Sobolev-type Embedding on Cauchy-Riemann Manifolds and Nonlinear Sub-elliptic PDE on Heisenberg Group

Xiaojie Shi

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Professor Paul Yang

June 2018

© Copyright by Xiaojie Shi, 2018.

All rights reserved.

Abstract

In this thesis, we study the optimal constants in the Sobolev-type inequality (AB inequality) on compact Cauchy-Riemann manifolds. We apply the results to give new proof to special case of CR Yamabe problem. We also consider the AB inequality for functions with constraints on CR manifolds. We show for those functions the optimal value can be smaller and use the analytic results to touch Nirenberg problem on CR sphere. In the last chapter, we discuss existence of positive solutions to some nonlinear sub-elliptic equations involving critical Sobolev exponents on Heisenberg group.

Acknowledgements

First of all, I would like to express my deepest gratitude to my advisor, Prof. Paul Chien-Ping Yang, for his patient guidance, generous support and encouragement; to Prof. Sun-Yung Alice Chang and Prof. Xingwang Xu for the kind help and suggestions; to Prof. Sagun Chanillo for joining the dissertation committee; to Dr. Ana Menezes for carefully reviewing the work; to Maths Department of Princeton University for providing such a great Ph.D. program.

Secondly, I would like to thank many friends and seniors I met in different periods of my life: Dr. Liangming Shen, Dr. Xuecheng Wang, Dr. Ziyang Gao, Dr. Chun-Hung Liu, Dr. Wen-Liang Tseng, Dr. Yao Wang, Dr. Ruixiang Zhang, Dr. Yang Xiu, Dr. Yu Zhang, Dr. Fan Zheng, Dr. Runpu Zong, Dr. Kathleen Emerson, Dr. Anibal Velozo, Dr. Daniel Kriz, Dr. Ruobing Zhang, Dr. Siyi Zhang, Dr. Eric Chen, Dr. Weibo Fu from Maths Department; Dr. Yuanda Xu from PACM; Dr. Bin Xu, Dr. Zhaoqi Leng, Dr. Huan He, Dr. Hao Zheng, Dr. Zhenbin Yang from Physics; Dr. Duyu Chen, Dr. Boxiao Zheng, Dr. Xiao Shi, Dr. Liye Chen from Chemistry; Dr. Yu Chen from Electrical Engineering; Dr. Liyu Dou, Dr. Mingyu Chen, Dr. Xiaoyang Dong from Economics; Dr. Wenjie Lei from Geosciences; Dr. Zhuhai Wang, Dr. Shuoguang Yang from Columbia University; Dr. Shi Luo from Caltech; Dr. Yunchao Xu from New York University; Dr. Zhang Zhang from Stanford University; Dr. Song Zhang from UC, Berkeley; Dr. Wei Zhou, Dr. Lei Fang, Mr. Lei Xu, Mr. Yizheng Guan, Ms. Candice Yan from New York; Dr. Brad Mann, Dr. Benjamin Blander, Ms. Jolene Qiu from Chicago; Dr. Haitao Wang, Dr. Feng Zhou, Dr. Rong Zhang, Dr. Jiajun Ma, Ms. Vivian Zhou, Ms. Jasmine Zhu, Ms. Valerie Sun from National University of Singapore; Ms. Cecilia Jin from Hai'an Middle School.

Finally, I want to give my sincere thanks to my family, especially my grandmother Ms. Cuihua You for their deep love and constant support.

To my family.

Contents

Abstract iii

Acknowledgements iv

1 Introduction **1**

2 Background **5**

3 AB Program on Compact CR Manifolds **16**

3.1 Answers to B Program 18

3.2 Compatible Coordinate Charts 26

3.3 Discussion on A Program 35

3.4 Case of the CR Sphere 40

3.5 Application to CR Yamabe Problem 41

3.6 AB Inequality with Constraints 51

3.7 Application to Nirenberg Problem on CR Sphere 62

4 Nonlinear Sub-elliptic PDE on Heisenberg Group **72**

4.1 Non-existence Results: Part 1 72

4.2 Existence Results 74

4.3 Non-existence Results: Part 2 84

5 Future Problems **86**

Bibliography **87**

Chapter 1

Introduction

In this chapter, we give brief description of contents of each chapter and each section, and outline the main results.

Chapter 2 is about background knowledge, we introduce the previous results of analysis on CR sphere, Heisenberg group or general CR manifolds and give relevant reference sources.

Chapter 3 is about AB program on compact CR manifolds and its applications. For $q \in (1, 2n+2)$ and q^* to be the conjugate number defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$, we first give the two forms of AB inequality on a compact CR manifold M: there exist positive constants A, B, A', B' such that for $u \in S_1^q(M)$ where $S_1^q(M)$ is the Sobolev-type space which will be defined later in Definition 2.0.17, we have:

$$\left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{\frac{1}{q^*}} \leq A \left(\int_M |\nabla_b u|^q d\mathrm{vol}\right)^{\frac{1}{q}} + B \left(\int_M |u|^q d\mathrm{vol}\right)^{\frac{1}{q}}$$

and

$$\left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \leq A' \int_M |\nabla_b u|^q d\mathrm{vol} + B' \int_M |u|^q d\mathrm{vol}.$$

Then we outline the AB program on CR manifolds to find the infimum values of A, B, A', B' such that the inequalities hold.

In Section 3.1 we completely answer the questions in B and $\hat{B}$ program: we prove if we denote the volume of the manifold by $V(M)$ and the dimension is $2n+1$, then

$$\inf B = (V(M))^{-\frac{1}{2n+2}} \quad \text{and} \quad \inf B' = (V(M))^{-\frac{q}{2n+2}}.$$

We also show that $\inf B$ is always attained. For the case $q \in (1, 2]$, $\inf B'$ is attained, while for $q \in (2, 2n+2)$, $\inf B'$ is not attained.

In Section 3.2 we construct compatible coordinate charts on the CR manifold to Heisenberg group which preserve the contact form and map the Hermitian frame close enough to the standard one on Heisenberg group.(see Theorem 3.2.2 and Theorem 3.2.5 for details)

In Section 3.3 we partially answer the questions in A program. We show for a compact CR manifold of dimension $2n+1$,

$$\inf A = \mathscr{A}(n, q) \quad \text{and} \quad \inf A' = \mathscr{A}(n, q)^q,$$

where $\mathscr{A}(n, q)$ is the optimal constant appears in the Sobolev-type embedding on Heisenberg group $\mathbb{H}^n$, as in Theorem 2.0.21.

In Section 3.4 we give example of the CR sphere $\mathbb{S}^{2n+1}$ and see the two optimal constants we find are both attained in AB inequality on $\mathbb{S}^{2n+1}$.

In Section 3.5 we apply the results of AB inequality to give new proof to the special case of CR Yamabe problem. We show that if we define μ_h for a function h as later in Definition 3.5.3, then we have the following theorem:

Theorem 1.0.1. *Let M be a compact CR manifold with the contact form θ. If there holds $0 < \mu_{\frac{n}{2n+2}R} < \frac{1}{\mathscr{A}(n,2)^2}$ where R is the Webster scalar curvature function on M, then there exists a positive smooth function u on M, such that under the CR* *conformal change $\theta' = u^{\frac{2}{n}}\theta$, the Webster scalar curvature R' of (M, θ') is constant on M.*

In Section 3.6, we continue to study the AB inequality for functions with constraints. In Theorem 3.6.1 and 3.6.2, we show for specific function u on CR sphere $\mathbb{S}^{2n+1}$ or compact CR manifold M satisfying certain orthogonality condition, the optimal constant A' in the AB inequality can be even smaller. Then based on that, we prove an estimate result uniformly in the power q, which is Theorem 3.6.3.

In Section 3.7, we first introduce the Nirenberg problem on a compact CR manifold M with contact form θ: to describe the set of functions which are Webster scalar functions of contact forms CR conformal to θ. We denote the set as $\mathcal{S}([\theta])$. We then apply the analytic results to touch Nirenberg problem on CR sphere $\mathbb{S}^{2n+1}$ with the standard contact form θ_1 which we will define later in Example 2.0.15. We first observe that although the constant function $1 \in \mathcal{S}([\theta_1])$, for any non-trivial first spherical harmonic ξ, $(1+\xi) \notin \mathcal{S}([\theta_1])$. Then we use the analytic estimates developed in previous section to show Lemma 3.7.5, which gives us a sufficient condition for a smooth function f on $\mathbb{S}^{2n+1}$ such that there exists a first spherical harmonic ξ_f satisfying $f - \xi_f \in \mathcal{S}([\theta_1])$. Finally we apply Lemma 3.7.5 to show two interesting theorems:

Theorem 1.0.2. *For a positive smooth function f on $\mathbb{S}^{2n+1}$, if we have*

$$\max_{\mathbb{S}^{2n+1}} f < 2^{\frac{1}{n}} \min_{\mathbb{S}^{2n+1}} f,$$

then there exists a first spherical harmonic function ξ such that $f - \xi \in \mathcal{S}([\theta_1])$.

Theorem 1.0.3. *Let f be a smooth function on $\mathbb{S}^{2n+1}$ which is positive somewhere, then there exists a first spherical harmonic ξ and a CR conformal transformation φ of $\mathbb{S}^{2n+1}$ such that $f - (\xi \circ \varphi) \in \mathcal{S}([\theta_1])$.*

Chapter 4 is about existence results of some nonlinear sub-elliptic PDE on Heisenberg group $\mathbb{H}^n$. We consider the following PDE problem:

$$\begin{cases} \Delta_b u = u^{2^*-1} + \lambda u & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{H}^n$ is a bounded domain, 2^* is the conjugate number of 2 and λ is a real number.

In Section 4.1, we show if either $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of Δ_b with zero Dirichlet condition, or $\lambda \leq 0$ combined with Ω is a smooth δ-starshaped domain (which we will define later in Definition 2.0.23), then (1.1) has no solution.

In Section 4.2, using the same subsequence technique as in Section 3.5, we show that if $\lambda \in (0, \lambda_1)$, there exists a solution of (1.1).

In Section 4.3, we show if Ω is a δ-starshaped domain, p is a real number with $p > 2^* - 1 = \frac{n+2}{n}$, $\lambda < \lambda_1 \cdot \frac{n}{n+1} \cdot \frac{p-(n+2)/n}{p-1}$, then there is no solution satisfying

$$\begin{cases} \Delta_b u = u^p + \lambda u & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

In Chapter 5 we outline several potential future problems related to the topic of the thesis.

Chapter 2

Background

In this chapter, we introduce some background knowledge with reference.

Definition 2.0.1 (CR Structures and CR Manifolds). *Let M be a C^∞ manifold of real dimension $(2n+1)$ for $n \geq 1$. Let $CT(M) = T(M) \otimes \mathbb{C}$ be the complexified tangent bundle over M. We consider a complex subbundle $T_{1,0}(M)$ of complex rank n. If*

$$T_{1,0}(M) \cap T_{0,1}(M) = \{0\}$$

with $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and for any open set $U \subset M$, there holds

$$\left[\Gamma^\infty(U, T_{1,0}(M)), \Gamma^\infty(U, T_{1,0}(M))\right] \subset \Gamma^\infty(U, T_{1,0}(M)),$$

then we say $(M, T_{1,0}(M))$ is a Cauchy-Riemann (abbreviated as CR) manifold with the CR structure, or simply denote as M.

Definition 2.0.2 (Levi Distribution). *Let $(M, T_{1,0}(M))$ be a CR manifold, the Levi distribution is the real rank $(2n)$ subbundle $H(M) \subset T(M)$ given by*

$$H(M) = Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It naturally carries the complex structure $J_b : H(M) \to H(M)$ *given by*

$$J_b(V + \overline{V}) = i(V - \overline{V}),$$

for any $V \in T_{1,0}(M)$.

Definition 2.0.3 (Pseudo-Hermitian Structure and Levi Form)**.** *Now consider the real line subbundle* $E \to M$ *of the cotangent bundle* $T^*(M) \to M$ *such that* $E_x = H_x^{\perp} \subset T_x^*$, *it follows that* M *is orentiable if and only if* E *has a global nonvanishing section, which is referred to as a pseudo-Hermitian structure. We will always assume this. Let* $\theta \in \Gamma^\infty(E)$ *be such a section, the Levi form* L_θ *is defined by*

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}),$$

for any $Z, W \in T_{1,0}(M)$. *Note that we can also view* $T_{1,0}(M)$ *as* $2n$*-dimensional real vector space and will denote the real inner product induced by* L_θ *by* $\langle Z, W\rangle$, *i.e.*

$$\big\langle Z, W \big\rangle = \frac{1}{2}\left(L_\theta(Z, \overline{W}) + L_\theta(W, \overline{Z})\right).$$

Definition 2.0.4 (Strictly Pseudoconvex CR Manifolds)**.** *With the same notations as above, we will always assume that* L_θ *is nondegenerate (equivalently,* θ *is a contact form). If in addition* L_θ *is positive definite for a suitable choice of* θ, *then* M *is said to be strictly pseudoconvex. Note that now the volume form on* M *is* $dvol = \theta \wedge (d\theta)^n$.

Definition 2.0.5 (Pseudo-Hermitian Frame)**.** *Consider a strictly pseudoconvex CR manifold* (M, θ), *let* $\{T_j\}_{j=1}^n$ *be a local orthonormal frame of* $T_{1,0}(M)$ *on an open subset* $V \subset M$. *If we set* $T_{\overline{j}} = \overline{T_j}$, *for* $1 \leq j, k \leq n$,

$$L_\theta(T_j, T_{\overline{k}}) = -i(d\theta)(T_j, T_{\overline{k}}) = \delta_j^k.$$

Such a frame $\{T_j\}_{j=1}^n$ is referred to as a pseudo-Hermitian frame. For convenience, we also let $X_j = Re(T_j)$ and $Y_j = -Im(T_j)$ for each $1 \le j \le n$, so together they will span $H(M)$.

Theorem 2.0.6 (Characteristic Direction)**.** *Let $(M, T_{1,0}(M))$ be a CR manifold with θ to be a fixed pseudo-Hermitian structure on M, there is a unique globally defined nowhere zero tangent vector field T on M such that*

$$\theta(T) = 1, \qquad T \rfloor d\theta = 0.$$

T is transverse to $H(M)$. Hence if we rewrite $\{X_j\}_{j=0}^{2n}$, with $X_0 = T$ and $X_j = X_j$, $X_{j+n} = Y_j$ for $1 \le j \le n$, they will span $T(M)$.

Proof. We refer the readers to Chapter 1.1 of [23]. □

Definition 2.0.7 (Admissible Coframe)**.** *With the notations above, there exists $\{\theta^j\}_{j=1}^n$ which forms a basis of $T_{1,0}^*(M)$ such that $\theta^j(T) = 0$ and $\theta^j(T_k) = \delta_{j,k}$ for $1 \le j, k \le n$. We call $\{\theta^j\}_{j=1}^n$ an admissible coframe. With $\theta^{\bar{j}} = \overline{\theta^j}$, now $\{\theta, \theta^1, \dots \theta^n, \theta^{\bar{1}}, \dots \theta^{\bar{n}}\}$ spans $CT^*(M)$.*
Note that now we have $d\theta = 2i\delta_j^k \theta^j \wedge \theta^{\bar{k}}$.

Definition 2.0.8 (Sub-Gradient)**.** *For any real C^1 function u on M, the sub-gradient of u is then defined by $\nabla_b u = \sum_{j=1}^n (T_j u) T_j$, which is in $T_{1,0}(M)$.*

Definition 2.0.9 (Sub-Laplacian and Integration by Parts)**.** *With the definition of sub-gradient, now we can define the operator sub-Laplacian Δ_b. For a C^2 real function u on M, $\Delta_b u$ is defined such that for any smooth compact-supported real function v on M, we have*

$$\int_M \langle \nabla_b u, \nabla_b v \rangle dvol = \int_M (\Delta_b u) v \; dvol.$$

Definition 2.0.10 (Webster Curvature)**.** *The definition and properties of Webster curvature on a CR manifold M can be easily found in [48]. If the Webster scalar* *curvature under the contact form θ is the scalar function R, then with a CR conformal transform $\tilde{\theta} = u^{2/n}\theta$ where $u > 0$, we have*

$$\tilde{R} = u^{-\frac{n+2}{n}} \left(\frac{2(n+1)}{n} \Delta_b u + Ru \right).$$

Definition 2.0.11 (Hörmander Condition)**.** *Let $(M, T_{1,0}(M))$ be a strictly pseudo-convex CR manifold with the pseudo-Hermitian structure θ, we know for any $x \in M$ and the local frame $\{X_j\}_{j=1}^{2n}$ of the subbundle $H(M)$ around x, the commutators of X_j's at x span the tangent space $T_x(M)$, which is often referred to as the Hörmander condition for the Levi distribution $H(M)$.*

Definition 2.0.12 (Carnot-Carathéodory Distance on CR Manifolds)**.** *A piecewise smooth path $\gamma : [0, T] \to M$ is called horizontal if it is tangent to $T_{1,0}(M)$ almost everywhere. The length of γ is defined to be*

$$\ell(\gamma) = \int_0^T \sqrt{L_\theta(\gamma', \overline{\gamma'})} dt.$$

Then for two points $x, y \in M$, the Carnot-Carathéodory distance is defined as

$$d_M(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is a horizontal path joining } x \text{ and } y\}.$$

By the classical result of W.L.Chow, see [1] or [18], the Hörmander condition implies that such path always exists. Besides, it induces the same topology on M.

Example 2.0.13 (Heisenberg Group)**.** *The Heisenberg group $\mathbb{H}^n$ (or simply $\mathbb{H}$) is the basic model of CR manifolds.*

Let us set $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the natural coordinate $(\mathbf{z}, t) = (z_1, \ldots, z_n, t)$. Then the group law is given by

$$(\mathbf{z}, t) \cdot (\mathbf{w}, s) = (\mathbf{z} + \mathbf{w}, t + s + 2Im\langle \mathbf{z}, \mathbf{w} \rangle),$$

where $\langle \mathbf{z}, \mathbf{w} \rangle = \delta_j^k z_j \overline{w}_k$.

On $\mathbb{H}^n$, *for each* $1 \leq j \leq n$, *the complex vectors fields (actually the pseudo-Hermitian frame) are given by*

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t}.$$

Let us then define $T_{1,0}(\mathbb{H}^n)$ *as the space spanned by those* Z_j*'s. Since*

$$[Z_j, Z_k] = 0, \quad 1 \leq j, k \leq n,$$

it follows that $(\mathbb{H}^n, T_{1,0}(\mathbb{H}^n))$ *is a CR manifold.*

So now

$$X_j = Re(Z_j) = \frac{1}{2}\frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \quad Y_j = -Im(Z_j) = \frac{1}{2}\frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial t}$$

for each $1 \leq j \leq n$.

The standard pseudo-Hermitian structure θ_0 *on* $\mathbb{H}^n$ *is then given by*

$$\theta_0 = dt + i \sum_{j=1}^{n} (z_j d\overline{z}_j - \overline{z}_j dz_j),$$

and hence

$$d\theta_0 = 2i \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j.$$

Also the standard volume form now becomes

$$\begin{aligned} dvol_0 = \theta_0 \wedge (d\theta_0)^n &= (2i)^n (n!) dt \wedge dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \\ &= 4^n (n!) dt \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \end{aligned}$$

Our choice of θ_0 *ensures that* $(\mathbb{H}^n, T_{1,0}(\mathbb{H}^n))$ *is a strictly pseudoconvex CR manifold.*

It is also easy to check $T = \frac{\partial}{\partial t}$ *is the characteristic direction of* $(\mathbb{H}^n, \theta_0)$.

With the notation above, we have on $\mathbb{H}^n$,

$$\Delta_b = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_jZ_j\right) = -\sum_{j=1}^{n}\left(X_j^2 + Y_j^2\right).$$

The Webster scalar curvature is 0 *on $\mathbb{H}^n$.*

Definition 2.0.14 (Dilation and Heisenberg Norm)**.** *For $\delta > 0$, the dilation map $D_\delta : \mathbb{H}^n \to \mathbb{H}^n$ is given by*

$$D_\delta(\mathbf{z}, t) = (\delta \cdot \mathbf{z}, \delta^2 \cdot t).$$

It is easy to check that each dilation is a group homomorphism and a CR isomorphism. The Heisenberg norm on $\mathbb{H}^n$ is given by

$$|\mathbf{x}| = \left(|\mathbf{z}|^4 + t^2\right)^{1/4}$$

for each $\mathbf{x} = (\mathbf{z}, t) \in \mathbb{H}^n$.

Example 2.0.15 (Hypersurface and CR Sphere)**.** *Let $\rho : \mathbb{C}^N \to \mathbb{R}$ be a real-valued smooth function which does not vanish on the hypersurface defined by $M = \{\mathbf{z} \in \mathbb{C}^N : \rho(\mathbf{z}) = 0\}$. If we define the subbundle by $T_{1,0}(M) = T_{1,0}(\mathbb{C}^N)\bigcap CT(M)$, it follows that $(M, T_{1,0}(M))$ is a CR manifold.*

The standard pesudo-Hermitian structure θ_1 on M is given by $i(\overline{\partial} - \partial)\rho$.

Then we define CR Sphere as the $(2n+1)$-dimensional hypersurface $\mathbb{S}^{2n+1}$ (or simply $\mathbb{S}$)in $\mathbb{C}^{n+1}$ with the corresponding subbundle and standard contact form given by

$$\theta_1 = i(\overline{\partial} - \partial)|\zeta|^2 = i\sum_{j=1}^{n+1}\left(\zeta_j d\overline{\zeta}_j - \overline{\zeta}_j d\zeta_j\right).$$

The volume form

$$
\begin{aligned}
dvol_1 =&\theta_1 \wedge (d\theta_1)^n = 2^{2n+1}(x_1dy_1 - y_1dx_1 + \cdots + x_{n+1}dy_{n+1} - y_{n+1}dx_{n+1})\\
&\wedge (dx_1 \wedge dy_1 + \cdots + dx_{n+1} \wedge dy_{n+1})^n\\
=&2^{2n+1}\,(n!) \sum_{j=1}^{n+1} \Big(x_j dx_1 \wedge dy_1 \wedge \cdots \wedge \widehat{dx_j} \wedge dy_j \wedge \cdots \wedge dx_{n+1} \wedge dy_{n+1}\\
&- y_j dx_1 \wedge dy_1 \wedge \cdots \wedge dx_j \wedge \widehat{dy_j} \wedge \cdots \wedge dx_{n+1} \wedge dy_{n+1} \Big).
\end{aligned}
$$

Recall that the surface area form on $\mathbb{S}^n$ induced from the Euclidean volume form on $\mathbb{R}^{n+1}$ is

$$d\sigma = \sum_{j=1}^{n+1} (-1)^{j+1} x_j dx_1 \wedge \ldots \widehat{dx_j} \wedge \ldots dx_{n+1}.$$

So we know the area of $\mathbb{S}^{2n+1}$ under the volume form $dvol_1$ is

$$\omega_{2n+1} = 2^{2n+1}\,(n!)\,\frac{2\pi^{n+1}}{\Gamma(n+1)} = 2^{2n+2}\pi^{n+1}. \tag{2.1}$$

Also for $j = 1, \ldots, n+1$, if we define the operators on $\mathbb{S}^{2n+1}$

$$T_j = \frac{\partial}{\partial \zeta_j} - \overline{\zeta}_j \sum_{k=1}^{n+1} \zeta_k \frac{\partial}{\partial \zeta_k}, \qquad \overline{T}_j = \frac{\partial}{\partial \overline{\zeta}_j} - \zeta_j \sum_{k=1}^{n+1} \overline{\zeta}_k \frac{\partial}{\partial \overline{\zeta}_k},$$

the sub-Laplacian on the CR sphere is then given by

$$\Delta_b = -\frac{1}{2} \sum_{j=1}^{n+1} \left(T_j \overline{T}_j + \overline{T}_j T_j \right).$$

The Webster scalar curvature is $\frac{1}{2}n(n+1)$ on $\mathbb{S}^{2n+1}$.

Definition 2.0.16 (Cayley Transformation)**.** *The Cayley transformation* $\mathcal{C} : \mathbb{H}^n \to \mathbb{S}^{2n+1}$ *and its inverse* $\mathcal{C}^{-1} : \mathbb{S}^{2n+1} \to \mathbb{H}^n$ *are given by*

$$\mathcal{C}(\mathbf{z}, t) = \left(\frac{2\mathbf{z}}{1 + |\mathbf{z}|^2 + it}, \frac{1 - |\mathbf{z}|^2 - it}{1 + |\mathbf{z}|^2 + it} \right),$$
$$\mathcal{C}^{-1}(\zeta) = \left(\frac{\zeta_1}{1 + \zeta_{n+1}}, \ldots, \frac{\zeta_n}{1 + \zeta_{n+1}}, Im\frac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}} \right).$$

We have that the Cayley transformation preserves the CR structure since it is restriction of holomorphic transformation and the Jacobian of the transformation is given by

$$J_{\mathcal{C}}(\mathbf{z}, t) = \frac{2^{2n+1}}{\left((1 + |\mathbf{z}|^2)^2 + t^2 \right)^{n+1}}.$$

Also the two standard contact forms θ_0 *and* θ_1 *are connected by*

$$\mathcal{C}^* (\theta_1) = \frac{4}{(1 + |\mathbf{z}|^2)^2 + t^2} \theta_0.$$

Definition 2.0.17 (Sobolev-type Norm)**.** *On a strongly pseudoconvex CR manifold* (M, θ)*, for* C^1 *functions on* M *and a positive* q*, we first define the Sobolev-type norm as follows:*

$$\|u\|^q_{S^q_1} = \int_M |u|^q dvol + \int_M |\nabla_b u|^q dvol.$$

Then the Sobolev-type space $S^q_1(M)$ *is defined to be the completion of* C^∞ *functions with finite norm. For convenience, we often write* $S^2_1(M)$ *as* $S_1(M)$*. Also we use the notations* $S^q_{1,0}(M), S_{1,0}(M)$ *to denote completion of compact-supported functions in the corresponding function space.*

Theorem 2.0.18 (Sobolev-type Embedding)**.** *For the compact strongly pseudoconvex CR manifold* (M, θ)*,* $q \in (1, 2n + 2)$*, if we define* q^* *by the relation*

$$1/q^* = 1/q - 1/(2n + 2),$$

then there exist positive constants A and B such that for $u \in S_1^q(M)$,

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{1}{q^*}} \leq A\left(\int_M |\nabla_b u|^q dvol\right)^{\frac{1}{q}} + B\left(\int_M |u|^q dvol\right)^{\frac{1}{q}},$$

or in other words,

$$S_1^q(M) \subset L^{q^*}(M).$$

And if $q < s < q^*$, the embedding of $S_1^q(M)$ into $L^s(M)$ is compact.

Proof. We refer the readers to Section 3.4 of [23], [27] and [39]. □

Theorem 2.0.19 (Poincaré-type Inequality)**.** *If we use $\overline{u}$ to denote the average value of the function u on the manifold M, i.e.,*

$$\overline{u} = \frac{\int_M u dvol}{\int_M dvol},$$

then for $p \in (1, +\infty)$, there exists a positive constant C such that for any $u \in S_1^p(M)$, there holds

$$\int_M |u - \overline{u}|^p dvol \leq C \int_M |\nabla_b u|^p dvol.$$

Proof. We refer the readers to Section 3.5 of [23] and [38]. □

Theorem 2.0.20 (Sobolev-Poincaré-type Inequality)**.** *With the notations as above, for $q \in (1, 2n+2)$, there exists a positive constant D such that for any $u \in S_1^q(M)$, we have*

$$\left(\int_M |u - \overline{u}|^{q^*} dvol\right)^{\frac{1}{q^*}} \leq D\left(\int_M |\nabla_b u|^q dvol\right)^{\frac{1}{q}}.$$

Proof. By Theorem 2.0.18, there exist positive constants A and B such that

$$\left(\int_M |u - \overline{u}|^{q^*} \mathrm{dvol}\right)^{\frac{1}{q^*}} \leq A\left(\int_M |\nabla_b u|^q \mathrm{dvol}\right)^{\frac{1}{q}} + B\left(\int_M |u - \overline{u}|^q \mathrm{dvol}\right)^{\frac{1}{q}}. \qquad (2.2)$$

Then by Theorem 2.0.19,

$$\left(\int_M |u-\overline{u}|^q d\text{vol}\right)^{\frac{1}{q}} \le C\left(\int_M |\nabla_b u|^q d\text{vol}\right)^{\frac{1}{q}}. \tag{2.3}$$

Combine (2.2) and (2.3) together, we obtain the result. □

Theorem 2.0.21 (Optimal Constants and Extremal Functions in Heisenberg Group)**.** *For $q \in (1, 2n+2)$, there exists $\mathscr{A}(n,q)$ such that for any function u in $D(\mathbb{H}^n)$, there holds*

$$\left(\int_{\mathbb{H}} |u|^{q^*} dvol_0\right)^{\frac{1}{q^*}} \le \mathscr{A}(n,q)\left(\int_{\mathbb{H}} |\nabla_b u|^q dvol_0\right)^{\frac{1}{q}}.$$

Here the constant $\mathscr{A}(n,q)$ is assumed to be optimal. Also note for the case $q=2$, $\mathscr{A}(n,2)$ is computed to be $\sqrt{\frac{1}{\pi n^2}}$ and the equality holds if and only if

$$u(\mathbf{z},t) = \frac{C}{\left(t^2+(1+|\mathbf{z}|^2)^2\right)^{\frac{n}{2}}}$$

up to left group translations and dilations on $\mathbb{H}^n$, here C is a constant.

Proof. We refer the readers to [29] and [40]. □

Theorem 2.0.22 (Maximum Principle)**.** *Let U be a relatively compact open subset of a normal coordinate neighborhood of the CR manifold M. Let $f, g \in C^\infty(U)$, u is a nontrivial function such that $u \in L^r(U)$ for some $r \ge \frac{2n+2}{n}$ and $u \ge 0$ on U. Assume that*

$$\Delta_b u + gu = fu^{q-1}$$

in the sense of distributions on U for some $2 \le q \le 2^*$, then $u \in C^\infty(U)$ and $u > 0$.

Proof. We refer the readers to Section 3.5 of [23] and [39]. □

Definition 2.0.23 (δ-starshaped Domain in $\mathbb{H}^n$). *On the Heisenberg group $\mathbb{H}^n$, we define the vector field*

$$X = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) + 2t \frac{\partial}{\partial t}.$$

A piecewise C^1 bounded domain $\Omega \subset \mathbb{H}^n$ is said to be δ-starshaped with respect to a point $\zeta \in \Omega$, if denoting by N the outer unit normal to the boundary of $\tau_{\zeta^{-1}}(\Omega)$, where $\tau_{\zeta^{-1}}$ is the left group action on $\mathbb{H}^n$ which maps ζ to 0, we have

$$X.N \geq 0$$

at every point of $\partial\left(\tau_{\zeta^{-1}}(\Omega)\right)$.

Theorem 2.0.24 (CR Pohozaev Identity). *Let Ω be a bounded domain in $\mathbb{H}^n$ with smooth boundary and the vector field X as defined above, we have for a C^2 function u on Ω, there holds*

$$\int_{\partial\Omega} |\nabla_b u|^2 X.N \; d\sigma = -2 \int_{\Omega} (Xu)\Delta_b u \; dvol_0 - 2n \int_{\Omega} |\nabla_b u|^2 dvol_0.$$

Hence if the domain Ω is δ-starshaped, RHS≥ 0.

Proof. We refer the readers to [30]. □

Chapter 3

AB Program on Compact CR Manifolds

Let (M, θ) be a compact strictly pseudoconvex CR manifold of real dimension $(2n+1)$ with given pseudo-Hermitian structure θ, where $n \geq 1$. We know the volume form $d\text{vol}$ is given by $\theta \wedge (d\theta)^n$. The Sobolev-type embedding (Folland-Stein embedding) is given in [27]:

$$\left(\int_M |u|^{q^*} d\text{vol}\right)^{\frac{1}{q^*}} \leq A\left(\int_M |\nabla_b u|^q d\text{vol}\right)^{\frac{1}{q}} + B\left(\int_M |u|^q d\text{vol}\right)^{\frac{1}{q}} \tag{3.1}$$

for $1 < q < 2n+2$, where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$ and the function u is in the Sobolev-type (Folland-Stein) space $S_1^q(M)$.

We remark that (3.1) is equivalent to existence of positive constants A and B (may be different values) such that

$$\left(\int_M |u|^{q^*} d\text{vol}\right)^{\frac{q}{q^*}} \leq A\int_M |\nabla_b u|^q d\text{vol} + B\int_M |u|^q d\text{vol}. \tag{3.2}$$

Let $q \in (1, 2n+2)$, q^* be defined as before, we know for $u \in C_0^\infty(\mathbb{H}^n)$, there holds

$$\left(\int_{\mathbb{H}} |u|^{q^*} d\mathrm{vol}_0\right)^{\frac{1}{q^*}} \le \mathscr{A}(n,q) \left(\int_{\mathbb{H}} |\nabla_b u|^q d\mathrm{vol}_0\right)^{\frac{1}{q}}, \tag{3.3}$$

here the constant $\mathscr{A}(n,q)$ is assumed to be optimal. In [29] and [40], the optimal constant $\mathscr{A}(n,2)$ is computed to be $\sqrt{\frac{1}{\pi n^2}}$. So we define

$$\mathbb{A}_q(M) = \{A \in \mathbb{R} : \text{ there exists } B \text{ such that (3.1) holds}\},$$

$$\widehat{\mathbb{A}}_q(M) = \{A \in \mathbb{R} : \text{ there exists } B \text{ such that (3.2) holds}\},$$

$$\mathbb{B}_q(M) = \{B \in \mathbb{R} : \text{ there exists } A \text{ such that (3.1) holds}\},$$

$$\widehat{\mathbb{B}}_q(M) = \{B \in \mathbb{R} : \text{ there exists } A \text{ such that (3.2) holds}\}.$$

Clearly from the definition, we have if $A \in \mathbb{A}_q(M)$ and $A' > A$, then $A' \in \mathbb{A}_q(M)$; if $A \in \widehat{\mathbb{A}}_q(M)$ and $A' > A$, then $A' \in \widehat{\mathbb{A}}_q(M)$; if $B \in \mathbb{B}_q(M)$ and $B' > B$, then $B' \in \mathbb{B}_q(M)$; if $B \in \widehat{\mathbb{B}}_q(M)$ and $B' > B$, then $B' \in \widehat{\mathbb{B}}_q(M)$. We let

$$\mathcal{A}_q(M) = \inf\{A : A \in \mathbb{A}_q(M)\},$$

$$\widehat{\mathcal{A}}_q(M) = \inf\{A : A \in \widehat{\mathbb{A}}_q(M)\},$$

$$\mathcal{B}_q(M) = \inf\{B : B \in \mathbb{B}_q(M)\},$$

$$\widehat{\mathcal{B}}_q(M) = \inf\{B : B \in \widehat{\mathbb{B}}_q(M)\}.$$

Now we outline the AB program and the related $\widehat{A}\widehat{B}$ program in this setting:

(i)Questions in A program: what is $\mathcal{A}_q(M)$? Is the set $\mathbb{A}_q(M)$ closed?

(ii)Questions in $\widehat{A}$ program: what is $\widehat{\mathcal{A}}_q(M)$? Is the set $\widehat{\mathbb{A}}_q(M)$ closed?

(iii)Questions in B program: what is $\mathcal{B}_q(M)$? Is the set $\mathbb{B}_q(M)$ closed?

(iv)Questions in $\widehat{B}$ program: what is $\widehat{\mathcal{B}}_q(M)$? Is the set $\widehat{\mathbb{B}}_q(M)$ closed?

3.1 Answers to B Program

In this section, we completely answer the questions in B and $\hat{B}$ program.

Theorem 3.1.1. *For any compact strictly pseudoconvex CR manifold* (M, θ) *with the pseudo-Hermitian structure* θ *and* $q \in (1, 2n+2)$*, we denote the volume of* M *by* $V(M)$*, then there exists* A *such that for any* $u \in S_1^q(M)$*, there holds*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{1}{q^*}} \leq A \left(\int_M |\nabla_b u|^q dvol\right)^{\frac{1}{q}} + (V(M))^{-\frac{1}{2n+2}} \left(\int_M |u|^q dvol\right)^{\frac{1}{q}}.$$

Here the value $(V(M))^{-\frac{1}{2n+2}}$ *is optimal.*

Proof. By taking the constant function 1 on M, we obtain $\mathcal{B}_q(M) \geq (V(M))^{-\frac{1}{2n+2}}$. Then by Theorem 2.0.20, there exists D such that

$$\left(\int_M |u - \overline{u}|^{q^*} \mathrm{dvol}\right)^{\frac{1}{q^*}} \leq D \left(\int_M |\nabla_b u|^q \mathrm{dvol}\right)^{\frac{1}{q}}.$$

By triangle inequality in $L^{q^*}(M)$, we obtain

$$\begin{aligned}\left(\int_M |u|^{q^*} \mathrm{dvol}\right)^{\frac{1}{q^*}} &\leq \left(\int_M |u - \overline{u}|^{q^*} \mathrm{dvol}\right)^{\frac{1}{q^*}} + \left(\int_M |\overline{u}|^{q^*} \mathrm{dvol}\right)^{\frac{1}{q^*}} \\ &\leq D \left(\int_M |\nabla_b u|^q \mathrm{dvol}\right)^{\frac{1}{q}} + (V(M))^{\frac{1}{q^*}-1} \left|\int_M u \mathrm{dvol}\right|.\end{aligned} \tag{3.4}$$

Apply Hölder's inequality,

$$\left|\int_M u \mathrm{dvol}\right| \leq \int_M |u| \mathrm{dvol} \leq (V(M))^{1-\frac{1}{q}} \left(\int_M |u|^q \mathrm{dvol}\right)^{\frac{1}{q}}. \tag{3.5}$$

Combine (3.4) and (3.5), we obtain the result. □

Theorem 3.1.2. *For a compact strictly pseudoconvex CR manifold* (M, θ) *with the pseudo-Hermitian structure* θ *and* $q \in (1, 2n+2)$*, we denote the volume of* M *by*

$V(M)$. Then for any $\epsilon > 0$, there exists A such that for any $u \in S_1^q(M)$, there holds

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \le A \int_M |\nabla_b u|^q dvol + \left((V(M))^{-\frac{q}{2n+2}} + \epsilon\right) \int_M |u|^q dvol.$$

Proof. We first recall some basic result: for $q \ge 1$ and any $\epsilon' > 0$, there exists constant C such that for nonnegative x, y, there holds

$$(x+y)^q \le (1+\epsilon')x^q + Cy^q.$$

Then by Theorem 3.1.1, there exists constant A such that

$$\left(\int_M |u|^{q^*} \mathrm{dvol}\right)^{\frac{q}{q^*}} \le \left((V(M))^{-\frac{q}{2n+2}} + \epsilon\right) \int_M |u|^q \mathrm{dvol} + CA^q \int_M |\nabla_b u|^q \mathrm{dvol},$$

which is exactly the result we want if we take CA^q to be the new constant. □

Lemma 3.1.3. *Let (M, θ) be a compact strictly pseudoconvex CR manifold with the pseudo-Hermitian structure θ and $q \in (1, 2n+2)$. We denote the volume of M by $V(M)$. If there exist A and B such that for any $u \in S_1^q(M)$, there holds*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \le A \int_M |\nabla_b u|^q dvol + B \int_M |u|^q dvol,$$

then $B \ge (V(M))^{-\frac{q}{2n+2}}$.

Proof. Similar to the proof to Theorem 3.1.1, we take the constant function 1 on M to obtain $\widehat{\mathcal{B}}_q(M) \ge V(M)^{-\frac{q}{2n+2}}$. □

Lemma 3.1.4. *For any compact strictly pseudoconvex CR manifold (M, θ) with the pseudo-Hermitian structure θ and $q \in [\frac{2n+2}{n+2}, 2]$, we denote the volume of M by $V(M)$, then there exists A such that for any $u \in S_1^q(M)$, there holds*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \le A \int_M |\nabla_b u|^q dvol + (V(M))^{-\frac{q}{2n+2}} \int_M |u|^q dvol.$$

Proof. For any function $v \in C^0(M)$ satisfies $\int_M v d\mathrm{vol} = 0$ and $\int_M v^2 d\mathrm{vol} = 1$, take

$$\varphi(t) = \left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}}.$$

Then we have $\varphi(0) = V(M)^{\frac{2}{q^*}}$ and since

$$\varphi'(t) = 2\left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}-1}\left(\int_M |1+tv|^{q^*-2}(1+tv)v d\mathrm{vol}\right),$$

there holds $\varphi'(0) = 0$. Then we compute $\varphi''(t)$ and obtain

$$\begin{aligned}\varphi''(t) = {} & 2q^*(\frac{2}{q^*}-1)\left(\int_M |1+tv|^{q^*-1} v d\mathrm{vol}\right)^2 \left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}-2} \\ & + 2(q^*-1)\left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}-1}\left(\int_M |1+tv|^{q^*-2} v^2 d\mathrm{vol}\right).\end{aligned}$$

For the choice of q, we have the first term is never positive. Then apply Hölder's inequality, there holds

$$\left|\int_M |1+tv|^{q^*-2} v^2 d\mathrm{vol}\right| \leq \left(\int_M |v|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}}\left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{1-\frac{2}{q^*}}.$$

So for $t \geq 0$,

$$\varphi''(t) \leq 2(q^*-1)\left(\int_M |v|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}},$$

and hence for any such v and $t \geq 0$,

$$\varphi(t) = \left(\int_M |1+tv|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}} \leq V(M)^{\frac{2}{q^*}} + t^2(q^*-1)\left(\int_M |v|^{q^*} d\mathrm{vol}\right)^{\frac{2}{q^*}}.$$

Consider now the function $u = 1 + tv$, such function v corresponds to function $u \in C^0(M)$ satisfying $\int_M u d\mathrm{vol} = V(M)$. In other words, we obtain for any $u \in C^0(M)$

satisfying $\overline{u}=1$, there holds

$$\left(\int_M |u|^{q^*}d\text{vol}\right)^{\frac{2}{q^*}} \leq V(M)^{-\frac{2(q^*-1)}{q^*}}\left(\int_M u d\text{vol}\right)^2 + (q^*-1)\left(\int_M |u-\overline{u}|^{q^*}d\text{vol}\right)^{\frac{2}{q^*}}. \tag{3.6}$$

By homogeneity and the fact that C^0 functions are dense in $L^{q^*}(M)$, we have (3.6) holds for any $u \in L^{q^*}(M)$. Applying Theorem 2.0.20 and Hölder's inequality, we obtain for any $u \in S_1^q(M)$,

$$\left(\int_M |u|^{q^*}d\text{vol}\right)^{\frac{2}{q^*}} \leq V(M)^{-\frac{2}{2n+2}}\left(\int_M |u|^q d\text{vol}\right)^{\frac{2}{q}} + C\left(\int_M |\nabla_b u|^q d\text{vol}\right)^{\frac{2}{q}}.$$

Since for $q \in [\frac{2n+2}{n+2}, 2]$ and nonnegative x, y, we have $(x+y)^{\frac{q}{2}} \leq x^{\frac{q}{2}} + y^{\frac{q}{2}}$, hence

$$\left(\int_M |u|^{q^*}d\text{vol}\right)^{\frac{q}{q^*}} \leq V(M)^{-\frac{q}{2n+2}}\int_M |u|^q d\text{vol} + C\int_M |\nabla_b u|^q d\text{vol},$$

which is just the result we want. □

Lemma 3.1.5. *For any compact strictly pseudoconvex CR manifold (M,θ) with the pseudo-Hermitian structure θ and $q \in (1, \frac{2n+2}{n+2})$, we denote the volume of M by $V(M)$, then there exists A such that for any $u \in S_1^q(M)$, there holds*

$$\left(\int_M |u|^{q^*}dvol\right)^{\frac{q}{q^*}} \leq A\int_M |\nabla_b u|^q dvol + (V(M))^{-\frac{q}{2n+2}}\int_M |u|^q dvol.$$

Proof. Consider a nontrivial function $v \in C^0(M)$ satisfying $\int_M v d\text{vol} = 0$, clearly we have

$$\int_M |1+v|^{q^*}d\text{vol} = \int_{\{v\geq 0\}} |1+v|^{q^*}d\text{vol} + \int_{\{-1\leq v<0\}} |1+v|^{q^*}d\text{vol} + \int_{\{v<-1\}} |1+v|^{q^*}d\text{vol}.$$

Since $\frac{2n+2}{2n+1} < q^* < 2$, there holds

$$\begin{aligned}\int_M |1+v|^{q^*} d\mathrm{vol} \leq & \int_{\{v\geq 0\}} \left(1+q^*v+v^{q^*}\right) d\mathrm{vol} + \int_{\{-1\leq v<0\}} \left(1+q^*v+|v|^{q^*}\right) d\mathrm{vol} \\ & + \int_{\{v<-1\}} |v|^{q^*} d\mathrm{vol} \\ \leq & \int_{\{v\geq 0\}} d\mathrm{vol} + \int_{\{v\geq 0\}} q^* v d\mathrm{vol} + \int_{\{v\geq 0\}} v^{q^*} d\mathrm{vol} + \int_{\{-1\leq v<0\}} d\mathrm{vol} \\ & + \int_{\{-1\leq v<0\}} q^* v d\mathrm{vol} + \int_{\{-1\leq v<0\}} |v|^{q^*} d\mathrm{vol} + \int_{\{v<-1\}} |v|^{q^*} d\mathrm{vol} \\ = & \int_{\{v\geq -1\}} d\mathrm{vol} + \int_M |v|^{q^*} d\mathrm{vol} + q^* \int_{\{v\geq -1\}} v d\mathrm{vol} \\ = & V(\{v\geq -1\}) + \int_M |v|^{q^*} d\mathrm{vol} + q^* \int_M v d\mathrm{vol} - q^* \int_{\{v<-1\}} v d\mathrm{vol} \\ = & V(\{v\geq -1\}) + \int_M |v|^{q^*} d\mathrm{vol} + q^* \int_{\{v<-1\}} |v| d\mathrm{vol}.\end{aligned}$$

Hence by Hölder's inequality, we obtain

$$\begin{aligned}\int_M |1+v|^{q^*} d\mathrm{vol} \leq & V(\{v\geq -1\}) + \int_M |v|^{q^*} d\mathrm{vol} \\ & + q^* \left(V(\{v<-1\})\right)^{1-\frac{1}{q^*}} \left(\int_M |v|^{q^*} d\mathrm{vol}\right)^{\frac{1}{q^*}}.\end{aligned}$$

Now for the real number $t \in [0, V(M))$, set

$$f(t) = t + \|v\|_{L^{q^*}}^{q^*} + q^* \|v\|_{L^{q^*}} \left(V(M) - t\right)^{\frac{q^*-1}{q^*}}.$$

Simple computation shows that

$$f'(t) = 1 - (q^*-1)\|v\|_{L^{q^*}} \left(V(M)-t\right)^{\frac{-1}{q^*}}$$

and

$$f''(t) = -\frac{q^*-1}{q^*} \|v\|_{L^{q^*}} (V(M)-t)^{-\frac{q^*+1}{q^*}} < 0,$$

and hence

$$\lim_{t\to V(M)} f'(t) = -\infty.$$

Then we discuss two cases. First we suppose that

$$f'(0) = 1 - (q^* - 1)\|v\|_{L^{q^*}} \left(V(M)\right)^{\frac{-1}{q^*}} < 0,$$

also since f' is non-increasing, so if we set $t_0 = V(\{v \geq -1\})$, we have $f(t_0) \leq f(0)$. Now

$$\begin{aligned}\int_M |1+v|^{q^*} d\mathrm{vol} \leq f(t_0) &\leq \|v\|_{L^{q^*}}^{q^*} + q^*\|v\|_{L^{q^*}} \left(V(M)\right)^{\frac{q^*-1}{q^*}} \\ &\leq \left(1 + q^*(q^*-1)^{q^*-1}\right) \|v\|_{L^{q^*}}^{q^*}.\end{aligned}$$

The other case is that

$$f'(0) = 1 - (q^* - 1)\|v\|_{L^{q^*}} \left(V(M)\right)^{\frac{-1}{q^*}} \geq 0.$$

Take t_m such that $f'(t_m) = 0$, we obtain

$$t_m = V(M) - (q^* - 1)^{q^*} \|v\|_{L^{q^*}}^{q^*}.$$

So in this case $f(t_m) \geq f(t_0)$. Direct computation leads to

$$\int_M |1+v|^{q^*} d\mathrm{vol} \leq V(M) + \left(1 + (q^*-1)^{q^*-1}\right) \|v\|_{L^{q^*}}^{q^*}.$$

In conclusion, in both cases there holds

$$\int_M |1+v|^{q^*} d\mathrm{vol} \leq V(M) + \left(1 + q^*(q^*-1)^{q^*-1}\right) \|v\|_{L^{q^*}}^{q^*}.$$

Since $(x+y)^{\frac{q}{q^*}} \le x^{\frac{q}{q^*}} + y^{\frac{q}{q^*}}$ for nonnegative x, y, we obtain

$$\left(\int_M |1+v|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \le V(M)^{\frac{q}{q^*}} + \left(1 + q^*(q^*-1)^{q^*-1}\right)^{\frac{q}{q^*}} \|v\|^q_{L^{q^*}}.$$

Consider now the function $u = 1+v$, such function v corresponds to function $u \in C^0(M)$ satisfying $\int_M u d\mathrm{vol} = V(M)$. In other words, we obtain for any $u \in C^0(M)$ satisfying $\overline{u} = 1$, there holds

$$\begin{aligned} \left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \le & V(M)^{\frac{q}{q^*}-q} \left(\int_M u d\mathrm{vol}\right)^q \\ & + \left(1 + q^*(q^*-1)^{q^*-1}\right)^{\frac{q}{q^*}} \left(\int_M |u-\overline{u}|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}}. \end{aligned} \tag{3.7}$$

By homogeneity and the fact that C^0 functions are dense in $L^{q^*}(M)$, we have (3.7) holds for any $u \in L^{q^*}(M)$. Applying Theorem 2.0.20 and Hölder's inequality, we obtain for any $u \in S_1^q(M)$,

$$\left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \le (V(M))^{-\frac{q}{2n+2}} \int_M |u|^q d\mathrm{vol} + C \int_M |\nabla_b u|^q d\mathrm{vol},$$

which is exactly the inequality we want. □

Theorem 3.1.6. *For any compact strictly pseudoconvex CR manifold (M, θ) with the pseudo-Hermitian structure θ and $q \in (1, 2]$, we denote the volume of M by $V(M)$, then there exists A such that for any $u \in S_1^q(M)$, there holds*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \le A \int_M |\nabla_b u|^q dvol + (V(M))^{-\frac{q}{2n+2}} \int_M |u|^q dvol.$$

Here the value $(V(M))^{-\frac{q}{2n+2}}$ is optimal.

Proof. It is direct conclusion from Lemma 3.1.3, Lemma 3.1.4 and Lemma 3.1.5. □

Theorem 3.1.7. *For any compact strictly pseudoconvex CR manifold (M, θ) with the pseudo-Hermitian structure θ and $q \in (2, 2n+2)$, we denote the volume of M by*

$V(M)$, then there doesn't exist A such that for any $u \in S_1^q(M)$, there holds

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \leq A \int_M |\nabla_b u|^q dvol + (V(M))^{-\frac{q}{2n+2}} \int_M |u|^q dvol.$$

Proof. Let $u \in C^\infty(M)$ be a nonconstant function. For $p > 2$, let's take

$$\varphi_p(t) = \int_M |1 + tu|^p d\text{vol}.$$

Simple calculation leads to

$$\varphi_p'(t) = p \int_M |1 + tu|^{p-2}(1 + tu)u d\text{vol}$$

and

$$\varphi_p''(t) = p(p-1) \int_M |1 + tu|^{p-2} u^2 d\text{vol}.$$

So we have

$$\varphi_p'(0) = p \int_M u d\text{vol} \text{ and } \varphi_p''(0) = p(p-1) \int_M u^2 d\text{vol}.$$

Hence

$$\varphi_p(t) = V(M) + p\left(\int_M u d\text{vol}\right) t + \frac{p(p-1)}{2}\left(\int_M u^2 d\text{vol}\right) t^2 + o(t^2).$$

Substitute $p = q$ and $p = q^*$, we obtain

$$\int_M |1 + tu|^q d\text{vol} = V(M) + q\left(\int_M u d\text{vol}\right) t + \frac{q(q-1)}{2}\left(\int_M u^2 d\text{vol}\right) t^2 + o(t^2)$$

and

$$\left(\int_M |1+tu|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} = (V(M))^{\frac{q}{q^*}} + q\,(V(M))^{\frac{q}{q^*}-1}\left(\int_M u d\mathrm{vol}\right)t$$
$$+\frac{q(q-q^*)}{2}(V(M))^{\frac{q}{q^*}-2}\left(\int_M u d\mathrm{vol}\right)^2 t^2$$
$$+\frac{q(q^*-1)}{2}(V(M))^{\frac{q}{q^*}-1}\left(\int_M u^2 d\mathrm{vol}\right)t^2 + o(t^2).$$

Now similar computation leads to

$$\int_M |\nabla_b(1+tu)|^q d\mathrm{vol} = o(t^2).$$

Let's assume there exists such A, substitute $1+tu$ into the inequality, we obtain

$$(V(M))^{\frac{q}{q^*}} + q\,(V(M))^{\frac{q}{q^*}-1}\left(\int_M u d\mathrm{vol}\right)t + \frac{q(q-q^*)}{2}(V(M))^{\frac{q}{q^*}-2}\left(\int_M u d\mathrm{vol}\right)^2 t^2$$
$$+\frac{q(q^*-1)}{2}(V(M))^{\frac{q}{q^*}-1}\left(\int_M u^2 d\mathrm{vol}\right)t^2 + o(t^2) \le$$
$$(V(M))^{\frac{q}{q^*}} + q\,(V(M))^{\frac{q}{q^*}-1}\left(\int_M u d\mathrm{vol}\right)t + \frac{q(q-1)}{2}(V(M))^{\frac{q}{q^*}-1}\left(\int_M u^2 d\mathrm{vol}\right)t^2.$$

Take $t \to 0$, there holds

$$V(M)\int_M u^2 d\mathrm{vol} \le \left(\int_M u d\mathrm{vol}\right)^2.$$

By Hölder's inequality, this is only possible when u is a constant function, so we obtain contradiction. □

3.2 Compatible Coordinate Charts

In this section, we construct compatible coordinate charts on the CR manifold.

Lemma 3.2.1. *Consider* $\mathbb{R}^3 = \{(x,y,t) : x,y,t \in \mathbb{R}\}$ *with the contact structure* $\theta = dt + 2xdy - 2ydx$*, the two vector fields* $X = \frac{1}{2}\partial_x + y\partial_t, Y = \frac{1}{2}\partial_y - x\partial_t$ *will span* $ker(\theta)$ *at each point. Suppose* $\widetilde{X}, \widetilde{Y}$ *are two linearly independent vector fields on* $\mathbb{R}^3$ *in* $ker(\theta)$ *at each point, and satisfies* $d\theta(\widetilde{X}|_0, \widetilde{Y}|_0) = d\theta(X|_0, Y|_0)$*. Then there exists a neighborhood* Ω *of* $0 = (0,0,0)$ *with a coordinate chart function* $\Phi = (\varphi, \phi, \tau) : \Omega \to \mathbb{R}^3$ *such that* $\Phi(0) = \widehat{0}$*,* $\Phi^*(\widehat{\theta}) = \theta$ *and*

$$d\Phi(\widetilde{X}) = (1 + o(1))\widehat{X} + o(1)\widehat{Y}\ ,\ d\Phi(\widetilde{Y}) = o(1)\widehat{X} + (1 + o(1))\widehat{Y}.$$

Here note we use the symbol $\widehat{}$ *to denote concepts in the range space* $\mathbb{R}^3$*.*

Proof. Suppose Φ is such a function on a neighborhood Ω and we will deduce the conditions it needs to satisfy.

Let $\widetilde{X} = AX + BY, \widetilde{Y} = CX + DY$, where A, B, C, D are real functions on $\mathbb{R}^3$ with $AD - BC \neq 0$ everywhere. Also let $A(0) = a, B(0) = b, C(0) = c, D(0) = d$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Since $d\theta(\widetilde{X}|_0, \widetilde{Y}|_0) = d\theta(X|_0, Y|_0)$, we have $ad - bc = 1$. Since $\Phi^*(\widehat{\theta}) = \theta$, there holds

$$\frac{\Phi^*(\widehat{\theta})(\partial_x)}{\theta(\partial_x)} = \frac{\Phi^*(\widehat{\theta})(\partial_y)}{\theta(\partial_y)} = \frac{\Phi^*(\widehat{\theta})(\partial_t)}{\theta(\partial_t)} = 1.$$

Since

$$\begin{cases} d\Phi(\partial_x) = \frac{\partial\varphi}{\partial x}\partial_{\widehat{x}} + \frac{\partial\phi}{\partial x}\partial_{\widehat{y}} + \frac{\partial\tau}{\partial x}\partial_{\widehat{t}} \\ d\Phi(\partial_y) = \frac{\partial\varphi}{\partial y}\partial_{\widehat{x}} + \frac{\partial\phi}{\partial y}\partial_{\widehat{y}} + \frac{\partial\tau}{\partial y}\partial_{\widehat{t}} \\ d\Phi(\partial_t) = \frac{\partial\varphi}{\partial t}\partial_{\widehat{x}} + \frac{\partial\phi}{\partial t}\partial_{\widehat{y}} + \frac{\partial\tau}{\partial t}\partial_{\widehat{t}} \end{cases}$$

direct computation leads to

$$\frac{\frac{\partial\tau}{\partial x}+2\varphi\frac{\partial\phi}{\partial x}-2\phi\frac{\partial\varphi}{\partial x}}{-2y}=\frac{\frac{\partial\tau}{\partial y}+2\varphi\frac{\partial\phi}{\partial y}-2\phi\frac{\partial\varphi}{\partial y}}{2x}=\frac{\frac{\partial\tau}{\partial t}+2\varphi\frac{\partial\phi}{\partial t}-2\phi\frac{\partial\varphi}{\partial t}}{1}=1. \tag{3.8}$$

We can also compute

$$\begin{aligned} d\Phi(\widetilde{X}|_0) =& a(d\Phi)(X|_0)+b(d\Phi)(Y|_0)=\frac{1}{2}a(d\Phi)(\partial_x|_0)+\frac{1}{2}b(d\Phi)(\partial_y|_0)\\ =&\frac{1}{2}\left(a\frac{\partial\varphi}{\partial x}|_0+b\frac{\partial\varphi}{\partial y}|_0\right)\partial_{\widehat{x}}|_{\widehat{0}}+\frac{1}{2}\left(a\frac{\partial\phi}{\partial x}|_0+b\frac{\partial\phi}{\partial y}|_0\right)\partial_{\widehat{y}}|_{\widehat{0}}\\ &+\frac{1}{2}\left(a\frac{\partial\tau}{\partial x}|_0+b\frac{\partial\tau}{\partial y}|_0\right)\partial_{\widehat{t}}|_{\widehat{0}}. \end{aligned}$$

Since Φ preserves the contact structure, now the condition

$$d\Phi(\widetilde{X})=(1+o(1))\widehat{X}+o(1)\widehat{Y}$$

is equivalent to

$$d\Phi(\widetilde{X}|_0)=\frac{1}{2}\partial_{\widehat{x}}|_{\widehat{0}}.$$

So we obtain

$$\begin{cases} a\frac{\partial\varphi}{\partial x}|_0+b\frac{\partial\varphi}{\partial y}|_0=1\\ a\frac{\partial\phi}{\partial x}|_0+b\frac{\partial\phi}{\partial y}|_0=0\\ a\frac{\partial\tau}{\partial x}|_0+b\frac{\partial\tau}{\partial y}|_0=0 \end{cases}$$

Similarly consider $d\Phi(\widetilde{Y})$, we have

$$\begin{cases} c\frac{\partial\varphi}{\partial x}|_0+d\frac{\partial\varphi}{\partial y}|_0=0\\ c\frac{\partial\phi}{\partial x}|_0+d\frac{\partial\phi}{\partial y}|_0=1\\ c\frac{\partial\tau}{\partial x}|_0+d\frac{\partial\tau}{\partial y}|_0=0 \end{cases}$$

Together with (3.8), we obtain

$$\begin{pmatrix} \frac{\partial\varphi}{\partial x}|_0 & \frac{\partial\varphi}{\partial y}|_0 & \frac{\partial\varphi}{\partial t}|_0 \\ \frac{\partial\phi}{\partial x}|_0 & \frac{\partial\phi}{\partial y}|_0 & \frac{\partial\phi}{\partial t}|_0 \\ \frac{\partial\tau}{\partial x}|_0 & \frac{\partial\tau}{\partial y}|_0 & \frac{\partial\tau}{\partial t}|_0 \end{pmatrix} = \begin{pmatrix} e & g & * \\ f & h & * \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

Inspired by this, we simply let

$$\begin{cases} \varphi = ex + gy \\ \phi = fx + hy \\ \tau = t \end{cases}$$

and check it satisfies (3.8) and (3.9). Hence we take a suitable neighborhood and obtain the right coordinate chart. □

Theorem 3.2.2. *Let (M, θ_M) be a strictly pseudoconvex CR manifold (not necessarily compact) of real dimension 3, with $\{T_1^M\}$ to be a pseudo-Hermitian frame. Denote $\mathbb{H}^1 = \{(\widehat{z}, \widehat{t}) : \widehat{z} \in \mathbb{C}, \widehat{t} \in \mathbb{R}\}$, with $\widehat{Z}_1 = \partial_{\widehat{z}} + i\widehat{\overline{z}}\partial_{\widehat{t}}$, $\widehat{Z_{\overline{1}}} = \partial_{\widehat{\overline{z}}} - i\widehat{z}\partial_{\widehat{t}}$ and $\widehat{X} = Re(\widehat{Z}_1)$, $\widehat{Y} = -Im(\widehat{Z}_1)$ on $\mathbb{H}^1$. For any point $m \in M$, there exists a neighborhood Ω and a coordinate chart function $\Phi : \Omega \to \mathbb{H}^1$ such that $\Phi(m) = 0$, $\Phi^*(\theta_{\mathbb{H}^1}) = \theta_M$ and*

$$d\Phi(T_1^M) = (1 + o(1))\widehat{Z}_1 + o(1)\widehat{Z}_{\overline{1}},$$
$$d\Phi(T_{\overline{1}}^M) = o(1)\widehat{Z}_1 + (1 + o(1))\widehat{Z}_{\overline{1}}.$$

Proof. Let's now denote $\mathrm{Re}(T_1^M) = X^M, -\mathrm{Im}(T_1^M) = Y^M$. By Darboux's Theorem, all contact structures are locally diffeomorphic. Hence we can take a smooth Φ_1 which maps a neighborhood of m onto an open set in $\mathbb{R}^3$, such that $\Phi_1(m) = 0$, $\Phi_1^*(dt + 2xdy - 2ydx) = \theta_M$. Take $d\Phi_1(X^M), d\Phi_1(Y^M)$ to be $\widetilde{X}, \widetilde{Y}$ as in Lemma 3.2.1, we obtain a neighborhood of 0 in $\mathbb{R}^3$ and a coordinate chart function Φ_2 onto an open subset of $\mathbb{H}^1$ (now considered as the range space $\mathbb{R}^3$ as in Lemma 3.2.1) such that

$\Phi_2(0) = 0$, $\Phi_2^*(\theta_{\mathbb{H}^1}) = dt + 2xdy - 2ydx$, and

$$d\Phi_2 \circ d\Phi_1(X^M) = (1 + o(1))\widehat{X} + o(1)\widehat{Y},$$
$$d\Phi_2 \circ d\Phi_1(Y^M) = o(1)\widehat{X} + (1 + o(1))\widehat{Y}.$$

Take $\Phi = \Phi_2 \circ \Phi_1$ and choose a suitable neighborhood of m, we get the result we want. □

Lemma 3.2.3. *Consider now $n \geq 2$, $\mathbb{R}^{2n+1} = \{(x^j, y^j, t) : 1 \leq j \leq n, x^j, y^j, t \in \mathbb{R}\}$ with the contact structure $\theta = dt + 2x^j dy^j - 2y^j dx^j$. The $2n$ vector fields are now*

$$X_j = \frac{1}{2}\partial_{x^j} + y^j \partial_t \ , \ Y_j = \frac{1}{2}\partial_{y^j} - x^j \partial_t.$$

Suppose there are also $2n$ linearly independent vector fields on $\mathbb{R}^{2n+1}$ given in the matrix form

$$\begin{pmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \\ \vdots \\ \widetilde{X}_n \\ \widetilde{Y}_1 \\ \widetilde{Y}_2 \\ \vdots \\ \widetilde{Y}_n \end{pmatrix} = \begin{pmatrix} A_1^1 & A_1^2 & \cdots & A_1^n & B_1^1 & B_1^2 & \cdots & B_1^n \\ A_2^1 & A_2^2 & \cdots & A_2^n & B_2^1 & B_2^2 & \cdots & B_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_n^1 & A_n^2 & \cdots & A_n^n & B_n^1 & B_n^2 & \cdots & B_n^n \\ C_1^1 & C_1^2 & \cdots & C_1^n & D_1^1 & D_1^2 & \cdots & D_1^n \\ C_2^1 & C_2^2 & \cdots & C_2^n & D_2^1 & D_2^2 & \cdots & D_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n^1 & C_n^2 & \cdots & C_n^n & D_n^1 & D_n^2 & \cdots & D_n^n \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix},$$

with $A_j^k, B_j^k, C_j^k, D_j^k$ to be smooth functions on $\mathbb{R}^{2n+1}$. We assume

$$\begin{pmatrix} A_1^1 & \cdots & A_1^n & B_1^1 & \cdots & B_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_n^1 & \cdots & A_n^n & B_n^1 & \cdots & B_n^n \\ C_1^1 & \cdots & C_1^n & D_1^1 & \cdots & D_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_n^1 & \cdots & C_n^n & D_n^1 & \cdots & D_n^n \end{pmatrix}^{-1} = \begin{pmatrix} E_1^1 & \cdots & E_1^n & F_1^1 & \cdots & F_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ E_n^1 & \cdots & E_n^n & F_n^1 & \cdots & F_n^n \\ G_1^1 & \cdots & G_1^n & H_1^1 & \cdots & H_1^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ G_n^1 & \cdots & G_n^n & H_n^1 & \cdots & H_n^n \end{pmatrix}.$$

Let $a_j^k \cdots h_j^k$ be values of those functions at 0. If there holds for any j, k,

$$\begin{cases} \sum_{\ell=1}^{n} (e_j^\ell f_k^\ell - e_k^\ell f_j^\ell) = 0 \\ \sum_{\ell=1}^{n} (g_j^\ell h_k^\ell - g_k^\ell h_j^\ell) = 0 \\ \sum_{\ell=1}^{n} (e_j^\ell h_k^\ell - f_j^\ell g_k^\ell) = \delta_k^j \end{cases} \tag{3.10}$$

then there exists a neighborhood Ω of 0 in $\mathbb{R}^{2n+1}$ and a coordinate chart function $\Phi = (\varphi^1 \cdots \varphi^n, \phi^1 \cdots \phi^n, \tau) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that $\Phi(0) = \widehat{0}$, $\Phi^*(\widehat{\theta}) = \theta$ and for any j,

$$\begin{cases} d\Phi(\widetilde{X}_j) = o(1)\widehat{X}_1 + \cdots + (1 + o(1))\widehat{X}_j + \cdots + o(1)\widehat{X}_n + o(1)\widehat{Y}_1 + \cdots + o(1)\widehat{Y}_n \\ d\Phi(\widetilde{Y}_j) = o(1)\widehat{X}_1 + \cdots + o(1)\widehat{X}_n + o(1)\widehat{Y}_1 + \cdots + (1 + o(1))\widehat{Y}_j + \cdots + o(1)\widehat{Y}_n \end{cases} \tag{3.11}$$

Here note we use the symbol $\widehat{}$ to denote concepts in the range space $\mathbb{R}^{2n+1}$.

Proof. Inspired by Lemma 3.2.1, we define the function Φ as follows:

$$\begin{cases} \varphi^j = e_\ell^j x^\ell + g_\ell^j y^\ell \\ \phi^j = f_\ell^j x^\ell + h_\ell^j y^\ell \\ \tau = t \end{cases}$$

for any $1 \le j \le n$. Then we check it satisfies all the requirements. First we notice that if $\Phi^*(\widehat{\theta}) = \theta$ (which will be checked later), to show (3.11), it suffices to check $d\Phi(\widetilde{X}_j|_0) = \frac{1}{2}\partial_{\widehat{x}^j}|_{\widehat{0}}$ and $d\Phi(\widetilde{Y}_j|_0) = \frac{1}{2}\partial_{\widehat{y}^j}|_{\widehat{0}}$. Since for each j,

$$\begin{cases} d\Phi(\partial_{x^j}) = \frac{\partial\varphi^\ell}{\partial x^j}\partial_{\widehat{x}^\ell} + \frac{\partial\phi^\ell}{\partial x^j}\partial_{\widehat{y}^\ell} + \frac{\partial\tau}{\partial x^j}\partial_{\widehat{t}} \\ d\Phi(\partial_{y^j}) = \frac{\partial\varphi^\ell}{\partial y^j}\partial_{\widehat{x}^\ell} + \frac{\partial\phi^\ell}{\partial y^j}\partial_{\widehat{y}^\ell} + \frac{\partial\tau}{\partial y^j}\partial_{\widehat{t}} \\ d\Phi(\partial_t) = \frac{\partial\varphi^\ell}{\partial t}\partial_{\widehat{x}^\ell} + \frac{\partial\phi^\ell}{\partial t}\partial_{\widehat{y}^\ell} + \frac{\partial\tau}{\partial t}\partial_{\widehat{t}} \end{cases}$$

we obtain

$$\begin{aligned} d\Phi(\widetilde{X}_j|_0) =& a_j^k d\Phi(X_k|_0) + b_j^k d\Phi(Y_k|_0) = \frac{1}{2}a_j^k d\Phi(\partial_{x^k}|_0) + \frac{1}{2}b_j^k d\Phi(\partial_{y^k}|_0) \\ =& \frac{1}{2}\left(a_j^k \frac{\partial\varphi^\ell}{\partial x^k}|_0 + b_j^k \frac{\partial\varphi^\ell}{\partial y^k}|_0\right)\partial_{\widehat{x}^\ell}|_{\widehat{0}} + \frac{1}{2}\left(a_j^k \frac{\partial\phi^\ell}{\partial x^k}|_0 + b_j^k \frac{\partial\phi^\ell}{\partial y^k}|_0\right)\partial_{\widehat{y}^\ell}|_{\widehat{0}} \\ &+ \frac{1}{2}\left(a_j^k \frac{\partial\tau}{\partial x^k}|_0 + b_j^k \frac{\partial\tau}{\partial y^k}|_0\right)\partial_{\widehat{t}}|_{\widehat{0}} \\ =& \frac{1}{2}\left(a_j^k e_k^\ell + b_j^k g_k^\ell\right)\partial_{\widehat{x}^\ell}|_{\widehat{0}} + \frac{1}{2}\left(a_j^k f_k^\ell + b_j^k h_k^\ell\right)\partial_{\widehat{y}^\ell}|_{\widehat{0}} = \frac{1}{2}\partial_{\widehat{x}^j}|_{\widehat{0}}. \end{aligned}$$

Similarly we can check $d\Phi(\widetilde{Y}_j|_0) = \frac{1}{2}\partial_{\widehat{y}^j}|_{\widehat{0}}$. Then it remains to check $\Phi^*(\widehat{\theta}) = \theta$. Actually for each j,

$$\begin{aligned} \Phi^*(\widehat{\theta})(\partial_{x^j}) = \widehat{\theta}\left(d\Phi(\partial_{x^j})\right) &= \frac{\partial\tau}{\partial x^j} + 2\sum_{\ell=1}^n \varphi^\ell \frac{\partial\phi^\ell}{\partial x^j} - 2\sum_{\ell=1}^n \phi^\ell \frac{\partial\varphi^\ell}{\partial x^j} \\ &= 2\sum_{\ell,k=1}^n (e_k^\ell x^k + g_k^\ell y^k) f_j^\ell - 2\sum_{\ell,k=1}^n (f_k^\ell x^k + h_k^\ell y^k) e_j^\ell \\ &= -2y^j = \theta(\partial_{x^j}). \end{aligned}$$

Similarly we can verify that $\Phi^*(\widehat{\theta})(\partial_{y^j}) = \theta(\partial_{y^j})$ and $\Phi^*(\widehat{\theta})(\partial_t) = \theta(\partial_t)$. Hence we conclude $\Phi^*(\widehat{\theta}) = \theta$. □

Lemma 3.2.4. *Let (M, θ_M) be a strictly pseudoconvex CR manifold (not necessarily compact) of real dimension $(2n+1)$, where $n \ge 2$, with $\{T_j^M\}_{j=1}^n$ to be a pseudo-Hermitian frame. We set $X_j^M = Re(T_j^M)$ and $Y_j^M = -Im(T_j^M)$ for each $1 \le j \le n$.*

For a $m \in M$, suppose there is a coordinate chart function Φ from a neighborhood Ω of m onto an open subset of $\mathbb{R}^{2n+1}$ such that $\Phi(m) = 0$ and $\Phi^*(\theta) = \theta_M$, where $\theta = dt + 2x^j dy^j - 2y^j dx^j$. If we denote $\widetilde{X}_j = d\Phi(X_j^M), \widetilde{Y}_j = d\Phi(Y_j^M)$, write them as linearly combinations of those X_j, Y_j in $\mathbb{R}^{2n+1}$ and then take the inverse matrix as in Lemma 3.2.3, then the values at 0 will satisfy (3.10) for any j, k.

Proof. We first rewrite

$$\widetilde{X}_j = A_j^k X_k + B_j^k Y_k \ , \ \widetilde{Y}_j = C_j^k X_k + D_j^k Y_k$$

in the inverse way:

$$X_j = E_j^k \widetilde{X}_k + F_j^k \widetilde{Y}_k \ , \ Y_j = G_j^k \widetilde{X}_k + H_j^k \widetilde{Y}_k. \tag{3.12}$$

Identify $\mathbb{R}^{2n+1}$ with $\mathbb{H}^n$, now $\theta = \theta_{\mathbb{H}^n}$ remains unchanged. We can take advantage of the complex structure. Let $Z_j = X_j - iY_j$, $Z_{\bar{j}} = X_j + iY_j$ and $\widetilde{Z}_j = \widetilde{X}_j - i\widetilde{Y}_j$, $\widetilde{Z}_{\bar{j}} = \widetilde{X}_j + i\widetilde{Y}_j$. Since $\{T_j^M\}_{j=1}^n$ is a pseudo-Hermitian frame, and $\Phi^*(\theta) = \theta_M$, we obtain for any $1 \le j, k \le n$,

$$\begin{aligned}
\delta_j^k &= -i(d\theta_M)(T_j^M, T_{\bar{k}}^M) = -i(d\theta)(\widetilde{Z}_j, \widetilde{Z}_{\bar{k}}), \\
0 &= -i(d\theta_M)(T_j^M, T_k^M) = -i(d\theta)(\widetilde{Z}_j, \widetilde{Z}_k), \\
0 &= -i(d\theta_M)(T_{\bar{j}}^M, T_{\bar{k}}^M) = -i(d\theta)(\widetilde{Z}_{\bar{j}}, \widetilde{Z}_{\bar{k}}).
\end{aligned} \tag{3.13}$$

We also have for any $1 \le j, k \le n$,

$$\delta_j^k = -i(d\theta)(Z_j, Z_{\bar{k}}) \ , \ 0 = -i(d\theta)(Z_j, Z_k) \ , \ 0 = -i(d\theta)(Z_{\bar{j}}, Z_{\bar{k}}).$$

Substitute (3.12) and (3.13), there holds

$$\begin{aligned}
\delta_j^k &= -i(d\theta)(Z_j, Z_{\overline{k}}) = -i(d\theta)(X_j - iY_j, X_k + iY_k) \\
&= -i(d\theta)\Big(E_j^\ell \widetilde{X}_\ell + F_j^\ell \widetilde{Y}_\ell - iG_j^\ell \widetilde{X}_\ell - iH_j^\ell \widetilde{Y}_\ell, E_k^\ell \widetilde{X}_\ell + F_k^\ell \widetilde{Y}_\ell + iG_k^\ell \widetilde{X}_\ell + iH_k^\ell \widetilde{Y}_\ell\Big) \\
&= -i(d\theta)\Big(\frac{(E_j^\ell - iG_j^\ell + iF_j^\ell + H_j^\ell)\widetilde{Z}_\ell + (E_j^\ell - iG_j^\ell - iF_j^\ell - H_j^\ell)\widetilde{Z}_{\overline{\ell}}}{2}, \\
&\qquad \frac{(E_k^\ell + iG_k^\ell + iF_k^\ell - H_k^\ell)\widetilde{Z}_\ell + (E_k^\ell + iG_k^\ell - iF_k^\ell + H_k^\ell)\widetilde{Z}_{\overline{\ell}}}{2}\Big).
\end{aligned}$$

Take the values at 0, we obtain

$$\begin{aligned}
&\sum_{\ell=1}^n (e_j^\ell - ig_j^\ell + if_j^\ell + h_j^\ell)(e_k^\ell + ig_k^\ell - if_k^\ell + h_k^\ell) - \\
&\qquad \sum_{\ell=1}^n (e_j^\ell - ig_j^\ell - if_j^\ell - h_j^\ell)(e_k^\ell + ig_k^\ell + if_k^\ell - h_k^\ell) = 4\delta_j^k.
\end{aligned}$$

Or equivalently,

$$\begin{aligned}
&\sum_{\ell=1}^n \left(e_j^\ell h_k^\ell + e_k^\ell h_j^\ell - f_j^\ell g_k^\ell - f_k^\ell g_j^\ell\right) = 2\delta_j^k, \\
&\sum_{\ell=1}^n \left(e_j^\ell f_k^\ell - e_k^\ell f_j^\ell + g_j^\ell h_k^\ell - g_k^\ell h_j^\ell\right) = 0.
\end{aligned} \tag{3.14}$$

Similarly, since $-i(d\theta)(Z_j, Z_k) = 0$ for any j, k, we obtain

$$\sum_{\ell=1}^n \left(e_j^\ell h_k^\ell - e_k^\ell h_j^\ell - f_j^\ell g_k^\ell + f_k^\ell g_j^\ell\right) = 0\ , \ \sum_{\ell=1}^n \left(e_j^\ell f_k^\ell - e_k^\ell f_j^\ell - g_j^\ell h_k^\ell + g_k^\ell h_j^\ell\right) = 0. \tag{3.15}$$

From (3.14) and (3.15), it is easy to deduce (3.10). □

Theorem 3.2.5. *Let (M, θ_M) be a strictly pseudoconvex CR manifold (not necessarily compact) of real dimension $(2n+1)$, where $n \geq 2$, with $\{T_j^M\}_{j=1}^n$ to be a pseudo-Hermitian frame. Denote $\mathbb{H}^n = \{(\widehat{z}^1 \cdots, \widehat{z}^n, \widehat{t}) : \widehat{z}^j \in \mathbb{C}, \widehat{t} \in \mathbb{R}\}$, with $\widehat{Z}_j = \partial_{\widehat{z}^j} + i\overline{\widehat{z}}^j \partial_{\widehat{t}}$, $\widehat{Z}_{\overline{j}} = \partial_{\overline{\widehat{z}}^j} - i\widehat{z}^j \partial_{\widehat{t}}$, and $\widehat{X}_j = Re(\widehat{Z}_j)$, $\widehat{Y}_j = -Im(\widehat{Z}_j)$ on $\mathbb{H}^n$. For any point $m \in M$,* *there exists a neighborhood* Ω *and a coordinate chart function* $\Phi : \Omega \to \mathbb{H}^n$ *such that* $\Phi(m) = 0$, $\Phi^*(\theta_{\mathbb{H}^n}) = \theta_M$ *and for any* $1 \leq j \leq n$,

$$d\Phi(T_j^M) = o(1)\widehat{Z}_1 + \cdots + (1+o(1))\widehat{Z}_j + \cdots + o(1)\widehat{Z}_n + o(1)\widehat{Z}_{\bar{1}} + \cdots + o(1)\widehat{Z}_{\bar{n}},$$
$$d\Phi(T_{\bar{j}}^M) = o(1)\widehat{Z}_1 + \cdots + o(1)\widehat{Z}_n + o(1)\widehat{Z}_{\bar{1}} + \cdots + (1+o(1))\widehat{Z}_{\bar{j}} + \cdots + o(1)\widehat{Z}_{\bar{n}}.$$

Proof. Let's now denote $\mathrm{Re}(T_j^M) = X_j^M$, $-\mathrm{Im}(T_j^M) = Y_j^M$. By Darboux's Theorem, all contact structures are locally diffeomorphic. Hence we can take a smooth Φ_1 which maps a neighborhood of m onto an open set in $\mathbb{R}^{2n+1}$, such that $\Phi_1(m) = 0$, $\Phi_1^*(dt + 2x^j dy^j - 2y^j dx^j) = \theta_M$. Take $d\Phi_1(X_j^M), d\Phi_1(Y_j^M)$ to be $\widetilde{X}_j, \widetilde{Y}_j$ as in Lemma 3.2.3, then by Lemma 3.2.3 and Lemma 3.2.4, we obtain a neighborhood of 0 in $\mathbb{R}^{2n+1}$ and a coordinate chart function Φ_2 onto an open subset of $\mathbb{H}^n$ (now considered as the range space $\mathbb{R}^{2n+1}$) such that $\Phi_2(0) = 0$, $\Phi_2^*(\theta_{\mathbb{H}^n}) = dt + 2x^j dy^j - 2y^j dx^j$, and for any $1 \leq j \leq n$,

$$\begin{aligned} d\Phi_2 \circ d\Phi_1(X_j^M) =& o(1)\widehat{X}_1 + \cdots + (1+o(1))\,\widehat{X}_j + \cdots + o(1)\widehat{X}_n \\ &+ o(1)\widehat{Y}_1 + \cdots + o(1)\widehat{Y}_n, \\ d\Phi_2 \circ d\Phi_1(Y_j^M) =& o(1)\widehat{X}_1 + \cdots + o(1)\widehat{X}_n + \\ & o(1)\widehat{Y}_1 + \cdots + (1+o(1))\,\widehat{Y}_j + \cdots + o(1)\widehat{Y}_n. \end{aligned}$$

Take $\Phi = \Phi_2 \circ \Phi_1$ and choose a suitable neighborhood of m, we get the result we want. □

3.3 Discussion on A Program

In this section, we prove $\mathcal{A}_q(M) = \mathscr{A}(n,q)$ and $\widehat{\mathcal{A}}_q(M) = \mathscr{A}(n,q)^q$ for any compact CR manifold M of dimension $2n+1$.

Lemma 3.3.1. *Let (M,θ) be a strictly pseudoconvex CR manifold (not necessarily compact) and let $q \in (1, 2n+2)$ be a positive number with q^* defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$. Suppose there exist positive constants A and B such that for any $u \in D(M)$, there holds*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{1}{q^*}} \leq A\left(\int_M |\nabla_b u|^q dvol\right)^{\frac{1}{q}} + B\left(\int_M |u|^q dvol\right)^{\frac{1}{q}}. \tag{3.16}$$

Then $A \geq \mathscr{A}(n,q)$.

Proof. We prove by contradiction. Suppose there exist $A < \mathscr{A}(n,q)$ and positive B such that (3.16) holds for any $u \in D(M)$. Take a $x \in M$, for any $\epsilon > 0$, we can take a small neighborhood Ω with the coordinate chart function Φ constructed in Theorem 3.2.2 and Theorem 3.2.5 such that $\Phi(\Omega) = B_\delta(0) \subset \mathbb{H}^n$, $\Phi^*(\theta_0) = \theta$ and

$$(1-\epsilon)\int_{B_\delta(0)} |\nabla_b u|^q d\mathrm{vol}_0 \leq \int_\Omega |\nabla_b u|^q d\mathrm{vol} \leq (1+\epsilon)\int_{B_\delta(0)} |\nabla_b u|^q d\mathrm{vol}_0.$$

Here $d$vol is the volume form on M and $d\mathrm{vol}_0$ is the standard volume form on $\mathbb{H}^n$. Taking ϵ small enough, we get there exist some $A' \in (A, \mathscr{A}(n,q))$ and B' such that for $u \in D(B_\delta(0))$,

$$\left(\int_{\mathbb{H}} |u|^{q^*} d\mathrm{vol}_0\right)^{\frac{1}{q^*}} \leq A'\left(\int_{\mathbb{H}} |\nabla_b u|^q d\mathrm{vol}_0\right)^{\frac{1}{q}} + B'\left(\int_{\mathbb{H}} |u|^q d\mathrm{vol}_0\right)^{\frac{1}{q}}.$$

By Hölder's inequality, we have

$$\left(\int_{B_\delta(0)} |u|^q d\mathrm{vol}_0\right)^{\frac{1}{q}} \leq V\left(B_\delta(0)\right)^{\frac{1}{2n+2}}\left(\int_{B_\delta(0)} |u|^{q^*} d\mathrm{vol}_0\right)^{\frac{1}{q^*}}.$$

Hence by choosing δ small enough, we obtain there exists $A'' < \mathscr{A}(n,q)$ such that for any $u \in D(B_\delta(0))$, there holds

$$\left(\int_{\mathbb{H}} |u|^{q^*} d\mathrm{vol}_0\right)^{\frac{1}{q^*}} \leq A'' \left(\int_{\mathbb{H}} |\nabla_b u|^q d\mathrm{vol}_0\right)^{\frac{1}{q}}. \tag{3.17}$$

Since (3.17) is invariant under dilation on $\mathbb{H}^n$, we have actually for any $u \in D(\mathbb{H}^n)$, (3.17) holds. This contradicts with the fact $\mathscr{A}(n,q)$ is the optimal constant, as we stated in Theorem 2.0.21. □

Lemma 3.3.2. *Let (M,θ) be a strictly pseudoconvex compact CR manifold and let $q \in (1, 2n+2)$ be a positive number with q^* defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$. For any $\epsilon > 0$, there exists positive number B such that for $u \in S_1^q(M)$,*

$$\left(\int_M |u|^{q^*} dvol\right)^{\frac{q}{q^*}} \leq (\mathscr{A}(n,q)^q + \epsilon) \int_M |\nabla_b u|^q dvol + B \int_M |u|^q dvol.$$

Proof. With the help of compatible coordinate charts, for any $\epsilon > 0$, we can take a finite partition of unity $(\Omega_j, \eta_j)_{j=1}^N$ of M such that if $u \in D(\Omega_j)$ for any $1 \leq j \leq N$, then we have

$$\left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \leq \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right) \int_M |\nabla_b u|^q d\mathrm{vol}.$$

So with the property of L^p spaces, we obtain for $u \in S_1^q(M)$,

$$\begin{aligned} \|u\|_{L^{q^*}}^q &= \|u^q\|_{L^{q^*/q}} = \left\| \sum_{j=1}^N \eta_j u^q \right\|_{L^{q^*/q}} \\ &\leq \sum_{j=1}^N \|\eta_j u^q\|_{L^{q^*/q}} = \sum_{j=1}^N \left(\int_M \eta_j^{\frac{q^*}{q}} |u|^{q^*} d\mathrm{vol}\right)^{\frac{q}{q^*}} \\ &\leq \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right) \sum_{j=1}^N \int_M |\nabla_b(\eta_j^{\frac{1}{q}} u)|^q d\mathrm{vol}. \end{aligned} \tag{3.18}$$

We choose positive constants α and β such that for any $t \geq 0$, there holds

$$(1+t)^q \leq 1 + \alpha t + \beta t^q.$$

Direct computation leads to

$$\begin{aligned} |\nabla_b(\eta_j^{\frac{1}{q}} u)|^q \leq & \left(\eta_j^{\frac{1}{q}} |\nabla_b u| + |u| |\nabla_b \eta_j^{\frac{1}{q}}| \right)^q \\ \leq & \eta_j |\nabla_b u|^q + \alpha \eta_j^{\frac{q-1}{q}} |\nabla_b u|^{q-1} |u| |\nabla_b \eta_j^{\frac{1}{q}}| + \beta |u|^q |\nabla_b \eta_j^{\frac{1}{q}}|^q. \end{aligned}$$

Combined with (3.18), we have if we take positive constant H with $H \geq |\nabla_b \eta_j^{\frac{1}{q}}|$ for all j, there holds

$$\begin{aligned} \left(\int_M |u|^{q^*} \right)^{\frac{q}{q^*}} \leq & \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2} \right) \times \\ & \int_M \sum_{j=1}^N \left(\eta_j |\nabla_b u|^q + \alpha \eta_j^{\frac{q-1}{q}} |\nabla_b u|^{q-1} |u| |\nabla_b \eta_j^{\frac{1}{q}}| + \beta |u|^q |\nabla_b \eta_j^{\frac{1}{q}}|^q \right) d\mathrm{vol} \\ \leq & \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2} \right) \left(\|\nabla_b u\|_{L^q}^q + \alpha N H \|u\|_{L^q} \|\nabla_b u\|_{L^q}^{q-1} + \beta N H^q \|u\|_{L^q}^q \right). \end{aligned}$$

Then we choose positive ϵ_0 and constant C such that

$$(1+\epsilon_0) \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2} \right) \leq \mathscr{A}(n,q)^q + \epsilon.$$

and

$$\alpha N H \|u\|_{L^q} \|\nabla_b u\|_{L^q}^{q-1} \leq \epsilon_0 \|\nabla_b u\|_{L^q}^q + C \|u\|_{L^q}^q.$$

Substitute, we obtain

$$\left(\int_M |u|^{q^*} \right)^{\frac{q}{q^*}} \leq (\mathscr{A}(n,q)^q + \epsilon) \int_M |\nabla_b u|^q d\mathrm{vol} + B \int_M |u|^q d\mathrm{vol}$$

with

$$B = \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2} \right) (C + \beta N H^q).$$

$\square$

Lemma 3.3.3. *Let (M,θ) be a strictly pseudoconvex CR manifold (not necessarily compact) and let $q \in (1, 2n+2)$ be a positive number with q^* defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$. Suppose there exist positive constants A and B such that for any $u \in D(M)$, there holds*

$$\left(\int_M |u|^{q^*} dvol \right)^{\frac{q}{q^*}} \leq A \int_M |\nabla_b u|^q dvol + B \int_M |u|^q dvol.$$

Then $A \geq \mathscr{A}(n,q)^q$.

Proof. We can prove by contradiction, and it is simple result if we combine Lemma 3.3.1 and the fact that for any nonnegative x and y,

$$(x+y)^{\frac{1}{q}} \leq x^{\frac{1}{q}} + y^{\frac{1}{q}}.$$

$\square$

Lemma 3.3.4. *Let (M,θ) be a strictly pseudoconvex compact CR manifold and let $q \in (1, 2n+2)$ be a positive number with q^* defined by $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}$. For any $\epsilon > 0$, there exists positive number B such that for $u \in S_1^q(M)$,*

$$\left(\int_M |u|^{q^*} dvol \right)^{\frac{1}{q^*}} \leq (\mathscr{A}(n,q) + \epsilon) \left(\int_M |\nabla_b u|^q dvol \right)^{\frac{1}{q}} + B \left(\int_M |u|^q dvol \right)^{\frac{1}{q}}.$$

Proof. It is conclusion from Lemma 3.3.2 and the simple fact that for any nonnegative x and y,

$$(x+y)^{\frac{1}{q}} \leq x^{\frac{1}{q}} + y^{\frac{1}{q}}.$$

$\square$

Theorem 3.3.5. *We have for any compact strictly pseudoconvex CR manifold* (M,θ),

$$\mathcal{A}_q(M) = \mathscr{A}(n,q).$$

Proof. It is now obvious from the results of Lemma 3.3.1 and Lemma 3.3.4. □

Theorem 3.3.6. *We have for any compact strictly pseudoconvex CR manifold* (M,θ),

$$\widehat{\mathcal{A}}_q(M) = \mathscr{A}(n,q)^q.$$

Proof. It is now obvious from the results of Lemma 3.3.2 and Lemma 3.3.3. □

3.4 Case of the CR Sphere

In this section, we give example of the two optimal constants in AB inequality on the CR sphere $\mathbb{S}^{2n+1}$.

Example 3.4.1. *As we can easily find in [29] or [40], on the CR sphere* $\mathbb{S}^{2n+1}$*, for the case* $q=2$*, for any* $u \in S_1^2(\mathbb{S}^{2n+1})$*, the AB inequality admits exactly optimal values of the two constants we find:*

$$\left(\int_{\mathbb{S}} |u|^{2^*} dvol_1\right)^{\frac{2}{2^*}} \leq \mathscr{A}(n,2)^2 \int_{\mathbb{S}} |\nabla_b u|^2 dvol_1 + \omega_{2n+1}^{-\frac{1}{n+1}} \int_{\mathbb{S}} u^2 dvol_1,$$

where $\mathscr{A}(n,2) = \sqrt{\frac{1}{\pi n^2}}$*,* ω_{2n+1} *is the volume of the sphere* $\mathbb{S}^{2n+1}$ *in* (2.1) *and the two constants are optimal. Here we use the volume form* $dvol_1$*, but in [29] the volume form is* $d\sigma$ *induced from* $\mathbb{R}^{2n+2}$*, so the constants may be different. Also note that the equality holds if and only if*

$$u(\zeta) = \frac{c}{|1-\bar{\xi}\cdot\zeta|^n}$$

for some constant c *and some* $\xi \in \mathbb{C}^{n+1}$ *with* $|\xi| < 1$.

Then the other form of AB inequality is also clear now. For any nonnegative numbers

x and y,

$$(x+y)^{\frac{1}{2}} \leq x^{\frac{1}{2}} + y^{\frac{1}{2}} \tag{3.19}$$

and the equality holds if $x = 0$ or $y = 0$. We have for any $u \in S_1^2(\mathbb{S}^{2n+1})$, there holds

$$\left(\int_{\mathbb{S}} |u|^{2^*} dvol_1\right)^{\frac{1}{2^*}} \leq \mathscr{A}(n,2) \left(\int_{\mathbb{S}} |\nabla_b u|^2 dvol_1\right)^{\frac{1}{2}} + \omega_{2n+1}^{-\frac{1}{2n+2}} \left(\int_{\mathbb{S}} u^2 dvol_1\right)^{\frac{1}{2}},$$

here the two constants $\mathscr{A}(n,2)$ and $\omega_{2n+1}^{-\frac{1}{2n+2}}$ are optimal and the equality holds if and only if $u \equiv c$ for some constant c. For if the equality holds, the equality in (3.19) also holds, we must have either $\int_{\mathbb{S}} u^2 dvol_1 = 0$ or $\int_{\mathbb{S}} |\nabla_b u|^2 dvol_1 = 0$, any will imply that u is constant on $\mathbb{S}^{2n+1}$.

3.5 Application to CR Yamabe Problem

In this section, we apply the analytic results to give new proof to special case of CR Yamabe problem.

Definition 3.5.1. *For some smooth function h on M, we define a functional $I_h(u)$ for each $u \in S_1(M)$:*

$$I_h(u) = \int_M |\nabla_b u|^2 dvol + \int_M hu^2 dvol.$$

Definition 3.5.2. *For $s \in (2, 2^*)$, we define*

$$\Lambda_s = \{u \in S_1(M) : \int_M |u|^s dvol = 1\},$$

and also

$$\Lambda = \{u \in S_1(M) : \int_M |u|^{2^*} dvol = 1\}.$$

Definition 3.5.3. *For $s \in (2, 2^*)$ and smooth h on M, we define*

$$\mu_{s,h} = \inf_{u \in \Lambda_s} I_h(u),$$

and also

$$\mu_h = \inf_{u \in \Lambda} I_h(u).$$

Lemma 3.5.4. *Let h be a smooth function on M, if $\mu_h > 0$, then the operator $\Delta_b + h$ is coercive. Or precisely, there exists $C > 0$, such that for any $u \in S_1(M)$, we have*

$$\int_M \left(|\nabla_b u|^2 + hu^2\right) dvol \geq C\|u\|_{S_1}^2.$$

Proof. By definition of μ_h and Hölder's inequality, there exist constants C_1 and C_2 such that for any $u \in S_1(M)$, there holds

$$\int_M \left(|\nabla_b u|^2 + hu^2\right) \mathrm{dvol} \geq C_1 \left(\int_M |u|^{2^*} \mathrm{dvol}\right)^{\frac{2}{2^*}} \geq C_2 \int_M u^2 \mathrm{dvol}.$$

Hence if we choose C small enough such that $0 < C < \frac{C_2}{2}$ and $(1 - C)C_2 + Ch \geq \frac{C_2}{2}$ everywhere, we will have

$$\begin{aligned}
\int_M \left(|\nabla_b u|^2 + hu^2\right) \mathrm{dvol} &\geq C \int_M \left(|\nabla_b u|^2 + hu^2\right) \mathrm{dvol} + (1 - C)C_2 \int_M u^2 \mathrm{dvol} \\
&\geq C \int_M |\nabla_b u|^2 \mathrm{dvol} + \frac{C_2}{2} \int_M u^2 \mathrm{dvol} \\
&\geq C\|u\|_{S_1}^2.
\end{aligned}$$

□

Lemma 3.5.5. *If for some smooth function h on M, we have $\mu_h > 0$, then for any $s \in (2, 2^*)$, there exists a positive function $\Phi_{s,h} \in C^\infty(M) \cap \Lambda_s$, such that*

$$\Delta_b \Phi_{s,h} + h\Phi_{s,h} = \mu_{s,h} \Phi_{s,h}^{s-1}. \tag{3.20}$$

In particular, $\mu_{s,h} > 0$.

Proof. Let $\{\Phi_j\}_{j=1}^{\infty}$ be a minimizing sequence in Λ_s to obtain $\mu_{s,h}$, i.e.

$$\lim_{j\to+\infty} I_h(\Phi_j) = \mu_{s,h}.$$

Without loss of generality, we may assume $\Phi_j \geq 0$ by replacing Φ_j by $|\Phi_j|$. Since (Φ_j) is bounded in $S_1(M)$, also by the fact that the embedding of $S_1(M)$ into $L^s(M)$ is compact, up to extraction of a subsequence, we have

$$\Phi_j \rightharpoonup \Phi_{s,h} \text{ in } S_1(M)$$

$$\Phi_j \to \Phi_{s,h} \text{ in } L^s(M)$$

$$\Phi_j \to \Phi_{s,h} \text{ a.e.}$$

It is obvious $\Phi_{s,h} \geq 0$ a.e. Since $\Phi_j \to \Phi_{s,h}$ in $L^s(M)$ and $\Phi_j \in \Lambda_s$, we have $\|\Phi_{s,h}\|_{L^s} = 1$. Hence $\Phi_{s,h} \not\equiv 0$. By weak convergence in $S_1(M)$, there holds

$$\|\Phi_{s,h}\|_{S_1} \leq \liminf_{j\to+\infty} \|\Phi_j\|_{S_1}.$$

Also since $L^s(M) \subset L^2(M)$, we obtain

$$I_h(\Phi_{s,h}) \leq \liminf_{j\to+\infty} I_h(\Phi_j) = \mu_{s,h}.$$

As a result we have $I_h(\Phi_{s,h}) = \mu_{s,h}$. By variational method, the fact that $\Phi_{s,h}$ is minimizer for I_h implies

$$\Delta_b \Phi_{s,h} + h\Phi_{s,h} = \mu_{s,h}\Phi_{s,h}^{s-1}.$$

Then by Theorem 2.0.22, we obtain $\Phi_{s,h}$ is smooth and positive. Hence multiply $\Phi_{s,h}$ on both sides and then take integral, it is easy to obtain $\mu_{s,h} > 0$ by Lemma 3.5.4. □

Lemma 3.5.6. *There holds*

$$\limsup_{s\to 2^*} \mu_{s,h} \le \mu_h.$$

Proof. For any $\epsilon > 0$, we can take nonnegative $v \in \Lambda$ such that

$$I_h(v) < \mu_h + \epsilon.$$

We now rescale v to obtain $v_s \in \Lambda_s$:

$$v_s = \left(\int_M v^s d\text{vol}\right)^{-\frac{1}{s}} v.$$

By definition of $\mu_{s,h}$, $I_h(v_s) \ge \mu_{s,h}$. We also have

$$\lim_{s\to 2^*} I_h(v_s) = I_h(v).$$

So we have

$$\limsup_{s\to 2^*} \mu_{s,h} \le I_h(v) \le \mu_h + \epsilon$$

for any $\epsilon > 0$. Hence the claim is proved. □

Lemma 3.5.7. *Let Ω be a smooth bounded domain of $\mathbb{H}^n$, let $\{u_j\}_{j=1}^{\infty}$ be a bounded sequence of functions in $S_{1,0}(\Omega)$. Assume $div_b(\nabla_b u_j)$ is bounded in $L^1(\Omega)$, then there exists $u \in S_{1,0}(\Omega)$ such that up to extraction of a subsequence,*

$$u_j \to u \ a.e.$$

$$\nabla_b u_j \to \nabla_b u \ a.e.$$

$$u_j \to u \ in \ L^2(\Omega)$$

$$\nabla_b u_j \rightharpoonup \nabla_b u \ in \ L^2(\Omega).$$

Proof. We have $\nabla_b u_j$ is bounded in $L^2(\Omega)$, also since the embedding of $S_1(\Omega)$ into $L^s(\Omega)$ is compact for $s \in (2, 2^*)$, so up to extraction of a subsequence, we may assume there exist $u \in S_{1,0}(\Omega)$ and some $\Sigma \in L^2(\Omega)$ such that

$$\begin{aligned}
&u_j \rightharpoonup u \text{ in } S_{1,0}(\Omega)\\
&u_j \to u \text{ in } L^2(\Omega)\\
&u_j \to u \text{ a.e.}\\
&\nabla_b u_j \rightharpoonup \Sigma \text{ in } L^2(\Omega).
\end{aligned}$$

For $\delta > 0$, now by Egoroff's theorem, there exists $\Omega_\delta \subset \Omega$ such that measure of $\Omega \backslash \Omega_\delta$ is less than δ, and u_j converges to u uniformly in Ω_δ. So let $\epsilon > 0$ be given, take j large enough, we will have $|u_j(x) - u(x)| < \epsilon$ for $x \in \Omega_\delta$.
Then we define a truncation function η_ϵ on $\mathbb{R}$ by

$$\eta_\epsilon(x) = \begin{cases} x \;\; if\; |x| < \epsilon \\ \frac{\epsilon x}{|x|} \;\; if\; |x| \geq \epsilon. \end{cases}$$

We have

$$\left\langle \nabla_b u_j - \nabla_b u, \nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) \right\rangle \geq 0$$

a.e. in Ω. For j large enough, there holds

$$\nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) = \nabla_b (u_j - u)$$

on Ω_δ, we have

$$\begin{aligned}
&\int_{\Omega_\delta} |\nabla_b (u_j - u)|^2 d\text{vol}_0 = \int_{\Omega_\delta} \left\langle \nabla_b u_j - \nabla_b u, \nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) \right\rangle d\text{vol}_0\\
&\leq \int_{\Omega} \left\langle \nabla_b u_j - \nabla_b u, \nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) \right\rangle d\text{vol}_0\\
&= -\int_{\Omega} div_b(\nabla_b u_j) \left(\eta_\epsilon \circ (u_j - u)\right) d\text{vol}_0 - \int_{\Omega} \left\langle \nabla_b u, \nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) \right\rangle d\text{vol}_0.
\end{aligned}$$

Since $\eta_\epsilon \circ (u_j - u)$ converges weakly to 0 in $S_{1,0}(\Omega)$, we obtain

$$\lim_{j\to+\infty} \int_\Omega \left\langle \nabla_b u, \nabla_b \left(\eta_\epsilon \circ (u_j - u)\right) \right\rangle d\mathrm{vol}_0 = 0.$$

Since $div_b(\nabla_b u_j)$ is bounded in $L^1(\Omega)$, there exists a positive constant C such that for any j,

$$\int_\Omega div_b(\nabla_b u_j) \left(\eta_\epsilon \circ (u_j - u)\right) d\mathrm{vol}_0 \leq \epsilon \int_\Omega |div_b(\nabla_b u_j)| d\mathrm{vol}_0 \leq C\epsilon.$$

Hence

$$\limsup_{j\to+\infty} \int_{\Omega_\delta} |\nabla_b(u_j - u)|^2 d\mathrm{vol}_0 \leq C\epsilon.$$

Since ϵ is arbitrarily chosen, there holds up to extraction of a subsequence, $\nabla_b(u_j - u)$ converges a.e. to 0 in Ω_δ as $j \to +\infty$. Since δ is also arbitrarily chosen, we have $\nabla_b u_j$ converges to $\nabla_b u$ a.e. in Ω. Since $\nabla_b u_j$ is bounded in $L^2(\Omega)$, so $\nabla_b u_j$ converges weakly to $\nabla_b u$, we conclude $\nabla_b u = \Sigma$. □

Lemma 3.5.8. *For $s \in (2, 2^*)$ and some smooth function h on M, assume that $0 < \mu_h < \frac{1}{\mathscr{A}(n,2)^2}$. let $\Phi_{s,h}$ be as in Lemma 3.5.5, with the additional condition that up to extraction of a subsequence, $\mu_{s,h}$ converges to some μ'_h as $s \to 2^*$. Then there exists smooth function $\Phi_h \in C^\infty(M) \cap S_1(M)$ with $\Phi_h > 0$, and*

$$\Delta_b \Phi_h + h\Phi_h = \mu'_h \Phi_h^{2^*-1}. \tag{3.21}$$

In particular, $\mu'_h > 0$.

Proof. For fixed function h and $s \in (2, 2^*)$, $\Phi_{s,h}$ is bounded in $S_1(M)$. Since for $s \in (2, 2^*)$, the embedding of $S_1(M)$ into $L^s(M)$ is compact, so up to extraction of a subsequence, we can assume as $s \to 2^*$,

$$\Phi_{s,h} \rightharpoonup \Phi_h \text{ in } S_1(M)$$

$$\Phi_{s,h} \to \Phi_h \text{ in } L^2(M)$$

$$\Phi_{s,h} \to \Phi_h \text{ a.e.}$$

So Φ_h is nonnegative. Since $|\nabla_b \Phi_{s,h}|$ is bounded in $L^2(M)$, we may assume up to extraction of a subsequence that as $s \to 2^*$,

$$\nabla_b \Phi_{s,h} \rightharpoonup \Sigma_h \text{ in } L^{\frac{2^*}{2^*-1}}(M).$$

Here, note that we have used the fact that weak convergence in $L^2(M)$ implies weak convergence in $L^{\frac{2^*}{2^*-1}}(M)$. Also, since $\Phi_{s,h}^{s-1}$ is bounded in $L^{\frac{2^*}{s-1}}(M) \subset L^{\frac{2^*}{2^*-1}}(M)$ and $\Phi_{s,h}^{s-1} \to \Phi_h^{2^*-1}$ a.e., we may assume up to extraction of a subsequence,

$$\Phi_{s,h}^{s-1} \rightharpoonup \Phi_h^{2^*-1} \text{ in } L^{\frac{2^*}{2^*-1}}(M).$$

Since

$$\Delta_b \Phi_{s,h} = -h\Phi_{s,h} + \mu_{s,h}\Phi_{s,h}^{s-1},$$

by Lemma 3.5.6, $\mu_{s,h}$ is bounded for fixed function h, so $\Delta_b \Phi_{s,h}$ is bounded in $L^{\frac{2^*}{2^*-1}}(M)$ and hence bounded in $L^1(M)$. We may also take a subsequence such that the limit of $\mu_{s,h}$ is μ'_h. Now take $s \to 2^*$ in (3.20), similar to Lemma 3.5.7, we obtain in the weak sense,

$$\Delta_b \Phi_h + h\Phi_h = \mu'_h \Phi_h^{2^*-1}. \tag{3.22}$$

By Theorem 2.0.22, Φ_h is positive smooth or identically 0. In the first case if we multiply Φ_h on both sides of (3.22) and take integral, it is easy to obtain $\mu'_h > 0$. So it suffices to show $\Phi_h \not\equiv 0$.

By definition of μ_h,

$$\mu_h = \inf_{u\in\Lambda} I_h(u) < \frac{1}{\mathscr{A}(n,2)^2}.$$

Take $\epsilon > 0$ such that

$$\left(\mathscr{A}(n,2)^2 + \epsilon\right) \inf_{u\in\Lambda} I_h(u) < 1. \tag{3.23}$$

By the result of AB inequality, there exists positive B such that for any $u \in S_1(M)$,

$$\left(\int_M |u|^{2^*} d\mathrm{vol}\right)^{\frac{2}{2^*}} \leq \left(\mathscr{A}(n,2)^2 + \epsilon\right) \int_M |\nabla_b u|^2 d\mathrm{vol} + B \int_M u^2 d\mathrm{vol}.$$

Substitute $u = \Phi_{s,h}$, there holds

$$\left(\int_M \Phi_{s,h}^{2^*} d\mathrm{vol}\right)^{\frac{2}{2^*}} \leq \left(\mathscr{A}(n,2)^2 + \epsilon\right) \int_M |\nabla_b \Phi_{s,h}|^2 d\mathrm{vol} + B \int_M \Phi_{s,h}^2 d\mathrm{vol}.$$

Applying Hölder's inequality, we obtain

$$\int_M \Phi_{s,h}^s d\mathrm{vol} \leq \left(\int_M \Phi_{s,h}^{2^*} d\mathrm{vol}\right)^{\frac{s}{2^*}} V(M)^{1-\frac{s}{2^*}},$$

where $V(M)$ represents volume of the manifold M. Since $\Phi_{s,h} \in \Lambda_s$, we have

$$V(M)^{\frac{2}{2^*}-\frac{2}{s}} \leq \left(\int_M \Phi_{s,h}^{2^*} d\mathrm{vol}\right)^{\frac{2}{2^*}}.$$

Now we have

$$\begin{aligned} V(M)^{\frac{2}{2^*}-\frac{2}{s}} &\leq \left(\mathscr{A}(n,2)^2 + \epsilon\right) \int_M |\nabla_b \Phi_{s,h}|^2 d\mathrm{vol} + B \int_M \Phi_{s,h}^2 d\mathrm{vol} \\ &\leq \left(\mathscr{A}(n,2)^2 + \epsilon\right) \left(\mu_{s,h} - \int_M h\Phi_{s,h}^2 d\mathrm{vol}\right) + B \int_M \Phi_{s,h}^2 d\mathrm{vol} \\ &\leq \left(\mathscr{A}(n,2)^2 + \epsilon\right) \mu_{s,h} + B' \int_M \Phi_{s,h}^2 d\mathrm{vol}. \end{aligned}$$

Then we take the limit, also since Lemma 3.5.6,

$$\mu'_h \le \inf_{u\in\Lambda} I_h(u) = \mu_h,$$

we get

$$1 \le \left(\mathscr{A}(n,2)^2 + \epsilon\right) \inf_{u\in\Lambda} I_h(u) + B' \int_M \Phi_h^2 d\mathrm{vol}.$$

By the choice of ϵ in (3.23), we have $\Phi_h \not\equiv 0$. This completes the proof. $\square$

Lemma 3.5.9. *We have*

$$\Phi_h \in \Lambda \qquad \textit{and} \qquad \mu'_h = \mu_h = \lim_{s\to 2^*} \mu_{s,h}.$$

Proof. Multiply Φ_h on both sides of (3.21), take integral, we obtain up to extraction of a subsequence,

$$\begin{aligned}
\mu'_h \int_M \Phi_h^{2^*} d\mathrm{vol} &= \int_M \left(|\nabla_b \Phi_h|^2 + h\Phi_h^2\right) d\mathrm{vol} \\
&\le \liminf_{s\to 2^*} \int_M \left(|\nabla_b \Phi_{s,h}|^2 + h\Phi_{s,h}^2\right) d\mathrm{vol} \\
&= \liminf_{s\to 2^*} \mu_{s,h} \le \mu'_h.
\end{aligned}$$

Hence $\int_M \Phi_h^{2^*} d\mathrm{vol} \le 1$. Take

$$v = \frac{\Phi_h}{\left(\int_M \Phi_h^{2^*} d\mathrm{vol}\right)^{\frac{1}{2^*}}},$$

clearly $v \in \Lambda$, hence we have

$$\begin{aligned}
\mu_h \le I_h(v) &= \frac{I_h(\Phi_h)}{\left(\int_M \Phi_h^{2^*} d\mathrm{vol}\right)^{\frac{2}{2^*}}} \\
&= \mu'_h \left(\int_M \Phi_h^{2^*} d\mathrm{vol}\right)^{\frac{2^*-2}{2^*}} \le \mu'_h.
\end{aligned}$$

With the help of Lemma 3.5.6, we conclude $\mu_h = \mu'_h$ and $\int_M \Phi_h^{2^*} d\text{vol} = 1$. Also since μ'_h is the limit of any subsequence, we obtain the result. $\square$

Theorem 3.5.10. *For some fixed smooth function h on M, if we have $0 < \mu_h < \frac{1}{\mathscr{A}(n,2)^2}$, then there exists a smooth function $\Phi_h \in C^\infty(M) \cap \Lambda$ with $\Phi_h > 0$, satisfying*

$$\Delta_b \Phi_h + h\Phi_h = \mu_h \Phi_h^{2^*-1}. \tag{3.24}$$

Here note that Φ_h is just the minimizer of the functional I_h we defined in Definition 3.5.1.

Proof. It is obvious from the several previous lemmas. $\square$

Theorem 3.5.11. *Let (M, θ) be a compact strictly pseudoconvex CR manifold with the pseudo-Hermitian structure θ. We denote the Webster scalar curvature of (M, θ) by R. If there holds $0 < \mu_{\frac{n}{2n+2}R} < \frac{1}{\mathscr{A}(n,2)^2}$, then there exists a CR conformal contact form θ', such that the Webster scalar curvature R' of (M, θ') is constant. Here by CR conformal contact form, we mean $\theta' = u^{\frac{2}{n}}\theta$, where u is a smooth positive function on M.*

Proof. It is well known that under the CR conformal change $\theta' = u^{\frac{2}{n}}\theta$, the Webster scalar curvature R' of (M, θ') is given by

$$R' = u^{-\frac{n+2}{n}}(\frac{2n+2}{n}\Delta_b u + Ru).$$

We denote $\mu_{\frac{n}{2n+2}R}$ by μ. Since $0 < \mu < \frac{1}{\mathscr{A}(n,2)^2}$, by Theorem 3.5.10, there exists a positive smooth function u on M, such that

$$\Delta_b u + \frac{n}{2n+2}Ru = \mu u^{2^*-1}.$$

The Webster scalar curvature R' under the CR conformal transformation $\theta' = u^{\frac{2}{n}}\theta$ now becomes

$$R' = u^{1-2^*}(\frac{2n+2}{n}\Delta_b u + Ru) = \frac{2n+2}{n}\mu,$$

which is a constant on M. Hence the CR Yamabe problem is solved in this setting. □

3.6 AB Inequality with Constraints

In this section, we give proof to AB inequality for functions with constraints on compact CR manifolds and CR sphere. Then we prove an analytic estimate based on that.

Theorem 3.6.1. *Let M be a compact CR manifold, with $q \in (1, 2n+2)$ and q^* defined such that*

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{2n+2}.$$

Let also $\xi_j,\ j = 1, \ldots, N$ be N changing sign C^1 functions which satisfies $\sum_{j=1}^{N} |\xi_j|^q = 1$. Then for any $u \in S_1^q(M)$ satisfying

$$\int_M \xi_j |\xi_j|^{q^*-1} |u|^{q^*} dvol = 0$$

for all $j = 1, \ldots, N$, and for any $\epsilon > 0$, there exists positive number B such that

$$\left(\int_M |u|^{q^*} dvol\right)^{q/q^*} \leq \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \epsilon\right) \int_M |\nabla_b u|^q dvol + B \int_M |u|^q dvol.$$

Proof. For each $j = 1, \ldots, N$, define $\xi_{j+} = \max(\xi_j, 0)$ and $\xi_{j-} = \max(-\xi_j, 0)$. So we have for all j,

$$\int_M (\xi_{j+})^{q^*} |u|^{q^*} \mathrm{dvol} = \int_M (\xi_{j-})^{q^*} |u|^{q^*} \mathrm{dvol}.$$

Clearly those $\xi_{j+}u$ and $\xi_{j-}u$ all belong to $S_1^q(M)$, so for $\epsilon > 0$, there exists B such that for any j, we have

$$\left(\int_M |\xi_{j\pm}u|^{q^*} d\text{vol}\right)^{q/q^*} \leq \left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right)\int_M |\nabla_b(\xi_{j\pm}u)|^q d\text{vol} + B\int_M |\xi_{j\pm}u|^q d\text{vol}.$$

Without loss of generality, we may also assume that for each j,

$$\int_M |\nabla_b\left(\xi_{j-}u\right)|^q d\text{vol} \geq \int_M |\nabla_b\left(\xi_{j+}u\right)|^q d\text{vol}.$$

Then since

$$|\nabla_b(\xi_{j+}u)|^q + |\nabla_b(\xi_{j-}u)|^q = |\nabla_b(\xi_j u)|^q$$

almost everywhere, we obtain

$$\begin{aligned}
\|\xi_j u\|_{L^{q^*}}^q =& \left(\int_M |\xi_j u|^{q^*} d\text{vol}\right)^{q/q^*} = 2^{q/q^*}\left(\int_M |\xi_{j+}u|^{q^*} d\text{vol}\right)^{q/q^*} \\
\leq& 2^{q/q^*}\left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right)\int_M |\nabla_b\left(\xi_{j+}u\right)|^q d\text{vol} + 2^{q/q^*}B\int_M |\xi_{j+}u|^q d\text{vol} \\
\leq& 2^{q/q^*-1}\left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right)\left(\int_M |\nabla_b(\xi_{j+}u)|^q d\text{vol} + \int_M |\nabla_b(\xi_{j-}u)|^q d\text{vol}\right) \\
&+ 2^{q/q^*}B\int_M |\xi_j u|^q d\text{vol} \\
\leq& 2^{-q/(2n+2)}\left(\mathscr{A}(n,q)^q + \frac{\epsilon}{2}\right)\int_M |\nabla_b(\xi_j u)|^q d\text{vol} + 2^{q/q^*}B\int_M |\xi_j u|^q d\text{vol}.
\end{aligned}$$

Now we have shown that for any $\epsilon > 0$, there exists constant B' such that for any j and u which satisfies the orthogonality condition, there holds

$$\left(\int_M |\xi_j u|^{q^*} d\text{vol}\right)^{q/q^*} \leq \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right)\int_M |\nabla_b(\xi_j u)|^q d\text{vol} + B'\int_M |\xi_j u|^q d\text{vol}. \tag{3.25}$$

Also we have

$$\|u\|^q_{L^{q^*}} = \|u^q\|_{L^{q^*/q}} = \Big\| \sum_{j=1}^N |\xi_j|^q u^q \Big\|_{L^{q^*/q}} \le \sum_{j=1}^N \Big\| |\xi_j|^q u^q \Big\|_{L^{q^*/q}} = \sum_{j=1}^N \|\xi_j u\|^q_{L^{q^*}}.$$

Now come back to (3.25), there holds

$$\begin{aligned}\left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{q/q^*} &\le \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right) \sum_{j=1}^N \int_M |\nabla_b(\xi_j u)|^q d\mathrm{vol} + B' \sum_{j=1}^N \int_M |\xi_j u|^q d\mathrm{vol} \\ &\le \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right) \sum_{j=1}^N \int_M \left(|\xi_j||\nabla_b u| + |\nabla_b \xi_j||u|\right)^q d\mathrm{vol} \\ &\quad + B' \int_M |u|^q d\mathrm{vol}\end{aligned}$$

Now we choose positive numbers α, β and H such that for $t \ge 0$

$$(1+t)^q \le 1 + \alpha t + \beta t^q$$

and for all j

$$|\nabla_b \xi_j| \le H.$$

Then we have

$$\begin{aligned}\|u\|^q_{L^{q^*}} = \left(\int_M |u|^{q^*} d\mathrm{vol}\right)^{q/q^*} &\le \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right) \times \\ &\int_M \sum_{j=1}^N \left(|\nabla_b u|^q |\xi_j|^q + \alpha |\nabla_b u|^{q-1} |\nabla_b \xi_j| |\xi_j|^{q-1} |u| + \beta |u|^q |\nabla_b \xi_j|^q\right) d\mathrm{vol} \\ &\quad + B' \int_M |u|^q d\mathrm{vol} \\ &\le \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right) \left(\|\nabla_b u\|^q_{L^q} + \alpha N H \|\nabla_b u\|^{q-1}_{L^q} \|u\|_{L^q} + \beta N H^q \|u\|^q_{L^q}\right) \\ &\quad + B' \int_M |u|^q d\mathrm{vol}.\end{aligned} \tag{3.26}$$

We can then choose constants ϵ_0 and C such that

$$\left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \frac{\epsilon}{2}\right)(1+\epsilon_0) \leq \frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \epsilon,$$

and also

$$\alpha N H \|\nabla_b u\|_{L^q}^{q-1} \|u\|_{L^q} \leq \epsilon_0 \|\nabla_b u\|_{L^q}^q + C\|u\|_{L^q}^q.$$

Substitute into (3.26), we obtain the result we want. □

Theorem 3.6.2. *Let $\mathbb{S}^{2n+1}$ be the CR sphere, with $q \in (1, 2n+2)$ and q^* defined as above. Also consider $\xi_j, j = 1, \ldots, 2n+2$ be the $(2n+2)$ coordinate functions of $\mathbb{R}^{2n+2}$ restricted to $\mathbb{S}^{2n+1}$. Then for any $u \in S_1^q(\mathbb{S}^{2n+1})$ satisfying*

$$\int_{\mathbb{S}} \xi_j |u|^{q^*} dvol_1 = 0$$

for all $j = 1, \ldots, 2n+2$, and for any $\epsilon > 0$, there exists positive number B such that

$$\left(\int_{\mathbb{S}} |u|^{q^*} dvol_1\right)^{q/q^*} \leq \left(\frac{\mathscr{A}(n,q)^q}{2^{q/(2n+2)}} + \epsilon\right) \int_{\mathbb{S}} |\nabla_b u|^q dvol_1 + B \int_{\mathbb{S}} |u|^q dvol_1.$$

Proof. Let Λ be the vector space of first spherical harmonics spanned by those ξ_j. Then for any point P on $\mathbb{S}^{2n+1}$, if we use r_P to denote the distance to P on the sphere measured by the metric induced from Euclidean metric on $\mathbb{R}^{2n+2}$. Then clearly $\xi_P = \cos(r_P)$ belongs to Λ and it is easy to check

$$\int_{\mathbb{S}} |\xi_P|^{q/q^*} \mathrm{dvol}_1 = \mathrm{const}$$

which is independent of the point P chosen.

From above facts, we can see for $\eta \in (0, \frac{1}{2})$ which will be chosen later, there exists a family of functions $(f_j)_{j=1,\dots,k} \in \Lambda$ such that $|f_j| < 2^{-q^*}$ for all j, and

$$1 + \eta < \sum_{j=1}^{k} |f_j|^{q/q^*} < 1 + 2\eta.$$

Then we let $(h_j)_{j=1,\dots,k}$ be a family of C^1 functions such that $h_j f_j \geq 0$ everywhere and

$$\max\left(\left||h_j|^q - |f_j|^{q/q^*}\right|, \left||h_j|^{q^*} - |f_j|\right|\right) < \left(\frac{\eta}{k}\right)^{q^*}.$$

Hence we have

$$1 < \sum_{j=1}^{k} |h_j|^q < 1 + 3\eta.$$

By the result of AB inequality, there exists B_η such that for $u \in S_1^q(\mathbb{S}^{2n+1})$, there holds

$$\left(\int_{\mathbb{S}} |u|^{q^*} d\text{vol}_1\right)^{q/q^*} \leq \mathscr{A}(n,q)^q (1+\eta) \int_{\mathbb{S}} |\nabla_b u|^q d\text{vol}_1 + B_\eta \int_{\mathbb{S}} |u|^q d\text{vol}_1.$$

Then for nonnegative $u \in S_1^q(\mathbb{S}^{2n+1})$, we have

$$\|u\|_{L^{q^*}}^q = \|u^q\|_{L^{q^*/q}} \leq \left\|\sum_{j=1}^{k} |h_j|^q u^q\right\|_{L^{q^*/q}} \leq \sum_{j=1}^{k} \left\||h_j|^q u^q\right\|_{L^{q^*/q}} = \sum_{j=1}^{k} \|h_j u\|_{L^{q^*}}^q.$$

Now we consider those nonnegative u such that for any $\xi \in \Lambda$, we have

$$\int_{\mathbb{S}} \xi u^{q^*} d\text{vol}_1 = 0.$$

Then if we let $f_{j+} = \max(f_j, 0)$ and $f_{j-} = \max(-f_j, 0)$, (similarly for h_j)we have for all j,

$$\int_{\mathbb{S}} f_{j+} u^{q^*} d\text{vol}_1 = \int_{\mathbb{S}} f_{j-} u^{q^*} d\text{vol}_1.$$

So if we let $\epsilon_0 = \frac{\eta}{k}$, there holds

$$\begin{aligned}\left(\int_{\mathbb{S}} |h_j|^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} &= \left(\int_{\mathbb{S}} (h_{j+})^{q^*} u^{q^*} d\mathrm{vol}_1 + \int_{\mathbb{S}} (h_{j-})^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \\ &\le \left(\int_{\mathbb{S}} f_{j+} u^{q^*} d\mathrm{vol}_1 + \epsilon_0^{q^*} \int_{\mathbb{S}} u^{q^*} d\mathrm{vol}_1 + \int_{\mathbb{S}} (h_{j-})^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \\ &\le 2^{q/q^*} \left(\int_{\mathbb{S}} \left((h_{j-})^{q^*} + \epsilon_0^{q^*}\right) u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \\ &\le 2^{q/q^*} \left(\int_{\mathbb{S}} (h_{j-} + \epsilon_0)^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \\ &\le 2^{q/q^*} \mathscr{A}(n,q)^q (1+\eta) \int_{\mathbb{S}} \Big|\nabla_b \left((h_{j-} + \epsilon_0) u\right) \Big|^q d\mathrm{vol}_1 \\ &\quad + B'_\eta \int_{\mathbb{S}} (h_{j-} + \epsilon_0)^q u^q d\mathrm{vol}_1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left(\int_{\mathbb{S}} |h_j|^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \le& 2^{q/q^*} \mathscr{A}(n,q)^q (1+\eta) \int_{\mathbb{S}} \Big|\nabla_b \left((h_{j+} + \epsilon_0) u\right) \Big|^q d\mathrm{vol}_1 \\ &+ B'_\eta \int_{\mathbb{S}} (h_{j+} + \epsilon_0)^q u^q d\mathrm{vol}_1.\end{aligned}$$

As in the proof of previous Theorem on arbitrary CR manifolds, we choose same positive numbers α, β and H such that for all j,

$$|\nabla_b h_j| \le H,$$

and for any $t \ge 0$

$$(1+t)^q \le 1 + \alpha t + \beta t^q.$$

Then similarly to the previous proof, we obtain for all j,

$$\begin{aligned}\Big|\nabla_b \left((h_{j\pm} + \epsilon_0) u\right) \Big|^q \le& \alpha |\nabla_b u|^{q-1} \Big|\nabla_b (h_{j\pm} + \epsilon_0)\Big| (h_{j\pm} + \epsilon_0)^{q-1} |u| \\ &+ |\nabla_b u|^q (h_{j\pm} + \epsilon_0)^q + \beta |u|^q \Big|\nabla_b (h_{j\pm} + \epsilon_0)\Big|^q.\end{aligned}$$

Then choose C_η such that for any positive x and y, we have

$$x^{q-1}y \leq \frac{\eta}{\alpha}x^q + C_\eta y^q.$$

Then there holds

$$\begin{aligned}\int_{\mathbb{S}} \Big|\nabla_b\left((h_{j\pm}+\epsilon_0)u\right)\Big|^q d\mathrm{vol}_1 \leq &(1+\eta)\int_{\mathbb{S}} |\nabla_b u|^q (h_{j\pm}+\epsilon_0)^q d\mathrm{vol}_1 \\ &+ (\beta+\alpha C_\eta)H^q \int_{\mathbb{S}} |u|^q d\mathrm{vol}_1.\end{aligned}$$

Note that for η small enough, $h_{j\pm}+\epsilon_0 \leq 1$, it is easy to check

$$(h_{j\pm}+\epsilon_0)^q \leq h_{j\pm}^q + q(h_{j\pm}+\epsilon_0)^{q-1}\epsilon_0 \leq h_{j\pm}^q + q\epsilon_0.$$

So now we have

$$
\begin{aligned}
\left(\int_{\mathbb{S}} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \leq & \sum_{j=1}^{k} \left(\int_{\mathbb{S}} |h_j|^{q^*} u^{q^*} d\mathrm{vol}_1\right)^{q/q^*} \\
\leq & \sum_{j=1}^{k} \Bigg[2^{q/q^*-1} \mathscr{A}(n,q)^q (1+\eta) \int_{\mathbb{S}} \Big| \nabla_b \left((h_{j+}+\epsilon_0) u \right) \Big|^q d\mathrm{vol}_1 \\
& \quad + 2^{q/q^*-1} \mathscr{A}(n,q)^q (1+\eta) \int_{\mathbb{S}} \Big| \nabla_b \left((h_{j-}+\epsilon_0) u \right) \Big|^q d\mathrm{vol}_1 \\
& \quad + \frac{1}{2} B'_\eta \int_{\mathbb{S}} (h_{j+}+\epsilon_0)^q u^q d\mathrm{vol}_1 + \frac{1}{2} B'_\eta \int_{\mathbb{S}} (h_{j-}+\epsilon_0)^q u^q d\mathrm{vol}_1 \Bigg] \\
\leq & 2^{q/q^*-1} \mathscr{A}(n,q)^q \left(1+\eta\right)^2 \sum_{j=1}^{k} \Bigg[\int_{\mathbb{S}} |\nabla_b u|^q (h_{j+}+\epsilon_0)^q d\mathrm{vol}_1 \\
& \quad + \int_{\mathbb{S}} |\nabla_b u|^q (h_{j-}+\epsilon_0)^q d\mathrm{vol}_1 \Bigg] + C' \int_{\mathbb{S}} |u|^q d\mathrm{vol}_1 \\
\leq & 2^{q/q^*-1} \mathscr{A}(n,q)^q \left(1+\eta\right)^2 \left(1+3\eta+2kq\epsilon_0\right) \int_{\mathbb{S}} |\nabla_b u|^q d\mathrm{vol}_1 \\
& + C' \int_{\mathbb{S}} |u|^q d\mathrm{vol}_1 \\
= & 2^{q/q^*-1} \mathscr{A}(n,q)^q \left(1+\eta\right)^2 \left(1+3\eta+2q\eta\right) \int_{\mathbb{S}} |\nabla_b u|^q d\mathrm{vol}_1 \\
& + C' \int_{\mathbb{S}} |u|^q d\mathrm{vol}_1 .
\end{aligned}
$$

Finally we can see for any ϵ in the statement, just choose η small is enough to show the result for such non-negative function u. For general u it is easy to see by simply replacing by $|u|$. □

Theorem 3.6.3. *Let $2^* = \frac{2n+2}{n}$, for any $\epsilon > 0$, there exists positive B_ϵ and $q_\epsilon \in (1, 2^*)$, such that for any $q \in (q_\epsilon, 2^*]$ and $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q dvol_1 = 0$ for all $\xi \in \Lambda$, we have*

$$
\begin{aligned}
\left(\int_{\mathbb{S}} |u|^q dvol_1\right)^{\frac{2n+2}{2n+2+q}} \leq & \left(\frac{\mathscr{A}(n,2)^2}{2^{1/(n+1)}} + \epsilon\right) \left(\int_{\mathbb{S}} |\nabla_b u|^2 dvol_1\right)^{\frac{q(n+1)}{2n+2+q}} \\
& + B_\epsilon \left(\int_{\mathbb{S}} u^2 dvol_1\right)^{\frac{q(n+1)}{2n+2+q}} .
\end{aligned}
$$

Proof. For $s \in [1, 2n+2)$, let s^* be such that

$$\frac{1}{s^*} = \frac{1}{s} - \frac{1}{2n+2}.$$

So s^* increases from $\frac{2n+2}{2n+1}$ to $+\infty$. Also for $q < 2^*$ close to 2^*, let s_q be such that $(s_q)^* = q$, i.e.

$$\frac{1}{s_q} = \frac{1}{q} + \frac{1}{2n+2}.$$

Given $q_0 < 2^*$ close to 2^*, for $\epsilon > 0$, there exists $B_0 > 0$ such that for any $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^{q_0} d\text{vol}_1 = 0$ for any $\xi \in \Lambda$, we have

$$\left(\int_{\mathbb{S}} |u|^{q_0} d\text{vol}_1\right)^{\frac{1}{q_0}} \le \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2}\right)^{\frac{1}{s_{q_0}}} \left(\int_{\mathbb{S}} |\nabla_b u|^{s_{q_0}} d\text{vol}_1\right)^{\frac{1}{s_{q_0}}} + B_0 \left(\int_{\mathbb{S}} |u|^{s_{q_0}} d\text{vol}_1\right)^{\frac{1}{s_{q_0}}}.$$

Then consider $q \in (q_0, 2^*]$, for $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q d\text{vol}_1 = 0$ for all $\xi \in \Lambda$, we set $\varphi = |u|^{q/q_0}$. Now the orthogonality condition becomes $\int_{\mathbb{S}} \xi \varphi^{q_0} d\text{vol}_1 = 0$ for all

$\xi \in \Lambda$. So there holds

$$
\begin{aligned}
\|u\|_{L^q}^{q/q_0} &= \left(\int_{\mathbb{S}} |u|^q d\mathrm{vol}_1\right)^{\frac{1}{q_0}} = \left(\int_{\mathbb{S}} \varphi^{q_0} d\mathrm{vol}_1\right)^{\frac{1}{q_0}} \\
&\leq \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2}\right)^{\frac{1}{s_{q_0}}} \left(\int_{\mathbb{S}} |\nabla_b \varphi|^{s_{q_0}} d\mathrm{vol}_1\right)^{\frac{1}{s_{q_0}}} \\
&\quad + B_0 \left(\int_{\mathbb{S}} |\varphi|^{s_{q_0}} d\mathrm{vol}_1\right)^{\frac{1}{s_{q_0}}} \\
&= \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2}\right)^{\frac{1}{s_{q_0}}} \left(\frac{q}{q_0}\right) \left(\int_{\mathbb{S}} |u|^{(\frac{q}{q_0}-1)s_{q_0}} |\nabla_b u|^{s_{q_0}} d\mathrm{vol}_1\right)^{\frac{1}{s_{q_0}}} \\
&\quad + B_0 \left(\int_{\mathbb{S}} |u|^{(\frac{q}{q_0}-1)s_{q_0}} |u|^{s_{q_0}} d\mathrm{vol}_1\right)^{\frac{1}{s_{q_0}}} \\
&\leq \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2}\right)^{\frac{1}{s_{q_0}}} \left(\frac{q}{q_0}\right) \times \\
&\quad \left(\int_{\mathbb{S}} |u|^{(\frac{q}{q_0}-1)s_{q_0} s_q/(s_q - s_{q_0})} d\mathrm{vol}_1\right)^{\frac{s_q - s_{q_0}}{s_q s_{q_0}}} \left(\int_{\mathbb{S}} |\nabla_b u|^{s_q} d\mathrm{vol}_1\right)^{\frac{1}{s_q}} \\
&\quad + B_0 \left(\int_{\mathbb{S}} |u|^{(\frac{q}{q_0}-1)s_{q_0} s_q/(s_q - s_{q_0})} d\mathrm{vol}_1\right)^{\frac{s_q - s_{q_0}}{s_q s_{q_0}}} \left(\int_{\mathbb{S}} |u|^{s_q} d\mathrm{vol}_1\right)^{\frac{1}{s_q}}.
\end{aligned}
$$

By definition of s_q, it is easy to check

$$
\frac{(\frac{q}{q_0} - 1)s_{q_0} s_q}{s_q - s_{q_0}} = q \qquad \text{and} \qquad \frac{s_q - s_{q_0}}{s_q s_{q_0}} = \frac{1}{q_0} - \frac{1}{q}.
$$

So we get for $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q d\mathrm{vol}_1 = 0$ for all $\xi \in \Lambda$ and for any $q \in (q_0, 2^*]$, there holds

$$
\begin{aligned}
\left(\int_{\mathbb{S}} |u|^q d\mathrm{vol}_1\right)^{\frac{1}{q}} \leq \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2}\right)^{\frac{1}{s_{q_0}}} \left(\frac{q}{q_0}\right) \left(\int_{\mathbb{S}} |\nabla_b u|^{s_q} d\mathrm{vol}_1\right)^{\frac{1}{s_q}} \\
+ B_0 \left(\int_{\mathbb{S}} |u|^{s_q} d\mathrm{vol}_1\right)^{\frac{1}{s_q}}.
\end{aligned} \tag{3.27}
$$

Applying Hölder's inequality, we have

$$
\int_{\mathbb{S}} |\nabla_b u|^{s_q} d\mathrm{vol}_1 \leq \omega_{2n+1}^{\frac{2-s_q}{2}} \left(\int_{\mathbb{S}} |\nabla_b u|^2 d\mathrm{vol}_1\right)^{\frac{s_q}{2}}
$$

and

$$\int_{\mathbb{S}} |u|^{s_q} d\mathrm{vol}_1 \le \omega_{2n+1}^{\frac{2-s_q}{2}} \left(\int_{\mathbb{S}} u^2 d\mathrm{vol}_1 \right)^{\frac{s_q}{2}}.$$

Substitute into (3.27) we obtain for any $q \in (q_0, 2^*]$, $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q d\mathrm{vol}_1 = 0$ for any $\xi \in \Lambda$, there holds

$$\begin{aligned} \left(\int_{\mathbb{S}} |u|^q d\mathrm{vol}_1 \right)^{\frac{1}{q}} \le & \omega_{2n+1}^{\frac{2-s_q}{2s_q}} \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{s_{q_0}/(2n+2)}} + \frac{\epsilon}{2} \right)^{\frac{1}{s_{q_0}}} \left(\frac{q}{q_0} \right) \left(\int_{\mathbb{S}} |\nabla_b u|^2 d\mathrm{vol}_1 \right)^{\frac{1}{2}} \\ & + B_0 \omega_{2n+1}^{\frac{2-s_q}{2s_q}} \left(\int_{\mathbb{S}} u^2 d\mathrm{vol}_1 \right)^{\frac{1}{2}}. \end{aligned}$$

We choose $q_0 < 2^*$ but close to 2^* such that for any $q \in (q_0, 2^*]$, there holds

$$\frac{q}{q_0} \omega_{2n+1}^{\frac{2-s_q}{2s_q}} \left(\frac{\mathscr{A}(n, s_{q_0})^{s_{q_0}}}{2^{\frac{s_{q_0}}{2n+2}}} + \frac{\epsilon}{2} \right)^{\frac{1}{s_{q_0}}} \le \left(\frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \frac{3\epsilon}{4} \right)^{\frac{1}{s_q}}$$

and

$$B_0 \omega_{2n+1}^{\frac{2-s_q}{2s_q}} \le 2B_0.$$

Then for $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q d\mathrm{vol}_1 = 0$ for all $\xi \in \Lambda$, we have

$$\left(\int_{\mathbb{S}} |u|^q d\mathrm{vol}_1 \right)^{\frac{1}{q}} \le \left(\frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \frac{3\epsilon}{4} \right)^{\frac{1}{s_q}} \left(\int_{\mathbb{S}} |\nabla_b u|^2 d\mathrm{vol}_1 \right)^{\frac{1}{2}} + 2B_0 \left(\int_{\mathbb{S}} |u|^2 d\mathrm{vol}_1 \right)^{\frac{1}{2}}.$$

Finally we choose $\eta > 0$ such that

$$\left(\frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \frac{3\epsilon}{4} \right) (1+\eta) \le \frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \epsilon.$$

For this specific η, there exists $C_\eta > 0$ such that for any $q \in [q_0, 2^*]$ and non-negative x, y, we have

$$(x+y)^{s_q} \le (1+\eta) x^{s_q} + C_\eta y^{s_q}.$$

So for $q \in (q_0, 2^*]$, $u \in S_1^2(\mathbb{S}^{2n+1})$ satisfying $\int_{\mathbb{S}} \xi |u|^q d\mathrm{vol}_1 = 0$ for all $\xi \in \Lambda$, we have

$$\begin{aligned}\left(\int_{\mathbb{S}} |u|^q d\mathrm{vol}_1\right)^{\frac{sq}{q}} \le& \left(\frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \epsilon\right)\left(\int_{\mathbb{S}} |\nabla_b u|^2 d\mathrm{vol}_1\right)^{\frac{sq}{2}} \\ &+ (2B_0)^{s_q} C_\eta \left(\int_{\mathbb{S}} u^2 d\mathrm{vol}_1\right)^{\frac{sq}{2}} \\ \le& \left(\frac{\mathscr{A}(n,2)^2}{2^{\frac{1}{n+1}}} + \epsilon\right)\left(\int_{\mathbb{S}} |\nabla_b u|^2 d\mathrm{vol}_1\right)^{\frac{sq}{2}} \\ &+ (1+2B_0)^2 C_\eta \left(\int_{\mathbb{S}} u^2 d\mathrm{vol}_1\right)^{\frac{sq}{2}}.\end{aligned}$$

We are done. □

3.7 Application to Nirenberg Problem on CR Sphere

In this section, we apply the analytic results of AB inequality for functions with constraints to touch Nirenberg problem on CR sphere.

Let M be a compact CR manifold with the contact form θ on it. The CR conformal class of θ, denoted by $[\theta]$, is

$$[\theta] = \{\tilde{\theta} = u^{2/n}\theta, u \in C^\infty(M), u > 0\}.$$

As we mentioned before, the transformation formula for Webster scalar curvature function is

$$\tilde{R} = u^{-\frac{n+2}{n}}\left(\frac{2(n+1)}{n}\Delta_b u + Ru\right). \tag{3.28}$$

Now we set

$$\mathcal{S}([\theta]) = \{f \in C^\infty(M) \text{ s.t. } f \text{ is the Webster scalar curvature of some contact form CR conformal to } \theta\}.$$

The CR version of Nirenberg problem, also called the Kazdan-Warner problem, is to describe the set $\mathcal{S}([\theta])$ of Webster scalar curvature functions of CR conformal contact forms to θ.

Then we restrict our interest to the CR sphere $\mathbb{S}^{2n+1}$. We know there is a standard contact form

$$\theta_1 = i(\overline{\partial} - \partial)|\zeta|^2 = i\sum_{j=1}^{n+1}\left(\zeta_j d\overline{\zeta}_j - \overline{\zeta}_j d\zeta_j\right)$$

and the Webster scalar curvature is constant $\frac{1}{2}n(n+1)$. So up to some harmless constant, now the Nirenberg problem is equivalent to finding conditions on f for the existence of positive $u \in C^\infty(\mathbb{S}^{2n+1})$, satisfying

$$\Delta_b u + \frac{n^2}{4}u = f u^{\frac{n+2}{n}}. \tag{3.29}$$

As we multiply u to (3.29) and apply integration by parts, it is easy to see a simple necessary condition for f belongs to $\mathcal{S}([\theta_1])$ is that $\max_{\mathbb{S}^{2n+1}} f > 0$.

Theorem 3.7.1. *Although it is clear that the constant function* $1 \in \mathcal{S}([\theta_1])$*, for any non-trivial first spherical harmonic* $\xi \in \Lambda$*, no matter how close* ξ *is to* 0*,* $(1+\xi) \notin \mathcal{S}([\theta_1])$*.*

Proof. Suppose $\tilde{R}$ is a function satisfies (3.28) with u smooth positive and $R = \frac{1}{2}n(n+1)$ which is the Webster curvature function on $\mathbb{S}^{2n+1}$ under the standard contact form θ_1. Fix any first spherical harmonic $\xi \in \Lambda$. Note that the integral

$$\int_{\mathbb{S}} \tilde{R} u^{\frac{2n+2}{n}} d\mathrm{vol}_1$$

is unchanged if we replace the function u by u_t where

$$u_t^{\frac{2}{n}} d\theta_1 = d(T_t)^* u^{\frac{2}{n}} d\theta_1$$

with T_t to be the 1-parameter group of CR conformal transformations which generate the vector field $\nabla_b \xi$. Then

$$\frac{d}{dt}\int_{\mathbb{S}} \tilde{R}_t u_t^{\frac{2n+2}{n}} \mathrm{dvol}_1 = 0,$$

which leads to

$$\int_{\mathbb{S}} \langle \nabla_b \tilde{R}, \nabla_b \xi \rangle \mathrm{dvol}_1 = 0.$$

The above result holds for arbitrary $\xi \in \Lambda$. So we know $\tilde{R}$ can never take the form of $1+\xi$ where ξ is a non-trivial first spherical harmonic. $\square$

Definition 3.7.2. *Let Λ be the space of first spherical harmonics on $\mathbb{S}^{2n+1}$, and for $f \in C^\infty(\mathbb{S}^{2n+1})$ and $q \in (1, \frac{2n+2}{n}]$, we define*

$$A_{f,q} = \{u \in S_1^2(\mathbb{S}^{2n+1}), u \geq 0, \int_{\mathbb{S}} f u^q dvol_1 = 1,$$
$$\int_{\mathbb{S}} \xi u^q dvol_1 = 0 \text{ for any } \xi \in \Lambda\}.$$

Also define the functional I on $S_1^2(\mathbb{S}^{2n+1})$ by

$$I(u) = \int_{\mathbb{S}} |\nabla_b u|^2 dvol_1 + \frac{n^2}{4}\int_{\mathbb{S}} u^2 dvol_1.$$

Now with the two notations above, for such f and q, we define

$$\lambda_{f,q} = \inf_{u \in A_{f,q}} I(u).$$

Lemma 3.7.3. *Let $f \in C^\infty(\mathbb{S}^{2n+1})$ and $q \in (1, \frac{2n+2}{n})$. Assume that either $\int_{\mathbb{S}} f dvol_1 > 0$ or f is positive at any two antipodal points of $\mathbb{S}^{2n+1}$, then $\lambda_{f,q}$ is attained. In particular, there exists smooth positive function $u_q \in A_{f,q}$ such that $I(u_q) = \lambda_{f,q}$ and*

u_q satisfies the Euler-Lagrange equation of I

$$\Delta_b u_q + \frac{n^2}{4} u_q = \lambda_{f,q}(f - \xi_{f,q}) u_q^{q-1} \tag{3.30}$$

for some $\xi_{f,q} \in \Lambda$.

Proof. The two conditions in the statement guarantee that the set $A_{f,q}$ is not empty. Actually, if $\int_{\mathbb{S}} f d\text{vol}_1 > 0$ implies that there is some positive constant function in $A_{f,q}$. If f is positive at two antipodal points, then we can construct symmetric function $u \in A_{f,q}$, i.e., u satisfies $u(x) = u(-x)$ for any x on $\mathbb{S}^{2n+1}$.

Now let (u_j) be a minimizing sequence for $\lambda_{f,q}$. Since for $q \in (1, \frac{2n+2}{n})$, the embedding of $S_1^2(\mathbb{S}^{2n+1}) \subset L^q(\mathbb{S}^{2n+1})$ is compact, we can take a subsequence, which we also denote by u_j, such that

$$u_j \rightharpoonup u_q \text{ in } S_1^2(\mathbb{S}^{2n+1})$$

$$u_j \to u_q \text{ in } L^2(\mathbb{S}^{2n+1})$$

$$u_j \to u_q \text{ in } L^q(\mathbb{S}^{2n+1})$$

$$u_j \to u_q \text{ a.e.}$$

The convergence a.e. and strong convergence in $L^q(\mathbb{S}^{2n+1})$ will guarantee $u_q \in A_{f,q}$. Also since the weak convergence in $S_1^2(\mathbb{S}^{2n+1})$ and strong convergence in $L^2(\mathbb{S}^{2n+1})$, we have

$$\begin{aligned} I(u_q) &= \int_{\mathbb{S}} |\nabla_b u_q|^2 d\text{vol}_1 + \frac{n^2}{4} \int_{\mathbb{S}} u_q^2 d\text{vol}_1 = \|u_q\|_{S_1^2} + (\frac{n^2}{4} - 1)\|u_q\|_{L^2} \\ &\leq \liminf_{j\to\infty} \|u_j\|_{S_1^2} + (\frac{n^2}{4} - 1) \lim_{j\to\infty} \|u_j\|_{L^2} = \lim_{j\to\infty} I(u_j) = \lambda_{f,q}. \end{aligned}$$

Then by the definition of $\lambda_{f,q}$, we have u_q actually achieves $\lambda_{f,q}$. Then by variational method, u_q must satisfies the Euler-Lagrange equation. Maximum principles and regularity results then finish the proof. □

Lemma 3.7.4. *Let* $2^* = \frac{2n+2}{n}$, $f \in C^\infty(\mathbb{S}^{2n+1})$, *we have*

$$\limsup_{q\to 2^*} \lambda_{f,q} \le \lambda_{f,2^*}.$$

Proof. For any $\epsilon > 0$, take $u \in A_{f,2^*}$, such that $I(u) \le \lambda_{f,2^*} + \epsilon$. We set $v_q = u^{2^*/q}$ for q close to 2^*. It is then easy to check that $v_q \in A_{f,q}$. So by definition of $\lambda_{f,q}$, we know $\lambda_{f,q} \le I(v_q)$. But

$$\lim_{q\to 2^*} I(v_q) = I(u).$$

So we have for any $\epsilon > 0$, there holds

$$\limsup_{q\to 2^*} \lambda_{f,q} \le \lambda_{f,2^*} + \epsilon.$$

That completes the proof. □

Lemma 3.7.5. *Let* $f \in C^\infty(\mathbb{S}^{2n+1})$, *assume that that either* $\int_{\mathbb{S}} f dvol_1 > 0$ *or* f *is positive at two antipodal points on* $\mathbb{S}^{2n+1}$, *also let* $2^* = \frac{2n+2}{n}$, *then if*

$$\lambda_{f,2^*} < \frac{2^{\frac{1}{n+1}}}{\mathscr{A}(n,2)^2 \left(\max_{\mathbb{S}^{2n+1}} f\right)^{n/(n+1)}},$$

then there exists $\xi_f \in \Lambda$ *such that* $f - \xi_f \in \mathcal{S}([\theta_1])$.

Proof. For $q \in (1, \frac{2n+2}{n})$, let u_q be as in Lemma 3.7.3. Since $(\lambda_{f,q})$ is bounded, multiply u_q to (3.30) and apply integration by parts, it is not difficult to check (u_q) is bounded in $S_1^2(\mathbb{S}^{2n+1})$. So we can choose a subsequence such that

$$u_q \rightharpoonup u \text{ in } S_1^2(\mathbb{S}^{2n+1})$$

$$u_q \to u \text{ in } L^2(\mathbb{S}^{2n+1})$$

$$u_q \to u \text{ a.e.}$$

$$u_q^{q-1} \rightharpoonup u^{\frac{n+2}{n}} \text{ in } L^{\frac{2n+2}{n+2}}(\mathbb{S}^{2n+1})$$

Note that the last line is because (u_q) is L^{2^*}-bounded and hence (u_q^{q-1}) is $L^{\frac{2^*}{2^*-1}}$-bounded, then there exists a weakly convergent subsequence. Combined with the fact we construct in such a way that u_q convergences to u a.e. will guarantee the weak limit is just $u^{\frac{n+2}{n}}$.

Then since $(\lambda_{f,q})$ is bounded, we can choose a subsequence u_q such that

$$\lim_{q\to 2^*} \lambda_{f,q} = \lambda.$$

By Theorem 3.6.3, for q close to 2^*, there exists constant B_ϵ which is independent of q, such that

$$\begin{aligned}
1 = \left(\int_{\mathbb{S}} f u_q^q d\mathrm{vol}_1\right)^{\frac{2n+2}{2n+2+q}} &\leq (\max_{\mathbb{S}^{2n+1}} f)^{\frac{2n+2}{2n+2+q}} \left(\int_{\mathbb{S}} u_q^q d\mathrm{vol}_1\right)^{\frac{2n+2}{2n+2+q}} \\
\leq (\max_{\mathbb{S}^{2n+1}} f)^{\frac{2n+2}{2n+2+q}} &\left(\frac{\mathscr{A}(n,2)^2}{2^{1/(n+1)}} + \epsilon\right) \lambda_{f,q}^{\frac{(n+1)q}{2n+2+q}} \\
+ (\max_{\mathbb{S}^{2n+1}} f)^{\frac{2n+2}{2n+2+q}} & B_\epsilon \left(\int_{\mathbb{S}} u_q^2 d\mathrm{vol}_1\right)^{\frac{q(n+1)}{2n+2+q}}.
\end{aligned} \tag{3.31}$$

By Lemma 3.7.4, we have

$$\begin{aligned}
\lim_{q\to 2^*} (\max_{\mathbb{S}^{2n+1}} f)^{\frac{2n+2}{2n+2+q}} &\left(\frac{\mathscr{A}(n,2)^2}{2^{1/(n+1)}} + \epsilon\right) \lambda_{f,q}^{\frac{(n+1)q}{2n+2+q}} \\
&\leq (\max_{\mathbb{S}^{2n+1}} f)^{\frac{n}{n+1}} \left(\frac{\mathscr{A}(n,2)^2}{2^{1/(n+1)}} + \epsilon\right) \lambda_{f,2^*}.
\end{aligned}$$

Under the assumption of the Lemma, we take $\epsilon > 0$ such that

$$(\max_{\mathbb{S}^{2n+1}} f)^{\frac{n}{n+1}} \left(\frac{\mathscr{A}(n,2)^2}{2^{1/(n+1)}} + \epsilon\right) \lambda_{f,2^*} < 1.$$

Combined with (3.31), there exists $C > 0$ such that for q close to 2^*,

$$\int_{\mathbb{S}} u_q^2 \, d\mathrm{vol}_1 \geq C. \tag{3.32}$$

In particular, $\lambda > 0$. Since if $\lambda = 0$, then weak convergence in $S_1^2(\mathbb{S}^{2n+1})$ will imply $u_q \to 0$ in $S_1^2(\mathbb{S}^{2n+1})$, which contradicts with $u_q \to u$ in $L^2(\mathbb{S}^{2n+1})$ and (3.32).
Then we write $\xi_{f,q} = \mu_q \xi_q$ with $\|\xi_q\|_{L^\infty} = 1$. Since Λ is finite dimensional, we may assume $\xi_q \to \xi$ in $L^\infty(\mathbb{S}^{2n+1})$ for some $\xi \in \Lambda$ with $\|\xi\|_{L^\infty} = 1$.
Again multiply $\xi_q u_q$ to (3.30) and apply integration, we obtain

$$\int_{\mathbb{S}} \xi_q u_q \left(\Delta_b u_q\right) d\mathrm{vol}_1 + \frac{n^2}{4}\int_{\mathbb{S}} \xi_q u_q^2 d\mathrm{vol}_1 = \lambda_{f,q}\int_{\mathbb{S}} f\xi_q u_q^q d\mathrm{vol}_1 - \lambda_{f,q}\mu_q \int_{\mathbb{S}} \xi_q^2 u_q^q d\mathrm{vol}_1.$$

Apply integration by parts and note that $\Delta_b \xi = \frac{n}{2}\xi$ for any $\xi \in \Lambda$, there holds

$$\begin{aligned}\int_{\mathbb{S}} \xi_q u_q \left(\Delta_b u_q\right) d\mathrm{vol}_1 &= \int_{\mathbb{S}} \left\langle \nabla_b(\xi_q u_q), \nabla_b u_q \right\rangle d\mathrm{vol}_1 \\ &= \int_{\mathbb{S}} \xi_q |\nabla_b u_q|^2 d\mathrm{vol}_1 + \frac{1}{2}\int_{\mathbb{S}} \left\langle \nabla_b \xi_q, \nabla_b \left(u_q^2\right) \right\rangle d\mathrm{vol}_1 \\ &= \int_{\mathbb{S}} \xi_q |\nabla_b u_q|^2 d\mathrm{vol}_1 + \frac{1}{2}\int_{\mathbb{S}} u_q^2 \left(\Delta_b \xi_q\right) d\mathrm{vol}_1 \\ &= \int_{\mathbb{S}} \xi_q |\nabla_b u_q|^2 d\mathrm{vol}_1 + \frac{n}{4}\int_{\mathbb{S}} \xi_q u_q^2 \; d\mathrm{vol}_1.\end{aligned}$$

Substitute we get

$$\begin{aligned}\mu_q \lambda_{f,q}\int_{\mathbb{S}} \xi_q^2 u_q^q d\mathrm{vol}_1 =& \lambda_{f,q}\int_{\mathbb{S}} f\xi_q u_q^q d\mathrm{vol}_1 - \int_{\mathbb{S}} \xi_q |\nabla_b u_q|^2 d\mathrm{vol}_1 \\ &- \frac{n(n+1)}{4}\int_{\mathbb{S}} \xi_q u_q^2 \; d\mathrm{vol}_1.\end{aligned}$$

Since $f \in C^\infty(\mathbb{S}^{2n+1})$, $\|\xi_q\|_{L^\infty} = 1$, (u_q) is bounded in $S_1^2(\mathbb{S}^{2n+1})$, it is not difficult to check every term on the right hand side is bounded. Then apply Hölder's inequality we will get

$$\mu_q \lambda_{f,q}\left(\omega_{2n+1}^{-\frac{q-2}{q}} \int_{\mathbb{S}} |\xi_q|^{4/q} u_q^2 d\mathrm{vol}_1\right)^{q/2} \leq \mu_q \lambda_{f,q}\int_{\mathbb{S}} \xi_q^2 u_q^q d\mathrm{vol}_1.$$

So there exists positive C' such that

$$\mu_q \lambda_{f,q} \left(\omega_{2n+1}^{-\frac{q-2}{q}} \int_{\mathbb{S}} |\xi_q|^{4/q} u_q^2 d\mathrm{vol}_1 \right)^{q/2} \leq C'.$$

We claim that will imply

$$\limsup_{q \to 2^*} \mu_q < +\infty,$$

for otherwise we will have

$$\int_{\mathbb{S}} |\xi|^{\frac{2n}{n+1}} u^2 \ d\mathrm{vol}_1 = 0.$$

This is impossible for non-trivial $\xi \in \Lambda$ attains 0 only on a set of measure 0. So we may now take a subsequence such that $\mu_q \to \mu$ for some $\mu < +\infty$ as $q \to 2^*$. Then we have u is a weak solution of

$$\Delta_b u + \frac{n^2}{4} u = \lambda(f - \mu\xi) u^{\frac{n+2}{n}}.$$

Actually for any $\phi \in C^\infty(\mathbb{S}^{2n+1})$, it is not difficult to see from those convergence and weak convergence properties of u_q. Finally the result follows by maximum principle, regularity results and transformation rule of Webster scalar curvature under CR conformal transformation. □

Theorem 3.7.6. *For $f \in C^\infty(\mathbb{S}^{2n+1})$ and positive on $\mathbb{S}^{2n+1}$, if*

$$max_{\ \mathbb{S}^{2n+1}} f < 2^{\frac{1}{n}} min_{\ \mathbb{S}^{2n+1}} f,$$

then there exists $\xi \in \Lambda$ such that $f - \xi \in \mathcal{S}([\theta_1])$.

Proof. Let the constant function u_0 be $u_0 \equiv \left(\int_{\mathbb{S}} f d\mathrm{vol}_1\right)^{-\frac{n}{2n+2}}$. It is not difficult to check $u_0 \in A_{f,2^*}$. Then by definition of $\lambda_{f,2^*}$, we have

$$
\begin{aligned}
\lambda_{f,2^*} \leq & I(u_0) = \frac{n^2}{4}\omega_{2n+1}\left(\int_{\mathbb{S}} f d\mathrm{vol}_1\right)^{-\frac{n}{n+1}} \leq \frac{n^2}{4}\omega_{2n+1}^{\frac{1}{n+1}}\left(\min_{\mathbb{S}^{2n+1}} f\right)^{-\frac{n}{n+1}} \\
< & 2^{\frac{1}{n+1}}\pi n^2 \left(\max_{\mathbb{S}^{2n+1}} f\right)^{-\frac{n}{n+1}} = \frac{2^{\frac{1}{n+1}}}{\mathscr{A}(n,2)^2}\left(\max_{\mathbb{S}^{2n+1}} f\right)^{-\frac{n}{n+1}}.
\end{aligned}
$$

The result follows immediately by Lemma 3.7.5. $\square$

Theorem 3.7.7. *Let $f \in C^\infty(\mathbb{S}^{2n+1})$ which satisfies max $_{\mathbb{S}^{2n+1}} f > 0$, then there exists a first spherical harmonic ξ and a CR conformal transformation φ of $\mathbb{S}^{2n+1}$ such that $f - (\xi \circ \varphi) \in \mathcal{S}([\theta_1])$.*

Proof. With the help of Cayley transformation and left group action on Heisenberg group, we may assume f attains its maximum at the south pole, i.e. the point $P_0 \in \mathbb{S}^{2n+1}$ with $\zeta_0 = \cdots = \zeta_n = 0$ and $\zeta_{n+1} = -1$. Then for $t \in (-\infty, +\infty)$, define a one-parameter subgroup $\{\varphi_t\}$ of the CR conformal transformations on $\mathbb{S}^{2n+1}$:

$$\varphi_t = \mathcal{C} \circ D_{e^t} \circ \mathcal{C}^{-1},$$

where D_δ is the dilation on $\mathbb{H}^n$ and $\mathcal{C} : \mathbb{H}^n \to \mathbb{S}^{2n+1}$ is the Cayley transformation. Since the Cayley transformation $\mathcal{C}$ maps $0 \in \mathbb{H}^n$ to the north pole, i.e. $-P_0 \in \mathbb{S}^{2n+1}$. For any $P \in \mathbb{S}^{2n+1}$ satisfying $P \neq \pm P_0$,

$$\lim_{t\to+\infty} \varphi_t(P) = P_0 \quad \text{and} \quad \lim_{t\to-\infty} \varphi_t(P) = -P_0.$$

Set $f_t = f \circ \varphi_t$. Then we have as $t \to +\infty$, $f_t \to f(P_0)$ uniformly on any compact subset of $\mathbb{S}^{2n+1}\backslash\{-P_0\}$ and in $L^s(\mathbb{S}^{2n+1})$ for any $s \in [1, +\infty)$. So we have

$$\lim_{t\to+\infty} \frac{1}{\omega_{2n+1}} \int_{\mathbb{S}} f_t \, d\mathrm{vol}_1 = \max_{\mathbb{S}^{2n+1}} f.$$

In particular, we know for large enough t, there holds

$$\int_{\mathbb{S}} f_t d\mathrm{vol}_1 > 0 \quad \text{ and } \quad \frac{\omega_{2n+1}}{2^{1/n}} \max_{\mathbb{S}^{2n+1}} f_t < \int_{\mathbb{S}} f_t d\mathrm{vol}_1.$$

Clearly $\max_{\mathbb{S}^{2n+1}} f_t = \max_{\mathbb{S}^{2n+1}} f$ for any t. Also the constant function u_0 defined by $u_0 \equiv \left(\int_{\mathbb{S}} f_t d\mathrm{vol}_1\right)^{-\frac{n}{2n+2}}$ belongs to $A_{f_t,2^*}$. By definition of $\lambda_{f_t,2^*}$, we have

$$\lambda_{f_t,2^*} \le I(u_0) = \frac{n^2}{4}\omega_{2n+1}\left(\int_{\mathbb{S}} f_t d\mathrm{vol}_1\right)^{-\frac{n}{n+1}} < \frac{2^{\frac{1}{n+1}}}{\mathscr{A}(n,2)^2}\left(\max_{\mathbb{S}^{2n+1}} f_t\right)^{-\frac{n}{n+1}}.$$

By Lemma 3.7.5, there exists $\xi \in \Lambda$ such that $f_t - \xi \in \mathcal{S}([\theta_1])$. Hence by CR conformal invariance, $f - (\xi \circ \varphi_{-t}) \in \mathcal{S}([\theta_1])$. $\square$

Chapter 4

Nonlinear Sub-elliptic PDE on Heisenberg Group

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. We consider the problem of existence of a function u satisfying:

$$\begin{cases} \Delta_b u = u^{2^*-1} + \lambda u & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $2^* = (2n+2)/n$ and λ is a real number. We also denote by λ_1 the first eigenvalue of Δ_b with zero Dirichlet condition on Ω. There holds

$$\lambda_1 = \inf_{u \in S_{1,0}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla_b u|^2 d\mathrm{vol}_0}{\int_\Omega u^2 d\mathrm{vol}_0}.$$

4.1 Non-existence Results: Part 1

In this section, we prove some non-existence results to (4.1).

Theorem 4.1.1. *Assume $\lambda \geq \lambda_1$, there exists no solution on Ω such that* (4.1) *holds.*

Proof. Suppose there exists such solution u of (4.1) for some real number λ. Let ϕ_1 be the eigenfunction of Δ_b corresponding to λ_1 with $\phi_1 > 0$ on Ω. Then there holds

$$\begin{aligned}\lambda_1 \int_\Omega u\phi_1 \, d\text{vol}_0 = \int_\Omega (\Delta_b \phi_1) u \, d\text{vol}_0 &= \int_\Omega (\Delta_b u)\phi_1 \, d\text{vol}_0 \\ &= \int_\Omega u^{2^*-1}\phi_1 \, d\text{vol}_0 + \lambda \int_\Omega u\phi_1 \, d\text{vol}_0.\end{aligned}$$

Since both u and ϕ_1 are positive, we have $\lambda < \lambda_1$ and hence obtain the result. □

Theorem 4.1.2. *Assume $\lambda \leq 0$ and Ω is a smooth δ-starshaped domain, there exists no solution on Ω such that* (4.1) *holds.*

Proof. Suppose there exists such solution u of (4.1) on the δ-starshaped domain Ω for some real number $\lambda \leq 0$ and we will get contradiction. By the CR Pohozaev identity in [30] (Theorem 2.0.24) we have

$$\int_\Omega (Xu)\, \Delta_b u \, d\text{vol}_0 + n \int_\Omega |\nabla_b u|^2 d\text{vol}_0 = -\frac{1}{2}\int_{\partial\Omega} |\nabla_b u|^2 \langle X, N\rangle d\sigma,$$

where N is the normal vector of $\partial\Omega$ and X is the vector field defined by $X = \sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right) + 2t\frac{\partial}{\partial t}$. Since Ω is a δ-starshaped domain, we have $\langle X, N\rangle \geq 0$ on $\partial\Omega$ and hence if we substitute (4.1), we obtain

$$\begin{aligned}&\int_\Omega (Xu)\, u^{2^*-1} d\text{vol}_0 + \lambda \int_\Omega (Xu)\, u d\text{vol}_0 + n \int_\Omega u\Delta_b u d\text{vol}_0 \\ =&\int_\Omega (Xu)\, u^{2^*-1} d\text{vol}_0 + \lambda \int_\Omega (Xu)\, u d\text{vol}_0 + n \int_\Omega u^{2^*} d\text{vol}_0 + n\lambda \int_\Omega u^2 d\text{vol}_0 \\ \leq& 0.\end{aligned}$$

Since $u = 0$ on $\partial\Omega$, applying integration by parts, we have

$$\int_\Omega \theta \frac{\partial u}{\partial \theta} u \, d\text{vol}_0 = -\frac{1}{2}\int_\Omega u^2 d\text{vol}_0$$

and

$$\int_\Omega \theta \frac{\partial u}{\partial \theta} u^{2^*-1} d\mathrm{vol}_0 = -\frac{1}{2^*}\int_\Omega u^{2^*} d\mathrm{vol}_0,$$

where θ can be any variable x_j, y_j or t. Take summation we get

$$\int_\Omega (Xu)\, u \; d\mathrm{vol}_0 = -(n+1)\int_\Omega u^2 d\mathrm{vol}_0$$

and

$$\int_\Omega (Xu)\, u^{2^*-1} d\mathrm{vol}_0 = -n\int_\Omega u^{2^*} d\mathrm{vol}_0.$$

Substitute we now obtain

$$\lambda \int_\Omega u^2 d\mathrm{vol}_0 \geq 0,$$

which contradicts with $\lambda \leq 0$. Hence we prove the result. □

4.2 Existence Results

In this section, we prove some existence results to (4.1).

Definition 4.2.1. *For $\lambda \in \mathbb{R}$, We define*

$$S_\lambda = \inf_{u \in S_{1,0}(\Omega),\ \|u\|_{L^{2^*}}=1} \left(\|\nabla_b u\|_{L^2}^2 - \lambda \|u\|_{L^2}^2 \right). \tag{4.2}$$

It is not difficult to check that the definition is equivalent to

$$S_\lambda = \inf_{u \in S_{1,0}(\Omega),\ u \neq 0} \left(\frac{\|\nabla_b u\|_{L^2}^2 - \lambda \|u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2} \right).$$

In particular, we denote

$$S = S_0 = \inf_{u \in S_{1,0}(\Omega),\ \|u\|_{L^{2^*}}=1} \left(\|\nabla_b u\|_{L^2}^2 \right), \tag{4.3}$$

which is just $\frac{1}{\mathscr{A}(n,2)^2}$*, related to the optimal constant for Sobolev-type embedding* $S_{1,0}(\Omega) \subset L^{2^*}(\Omega)$*.*

Remark 4.2.2. *Since the infimum of the quotient is invariant under dilation on* $\mathbb{H}^n$*,* S *is independent of* Ω *and depends only on* n*. By results of [29] and [40], when* $\Omega = \mathbb{H}^n$*, the infimum of* (4.3)*, up to left group translation, is achieved by the function*

$$u = C\left(t^2 + \left(|\mathbf{z}|^2 + 1\right)^2\right)^{-n/2},$$

or scaling of the function

$$u_\epsilon = C_\epsilon\left(t^2 + \left(|\mathbf{z}|^2 + \epsilon\right)^2\right)^{-n/2},$$

where C *and* C_ϵ *are just normalization constants. Note that here by* $\mathbf{z}$*, we actually mean complex variables in the vector form, i.e.* $(z_1, z_2, \ldots z_n)$*, where* $z_j = x_j + iy_j$*. So by* $|\mathbf{z}|^2$*, we actually mean* $\sum_{j=1}^n |z_j|^2$*, or* $\sum_{j=1}^n (x_j^2 + y_j^2)$*. For convenience, we also denote*

$$U = \left(t^2 + \left(|\mathbf{z}|^2 + 1\right)^2\right)^{-n/2}$$

and

$$U_\epsilon = \left(t^2 + \left(|\mathbf{z}|^2 + \epsilon\right)^2\right)^{-n/2}.$$

Actually we have

$$U_\epsilon(\mathbf{z}, t) = \epsilon^{-n} U\left(D_{1/\sqrt{\epsilon}}(\mathbf{z}, t)\right)$$

where $D_{1/\sqrt{\epsilon}}$ *is the dilation on* $\mathbb{H}^n$*.*

Definition 4.2.3. *Define*

$$f_\epsilon = \phi U_\epsilon = \frac{\phi}{\left(t^2 + \left(|\mathbf{z}|^2 + \epsilon\right)^2\right)^{n/2}},$$

where ϕ is a fixed function on $\mathbb{H}^n$ with support in Ω such that $0 \leq \phi(\mathbf{z}, t) \leq 1$ pointwisely and $\phi \equiv 1$ in some small neighborhood $B_r(0)$ of 0 defined by the Heisenberg norm.

Lemma 4.2.4. *There holds*

$$\|\nabla_b f_\epsilon\|_{L^2}^2 = \epsilon^{-n} K_1 + O(1) \tag{4.4}$$

where $K_1 = \|\nabla_b U\|_{L^2}^2$ is a positive constant.

Proof. We start with some basic computations of the functions U and U_ϵ.

$$U = \left(t^2 + \left(|\mathbf{z}|^2 + 1\right)^2\right)^{-n/2} = \left(t^2 + \left(1 + \sum_{j=1}^{n} \left(x_j^2 + y_j^2\right)\right)^2\right)^{-n/2}.$$

Then by direct computation

$$\begin{aligned}
\frac{\partial}{\partial x_j} U &= -2n\alpha^{-n/2-1}\beta x_j \\
\frac{\partial}{\partial y_j} U &= -2n\alpha^{-n/2-1}\beta y_j \\
\frac{\partial}{\partial t} U &= -n\alpha^{-n/2-1} t,
\end{aligned}$$

where $\beta = 1 + \sum_{j=1}^n (x_j^2 + y_j^2)$ and $\alpha = t^2 + \beta^2$. Because $X_j = \frac{1}{2}\frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}$ and $Y_j = \frac{1}{2}\frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial t}$. Now

$$\begin{aligned}
|\nabla_b U|^2 &= \sum_{j=1}^{n} \left((X_j U)^2 + (Y_j U)^2\right) \\
&= \sum_{j=1}^{n} \left(-n\alpha^{-n/2-1}\beta x_j - n y_j \alpha^{-n/2-1} t\right)^2 + \left(-n\alpha^{-n/2-1}\beta y_j + n x_j \alpha^{-n/2-1} t\right)^2 \\
&= n^2 \alpha^{-n-1}(\beta - 1) = \frac{n^2 |\mathbf{z}|^2}{\left(t^2 + (1 + |\mathbf{z}|^2)^2\right)^{n+1}} \\
&\leq \frac{n^2}{\left(t^2 + (1 + |\mathbf{z}|^2)^2\right)^{n+1/2}} \leq \frac{n^2}{\rho^{4n+2}(\mathbf{z}, t)},
\end{aligned}$$

where ρ is the Heisenberg norm. Since $U_\epsilon(\mathbf{z},t) = \epsilon^{-n}U\left(D_{1/\sqrt{\epsilon}}(\mathbf{z},t)\right)$, easy computation leads to

$$\nabla_b U_\epsilon(\mathbf{z},t) = \epsilon^{-n-1/2}\left(\nabla_b U\right)\left(D_{1/\sqrt{\epsilon}}(\mathbf{z},t)\right).$$

Substitute, we obtain

$$|\nabla_b U_\epsilon|^2 = \frac{n^2|\mathbf{z}|^2}{\left(t^2 + (\epsilon + |\mathbf{z}|^2)^2\right)^{n+1}} \leq \frac{n^2}{\left(t^2 + (\epsilon + |\mathbf{z}|^2)^2\right)^{n+1/2}} \leq \frac{n^2}{\rho^{4n+2}(\mathbf{z},t)}.$$

Then we come back to the problem itself. We have

$$\nabla_b f_\epsilon = (\nabla_b \phi)U_\epsilon + \phi(\nabla_b U_\epsilon).$$

Hence

$$|\nabla_b f_\epsilon|^2 = |\nabla_b \phi|^2 U_\epsilon^2 + \phi^2 |\nabla_b U_\epsilon|^2 + 2\phi U_\epsilon \langle \nabla_b \phi, \nabla_b U_\epsilon \rangle$$

Take integral there holds

$$\|\nabla_b f_\epsilon\|_{L^2}^2 = \int_\Omega |\nabla_b \phi|^2 U_\epsilon^2 d\mathrm{vol}_0 + \int_\Omega \phi^2 |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 + 2\int_\Omega \phi U_\epsilon \langle \nabla_b \phi, \nabla_b U_\epsilon \rangle d\mathrm{vol}_0. \quad (4.5)$$

Then we estimate each term in (4.5). For the first term, there exist positive numbers $0 < r < R$ such that

$$\begin{aligned}\int_\Omega |\nabla_b \phi|^2 U_\epsilon^2 d\mathrm{vol}_0 &\leq C\int_{r\leq\rho\leq R} U_\epsilon^2 d\mathrm{vol}_0 \leq C\int_{r\leq\rho\leq R} \rho^{-4n}(\mathbf{z},t) d\mathrm{vol}_0 \\ &= C\omega_0 \int_r^R \rho^{-2n+1} d\rho = O(1),\end{aligned}$$

where ω_0 is the area of the Heisenberg sphere.

For the third term, by previous estimate of $|\nabla_b U_\epsilon|$, for the same r and R there holds

$$\begin{aligned}\left|\int_\Omega \phi U_\epsilon \langle \nabla_b \phi, \nabla_b U_\epsilon\rangle d\mathrm{vol}_0\right| &\le \int_{r\le\rho\le R} \phi U_\epsilon |\nabla_b \phi||\nabla_b U_\epsilon| d\mathrm{vol}_0 \\ &\le C\int_{r\le\rho\le R} U_\epsilon |\nabla_b U_\epsilon| d\mathrm{vol}_0 \\ &\le C\int_{r\le\rho\le R} \frac{n}{\rho^{4n+1}(\mathbf{z},t)} d\mathrm{vol}_0 \\ &\le Cn\omega_0 \int_r^R \rho^{-2n} d\rho = O(1).\end{aligned}$$

For the last step, we consider the second term, actually

$$\begin{aligned}\int_\Omega \phi^2 |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 &= \int_{\mathbb{H}} \phi^2 |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 \\ &= \int_{\rho\le r} |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 + \int_{\rho>r} \phi^2 |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 \\ &= \int_{\mathbb{H}} |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 + \int_{\rho>r} \left(\phi^2-1\right) |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0.\end{aligned}$$

By the property of dilation in $\mathbb{H}^n$ and change of variable, we have

$$\int_{\mathbb{H}} |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 = \epsilon^{-n} \int_{\mathbb{H}} |\nabla_b U|^2 d\mathrm{vol}_0 = \epsilon^{-n} \|\nabla_b U\|_{L^2}^2.$$

While there also holds

$$\begin{aligned}\left|\int_{\rho>r} \left(\phi^2-1\right) |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0\right| \le \int_{\rho>r} |\nabla_b U_\epsilon|^2 d\mathrm{vol}_0 &\le \int_{\rho>r} \frac{n^2}{\rho^{4n+2}(\mathbf{z},t)} d\mathrm{vol}_0 \\ &= n^2 \int_r^\infty \rho^{-(2n+1)} d\rho = O(1).\end{aligned}$$

Combining together, we obtain the result. □

Lemma 4.2.5. *There holds*

$$\|f_\epsilon\|_{L^{2^*}}^2 = \epsilon^{-n} K_2 + O(\epsilon), \tag{4.6}$$

where $K_2 = \|U\|_{L^{2^*}}^2$ *is a positive constant.*

Proof. Since the support of ϕ is subset of Ω, we have

$$\begin{aligned} \|f_\epsilon\|_{L^{2^*}}^{2^*} &= \int_\Omega \frac{\phi^{2^*}}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^{n+1}} d\mathrm{vol}_0 = \int_{\mathbb{H}} \frac{\phi^{2^*}}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^{n+1}} d\mathrm{vol}_0 \\ &= \int_{\mathbb{H}} \frac{1}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^{n+1}} d\mathrm{vol}_0 - \int_{\mathbb{H}} \frac{1-\phi^{2^*}}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^{n+1}} d\mathrm{vol}_0. \end{aligned}$$

By the property of dilation on $\mathbb{H}^n$ and change of variable, the first term is just

$$\|U_\epsilon\|_{L^{2^*}}^{2^*} = \|\epsilon^{-n} U \circ D_{1/\sqrt{\epsilon}}\|_{L^{2^*}}^{2^*} = \epsilon^{-(n+1)} \|U\|_{L^{2^*}}^{2^*}.$$

Then we estimate the absolute value of second term, actually it can be controlled by

$$\int_{\rho>r} U_\epsilon^{2^*} d\mathrm{vol}_0 \leq \int_{\rho>r} \rho^{-4(n+1)}(\mathbf{z},t) d\mathrm{vol}_0 = \omega_0 \int_r^\infty \rho^{-(2n+3)} d\rho = O(1).$$

Combining the two terms we obtain $\|f_\epsilon\|_{L^{2^*}}^{2^*} = \epsilon^{-(n+1)}\|U\|_{L^{2^*}}^{2^*} + O(1)$. Take power of $2/2^*$, it is not difficult to see the result. □

Lemma 4.2.6. *There holds*

$$\|f_\epsilon\|_{L^2}^2 = \epsilon^{-(n-1)} K_3 + O(1) \tag{4.7}$$

where $K_3 = \|U\|_{L^2}^2$ *if the dimension* $n \geq 2$ *is a positive constant. Note here* $\frac{K_1}{K_2} = S$. *While for the case* $n = 1$, *there exist positive constants* $0 < C_1 < C_2$ *such that*

$$C_1 |log(\epsilon)| \leq \|f_\epsilon\|_{L^2}^2 \leq C_2 |log(\epsilon)| \tag{4.8}$$

if ϵ *is small enough.*

Proof. First we discuss the case $n \geq 2$. Since the support of ϕ is subset of Ω, we have

$$\begin{aligned}\|f_\epsilon\|_{L^2}^2 &= \int_\Omega \frac{\phi^2}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^n} d\text{vol}_0 = \int_{\mathbb{H}} \frac{\phi^2}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^n} d\text{vol}_0 \\ &= \int_{\mathbb{H}} \frac{1}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^n} d\text{vol}_0 - \int_{\mathbb{H}} \frac{1-\phi^2}{\left(t^2+(\epsilon+|\mathbf{z}|^2)^2\right)^n} d\text{vol}_0.\end{aligned}$$

By the property of dilation on $\mathbb{H}^n$ and change of variable, the first term is just

$$\|U_\epsilon\|_{L^2}^2 = \|\epsilon^{-n} U \circ D_{1/\sqrt{\epsilon}}\|_{L^2}^2 = \epsilon^{-(n-1)} \|U\|_{L^2}^2.$$

Then we estimate the absolute value of second term, actually it can be controlled by

$$\int_{\rho > r} U_\epsilon^2 d\text{vol}_0 \leq \int_{\rho > r} \rho^{-4n}(\mathbf{z}, t) d\text{vol}_0 = \omega_0 \int_r^\infty \rho^{-(2n-1)} d\rho = O(1).$$

Combining together, we obtain the result for $n \geq 2$.

Then we discuss the case $n = 1$. There exist positive numbers $0 < r < R$ such that

$$\int_{\rho \leq r} \frac{1}{t^2 + (|\mathbf{z}|^2+\epsilon)^2} d\text{vol}_0 \leq \int_\Omega \frac{\phi^2}{t^2 + (|\mathbf{z}|^2+\epsilon)^2} d\text{vol}_0 \leq \int_{\rho \leq R} \frac{1}{t^2 + (|\mathbf{z}|^2+\epsilon)^2} d\text{vol}_0.$$

Then we consider

$$\frac{1}{2} \int_{\rho \leq R} \frac{1}{\epsilon^2 + t^2 + |\mathbf{z}|^4} d\text{vol}_0 \leq \int_{\rho \leq R} \frac{1}{t^2 + (|\mathbf{z}|^2+\epsilon)^2} d\text{vol}_0 \leq \int_{\rho \leq R} \frac{1}{\epsilon^2 + t^2 + |\mathbf{z}|^4} d\text{vol}_0.$$

While for $\epsilon < 1$, there holds

$$\begin{aligned}\int_{\rho \leq R} \frac{1}{\epsilon^2 + t^2 + |\mathbf{z}|^4} d\text{vol}_0 = \omega_0 \int_0^R \frac{\rho^3}{\epsilon^2 + \rho^4} d\rho = \frac{1}{4} \omega_0 log(\epsilon^2 + \rho^4)|_{\rho=0}^{\rho=R} \\ = O(1) + \frac{1}{2}\omega_0 |log(\epsilon)|.\end{aligned}$$

Same result holds for the $\rho \leq r$ part. Hence we can take suitable C_1 and C_2 such that when ϵ is small enough the result holds. $\square$

Lemma 4.2.7. *We have*

$$S_\lambda < S \quad \textit{for all} \quad \lambda > 0.$$

Proof. By definition of S_λ,

$$S_\lambda \leq \frac{\|\nabla_b f_\epsilon\|^2_{L^2} - \lambda \|f_\epsilon\|^2_{L^2}}{\|f_\epsilon\|^2_{L^{2^*}}}$$

for any $\epsilon > 0$. Then we discuss two cases. If $n \geq 2$, by Lemma 4.2.4, 4.2.5, 4.2.6, we have

$$\begin{aligned}
\frac{\|\nabla_b f_\epsilon\|^2_{L^2} - \lambda \|f_\epsilon\|^2_{L^2}}{\|f_\epsilon\|^2_{L^{2^*}}} &= \frac{\epsilon^{-n} K_1 + O(1) - \lambda \left(\epsilon^{-(n-1)} K_3 + O\left(1\right)\right)}{\epsilon^{-n} K_2 + O(\epsilon)} \\
&= \frac{K_1 - \lambda K_3 \epsilon + O(\epsilon^n)}{K_2 + O(\epsilon^{n+1})} \\
&= \left(K_1 - \lambda K_3 \epsilon + O\left(\epsilon^n\right)\right) \left(\frac{1}{K_2} + O\left(\epsilon^{n+1}\right)\right) \\
&= \frac{K_1}{K_2} - \lambda \frac{K_3}{K_2} \epsilon + O(\epsilon^n) < S
\end{aligned}$$

if we take ϵ small enough. So we have proven the Lemma in this case.

Then we discuss the case $n = 1$. Also by Lemma 4.2.4, 4.2.5, 4.2.6, we have

$$\begin{aligned}
\frac{\|\nabla_b f_\epsilon\|^2_{L^2} - \lambda \|f_\epsilon\|^2_{L^2}}{\|f_\epsilon\|^2_{L^{2^*}}} &= \frac{\epsilon^{-1} K_1 + O(1) - \lambda \|f_\epsilon\|^2_{L^2}}{\epsilon^{-1} K_2 + O(\epsilon)} = \frac{K_1 + O(\epsilon) - \lambda \epsilon \|f_\epsilon\|^2_{L^2}}{K_2 + O(\epsilon^2)} \\
&\leq \left(K_1 + O\left(\epsilon\right) - C_1 \lambda \epsilon |log(\epsilon)|\right) \left(\frac{1}{K_2} + O(\epsilon^2)\right) \\
&= \frac{K_1}{K_2} - \frac{C_1 \lambda}{K_2} \epsilon |log(\epsilon)| + O(\epsilon) < S
\end{aligned}$$

if we take ϵ small enough. So we have also shown the Lemma in the second case. □

Lemma 4.2.8. *If $S_\lambda < S$, then the infimum in* (4.2) *is achieved.*

Proof. Let $\{u_j\}_{j=1}^{\infty}$ be a minimizing sequence for (4.2), i.e,

$$\begin{aligned} &\|u_j\|_{L^{2^*}} = 1, \\ &\|\nabla_b u_j\|_{L^2}^2 - \lambda \|u_j\|_{L^2}^2 = S_\lambda + o(1) \quad \text{as} \quad j \to \infty. \end{aligned} \tag{4.9}$$

(u_j) is bounded in $S_{1,0}(\Omega)$, there exists a weakly convergent subsequence. We also know that for $2 < s < 2^*$ the embedding of $S_{1,0}(\Omega)$ into $L^s(\Omega)$ is compact and convergence in $L^s(\Omega)$ will imply convergence in $L^2(\Omega)$. Combined with the fact that convergence in $L^2(\Omega)$ implies existence of subsequence which converges to the limit function a.e, we get a subsequence–still denoted by u_j, such that

$$\begin{aligned} &u_j \rightharpoonup u \quad \text{weakly in} \quad S_{1,0}(\Omega), \\ &u_j \to u \quad \text{strongly in} \quad L^2(\Omega), \\ &u_j \to u \quad \text{a.e. on} \quad \Omega. \end{aligned}$$

By Fatou's lemma, we can assume that $\|u\|_{L^{2^*}} \leq 1$. Set $v_j = u_j - u$, we have

$$\begin{aligned} &v_j \rightharpoonup 0 \quad \text{weakly in} \quad S_{1,0}(\Omega), \\ &v_j \to 0 \quad \text{strongly in} \quad L^2(\Omega), \\ &v_j \to 0 \quad \text{a.e. on} \quad \Omega. \end{aligned}$$

By the definition of S, we have $\|\nabla_b u_j\|_{L^2}^2 \geq S$. Combined with (4.9), there holds

$$\lambda \|u\|_{L^2}^2 = \lambda \lim_{j\to\infty} \|u_j\|_{L^2}^2 = \lim_{j\to\infty} \left(\|\nabla_b u_j\|_{L^2}^2 - S_\lambda \right) \geq S - S_\lambda > 0.$$

Therefore it is not possible that $u \equiv 0$. Again by (4.9) and the fact that $v_j \rightharpoonup 0$ weakly in $S_{1,0}(\Omega)$, we obtain

$$\|\nabla_b u\|_{L^2}^2 + \|\nabla_b v_j\|_{L^2}^2 - \lambda \|u\|_{L^2}^2 = S_\lambda + o(1). \tag{4.10}$$

By the famous result of Brezis and Lieb [9], since u_j are bounded in $L^{2^*}(\Omega)$ and $u_j \to u$ a.e., we have

$$\|u + v_j\|_{L^{2^*}}^{2^*} = \|u\|_{L^{2^*}}^{2^*} + \|v_j\|_{L^{2^*}}^{2^*} + o(1),$$

and since $2^* > 2$,

$$1 \leq \|u\|_{L^{2^*}}^2 + \|v_j\|_{L^{2^*}}^2 + o(1).$$

Since $\|v_j\|_{L^{2^*}}^2 \leq \frac{1}{S}\|\nabla_b v_j\|_{L^2}^2$, which leads to

$$1 \leq \|u\|_{L^{2^*}}^2 + \frac{1}{S}\|\nabla_b v_j\|_{L^2}^2 + o(1). \tag{4.11}$$

Now we claim that now we can show

$$\|\nabla_b u\|_{L^2}^2 - \lambda\|u\|_{L^2}^2 \leq S_\lambda \|u\|_{L^{2^*}}^2, \tag{4.12}$$

and since S_λ is the infimum by definition, the infimum is achieved by normalization of the function u.

Then we look at the proof of (4.12). We discuss by two cases. First case, if $0 < \lambda < \lambda_1$, then $S_\lambda > 0$, we have by (4.11),

$$S_\lambda \leq S_\lambda \|u\|_{L^{2^*}}^2 + \frac{S_\lambda}{S}\|\nabla_b v_j\|_{L^2}^2 + o(1).$$

Combining (4.10) and the assumption $S_\lambda < S$, we obtain (4.12).

Second case, if $\lambda \geq \lambda_1$, then $S_\lambda \leq 0$, now since $\|u\|_{L^{2^*}} \leq 1$, we have $S_\lambda \leq S_\lambda \|u\|_{L^{2^*}}^2$. Then we can deduce (4.12) directly from (4.10). □

Theorem 4.2.9. *Assume $\lambda \in (0, \lambda_1)$, then there exists a solution of* (4.1).

Proof. Let $u \in S_{1,0}(\Omega)$ be as in Lemma 4.2.8, there holds

$$\|u\|_{L^{2^*}} = 1 \quad \text{and} \quad \|\nabla_b u\|_{L^2}^2 - \lambda \|u\|_{L^2}^2 = S_\lambda. \tag{4.13}$$

Note that we may assume that $u \geq 0$ everywhere on Ω for otherwise we may replace u by $|u|$. Since u is minimizer for (4.2), by variational method we have there exists $\mu \in \mathbb{R}$ such that

$$\Delta_b u - \lambda u = \mu u^{2^*-1} \quad \text{on} \quad \Omega.$$

Substitute into (4.13), we get $\mu = S_\lambda$, and hence $\mu > 0$ since $\lambda < \lambda_1$. So multiply by a positive constant we have solutions to (4.1), which we still denote by u. We also note that $u > 0$ everywhere on Ω by the strong maximum principle. □

4.3 Non-existence Results: Part 2

In this section, we continue to prove some non-existence results to some related PDE.

Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. Now we consider the problem of existence of a function u satisfying:

$$\begin{cases} \Delta_b u = u^p + \lambda u & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.14}$$

where $p > 2^* - 1 = (n+2)/n$ and λ is a real number. As before, we denote by λ_1 the first eigenvalue of Δ_b with zero Dirichlet condition on Ω.

Theorem 4.3.1. *If Ω is a δ-starshaped domain, problem (4.14) has no solution if $\lambda < \lambda^*$, where λ^* is a positive constant depending on the domain Ω and p, which is defined by*

$$\lambda^* = \lambda_1 \cdot \frac{n}{n+1} \cdot \frac{p - (n+2)/n}{p-1}.$$

Proof. Suppose there exists such solution u to the problem (4.14). Since the domain Ω is δ-starshaped, applying the CR Pohozaev identity and integration by parts again, we obtain

$$\left(n - \frac{2n+2}{p+1}\right) \int_\Omega u^{p+1} d\mathrm{vol}_0 \leq \lambda \int_\Omega u^2 d\mathrm{vol}_0.$$

Since λ_1 is the first eigenvalue of Δ_b on Ω, there holds

$$\begin{aligned}
\lambda_1 \int_\Omega u^2 d\mathrm{vol}_0 &\leq \int_\Omega |\nabla_b u|^2 d\mathrm{vol}_0 = \int_\Omega (\Delta_b u) u d\mathrm{vol}_0 \\
&= \int_\Omega u^{p+1} d\mathrm{vol}_0 + \lambda \int_\Omega u^2 d\mathrm{vol}_0 \\
&\leq \left(n - \frac{2n+2}{p+1}\right)^{-1} \lambda \int_\Omega u^2 d\mathrm{vol}_0 + \lambda \int_\Omega u^2 d\mathrm{vol}_0.
\end{aligned}$$

Hence $\lambda \geq \lambda^*$ as desired and we have proven the result. □

Chapter 5

Future Problems

In this chapter, we outline several potential future problems.

(i)As we mentioned in Theorem 2.0.21, for the Sobolev-type embedding on Heisenberg group $\mathbb{H}^n$, if $q \neq 2$, what is the value of optimal constant $\mathscr{A}(n,q)$? Is the optimal constant attained? If attained, what are the extremal functions?

(ii)For the Riemannian case, E. Hebey and M. Vaugon showed that for any compact Riemannian manifold M, in AB inequality, the set $\hat{\mathbb{A}}_2(M)$ is closed using blow-up analysis ([34], [35] or [36]). Is it the same thing in the CR case?

Bibliography

[1] E. Stein A. Nagel and S. Wainger. Balls and metrics defined by vector fields I: basic properties. *Acta Math.*, 155:103–147, 1985.

[2] R. A. Adams and J. F. Fournier. *Sobolev Spaces.* Pure and Applied Mathematics. Elsevier, Singapore, 2003.

[3] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geometry*, 11:573–598, 1976.

[4] T. Aubin. Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire. *J. Functional Analysis*, 32:148–174, 1979.

[5] T. Aubin. *Best Constants in the Sobolev Embedding Theorem: the Yamabe Problem.* Seminar on Differential Geometry (S.-T. Yau, ed.). Princeton University, Princeton, 1982.

[6] D. Bakry. *L'hypercontractivté et son utilisation en théorie des semigroupes.* Lectures on Probability Theory. Springer, Berlin, 1994.

[7] A. Bellaiche and J. J. Risler. *Sub-Riemannian Geometry.* Progress in Mathematics. Birkhauser, Boston, 1996.

[8] T. P. Branson, L. Fontana, and C. Morpurgo. Moser-Trudinger and Beckner-Onofris inequalities on the CR sphere. *Ann. of Math.*, 177:1–52, 2013.

[9] H. Brezis and E. H. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88:486–490, 1983.

[10] H. Brezis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36:437–477, 1983.

[11] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem.* Progress in Mathematics. Birkhauser, Basel, 2007.

[12] S. Y. A. Chang. *The Moser-Trudinger Inequality and Applications to Some Problems in Conformal Geometry.* Nonlinear Partial Differential Equations in Differential Geometry, Mathematics Series. Amer. Math. Soc., Providence, 1996.

[13] S. Y. A. Chang, M. J. Gursky, and P. C. Yang. The scalar curvature equation on 2- and 3-spheres. *Calculus of Variations and Partial Differential Equations*, 1:205–229, 1993.

[14] S. Y. A. Chang and P. C. Yang. Prescribing Gaussian curvature on $\mathbb{S}^2$. *Acta. Math.*, 159:215–259, 1987.

[15] S. Y. A. Chang and P. C. Yang. A perturbation result in prescribing scalar curvature on $\mathbb{S}^n$. *Duke Math. J.*, 64:27–69, 1991.

[16] S. C. Chen and M. C. Shaw. *Partial Differential Equations in Several Complex Variables*. Studies in Advanced Mathematics. Amer. Math. Soc., Boston, 2001.

[17] S. S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta. Math.*, 133:219–271, 1974.

[18] W. L. Chow. über systeme von linearen partiellen differentialgleichungen erster ordnung. *Math. Ann.*, 117:98–105, 1939.

[19] W. S. Cohn and G. Lu. Best constants for Moser-Trudinger inequalities on the Heisenberg group. *Indiana Univ. Math. J.*, 50:1567–1591, 2001.

[20] W. S. Cohn and G. Lu. Sharp constants for Moser-Trudinger inequalities on spheres in complex space $\mathbb{C}^n$. *Comm. Pure Appl. Math.*, 57:1458–1493, 2004.

[21] L. Boutet de Monvel. *Intégration des équations de Cauchy-Riemann induites formelles*. Séminaire Goulaouic-Lions-Schwartz. École Polytechnique, Paris, 1974–1975.

[22] F. Demengel and E. Hebey. On some nonlinear equations involving the p-Laplacian with critical Sobolev growth. *Advances in Differential Equations*, 3:533–574, 1998.

[23] S. Dragomir and G. Tomassini. *Differential Geometry and Analysis on CR Manifolds*. Progress in Mathematics. Birkhauser, Basel, 2006.

[24] O. Druet and E. Hebey. *The AB Program in Geometric Analysis: Sharp Sobolev Inequalities and Related Problems*. Memoirs of the American Mathematical Society. Amer. Math. Soc., Providence, 2002.

[25] J. F. Escobar and R. Schoen. Conformal metrics with prescribed scalar curvature. *Inventiones Mathematicae*, 86:243–254, 1986.

[26] G. B. Folland. Compact Heisenberg manifolds as CR manifolds. *J. Geometric Analysis*, 14:521–532, 2004.

[27] G. B. Folland and E. Stein. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429–522, 1974.

[28] L. Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifold. *Comment. Math. Helvetic.*, 68:415–454, 1993.

[29] R. L. Frank and E. H. Lieb. Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math.*, 176:349–381, 2012.

[30] N. Gamara and A. B. Ahmed. A new proof of the CR Pohozaev identity and related topics. *CUBO A Mathematical Journal*, 14:43–59, 2012.

[31] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Classics in Mathematics. Springer, Berlin, 2001.

[32] E. Hebey. Scalar curvature on $\mathbb{S}^n$ and first spherical harmonics. *J. Differential Geometry and Its Applications*, 5:71–78, 1995.

[33] E. Hebey. *Sobolev Spaces on Riemannian Manifolds.* Lecture Notes in Mathematics, Research Monograph. Springer, Berlin-New York, 1996.

[34] E. Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities.* Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York, 1999.

[35] E. Hebey and M. Vaugon. The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds. *Duke Math. J.*, 79:235–279, 1995.

[36] E. Hebey and M. Vaugon. Meilleures constantes dans le théorème d'inclusion de Sobolev. *Annales de l'I. H. P.*, C:57–93, 1996.

[37] P. T. Ho. Prescribed Webster scalar curvature on $\mathbb{S}^{2n+1}$ in the presence of reflection or rotation symmetry. *Bull. Sci. Math.*, 140:506–518, 2016.

[38] D. Jerison. The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.*, 53:503–523, 1986.

[39] D. Jerison and J. M. Lee. The Yamabe problem on CR manifolds. *J. Differential Geometry*, 25:167–197, 1987.

[40] D. Jerison and J. M. Lee. Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. *J. Amer. Math. Soc.*, 1:1–13, 1988.

[41] D. Jerison and J. M. Lee. Intrinsic CR normal coordinates and the CR Yamabe problem. *J. Differential Geometry*, 29:303–343, 1989.

[42] J. L. Kazdan and F. W. Warner. Scalar curvature and conformal deformation of Riemannian structure. *J. Differential Geometry*, 10:113–134, 1975.

[43] J. M. Lee. The Fefferman metric and pseudo-Hermitian invariants. *Trans. Amer. Math. Soc.*, 296:411–429, 1986.

[44] J. M. Lee. Pseudo-Einstein structures on CR manifolds. *Amer. J. Math.*, 110:157–178, 1988.

[45] A. Nagel. *Introduction to Analysis on Carnot-Carathéodory Spaces.* Summer School Notes. Centro di Ricerca Matematica Ennio De Giorgi, Pisa, 2003.

[46] R. Schoen. *Variational Theory for the Total Scalar Curvature Functional for Riemannian Metrics and Related Topics.* Topics in Calculus of Variations, Lecture Notes in Mathematics. Springer, Berlin, 1989.

[47] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura. Appl.*, 110:353–372, 1976.

[48] S. M. Webster. Pseudohermitian structures on a real hypersurface. *J. Differential Geometry*, 13:25–41, 1978.