CONSIDERATIONS ON THE OPTIMAL AND EFFICIENT PROCESSING OF INFORMATION-BEARING SIGNALS

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Abstract

Noise is a fundamental hurdle that impedes the processing of information-bearing signals, specifically the extraction of salient information. Processing that is both optimal and efficient is desired; optimality ensures the extracted information has the highest fidelity allowed by the noise, while efficiency ensures limited resource usage. Optimal detectors and estimators have long been known, e.g., for maximum likelihood or minimum mean-squared error criteria, but might not admit an efficient implementation. A tradeoff often exists between the two goals.

This thesis explores the tradeoff between optimality and efficiency in a passive radar system and an analog-to-digital converter. A passive radar system opportunistically uses illuminating signals from the environment to detect and track targets of interest, e.g., airplanes or vehicles. As an opportunistic user of signals, the system does not have control over the transmitted waveform. The available waveforms are not designed for radar and often have undesirable properties for radar systems, so the burden is on the receiver processing to overcome these obstacles. A novel technique is proposed for the processing of digital television signals as passive radar illuminators that eases the need for complex detection and tracking schemes while incurring only a small penalty in detection performance. An analog-to-digital converter samples analog signals for digital processing. The Shannon-Nyquist theorem describes a sufficient sampling and recovery scheme for bandlimited signals from uniformly spaced samples taken at a rate twice the bandwidth of the signal. Frequency-sparse signals are composed of relatively few frequency components and have fewer degrees of freedom than a frequency-dense bandlimited signal. Recent results in compressed sensing describe sufficient sampling and recovery schemes for frequency-sparse signals that require a sampling rate proportional to the spectral density and the logarithm of the bandwidth, while providing high fidelity and requiring many fewer samples, which saves resources. A proposed sampling and simple recovery scheme is shown to efficiently
recover the locations of tones in a large bandwidth nearly-optimally using relatively few samples. The proposed sampling scheme is further optimized for full recovery of the input signal by matching the statistics of the scheme to the statistics of the input signal.
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To my family.
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<td>SNR</td>
<td>signal-to-noise ratio</td>
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<tr>
<td>MSE</td>
<td>mean squared error</td>
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<tr>
<td>AWGN</td>
<td>additive white Gaussian noise</td>
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<tr>
<td>ADC</td>
<td>analog-to-digital converter</td>
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<tr>
<td>ENOB</td>
<td>effective number of bits</td>
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<tr>
<td>FRI</td>
<td>finite rate of innovation</td>
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<tr>
<td>ML</td>
<td>maximum likelihood</td>
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<td>MAP</td>
<td>maximum a posteriori</td>
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<td>AF</td>
<td>ambiguity function</td>
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<tr>
<td>MF</td>
<td>matched filter</td>
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<td>MMF</td>
<td>mismatched filter</td>
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<tr>
<td>CS</td>
<td>compressed sensing</td>
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<td>KECoM</td>
<td>knowledge-enhanced compressive measurements</td>
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<tr>
<td>RIP</td>
<td>restricted isometry property</td>
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<td>OMP</td>
<td>orthogonal matching pursuit</td>
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<td>RD</td>
<td>random demodulator</td>
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<tr>
<td>CRD</td>
<td>constrained random demodulator</td>
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<tr>
<td>DFT</td>
<td>discrete Fourier transform</td>
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<tr>
<td>NUS</td>
<td>non-uniform sampler</td>
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<td>RLL</td>
<td>run-length limited</td>
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<td>RCS</td>
<td>repetition-coded sequence</td>
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<td>MRS</td>
<td>Markov-generated RLL sequence</td>
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<td>OST</td>
<td>one-step thresholding</td>
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<td>CP</td>
<td>coherence property</td>
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<td>Abbreviation</td>
<td>Description</td>
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<tr>
<td>CFAR</td>
<td>constant false alarm rate</td>
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<td>RCS</td>
<td>radar cross-section</td>
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<td>CPI</td>
<td>coherent processing interval</td>
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<tr>
<td>GPS</td>
<td>global positioning system</td>
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<tr>
<td>LOS</td>
<td>line-of-sight</td>
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<td>DP</td>
<td>direct path</td>
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<tr>
<td>DPI</td>
<td>direct path interference</td>
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<tr>
<td>RF</td>
<td>radio frequency</td>
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<tr>
<td>FM</td>
<td>frequency modulation</td>
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<td>AM</td>
<td>amplitude modulation</td>
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<td>MPEG</td>
<td>Motion Picture Experts Group</td>
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<tr>
<td>BPSK</td>
<td>binary phase shift keying</td>
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<tr>
<td>QPSK</td>
<td>quadrature phase shift keying</td>
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<td>QAM</td>
<td>quadrature amplitude modulation</td>
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<tr>
<td>PRBS</td>
<td>pseudorandom binary sequence</td>
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<tr>
<td>DVB-T</td>
<td>digital video broadcast – terrestrial</td>
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<td>OFDM</td>
<td>orthogonal frequency division multiplexing</td>
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<tr>
<td>TPS</td>
<td>transmission parameter signal</td>
</tr>
<tr>
<td>ITR</td>
<td>Institute for Telecommunications Research</td>
</tr>
<tr>
<td>UniSA</td>
<td>University of South Australia</td>
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<tr>
<td>DSTO</td>
<td>Defence Science and Technology Organisation</td>
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Chapter 1

Introduction

1.1 Motivation

The objective of signal processing is to extract salient information encoded in signals. Salient information can take a variety of forms depending on the specific application. For example, in a radar system the information is the location and velocity of various targets of interest; in a digital communication system the information is the transmitted symbols representing digital data (bits) that the transmitter sends to the receiver. In the petroleum industry, surveyors use seismic waves to search for pockets of petroleum and natural gas. In all of these systems noise is a fundamental impediment that complicates the extraction of information from signals. The goal of a signal processor is therefore to extract the useful information, which is corrupted by noise, as accurately and efficiently as possible. The terms accuracy and efficiency will be defined more rigorously over the course of the dissertation. Essentially, accuracy refers to how well the extracted information represents the actual information being sought, and efficiency refers to the resources (e.g., computation time, power consumption, memory usage) needed to extract the information. Often there is an inherent tradeoff between accuracy and efficiency that will be explored over the course of this
thesis. In fact, the main contribution is to explore the tradeoff by considering two specific examples.

The need for accurate information processing is obvious as it allows for better and more confident decisions to be made based on the extracted information. If a radar system is not concerned with accuracy when it detects and tracks targets, then its users cannot rely on any information delivered by that system, e.g., air traffic controllers might not know when planes are colliding, or military installations might not get notified of an incoming enemy. Similarly, the need for highly efficient processing is also very important in many applications. Applications that are very time-sensitive need to extract information as quickly as possible. A radar system that requires days or weeks to detect targets would not be very useful to warn of an impending attack from an enemy. Applications, such as remote sensing, that rely on small, mobile devices with small batteries, and therefore limited energy, also need to extract information as power-efficiently as possible. Fortunately, much of the salient information is highly structured in the acquired signals, and the structure can be exploited to more efficiently extract the information.

1.2 Extracting Information from Measurements

The fundamental challenge in extracting the useful information is summarized by Claude E. Shannon in his seminal 1948 paper [68] in regard to communication theory:

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.

Applying this idea more generally, the message in Shannon’s statement corresponds to any piece of information to be recovered, not necessarily limited to a communication setting. Rather than chosen by a communication transmitter, it might be selected by the physical environment or something partially or not at all under
the control of the system designer. For a concrete example, consider a radar system. The information to be extracted might be the location of moving objects, like cars or airplanes. A probing signal is transmitted and returned with modifications, e.g., a time delay and Doppler shift, by reflecting objects. The objects encode information onto the return of the probing signal through these modifications. The goal is therefore to extract the useful information, e.g., time delay and Doppler shift, from the received measurements of the probing signal. The measurements might also be corrupted by noise or other non-idealities in the acquisition system. Full or partial knowledge of the noise statistics and the measurement operation might be available. Processing algorithms generally perform better, and are easier to design, with more information available about noise or interference corrupting the signals. In addition, the probing signal can be designed taking noise and interference into account when prior knowledge is available.

To place this work in the context of the wider signal processing world, consider the abstract signal processing system in Figure 1.1. Here, \( \theta \) represents the underlying information that is to be extracted, and \( \hat{\theta} \) represents the information extracted by the signal processor. Both are a collection of parameters that are placed into a vector for notational convenience so that \( \theta = [\theta_1, \theta_2, \cdots] \). The Information Encoding block maps the information to a real-world signal (e.g., radio waves, light, sound waves) while the Channel block represents environmental influences on the encoded signal. Most often the Channel block is not in control of the designer and represents noise and interference added to the signal adversely affecting the extraction of information, while the Information Encoding block might be partially under the control of the designer. The Signal Acquisition block represents the processing front-end and interface between the real-world signal and the signal processor and most often includes an analog-to-digital converter (ADC). The Information Extraction block contains the

---

1. Encoding is not necessarily the same as in Communication Theory or Information Theory.
Figure 1.1: Block diagram of the abstract processing model. The quantity $\theta$ represents the information to extract while $\hat{\theta}$ is the extracted estimate of the information. The blocks in between represent different stages of the overall processing chain, which may or may not be accessible to the system designer.

main processing elements that attempt to undo the actions of the Information Encoder, Channel, and Signal Acquisition blocks to extract the desired information. This work concentrates on the last two blocks, Signal Acquisition and Information Extraction, because these blocks contain the most design freedom in the two problems considered. The output of the Signal Acquisition block is often referred to as *measurements*. The measurements contain the information that has been encoded onto real-world signals, corrupted by effects of the Channel, and acquired by the Signal Acquisition machinery. Because the Signal Acquisition often includes an ADC, the main challenge is to extract the information from these measurements by designing *digital* processing algorithms that are implemented on computer systems.\(^2\)

The performance of the processing algorithm is judged by two main criteria: how faithfully the extracted information represents the true information, and how many resources are used in the extraction process.

### 1.3 Optimal Detectors and Estimators

The performance of the processing algorithm is first judged by how well it extracts the desired information. A distinction between two classes of problems is important. In the first, $\theta$ contains parameters that take a finite number of values, e.g., $\theta_i \in \{0, 1\}$

\(^2\)Often more specialized digital signal processors are used in applications for further efficiency in the implementation. The operation of such processors is very similar to that of a general purpose computer. Specific hardware implementations are beyond the scope of this work.
representing off or on respectively. In this case, \( \hat{\theta} \) is called a detector. In the second class, \( \theta \) assumes a continuum of values, e.g., \( \theta_i \in \mathbb{R} \), and \( \hat{\theta} \) is called an estimator. Building an optimal estimator is often more difficult than building an optimal detector due to the larger parameter space. In many real-world applications, the measurements obtained by the Signal Acquisition block are corrupted by noise from various sources, so recovery of the information is expected to be inexact and contain some distortion. If the distortion in the recovered information is as small as possible, then the estimate is said to be optimal. Specifically, an optimal estimate means that \( \hat{\theta} \) matches \( \theta \) as closely as possible for all, or nearly all, possible signals.

Assume that a stochastic model for the measurements, denoted \( y \), can be written in terms of the information parameters \( \theta \). In this case, two well-known and related detectors (or estimators) are considered as the baseline for optimality: the maximum likelihood (ML) detector and the maximum a posteriori (MAP) detector \[56\]. Let \( p_{\theta}(y) \) be a probability distribution on the measurements \( y \) as a function of \( \theta \). The ML detector is the set of parameters \( \hat{\theta} \) that maximizes the likelihood of the measurements

\[
\hat{\theta}_{\text{ML}}(y) = \arg \max_{\theta} p_{\theta}(y). \tag{1.1}
\]

The MAP detector requires an additional probability distribution on \( \theta \), denoted \( p(\theta) \), that dictates the probability with which the parameters take a particular value. The MAP detector is the set of parameters that maximizes the likelihood of the measurements weighted by \( p(\theta) \)

\[
\hat{\theta}_{\text{MAP}}(y) = \arg \max_{\theta} \left( p_{\theta}(y) \cdot p(\theta) \right). \tag{1.2}
\]

Note that the ML detector is a special case of the MAP detector with \( p(\theta) \) a uniform distribution on \( \theta \).
1.4 Efficient Processing of Signals

Finding the ML or MAP detector (or estimator) is often difficult, and computing it is often resource intensive. Optimal detectors and estimators do not necessarily account for complexity in acquisition devices or post-processing at the receiver because the goal is to produce the most accurate detector possible. First, consider (1.1) or (1.2) computed by exhaustive search over all possible $\theta$. This would clearly be an optimal detector, but would be very inefficient and time-consuming to compute. As the sizes of $\theta$ and $y$ increase, the number of computations required for exhaustive search increases exponentially. More computing operations requires more computing resources such as computing time or processor cores. In many applications, such as radar processing, real-time or near real-time processing is highly desirable and therefore more efficient detectors are desired.

The most efficient detectors require many fewer computational operations than an exhaustive search. As another extreme example, consider a detector that outputs the same constant for all possible input measurements: $\hat{\theta}_c = C$. The detector will produce a lot of errors, but the result is obtained immediately. These two extremes highlight the inherent tradeoff between optimality and efficiency. Both are rarely achieved simultaneously.

Inefficiencies arise in the Signal Acquisition block as well. Consider the detection of frequencies in a bandlimited signal from samples of the signal. The Shannon-Nyquist sampling theorem \[69\] states that if a bandlimited signal is sampled at a rate of twice the highest frequency contained in the signal, then the samples contain all the information needed to reconstruct the original signal. The reconstruction is then carried out using an interpolating filter and the composite frequencies computed. However, this scheme becomes intractable for large bandwidth signals, i.e., with high frequency content spread across a wide range, because the required sampling rate increases linearly with the bandwidth. High rate converters are expensive, offer limited
resolution, and place large memory requirements on microchips [49, 83], so schemes that require a smaller sampling rate will, in general, allow more accurate sampling. Both of these examples show the need for careful consideration of processing efficiency when designing detectors for applications.

1.5 Main Contributions

The main contribution of this thesis is the analysis of the optimality-efficiency tradeoff in two specific examples. The first example considers radar processing and a modification to traditional radar processing that improves overall detection (and tracking) performance. The second example considers the detection and/or estimation of tones in a large bandwidth from far fewer samples than that required by traditional processing. A common thread develops in both cases. The processing algorithm is tailored to exploit the structure of the information-bearing signal.

1.5.1 Mismatched Filter Processing for Passive Radar

Radar processing is considered first. The goal in a radar system is two-fold:

1. to detect the presence of targets-of-interest and

2. to characterize the detected targets (e.g., position, velocity, size).

Consider the problem of detecting targets and identifying the range and velocity.

The range and velocity of a target are obtained by finding, respectively, the round-trip delay from transmitter to receiver and the Doppler offset imparted to the received signal relative to the transmitted signal. It is important to note that the radar processing chain needs access to the transmitted waveform as a reference, and it should be as clean as possible. Traditional radar processing relies on a well-known result in signal detection and estimation, the matched filter (MF) [56]. The MF is
Figure 1.2: Block diagram of the MF processing. A cross-correlation of the reference signal $r(t)$ with the surveillance signal $s(t)$ at various delay hypotheses $\tau$ and Doppler hypotheses $\nu$ produces the MF output $\chi(\tau,\nu)$. The properties of the function $\chi(\tau,\nu)$ are highly dependent on the choice of transmitted signal $x(t)$ as well as the targets present in $s(t)$.

the cross-correlation of the surveillance signal $s(t)$ with the known transmitted signal as a reference signal $r(t)$ at various delay and Doppler offset hypotheses. Figure 1.2 shows a block diagram of the MF processing. An important tool for analyzing the performance of the MF is the ambiguity function (AF), or the 2-D auto-correlation of the transmitted signal with time and frequency shifts. In other words, the AF is the matched filter for $s(t) = r(t) = x(t)$. Figure 1.4 shows an example of the AF if the transmitted signal is a stream of pulses modulated with a Barker sequence of length 13. The transmitted signal is shown in Figure 1.3 with a normalized time axis. When $s(t)$ is a delayed and Doppler-shifted version of $x(t)$, the delay and Doppler offset are located by finding the largest peak in the MF output. If the surveillance signal $s(t)$ also contains noise, then the floor, or the smallest values of the function $\chi(\tau,\nu)$, will no longer be exactly zero. The MF is known to be the ML detector for a single target in Gaussian noise because it offers optimal reconstruction of the received waveform, a delayed and Doppler shifted version of $x(t)$. Cross-correlating with any other reference signal will produce a smaller peak in $\chi(\tau,\nu)$.

The delay and Doppler offset are then translated into range and velocity data.

A problem may occur with this approach when returns from multiple targets are present in the received signal (or returns from stationary objects are received as well). Each target produces a characteristic signature in the MF output that is determined
Figure 1.3: A stream of pulses modulated with a length-13 Barker code.

Figure 1.4: The surface plot of the ambiguity function for a transmitted signal consisting of a stream of pulses modulated with a length-13 Barker code. Note the large peak in the middle of the plot at zero-delay and zero-Doppler. The spread in the main peak and the additional peaks at non-zero delay and Doppler make detection of multiple targets more difficult.
Figure 1.5: Block diagram of the MMF processing. A cross-correlation of the modified reference signal $r(t)$, which is modified by filtering with $h(t)$, with the surveillance signal $s(t)$ at various delay hypotheses $\tau$ and Doppler hypotheses $\nu$ produces the MMF output $\chi_m(\tau, \nu)$. The properties of the function $\chi_m(\tau, \nu)$ are still highly dependent on the choice of transmitted signal $x(t)$ as well as the targets present in $s(t)$.

by the AF. The AF might contain multiple peaks and/or place significant energy throughout the range-Doppler plane. This can reduce the reliability of detecting targets because target returns produce multiple peaks in the MF output. The additional peaks can lead to the detection of false targets or missed detections of true targets because some stronger returns might completely mask weaker target returns. In the case of multiple targets, a MF is no longer the guaranteed optimal detector.

Since radar systems often must deal with multiple targets, an AF with only a single peak in the region of interest is desirable. Energy placed outside of this region in the AF does not impact detection. The design of the transmit waveform $x(t)$ is therefore of paramount importance, and a large body of literature exists on radar waveform design [70]. However, at times it might be advantageous to use transmitters designed for other purposes, and the radar designer does not have the ability to design the transmit waveform. One such situation, known as Passive Radar, is described in Chapter 2. In this case, a mismatched filter (MMF), where the cross-correlation is performed with a modified reference signal, is shown to aid the detection of targets by reducing or eliminating undesirable peaks in the AF. Figure 1.3 shows a block diagram of the MMF where $r(t)$ is pre-filtered by $h(t)$. The MMF allows the detector

---

3The region of interest in delay-Doppler space is dictated by the receiver sensitivity and the nature of the targets of interest, among other things. For example, targets that are sufficiently far away will not return enough energy to the receiver for reliable detection. Specific limits on delay and Doppler are application specific.
to estimate the delay and Doppler of multiple targets directly from the MMF output rather than using a complicated scheme to extract the AF signature associated with each target from the MF output.

1.5.2 Compressed Sensing Applied to Sub-Nyquist Sampling

The Shannon-Nyquist theorem is at the heart of modern signal processing and provides a theoretical foundation for the digital processing of analog signals [69]. Unfortunately, technology provides a limit on the precision of the sampling operation. A tradeoff exists between the sampling rate and the resolution provided by each sample. The extent of the tradeoff has improved over time, but at a rather slow pace [83]. The limit on the sampling rate limits the useable bandwidth of the acquired signals through the Shannon-Nyquist theorem. Additionally, the information contained in the signals, the information which is actually to be extracted, is often highly structured, and the structure can be exploited to more efficiently extract the information, i.e., with fewer samples.

Compressed Sensing: Exploiting Signal Structure

In recent years, a technique to improve the efficiency of processing algorithms has received a great deal of attention. The technique is the exploitation of a particular structure, called *sparsity*, in acquired signals by the processing algorithms. A signal model has sparse structure if relatively few parameters, out of very many possible, completely (or nearly completely) describe a signal. For example, a signal composed of a small number of tones that can be placed at any frequency within a very large bandwidth (say, 5 tones in a bandwidth of 1 GHz) is sparse. A radar surveillance signal that contains returns from few targets, relative to the size of the entire delay-Doppler space, is also sparse. While other signal structure is important for some applications, e.g., the algebraic structure of codes in communications [9], sparsity
seems to be particularly prevalent and important in a wide range of applications \[6\]. Many real signals seem to fit naturally into sparse models.

The idea of exploiting sparse structure in signals can be traced back at least to the mid 1970’s with the work of Rife and Boorstyn \[62,63\]. They were interested in finding a single tone (and later several tones) contained in a large bandwidth contaminated with noise. They accomplished the recovery using fewer samples than that required by the Shannon-Nyquist theorem. The more recent fields of study known as sparse approximation or compressed sensing (CS), which have seen a great deal of attention over the past decade, have systematically explored the exploitation of sparse structure in signals. To gain some intuition for the sparse signal model, consider Figure 1.6. Here, a matrix with fewer rows than columns $\Phi$ (corresponding to an underdetermined system of equations) is multiplied by a long, sparse vector $x$. Sparsity in the vector $x$ means that relatively few entries are non-zero. If $x$ is not sparse, then recovering it from the measurements $y$ is not feasible in general because $\Phi$ is underdetermined. The CS literature has explored many efficient algorithms that can recover $x$ when it is sparse and if $\Phi$ satisfies certain conditions.
In the context of Figure 1.1, sparsity in the signal refers to the actions of the first block, Information Encoding. Relatively few parameters describe signals that live in a very large space of possible signals. The Signal Acquisition block is represented by the matrix $\Phi$. The Information Extraction corresponds to the processing algorithms. Some algorithms that extract information are reviewed in Chapter 2.

Chapter 2 reviews some of the key results from the CS literature, especially in regard to sub-Nyquist sampling. In particular, emphasis is placed on an architecture called the Random Demodulator (RD), which is described in detail in Chapter 4. A modification to the RD is first explored that helps with the practical implementation in hardware. Additionally, a technique is described to optimally exploit the sparse structure and other a priori information about the input signal. Finally, a very efficient processing algorithm is shown to recover spectrum usage nearly optimally under certain conditions, namely, when the signal-to-noise ratio is not too large.

1.6 Outline and Notation

The notational conventions that follow are briefly collected for convenience. When appropriate, the reader is reminded of these in the text.

Sets are denoted with curly braces $\{\cdots\}$, and the indicator function is denoted as $1(\cdot)$ where

$$1(\cdot) = \begin{cases} 1, & \text{the expression inside } (\cdot) \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

Matrices are denoted with upper case roman letters (A) and vectors with lower case roman letters (a). Scalars are denoted with italic lower case letters ($a$). The conjugate transpose, or Hermitian transpose, of a matrix, vector, and scalar are written $A^*$, $a^*$, and $a^*$. The $n \times n$ identity matrix, a square matrix with diagonal entries 1, is denoted $I_n$. For a matrix $A$, $A|_{\Omega \times \Omega}$ denotes the principal submatrix created from the columns
and rows contained in the set $\Omega \subset \{1, \ldots, n\}$. The spectral norm of a matrix $A$ is denoted $\|A\|_2 = \max\{\sqrt{\lambda}\}$ where $\lambda$ is an eigenvalue of $A^*A$, and $\|A\|_{\text{max}}$ denotes the maximum absolute entry of the matrix. For a vector $a$, the usually $\ell_2$ norm, or Euclidean norm, is $\|a\|_2 = \sqrt{\sum_{i=1}^{n} a_i^2}$, and the $\ell_1$ norm is $\|a\|_1 = \sum_{i=1}^{n} |a_i|$. The $\ell_0$ quasi-norm counts the number of non-zero entries $\|a\|_0 = \sum_{i=1}^{n} 1(a_i \neq 0)$.

The imaginary unit is denoted $i = \sqrt{-1}$. The letters $C$ and $c$ in roman font are reserved to denote universal constants that could change values at each instance. For a random variable $b$, let $\mathbb{E}[b]$ denote the expectation and $\mathbb{E}^p b = (\mathbb{E}|b|^p)^{1/p}$ is the $p^{th}$ absolute moment of $b$. Let $\mathbb{P}\{\cdot\}$ denote the probability of an event. The mutual information of two random variables, or vectors, $a$ and $b$ is denoted $I(a; b) = \int_{a} \int_{b} P_{ab} \log \left( \frac{P_{ab}}{P_a P_b} \right) da db$

where $P_{ab}$ denotes the joint probability distribution of $a$ and $b$ and $P_a$ and $P_b$ denote the marginal probability distributions. The short-hand $j \sim r$ means $(r - 1) \frac{n}{m} < j \leq r \frac{n}{m}$ for some $n$ and $m$ such that $m$ divides $n$.

The remainder of this thesis is outlined as follows. Chapter 2 opens with a brief description of the inversion of operators and moves to a brief discussion of matched filter processing and under what circumstances it is optimal. It then provides some background on compressed sensing, especially as applied to sub-Nyquist sampling using the Random Demodulator. Chapter 3 describes a passive radar system that uses digital television signals as the illuminator, or transmitter, source. Two processing techniques are proposed to improve the detection and tracking performance of passive radar systems. First, a demodulation and remodulation of the line-of-sight signal from the transmitter is described to obtain a (nearly) noise-free reference signal. Second, a mismatched filter is proposed to reduce the number of peaks present in the ambiguity function output, and consequently the matched filter output. The peak

\[4\] The integral is replaced by a sum in the case of discrete distributions.
arise from the structure of the digital television signal, and this process is described in detail. Chapter 4 turns to analog-to-digital conversion and describes how the Random Demodulator is used to recover frequency sparse bandlimited signals from sub-Nyquist samples. A modification to the Random Demodulator is then proposed that improves performance by taking into account additional structure in the input signal model and considering practical limitations on technology. Chapter 5 explores the utility of the Random Demodulator in spectrum sensing through consideration of the detection of the location of the active frequency components. The effect of noise on detection is explored by providing a lower bound on the measurement rate as a function of the noise present in the system. A low-complexity algorithm is then examined that accomplishes detection very successfully and is nearly optimal for some noise levels. Chapter 6 finally concludes with a summary of the main contributions and a brief discussion of future directions to consider.
Chapter 2

Inversion of the Measurement Operation

2.1 Introduction

Considering the signal processing model in Figure 1.1, the goal of the signal processor can be stated as follows: “The extracted information \( \hat{\theta} \) represents, as faithfully as possible, the true information \( \theta \).” Group the first three blocks – Information Encoding, Channel, and Signal Acquisition – together as an operation that produces discrete measurements \( y \in \mathbb{C}^m \). Let this operation be represented by the mapping \( f : \theta \mapsto y \).

The goal of the fourth block, the Information Extraction, might be to find the inverse of this mapping \( \hat{\theta} = f^{-1}(y) \). Alternatively, more limited information about \( \theta \) might suffice in some situations. For example, consider the detection of a tone in a large bandwidth corrupted with noise. The information describing the tone includes the frequency, the phase, and the amplitude. The most relevant piece of information might be the frequency if, for example, a transmitter wanted to avoid interfering with the tone. In this case, the whole inverse is not necessary, and just recovering the frequency suffices. The mapping \( f \) is often a stochastic mapping due to the noise.
introduced by the Channel and, possibly, the Signal Acquisition blocks. Finding an
exact inverse in this case is not possible, and the goal is instead to find a function
that best approximates the inverse.

Now, replace the abstract blocks of Figure 1.1 with some realistic models. The
output of the Information Encoding block is denoted by $x_\theta(t)$ where $t$ is the signal
index, e.g., time or a spacial dimension. The notation emphasizes the information-
bearing nature of the signal $x$ where the information is represented by $\theta$. The Channel
block corrupts the signal $x$ with additive noise, denoted by $\xi_x(t)$. The Signal Acquisi-
tion block is represented by the operator $\Phi$ and an additional additive noise denoted
by $\xi_y(t_y)$ and termed measurement noise. The measurements that appear at the
output of the Signal Acquisition block are

$$y(t_y) = \Phi (x_\theta(t) + \xi_x(t)) + \xi_y(t_y)$$  \hspace{1cm} (2.1)

where, again, $x_\theta(t)$ is the information-bearing signal, $t_y$ is the measurement index,
and $\theta$ the parameters that represent the useful information. The indices $t$ and $t_y$
are not necessarily the same. For example, the signal index in many cases is analog,
i.e., $t \in \mathbb{R}$ while the measurement index is discrete, i.e., $t_y \in \mathbb{Z}$. In this work, when the
measurement index is discrete the measurements are written with a discrete index,
e.g., $y[i]$.

Figure 2.1: Block diagram of the abstract measurement model (2.1). The opera-
tor $\Phi$ represents the error-free measurement operation. Corruptions in the signal
are introduced by the environment before, and by the measurement device after, the
measurement operation.
The additive noise $\xi_x$ and $\xi_y$ are often stochastic (e.g., Gaussian noise) and finding the true inverse is not possible. The designer of the Information Extraction block, in this case, wants to find a stable approximation to the inverse. A stable approximation means that the error of the approximation is proportional to the amount of noise added to the measurements. Depending on the application, the designer might have some level of control over the measurement operator $\Phi$ or the Information Encoding $x_\theta$.

### 2.2 Matched Filter Processing Applied to Radar

The goal of a radar system is to detect the presence of targets in space, estimate their range and velocity, and sometimes classify the targets based on other criteria. To accomplish these goals, the radar system *illuminates* the targets by transmitting a probing waveform that is reflected back to the receiver and processes the received signals from the reflecting targets. The range and velocity are encoded in the received signal by the delay and Doppler shift present relative to the transmitted waveform \[70\]. The delay is the total time required for the waveform to travel from the transmitter to the target to the receiver. The Doppler shift is imparted to the waveform by the velocity of the target relative to the transmitter and receiver.

Radar systems are classified into two broad categories based on the relative placement of the transmitter and receiver: monostatic and bistatic. In a monostatic radar system, the transmitter and receiver are collocated, while in a bistatic system they are separated. In each system the nature of the reference channel and the surveillance channel are different. In a monostatic radar system, the receiver has direct access to a clean (i.e., noise free) version of the transmitted signal. This is a major advantage for monostatic systems. In a bistatic system the transmitter and receiver are usually not close enough to have a direct connection, and the reference often comes from the
line-of-sight path between the transmitter and the receiver. This means the reference might contain noise and multi-path interference. A major advantage of bistatic systems, however, is that the receiver gets a different look at the target because it is receiving a different reflection of the transmitted signal than that of a monostatic system. The receiver also has the ability to move around in space to get diverse looks at the targets. In fact, if several receivers are spread around a transmitter, each one would receive a different reflection from the target. Figure 2.2 shows the geometry of a typical passive or bistatic radar system. One important thing to note about the geometry is that there are two receive antennas corresponding to two channels: a reference channel and a surveillance channel.

1In many systems the surveillance channel consists of multiple antennas in a phased array configuration for direction-of-arrival processing. This discussion simplifies the situation to a single antenna with the understanding that it can be extended to multiple antennas in a straightforward manner.
Figure 2.3: Block diagram of detecting a target using MF processing. A cross-correlation of the reference signal $r[n]$ with the surveillance signal $s[n]$ at various delay hypotheses $\tau$ and Doppler hypotheses $\nu$ produces the MF output $\chi(\tau, \nu)$. The properties of the function $\chi(\tau, \nu)$ are highly dependent on the choice of transmitted signal $x[n]$ as well as the targets present in $s[n]$. The MF output is then compared to a threshold $\lambda$ to detect targets. The threshold is set based on the noise present in the surveillance and reference channels and performance characteristics of the detector such as false alarms and missed detections.

### 2.2.1 Matched Filtering to Detect a Single Target in Noise

To make this discussion more rigorous, assume a radar system with a single antenna in the surveillance channel, and the received signal is sampled at a rate of $f_s = \frac{1}{T_s}$, which is greater than the Nyquist rate of the received waveform to avoid aliasing. The received signal from the surveillance channel with returns from $K$ targets present is

$$s[n] = \sum_{k=1}^{K} c_k x[n - \tau_k/T_s] e^{i2\pi\nu_k n} + \xi[n]$$  \hspace{1cm} (2.2)

where $x[n]$ is the (sampled) transmitted waveform, $\tau_k \geq 0$ is the delay of the $k^{th}$ target, $\nu_k$ is the Doppler shift of the $k^{th}$ target, $c_k \in \mathbb{C}$ is the (complex) amplitude of the $k^{th}$ target, and $\xi[n]$ is additive white Gaussian noise (AWGN). $N$ samples are collected, so $n = 0, \cdots , N-1$. The amplitude is determined by the channel defined by the path from transmitter to target to receiver, which includes, among other factors, the materials and shape of the target, the transmit power of the transmitter, the path length, and environmental factors in the channel. In the model (2.2), each target is assumed to be reasonably represented by a point scatterer, and the bandwidth of $x(t)$ is small relative to the carrier frequency of the signal so the Doppler effect can be represented by a single frequency shift.
With the received signal model (2.2) in hand, a detection problem can be set up, first in the simpler case of a single target (\(K = 1\)). In this case, the detection problem is binary detection. Under the null hypothesis, \(s[n] = s_0[n]\) where

\[
s_0[n] = \xi[n]
\]  
(2.3)

contains no signal and only noise. Under the alternative hypothesis, \(s[n] = s_1[n]\) where

\[
s_1[n] = c_1 x[n - \tau_1/T_s] e^{i2\pi \nu_1 n} + \xi[n]
\]  
(2.4)

contains a signal at delay \(\tau_1\) and Doppler \(\nu_1\) with additive noise. In this case, the optimal detector (in the maximum likelihood sense) is a matched filter (MF). The important details of this problem are summarized here, but please see [70, Chapter 10] or [56, Chapter III] for a more thorough analysis of the MF and this detection problem. The detector is described graphically in Figure 2.3. For clarity, the following analysis assumes that the surveillance channel signal does not contain a contribution direct from the transmit antenna, sometime called direct blast or direct path interference. This is addressed in Section 2.2.2 because it is particularly important in bistatic and passive radar systems. The received surveillance signal is first filtered with a reference signal \(r[n]\) that is a time-reversed, delayed, and Doppler-shifted version of the transmitted signal \(x[n]\), possibly corrupted by noise. Ideally the processor has access to a clean version of the transmitted signal, \(r[n] = x[n]\), but the analysis is valid regardless. The MF output is

\[
\chi(\tau, \nu) = \sum_{n=0}^{N-1} s[n] r[n - \tau/T_s] e^{i2\pi \nu n},
\]  
(2.5)

and the filter impulse response is \(h_{\tau, \nu}[n] = r[\tau/T_s - n] e^{-i2\pi \nu n}\). The MF output is then compared to a threshold to determine the location in the delay-Doppler plane of the
target. The threshold is set based on the desired performance characteristics such as false alarms and mis-detections. A false alarm occurs when the detector declares a target where one is not present due to the influence of noise, while a mis-detection occurs when the detector does not declare a true target when one is present also due to the influence of noise. The discussion here is primarily concerned with the structure of $\chi(\tau, \nu)$. Some brief discussions relate features of $\chi(\tau, \nu)$ to the detection problem, but the reader is referred to [56] and [70] for more details on the detection statistics. As a brief note, the real data in Chapter 3 uses an algorithm, which is called constant false alarm rate (CFAR), that adjusts the threshold to maintain a constant probability of false alarm based on a dynamic target, noise, and interference environment.

The preceding discussion provides the optimal detector in the case of a single target ($K = 1$) corrupted by AWGN with noise power $\sigma^2$. An important property of the MF, which it inherits because (2.5) is a cross-correlation, is that under the alternative hypothesis (2.4), and in the absence of noise, i.e., $\xi[n] \equiv 0$,

$$|\chi(\tau_1, \nu_1)| \geq |\chi(\tau, \nu)|$$

for all $\tau \neq \tau_1$ and $\nu \neq \nu_1$. Therefore, the target is located in delay and Doppler by finding the maximum value of $|\chi(\tau, \nu)|$ (in the absence of noise). Accounting for noise adds a noise floor to $\chi(\tau, \nu)$. Under hypothesis $s_1[n]$, (2.5) becomes

$$\chi(\tau, \nu) = \sum_{n=0}^{N-1} c_1 x[n - \tau_1/T_s] e^{j2\pi \nu_1 r[n - \tau/T_s]} e^{j2\pi \nu n} + \sum_{n=0}^{N-1} \xi[n] r[n - \tau/T_s] e^{j2\pi \nu n}$$  \hspace{1cm} (2.6)

and the term

$$\sum_{n=0}^{N-1} \xi[n] r[n - \tau/T_s] e^{j2\pi \nu n}$$  \hspace{1cm} (2.7)
provides the noise floor. Note that the noise is negligible if (2.7) is negligible in (2.6). If (2.6) is normalized by the number of samples $N$, then the average power of the noise floor is proportional to $\sigma^2/N$ while the peak at $\chi(\tau_1, \nu_1)$ is proportional to $|c_1|$. The noise floor is lowered by taking more samples, through an increase in the processing time, which is termed processing gain in the radar literature \cite{70}. The relative difference between the power of the peak in the MF output, due to the target return, and the power of the noise floor determines the detector performance.

2.2.2 The Ambiguity Function and Detection of Multiple Targets

Building the optimal detector is much more difficult when multiple targets, or direct blast, are present in the surveillance channel ($K > 1$) because each target can also affect the detection of the other targets. Targets affect each other because the MF output contains energy from each target that is spread out in the delay-Doppler plane due to the correlation properties of the transmitted waveform. The ambiguity function (AF), introduced in Chapter \cite{1}, is used to analyze the nature and extent of this interaction. The AF $\chi_{AF}(\tau, \nu)$ is the autocorrelation of the transmitted signal

$$\chi_{AF}(\tau, \nu) = \sum_{n=0}^{N-1} x[n]x[n - \tau]e^{j2\pi
u n}. \quad (2.8)$$

Two related features of the AF are important to understanding how targets affect the detection of each other:

1. local peaks, called ambiguity peaks, located at non-zero delay and Doppler, and

2. the average energy contained in the non-zero delay and Doppler hypotheses.

\footnote{If the reference signal $r[n]$ also contains noise, then additional cross terms are present in (2.6) and contribute to the noise floor.}
The second item is termed the *pedestal* and is similar to the noise floor. Both features hinder detection performance because strong targets for which $|c_k|$ is large can mask weak targets for which $|c_k|$ is small, even when the targets are well separated.

Figure 1.4 in Section 1.5.1 shows an example AF for a Barker-coded pulse stream; Figure 3.4 in Section 3.4.1 shows an example for a digital television signal described in detail in Chapter 3. Notice in each case the local peaks at non-zero delay and Doppler and the general presence of energy throughout the delay-Doppler plane.

The AF provides the key to understanding the MF output when multiple targets are present in the surveillance signal. The AF describes the *signature*, or characteristic placement of energy, of each target in the MF output. Each target places energy in $\chi(\tau, \nu)$ based on the pattern of energy distribution in $\chi_{AF}(\tau, \nu)$ with an appropriate shift by the delay $\tau_k$ and Doppler $\nu_k$ and a scaling by the complex amplitude $c_k$. The strongest return, i.e., the target with the largest $|c_k|$, determines the pedestal of the MF output, assuming the noise floor is negligible in comparison. Targets that are too weak to appear above the pedestal, or noise floor if not negligible, cannot be reliably detected. The difference between the main peak of the AF and the average power of the pedestal determines how reliably weak targets can be detected. The target with the strongest return also determines the most prominent ambiguity peaks. These can mask targets that are strong enough to otherwise appear above the pedestal and noise floor.

The astute reader will note that a sophisticated detector, or more appropriately a tracking algorithm, could be devised to disambiguate bad delay-Doppler scenes. For example, when tracking a target over time, the change in range should correspond to the velocity of the target, and targets that are masked by a large local peak in one view might not be masked at the following time instant. A good tracking algorithm would account for these issues, but tracking algorithms are beyond the scope of this analysis. Additionally, ambiguity peaks degrade detection performance,
and consequently tracking performance. In a real radar system, targets do not appear as single point scatters of the waveform, i.e., targets are not adequately described by a single delay and Doppler shift. A detector that must account for the whole signature of each target to make detections is much more complex, and in some cases is not possible to build, than a detector that directly uses the filter output and searches for single peaks.

2.3 Passive Radar Using Digital Television Signals

The design challenges in a specific example are analyzed taking into account MF processing and the problems encountered when detecting multiple targets. The radar system of interest has a bistatic geometry, shown in Figure 2.2, where the transmitter and receiver are not collocated. In addition, the system is passive, opportunistically using illuminators in the environment, and does not have direct control over the transmitted signal. The advantages to passive bistatic radar are discussed below. In the context of Figure 1.1, the designer does not have control over the Information Encoding block meaning the signal $x_\theta(t)$ is fixed. The transmitted signals of interest are digital television signals that are described in detail in Chapter 3. The measurement operation $\Phi$ is also very simple and consists of filters and modulators that shift the received signal to baseband and an ADC that samples the signal above the Nyquist rate to avoid aliasing. Because direct access to the reference signal is not available in a passive radar system, the transmitted signal must be acquired indirectly from the source. This is accomplished by obtaining the line-of-sight (LOS) channel from the transmitter and, since digital television signals are communication signals, utilizing the error-correcting structure in the signals to construct a (nearly) error-free reference signal for the cross-correlation processing.

Some systems actually modulate signals to an intermediate frequency for processing, which could offer some advantages in implementation. The net effect is the same, so baseband processing is assumed throughout.
Passive radars exploit readily available, non-cooperative sources of radio energy as illuminators of opportunity to measure reflections from the environment and targets of interest. Without the need for the deployment and operation of a dedicated transmitter, passive radar systems are significantly less expensive to implement and operate than their conventional counterparts. A key advantage is that with a suitable illuminator available, covert surveillance of an area of interest is possible. Another advantage is derived from the geometry of the passive radar system because it is bistatic. The bistatic radar cross-section (RCS) of a target differs from its monostatic RCS, and this may aid target detection and/or classification [23].

Many broadcast and communication signals have been considered and analyzed as possible passive radar illuminators including Digital Audio Broadcast (DAB), Digital Video Broadcast (DVB), FM radio, cellphone base-stations, and various satellite systems [8, 26, 32, 34, 52, 53, 57, 65]. Terrestrial digital television transmissions (e.g., DVB-T) provide an especially attractive opportunity for passive radar. Digital television transmitters offer a powerful, well-defined signal with sufficient bandwidth for reasonable precision in range and are noise-like, thereby allowing for good, consistent range compression and Doppler estimation of targets [60].

DVB-T signals employ orthogonal frequency division multiplexing (OFDM), and are designed to provide for good reception of television data. The signals are constructed based on a standard [30] which includes the data frame structure, guard intervals, and pilot information that is inserted for use by a DVB-T receiver in order to achieve synchronization and for channel estimation. The structure of the DVB-T signal gives rise to peaks in the radar ambiguity function.

Ambiguities in the range-Doppler domain are undesirable for two primary reasons:

- they directly mask returns from real targets, and
- some ambiguities will behave like real targets and could therefore erroneously be detected and tracked by the radar.
As these ambiguities are inherent to the transmitted signal, and the transmitted signal structure is beyond the control of the passive radar system designer, ambiguities must be managed at the receiver. Chapter 3 presents a method for mitigating ambiguities through modification of the LOS reference signal such that it becomes slightly mismatched to the surveillance signal.

The ambiguity function of DVB-T sources has been studied previously, albeit with a particular emphasis on the 2k-mode (2048 OFDM subcarriers). Various methods suited to the 2k-mode have been proposed that achieve improvement via manipulation, or filtering, of the reference signal prior to range-Doppler processing [14,33,65]. Their efficacy for 8k-mode (8192 OFDM subcarriers) was not considered, nor were these techniques validated using real data.

The study of the ambiguity function is extended to the 8k-mode DVB-T signals, and the analysis of the ambiguity function is formalized to fully identify and explain the source of the ambiguities. Due to the increase in the symbol length over the 2k-mode signal, ambiguities caused by pilots within the symbol (so-called intra-symbol ambiguities) dominate for typical target ranges of interest. As with the previously proposed methods for the 2k-mode signals, a mismatched reference approach is adopted for range-Doppler processing of the 8k-mode signals. Since hardware- and software-based demodulators for DVB-T signals are readily available, the assumption is that error-free digital data is available for use in the passive radar processing and that the system has direct access to the structure of the reference signal (as required for the methods of [33,65]). A key distinction between the approach described in Chapter 3 and [33,65] is that this approach modifies the reference signal to create a mismatched signal that attenuates the effect of the pilots in terms of the cross-correlation output rather than the input. The method of [14] also considers the output of a mismatched cross-correlation, but assumes that direct access to the structure of the reference
signal is not available and instead achieves the mismatch by filtering the measured reference signal directly.

Carefully crafted demodulation and remodulation of the reference signal are central to the approach described herein. As such, both the advantages and disadvantages presented by the structure of the DVB-T signal are considered. While the structure does introduce undesirable ambiguities, it can also be exploited to better estimate the transmitted signal and channel as well as any front-end mismatch, such as carrier offset between the transmitter and receiver. Based on this, a novel scheme for pre-processing the reference and surveillance signals obtained by the passive radar system is proposed to mitigate the effects of both the ambiguities and the clutter in range-Doppler processing. In contrast to other works, this scheme is also applied to real-world data from an 8k-mode DVB-T system [52] to demonstrate its effectiveness in enhancing target detection. Furthermore, the necessity of the additional signal processing undertaken to demodulate and remodulate the reference signal in achieving successful ambiguity mitigation is highlighted. A 29 dB reduction is observed in the residual ambiguity levels over existing mismatched techniques, while a greater than 36 dB reduction over standard matched filtering is observed. It should be noted that the focus here is on mitigating undesirable range-Doppler ambiguities and maximizing target signal-to-noise ratio (SNR) and system dynamic range.

### 2.4 Sub-Nyquist Sampling and Measurement Efficiency

Modern signal processing relies on the sampling of analog signals for discrete-time processing. The standard approach to sampling signals is based on the Shannon-Nyquist sampling theorem [69], which states that a bandlimited signal can be faithfully reconstructed from its samples collected uniformly at the Nyquist rate of twice the
highest frequency component. The appeal of Nyquist sampling is a simple sampling scheme and a linear reconstruction scheme. However, a very high sampling rate is required in order to sample high bandwidth signals, and this leads to poor resolution for such signals due to the physical constraints on analog-to-digital converter (ADC) technology. The rule of thumb in ADC technology is that a doubling of the sampling rate causes a 1 bit reduction in resolution $^4$ or, more explicitly, $P = 2^B \cdot f_s$ is a constant where $B$ is the effective number of bits (ENOB), a measure of resolution of an ADC, and $f_s$ is the sampling rate. This expression states that for a required sampling resolution, the sampling rate has a hard upper limit due to constraints on the ADC technology, and vice versa. The constant $P$ is dependent on the particular ADC architecture and has steadily increased over time as the technology has improved; the current state-of-the-art allows for sampling at 1 GHz with a resolution of approximately 10 ENOB $^{49,83}$. Unfortunately, this increase tends to happen rather slowly compared to the advancement seen in other areas of technology. For example, microprocessor technology generally follows Moore's law, which states that the transistor count (and effectively performance) doubles every two years $^{48}$. In particular, applications such as spectrum sensing for cognitive radios push modern ADC technology to its limit by requiring the acquisition of large bandwidth signals ($> 1$ GHz) in low SNR environments. The low ENOB at these sampling rates degrades the performance in terms of SNR.

Sub-Nyquist sampling schemes, those requiring rates lower than Nyquist, allow for higher resolution in the samples. In the context of the abstract measurement model, some additional constraints are added to the measurement operator $\Phi$ and slightly more complex reconstruction schemes are allowed in the Information Extraction block in order to decrease the number of measurements that are required connecting the two blocks.
Historically, signal recovery from sub-Nyquist measurements can be traced back at least to Rife and Boorstyn, who used parameter estimation to find the frequency, phase, and amplitude of a single tone \cite{62} and multiple tones \cite{63} in a large bandwidth from fewer measurements than that required by the Shannon-Nyquist theorem. More recently, a related line of research that can be traced back to Prony in the late 18th century has emerged known as finite rate of innovation (FRI) sampling \cite{81}. The work considers signal models that are not necessarily bandlimited but are described by a small number of parameters. For example, a stream of Dirac-delta functions, which satisfy some average density, are each described by two numbers describing the location and the amplitude, or more precisely the area under the curve, of each Dirac. This model could describe a stream of photons randomly striking a detector. For these signals, FRI provides the minimum number of samples that are necessary for recovery of the locations and amplitudes.

2.5 Compressed Sensing and Application to Sub-Nyquist Sampling

The systematic exploration of sparsity as a prior model on signal vectors has come to be called compressed sensing (CS) and has seen a great deal of attention over the past decade. In the context of the abstract processing model, CS limits the measurement operator $\Phi$ to correspond to linear measurements (meaning $\Phi$ can be represented by a matrix) and limits the Information Extraction block to problems that can be cast as convex optimization and/or greedy search problems that can be solved without the need for exhaustive search. Three basic ingredients are required to setup the proper model and perform recovery in a CS problem:

1. a sparse signal vector,
2. underdetermined (short, fat) matrices that satisfy certain properties, and
3. convex optimization or greedy algorithms to solve underdetermined problems.

2.5.1 Sparse Signal Vectors

The first key in any CS problem is casting the problem with a proper model. The input signal must be modeled as a vector with few non-zero entries relative to the total size of the vector. Often a signal must be represented in the proper basis to be described by a sparse signal vector. Consider the signal (column) vector \( x \) of size \( 1 \times n \), and suppose it can be represented in the basis \( \psi \), where \( \psi \) is an orthonormal matrix of size \( n \times n \):

\[
x = \psi u
\]

where \( ||u||_0 = S \). The quasi-norm \( ||u||_0 \) counts the number of non-zero entries in \( u \) and is called the \( \ell_0 \) norm. The matrix \( \psi \) is called the sparsifying basis.

2.5.2 Underdetermined Matrices

The second key ingredient in the CS framework is an underdetermined matrix. CS limits the measurement operator to linear measurements so each measurement can be written as

\[
y_i = \phi_i x
\]

for \( i = 1, \ldots, m \) where \( \phi_i \) is a row vector. The \( \phi_i \) are collected into a matrix and the \( y_i \) into a vector, so the measurements are written in a compact form as

\[
y = \Phi x. \tag{2.9}
\]

\[^4||\cdot||_0 \text{ is not technically a norm because } ||ax||_0 \neq a||x||_0. \text{ However, it does satisfy the triangle inequality and behaves like a norm in many ways. It is referred to as a norm with the understanding that this is not technically correct.}\]
In order to have fewer measurements than entries in the signal vector, the matrix must be underdetermined, i.e., \( m \ll n \). Recovering \( x \) from \( y \) if \( x \) is not sparse is infeasible because an infinite collection of vectors \( x \) can produce the measurements \( y \). The key is restricting \( x \) to be sparse and imposing \( \Phi \) to satisfy certain properties. Note that if \( x \) is sparse in the basis \( \Psi \), then (2.9) can be rewritten as \( y = \Phi \Psi u \), and the same restrictions on \( \Phi \Psi \) are imposed.

One such property is the restricted isometry property (RIP) \([19]\). A matrix satisfies RIP of order \( S \) with RIP constant \( 0 < \delta_S < 1 \) if it satisfies the following relation

\[
(1 - \delta_S)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_S)\|x\|_2^2 \tag{2.10}
\]

for all \( x \) such that \( \|x\|_0 \leq S \).

A key result from the CS literature is that generating matrices satisfying RIP is quite easy. In fact, if the entries of a matrix are drawn independently according to a Gaussian distribution each with variance \( 1/n \), then it satisfies the RIP with very high probability \([19]\).

### 2.5.3 Recovery Algorithms

The final key ingredient is recovery of \( x \) from \( y \) given the model (2.9). The recovery is performed via convex optimization or greedy algorithms. Please see \([15]\) for a thorough overview of convex optimization, but a brief review of a particularly useful problem is provided here.

In the absence of noise, the following problem, called basis pursuit, turns out to be very useful for recovering \( x \) from \( y \).

\[
\min_x \|x\|_1 \quad \text{subject to} \quad y = \Phi x \tag{2.11}
\]
where $\| \cdot \|_1$ is the $\ell_1$ norm or the sum of the magnitudes of the entries. The $\ell_1$ norm turns out to be a good relaxation of the $\ell_0$ norm, and (2.11) recovers $x$ from $y$ exactly provided the matrix $\Phi$ satisfies the RIP with a sufficiently small constant $\delta_S$ [19].

Convex optimization also performs well if noise is added to the measurements [18]. Consider the following problem closely resembling (2.11), and called *basis pursuit denoising*.

$$\min_x \| x \|_1 \quad \text{subject to} \quad \| y - \Phi x \|_2 \leq \gamma$$

(2.12)

where $\gamma$ is a parameter that controls the tradeoff between the $\ell_1$ norm of $x$ and the $\ell_2$ norm of the error in the measurements. The $\ell_2$ norm is the usual Euclidean norm. For an appropriately chosen $\gamma$, the solution to (2.12), denoted $\hat{x}$, satisfies

$$\| \hat{x} - x \|_2 \leq C \gamma$$

where $C$ is a well-behaved constant [18]. Other optimization problems built to handle noise, e.g., the lasso [74], can also be used to recover $x$ when noise is present.

Greedy algorithms operate by iteratively refining an estimate of $x$ when given $y$ and $\Phi$. Examples include iterative hard thresholding [13], orthogonal matching pursuit (OMP) [75, 77], and the related CoSaMP [50]. The details of these algorithms are omitted here, but suffice to say that greedy as well as convex optimization-based algorithms are a very active area of research and will continue to improve the efficiency of recovery in compressed sensing applications. The important characteristic of these algorithms is that the error in the recovery is well-controlled and the number of computations required is limited.
Chapter 3

Mismatched Filter Processing of DVB-T Signals for Passive Radar

3.1 Introduction

3.1.1 Passive Radar Systems

A passive radar system opportunistically uses illuminators that already exist in the environment. Consequently, the system lacks control over the transmit waveform and choosing illuminators with good autocorrelation properties is very important. Illuminators that produce an ambiguity function with a single, narrow peak at zero delay and zero Doppler are desirable. Many of the available illuminators transmit communication signals, e.g., broadcast radio, broadcast television, global positioning system (GPS), and cellular phones. These offer good geographic coverage and transmit essentially continuously, which results in good temporal coverage. Coverage in both senses is ideal for a radar system. Among available communication signals, digital signals tend to be more desirable than analog signals because the ambiguity function (AF) of the signal is practically invariant with respect to the communication content. For example, an AM radio signal contains just a carrier during periods of
silence and contains lots of energy when someone is talking. The content is highly structured and correlated, so the AF is dynamic with peaks appearing, disappearing, and moving during periods of voice activity. The AF of digital signals, on the other hand, are much less affected by the information content because the signal undergoes various operations to make it appear noise-like. The noise-like nature of digital signals produces ambiguity functions that are nearly ideal with a fairly narrow peak at zero delay and zero Doppler. However, redundant structure within the signals, such as pilot sequences that are needed for communication synchronization purposes, leads to undesirable effects. This chapter describes these undesirable effects and introduces methods to reduce their impact on the radar processing system. The ideas and main results in this chapter are published in [39, 55, 67], while the discussion here places them in a broader context of the optimality versus efficiency tradeoff.

3.1.2 Main Contributions

The main contribution of this chapter is the proposal and analysis of a novel mismatched processing technique that improves performance of a passive radar system. The technique essentially contains two parts. The first is a demodulation and remodulation of the signal obtained directly from the transmitter that leverages the properties of the signal useful in a communication context, e.g., the pilot structure for synchronization and error-correction coding. The imperfections of the transmitted signal and the mismatch between the transmitter and receiver hardware are estimated in much the same way as in a communication system. Accounting for these imperfections, the digital signal is demodulated to obtain the error-free digital symbols. Rather than throw away the estimated imperfections, the receiver remodulates the

---

1The digital symbols are obtained error-free as long as the corruptions caused by the channel direct from the transmitter are not too large. If the noise in the direct channel is too large, i.e., the SNR is too low, to reliably demodulate the symbols, then the AF degrades and the targets of interest are likely too weak to reliably detect as the target returns are usually much weaker than the signal direct form the transmitter.
transmitted signal and adds back the imperfections in order to match the waveform reflected by targets as closely as possible, while removing noise and clutter. The proposed method for generating a reference signal by demodulating and remodulating the received signal while retaining deterministic imperfections was first presented and analyzed in [67], with the analysis further refined in [55]. A special thanks to the authors of [67], especially S. Searle of the University of Melbourne, Australia, for the contribution and analysis of this technique.

The second contribution is a mismatched processing technique that mitigates ambiguous peaks in the AF of a DVB-T signal. The ambiguous peaks occur at non-zero delays and Dopplers due to the correlated structure, called pilot carriers, in the transmitted waveform. The pilot carriers aid in demodulation of the signal for communication purposes and are transmitted at a higher power than the other parts of the signal. The ambiguous peaks are mitigated by reducing the amplitude of the pilot carriers in the remodulated reference signal before it is used in the radar processing. The proposed method presented in this chapter for mitigating ambiguity peaks was first presented in [39] with further analysis provided in [55]. A special thanks to the fellow authors of [55], especially L.M. Davis of the Institute for Telecommunications Research (ITR) at the University of South Australia (UniSA), Australia, for collaboration on the analysis.

Finally, this chapter showcases the efficacy of these techniques by application to real radar data that was acquired by the experimental setup at the Defence Science and Technology Organisation (DSTO) in Australia [52, 54]. The experimental setup used the techniques described in this chapter to obtain the data shown in Figure 3.9. A special thanks to the folks at DSTO, and especially J. Palmer, for the contribution of their resources and the provided radar data and detection history plots.

\[\text{The techniques presented in this chapter likely can be applied to other digital communication waveforms used for passive radar systems. The specific processing would depend on the correlated structure in the signals and is beyond the scope of the present analysis.}\]
3.1.3 Organization

The remainder of this chapter is organized as follows. Section 3.2 begins with an overview of the digital television signal used as the illuminating waveform and describes the structure of these signals. Section 3.3 discusses the passive radar geometry and signal model, and why passive radar presents unique challenges to radar system designers. Section 3.4 analyzes the radar AF for digital television signals and identifies the sources of the ambiguous peaks, which are due to the correlated signal structure. The impact of these peaks on radar processing is also discussed. Section 3.5 presents the demodulation/remodulation technique used to obtain a clean reference signal and the mismatched processing technique that reduces the ambiguous peaks in the AF. Section 3.6 finally presents simulated numerical results and results using real-world data to showcase the advantage of using the proposed techniques in passive radar processing.

3.2 DVB-T Signal Overview

DVB-T signals have a very specific structure designed to provide for good reception of television signals as detailed in [30]. A brief summary of the relevant portions influencing a passive radar system is provided here.

3.2.1 OFDM Frame Structure

The DVB-T signal uses orthogonal frequency division multiplexing (OFDM) with either \( K = 2048 \) or \( K = 8192 \) subcarriers depending on the operating mode. The number of subcarriers determines whether the system operates in 2k-mode or 8k-mode. An OFDM symbol has a duration of \( T_S \) seconds and consists of \( K_C < K \) active carriers. Symbols are organized into frames, with each DVB-T frame consisting of \( L = 68 \) OFDM symbols. A super-frame consists of four frames and is used to match
the OFDM signaling with the framing for the error control coding in the system. The OFDM symbols carry three different types of data:

1. the MPEG-2 video and audio data stream,

2. the DVB-T transmission parameter signal (TPS), and

3. pilots.

**Video and Audio Data**

The raw MPEG-2 stream first passes through a series of stages including bit-randomization, outer-coding, and inner-coding before being mapped onto symbols from the signal constellation. The net result is that the information content appears as random, highly uncorrelated data and thus noise-like. The noise-like nature of the signal also produces a flat spectrum [39]. The data carriers are modulated with either quadrature phase shift keying (QPSK), 16-quadrature amplitude modulation (QAM), or 64-QAM depending on the operating mode as defined in the standard. The vast majority of the carriers in each DVB-T symbol contain video and audio data.

**Transmission Parameter Signal (TPS)**

The TPS carriers convey information about the parameters of the transmission scheme. The carrier locations are constant and are defined by the standard. All carriers convey the same information and use differential binary phase shift keying (BPSK). The initial symbol is derived from a pseudorandom binary sequence (PRBS).

**Pilots**

The pilot symbols aid the receiver in acquisition, demodulation, and decoding of the received signal. Two types of pilots are included: scattered pilots and continual pilots.
The scattered pilots are uniformly distributed among the carriers in any given symbol but occupy a different carrier from symbol to symbol. In contrast, the continual pilot signals occupy the same carrier in each symbol. The location of all pilot carriers is defined by the DVB-T standard. Figure 3.1 illustrates the pilot spacing for a single DVB-T OFDM frame where carriers are indexed horizontally and symbols are indexed vertically. The symbols modulated onto each and every pilot are based on a PRBS and use BPSK modulation at a boosted power level $\frac{16}{9}$ times greater than that used for the (average) data symbols and TPS symbols.

### 3.2.2 Signal Definition

The baseband structure of the transmitted OFDM signal is

$$x(t) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{L-1} \sum_{k=0}^{K-1} c_{m\ell k} \cdot \psi_{m\ell k}(t)$$  \hspace{1cm} (3.1)
where
\[
\psi_{m\ell k}(t) = e^{2\pi \frac{1}{T_U}(t-T_G-\ell T_S-m L T_S)} w_{m\ell}(t) \tag{3.2}
\]
and
\[ \begin{align*}
\bullet & \quad 0 \leq m < \infty \text{ denotes the OFDM frame number}, \\
\bullet & \quad 0 \leq \ell \leq (L - 1) \text{ denotes the OFDM symbol number}, \\
\bullet & \quad 0 \leq k \leq (K - 1) \text{ denotes the carrier number}, \\
\bullet & \quad c_{m\ell k} \text{ are the complex-valued modulation symbols}, \\
\bullet & \quad K \text{ is the number of subcarriers, } 2048 \text{ (2k-mode) or } 8192 \text{ (8k-mode)}, \\
\bullet & \quad K_C < K \text{ is the number of active subcarriers (for the inactive carriers, effectively } c_{m\ell k} = 0), \\
\bullet & \quad L \text{ is the number of symbols per frame}, \\
\bullet & \quad T_U \text{ is the duration of the useful part of the symbol, i.e., the part of the symbol excluding the guard interval,} \\
\bullet & \quad T_G \text{ is the guard interval duration (and } \frac{T_G}{T_U} \text{ is the guard interval fraction),} \\
\bullet & \quad T_S = T_U + T_G \text{ is the total OFDM symbol duration including the guard interval, and} \\
\bullet & \quad w_{m\ell}(t) = w(t - \ell T_S - m L T_S) \text{ is a rectangular window function representing the duration of symbol with the window function defined as}
\end{align*}\]

\[
w(t) = \begin{cases} 
1 & \quad 0 \leq t \leq T_S \\
0 & \quad \text{otherwise.}
\end{cases}
\]
Additionally, $N_U$, $N_G$ and $N_S$ are defined as the number of samples in the useful part, the guard interval, and the total symbol duration, respectively, when sampled with a given sampling period $T$. The delay to the start of the useful part of the $(m,\ell)$ symbol is defined as

$$\delta_{m\ell} = T_G + T_S(\ell + Lm).$$  \hspace{1cm} (3.3)

Because the real data presented in this chapter for confirmation of the ideas is from an Australian DVB-T system, the focus is on the Australian DVB-T standard [71], although the analysis and proposed scheme are suitable for other DVB-T systems in either the 2k or 8k modes of operation. The relevant parameters for the Australian standard are listed in Table 3.1. It should be noted that in Australia the channel spacing is 7 MHz, compared to the European standard that uses a channel spacing of 8 MHz. The bandwidth of the OFDM signal is correspondingly reduced by a factor of $\frac{7}{8}$ [30, 71]. DVB-T broadcasts in Australia were licensed on contiguous 7 MHz channels numbered 28 (centered at 529.5 MHz) through 69 (816.5 MHz). The passive radar demonstrator system of [52], which provided the real data presented in this chapter, used illuminators of opportunity employing the Australian 8k-mode DVB-T signal, most typically channel 33 (564.5 MHz), which has a guard interval fraction of $T_G/T_U = \frac{1}{8}$. Another interesting feature of the Australian implementation of the DVB-T standard is that, unlike the European version, a multi-frequency network implementation is the norm. This means that (typically) each broadcast channel at each broadcast site is transmitted on a unique frequency. In contrast, a single frequency network transmits a broadcast signal from multiple sites on the same frequency. This provides an added challenge for passive radar systems because each received reflection of the waveform must be matched to the correct source transmitter. While single frequency network implementations do exist in Australia, these are relatively rare and have limited overlapping coverage regions. As such, one of the major challenges faced
Table 3.1: Relevant parameters for the Australian 8k-mode DVB-T Signal

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>total subcarriers</td>
<td>$K$</td>
<td>8192</td>
</tr>
<tr>
<td>active subcarriers</td>
<td>$K_C$</td>
<td>6817</td>
</tr>
<tr>
<td>symbols per frame</td>
<td>$L$</td>
<td>68</td>
</tr>
<tr>
<td>useful part of symbol</td>
<td>$T_U$</td>
<td>1024 $\mu$s</td>
</tr>
<tr>
<td>guard interval fraction</td>
<td>$T_G/T_U$</td>
<td>$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$</td>
</tr>
<tr>
<td>carrier spacing</td>
<td>$1/T_U$</td>
<td>976.6 Hz</td>
</tr>
<tr>
<td>bandwidth</td>
<td>$(K_C - 1)/T_U$</td>
<td>6.65625 MHz</td>
</tr>
</tbody>
</table>

in the European context, that of transmitter ambiguity resolution, is typically not an issue in the Australian environment and is beyond the scope of the present analysis.

3.3 Passive Radar Received Signal Model

Figure 2.2 (reproduced in Figure 3.2 for convenience) depicts the passive bistatic radar geometry. The DVB-T source transmits the signal $x(t)$. The passive radar receiver collects both a reference signal $r(t)$ via a line-of-sight (LOS) path direct from the illuminator, and a surveillance signal $s(t)$ reflected via the target(s) of interest. These signals may be collected using individual antennas mechanically steered in the appropriate directions (as shown) or, alternatively, signals may be collected via electronic steering, i.e., beamforming, using an array of antennas. Electronic steering allows for multiple surveillance directions to be simultaneously considered. Assuming either physical or electronic steering, range-Doppler processing for a particular azimuth angle for a target of interest is considered. Thus, the model contains two signals: reference and surveillance.

The LOS reference signal $r(t)$ includes a direct path (DP) component from the transmitter, possibly several attenuated copies of the DP at short delay due to static multipath (clutter), and sensor noise introduced at the receiver. At baseband, the reference signal has the form
Figure 3.2: Passive radar geometry showing the transmitted signal \( x(t) \), received reference signal \( r(t) \), and received surveillance signal \( s(t) \). The transmitter (source) and receiver (reference, surveillance) are not collocated, and the designer does not have control over the source or transmitted signal. (Credit: [55]. Thanks to L.M. Davis of ITR, UniSA for the figure!)

\[
r(t) = \alpha_0(t)x(t - \tau_0) + \sum_{i=1}^{C} \alpha_i(t)x(t - \tau_i) + \xi_r(t) \tag{3.4}
\]

where the \( i^{th} \) resolvable component has a delay of \( \tau_i \) and \( \alpha_i(t) \in \mathbb{C} \) encompasses the attenuation and phase shift. The quantity \( \alpha_i(t) \) includes the channel attenuation as well as the combined effect of transmitter/receiver filtering and any unaccounted channel effects. The subscript 0 refers to the DP, and it is assumed that \( |\alpha_0| \gg |\alpha_i| \forall i \).

The final term \( \xi_r(t) \) is the additive white Gaussian noise (AWGN) at the LOS receiver as well as any diminished target return, which is assumed to be negligible.

The surveillance signal \( s(t) \) includes the target return(s) in the form of a (number of) delayed, Doppler-shifted, and attenuated version(s) of the transmitted signal, together with clutter and noise. It may also include a direct path component from the source, termed direct path interference (DPI). The attenuation, or filter factor, \( \beta_i \) in the surveillance channel is related to that of the reference channel \( \alpha_i \) via the relation

\[
\beta_i(t, \theta) = \zeta_i(\theta)\alpha_i(t) \tag{3.5}
\]
for \( i = 0, \ldots, C \) where \( \zeta_i \in \mathbb{C} \) accounts for the attenuation difference. This assumes a negligible separation between the two receivers and, therefore, that the channel and analog filtering effects are the same for both the reference and surveillance channels. As indicated, the complex gain factors will differ for different receiver beam patterns and/or steering angles \( \theta \). The surveillance signal is thus

\[
s(t) = \beta_0(t)x(t - \tau_0) + \sum_{i=1}^{C} \beta_i(t)x(t - \tau_i) + \sum_{p=0}^{P} \beta_{*p}(t)x(t - \tau_{*p}) + \xi_s(t) \quad (3.6)
\]

where there are \( p = 0, \ldots, P \) returns from a single target, highlighted by subscript \( * \), and \( \xi_s(t) \) is the AWGN in the surveillance channel. Note that \( s(t) \) looks very similar to \( r(t) \) and is only distinguished as a second signal in order to highlight the target returns. In many situations, the returns of the DPI and clutter are much stronger than the return from the target of interest \([40]\). This wide dynamic range imposes minimum performance requirements on the receiver radio frequency (RF) hardware and on the ADC technology \([40]\). While dynamic range considerations are very important in passive radar processing, these fall outside the scope of the current analysis and are not considered in detail. Suffice it to say that the real data target detections that are presented later in this chapter should reassure the reader that the appropriate considerations were made in the design and implementation of the system that was employed for this work.

### 3.4 Radar Ambiguity Function

The ambiguity function (AF) for the transmitted signal \( x(t) \) is a 2-dimensional autocorrelation function given by

\[
\chi_{xx}(\tau, \nu) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t - \tau)e^{-j2\pi\nu t} dt \quad (3.7)
\]
where \( \tau \) and \( \nu \) are the delay and Doppler hypotheses respectively. The AF in the discrete-time case is presented in (2.8) and discussed in Section 2.2.

The delay and Doppler profile of the target return is extracted by matched filtering (MF), introduced in (2.5) in discrete-time, or cross-correlating the reference signal \( r(t) \) with the surveillance signal \( s(t) \)

\[
\chi_{rs}(\tau, \nu) = \int_{-\infty}^{\infty} r(t) \cdot s^*(t - \tau) e^{-i2\pi\nu t} dt.
\]  

(3.8)

The range estimate of the target is derived from the delay hypothesis and the angles of arrival (azimuth and elevation). As discussed in Section 2.2.2, the ambiguous peaks in the target processing output (3.8) result from the correlation properties of \( x(t) \), captured by the AF (3.7), and the target returns and clutter described in (3.4) and (3.6).

In practical systems, the reference and surveillance signals are sampled, and the matched filter is implemented for a window of data. The size of the window is called the coherent processing interval (CPI) and determines the Doppler resolution and the processing gain of the system. Direct implementation of the matched filter according to (2.5), which is a discrete version of (3.8), is computationally expensive. In this work, the method used in [54] is adopted. This method is motivated by a typical two stage pulse-Doppler compression approach. The coherent processing interval \( T_{cpi} \) is represented as a series of \( N \) contiguous pulses of length \( T_c \) (i.e., \( T_{cpi} = NT_c \)). Assuming \( N \) is even and the intervals are symmetric about \( t = 0 \), the range-compressed pulse at zero-Doppler for the interval with index \( i \) is

\[
\chi_{rs:i}(\tau, 0) = \int_{0}^{T_c} r(t + iT_c) s^*(t + iT_c - \tau) dt.
\]  

(3.9)
From the full set of \( N \) range-compressed pulses, the Doppler dimension is recovered by applying a Fourier transform across \( i \)

\[
\chi'_{rs}(\tau, \mu) = \sum_{i=-N/2}^{N/2-1} \chi_{rs;i}(\tau, 0) \exp \left( -\frac{i2\pi}{N} \mu i \right) \approx \chi_{rs}(\tau, \nu) \tag{3.10}
\]

where \( \mu \) is the Fourier frequency (per correlation segment, \( T_c \)) estimate of \( \nu \).

Similar range-Doppler processing techniques are discussed elsewhere in the literature, most notably [8, 26, 57]. However, these typically are explicitly tailored to OFDM waveforms and require the cyclic prefix to be discarded. While processing the signal in this manner can offer some advantage, it is also a very particular implementation (in the data alignment sense) and involves discarding data. In contrast, the method proposed above is waveform independent and processes the data as contiguous pulses, i.e., with no discarded data. As a result, the method presented here is more flexible in implementation and encounters less SNR loss.

### 3.4.1 Features of the DVB-T Ambiguity Function

The ambiguities inherent in the transmitted DVB-T signal are investigated through mathematical analysis and numerical simulation. To highlight the features in the analysis that follows, the DVB-T system is numerically simulated at baseband, and the discrete-time AF (2.8), which is the sampled version of the continuous time (3.7), is calculated. Processing is implemented using one super-frame \( (T_{cpi} = 4 \times 68 \times T_S = 313.34 \text{ ms}) \) of data, and the results are shown in Figure 3.3, where the autocorrelation function energy is normalized relative to the zero-delay, zero-Doppler peak. The guard interval is \( T_G/T_U = \frac{1}{8} \) to match the signals typically used in Australia and used in the real-data presented in this chapter. Several ambiguities appear aside from the large peak at zero-delay and zero-Doppler. To make these clear, Figure 3.4 shows only the
Figure 3.3: Delay and Doppler projections of the ambiguity function of the full DVB-T signal in 8k-mode operation with a guard interval of $\frac{1}{8}$. The plot is normalized to the main peak at zero-delay and zero-Doppler. The peaks at non-zero delay and Doppler hamper target detection. (Credit: [55].)

portions of the ambiguity function that are greater than $-55$ dB as blue dots and further highlights those peaks greater than $-50$ dB with red squares.
Figure 3.4: Overhead view of the ambiguity function of the full DVB-T signal in 8k-mode operation with a guard interval of $\frac{1}{8}$. Only portions with energy greater than $-55$ dB are included. The peaks greater than $-50$ dB are indicated with larger red squares and are a result of the pilot structure of the DVB-T signal (see Section 3.4.2). The peaks greater than $-55$ dB and less than $-50$ dB are indicated with smaller blue dots. Select peaks are marked to aid in verifying the analysis of the pilot structure in Section 3.4.2. (Credit: [55].)

The analysis begins with the original autocorrelation function (3.7) expanded using (3.1) and (3.2) to obtain

$$
\chi_{xx}(\tau, \nu) = \int_{-\infty}^{\infty} \sum_{m, k, m', k'} c_{mL} c_{m'L} e^{j \frac{2\pi}{T_U} (t - T_G - \ell T_S - m L T_S)} w_m(t) w_{m'}(t - \tau) e^{-j2\pi k' T_U} dt
$$

$$
= \sum_{m, k, m', k'} c_{mL} c_{m'L} e^{j \frac{2\pi}{T_U} (k - k') T_G} e^{-j2\pi k' T_S (t - \ell') + (m - m') L} e^{-j2\pi \nu T_S (t + m L)}
$$

$$
= \chi_{ww} \left( \tau - (\ell - \ell') T_S - (m - m') L T_S, \nu - \frac{(k - k')}{T_U} \right)
$$

(3.11)
where

$$\chi_{ww}(\tau, \nu) = \int_{-\infty}^{\infty} w(t)w^*(t-\tau)e^{-i2\pi\nu t} dt.$$  \hspace{1cm} (3.13)

The data symbols have zero mean and unit variance, $E[|c_{m\ell k}|^2] = 1$, and are (nearly) independent if any of $k' \neq k$, $\ell' \neq \ell$, or $m' \neq m$ hold, meaning $E[c_{m\ell k}c_{m'\ell' k}^*] \approx 0$ if any of these conditions hold. The (normalized) sum over terms corresponding to data symbols with $k' \neq k$ will therefore approach zero for large $K$. Similarly for $\ell' \neq \ell$ and $m' \neq m$, the (normalized) sum over data symbols, including those on the same subcarrier ($k' = k$), approaches zero for large $K$. The term $\chi_{ww}\left(\tau - (\ell - \ell')T_S - (m - m')LT_S, \nu - \frac{(k-k')}{T_U}\right)$ is a cross-correlation of the window of each signal and captures the alignment of the reference and surveillance signals. Of course, $\chi_{xx}(\tau, \nu)$ has a large maximum when $\tau = 0$ and $\nu = 0$ that decays as $\tau$ and $\nu$ move away from 0 because $\chi_{ww}\left(\tau - (\ell - \ell')T_S - (m - m')LT_S, \nu - \frac{(k-k')}{T_U}\right)$ is maximum for the terms corresponding to $m' = m$, $\ell' = \ell$ and $k' = k$. This gives rise to the main peak in the AF. The term $\chi_{ww}\left(\tau - (\ell - \ell')T_S - (m - m')LT_S, \nu - \frac{(k-k')}{T_U}\right)$ also has peaks from the cross terms ($m' \neq m$, $\ell' \neq \ell$ and $k' \neq k$) when $\tau \geq T_S$. The data carriers sum incoherently in this case, but the pilot symbols can sum coherently. Other conditions give rise to peaks for $\tau < T_S$ and are described below. The key for the appearance of a peak in the AF is twofold: $\chi_{ww}(\tau, \nu)$ has a peak, and a sufficient number of terms in the sum (3.12) corresponding to that peak sum coherently.

Pilot carriers give rise to both intra-symbol ($\tau < T_S$) ambiguities, and inter-symbol ($\tau \geq T_S$) ambiguities. Pilot signals are boosted in power, and the value of $c_{m\ell k}$ is constant in $k$. Specifically for pilot positions $\ell, k$ defined by the standard, $E[c_{m\ell k}c_{m'\ell' k}^*] = \frac{16}{9}$ for $k = k'$ and $E[c_{m\ell k}c_{m'\ell' k}^*] = 0$ for $k' \neq k$. Thus terms corresponding to continual pilots, or scattered pilots on the same carrier, will contribute significantly to $\chi_{xx}(\tau, \nu)$. According to (3.12), the ambiguity peaks caused by

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3A reminder that the $c_{m\ell k}$ corresponding to data carriers are assumed independent. The exact statistics are not known, but the correlations are very small.

4The normalized sum specifically decays as $\frac{1}{K}$ for independent terms.
pilot signals occur when

\[ k = k' \quad \text{and} \quad \frac{k'}{T_U} (\tau - T_S (\ell - \ell' + (m - m')L)) - \nu T_S (\ell + mL) = d \]  

(3.14)

where \( d \in \mathbb{Z} \) is an integer, and the corresponding \( \chi_{\text{ww}}(\cdot, \cdot) \) is significant. The relation (3.14) is derived from (3.12) in Appendix A.1. The peaks in the ambiguity function arise when a significant number of pilot carriers satisfy the above conditions.

### 3.4.2 Sources of Ambiguity Peaks

**Continual Pilots**

Consider the continual pilots. The particular subcarriers \( k = k' \) are defined by the standard [30] and are not related to \( \ell \) or \( m \). According to (3.14), ambiguities can potentially occur for \( \tau = p T_U + z T_S \) and \( \nu = q \frac{1}{T_S} \) for integers \( p, q, \) and \( z \). First, the intra-symbol (\( \tau < T_S \)) ambiguities correspond to \( p = 0, 1 \) and \( z = 0 \). Ambiguity peaks are observed at \( T_U = 1024 \mu s \) and \( T_S = 1152 \mu s \) in Figure 3.3 as expected. Peaks are also observed at a Doppler of \( \frac{1}{T_S} = \pm 868 \) Hz, which correspond to \( q = \pm 1 \).

Note that a delay of 1024 \( \mu s \) corresponds to a target distance of at least 153.6 km. Figure 3.3 also clearly shows additional ambiguity peaks at zero-Doppler and these are from the scattered pilots. In fact the peak at 1024 \( \mu s \) and the peak at 868 Hz both result from contributions from both the continual and scattered pilots (as well as the guard interval), as discussed in the following subsections.

**Scattered Pilots**

The scattered pilots occur, according to the DVB-T standard, at carrier positions \( k = 3(\ell \mod 4) + 12q \) for integer \( q \) and \( 0 \leq k < K \). The contributions to the sum across \( k \) in (3.12) comes from every 12\(^{th} \) carrier and are significant since \( K/12 \gg 1 \) for both
2k- and 8k-mode signals. Let $\ell = 4a + c$ and $\ell' = 4a' + c'$ for integers $a$ and $a'$ and $c, c' = 0, 1, 2, 3$.

If $k = k'$, then $c = c'$ and, thus, $k = k' = 3b + 12q$ for integers $b$ and $q$. For a peak, substituting these relations into (3.14) yields

$$
\frac{(3b + 12q)}{T_U} (\tau - T_S(4(a - a') + (m - m')L)) - \nu T_S(4a + b + mL) = d
$$

where $w$ is an integer. Noting that $L = 68 = 4 \times 17$ yields

$$
\frac{(3b + 12q)}{T_U} (\tau - 4T_S[(a - a') + 17(m - m')]) - \nu T_S(4(a + 17m) + b) = d.
$$

Now consider the various solutions of (3.16). The largest contributions to the sum (3.12) occur for solutions that contribute for all $a$ and $c$ (i.e., for all $\ell$). The solutions to (3.16) are of the form

$$
\tau = T_U \left( \frac{f}{12} + \frac{g}{3} \right) + z4T_S, \quad \nu = \frac{1}{T_S} \left( \frac{f}{4} + h \right)
$$

for $f = 0, 1, 2, 3$, and integers $g, h$, and $z = (a - a') + 17(m - m')$. This relation is derived in Appendix A.2. Note that for $z > 0$, the delay is rather large ($\tau \geq 4T_S$), and targets at these ranges are likely to be very weak and beyond the range of interest.

For the simulated case with $T_G/T_U = \frac{1}{5}$, the analysis predicts intra-symbol ambiguities resulting from the scattered pilots at the following ($\tau, \nu$) combinations: $(85\mu s, 217 \text{ Hz})$, $(85\mu s, -651 \text{ Hz})$, $(171\mu s, 438 \text{ Hz})$, $(171\mu s, -438 \text{ Hz})$, $(256\mu s, 651 \text{ Hz})$, $(256\mu s, -217 \text{ Hz})$, $(341\mu s, 0 \text{ Hz})$, $(341 \mu s, \pm 868 \text{ Hz})$, and so on extending in the delay-Doppler space. These ambiguities are highlighted in Figure 3.4. Smaller ambiguity peaks also arise from less frequent solutions to (3.16). For example ambiguity peaks from scattered pilots at zero-Doppler and $\tau = \frac{T_U}{12}$, $\frac{T_U}{9}$, $\frac{T_U}{6}$, $\frac{T_U}{4}$, and so on, are clearly seen in Figure 3.3.
Equation (3.17) also shows that the pattern of ambiguities will be repeated at delays of multiples of $4T_S$. For 8k-mode DVB-T signals $4T_S$ corresponds to a distance ranging from at least 633 km (for a guard interval $\frac{1}{32}$) to 768 km (for a guard interval of $\frac{1}{4}$). As this distance is assumed to be outside of the feasible detection range for a passive radar system, these ambiguities may be ignored. It should be noted that for a 2k-mode DVB-T signal ($T_U = 224 \mu s$), inter-symbol ambiguity peaks at $4T_S$ occur near 1 ms delay and may therefore be within the feasible range of a passive radar system. These ambiguities, though important for 2k-mode DVB-T signals, are not considered practically significant for 8k-mode DVB-T signals.

**Frame and Super-Frame Structure**

Ambiguities arising from frame (68 symbols) and super-frame (4 frames) structure are also expected. For example, the TPS data will be correlated as well as pilot symbols. However, these correspond to very large time offsets (even beyond those of the inter-symbol ambiguities) and are therefore assumed to be outside the feasible operating region of the passive radar and are not considered in this treatment.

**Guard Interval**

Another source of ambiguity peaks is caused by the guard interval, which is a cyclic prefix, i.e., a repetition of the last portion of the useful DVB-T symbol inserted before the beginning of the useful part. Thus the repeated part of the symbol is separated by a delay of $T_U$, or $1024 \mu s$ for the Australian 8k-mode DVB-T signals. The power of these peaks is dependent on the length of the guard interval because a longer guard interval contributes more energy to the autocorrelation. As discussed in [39], the difference in power of these peaks can vary by a factor of up to 18 dB depending on the guard interval length. These ambiguities occur at ranges of greater than 153.6 km and can be mitigated by using the approaches suggested in [33, 65]. Alternatively, if
Figure 3.5: Constant false alarm rate (CFAR) detection history resulting from the use of a matched filter, i.e., a matched cross-correlation. The structure of the pilots causes ambiguities that do not arise from real targets. (Credit: [55]. The radar system and data were provided by J. Palmer (of DSTO) and DSTO [54]. Thanks to J. Palmer of DSTO for the figure!)

the desired system detection range is less than the range of these ambiguities, then no additional processing is required, and they can simply be ignored.

3.4.3 Effect of the Ambiguity Peaks on Target Detection, Tracking, and Classification

When considered in isolation, the ambiguities highlighted in Figure 3.3 may, at best, be considered minor annoyances because

1. they do not occupy a significant portion of the delay-Doppler domain and are therefore unlikely to mask target returns (at least not for very long), and
2. the additional Doppler offsets present in the majority of these ambiguities mean that they will not behave like true targets.

In other words, the range migration rate of a target is not consistent with its Doppler offset, and the detected target should therefore be disregarded by even a fairly rudimentary tracker. Unfortunately, real targets are not all point-like in the delay-Doppler plane and not all of the ambiguities have Doppler offsets. A number of ambiguous target peaks will therefore behave like real targets. With static clutter suppressed, as explained in Section 3.5.1, Figure 3.5 shows a range-Doppler detection history of several moving targets, including a strong return from a propeller driven aircraft (going from 5 km to 10 km), two helicopters (blade flash extending across all Doppler at approximately 2 km and the other obvious at 200 Hz to 600 Hz at 10 km) as well as a number of other aircraft (e.g., −200 Hz and 15 km, 300 Hz and 33 km, etc.). This figure reaffirms the fact that real targets are not all point-like and may even extend in Doppler in such a way as to have portions of their return deceive a tracker. It also highlights the fact that such false targets are much more likely to mask real target returns as they occupy a much greater portion of the range-Doppler domain of interest. Situations and targets such as these justify the need for ambiguity mitigation techniques, in contrast to focusing effort solely on, for example, increasing the sophistication of the tracker.

### 3.5 Proposed Mismatched Cross-Correlation

Mismatched cross-correlation has successfully been demonstrated to improve simulated detection for 2k-mode DVB-T passive radar systems. However, as Section 3.6 shows, the investigations based on directly applying these techniques to real-world signals from an Australian 8k-mode DVB-T system highlight that even greater discrimination is required. In this section, a scheme is proposed to pre-process the
reference and surveillance signals obtained by the passive radar to mitigate the effects of both the ambiguities and the clutter in range-Doppler processing. Like the existing methods for the 2k-mode signals, a mismatched cross-correlation approach is adopted for processing of the 8k-mode signal. A key distinction between the approach described herein and [33,65] is that this approach modifies the reference signal to create a mismatched signal that mitigates the effect of the pilots in terms of the cross-correlation output rather than the input. The method of [14] also considers the output of a mismatched cross-correlation, but assumes direct access to the structure of the reference signal is not available and instead achieves the mismatch by filtering the measured reference signal directly. Also, in contrast to these other methods, this approach uses carefully crafted demodulation and remodulation of the reference signal [3,4], allowing for additional benefits of clutter mitigation and noise attenuation in the reference channel. The techniques for demodulation and remodulation accounting for deterministic imperfections of the transmitted signal were first proposed in [67]. The clutter suppression technique was proposed and analyzed in [55].

A special thanks to S.J. Searle of the University of Melbourne, Australia for the bulk of the analysis of these techniques.

Figure 3.6 shows the block diagram for the proposed processing scheme. The reference signal is first demodulated via a standards-based DVB-T demodulator in order to retrieve an error-free replica of the transmitted data, as well as estimates of the channel and clock offset and a coarse estimate of the phase offset. This error-free replica is then remodulated as per the DVB-T standard to obtain a somewhat accurate estimate of the transmitted signal. This remodulated signal is compared to the original raw reference signal in order to estimate the fine phase offset, i.e., the relative drift that exists between the local oscillators of the transmitter and the passive radar receiver. The error-free digital data is then remodulated in parallel by two more sophisticated remodulation processes, which both take into account any
local oscillator drift. The first remodulation process uses full-power pilots to generate a clutter-free and noise-free reference signal for use in adaptive clutter cancellation in the surveillance signal. The second remodulation process attenuates the pilots to produce a mismatched reference signal for use in cross-correlation. These processes are described in detail in the following subsections.

3.5.1 Matched Processing: Demodulation, Remodulation and Clutter Suppression

The purpose of demodulating and remodulating the LOS reference signal is to provide a clean template for radar ambiguity processing, with no delayed DP multipath (clutter) or noise. While beneficial, unless caution must be taken with this processing as incoherencies in the ambiguity function may arise.
Demodulation

A second time variable $\varphi$ is introduced to allow for any (non-stationary) difference between the transmitter and receiver clocks. When the receiver believes the time to be $t$, the transmitter believes it to be $\varphi$. The relationship between the transmitter and receiver clocks is assumed to be piecewise affine, namely that

$$\varphi_{m\ell}(t) = \frac{T'_{m\ell}}{T_\Delta} t + \rho_{m\ell}$$

(3.18)

where $T_\Delta$ is the sampling period at the receiver and $T'_{m\ell}$ is the apparent sampling period at the transmitter during the $(m, \ell)$ symbol period, and $\rho_{m\ell}$ is the offset. Note that transmitter time $\varphi$ is thus a function of frame and symbol indices $(m, \ell)$, which in turn depend upon $t$. This model allows for the clock disparity to drift from one symbol to the next.

Revisiting the baseband reference signal $r(t)$ from (3.4), the $\alpha_i(t)$ factors are divided into complex gain $a_i$ and filter/channel effects $h(t)$. In the LOS signal the clutter returns are of relatively low power. For the purposes of demodulation, these components appear as noise on the (communication) information–bearing main return and are subsumed into the noise term $\xi_r(t)$. The received LOS reference signal $\tilde{r}(t)$ is thus

$$\tilde{r}(t) = a_0 h(t) x(\varphi_{m\ell}(t) - \tau_0) + \xi_r(t)$$

(3.19)

Express (3.19) in the same form as (3.1), in terms of its individual subcarriers, as

$$\tilde{r}(t) = \sum_{m, \ell} e^{j2\pi f_c(\varphi_{m\ell}(t) - \tau_0)} \sum_k d_k c_{m\ell k} \psi_{m\ell k} (\varphi_{m\ell}(t) - \tau_0) + \xi_r(t).$$

(3.20)

Each term now includes a phasor to account for the effect of carrier modulation and demodulation, where $f_c$ is the carrier frequency, $c_{m\ell k}$ and $\psi_{m\ell k}$ are as defined in (3.1) and (3.2), and $d_k$ is the frequency domain channel coefficient for the $k^{th}$ subcarrier.
The coefficient \( d_k \) encompasses the effect of \( a_0 \) and \( h(t) \), so \( d_k = a_0 H(k) \) where \( H(k) \) is the Fourier transform of \( h(t) \) at frequency \( k \). The coefficient \( d_k \) is assumed constant within the demodulation period \( T_{\text{demod}} \), which is at least as large as the OFDM symbol period \( T_s \). It is clear from (3.20) that an effect of clock disparity is to induce a small frequency offset \( f_{ml}^o \) upon carrier demodulation with

\[
f_{ml}^o = f_c \times \frac{d}{dt} \phi_{ml}(t)
\]

(3.21)

where the time difference is

\[
\phi_{ml}(t) = \varphi_{ml}(t) - t.
\]

(3.22)

In order to demodulate the signal it is necessary to synchronize to the beginning of an OFDM symbol both in time and in frequency. The delay to the start of an OFDM symbol can be estimated using the methods described in [67], and the signal is offset and has its phase compensated accordingly. The residual frequency offset described above must also be estimated and corrected in order for the individual OFDM subcarriers to align with the discrete Fourier transform (DFT) bins. For demodulation purposes, the clock drift is assumed small over \( T_{\text{demod}} \) and can be approximated by a constant (i.e., \( T_{ml}^r = T_{\text{const}} \) within \( T_{\text{demod}} \)). The resulting constant frequency offset is estimated to within a fraction of a bin via the methods described in [67]. The corrected signal becomes

\[
\tilde{r}(t) = \sum_{m\ell} e^{i2\pi f_c \tilde{\phi}_{ml}(t)} \sum_{k} d_k c_{m\ell k} \psi_{m\ell k}(t + \phi_{ml}(t)) + \xi_r(t)
\]

(3.23)
where \( \tilde{\phi}_{m\ell}(t) \) is the estimation error

\[
\tilde{\phi}_{m\ell}(t) = \phi_{m\ell}(t) - \hat{\phi}_{m\ell}(t).
\] (3.24)

The received signal \( \tilde{r}(t) \) is then sampled with a period of \( T_\Delta \). Let \( r_{m\ell}[n] \), \( n = 0, \ldots, N_U - 1 \) be the portion of \( \tilde{r}(t) \) corresponding to the useful samples of the \( (m, \ell) \)-th symbol, i.e., the portion beginning immediately after the guard interval samples. The contributions to this portion come solely from the \( (m, \ell) \)-th term of the summation, so the other terms may be ignored, and results in

\[
r_{m\ell}[n] = \tilde{r}(nT_\Delta + \delta_{m\ell})
\] (3.25)

\[
e^{2\pi f_c \tilde{\phi}_{m\ell}(nT_\Delta + \delta_{m\ell})} \sum_k d_k c_{m\ell k} \psi_{m\ell k}(\varphi_{m\ell}(nT_\Delta + \delta_{m\ell})) + \xi_r(nT_\Delta + \delta_{m\ell})
\] (3.26)

where \( \delta_{m\ell} \) is defined in (3.3). Since \( \frac{T_{m\ell}}{T_\Delta} \approx 1 \), the DFT of \( r_{m\ell} \) can be approximated as

\[
R_{m\ell}[k'] \approx \sum_k d_k c_{m\ell k} e^{i \frac{2\pi}{N_U} k' n} e^{i 2\pi f_c (nT_\Delta + \delta_{m\ell})} e^{i \frac{2\pi}{N_U} (k-k') n} + \Xi_{m\ell}[k']
\] (3.27)

where \( \Xi_{m\ell}[k'] \) is the DFT of \( \xi_r(nT_\Delta + \delta_{m\ell}) \). The phasors in the above arise solely due to the clock disparity between transmitter and receiver. The phase distortion is large enough to hamper correct recovery of the transmitted data \( c_{m\ell k} \). As previously mentioned, for demodulation the disparity is assumed to be constant across symbols within \( T_{\text{demod}} \). This constant offset is estimated and used to correct the Fourier bins. Any remaining phase artifacts due to the approximation tend to be small and may be subsumed into the \( d_k \) factor or otherwise ignored without affecting demodulation performance. Due to the orthogonality of the subcarriers, the sum over \( n \) is approximately zero when \( k \neq k' \), and has magnitude of \( N_U \) when \( k = k' \), so

\[
R_{m\ell}[k] \approx N_U d_k c_{m\ell k} + \Xi_{m\ell}[k].
\] (3.28)
Equalization of the channel is performed by averaging over the constant and scattered pilot bins to estimate the channel at these frequencies and interpolating to obtain a set of estimates between these frequencies. The Fourier coefficients are then scaled by these channel estimates \( \hat{d}_k \) before attempting a recovery of the message symbols at each carrier bin of the OFDM symbol. Specifically,

\[
\hat{c}_{m\ell k} = \arg \min_{c \in \mathcal{C}} \left( \frac{R_{m\ell}[k]}{N_T d_k} - c \right)^2
\]  

(3.29)

where \( \mathcal{C} \) is the set of all symbols in use (e.g., QPSK, 16-QAM, or 64-QAM as specified by [30]). Message symbol estimation via (3.29) is only possible when the LOS signal has a sufficiently high SNR. If the SNR is poor, then it is necessary to employ error correction as prescribed in the DVB-T standard [2, 30] in order to recover the transmitted symbols.

**Ideal Remodulation**

Having obtained an estimate of the transmitted message \( \hat{c}_{m\ell k} \), estimates of the channel response \( \hat{d}_k \), and an estimate of the clock disparity at each symbol (\( \hat{T}_{m\ell} \) and \( \hat{\rho}_{m\ell} \)), then one can produce a replica of the LOS signal at baseband with no noise or clutter. The samples of the reconstructed signal are

\[
q[n] = \sum_{m\ell} q_{m\ell}[n - \delta_{m\ell}/T_\Delta]
\]  

(3.30)

where \( q_{m\ell}[n] \) is the remodulated \((m, \ell)\) symbol

\[
q_{m\ell}[n] = \sum_k \hat{d}_k \hat{c}_{m\ell k} \psi_{m\ell k}(\varphi_{m\ell}(nT_\Delta + \delta_{m\ell})).
\]  

(3.31)
The cross-correlation of this reconstructed signal with the received (i.e., raw) LOS signal at baseband is written, ignoring noise for simplicity, as

\[ \chi_{rq}[\kappa, \nu] = \sum_{n=0}^{N_s-1} r[n] q^*[n - \kappa] e^{-i2\pi \nu n} \]  
\[ = \sum_{m} \sum_{k,k'} d_k \hat{d}_{k'}^* c_{m\ell k}^* \]  
\[ = N_s^{-1} \sum_{n=0}^{N_s-1} e^{i2\pi \left\{ f_{c} \hat{\phi}_{m\ell}(nT\Delta + \delta_{m\ell}) + \frac{k}{T\Delta} \left( \phi_{m\ell}(nT\Delta + \delta_{m\ell}) - \delta_{m\ell} \right) - \frac{\nu}{T\Delta} \left( \hat{\phi}_{m\ell}((n-\kappa)T\Delta + \delta_{m\ell}) - \delta_{m\ell} \right) \right\} + \nu n} \]

where the delay hypothesis is \( \tau = \kappa T\Delta \).

When the remodulated signal is perfectly matched to the LOS signal then \( \hat{c}_{m\ell k} = c_{m\ell k}, \hat{d}_k = d_k, \hat{T}_{m\ell}' = T_{m\ell}' \), and \( \hat{\rho}_{m\ell} = \rho_{m\ell} \) so that \( \hat{\phi}(\cdot) = 0 \). At zero-Doppler (\( \nu = 0 \)), the \( k = k' \) term will dominate, contributions from all other terms is small, and the ambiguity function is approximately

\[ \chi_{rq}[\kappa, 0] \approx \sum_{m} \sum_{k} |d_k c_{m\ell k}|^2 \sum_{n} e^{i2\pi k [\phi_{m\ell}(nT\Delta + \delta_{m\ell}) - \phi_{m\ell}(nT\Delta + \delta_{m\ell}) + \kappa T\Delta ]}. \]  

At zero-delay and zero-Doppler \( [\kappa = 0, \nu = 0] \) the main peak is

\[ \chi_{rq}[0, 0] = N_s \sum_{m\ell k} |d_k c_{m\ell k}|^2. \]  

Naïve Remodulation

The remodulation can also be performed by naïvely substituting the estimated symbols \( c_{m\ell k} \) back into the transmitted signal model (3.1). This is how the signal is remodulated in a communications application, e.g., in a repeater. In this case, clock disparity or channel effects are not considered, so the remodulation uses \( \hat{\rho}_{m\ell} = 0, \hat{d}_k = 1, \hat{T}_{m\ell}' = T_{\Delta} \), and assuming perfect extraction of the information symbols, \( \hat{c}_{m\ell k} = c_{m\ell k} \). The cross-correlation of the LOS signal with the remodulated baseband
signal at zero-Doppler ($\nu = 0$), ignoring the small $k \neq k'$ terms, is approximately

$$\chi_{r\kappa}[^{\kappa, 0}] \approx \sum_{m\ell} \sum_k d_k |c_{m\ell k}|^2 \sum_n e^{i2\pi \left[ f_c \phi_{m\ell}(nT\Delta + \delta_{m\ell}) + \frac{k}{T_U} (\phi_{m\ell}(nT\Delta + \delta_{m\ell}) + \kappa\Delta) \right]}$$  \hspace{1cm} (3.36)

$$= \sum_{m\ell} \sum_k d_k |c_{m\ell k}|^2 e^{i2\pi \left[ (f_c + \frac{k}{T_U})\phi_{m\ell}(\delta_{m\ell}) + \frac{k}{T_U} \kappa\Delta \right]} \sum_n e^{i2\pi (f_c + \frac{k}{T_U})(T'_{m\ell} - T\Delta)n}.$$  \hspace{1cm} (3.37)

Compared to the cross-ambiguity of the perfectly remodulated signal with the LOS signal (3.34), this is a sum of similar magnitude terms. However, the terms are each degraded and do not sum coherently due to the symbol-dependent complex phasors. Ideally, the quantity $f_c(T'_{m\ell} - T\Delta)$ is small, allowing (3.37) to be evaluated at the main lobe ($\kappa = 0$, $\nu = 0$)

$$\chi_{r\kappa}[0, 0] = N_S \sum_{m\ell} \sum_k d_k |c_{m\ell k}|^2 e^{i2\pi (f_c + \frac{k}{T_U})(\delta_{m\ell}) + \frac{N_S-1}{2}(T'_{m\ell} - T\Delta))}$$  \hspace{1cm} (3.38)

which unlike (3.35) is an incoherent sum. The implication is that using a na"{i}vely remodulated signal causes significant leakage of power from returns in the processing output.

**Actual Remodulation**

One cannot simply remodulate the recovered symbols as in a communication system; rather, one has to account for the channel effects and the transmitter effects (i.e., clock disparity) in order to obtain a properly matched template for radar ambiguity processing. To this end, an estimate of the clock rate errors ($\hat{T'}_{m\ell} - T\Delta$) and offsets $\hat{\rho}_{m\ell}$ for all symbols is required. An estimate of the channel response $\hat{d}_k$ at all frequencies is also required but is obtained from the channel equalization process during demodulation. Clearly, commercial off-the-shelf DVB-T receiver hardware cannot be directly used for demodulation and remodulation since these estimates are neither
recorded, nor used, in typical communications applications. Such hardware cannot
account for channel or clock-rate variations as required by the passive radar system.

One solution is to estimate $T'_\text{demod}$ and $\rho_{\text{m} \ell}$ independently at each symbol, or perhaps
track these values across symbols with a model of clock drift. However, a simpler
solution found in \cite{67} that works sufficiently well with real-world signals is to estimate
an average constant clock offset across the entire $T_{\text{demod}}$ (i.e., $\hat{T}'_{\text{m} \ell} = \hat{T}'_{\text{const}}$) and
remodulate the signal accordingly as $\hat{q}[n]$. The residual phase difference between this
improperly remodulated signal and a properly remodulated $q[n]$ is assumed to be
constant within a symbol, but varying from symbol to symbol. This is consistent
with the piecewise affine model of the clock discrepancy in (3.18). The residual
phase difference in the $(m, \ell)$ symbol $\theta_{\text{m} \ell}$ is estimated by correlating the improperly
remodulated symbol $\hat{q}_{\text{m} \ell}$ with the received symbol $r_{\text{m} \ell}$. The remodulated signal $q[n]$ is then formed by introducing the estimated phase disparity into the improperly
modulated signal. Finally,

$$\hat{\theta}_{\text{m} \ell} = \angle \sum_{n=-N_G}^{N_U-1} \hat{q}_{\text{m} \ell}[n] r_{\text{m} \ell}^*[n]$$

(3.39)

where $N_G$ is the number of samples in the guard band, and

$$q_{\text{m} \ell}[n] = \hat{q}_{\text{m} \ell}[n] e^{-i\hat{\theta}_{\text{m} \ell}}.$$

(3.40)

**Clutter and DPI Suppression**

The DVB-T waveform has an ambiguity function that contains a pedestal, which
is observed in the AF in Figure 3.3. This correlation floor extends uniformly in
all delay and Doppler at a level of approximately $10 \log_{10}(B \cdot T_{\text{cpi}})$ beneath the main
peak, where $B$ is the signal bandwidth. The DVB-T waveform is primarily comprised
of data carriers containing random data symbols. The pilot carriers produce the ambiguity peaks already discussed. For the following discussion, assume one super-frame of data is processed, so $T_{cpi} = 4 \times 68 \times T_S$. The random data symbols are spread in time across symbols, of duration $T_{cpi}$, and are spread in frequency across carriers, covering a bandwidth $B$. The number of independent symbols in the cross-correlation sum (at non-zero delay and Doppler) is roughly the product $B \cdot T_{cpi}$. The power of the cross-correlation at non-zero delay and Doppler is thus proportional to $\frac{1}{BT_{cpi}}$ and the magnitude of the channel coefficient of the strongest return. This means that the DPI and strong clutter limit the detection of weak targets (whether small and close, or at a distance) by the passive radar system. As a result, DPI and clutter suppression are required. In this work, filter weights are estimated for suppressing the DPI and clutter via the optimal (in a least-squares sense) Wiener-Hopf equation, or filter. These weights are used to filter the matched reference signal described above, and the result is subtracted from the surveillance signal. This effectively removes the DPI and strong static clutter signal components as evidenced by the clear line at zero-Doppler in Figure 3.5.

### 3.5.2 Mismatched Processing: Ambiguity Mitigation and Range-Doppler Processing

Matching the reference signal to the transmitted signal (and by extension the target returns) by estimating clock, frequency and phase offsets as described above is important for accurate range-Doppler estimation via cross-correlation. In addition, the scheme intentionally manipulates the pilots in the reference signal in order to mitigate the effect of ambiguities in the range-Doppler output caused by the pilots.

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5The data symbols are derived from actual data and deterministic randomization algorithms that produce symbols with very small correlations. The discussion assumes independent symbols, but symbols with small correlation produce results very similar to independent symbols.

6$T_{cpi}$ is chosen to match the simulated data presented in this chapter. Other choices might be more appropriate given practical considerations.
Figure 3.7: Cross-correlation of the full 8k-mode DVB-T signal with matched reference signal (top) and with mismatched reference signal (bottom). Here, any points greater than $-55$ dB are represented as blue dots, and any points greater than $-50$ dB are represented as red squares. Most ambiguities are suppressed to less than $-50$ dB. The ridge at $\pm 868$ Hz is suppressed, but a small number of points are still above $-50$ dB because a large number of carriers satisfy (3.14). (Credit: [55].)
A mismatched reference signal is constructed by remodulating the error-free data, including clock and phase offsets, and additionally directly setting the amplitude of the pilot symbols to \( \frac{3}{4} \) instead of the standard defined \( \frac{4}{3} \). Thus, the amplitude of the pilots in the mismatched reference signal is attenuated by a factor of \( \frac{9}{16} \) as compared to the DVB-T standard. This choice is motivated by a consideration of the cross-correlation output. Contributions to the cross-correlation arising from aligned pilots will have power 1 instead of \( \frac{16}{9} \), and are no longer preferential to contributions arising from DVB-T data. Note that this reduces the main zero-delay, zero-Doppler peak by approximately 1 dB, but this loss is very small relative to the reduction in ambiguity peaks.

The cross-correlation delay-Doppler output using the proposed mismatched reference signal is contrasted against using the original DVB-T signal (with boosted pilots as mandated by the standard) in Figure 3.7. The autocorrelation function output (Figure 3.4) is repeated for comparison and highlights the clear advantage of using the new reference signal in mitigating ambiguities. Most ambiguities have been suppressed to below the pedestal, but the ridge at \( \pm 868 \) Hz \( (\pm 1/T_S) \) still rises above \(-50\) dB because a sufficient number of carriers satisfy (3.14) at \( \pm 1/T_S \) at the corresponding delays.

### 3.6 Results

#### 3.6.1 Comparison of Mismatched Techniques

In [33, 65], a parallel processing approach consisting of three stages is employed to achieve a reduction in ambiguity peak power. The three stages are guard interval blanking, pilot equalization, and pilot blanking\(^7\). The guard interval blanking shifts

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\(^7\)Pilot equalization is in the sense of the reference signal only, i.e., reduced to the same level as the data signals, compared with the proposed technique in this chapter that additionally accounts for equalization in the surveillance signal.
the ambiguities resulting from the guard interval (which normally arise at delays of \( \tau \approx T_U \)) to zero-delay. The pilot equalization reduces the ambiguities associated with the intra-symbol pilots, corresponding to small delays of \( \tau < T_U \), and the pilot blanking reduces the ambiguities associated with inter-symbol pilots arising from large delays of \( \tau > T_S \). These methods are demonstrated for the 2k-mode DVB-T signals in [33, 65].

The results presented here concentrate on pilot equalization because these occur at small delays and are likely the most important in a passive radar system. The method of [33, 65] is applied to 8k-mode DVB-T signals, as well as the proposed method in this chapter. Notice that although the pilot equalization could be implemented as a filter, as proposed in [33, 65], here direct access to the pilots is available as the LOS signal is demodulated and remodulated to create the reference. As a result, and for fairness of comparison, direct specification of pilot amplitudes is the approach taken here for both the method of [33, 65] and for the proposed technique. The first mismatched reference signal, \( q_1(t) \), is obtained by specifying that the pilots in the LOS reference signal each have magnitude one, as per [33, 65]. This method provides a reduction in the ambiguity peaks, but does not eliminate them entirely. Figure 3.8 shows a cross-section in delay and Doppler through one of the ambiguities (85.3\( \mu \)s delay and 217 Hz Doppler). The autocorrelation output of a standard DVB-T signal is included as the baseline. The ambiguity peak power is reduced by 7.7 dB from \(-27.8\) dB to \(-35.5\) dB when comparing the matched reference to \( q_1(t) \). The proposed mismatched reference signal \( q_2(t) \) is then applied to the cross-correlation (i.e., pilots specified to have amplitude \( \frac{3}{4} \)). A reduction of more than 36 dB to below the pedestal \((-64\) dB) is seen when using \( q_2(t) \) in the cross-correlation instead of the matched reference signal. The ambiguity peak is now buried below the pedestal for the 8k-mode DVB-T signal.

The direct manipulation of pilot carriers before remodulation is essentially an ideal bandpass filter. Direct manipulation is thus likely to perform better than an implemented filter.
Figure 3.8: Comparison of methods for reducing ambiguities of a full 8k-mode DVB-T signal: (a) auto-correlation (matched filter), (b) cross-correlation with $q_1(t)$ \cite{33, 65}, and (c) cross-correlation with proposed method in this chapter $q_2(t)$. The pedestal at $10 \log_{10}(B \cdot T_{cpi}) = -64$ dB highlights how well the proposed method attenuates the ambiguity peak. It is attenuated to the pedestal or even below! The peak shown here is at $85.3 \mu s$ delay and $217$ Hz Doppler. Similar results are seen at other ambiguity peaks. (Credit: \cite{55}.)
3.6.2 Real-World Application

Figure 3.9 shows the effect of the proposed processing scheme applied to the real-world data described in Section 3.4.3. This figure shows the result of processing both with the matched reference, which is reproduced from Figure 3.5 for comparison, and with the proposed mismatched reference. The comparison clearly demonstrates the positive impact and value of ambiguity mitigation. For example, in the frame with matched filtering, the detections of the real targets located at approximately 50 Hz and 20 km range and at approximately 300 Hz and 30-35 km range are competing with, or are being masked by, ambiguous returns. In contrast, when the mismatched reference is used (bottom figure) the detections for both of these targets are unimpeded. Many other examples are also apparent in the figure. Furthermore, in the case of the matched filter, there are ambiguous returns of the helicopter and propeller-driven aircraft at zero-Doppler and 50 km delay that behave like real targets and would therefore likely be tracked by any non-DVB-T specific tracking algorithm. After processing via mismatched filtering, however, all traces of these returns (as well as all of the other ambiguous returns) are unobservable.

3.7 Summary

To conclude this chapter, the use of DVB-T signals is analyzed in passive radar systems, and a new mis-matched processing technique is proposed. A DVB-T signal has deterministic components that cause peaks in the ambiguity function at non-zero delay and varying Doppler offsets. These peaks reduce the effectiveness of the radar system as they mask targets at these offsets and/or contribute to false tracks in the radar system. A detailed analysis of the DVB-T waveform is presented that shows how the deterministic components of the waveform cause these ambiguities, and a scheme is proposed for pre-processing both the reference and surveillance signals obtained
Figure 3.9: Constant false alarm rate (CFAR) detection history with a matched filter (top) and with a mismatched filter (bottom). The mismatched filter produces a much cleaner detection history. The radar system and data were provided by J. Palmer (of DSTO) and DSTO [54]. (Credit: [55]. Thanks to J. Palmer of DSTO for the figure!)
by the passive radar to mitigate the effects of the ambiguities and clutter in range-Doppler processing. The effectiveness of the proposed scheme in enhancing target detection is demonstrated by using it on real-world data taken from an (Australian) 8k-mode DVB-T system. An approximately 29 dB reduction in residual ambiguity levels over existing techniques is observed along with an approximately 36 dB reduction over standard matched filtering while only incurring approximately a 1 dB reduction in the zero-delay, zero-Doppler peak. The small reduction in the main peak means that the detection performance for a single target, at least when viewed in isolation, is reduced. However, the performance of a radar system, and by extension the exact metric of optimality, is not just the detection of a single target. In a noisy, multi-target environment in the real-world, a radar system needs to detect and track multiple targets, and mitigating the ambiguous peaks in the cross-correlation output aids in that goal. It is, however, more difficult to place an exact metric on optimality in this case because it is dependent on the scene and environment, which is usually dynamic. This chapter presents a compelling case that the drastic reduction in ambiguous peaks is worth the slight decrease in main peak power, and is closer to optimal in a dynamic, noisy environment. Additionally, this technique places less of a burden on the tracking algorithm and allows the use of a less complicated one, with very little expense to change the amplitude of the pilots in the reference signal.
Chapter 4

Sub-Nyquist Sampling with the Constrained Random Demodulator

4.1 Introduction

4.1.1 Random Demodulation for Sub-Nyquist Sampling

Though Nyquist is the standard approach to sampling, other schemes have been considered that require a lower sampling rate for analog-to-digital conversion. The key to the success of these schemes is leveraging additional prior information about the class of signals to be sampled (perhaps in addition to being bandlimited). One such class of signals corresponds to complex-valued signals comprising a relatively small number of tones $S$ in a very large (two-sided) bandwidth $n$ such that $S \ll n$. These signals are said to have sparse spectral content. This class of signals is of significant interest in applications such as spectrum sensing and is the primary interest in this chapter. The reader is referred to Section 4.2 for a mathematically precise definition of this signal class. Two good architectures to sample such signals are the non-uniform sampler (NUS) [17,82] and the random demodulator (RD) [78] because both architectures assume spectrally sparse input signals and require many fewer samples.
than the Shannon-Nyquist theorem. This chapter concentrates exclusively on the RD because it offers a very general framework for sub-Nyquist sampling. The block diagram of the RD architecture is found in Figure 4.1 and is reviewed in more detail in Section 4.2. The major results for the RD are summarized as follows [78, Theorems 1 and 2]: let $C$ be a positive, universal constant and let $n$ be the Nyquist rate. The constituent tones of signals sampled by the RD can be recovered with high probability if the sampling rate $m$ scales as

- $m \geq C[S \log n + \log^3 n]$ for signals composed of $S$ randomly located tones
- $m \geq CS \log^6 n$ for signals composed of $S$ arbitrarily located tones.

Contrast these results to the Shannon-Nyquist sampling theorem, which guarantees recovery of the original signal from its samples if $m \geq n$.

A building block of the RD is a white noise-like, bipolar modulating waveform $p_c(t)$. This waveform switches polarity at the Nyquist rate of the input signal. An implicit assumption is that this waveform, in the analog domain, is made up of perfect square pulses with amplitude either $+1$ or $-1$. Hardware constraints, however, mean that a real waveform cannot switch polarity instantaneously and will encounter shape distortion. A non-zero time $\tau$ is required to switch polarity and is dictated by the circuits encountered in ADC architecture [20, Ch. 4]. The transitions therefore occur over this time-scale, and the square waveform can be viewed as passing through a low-pass filter with a bandwidth proportional to $1/\tau$. One implication is a reduction of the energy captured in the measurements that depends on $\tau$ and the number of transitions in the waveform. For a larger $\tau$, or for more transitions in the waveform, less energy is captured in the measurements.

Over 30 years ago a similar problem affected the peak detection of binary signals written on magnetic media. In magnetic recording, data is recovered by passing a

\footnote{While the focus is exclusively on a single-channel system, the analysis is readily extended to the multi-channel setting.}

\footnote{It is worth noting here that the NUS is shown to have similar results [17, Theorem 1.3].}
read head over the media; a higher recording density means there is greater interference between the read-back voltages of adjacent binary symbols. To reduce distortion in the read-back voltages, Tang and Bahl introduced run-length limited (RLL) sequences [72]. Run-length constraints specify the minimum separation \( d \) and the maximum separation \( k \) between transitions from one symbol to another (e.g., +1 to −1). Tang and Bahl proposed using these RLL sequences to increase the number of binary symbols written on the magnetic medium by a factor of \( d + 1 \) without affecting the read-back fidelity. Note that RLL sequences, compared to unconstrained sequences, require a longer length to store the same amount of information, so each binary symbol contains less information. Tang and Bahl nonetheless observed that for certain RLL sequences the fractional increase in length is smaller than \( d + 1 \), leading to a net increase in recording density because the closer spacing of the physical bits delineated on the magnetic medium overcomes the increase in bit-sequence length. The reader may refer to [43] for further details and a nice overview on this topic.

### 4.1.2 Main Contributions

This chapter is primarily based on work previously published in [35, 36, 38] and covers two major contributions to the area of sub-Nyquist sampling for signals with sparse spectral content. The first contribution is to apply the lessons learned from magnetic
recording to the RD. Specifically, the modulating waveform of the RD is replaced with a \((d, k)\)-constrained waveform generated from an RLL sequence (see Figure 4.2 for a comparison of these two waveforms). Such a sampling system is termed a \textit{Constrained Random Demodulator} (CRD). The use of an RLL sequence reduces the average number of transitions in the waveform by a factor of \(d+1\), which results in an increase in the signal energy captured by the hardware. From another viewpoint, if the acceptable energy loss (or average number of transitions in the waveform) is fixed, then using an RLL sequence allows a larger input signal bandwidth. A price has to be paid however as an RLL sequence introduces statistical dependence across the waveform. The first major contribution therefore establishes that the CRD still enjoys some theoretical guarantees for certain choices of waveform. In fact, the analysis explicitly shows that the power spectrum of the waveform is the key to understanding these guarantees and, hence, to choosing the best RLL sequence. Further, the tradeoff in acquirable bandwidth versus sparsity of the input signal is outlined, and numerical simulations show that a 20% increase in the bandwidth can be handled by the CRD with a negligible decrease in average performance. These results are mostly contained in [38], which builds upon the preliminary work in [35,36] that was primarily limited to introducing the idea of the CRD along with Theorem 1 without proof. This chapter more fully explores the results and explains their significance.

\textit{Remark} 1. Heuristically, the theoretical guarantees of the RD rely on two things:

1. each (active) tone leaves an identifiable signature that is extracted from the measurements by the recovery procedure, and

2. the measurements capture a significant amount of energy of each tone.

First, the identifiability is shown to depend on the modulating sequence power spectrum. Once this is established, the energy captured by the measurements should be maximized. Since an RLL waveform leads to an increase in the captured energy...
because of the switching constraints previously discussed, its use in a hardware implementation will lead to improved performance as long as it satisfies the identifiability criterion.

The second contribution is describing the foundations of a concept termed *knowledge-enhanced compressive measurements* (KECoM) for sub-Nyquist sampling, which is presented in [38] and preliminarily explored in [36] with limited numerical experiments. In the context of the CRD, the principle of KECoM assumes that some tones in the input signal are statistically more likely to appear than others. An immediate application of this is a spectrum sensing problem where some regions of the spectrum are assigned a higher detection priority than others, but none are deemed uninformative. Numerical simulations show that the distribution of the tones in the input signal has a profound effect on the reconstruction of input signals from samples collected using a CRD. Specifically, phase transition plots [28] show that if the prior distribution over the tones matches the power spectrum of the RLL sequence used by the CRD, then the reconstruction performance improves when
compared to a uniform distribution over the tones. Note that [46, 59] explore ideas along similar lines, albeit for a different class of sequences. In contrast to [46, 59], this presentation provides a theoretical analysis and a comprehensive numerical analysis of RLL sequences in the RD by examining the phase transition plots both in the case of noise-free measurements and noisy measurements.

4.1.3 Other Sub-Nyquist Sampling Schemes

The work of Rife and Boorstyn [62] in the mid-70’s is an early example of a successful sub-Nyquist sampling scheme. Their goal was to take samples of a sinusoid at a sub-Nyquist rate and then perform parameter estimation to determine the amplitude, frequency, and phase of a single, unknown tone. They also extended their work to the case of multiple tones in a large bandwidth [63]. Their work, however, becomes intractable when considering more than a couple tones. This is an early example of what has become known as compressed sensing of sparse signals. Compressed Sensing (CS) is the systematic exploration of sparsity as a prior model for input signals and recovery of these signals from a small number of linear measurements [29]. It has produced many analytical tools and algorithms for signal recovery. In addition to the RD, several other sub-Nyquist sampling architectures have taken advantage of ideas from CS including Chirp Sampling [1] and Xampling [47]. The Xampling architecture assumes that a signal is divided up into a large number of frequency bands, and only a small number of these bands contain signal energy. The architecture contains multiple channels in which the input signal is modulated with a random waveform, low-pass filtered, and finally sampled at a rate proportional to the number of active bands. The recovery procedure then recovers the locations of the active bands, and then reconstructs the signal.

While the RD considers a bandlimited input signal model with few active tones, several other classes of signals have been considered in the literature with the goal
of finding more efficient sampling methods. One such class contains signals with so-called finite rates of innovation \[81\]. Signals belonging to this class are described by a finite number of degrees of freedom over a given time interval, and it has been shown that they can be reconstructed from a small number of samples that is proportional to the degrees of freedom in that time interval. Another class constitutes signals in shift-invariant subspaces. These signals are composed of a superposition of shifted generator functions (e.g., splines or wavelets); see \[80\] for a nice overview of this signal class. In \[11\] and \[12\], this signal model is shown to provide an alternative to the bandlimited signal model; in particular, it allows the reconstruction of signals belonging to Sobolev spaces with an approximation error that scales polynomially with the sampling period.

One possible drawback to utilizing the RD for sampling is its assumed discrete-frequency signal model (cf. Section 4.2). Specifically, the RD assumes that the input signal can be described by a discrete set of integral frequencies, while real-world signals are likely to contain tones off this grid. While this signal model might not entirely describe real-world signals, the effectiveness of the RD architecture has been successfully demonstrated in the lab \[51, 86\]. To address signals with tones that do not conform to the integral-frequency assumption, energy leakage is considered in the frequency domain. A tone that does not fall exactly on the assumed frequency grid will leak energy across several tones due to the inherent windowing. The result is that a signal which is S-sparse in the analog domain becomes \((aS)\)-sparse after being sampled, where \(a > 1\). Other schemes, such as Xampling \[47\], offer an alternative approach assuming a different signal model; the pros and cons of both systems are examined in \[45\]. While the focus is exclusively on the RD, the for it could have implications for other sub-Nyquist architectures as well. Specifically, the Xampling architecture uses modulating sequences similar to the ones used in the RD/CRD, and
RLL sequences could very well benefit the Xampling architecture as well. A detailed analysis would make for interesting future work.

RLL sequences could also have utility of in the NUS. The implementation described in [82] requires a minimum and maximum spacing between sample points, while the analysis in [17] assumes the sample points are uniformly random without any constraints. The constraints in [82] can thus be described by an RLL sequence made up of 0’s and 1’s with 1’s representing sampling points and minimum and maximum constraints placed on the number of consecutive 0’s without the appearance of a 1. This interpretation of the physical limitations in [82] could help mathematical analysis of the architecture, but a detailed investigation is beyond the scope of the current analysis.

4.1.4 Organization

The remainder of this chapter is organized as follows. Some background is first provided on the RD in Section 4.2 and the challenges encountered by introducing RLL sequences into the RD architecture are explored in Section 4.3. The main theoretical results are found in Section 4.3 and two examples of constrained sequences follow, one with bad results in Section 4.4 and one with good results in Section 4.5 to illustrate the effectiveness of the analysis. Finally, Sections 4.6 and 4.7 present numerical simulations to offer some verification of the theoretical results.

4.2 Background: The Random Demodulator

The RD architecture of [78] is introduced, and the key components that allow sampling of sparse, bandlimited signals are highlighted. The reader is referred to [78] for a thorough overview. To start, the RD takes samples at a sub-Nyquist rate \( m \) while retaining the ability to reconstruct signals that are periodic, (two-sided) bandlimited...
to \( n \) Hz, and completely described by a total of \( S \ll n \) tones. These conditions describe a large class of wide-band analog signals comprised of frequencies that are small in number relative to the total bandwidth but are at unknown locations.

Formally, the input signal to a RD takes the parametric form

\[
f(t) = \sum_{\omega \in \Omega} a_{\omega} e^{-2\pi i \omega t}, \quad t \in [0, 1)
\]

where \( \Omega \subset \{0, \pm 1, \ldots, \pm n/2 - 1, n/2\} \) is a set of \( S \) integer-valued frequencies and \( \{a_{\omega} : \omega \in \Omega\} \) is a set of complex-valued amplitudes. Figure 4.1 illustrates the actions performed by the RD. The input \( f(t) \) is first multiplied by

\[
p_c(t) = \sum_{i=0}^{n-1} \varepsilon_i 1_{[\frac{i}{n}, \frac{i+1}{n})}(t),
\]

where

\[
1_{[\frac{i}{n}, \frac{i+1}{n})}(t) = \begin{cases} 
1, & \frac{i}{n} \leq t < \frac{i+1}{n} \\
0, & \text{otherwise}
\end{cases}
\]

is an indicator function, and the discrete-time modulating sequence \( \varepsilon = [\varepsilon_i] \) is a Rademacher sequence, a random sequence of independent entries taking values \( \pm 1 \) equally likely. The continuous-time product \( f(t) \cdot p_c(t) \) is then low-pass filtered using an integrate and dump filter. Finally, samples are taken at the output of the low-pass filter at a rate of \( m \ll n \) to obtain \( y[i] \).

\[\text{---} \]

3The integer \( n \) is assumed to be even. An appropriate change of the set \( \Omega \) would cover the case of \( n \) odd.

4The impulse response of such a filter is a box function of unit amplitude and duration \( \frac{1}{m} \). The frequency response of this filter is a sinc function with a main lobe width of \( m \) and tapering off at higher frequencies. Hence, it is a low-pass filter. The limited extent of this function in the time-domain makes it ideal for the analysis offered here.
4.2.1 Matrix Representation of the Random Demodulator

One of the major contributions of [78] is expressing the actions of the RD on a continuous-time, sparse, and bandlimited signal \( f(t) \) in terms of the actions of an \( m \times n \) matrix \( \Phi_{RD} \) on a vector \( \alpha \in \mathbb{C}^n \) that has only \( S \) nonzero entries. Specifically, let \( x \in \mathbb{C}^n \) denote a Nyquist-sampled version of the continuous-time input signal \( f(t) \) so that \( x_i = f(i/n), i = 0, \ldots, n-1 \). It is then easy to conclude from (4.1) that \( x \) can be written as \( x = F\alpha \), where the matrix

\[
F = \frac{1}{\sqrt{n}} \left[ e^{-2\pi i u \omega / n}\right]_{(u,\omega)}
\]

denotes a (unitary) DFT matrix and \( \alpha \in \mathbb{C}^n \) has only \( S \) nonzero entries corresponding to the amplitudes, \( a_\omega \), of the nonzero frequencies in \( f(t) \). Next, the effect of multiplying \( f(t) \) with \( p_c(t) \) in continuous-time is equivalent in the discrete-time Shannon-Nyquist world to multiplying an \( n \times n \) diagonal matrix \( D = \text{diag}(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}) \) with \( x = F\alpha \). Finally, the effect of the integrating filter on \( f(t) \cdot p_c(t) \) in the discrete-time Shannon–Nyquist setup is equivalent to multiplying an \( m \times n \) matrix \( H \), which has \( n/m \) consecutive ones starting at position \( rn/m + 1 \) in the \( r^{th} \) row of \( H \) and zeros elsewhere, with \( DF\alpha \). An example of \( H \) for \( m = 3 \) and \( n = 9 \) is

\[
H = \begin{bmatrix}
1 & 1 & 1 \\
& 1 & 1 & 1 \\
& & 1 & 1 & 1
\end{bmatrix}.
\]

The RD collects \( m \) samples per second, and therefore, the \( m \) samples collected over 1 second at the output of the RD can be collected into a vector \( y \in \mathbb{C}^m \). It follows from the preceding discussion that \( y = HDF\alpha = \Phi_{RD} \cdot \alpha \) with the complex-valued random demodulator matrix \( \Phi_{RD} = HDF \).

\[5\] Throughout this chapter, it is assumed that \( m \) divides \( n \); otherwise, a slight modification can be made to \( H \) as discussed in [78].
4.2.2 Signal Recovery

Given the discrete-time representation \( y = \Phi_{RD} \cdot \alpha \), recovering the continuous-time signal \( f(t) \) described in (4.1) is equivalent to recovering the \( S \)-sparse vector \( \alpha \) from \( y \). In this regard, the primary objective of the RD is to guarantee that \( \alpha \) can be recovered from \( y \) even when the sampling rate \( m \) is far below the Nyquist rate \( n \). Recent theoretical developments in the area of CS provide us with greedy as well as convex optimization-based methods that are guaranteed to recover \( \alpha \) (or a good approximation of \( \alpha \)) from \( y \), even in the presence of noise, as long as the sensing matrix \( \Phi_{RD} \) satisfies certain geometrical properties \[29\]. Tropp et al. \[78\] uses two properties from the CS literature to analyze the sensing matrix. The first is the coherence. The coherence \( \mu \) of a matrix \( \Phi \) with unit-norm columns is the largest inner product between its columns \( \phi_\omega \)

\[
\mu = \max_{\omega \neq \alpha} |\langle \phi_\omega, \phi_\alpha \rangle|.
\]

(4.4)

Many recovery algorithms rely on the coherence of the sensing matrix being sufficiently small \[76\]. The analysis in \[78\] in this regard also relies on the input signals conforming to a random signal model.

**Definition 1** (Random Signal Model). The random signal model describes the index of tones \( \Omega \) and set of amplitudes \( a_\omega \). Given the parametric form (4.1), the index \( \Omega \) is a set of \( S \) tones drawn uniformly at random from the set of \( n \) possible tones. The coefficients \( a_\omega \) are drawn uniformly at random from the complex unit circle. In the discretized model, the signal vector \( \alpha \) has \( S \) non-zero entries at locations specified by \( \Omega \), with the value of the entries described by \( a_\omega \).

Under this signal model, \[78\] shows that \( S \)-sparse signals are recoverable with high probability if the sampling rate scales as \( m \geq C(S \log n + \log^3 n) \) where \( C \) is a constant that does not depend on \( S \) or \( n \).
The second property used in [78] is the restricted isometry property (RIP) [19].

**Definition 2** (Restricted isometry property). The RIP of order \( S \) with restricted isometry constant \( \delta_S \in (0, 1) \) is satisfied for a matrix \( \Phi \) with unit-norm columns if

\[
(1 - \delta_S) \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_S) \|x\|_2^2
\]

or equivalently

\[
\left| \frac{\|\Phi x\|_2^2 - \|x\|_2^2}{\|x\|_2^2} \right| \leq \delta_S \quad (4.5)
\]

for every \( x \) with \( \|x\|_0 \leq S \). A reminder that \( \|x\|_0 \) counts the number of non-zero entries in \( x \).

Note that RIP-based analysis tends to be stronger than the coherence-based analysis because the RIP provides a better handle on worst-case performance as well as on performance in the presence of noise [18, Theorem 1]. It also provides stable recovery even if the signal is not exactly sparse, but is well-described by a sparse signal (so-called compressible signals) [18, Theorem 2]. The focus in this exposition is therefore on proving the RIP with the understanding that RIP automatically implies stable and robust recovery. [29] and the references therein provide an extensive list of results.

The *triple-bar norm* of [78] provides convenient notation to describe the RIP condition. Given a matrix \( A \) and set of indices \( \Omega \subset \{0, \ldots, n - 1\} \), the triple-bar norm captures the least upper bound on the spectral norm of any \( S \times S \) principal submatrix of \( A \)

\[
|||A||| = \sup_{|\Omega| \leq S} \|A|_{\Omega \times \Omega}\|. \quad (4.6)
\]

It is easily verified that \( ||| \cdot ||| \) is a norm. Positive homogeneity (\( |||cA||| = |c|||A||| \) for any constant \( c \)) and separation of points (\( |||0||| = 0 \) for \( 0 \) the zero matrix) follow immediately from the spectral norm. The triangle inequality also follows from the
spectral norm. Let the matrix $A = B + C$, then

$$
\sup_{||\Omega|| \leq S} ||A|_{\Omega \times \Omega}|| \leq \sup_{||\Omega|| \leq S} (||B|_{\Omega \times \Omega}|| + ||C|_{\Omega \times \Omega}||) \leq \sup_{||\Omega|| \leq S} ||B|_{\Omega \times \Omega}|| + \sup_{||\Omega|| \leq S} ||C|_{\Omega \times \Omega}||
$$

where the furthest term on the left is $|||A|||$ and the further term on the right is $||B|| + ||C||$. The last inequality follows because a separate supremum cannot be worse than a joint supremum over $\Omega$, and the first inequality is the triangle inequality for the spectral norm. It also follows that (4.5) is satisfied if and only if $|||\Phi^* \Phi - I||| \leq \delta S$. To show this, first restate (4.5)

$$
\left| \frac{x^* (\Phi^* \Phi - I_n) x}{x^* x} \right| \leq \delta S
$$

and remember that $||x||_0 \leq S$. The extreme values of (4.7) are captured by the spectral norm ($|||A||| = \max_x ||\frac{Ax}{||x||}||$), but since $X$ has only $S$ non-zero entries, it suffices to find the supremum over all $x$ satisfying $||x||_0 \leq S$, which means over all principal sub matrices of $A$ of size $S \times S$.

The main result of [78] in this respect is that the RD matrix satisfies the RIP of order $S$ as long as the sampling rate $m$ scales as $m \geq C S \log^6 n$ where $C$ is a constant that does not depend on $S$ or $n$.

### 4.3 The Constrained Random Demodulator

As described in the previous section, the RD uses a random waveform generated from a Rademacher sequence with transition density of $\frac{1}{2}$ (on average, one transition every two Nyquist periods). However, limitations of analog circuits imply that each transition in the waveform results in a loss of energy compared to a waveform with ideal square pulses [20]. RLL sequences are an attractive way to generate waveforms with a reduced transition density of $\frac{1}{d+2}$ and are also shown to lead to superior performance
for specific classes of input signals. If an RLL sequence is used, then the resulting system is termed a *constrained random demodulator* (CRD) and the corresponding system matrix is denoted $\Phi_{\text{CRD}} = HD$ where $D$ contains an RLL sequence $\varepsilon$ instead of a Rademacher sequence. The properties of the Rademacher sequence, in particular independence, are central to the analysis of the RD in $[78]$; therefore, the impact of using an RLL sequence that is inherently correlated must be carefully considered.

The strength of $[78]$ is showing that the RD matrix satisfies the RIP with high probability, allowing strong guarantees to be made about the recovery of signals sampled with the RD. The RIP is satisfied primarily because the RD matrix satisfies three criteria:

1. the Gram matrix averages (over realizations of the modulating sequence) to the identity matrix,

2. the rows are statistically independent, and

3. the entries are uniformly bounded.

All three properties rely on the independence of the modulating sequence. The CRD introduces dependence across $\varepsilon$ that must be addressed. Nevertheless, the last two properties are handled relatively easily. Specifically, some distance between entries in $\varepsilon$ exists such that any two entries, when separated by this distance, are independent, then the rows of $\Phi_{\text{CRD}}$ (or entries of $\varepsilon$) can be partitioned into sets of independent rows (entries). Bounds similar to those found in $[78]$ are then offered for these sets, and a union bound is taken over all the sets to obtain the desired properties.
4.3.1 Maximum Dependence Distance

To make the previous discussion more concrete, recall that the \((r, \omega)\) entry of \(\Phi_{\text{CRD}}\) is

\[
\varphi_{r\omega} = \sum_{j \sim r} \epsilon_j f_{j\omega}.
\]  

(4.8)

If \(\epsilon\) is an independent sequence, then each \(\varphi_{r\omega}\) is a sum of independent random variables, and each row of \(\Phi_{\text{CRD}}\) is independent. However, if a correlated sequence is used then the rows may not be independent, and it is important to know the extent of the dependence within the sequence.

**Definition 3.** The maximum dependence distance, \(\ell\), for a modulating sequence \(\epsilon\) is the smallest \(\ell\) such that \(E[\epsilon_j \epsilon_{j+k}] = 0\) for all \(j\) and \(|k| \geq \ell\) where the expectation is with respect to the distribution on the modulating sequence entries.\(^6\)

Define \(\rho = \lceil m/n (\ell - 1) \rceil \leq (\ell - 1)\); any two rows of \(\Phi_{\text{CRD}}\) separated by at least \(\rho + 1\) rows are independent. Given \(\rho\) and \(\ell\), it is possible to partition the rows of \(\Phi_{\text{CRD}}\) into \(\rho + 1\) subsets where the rows in each subset are independent.\(^7\) This partitioning scheme transforms the analysis to that of independent rows and allows a union bound to be taken over all subsets. Using \(\ell\), each entry of \(\Phi_{\text{CRD}}\) is similarly shown to be uniformly bounded. The details are in Appendices B.1 and B.2.

4.3.2 The Gram Matrix

Analysis of the Gram matrix of \(\Phi_{\text{CRD}}\) is a little more involved. To start, denote the columns of \(\Phi_{\text{CRD}}\) by \(\phi_\omega\) and note that the \((r, \omega)\) entry of \(\Phi_{\text{CRD}}\) is given by (4.8). The Gram matrix is a tabulation of the inner products between the columns, and using

\(^6\)Note that this is defined as a correlation distance, but uncorrelated implies independent for the bipolar sequences of concern. The details are provided in Appendix B.3.

\(^7\)The assumption is made for convenience that \(\rho + 1\) divides \(m\), but this assumption is readily relaxed by adjusting the size of the last subset.
the \((\alpha, \omega)\) entry is

\[
[\Phi^{\ast}_{\text{CRD}} \Phi_{\text{CRD}}]_{\alpha\omega} = \langle \phi_{\alpha}, \phi_{\omega} \rangle
\]

\[
= \sum_{j,k=0}^{n-1} \varphi_{j\alpha}^{\ast} \varphi_{k\omega}
\]

\[
= \sum_{j,k=0}^{n-1} \varepsilon_{j} f^{\ast}_{j\alpha} \varepsilon_{k} f_{k\omega} \eta_{jk}
\]

\[
= \sum_{j,k=0}^{n-1} \varepsilon_{j} f^{\ast}_{j\alpha} \varepsilon_{k} f_{k\omega} \eta_{jk} + \sum_{k=0}^{n-1} \varepsilon_{k} f^{\ast}_{k\alpha} \varepsilon_{k} f_{k\omega}
\]

\[
= \sum_{j,k=0}^{n-1} \varepsilon_{j} f^{\ast}_{j\alpha} \varepsilon_{k} f_{k\omega} \eta_{jk} + \delta_{\alpha\omega}
\]

where \([\varepsilon_{0}, \cdots, \varepsilon_{W-1}] = \varepsilon\) is the modulating sequence, \(\eta_{jk} = \langle h_{j}, h_{k} \rangle\) with \(h_{j}\) the \(j\)th column of \(H\), \(f_{j\alpha}\) is the \((j, \alpha)\) entry of the (unitary) DFT matrix \(F\), and \(\delta_{\alpha\omega}\) is the Dirac-delta function. In (4.9), the entries with \(j = k\) have been pulled out of the sum, and \(\eta_{jk} = 1\) for \(j = k\). This last sum is the inner product between rows of a unitary Fourier matrix, which is 1 for \(\alpha = \omega\) and 0 otherwise. This is captured with the Dirac-delta. Hence, the Gram matrix is decomposed as

\[
\Phi^{\ast}_{\text{CRD}} \Phi_{\text{CRD}} = I + X
\]

where the \((\alpha, \omega)\) entry of \(X\) is the sum

\[
x_{\alpha\omega} = \sum_{j \neq k} \varepsilon_{j} \varepsilon_{k} \eta_{jk} f^{\ast}_{j\alpha} f_{k\omega}.
\]

Expanding \(\eta_{jk}\) yields

\[
\eta_{jk} = \begin{cases} 
1, & \frac{n}{m} r \leq j, k < \frac{n}{m} (r + 1) \\
0, & \text{otherwise}
\end{cases}
\]

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\]

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\eta_{jk} = \begin{cases} 
1, & \frac{n}{m} r \leq j, k < \frac{n}{m} (r + 1) \\
0, & \text{otherwise}
\end{cases}
\]
for each \( r = 0, \ldots, m - 1 \). The term \( \eta_{jk} \) acts as a \textit{windowing function} in the sum with a window size of \( \frac{n}{m} \). In expectation, the Gram matrix is

\[
\mathbb{E}[\Phi_{\text{CRD}}^* \Phi_{\text{CRD}}] = I + \mathbb{E}[X] = I + \Delta
\]

(4.12)

where the matrix \( \Delta \equiv \mathbb{E}[X] \) has entries

\[
\Delta_{\alpha\omega} = \sum_{j \neq k} \eta_{jk} f_{ja}^* f_{kw} \mathbb{E}[\varepsilon_j \varepsilon_k].
\]

(4.13)

Note that \( \Delta \) is completely determined by the autocorrelation of \( \varepsilon \). If an independent \( \varepsilon \) is used (such as for the RD) then \( \mathbb{E}[\varepsilon_j \varepsilon_k] = 0 \) for \( j \neq k \); \( \Delta = 0 \), and \( \mathbb{E}[\Phi_{\text{RD}}^* \Phi_{\text{RD}}] = I \). As discussed in [78], this relation is taken to mean that the columns of \( \Phi_{\text{RD}} \) form an orthonormal system in expectation. This of course can never be true for any single instance if \( m < n \), and the RIP is proved by bounding the deviation from this expectation in \( ||| \cdot ||| \).

If \( \varepsilon \) has non-zero correlations, however, then \( \Delta \) does not disappear and the expectation of the Gram matrix is not the identity matrix. To establish the RIP in this case, the deviation of the Gram matrix from the identity matrix must still be bounded, but \( \Delta \) must also be bounded as well. Nevertheless, if this matrix is small in \( ||| \cdot ||| \) then the task is easier. Since the autocorrelation of \( \varepsilon \) determines \( \Delta \), choosing an \( \varepsilon \) that produces small \( ||| \Delta ||| \) is highly desirable. In particular, recall that the RIP of order \( S \) is satisfied if

\[
|||\Phi_{\text{CRD}}^* \Phi_{\text{CRD}} - I||| \leq \delta_S.
\]

(4.14)
Rearranging (4.12) yields \( I = \mathbb{E}[\Phi^*_{\text{CRD}} \Phi_{\text{CRD}}] - \Delta \), and the left-hand side of (4.14) is bounded as

\[
|||\Phi^*_{\text{CRD}} \Phi_{\text{CRD}} - I||| = |||\Phi^*_{\text{CRD}} \Phi_{\text{CRD}} - \mathbb{E}[\Phi^*_{\text{CRD}} \Phi_{\text{CRD}}] + \Delta|||
\leq |||\Phi^*_{\text{CRD}} \Phi_{\text{CRD}} - \mathbb{E}[\Phi^*_{\text{CRD}} \Phi_{\text{CRD}}]||| + |||\Delta|||
\]

(4.15)
due to the triangle inequality. Therefore, proving the RIP requires an upper bound on the two terms in (4.15). The first term is bounded using an argument very similar to that used in [78] but modified to deal with the correlations in \( \varepsilon \). The second term \( |||\Delta||| \) is determined by the autocorrelation of \( \varepsilon \), so an upper bound on \( |||\Delta||| \) is provided that directly relates to the choice of \( \varepsilon \) and the corresponding correlation structure of the sequence.

4.3.3 Main Results

Restricted Isometry Property for the CRD

The preceding discussion on \( \ell \) and \( \Delta \) enables a statement about the RIP of a CRD matrix that makes use of a correlated modulating sequence.

**Theorem 1** (RIP for the CRD). Let \( \Phi_{\text{CRD}} \) be an \( m \times n \) CRD matrix using a modulating sequence with maximum dependence distance \( \ell \) and matrix \( \Delta \) as defined by (4.13). Next, pick \( \delta, \delta' \in (0, 1) \) such that \( \delta' < \delta - |||\Delta||| \) and suppose that \( m \) divides \( n \), \( \ell \) divides \( \frac{n}{m} \), and \( m \) satisfies

\[
m \geq \ell^3 \delta'^{-2} \cdot C \cdot S \log^6(n)
\]

(4.16)

where \( C \) is a positive constant. Then with probability exceeding \( 1 - n^{-1} \) the CRD matrix \( \Phi_{\text{CRD}} \) satisfies the RIP of order \( S \) with constant \( \delta_S \leq \delta \).

---

8Throughout this chapter, the requirements on the divisibility of \( n, m, \) and \( \ell \) can be readily relaxed by careful accounting of terms in the analysis. The exercise is not particularly illustrative and omitted in this presentation.
The proof is provided in Appendix B.1. As with the RD, the sampling rate $m$ must scale linearly with the sparsity $S$ of the input signal and (poly)logarithmically with the bandwidth $n$. The sampling rate, however, also depends on the maximum correlation distance $\ell$ and on the matrix $\Delta$, both of which are determined by the choice of $\varepsilon$. Choosing an independent (i.e., unconstrained) $\varepsilon$ results in $\ell = 1$ and $\Delta = 0$, which are the results of the RD. For a constrained $\varepsilon$, the choice of $\varepsilon$ must be restricted to sequences such that $||\Delta|| < 1$. Obviously, sequences for which both $\ell$ and $||\Delta||$ are as small as possible are ideal. With these criteria in mind, the next two sections examine two classes of sequences and how well they each work in the CRD framework.

**Recovery of Randomly Generated Signals**

In addition to the RIP analysis, the coherence of the sensing matrix is used to provide results for the random signal model in Definition 1. The analysis of the coherence relies on a matrix from [66] that captures the dependence in $\varepsilon$ in a different way than $\Delta$. For a sequence $\varepsilon$, define the triangular matrix $\Gamma$ of mixing coefficients as $\Gamma = \{\gamma_{ij}\}$ with

$$
\gamma_{ij} = \begin{cases} 
0, & i > j \\
1, & i = j \\
|\mathbb{P}(\varepsilon_j = +1|\varepsilon_i = 1) - \mathbb{P}(\varepsilon_j = +1|\varepsilon_i = -1)|, & i < j.
\end{cases}
$$

(4.17)

The main result for the recovery of randomly generated signals is captured in the following theorem.

**Theorem 2** (Recovery under the random signal model). Suppose that the sampling rate satisfies

$$
m \geq C\ell^2[S \log n + \log^3 n]
$$

(4.18)
for some positive constant $C$, and that $m$ divides $n$ and $\ell$ divides $\frac{n}{m}$. Also suppose that $n$ satisfies

$$\frac{\log^2 n}{\sqrt{n}} \leq \frac{C}{16\sqrt{(\ell - 1)||\Gamma||^2}}$$

(4.19)

where $\Gamma$ is defined by (4.17). Let $\alpha$ be a vector with $S$ non-zero components drawn according to the random signal model in Definition 1, and let $\Phi_{\text{CRD}}$ be an $m \times n$ CRD matrix using a stationary modulating sequence with maximum dependence distance $\ell$. Let $y = \Phi_{\text{CRD}} \cdot \alpha$ be the samples collected by the CRD. The solution to the Basis Pursuit problem (2.11)

$$\hat{\alpha} = \arg \min_v ||v||_1 \text{ subject to } y = \Phi_{\text{CRD}} v$$

(4.20)

satisfies $\hat{\alpha} = \alpha$ with probability exceeding $1 - n^{-1}$.

The proof is given in Appendix B.2. Note that the optimization (4.20) is a convex program and can be solved efficiently [15]. The bounds offered here are similar to those for the RD in [78] with the rate scaling linearly with the sparsity $S$ and logarithmically with the bandwidth $n$ but more tightly constrained by the factor of $\ell^2$ and the extra constraint on $n$.

Because the choice of modulating sequence plays such a pivotal role in the analysis of the CRD, a natural question is what types of sequences offer good performance and what types offer bad performance. The remainder of this chapter analyzes two different types of sequences: one for which Theorems 1 and 2 (approximately) apply, and one for which they do not. Numerical experiments in Section 4.6 then show that these results appear to be tight and order-wise optimal, meaning that further analysis is needed to find the optimal constant $C$ in the preceding theorems but the scaling of $m$ in terms of $n$ and $S$ is optimal. Nevertheless, two points must be stressed. First, Theorems 1 and 2 are only sufficient conditions on the sampling rate and modulating sequence; further analysis is needed to prove necessary conditions.
Second, though the modulating sequences are shown to work well numerically, they satisfy Theorems 1 and 2 in an approximate sense. From an engineering perspective, however, the approximation, which is discussed in Section 4.5, is well justified and further validated by the numerical experiments that show near-optimal scaling of $m$.

### 4.4 Repetition-coded Sequences

The analysis of sequences begins with sequences that satisfy the RLL constraints and have a small value of $\ell$ but that have a large $||\Delta||$ and do not satisfy Theorem 1 or Theorem 2.

**Definition 4.** A repetition-coded sequence (RCS) is generated from a Rademacher sequence by repeating each element $d$ times. Let the repetition-coded sequence be denoted as $\varepsilon_{RCS} = [\varepsilon_0, \ldots, \varepsilon_{n-1}]$. Let $[\varepsilon_{(d+1)i}]$ for $0 \leq i \leq \left(\frac{n}{d+1} - 1\right)$ be a Rademacher sequence and require for $1 \leq z \leq d$ and each $i$ that

$$\varepsilon_{(d+1)i} = \varepsilon_{(d+1)i+z}. \quad (4.21)$$

Such a sequence switches at a rate of $\frac{n}{d+1}$. These sequences are one of the simplest forms of RLL sequences and have very small maximum dependence distance. To see this, notice that each group of repeated elements, $[\varepsilon_{(d+1)i+z}]$ for $0 \leq z \leq d$, is completely dependent while independent of every other element in the sequence. The maximum dependence distance is therefore $\ell = d + 1$.

The performance of the CRD also depends on $||\Delta||$, so its behavior must be understood. To begin, assume that $m$ divides $n$ and $\ell$ divides $\frac{n}{m}$ and denote by $\varepsilon_{RCS}$ a repetition-coded sequence. Let $\Phi_{RCS}$ be a CRD matrix that uses $\varepsilon_{RCS}$ as the modulating sequence: $\Phi_{RCS} = HDF$ where $D$ contains $\varepsilon_{RCS}$ on its diagonal. It is convenient
to rewrite the entries of $\Delta$, given in (4.13), in this case as

$$\Delta_{\alpha \omega} = \sum_{j, k \neq 0} \eta_j(j + k) f^*_{j \alpha} f_{j \omega} \mathbb{E}[\varepsilon_j \varepsilon_{j+k}].$$

To calculate $|||\Delta|||$, it is convenient to focus on the Gram matrix $\Lambda = \Delta^* \Delta$, which has entries

$$\Lambda_{\alpha, \omega} = \begin{cases} \frac{n}{d} \sum_{j=0}^{d-1} e^{-\pi \eta j \hat{F}(j, \omega) \hat{F}^*(j, \alpha)}, & \omega - \alpha = \frac{q}{\ell} \pi \\
0, & \text{otherwise} \end{cases} \quad (4.22)$$

where

$$\hat{F}(j, \omega) = \sum_{i \neq 0} \eta_j(j + i) f_{i \omega} \mathbb{E}[\varepsilon_j \varepsilon_{j+i}] \quad (4.23)$$

and $q = 0, \pm 1, \ldots, \pm (\ell - 1)$.

Studying the entries of $\Lambda$ leads to a bound on $|||\Delta|||$. Recall from the definition of the spectral norm that $||A||_{\Omega' \times \Omega'} \leq ||A||_{\Omega' \times \Omega}$ for any $\Omega' \subset \Omega$. In this case, a lower bound on $|||\Delta|||$ is found by taking $\Omega$ such that $|\Omega| = 1$, i.e., $S = 1$. For $S = 1$, $|||\Delta|||$ is the square root of the maximum entry on the diagonal of $\Lambda$. Applying (4.21) to the autocorrelation in (4.23), straightforward manipulation yields

$$\Lambda_{\omega, \omega} = \frac{n}{d + 1} \sum_{j=0}^{d} \sum_{i=-j}^{d-j} \sum_{k=-j}^{d-j} f^*_{i \omega} f_{k \omega}, \quad (4.24)$$

which is maximized by $\omega = 0$. This results in $\Lambda_{0,0} = d^2$, and $|||\Delta||| \geq d$ for any $S$.

Theorem 1 only applies if $d = 0$, which means the minimum run length cannot be constrained. In the context of Theorem 2, recall (4.10) for the case of $\alpha = \omega$. In this case, it is easy to see that $|x_{\alpha \omega}| \geq d + 1$. Theorem 2 on the other hand, relies on bounding $||X||_{\max}$ close to 0 (the details are in Appendix B.2) and this obviously cannot be done for an RCS.

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Theorems 1 and 2 do not hold for $\Phi_{\text{RCS}}$, and although this chapter does not present converse results, the tightness of the theory is demonstrated for a repetition-coded sequence through numerical experiments. The minimum and maximum singular values of the submatrices over an ensemble of matrices $\Phi_{\text{RCS}}$ generated using a repetition-coded sequence with $d = 1$ are calculated. The submatrices are chosen by picking 10 columns at random from $\Phi_{\text{RCS}}$, and the results are presented in Figure 4.3(a) where the minimum singular values are often at or very near zero for some values of $R$. This indicates the RIP is either not satisfied or barely satisfied with an extremely small isometry constant. Further, numerical experiments in Section 4.6 show that reconstruction performance is in general poor for $\Phi_{\text{RCS}}$.

4.5 Wide Sense Stationary RLL Sequences

The previous section demonstrates that $\Phi_{\text{RCS}}$ does not satisfy the requirements for Theorems 1 or 2, and Figure 4.3(a) offers numerical evidence that $\Phi_{\text{RCS}}$ likely does not satisfy the RIP, though checking RIP exactly is combinatorial and computationally impractical. A repetition-coded sequence is therefore not expected to perform well in the sampling and reconstruction of sparse signals. In this section, a different class of RLL sequences are proposed \[43\] that, although more complicated than a repetition-coded sequence, produce measurement matrices with better conditioned submatrices that perform much better in the sampling and reconstruction of frequency-sparse signals.

The analysis begins by examining the RIP for a modulating sequence, $\varepsilon$, that is wide sense stationary with autocorrelation function $R_\varepsilon(i) = \mathbb{E}[\varepsilon_j \varepsilon_{j+i}]$. The maximum dependence distance is assumed to be $\ell$, so $R_\varepsilon(i) = 0$ for $|i| \geq \ell$. Under these assumptions, an upper bound on $|||\Delta|||$ is found by focusing on the Gram matrix $\Lambda$ \[4.22\].
(a) RCS. The singular values very near to zero represent poor conditioning of the submatrices of $\Phi_{RCS}$. Here, $d = 1$ and $k = \infty$.

(b) MRS. The singular values are bounded away from 0 and 2 indicating good conditioning of the submatrices of $\Phi_{MRS}$. Here, $d = 1$ and $k = 20$.

Figure 4.3: The minimum and maximum singular values of submatrices with 10 (out of 512) randomly chosen columns averaged over 1000 realizations of the $m \times n$ measurement matrix with $n = 512$. The error bars represent 2 standard deviations above and below the average value. (Credit: [38].)
In this case, (4.23) is rewritten in terms of $R_\varepsilon (i)$

$$\hat{F}(j, \omega) = \sum_{i \neq 0} \eta_{j(j+i)} f_i \omega R_\varepsilon (i)$$

which is referred to as the windowed spectrum because $\eta_{j(j+i)}$ can be viewed as a windowing operation on $R_\varepsilon (i)$. From (4.11), the width of the window is $n/m$, which is quite large as $n$ increases and $m$ scales as in (4.16). $\hat{F}(j, \omega)$ also looks very much like the power spectrum of $\varepsilon$

$$F_\varepsilon (\omega) = \sum_i R_\varepsilon (i) e^{-\frac{2\pi i}{n} \omega}. \quad (4.25)$$

Note that $F_\varepsilon (\omega)$ is real-valued because $R_\varepsilon (i) = R_\varepsilon (-i)$. The significant differences in $\hat{F}(j, \omega)$ are the exclusion of $i = 0$ in the sum, a scaling by $n^{-\frac{1}{2}}$ from $f_i \omega$, and the windowing by $\eta_{j(j+i)}$. If $n/m \gg \ell$ then the windowing has negligible effect in $\hat{F}(j, \omega)$ because $R_\varepsilon (i) = 0$ for $|i| \geq \ell$, and $\hat{F}(j, \omega)$ and $\Lambda$ both simplify greatly. To see this, first notice that $R_\varepsilon (0) = 1$ because $|\varepsilon_i| = 1$. Separating the $i = 0$ term from the sum in (4.25) yields

$$F_\varepsilon (\omega) = \sum_{i \neq 0} R_\varepsilon (i) e^{-\frac{2\pi i}{n} \omega} + 1, \quad (4.26)$$

$\tilde{F}_\varepsilon (\omega)$ is termed the reduced spectrum of $\varepsilon$. Under the assumption that $n/m \gg \ell$, $\hat{F}(j, \omega)$ reduces to $n^{-1/2} \tilde{F}_\varepsilon (\omega)$ for all $j$ except $j$ satisfying $|rn/m + j| \leq \ell$ for $r = 0, \cdots, m-1$. This excludes all but a fraction $2\ell \frac{m}{n}$ corresponding to $j$ close to the edges of the windows $\eta_{j(j+i)}$, and this fraction becomes increasingly small as $n$ grows. In this case,
the entries of $\Lambda$ are approximately

$$\Lambda_{\alpha,\omega} \approx \frac{1}{n} \sum_{j=0}^{n-1} e^{-\frac{2\pi i}{n}(\omega-\alpha)j} \tilde{F}_\varepsilon(\alpha) \cdot \tilde{F}_\varepsilon(\omega)$$

$$= \delta_{\alpha,\omega} \tilde{F}_\varepsilon(\alpha) \cdot \tilde{F}_\varepsilon(\omega)$$

where $\delta_{\alpha,\omega}$ is the Kronecker delta. In words, $\Lambda$ is approximately a diagonal matrix with the square of the reduced spectrum on the diagonal: $\Lambda \approx \text{diag}[(\tilde{F}_\varepsilon(\omega))^2]$, and the eigenvalues of $\Lambda$ are approximately $(\tilde{F}_\varepsilon(\omega))^2$. Consequently, the singular values of $\Delta$ are approximately $|\tilde{F}_\varepsilon(\omega)|$, and $||\Delta|| \approx \max_\omega |\tilde{F}_\varepsilon(\omega)|$. The spectral norm of a submatrix is upper bounded by the spectral norm of the matrix, so finally

$$|||\Delta||| \leq ||\Delta|| \approx \max_\omega |\tilde{F}_\varepsilon(\omega)|. \quad (4.27)$$

Relation (4.27) provides a way to estimate whether or not a stationary $\varepsilon$ is well-suited for use within the CRD. A stationary $\varepsilon$ whose spectrum is bounded within $(0, 2)$ is good, while one with $F_\varepsilon(\omega) = 1 \ \forall \omega$ allows for the smallest isometry constants according to the criteria of Theorem 1.

Some examples make this discussion clearer. First, consider an unconstrained $\varepsilon$ with independently drawn entries, such as the one used in the RD. In this case, $F_\varepsilon(\omega) = 1$ and $\tilde{F}_\varepsilon(\omega) = 0 \ \forall \omega$. The Gram matrix exactly disappears ($\Lambda = 0$) and $\Delta = 0$ confirming the previous discussion and allowing the smallest isometry constants in Theorem 1. Next, consider the RLL sequences described in [72] and [43]. The Markov chain in Figure 4.4 provides a very generalized model for describing these sequences. To understand how well these sequences will work in the CRD, the power spectrum of sequences generated from the Markov chain in Figure 4.4 must be calculated.
Figure 4.4: State diagram of the Markov chain that describes the generation of the Markov-generated RLL sequences described in Definition 5. The transition probabilities are symmetric in the sense that $p_{(i+k)(j+k)} = p_{ij}$ where the sum is taken modulo $2k$. The top row of states outputs the symbol $+1$ while the bottom row outputs the symbol $-1$. (Credit: [38].)

4.5.1 Power Spectra of Sequences

The discussion begins with an explicit description of the RLL sequences in [43].

**Definition 5.** A $(d,k)$-constrained RLL sequence that is generated from the Markov chain whose state diagram is found in Figure 4.4 is called a *Markov-generated RLL sequence* (MRS), and denoted as $\varepsilon_{\text{MRS}} = [\varepsilon_0, \cdots, \varepsilon_{n-1}]$ with $\varepsilon_k \in \{+1, -1\}$. The transition probabilities are defined by the matrix $P = [p_{ij}]$ where $p_{ij}$ is the probability of transitioning from state $i$ to state $j$. The $p_{ij}$ also satisfy $p_{(i+k)(j+k)} = p_{ij}$ where the sum is modulo $2k$. $P$ is a stochastic matrix with rows summing to 1. The average of the symbols output from each state $i$ are collected in the vector $b = \{b_i\}$. The stationary distribution of the states is denoted by $\pi = [\pi_i]$ and satisfies $\pi^T = \pi^T P$. The autocorrelation function of these sequences is

$$R_{\varepsilon}(i) = a^T P^i b$$

where $a^T = b^T \cdot \text{diag}[\pi_1, \cdots, \pi_{2k}]$ and $R_{\varepsilon}(-i) = R_{\varepsilon}(i)$ [10]. The vector $a$ collects the average symbol output upon moving into each state in the Markov chain, while the vector $b$ captures the average symbol output upon leaving each state in the Markov chain. To understand the performance of a Markov-generated sequence in the CRD,
the behavior of $R_\epsilon(i)$ as $i$ increases must be characterized. Since $P$ is a stochastic matrix, the theory of non-negative matrices helps in understanding how $R_\epsilon(i)$ behaves. First note that $b$ is orthogonal to $w$, where $w = [1, 1, \cdots, 1]^T$ is the all ones vector, and that $a^Tb = 1$. The second largest eigenvalue $\lambda_2$ of $P$ satisfies $\lambda_2 < 1$ because $P$ is a stochastic matrix. The autocorrelation is bounded, in magnitude, using [42 Theorem 8.5.1] as

$$|R_\epsilon(i)| = |a^TP^ib| \leq \lambda_2^i,$$  

(4.28)

so $|R_\epsilon(i)|$ experiences geometric decay at a rate determined by $\lambda_2$. This is confirmed in Figure 4.5 where $10\log_{10}|R_\epsilon(i)|$ is plotted for several pairs $(d, k)$. Notice that the rate of decay, in magnitude, is smaller for larger values of $d$ and larger for larger values of $k$, while the decay is roughly the same for $k = 20$ and $k = \infty$. These trends are directly related to the eigenvalues of $P$ in each case.
Evaluation of the maximum dependence distance $\ell$ and the matrix $\Delta$ are critical to understanding the performance of sequences in the CRD. Addressing the maximum dependence distance first, (4.28) and the fact that $\lambda_2 < 1$ establish that $\lim_{i \to \infty} |R_\varepsilon(i)| = 0$. Hence, for any $\varrho > 0$, $|R_\varepsilon(i)| < \varrho$ for all $i \geq M$ where $M = M(\varrho) < \infty$. Though a Markov-generated sequence is not guaranteed to become completely uncorrelated for a finite $M$, $\varrho$ can be made as small as desired so that the sequence is nearly uncorrelated for large enough $M$. In this case, $\ell \approx M(\varrho)$ for some small $\varrho$, and a Markov-generated sequence satisfies the setting of Theorem 1 in an approximate sense. The approximation is justified from an engineering perspective because the correlation can be made very small; the numerical experiments in Section 4.6 add further justification.

Next, $|||\Delta|||$ is estimated from the reduced spectrum of the MRS. Using (4.27) and $\ell \approx M(\varrho)$ from above, $|||\Delta||| \leq ||\Delta|| \approx \max_\omega |\tilde{F}_\varepsilon(\omega)|$ where $\tilde{F}_\varepsilon(\omega)$ is the reduced spectrum. A reminder that $\varrho$ can be made as small as desired at the expense of a larger $\ell$. Consequently, it can be argued that an MRS that satisfies

$$\max_\omega |\tilde{F}_\varepsilon(\omega)| < 1$$

leading to a matrix $\Phi_{\text{MRS}}$ that approximately satisfies the RIP by virtue of Theorem 1.

Turning to Theorem 2, $|||\Gamma|||$ must be bounded independent of $n$. From the definition of the autocorrelation and for $i < j$,

$$\gamma_{ij} = \sqrt{|R_\varepsilon(j - i)|}/2 \leq \sqrt{\lambda_2^{-1}/2}$$

for a Markov-generated sequence. It is then also straightforward to show (see, e.g., the discussion after [66, Proposition 1]) that

$$|||\Gamma||| \leq 1/\sqrt{2}(1 - \lambda_2^{1/2})$$.
Since $||\Gamma||$ is independent of $n$, $n$ can be made large enough so that (4.19) is satisfied and Theorem 2 is approximately satisfied.

The argument for the use of an MRS within the CRD makes use of some approximations. To demonstrate the validity of these approximations, an MRS with $(d, k) = (1, 20)$ is considered. The spectrum of this MRS is shown in Figure 4.6(c). From this figure, $\max_\omega |\tilde{F}_\epsilon(\omega)| \approx 0.9$ corresponding to $\omega = \pm 0.5$. The theory, therefore,
predicts that the matrix $\Phi_{\text{CRD}}$ in this case satisfies the RIP. To verify this, the average minimum and maximum singular values of the submatrices of $\Phi_{\text{CRD}}$ are calculated with the results in Figure 4.3(b) for submatrices containing 10 columns. As $m$ decreases, the singular values approach 0 and 2 but remain bounded away from them. In Section 4.6 numerical reconstruction experiments further validate the theory.

4.6 Numerical Results: RD versus CRD

This section compares the numerical performance of the RD with that of the CRD. The CRD measurement matrices are built using both the RCS and MRS. The results presented are produced using the YALL1 software package, an $\ell_1$-solver that uses alternating direction algorithms [85]. The use of an RCS in the CRD is first examined and shown to give unsatisfactory results.

Recall that the argument in Section 4.4 that $\Phi_{\text{CRD}}$ shows that using a repetition-coded sequence in the CRD does not produce a matrix that satisfy the RIP. Consequently, if a sparse signal is sampled with such a measurement matrix, then the attempt at reconstruction is expected to produce poor results. This is indeed the case in the numerical experiments presented in Figure 4.7(a). To produce these results, the sampling rate is held constant at $m = 50$, and the bandwidth $n$ is varied. It is particularly noteworthy that sampling and reconstruction fails most of the time at $n = 100$ and $n = 200$. Note that the RCS performs relatively better at $n = 150$, owing to the splitting of some repeated entries of the RCS between successive rows of the composite matrix $HD$.

Sampling with a CRD that uses an MRS with $d = 1$ and $k = 20$ is shown to produce results similar to those for the RD using a Rademacher sequence. Recall that Section 4.5 argued for the usefulness of RLL sequences generated from the Markov chain of Figure 4.4 in the context of the CRD. Figure 4.7(b) validates this assertion.
Figure 4.7: Empirical probability of successful reconstruction over 1000 instances of the measurement matrix. The RCS does not offer good performance and, in fact, fails quite often. The MRS offers comparable performance to the Rademacher sequences of the RD. (Credit: [38].)
and shows the empirical probability of reconstruction sparse signals are sampled with a $\Phi_{\text{CRD}}$ that uses these sequences. The baseline for comparison is of course the RD. The figure shows that the performance using an MRS is very similar to the performance using the Rademacher sequences of the RD. In fact, the CRD allows a tradeoff between sparsity, bandwidth, and recovery success. In particular, concentrating on the RD curve at $n = 250$ and the CRD curve at $n = 300$, a 90% success rate requires only paying a sparsity penalty of 2 ($\approx 13\%$) when using the CRD. At the same time, however, the advantage in bandwidth $n$ is 20%. Comparing the CRD curve at $n = 300$ to the RD curve at $n = 150$ shows that at a 90% success rate, a sparsity penalty of approximately 28% is incurred for a 100% increase in bandwidth. Other tradeoffs are seen at different success rates, but it is reasonable to argue that most applications will operate A/D converters in the high success rate regions. At lower success rates, the advantage is even greater for the CRD. While the analysis concentrates on a high success rate, analysis at lower success rates could prove useful for future work.

4.7 Knowledge Enhanced Sub-Nyquist Sampling

This section presents an argument that the performance of a CRD is enhanced by leveraging a priori knowledge about the signal. Notice that two operations in Figure 4.1 are central to the functioning of the RD/CRD: the modulation by the random waveform and the subsequent low-pass filtering. The low-pass filtering operation allows the RD/CRD to operate at the sub-Nyquist rate $m$, while modulation by the random waveform—which smears the input signal tones across the spectrum, including in the low-pass region—results in a unique signature of each tone within the low-pass region. Theorem 11 states the sufficient conditions for uniqueness to hold for all possible input signals, and Section 4.5 explores how the RIP depends on the power spectrum of the random sequence. In addition to uniqueness of each tone’s
signature in the low-pass region, the performance of the RD/CRD depends on the energy smeared into the low-pass region because tones with a low-energy signature will be harder to recover.

Note that the modulation by the random waveform in time is equivalent to a convolution in the frequency domain. Therefore, the power spectrum of the random waveform tells us how much energy from each tone on average is smeared into the low-pass region (and thus collected in the measurements). Inspection of \( (4.27) \) says that the RIP depends on the worst-case deviation from a flat spectrum. However, if an MRS is used within the CRD and the input signal is statistically more likely to contain low frequencies, then this additional knowledge about the signal can be leveraged to improve the reconstruction averaged over many signals and random waveform realizations. Note that this is a different average case setup than the one in Theorem 2. Here, a nonuniform distribution on tones in the input signal is imposed. In this setting, the CRD is shown to perform better than the RD, provided the statistical distribution of the tones is matched to the power spectrum of the MRS. The CRD in this case will on average smear and capture more energy from the input tones in the low-pass region of the spectrum. In addition to the case of possessing prior knowledge about the input signal distribution, the exposition in this section is also of interest in other scenarios. Consider, for example, a spectrum sensing application in which one assigns a higher priority of detection to some regions and a lower priority of detection to other regions. Similarly, consider the case where one possesses knowledge about colored noise or narrowband interference injected into the signal. In both these settings, the CRD can be tailored through the choice of the modulating waveform to perform better than either the RD, which treats all spectral regions the same way, or a pure passband system, which completely throws away information in some spectral regions. Such usage of the CRD exploiting prior knowledge is termed a knowledge-enhanced CRD.
Somewhat similar ideas are briefly explored in [59] and [46], but without the explicit examination of the uniqueness of tone signatures. Recent work on model-based compressed sensing also attempts to leverage additional a priori information in the signal model [7], but the focus there is exclusively on the reconstruction side instead of the sampling side.

4.7.1 Phase Transitions of Reconstruction Success

To verify the understanding of the knowledge-enhanced CRD, extensive numerical simulations were conducted to compare reconstruction performance for signals sampled by a CRD (using an MRS) against the RD (using a Rademacher sequence). The focus here is on two classes of input signals. The first is generated by drawing a sparse set of tones uniformly at random; the second is generated with a distribution on the tones that matches the power spectrum of an MRS with \((d, k) = (1, 20)\) (see Figure 4.6(c)). The focus is on two measurement matrices: the RD and the CRD using an MRS with \((d, k) = (1, 20)\). Recall, the RD uses an (unconstrained) Rademacher sequence. The sequence is comprised of independent terms, resulting in a flat spectrum (see Figure 4.6(a)). Because the spectrum is flat, a Rademacher sequence will illuminate all tones equally well. That is to say, good reconstruction performance is expected for all sparse signals. On the other hand, the MRS used in the CRD has correlations between terms of the sequence that gives rise to the spectrum in Figure 4.6(c). Notice that the spectrum is close to 1 for the low frequencies (Region 1) and approximately 0.1 at high frequencies (Region 2). If low-frequency tones are statistically more likely in the input signal, then the CRD is expected, on average, to capture more energy in the measurements and offer better reconstruction performance. Note, the CRD using an RCS is not considered because it is shown in Section 4.6 to offer reconstruction performance that is very poor. To understand why it is poor for an RCS, the spectra of these sequences are examined. An RCS is
Figure 4.8: Empirical reconstruction success as a function of $S/m$ and $m/n$ for the RD. The phase transition is the transition of the success rate from 0 to 1. The transition is roughly the same for either input distribution. (Credit: [38]).
Figure 4.9: Empirical reconstruction success as a function of $S/m$ and $m/n$ for the CRD. The phase transition is the transition of the success rate from 0 to 1. The transition is shifted up significantly indicating improved reconstruction performance for the matched input distribution. (Credit: [38].)
not stationary but rather cyclo-stationary, so the spectrum is calculated by averaging over the cycle period. The resulting spectrum is shown in Figure 4.6(b) for \( d = 1 \). The spectrum approaches zero at high frequencies, so the CRD in this case is expected to capture very little energy from high frequency tones in the low-pass region. Consequently, poor reconstruction performance is also expected.

The results are presented in Figure 4.8 for the RD and in Figure 4.9 for the CRD and for the two input signal classes. For these experiments, an RD or CRD matrix is generated using a random instance of the modulating sequence 3000 times for each point, i.e., pixel, on the plot. The matrix is used to sample a new randomly generated \( S \)-sparse vector, and reconstruction of the original vector from its samples is carried out using the YALL1 software package. Success is defined as the two vectors being equal to each other to 6 decimal places in the \( \ell_\infty \) norm. The results in Figure 4.8 show that the RD performs (almost) equally well for the two input signal classes. On the other hand, Figure 4.9 shows that the CRD performs much better for the second class of input signals. Additionally, the CRD suffers slightly more at very small \( m/n \) ratios.

### 4.7.2 Reconstruction from Noisy Measurements

The phase transitions of Figures 4.8 and 4.8 correspond to a noiseless setting. Figure 4.10 (for the RD) and Figure 4.11 (for the CRD) present the results of reconstructing input signals from noisy samples \( y = \Phi \alpha + \sqrt{p} \cdot \xi \) where \( \xi \) is white Gaussian noise and \( p \) determines the noise power. The mean squared error (MSE) of the reconstruction is plotted as a function of \( S/m \) and \( n/m \) and use the SpaRSA software package, which solves an \( \ell_2/\ell_1 \) mixed-norm optimization termed \textit{lasso} [74] for noisy reconstruction purposes [84] [10]. Similar to the noiseless case, a transition from low

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9The model \( y = \Phi(\alpha + \sqrt{p} \cdot \xi) \) yields similar results, but \( \xi \) as colored noise could offer interesting future work.

10SpaRSA is better suited computationally for noisy reconstruction than YALL1. A regularization parameter of \( 1.9\sqrt{2p\log n} \) is used to produce the plots.
Figure 4.10: Reconstruction MSE (dB) as a function of $S/m$ and $m/n$ for the RD. The plots correspond to SNR 40 dB defined as the ratio of the power of the measurements to the noise variance. The transition from small MSE to large MSE is similar to the phase transition in the noiseless case. (Credit: [38].)
Figure 4.11: Reconstruction MSE (dB) as a function of $S/m$ and $m/n$ for the CRD. The plots correspond to SNR 40 dB defined as the ratio of the power of the measurements to the noise variance. The transition from small MSE to large MSE is similar to the phase transition in the noiseless case. (Credit: [38].)
MSE to high MSE is seen. The performance of the RD is also similar for each class of input signals while the CRD performs much better for the second class of input signals due to matching the prior to the power spectrum of the modulating sequence.

4.7.3 Reconstruction of Signals with Non-Integral Frequency Components

The signal model (4.1) assumes only integral-frequency tones, but real-world signals likely contain tones that can take a continuum of frequency values. These non-integral tones will leak energy to several integral tones based on the implicit windowing operation from the finite time of acquisition \( t \in [0, 1) \). The windowing produces a convolution of the input tones and the frequency response of the window but does not invalidate the signal model (4.1) \[58\]. Rather, the result is a scaling of the sparsity factor from \( S \) to \( aS \), where \( a \geq 1 \) determines the extent of the leakage. Figure 4.12 shows reconstruction results if non-integral tones in the input signal are allowed. Tones are drawn at random with frequencies from \([0, n]\) according to a distribution proportional to the spectrum in Figure 4.6(c). The coefficients in the input at the integral tones are determined by the frequency response of a Hamming window centered at the (randomly drawn) location of the tone. Compare Figure 4.12(a) with Figure 4.10(b) for the RD and Figure 4.12(b) with Figure 4.11(b) for the CRD. Both plots look similar, but Figure 4.12 has \( S \) scaled by a factor of 16. This suggests that the penalty for considering leakage in (4.1) is roughly a factor of 16 in input signal sparsity. In the worst-case, this mismatch can seriously degrade reconstruction performance \[24\]. However, the experiments described in this chapter often do not see the worst case, which is a tone occurring halfway between two integral tones, because the random drawing of frequencies does not pick the worst case very often and only a manageable decrease in performance is encountered.
Figure 4.12: Reconstruction MSE (dB) as a function of $S/m$ and $m/n$. The plots correspond to an SNR defined as in Section 4.7.2. In these experiments, non-integral frequencies are allowed and place energy at integral frequencies according to a Hamming window frequency response. Note the scaling on the y-axis indicating the sparsity penalty incurred from non-integral frequencies. (Credit: [38].)
4.8 Summary

In summary, the use of RLL sequences in the RD is proposed because of hardware constraints on generating high-fidelity, fast-switching waveforms. The analysis in this chapter shows both theoretically and numerically that for a fixed switching rate, certain classes of RLL sequences offer an increase in the observable bandwidth of the system. Specifically, a Markov-generated sequence works well, and a repetition-coded sequence does not. Insight into why each sequence succeeds or fails is found in the power spectrum of the sequence. Further, the numerical results show that matching the distribution of tones in the input signal to the power spectrum of the modulating sequence optimizes performance beyond that of the RD. The most obvious future directions to take are a better theoretical understanding of the knowledge-enhanced CRD and matching the modulating sequence to arbitrary distributions on the input tones. A more thorough understanding of the hardware system and the consideration of a more complex modulating waveform (e.g., with a pulse shape other than a square) would also be interesting and useful.
Chapter 5

Detection of Spectrum Usage with the Random Demodulator

5.1 Introduction

Chapter 4 explores the estimation or recovery of bandlimited signals from sub-Nyquist samples using the random demodulator (RD). It also explores the Constrained RD, which can offer improved performance by exploiting additional knowledge about the input signal. This chapter explores the utility of the RD in the detection of the spectrum occupancy of bandlimited signals, i.e., the locations of tones. Detection does not require recovering the signal exactly; rather, it only asks which tones are active in the signal. Two aspects of this question are explored. First, given a noise level and small number of allowed errors, what is the smallest sampling rate that still allows the detection of a large proportion of the active tones? Second, can a very simple and low-complexity algorithm offer nearly optimal performance? In other words, does the algorithm offer performance close to a converse bound? The results presented in this chapter are based on work first presented in [37] and more fully developed in this chapter.
5.1.1 Fundamental Limits on Spectrum Sensing

The first main contribution in this chapter is an analysis on the fundamental limits afforded when using the RD for sensing of wideband and underutilized spectrum, building off of work by Reeves and Gastpar [61] and extending it to the RD. In this regard, the focus is the scaling regime where the number of (noisy) acquired measurements and the number of active frequencies both scale linearly with the bandwidth. In the analysis, a small number of errors are allowed in the recovery while the bandwidth is allowed to grow to infinity. The ratio of samples to bandwidth is quantified beyond which recovery of the locations of the non-zero frequencies with a ratio of errors smaller than a parameter $\alpha$ is not possible.

5.1.2 Spectrum Sensing with Thresholding

The second main contribution is to show that a simple thresholding algorithm can recover the non-zero frequencies with high probability if the number of samples scales logarithmically with the bandwidth and linearly with the sparsity. In [78], signals are sampled with the RD and recovered, even in the presence of noise, if the number of samples $m$ satisfies $m \geq C \cdot S \log^6 n$ for some constant $C$ and where $n$ is the total number of tones and $S$ is the number of non-zero tones. The signal can be recovered using several low-complexity algorithms (e.g., CoSaMP [50] or Lasso [74]). The wideband sensing problem, however, is primarily interested in finding the locations of the frequencies containing the signal energy, not the values of the coefficients. To this end, a very simple and low-complexity algorithm called one-step thresholding (OST) (see Algorithm 1 in [4] or the description in Section 5.4.2) recovers the signal support in frequency space while requiring a small number of computations. OST is shown to recover the frequency support with high probability if the number of samples scales as $m \geq C \cdot \max\{S \log n, \log^2 n\}$. In particular, the OST algorithm performs near-
optimally, i.e., close to a converse lower bound, when either the non-zero entries have nearly the same magnitude or the signal-to-noise ratio (SNR) is not too large.

5.2 Problem Formulation

This chapter is concerned with the measurement model

\[ y = X\beta + w \]  

(5.1)

where \( X \) is an \( m \times n \) (complex) RD matrix, \( \beta \) is a length-\( n \) \( S \)-sparse complex signal vector (\( \|\beta\|_0 \leq S \)), and \( w \) is a length \( m \) complex white Gaussian noise vector with unit variance\(^1\). The sparsity pattern, or locations of the non-zero entries, of the vector is denoted \( \beta \) with \( \Omega = \{1 \leq i \leq n : \beta_i \neq 0\} \). The RD architecture is shown in Figure 5.1 (for convenience a duplicate of Figure 4.1) and a key aspect of \([78]\) is showing that the architecture can be represented by the RD matrix \( X = HDF \) where \( F \) is a unitary Fourier matrix, \( D = \text{diag}(\varepsilon_i) \) is a diagonal matrix with a Rademacher chipping sequence \( \varepsilon_i \) for \( i = 1, ..., n \) on the diagonal, and \( H \) is a matrix with \( n/m \) consecutive 1’s in each row and with each row orthogonal. The matrix \( H \) represents the low-pass filter, and an example is

\[
H = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]

A key result from \([78]\) guarantees that the columns of \( X \) are nearly unit-norm with high probability provided \( m \) is large enough.

\(^1\)A reminder that \( \| \cdot \|_0 \) counts the number of non-zero coefficients in a vector.
Figure 5.1: Block diagram of the random demodulator [78] (a duplicate of Figure 4.1 for convenience). The input signal is multiplied by a waveform generated from a Rademacher chipping sequence, then low-pass filtered, and finally sampled at a sub-Nyquist rate \( m \ll n \). (Credit: [38].)

**Theorem 3.** [78, Theorem 8] Suppose \( m \geq c\delta^{-2} \log n \). Then an \( m \times n \) RD matrix satisfies

\[
P\left[ \max_i \left| \|x_i\|_2^2 - 1 \right| \geq \delta \right] \leq n^{-1}
\]

where \( x_i \) is the \( i \)th column of \( X \) and \( c \) is a constant.

Section 5.3 presents an information-theoretic argument that lower bounds the measurement rate that still allows recovery of the signal support from noisy measurements. Section 5.4 presents thresholding as an achievable scheme and discusses when it is nearly optimal relative to the given information-theoretic bound as well as offering numerical verification.

### 5.3 Converse Bound on Reliable Sensing from Noisy Measurements

The RD matrix is analyzed in the linear scaling regime, the setting of [61], to find a lower bound on the measurement rate. Beginning with the measurement model (5.1), the relations \( m = \lceil \rho n \rceil \) with \( 0 < \rho < \infty \) and \( S = \lfloor \gamma n \rfloor \) with \( 0 \leq \gamma \leq \frac{1}{2} \) are imposed so that both \( m \) and \( S \) are linear in the problem size \( n \). In other words, \( \rho \) represents the measurement rate in the problem, and \( \gamma \) represents the sparsity ratio. The non-zero entries of \( \beta \) are drawn from a probability distribution \( F \) with mean \( \mu_F \) and variance...
\( \sigma^2_F \). The variance of the vector \( \beta \) is defined as \( V(\gamma, F) = \gamma (1 - \gamma) \mu^2_F + \gamma \sigma^2_F \). Further, each of the \( \binom{n}{S} \) realizations of \( \Omega \) is equally likely. Finally, the distortion for an estimate \( \hat{\Omega} \) of \( \Omega \) is defined, as in [61], as

\[
d(\Omega, \hat{\Omega}) = \max(\text{MDR}(\Omega, \hat{\Omega}), \text{FAR}(\Omega, \hat{\Omega}))
\]

with *missed detection rate*

\[
\text{MDR}(\Omega, \hat{\Omega}) = \frac{1}{|\Omega|} \sum_{i=1}^{n} 1(i \in \Omega, i \notin \hat{\Omega})
\]

and *false alarm rate*

\[
\text{FAR}(\Omega, \hat{\Omega}) = \frac{1}{|\hat{\Omega}|} \sum_{i=1}^{n} 1(i \notin \Omega, i \in \hat{\Omega}).
\]

The distortion is restricted to satisfy \( d(\Omega, \hat{\Omega}) \leq \alpha \) for a *distortion level* \( 0 \leq \alpha \leq 1 \). Finally, the RD matrix is rescaled so that the assumptions of [61] are satisfied, namely that the rows of \( X \) are unit norm in expectation, i.e., \( E[\text{trace}(XX^*)] = m \). For the RD matrix,

\[
XX^* = HDF(HDF)^* = HFF^*D^*H^* = HH^* = \frac{n}{m} I_m,
\]

and \( X \) must be scaled by \( \sqrt{\frac{m}{n}} \) to satisfy this condition.

With all this in mind, the main result is a lower bound on the measurement rate \( \rho \) beyond which recovery of \( \Omega \) is not possible with distortion level \( \alpha \).

**Theorem 4.** Suppose \( X \) is a (scaled) \( m \times n \) RD matrix, and \( \frac{m}{n} \to \rho \) as \( n \to \infty \). In order to have a distortion for the estimated sparsity pattern that satisfies \( d(\Omega, \hat{\Omega}) \leq \alpha \), \( \rho \) must satisfy

\[
\rho \geq \frac{2R(\gamma, \alpha)}{\log (1 + V(\gamma, F))}
\]

(5.2)
Figure 5.2: Plot of the lower bound on the measurement rate in Theorem 4 for sparsity ratio $\gamma = 0.01$ and distortion $\alpha = 0.1$. For a given SNR, all measurement rates below the curve are not achievable, i.e., the recovery of the sparsity pattern with distortion no larger than $\alpha$ is not possible. For small SNR, starting around $-10$ dB, the measurement rate $\rho > 1$ meaning that the number of measurements must be larger than the size of $\beta$ if too much noise is present. (Credit: [37].)

where

$$R(\gamma, \alpha) = \begin{cases} h(\gamma) - \gamma h(\alpha) - (1 - \gamma)h\left(\frac{\alpha}{1-\gamma}\right), & \alpha < 1 - \gamma \\ 0, & \alpha \geq 1 - \gamma \end{cases}$$

(5.3)

and $h(\alpha) = a \log\left(\frac{1}{a}\right) + (1-a) \log\left(\frac{1}{1-a}\right)$ is the binary entropy function.

The proof is supplied in Appendix C.1

To illustrate Theorem 4, Figure 5.2 plots the lower bound on the sampling rate as a function of SNR in dB for a sparsity ratio $\gamma = 0.01$ and distortion $\alpha = 0.1$. SNR
is given by the power of the distribution of $\beta$

$$P(\gamma, F) = \gamma(\mu_F^2 + \sigma_F^2) = V(\gamma, F) + \gamma^2 \mu_F^2$$

because the rows of X are unit norm. The lower bound on the sampling rate grows very large at small values of SNR. In particular, $\rho > 1$ for SNR smaller than about $-10$dB, meaning that the number of measurements required for recovery of $\Omega$ is larger than the size of $\beta$ if too much noise is present. The results of [61] also imply that this bound is loose at high SNR. The vanishing measurement rate at large SNR seems to verify that result. The next section shows that the thresholding algorithm considered is quite tight with the lower bound at low SNR.

### 5.4 Sufficient Conditions for Thresholding Success

This section considers the thresholding algorithm described in [4] to recover the locations of the constituent non-zero tones of a sparse, wideband signal that is sampled with the RD. The OST algorithm only requires two steps. First, a signal proxy $g$ is formed

$$g = X^*y.$$  

The locations of the non-zero coefficients are then recovered via thresholding of the signal proxy

$$\hat{\Omega} = \{1 \leq i \leq n : |g_i| > \lambda\}$$

where $g_i$ is the $i^{th}$ entry of $g$ and $\lambda$ is a threshold chosen according to the criteria in [4] and described in Section 5.4.1. Note that for the RD, computation of the signal proxy is less complex than a matrix multiplication because of the Fourier structure of $X$. The fast Fourier transform can replace the multiplication with $F$.  

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The theoretical key to ensure recovery is proving that the RD matrix satisfies the Coherence Property (CP). Let X be an (unscaled) \( m \times n \) RD matrix and \( x_i \) denote the \( i^{th} \) column of X. The CP is defined in terms of two measures of coherence on the sensing matrix.

**Definition 6** (Coherence Property). A matrix X with unit-norm columns satisfies the Coherence Property (CP) if the worst-case coherence and average coherence satisfy

\[
\mu \leq \frac{0.1}{\sqrt{2 \log n}}
\]

and

\[
\nu \leq \frac{\mu}{\sqrt{m}}
\]

The worst-case coherence is defined as

\[
\mu = \max_{i,j:i\neq j} |\langle x_i, x_j \rangle|,
\]

and the average coherence is defined as

\[
\nu = \frac{1}{n-1} \max_i \left| \sum_{j:j\neq i} \langle x_i, x_j \rangle \right|.
\]

Intuitively, the worst-case coherence measures the similarity between columns of X, while the average coherence measures the spread of the columns of X in the \( n \)-dimensional unit ball. Theorem \[8\] ensures that X has nearly unit-norm columns with high probability.

### 5.4.1 Coherence Metrics for the Random Demodulator

OST recovers the sparsity pattern of signals sampled with the RD if X satisfies the CP. The worst-case coherence is ensured small enough by the following result.
**Theorem 5.** Suppose $m \geq 200c_4^2 \log^2 n$. Then the worst-case coherence satisfies

$$P\left[ \mu \geq \frac{0.1}{\sqrt{2 \log n}} \right] \leq n^{-1}. \quad (5.6)$$

The proof is in Appendix C.2. The average coherence is bounded with a slightly stronger requirement on $m$.

**Theorem 6.** Suppose $m \geq 2000c_5^2 \log^2 n$ for constant $c_5 \geq \frac{0.1}{\sqrt{10}}$ and $n \geq m + 1$. Then the average coherence satisfies

$$P\left[ \nu \geq \frac{\bar{\mu}}{\sqrt{m}} \right] \leq 2n^{-1} \quad (5.7)$$

where $\bar{\mu} = \frac{0.1}{\sqrt{2 \log n}}$ is the upper bound on the worst-case coherence.

The proof is in Appendix C.3. Taking a union bound over both Theorem 5 and Theorem 6 holding, the CP is satisfied with probably exceeding $1 - 3n^{-1}$ if

$$m \geq \max\{200c_4^2, 2000c_5^2\} \log^2 n.$$ 

Because the CP is satisfied, [4, Theorem 1] can be used to show that OST is successful in recovering the locations of the non-zero frequencies as long as $m$ is large enough.

**Theorem 7 (Theorem 1, [4]).** Suppose that the RD matrix $X$ satisfies the coherence property, and let $\xi$ be independent complex Gaussian of unit variance. Next, choose the threshold

$$\lambda = \max\left\{ \frac{1}{t} 10\mu \sqrt{m \text{SNR}}, \frac{1}{1 - t} \sqrt{2} \right\} \sqrt{2\sigma^2 \log n}$$

for any $t \in (0, 1)$. Then, if $\mu = c_1 m^{-1/6}$ for some $c_1 > 0$ (which may depend on $n$) and $\delta \in \{0\} \cup [2, \infty)$, the OST algorithm satisfies $P[\hat{\Omega} \neq \Omega] \leq 6n^{-1}$ provided $n \geq 128$
and the number of measurements satisfies

\[ m > \max \left\{ 2S \log n, \frac{c_2S \log n}{\text{SNR}_{\text{min}}}, \left( \frac{c_3S \log n}{\text{MAR}} \right)^{\delta/2} \right\} = M(S, n). \]

The probability of failure is with respect to the true $\Omega$ and noise vector $\xi$. Here,

\[ \text{SNR}_{\text{min}} = S\beta_{\text{min}}^2/(\sigma^2 E[\|\xi\|_2^2]), \quad \text{MAR} = S\beta_{\text{min}}^2/\|\beta\|_2^2, \quad c_2 = 16(1 - t)^{-2}, \quad c_3 = 800c_1^2t^{-2}, \]

and $\beta_{\text{min}}$ is the smallest non-zero entry in $\beta$.

The OST algorithm recovers $\Omega$ from measurements taken by the RD with probability exceeding $1 - 9n^{-1}$ as long as $m > \max \left\{ M(S, n), c_6 \log^2 n \right\}$ for constant $c_6 = \max\{200c_4^2, 2000c_5^2\}$. Note that the $c_6 \log^2 n$ term means that the last term in $M(S, n)$ will dominate the maximum for large $n$.

5.4.2 Numerical Experiments

Numerical experiments using the RD and the OST algorithm verify the results of Section 5.4.1. For the experiments, the threshold is set with $t = \frac{1}{2}$ so that

\[ \lambda = \max\{20\mu \sqrt{m\text{SNR}}, 2\sqrt{2}\} \sqrt{2\sigma^2 \log n} \]

where $\sigma^2$ is the variance of the noise, and the signal vector is normalized to unit energy.

The results of the experiments are collected in Figure 5.3 along with the lower bound from Theorem 4 with $\gamma = 0.01$ and $\alpha = 0.1$. The empirical recovery success rate (successes/total trials) is plotted as a function of the measurement rate $\rho$ and the SNR in dB. The total numbers of trials for each set of measurement rate–SNR pairs is 100. A new RD matrix and a new signal vector are drawn for each experiment. The signal vector has a constant sparsity ratio $\gamma = 0.01$ to match the lower bound parameters, and all the coefficients have the same value. The SNR is computed.
Figure 5.3: One-step thresholding (OST) results using the RD to recover the locations of the active tones. A successful trial recovers the locations exactly. The empirical success rate, i.e., the ratio of successful trials to total trials, is plotted for various measurement rate and SNR (in dB) pairs. The lower bound from Theorem 4 (with $\gamma = 0.01$ and $\alpha = 0.1$) is also plotted as a reference and to highlight that OST is nearly optimal for smaller values of SNR. For all these experiments, the sparsity ratio is $\gamma = 0.01$. (Credit: [37].)

as $\text{SNR} = \frac{\|\beta\|^2}{\sigma^2}$. The results confirm that the OST algorithm is nearly optimal, relative to the lower bound, for small values of SNR, which confirms the results in [4] and [31] where the sensing matrix is composed of entries drawn uniformly from various probability distributions. The gap grows larger at higher SNR values.

### 5.5 Summary

The RD architecture is analyzed for the sensing of wideband, underutilized spectrum.

The best performance possible without considering a specific algorithm is first ana-
If the number of measurements and level of sparsity both scale linearly with the bandwidth, then recovery in the asymptotic regime is not possible if the number of measurements does not grow fast enough as the bandwidth increases. A specific algorithm, OST, that is very low-complexity is analyzed and simulated. OST works near-optimally if the SNR is not too large and if the non-zero frequency coefficients are not too small relative to the noise. In the case of OST, recovery of the locations of the non-zero coefficients succeeds if the number of measurements $m$ scales as $m \geq C \cdot \max\{S \log n, \log^2 n\}$ for some constant $C$. Finally, numerical experiments confirm that OST recovers the sparsity pattern and that OST is indeed near optimal if the SNR is small.

Some interesting directions for future work include improving the lower bound, especially in the large SNR regime, and finding upper bounds for the RD. An upper bound could include analyzing thresholding in the asymptotic regime or other achievable schemes that might perform better at larger SNR values. Additionally, considering a prior (non-uniform) distribution on $\beta$, similar to that explored in Chapter 4, could lead to some useful and interesting results.
Chapter 6

Conclusion

6.1 Summary

This thesis explores the notion of optimality in signal processing and the tradeoffs with efficiency. Optimality is primarily concerned with the fidelity of the extracted information by the signal processing system and is a very important consideration. Maximum likelihood and minimum error estimators offer good benchmarks for comparing the optimality of processing schemes. Efficiency is another important consideration in the processing of signals. A processing scheme that requires fewer resources and extracts information with sufficient fidelity in a given application is much preferred to one that requires many more resources. The tradeoff is examined in two specific applications: passive radar processing, and the acquisition of frequency-sparse signals.

In passive radar processing, the system designer does not have control over the illuminating waveform and non-idealities of the waveform must be managed at the receiver. In this case, sub-optimal processing (i.e., mismatched filtering) of the received signal suppresses these non-idealities, or ambiguous peaks, while only being slightly sub-optimal (within 1 dB of the optimal detector for a single target). The suppression
of the ambiguous peaks also allows simpler detection and tracking schemes. In the acquisition of frequency-sparse signals, the number of samples required for recovery is shown to scale linearly with the number of tones and logarithmically with the bandwidth while still offering recovery bounds with small error. Compare this to Nyquist sampling, which requires the number of samples to scale linearly with the bandwidth. In addition, a simple thresholding scheme is shown to recover the locations of tones nearly optimally for a given amount of noise if a small number of errors are allowed.

6.2 Future Work

This chapter concludes with a brief discussion of open problems and possible future directions.

6.2.1 Parameter Estimation versus Compressed Sensing

One major drawback to the compressed sensing framework is the inherent discrete model assumed for the input signal. Many problems assume a discrete grid is placed on a continuous parameter space, such as the grid of frequencies in the RD. Many times physical models do not respect these gridded assumptions, and mismatch can be detrimental to recovery [24]. A future direction of work is to use techniques that work directly on the continuous parameter space to recover the input signal parameters. This is a very old problem, dating back to the work of Prony [27]. Prony’s method and those based on it suffer from a lack of numerical stability, and more recent parametric recovery techniques, e.g., ESPRIT [64], MUSIC [22], and the annihilating filter method [79], use similar ideas but are more stable and robust to noise. This has led to leveraging these techniques in applications to reliably and efficiently recover information from signals (e.g., radar [5,32]).
Very recent work has also been pushing in this exact direction \cite{16, 21, 73}. In these works, the signal model is a superposition of continuous atoms, e.g., sinusoids indexed by a continuous frequency. A norm is defined in terms of these atoms, called the atomic norm \cite{21} or the total variation norm \cite{16}. The atomic norm is shown to reduce to the $\ell_1$ norm if the atoms are 1-sparse vectors and to the trace norm if the atoms are rank-1 matrices, but it offers a much more general framework. The idea of sparsity as a constraint on the input signal through the atomic norm and convex optimization in the recovery still seem to be viable ideas, and a convergence of these ideas that directly leverages the atomic norm seems to offer a great deal of potential.

### 6.2.2 Dynamic Range Considerations in Passive Radar

A primary drawback to passive radar systems is the limitations on the dynamic range of the receiver. The dynamic range is the difference in signal energy between the strongest and weakest signals that a receiver can discern. Because of automatic gain control circuits, the dynamic range is essentially referenced to the power of the strongest signal. Any received signals that are below the dynamic range, relative to the strongest signal, are buried in the noise of the receiver. In a radar system, targets with a weak return, whether due to large distance or small radar cross section, cannot be reliably detected or tracked.

Sources of receiver noise include thermal noise and quantization noise. Accurate characterization of this noise is essential to the performance of passive radar systems, and analyzing the effect of the dynamic range in the processing algorithms ensures that the processing is leveraging the entire dynamic range of the receiver. Some preliminary work along these lines is explored with respect to quantization noise \cite{40}, but a thorough analysis seems to be an open question.
Appendix A

Proofs for Chapter 3

A.1 Derivation of Ambiguity Peak Locations

The locations of the ambiguity peaks given in (3.14) are derived. Start with (3.12)

\[ \chi_{xx}(\tau, \nu) = \sum_{m'k, m'\ell'k'} c_{m'k} c_{m'\ell'k'}^* e^{-j \frac{2\pi}{T_U} (k-k') T_G} e^{-j \frac{2\pi}{T_U} k' T_S (\ell - \ell' + (m-m') L)} e^{-j \frac{2\pi}{T_U} \nu T_S (\ell + m L)} e^{-j \frac{2\pi}{T_U} \nu T_S (\ell - \ell' + (m-m') L)} \]

and concentrate on the exponentials

\[ e^{-j \frac{2\pi}{T_U} (k-k') T_G} e^{j \frac{2\pi}{T_U} k' T_S (\ell - \ell' + (m-m') L)} e^{-j \frac{2\pi}{T_U} \nu T_S (\ell + m L)}. \]  (A.1)

The interest is in the case of \( k = k' \) because for \( k \neq k' \) the \( c_{m'k} \) sum incoherently.

The first term in (A.1) for \( k = k' \) yields

\[ e^{-j \frac{2\pi}{T_U} (k-k') T_G} = 1. \]
In order for the remaining terms to sum coherently when \( k = k' \), the phase of the three remaining exponentials must sum to an integer multiple of \( 2\pi \). Removing the common \( 2\pi \) term from each phase expression, the necessary condition is

\[
\frac{1}{T_U} k'\tau - \frac{1}{T_U} k'T_S(\ell - \ell' + (m - m')L) - \nu T_S(\ell + mL) = d
\]

\[
\frac{k'}{T_U} (\tau - T_S(\ell - \ell' + (m - m')L)) - \nu T_S(\ell + mL) = d
\]  

(A.2)

where \( d \) is an integer. If \( \tau \) and \( \nu \) satisfy (A.2), then (A.1) satisfies

\[
e^{-\frac{2\pi}{T_U}(k-k')T_G} e^{\frac{2\pi}{T_U} k'\tau} e^{-\frac{2\pi}{T_U} k'T_S(\ell-\ell' +(m-m')L)} e^{-i2\pi\nu T_S(\ell+mL)} = e^{-\frac{2\pi\tau}{T_U} 0} e^{i2\pi d} = 1,
\]

and

\[
\chi_{xx}(\tau, \nu) = \sum_{m\ell k, m'\ell' k'} c_{m\ell k} c^*_{m'\ell' k'} \chi_{ww} \left( \tau - (\ell - \ell')T_S - (m - m')LT_S, \nu - \frac{(k - k')}{T_U} \right).
\]

Paired with the relation \( k = k' \), the conditions that \( \tau \) and \( \nu \) must satisfy are (3.14)

\[
k = k' \quad \text{and} \quad \frac{k'}{T_U} (\tau - T_S(\ell - \ell' + (m - m')L)) - \nu T_S(\ell + mL) = d. \quad \text{(A.3)}
\]

### A.2 Derivation of Scattered Pilot Ambiguities

The solutions to (3.16) are of the form specified in (3.17). Start by substituting the relations (3.17)

\[
\tau = T_U \left( \frac{f}{12} + \frac{g}{3} \right) + zT_S
\]

and

\[
\nu = \frac{1}{T_S} \left( \frac{f}{4} + h \right)
\]
\[
\frac{(3b + 12q)}{T_U}(\tau - 4T_S[(a - a') + 17(m - m')])
\]
\[- \nu T_S(4(a + 17m) + b) = d\]
\[
\frac{(3b + 12q)}{T_U} \left( T_U \left( \frac{f}{12} + \frac{g}{3} \right) + z 4T_S - 4T_S[(a - a') + 17(m - m')] \right)
\]
\[- \frac{1}{T_S} \left( \frac{f}{4} + h \right) T_S(4(a + 17m) + b) = d\]
\[
(3b + 12q) \left( \frac{f}{12} + \frac{g}{3} \right) - \left( \frac{f}{4} + h \right) (4(a + 17m) + b) = d\]

where \( z = (a - a') + 17(m - m') \). This is expanded into

\[
\frac{bf}{4} + bg + qf + 4gq - \left( \frac{bf}{4} + bh + (f + 4h)(a + 17m) \right) = d
\]
\[
bg + qf + 4gq - (bh + (f + 4h)(a + 17m)) = d
\]

where all the variables are now integers, and the expression always has a solution.
Appendix B

Proofs for Chapter 4

B.1 Proof of Theorem 1

To show that the CRD satisfies the RIP, the proof technique of [78] for the RD is followed with changes to account for correlations within ε. The entries of Φ_{CRD} are first bounded.

Lemma 1. [A Componentwise Bound] Let Φ_{CRD} be an m × n CRD matrix, and let ℓ be the maximum dependence distance of the corresponding modulating sequence. When 2 ≤ p ≤ 4 log n, then

\[ \mathbb{E}^p \| \Phi_{CRD} \|_{\max} \leq \sqrt{\frac{\ell \cdot 6 \log n}{m}} \]

and

\[ \mathbb{P} \left\{ \| \Phi_{CRD} \|_{\max} > \sqrt{\frac{\ell \cdot 10 \log n}{m}} \right\} \leq n^{-1}. \]

Proof. The first step is the following Lemma [78, Lemma 5].

Lemma 2. [Bounded Entries – RD] Let Φ_{RD} be an m × n RD matrix. When 2 ≤ p ≤ 4 log n, then

\[ \mathbb{E}^p \| \Phi_{RD} \|_{\max} \leq \sqrt{\frac{6 \log n}{m}} \]
\[
\mathbb{P}\left\{ \| \Phi_{\text{RD}} \|_{\text{max}} > \sqrt{\frac{10 \log n}{m}} \right\} \leq n^{-1}.
\]

Assume that \( m \) divides \( n \) and \( \ell \) divides \( \frac{n}{m} \), and write each entry of \( \Phi_{\text{CRD}} \) as

\[
\varphi_{r\omega} = \sum_{j \sim r} \varepsilon_j f_{j\omega} = \sum_{(j \sim r) \mod \ell} \varepsilon_j f_{j\omega} = \varphi_{r\omega}^{(0)} + \ldots + \varphi_{r\omega}^{(\ell-1)} \tag{B.1}
\]

where \([\varepsilon_j]\) is the modulating sequence, \([f_{j\omega}]\) are the entries of the DFT matrix \( F \), and \((j \sim r)_m\) denotes all \( j \) such that \( j \sim r \) and \( (j \mod \ell) = m \). Note that each \( \varphi_{r\omega}^{(i)} \) in (B.1) is a Rademacher series containing \( \frac{n}{m\ell} \) terms and proceed by applying the triangle inequality to (B.1)

\[
\mathbb{E}^p \varphi_{r\omega} = \mathbb{E}^p \sum_{i=0}^{\ell-1} \varphi_{r\omega}^{(i)} \leq \sum_{i=0}^{\ell-1} \mathbb{E}^p \varphi_{r\omega}^{(i)}.
\]

Applying Lemma 2 to each entry in the sum yields

\[
\mathbb{E}^p \| \Phi_{\text{CRD}} \|_{\text{max}} \leq \sum_{i=0}^{\ell-1} \sqrt{\frac{6 \log n}{\ell m}} = \sqrt{\frac{6 \ell \log n}{m}}.
\]

An application of Markov’s inequality provides the probability bound. Let \( M = \| \Phi_{\text{CRD}} \|_{\text{max}} \), then

\[
\mathbb{P}\left\{ M > u \right\} = \mathbb{P}\left\{ M^q > u^q \right\} \leq \left[ \frac{\mathbb{E}^q M}{u} \right]^q,
\]

and choosing \( u = e^{0.25} \mathbb{E}^q M \) yields

\[
\mathbb{P}\left\{ M > 2^{1.25} e^{0.25} \sqrt{\frac{\ell \log n}{m}} \right\} \leq e^{-\log n} = n^{-1}. \tag{B.2}
\]
Finally, a numerical bound yields the desired result.

To complete the proof of Theorem 1, recall that the RIP of order $S$ with constant $\delta_S \in (0, 1)$ holds if

$$|||\Phi^{\ast}_{\text{CRD}} \Phi_{\text{CRD}} - I||| < \delta_S.$$  

Using (4.15), it suffices to show that

$$|||\Phi^{\ast}_{\text{CRD}} \Phi_{\text{CRD}} - \mathbb{E}[\Phi^{\ast}_{\text{CRD}} \Phi_{\text{CRD}}]||| + |||\Delta||| < \delta_S. \quad \text{(B.3)}$$

The quantity $|||\Delta|||$ is bounded in Section 4.5. The first term is bounded by leveraging the results of [78] along with an argument similar to that used in [3] for proving the RIP of Toeplitz matrices. Before continuing, recall that the separation between two rows of $\Phi_{\text{CRD}}$ required for independence between the rows is $\rho = \lceil \frac{m}{n} (\ell - 1) \rceil \leq (\ell - 1)$.

In the following, let $z_r^{\ast}$ denote the $r^{\text{th}}$ row of $\Phi_{\text{RD}}$ or $\Phi_{\text{CRD}}$ depending on the context. Note that $z_r^{\ast} z_r^{\ast\ast}$ is a rank one matrix and that $\Phi_{\text{RD}}^{\ast} \Phi_{\text{RD}} = \sum_{r=1}^{R} z_r^{\ast} z_r^{\ast\ast}$. The following proposition, which is a corollary to [78, Theorems 16 and 18], bounds the triple bar norm of the Gram matrix.

**Proposition 1.** Let $\Phi_{\text{RD}}$ be an $m \times n$ random demodulator matrix and $z_r'$ be an independent copy of $z_r$. Define the random variable

$$Z_{\text{RD}} = |||\Phi_{\text{RD}}^{\ast} \Phi_{\text{RD}} - \mathbb{E}[\Phi_{\text{RD}}^{\ast} \Phi_{\text{RD}}]||| = \left|\left|\left| \sum_{r} (z_r^{\ast} z_r^{\ast\ast} - \mathbb{E} z_r'^{\ast}\mathbb{E} z_r'^{\ast\ast}) \right|\right| \right|.$$  

Then $Z_{\text{RD}}$ satisfies

$$\mathbb{E} Z_{\text{RD}} \leq (\mathbb{E} B^2)^{1/2} \sqrt{CS \log^4 n} \leq \sqrt{\frac{CS \log^5 n}{m}} < \delta$$

and

$$\mathbb{P}\{Z_{\text{RD}} > \delta\} \leq 8 n^{-1}$$
provided \( m \geq C\delta^{-2} \cdot S \log^6(n) \). Note that

\[
B = \max_{r,\omega} |\varphi_{r,\omega}| \leq \sqrt{\frac{10 \log n}{m}}
\]

with probability exceeding \( 1 - n^{-1} \).

The first term in (B.3) is bounded as

\[
Z_{\text{CRD}} = |||\Phi_{\text{CRD}}^* \Phi_{\text{CRD}} - \mathbb{E}\Phi_{\text{CRD}}^* \Phi_{\text{CRD}}|||
\]

\[
= ||| \sum_{r=1}^{R} z_r z_r^* - \mathbb{E} \sum_{r=1}^{R} z_r' z_r'^* |||\]

\[
= ||| \sum_{s=1}^{\rho+1} \left( \sum_{r \in R_s} z_r z_r^* - \mathbb{E} z_r' z_r'^* \right) |||
\]

where \( R_s = \{(\rho + 1)i + s\} \) for \( i = 0, 1, \ldots, m_{\rho+1} - 1 \). The triangle inequality yields

\[
Z_{\text{CRD}} \leq \sum_{s=1}^{\rho+1} \left| \sum_{r \in R_s} z_r z_r^* - \mathbb{E} z_r' z_r'^* \right| = \sum_{s=1}^{\rho+1} Z_s.
\]

Each \( Z_s \) is the norm of a sum of independent random variables, and Proposition 1 is applied to each term. Using Lemma 1 to obtain the value of \( B \) needed in Proposition 1 yields

\[
\mathbb{E} Z_{\text{CRD}} \leq \sum_{s=1}^{\rho+1} \mathbb{E} Z_s \leq \sum_{s=1}^{\rho+1} \sqrt{\frac{C \cdot \ell S \log^5 n}{m}}
\]

\[
= (\rho + 1) \sqrt{\frac{C \cdot \ell S \log^5 n}{m}}.
\]

The theorem requires that \( \mathbb{E} Z_{\text{CRD}} < \delta' \) for \( \delta' \in (0, 1) \), which is achieved as long as

\[
m \geq C\ell(\rho + 1)^2 (\delta')^{-2} S \log^5 n.
\]
A similar appeal to the probability bound in Proposition 1 yields

$$\mathbb{P}\{Z_s > \delta'/(\rho + 1)\} \leq 8n^{-1}$$

if \(m \geq C\ell(\rho + 1)^2(\delta')^{-2}\) and \(S \log^2 n \). Returning to (B.3) produces

$$||| \Phi^*_\text{CRD} \Phi_{\text{CRD}} - I ||| < \delta$$

if \(\delta' < (\delta - \|\Delta\|)\), and the RIP of order \(S\) is satisfied with constant \(\delta_S \leq \delta\) completing the proof of Theorem 1.

**B.2 Proof of Theorem 2**

The coherence and column norms of the matrix \(\Phi_{\text{CRD}}\) are first bounded. The coherence is bounded if the maximum absolute entry of \(X\) (4.10) is bounded

$$\max_{\alpha, \omega} |x_{\alpha, \omega}| = \max_{\alpha, \omega} \left| \sum_{j \neq k} \varepsilon_j \varepsilon_k \eta_{jk} f^*_j f_k \right|.$$  

If the sequence \(\varepsilon\) is not independent, but has maximum dependence distance \(\ell\), then the sum is broken up into smaller sums. Define the sets \(J_a = \{i\ell + a\}\) for \(0 \leq a \leq \ell - 1\) and \(0 \leq i \leq \frac{n}{\ell} - 1\) and the sets \(K_j = \{j - (\ell - 1), \ldots, j + (\ell - 1)\}\). Apply the triangle
inequality twice to $|x_{\alpha \omega}|$

$$
|x_{\alpha \omega}| = \left| \sum_{j \neq k} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} \right|
= \sum_j \left( \sum_{k \in K_j \setminus k \neq j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} \right) + \left( \sum_{k \notin K_j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} \right)
\leq \sum_j \sum_{k \in K_j \setminus k \neq j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} + \sum_{a=0}^{\ell-1} \left( \sum_{j \in J_a, k \notin K_j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} \right)
\leq \sum_j \sum_{k \in K_j \setminus k \neq j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega} + \sum_{a=0}^{\ell-1} \sum_{j \in J_a, k \notin K_j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega}
= E + \sum_{a=0}^{\ell-1} M_a.
$$

Each $M_a$ is a second-order Rademacher chaos because of the indices of summation, $J_a$ and $K_j$, and the following result deals with such a sum.

**Lemma 3.** [78, Lemma 6] Suppose that $m \geq 2 \log n$. Let $[\varepsilon_j]$ be an independent modulating sequence and define $x_{\alpha \omega} = \sum_{j \neq k} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} f_{k\omega}$ and $X = [x_{\alpha \omega}]$. Then

$$
\mathbb{E}^p[|X|_{\max}] \leq 8C \sqrt{\frac{\log n}{m}} \text{ and }
\mathbb{P}\left\{ |X|_{\max} > C \sqrt{\frac{\log n}{m}} \right\} \leq n^{-1}.
$$

Applying Lemma to each $M_a$ yields

$$
\mathbb{E}^p M_a \leq 8C \sqrt{\frac{\log n}{m}} \Rightarrow \mathbb{E}^p \left[ \sum_{a=0}^{\ell-1} M_a \right] \leq 8C \ell \sqrt{\frac{\log n}{m}}.
$$
It follows from Markov’s inequality that
\[
P \left\{ \sum_{a=0}^{\ell-1} M_a > C\ell \sqrt{\frac{\log n}{m}} \right\} \leq n^{-1}.
\]

The quantity \( E \) must also be bounded. Whenever \( n/m \geq \ell \), the term \( \eta_{jk} \) can be dropped because \( \eta_{jk} = 1 \) over the index of summation. In this case, \( E \) is rewritten as
\[
E = \left| \sum_j \sum_{k \in K, k \neq j} \varepsilon_j \varepsilon_k \eta_{jk} f^*_{j\alpha} \tilde{f}_{k\omega} \right|
\]
\[
= \left| \sum_j \varepsilon_j f^*_{j\alpha} \left( \sum_{k \in K, k \neq j} \varepsilon_k \tilde{f}_{k\omega} \right) \right|
\]
\[
= \left| \sum_j \varepsilon_j f^*_{j\alpha} E_2^{(j)} \right| \exp \left( i \cdot \text{phase} \left( E_2^{(j)} \right) \right)
\]
\[
= \left| \sum_j \varepsilon_j f^\prime_{j\alpha} \right| E_2^{(j)}
\]

where
\[
E_2^{(j)} = \sum_{k \in K, k \neq j} \varepsilon_k \tilde{f}_{k\omega},
\]
\( \text{phase} (\cdot) \) is the phase of the complex argument, and \( f^\prime_{j\alpha} = f^*_{j\alpha} \cdot \exp \left( i \cdot \text{phase} \left( E_2^{(j)} \right) \right) \).

In short order, \( E_2^{(j)} \) is bound as \( |E_2^{(j)}| \leq t_2 \forall j \) with high probability so that \( E \) is then bounded as
\[
E \leq \left| \sum_j \varepsilon_j f^\prime_{j\alpha} \right| \cdot t_2 = E_1 \cdot t_2
\]
with high probability. Bounding \( E_1 \) and finding \( t_2 \) requires a result that bounds the norm of a random series generated from a Markov chain.

**Proposition 2.** [66, Corollary 4] Let \( \varepsilon = [\varepsilon_j] \) be a sequence of random variables generated from a Markov chain with \( \varepsilon_j \in \{+1, -1\} \) equally likely. Let the matrix \( \Gamma \) be the matrix defined in Section 4.3.3. Let \( b_i \) for \( 1 \leq i \leq n \) be arbitrary complex numbers.
and let \( f = \left| \sum_{i=1}^{n} \varepsilon_i b_i \right| \). For every \( t \geq 0 \),

\[
P(|f - \mathbb{E}[f]| \geq t) \leq \exp \left( -\frac{t^2}{8\sigma^2 ||\Gamma||^2} \right)
\]

where

\[
\sigma^2 = \sum_{i=1}^{n} |b_i|^2.
\]

This proposition is applied to both \( E_1 \) and \( |E_2^{(j)}| \) with

\[
t_1 = \sqrt{\log n \cdot 8\sigma_1^2 ||\Gamma||^2}
\]

and

\[
t_2 = \sqrt{\log n \cdot 16\sigma_2^2 ||\Gamma||^2}
\]

respectively. As a result, \( \forall j \)

\[
P(E_1 \geq t_1) \leq \exp(-\log n) = n^{-1}
\]

and

\[
P\left(|E_2^{(j)}| \geq t_2\right) \leq n^{-2}.
\]

Finally, \( E \leq t_1 \cdot t_2 \) except with probability \( 2n^{-1} \). To finish the calculation, note that

\[
\sigma_1^2 = \sum_{j=0}^{n-1} |f_{ja}|^2 = 1
\]

and

\[
\sigma_2^2 = \sum_{k \in K, k \neq \ell} |f_{kw}|^2 = 2\frac{\ell - 1}{n}.
\]
Hence,
\[
t_1 \cdot t_2 = \sqrt{\log n \cdot 8\sigma_1^2 ||\Gamma||^2} \cdot \sqrt{\log n \cdot 16\sigma_2^2 ||\Gamma||^2}
\]
\[
= \log n \cdot 8\sqrt{2} ||\Gamma||^2 \sqrt{\sigma_1^2 \sigma_2^2}
\]
\[
= \frac{\log n}{\sqrt{n}} 16\sqrt{\ell - 1} ||\Gamma||^2.
\]

Finally, all of this results in the following probability bound for the matrix $X$
\[
P\left( ||X||_{\text{max}} \geq C\ell \sqrt{\frac{\log n}{m} + t_1 \cdot t_2} \right) \leq 3n^{-1}.
\]

Note that $\lim_{n \to \infty} (\log n / \sqrt{n}) = 0$, so the second term can be made arbitrarily small by requiring a large enough $n$. This leads to the following statements about the coherence, $\mu = \max_{\alpha \neq \omega} |\langle \phi_\alpha, \phi_\omega \rangle|$, and column norms of a CRD matrix.

**Lemma 4.** [Coherence] Suppose that $m \geq 2 \log n$. An $m \times n$ CRD matrix satisfies
\[
P\left( \mu \geq C\ell \sqrt{\frac{\log n}{m} + \frac{\log n}{\sqrt{n}} 16\sqrt{\ell - 1} ||\Gamma||^2} \right) \leq 3n^{-1}.
\]

**Lemma 5.** [Column Norms] Suppose the sampling rate satisfies
\[
m \geq 4 \cdot C\ell^2 \delta^{-2} \log n
\]
and that $n$ is large enough so that
\[
\frac{\log(n)}{\sqrt{n}} \leq \frac{\delta}{32 \sqrt{(\ell - 1) ||\Gamma||^2}}.
\]
Then, an $m \times n$ CRD matrix satisfies
\[
P\left\{ \max_{\omega} \left| ||\phi_\omega||_2^2 - 1 \right| \geq \delta \right\} \leq 3n^{-1}.
\]
Finally, the following theorem proves recovery results.

**Theorem 8.** [78, Corollary 15] Suppose that the sampling rate satisfies

\[ m \geq C[S \log n + \log^3 n]. \]

Draw an \( m \times n \) RD matrix such that

\[
\mathbb{P}\left\{ \mu \geq C \sqrt{\frac{\log n}{m}} \right\} \leq n^{-1} \quad \text{and} \quad \mathbb{P}\left\{ \max_{\omega} \| \phi_\omega \|^2 - 1 \geq \delta \right\} \leq n^{-1}.
\]

Let \( s \) be an \( S \)-sparse vector drawn according to the random signal model in Section 4.2. The solution \( \hat{s} \) to the convex program (4.20) satisfies \( \hat{s} = s \) except with probability \( 8n^{-1} \).

Theorem 2 is the result of applying Lemmata 4 and 5 to Theorem 8. The increased requirement on \( m \) and the additional requirement on \( n \) are needed to ensure the coherence and column norms are satisfactory to ensure recovery. Additionally, the probability of recovery failing increases slightly to \( 12n^{-1} \).

**B.3 Correlation and Dependence of Bipolar Sequences**

A brief proof is given to show that if two entries in the modulating sequence are uncorrelated then they are independent for the bipolar sequences used in the RD and CRD. The sequences, denoted by \( [\varepsilon_j] \) for \( j = 1, \ldots, n \), of concern have two defining characteristics

1. \( \varepsilon_j \in \{+1, -1\} \) and
2. \( \mathbb{P}\{\varepsilon_j = +1\} = \mathbb{P}\{\varepsilon_j = -1\} = 1/2. \)
The autocorrelation in this case is expressed as

$$\mathbb{E}[\varepsilon_j \varepsilon_{j+k}] = P\{\varepsilon_j = \varepsilon_{j+k}\} - P\{\varepsilon_j \neq \varepsilon_{j+k}\}.$$

A maximum dependence distance of $\ell$ means that $\mathbb{E}[\varepsilon_j \varepsilon_{j+k}] = 0$ for $|k| \geq \ell$, or

$$P\{\varepsilon_j = \varepsilon_{j+k}\} = P\{\varepsilon_j \neq \varepsilon_{j+k}\}$$

for $|k| \geq \ell$. This implies that

$$P\{\varepsilon_{j+k} = +1|\varepsilon_j = +1\} = P\{\varepsilon_{j+k} = +1|\varepsilon_j = -1\}$$

for $|k| \geq \ell$. Characteristic 2 requires that

$$P\{\varepsilon_{j+k} = +1|\varepsilon_j = +1\} + P\{\varepsilon_{j+k} = +1|\varepsilon_j = -1\} = 1,$$

and combining these two constraints results in

$$P\{\varepsilon_{j+k} = +1|\varepsilon_j = +1\} = P\{\varepsilon_{j+k} = +1|\varepsilon_j = -1\} = 1/2.$$

The same argument applies to $\varepsilon_{j+k} = -1$, and results in

$$P\{\varepsilon_{j+k} = -1|\varepsilon_j = +1\} = P\{\varepsilon_{j+k} = -1|\varepsilon_j = -1\} = 1/2.$$

Combining these yields

$$P\{\varepsilon_{j+k} = a|\varepsilon_j = b\} = P\{\varepsilon_{j+k} = a\}$$

for any $a, b \in \{+1, -1\}$, which is the condition for independence.
Appendix C

Proofs for Chapter 5

C.1 Proof of Theorem 4

The crucial result is [61, Lemma 2].

Lemma 6. [61, Lemma 2] Let \( \beta \) be a randomly generated vector, as described in Section 5.2, with sparsity pattern \( \Omega \), sparsity rate \( \gamma \), and variance \( V(\gamma, F) \). Fix \( 0 < \rho < \infty \), and let \( X_n \) be a sequence of \( m \times n \) sampling matrices with \( m = \lceil \rho n \rceil \) for \( n = 1, 2, \ldots \). In the recovery of the sparsity pattern \( \Omega \) with distortion \( d(\Omega, \hat{\Omega}) \leq \alpha \), the pair \((\rho, \alpha)\) is not achievable asymptotically (as \( n \to \infty \)) if

\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_{X_n} I(\Omega; y) < R(\gamma, \alpha) \tag{C.1}
\]

where \( I(\Omega; y) \) is the mutual information between the sparsity pattern \( \Omega \) and the noisy samples \( y \), and \( R(\gamma, \alpha) \) is given in (5.3).

The first step is to fix the problem size \( n \), and then condition on a realization of the matrix \( X_n \). The theorem thus requires bounding \( I(\Omega; y) \), which is first bounded by the mutual information between the noiseless samples and the noisy samples. The objects \( \Omega \to X_n \beta \to y \) form a Markov chain, so the data processing inequality [25]
yields
\[ I(\Omega; y) \leq I(X_n\beta; y). \quad \text{(C.2)} \]

The right hand side of (C.2) is bounded using (5.1) as
\[
I(X_n\beta; y) \leq \max_u I(u; u + w) \\
\leq \frac{1}{2} \log |I_m + V(\gamma, F)X_nX_n^*| 
\]
where \( u \) is a random vector with the same covariance as \( X_n\beta, \mathbb{E}[uu^*] = V(\gamma, F)X_nX_n^*. \)
A well-known result is that a Gaussian input maximizes the mutual information of a Gaussian noise channel, so the maximum occurs for \( u \) a jointly Gaussian vector with covariance \( V(\gamma, F)X_nX_n^*. \) The mutual information is evaluated as half the log of the determinant of the total covariance, leading to the second inequality \[25\].

Recall that \( X_nX_n^* = I_m \) for the scaled RD matrix. As a result
\[
I(X_n\beta; y) \leq \frac{1}{2} \log |I_m + V(\gamma, F)I_m| \\
\leq \frac{m}{2} \log (1 + V(\gamma, F))
\]
for any scaled RD matrix \( X_n \), and returning to (C.2)
\[
I(\Omega; y) \leq \frac{m}{2} \log (1 + V(\gamma, F)).
\]
Because \( \frac{m}{n} \to \rho \) as \( n \to \infty \), (C.1) is upper bounded as
\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_{X_n} I(\Omega; y) \leq \frac{m}{2n} \log (1 + V(\gamma, F)) \\
\leq \frac{\rho}{2} \log (1 + V(\gamma, F)).
\]
Combining this result with Lemma[1] and rearranging terms yields Theorem[4]
C.2 Proof of Theorem 5

The theorem is an application of the following theorem with a stricter requirement on $m$.

**Theorem 9.** [78, Theorem 9] Suppose $m \geq 2 \log n$ for an $m \times n$ RD matrix $X$. Then the worst-case coherence $\mu$ satisfies

$$P \left[ \mu \geq c \sqrt{\frac{\log n}{m}} \right] \leq n^{-1}$$

where $c$ is a constant.

Using Theorem 9 and requiring that $m \geq 200c^2 \log^2 n$ yields Theorem 5.

C.3 Proof of Theorem 6

The average coherence is bounded by bounding the sum in (5.5). The inner product is written as

$$\langle x_i, x_j \rangle = \delta_{ij} + \sum_{r,s: r \neq s} \varepsilon_r \varepsilon_s \eta_{rs} f_{ri}^* f_{sj}$$

where $\delta_{ij}$ is the Kronecker delta, $f_{ri}$ is the $(r, i)$ entry of the (unitary) DFT matrix $F$, $\eta_{rs} = \langle h_r, h_s \rangle$ is the inner product between the $r^{th}$ and $s^{th}$ columns of the matrix $H$, and $\varepsilon_i$ is the $i^{th}$ entry of the Rademacher sequence. Bounding (5.5) begins with rewriting the sum as

$$\left| \sum_{j: j \neq i} \langle x_i, x_j \rangle \right| = \left| \sum_{j: j \neq i} \sum_{r,s: r \neq s} \varepsilon_r \varepsilon_s \eta_{rs} f_{ri}^* f_{sj} \right|$$

\begin{align*}
&= \left| \sum_{r,s: r \neq s} \varepsilon_r \varepsilon_s \eta_{rs} f_{ri}^* \left( \sum_{j: j \neq i} f_{sj} \right) \right| \\
&= \left| \sum_{r,s: r \neq s} \varepsilon_r \varepsilon_s \eta_{rs} f_{ri}^* A_{si} \right| \quad \text{(C.3)}
\end{align*}
where

\[ A_{si} = \sum_{j \neq i} f_{sj} \]

\[ = \begin{cases} \sqrt{n} - f_{si}, & s = 0 \\ -f_{si}, & s \neq 0 \end{cases} \]

\[ = -f_{si} + \delta_s \sqrt{n} \]

and

\[ \delta_s = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0 \end{cases} . \]

Substituting this back into (C.3) yields

\[
\left| \sum_{j \neq i} \langle x_i, x_j \rangle \right| = \left| \sum_{r,s,r \neq s} \epsilon_r \epsilon_s \eta_{rs} f_{ri}^* f_{si} + \sum_{r \neq 0} \epsilon_r \epsilon_0 \eta_{r0} f_{ri}^* \right| \\
\leq \left| \sum_{r,s,r \neq s} \epsilon_r \epsilon_s \eta_{rs} f_{ri}^* f_{si} \right| + \sqrt{n} \left| \sum_{r \neq 0} \epsilon_r \epsilon_0 \eta_{r0} f_{ri}^* \right| \tag{C.4}
\]

by the triangle inequality. Using (C.4) in (5.5) yields

\[
\frac{1}{n-1} \max_i \left| \sum_{j \neq i} \langle x_i, x_j \rangle \right| \leq \frac{1}{n-1} \max_i \left| \sum_{r,s,r \neq s} \epsilon_r \epsilon_s \eta_{rs} f_{ri}^* f_{si} \right| + \frac{\sqrt{n}}{n-1} \max_i \left| \sum_{r \neq 0} \epsilon_r \epsilon_0 \eta_{r0} f_{ri}^* \right| \\
\leq \frac{1}{n-1} \max_i B_i^{(1)} + \frac{\sqrt{n}}{n-1} \max_i B_i^{(2)} \tag{C.5}
\]

where

\[ B_i^{(1)} = \left| \sum_{r,s,r \neq s} \epsilon_r \epsilon_s \eta_{rs} f_{ri}^* f_{si} \right| \]

and

\[ B_i^{(2)} = \left| \sum_{r \neq 0} \epsilon_r \epsilon_0 \eta_{r0} f_{ri}^* \right| . \]
Lemma 2 of Appendix B.1, which is [78, Lemma 6], is applied to $B_i^{(1)}$ to show that
\[ P \left[ \max_i B_i^{(1)} \geq c \sqrt{\frac{\log n}{m}} \right] \leq n^{-1} \]
where $c$ is a constant. Lemma 3 of Appendix B.2, which is [78, Lemma 5], is applied to $B_i^{(2)}$ to show that if $m \geq 2 \log n$, then
\[ P \left[ \max_i B_i^{(2)} \geq \sqrt{\frac{10 \log n}{m}} \right] \leq n^{-1}. \]
If $m \geq 2000c^2 \log^2 n$ with $c \geq \frac{0.1}{\sqrt{10}}$, then (C.5) is bounded as
\[
\nu = \frac{1}{n-1} \max_i \left| \sum_{j:j \neq i} \langle x_i, x_j \rangle \right| \leq \frac{1}{n-1} \frac{0.1}{\sqrt{20 \log n}} + \frac{\sqrt{n}}{n-1} \frac{0.1}{c \sqrt{2 \log n}}.
\]
If additionally $m \leq n - 1$, then
\[
\nu \leq \frac{\bar{\mu}}{\sqrt{m}}
\]
with probability exceeding $1 - 2n^{-1}$ where $\bar{\mu} = \frac{0.1}{\sqrt{2 \log n}}$ is the upper bound on $\mu$. This concludes the proof of Theorem 6. Note that the lower bound on $c$ is only important when considering small $n$, and the important setting for interesting problems is likely to be for large $n$. The condition $m \leq n - 1$ is likely to cover most interesting situations.
Appendix D

List of Publications

D.1 Journal Articles


D.2 Conference Papers


Bibliography


