Hyperbolic hypergeometric monodromy groups and geometric finiteness

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Abstract

The subject of this thesis is hyperbolic hypergeometric monodromy groups. I studied the monodromy groups $H(\alpha, \beta)$ for ${}_nF_{n-1}$ hypergeometric differential equation $Du = 0$, where $D = (\theta+\beta_1-1)\cdots(\theta+\beta_n-1)-z(\theta+\alpha_1)\cdots(\theta+\alpha_n)$ and $\theta = z\frac{d}{dz}, \alpha, \beta \in \mathbb{Q}^n$. $H(\alpha, \beta)$ is generated by the local monodromies A, B, C which correspond to $0, 1, \infty$ respectively. Let G is the zariski closure of $H(\alpha, \beta)$. In the self-dual case, G is either finite, $Sp(n)$, or $O(n)$. I focus on G orthogonal and of signature $(n-1, 1)$ over $\mathbb{R}$.

The first result is that the odd hyperbolic hypergeometric monodromy groups $H(\alpha, \beta)$ are of infinite index in $G(\mathbb{Z})$. In this case $H(\alpha, \beta)$ is called 'thin', which is equivalent to saying that the fundamental domain has infinite volume. It was shown that there are three families of odd hyperbolic hypergeometric monodromy groups, and we show that $H(\alpha, \beta)$ is thin for all but one case by showing that a finite index subgroup of $H(\alpha, \beta)$ is commensurable with a reflection subgroup of an infinite index subgroup of $G(\mathbb{Z})$. In the remaining case, I used an algorithm from [5] to show that the fundamental domain for a finite index reflection subgroup of $H(\alpha, \beta)$ is thin. This covers the second chapter.

In the third chapter, I investigated the full reflection subgroup of $G(\mathbb{Z})$, the 2-reflection subgroup containing all 2-reflections, and $H(\alpha, \beta)$. When Γ be one of them, I looked at limit set $\Lambda(\Gamma)$, the exponent of divergence $\delta(\Gamma)$ of the Poincare series, and the bottom of the spectrum of the Laplacian on $L^2(\Gamma\backslash\mathbb{H}^{n-1})$. The second result is 'geometric finiteness' for discrete groups generated by finitely many reflections, which means its fundamental domain has finitely many faces. It follows that $H(\alpha, \beta)$'s which were shown to be thin in [5] and in the second chapter are not only thin, but have 'sparse' limit sets.

In the last chapter, the first example for which the monodromy group is geometrically finite and thin is given. I found a fundamental domain using numerical com-

putation, and showed that it is the fundamental domain for $H(\alpha, \beta)$ using Poincare's polyhedron theorem.

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To my wife Ran and my daughter Jaelyn

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Chapter 1

Introduction

Suppose $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in \mathbb{C}^*$ with $a_j \neq b_k$ for all $j, k = 1, 2, \cdots, n$. A hypergeometric group with numerator parameters $a_1, a_2, \cdots, a_n$ and denominator parameters $b_1, b_2, \cdots, b_n$ is a subgroup of $GL(n, \mathbb{C})$ generated by elements

$$h_0, h_1, h_\infty \in GL(n, \mathbb{C}) \tag{1.1}$$

such that

$$det(t - h_\infty) = \prod_{j=1}^{n} (t - a_j) \tag{1.2}$$

$$det(t - h_0) = \prod_{j=1}^{n} (t - b_j) \tag{1.3}$$

and

$$h_1 = (h_0 h_\infty)^{-1} \tag{1.4}$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{C}$ and consider the linear differential equation

$$Du = 0 \tag{1.5}$$

where

$$D = (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n) \tag{1.6}$$

and $\theta = z\frac{d}{dz}$.

Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{C}$ be such that $a_j = e^{2\pi i \alpha_j}$ and $b_j = e^{2\pi i \beta_j}$ for $j = 1, \cdots, n$. Then the monodromy group of the hypergeometric equation

$$D(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n)u = 0 \tag{1.7}$$

is a hypergeometric group with parameters $a_1, \cdots, a_n, b_1, \cdots, b_n$.

Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{C}$ be defined by

$$P(t) = \prod_{j=1}^{n} (t - e^{2\alpha_j}) = t^n + A_1 t^{n-1} + \cdots + A_n,$$
$$\text{Q(t)}{=}\prod_{j=1}^{n} (t - e^{2\beta_j}) = t^n + B_1 t^{n-1} + \cdots + B_n$$

and let $A, B \in GL(n, \mathbb{C})$ be given by

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_n \\ 1 & 0 & \cdots & 0 & -A_{n-1} \\ 0 & 1 & \cdots & 0 & -A_{n-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_n \\ 1 & 0 & \cdots & 0 & -B_{n-1} \\ 0 & 1 & \cdots & 0 & -B_{n-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix} \tag{1.8}$$

Then $H(\alpha, \beta)$ is the group generated by A and B.(See reference [2].)

Equation (1.5) is regular away from $\{0, 1, \infty\}$, and its monodromy group $H(\alpha, \beta)$ is generated by the local monodromies A, B, C, where $C = A^{-1}B$. We restrict $\alpha_j, \beta_j \in \mathbb{Q}$, as examples from geometry are quasi-unipotent, and further that $H(\alpha, \beta)$ is contained in $GL_n(\mathbb{Z})$. This happens if the characteristic polynomial of A and B, whose roots are $e^{2\pi i \alpha_j}$ and $e^{2\pi i \beta_j}$ respectively, are products of cyclotomic polynomials. In particular for $n \geq 2$ there are only finitely many choices for the pairs α, β in $\mathbb{Q}^n$. Beukers and Heckman determined in [2] the Zariski closure $G(\alpha, \beta)$ of $H(\alpha, \beta)$ explicitly in terms of α, β. From the condition that $H(\alpha, \beta)$ is self-dual $G(\alpha, \beta)$ is either finite, $Sp(n)$ or $O(n)$. The signature of the quadratic form in the orthogonal case is determined by α, β.

Until recently, there were few cases for which the group $H(\alpha, \beta)$ itself is known, other than the cases when $H(\alpha, \beta)$ is finite. The interest in $H(\alpha, \beta)$ has been mainly as to whether it is of finite or infinite index in $G(\mathbb{Z}) = G(\alpha, \beta)[\mathbb{Z}]$. In the first case we say $H(\alpha, \beta)$ is *arithmetic*, and in the second case we say $H(\alpha, \beta)$ is *thin*. This distiction is important in various associated number theoretic problems(See [9]).

For $n = 2$ and $n = 3$, it is known that all the $H(\alpha, \beta)$'s are arithmetic. For $n = 4$ Brav and Thomas [3] showed that $\alpha = (0, 0, 0, 0), \beta = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$ with $G = Sp(4)$ is thin. In fact they show that $H(\alpha, \beta)$ is a free product of generators A and C, by showing A and C play generalized ping-pong on certain subsets of $\mathbb{P}^3$. Venkataramana [11] showed for n even and $\alpha = (\frac{1}{n+1}, \cdots, \frac{n}{n+1}), \beta = (\frac{1}{2}, \frac{1}{n}, \cdots, \frac{n-1}{n})$, $H(\alpha, \beta)$ is arithmetic in $Sp(n, \mathbb{Z})$. On the other hand, Fuchs, Meiri and Sarnak [5] is concerned with the case that $G(\alpha, \beta)$ is orthogonal of signature $(n-1, 1)$ over $\mathbb{R}$. They call such $H(\alpha, \beta)$ hypergeometric hyperbolic monodromy groups. They classified all the cases for which $H(\alpha, \beta)$ is a hypergeometric hyperbolic monodromy group. Also, they showed that some such families of $H(\alpha, \beta)$ are thin in $O_f(\mathbb{Z})$. $O_f(\mathbb{Z})$ could be thought as same as $O(L)$ with an integral quadratic lattice L. They used results of Vinberg [14] and Nikulin [8] on reflection groups $R(L)$ and $R_2(L)$, which are the full reflection group and subgroup of $R(L)$ generated by all the 2-root vectors in L in hyperbolic spaces, as well as their "distance graph" to relate hyperbolic hypergeometric monodromy groups and reflection groups.

All the results in this thesis are also concerned with the same cases as in [5]. One of the main results is to show that certain hyperbolic hypergeometric monodromy groups are not only thin, but have limit sets whose dimension is less than $n-2$. In particular this holds for all the examples that are shown to be thin in [5] as well as some new cases(See section 2.4 and 2.5) covering all of Table 3 in [5]. The proof that $H(\alpha, \beta)$ is small in this sense gives no information of the group itself. Our other main result is to show that one of these groups $\alpha = \left(0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\right), \beta = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}\right)$ is geometrically finite and has infinite index in $O_f(\mathbb{Z})$, and has an explicit presentation

$$H(\alpha, \beta) =< A, B | A^{10}, B^6, (A^{-1}B)^2 >.$$

(See section 4.2.) While this is the first such example and the only one that our research revealed, there may be many others.

The proof that the $H(\alpha, \beta)$s are thin is based on the certificate in [5] for the image of $H(\alpha, \beta)$ in $O(L)/R_2(L)$ being finite. While the limit set of $R_2(L)$ is all of S^{n-2} (See section 3.1), we show that a finite index subgroup of $H(\alpha, \beta)$ is contained in a geometrically finite infinite volume reflection group (See section 3.2). From this it follows that the dimension of the limit set Λ of $H(\alpha, \beta)$ is less than $n-2$. The completion of table 3 involves various algorithms for computing fundamental cells for groups of motions generated by finitely many reflections. The proof that $H\left(\left(0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\right), \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}\right)\right)$ is geometically finite goes via computing a Dirichlet fundamental domain in $\mathbb{H}^{n-1}$. It has 42 sides and it is shown to be a fundamental domain using Poincare's polyhedron theorem.

Chapter 2

Discrete groups generated by reflections in hyperbolic spaces

2.1 Convex polyhedra in hyperbolic spaces

The following is due to Vinberg[12].

In n-dimensional hyperbolic space $\mathbb{H}^n$ an intersection

$$P = \bigcap_{i \in I} \Pi_i^-$$

of a family of halfspaces is called convex polyhedron if

- Every bounded set is cut by only a finite number of hyperplanes Π_i
- P contains a nonempty open set.

The polyhedron P is called nondegenerate if

- The hyperplanes Π_i have no common point including a point at infinity.
- There is no hyperplane orthogonal to all of Π_i.

Given any vector $e \in V$ satisfying

$$(e,e) > 0,$$

denote by Π_e the hyperplane of $\mathbb{H}^n$ orthogonal to e. Let P be a convex polyhedron and e_i, $i \in I$, vectors of V such that $(e_i, e_i) > 0$ and $\Pi_i^- = \Pi_{e_i}^-$. The gram matrix $G = G(P)$ of the vector system $\{e_i\}$ is called Gram matrix of the polyhedron P. Its elements g_{ij} satisfy the condition $g_{ij} \leq (g_{ii}g_{jj})^{\frac{1}{2}}$ and have the following geometric significance :

1. If $g_{ij} \leq -(g_{ii}g_{jj})^{\frac{1}{2}}$, Π_i and Π_j do not intersect, and the distance ρ_{ij} between them is given by

$$\rho_{ij} = arccosh\left(-\frac{g_{ij}}{(g_{ii}g_{jj})^{\frac{1}{2}}}\right)$$

2. if $|g_{ij}| < (g_{ii}g_{jj})^{\frac{1}{2}}$, Π_i and Π_j intersect, and the angle ϕ_{ij} between them is given by

$$\phi_{ij} = arccos\left(-\frac{g_{ij}}{(g_{ii}g_{jj})^{\frac{1}{2}}}\right)$$

A symmetric matrix A with its elements $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$ will be called critical if it is not positive definite, yet every proper principal submatrix of it is positive definite. A critical matrix is obviously irreducible.

For given a symmetric matrix $A = (a_{ij})(i, j \in I$ and any set $S \subset I$, we denote by A_S the principal submatrix of the matrix A formed by elements $a_{ij},\ i, j \in S$.

Every symmetric matrix A is uniquely expandable into a direct sum of irreducible matrices, which we call its components. We denote by $A^+(A^0)$ the direct sum of all the positive definite(degenerate nonnegative definite) components of A, and by A^- the direct sum of all components which are not nonnegative definite.

Let P be a nondegenerate finite convex polyhedron without dihedral angle exceeding $\frac{\pi}{2}$, and G be its Gram matrix. P can be shown to be finite volume without looking at the fundamental domain itself, using the following proposition of Vinberg :

Proposition 2.1.1. *[13] The polyhedron P is finite volume if and only if for any critical principal submatrix G_S of the matrix G the following are true :*

- *If $G_S = G_S^0$, then a T exists such that $G_T = G_T^0$ and rank $G_T = n-1$*
- *If $G_S = G_S^-$, then $\{v \in P : (e_i, v) = 0$ for $i \in S\} = \{0\}$.*

2.2 Reflection groups in hyperbolic spaces

Let Γ be a discrete group of motions of $\mathbb{H}^n$ generated by reflections in hyperplanes. The mirrors of all the reflections in Γ divides $\mathbb{H}^n$ into convex polyhedra, which will be termed the cells of the group Γ. The cells are moved transitively by Γ, and every cell is a fundamental domain for Γ. Let P be a cell, and P_i $(i \in I)$ be its $(n-1)$-dimensional faces, and let R_i be the reflection in the hyperplane containing P_i. The angle between any pair of adjacent faces P_i, P_j is of the form $\frac{\pi}{n_{ij}}$, where $n_{ij} \in \mathbb{Z}^{>0} \cup \{\infty\}$. If we denote by $G = G(P)$ the Gram matrix of P, then $\frac{g_{ij}}{(g_{ii}g_{jj})^{\frac{1}{2}}} = -cos\frac{\pi}{n_{ij}}$. If P_i and P_j are not adjacent we have $g_{ij} < -(g_{ii}g_{jj})^{\frac{1}{2}}$ by Andreev [1].

If the faces P_i and P_j are not adjacent, we put $n_{ij} = \infty$. The group Γ is generated by reflections R_i $(i \in I)$, with the defining relations

$$R_i^2 = 1,\ (R_iR_j)^{n_{ij}} = 1,$$

i.e. it is an abstract Coxeter group with exponents n_{ij}.

The Coxeter group Γ with generators R_i $(i \in I)$ and exponents n_{ij} is described by a Coxeter graph, which is constructed as follows. For each $i \in I$ the graph has corresponding vertex v_i. If P_i and P_j are adjacent and $n_{ij} < \infty$, the vertices v_i and v_j are joined by an $(n_{ij} - 2)$-tuple branch or by a simple branch marked n_{ij}. If P_i and P_j are adjacent and $n_{ij} = \infty$, v_i and v_j are joined by a boldface branch, or by a simple branch marked ∞. If P_i and P_j are not adjacent, v_i and v_j are joined by a dotted line.

Definition 2.2.1. *A Euclidean lattice L is called reflective if $R(L)$ is of finite index in $O(L)$. It is also called 2-reflective if $R_2(L)$ is of finite index in $O(L)$.*

Now we need the following ingredients to show that $R(L)$ or $R_2(L)$ is thin.

Proposition 2.2.2. *[14] Let L be a hyperbolic lattice with $|O(L)/R(L)| < \infty$. Then for any isotropic vector $v \in L$ the Euclidean lattice $(\mathbb{Z}v)^{\perp}/\mathbb{Z}v$ reflective.*

Theorem 2.2.3. *[15] $O(L) = R(L) \rtimes H = R_2(L) \rtimes (R'(L) \rtimes H)$, where $R'(L)$ is the Coxeter group generated by reflections, which are not 2-roots, and H is the symmetry group of the fundamental domain for $R(L)$.*

Theorem 2.2.4. *[13] Let Γ be a discrete subgroup of $\mathbb{H}^n$ generated by reflections. Let $H_e^- = \{x \in V : (x, e) \leq 0\}$ and $H_e^0 = \{x \in V : (x, e) = 0\}$. The fundamental domain for Γ can be obtained as the following :*
Take an arbitrary point v_0 in V. Consider the vector set

$$R = \{e \in L : (e, e) > 0 \text{ and } R_e \in \Gamma\}.$$

The set of reflections $\{R_e : (e, v_0) = 0\}$ divides V into convex regions. Let P_0 be the closure of one of these components that contains v_0. $e_1, e_2, \cdots, e_k$ are found from

$$P_0 = \cap_{i=1}^{k} H_{e_i}^-.$$

When $n > k$, e_n is selected if $(e_n, e_i) \leq 0$ for all $i < n$,$(e_n, v_0) \leq 0$ and it minimizes the distance $d(v_0, H_{e_n}^0)$.

Then $P = \cap_{i=1} H_e^-$.

Note that in Theorem 2.2.4 if a R_e is given, we need to know if $R_e \in \Gamma$ to use this algorithm. For $\Gamma = R(L)$ or $R_2(L)$ we could use this algorithm perfectly. However, for example, we cannot use this algorithm for finitely generated subgroups of $R_2(L)$ in general.

2.3 Odd hyperbolic hypergeometric monodromy groups

We first examine which hyperbolic hypergeometric monodromy groups are odd. We use the notation and classification of [5].

Recall that $H(\alpha, \beta)$ is generated by two matrices

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & -b_{n-1} \\ 0 & 1 & \cdots & 0 & -b_{n-2} \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -b_1 \end{pmatrix} \tag{2.1}$$

where the characteristic polynomials of A and B are products of cyclotomic polynomials and denote

$$C := A^{-1}B = \begin{pmatrix} 1 & 0 & \cdots & 0 & -(a_{n-1}+b_{n-1}) \\ 0 & 1 & \cdots & 0 & -(a_{n-2}+b_{n-2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -(a_1+b_1) \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}, v := \begin{pmatrix} a_{n-1}+b_{n-1} \\ a_{n-2}+b_{n-2} \\ \vdots \\ a_1+b_1 \\ 2 \end{pmatrix} \tag{2.2}$$

Let L be the lattice generated by the vectors $v, Bv, \cdots, B^{n-1}v$. $H(\alpha, \beta)$ preserves the lattice L, and there exists a unique(up to a scalar product) non-zero integral quadratic form $(v, w) = v^tFw$, where $F \in M_n(\mathbb{Z})$ corresponds to a quadratic form of signature $(n-1, 1)$ and $A^tFA = B^tFB = F$. We normalize the quadratic form to have the minimal absolute integer value with $(v, v) < 0$. We call L and $H(\alpha, \beta)$ *even* if $(v, v) \in 2\mathbb{Z}$ for all $v \in L$, and *odd* if it is not even. Note that this definition is different from [5], where the quadratic form is normalized to have $(v, v) = -2$. We have L

is even when $(v,v)=-2$ and odd when $(v,v)=-1$. For any $w\in L$ (v,w) equals to $\frac{(v,v)w_n}{2}$, where w_n is the n-th coordinate of $w\in\mathbb{Z}^n$. If we let $f_{i,j}=(B^iv,B^jv)$, it depends on $|i-j|$, since $(Bv,Bw)=(Bv)^tF(Bw)=v^t(B^tFB)w=v^tFw=(v,w)$. Therefore L is odd if and only if the last coordinate of B^kv is even for $k=1,2,\cdots,n-1$.

Lemma 2.3.1. *Let $\{d_i\}$ be a sequence with $d_0=1, d_1=-b_1$ and defined inductively by the following equation :*

$$d_k=\sum_{i=0}^{k-1}-d_ib_{k-i}$$

for $k\leq n-1$. Then the last(n-th) row of B^k is given by

$$(0\ \cdots\ 0\ d_0\ d_1\ \cdots\ d_k)$$

and the last coordinate of B^kv is $\sum_{i=0}^{k-1}d_i(a_{k-i}-b_{k-i})$.

Proof. This can be easily shown by mathematical induction. □

Proposition 2.3.2. *$H(\alpha,\beta)$ is odd if and only if $a_i-b_i\in 2\mathbb{Z}$. for all $1\leq i\leq n-1$.*

Proof. For 'if' part, it is clear from Lemma 2.3.1. For 'only if' part, assume that some of a_i-b_i is odd. Let t be the minimum of such i. Then the last coordinate of B^tv is odd from Lemma 2.3.1. □

Now we can classify all odd hyperbolic monodromy groups.

Corollary 2.3.3. *All odd hyperbolic monodromy groups $H(\alpha,\beta)$ belong to one of the following families.*

1. $\mathcal{N}_1(1,1,n)$:
$$\alpha=\left(\frac{1}{2},\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n}\right)$$
$$\beta=\left(0,\frac{1}{2n},\frac{3}{2n},\cdots,\frac{2n-1}{2n}\right)$$

2. $\mathcal{N}_2(1,1,n) = \mathcal{N}_3(1,1,n)$
$$\alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}\right)$$
$$\beta = \left(0, \frac{1}{2n-2}, \frac{3}{2n-2}, \cdots, \frac{2n-3}{2n-2}\right).$$

3. $\mathcal{N}_4(1,1,n)$:
$$\alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2n-4}, \frac{3}{2n-4}, \cdots, \frac{2n-5}{2n-4}\right)$$
$$\beta = \left(0, 0, 0, \frac{1}{n-2}, \frac{2}{n-2}, \cdots, \frac{n-3}{n-2}\right).$$

Proof. By proposition 2.3.2, we need to have $P(z) - Q(z) = 0$ in $\mathbb{Z}/2\mathbb{Z}$. It is easily verified that this is satisfied by only the cases above. Also, there is no odd sporadic groups for $n \leq 9$ from the table 3 of [5].

□

2.4 Thinness of odd $H(\alpha, \beta)$ in $O(L)$

We first look at the odd lattices in the table 3 of [5].

1. $\alpha = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}\right), \beta = \left(0, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\right)$.

   $L =\, < x_1, x_2, x_3, x_4, x_5 >$, where $x_1 = (2\ 0\ -2\ 0\ 0)^t, x_2 = (4\ 0\ 6\ 0\ 0)^t, x_3 = (0\ 2\ 0\ -2\ 0)^t, x_4 = (0\ 4\ 0\ 6\ 0)^t, x_5 = (4\ 0\ 4\ 0\ 2)^t$. In this case F with respect to $\{x_1, x_2, \cdots, x_5\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

   .

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = x_5$, we get 9 vectors $e_1 = -x_1, e_2 = -x_3, e_3 = x_1 + 2x_2, e_4 = x_3 + 2x_4, e_5 = -x_2 + x_5, e_6 = -x_4 + x_5, e_7 = x_1 + x_5, e_8 = x_3 + x_5, e_9 = 2x_1 - x_2 + 2x_3 - x_4 + 5x_5$.

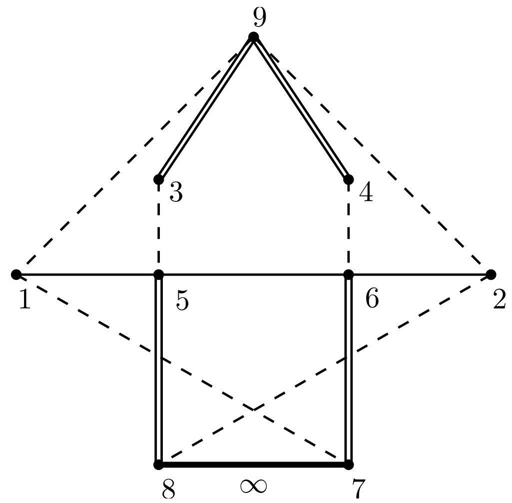

Figure 2.1: Coxeter graph for $R(L)$

We have Coxeter graph in Figure 2.1, and we have critical principal submatrices

$$S = \{1,7\}, \{2,8\}, \{1,9\}, \{2,9\}, \{3,5\}, \{4,6\}$$

such that $G_S = G_S^-$, and

$$S = \{3,4,9\}, \{7,8\}, \{1,5,6,8\}, \{2,5,6,7\}$$

such that $G_S = G_S^0$. They satisfy criteria of proposition 2.1.1, so L is reflective. However, among 9 faces of the fundamental domain, there are 5 faces corresonding to $\{e_3, e_4, e_7, e_8, e_9\}$ which are not in $R_2(L)$. By theorem, $R(L) = R_2(L) \rtimes < R_{e_3}, R_{e_4}, R_{e_7}, R_{e_8}, R_{e_9} >$, but $< R_{e_3}, R_{e_4}, R_{e_7}, R_{e_8}, R_{e_9} >$ is infinite since the subgraph restricted to $\{e_3, e_4, e_7, e_8, e_9\}$ is parabolic. $|O(L)/R_2(L)| = \infty$, $H_r(\alpha, \beta)$ has image in $O(L)/R_2(L)$ of order at most 2, and $H_r(\alpha, \beta)$ is a finite index subgroup of $H(\alpha, \beta)$. Therefore, $H(\alpha, \beta)$ is thin in $O(L)$.

2. $\alpha = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right), \beta = \left(0, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right)$.

$L =< x_1, x_2, x_3, x_4, x_5 >$, where $x_1 = (-2\ -6\ -4\ -4\ 0)^t x_2 = (2\ 0\ -2\ 0\ 0)^t, x_3 = (0\ 2\ 0\ -2\ 0)^t, x_4 = (4\ 4\ 6\ 2\ 0)^t, x_5 = (4\ 4\ 4\ 2\ 2)^t$. In this case F with respect to $\{x_1, x_2, \cdots, x_5\}$ is

$$F = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = (4\ 0\ 2\ 0\ 2)^t$, we select the following vectors :

$e_1 = -x_2, e_2 = -x_3, e_3 = x_1 + x_2 + x_3 + x_4, e_4 = -x_4 + x_5, e_5 = -x_1 + x_5, e_6 = x_2 + x_3 + x_5$.

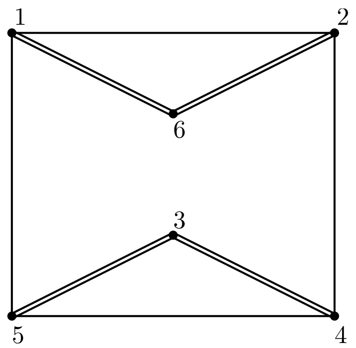

Figure 2.2: Coxeter graph for $R(L)$

We have Coxeter graph in Figure 2.2, and we have critical principal submatrices

$$S = \{1, 2, 6\}, \{3, 4, 5\}$$

such that $G_S = G_S^-$, and

$$S = \{1,3,5,6\}, \{2,3,4,6\}$$

such that $G_S = G_S^0$.

They satisfy criteria of proposition 2.1.1, so L is reflective. Also, among 6 faces of the fundamental domain, there are 2 faces corresond to $\{e_3, e_6\}$ which are not in $R_2(L)$. By theorem, $R(L) = R_2(L) \rtimes < R_{e_3}, R_{e_6} >$, and $< R_{e_3}, R_{e_6} >$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, L is 2-reflective. Thus we cannot use the same method to show that $H(\alpha, \beta)$ is thin.

We now use the same algorithm, not for $R(L)$ but for $R_2(L)$ to get the fundamental domain for $R_2(L)$. We have the following vectors :

$e_1 = x_2, e_2 = x_3, e_3 = x_4 + x_5, e_4 = x_1 + x_5, e_5 = x_1 + x_2 + x_3 + x_5, e_6 = x_2 + x_3 + x_4 + x_5, e_7 = -x_2 - 2x_3 + 2x_5, e_8 = -2x_2 - x_3 + 2x_5$.

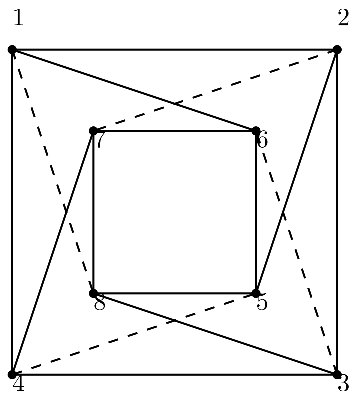

Figure 2.3: Coxeter graph for $R_2(L)$

We have Coxeter graph in Figure 2.3, and we have critical principal submatrices

$$S = \{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}$$

such that $G_S = G_S^-$, and

$$S = \{1,2,3,4\}, \{1,2,5,6\}, \{1,4,6,7\}, \{2,3,5,8\}, \{3,4,7,8\}, \{5,6,7,8\}$$

such that $G_S = G_S^0$.

They satisfy criteria of proposition 2.1.1, so $R_2(L)$ is generated by these 8 reflections.

$H_r(\alpha,\beta) =< R_v, R_{Bv}, \cdots, R_{B^4 v} >$ have an index 2 subgroup

$$< R_v R_{Bv}, R_{Bv} R_{B^2 v}, R_{B^2 v} R_{B^3 v}, R_{B^3 v} R_{B^4 v} >$$

which is contained in

$$< R_{w_1}, \cdots, R_{w_8} >$$

where $w_{2i-1} = B^{i-1}v - B^i v,\ w_{2i} = 3B^{i-1}v - B^i v$, by the Lemma 4.1 of [5]

We also use the algorithm in chapter 4 of [5] which determines the fundamental domain for discrete groups of motions generated by a finite number of reflections. We choose a point $x_1 + 2x_2 + 2x_3 + x_4 + 4x_5$ inside the fundamental domain of $R_2(L)$ determined above. Following the algorithm,

$$< R_{w_1}, \cdots, R_{w_8} >$$

is equal to

$$< R_{e_1}, \cdots, R_{e_6} >.$$

and 6 reflections form the fundamental domain. We use criteria of proposition 2.1.1 again, and we get the fundamental domain has infinite volume. Thus, $H_r(\alpha,\beta)$ is of infinite index in $O(L)$ and so is $H(\alpha,\beta)$.

3. $\alpha = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}\right), \beta = \left(0, 0, 0, \frac{1}{3}, \frac{2}{3}\right)$.
   $L =< x_1, x_2, x_3, x_4, x_5 >$, where $x_1 = (6\ 0\ 2\ 0\ 0)^t, x_2 = (-2\ 0\ \ -6\ 0\ 0)^t, x_3 =$

$(0\ 6\ 0\ 2\ 0)^t, x_4 = (0\ -2\ 0\ -6\ 0)^t, x_5 = (4\ 0\ 2\ 0\ 2)^t$. In this case F with respect to $\{x_1, x_2, \cdots, x_5\}$ is

$$F = \begin{pmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = (4\ 0\ 2\ 0\ 2)^t$, we select the following vectors :

$e_1 = x_1 + x_2, e_2 = x_1 - x_2, e_3 = x_3 + x_4, e_4 = x_3 - x_4, e_5 = -x_1 + x_5, e_6 = -x_3 + x_5, e_7 = -x_1 - x_2 - x_3 - x_4 + 2x_5, e_8 = -x_1 + x_2 - 2x_3 - 2x_4 + 4x_5, e_9 = -2x_1 - 2x_2 - x_3 + x_4 + 4x_5.$

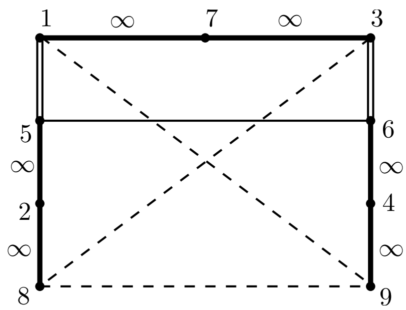

Figure 2.4: Coxeter graph for $R(L)$

We have Coxeter graph in Figure 2.4, and we have critical principal submatrices

$$S = \{1, 9\}, \{3, 8\}, \{8, 9\}$$

such that $G_S = G_S^-$, and

$$S = \{1, 7\}\{3, 7\}, \{2, 5\}, \{2, 8\}, \{4, 6\}, \{4, 9\}, \{1, 3, 5, 6\}$$

such that $G_S = G_S^0$.

They satisfy criteria of proposition 2.1.1, so L is reflective. However, among 9 faces of the fundamental domain, there are 7 faces corresond to $\{e_1, e_2, e_3, e_4, e_6, e_7, e_8\}$ which are not in $R_2(L)$. By theorem,

$$R(L) = R_2(L) \rtimes < R_{e_1}, R_{e_2}, R_{e_3}, R_{e_4}, R_{e_6}, R_{e_7}, R_{e_8} >.$$

However,

$$< R_{e_1}, R_{e_2}, R_{e_3}, R_{e_4}, R_{e_6}, R_{e_7}, R_{e_8} >$$

is infinite since the subgraph restricted to $\{e_1, e_2, e_3, e_4, e_6, e_7, e_8\}$ is hyperbolic. Therefore, $|O(L)/R_2(L)| = \infty$.

4. $\alpha = \left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}\right), \beta = \left(0, \frac{1}{14}, \frac{3}{14}, \frac{5}{14}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}\right).$

$L =< x_1, x_2, x_3, x_4, x_5, x_6, x_7 >$, where $x_1 = (2\ 0\ -2\ 0\ 0\ 0\ 0)^t, x_2 = (0\ 0\ 2\ 0\ -2\ 0\ 0)^t, x_3 = (4\ 0\ 4\ 0\ 6\ 0\ 0)^t, x_4 = (0\ 2\ 0\ -2\ 0\ 0\ 0)^t, x_5 = (0\ 0\ 0\ 2\ 0\ -2\ 0)^t, x_6 = (0\ 4\ 0\ 4\ 0\ 6\ 0)^t, x_7 = (4\ 0\ 4\ 0\ 4\ 0\ 2)^t$. In this case F with respect to $\{x_1, x_2, \cdots, x_7\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

For isotropic vector $v = x_1 + x_2 + x_4 + x_5 + 2x_7$, $a = a_1x_1 + \cdots + a_7x_7$ satisfies $(a, v) = 0$ if and only if $a_1 + a_2 - a_3 + a_4 + a_5 - a_6 - 2a_7 = 0$. Putting

$2a_7 = a_1 + a_2 - a_3 + a_4 + a_5 - a_6,$

$$
\begin{aligned}
2(a,a) &= 3(a_1 - a_2)^2 + 2(a_1 - a_2)a_3 + 5a_3^2 + 3(a_4 - a_5)^2 \\
&\quad + 2(a_4 - a_5)a_6 + 5a_6^2 + \frac{1}{2}(a_1 + a_2 - a_3 - a_4 - a_5 + a_6)^2 \\
&= \frac{1}{3}\left((3a_1 - 3a_2 + a_3)^2 + (3a_4 - 3a_5 + a_6)^2 + 14a_3^2 + 14a_6^2\right) \\
&\quad + \frac{1}{2}(a_1 + a_2 - a_3 - a_4 - a_5 + a_6)^2.
\end{aligned}
$$

Since $|L^*/L| = 7^2$, for any k-root of L, $k|2 \cdot 7^2$. Let $k = 2^p 7^q$. If $q = 2$, we get $7|a_1, a_2, \cdots, a_7$, and $\frac{a}{7}$ is a $\frac{k}{49}$-root. Therefore, we may assume without loss of generality that $q \leq 1$. $(a,a) = 1$ or 2 means that $a_3 = a_6 = 0$, $a_1 = a_2$, $a_4 = a_5$, and $|a_1 - a_4| = 1$. If $7|(a,a)$, $7|3a_1 - 3a_2 + a_3, 3a_4 - 3a_5 + a_6, a_1 + a_2 - a_3, a_4 + a_5 - a_6$. Also $2|a_1 + a_2 - a_3 - a_4 - a_5 + a_6$. Therefore $2(a,a) = \frac{1}{3}(49s_1^2 + 14s_2^2 + 49s_3^2 + 14s_4^2) + 98s_5^2 = 14$ or 28 for $s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}$. Multiplying both sides by $\frac{3}{7}$, we get $7s_1^2 + 2s_2^2 + 7s_3^2 + 2s_4^2 + 42s_5^2 = 6$ or 12, both of which have no integer solutions. So $(\mathbb{Z}v)^\perp/\mathbb{Z}v$ admits only one linearly independent reflection, and $|O(L)/R(L)| = \infty$. $|O(L)/R_2(L) = \infty$, $H_r(\alpha,\beta)$ has image in $O(L)/R_2(L)$ of order at most 2, and $H_r(\alpha,\beta)$ is a finite index subgroup of $H(\alpha,\beta)$. Therefore, $H(\alpha,\beta)$ is thin in $O(L)$.

5. $\alpha = \left(\frac{1}{12}, \frac{1}{4}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}\right), \beta = \left(0, 0, 0, \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\right).$

$L =< x_1, x_2, x_3, x_4, x_5, x_6, x_7 >$, where $x_1 = (2\ 0\ -2\ 0\ 0\ 0\ 0)^t, x_2 = (0\ -2\ 0\ 2\ 0\ 0\ 0)^t, x_3 = (0\ 0\ 2\ 0\ -2\ 0\ 0)^t, x_4 = (0\ 0\ 0\ -2\ 0\ 2\ 0)^t, x_5 = (4\ -4\ 4\ -4\ 6\ -2\ 0)^t, x_6 = (-4\ 0\ 0\ 0\ 0\ -4\ 0)^t, x_7 = (4\ -4\ 4\ -4\ 4\ -2\ 2)^t$. In this case

F with respect to $\{x_1, x_2, \cdots, x_7\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = x_7$, we select the following vectors :

$e_1 = -x_1, e_2 = -x_2, e_3 = -x_3, e_4 = -x_4, e_5 = x_6, e_6 = -x_5 + x_7, e_7 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7, e_8 = x_1 + x_2 + x_3 + x_4 + x_7, e_9 = x_1 + 2x_2 + 2x_3 + 2x_4 - x_6 + 3x_7, e_{10} = 2x_1 + 2x_2 + 2x_3 + x_4 - x_6 + 3x_7$. $R'(L)$ contains $< R_{e_9}, R_{e_{10}} >$, which is infinite, as it is a parabolic Coxeter subgroup of $R'(L)$. It means $|R(L)/R_2(L)| = \infty$, which leads to $|O(L)/R_2(L)| = \infty$. $H_r(\alpha, \beta)$ has image in $O(L)/R_2(L)$ of order at most 2, and $H_r(\alpha, \beta)$ is a finite index subgroup of $H(\alpha, \beta)$. Therefore, $H(\alpha, \beta)$ is thin in $O(L)$.

6. $\alpha = \left(\frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}\right), \beta = \left(0, 0, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$.

$L =< e_1, e_2, e_3, e_4, e_5, e_6, e_7 >$, where $e_1 = (-2\ 0\ 0\ 0\ 2\ 0\ 0)^t, e_2 = (2\ 0\ -4\ 0\ 2\ 0\ 0)^t, e_3 = (4\ 0\ 2\ 0\ 2\ 0\ 0)^t, e_4 = (0\ -2\ 0\ 0\ 0\ 2\ 0)^t, e_5 = (0\ 2\ 0\ -4\ 0\ 2\ 0)^t, e_6 = (0\ 4\ 0\ 2\ 0\ 2\ 0)^t, e_7 = (4\ 0\ 0\ 0\ 2\ 0\ 2)^t$. In this case F with respect to $\{e_1, e_2, \cdots, e_7\}$

is

$$F = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

. We use the algorithm of Vinberg[13]. Starting with the point $v_0 = (4\,0\,0\,0\,2\,0\,2)^t$, we select the following vectors :

$e_1 = x_1, e_2 = x_2, e_3 = x_4, e_4 = x_5, e_5 = x_1 + x_2 + 2x_3, e_6 = x_4 + x_5 + 2x_6, e_7 = -x_1 - x_2 - x_3 + x_7, e_8 = -x_4 - x_5 - x_6 + x_7, e_9 = -x_1 + x_7, e_{10} = -x_2 + x_7, e_{11} = -x_4 + x_7, e_{12} = -x_5 + x_7$. $R'(L)$ contains $< R_{e_9}, R_{e_{10}}, R_{e_{11}}, R_{e_{12}} >$, which is infinite, as it is a hyperbolic Coxeter subgroup of $R'(L)$. It means $|R(L)/R_2(L)| = \infty$, which leads to $|O(L)/R_2(L)| = \infty$.

7. $\alpha = \left(\frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}\right), \beta = \left(0, \frac{1}{18}, \frac{1}{6}, \frac{5}{18}, \frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{5}{6}, \frac{17}{18}\right)$.

$L =< e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 >$, where $e_1 = (2\ 0\ -2\ 0\ 0\ 0\ 0\ 0\ 0)^t, e_2 = (0\ 0\ 2\ 0\ -2\ 0\ 0\ 0\ 0)^t, e_3 = (0\ 0\ 0\ 0\ 2\ 0\ -2\ 0\ 0)^t, e_4 = (4\ 0\ 4\ 0\ 4\ 0\ 6\ 0\ 0)^t, e_5 = (0\ 2\ 0\ -2\ 0\ 0\ 0\ 0\ 0)^t, e_6 = (0\ 0\ 0\ 2\ 0\ -2\ 0\ 0\ 0)^t, e_7 = (0\ 0\ 0\ 0\ 0\ 2\ 0\ -2\ 0)^t, e_8 = (0\ 4\ 0\ 4\ 0\ 4\ 0\ 6\ 0)^t, e_9 = (4\ 0\ 4\ 0\ 4\ 0\ 4\ 0\ 2)^t$. In this case F with respect to

$\{e_1, e_2, \cdots, e_9\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

For isotropic vector $v = e_1 + e_2 + e_3 + e_5 + e_6 + e_7 + 2e_9$, $a = a_1e_1 + \cdots + a_9e_9$ satisfies $(a, v) = 0$ iff $a_1 + a_3 - a_4 + a_5 + a_7 - a_8 - 2a_9 = 0$. Putting $2a_9 = a_1 + a_3 - a_4 + a_5 + a_7 - a_8$, $(a, a) = (a_1 - 2a_2 + a_3)^2 + (a_5 - 2a_6 + a_7)^2 + \frac{1}{2}((2a_1 - 2a_3 + a_4)^2 + (2a_5 - 2a_7 + a_8)^2 + 9a_4^2 + 9a_8^2 + (a_1 + a_3 - a_4 - a_5 - a_7 + a_8)^2)$.

Since $|L^*/L| = 3^4$, for any k-root of L, $k|2 \cdot 3^4$. Let $k = 2^p3^q$ for some p, q and $p \leq 1$. If $q \geq 3$, we get $3|a_1, a_2, \cdots, a_9$, and $\frac{a}{3}$ is a $\frac{k}{9}$-root.Therefore, we may assume without loss of generality that $q \leq 2$. $(a, a) = 1$ or 2 means that $a_4 = a_8 = 0$,$a_1 = a_3$,$a_5 = a_7$, and $((a_1 - a_2)^2, (a_5 - a_6)^2, (a_1 - a_5)^2) = (1, 0, 0), (0, 1, 0)$, or $(0, 0, 1)$. $3|(a, a)$ means $3|a_1 - 2a_2 + a_3, a_5 - 2a_6 + a_7, 2a_1 - 2a_3 + a_4, 2a_5 - 2a_7 + a_8, a_1 + a_3 - a_4, a_5 + a_7 - a_8$, which makes $9|(a, a)$. Putting $(a, a) = 9$ or 18, since $a_1 - 2a_2 + a_3, a_5 - 2a_6 + a_7, 2a_1 - 2a_3 + a_4, 2a_5 - 2a_7 + a_8, a_1 + a_3 - a_4 - a_5 - a_7 + a_8$ are all multiple of 9, they are 0. $a_4^2 + a_8^2 = 2$ or 4, but $a_4 = 4(a_3 - a_2), a_8 = 4(a_7 - a_6)$, contradiction. So $(\mathbb{Z}v)^\perp/\mathbb{Z}v$ admits at most 3 linearly independent reflections. So $|O(L)/R(L)| = \infty$. $|O(L)/R_2(L)| = \infty$, $H_r(\alpha, \beta)$ has image in $O(L)/R_2(L)$ of order at most 2, and $H_r(\alpha, \beta)$ is a finite index subgroup of $H(\alpha, \beta)$. Therefore, $H(\alpha,\beta)$ is thin in $O(L)$.

8. $\alpha = \left(\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8}\right), \beta = \left(0,\frac{1}{16},\frac{3}{16},\frac{5}{16},\frac{7}{16},\frac{9}{16},\frac{11}{16},\frac{13}{16},\frac{15}{16}\right)$.
$L =< e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 >$, where $e_1 = (2\ 0\ \ -2\ 0\ 0\ 0\ 0\ 0\ 0)^t, e_2 = (0\ 2\ 0\ -2\ 0\ 0\ 0\ 0\ 0)^t, e_3 = (0\ 0\ 2\ 0\ -2\ 0\ 0\ 0\ 0)^t, e_4 = (0\ 0\ 0\ 2\ 0\ -2\ 0\ 0\ 0)^t, e_5 = (0\ 0\ 0\ 0\ 2\ 0\ \ -2\ 0\ 0)^t, e_6 = (0\ 0\ 0\ 0\ 0\ 2\ 0\ \ -2\ 0)^t, e_7 = (4\ 4\ 4\ 4\ 4\ 4\ 6\ 2\ 0)^t, e_8 = (-4\ 0\ 0\ 0\ 0\ 0\ 0\ 4\ 0)^t, e_9 = (4\ 4\ 4\ 4\ 4\ 4\ 4\ 2\ 2)^t$. In this case F with respect to $\{e_1, e_2, \cdots, e_9\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = (4\,4\,4\,4\,4\,4\,4\,2\,2)^t$, we select the following vectors :

$e_1 = x_1, e_2 = x_2, e_3 = x_3, e_4 = x_4, e_5 = x_5, e_6 = x_6, e_7 = -x_8, e_8 = -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 + x_9, e_8 = x_7 + x_8 + x_9 e_9 = -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 + x_9, e_{10} = -x_1 - 2x_2 - 3x_3 - 3x_4 - 2x_5 - x_6 + 2x_9, e_{11} = -2x_1 - 4x_2 - 4x_3 - 4x_4 - 4x_5 - 2x_6 + x_8 + 4x_9$. $R'(L)$ contains $< R_{e_7}, R_{e_{10}}, R_{e_{12}} >$, which is infinite, as it is a parabolic Coxeter subgroup of $R'(L)$ Therefore $|R(L)/R_2(L)| = \infty$, which leads to $|O(L)/R_2(L)| = \infty$. $H_r(\alpha,\beta)$ has image in $O(L)/R_2(L)$ of order at most 2, and $H_r(\alpha,\beta)$ is a finite index subgroup of $H(\alpha,\beta)$. Therefore, $H(\alpha,\beta)$ is thin in $O(L)$.

9. $\alpha = \left(\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right), \beta = \left(0,0,0,\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right).$
$L = < x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 >$, where $x_1 = (-2\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0)^t, x_2 = (2\ 0\ -4\ 0\ 2\ 0\ 0\ 0\ 0)^t, x_3 = (4\ 0\ 0\ 0\ 2\ 0\ 2\ 0\ 0)^t, x_4 = (0\ 0\ 2\ 0\ 0\ 0\ -2\ 0\ 0)^t, x_5 = (0\ -2\ 0\ 0\ 0\ 2\ 0\ 0\ 0)^t, x_6 = (0\ 2\ 0\ -4\ 0\ 2\ 0\ 0\ 0)^t, x_7 = (0\ 4\ 0\ 0\ 0\ 2\ 0\ 2\ 0)^t, x_8 = (0\ 0\ 0\ 2\ 0\ 0\ 0\ -2\ 0)^t, x_9 = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2)^t$. In this case F with respect to $\{x_1, x_2, \cdots, x_9\}$ is

$$F = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

We use the algorithm of Vinberg[13]. Starting with the point $v_0 = (4\ 0\ 0\ 0\ 0\ 0\ 2\ 0\ 2)^t$, we select the following vectors :

$e_1 = x_1, e_2 = x_2, e_3 = x_4, e_4 = x_5, e_5 = x_6, e_6 = x_8, e_7 = x_1 + x_2 + 2x_3 + 2x_4, e_8 = x_5 + x_6 + 2x_7 + 2x_8, e_9 = -x_3 + x_9, e_{10} = -x_7 + x_9, e_{11} = x_1 + x_2 + x_4 + x_9, e_{12} = x_5 + x_6 + x_8 + x_9$. $R'(L)$ contains $< R_{e_{11}}, R_{e_{12}} >$, which is infinite, as it is a parabolic Coxeter subgroup of $R'(L)$ Therefore $|R(L)/R_2(L)| = \infty$, which leads to $|O(L)/R_2(L)| = \infty$.

2.5 Thinness of $R_2(L)$ in $O(L)$ in odd hyperbolic hypergeometric monodromy groups

In this section we give another proof that $R_2(L)$ is of infinite index in $O(L)$, using Nikulin's result [8] on even lattices.

Theorem 2.5.1. *[8] If L is even, then $|O(L)/R_2(L)| = \infty$, unless L^*/L is two elementary, which means L^*/L is $(\mathbb{Z}/2\mathbb{Z})^a$ for some a, or L is isomorphic to $U \oplus K$ and K is one of*

$$A_3, A_1 \oplus A_2, A_1 \oplus A_2^2, A_1^2 3, A_2 \oplus A_3, A_1 \oplus A_4,$$
$$A_5, D_5, A_7, A_3 \oplus D_4, A_2 \oplus D_5, D_7, A_{18}, A_3 \oplus E_8$$

or L is isomorphic to

$$U(4) \oplus A_1^3, < -2^k > \oplus D_4 \text{ for } k = 2, 3, 4, \text{ or } < -2, 3 > \oplus A_2^2.$$

Proposition 2.5.2. *L is 2-reflective unless it is 2-elementary or $|L^*/L|$ is among the list below :*

dimension	$\|L^*/L\|$
5	*4,6,16,32,54,64,128*
7	*4,6,10,12,16,18*
9	*4,8,9,12,16*
13	*4*

Note that for odd hyperbolic lattice L, it contains the unique even sublattice

$$L^{even} = \{x \in L : (x, x) \in 2\mathbb{Z}\}. \tag{2.3}$$

of index 2 in L. Since any 2-root of L^{even} is a 2-root of L,

$$R_2(L) = R_2(L^{even}).$$

Also, from the fact $O(L^{even})$ and $O(L)$ are commensurable, we have that L is 2-reflective if and only if L^{even} is 2-reflective.

$|L^*/L| = det(F)$ where F is matrix realization of f with respect to a basis of L. We have $|L^*_{even}/L^*| = 2$, and

$$|L^*_{even}/L_{even}| = |L^*_{even}/L^*| \cdot |L^*/L| \cdot |L/L_{even}| = 4|L^*/L| \tag{2.4}$$

In Corollary 2.3.3 we have that every odd hyperbolic monodromy groups(up to scalar shifts) fall into one of the following three families :

1. $\mathcal{N}_1(1,1,n)$:
   $\alpha = \left(\frac{1}{2}, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\right)$
   $\beta = \left(0, \frac{1}{2n}, \frac{3}{2n}, \cdots, \frac{2n-1}{2n}\right)$
2. $\mathcal{N}_2(1,1,n) = \mathcal{N}_3(1,1,n)$
   $\alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}\right)$
   $\beta = \left(0, \frac{1}{2n-2}, \frac{3}{2n-2}, \cdots, \frac{2n-3}{2n-2}\right)$.
3. $\mathcal{N}_4(1,1,n)$:
   $\alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2n-4}, \frac{3}{2n-4}, \cdots, \frac{2n-5}{2n-4}\right)$
   $\beta = \left(0, 0, 0, \frac{1}{n-2}, \frac{2}{n-2}, \cdots, \frac{n-3}{n-2}\right)$.

Theorem 2.5.3. *For every odd hyperbolic hypergeometric groups with* $n \geq 5$, $|O(L)/R_2(L)| = \infty$, *except* $\mathcal{N}_2(1,1,5)$.

Proof. 1. $\mathcal{N}_1(1,1,n)$:

We have $v = (4\ 0\ 4\ 0\ \cdots\ 4\ 0\ 2)^t$ and we can show inductively that for $1 \leq i \leq n-1$, the last row of B^i is $(0, 0, \cdots, 0, 1, 2, 2, \cdots, 2)$, where 1 is $(n-i)$-th coordinate. So the matrix representing f with respect to $v, Bv, \cdots, B^{n-1}v$ is given by

$$f_{i,j} = \begin{cases} -1, & \text{if } i = j \\ -2|i-j|, & \text{if } i \neq j \end{cases} \tag{2.5}$$

We get $|L^*/L| = |det(f)| = n^2$.

From (3.2), $|L^*_{even}/L_{even}| = 4n^2$, so it is not 2-elementary, and it does not fall into the table of Proposition 3.2.2. Thus, $|O(L_{even}/R_2(L_{even})| = \infty$

2. $\mathcal{N}_2(1,1,n)$:

   We have $v = (4\,4\,\cdots\,4\,2\,2)^t$ and we can show inductively that for $1 \le i \le n-1$, for $1 \le i \le n-2$ the last row of B^i is $(0,0,\cdots,0,1,1,\cdots,1)$, where first 1 occurs in $(n-i)$-th coordinate, and the last row of B^{n-1} is $(1,1,\cdots,1,0)$. So the matrix representing f with respect to $v, Bv, \cdots, B^{n-1}v$ is given by

$$f_{i,j} = \begin{cases} -1, & \text{if } i = j \\ -2|i-j|, & \text{if } 1 \le |i-j| \le n-2 \\ -2n+3, & \text{if } |i-j| = n-1 \end{cases} \tag{2.6}$$

   We get $|L^*/L| = |det(f)| = 4(n-1)$.

   Firstly we can easily get

$$L = < e_1, e_2, \cdots, e_n >$$

   where $e_1 = (2\ 0\ -2\ 0\ 0\ \cdots\ 0\ 0\ 0\ 0)^t, e_2 = (0\ 2\ 0\ -2\ 0\ \cdots\ 0\ 0\ 0\ 0)^t, e_3 = (0\ 0\ 2\ 0\ -2\ \cdots 0\ 0\ 0\ 0)^t, \cdots, e_{n-3} = (0\ 0\ \cdots\ 0\ 0\ 0\ 2\ 0\ -2\ 0)^t, e_{n-2} = (4\,4\,\cdots\,4\,4\,4\,4\,6\,2\,0)^t, e_{n-1} = (-4\,0\,0\,\cdots\,0\,0\,0\,0\,4\,0)^t, e_n = (4\,4\,\cdots\,4\,4\,4\,4\,4\,2\,2)^t$.

In this case F with respect to $\{e_1, e_2, \cdots, e_n\}$ is

$$F = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & 0 & -1 & 2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \vdots & 0 & \ddots & \ddots & -1 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \ddots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

.

If we define

$$x_i = \begin{cases} e_i, & \text{if } i \leq n-3 \\ e_{n-2} + e_n, & \text{if } i = n-2 \\ e_{n-2} + e_{n-1} + e_n, & \text{if } i = n-1 \\ \sum_{i=1}^{n-2} 2ie_i + (n-2)e_{n-1} + (2n-2)e_n, & \text{if } i = n \end{cases} \tag{2.7}$$

$\{x_1, x_2, \cdots, x_n\}$ becomes a basis of L_{even} and F with respect to $\{x_1, x_2, \cdots, x_n\}$

is

$$F = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & 0 & -1 & 2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \vdots & 0 & \ddots & \ddots & -1 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \ddots & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4(n-1) \end{pmatrix}$$

so L_{even} is isomorphic to $< -4(n-1) > \oplus D_{n-1}$. From [8], $R_2(L_{even})$ is arithmetic only when $n = 5$.

3. $\mathcal{N}_4(1, 1, n)$:

We have $v = (4\ 0\ 0\ \cdots\ 0\ 2\ 0\ 2)^t$ and we can show inductively that for $1 \leq i \leq n-1$, for $1 \leq i \leq n-3$ the last row of B^i is $(0, 0, \cdots, 0, 1, 2, \cdots, i+1)$, where first 1 occurs in $(n-i)$-th coordinate, the last row of B^{n-2} is $(0, 1, 2, \cdots, n-2, n)$, and the last row of B^{n-1} is $(1, 2, \cdots, n-2, n, n+2)$. So the matrix representing f with respect to $v, Bv, \cdots, B^{n-1}v$ is given by

$$f_{i,j} = \begin{cases} -1, & \text{if } i = j \\ -2|i-j|, & \text{if } 1 \leq |i-j| \leq n-3 \\ -2n+3, & \text{if } |i-j| = n-2 \\ -2n-2, & \text{if } |i-j| = n-1 \end{cases} \tag{2.8}$$

We get $|L^*/L| = |det(f)| = 64$.

Since $|L^*_{even}/L_{even}| = 256$, $R_2(L)$ arithmetic if it is not two-elementary. We will show that it is not two-elementary, by picking an element $w \in L^*_{even}$ such that $2w \notin$

L_{even}.

- $n = 5$

$$L = < e_1, e_2, e_3, e_4, e_5 >$$

where $e_1 = (2\ 0\ 0\ 0\ -2)^t, e_2 = (-2\ 0\ 4\ 0\ -2)^t, e_3 = (0\ 4\ 0\ -4\ 0)^t, e_4 = (0\ 2\ 0\ 6\ 0)^t, e_5 = (4\ 0\ 2\ 0\ 2)^t$. Then F with respect to $\{e_1, e_2, \cdots, e_5\}$ is

$$F = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{pmatrix}$$

We have $L_{even} = < x_1, x_2, x_3, x_4, x_5 >$, where

$$x_1 = e_1, x_2 = e_2, x_3 = e_3, x_4 = e_4 + e_5, x_5 = 2e_5$$

Then F with respect to $\{x_1, x_2, \cdots, x_5\}$ is

$$F = \begin{pmatrix} 2 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 4 & 2 & 0 \\ 1 & 1 & 2 & 2 & -2 \\ 2 & 2 & 0 & -2 & -4 \end{pmatrix}$$

and we have a vector $w = \frac{1}{4}(x_1 + x_2 + x_3 + 2x_4 + 2x_5)$, with $w \in L^*_{even}$ and $2w \notin L_{even}$.

- $n \equiv 3\ (mod\ 4)$

  Let $n = 4k-1$.

$$L = < e_1, e_2, \cdots, e_n >$$

where $e_1 = (2\ 0\ 0\ 0\ \ -2\ 0\ \cdots\ 0)^t, e_2 = (-2\ 0\ 4\ 0\ \ -2\ 0\ \cdots\ 0)^t, e_3 = (0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ -2\ 0\ \cdots\ 0)^t, e_4 = (0\ 0\ 0\ 0\ -2\ 0\ 4\ 0\ -2\ 0\ \cdots\ 0)^t, \cdots, e_{2k-3} = (0\ 0\ \cdots\ 0\ 2\ 0\ 0\ 0\ -2\ 0\ 0)^t, e_{2k-2} = (0\ 0\ \cdots\ 0\ -2\ 0\ 4\ 0\ -2\ 0\ 0)^t, e_{2k-1} = (-4\ 0\ 0\ \cdots\ 0\ -2\ 0\ -2\ 0\ 0)^t, e_n = (4\ 0\ 0\ \cdots\ 0\ 2\ 0\ 2)^t$, and $e_{2k-1+i} = Je_i$ for $1 \leq i \leq 2k-1$, for $(2k-1) \times (2k-1)$ matrix J defined by

$$J_{ij} = \begin{cases} 1, & \text{if } i-j=1 \\ 0, & \text{otherwise} \end{cases}$$

In this case F with respect to $\{e_1, e_2, \cdots, e_n\}$ is

$$F = \begin{pmatrix} T & O_{2k-1,2k-1} & O_{2k-1,1} \\ O_{2k-1,2k-1} & T & O_{2k-1,1} \\ O_{1,2k-1} & O_{1,2k-1} & -1 \end{pmatrix}$$

. where

$$T = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 3 \end{pmatrix}$$

. is a $(2k-1)\times(2k-1)$ matrix and $O_{s,t}$ is $s\times t$ zero matrix.

We have

$$L_{even} =< x_1, x_2, \cdots, x_n >$$

where

$$x_i = \begin{cases} e_i + e_n, & \text{if } i = 2k-1 \text{ or } n-1 \\ 2e_n, & \text{if } i = n \\ e_i & \text{otherwise} \end{cases}$$

.

We have a vector $w = \dfrac{1}{4}(x_{2k-3} + x_{2k-2} + 2x_{2k-1} + x_{n-3} + x_{n-2} + 2x_{n-1}) \in L^*_{even}$, and $2w \notin L_{even}$.

- $n \equiv 1\ (mod\ 4)$ and $n \geq 9$

  Let $n = 4k+1$.

$$L =< e_1, e_2, \cdots, e_n >$$

where $e_1 = (-2\ 0\ 0\ 0\ 2\ 0\ \cdots\ 0)^t, e_2 = (2\ 0\ -4\ 0\ 2\ 0\ \cdots\ 0)^t, e_3 = (0\ 0\ 0\ 0\ -2\ 0\ 0\ 0\ 2\ 0\ \cdots\ 0)^t, e_4 = (0\ 0\ 0\ 0\ 2\ 0\ -4\ 0\ 2\ 0\ \cdots\ 0)^t, \cdots, e_{2k-3} = (0\ 0\ \cdots\ 0\ -2\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0)^t, e_{2k-2} = (0\ 0\ \cdots\ 0\ 2\ 0\ -4\ 0\ 2\ 0\ 0\ 0\ 0)^t, e_{2k-1} = (-4\ 0\ 0\ \cdots\ 0\ -2\ 0\ -2\ 0\ 0)^t, e_{2k} = (0\ 0\ \cdots\ 0\ 2\ 0\ 0\ 0\ -2\ 0\ 0)^t, e_n = (4\ 0\ 0\ \cdots\ 0\ 2\ 0\ 2)^t$, and $e_{2k+i} = Je_i$ for $1 \leq i \leq 2k$, for $2k \times 2k$ matrix J defined by

$$J_{ij} = \begin{cases} 1, & \text{if } i - j = 1 \\ 0, & \text{otherwise} \end{cases}$$

In this case F with respect to $\{e_1, e_2, \cdots, e_n\}$ is

$$F = \begin{pmatrix} T & O_{2k,2k} & O_{2k,1} \\ O_{2k,2k} & T & O_{2k,1} \\ O_{1,2k} & O_{1,2k} & -1 \end{pmatrix}$$

. where

$$T = \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & \cdots & 0 & -1 & -1 & -1 & 2 \end{pmatrix}$$

. is a $2k \times 2k$ matrix and $O_{s,t}$ is $s \times t$ zero matrix.

We have

$$L_{even} =< x_1, x_2, \cdots, x_n >$$

, where

$$x_i = \begin{cases} e_i + e_n, & \text{if } i = 2k-1 \text{ or } n-2 \\ 2e_n, & \text{if } i = n \\ e_i & \text{otherwise} \end{cases}$$

.

We have a vector $w = \dfrac{1}{4}(x_{2k-3} - x_{2k-2} + 2x_{2k} + x_{n-4} - x_{n-3} + 2x_{n-1} + x_n) \in L^*_{even}$, and $2w \notin L_{even}$.

□

Corollary 2.5.4. *The following two subfamilies of hyperbolic hypergeometric monodromy groups are thin :*

1. $\mathcal{N}_1(1, 1, n), n \geq 5$

2. $\mathcal{N}_2(1, 1, n), n \geq 5$

Note that $H_r(\alpha, \beta)$ is thin for $\mathcal{N}_4(1, 1, n)$ as well, since $H_r(\alpha, \beta)$ is commensurable with a subgroup of $R_2(L)$. But that does not imply that $H(\alpha, \beta)$ is thin, since in this case $H_r(\alpha, \beta)$ is not necessarily finite index in $H(\alpha, \beta)$.

Chapter 3

Geometric finiteness and measurements for discrete subgroups

3.1 Limit set, exponent of convergence and the bottom of the spectrum

Let $\mathbb{H}^{n-1}$ be a hyperbolic $(n-1)$-space. If $\Gamma \subset O_{\mathbb{R}}(n-1,1)$ is discrete it acts on $\mathbb{H}^{n-1}$ discontinuously, and its limit set $\Lambda(\Gamma)$ is defined to be the set of limit points in $\mathbb{S}^{n-2} = \partial(\mathbb{H}^{n-1})$ of Λv_0 for a point $v_0 \in \mathbb{H}^{n-1}$. Here $\Lambda(\Gamma)$ does not depend on v_0.

Lemma 3.1.1. *If Γ is not elementary, then every non-empty closed subset of S^{n-2} invariant by Γ contains $\Lambda(\Gamma)$.*

Proof. Let K be a closed set invariant by Γ. Since Γ is not elementary, K contains more than one element. Consider Klein model for $\mathbb{H}^{n-1}$, and let $C(K)$ denote the Euclidean convex hull of K. Clearly $C(K) \cap S^{n-2} = K$. Since K is invariant by Γ, $C(K)$ is also invariant by Γ. If x is any point in $\mathbb{H}^{n-1} \cap C(K)$, the limit set of the orbit Γ_x must be contained in the closed set $C(K)$. Therefore $\Lambda(\Gamma) \subset K$. $\square$

Theorem 3.1.2. *[4] If Γ is a discrete subgroup of $O_{\mathbb{R}}(n-1,1)$ and B is an infinite normal subgroup Γ, then*

$$\Lambda(B) = \Lambda(\Gamma).$$

Proof. Let p_0 be a point on $\mathbb{H}^{n-1}$. Since B is not finite, there is at least one accumulation point ξ on S^{n-2} and $b_j \in B$ such that $b_j p_0 \to \xi$. For any $\gamma \in \Gamma$, $d(b_j\gamma p_0, b_j p_0) = d(\gamma p_0, p_0)$ is finite, $b_j \gamma p_0$ also accumulate on ξ. Therefore, if we put $c_j = \gamma^{-1} b_j \gamma$, $c_j \in B$ and $c_j p_0 \to \gamma^{-1}\xi$. So, $\Gamma\xi \subset \Lambda(B)$. It follows from the lemma 2.1.2 that $\Lambda(B) \supset \Lambda(\Gamma)$. Since it is immediate that $\Lambda(B) \subset \Lambda(\Gamma)$, $\Lambda(B) = \Lambda(\Gamma)$. $\square$

Corollary 3.1.3. $\Lambda(R_{\pm 2}(L)) = S^{n-2}$, *and* $dim(\Lambda(R_{\pm 2}(L))) = n-2$.

Proof. $w \cdot r_v \cdot w^{-1}(x) = w\left(w^{-1}x - 2\dfrac{< v, w^{-1}x >}{< w^{-1}x, w^{-1}x >} \cdot v\right) = r_{wv}(x)$. So $R_{\pm 2}(L)$ are normal in $O(L)$. Since $O(L)$ is a lattice in $O_f(\mathbb{R})$, $\Lambda(O(L)) = S^{n-2}$. It follows from Theorem 1 that $\Lambda(R_{\pm 2}(L)) = S^{n-2}$. $\square$

The exponent of convergence([10]) $\delta(\Gamma)$ is the exponent for the series

$$\sum_{\gamma \in \Gamma} e^{-sd(\gamma p, q)}$$

where p and q are any fixed points in $\mathbb{H}^{n-1}$ and $d(p,q)$ is hyperbolic distance. If $\lambda_0(\Gamma)$ is the bottom of the spectrum of the Laplacian on $L^2(\Gamma \backslash \mathbb{H}^{n-1})$ and $\delta(\Gamma) > \frac{n-1}{2}$, then from [10] we have

$$\lambda_0(\Gamma) = \delta(\Gamma)(n-1-\delta(\Gamma)) \tag{3.1}$$

We are concerned with the sizes of $H(\alpha, \beta)$, $R(L)$ and $R_{\pm 2}(L)$ in terms of the limit set, the exponent of convergence and the bottom of the spectrum, not only thinness. There are two examples such that $R_2(L)$ is thin in $O(L)$, but $\lambda_0(R_2(L)\backslash\mathbb{H}^{n-1}) = 0$ in one example and $\lambda_0(R_2(L)\backslash\mathbb{H}^{n-1}) > 0$ in the other.

Example 3.1.4. *Let $f_n(x)$ be a quadratic form given by*

$$f_n(x) = -x_1^2 + x_2^2 + \cdots + x_n^2,\ x \in \mathbb{R}^{n+1}$$

and $O'_{f_n}(\mathbb{Z})$ the subgroup of index 2 in $O_{f_n}(\mathbb{Z})$, fixing each connected component of the set

$$C = \{x : f_n(x) < 0\}.$$

1. *Let $n = 20$, then according to [14], there is a fundamental domain P for $R(L)$ which has exactly 50 faces. Among them, 45 faces correspond to 2-roots and 5 faces correspond to 1-roots. From [13], $O_{f_{20}}(\mathbb{Z})/R_2(L)$ is isomorphic to the Coxeter group generated by 1-roots, which is semidirect product of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and H, where $'*'$ is free product and $H \subset SymP$. If we pick generators e_i of i-th $\mathbb{Z}/2\mathbb{Z}$, then $< e_1e_2, e_3e_4 >$ is a free group of rank 2. So $O'_{f_{20}}(\mathbb{Z})/R_2(L)$ is not amenable, and $\lambda_0(R_2(L)\backslash\mathbb{H}^{n-1}) > 0$ by theorem 2.1.4*

2. *Let $n = 17$, then according to [15], there is a fundamental domain P for $R(L)$ which has exactly 20 faces. Among them, 2 faces correpond to 1-roots. From[13] $O_{f_{17}}(\mathbb{Z})/R_2(L)$ is isomorphic to the Coxeter group generated by 1-roots, which is semidirect product of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and H, where $H \subset SymP$. Since $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ and $SymP$ is finite, $O_{f_{17}}(\mathbb{Z})/R_2(L)$ is amenable. Therefore, $\lambda_0(R_2(L)\backslash\mathbb{H}^{n-1}) = 0$*

So, when we have $R_2(L)$ is thin, $\lambda_0(R_2(L)\backslash\mathbb{H}^{n-1})$ could still be either 0 or not 0. Therefore, determining if λ_0 is 0 or not gives more information than just having it is thin or not. However, if the fundamental domain is geometrically finite, then they are equivalent. We will give the definition of geometric finiteness in next section.

Corollary 3.1.5. *If Γ is geometrically finite and $vol(\Gamma\backslash\mathbb{H}^{n-1}) = \infty$, then $\lambda_0(\Gamma\backslash\mathbb{H}^{n-1}) > 0$.*

Proof. Since Γ is geometrically finite, from [6] we have only finitely many(if any) positive eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$, and 0 is not the eigenvalue of the Laplace-Beltrami operator. Thus, $\lambda_0(\Gamma) = \mu_k > 0$. If it does not have positive eigenvalue, then $\lambda_0(\Gamma) = \left(\frac{n-1}{2}\right)^2 > 0$. □

3.2 Geometric finiteness

We start with the definition of *geometric finiteness.* Geometric finiteness are used in different ways. We stick to the following definition.

Definition 3.2.1. *Let a group Γ acts on $\mathbb{H}^n$ discretely. it is called geometrically finite if its fundamental domain has finitely many faces.*

Theorem 3.2.2. *Any discrete subgroup Γ of $O(L)$ which is generated by finitely many hyperbolic reflections is geometrically finite.* [1]

Proof. The fundamental domain P for the group generated by the reflections has dihedral angles between adjacent faces sub-multiples of π, and by proposition 2.1.1, its non-adjacent faces do not meet.

From this it follows that if T is a subset of faces of P, and P_T the polyhedral set gotten as the intersection of the corresponding half planes then P_T has all its adjacent faces meet at dihedral angles which are sub-mutiples of π. Thus the "parabolic" subgroup Γ_T of Γ generated by the reflections in the faces of P_T , that is by the elements of T, is itself a Coxeter group and has fundamental cell P_T.

Let

$$\Gamma = < r_1, r_2, \cdots , r_k >,$$

where $r_1, r_2, \cdots , r_k$ are hyperbolic reflections. We use theorem 2.2.4 for Γ, and let

[1]This theorem was recently shown by Fuchs, Meiri and Sarnak[5] using an algorithm. In addition to this theorem they also could show that the number of faces of the fundamental domain is at most the number of generating set.

$$s_1, s_2, \cdots$$

are chosen reflections. If the sequence $s_1, s_2, \cdots$ breaks off in finite terms, then the fundamental domain for $\Gamma \backslash \mathbb{H}^{n-1}$ has automatically finitely many faces. Let's assume that the sequence does not break off. If we define

$$\Gamma_t =< s_1, s_2, \cdots, s_t >,$$

We have $\Gamma_1 \subset \Gamma_2 \subset \cdots$, and $\cup_{t=1}^{\infty} \Gamma_t = \Gamma$, since the group Γ is generated by the reflections on the hyperplanes which include a face of the fundamental domain. Therefore, we can find Γ_{t_i} which contains r_i for $1 \leq i \leq k$. Let $T = max_i\{t_i\}$. Γ_T is a proper subgroup of Γ since its fundamental domain is strictly bigger than that of Γ. However Γ_T generate Γ, which is a contradiction. □

Corollary 3.2.3. *If at least one of A and B has finite order, $H_r(\alpha, \beta)$ has finite image in $O(L)/R_2(L)$ and $|O(L)/R_2(L)| = \infty$, then*

$$\delta(H(\alpha, \beta)) < n - 2, \ dim(\Lambda(H(\alpha, \beta))) < n - 2 \tag{3.2}$$

Note that this includes all hyperbolic hypergeometric groups which were shown to be thin in [5] and in this thesis.

Proof. An index 2 subgroup of the group $H_r(\alpha, \beta)$ is contained in a group $\triangle$ generated by finite number of 2-reflections. Since $R_2(L)$ is thin, we have $\delta(\triangle) < n - 2$ by corollary 3.1.5, and $dim(\Lambda(\triangle)) < n - 2$ by [10]. Since a finite index subgroup of $H(\alpha, \beta)$ is contained in $\triangle$, we have $\delta(H(\alpha, \beta)) < n - 2$ and $dim(\Lambda(H(\alpha, \beta))) < n - 2$ □

Chapter 4

Geometrically finite and thin $H(\alpha, \beta)$

4.1 Poincare's Polyhedron theorem

Let f be a non-degenerate quadratic form of signature $(n-1,1)$, Γ is a discrete subgroup of $SO_f(\mathbb{Z})$, and $v_0 \in \mathbb{H}^{n-1}$.

The dirichlet fundamental domain correspondind to v_0 is given by

$$F_{v_0} = \bigcap_{\gamma \in \Gamma} \{v \in \mathbb{H}^{n-1} : d(v, v_0) \leq d(v, \gamma v_0)\} = \bigcap_{\gamma \in \Gamma} H(\gamma, v_0) \tag{4.1}$$

One can show that F_{v_0} is compact(finite volume) if the intersection is compact(finite volume) even when taken over a finite number of elements $\gamma \in \Gamma$.

However, it is impossible to say that F_{v_0} is thin by using the same method. For an infinite covolume discrete subgroup, we need the Poincare's Polyhedron theorem :

Let $\mathbb{X}$ be one of $\mathbb{H}^n, \mathbb{R}^n$, or $\mathbb{S}^n$, with $n \geq 2$. Also let G be the group of isometries of $\mathbb{X}$, assume that we are given a polyhedron D. To show that D is the fundamental domain for G, we need to show the following conditions hold.

1. The sides of D are paired by elements of G. That is, for each side s of D, there is a side s', which is not necessarily distinct from s, and $g_s, g_{s'} \in G$ satisfies

$$g_s(s) = s',\ g_{s'} = g_s^{-1}.$$

2. $g_s(D) \cap D = \emptyset$

   The side-pairing transformations induce an equivalence relation on $\bar{D}$, where each point of D is equivalent only to itself. Let D^* be the space of equivalence class with the usual topology and $p : \bar{D} \to D^*$ be the projection which is continuous and open.

3. For every point $z \in D^*$, $p^{-1}(z)$ is a finite set.

   The next two conditions are related to the edges. Starting with an edge e_1. It lies on the boundary of two sides, call one of them s_1. Then there is a side s_1', and there is a side pairing transformation g_1, with $g_1(s_1) = s_1'$. Set $e_2 = g_1(e_1)$. Like e_1, e_2 lies on the boundary of exactly two sides, one of them is s_1', call the other s_2. Again, there is a side pairing transformation g_2 and side s_2' such that $g_2(s_2) = s_2'$. Set $e_3 = g_2(e_2)$. Continuing in this manner, we generate a sequence e_m of edges and g_m of side pairing transformations. This sequence is cyclic, as e_1 is equivalent to at most finitely many points in $\bar{D}$. The cyclically ordered sequence of edges $\{e_1, e_2, \cdots, e_k\}$ is called a *cycle of edges*. Observe that $g_k \circ g_{k-1} \circ \cdots \circ g_1(e_1) = e_1$, $h_{e_1} = g_k \circ g_{k-1} \circ \cdots \circ g_1$ is called the *cycle transformation* at e_1.

4. For each edge e, there is a positive integer t so that $h^t = 1$. The relations of G of the form $h^t = 1$ are called the *cycle relations*. There is essentially only one cycle relation for each equivalence class of cycles. If e' is equivalent to e, then $h(e')$ is a conjugate of $h(e)$ or $h(e)^{-1}$.

Let $\alpha(e)$ be the angle at the edge e, measured from the inside of D. We require that

5. $\sum_{m=1}^{k} \alpha(e_m) = \frac{2\pi}{t}$

In order to state the last condition, we need the following construction. We first form the group G^*, defined to be the abstract group generated by the side pairing transformations, and satisfying the reflection and cycle relations. We endow the discrete topology on G^*. There is an obvious homomorphism $\sigma : G^* \to G$.

We consider the equivalence relation on $G^* \times \bar{D}$ generated by the following. The pairs (g_1^*, x_1) and (g_2^*, x_2) are equivalent if there is a side pairing transformation f with $f(x_1) = x_2$, and if, $g_2^* = g_1^* \circ f^{-1}$. Let X^* be $G^* \times \bar{D}$, factored by this equivalence relation. We endow X^* with the usual identification topology, so that the natural projection from $G^* \times \bar{D}$ to X^* is continuous.

There is a natural map $q : X^* \to D^*$, defined by projection on the second factor of $G^* \times \bar{D}$, followed by the projection p from $\bar{D}$ to D^*. q is well defined and continuous.

It is easy to see that the distance $d(z, z')$ on D^* is naturally defined by, the infimum of the distances $d(x, x')$, where $q(x) = z, q(x') = z'$. Our last condition is

6. D^* is complete

Theorem 4.1.1. *[7] Assume that we are given a polyhedron D satisfying conditions (1) through (6). Then G, the group generated by the side pairing transformations is discrete, D is a fundamental polyhedron for G, and the reflection relations and cycle relations form a complete set of relations for G.*

4.2 A geometrically finite and thin hyperbolic hypergeometric monodromy group

As we mentioned before, there is no general method to see if the fundamental domain has finitely many sides. However, for a case $N_1(1,5,5)$ we could found that its fundamental domain is stable in (4.1), if the elements are listed in order of word length of A^2, B. In our case $H(\alpha,\beta)^+ =< A^2, B >$ is index 2 subgroup of $H(\alpha,\beta)$ and it acts on $\mathbb{H}^{n-1}$. In a numerical computation, we put $v_0 = (3\ 1\ 3\ 1\ 2)^t$, and the fundamental domain became stable after word length 10 to length 15. We found 42 sides and 160 edges. Since we have only finitely many sides, (c) and (f) are immediate. We checked (a),(b),(d) and (e) in the appendix. Theorem 4.1.1 leads us to our main result ; namely that in this case $H(\alpha,\beta)$ is inifnite volume and geometrically finite we have shown that $H(\alpha,\beta)^+$ has the fundamental domain which has finitely many sides.

Furthermore, we have found that

Theorem 4.2.1. *The complete set of relations for $H(\alpha,\beta)^+ =< A^2, B >$ are $A^{10} = B^6 = (B^{-1}A^2)^6$. So, we have the group presentation*

$$H(\alpha,\beta) =< A, B | A^{10}, B^6, (A^{-1}B)^2 >.$$

Note that one of the relation $(B^{-1}A^2)^6 = I$ can be obtained from other relations since $B^{-1}A^2 = (A^{-1}B)^{-1}A = (A^{-1}B)A = A^{-1}BA$. In other examples when α_i's and β_i's are all different, it seems that the geometric finiteness and explicit group presentation can be acquired in the same way.

Conjecture 4.2.2. *For the following families of hypergeometric hyperbolic monodromy groups, $H(\alpha,\beta)$ has fundamental domain with finite side, and the group presentation for $H(\alpha,\beta)$ can be given as below.*

- $\mathcal{M}_1(j,n)$, $H(\alpha,\beta) =< A, B | A^{2n}, B^{2lcm(j,n-j)}, (A^{-1}B)^2 >$

- $\mathcal{N}_1(j,k,n)$, $H(\alpha,\beta) =< A, B | A^{lj(k+1-j)}, B^{l(k+1)}, (A^{-1}B)^2 >$, *where* $lk = n$ *and* $(j, k+1) = 1$.

4.3 Proof of theorem 4.1.1

We give a proof for conditions (1),(2),(4), and (5) of theorem 4.1.1 for the example $\mathcal{N}_1(1,5,5)$ using numerical computation. Firstly, in equation (4.1), $H(\gamma, v_0)$ becomes a side of the fundamental domain when γ is one of the following :

1. A^2
2. A^{-2}
3. B
4. B^{-1}
5. A^4
6. $B^{-1}A^2$
7. A^{-4}
8. $B^{-1}A^{-2}$
9. A^2B
10. $A^{-2}B$
11. B^2
12. B^{-2}
13. $B^{-1}A^4$
14. $B^{-1}A^2B$
15. $A^{-4}B$
16. $B^{-1}A^{-2}B$
17. B^3
18. B^2A^4
19. $(B^{-1}A^2)^2$
20. B^3A^{-2}
21. $B^{-1}A^4B$
22. B^2A^2B
23. $B^{-1}A^{-4}B$
24. $(A^{-2}B)^2$
25. A^2B^{-3}
26. $A^{-4}B^{-2}$
27. $B^{-1}A^{-2}B^{-2}$
28. $A^4(B^{-1}A^2)^2$

29. $(B^{-1}A^2)^3$

30. $(A^{-2}B)^2A^{-4}$

31. $B^3A^{-4}B$

32. $B^{-1}A^4B^{-3}$

33. $(B^{-1}A^2)^3A^2$

34. $B^{-1}A^4(B^{-1}A^2)^2$

35. $A^{-2}(B^{-1}A^2)^{-3}$

36. $(A^{-2}B)^2A^{-4}B$

37. $B^2A^4(B^{-1}A^2)^2$

38. $B^{-1}A^{-2}(B^{-1}A^2)^3$

39. $(B^{-1}A^2)^3A^2B$

40. $(A^{-2}B)^2A^{-4}B^{-2}$

41. $B^3A^{-2}(B^{-1}A^2)^3$

42. $(B^{-1}A^2)^{-3}A^2B^{-3}$

and side-pairing relation for the following pairs - $(1,2),(3,4),(5,7),(6,10),(8,9),(11,12),(13,15),(14,16),(17),(18,26),(19,24),$ $(20,25),(21,23),(22,27),(28,30),(29),(31,32),(33,35),(34,36),(37,40),(38,39),$ $(41,42)$, where 17 and 29 are self-paired. In every case s is given by γ^{-1}, where γ is listed above. We could check with the computer if two sides are adjacent for any two sides, and there were 160 edges. By computation, there are 54 cycles of edges. For each cycle $\{e_1,e_2,\cdots,e_k\}$, $\{e_1,e_2,\cdots,e_k\}$,$h(e_1)$ and $\sum_{m=1}^{k}\alpha(e_m)$ are listed below. Each edge is describes as (a,b), if it is intersection of two sides a and b.

1. $\{(1,2),(7,2),(1,5)\}$, I, 2π

2. $\{(1,3),(10,2),(4,6)\}$, I, 2π

3. $\{(1,7),(5,2),(5,7)\}$, A^{-10}, 2π

4. $\{(1,9),(3,2),(8,4)\}$, I, 2π

5. $\{(1,10),(15,2),(6,13)\}$, I, 2π

6. $\{(1,15),(9,2),(14,8),(13,16)\}$, A^{-10}, 2π

7. $\{(1,25),(17,2),(20,17)\}$, B^{-6}, 2π

8. $\{(1,29),(35,2),(29,33)\}$, I, 2π

9. $\{(3,4),(12,4),(3,11)\}$, I, 2π

10. $\{(3,5),(13,4),(7,15)\}$, I, 2π

11. $\{(3,7),(21,4),(6,23),(5,10)\}$, A^{-10}, 2π

12. $\{(3,9),(14,4),(8,16)\}$, I, 2π

13. $\{(3,10),(16,4),(6,14)\}$, I, 2π

14. $\{(3,12),(17,4),(11,17)\}$, B^{-6}, 2π

15. $\{(3,15),(23,4),(13,21)\}$, I, 2π

16. $\{(3,17),(11,4),(17,12)\}$, B^{-6}, 2π

17. $\{(3,28),(34,4),(30,36)\}$, I, 2π

18. $\{(3,35),(38,4),(33,39)\}$, $(B^{-1}A^2)^{-6}$, 2π

19. $\{(5,9),(10,7),(8,6)\}$, I, 2π

20. $\{(5,12),(26,7),(11,18)\}$, I, 2π

21. $\{(5,15),(9,7),(13,8)\}$, A^{-10}, 2π

22. $\{(5,28),(19,7),(30,24)\}$, I, 2π

23. $\{(6,10),(24,10),(6,19)\}$, I, 2π

24. $\{(6,16),(15,10),(14,13)\}$, I, 2π

25. $\{(6,21),(9,10),(23,8)\}$, I, 2π

26. $\{(6,24),(29,10),(19,29)\}$, $(B^{-1}A^2)^{-6}$, 2π

27. $\{(6,27),(26,10),(22,18)\}$, I, 2π

28. $\{(6,29),(19,10),(29,24)\}$, $(B^{-1}A^2)^{-6}$, 2π

29. $\{(6,32),(25,10),(31,20)\}$, I, 2π

30. $\{(8,11),(25,9),(12,20)\}$, B^6, 2π

31. $\{(8,21),(15,9),(23,13)\}$, A^{10}, 2π

32. $\{(8,27),(12,9),(22,11)\}$, I, 2π

33. $\{(8,29),(28,9),(29,30)\}$, $(B^{-1}A^2)^{-6}$, 2π

34. $\{(8,38),(29,9),(39,29)\}$, $(B^{-1}A^2)^{-6}$, 2π

35. $\{(11,12)\}$, $B^{-2}, \frac{2\pi}{3}$

36. $\{(11,23),(31,12),(21,32)\}$, B^{-6}, 2π

37. $\{(11,37),(28,12),(40,30)\}$, I, 2π

38. $\{(11,38),(41,12),(39,42)\}$, B^{-6}, 2π

39. $\{(13,17),(26,15),(17,18)\}$, I, 2π

40. $\{(13,24),(35,15),(19,33)\}$, $(B^{-1}A^2)^{-6}$, 2π

41. $\{(13,32),(17,15),(31,17)\}$, B^{-6}, 2π

42. $\{(13,34),(19,15),(36,24)\}$, I, 2π

43. $\{(14,16),(23,16),(14,21)\}$, I, 2π

44. $\{(14,17),(27,16),(17,22)\}$, I, 2π

45. $\{(14,23),(21,16),(21,23)\}$, A^{-10}, 2π

46. $\{(14,34),(29,16),(36,29)\}$, I, 2π

47. $\{(17,34),(37,17),(36,40)\}$, I, 2π

48. $\{(17,35),(41,17),(33,42)\}$, B^{-6}, 2π

49. $\{(18,37),(19,26),(40,24)\}$, I, 2π

50. $\{(19,24)\}$, $(A^{-2}B)^2, \frac{2\pi}{3}$

51. $\{(19,39),(21,24),(38,23)\}$, $(B^{-1}A^2)^{-6}$, 2π

52. $\{(19,42),(32,24),(41,31)\}$, $(B^{-1}A^2)^{-6}$, 2π

53. $\{(20,41),(29,25),(42,29)\}$, I, 2π

54. $\{(22,37),(29,27),(40,29)\}$, I, 2π

I used three steps to determine which faces will be sides of the fundamental domain including specific point v_0. I used notations of [5], and set v_0 to be v in [5].

1. List all halfspaces $H(\gamma, v_0)$ for all γ of length less than or equal to certain length.

2. There is a non-degenerate real quadratic form f of signature $(n-1,1)$. I used the Klein model by setting $x_5 = 1$. Use linear programming to eliminate redundant halfspaces. In case of $\mathcal{N}_1(1,5,5)$, there are 312 halfspaces which are not redundant among about 10^8 halfspaces until length 16.

3. Use quadratic programming eliminate redundant halfspaces. Note that in Klein model every points in hyperbolic space can be represented by interior points of $(n-1)$-dimensional unit ball. I used 'OPTI toolbox' package in MATLAB, in which I could use quadratic programming only when the region determined by the quadratic equations are bounded.

I put C++ and matlab codes available in http://www.princeton.edu/ younghan.

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