electrons consist of < 1% of the plasma density, yet receive a large portion of the power. The phenomenon pushes models of a similar system, materials processing reactors, into lower-pressure and more-magnetized regimes, with implications on power balance and surface charging.

The electrons enter the center cell in the loss cone. There, even though the commonly used adiabatic parameter is small, \( \rho_e \nabla B / B \ll 1 \), they accumulate and persist for hundreds of transits due to the non-adiabaticity of magnetic moment. The same dynamics also lead to de-trapping in magnetic mirror-based fusion reactors.

Under low-pressure conditions, \( \sim 10\% \) of these electrons are accelerated still further, up to 3 keV temperature, some electrons above 30 keV, by a form of Fermi-Ulam acceleration. I measure a voltage oscillation consistent with two-stream instability caused by electrons from the last end cell re-entering as a beam. Non-adiabaticity of magnetic moment is essential to destroy resonances between mirror transit time and oscillation period, destroying barriers in phase space.
I compare the proposed mechanisms to approximate models. The proposed mechanism for creation is compared to an approximate kinetic model which includes confinement by a plasma-terminating plate with a fluctuating potential. The proposed mechanism for accumulation in the center cell is compared to the nonlinear-resonance-overlap model of Chirikov, and ground-truthed with a Boris algorithm simulation. The proposed mechanism for their acceleration is compared to an energy diffusion model. Their mechanism for Fermi-Ulam voltage fluctuation is compared to a nonlinear saturation model. The mechanism for resonance breaking is compared to a 2D numerical map model.
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# Contents

Abstract iii  
Acknowledgements v  
List of Figures xii  
Abbreviations xx  
Symbols xxii  

1 Introduction  
   1.1 The chapters in this dissertation . . . . . . . . . . . . . . . . 3  
   1.2 Personal note . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8  

2 Reconstruction of EEDFs from x-ray pulse-height detectors  
   2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9  
   2.2 Physical principle of x-ray pulse-height detectors and SDDs . . . 11  
   2.3 Lines of sight, view factors, and mounting to the PFRC-II apparatus . 15  
   2.4 Physical processes of plasma x-ray emission . . . . . . . . . . . . . . . . 20  
   2.4.1 Bremsstrahlung . . . . . . . . . . . . . . . . . . . . . . . . . . 21  
   2.4.2 Spectral Lines . . . . . . . . . . . . . . . . . . . . . . . . . . . 26  
   2.5 Spectral Inversion . . . . . . . . . . . . . . . . . . . . . . . . . . . 28  
   2.6 Derivation and justification of the cost function . . . . . . . . . . . . . 29  
   2.7 The numerical algorithm implemented in MATLAB . . . . . . . . . . . . 34  
   2.8 Current state of the art in spectral inversion . . . . . . . . . . . . . . . . 37  
   2.8.1 Piana and RHESSI . . . . . . . . . . . . . . . . . . . . . . . . . 37  
   2.8.2 DeGaSum . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38  
   2.9 Detector response function . . . . . . . . . . . . . . . . . . . . . . . 39  
   2.9.1 Transmission . . . . . . . . . . . . . . . . . . . . . . . . . . . 39  
   2.9.2 Resolution . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40  
   2.9.3 Bremsstrahlung . . . . . . . . . . . . . . . . . . . . . . . . . . 42  
   2.10 Calibration with electron beam . . . . . . . . . . . . . . . . . . . . . . 43  
   2.10.1 Energy calibration . . . . . . . . . . . . . . . . . . . . . . . . . 45  
   2.10.2 Transmission calibration . . . . . . . . . . . . . . . . . . . . . . . 46  
   2.11 Spectral line subtraction . . . . . . . . . . . . . . . . . . . . . . . . 48  
   2.12 Example SDD data collection procedure . . . . . . . . . . . . . . . . . 50
2.13 Comparison with Piana et. al. ........................................... 52
2.14 Example use case .......................................................... 55
2.15 Conclusion ................................................................. 58

3 Creation of high-energy electrons: Secondary emission by ion bombardment of the plasma source ................................................................. 59
3.1 Capacitively coupled and other low-pressure, low-temperature RF plasma reactors ................................................................. 61
  3.1.1 Plasma reactors for semiconductor processing ......................... 61
  3.1.2 Capacitive coupling introduction ........................................ 62
  3.1.3 Plasma reactor sheath heating experiments and models ............... 63
  3.1.4 Experimental results from similar experiments ......................... 65
  3.1.5 Novelty of phenomenon: 10x higher energy, lower pressure, magnetic field ................................................................. 66
3.2 Apparatus ................................................................. 68
3.3 Experimental data from plasma source ........................................ 70
  3.3.1 EEDF ................................................................. 70
  3.3.2 Glowing Langmuir probe ............................................... 73
  3.3.3 Dependence on parameters ............................................... 74
    3.3.3.1 Power .......................................................... 74
    3.3.3.2 Radial profile ................................................ 78
    3.3.3.3 Magnetic field ................................................. 80
    3.3.3.4 Gas species .................................................. 81
    3.3.3.5 RF Frequency ............................................... 83
  3.3.4 Waves in the SEC ..................................................... 85
3.4 Mechanism for the creation of fast electrons ............................... 86
  3.4.1 SEE from the self-biased stainless steel backplate .................... 86
  3.4.2 Analytic model ........................................................ 90
    3.4.2.1 Ion flux ...................................................... 91
    3.4.2.2 SE flux at one instant ....................................... 91
    3.4.2.3 Time-averaged SE flux ...................................... 92
    3.4.2.4 Single-bounce EEDF ......................................... 92
    3.4.2.5 Multiple-bounce EEDF ...................................... 94
    3.4.2.6 Density and Temperature ..................................... 96
    3.4.2.7 Electron-impact secondary electron emission ................... 97
  3.4.3 Application to the SEC of the PFRC-II ................................ 98
3.5 Conclusion ................................................................. 101

4 SEE from complex surface geometry ............................................ 103
4.1 Background ........................................................................ 104
4.2 Monte-Carlo tool implemented in MATLAB .................................. 106
4.3 Weighted view-factor model for SEY from geometry ....................... 110
  4.3.1 Mathematical formulation .............................................. 111
  4.3.2 Constraints on SEY from geometry and suggestions for optimization ................................................................. 115
### Contents

| 4.4  | Monte-Carlo results and analytic theory for surfaces | 116 |
| 4.4.1 | Velvet | 116 |
| 4.4.2 | Foam | 123 |
| 4.4.3 | Feathers | 130 |
| 4.5  | Conclusion | 132 |

| 5  | Trapping of high energy electrons: Magnetic moment non-adiabaticity in a magnetic mirror | 134 |
| 5.1  | Relevance: Magnetic mirrors and constant $\mu$ | 136 |
| 5.1.1 | Background: Magnetic mirror machines | 136 |
| 5.1.2 | The ubiquity of constant magnetic moment in foundational, popular, and productive models of plasma behavior | 138 |
| 5.1.3 | Early experimental and numerical evidence for non-adiabicity in mirror machines | 140 |
| 5.1.4 | Accurate models for the size of a single change in $\mu$ | 142 |
| 5.1.5 | Chirikov and the modern understanding of quasiadiabatic systems | 144 |
| 5.2  | PFRC-II experimental results | 146 |
| 5.2.1 | Apparatus | 146 |
| 5.2.2 | Steady-state density versus expected from transit time | 148 |
| 5.2.3 | Pulsed operation decay time | 151 |
| 5.3  | Modeling of magnetic moment non-adiabaticity | 153 |
| 5.3.1 | Single-particle numerical integration | 153 |
| 5.3.1.1 | Example trajectories | 154 |
| 5.3.1.2 | Ensembles | 155 |
| 5.3.2 | Analytic formulae for magnetic moment change | 156 |
| 5.3.2.1 | Single transit $\mu$ change | 156 |
| 5.3.2.2 | Chaotic map and phase-space separatrices | 163 |
| 5.3.2.3 | Expected Density Increase of Passing Particles | 170 |
| 5.3.3 | Expected Density Increase Applied to the PFRC-II | 171 |
| 5.4  | Summary and conclusion | 172 |

| 6  | Acceleration of high energy electrons: Fermi acceleration through spontaneous electrostatic oscillations at mirror nozzles | 174 |
| 6.1  | Background | 176 |
| 6.1.1 | Fermi Acceleration | 176 |
| 6.1.2 | Breaking the periodic forcing of electrostatic waves | 177 |
| 6.1.3 | Electrostatic wave heating in a magnetic mirror | 178 |
| 6.2  | Experimental results from central mirror chamber at low pressure | 179 |
| 6.2.1 | Apparatus | 179 |
| 6.2.2 | Characteristic parameters of the accelerated CC electrons | 183 |
| 6.2.3 | Typical EEDF | 186 |
| 6.2.4 | Parameter scans | 187 |
| 6.2.4.1 | CC pressure | 187 |
| 6.2.4.2 | RF power | 189 |
| 6.2.4.3 | Nozzle current | 190 |
| 6.2.5  | Radial profile  | 191 |
| 6.2.6  | Ramp-up and decay of EEDF | 193 |
| 6.2.6.1 | Decay time of each energy | 193 |
| 6.2.6.2 | Rise time | 196 |
| 6.2.7  | Magnetic oscillation probe at nozzle | 198 |
| 6.2.8  | Electrostatic oscillation probe at nozzle | 199 |
| 6.2.8.1 | With FEC pressure | 203 |
| 6.2.9  | Conditions in FEC | 204 |
| 6.3    | Model of acceleration | 208 |
| 6.3.1  | Origin of the oscillation | 209 |
| 6.3.2  | Energy balance | 212 |
| 6.3.3  | Non-phase-correlated energy diffusion model | 215 |
| 6.3.3.1 | Differences between mirror-bounce and rigid-wall bounce | 215 |
| 6.3.3.2 | EEDF expected from this process | 218 |
| 6.3.4  | Phase-correlated resonances: Fermi-Ulam map | 222 |
| 6.3.5  | Resonance-destroying effects | 225 |
| 6.3.5.1 | Turbulence | 225 |
| 6.3.5.2 | Close approaches to nozzles | 225 |
| 6.3.5.3 | Non-adiabaticity of magnetic moment | 226 |
| 6.4    | Conclusion | 228 |

| 7     | Conclusion | 230 |
| 7.1   | Very brief summary | 230 |
| 7.2   | Future work | 231 |
| 7.2.1 | Mirror experiments | 231 |
| 7.2.2 | PFRC-II | 232 |

| A      | Electron Equations of Motion in B-Following Coordinates | 234 |
| B      | Visualizations of the non-adiabaticity of magnetic moment in various magnetic fields | 238 |

| Bibliography | 246 |
List of Figures

1.1 Schematic representation of quantities of relevance to the phenomenon observed in the PFRC-II: a) The field-line map. b) The strength of the axial magnetic field. c) Representative change in the magnetic moment $\partial_t \mu$ of a fast electron. d) Plausible electrostatic potential along the axis of the PFRC-II. Red $\times$ signs represent measured points; other locations are speculation. Far left: RF self-bias of the plasma-terminating cup. Far right: The magnetic nozzle at $z \approx 44$ cm decreases the space potential many hundreds of volts. At $z \approx 33$ cm: Two-stream instability. 4

2.1 Schematic of the relevant dimensions of the detector/aperture/target system 15

2.2 Schematic of the relevant dimensions of the detector/baffle/aperture system 20

2.3 Example Bremsstrahlung x-ray spectral count rates produced by monoenergetic EEDFs, as predicted by Equation 2.11. X-ray spectral count rate is zero when $E_x > E_e$, as an electron cannot produce an x-ray with more energy than itself. Y-axis is in arbitrary units. 25

2.4 The gas-target x-ray tube used for calibration of the detector 43

2.5 Transmission Calibration for SDD1. In blue, the XEDF as directly reported by the SDD. In green, the XEDF expected from a monoenergetic EEDF assuming the published transmission efficiencies. In red, assuming the transmission efficiency produced by the fit parameters given in the text. Sudden jumps occur because of the K-edge energies of the elements in the window: Aluminum exhibits a jump in x-ray absorptivity at 1559 eV and Silicon exhibits a jump in x-ray absorptivity at 1839 eV. 47

2.6 Illustration of two Argon spectral lines, their effect on derived EEDF, and two potential methods of correcting for them. a) shows the effect of an un-corrected spectral line. This spectrum cannot be the result of Bremsstrahlung processes, so the fit fails and the resultant EEDF has large non-physical features. b) shows the method of correction used in this paper: the elements of $i$, the discretized energy vector, which correspond to the energy range obscured by the spectral line have been removed from the sum in Equation 2.18. c) shows the method of manually subtracting two Gaussian functions from the observed x-ray spectrum until the result appears subjectively correct. 48
2.7 An example comparison of Poisson-regularized Inversion and the Piana algorithms for recovering an EEDF. a) The true, supposed EEDF and the synthetic x-ray data generated from it. b) Computed EEDFs compared to the true EEDF. c) Discrepancy factor between each computed EEDF and the true value. ..........53

2.8 A typical XEDF and EEDF from the CC in tandem mirror mode. The red line is the measured XEDF, directly taken from the SDD. The black line and associated dotted lines are the derived EEDF and uncertainty, corresponding to 1σ. The blue dashed line is this EEDF re-transformed into the expected measured XEDF for reasons of comparison. The EEDF is mostly exponential with an e-folding energy of 340 eV. The large uncertainty of the region around 500 eV is caused by spectral lines obscuring Bremsstrahlung spectrum, excised as in Figure 2.6b. .......57

2.9 XEDF and EEDF from the CC in tandem mirror mode with a high pressure in the far end cell. The red line is the measured XEDF, directly taken from the SDD. The black line and associated dotted lines are the derived EEDF and uncertainty. The blue dashed line is this EEDF re-transformed into the expected measured XEDF for reasons of comparison. The EEDF is split into two domains: below 1800 eV it appears flat. Above 1800 eV it appears exponential with e-folding energy 220 eV. .....57

3.1 The RF self-bias effect. While the electrode spends most of its RF cycle at a negative bias, collecting ions, the small time it spends at a positive bias collects the same number of electrons due to their larger mobility. .....62

3.2 Schematic of the PFRC-II experiment run in seed plasma mode, with emphasis on the SEC. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The |B| profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate. .....................69

3.3 Example inverted Bremsstrahlung spectrum from the SEC during PFRC-II seed plasma operation. The parameters of the Maxwellian fit are \(n_e = 6.18 \times 10^8 / \text{cm}^3, T_e = 467 \text{eV}\). Recall that this is a line-averaged density over the plasma diameter. The conditions and features are discussed in the text. .................................71

3.4 Maxwellian fit temperature and extrapolated density, as well as observed density above 600 eV, by spectral inversion of the Bremsstrahlung spectrum, during a scan in net RF power. Because the EEDF is not a Maxwellian, I have chosen not to include error bars, as the density and temperature depends on which energy region is fit. Examples of fit quality can be seen in Figure 3.3. The directly observed density uncertainty is a few percent. ..........................75

3.5 Antenna and backplate voltages during a scan in net RF power. The quantity “\(\sim \sqrt{P}\)” is proportional to \(\sqrt{P}\) and has a coefficient which was fit to the data. .................................77

3.6 Bulk density, as determined by ion saturation current, during a scan in net RF power. The red trend line is power law 0.71, \(n \propto P^{0.71}\) .......77
3.7 Photograph from the Far End Cell (FEC), opposite the SEC, of the plasma terminating on a sapphire plate. At left: detail of the visibly hollow profile of light produced. At right: From an angle, including the nozzle through which the Center Cell (CC) of the PFRC-II is visible. The sapphire plate is partially obscured by a fiber optic mount.

3.8 Maxwellian fit densities from the SEC, where available, and CC. We generally observe CC Maxwellian fit density to be proportional to SEC Maxwellian fit density.

3.9 Maxwellian fit parameters from the SEC Bremsstrahlung while changing nozzle current and therefore mirror ratio.

3.10 Raw x-ray spectra from the SEC using different fill gases. The energies called out are the K-α energies of those elements.

3.11 Raw x-ray spectra and Maxwellian fit parameters from the SEC using different seed RF frequencies. Spectra were recorded months apart in some cases, although the parameters were similar. Each run is described in the table. 7 MHz used a higher pressure and lower power. 27 MHz spectrum used a different mount than the others. Because the EEDF is not a Maxwellian, I have chosen not to include error bars, as the density and temperature depends on which energy region is fit. Examples of fit quality can be seen in Figure 3.3.

3.12 Correlation between backplate floating potential, filtered to remove 19 MHz, and the electron saturation current at the SEC probe, 30 cm away. This analysis enhances the periodic part of the signal, which appears at 22.7 kHz. On the right, the raw signals are plotted.

3.13 Detail of plasma source.

3.14 Schematic of SEC, including plasma source, B-field, and B-coils.

3.15 On-axis magnetic field profile in SEC. The specific currents which produce this profile are $I_{L2} = 135A$, $I_N = 380A$.

3.16 Example EEDFs produced in this model. a) EEDF exiting the sheath b) EEDF after homogenizing c) $N$ quantity, number of transits d) EEDF including multiple transits. In the arbitrary energy units used, $E_0 = 1100$ and $E_1 = 800$. Figure (d) shows a long approximately exponential region, allowing an effective temperature to be defined.

4.1 Flowchart of the Monte Carlo simulation algorithm.

4.2 Schematics of the velvet surface: the whisker geometrical quantities radius, $r$, height, $h$, and spacing, $(2s)^2 \equiv n^{-1}$. Also shown are electron velocity polar angle $\theta$, and velocity azimuthal angle, $\phi$. Numerical calculations include three contributions to secondary electron emission from velvet: electrons emitted by the side, top, and bottom surface of the whiskers.
4.3 SEY reduction from the case of normal incidence on a flat surface. SEY reduction is given as a function of incident angle, \( \theta \), for different values of whisker aspect ratio \( A \), and packing density \( D \). Figure (a) shows SEY for 4 different \( D \) values and the same \( A = 1000 \). Figure (b) shows SEY for 3 different values of \( A \) and the same \( D = 4\% \). Solid lines show the result of an analytic approximation. Points with error bars are the result of these Monte Carlo simulations.

4.4 Top: \( f(u, \theta) \) vs \( \theta \) for several \( u \) (curves) that determines the net SEY in equation \( \gamma_{eff} = \gamma_{flat}\{D + (1 - D)f(u, \theta)\} \). Bottom: Relative contribution to the SEY of the sides of the whiskers. Pointed out in both are the points at which the quantity \( u \tan \theta \) crosses unity.

4.5 Rendering of an example of the foam surface used in this section. This foam had 80 fibers, aspect ratio \( A = 10 \), volume fill fraction \( D = 4.3\% \), and foam parameter \( \bar{u} = 2.2 \). The absolute length scale is not defined for our analytic model.

4.6 Results of analytic theory compared to full numerical Monte Carlo model.

4.7 Results of analytic theory. Total SEY is \( \gamma[D + (1 - D)f_{geom}] \).

4.8 a) Drawing of the feathered, “whisker on a whisker” geometry and schematic representations of the suppression mechanism. This geometry corresponds to a shorter, fatter \((D = 16\%, A = 10\) rather than \(D = 4\%, \ A = 80\)) geometry than the one calculated. At right are shown the effects that lead to SEY reduction: b) increase in effective capture area. c) Normal and shallow incident primary electrons on a velvet geometry. d) Normal and shallow incident primary electrons on a feathered geometry. Red arrows correspond to primary electron trajectories. Yellow arrows correspond to example secondary electron trajectories.

4.9 Solid lines show the result of the numerical Monte Carlo calculation: reduction in SEY of the considered \( u = 2 \) graphite velvet either without another recursive velvet grown onto the whisker sides (“Primary whiskers, \( u = 2\)”), or with this smaller velvet (“Secondary whiskers, \( u = 2\)”). Also shown (2 dashed lines) are the result of the analytic model for velvet with \( u = 4 \) and \( u = \infty \) for velvets with \( D = 4\% \). Also shown (last dashed line) is the result of the analytic model for \( u = 4, D = 4\% \), but with the emission from the sides of the whisker reduced by half.

5.1 Schematic of the PFRC-II experiment run in seed plasma mode. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The \(|\vec{B}|\) profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate.

5.2 Raw x-ray spectra and Maxwellian fits for concurrent spectra in the SEC and CC. The CC spectrum was recorded with SDD1, which has a wider FWHM than SDD2. Though the spectra are coincident in x-ray count rate over a wide range of energies, the effective volume from which each detector collects is different. See the table in Section 2.3.
5.3 X-ray count rate as a function of time in the CC and SEC while the RF power was being pulsed. The characteristic fall-off time after cessation of RF power was 17 $\mu$s in the CC and 5.2 $\mu$s in the SEC. A 5.6 $\mu$s peaking time was used for both detectors, so the SEC decay time is not resolvable. 152

5.4 Boris-algorithm numerically calculated trajectory of a single electron in the PFRC-II run as a tandem mirror. a) the trajectory in space of the electron, superimposed over the mirror geometry. b) the trajectory in $\mu$ of this particle. The energy of this electron is 5.4 keV. It starts marginally trapped at $t = 0$. The magnetic field is the PFRC-II magnetic field assuming $I_{L2} = 60A, I_N = 350A$. 154

5.5 Ensemble of marginally trapped 5.4 keV electrons in the PFRC-II mirror field. a) The trajectory in space of the electrons, superimposed over the mirror geometry. b) the $\mu$ trajectory of each particle in time. c) the values of $\Delta \mu$ given by the Boris algorithm, by the approximate model derived later this chapter, and by the model of Hastie, Taylor, and Hobbs. Other models are described briefly in Section 5.1.4. 155

5.6 Left: $\mu$ trajectories for two ensembles of particles in the PFRC-II mirror field. Right: $\Delta \mu$ values vs the gyrophase at the midplane obtained from: the full-orbit Boris algorithm, Equations 5.24, 5.27, and 5.28. Top: electrons begin with energy 1.5 keV. Bottom: electrons begin with energy 15 keV. 161

5.7 Plots of $p, q$ points produced by applying Equations 5.38, 5.39, and 5.40 to 20 points starting at $p = 10$, $q$ evenly spaced. $K$ values are 0.9, 1.1, and 1.3. Color describes the initial $q$ of the point. 2000 time steps were performed. 165

5.8 Boris-algorithm results of $\mu$ trajectories of two ensembles of particles. One ensemble of 32 particles was initialized with a $\mu$ of 10 times the passing $\mu$, and one ensemble was initialized with a $\mu$ of 6 times the passing $\mu$. It appears that there is a boundary between them. The lines which stay constant from 10 $\mu$s onward are lost at that time. 168

5.9 Plots of the ratio between $\mu_c$, defined in Equation 5.41, and $\mu_p$, the passing $\mu$ value, versus energy and initial radius ($z_0 = 44$cm). This defines the effective loss cone of the PFRC-II with the specified currents. 169

6.1 Schematic of the PFRC-II experiment run in seed plasma mode. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The $|\vec{B}|$ profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate. 180

6.2 X-ray energy distribution function from a typical PFRC-II plasma during electron Fermi acceleration. The parameters for this run are given in the text. The SEC x-ray fit parameters for this condition were $T_e = 394eV, n_e = 1.3 \times 10^9/cm^3$. Clearly, there is a roughly Maxwellian population of electrons characterized by an effective temperature much higher than those produced in the SEC. 184
6.3 Detail of a rich section of x-ray spectrum, with Argon K-\(\alpha\) and K-\(\beta\) lines and Chromium and Iron K-\(\alpha\) lines definitively identified, and Cobalt and Nickel K-\(\alpha\) lines tentatively identified. 185

6.4 Example EEDF obtained by inverting an x-ray spectrum. The Maxwellian EEDF (magenta line) has parameters \(n_e = 3.2 \times 10^7/\text{cm}^3\), \(T_e = 2550\text{eV}\). 187

6.5 The dependence on effective temperature (“T 3keV+) of the accelerated population on the CC neutral gas pressure. This temperature was obtained by fitting the EEDF above 3 keV to a Maxwellian. To explain these results, another derived quantity (“Vfloat*sqrt(tau/0.5ns”) is plotted. It is the expected result as derived by quantities measured in Section 6.2.8 and modeled in Section 6.3. The Y-axis is “Energy” because the derived quantity is not a temperature, but has energy units. 188

6.6 The effect on Maxwellian fit parameters of the accelerated population (“CC, hot”) of the RF power. “CC, 3keV+” is a fit to only those x-rays of greater than 3 keV of energy; the results agree with the full domain. To explain these results, another quantity is plotted (“DL float pkpk x 900”). It is the expected result as derived by quantities measured in Section 6.2.8 and modeled in Section 6.3. 189

6.7 Maxwellian fit parameters for the SEC electrons (“SEC”), colder population of electrons in CC (“CC, cold”), and accelerated population of electrons in the CC (“CC, hot”). The nozzle currents specified span a mirror ratio of R=10-19. Changing the mirror ratio changes only the density of the accelerated electrons. 190

6.8 Abel-inverted radial profiles of two EEDF-derived quantities of the electrons in the CC at two values of the L-2 current. Density above 600 eV, left, and average energy per electron above 600 eV, right. For a description of the two conditions plotted, refer to the text. The results can be explained from the perspective of the behavior of the non-adiabaticity of the magnetic moment, described in Chapter 5. 191

6.9 X-ray count vs modulation phase histograms for four different bands of x-ray energies. The oscilloscope was triggered on the RF modulation signal. Each time-axis division is 100 \(\mu\text{s}\), showing that very high energies of electron persist for many hundreds of \(\mu\text{s}\) after the cessation of the RF power. This corresponds to tens of thousands of transits. 194

6.10 e-folding decay times derived from the full EEDFs taken after several delay times from the cessation of RF power. Clearly, higher-energy electrons persist in the CC for longer than lower-energy electrons, as long as hundreds of \(\mu\text{s}\). Above 4 keV, the uncertainty is large as not enough x-rays were counted for an accurate decay time to be computed. 195

6.11 The rise after the initiation of RF of two EEDF-derived quantities. At left: The density of electrons above 600 eV. At right: The average electron energy above 600 eV. This shows that the CC accelerated population takes hundreds of \(\mu\text{s}\) to reach its full density and effective temperature. 197
6.12 Auto-correlation of the Langmuir probe voltage for each delay time at each microsecond. A white horizontal band indicates the presence of a periodic signal with that period. The persistent band at 53 µs is the 19 MHz RF signal. Higher harmonics (5-10) are often strongest, as from 40 µs to 55 µs, but this visualization resolves them only poorly. 201

6.13 Schematic representation of the proposed mechanism for high harmonics of the RF frequency (here called “RF signal”) to saturate. While the most unstable mode frequency is not a harmonic of the RF frequency, the growth rate at a harmonic of the RF frequency is sufficient to amplify it to saturation. Once any mode reaches saturation, no more amplification occurs as the source of free energy is used up. 202

6.14 X-ray fit parameters and the scaled Langmuir probe voltage oscillation from the CC-FEC nozzle as a function of FEC pressure. The temperature of the accelerated population (“CC, hot”) increases linearly with the scaled probe voltage amplitude (“DL float pkpk x 700”), and both increase with increasing FEC pressure. 203

6.15 Plot of Langmuir characteristic derived space potential in the FEC. At low pressure, a sheath forms and downstream of it the potential is very negative. At high pressure, ionization in the FEC allows this sheath to be less extreme. This plot was prepared by Tony Qian. 205

6.16 Plot of Langmuir characteristic derived electron density in the FEC. At low pressure, a sheath forms and only fast electrons may penetrate to the FEC. At high pressure, ionization in the FEC increases electron density. This plot was prepared by Tony Qian. 206

6.17 Example EVDFs before and after the onset of two-stream instability. SEC-born electrons in the CC start with $n_e = 5 \times 10^8$ cm$^{-3}$, $T_e = 350$ eV and FEC-born electrons in the CC start with $n_e = 3.5 \times 10^7$ cm$^{-3}$, $T_e = 5$ eV, with a 300 eV drift energy. The bulk EVDF is not shown. A 50 V$_{pkpk}$ oscillation flattens the EVDFs over the velocity interval which is trapped. This oscillation is sufficient to make $f'(v) \leq 0$ everywhere, the condition for two-stream instability saturation in the inverse Landau damping limit. The bulk electron EVDF is not shown. 210

6.18 Schematic representation of a particle with velocity $v_0$ bouncing off of a sheath moving at $v_w$. 216

6.19 Plots of the map defined by Equations 6.30 and 6.31. Each plot is for $K = 0.1$. From left to right, these plots have $R = 0, 0.1, 0.2$. 1600 time steps are depicted. 400 points were initialized with $p = 15$, evenly spaced in $q$. The color of the point corresponds to its initial $q$ value. 228

B.1 Visualization of the non-adiabaticity of electrons in the PFRC-II run in seed plasma mode. This figure has intermediate plots of the dependencies of $E_{||,pe}$, specifically the amplitude of the magnetic field $B$ and the inverse radius of curvature of the magnetic field $1/R_c$. 240

B.2 Visualization of the non-adiabaticity of electrons in the PFRC-II run in seed plasma mode. This figure has the field lines, dark blue, overlaid upon the parameter which describes adiabaticity, $E_{||,pe}$, defined in Equation B.5 for $p_c = 1\%$. 240
B.3 Visualization of the non-adiabaticity of Deuterons in a modest magnetic mirror. The features at about $z = \pm 0.5$ meters are artifacts from a region of zero curvature. ........................................... 241

B.4 Visualization of the non-adiabaticity of Deuterons in a modest magnetic mirror, detail. The features at about $z = \pm 0.5$ meters are artifacts from a region of zero curvature. ........................................... 242

B.5 Visualization of the non-adiabaticity of Deuterons in another magnetic mirror, reminiscent of the GOL-3. The features at about $z = \pm 0.3$ meters are artifacts from a region of zero curvature. ......................... 243

B.6 Visualization of the non-adiabaticity of electrons in a dipole field, reminiscent of Proto-RT. ................................................................. 244

B.7 Visualization of the non-adiabaticity of electrons in a Hill’s Vortex FRC, reminiscent of the PFRC-II run in RMF-FRC mode. ....................... 244

B.8 Visualization of the non-adiabaticity of Deuterons in a Hill’s Vortex FRC, similar to concepts of FRC-based power reactors. ................. 245
## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CC</td>
<td>Center Cell</td>
</tr>
<tr>
<td>EEDF</td>
<td>Electron Energy Distribution Function</td>
</tr>
<tr>
<td>EVDF</td>
<td>Electron Velocity Distribution Function</td>
</tr>
<tr>
<td>FEC</td>
<td>Far End Cell</td>
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<tr>
<td>FPGA</td>
<td>Field-Programmable Gate Array</td>
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<tr>
<td>FRC</td>
<td>Field-Reversed Configuration</td>
</tr>
<tr>
<td>FWHM</td>
<td>Full Width at Half Maximum</td>
</tr>
<tr>
<td>ID</td>
<td>Inner Diameter</td>
</tr>
<tr>
<td>MCA</td>
<td>Multi Channel Amplifier</td>
</tr>
<tr>
<td>MHD</td>
<td>Magnetohydrodynamics</td>
</tr>
<tr>
<td>MNX</td>
<td>Magnetic Nozzle Experiment</td>
</tr>
<tr>
<td>OD</td>
<td>Outer Diameter</td>
</tr>
<tr>
<td>PFRC-II</td>
<td>Princeton Field-Reversed Configuration II</td>
</tr>
<tr>
<td>RF</td>
<td>Radio Frequency</td>
</tr>
<tr>
<td>RHESSI</td>
<td>Reuven Ramaty High Energy Solar Spectroscopic Imager</td>
</tr>
<tr>
<td>RMF</td>
<td>Rotating Magnetic Field</td>
</tr>
<tr>
<td>SDD</td>
<td>Silicon Drift Detector</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
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<tr>
<td>-------------</td>
<td>---------------------------------</td>
</tr>
<tr>
<td>SE</td>
<td>Secondary Electron</td>
</tr>
<tr>
<td>SEC</td>
<td>Source End Cell</td>
</tr>
<tr>
<td>SEE</td>
<td>Secondary Electron Emission</td>
</tr>
<tr>
<td>SEY</td>
<td>Secondary Electron Yield</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>UV</td>
<td>Ultra Violet</td>
</tr>
<tr>
<td>XEDF</td>
<td>X-ray Energy Distribution Function</td>
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</table>
## Symbols

<table>
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<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Unit</th>
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<tr>
<td>$A$</td>
<td>Aspect ratio</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>RF phase</td>
<td>rad</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic field strength</td>
<td>G or T</td>
</tr>
<tr>
<td>$\vec{B}$</td>
<td>Magnetic field</td>
<td>G or T</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Secondary electron yield</td>
<td></td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Flux</td>
<td>cm$^{-2}$s$^{-1}$</td>
</tr>
<tr>
<td>$D$</td>
<td>Volume fill fraction</td>
<td></td>
</tr>
<tr>
<td>$D_q$</td>
<td>Diffusivity in $q$</td>
<td>$q^2$/s</td>
</tr>
<tr>
<td>$D_t$</td>
<td>Total material time derivative</td>
<td>s$^{-1}$</td>
</tr>
<tr>
<td>$e$</td>
<td>Electron charge</td>
<td>$e$ or Coulomb</td>
</tr>
<tr>
<td>$e$</td>
<td>Base of the natural logarithm</td>
<td></td>
</tr>
<tr>
<td>$E$</td>
<td>Energy</td>
<td>eV</td>
</tr>
<tr>
<td>$\vec{E}$</td>
<td>Electric field</td>
<td>V/cm</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Adiabatic parameter</td>
<td></td>
</tr>
<tr>
<td>$f(E)$</td>
<td>EEDF</td>
<td>cm$^{-3}$eV$^{-1}$</td>
</tr>
<tr>
<td>$f(\vec{v})$</td>
<td>EVDF</td>
<td>s$^3$cm$^{-6}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Chirikov parameter</td>
<td>rad</td>
</tr>
<tr>
<td>Symbol</td>
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</tr>
<tr>
<td>--------</td>
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</tr>
<tr>
<td>(\lambda)</td>
<td>Mean free path</td>
<td>cm</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Magnetic moment</td>
<td>eV/T</td>
</tr>
<tr>
<td>(n)</td>
<td>Density</td>
<td>cm(^{-3})</td>
</tr>
<tr>
<td>(R)</td>
<td>Mirror ratio</td>
<td></td>
</tr>
<tr>
<td>(R_0)</td>
<td>Total particle rate</td>
<td>s(^{-1})</td>
</tr>
<tr>
<td>(R_c)</td>
<td>Radius of curvature</td>
<td>cm</td>
</tr>
<tr>
<td>(\rho_e)</td>
<td>Electron Larmor radius</td>
<td>cm</td>
</tr>
<tr>
<td>(\rho)</td>
<td>X-ray production rate</td>
<td>1/cm(^3)/s</td>
</tr>
<tr>
<td>(\Phi)</td>
<td>Enclosed magnetic flux</td>
<td>T/m(^2) or G/cm(^2)</td>
</tr>
<tr>
<td>(\theta)</td>
<td>Pitch angle or polar angle</td>
<td>rad</td>
</tr>
<tr>
<td>(\phi)</td>
<td>Azimuthal angle</td>
<td>rad</td>
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<tr>
<td>(\psi)</td>
<td>Cyclotron gyrophase</td>
<td>rad</td>
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<tr>
<td>(t_t)</td>
<td>Transit time</td>
<td>s</td>
</tr>
<tr>
<td>(T)</td>
<td>Temperature</td>
<td>eV</td>
</tr>
<tr>
<td>(\tau_l)</td>
<td>Loss time</td>
<td>s</td>
</tr>
<tr>
<td>(U(x,t))</td>
<td>Effective potential</td>
<td>J</td>
</tr>
<tr>
<td>(v)</td>
<td>Velocity</td>
<td>cm/s</td>
</tr>
<tr>
<td>(v_{</td>
<td></td>
<td>})</td>
</tr>
<tr>
<td>(v_{\perp})</td>
<td>Velocity perpendicular to (\vec{B})</td>
<td>cm/s</td>
</tr>
<tr>
<td>(\omega)</td>
<td>Angular frequency</td>
<td>rad/s</td>
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<tr>
<td>(\Omega)</td>
<td>Cyclotron frequency</td>
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Chapter 1

Introduction

In this dissertation, I discover and present a novel mechanism for electron acceleration by an electrostatic oscillation in a magnetic mirror. For reasonable values of the electron adiabatic parameter, I find that the non-adiabaticity of magnetic moment is sufficient to break resonances between the mirror transit time of the electron and the oscillation period. In a nontrivial finding, I calculate that the effect is valid even for a perfectly periodic oscillation. This finding supplements the known mechanism for electrostatic heating in a mirror, that of a turbulent wave spectrum.

I use x-ray spectrometry and confirmatory electrostatic measurements to measure the behavior of fast electrons in a tandem magnetic mirror. The plasma is generated by a double-saddle antenna in one end cell, and is mostly composed of a cold, tenuous bulk plasma. For a plasma of this type, electrostatic probe measurements are sensitive to changes in temperature, density, and space potential. Via Langmuir characteristic analysis, they are sensitive to some non-Maxwellian features in the Electron Energy
Distribution Function (EEDF). Langmuir probes are insensitive to EEDF features at electron energies of more than a few dozen eV.

X-ray pulse-height detectors are sensitive to EEDF features at electron energies from a few hundred eV to dozens of keV. Electrons in this range emit Bremsstrahlung x-rays with energies that the detectors can measure. Using x-ray detectors, I measure that alongside the cold bulk plasma is a hot minority component with $100\times$ higher temperature but 1% of the density. These electrons are created in one end cell of the tandem mirror, the Source End Cell (SEC) which contains the antenna. They are created by ion-induced secondary electron emission, when ions fall through a large sheath and cause electrons to be emitted from surfaces near the antenna.

These electrons are then observed via x-ray detector to pass into the Center Cell (CC) of the mirror and increase in density. Calculations indicate that this increase in density is caused by non-adiabaticity of the magnetic moment of the electron, which allows it to persist in the mirror cell. This is a novel phenomenon which had not been measured before.

Finally, under certain circumstances approximately 10% of these electrons may be accelerated another $10\times$ in temperature, up to 3 keV, by a novel form of acceleration in quasi-adiabatic magnetic mirrors: a kind of Fermi-Ulam acceleration whose phase correlations are broken by the non-adiabaticity of the magnetic moment. The periodic forcing of the Fermi-Ulam acceleration is a sinusoidal electrostatic potential oscillation in the CC that comes from electrons born in the last mirror end cell, the Far End Cell (FEC), re-entering the CC as a beam and causing two-stream instability. The FEC has a
much more negative space potential than the CC because of a voltage drop established by the original SEC-born fast electrons.

Supporting these claims, I describe several measurements I performed and models I fit to these measurements from the PFRC-II in seed plasma mode. The PFRC-II in seed plasma mode is a 3-cell tandem magnetic mirror, with a mirror ratio 10 - 40 and $B_{min}$ 50 - 200 Gauss. The plasma is created by a double-saddle antenna in one end cell with net power 300 - 500 W. The bulk plasma created thereby has density in that cell $n_e \sim 10^{11}/cm^3$ and temperature $T_e \sim 5$ eV. Using x-ray measurements and confirming via electrostatic measurements, I have measured fast electron components at densities and temperatures varying from $n_e \sim 3 \times 10^8/cm^3$, $T_e \sim 300$ eV to $n_e \sim 3 \times 10^7/cm^3$, $T_e \sim 3$ keV, with individual electrons attaining energies above 30 keV.

1.1 The chapters in this dissertation

In Chapter 2, I describe a tool which I developed to study this population of fast electrons: a method for inverting x-ray Bremsstrahlung spectra into full Electron Energy Distribution Functions (EEDFs). This method is essential for characterizing accurately not just bulk parameters like the effective temperatures and densities of the fast electrons, but energy-specific parameters like decay times, risetimes, and spatial profiles of the electrons. The inversion tool is based on numerically extremizing the Poisson-derived log-likelihood. The x-ray detectors are Amptek Silicon Drift Detectors (SDDs) which I specially calibrated using a purpose-built gas-target x-ray tube.
FIGURE 1.1: Schematic representation of quantities of relevance to the phenomenon observed in the PFRC-II: a) The field-line map. b) The strength of the axial magnetic field. c) Representative change in the magnetic moment $\partial_t \mu$ of a fast electron. d) Plausible electrostatic potential along the axis of the PFRC-II. Red $\times$ signs represent measured points; other locations are speculation. Far left: RF self-bias of the plasma-terminating cup. Far right: The magnetic nozzle at $z \approx 44$ cm decreases the space potential many hundreds of volts. At $z \approx 33$ cm: Two-stream instability.
Chapter 1.1

Other algorithms for spectral inversion of Bremsstrahlung do exist. I build upon them by treating each measurement as a Poisson-distributed random variable. I also consider the effects of x-ray transmission through a window, detector resolution, and EEDF smoothness self-consistently from a Bayesian statistics perspective. This allows self-consistent uncertainty error bars to be computed and allows small EEDF features such as beams and cutoffs to be measured accurately.

In Chapter 3, I describe findings from the end cell in which the plasma is formed, the Source End Cell (SEC). The SEC contains the double-saddle antenna which produces the bulk plasma and the initial population of fast electrons which is later accelerated further. I find that ion-induced secondary electron emission from a surface which has a large RF self-bias injects electrons already at hundreds of eV of energy, rather than accelerating them over time. I model this behavior using an approximate kinetic model.

Putting the findings of Chapter 3 into perspective requires an introduction to plasma materials processing reactors of the kind used in semiconductor manufacturing. I give an overview in that chapter. Fast electrons produced from RF sheaths have been seen in these reactors; indeed, in one case a DC bias was applied to an electrode to deliberately produce fast electrons for more even surface charging and etching properties. The novelty of the phenomenon described in Chapter 3 lies in its extremity and its magnetic configuration. The low pressure and collisionality of the SEC allows the effective density and temperature of these fast electrons to be many times higher than those typically seen in industry, as measured via x-ray detectors. Furthermore, a result of the model I present implies that another population of secondary electrons allows a stainless steel
plasma-terminating surface to float at a very negative potential, electrostatically confining axially directed fast electrons in a density-enhancing configuration.

In Chapter 4, I depart somewhat from the central thrust of the dissertation and propose a means by which the creation of secondary electrons may be suppressed. Reporting results of three papers I co-authored, I consider three complex surface geometries using a Monte-Carlo model and an analytic model and find that long, fibrous structures can reduce the flux of secondary electrons by an order of magnitude under the right circumstances. This is part of a growing field of research into secondary electron suppressing surfaces. Generally it is not known \textit{a priori} whether a surface will be suitable for this suppression; because of this, I derive a previously un-published “weighted view factor” model which provides intuitive guidelines for selecting such a surface. I find that surfaces with a large total surface area squeezed as tightly as possible into as small a cross-sectional area as possible, such as a fractal or dendritic surface, are appropriate for suppressing secondary electrons.

In Chapter 5, I describe measurements of a novel phenomenon: the non-adiabatic accumulation of fast, passing electrons in the central mirror cell. Because there persist misconceptions about the adiabatic conservation of the magnetic moment, \(\mu\), of an electron in a magnetic mirror, I give a novel, elementary derivation of a known result: the size of the non-adiabatic change in \(\mu\) and the long-time mobility of \(\mu\) over many mirror transits. In Appendix B I visualize the mobility of \(\mu\) over a single transit in many magnetic configurations, including the surprising result that large changes in \(\mu\) can be obtained from relatively conservative mirror parameters.

The truly novel results in Chapter 5 are the measurement and description of the
accumulation of fast, passing electrons in a magnetic mirror. However, I spend many
pages of that chapter on a new derivation of the size of changes in $\mu$, a known result.
I do this because, in my experience, knowledge of extreme changes to the magnetic
moment of particles in a magnetic field is confined to specific silos or niches of plasma
physics. While $\mu$ is well-conserved of thermal particles in most Tokamaks, such mag-
netic configurations as FRCs, mirrors, and dipoles may exhibit extreme non-adiabaticity
unbeknownst to their experimental and modeling communities.

In Chapter 6, I describe the most extreme measurements of electron temperature
in the PFRC-II and I present a model for the acceleration of electrons to these extreme
energies. I find that the fast electrons born in the SEC expanding into the opposing
dead end cell, the Far End Cell (FEC), induces a large voltage drop ($\sim 600$ V) in the FEC.
Ionization in the FEC causes cold electrons to be born which fall back across the voltage
drop into the Center Cell (CC) in a beam. This beam excites an electrostatic fluctuation
in the CC, which is mostly periodic at $\sim 200$ MHz. I present measurements of this
voltage drop and the electrostatic fluctuation. Mirror-bouncing electrons passing into
and out of the region of fluctuation are stochastically accelerated up to a characteristic
energy of $2 - 3$ keV, with individual electrons attaining 30 keV, over the course of tens
of thousands of mirror transits. However, a periodic signal alone would not accelerate
electrons beyond a resonance in their transit time and the oscillation period. I show that
non-adiabaticity of $\mu$, studied in-depth in Chapter 5 is sufficient to break this resonance
and allow electron acceleration up to $30$ keV.

Electron acceleration by electrostatic fluctuation in mirrors is not novel; it has been
seen before in many experiments. It is typically understood to be the result of a turbulent
wave spectrum. The novel phenomenon measured in the PFRC-II is acceleration from a mostly periodic electrostatic spectrum. It was not previously known that a sinusoidal electrostatic potential, either from an antenna or spontaneous as in the PFRC-II, can produce stochastic Fermi-Ulam acceleration in this way.

1.2 Personal note

This dissertation grew out of a project to characterize and describe the seed plasma of the PFRC-II experiment. The PFRC-II experiment is a small-scale testbed for a suite of technologies which may someday make compact, Helium-3 burning fusion reactors possible. However, in order to understand the hot, dense Field-Reversed Configuration (FRC) plasma in the PFRC-II, we first undertook a project to characterize its initial condition, a cold, tenuous seed plasma in a magnetic mirror.

As is common in research, while studying one phenomenon we were surprised by another. What we thought was a quiescent, thermalized, cold plasma of the kind in everyday objects such as fluorescent lightbulbs and materials processing reactors was in fact a case study in collisionless mirror dynamics, with populations of electrons reaching effective temperatures of 3 keV.
Chapter 2

Reconstruction of EEDFs from x-ray pulse-height detectors

Some of the most compelling data presented in this thesis are x-ray spectra that have been recorded with a Silicon Drift Detector (SDD). It is worthwhile to discuss this x-ray detector, its use, and the analysis of its data, as diagnosing the energy distribution of a population of fast electrons in a laboratory plasma is an atypical use-case for this detector. Some of this chapter was first published in my paper with Peter Jandovitz and Samuel A Cohen.[1] Because of that, they should also be considered co-authors of parts of this chapter.

2.1 Introduction

The Amptek X-123SDD units used here are part of the broader family of x-ray pulse-height detectors. These detectors operate by collecting the mobile charges produced in a
semiconductor by the passage of x-rays, relating the number of electron-hole pairs to the
energy of the x-ray. These stand in comparison to x-ray Bragg-crystal spectrometers,
which use the Bragg scattering condition to transform a dependence on energy into a
dependence on angle.[2]

Bragg-crystal spectrometers are able to measure x-ray spectra with excellent resolution,
$10^{-4}$ of the x-ray energy in cases, but are limited to detecting only a narrow range
of x-ray energies. This makes them well-suited to measuring spectral lines from atomic
emission, and since the 1970s they have been used for this purpose on thermonuclear
plasma experiments.[3]

Pulse-height detectors have resolutions limited by thermal noise[4] and counting
statistics of the electron-hole pairs liberated[5]. The resolution of the Amptek SDD
we used is 125 eV at 5.9 keV, making measurements of broadening and satellite peaks
impossible. The energy detection range, however, can extend from 200 eV to 100 keV.
This makes pulse-height detectors well-suited to measuring the amplitudes of many
spectral lines at once, or even broad-spectrum emission such as Bremsstrahlung. This
is the use to which we put the detector.

In one sense, our use of SDDs is the latest step in a renaissance of self-contained,
Peltier-cooled x-ray detectors since the 1990s. The industry that has driven their devel-
opment is manufacturing and quality assurance, allowing in-situ identification of alloy
and metal purity.[4] Plasma physics is also not the only science piggy-backing off of this
development; detectors have been used to study the chemical composition of Michelan-
gelo’s David[6] and rocks on the planet Mars.[7]
In this chapter, I describe the physical principles on which pulse-height detectors operate. I describe practical matters of mounting the detectors on apparatus, including angular resolutions, lines of sight, and view factors. I describe the physical processes by which plasma may produce x-rays. I describe the process of spectral inversion, the recovery of the Electron Energy Distribution Function (EEDF) from x-ray spectra, both theoretically and numerically. I derive my own spectral inversion algorithm from the condition of maximal log-likelihood computed using Poisson statistics. I compare the algorithm presented in this chapter to the current state-of-the-art. I describe the detector’s response function and how I calibrated it with a custom-made gas-target x-ray tube. I describe how spectral lines are treated in the inversion process. Finally I give examples of the data collection procedure and its application to previously recorded data.

2.2 Physical principle of x-ray pulse-height detectors and SDDs

At their core, x-ray pulse-height detectors are identical in operation to photodiodes. In fact, Si-PIN detectors, the state-of-the-art until the early 2000s, are built around commercially available photodiodes marketed for visible and UV sensitivity.[8–11] Since then, purpose-fabricated SDD x-ray detectors have surpassed repurposed photodiodes, but the operating principle is much the same.
Both SiPIN detectors and SDDs employ the property of ionizing radiation to liberate charge carriers, even in non-conducting media. Semiconductors are made non-conducting by reverse-biasing them, a process which leads to the creation of a layer in which all mobile charge carriers are depleted, called the “depletion layer.” X-rays traversing the depletion layer liberate some number of electrons, statistically proportional to their energy.[5] The charge carriers are swept by an electric field toward an electrode and the net charge is measured by a low-noise amplifier circuit.

In SiPIN detectors, or identically photodiodes, the semiconductor is arranged into a thin disk with electrodes on either face. One electrode attaches to n-type doped semiconductor, and one electrode attaches to p-type doped semiconductor. A reverse bias depletes these doped regions of charge carriers and renders them non-conducting, so no current flows in equilibrium. X-ray flux is incident parallel to the axis of the disk, penetrates the electrode, and releases charge carriers in the depletion layer.

SDDs seek to obtain lower noise and higher count rates than SiPIN detectors by lowering the collection electrode’s capacitance.[4, 12–15] They are similar in operating principle, but leverage an additional property of the depletion layer: it can support significant space charge. Uniform positive volumetric space charge furnished by a depleted n-type doped semiconductor forms a 2-D potential well surface in its center, trapping liberated charges. A secondary, externally-imposed potential then causes the electrons to drift toward a much smaller anode.

As an example, the Hamamatsu S1772-N photodiode used in an early SiPIN detector had 12pF of capacitance due to its anode spanning the entire active face of the detector. This detector was able to achieve a 1.43 keV FWHM for a 59.55 keV peak,
with the commonly used Fe-55 5.9 keV calibration peak just visible from the noise.\[9\] Later SiPINs were able to resolve the 5.9 keV peak with a 660 eV FWHM.\[10\] An early commercial SDD had 250 fF of anode capacitance and achieved 145 eV FWHM for the 5.9 keV peak.\[14\] State-of-the-art SDDs achieve 122 eV FWHM for the 5.9 keV peak.\[16\]

SDDs were developed for particle detection primarily for their position sensing, rather than their low noise. The time delay between cathode current (holes immediately lost to the surface of the silicon) and the anode current (electrons which are trapped in the silicon and must drift to the anode) determined the radius at which a particle produced charge carriers. The name “Silicon Drift Detector” comes from analogy to drift chamber particle detectors. We do not use this position-sensing feature.

It may be counterintuitive to increase signal to noise ratio by decreasing capacitance, as thermal noise varies as \( V_t = \sqrt{\frac{k_B T}{C}} \). However, the charge released by an x-ray is constant, and the dependence of the signal on the capacitance is still stronger, \( V = \frac{Q}{C} \).

The SDDs used for x-ray spectroscopy are generally 300 \( \mu \text{m} \) in thickness with an area of around 20 \( \text{mm}^2 \). They are protected by the elements and visible light by a window which may be 1/3 to 1 mil of Beryllium, or for specialized applications a layer of \( \text{Si}_3\text{N}_4 \) on a grid of Aluminum. The detector thickness determines the upper limit of x-ray energy detection at around 50 - 100 keV. The window thickness determines the lower limit of x-ray energy detection at around 200 - 600 eV.

The maximum SDD count rate is limited by its collection electronics rather than the detector itself. In order to determine the number of charges liberated by the passage of
an x-ray, the anode signal is immediately coupled to FET and a low-noise preamplifier. After this, pulse-processing occurs. In the Amptek X-123SDD used for much of this thesis, two channels of pulse-processing occur: a fast channel and a slow channel.

The fast channel does not participate in energy determination. It applies a high-pass filter of configurable time constant, typically 50 - 100 ns, to determine most precisely the timing of an x-ray detection. This channel is most useful for determining RF phase dependence of x-ray detection, determining total real count rate of x-rays, and detecting and rejecting double-counts of x-rays (pulse pileup rejection).

The slow channel performs the energy determination. Pulse-shaping occurs using digital filters implemented in an FPGA to form a trapezoidal pulse. A multi-channel amplifier (MCA) detects the value of the peak of this pulse, and increments that channel’s counter by 1. The rise time and flat-top time of the trapezoidal pulse are configurable digitally. Choice of times constitutes a tradeoff between good resolution (long times, $5.6\mu s$) and good count rates (short times, 100 ns). If two pulses occur within a slow filtering time of each other, the pulse is rejected by logic performed on the fast channel, as the slow channel would erroneously register a single x-ray with their summed energy.

The resolution of the SDD is determined by electronic noise and counting statistics. The electronic noise includes thermal noise, shot noise, and other sources of noise inherent to the electronics. The counting statistics, also called Fano noise,[5] comes from the fact that an x-ray liberates only a finite number of charge carriers when it stops in silicon. This finite number gives rise to a Poisson-like distribution whose variance is proportional to the expected number of counts. At 5.9 keV, this irreducible source of noise produces a peak of 119 eV Full Width at Half of Maximum (FWHM).
2.3 Lines of sight, view factors, and mounting to the PFRC-II apparatus

The mounting that attaches the SDD to the PFRC-II apparatus must be carefully designed with several criteria in mind: Obviously, it must allow the SDD to see the portion of the plasma being studied. But it must also have the correct positional resolution. It must have narrow enough view that the count rate does not exceed the maximum, but wide enough that a high-quality spectrum can be obtained during normal run conditions.

For this analysis we will consider circular detector and aperture areas, with an axis of symmetry running from the center of the detector through the center of the aperture to the center of the target, which is a uniform slab of plasma with some thickness. We consider detector radius $r_1$, aperture radius $r_2$, for aperture $l_1$ from the detector and the target (plasma slab) $L$ in thickness and $l_2$ from the detector. $r_i \ll l_j$ for all $i, j$.

We consider a plasma which emits x-rays uniformly and isotropically with volumetric...
emission density $\rho$. This system is depicted in Figure 2.1. The positional resolution $r_3$ of the system is as follows:

$$r_3 = r_2 + (r_1 + r_2)(l_2/l_1 - 1) \quad (2.1)$$

This is derived from the method of similar triangles; $(r, z) - (r, z)$ point $(r_1, 0)$ – $(-r_2, l_1)$ crosses the axis of symmetry at some $z$-point. This makes two similar right triangles, that from this point to $(r_1, 0)$ and that from this point to $(-r_2, l_1)$. Extending this most extreme line of sight of the detector-aperture system to the target at $l_2$, one has a similar triangle between the axis-crossing point and $(-r_3, l_2)$.

Each point on the detector has line-of-sight to a volume of plasma $V_{los} = L(\pi r_2^2)(l_2^2/l_1^2)$. Each point on the detector has line of sight to a different subset of the plasma volume, but every point has line of sight to the same amount of plasma. Because of this, we may find the x-rays which are incident the detector by asking how many x-rays a plasma volume $V_{los}$ emits into a solid angle that intersects the detector area. Because of the property of optical systems to conserve étendue, or more intuitively because of conservation of x-rays, it is valid to determine the x-rays incident on a detector limited in line-of-sight to some solid angle by determining the x-rays emitted by the effective volume that each point sees into the solid angle subtended by the detector.

Since emission is assumed to be isotropic, the rate of x-rays emitted by a volume $V_{los}$ into a solid angle $\Omega$ is $\rho V_{los} \frac{\Omega}{4\pi}$. This solid angle is calculable geometrically $\Omega = \pi r_1^2/l_2^2$. Thus the rate of x-rays received by the detector-aperture system is
\begin{equation}
CR = \frac{\rho L (\pi r_2^2 \times \pi r_1^2)}{4\pi l_1^2} = \rho V_{eff}
\end{equation}

\begin{equation}
V_{eff} = \frac{L(A_1A_2)}{4\pi l_1^2}
\end{equation}

where $A_1$ is the active area of the detector, $A_2$ is the area of the aperture, $\rho$ is the volumetric x-ray emission rate, and $V_{eff}$ is the effective volume of plasma from which the detector collects all x-rays. This does not correspond to any specific, physical subset of plasma; it is rather the amount that, if the detector completely encompassed it, would produce the same count rate. Using Equation 2.2, one can ensure that the count rate conditions mentioned at the beginning of this section are satisfied, that the count rate is neither too fast for the peaking time chosen nor too slow to accumulate a spectrum in the required time.

The SDDs in this thesis collected data via four detector-aperture systems: One mounted to the Source End Cell (SEC) of the PFRC-II apparatus with a radially scanning aperture, one mounted to the SEC of the PFRC-II with a large aperture for high count-rate data collection, one mounted to the Center Cell (CC) of the PFRC-II looking through the midpoint of the machine with five selectable apertures of various sizes, and one mounted to the CC of the PFRC-II with a radially scanning aperture. Their relevant parameters are as follows:
<table>
<thead>
<tr>
<th>Name</th>
<th>Variable</th>
<th>SEC Radial</th>
<th>SEC Short</th>
<th>CC Midpoint</th>
<th>CC Radial</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detector Area</td>
<td>$A_1$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>cm$^2$</td>
</tr>
<tr>
<td>Aperture Area</td>
<td>$A_2$</td>
<td>0.050</td>
<td>9.2</td>
<td>1.0 - 0.027</td>
<td>0.038</td>
<td>cm$^2$</td>
</tr>
<tr>
<td>l to aperture</td>
<td>$l_1$</td>
<td>18.0</td>
<td>24.5</td>
<td>25.4</td>
<td>10.9</td>
<td>cm</td>
</tr>
<tr>
<td>Plasma max OD</td>
<td>$L$</td>
<td>6.0</td>
<td>6.0</td>
<td>16</td>
<td>16</td>
<td>cm</td>
</tr>
<tr>
<td>Effective Volume</td>
<td>$V_{eff}$</td>
<td>1.84</td>
<td>184</td>
<td>49.3 - 0.134</td>
<td>10.2</td>
<td>$\times 10^{-5}$cc</td>
</tr>
</tbody>
</table>

“Plasma max OD” stands for the maximum outer diameter of the plasma. A radial scan is accomplished by performing a vertical scan, then applying the Abel transform. The CC Midpoint detector has a range of effective volumes because it has several selectable aperture sizes.

The SEC radial scan is accomplished by cutting a rectangular aperture into 5 mil stainless steel shimstock, bending it into a drum, and mounting it to a rotating vacuum feedthrough. The CC radial scan is accomplished by mounting the entire detector-aperture apparatus to a vacuum-tight welding bellows in a Lexan frame. The CC variable aperture (midpoint mount) is accomplished by punching holes of various sizes into a 5 mil stainless steel shimstock pie wedge offset from the axis of the detector whose pie center is attached to a Wilson seal to allow rotation.

The CC midpoint mount is constructed of Lexan. 10+ keV x-rays are observed to penetrate the Lexan of the CC and mount, and so give an artificially high count rate at these energies. To prevent this, the CC radial scanning mount is constructed out of 35 mil thick stainless steel tubing. This was evaluated to be opaque to x-rays beyond 35 keV using the NIST x-ray mass attenuation coefficients table.[17, 18]
Another phenomenon that we are required to mitigate is x-ray fluorescence of the walls of the mount. The structure of the detector-aperture mount is often irradiated with x-ray emissions from the bulk plasma; these x-rays can cause fluorescence of the structure with line-of-sight to the detector. To mitigate this, baffles are added to the CC midpoint and CC radial scanning mounts. The purpose of the baffle is to ensure that no point in the active area of the detector can see any part of the structure which is directly illuminated by x-rays.

A baffle is an aperture which is between the detector and the ultimate aperture which does not restrict line-of-sight to the target. To analyze the optimal position of the baffles, we require another radius $r_w$, the radius of the walls of the tube that holds the detector and aperture. The first baffle should be placed such that the most extreme point on the detector active area, at $(r, z)$ point $(r_1, 0)$ should not be able to have line-of-sight to the most extreme point which could conceivably be illuminated by x-rays, $(-r_w, l_1)$. The inner baffle radius $r_b$ is a free parameter, but must be larger than $r_1$ and $r_2$.

For a baffle to only just block the line-of-sight between these points, it must be at $z$ position

$$d_1 = l_1(1 - (r_w - r_b)/(r_w + r_1))$$  \hspace{2cm} (2.3)

The geometry is depicted in Figure 2.2. Depending on these values, there may still be illuminated wall visible to the detector. If so, another baffle should be inserted at

$$d_{n+1} = d_n(1 - (r_w - r_b)/(r_w + r_1))$$  \hspace{2cm} (2.4)
As an example, for the CC radially scanning mount, the walls were 35 mil thick, 3/4” outer diameter stainless steel tubing, giving the value of $r_w = 0.86\text{cm}$. $r_b$ was chosen to be 0.28 cm, as this was close to the size of an available punch. $r_1 = 0.28\text{ cm}$ also. Thus baffles were placed at 5.5 cm and 2.7 cm from the detector, ensuring that no point in the active area of the SDD had line-of-sight to a part of the tube which was directly illuminated by the plasma.

### 2.4 Physical processes of plasma x-ray emission

Plasmas emit x-rays via two mechanisms: Bremsstrahlung, in which a free electron incident on a nucleus emit a photon with a continuum of possible energies, and atomic spectral-line excitation, in which a nucleus and its bound electron emit a photon at a
specific, fixed energy. As in Hutchinson, we use the term Bremsstrahlung to refer to both free-free and free-bound radiation.[19]

### 2.4.1 Bremsstrahlung

Bremsstrahlung, from the German for “braking radiation,” occurs when electrons are incident on the Coulomb potential of an atomic nucleus. The acceleration produced by this interaction produces broad-spectrum x-rays. Incident electrons are separated into several regimes for the purpose of analysis. The case in which an incident electron has much higher kinetic energy than the highest binding energy of any bound electrons present is well known and is further split into several more regimes.[2, 19] The case in which the incident electron and a bound electron have similar energies is in general calculable only numerically, though approximate models are available.[20–22]

Hutchinson’s *Principles of Plasma Diagnostics* gives an excellent pedagogical tutorial on the various parameter regimes of the spectral emission due to Bremsstrahlung.[19] The simplest case is for classical, non-Quantum, non-relativistic electrons, which does not apply here but is nonetheless informative. In this case, the spectrally-resolved energy (differential energy *versus* frequency, $\partial W/\partial \nu$) emitted by the electron is as follows:

$$\frac{\partial W}{\partial \nu} = \frac{e^2}{4\pi\epsilon_0} \frac{4}{3c^3} \left| \int_{-\infty}^{\infty} dt \tilde{v} \tilde{e}^{i\omega t} \right|^2$$  \hspace{1cm} (2.5)

where $W$ is the energy emitted and the other symbols have their customary meanings: $\nu$ is the frequency of x-rays being considered, $e$ is the electron charge, $\epsilon_0$ is the
vacuum permittivity, \( v \) is the velocity of the electron, \( e \) is the base of the natural logarithm, and \( \omega \) is the angular frequency of the x-rays being considered. This equation is an extension of the standard Larmor formula.

This quantity is an energy per interaction with a target. In a field of targets, electrons incident with a distributions of impact parameters \( b \) will radiate the following spectrally resolved power

\[
\frac{\partial P}{\partial \nu} = n_\sigma v \int_0^\infty db \frac{\partial W}{\partial \nu}(\nu, b) 2\pi b \tag{2.6}
\]

where \( n_\sigma \) is the density of targets.

The trajectory of an electron in a Coulomb potential, and therefore \( \dot{\vec{v}} \), are susceptible to classical orbit theory. One familiar with Hankel functions will determine that the solution to the orbital differential equation will produce this spectrally resolved power:

\[
\frac{\partial P}{\partial \nu} = \frac{\partial P}{\partial \nu}\big|_c G(u_{90}) \tag{2.7}
\]

\[
\frac{\partial P}{\partial \nu}\big|_c = \frac{Z^2 e^6}{(4\pi\epsilon_0)^3} \frac{32\pi^2}{3\sqrt{3}} \frac{n_\sigma}{m_e^2 c^3 \nu} \frac{G(u)}{4} u H_0(u) H'_u(u)
\]

where \( u_{90} = i\omega b_{90}/v \), \( b_{90} \) is the impact parameter for \( 90^\circ \) scattering, \( H_u \) is the Hankel function of the 1st kind of order \( u \). The prefactor in Equation 2.7 is known as the classical result and will occur in successively more accurate analyses. The postfactor
in Equation 2.7 is known as the Gaunt factor and has a different dimensionless form for each parameter regime.

The Gaunt factor is always a decreasing function of $\nu$, usually of order unity.

The parameter which governs whether the classical approach can be used is the balance between the electron incident kinetic energy and the binding energy of the lowest energy level of the target nucleus,

$$\eta_q^{-2} = \frac{\frac{1}{2} m_e v_q^2}{Z^2 R_y^2}$$  \hspace{1cm} (2.8)

where $q = 1, 2$ for incoming and outgoing electron velocity, respectively, $Z$ is the nuclear charge of the target nucleus, and $R_y$ is Rydberg’s constant, $R_y \sim 13.6\text{eV}$. If $\eta_1^{-2} \ll 1$, the classical treatment is valid. However, if $\eta_1^{-2} \gg 1$, quantum mechanical effects become large and the Born approximation for scattering may be applied. This produces the Gaunt factor

$$G_{\text{Born}} = \frac{\sqrt{3}}{\pi} \ln \left( \frac{\eta_2 + \eta_1}{\eta_2 - \eta_1} \right)$$  \hspace{1cm} (2.9)

The Born approximation does not consider the case that the electron may lose most of its energy to an x-ray. Because of this, a modified form, valid when $v_2 << v_1$, was found by Elwert:[23]

$$G_{\text{Elwert}} = \frac{\sqrt{3}}{\pi} \frac{\eta_2 [1 - e^{-2\pi \eta_1}]}{\eta_1 [1 - e^{-2\pi \eta_2}]} \ln \left( \frac{\eta_2 + \eta_1}{\eta_2 - \eta_1} \right)$$  \hspace{1cm} (2.10)
Equation 2.10 is valid in the parameter regime most applicable to the data reported in this thesis. However, there is one more case which is often considered: The case that the electron’s energy is still yet higher, in the relativistic regime. This Gaunt factor is called the Bethe-Heitler formula and is not repeated here.[24]

\( G_{Elwert} \) is zero when the x-ray energy is higher than the incident electron energy. This is because it is impossible for an electron to give more energy than it has to an x-ray.

Converted into the more useful units of x-ray volumetric production rate from some electron energy distribution (EEDF), Equation 2.7 with the Born-Elwert Gaunt factor defined in Equation 2.10 gives the following:

\[
\frac{\partial \rho}{\partial E_x} = n_\sigma n_e Z^2 \frac{32\pi^2 (m_e c^2)^{3/2}}{3\sqrt{3}} \frac{r_e^3}{h} \int dE_e \frac{f(E_e) G(E_e, E_x)}{E_x \sqrt{E_e}}
\]  

(2.11)

where \( \rho \) is the volumetric x-ray production rate of the plasma, \( E_x \) is the energy of the x-ray under consideration, \( n_\sigma \) is the number density of targets (ions), \( n_e \) is the number density of electrons, \( h \) is Planck’s constant, \( r_e \) is the classical electron radius \( \sim 2.82 \times 10^{-13} \text{ cm} \), \( E_e \) is the energy of an electron, \( f(E_e) \) is the electron energy distribution normalized to 1. This formula is plotted for monoenergetic EEDFs in Figure 2.3.

For the purposes of interpreting x-ray spectral data, we are interested in the number of x-rays within an energy range \( E_x \rightarrow E_x + \Delta E_x \) that hit an active area within a time period \( \tau \) from a plasma with electron density \( n_e \) and EEDF \( f(E_e) \). Transforming Equation 2.11 into these quantities, we get
Figure 2.3: Example Bremsstrahlung x-ray spectral count rates produced by monoenergetic EEDFs, as predicted by Equation 2.11. X-ray spectral count rate is zero when $E_x > E_e$, as an electron cannot produce an x-ray with more energy than itself. Y-axis is in arbitrary units.

\[ b_i = \tau V_{eff} \Delta E_x n_\alpha n_e Z^2 \frac{32 \pi^2 \left( m_e c^2 \right)^{3/2}}{3\sqrt{3}} \frac{e^3}{h} \int dE_e \frac{f(E_e) G(E_e, E_{x,i})}{E_{x,i} \sqrt{E_e}} \] (2.12)

where $b_i$ is the number of x-rays that enter the detector in energy bin $i$, $\tau$ is the time that data is collected, $V_{eff}$ is the effective volume defined in 2.2, $\Delta E_x$ is the interval of energy that each energy bin spans, $E_{x,i}$ is the central energy of x-ray energy bin $i$. 
2.4.2 Spectral Lines

Spectral emission of x-rays occurs when free electrons are incident on a target atom or ion with one or more bound electrons. The interaction has a certain probability to excite the bound electron to a higher energy state. When it decays back down to its lowest energy state, an x-ray in a very narrow range of energies is emitted. This is the same process which causes visible light emission from a plasma, but at much higher energies. For example, an electronic transition that produces visible light from a Hydrogen plasma is the H-α transition, which produces a photon with wavelength 656 nm, or 1.9 eV. An electronic transition detected by SDD in this thesis was the Carbon K-α transition, which produces a photon with 277 eV.

While the methods given later in this section are in principle applicable to spectral emission, we were not able to model the spectral emission from our plasma with enough accuracy to produce quantitative results. Such quantitative results derivable from spectral lines would be: impurity concentration and electron temperature.

Hutchinson’s *Principles of Plasma Diagnostics* provides a good introduction to models of spectral emission from plasmas.[19] The models are divided into different parameter regimes, with marginal conditions being: whether the plasma is optically thin or thick to the produced x-rays, and whether an excited state can be collisionally de-excited before it spontaneously emits an x-ray.

Laboratory plasmas are almost never optically thick to the x-rays produced, and the PFRC-II is no exception. If excited states spontaneously emit x-rays before they are collisionally de-excited, the plasma is well described by the Coronal Equilibrium
Model, so named for its application to the optically thin corona of the Sun. In this model, all upward transitions are by electron-impact collision and all downward transitions are by radiation. The relevant timescale is the first Einstein coefficient, $A_{i,j}$, which is the e-folding time for state $i$ to spontaneously decay into state $j$. $A_{i,j}$ are on the order of 0.1-10 ns.

If collisional de-excitations make up a significant proportion of de-excitations, the Collisional Radiative Model must be used. This model takes as degrees of freedom the density of each excited state $n_i$ and the radiation density at all frequencies $\rho(\nu_{i,j})$. From the second Einstein Coefficients $B_{i,j}$ and the averaged electron excitation rate densities $\langle \sigma_{i,j} v_e \rangle$, the relationship between the degrees of freedom is self-consistently determined. In general this must be done numerically.

The Collisional Radiative code FLYCHK, maintained by NIST, is available online.[25, 26] It is an implementation available for academic use. It produces expected x-ray spectral line and Bremsstrahlung spectra from Maxwellian or Bi-Maxwellian EEDFs. Other resources are the Flexible Atomic Code (FAC), the Hebrew University Lawrence Livermore Atomic Code (HULLAC), and the Sandia Central Receiver Approximation Model (SCRAM) code.[27–30]

We find that the x-ray volumetric emission from a specific transition (energy) can be approximated to within a factor of a few by applying the analytically tractable Coronal Equilibrium Model to the PFRC seed plasmas studied.

Approximate formulae can be found in Hutchinson for ionization from a specific state (Hutchinson 6.3.24 & 6.3.26) from a beam of electrons or a Maxwellian distribution respectively.[19] The same is true of excitation between states (Hutchinson 6.3.25
& 6.3.27). Both of these processes, evaluated from \( i = 1 \), produce a vacancy in the K-shell of a target atom or ion. Comparing them, we find that vacancies in the K-shell are more often produced by ionization than excitation in the PFRC-II.

When a vacancy in the K-shell occurs, an electron from an upper state fills it. When it does so, it gives its energy either to another electron or an x-ray. It gives its energy to an x-ray with a probability known and tabulated as the Fluorescence Yield.[31–33] We can therefore apply Hutchinson 6.3.24 to determine a rate density of K-shell vacancy events, then multiply by Fluorescence yield to determine a rate density of x-rays emitted from this vacancy. This procedure was applied to the data collected for this dissertation; its predicted count rate of x-rays at the Argon K-\( \alpha \) energy is generally 75% higher than the measured value.

As well as Fluorescence yields, K-shell ionization rate densities and x-ray K-\( \alpha \) emission rate densities are available tabulated in the literature.[31–34]

2.5 Spectral Inversion

The transformation between EEDF and X-ray Energy Distribution Function (XEDF) given in Equation 2.11 is invertible. In the language of linear algebra, this means that the matrix which constitutes the linear transformation from EEDF vector to XEDF vector is nonsingular. In practice, this means that we can determine information about the entire EEDF from careful measurements of the x-ray spectrum emitted by the plasma.
This capability is unique to broad-spectrum x-ray measurements; more than one EEDF can produce a given collection of spectral lines. Because of this, the transformation from EEDF to spectral line XEDF is non-invertible.

### 2.6 Derivation and justification of the cost function

The spectral inversion algorithm we developed to analyze SDD data recorded from the PFRC-II seed plasma can be described as a Poisson-Regularized Inversion. “Regularized” refers to the fact that its shape differs from the direct matrix inversion in favor of physics-based criteria like positive-definiteness and smoothness. I first published the precursor to this algorithm at the American Physical Society’s Division of Plasma Physics conference in 2016.[35]

Regularization of data is a well-developed field with fundamental basis lying in Bayes’s Theorem.[36] “Poisson” refers to the fact that we derive our regularization from the fact that the measurement reported by the SDD is a vector of independent Poisson random variables.

So why regularize at all? If the transformation between XEDF and EEDF is a matrix, and that matrix can be inverted, why not report the EEDF as $M^{-1} \cdot \vec{b}$, where $M$ is the transformation matrix and $\vec{b}$ is the measured XEDF? The answer is noise. Numerical and experimental noise is drastically amplified by this $M^{-1}$ matrix; consider the case that the XEDF is a monotone decreasing function, as it should be, except for one energy bin in which a fluctuation caused the XEDF to briefly have a positive derivative. This could not happen in the real world; Bremsstrahlung can only produce monotonic
decreasing XEDFs. A solution would require a portion of the EEDF to be negative. Unfortunately the $M^{-1}$ procedure is happy to return a negative answer, and so the method fails. The results returned by the $M^{-1}$ method are often oscillatory and exponentially divergent with number of bins.

Poisson-regularized inversion is most easily understood by contrast with the Least-Squares method of curve fitting (LSF). LSF is not relevant to the method presented in this chapter, but it is instructional.

It may be shown that the Least Squares solution to a fitting problem is the solution which has the Maximum A Posteriori (MAP) probability assuming that the measurements are Gaussian-distributed random variables centered on some function of the model vector.[36] Bayes’s Theorem is used to determine which model vector ($\vec{a}$) has the highest probability given the observation vectors ($\vec{b}$)

$$P(\vec{a}|\vec{b}) = \frac{P(\vec{b}|\vec{a}) \cdot P(\vec{a})}{P(\vec{b})}$$

(2.13)

where $P(q)$ is the probability of event $q$, and $P(q|p)$ is the probability of event $q$ given that event $p$ is true. Interpreting this in the context of spectral inversion, $\vec{a}$ is the vector of discretized EEDF values and $\vec{b}$ is the vector of observed x-ray counts.

This is an optimization problem in $\vec{a}$. Assuming that the prior ($P(\vec{a})$) is uniform, $P(\vec{a}|\vec{b}) \propto P(\vec{b}|\vec{a})$. In LSF, the measurements ($\vec{b}$) are assumed to have Gaussian probability density function centered on some function of the model vector $\vec{l}(\vec{a})$. Interpreting this as a spectral inversion problem, $\vec{l}(\vec{a})$ is the expected Bremsstrahlung measurement given a model EEDF, and Gaussian measurements.
As the logarithmic function is monotonic in its argument, maximizing Equation 2.14 with respect to \( \vec{a} \) is the same as minimizing the negative logarithm (the log-likelihood)

\[
\min (-\ln P_{\text{Gaus}}(\vec{b}|\vec{a})) = \min \left( \sum_i \frac{(b_i - l_i(\vec{a}))^2}{2\sigma_i^2} \right)
\]  

(2.15)

The \( \vec{a} \) at which this is achieved is said to be the LSF solution or the \( \chi^2 \) minimum solution. Tikhonov regularization, as used by Piana for spectral inversion, combines this cost function and a set of priors that allows the absolute amplitude, slope, curvature, etc. to be constrained, if desired.[36, 37]

We now turn back to the case of Poisson random variables. We wish to find the equivalent of Equation 2.15 for measurements \( \vec{b} \) distributed as Poisson random variables around some function of the model \( \vec{l}(\vec{a}) \). We find that

\[
P_{\text{Pois}}(\vec{b}|\vec{a}) = \prod_i \frac{[l_i(\vec{a})]^{b_i} e^{-l_i(\vec{a})}}{b_i!}
\]  

(2.16)

\[
\min(-\ln P_{\text{Pois}}(\vec{b}|\vec{a})) = \min \left( \sum_i \left( l_i(\vec{a}) - b_i \ln(l_i(\vec{a})) \right) \right)
\]  

(2.17)

Here \( \ln b_i! \) has been neglected as it does not contribute to the optimization.

The XEDF (\( \vec{l}(\vec{a}) \)), Equation 2.12, is linear in EEDF (\( \vec{a} \)). Thus, in discretized form the transformation is a matrix \( \vec{l}(\vec{a}) \rightarrow M\vec{a} \). In the language of matrices, we wish to
minimize the following function

\[ C(\vec{a}) = \sum_i [(M\vec{a})_i - b_i \ln(M\vec{a})_i] \]  \hspace{1cm} (2.18)

with respect to \( \vec{a} \), where \( \vec{a} \) is the vector of EEDF values, \( \vec{b} \) is the matrix of measured XEDF values, and \( M \) is the matrix which transforms EEDF into XEDF given by Equation 2.12

In practice, we solve this optimization problem numerically, as described in the next section. As in Tikhonov regularization, we add a small “smoothness” prior to the cost function,

\[ C_s(\vec{a}) = \sum_j \left( \frac{a_{j+1} - a_j}{a_j} \right)^2 E_{\text{corr}}^2 2\Delta E_c^2 \]  \hspace{1cm} (2.19)

where \( E_{\text{corr}} \) is some correlation energy and \( \Delta E_c \) is the energy spacing of the EEDF vector. This is a Bayesian prior that neighboring EEDF points will be correlated. We chose \( E_{\text{corr}} \) to be small, \( \sim 5eV \), to avoid introducing unphysical results. The addition of this smoothness prior ensures that, in regions of little information about the EEDF, it will behave in a plausible manner. Smoothness is not included in uncertainty computation.

Note that the number of elements in \( \vec{a} \), which is the same as the number of columns in \( M \), is arbitrary. The error of \( a_j \) values will be discussed shortly. Choosing a small value of the dimensionality of \( \vec{a} \) will decrease the uncertainty of each \( a_j \) but not resolve features of the derived EEDF. Choosing a large value of the dimensionality of \( \vec{a} \) will increase the uncertainty of each \( a_j \) but resolve better the energy-space of the EEDF. Natural choices for the dimensionality of \( \vec{a} \) are the number of measured points in \( \vec{b} \) for
maximum resolution, or the number that gives the energy spacing of $\vec{a}$ the same resolution as the energy-resolution of the measurement, to balance resolution and uncertainty.

This Poisson-regularized inversion is well-suited to XEDFs that suffer from poor counting statistics. If Tikhonov-regularized inversion is used in cases of low signal-to-noise ratio (SNR), positive-definiteness is not assured and the derived EEDF can be negative, a non-physical result. This is because the assumption that the measurement is Gaussianly distributed is only justified at high numbers of x-rays counted.

Poisson-regularized inversion is also well-suited to low-energy-resolution data. The Fano noise in energy described earlier this section can be made into a response matrix when the response functions to many different energies are concatenated as the columns of a matrix. This will be discussed later. Multiplying this matrix $M_{res}$ into the EEDF - XEDF transformation $M_{Brem}$ yields a resolution-included response matrix $M_{res\cdot Brem}$ which allows the resolution of the detector to contribute to the derived EEDF and its reported uncertainties in a correct and self-consistent way. This procedure is not practiced in the most widely used Tikhonov-regularized inversions.[38, 39]

Poisson-regularized inversion also yields natural uncertainties and error bars in a self-consistent way that Tikhonov-regularized inversion does not. The systematic uncertainty of each point $a_j$ may be determined by finding the value of $a_j + \Delta a_j$ which increases $C(\vec{a})$ by the value $1/2$, equivalent to a $1\sigma$ deviation for Gaussian statistics. The statistical uncertainty of a channel $j$ may be found in the following way: all points except $a_j$ are optimized to minimize $C(\vec{a})$. When this new $C$ is larger than the old $C$ by $1/2$, $a_j$ is $1\sigma$ from its nominal value.
Chapter 2.7

2.7 The numerical algorithm implemented in MATLAB

The field of high-dimensional optimization has advanced in recent years due to the commercial application of machine learning. The method implemented here is not the state of the art; rather it is the point at which further development of the algorithm produced diminishing returns. The most correct answer to the question of how to fully generally optimize a high-dimensional scalar function is with stochastic gradient descent. Such a method is now a standard tool in the repertoire of Artificial Intelligence researchers. It is even one step in the training algorithm of Alpha Go Zero, the algorithm that famously learned how to play the board game Go at expert levels without human-generated training sets.[40]

As implemented for analysis of PFRC-II seed plasma x-ray data, the algorithm is not a stochastic gradient descent. It is rather a successive Quasi-Newton solver. “Successive” in this case means it considers one “dimension” of the data at a time. The algorithm loops over electron energy and optimizes the value of EEDF at that energy at each step.

A Quasi-Newton iteration step involves evaluating the scalar function at 3 points, assuming a parabolic functional form of the function, and finding the analytic extremum of the parabola.

\[ x_{n+1} = x_n (1 + \epsilon \frac{f_+ - f_-}{2(f_+ + f_- - 2f_0)}) \]  

\[ f_{\pm} = f(x_n \cdot (1 \pm \epsilon)) \]  

(2.20)
\[ f_0 = f(x_n) \]

where \( x_n \) is the iteration step and not the dimension. This example is for one-dimensional function \( f(x) \). \( \epsilon \) is a free parameter of the solver.

This algorithm is not the classical Quasi-Newton algorithm given in textbooks; that is usually formulated in terms of a free parameter which is added to \( x_n \) to form the guesses rather than a free parameter \((1 \pm \epsilon)\) which multiplies \( x_n \) to form the guesses. Having the guesses formed multiplicatively rather than additively allows the same free parameter \( \epsilon \) to serve equally well for \( x_n \) spaced over many orders of magnitude.

Even with our adjustment (multiplicative \( \epsilon \)), choosing an appropriate value of \( \epsilon \) remains a problem. For the best results, \( \epsilon \) should be near that value which makes \( x_n \pm \epsilon \) an extremum. However \textit{a priori} this \( \epsilon \) is not known. The following two measures are taken to ensure good results in light of this problem.

The MATLAB language has wonderful support for vectorization of problems, or performing identical operations on different data at a large speed-up compared to serialized operations. This can be achieved through parallelization, but can also be achieved through optimization of single threaded processes. It is therefore efficient to have the algorithm applied to four different \( \epsilon \)-values simultaneously. The scales are chosen in an ad-hoc manner and are observed to work well. They are \( \epsilon = 4^{-[1,2,3,4] + 1/2} = [0.5, 0.125, 0.03125, 0.0078125] \).

Even this measure is not always sufficient to choose an appropriate \( \epsilon \). Another optimization step is therefore performed using the values \(|x_{n+1} - x_n|/x_n\) for each \( \epsilon \)
as a new $\epsilon$. This can be seen as not only iterating to determine optimal $x$ but also to determine optimal $\epsilon$ to better find the optimal $x$.

$$x_{n+1} \geq 0.1x_n$$ is enforced to prevent $x$ from temporarily becoming negative or zero, as the domain of the cost function is positive definite.

For each dimension of the scalar function, in this section called $f$, a total of 21 values of $f$ are computed per iteration step. These are $f_0$, the eight $f_\pm$ for each prior value of $\epsilon$, the eight $f_\pm$ for each posterior value of $\epsilon$, and the four $f$ values evaluated from those posterior $f_\pm$. $x_{n+1}$ is chosen to be the $x$-value which produces the most extreme $f$, regardless of which stage in the algorithm produces the $f$.

In pseudocode, the algorithm is:

$$\epsilon_{1,i} = 4^{-[1,2,3,4]+1/2}$$ (2.21)

$$f_0 = f(x_n)$$

$$f_{\pm,1,i} = f(x_n \cdot (1 \pm \epsilon_{1,i}))$$

$$x_{n+1,1,i} = x_n(1 + \epsilon_{1,i}\frac{f_{-,1,i} - f_{+1,i}}{2(f_{+,1,i} + f_{-,1,i} - 2f_0)})$$

$$\epsilon_{2,i} = |x_{n+1,1,i} - x_n|/x_n$$

$$f_{\pm,2,i} = f(x_n \cdot (1 \pm \epsilon_{2,i}))$$

$$x_{n+1,2,i} = x_n(1 + \epsilon_{2,i}\frac{f_{-,2,i} - f_{+2,i}}{2(f_{+,2,i} + f_{-,2,i} - 2f_0)})$$

$$f_{u,2,i} = f(x_{n+1,2,i})$$
where the final value of $x_{n+1}$ are drawn from the $x$ that, when evaluated, produces whichever is most extreme of $f_0, f_{\pm,1,i}, f_{\pm,2,i}, f_{u,2,i}$. Subscript 1 and 2 are named “prior” and “posterior” in the preceding paragraph.

### 2.8 Current state of the art in spectral inversion

#### 2.8.1 Piana and RHESSI

The RHESSI solar observation satellite is an example of a well supported experiment for which x-ray spectral inversion has yielded key insights into the physics being studied.[41] RHESSI contains an x-ray spectrometer built on the pulse-height detector plan. Its chemistry is n-type doped Germanium rather than the n-type doped Silicon of the SDD. It employs a rotating collimator system to resolve x-rays both in energy and angle to not only take spectra but also images of solar flares. The Ge detectors can record x-rays from 2.7 keV to 17 MeV (highly relativistic) with energy resolution of 1 keV FWHM for much of that range.[42]

RHESSI is a wonderful sandbox for ideal spectral inversion. Its energy range spans 3 orders of magnitude, its spectrum is energy resolved to better than 1%, its spectrum is devoid of spectral lines, and its number of counts make Poisson (counting) statistics a negligible part of the overall error.

The algorithm that researchers have used to spectrally invert RHESSI data was formulated in two papers by Piana et. al.[37, 38] It was favorably evaluated by another RHESSI researcher.[43] It is based on Tikhonov regularization of the Bremsstrahlung
transformation. This is equivalent to evaluating the LSF solution, Equation 2.15, but where \( l_i(\vec{a}) = (M_{Brem} \vec{a})_i \) rather than the full \( l_i(\vec{a}) = (M_{tot} \vec{a})_i \). There is also a smoothness term added, and each \( \sigma_i \) is assumed to be 1. For these reasons, as will be seen later, the inversion performs less well under the non-ideal conditions of the PFRC-II experiment.

Before this version of spectral inversion was applied, the most that could be said of a particularly rich spectrum from a solar flare of July 23, 2002 was that it had strong non-power-law behavior.[44] With Tikhonov spectral inversion, the full EEDF of the plasma that constituted the flare was recovered.[38]

### 2.8.2 DeGaSum

The DeGaSum code was developed to invert gamma-ray emissions from fusion plasmas like Tokamaks.[45, 46] It is based on an algorithm which does strictly maximize Poisson likelihood, Equation 2.17.[47] It has been used to great effect in the JET Tokamak.

The algorithm as implemented is a truncated gradient descent with a large step size. Rather than include smoothness as a Prior factor in Bayes’s Theorem, every few iterations the spectrum is numerically smoothed. Because of this there is no true optimal solution to which it converges.

The error bars are found by using a Monte-Carlo algorithm which permutes the measured XEDF and produces one EEDF value, then produces statistics from those EEDF values, rather than producing a probability distribution for the high-dimensional EEDF and drawing statistics from this function.
The EEDF - XEDF transformation matrix is found with using a Monte Carlo radiation effects code, rather than an experimental calibration.

### 2.9 Detector response function

Essential to the algorithm is the specific transformation function, $\vec{l}(\vec{a})$ in Equation 2.17 and equivalently the $M$ matrix in Equation 2.18. This is the detector’s response to the presence of a mono-energetic EEDF.

$M$ consists of three response matrices, multiplied together.

$$M = M_{\text{res}} \cdot M_{\text{trans}} \cdot M_{\text{Brem}}$$  \hspace{1cm} (2.22)

where $M_{\text{Brem}}$ is the Bremsstrahlung production matrix, $M_{\text{trans}}$ is the window transmission matrix, and $M_{\text{res}}$ is the resolution response matrix. They must be multiplied in this order, as these transformations are not commutative.

If the physics and detector are well known, then $M$ is specifiable analytically. In practice, all or some of $M_i$ matrix factors in Equation 2.22 must be measured.

#### 2.9.1 Transmission

$M_{\text{trans}}$ is the transmission efficiency matrix that transforms the XEDF produced by the plasma to the XEDF which penetrates the detector window and is incident on the detector. As this process does not change the energy of the x-rays, $M_{\text{trans}}$ is a diagonal
matrix. The value of element $M_{\text{trans},i,i} = \vec{m}_{\text{trans},i}$ is the value of the transmission efficiency of the window and detector at energy $E_{x,j}$.

This transmission efficiency is published for each pair of Amptek detectors and windows by Amptek. [16] We used a X-123 FAST SDD detector with a $\text{Si}_3\text{N}_4$ C1 window. The Amptek C1 window is composed of a 90-nm layer of silicon nitride, a 250-nm layer of grounded aluminum, and a 78%-open grid made of 15-$\mu$m-thick silicon. The SDD is 500-$\mu$m thick which limits the detector's upper energy.

The transmission efficiencies of this combination of window and detector are published, or can be calculated from the given dimensions using NIST's database of x-ray attenuation coefficients, which tabulate calculations of Seltzer.[17, 18]

$$M_{\text{trans}} = \text{Diag}[\vec{m}_{\text{trans}}]$$

(2.23)

where $\vec{m}_{\text{trans}}$ is the vector of transmission efficiencies at energies $E_{x}$.

The windows are manufactured with a $\pm 10\%$ thickness tolerance of the silicon nitride film; when instructing how to determine line amplitudes for x-ray material assaying, Amptek instructs the user to first calibrate with a known source. In Section 2.10 I extend this procedure to broad-spectrum x-ray emissions.

### 2.9.2 Resolution

The resolution of pulse-height detectors are limited by two phenomena: electrical (thermal, shot, etc.) noise of the counting electronics and counting statistics of the finite number of electron-hole pairs produced by the incident x-ray.
This latter produces a full-width-at-half-maximum (FWHM) which is proportional to the square root of the signal. This is so-called Fano noise [5] and, in the Amptek X-123 FAST SDD detector, has form[16]

\[
FWHM_{\text{Fano}} = \sqrt{2.404 \text{eV} \cdot E}
\]  
(2.24)

where all quantities are in eV.

The electrical noise is constant with respect to incident x-ray energy. According to the Amptek calibration procedures, the response function to a monoenergetic x-ray distribution is mostly Gaussian centered at the x-ray energy and with FWHM contributions from electrical and Fano noise.[16] The FWHM of the electrical noise is measured each time data is recorded by first recording the measured XEDF from no x-ray signal, yielding a Gaussian near 0 eV with the correct FWHM.

\[
FWHM(E_x) = \sqrt{FWHM_{\text{elec}}^2 + FWHM_{\text{Fano}}^2(E_x)}
\]  
(2.25)

Thus the resolution response matrix \(M_{\text{res}}\) which transforms the XEDF vector which is incident on the detector into the measured XEDF vector is many column vectors of discretized Gaussian values concatenated into rows, the central values and FWHMs of which depend on the row.

\[
M_{\text{res},i,j} = \frac{1}{\sqrt{2\pi\sigma(E_{x,j})}} e^{-\frac{(E_{x,i}-E_{x,j})^2}{2\sigma^2(E_{x,i})}}
\]  
(2.26)

where \(\sigma(E_{x,i}) = FWHM(E_{x,i})/(2\sqrt{2\ln 2})\).
2.9.3 Bremsstrahlung

$M_{\text{Brem}}$ was given during the discussion of the phenomenon of Bremsstrahlung. It is Equation 2.12 with the Gaunt factor described in 2.10.

Equations 2.12 and 2.10 assume that either the target atom is fully ionized or that the energy of the x-ray is larger than the K-edge energy of the inner electron shell of the target. Atoms with bound electrons emit x-rays with the same energy as the bound electrons in a very different functional form, which I do not implement as there is not an accurate closed-form expression.[21]

The criterion that the energy of the x-ray is larger than the K-edge energy of the inner electron shell of the target is always valid for Hydrogen (K-edge 13.6 eV). Given the energy range of interest for recorded x-rays in this dissertation, 600 eV - 30 keV, it is also valid for Carbon, Nitrogen, and Oxygen (284 eV, 410 eV, and 543 eV respectively). It is not valid below 870 eV for target gases with a significant portion of Neon, nor below 3206 eV for target gases with a significant portion of Argon.[31]

The $Z^2$ factor in Equation 2.12 has important practical implications: For accurate calculation of the electron density, $Z_{\text{eff}} = \sqrt{\langle Z^2 \rangle}$ must be known. It does not take a large concentration of Carbon or Oxygen to increase the emissivity of x-rays by a factor of two over pure Hydrogen; 0.7% of Carbon or 0.4% of Oxygen is sufficient. As these levels are commonly exceeded in the PFRC-II, careful measurement of $Z_{\text{eff}}$ must be carried out.

The strength of the K-α spectral line can in principle determine the concentration of these elements. However as I discuss in Section 2.4.2, elementary models of spectral
line emission do not have the required accuracy. In the PFRC-II, we therefore record the concentration of gases measured with an SRS Residual Gas Analyzer in the FEC and calculate the concentration of gases in the target chamber based on gas conductance between the chambers.

2.10 Calibration with electron beam

I first published the calibration procedure in this section at the American Physical Society Division of Plasma Physics conference in 2017.[48]

To calibrate the SDD, a gas-target x-ray tube was built using a 20-cm total length Pyrex cross with 5-cm inner diameter, depicted in Figure 2.4. At the far right port of the cross, a 0.010” diameter tungsten filament is heated by a DC power supply to thermionically emit electrons. The potential along the filament’s length varies about
3 V with respect to ground. The electrons are extracted by a nearby stainless steel extractor grid at +50 V and then accelerated into the gas target volume. At the left port of the cross, a triangular prism carbon target is biased up to 5 kV. So that the electrical potential and electron energy are constant in the gas target volume, a stainless steel drift-space liner is placed around the gas-target volume, with two 5-cm holes at the top and bottom to allow line-of-sight of the detector through the liner and pumping. This drift-space liner is biased at the carbon target voltage. Wrapped around the horizontal arms of the cross are 25 turns each of wire carrying 4 A of current, providing a \( \sim 4G \) axial magnetic field to contain the electron beam. The current collected by the carbon target and drift space liner is typically 100\( \mu \)A.

At the bottom port of the cross, gas is fed into the gas target volume by an adjustable needle valve. A Leybold-Heraeus Turobovac 150 turbomolecular pump maintains a base pressure below \( 10^{-5} \) Torr, as measured by a Granville-Phillips ion gauge. The gas pressure is kept low enough that the thermionic current is not noticeably supplemented by current from ionization of neutral gas, i.e. the current collected by the target at base pressure is not smaller than the current collected by the target at operational pressure.

At the top port of the cross, the X-123 FAST SDD detector is mounted, with a grounded stainless steel 3-baffled collimator to restrict line-of-sight.

For the summer of 2016, I supervised SULI intern Alexandra Bosh. She performed some of the measurements described in this section.
2.10.1 Energy calibration

The energy scale of each detector must be calibrated for each combination of pulse-shaping parameters. At its most basic level, the data obtained in every run is a list of counts per energy channel, \( b \). The energies to which they correspond and the width of the energy bin are not specified. Amptek instructs the user to use known, calibrated signals from radioisotopes such as Iron-55 to calibrate this energy scale.[16]

We produce x-ray signals at known energy by using the gas-target x-ray tube that we constructed for this purpose. Using different fill gases with different K-\( \alpha \) spectral line energies we are able to calibrate the energy scale over a large range of energies, from the K-\( \alpha \) line of Carbon at 277 eV to the K-\( \alpha \) line of Argon at 2958 eV. We consider the position of six elemental K-\( \alpha \) lines: Carbon from methane gas fill, Nitrogen from air gas fill, Oxygen from air gas fill, Neon from neon gas fill, Aluminum from x-ray fluorescence from aluminum foil, and Argon from air gas fill.

The detector has configurable time constants, including pulse rise time and flat top, that affect the tradeoff between resolution and count rate [16]. The pulse peaking time and flat-top time also affect the energy calibration of the detector. We calibrated the energy scale of the detector under 7 peaking and flat-top time combinations applicable to our experiment. In each case the calibration function is found to be linear; that is the \( N \)th energy bin corresponds to energy \( E = A + B.N \). Energy calibrations performed months apart vary by < 0.5%.

The detector known to our group as SDD1 can be absolutely identified by its serial number 15555. SDD2 has serial number 16645. The energy calibration of SDD1 began
on 2017/02/02. The energy calibration of SDD2 began on 2017/10/31. The results are shown in the table below. The slope and offset are applied in slope-intercept form, 
\[ y = mx + b \] where \( y \) is the central energy of the bin, \( m \) is the slope given below, \( x \) is the bin number starting at 1 (not Amptek’s convention of starting at 0), and \( b \) is the offset given below.

<table>
<thead>
<tr>
<th>Timing (rise/flat)</th>
<th>SDD1 Slope</th>
<th>SDD1 Offset</th>
<th>SDD2 Slope</th>
<th>SDD2 Offset</th>
</tr>
</thead>
<tbody>
<tr>
<td>100ns/12ns</td>
<td>9.62 eV/ch</td>
<td>-25.8 eV</td>
<td>9.05 eV/ch</td>
<td>-21.6 eV</td>
</tr>
<tr>
<td>200ns/12ns</td>
<td>9.13 eV/ch</td>
<td>-27.0 eV</td>
<td>8.80 eV/ch</td>
<td>-25.3 eV</td>
</tr>
<tr>
<td>400ns/12ns</td>
<td>8.87 eV/ch</td>
<td>-22.5 eV</td>
<td>8.60 eV/ch</td>
<td>-18.7 eV</td>
</tr>
<tr>
<td>1us/12ns</td>
<td>8.70 eV/ch</td>
<td>-13.4 eV</td>
<td>8.50 eV/ch</td>
<td>-15.1 eV</td>
</tr>
<tr>
<td>2us/200ns</td>
<td>8.63 eV/ch</td>
<td>-19.3 eV</td>
<td>8.44 eV/ch</td>
<td>-16.0 eV</td>
</tr>
<tr>
<td>5.6us/200ns</td>
<td>8.63 eV/ch</td>
<td>-15.7 eV</td>
<td>8.43 eV/ch</td>
<td>-13.2 eV</td>
</tr>
<tr>
<td>9.6us/200ns</td>
<td>8.64 eV/ch</td>
<td>-16.5 eV</td>
<td>8.43 eV/ch</td>
<td>-12.3 eV</td>
</tr>
<tr>
<td>25.6us/200ns</td>
<td>8.65 eV/ch</td>
<td>-16.5 eV</td>
<td>8.43 eV/ch</td>
<td>-11.4 eV</td>
</tr>
</tbody>
</table>

### 2.10.2 Transmission calibration

As depicted in Figure 2.5, we recorded the XEDF from the monoenergetic EEDF produced by the gas-target x-ray tube. Its shape differs slightly from the shape expected from the published x-ray transmission efficiency. This is because the C1 window used in our detector was manufactured to some finite tolerance, with each of its thicknesses subject to some variation. The four free parameters that characterize the C1 window are: Silicon Nitride layer thickness (nominally 90 nm), Aluminum backing thickness
Figure 2.5: Transmission Calibration for SDD1. In blue, the XEDF as directly reported by the SDD. In green, the XEDF expected from a monoenergetic EEDF assuming the published transmission efficiencies. In red, assuming the transmission efficiency produced by the fit parameters given in the text. Sudden jumps occur because of the K-edge energies of the elements in the window: Aluminum exhibits a jump in x-ray absorptivity at 1559 eV and Silicon exhibits a jump in x-ray absorptivity at 1839 eV.

(nominally 250 nm), Silicon grid thickness (nominally 15 μm), and Silicon grid open fraction (nominally 78%).

For SDD1, as depicted in Figure 2.5, thicknesses and open fractions slightly different than this give improved fit with the measured XEDF. The best fit is produced by Silicon Nitride layer of 64 nm, Aluminum layer of 244 nm, Silicon grid thickness of 23 μm, and Silicon grid open fraction of 80%.

The agreement between the calculated monoenergetic XEDF and measured XEDF gives us confidence in our Bremsstrahlung production model.
2.11 Spectral line subtraction

$M_{Brem}$ from Equation 2.12 was formulated only considering x-rays from Bremsstrahlung emission. Because of this, the presence of line radiation in the energy domain-of-interest causes the algorithm to fail. See Figure 2.6a.

Because of the large, obvious unphysical features in the EEDF of Figure 2.6a, this method can be used to find spectral lines. The failure of a fit indicates that the measured x-ray spectrum could not have been produced by Bremsstrahlung only, and that a spectral line must be present.

To invert XEDFs with prominent spectral lines in the domain-of-interest, we use the following procedure: We determine the energy range over which the spectral line
extends. This corresponds to some range of $i$ values in Equation 2.18. We remove this interval from the summation in that equation. This method removes the unphysical features of the spectral line, and produces a plausible but uncertain EEDF in the vicinity of the excised line. To understand why this is the case, we must first discuss some of the properties of the reconstructed EEDF’s dependence on the measured XEDF.

Normally the EEDF in some energy range $E_i$ is determined mostly by the local behavior of the XEDF in the vicinity of $E_i$ and partly by the behavior of the XEDF at lower energy than $E_i$. Both behaviors of the XEDF are “evidence” in the Bayesian sense of the word which specify the value and uncertainty of the EEDF. If a spectral line is excised with this method, however, EEDF values are determined by the XEDF only at lower energies. These EEDF points have larger uncertainty because the lower-energy XEDF points are less sensitively dependent on these EEDF values. Large EEDF features may produce only small XEDF changes when the energy range of interest is excised. This method is depicted in Figure 2.6b.

Spectral lines can also be removed “manually,” by choosing for each line a candidate peak location ($E_p$), height ($h_p$), and width ($w_p$), subtracting that peak from the measured XEDF and varying $E_p$, $h_p$, and $w_p$ to minimize the XEDF deviation from a “smooth line” in the vicinity of the peak. Essentially, the resultant XEDF is chosen to appear, “by eye,” as though there were no spectral line. Because of the shape of $M$, small differences in the XEDF correspond to large differences in the EEDF, so care must be taken not to introduce artificial excursions from the correct EEDF. This method has the benefits of corroborating the SDD’s energy resolution and measuring the impurity content. This method has the drawback that human bias is introduced. This method is
depicted in Figure 2.6c. I do not employ this method in this dissertation.

The Gaussian peaks removed to create Figure 2.6c are centered at 2964.5 eV and 3196.6 eV, very close to Argon spectral lines, indicating that our energy calibration is correct to 0.2%. The peaks have FWHMs of 91.4 eV and 93.4 eV, within Amptek’s specifications for this range. The XEDF falls over this energy range while the EEDF appears flat. This is not a contradiction, as a constant EEDF means that more and more electrons can produce x-rays of a given energy as that energy decreases. The dip in EEDF at 3600 eV may be due to an effect discussed in Avdonina et al.: the complicated behavior of the Gaunt factor of Avdonina and Pratt around an electron shell energy.[21]

The most correct way to add spectral line considerations is to include a model for their production from a general EEDF. This model would modify the x-ray response function $M_{\text{Brem}}$. This requires the concentrations of the emitting gases to be known. For our plasmas, the correct model is the collisional radiative model, which is non-linear in the EEDF and requires knowledge of particle confinement, and cannot be subjected to this analysis. We did not implement this feature.

## 2.12 Example SDD data collection procedure

This section is for future PFRC students who wish to collect x-ray data using the mounted SDDs.

A data collection run begins with turning on the SDD. All that’s necessary for this is for the power and USB cables to be plugged into the back of the X-123SDD unit.
The power port is labeled “5V” and the USB port is labeled “USB.” The Peltier cooler immediately starts cooling the detector.

The SDD is controlled through the software DppMCA (Digital pulse processor Multi Channel Analyzer). It is available for download on the Amptek website.[16]

Start DppMCA and wait until the temperature of the SDD is stabilized. In the Acquisition Setup menu, choose the timing conditions that meet your dual criteria of resolution and count rates. Set the slow channel threshold high enough that the Gaussian noise counts are not collected. Channel 20, or 2%, is typical for a 5.6 \( \mu \text{s}/200 \text{ ns rise/flat top time at 35x gain} \). Set the fast threshold such that the Input Count Rate is about 1 count per second. The fast channel and Input Count Rate are not used to record x-ray energy spectra; they are only used to measure raw count rates and reject pulse pileup. Channel 13 is typical.

Save a noise spectrum, slow threshold of zero, for later deconvolution. It is this spectrum from which the electrical noise is evaluated.

During collection, be constantly vigilant regarding the Total count rate and the Input Count Rate. For count rate \( CR \), the proportion \( CR \times \tau_{\text{fast}} \) is the proportion of counts which are double-counts, and therefore in error. This number should be very small. The proportion \( CR \times \tau_{\text{slow}} \) is the proportion of counts which are thrown out, potentially distorting your spectrum. This number should be small depending on how tolerant your application is. 1% is generally too high.

Noise conditions in the PFRC-II are constantly changing. Noise affects the fast and slow channels differently. Remain constantly vigilant to ensure that the Input Count and
Chapter 2.13

The default gain is 35x. At typical conditions, a gain of 35x makes the energy range extend up to a maximum of 8800 eV. Different gain values change this extent recipro-
cally. The gain divides the energy calibration and offset, effectively scaling the channel number. However, at rise times faster than 100 ns, gain is not perfectly reciprocal. You must perform a calibration in-situ in this situation.

Record the gas pressure in the target chamber, as this is a factor in Equation 2.12. Record the concentration of the various impurity gases from the SRS Residual Gas analyzer for the $Z_{\text{eff}}$ analysis described in Section 2.9.3.

2.13 Comparison with Piana et. al.

In order to illustrate the differences between the Poisson-regularized inversion of this paper and the Tikhonov-regularized inversion of Piana et. al., we suppose a functional form for an EEDF, depicted in Figure 2.7a. From this EEDF, we generate a synthetic XEDF by applying the transformation matrix we found in Section 2.9, $\tilde{m}_{\text{true}} = M \tilde{f}_{\text{true}}$. For each energy bin $E_{x,j}$, a random value is generated from a Poisson distribution centered at $m_{\text{true},j}$. This synthetic XEDF, $m_{\text{Poisson}}$, should be thought of data that could plausibly come from a laboratory plasma. It incorporates the effect of Bremsstrahlung, window transmission, finite resolution, and counting statistics.

The specific functional form of $f_{\text{true}}$ is based on a Maxwellian distribution with density $n_e = 5 \times 10^8/\text{cc}$ with temperature 320eV. We assume a target of 0.30 mTorr Hydrogen gas. Data is “accumulated” for 45 minutes. These are all plausible conditions for
Figure 2.7: An example comparison of Poisson-regularized Inversion and the Piana algorithms for recovering an EEDF. a) The true, supposed EEDF and the synthetic x-ray data generated from it. b) Computed EEDFs compared to the true EEDF. c) Discrepancy factor between each computed EEDF and the true value.
the PFRC-II experiment. As can be seen in Figure 2.7, both the Poisson-regularized inversion and that of Piana capture this behavior as far as it persists in energy. Piana’s EEDF exceeds 30% discrepancy at 1600eV. Poisson-regularized EEDF does not exceed 30% discrepancy until 2500eV, where the Maxwellian character ends.

To demonstrate resolution of sharp features, we add a jump in EEDF at 2500eV. As can be seen in Figure 2.7a, counting statistics (Poisson error) have become significant in this energy range. Despite this complication, both computed EEDFs show a jump of comparable amplitude at 2500eV, with similar artifacts due to resolution.

To demonstrate sensitivity, a tenuous beam is added. We chose 3550eV with density \( n_{\text{beam}} = 9 \times 10^5/\text{cc} \). In this energy range, the Poisson error in the synthetic XEDF are extreme, with variance > 35%. There are only 67 synthetic x-rays “measured” above 3550eV. The Poisson-regularized EEDF clearly shows a beam centered at 3550eV, however the density excess is 15% higher than the true density of the beam, and the FWHM of the beam is 250eV, larger than the 100eV of the true beam. The EEDF calculated from the algorithm of Piana shows no detectable beam-vs-background relationship from which a density could be calculated.

The ability to detect a beam of \( \sim 0.1\% \) of the bulk density is potentially of great use. A beam with these parameters can be injected from an electron gun, providing a valuable diagnostic of confinement time and slowing down profile. A beam of these parameters can be a source of free energy for instabilities such as two-stream instability.

These data confirm the claim in Section 2.6 that Poisson-regularized inversion is more suitable than Tikhonov-derived inversions in the case that the relevant EEDF feature is in an energy range in which uncertainties and resolutions are high. When the
signal-to-noise ratio is low, the Tikhonov-derived Piana inversion produces unreliable results where Poisson-regularized inversion reproduces the energies and amplitudes of important EEDF features.

The first direct comparison between inversion methods by the PFRC group was performed by Himawan Winarto, an undergraduate whom I supervised who joined the PFRC group for the fall of 2016 for the purpose of writing his Junior Paper for Princeton University.

2.14 Example use case

Two raw XEDF spectra and their inverted EEDFs are shown in Figures 2.8 and 2.9. The precise conditions of this run and the physical relevance of the observed EEDF will be discussed in subsequent chapters; these are intended as examples only.

Figure 2.8 was obtained with 350 W of forward RF power being deposited into the antenna and hydrogen plasma. The gas pressure in the CC was 0.43 mTorr and $1.4 \times 10^{-5}$ Torr in the FEC. The magnetic field in the midplane of the CC was 70 G and 2.2 kG at the nozzle. The floating potential of the tantalum paddle in the FEC was -1.2 kV.

The raw spectrum shows N and O K-α x-rays, necessitating the removal of that portion of the spectrum as in Section 2.11. Figure 2.8 shows a mostly exponential EEDF. This population of electrons could well be thermal with an effective temperature of 340 eV. If this Maxwellian distribution continued down to 0 eV, this population would have a density of $6.7 \cdot 10^9$/cm$^3$. The low-energy portion of the spectrum is made uncertain
by the presence of the N, O lines, but it appears that this Maxwellian behavior does not extend down to 500eV in EEDF, instead becoming “colder” and steeper at this energy.

By increasing the pressure in the FEC to $4.9 \cdot 10^{-5}$ Torr, a factor of 3.5 larger than in Figure 2.8, the floating potential of the Tantalum paddle in the FEC becomes far less negative, -20 V, and the paddle glows cherry red hot.

The increase in potential, from -1.2 kV to -20 V, and power, dark to red-hot, to the Tantalum paddle is indicative of a fast-electron created cold plasma in the FEC. Before the pressure is increased, the fast electron population is sufficient to balance the bulk ion current to the paddle, keeping its floating potential at -1.2 kV and keeping the power in the bulk plasma from heating the paddle. After the pressure is increased, we suspect that the fast electrons produce a plasma whose larger density is sufficient to set the paddle floating voltage to the measured -20 V, allowing energy in the bulk plasma to heat the paddle. I describe this fast-electron produced plasma more fully in Chapter 6.

The EEDF derived from this configuration is shown in Figure 2.9. It displays a very different EEDF; at higher energies than 1800 eV, the EEDF has a much sharper fall than the low pressure condition, with an e-folding energy of 220 eV. This may be attributed to the effect of the Tantalum paddle no longer providing any electrostatic confinement of particles in the FEC. At energies lower than 1800 eV, the EEDF shallowed, appearing flat. Flattening of the EEDF can arise from slowing-down of a beamlike distribution, [19] or significant wave activity in the range of wave phase velocities resonant with that range of electron energies.
Figure 2.8: A typical XEDF and EEDF from the CC in tandem mirror mode. The red line is the measured XEDF, directly taken from the SDD. The black line and associated dotted lines are the derived EEDF and uncertainty, corresponding to 1σ. The blue dashed line is this EEDF re-transformed into the expected measured XEDF for reasons of comparison. The EEDF is mostly exponential with an e-folding energy of 340 eV. The large uncertainty of the region around 500eV is caused by spectral lines obscuring Bremsstrahlung spectrum, excised as in Figure 2.6b.

Figure 2.9: XEDF and EEDF from the CC in tandem mirror mode with a high pressure in the far end cell. The red line is the measured XEDF, directly taken from the SDD. The black line and associated dotted lines are the derived EEDF and uncertainty. The blue dashed line is this EEDF re-transformed into the expected measured XEDF for reasons of comparison. The EEDF is split into two domains: below 1800eV it appears flat. Above 1800eV it appears exponential with e-folding energy 220eV.
2.15 Conclusion

In this section, I have described the specific pulse-height detector that I used to collect x-ray data from the PFRC-II apparatus, the Amptek SDD. I derived an algorithm by which one may reproduce full EEDF data from the collected x-ray data using a Poisson-derived log-likelihood function. I furthermore described how I used a gas-target x-ray tube to calibrate the SDDs, both in energy and in x-ray transmission. Finally, I applied the algorithm to some data and showed how strongly non-Maxwellian features can be recovered.

This process will be applied again and again in this dissertation. Chapters 3, 5, and 6 present EEDFs and EEDF-derived quantities that are obtained via the procedure detailed in this chapter. The energy-resolved nature, accuracy, and sensitivity of the algorithm is essential to the interpretation of data in those chapters; many of the measurements of EEDF features and radial profiles could be obtained in no other way.
Chapter 3

Creation of high-energy electrons:

Secondary emission by ion bombardment of the plasma source

In measuring the x-rays emitted from the plasma in the Source End Cell (SEC) of the PFRC-II device, we discovered a population of electrons of vastly higher energy than the bulk population. While a typical temperature and density of the bulk population in the SEC is 5 eV, $10^{11}/\text{cm}^3$, this population, when fit to a Maxwellian EEDF, could have effective temperatures of 650 eV and extrapolated Maxwellian densities of $10^9/\text{cm}^3$.

The discovery of this population was reported in my paper with first author Peter Jandovitz and Samuel A. Cohen in 2018.[49] Some preliminary data was published in conference papers beforehand.[48, 50]

Calculations in this chapter indicate that the source of these electrons is plasma ions incident on the walls of the chamber near the antenna which are charged very
negative via the RF self-bias effect. These ions cause secondary electron emission, and the electrons so produced gain energy while traversing the sheath into the bulk. The magnetic configuration appears to allow another population of electrons to maintain the plasma terminating component, a stainless steel plate, at a very negative potential, electrostatically confining axially-directed electrons.

This population is the source of the high-energy electrons in the Center Cell (CC) of the PFRC-II. Understanding their creation and characteristics is essential to understanding the origin of the electrons which are accelerated to temperatures of 3 keV in the CC.

The novel creation mechanism is also relevant to the process of electron heating at a high-voltage RF sheath, an important process for determining the plasma parameters of capacitively coupled plasma reactors. These are used to manufacture semiconductor integrated circuits. Because the PFRC-II runs at orders-of-magnitude lower neutral gas pressure than is common in these reactors, this chapter demonstrates generally the wealth and complexity of phenomena which can occur when energetic electrons are permitted to penetrate deeply into the otherwise collisional plasma. Specifically, we discuss a case in which the magnetic configuration allows enhanced electrostatic trapping of these electrons. Such enhanced fast electron creation in a capacitively coupled plasma reactor would critically affect the power balance and surface charging in the device.

In this chapter, I give background information on capacitively coupled plasma reactors and other areas in which high-voltage RF sheaths are important. I describe the
measurements which characterize the fast electrons in the SEC of the PFRC-II as ion-induced secondary electrons from such a sheath. Finally, I present a simple model which reproduces the temperatures and densities of these electrons.

3.1 Capacitively coupled and other low-pressure, low-temperature RF plasma reactors

3.1.1 Plasma reactors for semiconductor processing

Lieberman and Lichtenberg’s *Principles of Plasma Discharges and Materials Processing* gives a detailed and accessible introduction to the field.[51] They note that, of the tens or hundreds of operations required to manufacture a semiconductor integrated circuit, fully 30% of them are performed using plasma. Sputtering, etching, and doping are all operations commonly achieved using plasma. Thus, plasma reactors are vital to a $400 Billion per year industry.[52]

A minority population of fast electrons, or in the field’s specific nomenclature Ballistic Electrons (BEs), can be desirable in specific semiconductor operations of capacitively coupled plasma reactors.[53] Fast electrons allow electrons to penetrate to the bottom of deep nanostructures on semiconductor wafers, allowing more uniform space charge on the surface. This in turn results in a more uniform etch.

Modeling commercial plasma reactors is also an active field. Pushing the phenomenology of electron heating at a high-voltage RF sheath into regimes of lower pressure could have immediate applicability.
3.1.2 Capacitive coupling introduction

Essential to understanding the PFRC-II SEC in seed plasma mode is the delineation between capacitive, inductive, and helicon modes. In a capacitively coupled plasma, most of the power coupled from the antenna into the electrons comes from the capacitive electric field pointing locally radially from the antenna. In an inductively coupled plasma, most of the power is coupled via the inductive electric field pointing locally along the antenna. In a helicon plasma, a left-circularly polarized electromagnetic wave is transmitted into the plasma and propagates, damping on an electron resonance.

The PFRC-II in seed plasma mode primarily couples power to the plasma capacitively, with a minority coupled inductively. In systems like this, surfaces close to the
plasma are charged due to the RF self-bias effect. This effect is described in an accessible way by Lieberman and Lichtenberg in their Chapter 11, or in a complete way by Godyak and Sternberg.[51, 55] Essentially, The RF electrode must collect zero time-averaged current. However, the electron mobility is much higher than the ion mobility. To maintain zero current, the electrode must pick up a very negative DC voltage so that it can only collect electron current a tiny minority of the time, balancing the slower-collecting ion current. This process is depicted in Figure 3.1.

3.1.3 Plasma reactor sheath heating experiments and models

The classical pedagogical derivation of plasma electron heating at a high-voltage capacitive sheath is given by Lieberman and Lichtenberg in their Chapter 11: Capacitive Discharges.[51] The name given to the primary mechanism is stochastic heating. The model treats the sinusoidally oscillating high-voltage sheath as a moving rigid wall, off of which electrons bounce. This model with a Maxwellian incident electron distribution produces a power per electrode area (Lieberman and Lichtenberg 11.1.33)

\[ \bar{S}_{stoc} = \frac{1}{2} m u_0^2 n \bar{v}_e \]  

(3.1)

where \( \bar{S}_{stoc} \) is the time-averaged power into the plasma per electrode area of this mechanism, \( m \) is the electron mass, \( u_0 \) is the motion of the nominal “wall” which constitutes the sheath, \( n \) is the electron density of the bulk, and \( \bar{v}_e \) is the average speed of the Maxwellian, \( \Gamma_e = \frac{1}{3} n \bar{v}_e \). This model is put into the context of a macroscopic model by Misium et al.[56]
In actuality, electron inertia and kinetic effects change this picture.

A paper by I. D. Kaganovich in 2002 argues that the local approximation is never valid for this analysis, and that the phenomenon requires a kinetic model.[57] He performs a calculation that shows that, even in the parameter regime common to plasma reactors, fast electrons penetrate a significant distance into the plasma before the electric field associated with their “bunches” gives their energy to colder electrons.

An experiment performed by Godyak et. al. in 1990 reveals that a Bi-Maxwellian distribution is more accurate than a thermalized, fluid model.[58] They measure temperatures \( T_1 = 0.34 \text{ eV} \) and \( T_2 = 3.1 \text{ eV} \).

Many contemporary PIC studies reproduced the result of Godyak et. al.’s 1990 measurement, finding Bi-Maxwellian EEDFs.[59–62] They each consider common reactor parameters, whose pressures are high enough that electrons scatter off of neutral atoms before making a transit. The highest temperature observed was 35 eV.

Closer to the model we will present shortly of our unique system, Godyak et. al. in 1986 performed measurements and analytical analyses on the effect of Secondary Electron Emission (SEE).[63] They find that the discharge transitions to a fully Secondary Electron (SE) sustained mode at high pressure (1 Torr), where most of the ionization that occurs does so in the sheaths from fast electrons. However, their model is purely local and does not capture penetration of SEs into the bulk, which we expect to be important at the \(~5,000\text{x}\) lower pressure that we consider.

One PIC study, that of Surendra et. al. in 1990, also studied whether SEE affects
capacitively coupled discharges.[60] They report that a pulse of ionization, and presumably fast electrons, penetrates the plasma all the way to the opposing electrode.

The PIC study which most closely fit our conditions was that of Krimke et al., who studied secondary electron emission in a plasma reactor operated at 1.36 mTorr.[64] They find a large non-Maxwellian excess in the EEDF above 50 eV and extending beyond 100 eV. Their electric field profile was such that a SE emitted during the cathodic portion of the RF cycle would pick up 170 eV of energy. The absolute EEDF was not reported.

Finally, the most directly relevant plasma reactor study to our paper is that of Xu et al. in 2008.[53] They report on a beam of 800 eV, $10^7$/cm$^3$ electrons produced by ion-induced secondary electron emission, mostly from an electrode externally maintained at -800 V DC bias but possibly partially from a 2100 V$_{pk}$ RF electrode. This experiment inspired a PIC study of the same system.[65]

### 3.1.4 Experimental results from similar experiments

Along with the Xu paper listed above, the following experiments report results closer to our own. The parameters are more PFRC-like than plasma reactor like, going down to 1 mTorr, and including hundreds of Gauss magnetic fields.

Boswell and Vender report a burst of 200eV electrons at the beginning of a helicon discharge.[66] This pulse lasted 1 $\mu$s. Ion-induced SEE was thought to be the cause.
Chen et. al. found beams of electrons from a similar antenna configuration.[67] They found multiple beam energies, up to 100 eV, which corresponded to electrons with the phase velocity of propagating helicon modes excited by their antenna.

Charles et. al. performed a power-balance analysis for a system very like the PFRC-II.[68] They found that about 15% of the power deposited into the plasma was as ion-induced SEs from the walls of an insulator next to their antenna.

Takahashi et. al. in 2017 reported on 10 eV electrons at the very edge of their plasma column, near their helicon antenna.[69] They were apparently energized by the parallel inductive electric field a skin-depth into the plasma.[54]

The MNX experimental program used the same apparatus as the PFRC-II, but in a different magnetic configuration. The SEC and antenna are unchanged. Cohen et. al. reported in 2006 the detection of 0.1% density, 200 eV temperature electrons in the SEC of MNX.[70] At the time, these were attributed to ionization downstream of an electrostatic double layer traveling back upstream. They may have been an early detection of the phenomenon described in this chapter. MNX was also simulated in PIC by Sefkow and Cohen in 2009.[71]

3.1.5 Novelty of phenomenon: 10x higher energy, lower pressure, magnetic field

The highest temperature from plasma reactor parameters which did not deliberately produce fast electrons was 35 eV.[62] When auxiliary considerations were added to produce fast electrons, their density was $10^7$/cm$^3$.[53]
We see electrons in the SEC with effective temperature up to 650 eV, and maximum energy up to 5 keV. These electrons, if their Maxwellian character extended down to 0 eV, would have density $> 10^9$/cm$^3$. We directly observe the density above 600 eV to be up to $2.5 \times 10^8$/cm$^3$. These fast electrons have drastic implications for the power balance of low-pressure plasma reactors; they make up $< 1\%$ of the plasma density but carry a large proportion of the energy. As fast electrons, these electrons affect the surface charging profile of the semiconductor wafer, for better (Xu et. al.) or worse. Clearly, their presence is essential to include in a model of the system.

The major differences between our apparatus and those which have not observed keV x-rays are pressure and magnetic field. Plasma reactor parameters are designed to cause some chemical reaction, and so typically operate at higher densities of the reactants.

We find fast electrons when the SEC has 0.4 mTorr of gas pressure, 20-5,000x lower than those typically studied for plasma semiconductor processing. For plasma reactors running at several Torr, the mean free path for electron-neutral scattering can be shorter than the sheath length. Secondary electrons do not pick up the full sheath amplitude of energy. At lower pressure of the PFRC-II, the mean free path is many m, allowing electrons to accelerate to keV of energy.

Furthermore, the magnetic configuration of the PFRC-II SEC acts to sweep a subset of these fast electrons into one floating electrode. This electrode can be maintained by these electrons to float at significantly more negative potentials than would be possible without fast electrons, never crossing the plasma potential and collecting bulk electrons. Because of this, it acts to electrostatically trap a subset, increasing their density to the
values that we measure. There are plasma semiconductor processing devices which employ magnetic fields, such as HiPIMS magnetron sputtering.[72] However they are not used to suppress or enhance fast electrons, with their associated charging and power balance implications.

3.2 Apparatus

We will discuss the Source End Cell (SEC) of the PFRC-II apparatus in this section. Electron dynamics in the Center Cell (CC) and Far End Cell (FEC) are complex and will be discussed in later chapters. They do not strongly affect the SEC.

The PFRC-II apparatus and detail of the seed plasma formation region is depicted in schematic form in Figure 3.2. The SEC is a 13” stainless steel port cross, attached via welded bellows to the CC. Gas is bled into the SEC through a needle valve. The SEC pressure is monitored by a Granville-Phillips ion gauge and two Baratron capacitive manometers during experimentation. The SEC is pumped through its aperture to the CC and FEC, where a 100 l/s Varian TV 141 NAV and a 510 l/s Agilent TV 551 NAV turbomolecular pumps exhaust the gas, respectively. The base pressure varies from day to day, often $10^{-7}$ Torr. Working gases Hydrogen, Helium, Neon, and Argon have been used.

Two sets of 10” ID pancake coils form a weak, $\sim 2:1$ magnetic mirror in the SEC. These are called the L-2 coils. An additional, 0.700”-bore set of coils forms a magnetic nozzle, raising the mirror ratio on one side only to $\sim 10:1$. This is the nozzle (N) coil, and demarcates the boundary between the SEC and the CC.
Figure 3.2: Schematic of the PFRC-II experiment run in seed plasma mode, with emphasis on the SEC. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The $|\vec{B}|$ profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate.
Outside of the weaker of the magnetic maxima is the plasma formation region. It consists of a 3.8 cm ID Pyrex pipe. 11 cm of its length has a double-saddle type helicon antenna wrapped around it. This is the antenna which produces the plasma. The antenna is capable of exciting a helicon mode, or inductive or capacitive modes. In this chapter, the antenna couples mostly capacitively to the plasma.

The antenna is powered by a Henry 2KD Classic RF amplifier capable of 7-30 MHz frequency at 1 kW of power. The forward and reverse power are monitored by a Bird RF meter and a Werlatone directional coupler. A two variable capacitor tank circuit, serial and parallel, matches the inductive antenna load to the resistive amplifier load.

Specific parameters will be listed for each of the recorded measurements below, but typical operating parameters are: 27 MHz frequency, 200 G magnetic minimum, 0.4 mTorr H\textsubscript{2} neutral pressure, 350 W net (forward minus reverse) power transmitted to the antenna, which produces antenna end-to-end inductive voltage drops of 2 kV. This produces a bulk Hydrogen plasma with temperature 5eV and density $10^{11}/\text{cm}^3$.

### 3.3 Experimental data from plasma source

#### 3.3.1 EEDF

An example of the phenomenon discussed in this chapter is given in Figure 3.3. This x-ray spectrum was recorded using SDD2 during the run of 2018/01/09. At the time, the parameters were: 320 A of current in the L-2 coils, 170 A of current in the Nozzle coils, 0.6 mTorr of Hydrogen gas pressure in the SEC, 485 W of net RF power at 27
Figure 3.3: Example inverted Bremsstrahlung spectrum from the SEC during PFRC-II seed plasma operation. The parameters of the Maxwellian fit are $n_e = 6.18 \times 10^8$/cm$^3$, $T_e = 467$eV. Recall that this is a line-averaged density over the plasma diameter. The conditions and features are discussed in the text.

MHz frequency. This caused the antenna voltage to be 3.5 kV peak to peak, and the floating stainless steel cup at the end of the Pyrex pipe to float at -1.8 kV.

Figure 3.3 has many features to discuss. First and foremost, a discussion of the spectrum itself (red line on the plot). There are four peaks at low energy. The higher three are, in ascending order, K-$\alpha$ emission from carbon, nitrogen, and oxygen impurities. The FWHM of these peaks is set by the detector; the width of the peaks are not resolvable. The peak at the very lowest energy is thermal noise, and shows that the slow threshold was set improperly for this run. Every x-ray above 600 eV is assumed to be the result of Bremsstrahlung emission. A Maxwellian EEDF (magenta line on the plot) to the XEDF would indicate a population of electrons with density $n_e = 6.18 \times 10^8$/cm$^3$, temperature $T_e = 467$eV. This density is line-averaged over the diameter of the plasma
467 eV is clearly much higher than the bulk temperature, as measured with Langmuir probes. If the Maxwellian fit fully characterized this population of electrons, they would be 0.5% of the total population, yet carry 30% of its total energy.

There are x-rays detected of energy $E_x > 3000$ eV. From this, we know that there are at least some electrons with energy $E_e > 3000$ eV. They are a factor of a few less dense than those that would be expected from the 467 eV Maxwellian given above, but many orders of magnitude more dense than those expected from the bulk 5 eV distribution.

The full, inverted EEDF (black line and associated dotted line) is recovered using the procedure in Chapter 2. It is only available above 600 eV, as at lower energies line emission obscures the Bremsstrahlung spectrum. Between 600 eV and 2200 eV, the EEDF is well characterized by a Maxwellian distribution. While the Maxwellian distribution which fits this portion of the EEDF has density $6.18 \times 10^8$/cm$^3$, only $2.6 \times 10^8$/cm$^3$, those above 600 eV, are directly observed. This quantity is obtained by integrating the depicted EEDF.

Above 2200 eV, there is a clear deficit between the recovered EEDF and a Maxwellian distribution. Above 3000 eV, the discrepancy is an order of magnitude, and above statistical significance. We say that this distribution is truncated, indicating that either electrons are not energized to this level, or that electrons which are energized to this level are lost promptly. This feature is universal in the x-ray spectra of the fast electron population of the SEC.
As we will discuss quantitatively later in this chapter, we have determined that these electrons likely result from ion-impact secondary electron emission from the surface of the self-biased stainless steel backplate. Thus they are born at already high energy, not accelerated up to this energy from the bulk. The EEDF of their creation is discussed in the modeling section, Section 3.4.2.

### 3.3.2 Glowing Langmuir probe

An order-of-magnitude check on the parameters determined from Figure 3.3 can be deduced from the behavior of a Langmuir probe in the SEC. There is a 0.010” diameter Tungsten wire Langmuir probe in the SEC of the PFRC-II, on a radially scannable mount. The low density and high temperature of the population of fast electrons makes it impossible to measure a reliable Langmuir characteristic. Furthermore, when the Langmuir probe is inserted, the x-ray signal is observed to decrease, indicating that the probe is intercepting many of the fast electrons and changing the equilibrium of the discharge.

However, two measurements are possible from the Langmuir probe. The first is the floating potential. The floating potential is uniformly around -10 V, not at negative hundreds of volts. The floating potential of a Langmuir probe in a two-temperature electron population is set by the ratio of fluxes; if the flux of the cold population is much larger, the potential seeks the floating potential of that population. If the flux of the hot population is much larger, the potential seeks the floating potential of that population. In actuality there is a continuum of solutions between these two cases, but these are the limiting cases.[19] A floating potential of -10 V indicates that the flux of
the fast electrons is smaller than that of the bulk electrons. The flux is proportional to the product $n\sqrt{T_e}$. This puts an upper limit on the fast electron density of around $10^{10}/\text{cm}^3$.

The second is a rough estimate of the power deposited by the fast electrons. When the SEC probe is inserted, I observe it to glow with blackbody radiation, appearing by eye to be yellow-white in color. Consulting Planck’s Law and steel welder’s charts, this puts the temperature at around $1300 \pm 400$ K. Assuming an emissivity of Tungsten around 0.3 and a floating potential around 0 V, this implies that the density of 300 eV electrons is around $2 \times 10^8/\text{cm}^3$. However, the inaccuracy of the temperature measurement (made by eye as it was) makes it hard to say with anything but order-of-magnitude certainty.

This density is the density after the probe is inserted. As we will see in the analytic model presented in this chapter, we expect the presence of a probe to decrease the electron density in that flux tube by a factor of about 6, so this factor should be multiplied by the density given in the preceding paragraph. The glowing Langmuir probe measurements therefore roughly agree with the x-ray-derived EEDF measurements, but do not add any new information.

3.3.3 Dependence on parameters

3.3.3.1 Power

As net RF power changed, so too did the x-ray spectra. So too did other quantities of interest: Bulk density and antenna/backplate voltages. I present these data as a function
Figure 3.4: Maxwellian fit temperature and extrapolated density, as well as observed density above 600 eV, by spectral inversion of the Bremsstrahlung spectrum, during a scan in net RF power. Because the EEDF is not a Maxwellian, I have chosen not to include error bars, as the density and temperature depends on which energy region is fit. Examples of fit quality can be seen in Figure 3.3. The directly observed density uncertainty is a few percent.

During the PFRC-II seed plasma run of 2018/04/09, we recorded high-quality x-ray spectra from the SEC while varying net RF power. In Figure 3.4, we plot three parameters extracted. The densities depicted require some explanation: The Maxwellian Fit density assumes that the EEDF above 600 eV, when fit to a Maxwellian, extends all the way down to 0 eV, where our SDDs cannot detect x-rays. Thus, this density is much larger than the “directly observed” density of electrons above 600 eV, which is the integral of the EEDF obtained through spectral inversion.
Recall that densities are line averaged over the diameter of the plasma column, as described in Chapter 2.

The temperature in Figure 3.4 is a monotone and slightly sublinear function of power. The directly observed density is a sharply increasing function of power, as the number of electrons above \( E \gg T_e \) is sharply increasing in \( T_e \). The Maxwellian fit density has a strange feature at low energy; it appears to increase with decreasing RF net power. We find this unlikely. As we are only detecting \( \sim 3\% \) of electrons in this population, a Maxwellian fit is less justified for low power than for high.

During the PFRC-II seed plasma run of 2016/12/15, we recorded the inductive end-to-end voltage peak-to-peak amplitude of the RF antenna using a Tektronix P6015 high voltage probe. We also recorded the mean value at which the stainless steel backplate floats, and its peak-to-peak fluctuation. The RF operating frequency was 27 MHz. The Hydrogen gas pressure in the SEC was 1.4 mTorr. The current into the L-2/Nozzle coils was 82/300 A, creating a minimum B in the SEC of 215 G. The RF net power was changed from 12 to 450 W in six steps over the course of 90 minutes.

As depicted in Figure 3.5, the peak-to-peak antenna voltage varies as the square root of the net RF power. Furthermore, the backplate floats to a potential within 10% of negative one-half the antenna peak-to-peak voltage. The backplate’s peak-to-peak voltage is similar to its floating potential, never approaching zero or the plasma potential. This is significant; as we will see in the model, this measurement shows that the fast electron flux to the backplate balances the ion flux, and the bulk electron flux is never collected.
During the PFRC-II seed plasma run of 2017/11/22, we recorded the ion saturation current in the SEC during a similar power scan. A Langmuir probe made of 1/8” of exposed 10 mil Tungsten wire supported by an Alumina structure is mounted on a dog-leg rotatable via Wilson seal. The probe was biased to -75 V. A bulk temperature of 5 eV was assumed.
The results are depicted in Figure 3.6. We find that bulk electron density is a monotone but sublinear function of net RF power. The best fit power for a power law is 0.71. This result, compared against the directly observed density in Figure 3.4, shows that bulk density and fast electron density vary together, which is a feature of the model given in Section 3.4.2.

3.3.3.2 Radial profile

During the PFRC-II seed plasma run of 2018/01/16, we terminated the plasma in the Far End Cell (FEC), opposite the SEC, on a sapphire plate instead of a Tantalum paddle. This allowed us to see a spatially resolved profile of visible light emitted by the plane of termination of the plasma. A photograph is reproduced in Figure 3.7. A visibly hollow profile is evident.

This light is from excitation or recombination of molecular or atomic hydrogen, and is proportional in either way to the ion flux to the sapphire. The space potential in the
FIGURE 3.8: Maxwellian fit densities from the SEC, where available, and CC. We generally observe CC Maxwellian fit density to be proportional to SEC Maxwellian fit density.

FEC is negative enough (-600 V) to exclude bulk electrons, an effect which is explored in detail in Chapter 6. Only the fast population of electrons penetrate into the FEC to cause this visible ring of light. Figure 3.7 is therefore evidence of a hollow radial profile of fast electron density.

During the PFRC-II seed plasma run of 2018/02/27, we used the SEC Langmuir probe to determine the radial profile of the population of fast electrons in the SEC. This is the same Langmuir probe that was used to record Figure 3.6. However, rather than placing it in the center of the plasma column and collecting ion saturation, we swept it radially and allowed it to collect the fast electrons and deplete them from that flux tube.
The resulting x-ray-derived quantities are depicted in Figure 3.8. At the intermediate probe radius of 1.8 cm, we do not show SEC parameters. This is because the probe was in the line-of-sight of the SDD, and so this x-ray spectrum was contaminated by x-ray emission from fast electrons hitting the Tungsten probe tip and Alumina probe structure. The Center Cell (CC) density is shown because, as will be seen in subsequent chapters, we find it to be generally proportional to the SEC density, as fast particles are lost from the SEC and stream through the CC. A lower CC density implies that more particles were intercepted by the probe in the SEC.

As the CC fast particle density is reduced by 45% when the probe is at 1.8 cm radius, but reduced by only 10% when the probe is at 0 radius, we deduce that the fast particle density in the SEC is a hollow shell profile. This is expected, as we expect the bulk plasma to also form a hollow shell, as proximity to the antenna causes larger capacitive power deposition.

### 3.3.3.3 Magnetic field

One main difference between the PFRC-II apparatus and a capacitively coupled plasma reactor is the magnetic field. During the run of 2018/01/09, we recorded x-ray emissions from the SEC as we changed the current into the nozzle coils, and so therefore the magnetic mirror ratio. The results are depicted in Figure 3.9. The parameters do not appreciably change, indicating that the confinement limiting mechanism is electrostatic rather than magnetic mirror.
3.3.3.4 Gas species

During the PFRC-II seed plasma run of 2016/12/22, we used Neon and Argon fill gas as well as Hydrogen, and recorded x-ray spectra in the SEC. Bremsstrahlung from targets with bound electrons is complex.\cite{21} It is in general not possible to determine with confidence effective densities and temperatures from these spectra. However, in each case, x-rays were observed above 3500 eV of energy, a sure indicator that there were electrons above 3500 eV of energy. The raw spectra are depicted in Figure 3.10.

Bremsstrahlung emission from targets with bound electrons from incident electrons with higher energy than the K-shell energy of the target is similar to fully ionized Bremsstrahlung, which has emission larger by the factor $Z^2$, where $Z$ is the atomic number of the target. Because the K-shell energy of Neon is 870 eV, we may extract
with reasonable confidence EEDF information above this energy.[31] At the most extreme conditions of this run, we found that the Bremsstrahlung spectrum above 870 eV was well fit by a Maxwellian with $n_e = 8.50 \times 10^8$/cm$^3$, $T_e = 366$ eV, similar to typical Hydrogen fast electron populations.

The K-shell energy of Argon is 3205 eV. Even in the hottest spectrum, there were not enough x-ray counts above this energy to fit a Maxwellian distribution with good confidence. The temperature was only specifiable to 277 - 324 eV, and the density was only specifiable to 3.2 - 25 $\times 10^9$/cm$^3$. 

**Figure 3.10:** Raw x-ray spectra from the SEC using different fill gases. The energies called out are the K-$\alpha$ energies of those elements.
This section shows that the generation of keV electrons is not dependent on the neutral fill gas or the plasma species.

### 3.3.3.5 RF Frequency

The procedure for changing RF frequency is too cumbersome to perform multiple times per day. Because of this, seed plasma experiments at different frequencies were not performed at exactly the same parameters. We recorded SEC x-ray spectra from 27 MHz RF frequency to 7 MHz RF frequency from 2017/11/15 to 2018/04/09.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Date</th>
<th>L-2 Current</th>
<th>Nozzle current</th>
<th>H\textsubscript{2} pressure</th>
<th>Net RF</th>
</tr>
</thead>
<tbody>
<tr>
<td>27 MHz</td>
<td>2017/11/15</td>
<td>290 A</td>
<td>380 A</td>
<td>0.5 mTorr</td>
<td>350 W</td>
</tr>
<tr>
<td>19 MHz</td>
<td>2018/03/20</td>
<td>270 A</td>
<td>380 A</td>
<td>0.4 mTorr</td>
<td>350 W</td>
</tr>
<tr>
<td>14 MHz</td>
<td>2018/04/09</td>
<td>270 A</td>
<td>375 A</td>
<td>0.3 mTorr</td>
<td>350 W</td>
</tr>
<tr>
<td>7 MHz</td>
<td>2018/03/29</td>
<td>270 A</td>
<td>350 A</td>
<td>0.7 mTorr</td>
<td>180 W</td>
</tr>
</tbody>
</table>

As can be seen in this table, magnetic fields and gas pressures were not identical, making the comparison in Figure 3.11 imperfect. Even if they had been identical, net RF power contains power coupled to the plasma and power dissipated in the antenna. 350 W at 27 MHz is not the same power as 350 W at 14 MHz. At lower frequency, the amplitude of the voltage across the antenna was generally smaller, as $V \propto LI\omega$.

At 27 MHz, the SEC Radial mount was used to mount the SDD to the chamber, but for all other frequencies, the SEC Short mount was used. This explains the two order of magnitude discrepancy between the 27 MHz count rate and the 19 MHz count rate, even though their fit densities and temperatures are similar.
Figure 3.11: Raw x-ray spectra and Maxwellian fit parameters from the SEC using different seed RF frequencies. Spectra were recorded months apart in some cases, although the parameters were similar. Each run is described in the table. 7 MHz used a higher pressure and lower power. 27 MHz spectrum used a different mount than the others. Because the EEDF is not a Maxwellian, I have chosen not to include error bars, as the density and temperature depends on which energy region is fit. Examples of fit quality can be seen in Figure 3.3.

At 7 MHz, it was difficult to maintain a plasma. No run at high power, low pressure was possible at this frequency.

In Figure 3.11, it appears that temperature of the fast electron population is generally proportional to frequency. Antenna voltages are higher at higher frequency, so this is to be expected. Density does not show a strong trend with frequency, and is generally dependent on parameters which were not held constant.
3.3.4 Waves in the SEC

The SEC plasma is not quiescent. During the run of 2018/02/22, the SEC Langmuir probe and the high-voltage probes on the stainless steel backplate showed that the plasma is electrostatically active in the dozens of kHz range of frequencies. From the frequency we suspected that this was a MHD interchange mode or an ion acoustic wave. Examples of $\sim 20$ kHz behavior is depicted on the right side of Figure 3.12. As can be seen, the motion is not periodic, appearing more chaotic or stochastic than sinusoidal. The backplate floating potential has been filtered to remove the 19 MHz RF signal.

We performed a correlation study between the floating potential of the stainless steel backplate and the electron saturation current of the SEC probe, 30 cm away, depicted in Figure 3.12. This analysis extracts the periodic character of the data from the stochastic or chaotic character. We find features of large correlation that have periodicity 22.7 kHz, and maximum correlation at $-22 \mu s$. This is approximately the ion sound speed.
transit time of $\text{H}_2^+$ ions in 5 eV electrons over the 50 cm distance between the plate and probe. In this figure, negative time delay means that we receive the signal on the backplate before the Langmuir probe. It seems likely that the oscillation is excited at the backplate.

### 3.4 Mechanism for the creation of fast electrons

#### 3.4.1 SEE from the self-biased stainless steel backplate

As discussed in Section 3.2 and depicted in Figure 3.2, a floating stainless steel backplate terminates the plasma column in the SEC. As discussed in Section 3.3.3.1 and depicted in Figure 3.5, the backplate and antennae have large-amplitude voltage fluctuations at the RF frequency. This 7 - 27 MHz signal is capacitively coupled to the backplate via its proximity to the antenna. A detail of the backplate’s and antenna’s position in the SEC is depicted in Figure 3.13, and put into larger context in Figures 3.14 and 3.15.
Figure 3.14: Schematic of SEC, including plasma source, B-field, and B-coils.

Figure 3.15: On-axis magnetic field profile in SEC. The specific currents which produce this profile are $I_{L2} = 135\, \text{A}, I_N = 380\, \text{A}$

The self-bias effect by electron bombardment causes this backplate and the inner surface of the antenna-backed Pyrex to float at a very negative potential. An overview and references for the self-bias effect were given in Section 3.1.2. Ions stream into a high-voltage sheath at the Bohm velocity, $v_b = \sqrt{\frac{T_e}{m_i}}$. High-energy ions hitting a solid surface have some probability to produce a secondary electron, $\gamma_s$. This secondary electron is accelerated by falling back through the sheath and picks up a large energy. It is these electrons that we observe with our x-ray detector.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Equation</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic field</td>
<td>$B$</td>
<td>300 G</td>
<td></td>
<td>Biot-Savart</td>
</tr>
<tr>
<td>Bulk density</td>
<td>$n$</td>
<td>$\sim 10^{11}$/cm$^3$</td>
<td></td>
<td>Measured</td>
</tr>
<tr>
<td>Neutral gas density</td>
<td>$n_\sigma$</td>
<td>$3 \times 10^{13}$/cm$^3$</td>
<td>PV=NRT</td>
<td>Ideal gas law</td>
</tr>
<tr>
<td>Backplate Potential</td>
<td>$V_b$</td>
<td>$1000V \pm 500V$</td>
<td></td>
<td>Measured</td>
</tr>
<tr>
<td>Antenna Potential</td>
<td>$V_a$</td>
<td>$\pm 1000V$</td>
<td></td>
<td>Measured</td>
</tr>
<tr>
<td>Thermal gyroradius</td>
<td>$\rho_{geV}$</td>
<td>0.17 mm</td>
<td>$v/\Omega$</td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>SE gyroradius</td>
<td>$\rho_{1keV}$</td>
<td>2.5 mm</td>
<td>$v/\Omega$</td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>H. V. Sheath</td>
<td>$\bar{S}$</td>
<td>0.7 mm</td>
<td>$\sqrt{\frac{e\lambda}{2\pi n\sigma}}$</td>
<td>L&amp;L[51]</td>
</tr>
<tr>
<td>Skin depth</td>
<td>$\delta$</td>
<td>17 mm</td>
<td>$c/\omega_p$</td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>Pipe ID</td>
<td>ID</td>
<td>38 mm</td>
<td></td>
<td>Measured</td>
</tr>
<tr>
<td>Axial electric field</td>
<td>$\vec{E}_{</td>
<td></td>
<td>}$</td>
<td>$\pm 30V/cm$</td>
</tr>
<tr>
<td>Mu-grad-B Force</td>
<td>$\mu \nabla B$</td>
<td>$0 - 60eV/cm$</td>
<td>$\mu \nabla B$</td>
<td></td>
</tr>
<tr>
<td>SE Ionization MFP</td>
<td>$\lambda_{iz}(1keV)$</td>
<td>35 m</td>
<td>$(n_\sigma \sigma)^{-1}$</td>
<td>NIST[73]</td>
</tr>
<tr>
<td>SE Pitch-angle MFP</td>
<td>$\lambda_p(1keV)$</td>
<td>500 m</td>
<td>$(n_\sigma \sigma)^{-1}$</td>
<td>M&amp;M [74]</td>
</tr>
<tr>
<td>SE Pyrex transit time</td>
<td>$\tau_P$</td>
<td>16 ns</td>
<td>$l_p/v(1keV)$</td>
<td></td>
</tr>
<tr>
<td>SE SEC transit time</td>
<td>$\tau_{SEC}$</td>
<td>40 ns</td>
<td>$l_{SEC}/v(1keV)$</td>
<td></td>
</tr>
<tr>
<td>RF period</td>
<td>$\tau_{RF}$</td>
<td>37 ns - 142 ns</td>
<td>$1/f_{RF}$</td>
<td></td>
</tr>
<tr>
<td>Electron plasma freq.</td>
<td>$\omega_{pe}$</td>
<td>$1.8 \times 10^{10}$ rad/s</td>
<td>$\sqrt{\frac{ne^2}{m_e\omega}}$</td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>Electron cycl. freq.</td>
<td>$\Omega_e$</td>
<td>$5.3 \times 10^9$ rad/s</td>
<td>$\frac{e}{m}B$</td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>Bulk-bulk $e$ coll. rate</td>
<td>$\nu_{b,e,e}$</td>
<td>$3.9 \times 10^5$ s</td>
<td></td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>SE-bulk $e$ coll. rate</td>
<td>$\nu_{SE,e,e}$</td>
<td>360/s</td>
<td></td>
<td>NRL Formulary</td>
</tr>
<tr>
<td>SE-gas $e$ coll. rate</td>
<td>$\nu_{SE,e,a}$</td>
<td>$3.8 \times 10^4$ s</td>
<td>$v/\lambda$</td>
<td>M&amp;M [74]</td>
</tr>
</tbody>
</table>
Typical characteristic parameters of the plasma in the proximity of the antenna are given in the preceding table.

The picture painted by these parameters is one of collisionless, magnetized Secondary Electrons (SEs) emitted by high-voltage, thin sheaths. SEs are emitted with only a few eV of energy themselves, so the direction of their motion after falling through the sheath is almost purely normal.

Those SEs emitted by the Pyrex are accelerated by the sheath in the perpendicular direction. Because of this, their magnetic moment $\mu$ is large. The diverging $B$ of the SEC pushes them upstream toward the backplate, the $\mu \nabla B$ force exceeding the axial inductive electric field. They hit either more Pyrex or the stainless steel backplate. It is those Pyrex-born SEs that allow the stainless steel backplate to float at a more negative potential than its RF amplitude; the backplate absorbs more Pyrex-born SEs than it absorbs bulk ions.

To address the concern that secondary electrons simply execute half a gyro-orbit and impact the pyrex again, I point out that during half of the RF cycle, the potential in the vicinity of the antenna is becoming more negative and reflects these electrons.

Those SEs emitted by the backplate are accelerated in the field-parallel direction, so their $\mu$ is small. During the course of one transit they can pick up or lose as much as 300 eV of energy from the inductive electric field of the antenna. This effect is not considered in the analysis to follow. It is a source of discrepancy between measured and modeled EEDF.
As will be shown in the following analysis, these electrons persist for about six transits of the chamber, 250 ns, before hitting the backplate at an RF phase less negative than their energy and being lost.

Some of these SEs have such parallel energy that they can overcome even the $R=10$ mirror ratio of the nozzle field, and make it into the Center Cell (CC) and eventually Far End Cell (FEC). There, they undergo still more complex dynamics, which will be discussed in the following chapters.

### 3.4.2 Analytic model

Consider an electrically isolated conducting slab electrode of a secondary-electron-emitting material in contact normal to a magnetic field in a uniform, thermal plasma. Suppose this electrode has a large ($V_1 \gg T_e/e$) sinusoidal voltage signal applied. Suppose the frequency of the voltage signal is fast enough that ions interact only with a mean field, but electrons transit the sheath much faster than the RF frequency. There is some floating potential, $V_0$.

$$V(t) = -V_0 + V_1 \cos(\omega t) \quad (3.2)$$

We will consider $V_1 < V_0$, as this is observed in our system, but the theory is also generalizable to the more common case that $V_1 > V_0$. 
3.4.2.1 Ion flux

The ion flux is determinable from high-voltage sheath theory, a good introduction of which is given in Lieberman and Lichtenberg.\[51\]

\[ \Gamma_i = 0.61 n_b v_b \]  \hspace{1cm} (3.3)

where $n_b$ is the bulk plasma density and $v_b$ is the Bohm velocity, $v_b = \sqrt{\frac{T_e}{m_i}}$.

3.4.2.2 SE flux at one instant

The Secondary Electron (SE) flux is related to the primary ion flux via the secondary electron yield $\gamma_i$:

\[ \Gamma_{2e} = \gamma_i \Gamma_i = \gamma_i 0.61 n_b v_b \]  \hspace{1cm} (3.4)

Secondary electrons are born with at most a few eV of energy. We neglect their initial energy. At some time $t$ at which the slab has voltage $V(t)$, SEs exiting the sheath have energy $E(t) = -eV(t)$. If SEs were only ever emitted at this voltage, we would have:

\[ \Gamma_{2e} = n_{2e,1} v_E \]  \hspace{1cm} (3.5)
where $v_E = \sqrt{2E/m_e}$. Here, we have defined the density $n_{2e,1}$ because this expression for flux assumes that no SEs reflect magnetically or electrostatically and return once leaving the sheath. This assumption will be relaxed later.

3.4.2.3 Time-averaged SE flux

Electrons are not born with only one energy $E$. $E(t) = -eV(t)$. The differential flux per unit energy $d\Gamma_{2e,1}/dE$ is calculable from the differential time that the antenna spends creating electrons of each energy:

$$d\Gamma_{2e,1} dE(E) = \frac{1}{\tau} \frac{dt}{dE} (E) \gamma_i \Gamma_i$$

(3.6)

where $\tau$ is half the RF period and $\frac{dt}{dE} (E)$ is either the reciprocal of the derivative of $E(t)$ evaluated at the inverse $t(E)$, or equivalently the derivative of the inverse $t(E)$. $\gamma_i$ is not assumed to be a function of energy for this model.

3.4.2.4 Single-bounce EEDF

The EEDF is relatable to the differential flux by the energy derivative of Equation 3.5:

$$d\Gamma_{2e,1} dE(E) = f_{2e,1}(E) \sqrt{\frac{2E}{m_e}}$$

(3.7)

Evaluating Equation 3.6 and $\frac{dt}{dE} (E)$ from the voltage time behavior in Equation 3.2, we obtain the EEDF of SEs leaving the electrode sheath which have not undergone a reflection:
Figure 3.16: Example EEDFs produced in this model. a) EEDF exiting the sheath b) EEDF after homogenizing c) $N$ quantity, number of transits d) EEDF including multiple transits. In the arbitrary energy units used, $E_0 = 1100$ and $E_1 = 800$. Figure (d) shows a long approximately exponential region, allowing an effective temperature to be defined.

$$f_{2e,1}(E) = \frac{1}{\pi E_1 \sqrt{1 - \left(\frac{E - E_0}{E_1}\right)^2}} \sqrt{\frac{T_e m_e}{2 E m_i}} 0.61 n_i$$

where $E_0 = -eV_0$, $E_1 = -eV_1$. $f_{2e,1}$ is plotted in Figure 3.16a.

This EEDF is manifestly unstable to electrostatic modes. Not only is the derivative positive in places, it diverges. Because of this, I make a qualitatively justified but quantitatively suspect approximation: this EEDF will homogenize ($f_{2e,h}$) shortly after leaving the sheath. I do not attempt a quantitative analysis of the growth rate, but the growth rate is likely on the order of the plasma frequency.[185] This is because the integral
of the positive-slope region of the EEDF is almost half of the density, and the positive slope is on the order of the energy range (not shallow). Equation 5 of that reference is approximated here:

\[ \gamma_g \approx \frac{\pi}{2} \omega_p \left[ \frac{\omega_p^2}{k^2} \partial_v f \bigg|_{v=\omega/k} \right] \]  

(3.9)

Suppose that the growth rate \(\gamma_g\) is very fast and the EEDF becomes completely uniform between \(E_0 - E_1\) and \(E_0 + E_1\):

\[ f_{2e,h}(E) = \frac{1}{2E_1} \int_0^\infty dE f_{2e,1}(E) \quad \text{for} \quad E_0 - E_1 < E < E_0 + E_1 \]  

(3.10)

\[ f_{2e,h}(E) \approx \frac{1}{2E_1} 0.61 n_i \times \gamma_i \sqrt{\frac{T_e m_e}{2E_0 m_i}} \quad \text{for} \quad E_0 - E_1 < E < E_0 + E_1 \]  

(3.11)

\(f_{2e,h}\) is depicted in Figure 3.16b.

### 3.4.2.5 Multiple-bounce EEDF

Suppose now that electrons flowing downstream are reflected by some means. In the PFRC experiment, this could be the strong mirror field or electrostatic field of the floating paddle in the FEC. The final EEDF \(f_{2e}\) is now obtained by multiplying by the average number of transits, \(N(E)\) of an electron with energy \(E\):

\[ f_{2e}(E) = f_{2e,h}(E) N(E) \]  

(3.12)
$N(E)$ is calculable with the additional assumption that the RF phase ($t$) upon the electron’s return to the electrode is random ($t = U(0, \tau)$). This assumption is not justified \textit{a priori} by our system, as transit time and RF period are of similar order and resonances are expected, but we assume it for analytic tractability. If the voltage upon return is more negative than the electron’s energy, $V(t) < -eE$, the electron is reflected and makes another transit. If the voltage upon return is less negative, the electron impacts the electrode.

The electron-impact secondary electron yield, $\gamma_e$, of stainless steel is above 1 from a few dozen eV of energy until over 1500 eV of energy.[75] That is, for every electron that hits the electrode, we expect more than one to be released at a lower energy. In the vicinity of 350 eV of primary energy, a primary electron can liberate as many as 2 SEs. We include these electron-impact SEs only after this section, as a correction. For the analysis at hand, we assume that electrons that impact the electrode are lost.

At each transit, the electron is lost with probability equal to the fraction of the RF phase during which the electrode is at a less negative potential than the electron’s energy

$$P_{\text{lost}}(E) = 1 - \frac{1}{\pi} \cos^{-1} \frac{E - E_0}{E_1}$$ \hspace{1cm} (3.13)

and

$$N(E) = \frac{1}{P_{\text{lost}}(E)}$$ \hspace{1cm} (3.14)

Thus, this model predicts that the steady-state EEDF of SEs is
Chapter 3.4

\[ f_{2e}(E) = \frac{1}{E_1} 0.61 n_i \times \gamma \sqrt{\frac{T_e m_e}{2 E_0 m_i}} \frac{1}{1 - \frac{1}{\pi} \cos^{-1} \frac{E - E_0}{E_1}} \] \quad \text{for } E_0 - E_1 < E < E_0 + E_1

(3.15)

\[ N(E) \text{ and } f_{2e} \text{ are depicted in Figures 3.16c and 3.16d.} \]

3.4.2.6 Density and Temperature

Integrating Equation 3.15, the predicted density is larger than the single-bounce density by the quantity \( \langle N \rangle \),

\[ n_{2e} = 0.61 n_i \times \gamma_i \sqrt{\frac{T_e m_e}{2 E_0 m_i}} \langle N \rangle \]

(3.16)

\[ \langle N \rangle = \int_{E_0 - E_1}^{E_0 + E_1} \frac{dE}{E_1} \frac{1}{1 - \frac{1}{\pi} \cos^{-1} \frac{E - E_0}{E_1}} \]

(3.17)

which is a function of a single variable, the ratio \( b_1 \equiv E_1 / E_0 \). \( \langle N \rangle \) is between 5.6 and 5.9 for all \( b_1 \) from zero until 1, dropping to 4.5 when \( b_1 = 1.1 \) (for \( b_1 > 1 \), the integral must be taken from zero and not \( E_0 - E_1 \), this being the generalization to \( V_1 > V_0 \)).

This finding is important to the scope of the phenomenon. It means that when there are enough secondary electrons born of Pyrex to allow the backplate to float at a potential which never collects bulk electrons, the density of the secondary electrons born of the backplate is enhanced by a large factor. Because of this condition, secondary electrons bounce about six times before being lost to the electrode.
The effective temperature (slope) of the EEDF given in Equation 3.15 is determined by this $N(E)$ function, as we assumed that the initial EEDF is flat. The effective temperature (negative reciprocal logarithmic slope) of this EEDF analytically solvable, evaluated at $E_0$ being

$$T_{\text{eff}} = \frac{\pi}{2} E_1 \approx 1.57 E_1$$

(3.18)

where $E_1$ is $-eV_1$, the voltage oscillation amplitude. As we can see from Figure 3.16d, $f_{2e}$ is approximately exponential over a range of energies, justifying our use of an effective temperature.

### 3.4.2.7 Electron-impact secondary electron emission

In this section, we will consider electron-impact secondary electron emission as well as ion-impact secondary electron emission. The full dependence on energy, $\gamma_e(E)$, is complex.\[75\] We will suppose that the secondary electron yield is constant, $\gamma_e(E) = \gamma_e$, over the energy range of interest.

Equation 3.6 picks up an additional term:

$$\frac{d\Gamma_{2e,1}}{dE}(E) = \frac{1}{\tau} \frac{dt}{dE}(E) \left[ \gamma_i \Gamma_i + \gamma_e \int_{E}^{\infty} dE' \frac{d\Gamma_{2e,1}}{dE'}(E') \right]$$

(3.19)

This, when solved, produces an EEDF which has a non-integrable divergence. This is non-physical and occurs because of our assumed $V(t)$ form and $\gamma_e(E)$ form.
However this equation is solvable if the voltage signal is triangular rather than sinusoidal. This is not true of the PFRC-II, but allows us to qualitatively describe the effects of electron-induced SEs.

This model with triangular voltage signal predicts that the density is increased by a factor of \((e^{\gamma_e} - 1)/\gamma_e\) by electron-induced SEE. This model predicts that the effective temperature measured at \(E_0\) of the single-bounce distribution including electron-induced SEE is \(T_{eff}(E_0) = 2E_1/\gamma_e\).

Applied to the results of the model given above, it may be said that our estimate for the SE density in the SEC is:

\[
n_{2e} = 0.61n_i \times \gamma_i \sqrt{\frac{T_e m_e}{2E_0 m_i}} \langle N \rangle (e^{\gamma_e} - 1)/\gamma_e \tag{3.20}
\]

where \(\langle N \rangle \approx 6\) if \(E_1 < E_0\) and smaller if voltage crosses the plasma potential. Our estimate for the effective temperature is (for \(f_{12} = f_1 \times f_2\), \(T_{12}^{-1} = T_1^{-1} + T_2^{-1}\)):

\[
T_{eff} = E_1 \times \frac{1}{2/\pi + \gamma_e/2} \tag{3.21}
\]

### 3.4.3 Application to the SEC of the PFRC-II

In a typical run, the Pyrex pipe contains a plasma with bulk density and temperature \(T_e = 5\text{eV}, n_e = 10^{11}/\text{cm}^3\). The backplate has 1000 kV peak-to-peak signal and floats around -1000 kV.
The secondary electron yield of H$_2^+$ on Stainless Steel is around $\gamma_i = 0.1$.\[76] The mass ratio between $m_e$ and $m_{H^+}$ is $\sim 511\text{keV}/(2 \times 938\text{MeV})$. From the voltage measurements, $E_0 = 1\text{keV}, E_1 = 500\text{eV}$. The maximum $\gamma_e$ for stainless steel is 2.\[75]

By applying the final density equation above, Equation 3.20, we would expect the SE population to have a density of about $1.0 \times 10^8/\text{cm}^3$, all above 500 eV of energy and truncated at 1500 eV.

By applying the final temperature equation above, Equation 3.21, we would expect the SE population to have an effective temperature of about 305 eV.

These run conditions were chosen because they approximate the parameters of the data shown in Figures 3.3, 3.5, 3.6, and 3.4, respectively showing an inverted EEDF, the potentials of the backplate, the bulk density, and the EEDF-derived parameters of the x-ray spectrum.

The conditions most similar to the potentials assumed, the 350 W point of Figure 3.4, produced a density above 600 eV of $1.8 \times 10^8/\text{cm}^3$, compared to $1.0 \times 10^8/\text{cm}^3$ from the model. This condition achieved an effective temperature of 275 eV, compared to 305 eV from the model. This point and others like it also showed a truncation appearing at about 2200 eV, rather than the expected 1500 eV.

We attribute these discrepancies to the assumptions of our model that are the least well justified:

We neglected the inductive electric field. In the SEC, the inductive electric field penetrates a skin depth, almost all the way into the plasma and electrons can pick up as much as 300 eV on each transit of the vessel. Some portion of those electrons will be
phased such that they gain multiple of these inductive accelerations on subsequent RF phases, and so gain more than 2200 eV of energy.

We neglected resonances between the RF phase and the electron transit time. At 1 keV, a particle would transit the SEC, bounce, and return in roughly 100 ns. At 14 MHz, the RF cycle takes 71 ns. We expect significant effects from chaotic quasi-periodicity, as this system looks superficially like the classically chaotic Standard Mapping. We did not include these effects at all.

We assumed that the single-bounce EEDF was completely homogenized into a uniform value shortly after leaving the high-voltage sheath. In experiments in which electrostatically unstable plasmas are deliberately created and studied, there is usually some finite slope to the resultant EEDF. See references in Chapter 6.

This model explains the dependence on RF power depicted in Figure 3.4 through the effect of RF power on bulk density (Figure 3.6) and the backplate potentials (Figure 3.5). The directly observed SE density in Figure 3.4 increases with increasing power because of the larger bulk density and greater proportion of electrons with energy above 600 eV, which we can observe via Bremsstrahlung detection.

This model is consistent with the hollow radial profile given in 3.8. The bulk plasma may have a hollow density profile because of the enhanced ionization in the stochastically heated sheaths.[51] The ion flux to the stainless steel cup would therefore higher radially outwards, liberating SEs preferentially in a hollow shell.

This model is consistent with the lack of a strong dependence on the nozzle field depicted in Figure 3.9. It shows that the limit to SE confinement is may be electrostatic
from the FEC paddle rather than magnetic from the nozzle.

The persistence of the phenomenon through multiple species of fill gas depicted in Figure 3.10 is consistent with this model also.

This model explains the dependence on RF frequency to within the limited measurements given in Figure 3.11. The net RF power was held roughly constant. At lower frequencies, the peak-to-peak voltage across the antenna was decreased as the \( V = L \dot{I} \) inductive voltage drop decreased. Because of this, the effective temperature decreased proportionally.

## 3.5 Conclusion

In this chapter, I showed measurements that indicate that a minority population of fast electrons with temperatures 300 - 650 eV and extrapolated Maxwellian densities of \( 10^9 / \text{cm}^3 \) are created in the Source End Cell (SEC) of the PFRC-II when run in seed plasma mode. I explain the existence of this population as a consequence of the low-collisionality, high-magnetization regime and the amplitude of a large voltage signal capacitively applied to the plasma from a double-saddle RF antenna at 8 - 27 MHz. This RF signal induces a large-amplitude sheath via the self-bias effect, and secondary electrons born on the low-potential side of this sheath are accelerated into the bulk.

The phenomenon is of primary interest to this dissertation because these electrons go on to exhibit more unexpected phenomena, such as extreme non-adiabaticity of magnetic moment and electrostatic Fermi acceleration up to keV of effective temperature. The phenomenon is also of interest because this parameter regime is near the extremes
of the conditions commonly used in capacitively-coupled reactors for semiconductor processing. The collisionality of the SEC is lower than in plasma reactors, so fast electrons may be accelerated by sheaths without collisions and penetrate deeply into the plasma for multiple bounces. The magnetic field of the SEC is fortuitously configured to allow the axially-directed SEs emitted by the stainless steel backplate to be electrostatically confined and accumulate.

In Chapter 4, a means of suppressing these electrons is presented. By texturing the walls of the RF antenna region into complex, fibrous shapes, SEY may be suppressed. In Chapter 5, these electrons are used to probe the behavior of particles with marginally adiabatically-conserved magnetic moments. In Chapter 6, these electrons are shown to be accelerated to many times their initial energy via a form of electrostatic Fermi acceleration.
Chapter 4

SEE from complex surface geometry

In Chapter 3, I showed that ion-induced Secondary Electron Emission (SEE) from a large RF sheath produces a minority population of $T_e \sim 300 - 650$ eV electrons in the Source End Cell (SEC) of the PFRC-II run in seed plasma mode. This fortuitous discovery provides the tool for two other remarkable phenomena to explore, the dynamics of marginally adiabatically-conserved magnetic moment in Chapter 5 and Fermi acceleration in a magnetic mirror in Chapter 6.

However, this population may prove deleterious for other applications of a SEC-like chamber. A capacitively-coupled semiconductor processing reactor, for example, exhibits very different surface charging and etching when a population of fast electrons is present.\cite{53} Closer to the purpose of the PFRC-II machine, fast electrons may affect the formation of a rotating-magnetic-field driven field-reversed configuration in the seed plasma.
For this reason, I present a method by which SEE may be suppressed: Texturing the plasma-facing components into complex, fibrous geometries. Much of this chapter was published by myself and Igor D. Kaganovich in 2016, 2017, and 2018.[77–79] Because of that, Igor should be considered a co-author for much of this chapter. In nascent form, some of the analyses were first published in conference posters.[80–82]

In this chapter, I describe a Monte Carlo tool I created in MATLAB to directly simulate the process of SEE from a complex surface. I also present a model from which physical insight and engineering best-practices can be drawn for suppressing SEE using complex surfaces, the “Weighted View Factor Model.” Finally, I apply the Monte Carlo tool and analytic analysis to three candidate geometries for suppressing SEE: Velvet, which consists of normally-aligned, long, thin fibers; foam, which consists of isotropically-aligned, long, thin fibers; and feathers, which consists of large, normally aligned primary fibers with smaller, secondary fibers grown onto the primary fibers.

### 4.1 Background

Secondary Electron Emission (SEE) occurs when a high-energy ion or electron impacts a material surface and causes electrons to be emitted from that surface. Both ion-impact SEE and electron-impact SEE can be leading-order effects in discharges: Ion-impact SEE is the source of electrons from the cathode that allows a steady-state glow discharge to be maintained,[83] and the electron-impact Secondary Electron Yield (SEY) can, for primary electrons at hundreds of eV of energy, exceed unity and strongly affect the
shape of potential structures. SEE conditions can also affect instabilities in plasmas
and electron energy distribution functions [85, 86].

Some applications have well-known sensitivities to SEE. Clouds of secondary elec-
trons have been found to affect particle beam transport in accelerators. RF ampli-
fiers often employ cavities whose gain is limited by the Multipactor effect. SEE
processes are also known to affect Hall thruster operation due to contribution to so-
called near-wall conductivity or due to reducing wall potential and increasing plasma
energy losses [89–91]. The SEY of the Tungsten divertor in ITER is expected to be near
unity. [92, 93]

Many surface geometries have been analyzed for their potential for suppressing
SEE. Accelerator communities were early proponents of the concept, analyzing grooved
accelerator walls. Velvets, prior to and since the publication of my 2016 paper
with Igor D. Kaganovich, have received theoretical and experimental attention. Foams, of the kind that spontaneously generate when Helium plasma is incident on
Tungsten, are also surfaces of interest. Baglin et. al. experimentally
characterized the secondary electron emission of dendritic copper. Nguyen et. al.
thorize a reduced SEY for surfaces that have been coated in graphene.

Some groups have studied specific geometries analytically and determined detailed
parametric dependencies on SEY, like I do in this chapter. Other groups also
use a Monte-Carlo tool, like I do in this chapter. Huerta, Patino, and Wirz have per-
formed Monte Carlo modeling to characterize the SEY from velvet structures, rectilin-
ear cage-like structures, and foam structures. Alvarado et. al. have produced
a technically advanced Monte Carlo algorithm for calculating the SEY of surfaces.
My analyses in this chapter are novel because they combine Monte-Carlo ground-truth with analytic models for parametric dependence of velvet, foam, and feathers, and I propose a novel weighted view-factor model.

### 4.2 Monte-Carlo tool implemented in MATLAB

I numerically simulated the emission of secondary electrons by using the Monte Carlo method, initializing many particles and allowing them to follow ballistic, straight-line trajectories until they interact with surface geometry. A flowchart of the algorithm is in Fig. 4.1. The simulation tool was introduced first in my paper with Igor D. Kaganovich in Journal of Applied Physics in 2016.[77]

In the results presented here, I use $10^5$ particles. Each particle object keeps track of seven quantities: its three spatial positions $x, y, z$, its energy, $E$, and velocity angles $\theta, \phi$, and its “weight,” meaning how many particles it stands for. All weights start at a value of 1. Weights are changed upon interaction with a surface.

An alternate approach would be to start with fewer particles and have them stand for a fixed number of particles, all with weight 1. When SEY occurs, in this approach one would instantiate more particles until one tallies $10^5$. Starting with $10^5$ and instead changing the weights upon SEY produces identical counting statistics, with error associated with counting statistics being $N^{-1/2} = 0.3\%$.

The surface geometry and initial distribution of incident particles are the simulation’s main input parameters. The surface geometry is implemented as an isosurface of a function of space, as collisions with an isosurface are trivially detectable by a particle
Initialize particles at top of simulation domain with weight 1

Move particles

Check to see if particles interacted with a surface

Yes No
Yes No

Give the particles new weight, energy, and velocity

Check to see whether particles escaped from the top of the simulation

Yes No

Tally weight and add to SEY

**Figure 4.1**: Flowchart of the Monte Carlo simulation algorithm.

object which stores its spatial location. \( F_{iso}(\vec{x}) < 0 \) defines space inside the geometry and \( F_{iso}(\vec{x}) > 0 \) defines space outside the geometry.

At every time step, the algorithm checks to see whether particles have passed into the surface. If they have, their local normal angle is determined relative to the gradient of \( F_{iso} \), and the SEY of their energy and local normal angle is computed. Their weight is multiplied by the SEY.

Emitted particles are given a new velocity angle. Secondary electrons are emitted with probability linearly weighted by the cosine of the normal angle \([109]\). Thus
Chapter 4.2

\[ P(\Omega)d\Omega = 2 \cos \vartheta d\cos \vartheta \frac{d\phi}{2\pi} \]  

(4.1)

where \( \vartheta \) is computed relative to the local normal, \( \mathbf{\hat{\nabla}} F_{iso} \). Specifically in the code, \( \cos \vartheta = R^{1/2} \), where \( R \) is a uniform random variable from 0 to 1. The azimuthal angle in the local normal frame is uniformly distributed from 0 to \( 2\pi \).

Because of confusion in the literature, I note the differences between flux and velocity distribution function, and how they characterize the number of electrons with a certain velocity. As flux is total number of particles that pass through a differential cross sectional area oriented along some normal \( \mathbf{\hat{n}} \), flux is

\[ \Gamma = \int d^3v \mathbf{\hat{v}} \cdot \mathbf{\hat{n}} f(\mathbf{\hat{v}}) \]  

(4.2)

for distribution function \( f(\mathbf{\hat{v}}) \). That is, flux counts particles passing through a surface while distribution function counts particles within a volume. Thus, though flux is weighted by \( \cos \vartheta \), this is the condition that the distribution function \( f(\mathbf{\hat{v}}) \) is isotropic in angle.

The SEY of the incident electron is computed using several different \textit{a priori}, empirical, and semi-empirical expressions. We use one of the latter, that of Scholtz, [110]

\[ \gamma(E_p, \theta) = \gamma_{max}(\theta) \times \exp \left[ - \left( \frac{\ln[E_p/E_{max}(\theta)]}{\sqrt{2}\sigma} \right)^2 \right]. \]  

(4.3)
where the parameters $E_{\text{max}}, \gamma_{\text{max}}, \sigma$ are free parameters of the Scholtz model. Angular dependence is taken from Vaughan [88],

$$\gamma_{\text{max}}(\theta) = \gamma_{\text{max}0} \left(1 + \frac{k_s \theta^2}{2\pi}\right)$$  \hspace{1cm} (4.4)$$

$$E_{\text{max}}(\theta) = E_{\text{max}0} \left(1 + \frac{k_s \theta^2}{\pi}\right).$$  \hspace{1cm} (4.5)$$

The specific constants in the Scholtz model are taken from the graphite experimental data given in Patino et al.[111], $\gamma_{\text{max}0} = 1.2$, $E_{\text{max}0} = 325\text{eV}$, $\sigma = 1.6$, $k_s = 1$. The semi-empirical model of Scholtz is chosen because it agrees well with Patino et al. experimental data for graphite.

Emitted electrons have three energy-groups: true, elastic, and rediffused. True secondary electrons are given a low few eV temperature. Elastically scattered electrons are given the same energy as the primary electrons. Rediffused electrons are given energy uniformly distributed between zero and the primary electron energy. True secondary electrons, elastically scattered electrons, and inelastically scattered (rediffused) electrons are simulated with the energy-dependent probabilities of emission reported in Ref. [111]. In accordance with that experiment, we use following empirical formula for fraction of elastically scattered electrons, $f_{el}$:

$$f_{el}(E_p) = \exp\{1.59 + 3.75 \ln(E_p) - 1.37 [\ln(E_p)]^2$$

$$+ 0.12 [\ln(E_p)]^3\}.$$  \hspace{1cm} (4.6)$$
for \( E_p = 6 - 390 \text{eV} \), and \( f_{el}(E_p) = 100\% \) for \( E_p < 6 \text{eV} \). For \( E_p > 390 \text{eV} \), \( f_{el}(E_p) = 2\% \). The fraction of inelastic electrons is assumed to be equal to 7\%, where permitting by \( f_{el} < 93\% \). The values of SEY calculated using this formula prove to be sensitive to the fraction of elastically scattered electrons, as these electrons are not absorbed by surfaces, and can still contribute their full number to the SEY.

The remaining, true secondary electrons are given a Maxwellian EVDF with temperature \( T_{true} = 5.4 \text{eV} \) [111].

The simulations presented here were performed for various electron incident energies, mostly 200 eV - 350 eV. The sensitivity of the resulting SEY to this primary energy is not large; simulations performed at significantly higher energies had the effect of increasing the tertiary electrons created by elastically scattered secondary electrons, but these only account for 2\% of secondary electrons in this range.

### 4.3 Weighted view-factor model for SEY from geometry

Because of the complexity and inaccessibility of the analytic calculations given later in this chapter, I formulated a novel and intuitive model of SEE from complex surfaces. In this section I will derive the model and give some of its important consequences.
4.3.1 **Mathematical formulation**

The model imagines an infinite surface of material which is asymptotically horizontal, though at small scale can have features of any size, including fractal. Consider a section of the surface which has cross-sectional area in the x-y plane \( a \), but total integrated area \( A \). Consider this surface to be made of a material which has flat secondary electron yield \( \gamma \), independent of angle of incidence. These secondary electrons are emitted with probability linearly weighted in the local normal direction, as in Equation 4.1.

Suppose that there are electrons incident on this surface from above, \( z > 0 \). Suppose that electrons travel in straight-line trajectories. Suppose that the electron velocity distribution function (EVDF) is isotropic in the downward hemisphere,

\[
f(x, \Omega, v) = f(v)\Theta(-\hat{\Omega} \cdot \hat{z})
\]

(4.7)

where vector velocity \( \vec{v} = v\hat{\Omega} \) for solid angle \( \Omega \). An electron flux in a specified direction \( \hat{n} \) is related to the EVDF by

\[
\Gamma_{\hat{n}} = \int_{\Omega} \frac{d\Omega}{2\pi} \int dv f(x, \Omega, v)v\hat{\Omega} \cdot \hat{n}
\]

(4.8)

For flux across the \( x - y \) plane in the \( z \) direction, then, the total primary flux is

\[
\Gamma_0 = \frac{1}{2} \int dv f(v)
\]

(4.9)
By phase-space conservation, a value of the EVDF $f(x(t_0), v(t_0), t_0)$ in the bulk is the same when its trajectory (straight-line in this case) intersects a surface $f(x(t_1), v(t_1), t_1)$. In steady state, this means that the EVDF at a surface element $u, v$ is the same as the EVDF in the bulk along that line-of-sight.

$$f(\vec{x}_{u,v}, \hat{\Omega}_{los}, v) = f(v) \quad (4.10)$$

where $\hat{\Omega}_{los}$ goes from the point $\vec{x}_{u,v}$ into the bulk without intersecting another particle source/sink (surface element).

For each surface element, applying Equations 4.8, 4.9, and 4.10 the primary flux to that surface is

$$\Gamma_1(u, v) = 2\int_{\Omega_p(u,v)} \frac{d\Omega}{2\pi} \Gamma_0 \hat{\Omega} \cdot \hat{n} \quad (4.11)$$

where $\hat{n}$ is the normal vector of the surface and $\Omega_p(u, v)$ is only that subset of solid angles that have line-of-sight to the plasma, i.e. are not shadowed by another surface element along that solid angle. The secondary flux from that surface is

$$\Gamma_2(u, v) = \gamma \Gamma_1(u, v) \quad (4.12)$$

where $\gamma$ is the SEY and SEE flux has angular distribution

$$\frac{d\Gamma_2}{d\Omega} = 2\hat{\Omega} \cdot \hat{n} \quad (4.13)$$
Again applying Equations 4.8 and 4.10, the secondary flux which escapes to the plasma without hitting another surface element is

$$\Gamma_f(u, v) = 2 \int_{\Omega_p(u,v)} \frac{d\Omega}{2\pi} \Gamma_2(u, v) \hat{\Omega} \cdot \hat{n}$$  \hspace{1cm} (4.14)

where again $\Omega_p(u, v)$ is that subset of the solid angle that has line-of-sight to the bulk plasma without hitting another surface element. We now truncate the generation of electrons by assuming that secondary electrons produced by secondary electrons, tertiary electrons, are negligible. Physically, this is motivated by the observation that $\gamma$ is usually significant for energies of 100s of eV, but the temperature of emitted electrons is single-digit eV.

By defining the “Weighted View Factor,” $b$, of the surface element $(u, v)$:

$$b(u, v) = 2 \int_{\Omega_p(u,v)} \frac{d\Omega}{2\pi} \hat{\Omega} \cdot \hat{n}$$  \hspace{1cm} (4.15)

we may relate all of these fluxes

$$\Gamma_f(u, v) = b(u, v) \Gamma_2(u, v) = \gamma b(u,v) \Gamma_1(u, v) = \gamma b^2(u,v) \Gamma_0$$  \hspace{1cm} (4.16)

Integrate over the entire surface to determine bulk properties. The total particle rate, electrons per time, to the surface can be posed in terms of both the primary flux through the cross section $x - y$ or to the surface $A$: 
Substituting in the $b$-relations for these fluxes, we can see that $b$ integrates over $A$ to $a$:

$$\int dA b(u, v) = a \quad (4.18)$$

The total particle rate from the surface can be posed in terms of secondary flux

$$R_f = \int dA \Gamma_f (u, v) \quad (4.19)$$

Substituting the $b$-relations above,

$$\gamma_f = \frac{R_f}{R_0} = \gamma \frac{\int dA b^2 (u, v)}{\int dA b(u, v)} = \gamma \frac{\int dA b^2 (u, v)}{a} \quad (4.20)$$

Equation 4.20 is the mathematical result of the Weighted View Factor model. It is this equation from which the physics insights are drawn.

As we can see from Equation 4.20, $\gamma_f$ can be arbitrarily small when $b^2$ can integrate to arbitrarily small. The implications of this fact on engineering guidelines for surfaces is explored in the following section.
4.3.2 Constraints on SEY from geometry and suggestions for optimization

As mentioned just above, according to this model $\gamma_f$ may be arbitrarily small. However, $b$ must integrate to $a$. For both of these criteria to be true, it is necessary that $A$ be arbitrarily large. In fact, as we will see, the smallest possible $\gamma_f$ of a surface is the value $\gamma_f a/A$.

Suppose the values of $a$ and $A$ are set. Is there now an optimal surface to extinguish the SEY? What would some properties of this surface be?

$b(u, v)$ would be uniform. To prove this, assume there is some surface with $b(u, v) = b_{opt}$, uniform, that fulfills our criteria above, i.e. integrates to $a$. Perturb this surface preserving the surface area, infinitesimally or not, and expand around this uniform $b_{opt}$, $b(u, v) = b_{opt} + \epsilon(u, v)$. For $b$ to integrate to $a$, $\epsilon$ must integrate to zero

$$a = \int dA b(u, v) = Ab_{opt} + \int dA \epsilon(u, v) \quad (4.21)$$

and $Ab_{opt} = a$. Finding the $\gamma_f$ of this surface

$$\gamma_f = \gamma \frac{\int dA b^2(u, v)}{a} = \gamma \frac{\int dA [b_{opt}^2 + 2b_{opt} \epsilon(u, v) + \epsilon^2(u, v)]}{a} \quad (4.22)$$

The first term in the integral is $\gamma_f$ given a uniform $b$. The second term integrates to zero. The third term is positive. Thus to minimize SEY create a uniform which has a uniform $b(u, v) = b$. 
This uniform $b$ has a geometric relationship to $a,A$. $Ab_{opt} = a$, so $b_{opt} = \frac{a}{A}$. This is also the factor by which $\gamma_f$ is reduced, $\gamma_f = b_{opt}\gamma$.

\[
\gamma_{f,\text{min}} = \frac{a}{A}\gamma
\]  \hspace{1cm} (4.23)

the most suppression possible from a surface with cross sectional and integrated areas $a,A$ is the ratio of the two, with very general applicability.

In terms of engineering guidelines for a surface, succinctly put they are twofold: 1) For a given surface area, the primary flux should illuminate the surface uniformly. 2) The larger the ratio of the integrated surface area to the cross-sectional surface area, the lower the SEY of the surface can be. This implies that complex, fractal-like, dendritic, or fibrous structures will be the most efficient surfaces for suppressing SEE.

4.4 Monte-Carlo results and analytic theory for surfaces

4.4.1 Velvet

We simulated a velvet surface using the Monte-Carlo algorithm. These results were first reported in my paper with Igor D. Kaganovich in Journal of Applied Physics in 2016.[77]

Velvet consists of many long, thin whiskers grown onto the surface of a flat substrate. In this thesis, the substrate would be the Pyrex antenna tube and the stainless steel endplate in the SEC.
We characterize our velvet with two dimensionless parameters, aspect ratio, 

\[ A = \frac{h}{r} \]

and packing density, 

\[ D = \frac{\pi r^2}{n}, \]

which is the proportion of surface area of the bottom taken up by the base of the whiskers. \( h \) is the length of the whiskers, \( r \) is the radius of the whiskers, \( n \) is the areal density of whiskers, i.e. \( n \) whiskers grown per square centimeter of substrate. This geometry is depicted in Figure 4.2.

The results of the Monte-Carlo calculation are depicted in Figure 4.3. We found that adding velvet to a surface can significantly decrease the net SEY of velvet surface as much as 90% from the case of normal incidence on a flat surface. The reduction in SEY depends strongly on the velvet parameters: packing density, \( D \), and aspect ratio, \( A \). However, the net SEY of velvet is a strong function of the incident angle \( \theta \), increasing up to almost 1/2 for more shallow angles of incidence.

To understand the dependencies of the SEY on the velvet parameters shown in Figure 4.3, an analytical model was developed and discussed:

The analytic model considers only one generation of secondary electrons. It centers around the approximation that the electrons have finite probability of intersecting a whisker surface per unit distance traveled perpendicular to their axis, \( l_\perp \)

\[ P_{\text{free}}(l_\perp) = e^{-l_\perp/\lambda_\perp}, \quad (4.24) \]
\[ \lambda_\parallel = \frac{1}{2r n} \]  

(4.25)

Figure 4.2: Schematics of the velvet surface: the whisker geometrical quantities radius, \( r \), height, \( h \), and spacing, \((2s)^2 \equiv n^{-1}\). Also shown are electron velocity polar angle \( \theta \), and velocity azimuthal angle, \( \phi \). Numerical calculations include three contributions to secondary electron emission from velvet: electrons emitted by the side, top, and bottom surface of the whiskers.

\[ P(\Delta z) = e^{-2ruz \tan \theta} \]  

(4.26)

Rearranging into a dependence on axial coordinate \( z \) and polar velocity angle \( \theta \), the probability that an electron will traverse axial distance \( \Delta z \) within the layer of whiskers and not intersect a whisker surface is
\[ u = 2 \rho n h = \frac{2}{\pi} D A \]  

Figure 4.3: SEY reduction from the case of normal incidence on a flat surface. SEY reduction is given as a function of incident angle, $\theta$, for different values of whisker aspect ratio $A$, and packing density $D$. Figure (a) shows SEY for 4 different $D$ values and the same $A = 1000$. Figure (b) shows SEY for 3 different values of $A$ and the same $D = 4\%$. Solid lines show the result of an analytic approximation. Points with error bars are the result of these Monte Carlo simulations.

It will become apparent that the dimensionless parameter which governs the regime of SEY suppression is

The SEY has three components consisting of those particles which are emitted by the tops of the whiskers, $\gamma_{\text{tops}}$, those which are emitted from the sides of the whiskers, $\gamma_{\text{sides}}$, and those particles which are emitted by the bottom substrate, $\gamma_{\text{bottom}}$. SEYs can also be described in the language of probability, as the reduced SEY is just the
SEY multiplied by the probability that a secondary electron escapes to contribute to secondary flux:

\[
\gamma_{tot} = \gamma_{tops} + \gamma_{sides} + \gamma_{bottom} \quad (4.28)
\]

\[
= \gamma_{flat} \times [P(esc|top)P(top) + P(esc|bottom)P(bottom) + \int dz P(esc|sides, z)P(sides, z)]
\]

where \( P(esc) \) is the probability that an electron escapes without intersecting another whisker and \( P(surface) \) is the probability that a primary electron produces a secondary electron at that surface.

\( P(esc|top) = 1 \) is simple, as any electron which is produced on the whisker tops escapes unimpeded. \( P(top) = D \), as the whisker tops are simply circles of radius \( r \) with areal density \( n \). Thus

\[
\gamma_{tops} = \gamma_{flat}(\theta)D \quad (4.29)
\]

\( P(bottom) \) can be calculated using Equations 4.26 and 4.27, \( P(bottom) = e^{-u \tan \theta} \).

\( P(esc|bottom) \) requires an integral over the emitted electron polar velocity angle, \( \theta_2 \); an electron emitted straight-up will not hit any whisker sides but an electron emitted at a shallow angle will have a much higher probability of hitting a whisker side.

\[
P(esc|bottom) = \int_0^1 d \cos \theta_2 P(esc|bottom, \theta_2)P(\theta_2|bottom) \quad (4.30)
\]
\( P(\theta_2) \) is derivable from Equation 4.1 and \( P(esc|bottom, \theta_2) \) is derivable from Equation 4.26.

\[
P(esc|bottom) = \int_0^1 d\cos \theta_2 \cos \theta_2 e^{-u \tan \theta_2} \tag{4.31}
\]

Thus

\[
\gamma_{bottom} = \gamma_{flat}(\theta)2(1 - D) \int_0^1 d\cos \theta_2 \cos \theta_2 e^{-u(\tan \theta + \tan \theta_2)} \tag{4.32}
\]

\[
= \gamma_{flat}(\theta)2(1 - D) \int_0^\infty dt \frac{e^{-u(t + \tan \theta)}}{(1 + t^2)^2}
\]

In the second form, the integration variable \( t = \tan \theta_2 \).

\( \gamma_{sides} \) is similarly derivable, with the added wrinkles that \( P(esc) \) must be integrated over the axial coordinate \( z \) of emission, \( P(\cos \theta_2) \) must be computed considering that the substrate is now at a 90° angle to the normal, and \( \gamma_{flat} \) is of a different angle of impact:

\[
\gamma_{sides} = \langle \gamma_{flat,loc}(\theta) \rangle \int_0^h dz \left[ \int_0^1 d\cos \theta_2 P(esc|side, z, \theta_2) P(\theta_2|side) \right] P(side, z) \tag{4.33}
\]

\( P(side, z)dz \), the probability that an electron does hit within \( dz \), is the negative derivative of Equation 4.26, the probability that an electron has not hit by height \( z \). 

\[
P(side, z)dz = 2rn \tan \theta e^{u \tan \theta z/h}.
\]
$P(\theta_2|\text{side})$ is derivable from transforming Equation 4.1 into a coordinate system turned $90^\circ$ on its side.

\[
P(\theta_2|\text{side}) = \frac{2}{\pi} \sin^2 \theta_2
\]  \hspace{1cm} (4.34)

$P(\text{esc}|\text{side}, z, \theta_2)$ is derivable from Equation 4.26. $P(\text{esc}|\text{side}, z, \theta_2) = e^{-ut\tan\theta_2z/h}$

$\langle \gamma_{\text{flat,loc}}(\theta) \rangle$ is flat values of the SEY at the local normal angle of incidence, integrated over normalized impact parameter

\[
\langle \gamma_{\text{flat,loc}}(\theta) \rangle = \int_0^1 db \gamma(\cos^{-1}(\sin \theta \sqrt{1-b^2})). \hspace{1cm} (4.35)
\]

Using Scholtz’s and Vaughan’s expressions for SEY, this averaged SEY, $\langle \gamma_{\text{flat,loc}}(\theta) \rangle$, was never more than about 30% higher than the flat value, $\gamma(\theta)$.

Plugging all of these substitutions into Equation 4.33 and integrating over $z$,

\[
\gamma_{\text{side}}(\theta) = \frac{2}{\pi}(1 - D) \langle \gamma_{\text{flat,loc}}(\theta) \rangle \tan \theta \times \int_0^{\infty} dt \frac{t^2}{(1 + t^2)^2} \frac{1 - e^{-2rnh(t + \tan \theta)}}{t + \tan \theta} \hspace{1cm} (4.36)
\]

The summation of Equations 4.29, 4.32, and 4.36 constitutes the model. It can be split into a dependency only on packing density $D$ and a dependency only on velvosity $u$:

\[
\gamma_{\text{tot}} = D + (1 - D)f(u, \theta) \hspace{1cm} (4.37)
\]
The quantity \( f(u, \theta) \) is plotted in Figure 4.4 and is compared to the numerical result in Figure 4.3.

The purpose of this lengthy exercise was to gain physical insight into the process of SEE from a whiskered surface. From examination of Figure 4.4 and Equations 4.29, 4.32, and 4.36, we can do just that:

We see that, at electron incident angles that are very normal, electrons are suppressed efficiently by a velvose velvet, \( u >> 1 \). However, even for an infinitely velvose surface, \( u \to \infty \), electrons with some finite off-normal velocity component are relatively un-suppressed. From this we should take the practical lesson: For normal angles of incidence, only modest velvets are required for SEE suppression, but for shallow angles of incidence, even a perfect velvet cannot suppress by a large factor.

In the PFRC-II experiment, for the ions incident on the Pyrex antenna tube, we expect a very narrow angle of incidence and expect velvet to be an appropriate choice for suppression of the fast population of electrons. However for the Pyrex-born electrons incident on the backplate, we expect very shallow angles of incidence, and so must turn to other surface geometries to suppress SEE.

### 4.4.2 Foam

A foam surface is like a velvet surface except that the axes of the whiskers are isotropically distributed rather than perfectly normal. An example geometry is presented in Figure 4.5. The hope was that a more isotropic distribution of whisker axes would produce a more isotropic primary electron angular dependence of SEY suppression.
Figure 4.4: Top: $f(u, \theta)$ vs $\theta$ for several $u$ (curves) that determines the net SEY in equation $\gamma_{eff} = \gamma_{flat}\{D + (1 - D)f(u, \theta)\}$. Bottom: Relative contribution to the SEY of the sides of the whiskers. Pointed out in both are the points at which the quantity $u \tan \theta$ crosses unity.
Our numerical and analytic calculations were first presented in my paper with Igor D. Kaganovich in 2018.[79]

The results of the Monte Carlo and analytic calculations of SEY from a foam are given in Figures 4.6 and 4.7. To interpret them, you must know that a generalized velvosity $\bar{u}$ is defined $\bar{u} \equiv \frac{\pi}{2} r h n = A D / 2$, where $A$ is the aspect ratio $A \equiv h / r$ and $D$ is the volume fill density. $\bar{u}$ is a measure of how much whisker there is: the more dense, or long, or wide the whiskers, the higher $\bar{u}$. It is related ($\bar{u} = \frac{\pi}{2} u$) to the velvosity $u$ found for velvet, with the differences in geometry accounting for the numerical coefficient. The quantity plotted is the $f(u, \theta)$ function which was defined in the last section, $\gamma_{tot} = \gamma_{flat}(D + (1 - D)f)$. 

**Figure 4.5:** Rendering of an example of the foam surface used in this section. This foam had 80 fibers, aspect ratio $A = 10$, volume fill fraction $D = 4.3\%$, and foam parameter $\bar{u} = 2.2$. The absolute length scale is not defined for our analytic model.
Figure 4.6: Results of analytic theory compared to full numerical Monte Carlo model.

Figure 4.7: Results of analytic theory. Total SEY is $\gamma[D + (1 - D)f_{geom}]$. 
The analytic model for foam was adapted from that of velvet. Equations 4.29, 4.30, and 4.33 remain true, but the equations for probability of intersecting a whisker surface while traversing the axial direction, \( P(\Delta z) \) (Equation 4.26) and the probability distribution of a side-emitted electron’s polar angle \( P(\theta_2|\text{sides}) \) (Equation 4.34) are different.

Electrons inside a uniform-axis whisker layer (velvet, not a foam) hit the whiskers with uniform probability per unit distance traveled perpendicular to the whiskers’ axes, as in Equation 4.24 for foam. The differential form of this equation is

\[
P(\text{hit})dS_\perp = 2rndS_\perp,
\]
where \( dS_\perp \) is the distance traveled perpendicular to the axis. If the whiskers are aligned along \( \hat{a} \), this becomes

\[
P(\text{hit})dz = 2rn\frac{\sqrt{1 - \hat{\nu}\cdot\hat{a}^2}}{\hat{\nu}\cdot\hat{z}}dz,
\]
where \( z \) is the direction normal to the solid surface and \( \hat{\nu} \) is the direction of primary electron incidence. This equation is linear with density \( n \) of whiskers. Differentially, multiple populations of whiskers 1, 2 simply add:

\[
P_{1+2}(\text{hit})dz = 2rn_1\frac{\sqrt{1 - \hat{\nu}\cdot\hat{a}_1^2}}{\hat{\nu}\cdot\hat{z}}dz + 2rn_2\frac{\sqrt{1 - \hat{\nu}\cdot\hat{a}_2^2}}{\hat{\nu}\cdot\hat{z}}dz.
\]

In a foam, \( \hat{a} \) is isotropic. \( l \equiv \hat{a} \cdot \hat{\nu} \) is uniformly distributed. Thus in a field of
infinitely many infinitesimally dense fields of isotropically aligned whiskers, the probability of hitting one is

\[ P(\text{hit})dz = 2\pi n \frac{dz}{\mu} \int_{-1}^{1} dl \frac{\sqrt{1-l^2}}{2} = \frac{\pi}{2} \pi n \frac{dz}{\mu}, \quad (4.41) \]

where \( \mu \equiv \hat{v} \cdot \hat{z}, \mu = \cos \theta. \)

The probability that an electron will traverse \( \Delta z \) without having hit a whisker is this value integrated over \( z \)

\[ P(\Delta z) = e^{-\frac{\pi}{2} r n \frac{\Delta z}{\mu}} = e^{\frac{\Delta z}{\mu}}. \quad (4.42) \]

This is the equivalent of velvet’s Equation 4.26 for foam. It replaces factors in Equations 4.30 and 4.33 to adapt the analytic velvet model into a foam model.

Because foam whisker axes are isotropic, the vector field of normals to the whiskers’ sides is also isotropic. From this fact we may produce the quantity \( P(\mu_2|\mu, \text{side}) \), the probability that a primary electron with axial velocity component \( \mu = \cos \theta \) will, upon impact with a whisker side, produce a secondary electron with axial velocity component \( \mu_2 = \cos \theta_2 \). The probability that a primary electron hits a surface element with normal vector \( \hat{n} \) will be linearly weighted by the factor \(-\hat{v} \cdot \hat{n}\). Integrating over all normal vectors \( \hat{n} \) and leaving out some tedious steps,

\[ P(\mu_2|\mu) = \frac{4}{\pi} \int_{-1}^{1} dm (A_1 \sin \phi_1 + B_1 \phi_1) (A_2 \sin \phi_2 + B_2 \phi_2) \quad (4.43) \]
\[ A_{1,2} = \sqrt{(1 - m^2)(1 - \mu_{1,2}^2)}, \quad B_{1,2} = m\mu_{1,2} \]

\[ \phi_{1,2} = \text{Re}\left[\cos^{-1}\left(-\frac{B_{1,2}}{A_{1,2}}\right)\right], \]

where \( \text{Re}(x) \) is the real part of \( x \). \( m \) is an integration variable, but it may be informative to know that \( m = \hat{n} \cdot \hat{z} \). This equation replaces a factor in Equation 4.33 to adapt the velvet model into a foam model.

The primary angle of incidence distribution will be different also. \( \langle \gamma \rangle \) is the SEY averaged over these many local primary angles of incidence. According to Eq. 4.4 and the value of \( k_s = 1 \) of a smooth surface, the average SEY from isotropically aligned surface elements will be larger than that of the flat value by

\[ \langle \gamma \rangle / \gamma = \int_0^1 d(\cos \theta) 2 \cos \theta (1 + \frac{\theta^2}{2\pi}) \approx 1.12. \] (4.44)

The entire model may be written:

\[ \gamma_{tot} = \gamma D + (1 - D)[\gamma \int_0^1 d\mu_2 \mu_2 e^{-\left(\frac{\mu_1}{\mu_2} + \frac{1}{\mu_2}\right)\bar{u}} + \langle \gamma \rangle \int_0^1 d\mu_2 \frac{1 - e^{-\bar{u}\left(\frac{1}{\mu} + \frac{1}{\mu_2}\right)}}{1 + \frac{\mu}{\mu_2}} P(\mu_2 | \mu)], \] (4.45)

where \( \langle \gamma \rangle \) is defined in Eq. 4.44 and \( P(\mu_2 | \mu) \) is defined in Eq. 4.43. Recall that \( \mu = \cos \theta \), where \( \theta \) is the polar angle. Also recall that \( D \) is the volumetric fill ratio and \( \bar{u} = AD/2 \) where \( A = h/r \) the ratio between the whisker layer thickness and the whisker radius.
The factor in the square brackets is a function only of \( \bar{u} \) and \( \theta \). It is plotted in Fig. 4.7. If the foam is very deep or very dense, \( \bar{u} \) is infinity and there is a limiting case:

\[
\gamma_{\text{eff}} = \gamma D + (1 - D) \langle \gamma \rangle \int_0^1 d\mu_2 \frac{1}{1 + \frac{\mu}{\mu_2}} P(\mu_2 | \mu), \tag{4.46}
\]

which is also plotted in that figure.

From the plotted result and analytic formula, it can be seen that the SEY response to primary angle of incidence is in fact more isotropic than that of velvet. Unfortunately it is higher than that of velvet at every angle, making it not useful for mitigating SEE in the PFRC-II or other devices which hope to minimize SEE.

### 4.4.3 Feathers

To address the shortcomings of both velvet and foam, we propose a “feathered” surface geometry to suppress SEE. A “feathered” surface is a velvet surface onto whose whiskers a smaller layer of velvet whiskers is grown. An example is given in Figure 4.8. The Monte-Carlo and analytic calculations of SEY from feathers were first presented in my paper with Igor D. Kaganovich in 2017.[78]

The motivation behind proposing this shape was that, to first order, the shape is identical to velvet and will have velvet’s strength in suppressing SEE from normally incident primary electrons. To second order, at a smaller scale, shallowly incident primary electrons will encounter the smaller velvet at a near-normal angle of incidence and also be suppressed strongly.
Figure 4.8: a) Drawing of the feathered, “whisker on a whisker” geometry and schematic representations of the suppression mechanism. This geometry corresponds to a shorter, fatter ($D = 16\%, A = 10$ rather than $D = 4\%, A = 80$) geometry than the one calculated. At right are shown the effects that lead to SEY reduction: b) increase in effective capture area. c) Normal and shallow incident primary electrons on a velvet geometry. d) Normal and shallow incident primary electrons on a feathered geometry. Red arrows correspond to primary electron trajectories. Yellow arrows correspond to example secondary electron trajectories.

The results of the calculation, and some illuminating analytic approximate results, are presented in Figure 4.9.

As can be seen in that figure, the SEY from the feathered surface is much reduced from the velvet surface, especially at shallow angles of incidence. In fact, the feathered surface suppresses SEE more effectively than an infinitely velvose ($u \to \infty$) surface at high angles of incidence. The most accurate ad-hoc analytic model to characterize the one geometry tested in the Monte-Carlo code is that of a simple velvet with the
Figure 4.9: Solid lines show the result of the numerical Monte Carlo calculation: reduction in SEY of the considered $u = 2$ graphite velvet either without another recursive velvet grown onto the whisker sides (“Primary whiskers, $u = 2$”), or with this smaller velvet (“Secondary whiskers, $u = 2$”). Also shown (2 dashed lines) are the result of the analytic model for velvet with $u = 4$ and $u = \infty$ for velvets with $D = 4\%$. Also shown (last dashed line) is the result of the analytic model for $u = 4, D = 4\%$, but with the emission from the sides of the whisker reduced by half.

SEY from the sides of the whiskers reduced by one-half. We expect that SEY suppression factors will be more extreme at more extreme velosity values, but simulation time scales linearly with dynamic range of length scales and so these regimes were not possible to simulate.

### 4.5 Conclusion

In this chapter, I evaluate several complex, fibrous surface geometries for their effect in reducing Secondary Electron Yield (SEY). I use two models: a numerical Monte Carlo model based on representing the surface as an isosurface of a scalar function of space, and an analytic model assuming a continuous probability of interacting with a fiber.
From these models, I determine that a velvet, which is a lattice of perfectly normally aligned fibers, is suitable for reducing SEY from perfectly normal electrons. To reduce SEY from isotropically or shallowly incident electrons, I evaluate the SEY reduction from a foam or fuzz. This produces more isotropic results. For yet still improved SEY reduction, I evaluate a fractal velvet, or a feather, which includes both large primary whiskers and small secondary whiskers grown onto the primary whiskers. I find that this general strategy is quite promising for the reduction of SEY.

I also gave some insight into how complex geometries can suppress SEY based on a “weighted view factor” model. The consequence of this model is that a SEY-suppressing surface should have as much integrated surface area in as small a cross-sectional surface area as possible, as the maximum possible SEY-reduction is the ratio of these surface areas. I also found that, for a surface-area-optimal SEY-suppressing surface, each surface element should be equally illuminated by the bulk plasma.

Based on these findings, I support the claim that the effect measured in Chapter 3 is possible to mitigate, were it to be found deleterious for some purpose.
Chapter 5

Trapping of high energy electrons: Magnetic moment non-adiabaticity in a magnetic mirror

This chapter describes several important behaviors of a single particle in a magnetic mirror machine in which the particle’s magnetic moment, $\mu$, is only poorly adiabatically conserved. This behavior is sometimes called quasadiabatic.[112] Quasadiabatic systems have received constant theoretical and experimental attention since the 1940s, but there appear to be many pervasive misconceptions about such systems.

In this chapter, I describe measurements of the PFRC-II run as a tandem mirror that show surprising and potentially useful phenomena that are only possible in the quasadiabatic regime. I explain these behaviors both with elementary analytic models and high-fidelity numerical calculations.
In measuring the x-rays emitted from the CC, we discovered that the population of electrons produced in the Source End Cell (SEC) extended into the CC. This is expected, as we found in the Chapter 3 that they leave the sheath with velocity almost parallel with the magnetic field. We would expect the density of fast electrons to be smaller in the CC than in the SEC by the ratio of magnetic flux densities at the measurement points, about a factor of 3.5, as the particles spread out as they follow less dense field lines.

Instead, we measured that the fast electrons in the CC can be as dense or denser than those found in the SEC. Furthermore we found that, after the cessation of the RF power in the SEC, the x-ray signal decays much more slowly in the CC than in the SEC, having an $e$-folding time of more than $10 \mu s$. This indicates that fast electrons that enter inside the loss cone of the CC mirror are persisting for on average hundreds of transits, with some persisting for thousands. The collisionality and electrostatic activity of the plasma are insufficient to produce this effect. Furthermore, based upon a commonly posited definition of the adiabatic parameter $\epsilon = \rho \nabla B / B$, we expect that $\mu$ should be very well conserved, in contrast to the observed result.

Run at lower gas pressure, we measured the fast electrons in the CC to be much hotter than those in the SEC and persist for hundreds of thousands of transits. This behavior will be explained in Chapter 6.

There is a misconception that non-adiabaticity of $\mu$ is not of practical concern in thermonuclear plasma confinement devices. Indeed, popular models are often formulated assuming a conserved $\mu$. However, quasiadiabatic research shows that non-adiabaticity of $\mu$ can be a leading-order effect in some of these devices, especially of magnetic mirrors fueled from an end cell. In Tokamaks, the approximation of conserved
\( \mu \) is well justified for thermal particles. See Appendix B for three examples of devices in which quasiadiabatic effects are important.

Part of this misconception is confusion around the term “conserved to all orders.” Part comes from lack of appreciation for the effect of parallel velocity \( v_\parallel \), and the radius of curvature of the magnetic field lines \( R_c \) and its gradient on the non-adiabaticity of \( \mu \).

I will address these in this chapter.

This chapter is much improved for having read the master’s thesis of Xiang Chen.[113] Xiang was inspired by un-published simulations by Samuel A. Cohen of electrons in the PFRC-II magnetic field exhibiting \( \mu \)-jumps. He modeled the process numerically and analytically.

This chapter is also much improved for having read a summary of \( \mu \) non-conservation results by Samuel A. Cohen prepared for the PFRC group.[114]

### 5.1 Relevance: Magnetic mirrors and constant \( \mu \)

#### 5.1.1 Background: Magnetic mirror machines

Magnetic mirror machines were once a front-runner for the shape of future fusion power reactors. Initially they offered confinement performance competitive with Tokamaks. However interest has waned; in 1986 the largest magnetic mirror research project was cancelled.[115] The largest magnetic mirror experiment today is the Gas Dynamic Trap in Novosibirsk.[116]
Magnetic mirror machines can be as simple as two magnetic loops with a space in between. In a magnetic mirror, particles are confined radially by a strong axial magnetic field such that their gyroradius is small compared to the vessel radius. Particles are confined axially by the $\mu \nabla B$ force, as $|\vec{B}|$ increases in amplitude toward the ends of the machine.

Because of this, only particles with a larger $\mu B_{max}$ than their energy $E$ are confined. If $E$ exceeds the maximum of the effective potential of the $\mu \nabla B$ force, $\mu B_{max}$, the particle is not confined and exits through one of the mirror coils. In velocity-space, this un-trapped population corresponds to a cone around the $v_||$ axis with half-angle $\theta = \sin^{-1}(\sqrt{1/R})$, the “loss cone,” where $R = B_{max}/B_{min}$, the “mirror ratio.”

Trapped particles can become un-trapped and vice versa through collisions which change their pitch angle. The size of this effective cross section for loss has plagued mirror machines and required technologically complex measures to be adopted.

Mirrors confinement and dynamics get significantly more complicated than this. For a discussion of how faster electron losses lead to an ambipolar potential which causes ion losses, read Pastukhov.[117] For a discussion of particle loss through collective modes, read Rosenbluth and Longmire.[118] For a theoretical understanding on the most promising axial confinement scheme, end plugs with thermal barriers, read Baldwin et. al.[119] For experimental results see Simonen.[120] For a good review, read Hershkowitz et. al.[121]
5.1.2 The ubiquity of constant magnetic moment in foundational, popular, and productive models of plasma behavior

In his foundational 1950 monograph *Cosmical Electrodynamics*, Hannes Alfvén derived constraints of the orbits of energetic, collisionless electrons and ions in the Earth’s dipole magnetic field.[122] This work formed the basis of modeling single-particle motion in a magnetic mirror, and remains a classic pedagogical tool for first-year plasma physics students. In deriving the motion of particles in an inhomogeneous magnetic field, Alfvén claims that, for the condition

$$\rho \nabla B \ll B$$

is a constant of the motion. Here $\rho$ is the gyroradius and has no dependence on $v_{||}$.

This claim was developed and given specific functional form by a pair of theoretical papers in 1957. Russell Kulsrud showed that in a harmonic oscillator, an adiabatic invariant is conserved to “all orders” in the adiabatic parameter.[123] He is careful to note that “This does not imply that it must be a rigorous constant.” This result was applied to a charged particle’s motion in a paper by Martin Kruskal, who determined that the change in adiabatic invariant has the shape of the essential singularity, $e^{-1/\epsilon}$.[124]

An early paper on plasma confinement by the “Pyrotron” experiment by Richard Post in 1958 applied these results to single-particle motion in a magnetic mirror geometry.[125] He discussed and investigated the non-adiabaticity of $\mu$, taking Kruskal’s equation:

$$|\delta \mu / \mu| = ae^{-b/\nu}$$  \hspace{1cm} (5.1)
where $\nu = 2\pi \rho / L$, a ratio of the gyroradius to the mirror length. $a, b$ were fit parameters of order unity found numerically. Like Alfvén’s equation, it has no dependence on $v_{||}$. Using this equation, Post claimed that $\delta \mu$ was not worth consideration in practical mirror machines. In his model, losses occur only from collisional diffusion into the loss cone of the mirror machine. Subsequent analyses of confinement in magnetic mirror machines do not consider $\delta \mu$ at all, assuming $\mu$ is constant without directly stating it.[117]

Magnetohydrodynamics (MHD) is among the most abstracted, reduced models of plasma behavior, modeling it only as one fluid and a magnetic field.[126] Its appeal comes from its simplicity. MHD is the first model applied to any new fusion reactor concept, as MHD stability is simple, intuitive, and often analytically tractable.

In its most idealized form, MHD considers only a scalar pressure, $p$. However, as a first cut into more collisionless dynamics, MHD is often formulated with an anisotropic pressure, allowing pressure parallel and perpendicular to the magnetic field to be unequal ($p_{\perp} \neq p_{||}$). This step is essential to modeling magnetic mirrors, as from an MHD perspective, pressure anisotropy is what confines the particles axially. In each formulation of MHD allowing anisotropic pressure, absolute conservation of $\mu$ is assumed.[118, 127–129]

Gyrokinetics is a model of plasma behavior in which the total particle distribution function $f(\vec{x}, \vec{v}, t)$ is considered instead a function of gyrocenter variables $f(x_{gc}, v_{||}, \bar{\mu})$.[130] Gyrokinetics has been famously successful in recent years. It is the go-to simulation tool for microinstabilities like the Ion Thermal Gradient mode and Trapped Electron modes, all the way up to macro-features like zonal flows and streamers.[131–134]
\( \bar{\mu}_n \) is a quantity which is defined to some order in an adiabatic ordering parameter \( \epsilon \). The zeroth order \( \bar{\mu}_0 = \frac{mv^2}{2B} \) is the definition of \( \mu \) that we take in this chapter. Under \( n^{th} \)-order gyrokinetics, \( \bar{\mu}_n \) is conserved. [135] Because it is possible to formulate gyrokinetics to any order in \( \epsilon \), it is tempting to believe that there exists some arbitrarily high order to which \( \bar{\mu} \) can be defined that is arbitrarily-well conserved. However, this is not accurate:

If there were some well-conserved order \( N \) of \( \bar{\mu}_N \), then we would expect \( \bar{\mu}_0 \) to oscillate around this constant value of \( \bar{\mu}_N \), with its largest excursions being at the magnetic minimum of its trajectory. This is indeed observed; \( \bar{\mu}_0 \) exhibits large oscillations at the midplane of a magnetic mirror where \( \bar{\mu}_1 \) exhibits almost none at all. However, at the turning point of a mirror-bounce trajectory, at its magnetic maximum where \( v_{||} = 0 \), we would expect all definitions of \( \bar{\mu} \) to converge once again on the constant \( \bar{\mu}_N \), including \( \bar{\mu}_0 \). As will be shown later in this chapter, we do not observe this to be the case; in less abstract calculations than gyrokinetics, we see a permanent change in the value of \( \bar{\mu}_0 \) which persists to the turning point of the electron in the high-field region. Therefore, the assumption of a perfectly conserved adiabatic \( \bar{\mu} \) in gyrokinetics is an approximation that leaves out the important phenomena discussed in this chapter.

### 5.1.3 Early experimental and numerical evidence for non-adiabaticity in mirror machines

Ours is not the first observation of dynamics caused by non-adiabaticity of \( \mu \). There have been several experiments and theoretical analyses dating back to the earliest days of magnetic fusion energy which show large jumps in \( \mu \) in magnetic mirror machines.
L. R. Henrich’s submission to the Conference on Thermonuclear Reactions in 1956, which at the time was a classified military secret, was entitled “Departure of Particle Orbits from the Adiabatic Approximation.”\cite{136} It was a numerical integration of the equations of motion of a particle in the magnetic field of a contemporary experiment called Felix. The integration was performed on a UNIVAC computer. Henrich found $|\delta \mu / \mu|$ jumps as high as 40%. His definition of the adiabatic parameter included $v_{||}$, the velocity parallel to $\vec{B}$, which was uncommon for the time and presaged later work. If the usual adiabatic parameter were assumed, one would expect orders of magnitude less mobility in $\mu$.

Garren et. al. also performed numerical calculations of particle orbits in a magnetic mirror in 1958.\cite{137} They described the size of their $\mu$ jumps as “disturbingly large.” Their empirically approximated $|\delta \mu / \mu|$ expression includes only $v_{||}$, again contradicting the commonly used adiabatic parameter. They plotted recurrence figures, that is they plotted $(\mu, \psi_0)$ points for each subsequent transit of the midplane of a single trajectory, where $\psi_0$ is the particle’s gyrophase when crossing the midplane. Presaging chaos, they found regions where $\mu$ is bounded and $(\mu, \psi_0)(t)$ defines a closed curve, and regions where $\mu$ is unbounded and $(\mu, \psi_0)(t)$ is volume-filling. Their interest was predicting the performance of a magnetic mirror device with loss cone angle 55°, so the bounds of this stochastic $(\mu, \psi_0)$ region became the new loss cone, which could be as high as 80° for higher-energy particles. Far from being negligible, the non-adiabaticity of $\mu$ had a leading-order effect on the performance of the notional device.

Gibson et. al. in 1963 performed an experiment in which they flooded the interior of a magnetic mirror device with radioactive Neon-19.\cite{138} Neon-19 has a half-life of
17 seconds and emits a positron at 2.2 MeV. These positrons stood in for very energetic electrons, and were the instrument for tracing collisionless trajectories of electrons in a magnetic mirror for \(10^{10}\) cyclotron periods. They used a definition of the adiabatic parameter that included total velocity, \(v\), not just \(v_\perp\). The found that, for adiabatic parameter \(\epsilon > 0.1\), there was a noticeably larger effective loss cone than that predicted by adiabatic \(\mu\).

The 1960s saw more experiments measuring \(\delta\mu\), with Reece Roth measuring the non-adiabaticity of ion in a plasma flow across a sharp magnetic barrier,[139] and Balebanov et. al. observing a decrease in lifetime when a sharp magnetic constriction was added to the throat magnetic mirror to protect an electron gun there.[140]. These experiments observed large \(\delta\mu\) down to values of the adiabatic parameter of \(\sim 0.1\).

### 5.1.4 Accurate models for the size of a single change in \(\mu\)

Hastie et. al. formulated an analytic approximation of \(\delta\mu\) dependent on the local derivatives of \(\vec{B}\) at the midplane of a magnetic mirror, both components of \(v (v_\perp, v_\parallel)\), and the gyrophase.[141] Their analytic \(\delta\mu\) has been compared favorably to numerical calculations both in their own paper and in subsequent analyses.[113] The adiabatic parameter they discovered is

\[
\epsilon = \frac{3}{2\sqrt{2} \Omega_e} \sqrt{\frac{1}{B} \frac{\partial^2 B}{\partial l^2}} \tag{5.2}
\]

The full expression of Hastie, Taylor, and Hobbs is
\[ \Delta \mu = A \sin \phi_0 e^{-1/\epsilon} e^{-\beta_H \cos \phi_0} \] (5.3)

where \( \phi_0 \) is the gyrophase at the midplane and \( \beta_H \) and \( A \) are functions of the magnetic field, particle velocity, magnetic radius of curvature, and second derivatives thereof:

\[ \beta_H = \frac{1}{3} \frac{v_{\perp} z_B}{v_{\parallel} R_c} \left( 1 - \frac{z_B^2}{z_R^2} \right) \] (5.4)

\[ A = -\frac{\sqrt{\pi} e}{4} \left[ \frac{v_{\perp}}{R_c B} \left( \frac{3}{2} v_{\perp}^2 \right) + v_{\perp} \frac{z_B^2}{z_R^2} \frac{1}{R_c B} \left( v_{\perp}^2 + 2 v_{\parallel}^2 \right) \right] \frac{z_B}{v_{\parallel}} \] (5.5)

where \( R_c \) is the magnetic radius of curvature and

\[ z_Q^{-2} = \frac{\partial^2 z \cdot Q}{2 Q_0} \] (5.6)

where \( Q \) is either the magnetic field or the radius of curvature, depending on the subscript of \( z_Q \), \( Q_0 \) is the value of \( Q \) at the midplane, and \( z \) is the spatial dimension along the magnetic field line.

Other models for the size of \( \Delta \mu \) have also been computed with different asymptotic techniques and different approximations from those of Hastie, Taylor, and Hobbs.[142–145]

A clear physical picture of the source of the \( \mu \) jumps is given in Delcourt et al.[146] They derive an approximate formula for the size of \( \Delta \mu \) based on the model that the
particle, while passing a region of sharply changing magnetic curvature, exhibits an impulse-like moment of centrifugal force. While their formula is approximate, more rigorously defined models of $\Delta \mu$ like that of Hastie, Taylor, and Hobbs and the one derived later in this chapter exhibit sensitive dependences on parameters which control the magnitude and sharpness of magnetic-curvature-induced centrifugal force, $v_{||}, R_c,$ and $z_R,$ the second spatial derivative of the radius of curvature.

5.1.5 Chirikov and the modern understanding of quasadiabatic systems

In 1978, B. V. Chirikov reviewed the study of non-adiabaticity of the magnetic moment of particles in a magnetic mirror.[147] For the first time, all findings were in place to characterize their long-time behavior. By this time, many studies had determined approximate forms of the size of $\mu$ jumps based on well-founded asymptotics. They showed the correct dependency on $v_{||}$ and $R_c.$ Furthermore, the numerical and analytic study of chaotic maps such as the Standard Map (sometimes called the Chirikov Map) had been accomplished over the preceding decade. With access to these two tools, Chirikov was able to formulate the condition which determines the long-time constancy or not of $\mu.$ The condition, which I will explain in detail later in this chapter, is that the $\mu$-jump should change the gyrophase upon the particle’s next midplane passing by about 1 radian. If the $\mu$ jump changes it by more than this, the particle is not confined and $\mu$ is free to diffuse. If the $\mu$ jump is not large enough to cause such a change, $\mu$ is confined to a narrow interval around its initial value.
Chirikov did not appear to realize that this implies that there is always a chaotic ($\mu$-diffusing) region around the loss cone. This was discovered by S. G. Tagare in 1986.[148] This implies that Chirikov’s criterion describes the size of the effective loss cone found by Garren et al. in 1958.

Chirikov and others described the phenomenon of quasiadiabatic loss from a magnetic mirror. The inverse problem, that of passing particles becoming quasiadiabatically trapped in a magnetic mirror, has not been studied. But a similar process has been experimentally studied in dipole machines in the University of Tokyo.[149–151] Observed are very long particle lifetimes, with very few particles returning to the injection mechanism after one transit. Because of this, Nakashima et al. suggested that quasiadiabatic trapping could be used to make efficient particle injectors, for placing particles into trapped trajectories in a dipole field.[150]

The analysis tool of treating particles’ trajectories as perfectly adiabatic between instantaneous piecewise jumps, and so making discrete numeric maps from the dynamics, has been generalized from Chirikov’s treatment of magnetic mirrors. It is a productive practice in the study of space plasmas.[112][152]

This larger loss cone is not a small effect. In a review in 1987, Richard Post laid out some very modest conditions under which non-adiabatic loss due to the Chirikov condition exceeded Coulomb collision-induced loss.[153]

Despite the importance of the finding, it does not appear well known. I could find no magnetic mirror review article after the year 1987 which describes non-adiabaticity and its effects on end-loss.[121, 128, 154, 155] The loss due to the phenomenon described
by Chirikov does not appear to be part of the standard institutional body of knowledge of plasma physics.

5.2 PFRC-II experimental results

5.2.1 Apparatus

This chapter discusses what happens when fast electrons produced in the Source End Cell (SEC) of the PFRC-II enter the Center Cell (CC). For information on how these electrons are produced and descriptions of their characteristics, see Chapter 3.

The PFRC-II in seed plasma mode is a tandem mirror, with a center cell and two end cells. This is depicted in Figure 5.1. The Center Cell is a magnetic mirror cell with a mirror ratio of $10^{-40}$. The vessel of the CC is made of transparent polycarbonate Lexan. Its inner diameter (ID) is 22.7 cm. The vessel wall is 16 mm in thickness, with 87 penetrations for diagnostics to be inserted. The vessel is 84 cm long. Use of Lexan for a vacuum vessel is atypical; for a discussion, see Berlinger et al.[156] Gas is introduced only in the SEC. It enters the CC through the SEC-CC nozzle. The CC is pumped by a 100 l/s Varian TV 141 NAV turbomolecular pump.

The magnetic field in the midplane, the minimum-B point, is determined by four sets of water-cooled pancake coils, called the L-2 coils. In Figure 5.1, these are called the Axial Field Coils. The maximum of the magnetic field, at the nozzles, is determined by both the L-2 coil current and two sets of small-bore, water-cooled Nozzle coils or N coils.
Figure 5.1: Schematic of the PFRC-II experiment run in seed plasma mode. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The $|\mathbf{B}|$ profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate.
At the magnetic minimum at the midpoint, the magnetic field has strength $0.74 \frac{G}{A} \times I_{L-2}$, where the units of that numerical constant are Gauss per Ampere, and $I_{L-2}$ is the current carried by the L-2 coils. At the magnetic maximum at the nozzle, the magnetic field has strength $6.6 \frac{G}{A} \times I_{L-2} + 5.9 \frac{G}{A} \times I_N$, where $I_N$ is the current carried by the N coils. The L-2 coils carry current between 40 A and 300 A. The N coils carry current between 0 A and 400 A.

The x-ray data presented in this chapter was recorded in both the CC and SEC. In the SEC, the SDD used the SEC Radial mount, whose parameters were reported in the table in Section 2.3. Along the line-of-sight of the SEC Radial mount, the magnetic field has strength $2.5 - 3.0 \frac{G}{A} \times I_{L-2}$. In the CC, the SDD used the CC Midpoint mount, whose parameters were reported in the same table. The line-of-sight of the CC Midpoint mount goes through the magnetic minimum of the CC. The procedure for operating and calibrating the SDD x-ray detectors and analyzing x-ray data was described in Chapter 2.

### 5.2.2 Steady-state density versus expected from transit time

The PFRC-II seed plasma run of 2017/11/15 was the first time that two SDD detectors, SDD1 and SDD2, were used concurrently in the CC and SEC respectively. The resulting spectra are depicted in Figure 5.2. For this condition, the CC pressure was 0.28 mTorr of $H_2$ gas, the SEC pressure was 0.5 mTorr of $H_2$ gas, the net power to the antenna was 350 W at 27 MHz, the L-2 coils had 130 A and the N coils had 225 A. The CC mirror ratio was 23. The minimum mirror B was 96 G.
The “observed” density, \(1.2 \times 10^9/\text{cm}^3\) in the CC and \(9.4 \times 10^8/\text{cm}^3\) in the SEC, is not from a Maxwellian fit to the distribution. As in previous chapters, it is the integral of the inverted EEDF, which corresponds to density above 600 eV of electron energy. The detailed procedure for recovering this density was described in Chapter 2.

Between the SEC and the CC, there is a magnetic mirror with SEC mirror ratio of \(4 - 10\). For the conditions reported in Figure 5.2, the SEC mirror ratio was 6.2.

We model the creation of fast electrons in the SEC in Chapter 3. Our model relies on fast electrons returning back upstream after a reflection, either from the magnetic mirror or from the floating paddle in the Far End Cell (FEC). To demonstrate that there...
must be some trapping of fast electrons in the CC, let us now assume that all of the
SEC-born fast electrons pass through the CC. The collisionless electrons travel along
magnetic field lines, whose density is proportional to $B$. $B$ along the CC Midpoint
mount line-of-sight is $\sim 3.5$ times smaller than $B$ along the SEC Radial mount line-
of-sight. The larger area of the plasma column in the CC was taken into account in the
computation of the effective volume reported in the table in Section 2.3.

For distributions of particles which pass through two surfaces, the densities can be
related by equating the particles passing through each surface:

$$n_1 \bar{v}_{||,1} A_1 = \Gamma_1 A_1 = \Gamma_2 A_2 = n_2 \bar{v}_{||,2} A_2$$  \hspace{1cm} (5.7)

where $n_i$ is the density of particles at surface $i$, $\bar{v}_{||,i}$ is the average velocity in the
direction normal to surface $i$, and $A_i$ is the area of surface $i$. From Figure 5.2 we can
see that the energy dependence on the EEDF of both chambers is similar, so $\bar{v}_{||,1} \approx \bar{v}_{||,2}$. $A_i$ are such that total enclosed magnetic fluxes are equal, as particles follow field lines.
Thus, the following relation should be at least approximately true:

$$n_{CC} \approx n_{SEC} \frac{B_{CC}}{B_{SEC}} \approx n_{SEC} \frac{1}{3.5}$$  \hspace{1cm} (5.8)

The smaller $v_{||}$ in the SEC expected from the $\mu \nabla B$ force would decrease this frac-
tion further. A subset of the fast electrons from the SEC passing into the CC would also
decrease this fraction. The approximate equality “$\approx$” should therefore be a less than,
“$<$.”
However this is not observed to be the case. In Figure 5.2, the density of fast electrons in the CC is larger than that in the SEC, though this is not always true. I chose to reproduce data from the run of 2017/11/15 because it shows this in its most extreme ratio, but a value of $n_{CC}$ measured to be larger than the expected value is universal in PFRC-II seed plasma experiments. It is typically larger than the value predicted in Equation 5.8 by a factor of about 3 - 5.

I claim that a subset of these electrons become trapped in the CC and persist for hundreds of mirror bounces, with some persisting for thousands. The evidence is given in the next section.

### 5.2.3 Pulsed operation decay time

During the PFRC-II seed plasma run of 2017/11/29, we pulsed the seed plasma RF power at 1 kHz with a 50% duty cycle. The RF power as a function of time was 500 µs at 440 W, 500 µs at 0 W, repeating. The seed antenna frequency was 27 MHz. The pressure in the CC was 0.37 mTorr of H$_2$. The pressure in the SEC was 0.6 mTorr of H$_2$. The L-2 current was 200 A. The N current was 240 A. This produced a mirror ratio of 18 and a minimum B of 148 G.

The SDD is able to produce a logic pulse on its AUX out pin whenever it collects an x-ray. We recorded these pulses as a function of time measured since the RF shut off. The result is depicted in Figure 5.3. In it, the SEC x-ray count rate as measured decays with time constant 5.2 µs, while the CC x-ray count rate decays with time constant 17 µs. The SDDs were set to a peaking time of 5.6 µs, so this is their effective resolution.
FIGURE 5.3: X-ray count rate as a function of time in the CC and SEC while the RF power was being pulsed. The characteristic fall-off time after cessation of RF power was 17 µs in the CC and 5.2 µs in the SEC. A 5.6 µs peaking time was used for both detectors, so the SEC decay time is not resolvable.

The x-ray signal is roughly proportional to the density of electrons above the Oxygen K-edge at 541 eV.[31]

In the SEC, the decay time was 5.2 µs, which is on the same order as the SDD peaking time as it was set. Because of this the SEC fast electron density decay time is not resolvable, but is shorter than ~5 µs.

In the CC, the decay time was 17 µs, larger than the 5.6 µs resolution. A 600 eV electron with zero pitch angle has a transit time of the 84 cm vessel of 58 ns. A decay time of 17 µs means that the particles that we detected in the CC persisted an average of about 300 transits between the nozzles.

As the fast electrons start in the SEC, those that make it into the CC do so already
in the loss cone of the CC mirror. To make 300 transits, they must become trapped by some mechanism. Only about 1 - 2 % of electrons are trapped, as the density measured in the last section is only larger by about 3 - 5 times.

5.3 Modeling of magnetic moment non-adiabaticity

We seek to determine the behavior of $\mu$ of a particle exhibiting collisionless motion in a magnetic mirror configuration. The particle’s motion is deterministic, and so characterized only by an initial position and velocity. With a coordinate transformation, these become the more illustrative coordinates of initial gyrocenter radius and axial position, energy, $\mu$, and gyrophase.

5.3.1 Single-particle numerical integration

Of all the methods for computing the change in $\mu$ cited in this chapter, the most trusted is the Boris algorithm.[157] The Boris algorithm is an explicit, discretized equation-of-motion integrator. While it is not symplectic, it preserves phase space volume. In the case of no electric field, it conserves energy.

The Boris algorithm, and other phase-space-volume-preserving equation-of-motion integrators, exhibit excellent bounding of error from constants of motion such as energy and canonical angular momentum.[158] However, gyro-phase error is not bounded.

For this section, we choose a timestep of $\delta t = 1/\Omega_{\text{max}}/100$. Convergence studies indicate that this is well within the envelope of fidelity of the Boris algorithm.
Figure 5.4: Boris-algorithm numerically calculated trajectory of a single electron in the PFRC-II run as a tandem mirror. a) the trajectory in space of the electron, superimposed over the mirror geometry. b) the trajectory in $\mu$ of this particle. The energy of this electron is 5.4 keV. It starts marginally trapped at $t = 0$. The magnetic field is the PFRC-II magnetic field assuming $I_{L2} = 60\, \text{A}, I_N = 350\, \text{A}$.

5.3.1.1 Example trajectories

Plotted in Figure 5.4 is an example of a non-adiabatic electron trajectory in a magnetic mirror. Specifically, this is an electron which at $t = 0$ has energy 5.4 keV, with gyro-center starting at $r = 4$ mm, $z = 44$ cm in a PFRC-II magnetic mirror field assuming $I_{L2} = 60\, \text{A}, I_N = 350\, \text{A}$. It starts with velocity only in the perpendicular direction, and since the magnetic field at $z = 44$ cm is very near the maximum, it starts very near to marginally trapped.

Figure 5.4 is exemplary of $\mu$-trajectories in magnetic mirrors. The only change in $\mu$ occurs at the magnetic minimum, with $\mu$ constant toward the nozzles. Because of the shortness of the period of $\mu$ change, we call this a $\mu$-bounce, or a $\Delta \mu$. 
5.3.1.2 Ensembles

Plotted in Figure 5.5 is an ensemble of particles which start like the one depicted in Figure 5.4, except that they start evenly spaced in initial gyrophase. This figure elucidates another important property of $\mu$-jumps: That they are periodic in midplane-crossing gyrophase. Indeed, by eye the Boris-derived values of $\Delta \mu$ appear to be roughly proportional to $\sin \psi_0$, where $\psi_0$ is the gyrophase of the particle at the midplane. Also depicted are a model derived later in this chapter and the model of Hastie, Taylor, and Hobbs.
5.3.2 Analytic formulae for magnetic moment change

5.3.2.1 Single transit $\mu$ change

This section derives an approximate formula for the size of the change in $\mu$ when crossing the midplane of a magnetic mirror. It does not re-trace the steps of Hastie, Taylor, and Hobbs;[141] rather it proceeds in a more elementary manner yet produces a largely similar answer.

The equations of motion for an electron in an axisymmetric mirror field ($B_\phi = 0$) are Newton’s second law and the Lorentz force equation. In cylindrical coordinates these are:

\[
\dot{v}_r = \frac{e}{m} v_\phi B_z + \frac{v_\phi^2}{r} \tag{5.9}
\]

\[
\dot{v}_\phi = \frac{e}{m} (v_z B_r - v_r B_z) - \frac{v_r v_\phi}{r} \tag{5.10}
\]

\[
\dot{v}_z = -\frac{e}{m} v_\phi B_r \tag{5.11}
\]

Just as non-cyclotron terms, the centrifugal term and the Coriolis term, in Equation 5.9 and 5.10 are inertial terms which come from the curvilinear coordinates changing direction, putting Equations 5.9-5.11 into a coordinate system which follows $\vec{B}$ yields both cyclotron and inertial terms:

\[
\begin{align*}
\dot{v}_n &= \Omega v_\phi + v_\phi^2 \frac{\hat{n} \cdot \hat{r}}{r} + v_nv_b (\frac{1}{l_B} + \frac{\hat{b} \cdot \hat{r}}{r}) + v_{n}^2 \frac{1}{R_c} \tag{5.12} \\
\dot{v}_\phi &= -\Omega v_n - v_\phi v_n \frac{\hat{n} \cdot \hat{r}}{r} - v_\phi v_b \frac{\hat{b} \cdot \hat{r}}{r} \tag{5.13}
\end{align*}
\]
\[ \dot{v}_b = v^2_\phi \frac{\hat{b} \cdot \hat{r}}{r} - v^2_n \left( \frac{1}{l_B} + \frac{\hat{b} \cdot \hat{r}}{r} \right) - v_b v_n \frac{1}{R_c} \] (5.14)

where \( \hat{b} = \vec{B}/|\vec{B}|, l_B^{-1} = \frac{\partial \vec{B}}{\vec{B}}, R_c \) is the radius of curvature of the magnetic field line, \( \hat{n} \) is the radially outward direction perpendicular to both \( \hat{b}, \phi \). Notice that \( B \) only serves to trade velocity between the \( \phi \) and \( n \) directions. The details of this calculation are given in Appendix A.

An equation equivalent to Equation 5.14 was also derived by Putvinskii in 1982 to study the effect of resonant magnetic perturbations on particle transport in Tokamaks.[159]

According to Equation 5.14, \( v_b \) only changes due to inertial terms brought about by curvature of coordinates, in this case the curvature of the magnetic field lines and the \( \phi \) coordinate. This is our first hint that \( \Delta \mu \) is mediated by curvature.

The second term in Equation 5.14, proportional to \( l_B^{-1} \), is the mirror bounce term. It is responsible for conserving \( \mu \). The fourth term in Equation 5.14, proportional to \( R_c^{-1} \), is primarily responsible for changes in \( \mu \). The centrifugal (first and third) terms, proportional to \( \frac{\hat{b} \cdot \hat{r}}{r} \), tend to average away, as they yield only higher harmonics of gyrophase.

For ease of numerical computation, I give these quantities in terms of the scaled enclosed flux, \( \zeta = \frac{e}{2\pi m} \Phi \):

\[ \Omega = \frac{\partial_n \zeta}{r} \] (5.15)

\[ R_c^{-1} = \frac{\partial^2_{\phi, \phi} \zeta}{\partial_n \zeta} \] (5.16)

\[ l_B^{-1} = \frac{\partial^2_{\phi, n} \zeta}{\partial_n \zeta} - \frac{\hat{b} \cdot \hat{r}}{r} \] (5.17)
Introducing the definition of $\mu = \frac{mv_{\perp}^2}{2B}$, conservation of energy $v_{\parallel} = \sqrt{v^2 - v_{\perp}^2}$, and a phase angle $v_{\parallel} = v_{\perp} \sin \psi$, $v_{\phi} = v_{\perp} \cos \psi$, Equation 5.14 becomes

$$\partial_t \mu = \frac{mv_{\perp}v_{\parallel}^2}{B} \frac{1}{R_c} \sin \psi - \frac{mv_{\parallel}v_{\parallel}^2}{2B} \frac{1}{l_B} \cos 2\psi - \frac{mv_{\parallel}v_{\parallel}^2}{B} \frac{1}{r} \cos 2\psi - \frac{mv_{\perp}^3}{2B} \frac{1}{l_{\perp}} \sin \psi$$

(5.18)

and the gyrophase evolves at rate

$$\dot{\psi} = \dot{\Omega} + 2v_{\parallel} \frac{\hat{b} \cdot \hat{r}}{r} \cos \psi \sin \psi + v_{\perp} \frac{\hat{n} \cdot \hat{r}}{r} \cos \psi + v_{\parallel} \frac{1}{l_B} \sin \psi \cos \psi + \frac{v_{\perp}^2}{v_{\perp} \hat{R}_c} \cos \psi$$

(5.19)

In a vacuum field, this equation reduces to the one used by Hastie et. al., Chirikov, Tagare, and others.[141, 147, 148]

Neglecting the higher-order cyclotron terms and choosing $v_{\parallel} >> v_{\perp}$, Equation 5.14 becomes the much simpler

$$\partial_t \mu = \frac{mv_{\perp}v_{\parallel}^2}{BR_c} \sin \psi$$

(5.20)

Because we know that $\Delta \mu$ occurs in a small region around the midplane, we choose the following functional forms of $B, R_c$. These functional forms capture first-order effects and produce an analytically tractable integral:
\[ R_c(z) = R_{c,0} e^{z^2/z_R^2} \]  \hspace{1cm} (5.21)

\[ B(z) = B_0 e^{z^2/z_B^2} \]  \hspace{1cm} (5.22)

\[ \psi(z) = \psi_0 + \frac{e}{m} B_0 z / v_{||} \]  \hspace{1cm} (5.23)

The integral \( \int_{-\infty}^{\infty} dt \partial_t \mu \) is then solvable by completing the square in the exponent:

\[ \Delta \mu_0 = \frac{mv_{\perp,0} v_{||,0} z_0 \sqrt{\pi}}{B_0 R_{c,0}} e^{-\frac{z_0^2 \mu^2}{v_{||,0}^2}} \sin \psi_0 \]  \hspace{1cm} (5.24)

\[ z_0^{-2} = z_R^{-2} + z_B^{-2} \]  \hspace{1cm} (5.25)

\[ z_Q^{-2} = \frac{\partial^2 z_Q}{2 Q_0} \]  \hspace{1cm} (5.26)

where \( \Delta \mu_0 \) is has the subscript 0 because we have yet to add two corrections, and any other quantity \( Q_0 \) is the quantity \( Q \) at the midplane. Equation 5.24 is not what the model of Hastie, Taylor, and Hobbs reduces to in the limit of \( v_{||} >> v_{\perp} \).[141]

The exponential factor in Equation 5.24 is the cause of some of the mystery surrounding the adiabaticity of \( \mu \). Expanded around \( v_{||} = 0 \), each derivative of \( \Delta \mu_0 \) is zero, and so \( \mu \) is “conserved to all orders.” An approximate form of Equation 5.24 can not be obtained by any order of expansion around a small parameter.

Comparing \( \Delta \mu_0 \) to the values of \( \Delta \mu \) from the full-orbit Boris algorithm reveals that they agree within a factor of 2 for the energy range of interest. For more accuracy we must look more closely at the approximations that went into Equation 5.24. The least justified approximation in Equation 5.24 is that \( \psi(z) \) is governed by the cyclotron
frequency at the midplane, Equation 5.23. Because of this, I add two correction terms to Equation 5.24, $\Delta \mu_{c,1}$ to account for the changing cyclotron frequency with $z$ and $\Delta \mu_{i,1}$ to account for the effect of the inertial terms on $\dot{\psi}$.

\[ \Delta \mu_{i,1} = -\epsilon_i \frac{\cos 2\psi_0}{\sin \psi_0} \Delta \mu_0 \]  \hspace{1cm} (5.27)

\[ \Delta \mu_{c,1} = \frac{\epsilon_B^2}{3\epsilon_0} \left( -\frac{3}{4} + \frac{1}{8\epsilon_0} \right) \Delta \mu_0 \]  \hspace{1cm} (5.28)

where

\[ \epsilon_0 = \frac{v_{||,0}}{\Omega z_0} \]  \hspace{1cm} (5.29)

\[ \epsilon_B = \frac{v_{||,0}}{\Omega z_B} \]  \hspace{1cm} (5.30)

\[ \epsilon_i = \frac{1}{\Omega} \left( \frac{v_{\perp,0}}{r_0} + \frac{v_{||,0}^2}{R_{e,0} v_{\perp,0}} \right) \]  \hspace{1cm} (5.31)

These equations were obtained by Taylor-expanding the argument of the $\sin$ in Equation 5.24. The expression for $\Delta \mu_{c,1}$ came from expanding $\Omega$ around the midplane using $z_B$. The expression for $\Delta \mu_{i,1}$ came from including to first order the terms which contain $\cos \psi$, but not those containing $\sin \psi \cos \psi$, in Equation 5.19.

The results of this exercise are depicted in Figure 5.6, compared against the full-orbit Boris algorithm results and the formula of Hastie, Taylor, and Hobbs.[141] Over a factor of 10 in energy in which $\delta \mu/\mu$ goes from 2% to 270%, the results of Equations 5.24, 5.27, and 5.28 predict to reasonable accuracy the size of the $\Delta \mu$ in the PFRC-II mirror field. At the high-energy end of the range, $\Delta \mu_{i,1}$ is the more important correction.
At the low-energy range, $\Delta \mu_{c,1}$ is more important. Below 1 keV in the PFRC-II, no analytic model agrees with the Boris algorithm $\Delta \mu$ to within a factor of 2.

Those familiar with the model of Hastie, Taylor, and Hobbs may be surprised to see how sinusoidal the model’s predicted $\Delta \mu$ appears in Figure 5.6. Figures of $\Delta \mu(\psi)$ in Hastie et. al. show a sinusoid strongly distorted by an exponential factor $e^{\beta_H \cos \psi}$. By eye, this factor does appear to be in the correct direction to force agreement between the model of Hastie et. al. and the Boris algorithm results. However in the PFRC-II, $\beta_H$ is the product of 3 small factors and one of order unity. A typical value is $\beta_H \approx 0.016$.

In the terminology of Equation 5.24, $\beta_H = \frac{1}{3} \frac{v_\perp}{v_\parallel} \frac{z_H}{R_c} (1 + \frac{z_H^2}{z_R^2})$. It is only systems of larger and more sharply changing curvature that $\beta_H$ distorts the sinusoid dependence.
For a field line starting at $r = 4 \text{ mm}$ and $z = 44 \text{ cm}$, the PFRC-II mirror field from
the currents specified produce: $B_0 = 46 \text{ G}$, radius of curvature at the midplane $R_{c,0} = 2.23 \text{ m}$, curvature rate of change $z_R = 17 \text{ cm}$, and magnetic field rate of change $z_B = 25 \text{ cm}$. Every magnetic mirror will have its own values of these parameters.

I would like to stress that, according to the most accurate models of $\Delta \mu$ both historical and novel, it is the parallel velocity $v_{||}$ which determines the adiabaticity of $\mu$ rather than the perpendicular velocity. Furthermore, the radius of curvature $R_c$ and its characteristic lengthscale of change $z_R$ are essential to the process and cannot be neglected. These findings are not captured by the commonly used adiabatic parameters such as $\epsilon = \rho \nabla B / B$. They can be understood from the physical picture of Delcourt et al., in which $\Delta \mu$ is caused by an impulse-like moment of centrifugal force.$^{[146]} m v_{||}^2 / R_c$ is the magnitude of this force, and the sharpness of the “impulse” is mediated by $z_R$, how fast $R_c$ changes.

In Appendix B, I apply Equation 5.24 to various other magnetic configurations, including a modest thermonuclear magnetic mirror, a magnetic mirror reminiscent of the GOL-3 device in Novosibirsk, a dipole reminiscent of the Proto-RT device at the University of Tokyo, the PFRC-II in RMF-driven FRC mode, and a notional FRC-based compact fusion reactor. In that appendix, I distill the model of Equation 5.24 into one scalar function of space, and plot it inside the above configurations. If you wish to visualize the effect of a region of high curvature or low-field on the constancy of $\mu$, that appendix will attempt to satisfy. Among other things, the results suggest that quasiadiabatic change in $\mu$ can be a leading-order effect in very reasonable magnetic mirrors.
5.3.2.2 Chaotic map and phase-space separatrices

The size of $\Delta \mu$ is only half of the phenomenon. The other half is the long-term behavior of a $\mu$ trajectory, as studied by Chirikov in 1978.[147] If $\mu$ behaves diffusively, an electron with any $\mu$ may eventually escape the mirror trap, and electrons injected passing may become arbitrarily trapped. On the other hand, if $\mu$ behaves quasi-periodically, there will only be a narrow range of $\mu$ that can trap and de-trap non-adiabatically. To determine which is true, we will construct a simplified map between $(\mu, \psi_0)$ for each midplane crossing. Because $\mu$ at the midplane is changing, we will take $\mu$ at the turning point of the mirror bounce trajectory and $\psi_0$ at the midplane.

$$\begin{align*}
(\mu_n, \psi_{0,n}) &\rightarrow (\mu_{n+1}, \psi_{0,n+1}) \quad (5.32)
\end{align*}$$

$\mu$ evolution is determined by Equations 5.24, 5.27, and 5.28. For simplicity, we will consider only 5.24 and 5.28.

$$\mu_{n+1} = \mu_n + \Delta \mu_0(\mu_n, \psi_{0,n}) = \mu_n + \delta \mu(\mu_n) \sin \psi_{0,n} \quad (5.33)$$

where $\delta \mu(\mu)$ is the prefactor of $\sin \psi_0$ in Equations 5.24 and 5.28.

$\psi_0$ evolution is determined by the number of gyroperiods between each midplane passing, $\Delta \psi$, which is a function of $\mu$:

$$\Delta \psi(\mu) = 2 \frac{e}{m} \int_0^{z_t} dz \frac{B(z)}{\sqrt{v^2 - 2B(z)\mu/m}} \quad (5.34)$$
In the map, the gyrophase between two midplane transits is given by:

$$\psi_{0,n+1} = \psi_{0,n} + \Delta \psi(\mu_{n+1}) \approx \psi_{0,n} + \Delta \psi(\mu_0) + \partial_\mu \Delta \psi \cdot (\mu_{n+1} - \mu_0) \quad (5.35)$$

By transforming into coordinates

$$p_n = \partial_\mu \Delta \psi_0(\mu_0) \mu + \Delta \psi_0(\mu_0) - \partial_\mu \Delta \psi_0(\mu_0) \mu_0 \quad (5.36)$$

$$q_n = \psi_{0,n} + \pi \quad (5.37)$$

and assuming that $\delta \mu(\mu)$ changes slowly enough as a function of $\mu$ that the map’s local structure is preserved by taking it constant, the map becomes

$$\begin{align*}
(p_n, q_n) & \rightarrow (p_{n+1}, q_{n+1}) \quad (5.38) \\
p_{n+1} & = p_n + K \sin q_n \quad (5.39) \\
q_{n+1} & = q_n + p_{n+1} \quad (5.40)
\end{align*}$$

where $K = -\partial_\mu \Delta \psi \cdot \delta \mu(\mu_0)$, $K$ a positive number.

The map that Equations 5.38, 5.39, and 5.40 constitute is the Standard Map, sometimes called the Chirikov Map.[147] It is well known that the Standard Map retains a phase-space separatrix until $K \sim 1$. 
FIGURE 5.7: Plots of $p, q$ points produced by applying Equations 5.38, 5.39, and 5.40 to 20 points starting at $p = 10$, $q$ evenly spaced. $K$ values are 0.9, 1.1, and 1.3. Color describes the initial $q$ of the point. 2000 time steps were performed.

The map that Equations 5.38, 5.39, and 5.40 constitute has stable stationary points at $(p, q) = (2\pi N, 0)$, $N \in \mathbb{Z}$ and unstable stationary points at $(p, q) = (2\pi N, \pi)$. These correspond to the $N$th resonance between the cyclotron frequency and the bounce frequency. Around the stable points, there is an island of closed $(p, q)$ trajectories, indicating that there are always particles which exhibit quasiperiodic orbits in $\mu$.

A visualization of the behavior of the map can be seen in Figure 5.7. The $K$ values of 0.9, 1.1, and 1.3 are depicted. At $K = 1.3$, particles in the stochastic region can travel, apparently diffusively, to arbitrary $p$. At $K = 0.9$, particles are limited to traverse at most $\pi$ around $p_0$. The condition $K = 1$ is called the nonlinear resonance overlap condition, or sometimes the Chirikov condition.
Relating this result to the problem of $\mu$ in a magnetic mirror, we can make the following statement: if the maximum possible change in $\mu$ increments the gyrophase upon next midplane intersect by less than 1 radian, then $\mu$ does not diffuse. If instead the maximum possible change in $\mu$ increments the gyrophase upon next midplane intersect by more than 1 radian, then the particle’s $\mu$ is free to diffuse.

If a region in $\mu$ space for which the maximum midplane gyrophase increment is more than 1 radian meets a region in $\mu$ space for which the maximum midplane gyrophase increment is less than 1 radian, then a barrier separates these two regions, and the particle’s $\mu$ cannot diffuse over this region. However, this region must be more than $\pi$ in $p$-space wide, or there may not be a boundary.

Reproducing Equation 5.34 here, the functional form of $\Delta \psi(\mu)$ is:

$$\Delta \psi(\mu) = 2e \int_0^{z_t} dz \frac{B(z)}{\sqrt{v^2 - 2B(z)\mu/m}}$$

where $z_t$ is reached when the denominator is zero. At the marginal passing $\mu$, $\mu_p$, this integral diverges. Because of this, there is always a region around $\mu_p$ for which the maximum midplane gyrophase increment due to $\Delta \mu$ is larger than 1 radian. As $\mu$ increases, there may or may not be a critical $\mu_c$ at which the values becomes less than 1 radian, depending on particle energy and magnetic field configuration.

$\mu_c$ is defined so that

$$- \partial_\mu \Delta \psi \cdot \delta \mu(\mu_c) = 1 \quad (5.41)$$
Particles with $\mu < \mu_c$ correspond to the large $K$ case in Figure 5.7, and can diffuse all the way from passing to $\mu_c$ and back freely. Particles with $\mu > \mu_c$ correspond to the small $K$ case in Figure 5.7 and are confined to quasiperiodic orbits around their initial $\mu$.

Because particles with $\mu < \mu_c$ can become passing and only particles with $\mu > \mu_c$ are well trapped, this larger $\mu_c$ is the effective loss cone discovered by Garren et. al. in 1958, and the boundary between quasiperiodicity and chaos found by Saitoh et. al. in 2016.[137, 151] Garren et. al. numerically studied particles that start trapped. For trapped particles, a larger effective loss cone decreases confinement, as particles which would otherwise stay in the machine may exit.

In the novel contribution of this dissertation, for particles which start passing and are allowed to diffuse over the passing/trapped boundary, such as those in the PFRC-II, a larger effective loss cone increases confinement, as particles which would otherwise transit the machine once and exit may persist for many transits.

To verify the existence of a boundary in $\mu$-space, I performed Boris algorithm calculations of the $\mu$ trajectories of two ensembles of particles. Both ensembles have an energy of 3.6 keV. Both ensembles have gyrocenters that start on the field line that passes through $r = 4$ mm, $z = 44$ cm. The particles have two different initial values of $\mu$: one at 10x the passing $\mu$, and one at 5.5x the passing $\mu$. The results are shown in Figure 5.8. The $\mu_0 = 5.5\mu_p$ particles are able to traverse the entire $\mu$-space, from passing up to a maximum value. The $\mu_0 = 10\mu_p$ particles are contained to a narrow range of $\mu$. The boundary between them appears to occur at approximately $8.8\mu_p$. 
Figure 5.8: Boris-algorithm results of $\mu$ trajectories of two ensembles of particles. One ensemble of 32 particles was initialized with $\mu$ of 10 times the passing $\mu$, and one ensemble was initialized with $\mu$ of 6 times the passing $\mu$. It appears that there is a boundary between them. The lines which stay constant from 10 $\mu$s onward are lost at that time.

Including the 1st order cyclotron correction, Equations 5.24, 5.28, 5.34, and 5.41 predict that $\mu_c = 10.7\mu_p$ for a 3.6 keV particle in the specified mirror field that starts at the specified position. This is compared to the $8.8\mu_p$ produced by the Boris algorithm calculation. I attribute this error to the approximate form of Equations 5.24 and 5.28. Nonetheless, it appears that the Boris algorithm results support the finding that there is a value of $\mu$ that separates a diffusive region from a quasiperiodic region.

For the case depicted in Figure 5.8, $\delta\mu(\mu_p) \approx 0.18\mu_p$ and $\delta\mu(\mu_c) \approx 0.37\mu_p$, indicating that about a hundred transits are required on average to diffuse from the initial $\mu$ to the boundary $\mu_c$, or about 500 to diffuse from the initial $\mu$ to $\mu_p$. Figure 5.8 shows
Approximately 400 transits.

Applying Equation 5.41 to the PFRC geometry, the new, larger effective loss cone can be computed. I computed $\mu_c$ versus energy of the electron and versus the radius of the particles’ field lines at the nozzle ($z_0 = 44$ cm), from 0 mm to 9 mm, the inner radius of the nozzle. I also averaged the energy profiles over $dr^2$ to produce an area-averaged result. The result is depicted in Figure 5.9. Recall that the region of validity of our expression of $\delta\mu$ is approximately $E > 1.5\text{keV}$.

Visualizing the loss cone in $(\mu, E)$ space or $(v_\perp, v_\parallel)$ space, Figure 5.9 reveals that the loss cone is actually a super-linear function of the parallel coordinate. Visualized, it

![Figure 5.9: Plots of the ratio between $\mu_c$, defined in Equation 5.41, and $\mu_p$, the passing $\mu$ value, versus energy and initial radius ($z_0 = 44$ cm). This defines the effective loss cone of the PFRC-II with the specified currents.](image)
is rather a loss trumpet, increasing in radius super-linearly but sub-quadratically with $E$ or $v_{||}$.

### 5.3.2.3 Expected Density Increase of Passing Particles

Quasiadiabatic dynamics in a mirror machine have most often been studied from the perspective of trapped particles becoming un-trapped and escaping. The present case in the PFRC-II is the inverse of this process, in which passing particles become quasiadiabatically trapped and persist for many transits of the mirror. This perspective is novel to this dissertation.

The density increase in particles due to this behavior can be estimated. Consider a distribution function in $\mu$ such that $\int d\mu f(\mu) = n_e$. We expect the behavior of $f(\mu)$ to be approximately diffusive as seen from Figures 5.7 and 5.8. We expect this behavior to hold between $\mu_p$ and $\mu_c$.

If particles are injected at a constant rate and can only be lost to the mirror nozzles, we may suppose that the value of $f(\mu) = f_0$ is constant between $\mu = 0$ and $\mu = \mu_p$, regardless of the effect of non-adiabaticity of $\mu$. This is the same as saying that, if there is no other loss mechanism for the quasiadiabatically trapped particles, their density comes to equilibrium with that of the passing particles.

If the diffusivity is constant between $\mu_p$ and $\mu_c$ and zero above $\mu_c$,

$$D_\mu = \frac{\langle \delta \mu \rangle^2}{\langle \Delta t \rangle} \times \begin{cases} 1 & \mu_p < \mu < \mu_c \\ 0 & \mu > \mu_c \end{cases}$$

(5.42)
then the long-time behavior of this system is to approach \( f(\mu) = f_0 \) between \( \mu = 0 \) and \( \mu = \mu_c \). Integrating the distribution function,

\[
ne = n_0 \frac{\mu_c}{\mu_p}
\]  

(5.43)

where \( n_0 \) is the density of injected passing particles assuming only one transit. This is to say that the density in the mirror cell is enhanced over the perfectly adiabatic density by the factor \( \mu_c/\mu_p \). This quantity is depicted in Figure 5.9, and increases for increasing energy.

Equation 5.42 is not the Diffusion constant assumed by Chirikov. Rather, he uses an empirically determined coefficient which comes from numerical analysis of the Standard map:

\[
D_\mu = \frac{\langle \delta \mu \rangle^2}{\langle \Delta t \rangle^2} (K - 1)^2
\]  

(5.44)

If no particles are passing \( \mu_c \) via means other than quasiadiabaticity, then the specific shape of \( D_\mu \) does not matter and Equation 5.43 is recovered.

### 5.3.3 Expected Density Increase Applied to the PFRC-II

Over the region of applicability, \( \mu_c/\mu_p > 5 \) (see Figure 5.9). Recall from Equation 5.43 that this ratio is also the expected enhancement factor of the density of passing particles.
Compare this result to the experimentally measured density. Recall from Equation 5.8 that we expect the ratio of $B$ in the SEC and CC to reduce the CC density by a factor of 3.5. Recall from Figure 5.2 that the measured density in the CC can approach and even exceed that in the SEC; the measured enhancement factor is approximately 3 - 5.

For the case of 3.6 keV electrons depicted in Figure 5.8, $\delta\mu/\mu_c$ is about 25, so the expected lifetime of the particles is a few hundred transits. This agrees well with the approximately 300 transit lifetimes measured in Figure 5.3.

Both the density predicted by quasiadiabatic analysis and the average lifetime agree, at least to the accuracy to which the theory was formulated.

5.4 Summary and conclusion

Dynamics of very adiabatic and very non-adiabatic particles each have their own tools for the analysis of long time behavior. The marginal case, in which the particle is mostly adiabatic but at times exhibits sharp jumps in the adiabatically conserved parameter, is called quasiadiabatic. Quasiadiabatic systems are also amenable to theoretical analysis and a wealth of surprising and potentially important phenomena is illuminated thereby.

One such system is a high-mirror-ratio mirror which is fueled through the mirror coils. The PFRC-II in seed plasma mode is such a mirror. In this chapter, I described a quasiadiabatic phenomenon observed in the PFRC-II: fast electrons produced in an end cell of the mirror become trapped and persist in the center cell for hundreds of transits. I explain quantitatively both the lifetime of these electrons in the mirror and their density.
In this chapter, I also explain from an elementary starting point the origin of the jumps in the magnetic moment, $\mu$, that occur at the midplane. Essential to the process is the speed $v_{||}$ with which a particle traverses a sharp change in the curvature of the magnetic field lines $R_c$. Some visualizations of the parameters which lead to large jumps in $\mu$ are provided in Appendix B.

These jumps in $\mu$, if they are small enough, act to keep $\mu$ to a narrow range around its starting value. However, when a condition first derived by Chirikov is met and the jump in $\mu$ can change the gyrophase upon subsequent midplane intersects by 1 radian, $\mu$ is free to diffuse.

There always exists a region in which this condition, $K > 1$, is true at $\mu$ values very close to the marginal passing $\mu_p = E/B_{max}$. This is the effective loss cone, as particles within this region are adiabatically trapped but may become passing. The size of this region depends on the particle’s energy and the profiles of $|B|, R_c$ along the field line that the particle follows. In the PFRC-II, a typical value of the size of this region is $\mu_c/\mu_p \approx 5-10$.

For the fusion-relevant case of particles trapped in a mirror becoming un-trapped, others have shown that very reasonable values of the relevant parameters cause quasiadiabatic losses to be a leading-order effect.

For the inverse case of passing particles becoming trapped, quasiadiabatic motion causes particles to build up in the quasiadiabatic effective loss cone, $\mu_p < \mu < \mu_c$. The density of these particles is larger than the density of passing particles by approximately the factor $\mu_c/\mu_p$. 
Chapter 6

Acceleration of high energy electrons:

Fermi acceleration through spontaneous electrostatic oscillations at mirror nozzles

In this chapter, I describe a mechanism by which fast electrons trapped in the center mirror cell (CC) of the PFRC-II in seed plasma mode are accelerated still further, from an effective temperature of a few hundred eV to a few keV, with individual electrons reaching energies above 30 keV. I describe the process as a kind of Fermi acceleration, after the mechanism suggested by Enrico Fermi in 1949 for the origin of high-energy cosmic rays.[160] As in that mechanism, electrons periodically interact with a fluctuation at the turning points of their trajectories, which causes them to statistically gain energy. In the CC of the PFRC-II, that fluctuation is an electrostatic oscillation, caused
by an instability driven by a spontaneous beam of electrons entering from the far end cell (FEC) of the PFRC-II.

Typical parameters of this accelerated population are: Maxwellian fit density of $3 \times 10^7$/cm$^3$, Maxwellian fit temperature 2 keV. All energy into the system comes from same $\sim$ 300 Watt double-saddle antenna as was described in Chapter 3; while the PFRC-II apparatus is fitted with a 200 kW Rotating Magnetic Field (RMF) current drive system, it was not turned on for the experiments described in this thesis.

This chapter is relevant because of two novel areas of exploration:

First, the form that Fermi acceleration takes in a quasadiabatic magnetic mirror with a weak electrostatic oscillation can be quite different from the form that Fermi acceleration takes between clouds of magnetized interstellar media. It ultimately produces an electron energy distribution function (EEDF) which is exponential with energy, rather than a power law. Exponential EEDFs are known to be created by turbulent electrostatic wave spectra, but even if the forcing is periodic rather than turbulent, non-adiabaticity of magnetic moment can act to break the resonances which are typically thought to produce absolute barriers to energy gain.

Second, the magnetic configuration that gives rise to Fermi acceleration in the PFRC-II is common in the world of electric spacecraft propulsion. All that is required is a population of superthermal electrons, a magnetic mirror containing a plasma, an open end which is terminated on an electrically floating electrode (or empty space), and the possibility of electron-impact ionization (gas) in the cell containing the floating electrode. These criteria are typically fulfilled by systems which employ a magnetic nozzle.
Chapter 6.1

6.1 Background

6.1.1 Fermi Acceleration

In 1949, Enrico Fermi proposed a mechanism for the acceleration of cosmic rays, now called Fermi Acceleration.[160] Having just hosted Hannes Alfvén at the University of Chicago, Fermi knew that large, magnetized clouds of denser-than-average interstellar media were common in the galaxy, streaming haphazardly at characteristic velocities of 30 km/s. He knew that there were two mechanisms by which protons which had already attained an energy higher than 200 MeV could magnetically “bounce” off of these clouds: Firstly they could undergo a mirror-bounce due to their magnetic moment, and secondly they could be following a magnetic field which undergoes a hairpin turn within one of the clouds. In the very relativistic case, this interaction statistically increases the energy of the particle.

Assuming a relativistic particle and a constant probability per unit time of loss, the EEDF \( f(E) \) of protons accelerated by this process is a power law with \( f(E) \propto E^{-p} \), \( p = 1 + t_a/t_l \), \( t_a \) the acceleration timescale and \( t_l \) the loss timescale. Indeed, the EEDF of high-energy cosmic rays is observed to be a power law.

Stanislaw Ulam analyzed a specific form of the momentum increment that each bounce gives.[161] He imagined that the motion of a proton bouncing from a magnetized oscillation was the same as a ball bouncing from a moving wall, and for specificity gave the wall sinusoidal velocity in time (“periodic forcing”). The so-called Fermi-Ulam map proved to be rich in complicated phenomena.[162–165] Locally, it reduces
to the famously chaotic Standard Map, sometimes called the Chirikov Map. For another incarnation of the Standard Map, see the analysis in Chapter 5.

As with the $\mu$-jumping case described in Chapter 5, it developed that the behavior of particles following such a map is split into ranges of velocity: a low-energy range in which the dynamics of Fermi’s stochastic analysis are recovered, an intermediate-energy range in which particles are free to traverse the entire range but there are islands in energy-phase space which are denied, and a forbidden high-energy range which low-energy particles cannot enter. This analysis shows that, for periodic forcing, Fermi acceleration cannot cause particles to be accelerated to arbitrary energy. Rather, there is a maximum energy that can be gained via this mechanism.

Periodic forcing is of great interest to this chapter, but is not of particular relevance to the origin of cosmic rays. In Fermi’s own original paper, it was suggested that particles were not bouncing between two sinusoidally varying clouds, but between many independent clouds. Later, the thinking turned to turbulent Alfvén wave spectra and Fermi’s result was recovered through Quasi-linear analysis.[166] The current thinking is that protons are accelerated by a relatively small number of interactions with independent shocks, though the matter is not settled.[167, 168]

### 6.1.2 Breaking the periodic forcing of electrostatic waves

This section will discuss the behavior of particles in uniform electrostatic oscillations. There is not a deep connection between particle trapping in energy-space of periodic forcing and particle trapping in space of electrostatic oscillations, but the breaking of both phenomena occurs in turbulence.
Perfectly periodic electrostatic waves lead to trapping of particles in real space. This can be seen quite simply: in the frame moving at the phase velocity of the wave, the oscillation is a static potential, and particles may not have enough energy to travel from the trough to the peak. Spatial trapping also limits the amount of energy that particles can gain or lose from the wave. Spatial trapping of particles in an electrostatic wave is essential to such phenomena as Landau damping and the hot-plasma limit of the two-stream instability, sometimes called inverse Landau damping.\[169, 170\] Simple alterations such as adding another wave or randomizing the phase at a fixed time interval are sufficient to allow diffusion in space and energy.\[171, 172\]

The behavior of a particle in a continuous wave spectrum is modeled by Quasilinear Diffusion. This is the limit of many individual waves, and a stepping stone to turbulence.

### 6.1.3 Electrostatic wave heating in a magnetic mirror

From the perspective of Fermi acceleration, the Fermi-Ulam map appears superficially to describe particles in a magnetic mirror trap which periodically dip into a region of electrostatic oscillation. In 1963, Alexeff et. al. discovered that an electrostatic oscillation produced by two-stream instability was heating electrons far beyond the nominal barrier, to effective temperatures of 130 keV.\[173–175\]

The Alexeff apparatus was a six-inch-long, R=3 mirror with $B_{\text{min}}=1500$ Gauss. They injected 5 keV electrons with a current of 0.5 Amperes, and to their surprise measured X-rays with energies up to 250 keV.\[173\] They did not immediately understand how periodic forcing could lead to electrons of such high energies.\[174\] The apparent solution was a turbulent wave spectrum, rather than periodic.\[175\]
Turbulence is the well-known mechanism by which electrostatic oscillations in a mirror may heat particles to high energy.\cite{171, 172, 176}. From a theoretical point of view, it is easy to see why; rather than the periodic forcing of the Fermi-Ulam map which allows particles’ bounce phase and forcing phase to correlate, the forcing is random and the particle can gain or lose energy regardless of its phase. From an experimental point of view, many machines other than the Alexeff apparatus have seen turbulent heating in magnetic mirrors.\cite{177–180} The largest existent today is the GOL-3 Multimirror machine in Novosibirsk, which regularly makes measurements of plasmas heated entirely by the turbulent wave spectrum of two-stream instability induced by a relativistic electron beam.\cite{181, 182}

Section 6.3.5 describes two more mechanisms by which a perfectly periodic electrostatic oscillation may heat particles: close approaches to the magnetic maximum effectively randomizing RF phase, and changes to the magnetic moment effectively randomizing bounce phase. The latter is more important to the PFRC-II.

## 6.2 Experimental results from central mirror chamber at low pressure

### 6.2.1 Apparatus

This chapter describes a phenomenon which affects the population of trapped electrons in the Center Cell (CC) of the PFRC-II device. For information on how these electrons
FIGURE 6.1: Schematic of the PFRC-II experiment run in seed plasma mode. A) Schematic representation of the locations of vessels, coils, and diagnostics. B) The $|\vec{B}|$ profile along the axis of the PFRC-II. C) Detail of the seed plasma formation region, the antenna, the Pyrex pipe, and the stainless steel backplate.
were produced, see Chapter 3. For information on how these electrons became trapped and the inconstancy of their magnetic moment, see Chapter 5.

The PFRC-II in seed plasma mode is a tandem mirror, with a center cell and two end cells. This is depicted in Figure 6.1. The Center Cell is a magnetic mirror cell with a mirror ratio of $10^{-40}$. The vessel of the CC is made of transparent polycarbonate Lexan. Its inner diameter (ID) is 22.7 cm. The vessel wall is 16 mm in thickness, with 87 penetrations for diagnostics to be inserted. The vessel is 84 cm long. Use of Lexan for a vacuum vessel is atypical; for a discussion, see Berlinger et. al.[156] Gas is introduced only in the SEC. It enters the CC through the SEC-CC nozzle. The CC is pumped by a 100 l/s Varian TV 141 NAV turbomolecular pump.

The magnetic field in the midplane, the minimum-B point, is determined by four sets of water-cooled pancake coils, called the L-2 coils. In Figure 6.1, these are called the Axial Field Coils. The maximum of the magnetic field, at the nozzles, is determined by both the L-2 coil current and two sets of small-bore, water-cooled Nozzle coils or N coils.

At the magnetic minimum at the midpoint, the magnetic field has strength $0.74G/A \times I_{L-2}$, where the units of that numerical constant are Gauss per Ampere, and $I_{L-2}$ is the current carried by the L-2 coils. At the magnetic maximum at the nozzle, the magnetic field has strength $6.6G/A \times I_{L-2} + 5.9G/A \times I_N$, where $I_N$ is the current carried by the N coils. The L-2 coils carry current between 40 A and 300 A. The N coils carry current between 0 A and 400 A.

The x-ray data presented in this chapter were recorded in the CC. For details on the SDD x-ray detectors, including placement, calibration, and analysis of the data into
full EEDFs and derived quantities, see Chapter 2. In this chapter, two mounts were
used: the CC Midpoint mount and the CC Radial Scan mount, whose parameters are
described in the table in Section 2.3.

The probe data presented in this chapter were recorded in the CC and the Far End
Cell (FEC) of the PFRC-II. The FEC is depicted at the far right of Figure 6.1(a). The
FEC is a 13” stainless steel port cross, attached via welding bellows to the CC. The FEC
is pumped by a 510 l/s Agilent TV 551 NAV turbomolecular pump.

Magnetically, the FEC is the mirror-image of the Source End Cell (SEC) described
in Chapter 3 and depicted in Figures 3.14 and 3.15. The L-2 coils form a weak, \( \sim 2 : 1 \)
magnetic mirror in the FEC. On the upstream side of the FEC, the Nozzle coils raise the
mirror ratio on that side only to \( \sim 10 : 1 \). This demarcates the boundary between the
CC and the FEC.

Electrically, the FEC is very different from the SEC. Rather than a double-saddle
antenna and backplate, there is a radially-scannable, biasable Tantalum paddle of 6
cm diameter and 8 mm thickness. This serves to terminate the plasma column. The
Tantalum paddle serves as an order-of-magnitude check on the energies and densities of
electrons assumed to be measured by the x-ray detectors. When the paddle is allowed
to float, it may achieve very negative floating potentials, surpassing -1 kV.

Also in the center of the FEC is a radially-scannable Tungsten Langmuir probe
with diameter 0.010” and an Alumina shield. This probe is able to determine the space
potential, density, and effective temperature of the plasma at the center of the FEC. This
acts to corroborate our model of the origin of the electrostatic oscillation in the CC.
The accelerated CC electrons were discovered in October of 2016, when my advisor S. A. Cohen suggested that we run the PFRC-II in seed plasma mode with LN2 cooling several copper rings coated in Boron Nitride inside the CC vacuum vessel. These served to reduce the base pressure in the CC by cryopumping water vapor, a product of Lexan outgassing, and ammonia synthesized by the plasma from atmospheric nitrogen. To our surprise, we observed a new, distinct population of electrons in the CC with electron temperature several times that of the SEC-born secondary electrons.

Since then, we have studied this population extensively and learned its parametric dependence on CC pressure, RF power, the magnetic mirror ratio, the radius, the time after the RF power was switched on and off, and the FEC pressure. This led us to suspect an electrostatic or magnetic oscillation near one of the nozzles as the accelerating mechanism, either at the RF frequency or a spontaneous frequency. We looked for and found a spontaneous electrostatic oscillation at the FEC-CC nozzle, and ruled out a magnetic oscillation at the same. By measuring the parametric dependence of this oscillation on the same parameters, we find that our measured quantities are all in agreement with the model presented in this chapter.

I supervised Tony Qian in the spring of 2018 as an intern in the SULI program. He made some of the electrostatic measurements in the FEC that are presented in this chapter.

### 6.2.2 Characteristic parameters of the accelerated CC electrons

A characteristic spectrum recorded from the CC during acceleration is shown in Figure 6.2, along with fits to two Maxwellian distributions. This spectrum was recorded on
Figure 6.2 shows that, in the CC, there is a population with temperature somewhat warmer than that of the SEC-born secondary electrons, with $T_e = 570$ eV, at Maxwellian fit density $n_e = 2.2 \times 10^8$/cm$^3$, compared to $T_e = 394$ eV, $n_e = 1.3 \times 10^9$/cm$^3$ in the SEC. In the CC, there is another population, less dense but hotter by
Figure 6.3: Detail of a rich section of x-ray spectrum, with Argon K-\(\alpha\) and K-\(\beta\) lines and Chromium and Iron K-\(\alpha\) lines definitively identified, and Cobalt and Nickel K-\(\alpha\) lines tentatively identified.

far, whose Maxwellian fit parameters are \(T_e = 2485\text{eV}, n_e = 5.1 \times 10^7/\text{cm}^3\). The Maxwellian is not a perfect fit, however; a typical EEDF is shown in the next section. The EEDF in the FEC was not measured.

The PFRC-II can reliably be made to produce such x-ray spectra by reducing the neutral gas pressure in the CC to below 0.2 mTorr of hydrogen gas. LN2 is not necessary.

The base pressure on 2017/11/15 was particularly poor, which had the fortuitous effect of producing a rich spectrum of atomic lines from impurities in the plasma. Figure 6.3 shows a detail of the x-ray spectrum, with Argon K-\(\alpha\) and K-\(\beta\) lines and Chromium and Iron K-\(\alpha\) lines definitively identified, and Cobalt and Nickel K-\(\alpha\) lines tentatively
Chapter 6.2

identified. Such features serve as a sanity check on the measured effective temperature, as to produce K-\(\alpha\) light, an iron atom must be impacted by a 7112 eV electron.[31] The Argon is from an air leak into the vessel. The metals are presumably from stainless steel which has sputtered into the plasma, either from the SEC plasma-terminating cup or the nozzle coil structure.

The highest energy x-ray seen on 2017/11/15 was above 30 keV. Therefore there must have been at least some electrons with energy higher than 30 keV. On an operational note, 30 keV x-rays can penetrate many inches of Lexan with ease, and so the PFRC-II lab needed to be surveyed and personal film dosimeters were required. No additional shielding was required, as the calculated and measured flux of x-rays is very low.

6.2.3 Typical EEDF

The Maxwellian fit of Figure 6.2 is not perfect. Because of this, I have reproduced in Figure 6.4 the full EEDF by performing the procedure described in Chapter 2. This x-ray spectrum was recorded on 2018/01/16. The CC pressure was 0.250 mTorr of hydrogen gas. The FEC pressure was \(7.7 \times 10^{-6}\) Torr. The forward RF power was 425 W at 27 MHz. The L-2 current was 320 A and the Nozzle current was 380 A, giving mirror ratio 18 and minimum magnetic field 235 Gauss. LN2 cooled plasma-facing Boron Nitride was used to pump condensable gases, presumably water and ammonia.

Figure 6.4 shows that the x-ray spectrum is consistent with a Maxwellian EEDF at energies above 4 keV. Below this, the lower-energy population of SEC-born electrons
Figure 6.4: Example EEDF obtained by inverting an x-ray spectrum. The Maxwellian EEDF (magenta line) has parameters $n_e = 3.2 \times 10^7$ cm$^{-3}$, $T_e = 2550$ eV. Obscures the EEDF. A power-law, $f(E) \propto E^p$ is marginally consistent for $p = -2.3$. I discuss what shape of EEDF is produced by what process in Section 6.3.3.2.

6.2.4 Parameter scans

6.2.4.1 CC pressure

When the gas pressure in the CC is increased, the x-ray derived effective temperature of the accelerated CC population decreases. This is depicted in Figure 6.5. These data were recorded on 2018/04/09, with conditions $5.0 \times 10^{-6}$ Torr in the FEC, 340 W peak RF power at 14 MHz modulated with a 1 kHz square wave, $I_{L-2} = 270$ A, $I_N = 370$ A for mirror ratio 20, minimum B 200 Gauss.
Figure 6.5: The dependence on effective temperature (“T 3keV+”) of the accelerated population on the CC neutral gas pressure. This temperature was obtained by fitting the EEDF above 3 keV to a Maxwellian. To explain these results, another derived quantity (“$V_{float}*\sqrt{\text{tau}/0.5\text{ns}}$”) is plotted. It is the expected result as derived by quantities measured in Section 6.2.8 and modeled in Section 6.3. The Y-axis is “Energy” because the derived quantity is not a temperature, but has energy units.

To evaluate the agreement of these data and the model described in Section 6.3, a derived quantity is presented based on two other measurements: The first is the decay time of the x-ray signal after the RF has been shut off. The second is the direct measurement via Langmuir probe of an electrostatic oscillation (∼200 MHz) on the CC side of the CC-FEC nozzle. These measurements are described in Sections 6.2.6.1 and 6.2.8, and modeled in Section 6.3. The essence of the findings is that the shorter confinement time at higher pressure causes electrons to spend less time accelerating, and so accumulate less energy before being lost.
6.2.4.2 RF power

When the RF power is increased, the effective temperature of the accelerated CC population increases. This is depicted in Figure 6.6. These data were recorded on 2018/04/09, with conditions 0.125 mTorr in the CC, $5.0 \times 10^{-6}$ Torr in the FEC, RF at 14 MHz, $I_{L-2} = 270$ A, $I_N = 370$ A for mirror ratio 20, minimum B 200 Gauss.

To evaluate the agreement of these data and the model in Section 6.3, another quantity is presented: It is the scaled amplitude of an electrostatic oscillation measured directly via Langmuir probe on the CC side of the CC-FEC nozzle. These points were actually measured at a different time than the x-ray measurements; inserting the probe
decreased the x-ray signal. This can be seen by the 350 W point, whose x-ray derived quantities are plotted once for no probe, and once for the probe inserted to a radius of 13 mm. This measurement is described in Section 6.2.8 and modeled in Section 6.3. The of that model is that the stronger electrostatic oscillation at higher power is able to accelerate electrons more strongly.

6.2.4.3 Nozzle current

When the mirror ratio of the CC is changed, the Maxwellian fit density of the CC accelerated population increases but its effective temperature changes little. This is depicted in Figure 6.7. These data were recorded on 2018/01/09, with conditions 0.240 mTorr H\textsubscript{2} gas in CC, $7.2 \times 10^{-6}$ Torr in FEC, 410 Watts forward RF at 27 MHz, $I_{L-2} =$
FIGURE 6.8: Abel-inverted radial profiles of two EEDF-derived quantities of the electrons in the CC at two values of the L-2 current. Density above 600 eV, left, and average energy per electron above 600 eV, right. For a description of the two conditions plotted, refer to the text. The results can be explained from the perspective of the behavior of the non-adiabaticity of the magnetic moment, described in Chapter 5.

The changing nozzle current produced a mirror ratio from 10 to 19. LN2 cooled plasma-facing Boron Nitride was used to pump impurities.

6.2.5 Radial profile

In January of 2018, we added the CC Radial Scanning mount for the SDD x-ray detectors to the PFRC. This mount employs a welding bellows to allow a long collimator to sweep its line-of-sight along the plasma radius, allowing radial profiles of x-ray measurements to be obtained. The resolution is 1.2 cm (±0.6 cm around the measurement
point), scannable from 0.6 cm to 8.1 cm radius.

An Abel inversion of an entire EEDF would be overcome by the error of counting statistics, as both spectral inversion and Abel inversion amplify this error. Because of this, I have plotted only the Abel inversion of two derived quantities from the EEDF in Figure 6.8; the first is the electron density (integral of the EEDF) above 600 eV, and the second is the average electron energy (first moment of the EEDF divided by integral of the EEDF) above 600 eV.

The uncertainty in quantities is so large because of the Abel inversion. The Abel inversion requires a numerical derivative to be taken, which amplifies the uncertainty of the measured quantities.

Plotted are data from two different L-2 axial coil current conditions, corresponding to minimum B fields of 240 Gauss and 100 Gauss. The 240 Gauss case was measured on 2018/01/24. The 100 Gauss case was measured on 2018/01/29. Conditions were otherwise similar, with $I_N = 380$ A, 0.250 mTorr in the CC of H$_2$ gas, RF net power 425 Watts at 27 MHz.

The maximum gyroradius of 1 keV electrons in 100 Gauss is 7.5 mm. The maximum gyroradius of 1 keV electrons in 240 Gauss is 3.1 mm. For the 100 Gauss case, the nozzle-bore-limiting field line is at 3.3 cm at the axial position of the x-ray detector. For the 240 Gauss case, the nozzle-bore-limiting field line is at 2.5 cm at the axial position of the x-ray detector. The nozzle-bore-limiting field line is the outermost field line to which plasma can flow from the SEC.
The difference in position of the nozzle-bore-limiting field line explains the profile of densities above 600 eV in Figure 6.8. Both cases exhibit sharp decreases in density radially outward of their limiting field line. Furthermore, the density appears slightly peaked off-axis, though the resolution and uncertainty make this hard to assert definitively. This makes sense in the context of the $\mu$-jumps described in Chapter 5; the more radially outward the electron is, the larger the non-adiabaticity of its magnetic moment.

The average kinetic energy above 600 eV also has features which must be explained. The average kinetic energy increases much more strongly with radius in the higher-B case. From the perspective of Chapter 5, this also makes sense. The size of a $\mu$-jump is a very strongly varying function of energy, magnetic field, and magnetic curvature; the higher magnetic field pushes the threshold in energy for a $\mu$-jump of a certain size much higher in energy and curvature. In the strong-field case, only those particles with very high energy are able to change $\mu$ at all and be trapped, and only at large radius where the curvature of the field line is stronger. At the lower field, particles of many more energies and radii may participate in the nonadiabatic process.

### 6.2.6 Ramp-up and decay of EEDF

#### 6.2.6.1 Decay time of each energy

By modulating the RF power with a 1 kHz square wave, 500 $\mu$s of RF power followed by 500 $\mu$s of zero RF power, we may measure the behavior of the accelerated population of electrons in time. A very basic measurement is depicted in Figure 6.9. These are four screenshots from oscilloscopes set to producing a histogram in time of the voltage signal.
Figure 6.9: X-ray count vs modulation phase histograms for four different bands of x-ray energies. The oscilloscope was triggered on the RF modulation signal. Each time-axis division is 100 µs, showing that very high energies of electron persist for many hundreds of µs after the cessation of the RF power. This corresponds to tens of thousands of transits.

from the x-ray detector. The oscilloscopes are triggered on the modulation signal. The x-ray detector is commanded for each sub-figure to produce a voltage signal if it detects an x-ray within four bands of energy: 500 eV - 1.2 keV, 1.2 keV - 4 keV, 4 keV - 8 keV, and 8 keV and higher.

The data in 6.9 were recorded on 2017/12/12, under the following conditions: 0.220 mTorr of H$_2$ gas in the CC, 320 W net RF during the RF pulse, RF modulated by a 1 kHz square wave, 300 A into the L-2 coils, 380 A into the Nozzle coils, for a mirror ratio of 19 and a minimum B of 220 Gauss, and LN2-cooled Boron Nitride facing the plasma.

Figure 6.9 illustrates that there is very different time-behavior for each energy band
Figure 6.10: e-folding decay times derived from the full EEDFs taken after several delay times from the cessation of RF power. Clearly, higher-energy electrons persist in the CC for longer than lower-energy electrons, as long as hundreds of µs. Above 4 keV, the uncertainty is large as not enough x-rays were counted for an accurate decay time to be computed.

of accelerated electrons, that higher-energy electrons persist for longer after the cessation of RF power than lower-energy electrons. High-energy electrons may therefore persist for tens of thousands of transits of the CC.

With the help of the full EEDF at different times, we may quantify this behavior. It is plotted in Figure 6.10. Rather than measuring the times at which energies within an energy band are recorded as in Figure 6.9, the x-ray detector was commanded to accumulate counts only within an interval of 100 µs, with some delay after the cessation of the RF power. By inverting the spectra from several of these delay times into EEDFs and assuming that electrons are only lost, that they do not change energy over this interval, the logarithm of the ratio of the value of the EEDF at each time can tell us the
characteristic e-fold decay time of the EEDF at each energy.

Figure 6.10 reports the result of this analysis. The uncertainty is quite large, as counting statistics are compounded by both the spectral inversion and the decay time analysis, but the increasing value is clear. Electrons with higher energy persist for longer in the CC before they are lost, staying in the trap for hundreds of $\mu$s. We expect the loss from the trap to be a combination of non-adiabatic dynamics and pitch angle scattering; these results qualitatively agree. The model is explained in Section 6.3.

Figure 6.10 does not have the $E^2$ dependence expected from Coulomb collisions into the loss cone because the loss processes of this population are complex. As seen in Chapter 5, they are not strictly adiabatically confined, and their diffusivity in $\mu$ and the range of $\mu$ over which they may diffuse are both functions of energy. Pitch angle scattering does play a role, however, as can be seen in Figure 6.5, as the decay time decreases for increasing CC gas pressure.

The data reported in Figure 6.10 were recorded on 2018/01/16, with conditions: 0.270 mTorr of H$_2$ in the CC, 300 net RF during the RF pulse, RF modulated by a 500 Hz square wave, 340 A into the L-2 coils, 370 A into the Nozzle coils, producing a mirror ratio 18 and a minimum B 250 Gauss. The data took 2.5 hours to accumulate to the quality depicted.

6.2.6.2 Rise time

The same analysis was performed on delay times following the *initiation* of RF power rather than the *cessation* of RF power during square wave modulation. This measures the rise of the x-ray signal rather than the decay of the x-ray signal. Two quantities
were computed from the EEDFs that resulted, which were not of high enough quality to prepare an e-folding time at each energy. The quantities are density above 600 eV and average kinetic energy of electrons above 600 eV. They are depicted in Figure 6.11. From that figure, it can be seen that the density of the accelerated population evolves over the course of hundreds of µs, as does the average energy thereof. This can be explained as the slow accumulation of energy over many hundreds of transits of the mirror machine.

The data in Figure 6.11 were recorded on 2018/02/01. The conditions were: 0.250
mTorr of H$_2$ gas in the CC, 320 W net RF power during the RF pulse, RF power modulated by a 500 Hz square wave, 320 A into the L-2 coils, 380 A into the nozzle coils, for a mirror ratio of 18 and a minimum B of 235 Gauss. The Boron Nitride plasma-facing components were cooled by LN2.

### 6.2.7 Magnetic oscillation probe at nozzle

One theory we evaluated was that a magnetic oscillation at the nozzles is pumping the electrons to high energy. To directly measure such an oscillation, I designed and technician Bruce Berlinger constructed a magnetic oscillation probe for insertion into the PFRC-II. It consisted of three magnetic pickup coils, one for each of $r, \phi, z$ directions.

The magnetic pickup coils are made of 16 turns of 10 mil copper magnet wire wrapped around a 5 mm Boron Nitride spool. A sheet of 2 mil stainless steel shimstock covers the pickup coils for electrostatic shielding, leaving a 1 mm electrical break to prevent the shielding from reflecting magnetic oscillation signals. Into a 50Ω cable or oscilloscope, such a probe should have a sensitivity above 3.5 MHz of 680 mV/G, and below 3.5 MHz have a sensitivity linear with frequency down to 1 mV/G at 5.7 kHz.

In actuality, the twisted pair leading from the probe to the 50 Ω cable acts as a transmission line with measured characteristic impedance $Z_c = 180\Omega$. The circuit was simulated in LTSpice and was calculated to abide by the above sensitivities up to 100 MHz.

The magnetic sensitivity was measured by placing two of the probe heads 1 cm from each other and transmitting a signal of a variety of amplitudes and frequencies
from one to the other. The sensitivity was found to abide by the above calculation, in
agreement with the LTSpice simulation.

The electrostatic shielding is grounded through a 10 $\mu$F capacitor, which serves the
double-duty of allowing the shield do be electrically floating and to pass electrostatic
signals above 5 kHz to ground and therefore shield against them. The electrostatic
shielding was tested by touching the bare output of a function generator to it at a variety
of frequencies and amplitudes and measuring the probe output.

The magnetic probe is placed on a dog-leg of Alumina and able to rotate via Wilson
seal into a position which is 1/8” from the nozzle.

The magnetic probe did not detect a signal above the noise floor. This determines
definitively that there is no magnetic oscillation that could produce the accelerated pop-
ulation of electrons in the CC within a factor of 10,000.

6.2.8 Electrostatic oscillation probe at nozzle

To measure the level of electrostatic activity at the nozzle between the CC and the
FEC I added a Langmuir probe to the CC whose tip is radially insertable. The probe’s
axial position is 13 cm from the CC-FEC nozzle. The probe is Tungsten wire, 0.010”
in diameter. The probe is inserted to a radius of 13 mm when data is collected. At
this radius, the measured x-ray density decreases, so x-ray spectra and electrostatic
measurements are not made concurrently.

It is hard to accurately measure an oscillating space potential at dozens or hundreds
of MHz. One established technique is to measure a signal which is injected at a known
amplitude with an identical probe, and determine the effective “coupling constant,”
(square root of the measured gain) as was performed by Malmberg *et. al.* in 1967.[183]
Even that procedure was not able to determine the coupling constant to better than a
factor of 2. Typical coupling constants at hundreds of MHz were -30 dB, though the
plasma was very different from the PFRC-II.[184] I did not perform this calibration on
the PFRC-II, and so all measurements are assumed to have some constant factor reduc-
tion. The factor 7-10 causes agreement between the model and the experiment. This is
not an atypical correction factor for high-frequency probe measurements.[183]

In this section I report several values of the peak-to-peak voltage from this probe.
The frequency spectra of the Langmuir probe trace for each run condition are all qual-
itatively the same, with a very broad peak at 200 MHz, 100 MHz in width, with high
harmonics (≈ 200 MHz) of the RF frequency superimposed. The exact harmonic of
the RF frequency which has the largest amplitude and sets the measured peak-to-peak
voltage changes on a 10 µs timescale. This effect is not captured in a time-averaged
Fast Fourier Transform.

This is shown in Figure 6.12. The plotted quantity is the autocorrelation function at
several delay times averaged over each microsecond. The quantity plotted is a function
of delay time \( t_d \) and signal time \( t_s \). It is:

\[
\int_{t_s}^{t_s + \Delta t} dt [v(t) \times v(t + t_d)]
\]  

(6.1)

A white region at some delay time and signal time indicates that, at the signal time,
there is a periodic signal with period equal to the delay time. This trace, from 0 µs to 100
Figure 6.12: Auto-correlation of the Langmuir probe voltage for each delay time at each microsecond. A white horizontal band indicates the presence of a periodic signal with that period. The persistent band at 53 $\mu$s is the 19 MHz RF signal. Higher harmonics (5-10) are often strongest, as from 40 $\mu$s to 55 $\mu$s, but this visualization resolves them only poorly.

$\mu$s, shows that different harmonics of the seed RF frequency were strongest at different times. This visualization is only sensitive to about the fourth harmonic, however. The persistent white region at a delay time of 53 ns is the 19 MHz fundamental.

Regions of visible fundamental, second harmonic, third harmonic, and fourth harmonic are clearly visible in Figure 6.12. Higher harmonics (5-10) are present from 40 $\mu$s to 55 $\mu$s but this visualization resolves them only poorly. They are clearly visible instantaneously on oscilloscope time traces.

Harmonics of 19 MHz in the vicinity of 100 - 200 MHz are the strongest signal, which usually set the peak-to-peak voltage. I attribute this to the following phenomenon: Two-stream instability causes mode growth in the plasma. While the mode of fastest growth is generally not a multiple of 19 MHz, the RF antenna in the SEC
produces a signal at multiples of 19 MHz which grows marginally more slowly but starting at a larger amplitude than a random thermal oscillation at the frequency of fastest growth. This harmonic frequency saturates more quickly than the mode of fastest growth, as it has a head start. This process is depicted schematically in Figure 6.13. As is shown in Figure 6.12, it is not always the same harmonic of the fundamental which is the strongest signal.

Figure 6.12 was recorded on 2018/03/27. The FEC pressure was \(1.5 \times 10^{-5}\) Torr. There was 0.133 mTorr of \(\text{H}_2\) gas in the CC, 350 W net RF at 19 MHz, 270 A into the L-2 coils, and 380 A into the Nozzle coils, producing a mirror ratio of 20 with a
minimum B of 200 Gauss.

I have already shown two plots including the peak-to-peak voltage from this probe. One is Figure 6.6, which shows that the voltage oscillation amplitude increases when RF power is increased. One is Figure 6.5, which shows a derived quantity that includes the voltage oscillation as a function of gas pressure in the CC.

### 6.2.8.1 With FEC pressure

The origin of the oscillation is a beam of electrons produced in the FEC flowing into the CC. This will be discussed more completely in Section 6.3. Evidence for this claim
is given in Figure 6.14. It shows that, as the FEC neutral gas pressure is increased, the amplitude of the voltage oscillation increases. Furthermore, it shows that the temperature of the CC accelerated population increases linearly with the amplitude of the oscillation.

The x-ray measurements and probe measurements in Figure 6.14 were not recorded at the same time. When the probe is inserted into the plasma, the x-ray derived density decreases. One such point at 5 $\mu$Torr is shown in that figure.

The temperature behavior will be explained in Section 6.3 as the effect of the stronger oscillation accelerating electrons more effectively. The probe peak-to-peak measurement can be understood as the effect of more ionization in the FEC. As more electrons stream back into the CC at uniform energy, the strength of the two-stream instability increases and causes a larger voltage oscillation.

Figure 6.14 was recorded on 2018/03/27. The conditions were: 0.133 mTorr of H$_2$ gas in the CC, 350 W net RF at 19 MHz, 270 A into the L-2 coils, and 380 A into the Nozzle coils, producing a mirror ratio of 20 with a minimum B of 200 Gauss.

### 6.2.9 Conditions in FEC

Clearly, the conditions in the FEC are vital to producing the electrostatic oscillation. This is why I supervised SULI intern Tony Qian in the Spring of 2018 to use Langmuir probes in the FEC to characterize these conditions.
Figure 6.15: Plot of Langmuir characteristic derived space potential in the FEC. At low pressure, a sheath forms and downstream of it the potential is very negative. At high pressure, ionization in the FEC allows this sheath to be less extreme. This plot was prepared by Tony Qian.

Tony produced the data reported in Figures 6.16 and 6.15. Two quantities are presented: The plasma potential in the FEC determined from the Langmuir probe characteristic and the electron density in the FEC determined from the Langmuir probe characteristic.

The data at the low-pressure end can be understood in the following way: Because the Tantalum paddle in the FEC is electrically floating, it can collect no net current. This is the same condition which produces the Bohm sheath in a Langmuir probe. It causes a large voltage drop (the floating potential) proportional to the temperature of the SEC-born fast electrons.

In a plasma with two populations of electrons, the ratio of the product \( n_e \sqrt{T_e} \) of each population determines which temperature the floating potential more resembles,
as this flux must balance the ion flux across the sheath. If the fast electron flux is able to completely balance the bulk ion flux, the floating potential seeks the fast electron temperature. If instead the fast electron flux is insufficient to balance the bulk ion flux, the floating potential seeks the bulk electron temperature. In reality there is a smooth transition between the two cases.

Given the ratio of bulk to fast $n_e, T_e$ given in Chapter 3, the space potential in the SEC seeks the bulk electron temperature. However, as reported in Figure 6.15, the space potential in the FEC is very negative. Therefore the fast electron flux from the SEC-born electrons out of the magnetic mirror into the FEC is sufficient to balance the flux from ions out of the magnetic mirror into the FEC.

**Figure 6.16:** Plot of Langmuir characteristic derived electron density in the FEC. At low pressure, a sheath forms and only fast electrons may penetrate to the FEC. At high pressure, ionization in the FEC increases electron density. This plot was prepared by Tony Qian.
This can be seen from adapting Hutchinson’s equation for the Langmuir characteristic, Hutchinson Equation 3.2.33, to include two populations of electrons, setting the current to zero, and solving for the downstream potential $V$: \[19\]

\[
2 \sqrt{\frac{\pi m_e}{2m_i} e^{-1/2}} = \frac{n_b}{n_b + n_h} e^{eV/T_b} + \sqrt{\frac{T_h}{T_b}} \frac{n_h}{n_b + n_h} e^{eV/T_h}
\]  \hfill (6.2)

where $n_b, T_b$ are the density and temperature of the bulk, $n_h, T_h$ are the density and temperature of the hot population. I assumed that $n_b >> n_h$ and $T_h >> T_b$, but I assumed no relative ordering of their ratios.

The space potential in the CC was not measured on the same day that the data in the FEC depicted in Figure 6.15 were measured, but on other days at the same conditions it is measured to be within 20 V of zero. This demonstrates a ~600 V voltage drop across the CC-FEC nozzle.

When the pressure in the FEC is allowed to rise, the fast electrons in the FEC are able to ionize neutral gas into ions and cold, ~5 eV electrons. This accounts for the increase in density seen in Figure 6.16. These colder electrons are accelerated by the potential drop across the nozzle back into the CC, as they do not have enough energy to travel downstream toward the paddle. This relaxes the current-free condition that produces the sheath in exactly the same way that an Emissive probe does; fast electron flux downstream across the nozzle may be balanced by emitted electron flux back upstream, rather than by ion flux downstream. This explains the less extreme space potentials seen in Figure 6.15.
The combination of these two effects explains the measured electrostatic oscillation on the CC side of the CC-FEC nozzle, Figure 6.14. The cold electrons, accelerated by the potential drop, form a beam with narrow energy when they re-enter the CC and cause two-stream instability, whose saturation amplitude depends both on the density and energy of the beam particles. This is discussed more in Section 6.3.

The measurements reported in Figures 6.16 and 6.15 occurred on 2018/02/14, at conditions: 0.451 mTorr of H₂ gas in the CC, 325 W net RF at 19 MHz, 240 A into the L-2 coils, 380 A into the Nozzle coils, creating a mirror ratio of 21.5 with minimum B 177 Gauss. At this higher CC pressure, the accelerated population of electrons was not observed. I do not expect the conditions in the FEC to have different behavior at lower CC pressure.

### 6.3 Model of acceleration

The model that will be described in this section is that the SEC-born electrons establish a voltage drop between the CC and the FEC, ionization-produced electrons flow upstream back across this voltage drop, and the resulting beam produces an electrostatic oscillation through two-stream instability. Electrons trapped in the CC via non-adiabaticity of magnetic moment are accelerated by this oscillation beyond the maximum energy expected from the Fermi-Ulam map by a combination of turbulence, close approach to the magnetic maximum, and further non-adiabaticity of magnetic moment. The specific form of the acceleration results in a mostly exponential EEDF.
6.3.1 Origin of the oscillation

Two-stream instability in which there are resonant particles of both populations with the wave is called “inverse Landau damping.”[169] In which there are no resonant particles of one population, it is called “beam-plasma instability.” These are two limiting behaviors of a continuous solution.[185] The FEC ionization-born electrons in the CC are in the inverse Landau damping limit.

The fully kinetic equation which determines the stability of an electron velocity distribution function (EVDF) is the Nyquist theorem. It is described in detail in Chapter 7 of Thomas Stix’s book, Waves in Plasmas.[170] In the inverse Landau damping limit, the instability condition reduces to:

\[ f'(v) > 0 \]  

where \( f(v) \) is the EVDF and \( f'(v) \) is the derivative of the EVDF with respect to velocity. We will see visually that this is satisfied, but for completeness the Nyquist theorem was also evaluated for the EVDFs in this chapter and they were found to be unstable.

The amplitude at which two-stream instability saturates is calculable in the full kinetic model.[186] That reference finds that the kinetic model “roughly agrees” with the inverse Landau damping limit of the saturation condition. The inverse Landau damping saturation condition is that the voltage oscillation should be large enough that, if the EVDF is flattened over the range of velocities which may be trapped by the oscillation, \( f'(v) \) is nowhere positive. In this section I will use the latter, simpler criterion.
Figure 6.17: Example EVDFs before and after the onset of two-stream instability. SEC-born electrons in the CC start with \( n_e = 5 \times 10^8 / \text{cm}^3, T_e = 350 \text{eV} \) and FEC-born electrons in the CC start with \( n_e = 3.5 \times 10^7 / \text{cm}^3, T_e = 5 \text{eV} \), with a 300 eV drift energy. The bulk EVDF is not shown. A 50 V \( \text{pkpk} \) oscillation flattens the EVDFs over the velocity interval which is trapped. This oscillation is sufficient to make \( f'(v) \leq 0 \) everywhere, the condition for two-stream instability saturation in the inverse Landau damping limit. The bulk electron EVDF is not shown.

Plotted in Figure 6.17 is an example of the EVDF before and after the onset, growth, and saturation of the instability. Depicted are the EVDFs from the SEC-born secondary electrons in the CC and the FEC-born ionization-produced electrons in the CC. In this example, the SEC-born population in the CC start with \( n_e = 5 \times 10^8 / \text{cm}^3, T_e = 350 \text{eV} \) and the FEC-born population in the CC start with \( n_e = 3.5 \times 10^7 / \text{cm}^3, T_e = 5 \text{eV} \) with a 300 eV drift energy. The populations’ densities have not changed after the onset of the instability but a 50 \( \text{V}_{\text{pkpk}} \) electrostatic oscillation has flattened both EVDFs in a \( 4.2 \times 10^8 \text{cm/s} \) velocity range around the drift speed of the FEC-born population. The result is that the EVDF has \( f'(v) \leq 0 \) everywhere, the condition for two-stream
instability saturation in the inverse Landau damping limit. The bulk population is not shown. EVDFs depicted are one-dimensional.

By taking the EVDF of the SEC-born electrons as linear around the velocity of the entering FEC-born electrons, we may derive an approximate equation for the amplitude of the oscillation:

$$\epsilon V_{pkpk} \approx T_{SEC} \frac{1}{2} n_{SEC} \sqrt{\frac{\pi T_{SEC}}{E_{FEC}}} e^{\frac{E_{FEC}}{T_{SEC}}}$$

(6.4)

where \( V_{pkpk} \) is the peak-to-peak amplitude of the oscillation at saturation, \( T_{SEC} \) is the temperature of the SEC-born secondary electrons trapped in the CC, \( n_{SEC} \) is the density of the SEC-born electrons trapped in the CC, \( E_{FEC} \) is the drift energy of the FEC-born ionization-produced energy in the CC (approximately the voltage drop across the CC-FEC nozzle), and \( n_{FEC} \) is the density of the FEC-born electrons in the CC. The oscillation amplitude is an increasing function of \( n_{FEC} \) and \( E_{FEC} \), though the \( n_{FEC} \) is the stronger dependence at \( E_{FEC} \sim T_{SEC} \).

Equation 6.4 qualitatively agrees with the measured oscillation strength as a function of pressure in the FEC reported in Figure 6.14. In that figure, the oscillation strength appears roughly linear with the FEC pressure.

In the inverse Landau damping limit of two-stream instability, the fastest-growing mode occurs at the plasma frequency of the participating populations. For the SEC-born electrons in the CC, this is about 200 MHz. For an otherwise quiescent plasma, we would expect thermal oscillations at this frequency to be amplified and saturate. However, if there is an electrostatic oscillation at slightly different frequency that starts
with much larger amplitude, it will grow more slowly but can saturate before the mode with fastest growth, limiting the growth of this mode. This is what is observed in the PFRC-II: the largest signal is a high harmonic of the RF frequency around 200 MHz with approximately 50 V peak-to-peak amplitude.

The electron cyclotron frequency in the vicinity of the electrostatic probe is also around 200 MHz. We would expect that the longitudinal electrostatic mode would be the most unstable, but do not rule out a mode partially mediated by the magnetic field.[176]

### 6.3.2 Energy balance

Here, I tally approximate figures for the power lost from each population of fast electrons, for consistency.

The population of particles born in the SEC via the process described in Chapter 3 lose energy at the following rate:

\[
P_{SEC\text{-born}} = A \times v \times E_{\text{lost}} \times n_{SEC}/N
\]  

(6.5)

where \( P_{SEC\text{-born}} \) is the power lost by the SEC-born electrons, \( A \) is the area over which the SEC-born electrons are created, \( v \) is the characteristic velocity with which electrons move through an axially-oriented surface, \( E_{\text{lost}} \) is the energy that each particle loses over the course of its lifetime \( E_{\text{lost}} \approx T_e \), \( n_{SEC} \) is the measured total density of the SEC-born electrons which is the single-transit density multiplied by the average number of transits, and \( N \) is the average number of transits.
Chapter 6.3

Taking \( A = 11 \text{cm}^2 \) from the inner diameter of the pyrex pipe in the SEC, \( v = 1 \times 10^9 \text{cm/s} \) is the velocity of a 300 eV electron, \( E_{\text{lost}} = 300 \text{ eV} \) based on temperature measurements, \( n_{\text{SEC}} = 3.5 \times 10^8 / \text{cm}^3 \) is the measured fast electron density in the SEC, \( N = 6 \) is the approximate number of transits found in the model in Chapter 3, we find that \( P_{\text{SEC-born}} \sim 30 \text{ Watts} \).

The population of particles born in the FEC via the process described in this section, ionization by SEC-born electron impact, lose energy at the following rate:

\[
P_{\text{FEC-born}} = A \times v \times E_{\text{lost}} \times n_{\text{FEC}}
\]

where \( P_{\text{FEC-born}} \) is the power lost by the FEC-born electrons, \( A \) is the area of a surface on the CC side of the CC-FEC nozzle at which we know approximate values of the rest of the parameters, \( v \) is the velocity with which electrons move through the surface, \( E_{\text{lost}} \) is the energy that each particle loses over the course of its lifetime, \( n_{\text{FEC}} \) is the supposed density of the FEC-born particles at the surface under consideration.

I choose the surface 11 cm upstream of the CC-FEC nozzle, as the electrostatic probe there gives potential fluctuation measurements which may be used to determine \( n_{\text{FEC}} \) from Equation 6.4. Taking \( A = 5.3 \text{cm}^2 \) from Biot-Savart flux mapping, \( v = 1 \times 10^9 \text{cm/s} \) for a 300 eV electron, \( E_{\text{lost}} = 300 \text{ eV} \) from a plausible number for the potential drop from CC to FEC, and \( n_{\text{FEC}} = 3.5 \times 10^7 / \text{cm}^3 \) from Equation 6.4 and measurements of the voltage fluctuation, \( P_{\text{FEC-born}} \sim 10 \text{ Watts} \).

A general argument for why \( P_{\text{SEC-born}} > P_{\text{FEC-born}} \) is that we know that electron current from CC to FEC is larger than electron current from FEC to CC because a large
voltage drop exists there, and SEC-born electrons which lose energy in the FEC lose uniformly more than the CC-FEC potential drop, but electrons born in the FEC gain almost exactly the CC-FEC potential drop in energy. These two arguments together show that $P_{SEC-born}$ is always more than $P_{FEC-born}$. This is an important sanity check, as by this model, the FEC-born electrons get their energy from the SEC-born electrons.

The population of SEC-born particles which have been accelerated to much higher energies than the SEC-born temperature lose energy at the following rate:

$$P_{fermi} = V \times n_{fermi} \times E_{lost}/\tau$$  \hspace{1cm} (6.7)

where $P_{fermi}$ is the power lost by the Fermi-accelerated electrons, $V$ is the volume that the Fermi-accelerated electrons occupy, $n_{fermi}$ is the measured density of Fermi-accelerated electrons in this volume, $E_{lost}$ is the average energy lost by the particle over its lifetime, and $\tau$ is the loss time of the particle.

Taking $V = 2,300\, \text{cm}^3$ from the volume of double-cone with the approximate dimensions of the last closed flux surface of the CC, $n_{fermi} = 3 \times 10^7/\text{cm}^3$ from the x-ray measurements in this chapter, $E_{lost} = 3$ keV from the temperature measurements in this chapter, $\tau = 100\, \mu\text{s}$ from the x-ray measurements in Section 6.2.6.1, we find that $P_{fermi} \sim 3$ Watts.

That $P_{fermi} < P_{FEC-born}$ is expected because the FEC-born electrons are the source of the energy to drive Fermi acceleration.
6.3.3 Non-phase-correlated energy diffusion model

This section will consider the simple dynamics of the electron population under random phase of the electrostatic oscillation at every interaction. This is not what actually happens; in actuality the oscillation is mostly periodic, so the phase of the oscillation at each interaction is correlated. However, random phase is a good first approximation. Phase-correlated bounces will be considered later in this chapter.

6.3.3.1 Differences between mirror-bounce and rigid-wall bounce

The action of a weak electrostatic oscillation on a mirror-trapped electron is very different from the rigid walls of the Fermi-Ulam map. For a particle bouncing off of a moving wall, the particle’s velocity is incremented by twice the wall’s velocity. For a mirror-trapped electron undergoing a weak electrostatic oscillation while following its mirror bounce trajectory, the particle’s energy is incremented by the change in space potential over the time interval that the particle was within its spatial extent.

To see this, consider a specific form of a changing electrostatic potential, with no magnetic mirror field, which is reminiscent of a rigid wall:

\[
U(x, t) = E_0 e^{-x/x_c + v_w t/x_c} \tag{6.8}
\]

where \(U\) is the potential field felt by the particle, \(E_0\) is the strength of the potential chosen to be the initial energy of the particle so that \(x = 0\) is the reflection point, \(x\) is an axial coordinate, \(x_c\) is a characteristic spatial dimension of the field, and \(v_w\) is a
characteristic velocity of the potential. This potential field and the particle’s reflection is represented schematically in Figure 6.18.

Assume the particle’s initial velocity is much higher than $v_w$. In this toy, electrostatic-sheath-bouncing model, the particle is reflected at the point at which $U = E_0$. This point starts at $x = 0$ and moves along $x$ at velocity $v_w$. Because of this behavior, the potential acts as a moving “wall” off of which the particle “bounces.”

The energy gain of a particle bouncing from this potential is calculable. Taking the time derivative of the kinetic energy of the particle
\[ D_t E_k = \vec{v} \cdot \vec{F} = -\vec{v} \cdot \nabla U(x_p(t), t) = -D_t U(x_p(t)) + \partial_t U(x_p(t), t) \]  

(6.9)

where \( D_t = \partial_t + \vec{v} \cdot \vec{\nabla} \) is the total time derivative of the particle, \( E_k \) is the kinetic energy of the particle, \( \vec{F} \) is the force on the particle, \( U(x, t) \) is the potential field for the particle, and \( x_p(t) \) is the position of the particle at time \( t \). The first term on the right-hand side integrates to zero, as the particle eventually escapes into the bulk plasma at constant potential.

The second term on the right-hand side can be integrated to find the kinetic energy gain integral:

\[ \Delta E = \int dt \times \partial_t U(x_p(t), t) \]  

(6.10)

Applying Equation 6.10 to Equation 6.9, we recover the case of a particle bouncing off of a rigid wall:

\[ \Delta E = 4\sqrt{E_0} \sqrt{\frac{1}{2} m_e v_w^2} \]  

(6.11)

However, the result is very different if the particle is undergoing a mirror bounce in a static magnetic potential \( (U_m = \mu B) \) with a superimposed small electrostatic potential. For specificity, imagine the combined magnetic and electrostatic potential:
\[ U(x, t) = \mu B_{\text{max}} e^{-x/x_c} + eV_1 e^{-x/x_c + v_w t/x_c} \]  

(6.12)

where \( B_{\text{max}} \) is the magnetic field at the particle’s turning point, and \( V_1 \) is the amplitude of the electrostatic oscillation. Applying Equation 6.10 to Equation 6.12:

\[ \Delta E = 4 \sqrt{\frac{1}{2} m_e v_w^2 \frac{eV_1}{\sqrt{\mu B_{\text{max}}}}} \]  

(6.13)

By the definition of \( B_{\text{max}}, \mu B_{\text{max}} = E_0 \). Therefore, rather than expecting an energy increment which increases in energy as with the rigid wall case, we expect that the particles’ energy increment decreases with energy. At higher energy, the particle’s velocity is larger, and it spends less time near the localized electrostatic fluctuation.

### 6.3.3.2 EEDF expected from this process

The energy diffusion equation is:

\[ \partial_t f(E) = \partial_E D_E \partial_E f(E) \]  

(6.14)

where \( D_E \) is the diffusivity in energy, \( D_E = \langle \Delta E^2 \rangle / t_t \), where \( t_t \) is the mirror transit time, and \( \langle Q \rangle \) is the average of quantity \( Q \). The average is taken over the phase of the oscillation when the particle is incident; \( \Delta E \) of a particle incident on the nozzle when the oscillation is increasing is different from \( \Delta E \) of a particle incident on the nozzle when the oscillation is flat. If there is some loss time independent of energy \( \tau_l \) which
characterizes a constant probability in time of being lost from the system, the equation becomes:

\[ \partial_t f(E) = \partial_E D_E \partial_E f(E) - \frac{f(E)}{\tau_l} \]  \hspace{1cm} (6.15)

Far from sources of particle, and at steady state, the Green’s function for the EEDF is

\[ \partial_E D_E \partial_E f(E) = \frac{f(E)}{\tau_l} \]  \hspace{1cm} (6.16)

Here I say the Green’s function because the solution to Equation 6.16 is the response of the system to a source of particles at some energy. There are two solutions: a decreasing function of energy on the \( E > E_i \) side of the energy at which particles are injected \( E_i \) and an increasing function of energy on the \( E < E_i \) side. From the perspective of the CC electron acceleration process, the “injected” particles are the SEC-born secondary electrons trapped in the CC.

As a first approximation, taking \( D_E \) and \( \tau_l \) constant in energy, the resultant EEDF is:

\[ f(E) = A e^{\pm \frac{E}{T_{\text{eff}}}} \]  \hspace{1cm} (6.17)

\[ T_{\text{eff}} = \Delta E \sqrt{\frac{\tau_l}{t_t}} \]  \hspace{1cm} (6.18)
where above the injection energy, the (-) sign is taken. \( \Delta E = \sqrt{\langle \Delta E^2 \rangle} \). At \( E > T_{\text{eff}} \), this looks like a Maxwellian with temperature \( T_{\text{eff}} \).

As a general approximation, plugging in plausible and measured values for \( \tilde{\Delta}E \approx 50\text{eV} \) from the measured probe voltage fluctuation, \( \tau_l \approx 100\mu\text{s} \) from the measured decay time, \( t_t \approx 50\text{ns} \) from the transit time, the effective temperature of this distribution is \( T_{\text{eff}} = 2.2 \text{ keV} \), which is close to what has been measured.

In actuality, each of these parameters is a function of \( E \). Particle loss time \( \tau_l \) was measured to be approximately proportional to \( E \) or \( \sqrt{E} \) in Figure 6.10. Mirror transit time \( t_t \) is a function of both \( E \) and \( \mu \) but is generally decreasing with \( t_t \propto E^{-1/2} \).

In the last section we found that \( \Delta E \) depends on the time that the particle spends near the region of oscillation, so \( \Delta E \propto E^{-1/2} \). However, this analysis was performed in the context of a moving \( V \) rather than a sinusoidal \( V \).

If the potential frequency is low, the electron will not be in the interaction region for long enough for the potential to change appreciably. In this regime, the maximum energy change is

\[
\Delta E \approx V \omega \tau_l \approx V \omega \frac{r}{v_e} \quad (6.19)
\]

where \( V \) is the amplitude of the oscillation, \( r \) is the spatial extent of the oscillation, \( \omega \) is the angular frequency of the oscillation, and \( v_e \) is the velocity of the electron. So \( \Delta E \propto E^{-1/2} \) as before, but with more physically relevant parameters determining the constant. In evaluating the model in this section, we will assume that \( \Delta E \approx V \) for
$E \approx 2\text{keV}$ and $\Delta E \propto E^{-1/2}$ above this. This is plausible given the value of $\omega$ and considering that $r$ is a few centimeters.

Because it is not clear in Figure 6.10 whether $\tau_l$ is proportional to $E$, $E^{1/2}$, or somewhere in between, we may only say that $T_{\text{eff}} = \Delta E \sqrt{\tau_l/t_t}$ does not have a particularly strong dependence on $E$. If $\tau_l \propto \sqrt{E}$, there is no dependence at all, as all dependencies cancel.

The plausibility of Equation 6.18 can be seen in Figures 6.5 and 6.6, and especially 6.14. These are the three plots which show quantities derived from the measured Langmuir probe oscillations from the CC-FEC nozzle.

In Figures 6.6 and 6.14, the effective temperature of the accelerated population can be seen to vary linearly with the measured oscillation amplitude. An absolutely calibrated determination of the oscillation amplitude is not available, but assuming a probe coupling factor of 7 - 10, there is agreement between $T_{\text{eff}}$ as predicted in this model and Figures 6.6 and 6.14.

For Figure 6.5, three quantities were measured: the effective temperature of the CC accelerated population measured via x-ray spectrum, the decay time of the CC accelerated population measured via the x-ray signal decay time, and the voltage oscillation measured via the CC-FEC nozzle Langmuir probe. The quantities $T_{\text{eff}}$ and $V_{\text{pkpk}} \sqrt{\tau_l/0.5\text{ns}}$ are plotted and agree. As a characteristic transit time $t_t$ is 30 ns rather than 0.5 ns, this implies agreement between Equation 6.18 and the data under the condition that the coupling coefficient of the Langmuir probe at 200 MHz is $\sqrt{30/0.5} \approx 8$, the same coupling coefficient that caused agreement with Figures 6.6 and 6.14.
There is agreement of each dependency of Equation 6.18:

\[ \Delta E \] agreement is demonstrated with Figures 6.6 and 6.14, showing that the temperature of the accelerated population is proportional to the voltage amplitude

\[ \tau_l \] agreement is demonstrated with Figure 6.5, showing that the voltage amplitude and the particle confinement time act in concert to produce the accelerated population temperature in the way that is predicted in Equation 6.18.

\( t_t \) is analytically calculable, as it depends only on the trajectory of a particle in a mirror field. Nonetheless, agreement is demonstrated by the rough constancy of \( T_{eff} \) in Figure 6.4.

Because of this, I propose the diffusion of electron energy due to mirror-bounce motion passing through a region of two-stream instability-produced electrostatic oscillation as the mechanism for acceleration of particles in the CC of the PFRC-II in seed plasma mode.

### 6.3.4 Phase-correlated resonances: Fermi-Ulam map

In formulating the diffusion equation, we used the approximation that subsequent interactions with the electrostatic oscillation are not correlated in phase. A more accurate model is a discrete, 2D map relating the energy and electrostatic oscillation phase at each interaction with the CC-FEC nozzle:

\[
E_{n+1} = E_n + \Delta E(E_n) \sin \alpha_n
\]  

(6.20)
\[ \alpha_{n+1} = \alpha_n + \Delta \alpha(E_{n+1}) \]  

(6.21)

where \( E \) is the particle’s energy and \( \alpha \) is the RF phase of the electrostatic oscillation.

This map looks like several others that we’ve considered: It looks like Equation 5.33 and 5.35 from Chapter 5, which governed the evolution of the magnetic moment \( \mu \) in a quasiadiabatic magnetic mirror, studied by Chirikov.[147] This map also looks like the Fermi-Ulam map.[161] Both of these maps reduce locally under correct coordinate transformation to the Standard Map, sometimes called the Chirikov Map.[164]

As we saw in Chapter 5, the correct parameter to describe the quasiperiodic or chaotic nature of a map such as this is Chirikov’s \( K \) parameter. If \( K > 1 \), the map is chaotic and the random-phase approximation is somewhat justified. If \( K < 1 \), the map has boundaries in \( E \) over which particles cannot diffuse. Physically, \( K \) describes the extent to which resonances between the transit time \( t \) and the electrostatic oscillation period are able to trap particles.

The coordinate transformation to put the map given by Equations 6.20 and 6.21 into the form of the Standard Map is:

\[ b = \partial_E(\Delta \alpha) \times E + \Delta \alpha_0(E_0) - \partial_E(\Delta \alpha_0) \times E_0 \]  

(6.22)

\[ q = \alpha \]  

(6.23)

where \( \Delta \alpha(E) \approx \Delta \alpha(E_0) + \partial_E(\Delta \alpha) \times (E - E_0) \).

This produces the map:
\begin{align*}
b_{n+1} &= b_n + K \sin \alpha_n \quad (6.24) \\
\alpha_{n+1} &= \alpha_n + b_{n+1} \quad (6.25)
\end{align*}

which is the Standard Map. Formulated for the map given by Equations 6.20 and 6.21, Chirikov’s $K$ parameter is

\[ K = \partial_E (\Delta \alpha) \times \Delta E \quad (6.26) \]

From the definition of $\alpha$,

\[ \Delta \alpha = \omega t_t \quad (6.27) \]

where $\omega$ is the angular frequency of the electrostatic oscillation and $t_t$ is the transit time of the electron. $\omega$ was measured to be 200 MHz, and $t_t$ is determinable via particle motion in a mirror field.

For the conditions typical of the PFRC-II, $K \approx 0.1$. Clearly, electrons should not be able to gain energy beyond a narrow band around their initial energy. There must be some mechanism by which the resonance between bounce phase and electrostatic oscillation phase is broken.
6.3.5 Resonance-destroying effects

6.3.5.1 Turbulence

Turbulence is the mechanism which is most well known for breaking the resonances between $t_t$ and a wave spectrum. Background information was given in Sections 6.1.2 and 6.1.3. However, the wave spectrum recorded from the PFRC-II is not turbulent, as was reported in Section 6.2.8. We must examine other possible identities of the mechanism which breaks the resonance between bounce phase and voltage oscillation phase.

6.3.5.2 Close approaches to nozzles

Recall from Chapter 5 that there is a narrow range around the marginal passing $\mu_p = B_{max}/E$ for which the behavior of $\mu$ is always chaotic. Recall that this is because the transit time of a particle diverges as $\mu$ approaches the passing $\mu_p$, so the derivative of the gyrophase at midplane crossing $\partial_\mu \Delta \psi_0$ is very large around this $\mu$ value.

The same is true from the perspective of $E$. As $\mu$ is kept constant and $E$ is increased to approach the marginal passing $E_p = \mu B_{max}$, the transit time diverges. Thus there is a narrow range of energy around $E_p$ for which $K > 1$.

For the PFRC-II, this range is not very large. In fact, it is smaller in width than $\Delta E$. It is not large enough to explain the diffusivity of $E$ that is measured.
6.3.5.3 Non-adiabaticity of magnetic moment

Jumps in $\mu$ have been seen to aid in energy gain due to Fermi acceleration in simulations of astrophysical plasmas.[187]

Jumps in $\mu$ can affect the transit time of a particle. This was explored in Chapter 5. Because the voltage oscillation phase $\alpha$ is sensitively dependent on the transit time of the electron, $\mu$-jumps can radically change the map that Equations 6.20 and 6.21 constitute. Furthermore, while the change in $\alpha$ on each transit depends on the $\alpha$ of the previous transit, oscillation phase $\alpha$ and gyrophase at the midplane $\psi_0$ are entirely decoupled. From the perspective of $(E, \alpha)$, the quantities $(\mu, \psi_0)$ are random.

The map that Equations 6.24 and 6.25 constitute are adjusted in the following way to include $\mu$ jumps:

\begin{align*}
b_{n+1} &= b_n + K \sin \alpha_n \quad (6.28) \\
\alpha_{n+1} &= \alpha_n + b_{n+1} + R r_{n+1} \quad (6.29)
\end{align*}

where $r_n = \Delta \mu_n/\delta \mu$ is treated as a random number between -1 and 1.

The quantity $R = \partial_{\mu} (\Delta \alpha) \times \delta \mu$ is the governing parameter. It describes how $\mu$ jumps affect subsequent values of the electrostatic oscillation phase at the electron’s CC-FEC nozzle turning point. In the PFRC-II, for typical values of the relevant parameters, $R \approx 0.1$. 
Parametrized in this way, it is not immediately obvious whether $R = 0.1$ is sufficient to break the quasiperiodic resonance behavior characterized by $K = 0.1$ in the Standard Map. I therefore define $p = b/K$ to make the effect of $R$ clearer:

$$p_{n+1} = p_n + \sin \alpha_n$$  
(6.30)

$$\alpha_{n+1} = \alpha_n + Kp_{n+1} + Rr_{n+1}$$  
(6.31)

Now, it is clear that when $R \sim K$, the effect that $p$ has on the evolution of $q$ (the second term on the right-hand side of Equation 6.31) is overcome by randomness. Plots of these maps are depicted in Figure 6.19. Each plot is for $K = 0.1$. Plots of $R = 0, 0.1, 0.2$ are presented. 1600 time steps were performed. 400 points were initialized with $p = 15$, evenly spaced in $q$. The color of the point corresponds to its initial $q$ value.

For $R = 0$, the Chirikov condition $K < 1$ is well met and the particle does not diffuse. For $R = 0.1$ and $R = 0.2$, the particle diffuses. For $R = 0.2$, the particle diffuses approximately the amount expected by the random-phase approximation.

Thus, the condition for a random term added to the phase of the Standard Map to have a large effect on the behavior of the map is $R \gtrsim K$.

Physically, this means that in the PFRC-II, the magnetic moment is sufficiently mobile to allow resonances between the mirror transit time and the electrostatic oscillation to be broken. This allows the particle’s energy to diffuse without limit.
Figure 6.19: Plots of the map defined by Equations 6.30 and 6.31. Each plot is for $K = 0.1$. From left to right, these plots have $R = 0, 0.1, 0.2$. 1600 time steps are depicted. 400 points were initialized with $p = 15$, evenly spaced in $q$. The color of the point corresponds to its initial $q$ value.

6.4 Conclusion

In this chapter, I have reported on measurements which indicate that a population of particles in the CC of the PFRC-II in seed plasma mode is accelerated to effective temperatures of 2 - 3 keV. I have reported on measurements which indicate that the cause of this acceleration is an electrostatic oscillation on the CC side of the nozzle between the CC and FEC. The electrostatic oscillation is caused by a beam of electron-impact-born electrons created by fast electrons downstream of a large voltage drop between the CC and FEC.
One might initially assume that a mostly-periodic electrostatic oscillation is incapable of accelerating electrons past energy boundaries which correspond to resonances between the particle’s bounce motion and the electrostatic oscillation. I have demonstrated in this chapter that there are mechanisms by which these resonances can be broken. The one primarily responsible for acceleration of electrons in the PFRC-II appears to be the change in transit times due to the non-conservation of the magnetic moment.
Chapter 7

Conclusion

In this conclusion, I very briefly summarize the findings of this dissertation. I then discuss a few categories of future work which arise from this project. I discuss the applicability of the acceleration model to current machines for later analysis. I discuss the possible utility of deliberately causing electron acceleration in current and purpose-built machines to perform small-scale experiments of effects which occur at near thermonuclear temperatures. Finally, I discuss how the results inform our understanding of the seed plasma from which a Field-Reversed Configuration (FRC) is created during PFRC-II Rotating Magnetic Field (RMF) experiments, the original goal of this project.

7.1 Very brief summary

This dissertation describes measurements and models of fast electron dynamics in the seed plasma of the PFRC-II. The standout result is a mechanism by which fast electrons in a magnetic mirror are accelerated to temperatures of 2 - 3 keV by many interactions
with a region of sinusoidal electrostatic potential whose bounce-period resonances are broken by the non-adiabaticity of the magnetic moment. To my knowledge this is a new mechanism for acceleration, standing alongside and supplementing the known mechanism of turbulent electrostatic heating.

Along the path to explaining the acceleration mechanism in full, I describe a small number of ancillary novel contributions: a means by which x-ray Bremsstrahlung spectra may be inverted into EEDFs, a mechanism of fast electron creation by ion-induced secondary electron emission from an RF self-biased surface in a magnetic field, a means by which secondary electrons may be suppressed by texturing a surface, and a mechanism by which passing electrons may non-adiabatically persist in a magnetic mirror.

7.2 Future work

7.2.1 Mirror experiments

The acceleration mechanism discovered here is applicable to some of today’s magnetic mirror experiments and concepts, but not all. The Gas Dynamic Trap in Novosibirsk, for example, is more collisional than would allow this effect to be leading-order.[116] The GOL-3 multimirror is also very collisional, but is already heated by electrostatic fluctuations that arise from injected electron beams; the results herein imply that even if it were to become significantly less dense, electron beam injection would still provide electron heating.[182] Conditions which allow the non-adiabatic resonance breaking are common in tabletop, small-scale experiments.[188]
The acceleration mechanism discovered here is not just applicable to existing experiments; it is a method which may be deliberately exploited to generate electron populations which approach thermonuclear temperatures in a small, inexpensive, steady-state apparatus. This allows small-scale experiments to make physically meaningful contributions to areas of research which were previously the domain of much larger budgets. The PFRC-II is a perfect example; despite having a center cell less than one meter in length, fast electrons are able to provide a means to measure the non-adiabaticity of magnetic moment. With electron temperatures as high as 3 keV, there is the possibility of using the PFRC-II or a PFRC-like mirror in this configuration as a kind of Electron Beam Ion Trap (EBIT) or Electron Beam Ion Source (EBIS) to study the spectra of highly charged, high-Z ions.[189]

### 7.2.2 PFRC-II

Despite the success of this project in discovering and characterizing new fast electron phenomena in magnetic mirror machines, that was not its original purpose. The original goal of the project was to characterize the PFRC-II seed plasma as the initial state of the Field-Reversed Configuration (FRC) at the heart of the PFRC-II experiment. Now that the presence of fast electrons is known, further research must be conducted to determine their relevance or lack thereof to the FRC.

We will research, first theoretically, then experimentally, whether the Rotating Magnetic Field (RMF) which creates the FRC couples well or poorly to this minority population of energetic electrons. If the RMF couples well, the injection of fast electrons will become part of the experimental protocol for producing FRCs. If it couples poorly,
the RF input power to the antenna in the Source End Cell will be reduced to powers at
which no fast electrons are detected.

We will test whether previously discovered phenomena can be attributed to the
fast electrons. Previously, the PFRC has produced long-lived, stable FRCs.[190] These
results were interpreted to be a combination of the kinetic tilt-mode stability criterion
and the stabilizing effect of RMF. The experiments will be repeated with and without
fast electrons to ensure that this interpretation is in agreement.

The Scrape-Off Layer (SOL) of the PFRC concept is expected to be of great interest.
In the working concept for a notional PFRC-based fusion reactor, the SOL has the
crucial job of removing fusion products and power from the core of the FRC.[191] In
the PFRC-II, the population of fast electrons gives us a degree of freedom to test our
understanding of the SOL, as when the FRC is present, the SEC-born fast electrons
flow around it in the SOL. We expect to begin diagnosing the SOL of the PFRC-II with
Langmuir probes as soon as this year. Having control over fast electrons to change
power flow and the axial space potential profile will allow us to more completely test
our models of SOL dynamics.
Appendix A

Electron Equations of Motion in B-Following Coordinates

The equation at the end of this appendix is given by Putvinskii in 1982 but not derived there. [159] Hopefully this will elucidate its origin for any confused readers.

$B$-following coordinates are also enclosed-flux-gradient-following coordinates, as $B$ points along an isocontour of the enclosed flux $\Phi$ and $\nabla \zeta$ is perpendicular to this. Define $\hat{n}$ to be along the gradient of $\zeta$ and $\hat{b}$ to be orthonormal to this in the right-hand sense. $\zeta = \frac{e}{2\pi m} \Phi$.

Suppose there are 2D Cartesian coordinate $(x, y)$ equations of motion

\[ \dot{\vec{v}} = \vec{F} \]  \hspace{1cm} (A.1)

\[ v_x = F_x \]  \hspace{1cm} (A.2)
\[ v_y = F_y \]  \hspace{1cm} (A.3)

In transforming into curvilinear coordinates \((n, b)\), there are inertial terms that appear due to \(\dot{n}, \dot{b}\) nonzero:

\[ \dot{v} = v_n \dot{n} + v_n' \dot{\hat{n}} + v_b \dot{\hat{b}} = F_n \dot{n} + F_b \dot{\hat{b}} \]  \hspace{1cm} (A.4)

Furthermore

\[ \dot{Q} = \partial_t Q = v_n \partial_n Q + v_b \partial_b Q \]  \hspace{1cm} (A.5)

So each component of Equation A.4 becomes

\[ v_n = F_n - v_b'^2 (\hat{n} \cdot \partial_b \hat{b}) - v_b v_n (\hat{n} \cdot \partial_n \hat{b}) \]  \hspace{1cm} (A.6)

\[ v_b = F_b - v_n'^2 (\hat{b} \cdot \partial_n \hat{n}) - v_b v_n (\hat{b} \cdot \partial_b \hat{n}) \]  \hspace{1cm} (A.7)

In terms of \((x, y), (\hat{n}, \hat{b})\) are

\[ \hat{n} = \frac{\partial_x \zeta \hat{x} + \partial_y \zeta \hat{y}}{\sqrt{(\partial_x \zeta)^2 + (\partial_y \zeta)^2}} \]  \hspace{1cm} (A.8)

\[ \hat{b} = \frac{\partial_y \zeta \hat{x} - \partial_x \zeta \hat{y}}{\sqrt{(\partial_x \zeta)^2 + (\partial_y \zeta)^2}} \]  \hspace{1cm} (A.9)

Because we are free to use any rotation of the coordinates \(x, y\), for each point we will choose one for which locally \(\hat{x} = \hat{b}\) and \(\hat{y} = \hat{n}\).
With this simplification, the derivatives in Equations A.8 and A.9 become easier, giving us the equations

\[
\hat{n} \cdot \partial_n \hat{b} = -\frac{\partial^2 b}{\partial_n \zeta} \quad (A.10)
\]

\[
\hat{n} \cdot \partial_n \hat{b} = -\frac{\partial^2 n}{\partial_n \zeta} \quad (A.11)
\]

\[
\hat{b} \cdot \partial_n \hat{n} = \frac{\partial^2 b}{\partial_n \zeta} \quad (A.12)
\]

\[
\hat{b} \cdot \partial_n \hat{n} = \frac{\partial^2 b}{\partial_n \zeta} \quad (A.13)
\]

Choosing \((r, z)\) as our Cartesian coordinates, and including the centrifugal and Coriolis terms in \(\vec{F}\), plugging Equations A.10 through A.13 into Equations A.6 and A.7 constitutes the complete equations of motion of a particle in \(\vec{B}\)-following coordinates.

Because \(\hat{n} \cdot \partial_n \hat{b}\) is the change in the \(\vec{B}\) vector along its length, it is \(1/R_c\), the radius of curvature of the magnetic field line. Because positive and negative are defined arbitrarily, we will define

\[
R_c^{-1} = \frac{\partial^2 b}{\partial_n \zeta} \quad (A.14)
\]

Likewise, \(\frac{\partial^2 n}{\partial_n \zeta}\) has a physical interpretation, but it is the combination of two effects. By performing the derivative and invoking the magnetic Gauss’s Law,

\[
\frac{\partial^2 n}{\partial_n \zeta} = \frac{\partial_b B}{B} + \frac{\hat{b} \cdot \hat{r}}{r} \quad (A.15)
\]
The full equations of motion in a $\vec{B}$ following coordinate system are therefore:

\[
\dot{v}_n = \Omega v_\phi + v_\phi^2 \frac{\dot{n} \cdot \hat{r}}{r} + v_n v_b \left( \frac{1}{l_B} + \frac{\hat{b} \cdot \hat{r}}{r} \right) + \frac{1}{R_c} 
\tag{A.16}
\]

\[
\dot{v}_\phi = -\Omega v_n - v_\phi v_n \frac{\hat{n} \cdot \hat{r}}{r} - v_\phi v_b \frac{\hat{b} \cdot \hat{r}}{r} 
\tag{A.17}
\]

\[
\dot{v}_b = v_\phi^2 \frac{\hat{b} \cdot \hat{r}}{r} - v_\phi^2 \left( \frac{1}{l_B} + \frac{\hat{b} \cdot \hat{r}}{r} \right) - v_b v_n \frac{1}{R_c} 
\tag{A.18}
\]

where $\hat{b} = \vec{B}/|\vec{B}|$, $l_B^{-1} = \frac{\partial B}{\partial r}$, $R_c$ is the radius of curvature of the magnetic field line, $\hat{n}$ is the radially outward direction perpendicular to both $\hat{b}$, $\hat{\phi}$. 
Appendix B

Visualizations of the non-adiabaticity of magnetic moment in various magnetic fields

Here, I reproduce Equation 5.24:

$$\Delta \mu_0 = \frac{m v_{\perp,0} v_{\parallel,0} z_0 \sqrt{\pi}}{B_0 R_{c,0}} e^{-\frac{z_0^2 a^2}{4 v_{\parallel,0}^2}} \sin \psi_0 = \delta \mu \sin \psi_0 \quad (B.1)$$

$$z_0^{-2} = z_R^{-2} + z_B^{-2} \quad (B.2)$$

$$z_Q^{-2} = \frac{\partial^2 z_Q}{2 Q_0} \quad (B.3)$$

Normalized to the original $\mu$, the size of the $\mu$ jump is:
\[
\frac{\delta \mu}{\mu} = \sqrt{\frac{\pi}{2}} \frac{z_0}{R_{c,0} v_{\perp}} e^{-\frac{z_0^2 \Omega^2}{v_{\parallel,0}^2}} \tag{B.4}
\]

The parallel energy \( E_{\parallel} = m_e v_{\parallel}^2/2 \) which makes this fraction equal to some proportion \( p_c v_{\parallel}/v_{\perp} \) is:

\[
E_{\parallel, p_c} = \frac{1}{8} m_e z_0^2 \Omega^2 \frac{1}{-\ln(p_c R_{c,0} z_0 \sqrt{\pi})} \tag{B.5}
\]

That is, if a particle is on a field line which passes through a region for which its perpendicular energy is larger than \( E_{\parallel, p_c} \), then its \( \mu \) can undergo a change of more than \( p_c \mu v_{\parallel}/v_{\perp} \). This is a measure of how non-adiabatic \( \mu \) is.

\( E_{\parallel, p_c} \) can be plotted as a function of space. We can see which field lines conserve \( \mu \) to better than \( p_c \) at a given \( E_{\parallel} \). Because of this, I contend that Equation B.5 is a good visualization for the extent of non-conservation of \( \mu \).

This value, for a \( p_c \) of 1% is shown in Figures B.1 and B.2. The auxiliary quantities that make up \( E_{\parallel, p_c} \) are also shown, to build intuition for where in a mirror there is high curvature and low field.

Here’s how to read Figures B.1 and B.2: Choose a field line on the field line plot, the upper lefthand plot in Figure B.1 or the dark blue lines in Figure B.2. As you follow the field line, take note of the smallest value of \( E_{\parallel, p_c} \) through which it passes. This is the energy of particle which may undergo a \( \mu \) jump as high as \( p_c \mu v_{\parallel}/v_{\perp} \).

It is possible to see many of the features derived in Chapter 5 in the plots of \( E_{\parallel, p_c} \). Visibly, \( \mu \) jumps occur at the midplane. Also visibly, the size of the \( \mu \) jump increases
FIGURE B.1: Visualization of the non-adiabaticity of electrons in the PFRC-II run in seed plasma mode. This figure has intermediate plots of the dependencies of $E_{||,p_c}$, specifically the amplitude of the magnetic field $B$ and the inverse radius of curvature of the magnetic field $1/R_c$.

FIGURE B.2: Visualization of the non-adiabaticity of electrons in the PFRC-II run in seed plasma mode. This figure has the field lines, dark blue, overlaid upon the parameter which describes adiabaticity, $E_{||,p_c}$, defined in Equation B.5 for $p_c = 1\%$. 
with radius. Visibly, there are no other regions accessible to a passing electron in the PFRC-II that are non-adiabatic.

We may plot the same value, $E_{\parallel,pe}$, for many such configurations and so gain insight into the regions of space and energy which will be adiabatic in them.

I have plotted the same quantities for a modest mirror machine which may be used for fusion experiments. It consists of only two coils, spaced 2 meters apart. The mirror ratio is 4. The minimum $B$ is 1 Tesla. The intermediate quantities of $B$ and $R_c$ can be seen in Figure B.3, and a detail of the field lines overlaid with $E_{\parallel,pe}$ can be seen in Figure B.4. These quantities are calculated for Deuterium ions.

From Figures B.3 and B.4, we can see that 100 keV ions in this mirror are not particularly adiabatic. Each transit, they may gain or lose a large proportion of their $\mu$. 

**Figure B.3:** Visualization of the non-adiabaticity of Deuterons in a modest magnetic mirror. The features at about $z = \pm 0.5$ meters are artifacts from a region of zero curvature.
For reasonable values of $n, T$, this is a leading-order effect.

From now on, I will omit the plots of the intermediate values of $R_c$ and $B$, showing only the field lines with $E_{||,pc}$ overlay.

Figure B.5 was inspired by the GOL-3 mirror in Novosibirsk.[181, 182, 192] The GOL-3 supports the multimirror concept, which paradoxically improves in confinement with increasing ion collisionality. Visible in the figure is the fact that, for the less extreme mirror ratio $R = 2$ and higher magnetic field 6 Tesla, the onset of non-adiabaticity does not begin until several hundred keV. However, if researchers should want to deliberately stimulate diffusion in $\mu$, it should be noted that the curvature of the magnetic field is easy to increase to any arbitrary degree by the addition of a counter-polarized coil between the main coils.
Chapter B.0

Figure B.5: Visualization of the non-adiabaticity of Deuterons in another magnetic mirror, reminiscent of the GOL-3. The features at about \( z = \pm 0.3 \) meters are artifacts from a region of zero curvature.

Figure B.6 was inspired by the Proto-RT device at the University of Tokyo.[150] It supports and visualizes the finding that those electrons which are more radially outward exhibit larger jumps in \( \mu \). The energy scales at which \( \mu \) jumps occur are comparable to the energies of 2 keV injected into the dipole trap during operation of the Proto-RT, though Figure B.6 should not be taken to be quantitatively similar to that experiment.

Figure B.7 describes electrons in a Hill’s Vortex configuration which has approximately the parameters that we observe from the PFRC-II when it is run in a Rotating Magnetic Field (RMF) driven Field Reversed Configuration (FRC) mode. It shows that the combination of a surface of low field amplitude and high curvature create a boundary that intersects every field line in the FRC region and effectively randomizes \( \mu \) of electrons, even of very low energy. This has been observed previously in single-particle Hamiltonian simulations of the FRC.
Figure B.6: Visualization of the non-adiabaticity of electrons in a dipole field, reminiscent of Proto-RT.

Figure B.7: Visualization of the non-adiabaticity of electrons in a Hill’s Vortex FRC, reminiscent of the PFRC-II run in RMF-FRC mode.
Figure B.8 describes Deuterons in a much larger and stronger Hill’s Vortex FRC. This Hill’s Vortex size and strength was chosen to mimic those under consideration for FRC-based compact fusion reactors for burning Deuterium and Helium-3.[193] It shows that the same effect as is observed in the smaller, weaker FRC for electrons in Figure B.7 is visible for ions in this FRC. The FRC size and strength I chose is expected to be very kinetic in character, and the non-adiabaticity of the Deuterons agrees; even Deuterons of only a few eV exhibit large changes in $\mu$ upon each crossing of the low-$B$ surface.
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