Backward Stochastic Differential Equations with Superlinear Drivers

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Abstract

This thesis focuses mainly on the well-posedness of backward stochastic differential equations:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

The most prevalent method for showing the well-posedness of BSDE is to use the Banach fixed point theorem on a space of stochastic processes. Another notable method is to use the comparison theorem and limiting argument. We present three other methods in this thesis:

1. Fixed point theorems on the space of random variables

2. BMO martingale theory and Girsanov transform

3. Malliavin calculus

Using these methods, we prove the existence and uniqueness of solution for multidimensional BSDEs with superlinear drivers which have not been studied in the previous literature. Examples include quadratic mean-field BSDEs with L^2 terminal conditions, quadratic Markovian BSDEs with bounded terminal conditions, subquadratic BSDEs with bounded terminal conditions, and superquadratic Markovian BSDEs with terminal conditions that have bounded Malliavin derivatives.

Along the way, we also prove the well-posedness for backward stochastic equations, mean-field BSDEs with jumps, and BSDEs with functional drivers. In the last chapter, we explore the relationship between BSDEs with superquadratic driver and semilinear parabolic PDEs with superquadratic nonlinearities in the gradients of solutions. In particular, we study the cases where there is no boundary or there is a Dirichlet or Neumann lateral boundary condition.

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To My Lovely Wife, Soojin

Contents

Prior Publications and Presentations

This thesis is based on the following publications and presentations

Publications

- Cheridito, P. and Nam, K., 2014. BSDEs, BSEs, and fixed points. *in preparation*.
- Cheridito, P. and Nam, K., 2013. Multidimensional quadratic and subquadratic BSDEs with special structure. *arXiv.org*.
- Cheridito, P. and Nam, K., 2014. BSDEs with terminal conditions that have bounded Malliavin derivative. *Journal of Functional Analysis*, 266(3), pp.1257–1285.

Presentations

- *PACM Graduate Student Seminar*. May 2011. Princeton University, Princeton, NJ, USA.
- *Young Researchers Meeting on BSDEs, Numerics, and Finance*. July 2012. Oxford University, Oxford, UK.
- *Perspectives in Analysis and Probability: Workshop 3 Backward Stochastic Differential Equation*. (poster). May 2013. University of Rennes, Rennes, France.
- *Mathematical Finance Seminar*. Sep 2013. University of Texas–Austin, Austin, TX, USA.
- *Center for Computational Finance Seminar*. Feb 2014. Carnegie Mellon University, Pittsburgh, PA, USA.

Frequently Used Notation

Probability Space

Let W be a $\mathbb{R}^n$-valued Brownian motion for time $[0,T]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We assume $\mathbb{F}$ to be right-continuous and complete. For a stochastic process X, $\mathbb{F}^X$ is the the filtration which is generated by X and augmented. In particular, $\mathbb{F}^W$ is the Brownian filtration. We let $\mathcal{P}$ be the predictable σ-algebra on $[0,T]\times\Omega$. We will denote by $\mathbb{E}_t$, the conditional expectation with respect to $\mathcal{F}_t$, that is, $\mathbb{E}(\cdot|\mathcal{F}_t)$. We identify random variables that are equal $\mathbb{P}$-almost surely. Accordingly, we will understand the equality and inequality in the $\mathbb{P}$-almost sure sense.

Vectors and Matrices

For a given matrix $X \in \mathbb{R}^{d\times n}$, we let X^i be the ith row of X and X^{ij} to be the component at row i and column j. We identify $\mathbb{R}^d$-valued vectors with $\mathbb{R}^{d\times 1}$-valued matrices and understand multiplication as matrix multiplication if the dimensions are right. For $X, Y \in \mathbb{R}^d$, we will denote $XY := X^T Y$ where X^T is the transpose of X.

For a vector valued function $f = (f^1, \cdots, f^d)^T$, we understand ∇f as a $(d\times n)-$matrix valued function, that is,

$$\nabla f := \begin{pmatrix} \nabla f^1 \\ \cdots \\ \nabla f^d \end{pmatrix} = \begin{pmatrix} \partial_{x^1} f^1 & \cdots & \partial_{x^n} f^1 \\ \vdots & & \vdots \\ \partial_{x^1} f^d & \cdots & \partial_{x^n} f^d \end{pmatrix}.$$

Banach Spaces

The norm $|\cdot|$ is defined as the Euclidean norm, that is

$$|X| := \sqrt{\mathrm{tr}(XX^T)}.$$

First, let us define $\mathbb{L}^p$ space of random variables.

$\mathbb{L}^p$ the set of random variables X with

$$\|X\|_p := (\mathbb{E}|X|^p)^{1/p} < \infty \quad \text{if} \quad p < \infty$$
$$\|X\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} |X(\omega)| < \infty \quad \text{if} \quad p = \infty.$$

For a given σ-algebra $\mathcal{G}$, let $\mathbb{L}^p(\mathcal{G})$ denote the $\mathcal{G}$-measurable random variable in $\mathbb{L}^p$.

Now, let us define Banach spaces of stochastic processes for $p \geq 2$.

$\mathbb{S}^p$ is the set of $\mathbb{F}$-adapted, right-continuous left-limit (RCLL) processes X with

$$\|X\|_{\mathbb{S}^p} := \left\| \sup_{0\leq t\leq T} |X_t| \right\|_p < \infty.$$

In the case where $\mathbb{F}$ is the filtration generated by W and augmented, $\mathbb{S}^p$ is defined to be the set of continuous adapted processes with $\|\cdot\|_{\mathbb{S}^p}$.

$\mathbb{M}^p$ is the set of martingale in $\mathbb{S}^p$.

$\mathbb{H}^p$ is the set of $\mathbb{F}$-predictable stochastic processes X with

$$\|X\|_{\mathbb{H}^p} := \left(\mathbb{E}\left(\int_0^T |X_t|^2 dt \right)^{p/2} \right)^{1/p} < \infty \quad \text{if} \quad p < \infty$$
$$\|X\|_{\mathbb{H}^\infty} := \operatorname*{ess\,sup}_{(t,\omega)} |X_t(\omega)| < \infty \quad \text{if} \quad p = \infty.$$

$\mathbb{W}^{1,2}$ is the Sobolev space of random variables defined as in Da Prato [23]. See Definition 2.3.2. We use $\mathbf{D}$ for derivative operator.

$\mathbb{D}^{1,p}$ is the Wiener-Sobolev space where Malliavin calculus is defined. See Definition 4.1.1. We use D for Malliavin derivative operator.

$\mathbb{L}_a^{1,2}$ is the space of $\mathbb{R}^d$-valued progressively measurable processes X satisfying

(i) $X_t \in (\mathbb{D}^{1,2})^d$ for almost all t,

(ii) $(t,\omega) \mapsto DX_t(\omega) \in (L^2[0,T])^{d\times n}$ admits a progressively measurable version

(iii) $\|X\|^2_{\mathbb{L}_a^{1,2}} := \|X\|_{\mathbb{H}^2} + \left\| \left(\int_0^T \int_0^T |D_r X_t|^2 dr dt \right)^{1/2} \right\|_2 < \infty,$

where processes X, Y are identified if $\|X - Y\|_{\mathbb{L}_a^{1,2}} = 0$.

BMO is the set of $X \in \mathbb{M}^2$ such that

$$\|X\|_{BMO} := \sup_{\tau \in \mathcal{T}} \left\| \left(\mathbb{E}_\tau \left(\langle X \rangle_T - \langle X \rangle_\tau \right) \right)^{1/2} \right\|_\infty < \infty.$$

$\mathbb{H}^{\mathrm{BMO}}$ is the set of $X \in \mathbb{H}^2$ satisfying

$$\|X\|_{\mathrm{B}} := \sup_{\tau \in \mathcal{T}} \left\| \mathbb{E}_\tau \int_\tau^T |X_s|^2 ds \right\|_\infty^{1/2} < \infty,$$

where $\mathcal{T}$ denotes the set of all $[0, T]$-valued stopping times τ and $\mathbb{E}_\tau$ the conditional expectation with respect to $\mathcal{F}_\tau$.

Let Ξ be one of the above Banach spaces of stochastic processes. Then we use the following notation.

$\Xi(\mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued stochastic processes in Ξ. Sometimes, we use a simplified notation Ξ^d. In the case where d is obvious, we will keep using Ξ instead of $\Xi(\mathbb{R}^d)$ or Ξ^d.

Ξ_0 is the set of stochastic processes $X \in \Xi$ with $X_0 = 0$.

$\Xi_{[a,b]}$ is the Ξ defined on $[a, b] \subset [0, T]$ instead of the original Ξ.

For example, $\mathbb{S}_0^p(\mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued stochastic processes $X \in \mathbb{S}^p$ with $X_0 = 0$.

Borel algebra

For a Banach space Ξ, we let $\mathcal{B}(\Xi)$ be the Borel algebra on Ξ.

Standard Parameters

Let $p \in \mathbb{N}$ such that $p \geq 2$. For a random variable ξ and a random function $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \to \mathbb{R}^d$, (ξ, f) are called p-standard parameters if they satisfy the following three conditions:

(S1) $\xi \in \mathbb{L}^p(\mathcal{F}_T)$

(S2) $|f(t, y, z) - f(t, y', z')| \leq L(|y - y'| + |z - z'|)$ for a constant $L \in \mathbb{R}_+$

(S3) $f(., 0, 0) \in \mathbb{H}^p(\mathbb{R})$.

Chapter 1

Introduction

1.1 Introduction to Backward Stochastic Differential Equations

What is Backward Stochastic Differential Equations?

The most classical form of *backward stochastic differential equation* (BSDE) is

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{1.1.1}$$

where $\mathbb{F} = \mathbb{F}^W$, the terminal condition ξ is a $\mathbb{R}^d$-valued $\mathcal{F}_T^W$-measurable random variable, and the driver $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{d\times n})$-measurable function. A solution of BSDE (1.1.1) is a pair of predictable processes (Y, Z) taking value in $\mathbb{R}^d \times \mathbb{R}^{d\times n}$ such that $\int_0^T \left(|f(t, Y_t, Z_t)| + |Z_t|^2\right) dt < \infty$ and (1.1.1) holds for all $0 \le t \le T$. We call the BSDE is multidimensional if $d \ge 1$ and one-dimensional if $d = 1$. We assume $f(t, y, z)$ is Lipschitz with respect to y unless otherwise indicated. Quadratic BSDE is a BSDE that has at most quadratic growth in Z. Subquadratic and superquadratic BSDE are defined analogously. In the same spirit, superlinear driver is the driver $f(s, y, z)$ that is Lipschitz in y and has superlinear growth in z.

There are numerous generalizations of the classical BSDEs. First of all, the driver may depend on a random vector (Y_s, Z_s) itself rather than the value $(Y_s(\omega), Z_s(\omega))$ of random variables. This generalization includes McKean-Vlasov BSDEs and mean-field BSDEs. In addition, $\int f(s, Y_s, Z_s)ds$ can be generalized to a mapping $F(Y, Z)$ which might not be absolute continuous with respect to Lebesgue measure ds. Well-known example is a BSDE with reflecting barriers. Also, we can generalize the Brownian motion into a semimartingale and consider a general filtration $\mathbb{F}$. BSDEs with jumps are one such generalizations.

All such generalizations can be called BSDEs but we will use the term BSDE for the classical BSDE unless stated otherwise.

Applications of Backward Stochastic Differential Equations

BSDEs have been intensively studied for the last 20 years regarding its application to many areas of mathematics. In this subsection, we provide some examples of its application.

As El Karoui et al. emphasized in their survey paper [34], BSDEs have been used for many problems in financial mathematics. Indeed, BSDEs with linear drivers were first introduced by Bismut [9] for the application to stochastic control problem using convex duality. Since then, BSDEs have been one of the main methods to solve stochastic optimization problems.

First, BSDE is naturally related to the option pricing in complete market. The price of a contingent claim is determined by constructing a replicating portfolio. Consider an European call option which pays an amount ξ at time T. If we let Y be the price of its replicating portfolio which is governed by $dY_t = -f(t, Y_t, Z_t) + Z_t dW_t$ for the investment strategy Z, then (Y, Z) becomes the solution of BSDE since we require $Y_T = \xi$ as the terminal condition. In this context, El Karoui et al. [34] pointed out that the works by Black and Scholes [10], Merton [61], Harrison and Kreps [43], Harrison and Pliska [44], Duffie [30], and Karatzas [50] can be reformulated as BSDEs.

Another application of BSDE is the utility-based pricing problem for incomplete market. For example, Rouge and El Karoui [75], Hu et al. [45], Sekine [77], Mocha [62], and Cheridito et al. [16] used BSDEs in utility maximization in incomplete market.

The application of BSDEs is not restricted to optimization problems of a single agent. One can also use BSDEs to study stochastic differential games. Hamadéne and Lepeltier [40] applied BSDE results to show the existence of a saddle point for a given zero-sum game. Cvitanic and Karatzas [22] used a BSDE with double reflecting barrier to study zero-sum Dynkin game. Their result is further generalized by Hamadéne and Lepeltier [41] and Hamadéne [39] using reflected BSDEs. Non-zero-sum games are also studied using BSDEs (see Hamaéne et al. [42] and Karatzas and Li [51]).

A BSDE defines g-expectation that can be used as a coherent or convex risk measure as suggested by Artzner et al. [3]. For a random variable ξ, Peng [70] defined g-expectation of ξ as the solution Y_0 of BSDE where the driver is g and the terminal condition is ξ. Gianin [36] showed that if g is sublinear, g-expectation corresponds to a coherent risk measure and if g is convex, g-expectation corresponds to a convex risk measure. Moreover, since a solution Y of BSDE is a stochastic process, the author suggested a conditional g-expectation as a dynamic risk measure. Moreover, the author proved that almost any dynamic coherent or convex risk measure can be represented as a conditional g-expectation.

In addition to its applications in financial mathematics, PDEs are closely related to BSDEs. Brief introductions to this relationship are provided by Barles and Lesigne [8], Section 4 of El Karoui et al. [34], and Pardoux [64]. One of the earliest results in this relationship was done by Peng [69]. He showed that if the randomness of the terminal condition and the driver comes from the value of diffusion process, that is, if a BSDE is Markovian, then a solution of the BSDE with a random terminal time is a probabilistic representation of a solution for a semilinear parabolic PDE with Dirichlet lateral boundary condition. Pardoux and Peng [66] showed that the Markovian BSDE solution Y becomes a viscosity solution of a quasilinear parabolic PDE with the nonlinearity being given by the driver of the BSDE. Moreover, they also provided a set of sufficient conditions that guarantees the solution obtained by BSDE to be, in fact, a $C^{1,2}$ solution of the corresponding PDE. Darling and Pardoux [26] showed results on BSDE with random terminal time can be used to construct a viscosity solution of elliptic PDE with Dirichlet boundary condition. Pardoux and Zhang [68] studied semilinear parabolic PDE with nonlinear Neumann lateral boundary condition using BSDE. When $d = 1$, PDE-BSDE relationships are generalized in the recent paper by Cheridito and Nam [17] and will be presented in Section 4.4 of this thesis. In addition to the relationship between Markovian BSDEs and PDEs, the relationship between non-Markovian BSDEs and path-dependent PDEs was studied by Peng [72], Peng and Wang [73], and Ekren et al. [32].

Brief History of Well-Posedness Theory for Backward Stochastic Differential Equations

The first significant breakthrough was achieved by Pardoux and Peng [65] for 2-standard parameter and then generalized to p-standard parameters for $p \geq 2$ by El Karoui et al. [34]. They showed there exist a unique solution $(Y, Z) \in \mathbb{S}^p(\mathbb{R}^d) \times \mathbb{H}^p(\mathbb{R}^{d\times n})$ using the Banach fixed point theorem and martingale representation theorem. The authors constructed a contraction mapping

$$\phi : (Y, Z) \in \mathbb{S}^p \times \mathbb{H}^p \mapsto (y, z) \in \mathbb{S}^p \times \mathbb{H}^p$$

by the following BSDE:

$$y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T z_s dW_s$$

Given (Y, Z), if we take conditional expectation $\mathbb{E}_t$ on both sides, we have

$$y_t + \int_0^t f(s, Y_s, Z_s)ds = \mathbb{E}_{\mathcal{F}_t}\left(\xi + \int_0^T f(s, Y_s, Z_s)ds\right)$$

and then z is determined by the martingale representation theorem. Then, they used the Banach fixed point theorem for ϕ when T is small enough. The argument can be iterated to get the global solution by partitioning $[0, T]$ to small time intervals.

Lipschitz condition on $f(s, y, z)$ with respect to y can be relaxed to monotonicity condition

$$\exists C \geq 0 \quad s.t. \quad (y - y')^T(f(s, y, z) - f(s, y', z)) \leq C|y - y'|^2 \quad \forall y, y' \in \mathbb{R}^d$$

and continuity condition because the fixed point mapping defined above still remains a contraction under this relaxed conditions. Using this property, Pardoux [64] showed the existence and uniqueness of solution for BSDEs with drivers which are non-Lipschitz in y.

Hamadéne [38] was also able to relax Lipschitz condition of the driver to uniform continuity condition with linear growth. In particular, when the ith coordinate of the driver $f(s, y, z)$ does not depend on z^j for $j \neq i$, he proved the existence and uniqueness of solution when $f : y \mapsto f(s, y, z)$ and $f : z \mapsto f(s, y, z)$ are uniformly continuous with linear growth.

On the other hand, when $d = 1$ and the terminal condition is bounded, Kobylanski [55] showed that there exists a unique solution for BSDE with a driver that grows quadratically in z. The main techniques she used are exponential change of variable, comparison theorem, and monotone stability property of solution. Moreover, she presented the stability result and its relationship with semilinear parabolic PDE as well. Briand and Hu [12, 13] and Delbaen et al. [29] extended her result to the case of unbounded terminal condition with an additional convexity assumption on the driver.

When the driver has superquadratic growth in Z, Delbaen et al. [28] showed that the BSDE is ill-posed even when the terminal condition is bounded and the driver is a deterministic function of Z. Such BSDE may have an infinite number of solutions or have no solution at all. However, in Markovian settings, they proved that a superquadratic BSDE with bounded terminal condition has a unique bounded solution. Cheridito and Stadje [18] generalized the existence and uniqueness result to the one-dimensional non-Markovian superquadratic BSDE with a Lipschitz terminal condition and a convex driver that is random and depends on Y. In Markovian settings, Richou [74] was able to remove the Lipschitz assumption on the terminal condition and the convexity assumption on the driver. For non-Markovian BSDEs, Cheridito and Nam [17] removed the convexity assumption on the driver and relaxed the Lipschitz assumption on the terminal condition using Malliavin calculus.

People have sought existence and uniqueness results for multidimensional quadratic BSDEs due to both theoretical and practical interests. For example, Peng, one of the founders of BSDE theory, chose the existence and uniqueness of solution for multidimensional quadratic BSDE as one of the main open problems in BSDE in his article [71]. The main difficulty in this multidimensional case is the lack of a comparison theorem which holds when $d = 1$. Kobylanski used the comparison theorem to prove monotone stability. Therefore the comparison theorem is essential in order to use similar proof technique for multidimensional quadratic BSDE. In 2006, Hu and Peng [47] published a short article about the necessary and sufficient conditions for the existence of the multidimensional comparison theorem. They proved that, when the drivers are the same, the ith coordinate of $f(s, y, z)$ should depend only on the ith row of z in order to have the comparison theorem for multidimensional BSDEs. If $f(s, y, z)$ does not depend on y, their condition implies that one can decouple the multidimensional BSDE to multiple one-dimensional BSDEs. Therefore, one cannot expect a naive comparison theorem for multidimensional quadratic BSDEs.

However, in 2008, Tevzadze [80] proved that when ξ is small enough, one has a unique solution for multidimensional quadratic BSDE. To show this, he used the Banach fixed point theorem in $\mathbb{S}^\infty \times \mathrm{BMO}$. This is an interesting result because this is the first general result on multidimensional quadratic BSDEs. However, the bound on the terminal conditions should be tiny compared to the growth of the driver, that is, $\|\xi\|_\infty \lesssim |\partial_{zz} f|^{-2}$ if f is a deterministic differentiable function of z with at most quadratic growth. When $d = 1$, he was able to prove the existence and uniqueness of solution by decomposing the BSDE into BSDEs with small terminal conditions and solving BSDEs iteratively. This recovers the result in Kobylanski [55]. This scheme is also used by Kazi-Tani et al. [54] to study quadratic BSDEs with jumps.

Even though Tevzadze's work supports the existence of a solution for the multidimensional quadratic BSDE, Frei and Dos Reis [35] showed that there is a multidimensional quadratic

BSDE with bounded terminal conditions with no solution. They constructed a counterexample when $d = 2$, $n = 1$, and the driver is

$$f(z^1, z^2) = \left(0, (z^1)^2 + \frac{1}{2}(z^2)^2\right)^T.$$

The main argument is that, for a carefully chosen terminal condition, one can find a unique explicit solution (Y^1, Z^1) and this makes the solution Y^2 to blow up. Since Y should be continuous, this leads to nonexistence. The key is to select the terminal condition $\xi = (\xi^1, 0)^T$ where ξ^1 is in $\mathbb{L}^\infty$ and satisfies

$$\mathbb{E}\exp\left(\langle \mathbb{E}(\xi|\mathcal{F}_\cdot)\rangle_T\right) = \infty.$$

This counterexample shows that we need to assume more restrictive conditions on the driver or the terminal condition in order to guarantee the existence of solution. Moreover, since the conditions of this counterexample are hard to be satisfied in reality, finding appropriate conditions for the existence and uniqueness of solutions for multidimensional quadratic BSDEs is still a big challenge in BSDE theory.

1.2 Thesis Overview

BSDEs with quadratic drivers appears in many situation: risk sensitive control by El Karoui and Hamadene [33], utility maximization in incomplete market by Hu et al. [45], equilibrium pricing in incomplete market by Cheridito er al [16], the construction of gamma martingale by Darling [25], and so on. As a result, Peng presented well-posedness question about multidimensional quadratic BSDE as one of main open problem in BSDE: see [71].

Even though multidimensional quadratic BSDE do not have a solution in general, one may assume restrictive conditions to guarantee the existence of solution. In this thesis, we will prove the existence of solution for some multidimensional quadratic mean-field BSDE, multidimensional quadratic and subquadratic BSDEs with special structure, and one-dimensional superquadratic BSDE. We will also prove uniqueness if possible.

This thesis consists of three main chapters (Chapter 2, 3, and 4) and two appendices (Appendix A and B). Three main chapters are independent and can be read separately. Given the small number of alphabets, every coefficient will be defined a new in each chapter.

In Chapter 2, we will assume $d \geq 1$ and $\mathbb{F}$ to be a filtration that satisfies the usual conditions. We will study the existence and uniqueness of solution (Y, M) for *backward stochastic* *equations* (BSEs)

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T \tag{1.2.2}$$

where F is a mapping from $\mathbb{S}^p \times \mathbb{M}_0^p$ to $\mathbb{S}_0^p$. Here, we are trying to find an adapted process Y and a martingale M where F is a mapping from stochastic process to stochastic process. Therefore, BSEs can be thought as a generalized version of BSDEs with functional drivers. For a given BSE, we will define a fixed point mapping in $\mathbb{L}^p$ space and relate the fixed points to solutions of the BSE. By applying the Banach fixed point theorem, we will prove that a unique solution exists for a BSDE with a Lipschitz functional driver. In particular, there is a unique solution for mean-field BSDEs driven by Brownian motion and compensated poisson process. This extends the well-posedness results of the classical BSDEs with 2-standard parameters and mean-field BSDEs of Buckdahn et al. [14]. Then, we will use Schauder-type (Krasnoselskii) fixed point theorem to study BSDEs with drivers that have superlinear growth in Z. We will use the fact that Sobolev space of random variables can be compactly embedded in $\mathbb{L}^2$ space. As a result, we will prove the existence of solutions for mean-field BSDEs with quadratic drivers when $\mathbb{F} = \mathbb{F}^W$.

In Chapter 3 and 4, we will study the existence and uniqueness of solution for BSDEs assuming $\mathbb{F} = \mathbb{F}^W$ and $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$.

Chapter 3 will be devoted to multidimensional BSDEs with drivers that have quadratic and subquadratic growth. We exploit the BMO martingale theory and the Girsanov transform to remove superlinear part of the driver. The main obstacle for quadratic BSDEs is that the filtration generated by a Girsanov-transformed Brownian motion $\tilde{W}$ may be strictly coarser than $\mathbb{F}^W$. As a result, a naive application of Girsanov transform only gives a weak solution (see Liang et al. [58]). We use the results from forward and backward stochastic differential equations (FBSDEs) to show the well-posedness of Markovian quadratic BSDEs. If non-Markovian multidimensional quadratic BSDEs can be projected to one-dimensional subspace, one can use the result in Kobylanski [55] to prove the existence and uniqueness of solution. If the driver is strictly subquadratic, we can show the existence and uniqueness of solution for a short time interval using the Banach fixed point theorem and under certain conditions, the argument can be repeated to show the existence of a unique global solution.

In Chapter 4, we will use Malliavin calculus to study BSDEs with drivers that has superquadratic growth in z. For BSDEs with 4-standard parameters, El Karoui et al. [34] showed that $Z_t = D_tY_t$ and (D_rY, D_rZ) is a solution of a differentiated BSDE. Then, when the terminal condition has bounded Malliavin derivative, we can bound D_rY and Z uniformly for a BSDE with standard parameters. When the terminal condition is bounded and the driver is locally Lipschitz and has superquadratic growth in Z, a conventional cutoff argument applies and we can prove the existence and uniqueness of a bounded solution (Y, Z). The main step is to find a good bound for Z of BSDE with localized driver which is Lipschitz. The bound on Z can be found by change of variables for the differentiated BSDE. Since the bound on Z depends on the Lipschitz coefficient of $f(s, y, z)$ with respect to z, we only have a unique solution when T is small enough. We can stretch the small time solution to any finite time solution by assuming more restrictive conditions on the driver. These results are shown in Section 4.2. On the other hand, when $d = 1$, the comparison theorem tells us that Z is bounded by a constant which does not depend on the Lipschitz coefficient of $f(s, y, z)$ with respect to z. This enables us to get a solution for any finite T. In both cases, the driver is virtually Lipschitz since Z is bounded. In turn, uniqueness results follow by the classical results on Lipschitz BSDEs. Moreover, in the case where BSDEs are Markovian, one can apply its usual relationship with semilinear parabolic PDEs with non-Lipschitz nonlinearities using the generalized Feynman-Kac formula. We discuss three cases: when there is no lateral boundary, when there is a lateral boundary condition of Dirichlet type, and when there is a lateral boundary condition of Neumann type.

Chapter 2

Fixed Point Methods for BSDEs and Backward Stochastic Equations

The first breakthrough in BSDE was achieved by Pardoux and Peng [65]. They proved the well-posedness of BSDE when f is Lipschitz using the Banach fixed point theorem on stochastic Hardy space. Later, Tevzadze [80] proved the well-posedness of multidimensional quadratic BSDE when its terminal condition is tiny using the Banach fixed point theorem in BMO space. Even though it is tempting to apply a Schauder-type fixed point theorem to produce a new result, this is not easy because a space of stochastic processes is infinite dimensional and, in turns, it is not locally compact in general. There are two sources of infinite dimensionality in stochastic processes; the time variable and the randomness of functions. If we consider the space of deterministic functions, Arzela-Ascoli theorem yields the necessary and sufficient conditions for the compactness in this function space with respect to a uniform convergence topology. For the space of random variables, Da Prato et al. [24] give a set of sufficient conditions for the compactness in $\mathbb{L}^2$ and Wiener-Sobolev space using Malliavin calculus. Recently, Bally and Saussereau [6] also provided sufficient conditions for the compactness in Wiener-Sobolev space. However, their results are not tenable to use to prove the existence of BSDE solution.

Kobylanski [55] was able to detour this compactness issue using the comparison principle.

However, when $d > 1$, it is known that comparison principle does not hold in general (Hu and Peng [47]). Therefore, multidimensional quadratic BSDEs cannot be solved via Kobylanski's method. Moreover, Frei and Dos Reis [35] found a multidimensional quadratic BSDE with a bounded terminal condition which does not have a solution.

On the other hand, Liang et al. [59] pointed out that a BSDE (1.1.1) can be understood at two levels: martingale representation and backward stochastic dynamics. In other words, BSDE can be viewed as an equation of an adapted process Y and a martingale M that satisfies

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathcal{R}_s(M))ds + M_T - M_t$$

where $\mathcal{R}$ is the martingale representation operator, that is, $\mathcal{R}(\int_0^\cdot Z_s dW_s) = Z$. Since $\mathcal{R}$ does not have to be the martingale representation operator, they generalized BSDEs into backward stochastic dynamics

$$dY_t = (f(t, Y_t, \mathcal{L}(M)_t) + g(t, Y_t))\, dt + dM_t; \quad Y_T = \xi$$

where $\mathcal{L}$ is a Lipschitz mapping from $\mathbb{M}_0^2$ to $\mathbb{H}^2$. In the case where f and g are Lipschitz, they proved the existence of global solution using the Banach fixed point theorem on the space of stochastic processes. Casserini [15] combined Liang et al. [59] and Tevzadze [80] to study quadratic backward stochastic dynamics for small terminal conditions when $d > 1$ and arbitrary bounded terminal conditions when $d = 1$.

Note that this idea can be extended further since the information of uniformly integrable martingale M can be stored by its terminal random variable M_T and the information of Y can be encoded by its initial value Y_0 and M by the forward stochastic differential equation given by the driver. Consider a special case of BSDEs with the driver $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$. The backward stochastic differential equation (BSDE)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

can be view as backward stochastic equation (BSE)

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T$$

where

$$M_t = -\int_0^t Z_s dW_s \quad \text{and} \quad F_t(Y,M) = \int_0^t f(s,Y_s,Z_s)ds.$$

BSE can be reduced further to an equation of random variable in $\mathbb{L}^p(\mathcal{F}_T)$

$$G(Y_0 - M_T) = Y_0 - M_T$$

where $G : \mathbb{L}^p(\mathcal{F}_T) \to \mathbb{L}^p(\mathcal{F}_T)$ is defined by

$$G(V) := \xi + F_T(Y^V, M^V)$$

where $M_t^V = \mathbb{E}_0 V - \mathbb{E}_t V$ and Y^V is a unique solution of $Y_t^V = \mathbb{E}_0 V - F_t(Y^V, M^V) - M_t^V$: see Section 2.1 of this chapter. Note that if we know $Y_0 - M_T$, then (Y_0, M_T) is given by

$$Y_0 = \mathbb{E}_0 V \quad \text{and} \quad M_T = \mathbb{E}_0 V - V.$$

in this equation. Let us call G a random variable mapping.

There are many advantages if we remove the time variable. First of all, we can remove the infinite dimensionality of stochastic process which arises from the time variable. Then, we can use the compact embedding theorem of Da Prato [23] to find a compact set in $\mathbb{L}^2(\mathcal{F}_T)$. In addition, for a fixed (s,ω), the driver f of BSDE can be a function of the random vector (Y_s, Z_s) rather than simply of the deterministic vector $(Y_s(\omega), Z_s(\omega))$ as in standard BSDEs. For instance, it could depend on the distribution of (Y_s, Z_s). Moreover, as F is a function on $\mathbb{S}^p \times \mathbb{M}_0^p$, the corresponding BSDE may depend on the solution path as well.

In this chapter, we use the Banach fixed point theorem and the Krasnoselskii fixed point theorem on random variable mappings to prove the existence and uniqueness of solutions for the corresponding BSEs. As a result, we obtain the existence and uniqueness of solution for the corresponding BSDEs with functional drivers. The cases we study in this chapter include BSDEs with drivers of McKean-Vlasov type and BSDEs with solution-path-dependent drivers. In particular, we will also prove the existence of solutions for BSDEs with drivers that have quadratic growth in density process Z: see examples 2.3.19 and 2.3.20. Our main contributions are

- to develop a new method for solving BSDEs: mapping a BSDE into a fixed point problem on the space of $\mathbb{L}^p$ random variables (see Lemma 2.1.2).

- to provide new results on BSEs and BSDEs with functional drivers.

With this new framework, solutions of BSDEs are simply fixed points of the corresponding random variable mappings. In order to show the existence (and uniqueness) of solution, one only needs to check the sufficient conditions for a fixed point theorem through calculation. As far as we know, most of the theorems, propositions, and examples in Section 2.2 and 2.3 are novel to the previous literature.

The chapter is organized as follows. In Section 2.1, we study the relationship between the solutions of BSEs and fixed points of the corresponding random variable mappings. Then, using Banach fixed point theorems, we study BSEs and BSDEs with Lipschitz functional drivers in Section 2.2. In particular, we generalize the classical result proved by Pardoux and Peng [65] and Buckdahn et al. [14]. Section 2.3 is devoted to the case where F is not Lipschitz using the Krasnoselskii fixed point theorem. This gives us the existence of solutions for the random variable mappings which corresponds to multidimensional quadratic mean-field BSDEs.

2.1 Backward Stochastic Equations and Fixed Points in $\mathbb{L}^p$

In this section we introduce our notion of a BSE, which extends the concept of a BSDE, and relate it to fixed point problems in $\mathbb{L}^p$-spaces.

Throughout this chapter, we let $\mathbb{S}^p := \mathbb{S}^p(\mathbb{R}^d)$, $\mathbb{H}^p := \mathbb{H}^p(\mathbb{R}^{d\times n})$, and $\mathbb{M}_0^p := \mathbb{M}^p(\mathbb{R}^d)$ and we consider a mapping $F : \mathbb{S}^p \times \mathbb{M}_0^p \to \mathbb{S}_0^p$ and a terminal condition $\xi \in \mathbb{L}^p(\mathcal{F}_T)^d$.

Definition 2.1.1. *A solution to the BSE*

$$Y_t + F_t(Y, M) + M_t = \xi + F_T(Y, M) + M_T \tag{2.1.1}$$

consists of of a pair $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ *such that* (2.1.1) *holds for all* $t \in [0, T]$.

We say that F satisfies the condition (S) if for all for all $y \in \mathbb{L}^p(\mathcal{F}_0)^d$ and $M \in \mathbb{M}_0^p$ the SDE

$$Y_t = y - F_t(Y, M) - M_t \tag{2.1.2}$$

has a unique solution $Y \in \mathbb{S}^p$.

For a given $V \in \mathbb{L}^p(\mathcal{F}_T)^d$, we will denote $y^V := \mathbb{E}_0 V$ and $M^V_t := \mathbb{E}_0 V - \mathbb{E}_t V$. Note that $y^V \in \mathbb{L}^p(\mathcal{F}_0)^d$ and $M \in \mathbb{M}^p_0$ by Doob's maximal inequality (Theorem I.3.8 of Karatzas [52]). If F satisfies (S), we denote Y^V the solution of $Y_t = y^V - F_t(Y, M^V) - M^V_t$ and define the mapping $G : \mathbb{L}^p(\mathcal{F}_T)^d \to \mathbb{L}^p(\mathcal{F}_T)^d$ by

$$G(V) := \xi + F_T(Y^V, M^V).$$

To relate solutions of the BSE to fixed points of G, we define the mappings $\psi : \mathbb{L}^p(\mathcal{F}_T)^d \to \mathbb{S}^p \times \mathbb{M}^p_0$ and $\pi : \mathbb{S}^p \times \mathbb{M}^p_0 \to \mathbb{L}^p(\mathcal{F}_T)^d$ by

$$\psi(V) := (Y^V, M^V) \quad \text{and} \quad \pi(Y, M) := Y_0 - M_T.$$

The following result relates solutions of the BSE (2.1.1) to fixed points of G.

Lemma 2.1.2. *Assume F satisfies (S). Then the following hold:*

a) $V = \pi \circ \psi(V)$ *for all* $V \in \mathbb{L}^p(\mathcal{F}_T)^d$*. In particular,* ψ *is injective.*

b) *If* $V \in \mathbb{L}^p(\mathcal{F}_T)^d$ *is a fixed point of* G*, then* $\psi(V)$ *is a solution of the BSE* (2.1.1).

c) *If* $(Y, M) \in \mathbb{S}^p \times \mathbb{M}^p_0$ *solves the BSE* (2.1.1)*, then* $\pi(Y, M)$ *is a fixed point of* G *and* $(Y, M) = \psi \circ \pi(Y, M)$.

d) V *is a unique fixed point of* G *in* $\mathbb{L}^p(\mathcal{F}_T)^d$ *if and only if* $\psi(V)$ *is a unique solution of the BSE* (2.1.1) *in* $\mathbb{S}^p \times \mathbb{M}^p_0$.

Proof. a) is clear.

b) If $V \in \mathbb{L}^p(\mathcal{F}_T)^d$ is a fixed point of G, then

$$y^V - M^V_T = \pi \circ \psi(V) = V = G(V) = \xi + F_T(Y^V, M^V). \tag{2.1.3}$$

Since Y^V satisfies $Y^V_t = y^V - F_t(Y^V, M^V) - M^V_t$ for all t, (2.1.3) is equivalent to

$$Y^V_t + F_t(Y^V, M^V) + M^V_t = \xi + F_T(Y^V, M^V) + M^V_T \quad \text{for all } t,$$

which shows that $\psi(V) = (Y^V, M^V)$ solves the BSE (2.1.1).

c) Let $(Y, M) \in \mathbb{S}^p \times \mathbb{M}^p_0$ be the solution of the BSE (2.1.1), and let $V := \pi(Y, M) = Y_0 - M_T$. Then, $y^V = Y_0$ and $M^V_t = M_t$. In particular,

$$Y_t = Y_0 - F_t(Y, M) - M_t = y^V - F_t(Y, M^V) - M^V_t$$

for all t. It follows that $(Y,M)=(Y^V,M^V)=\psi(V)$ and that

$$y^V=Y_0^V=\xi+F_T(Y^V,M^V)+M_T^V=G(V)+M_T^V.$$

Since $y^V-M_T^V=V$, we have $V=G(V)$.

d) follows from a)–c). □

The following lemma shows that F satisfies condition (S) under a standard Lipschitz assumption.

Lemma 2.1.3. *The mapping F satisfies (S) if for every $M\in\mathbb{M}_0^p$, there exists a non-negative constant $C_F<1$ such that*

$$\|F(Y,M)-F(Y',M)\|_{\mathbb{S}^p}\le C_F\,\|Y-Y'\|_{\mathbb{S}^p}\quad \textit{for all } Y,Y'\in\mathbb{S}^p.$$

Proof. For given $y\in\mathbb{R}^d$ and $M\in\mathbb{M}_0^p$, the mapping $Y\mapsto y-F(Y,M)-M$ is a contraction in $\mathbb{S}^p$. It follows from the Banach fixed point theorem that the SDE (2.1.2) has a unique solution in $\mathbb{S}^p$. □

Remark 2.1.4. In the case where $F(Y,M)$ does not depend on Y, condition (S) is satisfied trivially and finding a fixed point of G is equivalent to finding a fixed point of $H(V):=G(V)-\mathbb{E}_0G(V)$ in $\mathbb{L}^p(\mathcal{F}_T)^d$. Indeed if $V'=G(V')-\mathbb{E}_0G(V')$, then for $V=V'+\mathbb{E}_0G(V')$, it is easy to check $M^V=M^{V'}$, and therefore,

$$V=V'+\mathbb{E}_0G(V')=G(V')=\xi+F_T(M^{V'})=\xi+F_T(M^V)=G(V).$$

Then, Y^V is determined by

$$Y_t^V=\mathbb{E}_0V-F_t(M^V)-M_t^V=\mathbb{E}_0\xi+\mathbb{E}_0F_T(M^{V'})-F_t(M^{V'})-M_t^{V'}.$$

If F is of the form $F_t(Y,M)=\int_0^t f(s,Y,M)ds$, one needs assumptions on the function f to ensure that F maps $\mathbb{S}^p\times\mathbb{M}_0^p$ into $\mathbb{S}_0^p$.

Proposition 2.1.5. *Assume F is of the form $F_t(Y,M)=\int_0^t f(s,Y,M)ds$ for a function $f:[0,T]\times\Omega\times\mathbb{S}^p\times\mathbb{M}_0^p\to\mathbb{R}^d$ such that $f(\cdot,Y,M)$ is progressively measurable for fixed $(Y,M)\in\mathbb{S}^p\times\mathbb{M}_0^p$. If $\mathbb{E}\left(\int_0^T|f(s,Y,M)|ds\right)^p<\infty$, then $F(Y,M)\in\mathbb{S}_0^p$.*

Proof.

$$\left\|\int_0^{\cdot} f(s,Y,M)ds\right\|_{\mathbb{S}^p}^p = \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t f(s,Y,M)ds\right|^p$$
$$\le \mathbb{E}\sup_{0\le t\le T}\left(\int_0^t |f(s,Y,M)|ds\right)^p = \mathbb{E}\left(\int_0^T |f(s,Y,M)|ds\right)^p < \infty.$$

□

If the conditions of Proposition 2.1.5 are met, then we can apply Lemma 2.1.2 and Remark 2.1.4 to BSE (2.1.1), and the BSE becomes

$$Y_t = \xi + \int_t^T f(s,Y,M)ds + M_T - M_t.$$

This is a generalized version of BSDE, that is, the driver can depend on the path of solution as well as the law of solution. Moreover, the martingale part does not have to be continuous and can be stochastic integrals of a Brownian motion and a compensated Poisson process. Using Lemma 2.1.2 with fixed point theorems, we will study these generalized BSDEs in subsequent sections.

2.2 Contraction Mappings and Banach Fixed Point Theorem

One of the most well-known fixed point theorems is the Banach fixed point theorem. If one can construct a contraction mapping, the existence of a unique fixed point is given by the completeness of Banach space. Main advantage of the Banach fixed point theorem is that one can work with infinite dimensional space. In this subsection, we will apply the Banach fixed point theorem to random variable mapping in order to show our framework for BSDE. This will recover and generalize the existence and uniqueness results by Pardoux and Peng [65] and Buckdahn et al. [14].

The following theorem is an easy application of the Banach fixed point theorem to BSEs using Lemma 2.1.2.

Theorem 2.2.1. *Let $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ for $p > 1$. Assume that $F : \mathbb{S}^p \times \mathbb{M}_0^p \to \mathbb{S}_0^p$ satisfies*

$$\|F(Y,M) - F(Y',M')\|_{\mathbb{S}^p} \le C_F\left(\|Y-Y'\|_{\mathbb{S}^p} + \|M-M'\|_{\mathbb{S}^p}\right).$$

If $C_F < (p-1)/(4p-1)$*, then there exists a unique solution* $(Y, M) \in \mathbb{S}^p \times \mathbb{M}_0^p$ *of BSE* (2.1.1).

Proof. Since $C_F < 1$, it follows from Lemma 2.1.3 that F satisfies (S). So by Lemma 2.1.2, it is enough to prove that G has a unique fixed point in $\mathbb{L}^p(\mathcal{F}_T)^d$. This follows if we can show that G is a contraction mapping in $\mathbb{L}^p(\mathcal{F}_T)^d$. Choose $V, V' \in \mathbb{L}^p(\mathcal{F}_T)^d$. Then

$$\sup_{0\le t\le T} |Y_t^V - Y_t^{V'}| \le \sup_{0\le t\le T} |\mathbb{E}_t(V - V')| + \sup_{0\le t\le T} |F_t(Y^V, M^V) - F_t(Y^{V'}, M^{V'})|,$$

and therefore,

$$\begin{aligned}
\left\|Y^V - Y^{V'}\right\|_{\mathbb{S}^p} &\le \left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p + \left\|F(Y^V, M^V) - F(Y^{V'}, M^{V'})\right\|_{\mathbb{S}^p} \\
&\le \left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p + C_F \left\|Y^V - Y^{V'}\right\|_{\mathbb{S}^p} + C_F \left\|M^V - M^{V'}\right\|_{\mathbb{S}^p}.
\end{aligned}$$

This implies that

$$\left\|Y^V - Y^{V'}\right\|_{\mathbb{S}^p} \le \frac{1}{1 - C_F}\left(\left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p + C_F \left\|M^V - M^{V'}\right\|_{\mathbb{S}^p}\right),$$

and one obtains

$$\begin{aligned}
&\|G(V) - G(V')\|_p = \left\|F_T(Y^V, M^V) - F_T(Y^{V'}, M^{V'})\right\|_p \\
&\le \frac{C_F}{1 - C_F}\left(\left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p + C_F \left\|M - M^{V'}\right\|_{\mathbb{S}^p}\right) + C_F \left\|M^V - M^{V'}\right\|_{\mathbb{S}^p} \\
&= \frac{C_F}{1 - C_F}\left(\left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p + \left\|M^V - M^{V'}\right\|_{\mathbb{S}^p}\right).
\end{aligned}$$

By Doob's maximal inequality,

$$\left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p \le \frac{p}{p-1} \left\|V - V'\right\|_p.$$

Therefore,

$$\left\|M^V - M^{V'}\right\|_{\mathbb{S}^p} \le 2\left\|\sup_{0\le t\le T} |\mathbb{E}_t(V - V')|\right\|_p \le \frac{2p}{p-1} \left\|V - V'\right\|_p,$$

and

$$\|G(V) - G(V')\|_p \le \frac{3p}{p-1}\frac{C_F}{1 - C_F} \left\|V - V'\right\|_p.$$

It follows from the assumption that $\frac{3p}{p-1}\frac{C_F}{1-C_F} < 1$ that G is a contraction. □

For general F, C_F in above theorem should be small enough. In the case where C_F goes to 0 as T decreases, then for small enough T, we have a unique solution. If $F(Y,M) := \int_0^\cdot f(s,Y,M)ds$ satisfies certain conditions, we may iterate fixed point argument on small time intervals then paste the small time solutions to get a global solution. For example, if $f(s,Y,M)$ depends only on Y_s and the density process of M at time s, then iteration is possible because we can divide the BSDE to many BSDEs which have drivers with support on small time intervals. On the other hand, if f depends on the whole path of (Y,M), then such iteration is not possible. Let us consider BSDEs driven by Brownian motion and compensated Poisson process in the case where $p=2$. Consider the two mutually independent processes

- a n-dimensional Brownian motion W, and
- a Poisson random measure μ on $[0,T]\times E$, where $E := \mathbb{R}^m \diagdown \{0\}$ is equipped with Borel σ-algebra $\mathcal{B}(E)$, with a compensator $\nu(dt,de) = dt\lambda(de)$, such that

  $$(\tilde{\mu}([0,t]\times A))_{t\geq 0} := ((\mu-\nu)([0,t]\times A))_{t\geq 0}$$

  is a martingale for all $A\in\mathcal{B}(E)$ satisfying $\lambda(A)<\infty$. Here, λ is assumed to be a σ-finite measure on $(E,\mathcal{B}(E))$ satisfying

  $$\int_E (1\wedge |e|^2)\lambda(de) < \infty.$$

Let us denote $L^2(\tilde{\mu})$ be the set of mappings $U : \Omega\times[0,T]\times E \to \mathbb{R}^d$ which are $\mathcal{P}\otimes\mathcal{B}(E)$ measurable and such that

$$\|U\|_{L^2(\tilde{\mu})} := \left(\mathbb{E}\int_0^T\int_E |U_t(e)|^2\lambda(de)dt\right)^{1/2} < \infty.$$

Here $\mathcal{P}$ is the σ-algebra of $\mathbb{F}$-predictable subset of $\Omega\times[0,T]$.

Now it is well known that any square integrable $\mathbb{F}$-martingale has the following martingale representation: see Jacod [49], Tang and Li [79] or Kunita [56].

Lemma 2.2.2 (Martingale Representation)**.** *For all $M\in\mathbb{M}_0^2$, there exists a unique $(Z^M,U^M,N^M)\in\mathbb{H}^2\times L^2(\tilde{\mu})\times\mathbb{M}_0^2$ such that*

- *N^M is strongly orthogonal to W and $\tilde{\mu}$, that is, $(N^M)^iW^j$ and $(N^M)^i\tilde{\mu}([0,t],A)$ are martingale on $[0,T]$ for all $i=1,2,\cdots,d$, $j=1,2,\cdots,n$, and $A\in\mathcal{B}(E)$,*

- *and*

$$M_t = \int_0^t Z_s^M dW_s + \int_0^t \int_E U_s^M(e)\tilde{\mu}(ds, de) + N_t^M.$$

Moreover, $N^M = 0$ *if* $\mathbb{F}$ *is the filtration generated by* W *and* $\tilde{\mu}$ *and augmented.*

Now, let us define martingale representation operator $\mathcal{D} : \mathbb{M}_0^2 \to \mathbb{H}^2 \times L^2(\tilde{\mu})$ by $\mathcal{D}_t(M) := (Z_t^M, U_t^M)$ and consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathcal{D}_s(M))ds + M_T - M_t \tag{2.2.1}$$

where

$$f : [0,T] \times \Omega \times \mathbb{L}^2(\mathcal{F}_T)^d \times \mathbb{L}^2(\mathcal{F}_T)^{d\times n} \times L^2(\Omega \times E, \mathcal{F} \otimes \mathcal{B}(E), \mathbb{P} \otimes \lambda; \mathbb{R}^d) \to \mathbb{R}^d.$$

As a consequence of Theorem 2.2.1 we obtain the following result for BSDEs with generalized drivers.

Theorem 2.2.3. *Let* $\xi \in \mathbb{L}^2(\mathcal{F}_T)^d$ *and* f *satisfies the following conditions.*

(i) *For all* $(Y, Z, U) \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{\mu})$*, the process* $\int_0^t f(s, Y_s, Z_s, U_s)ds$*,* $0 \le t \le T$*, belongs to* $\mathbb{S}_0^2$.

(ii) *There exists a constant* $C \ge 0$ *such that*

$$\begin{aligned}&\|f(s, Y_s, Z_s, U_s) - f(s, Y_s', Z_s', U_s')\|_2\\ &\le C\left(\|Y_s - Y_s'\|_2 + \|Z_s - Z_s'\|_2 + \left(\mathbb{E}\int_E |U_s(e) - U_s'(e)|^2 \lambda(de)\right)^{1/2}\right)\end{aligned}$$

for all $s \in [0,T], (Y, Z, U), (Y', Z', U') \in \mathbb{S}^2 \times \mathbb{H}^2 \times L^2(\tilde{\mu})$.

Then the BSDE (2.2.1) *has a unique solution* (Y, M) *in* $\mathbb{S}^2 \times \mathbb{M}_0^2$.

Proof. Note that

$$\mathbb{E}|M_T|^2 = \mathbb{E}\int_0^T |Z_s^M|^2 ds + \mathbb{E}\int_0^T \int_E \left|U_s^M(e)\right|^2 \lambda(de)ds + \mathbb{E}|N_T|^2 \le \|M\|_{\mathbb{S}^2}^2$$

for all $M \in \mathbb{M}_0^2$. It follows from (ii) that there exists a constant $C' \ge 0$ such that

$$\|f(s, Y_s, Z_s, U_s) - f(s, Y_s', Z_s', U_s')\|_2^2 \le C'\left(\|Y_s - Y_s'\|_2^2 + \|Z_s - Z_s'\|_2^2 + \mathbb{E}\int_E |U_s(e) - U_s'(e)|^2 \lambda(de)\right)$$

for all $s \in [0,T]$. Choose $\delta > 0$ small enough so that

$$(C'\delta(\delta+1))^{1/2} < \frac{1}{7} \quad \text{and} \quad l := T/\delta \in \mathbb{N}.$$

Define

$$F_t(Y,M) := \int_0^t f(s, Y_s, \mathcal{D}_s(M)) 1_{[T-\delta,T]}(s) ds.$$

Then, for all $(Y,M), (Y',M') \in \mathbb{S}^2 \times \mathbb{M}_0^2$,

$$\begin{aligned}
&\|F(Y,M) - F(Y',M')\|_{\mathbb{S}^2}^2 = \mathbb{E} \sup_{0\le t\le T} \left| \int_0^t \left(f(s,Y_s,\mathcal{D}_s(M)) - f(s,Y'_s,\mathcal{D}_s(M')) \right) 1_{[T-\delta,T]}(s) ds \right|^2 \\
&\le \mathbb{E} \left(\int_{T-\delta}^T \left| f(s,Y_s,\mathcal{D}_s(M)) - f(s,Y'_s,\mathcal{D}_s(M')) \right| ds \right)^2 \\
&\le \delta \mathbb{E} \int_{T-\delta}^T \left| f(s,Y_s,\mathcal{D}_s(M)) - f(s,Y'_s,\mathcal{D}_s(M')) \right|^2 ds \\
&\le C'\delta \int_{T-\delta}^T \left(\|Y_s - Y'_s\|_2^2 + \left\| Z_s^M - Z_s^{M'} \right\|_2^2 + \mathbb{E} \int_E \left| U_s^M(e) - U_s^{M'}(e) \right|^2 \lambda(de) \right) ds \\
&\le C'\delta \left(\delta \|Y - Y'\|_{\mathbb{S}^2}^2 + \mathbb{E} \int_0^T |Z_s^M - Z_s^{M'}|^2 ds + \mathbb{E} \int_0^T \int_E \left| U_s^M(e) - U_s^{M'}(e) \right|^2 \lambda(de) ds \right) \\
&\le C'\delta(\delta+1) \left(\|Y - Y'\|_{\mathbb{S}^2}^2 + \|M - M'\|_{\mathbb{S}^2}^2 \right).
\end{aligned}$$

It follows that

$$\|F(Y,M) - F(Y',M')\|_{\mathbb{S}^2} \le (C'\delta(\delta+1))^{1/2} \left(\|Y - Y'\|_{\mathbb{S}^2} + \|M - M'\|_{\mathbb{S}^2} \right)$$

for all $(Y,M), (Y',M') \in \mathbb{S}^2 \times \mathbb{M}_0^2$. Since $(C'\delta(\delta+1))^{1/2} < 1/7$, one obtains from Theorem 2.2.1 that the BSDE

$$Y_t^{(1)} = \xi + \int_t^T f(s, Y_s^{(1)}, \mathcal{D}_s(M^{(1)})) 1_{[T-\delta,T]}(s) ds + M_T^{(1)} - M_t^{(1)}$$

has a unique solution $(Y^{(1)}, M^{(1)})$ in $\mathbb{S}^2 \times \mathbb{M}_0^2$. By the same argument it follows that the BSDE

$$Y_t^{(2)} = Y_{T-\delta}^{(1)} + \int_t^{T-\delta} f(s, Y_s^{(2)}, \mathcal{D}_s(M^{(2)})) 1_{[T-2\delta,T-\delta]}(s) ds + M_{T-\delta}^{(2)} - M_t^{(2)}$$

has a unique solution $(Y^{(2)}, M^{(2)})$ in $\mathbb{S}^2 \times \mathbb{M}_0^2$. Iterating this procedure, we get $(Y^{(k)}, M^{(k)})_{k=1,2,\cdots,l}$.

Now, let $(Y_t, M_t) := (Y_t^{(l)}, M_t^{(l)})$ for $0 \le t \le \delta$ and define

$$\begin{aligned} Y_t &:= Y_t^{(k)}; && T - k\delta \le t \le T - (k-1)\delta, \\ M_t - M_{k\delta} &:= M_t^{(l-k)} - M_{k\delta}^{(l-k)}; && k\delta < t \le (k+1)\delta. \end{aligned}$$

for $k = 1, 2, \cdots, l-1$. Since M is a martingale and $\mathbb{F}$ satisfies the usual conditions, M has a right-continuous version; we will maintain the notation M for this right-continuous version. Then, automatically, Y becomes right-continuous. It is easy to check that $\mathcal{D}_s(M) = \mathcal{D}_s(M^{(l-k)})$ and $dM_s = dM_s^{(l-k)}$ for $k\delta < s \le (k+1)\delta$. Therefore,

$$Y_t = Y_{T-k'\delta} + \int_t^{T-k'\delta} f(s, Y_s, \mathcal{D}_s(M))ds + M_{T-k'\delta} - M_t \quad \text{where } t \in (T-(k'+1)\delta, T-k'\delta]$$

for all $k' = 0, 1, \cdots, l-1$. This implies (Y, M) is a global solution to (2.2.1) in $\mathbb{S}^2 \times \mathbb{M}_0^2$. $\square$

For a fixed (s, ω), the driver f in Theorem 2.2.3 is a function of the random vector (Y_s, Z_s, U_s) and not only the deterministic vector $(Y_s(\omega), Z_s(\omega), U_s(\omega))$ as in standard BSDEs. For instance, it could depend on the distribution of (Y_s, Z_s, U_s). As an example, we derive an existence and uniqueness result for Mckean–Vlasov BSDEs, whose drivers depend on the distributions of Y_s and Z_s. We recall the definition of Wasserstein metric.

Definition 2.2.4. *Denote by $\mathcal{P}(\Xi)$ the set of all probability measures on a normed vector space $(\Xi, \|\cdot\|)$. The p-Wasserstein metric on*

$$\mathcal{P}_p(\Xi) := \left\{ \mu \in \mathcal{P}(\Xi) : \int_\Xi \|x\|^p \mu(dx) < \infty \right\}$$

is given by

$$\mathcal{W}_p(\mu, \mu') := \inf \left\{ \int_{\Xi\times\Xi} \|x - x'\|^p \nu(dx, dx') : \nu \in \mathcal{P}_p(\Xi \times \Xi) \text{ with marginals } \mu \text{ and } \mu' \right\}^{1/p}.$$

Let us denote the law of X as $\mathcal{L}(X)$. Then we can easily deduce the following remark.

Remark 2.2.5. *If $X, X' \in \mathbb{L}^p(\mathbb{R}^n)$ for some $n \in \mathbb{N}$, then*

$$\mathcal{W}_p(\mathcal{L}(X), \mathcal{L}(X')) \le \|X - X'\|_p.$$

If $U, U' \in L^p(\Omega \times E, \mathbb{P} \otimes \lambda; \mathbb{R}^n)$ *for some* $n \in \mathbb{N}$*,*

$$\mathcal{W}_p(\mathcal{L}(U), \mathcal{L}(U')) \leq \left(\mathbb{E} \int_E |U(e) - U'(e)|^p \lambda(de) \right)^{1/p}$$

The following result is a generalization of Buckdahn et al. [14] to the BSDE driven by a Brownian motion and a compensated Poisson process.

Corollary 2.2.6. *Let* $\xi \in \mathbb{L}^2(\mathcal{F}_T)^d$ *and consider a function*

$$f : [0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^{d\times n}) \times \mathcal{P}_2(L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^d)) \to \mathbb{R}^d$$

such that $f(\cdot, \cdot, y, z, u, \mu, \nu, \kappa)$ *is progressively measurable for fixed* $(y, z, u, \mu, \nu, \kappa) \in \mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^{d\times n}) \times \mathcal{P}_2(L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^d))$ *and satisfies the following two conditions:*

(i) $\int_0^T |f(., 0, 0, 0, \mathcal{L}(0), \mathcal{L}(0), \mathcal{L}(0)| ds \in \mathbb{L}^2(\mathcal{F}_T)^d$

(ii) *There exists a constant* $C \geq 0$ *such that*

$$\begin{aligned} &|f(s, y, z, u, \mu, \nu, \kappa) - f(s, y', z', u', \mu', \nu', \kappa)| \\ &\leq C \left(|y - y'| + |v - v'| + \left(\int_E |u(e) - u'(e)|^2 \lambda(de) \right)^{1/2} + \mathcal{W}_2(\mu, \mu') + \mathcal{W}_2(\nu, \nu') + \mathcal{W}_2(\kappa, \kappa') \right). \end{aligned}$$

Then the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, \mathcal{D}_s(M), \mathcal{L}(Y_s), \mathcal{L}(\mathcal{D}_s(M)))ds + M_T - M_t$$

has a unique solution (Y, M) *in* $\mathbb{S}^2 \times \mathbb{M}_0^2$*. Here,* $\mathcal{L}(\mathcal{D}_s(M))$ *denotes* $(\mathcal{L}(Z_s^M), \mathcal{L}(U_s^M))$.

Proof. Note that $f(s, \omega, y, z, u, \mu, \nu, \kappa)$ is jointly measurable because it is predictable (measurable) in (s, ω) and continuous in $(y, z, u, \mu, \nu, \kappa)$ on separable space. It follows from the assumptions that the condition (i) of Theorem 2.2.3 holds. So it is enough to show that

$$\begin{aligned} &\|f(s, Y_s, Z_s, U_s, \mathcal{L}(Y_s), \mathcal{L}(Z_s), \mathcal{L}(U_s)) - f(s, Y_s', Z_s', , U_s', \mathcal{L}(Y_s'), \mathcal{L}(Z_s'), \mathcal{L}(U_s')\|_2 \\ &\leq D \left(\|Y_s - Y_s'\|_2 + \|Z_s - Z_s'\|_2 + \left(\mathbb{E} \int_E |U_s(e) - U_s'(e)|^2 \lambda(de) \right)^{1/2} \right) \end{aligned}$$

for some constant $D \geq 0$. But this follows from condition (ii) since for $\nu = \mathcal{L}(Y_s, Y'_s)$ one has

$$\mathcal{W}_2^2(\mathcal{L}(Y_s), \mathcal{L}(Y'_s)) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - x'|^2 \nu(dx, dx') = \|Y_s - Y'_s\|_2^2 \,,$$

and analogously,

$$\mathcal{W}_2^2(\mathcal{L}(Z_s), \mathcal{L}(Z'_s)) \leq \|Z_s - Z'_s\|_2^2$$
$$\mathcal{W}_2^2(\mathcal{L}(U_s), \mathcal{L}(U_s)) \leq \mathbb{E} \int_E |U_s(e) - U'_s(e)|^2 \lambda(de).$$

□

2.3 Compact Mappings and Krasnoselskii Fixed Point Theorems

Another famous fixed point theorem is the Schauder fixed point theorem and its variants. Schauder-type fixed point theorems use the compactness and the continuity of a fixed point mapping to prove the existence of a fixed point. Unlike the Banach fixed point theorem, the uniqueness of fixed point is not automatically given. The main difficulty of using the Schauder fixed point theorem is to construct a compact mapping because infinite dimensional space is not locally compact. Even though it is hard to find general criterions for compactness in the space of stochastic processes, Da Prato proved that Sobolev space for random variables can be compactly embedded in $\mathbb{L}^2$ space. In this section, we will use this result to use the Krasnoselskii fixed point theorem.

We will consider the case where $p = 2$ and assume $\xi \in \mathbb{L}^2(\mathcal{F}_T)^d$ throughout this section. We follow the notions and assumptions provided in Da Prato (2006). Let $\Omega := L^2([0,T]; \mathbb{R}^n)$ be the Hilbert space of functions from $[0,T]$ to $\mathbb{R}^n$ endowed with the inner product $\langle x, y \rangle := \int_0^T x_t \cdot y_t dt$ for $x, y \in \Omega$. Here, $\cdot$ is the usual inner product in $\mathbb{R}^n$. We let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of Ω and we define a linear operator $Q : \Omega \to \Omega$ by $Qe_k = \lambda_k e_k$ where λ_k are positive with $\sum_{k=1}^{\infty} \lambda_k < \infty$. On Ω, we endow Borel algebra $\mathcal{F}$ and Gaussian measure $\mathbb{P}$ such that mean zero and covariance Q with $\ker Q = \{0\}$. Moreover, we assume $\mathcal{F}_T \subset \mathcal{F}$.

Definition 2.3.1. *Consider the mapping*

$$W : Q^{1/2}(\Omega) \subset \Omega \to \mathbb{L}^2(\mathcal{F}), z \mapsto W_z, W_z(x) = \left\langle x, Q^{-1/2} z \right\rangle.$$

Since W is an isometry and $Q^{1/2}(\Omega)$ is dense in Ω, W can be uniquely extended to Ω and we will keep notation W for this extended mapping. This W is called a white noise mapping.

Consider an orthonormal basis $\left\{f_j \in \mathbb{R}^n : (f_j)^i = 1 \text{ if } i = j \text{ and } 0 \text{ otherwise}\right\}$. On $(\Omega, \mathcal{F}, \mathbb{P})$, $W = (W^1, \cdots, W^n)$ where $W_t^j := W_{1_{[0,t]}f_j}$ is a well-defined Brownian motion (see Theorem 3.17 of Da Prato [23] for more detail).

Now, let us define Sobolev space in $\mathbb{L}^2$. Let $C_b(\Omega;\mathbb{R})$ be the Banach space of all uniformly continuous and bounded mappings $\varphi : \Omega \to \mathbb{R}$ endowed with the sup-norm

$$\|\varphi\|_\infty = \sup_{\omega\in\Omega} |\varphi(\omega)| .$$

Definition 2.3.2. *Let $\mathcal{E}(\Omega)$ be the linear span of all real and imaginary parts of functions $\varphi_h, h \in \Omega$ in $C_b(\Omega;\mathbb{R})$, where*

$$\varphi_h(\omega) = e^{i\langle h,\omega\rangle}.$$

For any $\varphi \in \mathcal{E}(\Omega)$ and any $k \in \mathbb{N}$, we denote $\mathbf{D}_k\varphi$ to be the derivative of φ in the direction of e_k, namely

$$\mathbf{D}_k\varphi(\omega) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\varphi(\omega + \varepsilon e_k) - \varphi(\omega)\right), \quad \omega \in \Omega.$$

Then, the mapping

$$\mathbf{D} : \mathcal{E}(\Omega) \subset \mathbb{L}^2 \to L^2(\Omega, \mathbb{P}; \Omega), \varphi \mapsto \mathbf{D}\varphi$$

is closable. We will maintain the notation $\mathbf{D}$ for the closure of $\mathbf{D}$. We shall denote the domain of $\mathbf{D}$ by $\mathbb{W}^{1,2}$ and call it Sobolev space. The Sobolev space $\mathbb{W}^{1,2}$, endowed with the inner product

$$\langle\varphi, \psi\rangle_{\mathbb{W}^{1,2}} := \mathbb{E}\left(\varphi\psi + \langle\mathbf{D}\varphi, \mathbf{D}\psi\rangle\right)$$

is a Hilbert space.

The closedness of $\mathbf{D}$ is proved in Appendix A.

Remark 2.3.3. *If one consider Wiener-Chaos decomposition for a random variable, one can easily check that $\mathbf{D}$ is not Malliavin derivative. Moreover, one can prove that $\mathbb{W}^{1,2} \subset \mathbb{D}^{1,2}$ where $\mathbb{D}^{1,2}$ is Wiener-Sobolev space with Malliavin derivative; see Appendix A.2.*

Note that above definitions can be easily extended to the case where $\varphi : \Omega \to \mathbb{R}^d$ in coordinate-by-coordinate sense. After this extension, we denote $\mathbb{L}^2 := \left(\mathbb{L}^2(\Omega, \mathbb{P})\right)^d$ and $\mathbb{W}^{1,2} :=$

$\left(\mathbb{W}^{1,2}(\Omega,\mathbb{P})\right)^d$ for appropriate dimension d. The norm in $\mathbb{L}^2$ and $\mathbb{W}^{1,2}$ are defined by

$$\|\varphi\|_2^2 := \sum_{i=1}^{d} \mathbb{E}\left|\varphi^i\right|^2$$
$$\|\psi\|_{\mathbb{W}^{1,2}}^2 := \sum_{i=1}^{d} \mathbb{E}\left(|\psi^i|^2 + \left\langle \mathbf{D}\psi^i, \mathbf{D}\psi^i \right\rangle\right)$$

for $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^d) \in \mathbb{L}^2$ and $\psi = (\psi^1, \psi^2, \cdots, \psi^d) \in \mathbb{W}^{1,2}$. We will need the following propositions which are Proposition 10.11, Theorem 10.16, and Theorem 10.25 of Da Prato [23], respectively. The most important Proposition 2.3.6 will be proved in Appendix A. For the proofs of Proposition 2.3.5 and 2.3.8, see Da Prato [23].

Definition 2.3.4. *We call a random variable φ is L-Lipschitz in ω if*

$$|\varphi(\omega) - \varphi(\omega')| \leq L\sqrt{\langle \omega - \omega', \omega - \omega' \rangle}$$

for all $\omega, \omega' \in \Omega$.

Proposition 2.3.5. *If φ is a L-Lipschitz random variable, then φ is in $\mathbb{W}^{1,2}$ with $\mathbb{E}\langle \mathbf{D}\varphi, \mathbf{D}\varphi \rangle \leq L^2$.*

Proposition 2.3.6 (Compact Embedding Theorem)**.** *$\mathbb{W}^{1,2}$ is compactly embedded to $\mathbb{L}^2$. That is, any bounded sequence in $\mathbb{W}^{1,2}$ has a subsequence which is convergent in $\mathbb{L}^2$.*

Remark 2.3.7. *The above proposition is equivalent to the following statement:*
For any $C \in \mathbb{R}_+$, there exists a compact set $\mathcal{K}$ in $\mathbb{L}^2$ such that $\left\{V \in \mathbb{W}^{1,2} : \|V\|_{\mathbb{W}^{1,2}} \leq C\right\} \subset \mathcal{K}$.

Proposition 2.3.8 (Poincare inequality)**.** *For all $\varphi \in \mathbb{W}^{1,2}$, we have*

$$\mathbb{E}\left|\varphi - \mathbb{E}\varphi\right|^2 \leq \lambda \mathbb{E}\langle \mathbf{D}\varphi, \mathbf{D}\varphi \rangle .$$

The following is an obvious corollary of above propositions.

Corollary 2.3.9. *The set of L-Lipschitz random variables with mean zero is compact in $\mathbb{L}^2$.*

Let us remind the Krasnoselskii fixed point theorem (Smart [78]).

Theorem 2.3.10 (Krasnoselskii fixed point theorem)**.** *Assume that $\mathcal{C} \subset \mathbb{L}^2(\mathcal{F})$ is a closed convex nonempty set. Assume that $G^1, G^2 : \mathcal{C} \to \mathbb{L}^2(\mathcal{F})$ satisfy the following conditions*

- *$G^1(v) + G^2(v') \in \mathcal{C}$ for all $v, v' \in \mathcal{C}$.*

- *G^1 is a contraction.*
- *G^2 is continuous and $G^2(\mathcal{C})$ is contained in a compact set.*

Then, $G^1 + G^2$ has a fixed point in $\mathcal{C}$.

Using above previous results, we may proceed to the main results of this section. Let us consider the following conditions for a constant $C_F \in [0, 1/4)$.

(A1) For all $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$, $F(Y, M) \in \mathbb{S}_0^2$ and

$$F(Y, M) = F^1(Y, M) + F^2(Y, M).$$

(A2) For all $Y, Y' \in \mathbb{S}^2$, and $M \in \mathbb{M}_0^2$,

$$\|F(Y, M) - F(Y', M)\|_{\mathbb{S}^2} \leq C_F \|Y - Y'\|_{\mathbb{S}^2}$$

(A3) For all $(Y, M), (Y', M') \in \mathbb{S}^2 \times \mathbb{M}_0^2$,

$$\left\|F_T^1(0,0)\right\|_2 < \infty \quad \text{and}$$
$$\left\|F_T^1(Y, M) - F_T^1(Y', M')\right\|_2 \leq C_F \left(\|Y_0 - Y_0'\|_2 + \|M - M'\|_{\mathbb{S}^2}\right).$$

(A4) There exist $k \in \mathbb{R}_+$ and nondecreasing functions $\rho, \rho' : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $(Y, M) \in \left\{(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2 : \|Y_0\|_2 + \|M\|_{\mathbb{S}^2} \leq k\right\}$,

$$F_T^2(Y, M) \in \mathbb{W}^{1,2}, \quad \left|\mathbb{E} F_T^2(Y, M)\right| \leq \rho(k),$$
$$\text{and} \quad \sum_{i=1}^d \mathbb{E}\left\langle \mathbf{D} F_T^{2,i}(Y, M), \mathbf{D} F_T^{2,i}(Y, M)\right\rangle \leq \rho'(k).$$

In addition, $(F_T^2 \circ \psi) : \mathbb{L}^2(\mathcal{F}_T)^d \to \mathbb{L}^2(\mathcal{F}_T)^d$ is continuous.

Proposition 2.3.11. *Assume* (A2) and (A3). *If $F : (Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2 \mapsto F(Y, M) \in \mathbb{S}_0^2$ is continuous in M , then (S) is satisfied and $(F_T^2 \circ \psi) : L^2(\mathcal{F}_T)^d \to L^2(\mathcal{F}_T)^d$ is continuous.*

Proof. By Lemma 2.1.3, (S) is satisfied. For given $(y, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$, let $Y^{y,M}$ be the solution of the following stochastic equation.

$$Y_t^{y,M} = y - F_t(Y^{y,M}, M) - M_t.$$

Note that $Y^{y,M}$ is well-defined because of (A2). By the definition of $Y^{y,M}$ and the assumption (A2), for $(y, M), (y, M') \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$, we have

$$
\begin{aligned}
Y_t^{y,M} - Y_t^{y',M'} &= (y - y') - \Big(F_t(Y^{y,M}, M) - F_t(Y^{y',M'}, M')\Big) - (M_t - M_t') \\
\|Y^{y,M} - Y^{y',M'}\|_{\mathbb{S}^2} & \\
&\leq \|y - y'\|_2 + \|M - M'\|_{\mathbb{S}^2} + C_F \left\|Y^{y,M} - Y^{y',M'}\right\|_{\mathbb{S}^2} + \left\|F(Y^{y',M'}, M) - F(Y^{y',M'}, M')\right\|_{\mathbb{S}^2} \\
\|Y^{y,M} - Y^{y',M'}\|_{\mathbb{S}^2} &\leq \frac{1}{1 - C_F}\left(\|y - y'\|_2 + \|M - M'\|_{\mathbb{S}^2} + \left\|F(Y^{y',M'}, M) - F(Y^{y',M'}, M')\right\|_{\mathbb{S}^2}\right)
\end{aligned}
$$

Since we assumed $F : \mathbb{S}^2 \times \mathbb{M}_0^2 \to \mathbb{S}_0^2$ is continuous, $(y, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2 \mapsto Y^{y,M} \in \mathbb{S}^2$ is continuous. Moreover, $V \in \mathbb{L}^2(\mathcal{F}_T)^d \mapsto (\mathbb{E}_0 V, (\mathbb{E}_0 V - \mathbb{E}_t V)_{t\in[0,T]}) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$ is continuous by Doob's maximal inequality. Therefore,

$$
\psi : V \mapsto \psi(V) := \left(Y^{\mathbb{E}_0 V, (\mathbb{E}_0 V - \mathbb{E}_t V)_{t\in[0,T]}}, (\mathbb{E}_0 V - \mathbb{E}_t V)_{t\in[0,T]}\right)
$$

is continuous. Moreover, note that

$$
\begin{aligned}
\left\|F_T^2(\psi(V)) - F_T^2(\psi(V'))\right\|_2 &\leq \|F_T(\psi(V)) - F_T(\psi(V'))\|_2 + \left\|F_T^1(\psi(V)) - F_T^1(\psi(V'))\right\|_2 \\
&\leq \|F(\psi(V)) - F(\psi(V'))\|_{\mathbb{S}^2} + C_F\left(2\,\|\mathbb{E}_0 V - \mathbb{E}_0 V'\|_2 + \|\mathbb{E}_t V - \mathbb{E}_t V'\|_{\mathbb{S}^2}\right)
\end{aligned}
$$

and F is continuous. Therefore, $F_T^2 \circ \psi$ is continuous and our claim is proved. □

Theorem 2.3.12. *Assume that* (A1)–(A4). *In addition, assume that*

$$
\|\xi\|_2 + \left\|F_T^1(0,0)\right\|_2 + C_F k + \sqrt{\lambda\rho'(k)} + \rho(k) \leq k/4. \tag{2.3.1}
$$

Then, BSE (2.1.1) *has a solution* $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$.

Proof. Note that (S) is satisfied by (A2). Define

$$
\begin{aligned}
\mathcal{C} &:= \left\{V \in \mathbb{L}^2(\mathcal{F}_T)^d : \|V\|_2 \leq l\right\} \\
G^1(V) &:= \xi + F_T^1(Y^V, M^V) \\
G^2(V) &:= F_T^2(Y^V, M^V) \\
G(V) &:= G^1(V) + G^2(V)
\end{aligned}
$$

where $l = k/4$.

(Step 1) Let us show $G^1(V) + G^2(V') \in \mathcal{C}$ for all $V, V' \in \mathcal{C}$. Note that

$$\left\|G^1(V)\right\|_2 \leq \|\xi\|_2 + \left\|F_T^1(Y^V, M^V)\right\|_2 \leq \|\xi\|_2 + \left\|F_T^1(0,0)\right\|_2 + C_F \left\|y^V\right\|_2 + C_F \left\|M^V\right\|_{\mathbb{S}^2}$$
$$\left\|M^V\right\|_{\mathbb{S}^2} \leq \left(\mathbb{E}\left|\mathbb{E}_0 V\right|^2\right)^{1/2} + \left(\mathbb{E} \sup_{0\leq t\leq T} \left|\mathbb{E}_t V\right|^2\right)^{1/2} = 3\left\|V\right\|_2 \leq 3l$$

by (A3) and Doob's maximal inequality. Therefore, for $V \in \mathcal{C}$,

$$\left\|y^V\right\|_2 + \left\|M^V\right\|_{\mathbb{S}^2} \leq 4l = k$$
$$\left\|G^1(V)\right\|_2 \leq \|\xi\|_2 + \left\|F_T^1(0,0)\right\|_2 + C_F l + 3C_F l \leq \|\xi\|_2 + \left\|F_T^1(0,0)\right\|_2 + C_F k.$$

On the other hand,

$$\left\|G^2(V')\right\|_2 \leq \left\|G^2(V') - \mathbb{E}G^2(V')\right\|_2 + \left\|\mathbb{E}G^2(V')\right\|_2 \leq \sqrt{\lambda\rho'(k)} + \rho(k)$$

Therefore, the claim is proved.

(Step 2) $G^1(V)$ is a contraction mapping in $\mathbb{L}^2(\mathcal{F}_T)^d$ because

$$\begin{aligned}\left\|G^1(V) - G^1(V')\right\|_2 &\leq \left\|F_T^1(Y^V, M^V) - F_T^1(Y^{V'}, M^{V'})\right\|_2 \\ &\leq C_F\left(\left\|Y_0^V - Y_0^{V'}\right\|_2 + \left\|M^V - M^{V'}\right\|_{\mathbb{S}^2}\right) \\ &\leq C_F\left(2\left\|V - V'\right\|_2 + \left(\mathbb{E}\sup_t \left|\mathbb{E}_t(V - V')\right|^2\right)^{1/2}\right) \\ &\leq 4C_F\left\|V - V'\right\|_2.\end{aligned}$$

(Step 3) Lastly, from the condition (A4), $G^2(\mathcal{C})$ is contained in a compact set of $\mathbb{L}^2(\mathcal{F}_T)^d$. Moreover, $G^2 = F_T^2 \circ \psi$ is continuous by our assumption (A4). Therefore, by the Krasnoselskii fixed point theorem and Lemma 2.1.2, there exists a solution to BSE (2.1.1). □

The condition (A2) and (2.3.1) are required to guarantee the well-posedness of stochastic equation

$$Y_t = y^V - F_t(Y, M^V) - M_t^V.$$

for given $(y^V, M_t^V) := (\mathbb{E}_0 V, \mathbb{E}_0 V - \mathbb{E}_t V)$ and to show $G^1(\mathcal{C}) + G^2(\mathcal{C}) \in \mathcal{C}$ for $\mathcal{C} := \{V : \|V\|_2 \leq k/4\}$. In the case where $F(Y, M)$ depends only on (Y_0, M), ρ has sublinear growth, and ρ' has subquadratic growth, we can omit these conditions.

Proposition 2.3.13. *Assume* (A1) *and* (A3)*. In addition, assume that* $F_T^2(Y, M) = H(Y_0, M)$ *where* H *is continuous with respect to the norm in* $\mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{L}^2(\mathcal{F}_T)^d$*,* H *is* L*-Lipschitz for any given* $(Y_0, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$*, and* $|\mathbb{E}H(\mathbb{E}_0 V, (\mathbb{E}_0 V - \mathbb{E}_t V)_t)|$ *has sublinear growth with respect to* $\|V\|_2$*. Then, BSE* (2.1.1) *has a solution* $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$*.*

Proof. Since H is L-Lipschitz for any given (Y_0, M), $H(Y_0, M) \in \mathbb{W}^{1,2}$ and

$$\sum_{i=1} \mathbb{E}\left\langle \mathbf{D}H^i(Y_0, M), \mathbf{D}H^i(Y_0, M)\right\rangle \leq L^2$$

by Proposition 2.3.5. Also, since (y^V, M^V) are continuous in V, $(F_T^2 \circ \psi)(V)$ is continuous. Therefore, (A4) is satisfied. By letting k large enough, (2.3.1) is satisfied. $\square$

If F^1 and F^2 do not depend on Y, then we have the following simple version of the above theorem.

Theorem 2.3.14. *Assume that there exist* $k \in \mathbb{R}_+$*, nondecreasing functions* $\rho, \rho' : \mathbb{R}_+ \to \mathbb{R}_+$*, and a constant* $C_F \in [0, 1/2)$ *which satisfy the following conditions.*

(B1) For all $M \in \mathbb{M}_0^2$*,* $F^1(M), F^2(M) \in \mathbb{S}_0^2$ *and*

$$F(M) = F^1(M) + F^2(M).$$

(B2) For all $M, M' \in \mathbb{M}_0^2$*,*

$$\left\|F_T^1(M) - F_T^1(M')\right\|_2 \leq C_F \left\|M - M'\right\|_{\mathbb{S}^2}.$$

(B3) For all $M \in \left\{M \in \mathbb{M}_0^2 : \|M\|_{\mathbb{S}^2} \leq k\right\}$*,* $F_T^2(M)$ *is continuous in* M *and*

$$F_T^2(M) \in \mathbb{W}^{1,2} \quad \textit{and} \quad \sum_{i=1} \mathbb{E}\left\langle \mathbf{D}F_T^{2,i}(M), \mathbf{D}F_T^{2,i}(M)\right\rangle \leq \rho'(k)$$

In addition, assume that

$$\|\xi\|_2 + \left\|F_T^1(0)\right\|_2 + C_F k + \sqrt{\lambda \rho'(k)} \leq k/2 \tag{2.3.2}$$

Then, BSE (2.1.1) *has a solution* $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$*.*

Proof. Let

$$\begin{aligned}
\mathcal{C} &:= \left\{V \in \mathbb{L}^2(\mathcal{F}_T)^d : \|V\|_2 \le k/2\right\} \\
M^V_t &:= \mathbb{E}_0 V - \mathbb{E}_t V \\
H^1(V) &:= \xi + F^1_T(M^V) - \mathbb{E}\left(\xi + F^1_T(M^V)\right) \\
H^2(V) &:= F^2_T(M^V) - \mathbb{E}F^2_T(M^V)
\end{aligned}$$

If there exists a $V \in \mathbb{L}^2(\mathcal{F}_T)^d$ such that $H^1(V) + H^2(V) = V$, (Y^V, M^V) is the solution of BSE (2.1.1) by Remark 2.1.4 .

(Step 1) Let us show $H^1(V) + H^2(V') \in \mathcal{C}$ for all $V, V' \in \mathcal{C}$. For $V \in \mathcal{C}$, we have $\left\|M^V\right\|_{\mathbb{S}^2} \le 2\|V\|_2$ by Doob's maximal inequality and

$$\left\|H^1(V)\right\|_2 \le \left\|\xi + F^1_T(M^V)\right\|_2 \le \|\xi\|_2 + \left\|F^1_T(0)\right\|_2 + C_F k$$

and, by Proposition 2.3.8 (see Step 3 of this proof),

$$\left\|H^2(V')\right\|_2 \le \sqrt{\lambda \rho'(k)}$$

Therefore, by (2.3.2), we get $H^1(V) + H^2(V') \in \mathcal{C}$.

(Step 2) H^1 is a contraction mapping in $\mathbb{L}^2(\mathcal{F}_T)^d$ because

$$\left\|H^1(V) - H^1(V')\right\|_2 \le \left\|F^1_T(M^V) - F^1_T(M^{V'})\right\|_2 \le C_F \left\|M^V - M^{V'}\right\|_{\mathbb{S}^2} \le 2C_F \|V - V'\|_2 .$$

(Step 3) $H^2(V)$ is continuous because $V \in \mathcal{C} \mapsto M^V \in \{M : \|M\|_{\mathbb{S}^2} \le k\}$ is continuous and F^2_T is continuous in $\{M : \|M\|_{\mathbb{S}^2} \le k\}$. Also, note that for all $V \in \mathcal{C}$, $H^2(V) \in \mathbb{W}^{1,2}$ and

$$\sum_{i=1}^d \mathbb{E}\left\langle \mathbf{D}H^{2,i}(V), \mathbf{D}H^{2,i}(V)\right\rangle = \sum_{i=1}^d \mathbb{E}\left\langle \mathbf{D}F^{2,i}_T(M^V), \mathbf{D}F^{2,i}_T(M^V)\right\rangle \le \rho'(k).$$

Moreover, since $\mathbb{E}H^2(V) = 0$ for all $V \in \mathcal{C}$,

$$\left\|H^2(V)\right\|^2_{\mathbb{W}^{1,2}} \le \mathbb{E}\left|H^2(V)\right|^2 + \sum_{i=1}^d \mathbb{E}\left\langle \mathbf{D}H^{2,i}(V), \mathbf{D}H^{2,i}(V)\right\rangle \le (\lambda+1)\rho'(k)$$

by Proposition 2.3.8. This implies $\left\{H^2(V) : V \in \mathcal{C}\right\}$ is contained in a compact set of $\mathbb{L}^2(\mathcal{F}_T)^d$. In sum, by application of the Krasnoselskii fixed point theorem, there exists a solution to BSE

(2.1.1). ☐

Let us apply above results to BSDEs.

Proposition 2.3.15. *Assume that*

- $f_1 : \Omega \times [0,T] \times \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2 \to \mathbb{R}^d$ *satisfies*

$$\|f_1(s,u,v) - f_1(s,u',v')\|_2^2 \le C^2 \left(\|u-u'\|_2^2 + \|v-v'\|_{\mathbb{S}^2}^2 \right)$$

  with $CT < 1/4$*,* $f_1(\cdot, Y_0, M)$ *is predictable for any* $(Y_0, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$*, and* $\|f_1(s,0,0)\|_{\mathbb{H}^2} < \infty$.

- $f_2 : (\omega, s, Y_0, M) \in \Omega \times [0,T] \times \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2 \mapsto f_2(\omega, s, Y_0, M) \in \mathbb{R}^d$ *is continuous in* (Y_0, M) *and uniformly L-Lipschitz for all* $(s, Y_0, M) \in [0,T] \times \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$*. Moreover,*

$$\left\| \int_0^T |f_2(s, Y_0, M)| ds \right\|_2 < \infty$$

  for all $(Y_0, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$ *and* $f_2(\cdot, Y_0, M)$ *is predictable for any* $(Y_0, M) \in \mathbb{L}^2(\mathcal{F}_0)^d \times \mathbb{M}_0^2$.

- $\sup_{s\in[0,T]} |\mathbb{E} f_2(s, \mathbb{E}_0 V, (\mathbb{E}_0 V - \mathbb{E}_t V)_t)|$ *has sublinear growth with respect to* $\|V\|_2$.

Then, there is a solution (Y, M) *to the following BSDE.*

$$Y_t = \xi + \int_t^T (f_1(s, Y_0, M) + f_2(s, Y_0, M))\, ds + M_T - M_t$$

Proof. Let

$$\begin{aligned}
F_t(Y,M) &:= \int_0^t f_1(s, Y_0, M) ds + \int_0^t f_2(s, Y_0, M) ds \\
F_t^1(Y,M) &:= \int_0^t f_1(s, Y_0, M) ds \\
F_t^2(Y,M) &:= \int_0^t f_2(s, Y_0, M) ds
\end{aligned}$$

First of all, note that we can easily check the condition (A1) by our assumptions. Also, we can remove the condition (A2) by the same argument follows after the proof of Theorem 2.3.12. On the other hand,

$$\begin{aligned}\left\|F_T^1(Y,M)-F_T^1(Y',M')\right\|_2^2 &\leq T\mathbb{E}\int_0^T \left|f_1(s,Y_0,M)-f_1(s,Y_0',M')\right|^2 ds\\ &\leq C^2T\int_0^T \left(\|Y_0-Y_0'\|_2^2+\|M-M'\|_{\mathbb{S}^2}^2\right) ds\\ &\leq C^2T^2\left(\|Y_0-Y_0'\|_2^2+\|M-M'\|_{\mathbb{S}^2}^2\right)\end{aligned}$$

and

$$\left\|F_T^1(0,0)\right\|_2^2=\mathbb{E}\left|\int_0^T f_1(s,0,0)ds\right|^2 \leq T\mathbb{E}\int_0^T |f_1(s,0,0)|^2\, ds = T\,\|f_1(s,0,0)\|_{\mathbb{H}^2}^2<\infty.$$

Therefore, (A3) is satisfied with $C_F = CT < 1/4$. Let us check (A4). Since f_2 is L-Lipschitz and continuous in (Y_0, M), $F_T^2(Y,M)$ is LT-Lipschitz and $F_T^2 \circ \psi$ is continuous. Therefore, all the conditions in Proposition 2.3.13 are satisfied. Since (2.1.1) is the BSDE we want, the claim is proved. □

Let us consider a special case where $f_1(s, Y_0, M)$ and $f_2(s, Y_0, M)$ does not depend on Y_0.

Proposition 2.3.16. *Assume that*

- $f_1 : \Omega \times [0,T] \times \mathbb{M}_0^2 \to \mathbb{R}^d$ *satisfies*

$$\|f_1(s,v)-f_1(s,v')\|_2 \leq C\,\|v-v'\|_{\mathbb{S}^2}$$

  with $CT < 1/2$*,* $f_1(\cdot, M)$ *is predictable for any* $M \in \mathbb{M}_0^2$*, and* $\|f_1(s,0)\|_{\mathbb{H}^2} < \infty$*.*

- $f_2 : (\omega, s, M) \in \Omega \times [0,T] \times \mathbb{M}_0^2 \mapsto f_2(\omega, s, M) \in \mathbb{R}^d$ *is continuous in* M *and uniformly* L*-Lipschitz for all* $(s,M) \in [0,T] \times \mathbb{M}_0^2$*. Moreover,*

$$\left\|\int_0^T |f_2(s,M)|ds\right\|_2 < \infty$$

  for all $M \in \mathbb{M}_0^2$ *and* $f_2(\cdot, M)$ *is predictable for any* $M \in \mathbb{M}_0^2$*.*

Then, BSDE

$$Y_t = \xi + \int_t^T \left(f_1(s,M)+f_2(s,M)\right) ds + M_T - M_t$$

has a solution $(Y,M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$*.*

Proof. Let

$$F_t^1(M) := \int_0^t f_1(s, M)ds$$
$$F_t^2(M) := \int_0^t f_2(s, M)ds$$
$$F(M) := F^1(M) + F^2(M).$$

Then, (B1)–(B2) are satisfied because

$$\left\|F_T^1(M) - F_T^1(M')\right\|_2^2 \le T\mathbb{E}\int_0^T |f(s,M) - f(s,M')|^2 ds \le C^2 T \int_0^T \|M - M'\|_{\mathbb{S}^2}^2 \, ds$$
$$\le C^2 T^2 \|M - M'\|_{\mathbb{S}^2}^2 .$$

Moreover, since $F_T^2(M)$ is continuous in M and uniformly LT-Lipschitz for any given M, (B3) is satisfied with $\rho' \equiv (LT)^2$. Therefore, (2.3.2) is satisfied if we take k large enough. □

Now let us prove the uniqueness result. Note that the driver is path dependent functional and therefore, the conventional Banach fixed point method only works when T is small enough.

Proposition 2.3.17. *Assume that $f : (\omega, s, M) \in \Omega \times [0,T] \times \mathbb{M}_0^2 \mapsto f(\omega, s, M) \in \mathbb{R}^d$ is uniformly L-Lipschitz for all $(s, M) \in [0,T] \times \mathbb{M}_0^2$, $f(s, M)$ is continuous in M, $f(\cdot, M)$ is predictable for any $M \in \mathbb{M}_0^2$, $f(\cdot, 0) \in \mathbb{H}^2$, and*

$$\|f(s,M) - f(s,M')\|_2^2 \le C^2 \mathbb{E} \sup_{s \le u \le T} |(M_u - M'_u) - (M_s - M'_s)|^2$$

for all $(s, M, M') \in [0,T] \times \mathbb{M}_0^2 \times \mathbb{M}_0^2$. Then, BSDE

$$Y_t = \xi + \int_t^T f(s, M)ds + M_T - M_t$$

has a unique solution $(Y, M) \in \mathbb{S}^2 \times \mathbb{M}_0^2$.

Proof. Note that

$$\left\|\int_0^T |f(s,M)|ds\right\|_2^2 \leq T\int_0^T \mathbb{E}|f(s,M)|^2 ds \leq 2C^2\int_0^T \mathbb{E}\sup_{u\in[s,T]}|M_u - M_s|^2 ds + 2\left\|f(\cdot,0)\right\|_{\mathbb{H}^2}^2$$
$$\leq 4C^2\int_0^T \mathbb{E}\left(\sup_{u\in[s,T]}|M_u|^2 + |M_s|^2\right) ds + 2\left\|f(\cdot,0)\right\|_{\mathbb{H}^2}^2$$
$$\leq 8C^2\int_0^T \mathbb{E}\sup_{u\in[s,T]}|M_u|^2 ds + 2\left\|f(\cdot,0)\right\|_{\mathbb{H}^2}^2$$
$$\leq 32C^2\int_0^T \mathbb{E}|M_T|^2 ds + 2\left\|f(\cdot,0)\right\|_{\mathbb{H}^2}^2$$
$$\leq 32C^2 T\mathbb{E}|M_T|^2 + 2\left\|f(\cdot,0)\right\|_{\mathbb{H}^2}^2 < \infty.$$

Therefore, the existence of solution follows from Proposition 2.3.16. Let (Y,M) and (Y',M') be solutions of the BSDE. Then, for $\varepsilon = 1/(8C)$ and $t = T-\varepsilon$,

$$Y_t - Y_t' = \int_t^T (f(s,M) - f(s,M'))\, ds + (M_T - M_t) - (M_T' - M_t')$$

If we take $\mathbb{E}_t$ on both side, we get $Y_t - Y_t' = \mathbb{E}_t\int_t^T (f(s,M) - f(s,M'))\, ds$ and therefore,

$$(M_T - M_t) - (M_T' - M_t') = \int_t^T (f(s,M) - f(s,M'))\, ds - \mathbb{E}_t\int_t^T (f(s,M) - f(s,M'))\, ds$$
$$\|(M_T - M_t) - (M_T' - M_t')\|_2 \leq \left\|\int_t^T (f(s,M) - f(s,M'))\, ds\right\|_2 + \left\|\mathbb{E}_t\int_t^T (f(s,M) - f(s,M'))\, ds\right\|_2$$
$$\leq 2\sqrt{\varepsilon}\sqrt{\mathbb{E}\int_t^T |f(s,M) - f(s,M')|^2\, ds}$$
$$\leq 2C\sqrt{\varepsilon}\sqrt{\int_t^T \mathbb{E}\sup_{s\leq u\leq T}|(M_u - M_u') - (M_s - M_s')|^2\, ds}$$
$$\leq 4C\sqrt{\varepsilon}\sqrt{\int_t^T \mathbb{E}\,|(M_T - M_T') - (M_s - M_s')|^2\, ds}$$
$$\leq 4C\sqrt{\varepsilon}\sqrt{\int_t^T \mathbb{E}\,|(M_T - M_T') - (M_t - M_t')|^2\, ds}$$
$$\leq 4\varepsilon C\,\|(M_T - M_T') - (M_t - M_t')\|_2$$

Since $4C\varepsilon = 1/2 < 1$, we know that $M_T - M_t = M_T' - M_t'$. Therefore, $f(s,M) = f(s,M')$ for $s\in[t,T]$ and this implies $Y_t = Y_t'$. We can repeat above argument for $[T-2\varepsilon, T-\varepsilon]$ using the fact that $f(s,M) = f(s,M')$ for $s\in[T-\varepsilon,T]$. By repeating the argument to time 0, we get $(Y,M) = (Y',M')$ and the uniqueness of solution is proven. □

Assume that $\mathbb{F} = \mathbb{F}^W$ and Z be the density process of martingale representation of M. If the driver $f(s, M)$ depends only on $Z_s \in \mathbb{L}^2(\mathcal{F}_T)$, then one can use similar argument used in Theorem 2.2.3 to remove the condition $CT < 1/2$ from Proposition 2.3.16.

Proposition 2.3.18. *Assume the following conditions*

- $\mathbb{F} = \mathbb{F}^W$
- $g_1 : \Omega \times [0,T] \times \mathbb{L}^2(\mathcal{F}_T)^{d\times n} \to \mathbb{R}^d$ *satisfies*

$$\|g_1(s,v) - g_1(s,v')\|_2 \leq C \|v - v'\|_2 ,$$

  $g_1(\cdot, Z.)$ *is predictable for any* $Z \in \mathbb{H}^2$, *and* $\|g(s,0)\|_{\mathbb{H}^2} < \infty$.

- $g_2 : (\omega, s, Z_s) \in \Omega \times [0,T] \times \mathbb{L}^2(\mathcal{F}_s) \mapsto g_2(\omega, s, Z_s) \in \mathbb{R}^d$ *is uniformly* L*-Lipschitz for all* $(s, Z_s) \in [0,T] \times \mathbb{L}^2(\mathcal{F}_T)^{d\times n}$, $g_2(\cdot, Z.)$ *is predictable for any* $Z \in \mathbb{H}^2$, $g_2(\cdot, 0) \in \mathbb{H}^2$, *and for all* $u, v \in \mathbb{H}^2$,

$$\left\| \int_0^T |g_2(s,u_s) - g_2(s,v_s)|\, ds \right\|_2 \leq \rho(\|u\|_{\mathbb{H}^2} + \|v\|_{\mathbb{H}^2}) \|u - v\|_{\mathbb{H}^2}$$

  for some increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$.

Then, there is a solution (Y, Z) *to the following BSDE.*

$$Y_t = \xi + \int_t^T (g_1(s,Z_s) + g_2(s,Z_s))\, ds - \int_t^T Z_s dW_s$$

Proof. First, partition $[0,T]$ into intervals of size $\varepsilon < (2C)^{-2}$ such that T/ε is a natural number. For each given $j \in \{1, 2, \cdots, T/\varepsilon\}$, consider BSDEs

$$Y_t^j = Y_{T-(j-1)\varepsilon}^{j-1} + \int_t^T \left(g_1(s, Z_s^j) 1_{[T-j\varepsilon, T-(j-1)\varepsilon]}(s) + g_2(s, Z_s^j) \right) ds - \int_t^T Z_s^j dW_s.$$

where $Y_T^0 := \xi$. For $Z \in \mathbb{H}^2$, let

$$\begin{aligned} F_t^{1,j}\left(\int_0^\cdot Z_s dW_s\right) &:= \xi + \int_0^t g_1(s,Z_s) 1_{[T-j\varepsilon, T-(j-1)\varepsilon]}(s) ds \\ F_t^2\left(\int_0^\cdot Z_s dW_s\right) &:= \int_0^t g_2(s,Z_s) ds \end{aligned}$$

Then, it is obvious that (B1)–(B3) are satisfied because

$$\left\|F_T^{1,j}\left(\int_0^{\cdot} Z_s dW_s\right) - F_T^{1,j}\left(\int_0^{\cdot} Z'_s dW_s\right)\right\|_2^2 \leq C^2 \varepsilon \mathbb{E}\int_0^T |Z_s - Z'_s|^2 ds \leq C^2 \varepsilon \left\|\int_0^{\cdot} Z_s dW_s - \int_0^{\cdot} Z_s dW_s\right\|_{\mathbb{S}^2}^2.$$

On the other hand, $F_T^2(M)$ is in $\mathbb{W}^{1,2}$ and uniformly LT-Lipschitz random variable for any given $M \in \mathbb{M}_0^2$. Moreover, by our assumption,

$$\begin{aligned}
&\left\|F_T^2\left(\int_0^{\cdot} Z_s dW_s\right) - F_T^2\left(\int_0^{\cdot} Z'_s dW_s\right)\right\|_2 \\
&\leq \rho(\|Z\|_{\mathbb{H}^2} + \|Z'\|_{\mathbb{H}^2})\, \|Z - Z'\|_{\mathbb{H}^2} \\
&\leq \rho\left(\left\|\int_0^{\cdot} Z_s dW_s\right\|_{\mathbb{H}^2} + \left\|\int_0^{\cdot} Z'_s dW_s\right\|_{\mathbb{H}^2}\right) \left\|\int_0^{\cdot} Z_s dW_s - \int_0^{\cdot} Z'_s dW_s\right\|_{\mathbb{S}^2},
\end{aligned}$$

and therefore, $F_T^2(M)$ is continuous in M. By Proposition 2.3.16, we have a solution (Y^1, Z^1) for this BSDE. We can repeat this argument to define (Y^j, Z^j) for $j = 1, 2\cdots, T/\varepsilon$. Define $(Y_t, Z_t) = (Y_t^j, Z_t^j)$ for $t \in [T - j\varepsilon, T - (j-1)\varepsilon]$. Then, (Y, Z) is a solution of the BSDE in the proposition. □

Let us provide two examples of the above proposition. Note that we are considering multidimensional mean-field BSDEs with quadratic drivers. In particular, these example shows that the existence of solutions can persist if the superlinearity of driver comes from the law of solutions. To our best knowledge, the existence of solution is not proved in any other literature for these BSDEs.

Example 2.3.19. *Assume the following conditions.*

- $\mathbb{F} = \mathbb{F}^W$
- $h : (\omega, s, u, v) \in \Omega \times [0, T] \times \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} \mapsto h(\omega, s, u, v) \in \mathbb{R}^d$ *is uniformly Lipschitz in* (u, v) *with coefficient* C*,* $h(\cdot, u, v)$ *is predictable for all* $(u, v) \in \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$*, and* $h(\cdot, 0, 0) \in \mathbb{H}^2$.
- *Let* $G : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ *and* $g : \Omega \times [0, T] \times \mathbb{R}^{d\times n} \to \mathbb{R}^m$ *where*

$$|g(s, a) - g(s, b)| \leq C(1 + |a| + |b|)|a - b|$$

$$|G(s, x) - G(s, y)| \leq C|x - y|$$

$$|G(s, 0)|, |g(s, 0)| \leq C$$

for all $a, b \in \mathbb{R}^{d\times n}, x, y \in \mathbb{R}^m$*. In addition, we assume that* $G(s, x)$ *is uniformly* L*-Lipschitz* *for any given* $(s,x) \in [0,T] \times \mathbb{R}^m$ *and that* $g(\cdot,x)$ *and* $G(\cdot,x)$ *are predictable for any given* $x \in \mathbb{R}^m$.

Then, there exists a solution to the following BSDE

$$Y_t = \xi + \int_t^T \left(\mathbb{E}' h(s, Z_s, Z'_s) + G(s, \mathbb{E} g(s, Z_s))\right) ds - \int_t^T Z_s dW_s$$

where

$$\mathbb{E}' h(s, Z_s, Z'_s)(\omega) := \int_\Omega h(\omega, s, Z_s(\omega), Z_s(\omega'))\mathbb{P}(d\omega')$$

Proof. Note that

$$\begin{aligned}
&\mathbb{E}|\mathbb{E}' h(s, u_s, u'_s) - \mathbb{E}' h(s, v_s, v'_s)|^2 \\
&\quad \leq \int_\Omega \left(\int_\Omega |h(\omega, s, u_s(\omega), u_s(\omega')) - h(\omega, s, v_s(\omega), v_s(\omega'))|^2 \,\mathbb{P}(d\omega') \right) \mathbb{P}(d\omega) \\
&\quad \leq 2C^2 \int_\Omega \left(\int_\Omega \left(|u_s(\omega) - v_s(\omega)|^2 + |u_s(\omega') - v_s(\omega')|^2\right) \mathbb{P}(d\omega') \right) \mathbb{P}(d\omega) \\
&\quad \leq 4C^2 \mathbb{E}|u_s - v_s|^2.
\end{aligned}$$

Therefore, $g_1(s, Z_s) := \mathbb{E}' h(s, Z_s, Z'_s)$ satisfies the condition in the previous proposition. On the other hand,

$$\begin{aligned}
\left\| \int_0^T |G(s, \mathbb{E} g(s, u_s)) - G(s, \mathbb{E} g(s, v_s))| \, ds \right\|_2^2 &\leq C^2 \mathbb{E} \left| \int_0^T |\mathbb{E} g(s, u_s) - \mathbb{E} g(s, v_s)| \, ds \right|^2 \\
&\leq C^4 \mathbb{E} \left| \int_0^T \mathbb{E}(1 + |u_s| + |v_s|)|u_s - v_s| ds \right|^2 \\
&\leq C^4 \left\| 1 + u + v \right\|_{\mathbb{H}^2}^2 \left\| u - v \right\|_{\mathbb{H}^2}^2 \\
&\leq C^4 (T + \|u\|_{\mathbb{H}^2} + \|v\|_{\mathbb{H}^2})^2 \left\| u - v \right\|_{\mathbb{H}^2}^2
\end{aligned}$$

and therefore, the conditions for $g_2(s, Z_s) := G(s, \mathbb{E} g(s, Z_s))$ is satisfied. □

Example 2.3.20. *Assume the following conditions.*

- $\mathbb{F} = \mathbb{F}^W$
- $h : (\omega, s, u, v) \in \Omega \times [0,T] \times \mathbb{R}^{d\times n} \times \mathbb{R}^{d \times n} \mapsto h(\omega, s, u, v) \in \mathbb{R}^d$ *is uniformly Lipschitz in* (u, v) *with coefficient* C*,* $h(\cdot, u, v)$ *is predictable for all* $(u, v) \in \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$*, and* $h(\cdot, 0, 0) \in \mathbb{H}^2$.

- *Let* $G : \Omega \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *and* $g : \Omega \times [0,T] \times \mathbb{R}^{d\times n} \to \mathbb{R}^m$ *where*

$$
\begin{aligned}
&|G(s,a) - G(s,b)| \leq C(1 + |a| + |b|)|a - b| \\
&|g(s,x) - g(s,y)| \leq C|x - y| \\
&|G(s,0)|, |g(s,0)| \leq C
\end{aligned}
$$

for all $a, b \in \mathbb{R}^{d\times n}, x, y \in \mathbb{R}^m$*. In addition, we assume that* $G(s,x)$ *is uniformly* L*-Lipschitz for any given* $(s,x) \in [0,T] \times \mathbb{R}^m$ *and that* $g(\cdot, x)$ *and* $G(\cdot, x)$ *are predictable for any given* $x \in \mathbb{R}^m$*.*

Then, there exists a solution to the following BSDE

$$
Y_t = \xi + \int_t^T \left(\mathbb{E}' h(s, Z_s, Z'_s) + G(s, \mathbb{E}g(s, Z_s))\right) ds - \int_t^T Z_s dW_s
$$

where

$$
\mathbb{E}' h(s, Z_s, Z'_s)(\omega) := \int_\Omega h(\omega, s, Z_s(\omega), Z_s(\omega'))\mathbb{P}(d\omega')
$$

Proof. Note that

$$
\begin{aligned}
&\left\| \int_0^T |G(s, \mathbb{E}g(s,u_s)) - G(s, \mathbb{E}g(s, v_s))| \, ds \right\|_2^2 \\
&\leq C^2 \mathbb{E} \left| \int_0^T (1 + \mathbb{E}g(s,u_s) + \mathbb{E}g(s,v_s)) \, |\mathbb{E}g(s,u_s) - \mathbb{E}g(s,v_s)| \, ds \right|^2 \\
&\leq C^4 \left| \int_0^T \mathbb{E}(1 + 2C + C|u_s| + C|v_s|)\mathbb{E}|u_s - v_s| ds \right|^2 \\
&\leq C^4 \int_0^T |\mathbb{E}(1 + 2C + C|u_s| + C|v_s|)|^2 \, ds \int_0^T |\mathbb{E}|u_s - v_s||^2 \, ds \\
&\leq 3C^4((1+2C)^2 + C^2 \|u\|_{\mathbb{H}^2}^2 + C^2 \|v\|_{\mathbb{H}^2}^2) \|u - v\|_{\mathbb{H}^2}^2 .
\end{aligned}
$$

Other conditions of the above proposition can be checked as in the above example. □

Chapter 3

BMO Martingale and Girsanov Transform

In this chapter, we assume $\mathbb{F} = \mathbb{F}^W$ and $d \geq 1$. If the terminal condition ξ is square-integrable and the driver $f(t, y, z)$ Lipschitz continuous in (y, z), the existence of a unique solution can be shown with a Picard–Lindelöf iteration argument, see, see for example, El Karoui et al. [34]. Kobylanski [55] proved that one-dimensional quadratic BSDE has a unique solution if ξ is bounded. Moreover, if ξ has a bounded Malliavin derivative, the growth of $f(s, y, z)$ in z can be arbitrary (see Cheridito and Nam [17]). For multidimensional BSDEs, the situation is more intricate because one cannot use the comparison results; see Hu and Peng [47]. In fact, multidimensional BSDEs with drivers of quadratic growth in z do not always admit solutions even if the terminal condition ξ is bounded; see Frei and dos Reis [35] for an example. An early result for superlinear multidimensional BSDEs was given by Bahlali [4], which assumed that the growth of $f(s, y, z)$ in z is of the order $|z|\sqrt{\log|z|}$. It was generalized by Bahlali et al. [5] to the case where $f(s, y, z)$ has a strictly subquadratic growth in z and satisfies a monotonicity condition. Tevzadze [80] proved the well-posedness for multidimensional quadratic BSDE in the case where the terminal condition has a small enough $\mathbb{L}^\infty$-norm.

Suppose we already have a solution (Y, Z) of (1.1.1) and $G(\cdot, Y_\cdot, Z_\cdot) \in \mathbb{H}^{\mathrm{BMO}}$. By Kazamaki [53], we can change the measure using the Girsanov transform, that is,

$$\tilde{\mathbb{P}} = \mathcal{E}_T^{G(\cdot, Y_\cdot, Z_\cdot)} \cdot \mathbb{P}$$

so that

$$\tilde{W}_t := W_t - \int_0^t G(s, Y_s, Z_s)ds$$

is a $\tilde{\mathbb{P}}$-Brownian motion. Then, the following equation holds:

$$Y_t = \xi + \int_t^T [f(s, Y_s, Z_s) + Z_s G(s, Y_s, Z_s)]\, ds - \int_t^T Z_s d\tilde{W}_s \tag{3.0.1}$$

As Liang et al. stated in their preprint [58], (Y, Z) is a weak solution to (3.0.1) because it might not be adapted to the filtration $\mathbb{F}^{\tilde{W}}$ which might be strictly coarser than $\mathbb{F}^W$ (see the example of Cirel'son [20]). Naturally, one may ask whether (3.0.1) has a strong solution. In this chapter we provide sufficient conditions to address this question. This leads to the well-posedness of multidimensional quadratic and subquadratic BSDEs.

Three different cases are considered in this chapter. In all three we assume ξ to be bounded and use the BMO martingale theory together with the Girsanov theorem to construct an equivalent probability measure that can be used to prove the existence of a solution.

In Section 3.1 we assume the BSDE to be Markovian and related to an FBSDE of the form

$$dP_t = G(t, P_t, Q_t, R_t)dt + dW_t, \quad P_0 = 0 \tag{3.0.2}$$

$$dQ_t = -F(t, P_t, Q_t, R_t)dt + R_t dW_t, \quad Q_T = h(P_T) \tag{3.0.3}$$

for a bounded function h. If the FBSDE has a solution, we change the probability measure to obtain a solution to a different FBSDE, from which a solution to the BSDE (1.1.1) can be derived. We discuss two different sets of sufficient conditions under which the FBSDE (3.0.2) has a solution. Mania and Schweizer [60] and Ankirchner et al. [2] also studied the transformation of one-dimensional quadratic BSDEs under a change of measure, but not directed at proving the existence of a classical solution. In Section 3.2, we give conditions under which equation (1.1.1) can be turned into a one-dimensional quadratic BSDE by projecting it onto a one-dimensional subspace of $\mathbb{R}^d$. Results of Kobylanski [55] guarantee that the resulting one-dimensional equation has a solution. From there a solution to the multidimensional equation can be obtained by changing the probability measure and solving a linear equation. The Markovian assumption or projectability assumption can be relaxed if the growth of $f(s, y, z)$ in z is assumed to be strictly subquadratic. This is studied in Section 3.3. The subquadratic growth assumption allows to prove the existence of a unique solution on a short time interval with the Banach fixed point theorem. Under an additional structural assumption, the solution can be estimated by taking conditional expectation with respect to an equivalent probability measure. Then, by iterating the argument, the short-time solution can be extended to a global solution.

We will use the following remark throughout this chapter.

Remark 3.0.21. *For $H \in \mathbb{H}^{\text{BMO}}(\mathbb{R}^{n\times 1})$, $\int_0^t H_s^T dW_s$ is a BMO martingale and*

$$\mathcal{E}_t^H := \exp\left(\int_0^t H_s^T dW_s - \frac{1}{2}\int_0^T |H_s|^2 ds\right)$$

a martingale; see Kazamaki [53]. One obtains from the Girsanov theorem that $\mathcal{E}_T^H \cdot \mathbb{P}$ defines an equivalent probability measure, under which $W_t - \int_0^t H_s ds$ is a Brownian motion. Moreover, every $Z \in \mathbb{H}^{\text{BMO}}(\mathbb{R}^{d\times n})$ with respect to $\mathbb{P}$ is also in $\mathbb{H}^{\text{BMO}}(\mathbb{R}^{d\times n})$ with respect to $\mathcal{E}_T^H \cdot \mathbb{P}$.

3.1 Markovian Quadratic BSDEs

In this section we consider BSDEs of the form

$$Y_t = h(W_T) + \int_t^T \{F(s, W_s, Y_s, Z_s) + Z_s G(s, W_s, Y_s, Z_s)\} ds - \int_t^T Z_s dW_s \tag{3.1.1}$$

for functions $h : \mathbb{R}^n \to \mathbb{R}^d$, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^d$ and $G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to \mathbb{R}^n$.

The following theorem gives conditions under which (3.1.1) has a solution if there is a solution to a related FBSDE.

Theorem 3.1.1. *Assume that there exists a constant $C \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following conditions hold:*

(A1) $|h(x)| \leq C$.

(A2) $y^T F(t,x,y,z) \leq C|y|(1 + |y| + |z|)$ *for all* $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times n}$.

(A3) $|G(t,x,y,z)| \leq \rho(|y|)(1 + |z|)$ *for all* $(t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n}$.

(A4) *The FBSDE*

$$dP_t = G(t, P_t, Q_t, R_t)dt + dW_t, \quad P_0 = 0$$
$$dQ_t = -F(t, P_t, Q_t, R_t)dt + R_t dW_t, \quad Q_T = h(P_T)$$

has a solution $(P, Q, R) \in \mathbb{H}^2(\mathbb{R}^n) \times \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$ *of the form* $Q_t = q(t,p)$ *and* $R_t = r(t,p)$ *for predictable functions* $q : [0,T] \times C([0,T], \mathbb{R}^n) \to \mathbb{R}^d$ *and* $r : [0,T] \times C([0,T], \mathbb{R}^n) \to \mathbb{R}^{d\times n}$.

Then $(Y_t, Z_t) = (q(t,W), r(t,W))$ *is a solution of the BSDE* (3.1.1) *in* $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$*, and* Z *is bounded if* R *is bounded.*

Proof. One obtains from Itô's formula that for every $a \in \mathbb{R}_+$ and $[0,T]$-valued stopping time τ,

$$e^{a\tau}|Q_\tau|^2 = \mathbb{E}_\tau\left(e^{aT}|h(P_T)|^2 + \int_\tau^T e^{as}\left(2Q_s^T F(s,P_s,Q_s,R_s) - |R_s|^2 - a|Q_s|^2\right)ds\right).$$

By assumption (A2),

$$\begin{aligned}&2Q_s^T F(s,P_s,Q_s,R_s) - |R_s|^2 - a|Q_s|^2 \le 2C|Q_s|(1+|Q_s|+|R_s|) - |R_s|^2 - a|Q_s|^2\\ &\le C^2 + (2C^2+2C+1-a)|Q_s|^2 - \frac{1}{2}|R_s|^2.\end{aligned}$$

So for $a = 2C^2 + 2C + 1$, one obtains

$$\begin{aligned}&|Q_\tau|^2 + \frac{1}{2}\mathbb{E}_\tau\int_\tau^T |R_s|^2 ds \le e^{a\tau}|Q_\tau|^2 + \frac{1}{2}\mathbb{E}_\tau\int_\tau^T e^{as}|R_s|^2 ds\\ &\le \mathbb{E}_\tau\left(e^{aT}|h(P_T)|^2 + C^2\int_\tau^T e^{as}ds\right) \le C^2 e^{aT}(1+T).\end{aligned}$$

In particular, Q is in $\mathbb{S}^\infty(\mathbb{R}^d)$ and R in $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$. Set $K := \rho(C^2 e^{aT}(1+T))$. By assumption (A3), one has

$$|G(s,P_s,Q_s,R_s)| \le K(1+|R_s|),$$

from which it follows that $G(s,P_s,Q_s,R_s)$ belongs to $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{n\times 1})$. Therefore, P is a Brownian motion under the measure $\mathcal{E}_T^{-G}\cdot\mathbb{P}$, and R is still in $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$ under $\mathcal{E}_T^{-G}\cdot\mathbb{P}$. The backward equation from (A4) can be written as

$$dQ_t = -\left(F(t,P_t,Q_t,R_t) + R_t G(t,P_t,Q_t,R_t)\right)dt + R_t dP_t, \quad Q_T = h(P_T).$$

But since $Q_t = q(t,P)$ and $R_t = r(t,P)$, one has

$$dq(t,P) = -\left(F(t,P_t,q(t,P),r(t,P)) + r(t,P)G(t,P_t,q(t,P),r(t,P))\right)dt + r(t,P)dP_t.$$

So $(Y, Z) = (q(\cdot, W), r(\cdot, W))$ is in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d \times n})$ and satisfies

$$dY_t = -\left(F(t, W_t, Y_t, Z_t) + Z_t G(t, W_t, Y_t, Z_t)\right) dt + Z_t dW_t, \quad Y_T = h(W_T).$$

Moreover, if R is bounded, then so is Z. □

Remark 3.1.2. Since the BSDE (3.1.1) is Markovian, it is related to the semilinear parabolic PDE with terminal condition

$$u_t + \frac{1}{2}\triangle u + F(t, x, u, \nabla u) + (\nabla u) g(t, x, u, \nabla u) = 0, \quad u(T, x) = h(x).$$

For example, if it has a $C^{1,2}$-solution $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$, it follows from Itô's formula that $(Y_t, Z_t) = (u(t, W_t), \nabla u(t, W_t))$ solves the BSDE (3.1.1). But the standard construction of a viscosity solution to the PDE from a BSDE solution does not work because the necessary comparison results do not extend from the one- to the multidimensional case; see Peng [71].

The main assumption of Theorem 3.1.1 is (A4). There exist different results in the FBSDE literature from which it follows. In the following we use conditions of Pardoux and Tang [67] and Delarue [27].

Corollary 3.1.3. *In addition to* (A1)–(A3)*, assume that* F, G *and* h *are continuous and there exist constants* $\lambda_1, \lambda_2 \in \mathbb{R}$, $k, k_1, k_2, k_3, k_4, k_5, C_1, C_3, C_4, \theta, \alpha \in \mathbb{R}_+$ *such that for all* t, x, x', y, y', z, z' *the following conditions hold:*

- $(x - x')^T \left(G(t, x, y, z) - G(t, x', y, z)\right) \le \lambda_1 |x - x'|^2$
- $(y - y')^T \left(F(t, x, y, z) - F(t, x, y', z)\right) \le \lambda_2 |y - y'|^2$
- $|G(t, x, y, z) - G(t, x, y', z')| \le k_1 |y - y'| + k_2 |z - z'|$
- $|G(t, x, y, z)| \le |G(t, 0, y, z)| + k(1 + |x|)$
- $|F(t, x, y, z) - F(t, x', y, z')| \le k_3 |x - x'| + k_4 |z - z'|$
- $|F(t, x, y, z)| \le |F(t, x, 0, z)| + k(1 + |y|)$
- $|h(x) - h(x')| \le k_5 |x - x'|$
- $C_4 < k_4^{-1}$
- $\lambda_1 + \lambda_2 <$
  $$-\frac{1}{2}\left((1+\alpha)\left(k_1 C_1 + \frac{k_2^2}{\alpha(1 - k_4 C^4)}\right)\left(k_5^2 + \frac{k_3 C_3}{\theta}\right) + k_1 C_1^{-1} + k_3 C_3^{-1} + k_4 C_4^{-1} + \theta\right).$$

Then the BSDE (3.1.1) *has a unique solution* (Y, Z) *in* $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$*, and it is of the form* $Y_t = y(t, W_t)$*,* $Z_t = \nabla_x y(t, W_t)$*, where* $y : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ *is a continuous function that is uniformly Lipschitz in* $x \in \mathbb{R}^n$ *and* ∇_x *denotes the weak derivative with respect to* x *in the Sobolev sense.*

Proof. It is shown in Pardoux and Tang [67] that for each pair $(t, x) \in [0,T] \times \mathbb{R}^n$, the FBSDE

$$P_s^{t,x} = x + \int_t^s G(u, P_u^{t,x}, Q_u^{t,x}, R_u^{t,x})du + \int_t^s dW_u$$
$$Q_s^{t,x} = h(P_T^{t,x}) + \int_s^T F(u, P_u^{t,x}, Q_u^{t,x}, R_u^{t,x})du - \int_s^T R_u^{t,x} dW_u.$$

has a unique solution $(P^{t,x}, Q^{t,x}, R^{t,x}) \in \mathbb{H}^2_{[t,T]}(\mathbb{R}^n) \times \mathbb{H}^2_{[t,T]}(\mathbb{R}^d) \times \mathbb{H}^2_{[t,T]}(\mathbb{R}^{d\times n})$ adapted to the filtration generated by $(W_s - W_t)$, $t \le s \le T$. So one can set $q(t,x) := Q_t^{t,x}$, and it can be seen from the proof of Theorem 5.1 of Pardoux and Tang [67] that $Q_s^{t,x} = q(s, P_s^{t,x})$. This shows that the FBSDE in (A4) has a unique solution (P, Q, R) in $\mathbb{H}^2(\mathbb{R}^n) \times \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$, and Q is of the form $Q_s = q(s, p_s)$. Moreover, it follows from Theorem 4.2 of Pardoux and Tang [67] that $q(t,x)$ is continuous in (t,x) and uniformly Lipschitz in x. As in the proof of Theorem 3.1.1, one obtains that P is an $\mathbb{F}$-adapted n-dimensional Brownian motion with respect to a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$. It can be seen from the representation

$$Q_t = q(t, P_t) = Q_0 - \int_0^t \{F(s, P_s, Q_s, R_s) + R_s G(s, P_s, Q_s, R_s)\} ds + \int_0^t R_s dP_s$$

that Q is a continuous $\mathbb{F}$-semimartingale. By Stricker's theorem, it is also a continuous semimartingale with respect to the filtration $\mathbb{F}^P$ generated by P. In particular, it has a unique $\mathbb{F}^P$-semimartingale decomposition $Q_t = Q_0 + M_t + A_t$. By the martingale representation theorem, M_t can be written as $M_t = \int_0^t H_s dP_s$ for a unique $\mathbb{F}^P$-predictable process H. But since P is an $\mathbb{F}$-Brownian motion, $Q_t = Q_0 + M_t + A_t$ is also the unique $\mathbb{F}$-semimartingale decomposition of Q. It follows that $R = H$, and therefore, $R_t = r(t, P)$ for a predictable function $r : [0,T] \times C([0,T], \mathbb{R}^n) \to \mathbb{R}^{d\times n}$. This shows that (A4) holds. So it follows from Theorem 3.1.1 that $(Y_t, Z_t) = (q(t, W), r(t, W))$ is a solution of the BSDE (1.1.1). Moreover, since q is continuous and $q(t, P_t)$ an Itô process, one obtains from Theorem 1 of Chitashvili and Mania [19] that $r(t, P) = \nabla_x q(t, P_t)$, where $\nabla_x q$ is a bounded weak derivative of q with respect to x. It follows that $(q(t, W_t), \nabla_x q(t, W_t))$ is a solution of (3.1.1) in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$.

Now assume $(\tilde{Y}, \tilde{Z})$ is another solution of (3.1.1) in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$ and let L be a common bound for Y, Y', Z, Z'. Then (Y, Z) and (Y', Z') are both solutions of the modified BSDE

$$Y_t = h(W_T) + \int_t^T f(s, W_s, \pi_L(Y_s, Z_s))ds - \int_t^T Z_s dW_s,$$

where

$$\begin{aligned} f(t,x,y,z) &:= F(t,x,y,z) + zG(t,x,y,z) \\ \pi_L(y,z) &:= \left(\min\left\{1, L/|y|\right\} y, \min\left\{1, L/|z|\right\} z\right). \end{aligned}$$

Since this BSDE satisfies the conditions of Pardoux [64], it has a unique solution, and it follows that $(Y, Z) = (\tilde{Y}, \tilde{Z})$. □

In the next corollary we use conditions of Delarue [27] ensuring that the FBSDE in (A4) has a solution.

Corollary 3.1.4. *Assume that there exists a constant $C \in \mathbb{R}_+$ such that for all t, x, x', y, y', z, z' the following hold:*

- $|F(t,x,y,z) - F(t,x',y,z')| \le C(|x - x'| + |z - z'|)$
- $(y - y')^T (F(t,x,y,z) - F(t,x,y',z)) \le C|y - y'|^2$
- $|F(t,x,y,z)| \le C(1 + |y| + |z|)$
- $F(t,x,y,z)$ *is continuous in* y
- $|G(t,x,y,z) - G(t,x,y',z')| \le C(|y - y'| + |z - z'|)$
- $(x - x')^T (G(t,x,y,z) - G(t,x',y,z)) \le C|x - x'|^2$
- $|G(t,x,y,z)| \le C(1 + |y| + |z|)$
- $G(t,x,y,z)$ *is continuous in* x
- $|h(x) - h(x')| \le C|x - x'|$
- $|h(x)| \le C$.

Then the BSDE (3.1.1) *has a unique solution (Y, Z) in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$, and it is of the form $Y_t = y(t, W_t)$, $Z_t = \nabla_x y(t, W_t)$, where $y : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a continuous function that is uniformly Lipschitz in $x \in \mathbb{R}^n$ and ∇_x denotes the weak derivative with respect to x in the Sobolev sense.*

Proof. By Theorem 2.6 of Delarue [27], the FBSDE in (A4) has a unique bounded solution (P, Q, R). Moreover, by Proposition 2.4 of the same paper, Q is of the form $Q_t = q(t, P_t)$ for a continuous function $q : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ that is uniformly Lipschitz in $x \in \mathbb{R}^n$. Now the corollary follows like Corollary 3.1.3. □

Example 3.1.5. If F, G, h are uniformly Lipschitz in (x, y, z) and $|F(t, x, 0, 0)| + |G(t, x, 0, 0)| + |h(x)|$ is bounded, then the conditions of Corollary 3.1.4 hold. So the BSDE (3.1.1) has a unique solution $(Y, Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$ of the form $Y_t = y(t, W_t)$, $Z_t = \nabla_x y(t, W_t)$ for a continuous function $y : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ and its weak derivative $\nabla_x y$.

3.2 Projectable Quadratic BSDEs

Definition 3.2.1. *We call a multidimensional BSDE projectable if its driver can be written as*

$$f(s, y, z) = P(s, a^T y, a^T z) + yQ(s, a^T y, a^T z) + zR(s, a^T y, a^T z) \tag{3.2.1}$$

for a constant vector $a \in \mathbb{R}^d$ and predictable functions

$$\begin{aligned} P &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{d\times 1} \\ Q &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \\ R &: [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times 1}. \end{aligned}$$

A projectable BSDE becomes one-dimensional if projected to the vector a:

$$\begin{aligned} a^T Y_t =& a^T \xi + \int_t^T a^T \left\{ P(s, a^T Y_s, a^T Z_s) + Y_s Q(s, a^T Y_s, a^T Z_s) + Z_s R(s, a^T Y_s, a^T Z_s) \right\} ds \\ & - \int_t^T a^T Z_s dW_s. \end{aligned}$$

In the following theorem we consider a projectable BSDE under conditions ensuring that the projected BSDE has a solution. This allows to derive the existence of a solution to the multidimensional BSDE.

Theorem 3.2.2. *Consider a bounded terminal condition $\xi \in \mathbb{L}^\infty(\mathcal{F}_T)^d$ and a driver f of the form* (3.2.1) *such that*

$$|P(s, u, v)| \le C(1 + |y|), \quad |Q(s, u, v)| \le C, \quad |R(s, u, v)| \le C + \rho(|u|)|v|$$

for a constant $C \in \mathbb{R}_+$ *and a nondecreasing function* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. *Then the BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \tag{3.2.2}$$

has a solution $(Y, Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$.

Moreover, if

$$F(s, u, v) = a^T P(s, u, v) + uQ(s, u, v) + v^T R(s, u, v)$$

satisfies

$$|F(s, u, v) - F(s, u', v') \leq C|u - u'| + C(1 + |v| \vee |v'|)|v - v'|, \tag{3.2.3}$$

then there is only one solution $(Y, Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$.

Proof. By Theorem 2.3 of Kobylanski [55], the one-dimensional BSDE

$$U_t = a^T \xi + \int_t^T F\left(s, U_s, V_s\right) ds - \int_t^T V_s dW_s \tag{3.2.4}$$

has a solution $(U, V) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{1\times n})$. So there exists a constant $K \geq 0$ such that $|F(s, U_s, V_s)| \leq K(1+|V_s|^2)$, and it follows like in the proof of Proposition 2.1 in Briand and Elie [11] that V is in $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times n})$. Denote $P_s := P(s, U_s, V_s)$, $Q_s := Q(s, U_s, V_s)$, $R_s = R(s, U_s, V_s)$ and assume the multidimensional linear BSDE

$$Y_t = \xi + \int_t^T \{P_s + Y_s Q_s + Z_s R_s\}\, ds - \int_t^T Z_s dW_s \tag{3.2.5}$$

has a solution $(Y, Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$. It follows from the assumptions that $|R_s| \leq C + \rho(\|U\|_{\mathbb{S}^\infty})|V_s|$. So R is in $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{n\times 1})$, and $\tilde{\mathbb{P}} := \mathcal{E}^R_T \cdot \mathbb{P}$ is an equivalent probability measure under which $\tilde{W}_t = W_t - \int_0^t R_s ds$ is a Brownian motion. Now one can write

$$Y_t = \xi + \int_t^T \{P_s + Y_s Q_s\}\, ds - \int_t^T Z_s d\tilde{W}_s, \tag{3.2.6}$$

from which it follows that

$$e^{\int_0^t Q_u du} Y_t = e^{\int_0^T Q_u du} \xi + \int_t^T e^{\int_0^s Q_u du} P_s ds - \int_t^T e^{\int_0^s Q_u du} Z_s d\tilde{W}_s,$$

and therefore,

$$e^{\int_0^t Q_s ds} Y_t = \tilde{\mathbb{E}}_t \left[e^{\int_0^T Q_u du} \xi + \int_t^T e^{\int_0^s Q_u du} P_s ds \right], \tag{3.2.7}$$

where $\tilde{\mathbb{E}}$ denotes expectation with respect to $\tilde{\mathbb{P}}$. This uniquely determines Y. Now Z is uniquely given by (3.2.6). To show that (3.2.5) has a solution in $\mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$, one can define Y by (3.2.7), which is equivalent to

$$\Gamma_t Y_t = \mathbb{E}_t \left[\Gamma_T \xi + \int_t^T \Gamma_s P_s ds \right],$$

where Γ is the unique solution of the SDE

$$d\Gamma_t = \Gamma_t (Q_s ds + R_s^T dW_s), \quad \Gamma_0 = 1.$$

Then Y belongs to $\mathbb{S}^\infty(\mathbb{R}^d)$, and by the martingale representation theorem, there exists a unique predictable process Z such that ΓZ belongs to $\mathbb{H}^2(\mathbb{R}^{d\times n})$ and

$$\int_0^T \Gamma_s (Y_s R_s^T + Z_s) dW_s = \Gamma_T \xi + \int_0^T \Gamma_s P_s ds - \mathbb{E}\left[\Gamma_T \xi + \int_0^T \Gamma_s P_s ds \right].$$

Since $Y_0 = \mathbb{E}\left[\Gamma_T \xi + \int_0^T \Gamma_s P_s ds \right]$, one has

$$Y_0 + \int_0^t \Gamma_s (Y_s R_s^T + Z_s) dW_s = \mathbb{E}_t \left[\Gamma_T \xi + \int_0^T \Gamma_s P_s ds \right] = \Gamma_t Y_t + \int_0^t \Gamma_s P_s ds.$$

Therefore,

$$Y_t = \Gamma_t^{-1} \left(Y_0 + \int_0^t \Gamma_s (Y_s R_s^T + Z_s) dW_s - \int_0^t \Gamma_s P_s ds \right),$$

and one obtains

$$dY_t = -\left\{ P_s + Y_s Q_s + Z_s R_s \right\} ds + Z_s dW_s, \quad Y_T = \xi.$$

From (3.2.6) one deduces that Z is in $\mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$ with respect to $\tilde{\mathbb{P}}$ and hence, also with respect to $\mathbb{P}$. So we have shown that for a given solution $(U, V) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times n})$ of the one-dimensional BSDE (3.2.4), the linear BSDE (3.2.5) has a unique solution $(Y, Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$. It follows that $(a^T Y, a^T Z)$ solves the one-dimensional linear BSDE

$$\tilde{U}_t = a^T \xi + \int_t^T \left\{ a^T P_s + \tilde{U}_s Q_s + \tilde{V}_s R_s \right\} ds - \int_t^T \tilde{V}_s dW_s,$$

which, like (3.2.5), can be shown that have a unique solution in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times n})$. In turns, $(a^T Y, a^T Z) = (U, V)$, from which it follows that (Y, Z) solves the original BSDE (3.2.2).

If the additional condition (3.2.3) holds, it follows from Theorem 2.6 in Kobylanski [55] that the one-dimensional BSDE (3.2.4) admits only one solution $(U,V) \in \mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{1\times n})$. So (3.2.2) has a unique solution $(Y,Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$. □

3.3 Subquadratic BSDEs

In the case where our BSDE is not Markovian or projectable, we assume the driver $f(s,y,z)$ to be of strictly subquadratic growth in z. For constants $C_i \in \mathbb{R}_+$, $\varepsilon \in (0,1)$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, set $C := \sqrt{\sum_{i=1}^d |C_i|^2}$ and consider the following conditions:

(B1) For every i, ξ^i is an $\mathcal{F}_T$-measurable random variable bounded by C_i.

(B2) $|f(s,y,z)| \le C\left(1 + |y| + \rho(|y|)|z|^{2-\varepsilon}\right)$

(B3) $|f(t,y,z) - f(t,y',z')| \le \rho\left(|y| \vee |y'|\right)\left(|y-y'| + \left(1 + (|z| \vee |z'|)^{1-\varepsilon}\right)|z-z'|\right)$

(B4) $f^i(s,y,z) = F^i(s,y,z) + G^i(s,y,z)$, where

$$|F^i(s,y,z)| \le C_i(1 + |y| + |z|) \quad \text{and} \quad |G^i(s,y,z)| \le C_i\rho(|y|)|z^i||z|.$$

The main result of this section is the following

Theorem 3.3.1. *A BSDE*

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s \tag{3.3.1}$$

satisfying (B1)–(B4) *has a unique solution in* $(Y,Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$. *Moreover,*

$$|Y_t| \le (C+1)\exp\left(\frac{(C+1)^2}{2}(T-t)\right).$$

We prove Theorem 3.3.1 by first showing that the BSDE (3.3.1) has a unique solution for short time intervals and then constructing a solution on $[0,T]$ recursively.

For small $\delta > 0$ we use the Banach fixed point theorem to prove the existence and uniqueness of a solution on $[T-\delta,T]$. For $R \in \mathbb{R}_+$, define

$$\mathcal{B}_R := \left\{(Y,Z) \in \mathbb{S}^\infty_{[T-\delta,T]}(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}_{[T-\delta,T]}(\mathbb{R}^{d\times n}) : \|Y\|_{\mathbb{S}^\infty_{[T-\delta,T]}} \le R, \|Z\|_{\mathrm{B}_{[T-\delta,T]}} \le R\right\}.$$

Lemma 3.3.2. *Assume that* $R \ge 3C$ *and* (B1)–(B3) *hold. Then for small enough* $\delta > 0$*, the BSDE* (3.3.1) *has a unique solution* $(Y,Z) \in \mathcal{B}_R$ *on* $[T-\delta,T]$.

Proof. In the whole proof we assume $t \in [T-\delta, T]$ and treat $\rho = \rho(R)$ as a constant. This is possible because Y will turn out to be bounded by R. For $(y,z) \in \mathcal{B}_R$, define $\phi(y,z) := (Y,Z)$, where (Y,Z) is the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T Z_s dW_s, \quad T - \delta \leq t \leq T. \tag{3.3.2}$$

Since for every $X \in \mathbb{L}^2(\mathcal{F}_T)^d$, there exists a unique $H \in \mathbb{H}^2_{[T-\delta,T]}(\mathbb{R}^{d \times n})$ such that

$$\int_{T-\delta}^T H_s dW_s = X - \mathbb{E}_{T-\delta} X,$$

it follows from standard arguments (see El Karoui et al., 1997) that (3.3.2) has a unique solution (Y,Z) in $\mathbb{S}^2_{[T-\delta,T]}(\mathbb{R}^d) \times \mathbb{H}^2_{[T-\delta,T]}(\mathbb{R}^{d\times n})$. Moreover, if one takes $\mathbb{E}_t$, one obtains

$$|Y_t| \leq \left| \mathbb{E}_t \left(\xi + \int_t^T f(s, y_s, z_s) ds \right) \right| \leq C + \mathbb{E}_t \int_t^T |f(s, y_s, z_s)| ds.$$

Let τ be a stopping time with values in $[T-\delta, T]$. It follows from (B2) and Hölder's inequality that

$$\begin{aligned} & \mathbb{E}_\tau \int_\tau^T |f(s, y_s, z_s)| ds \leq C \mathbb{E}_\tau \int_\tau^T \left(1 + |y_s| + \rho |z_s|^{2-\varepsilon}\right) ds \\ \leq \;& C\delta \left(1 + \|y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\right) + C\rho \mathbb{E}_\tau \int_\tau^T |z_s|^{2-\varepsilon} ds \leq C\delta \left(1 + \|y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\right) + C\rho\delta^{\varepsilon/2} \|z\|^{2-\varepsilon}_{\mathrm{B}_{[T-\delta,T]}} \\ \leq \;& C\delta (1 + R) + C\rho\delta^{\varepsilon/2} R^{2-\varepsilon}. \end{aligned}$$

Choose $\delta > 0$ so small that $C\delta(1+R) + C\rho\delta^{\varepsilon/2}R^{2-\varepsilon} \leq C$. Then $\|Y\|_{\mathbb{S}^\infty_{[T-\delta,T]}} \leq 2C \leq R$. Moreover, Itô's formula gives

$$|Y_\tau|^2 + \int_\tau^T |Z_s|^2 ds = |\xi|^2 + 2\int_\tau^T Y_s^T f(s, y_s, z_s) ds - 2\int_\tau^T Y_s^T Z_s dW_s,$$

and therefore,

$$\mathbb{E}_\tau \int_\tau^T |Z_s|^2 ds \leq C^2 + 2\|Y\|_{\mathbb{S}^\infty_{[T-\delta,T]}} \mathbb{E}_\tau \int_\tau^T |f(s, y_s, z_s)| ds \leq 5C^2,$$

which shows that $\|Z\|_{\mathrm{B}_{[T-\delta,T]}} \leq 3C \leq R$. So ϕ maps $\mathcal{B}_R$ into itself.

Next, we show that ϕ is a contraction on $\mathcal{B}_R$. Choose $(y,z), (y',z') \in \mathcal{B}_R$ and denote $(Y,Z) =$

$\phi(y,z)$, $(Y',Z')=\phi(y',z')$, $\Delta y=y-y'$, $\Delta z=z-z'$, $\Delta Y=Y-Y'$, and $\Delta Z=Z-Z'$. Then,

$$\Delta Y_t=\int_t^T\left(f(s,y_s,z_s)-f(s,y'_s,z'_s)\right)ds-\int_t^T\Delta Z_s dW_s,$$

and by Itô's formula,

$$|\Delta Y_\tau|^2+\int_\tau^T|\Delta Z_s|^2ds=2\int_\tau^T\Delta Y_s^T\left(f(s,y_s,z_s)-f(s,y'_s,z'_s)\right)ds-2\int_\tau^T\Delta Y_s^T\Delta Z_s dW_s.$$

It follows from (B3) that

$$\begin{aligned}
&\mathbb{E}_\tau\int_\tau^T|f(s,y_s,z_s)-f(s,y'_s,z'_s)|\,ds\le\mathbb{E}_\tau\int_\tau^T\rho\left(|\Delta y_s|+\left(1+(|z_s|\vee|z'_s|)^{1-\varepsilon}\right)|\Delta z_s|\right)ds\\
&\le\rho\delta\left\|\Delta y\right\|_{\mathbb{S}^\infty_{[T-\delta,T]}}+\rho\sqrt{\mathbb{E}_\tau\int_\tau^T\left(1+(|z_s|\vee|z'_s|)^{1-\varepsilon}\right)^2ds}\sqrt{\mathbb{E}_\tau\int_\tau^T|\Delta z_s|^2\,ds}\\
&\le\rho\delta\left\|\Delta y\right\|_{\mathbb{S}^\infty_{[T-\delta,T]}}+\rho\sqrt{2\left(\delta+\mathbb{E}_\tau\int_\tau^T(|z_s|+|z'_s|)^{2-2\varepsilon}\,ds\right)}\left\|\Delta z\right\|_{\mathrm{B}_{[T-\delta,T]}}\\
&\le\rho\delta\left\|\Delta y\right\|_{\mathbb{S}^\infty_{[T-\delta,T]}}+\rho\sqrt{2\left(\delta+\delta^\varepsilon(2R)^{2-2\varepsilon}\right)}\left\|\Delta z\right\|_{\mathrm{B}_{[T-\delta,T]}}.
\end{aligned}$$

So for $\delta>0$ is small enough, one has

$$\begin{aligned}
&\mathbb{E}_\tau\int_\tau^T|f(s,y_s,z_s)-f(s,y'_s,z'_s)|\,ds\le\frac{1}{4}\left(\|\Delta y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}+\|\Delta z\|_{\mathrm{B}_{[T-\delta,T]}}\right)\\
&\le\frac{1}{2}\left(\|\Delta y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\vee\|\Delta z\|_{\mathrm{B}_{[T-\delta,T]}}\right).
\end{aligned}$$

Therefore,

$$\|\Delta Y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\le\frac{1}{2}\left(\|\Delta y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\vee\|\Delta z\|_{\mathrm{B}_{[T-\delta,T]}}\right)$$

and

$$\begin{aligned}
&\|\Delta Z\|^2_{\mathrm{B}_{[T-\delta,T]}{}_{[T-\delta,T]}}\le2\,\|\Delta Y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}\sup_\tau\mathbb{E}_\tau\int_\tau^T|f(s,y_s,z_s)-f(s,y'_s,z'_s)|\,ds\\
&\le\frac{1}{2}\left(\|\Delta y\|_{\mathbb{S}^\infty_{[T-\delta,T]}}+\|\Delta z\|_{\mathrm{B}_{[T-\delta,T]}}\right)^2.
\end{aligned}$$

This shows that for $\delta>0$ small enough, ϕ is a contraction on $\mathcal{B}_R$. So on $[T-\delta,T]$, (3.3.1) has a unique solution in $\mathcal{B}_R$. □

In the next step we show that under the additional condition (B4), the solution of Lemma 3.3.2 satisfies

$$|Y_t| \leq (C+1)\exp\left(\frac{(C+1)^2}{2}(T-t)\right).$$

This allows one to construct a solution on the whole time interval $[0,T]$ recursively backwards in time. Note that if the driver f satisfies (B4), one can write

$$G^i(s,y,z) = z^i g^i(s,y,z)$$

for

$$g^i(s,y,z) := \begin{cases} \frac{(z^i)^T G^i(s,y,z)}{|z^i|^2} & \text{if } z^i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (B4) that $\left|g^i(s,y,z)\right| \leq C_i \rho(|y|)|z| \leq C\rho(|y|)|z|$.

Lemma 3.3.3. *Assume* (B1) *and* (B4) *hold and for some* $\delta > 0$*, the BSDE* (3.3.1) *has a solution* $(Y,Z) \in \mathbb{S}^\infty_{[T-\delta,T]}(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}_{[T-\delta,T]}(\mathbb{R}^{d\times n})$*. Then*

$$|Y_t| \leq (C+1)\exp\left(\frac{(C+1)^2}{2}(T-t)\right).$$

Proof. For $a \in \mathbb{R}_+$ one obtains from Itô's formulas that

$$\begin{aligned} e^{at}|Y_t|^2 &= e^{aT}|\xi|^2 + 2\int_t^T e^{as}\left(Y_s^T f(s,Y_s,Z_s) - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2\right)ds - 2\int_t^T e^{as} Y_s^T Z_s dW_s \\ &= e^{aT}|\xi|^2 + 2\int_t^T e^{as}\left(Y_s^T f(s,Y_s,Z_s) - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2 - Y_s^T Z_s\left(C\rho(|Y_s|)|Z_s|\frac{Y_s^T Z_s}{|Y_s^T Z_s|}\right)^T\right)ds \\ &\quad - 2\int_t^T e^{as} Y_s^T Z_s d\tilde{W}_s, \end{aligned}$$

where

$$\tilde{W}_s = W_s - \int_0^t \left(C\rho(|Y_s|)|Z_s|\frac{Y_s^T Z_s}{|Y_s^T Z_s|}\right)^T ds$$

is a Brownian motion under the equivalent probability measure

$$\tilde{\mathbb{P}} := \exp\left(\int_0^T \left(C\rho(|Y_s|)|Z_s|\frac{Y_s^T Z_s}{|Y_s^T Z_s|}\right) dW_s - \frac{1}{2}\int_0^T \left(C\rho(|Y_s|)|Z_s|\right)^2 ds\right)\cdot \mathbb{P}.$$

Denote the expectation with respect to $\tilde{\mathbb{P}}$ as $\tilde{\mathbb{E}}$ and notice that

$$
\begin{aligned}
&Y_s^T f(s,Y_s,Z_s) - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2 - Y_s^T Z_s \left(C\rho(|Y_s|)|Z_s| \frac{Y_s^T Z_s}{|Y_s^T Z_s|} \right)^T \\
&\le \left| Y_s^T F(s,Y_s,Z_s) \right| - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2 + \left| \sum_i Y_s^i Z_s^i g^i(s,Y_s,Z_s) \right| - |Y_s^T Z_s| C\rho(|Y_s|)|Z_s| \\
&\le |Y_s|\,|F(s,Y_s,Z_s)| - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2 \\
&\le C(|Y_s| + |Y_s|^2 + |Y_s||Z_s|) - \frac{a}{2}|Y_s|^2 - \frac{1}{2}|Z_s|^2 \\
&\le \frac{C^2}{2} + \frac{1}{2}\left((C+1)^2 - a \right)|Y_s|^2.
\end{aligned}
$$

So for $a = (C+1)^2$ one gets

$$
e^{at}|Y_t|^2 \le C^2 e^{aT} + C^2 \int_t^T e^{as} ds - 2\int_t^T e^{as} Y_s^T Z_s d\tilde{W}_s,
$$

which by taking $\tilde{\mathbb{E}}_t$ gives $|Y_t|^2 \le C^2 e^{a(T-t)}(a+1)/a$, and therefore,

$$
|Y_t| \le C \exp\left(\frac{(C+1)^2}{2}(T-t) \right) \sqrt{\frac{C^2+1}{C^2}} \le (C+1)\exp\left(\frac{(C+1)^2}{2}(T-t) \right).
$$

□

Proof of Theorem 3.3.1

Fix $N \in \mathbb{N}$ such that Lemma 3.3.2 holds with $\delta = T/N$ and $(C+1)\exp\left((C+1)^2 T/2\right)$ instead of C. Then (3.3.1) has a unique solution $(Y^{(1)}, Z^{(1)}) \in \mathbb{S}^\infty_{[T-\delta,T]}(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}_{[T-\delta,T]}(\mathbb{R}^{d\times n})$ on $[T-\delta, T]$, and it follows from Lemma 3.3.3 that $|Y_t^{(1)}| \le \varphi(t) := (C+1)\exp\left((C+1)^2(T-t)/2\right)$. Another application of Lemma 3.3.2 yields that on $[T-2\delta, T-\delta]$, the BSDE (3.3.1) with terminal condition $Y^{(1)}_{T-\delta}$ has a unique solution $(Y^{(2)}, Z^{(2)}) \in \mathbb{S}^\infty_{[T-2\delta,T-\delta]}(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}_{[T-2\delta,T-\delta]}(\mathbb{R}^{d\times n})$. Pasting together the two solutions gives a solution $(Y,Z) \in \mathbb{S}^\infty_{[T-2\delta,T]}(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}_{[T-2\delta,T]}(\mathbb{R}^{d\times n})$ on $[T-2\delta, T]$. By Lemma 3.3.3, it satisfies $|Y_t| \le \varphi(t)$. Continuing like this gives a unique solution $(Y,Z) \in \mathbb{S}^\infty(\mathbb{R}^d) \times \mathbb{H}^{\mathrm{BMO}}(\mathbb{R}^{d\times n})$ on $[0,T]$, which by Lemma 3.3.3, satisfies $|Y_t| \le \varphi(t)$. □

3.4 Further Discussions

The first important remaining question is whether we can prove the existence of solution for multidimensional non-Markovian quadratic BSDEs. This is plausible, since we can prove it for any arbitrary small ε in Theorem 3.3.1. Moreover, if the BSDE is projectable, then we can prove it for $\varepsilon = 0$. Let us explain the important obstacle for extending $\varepsilon \in (0, 1)$ to $\varepsilon \in [0, 1)$. Since the terminal condition is bounded and ε is strictly positive in (B2) and (B3), we are able to use the Banach fixed point theorem to prove the local existence of a solution. Then we remove the superlinear term of the driver by the Girsanov transform using (B4). This allows us to estimate the process Y and iterate our local scheme to prove the existence of a global solution. Therefore, if one can prove the existence of a local solution for a quadratic driver and a bounded terminal condition, our conditions (B1) and (B4) guarantee the existence of a global solution. However, the existence of a local solution for strictly quadratic driver is not known except for the case where the quadratic part of the driver can be removed using change of variables.

Another important question is whether we can relax the condition (B4). The condition (B4) is used to express f as a sum of a Lipschitz term and a superlinear term so that we can subtract the superlinear term $G(s, y, z) = Zg(s, Y, Z)$ by the Girsanov transform. Consider G defined by

$$G(s, y, z) = \sum_{i=1}^{d} \sum_{j=1}^{n} g^{ij}(s, y, z) z^{ij}$$

for a Lipschitz function g. If $d = 1$, $G(s, y, z) = zg(s, y, z)$; therefore, G can be removed by the Girsanov theorem. However, if $d > 1$, we have remaining the following term,

$$\sum_{i \neq j} g^{ij}(s, y, z) z^{ij}$$

after the Girsanov transform. Therefore, our Girsanov transform method cannot be generalized further.

Chapter 4

Malliavin Calculus Technique

In this chapter, we assume $\mathbb{F} = \mathbb{F}^W$ and show the existence and uniqueness of solutions to (1.1.1) in the case where the terminal condition ξ has a bounded Malliavin derivative.

If the driver $f(s, y, z)$ is Lipschitz in (y, z), it can be shown with a Picard iteration argument that (1.1.1) has a unique solution for any square-integrable terminal condition ξ (see Pardoux and Peng [65]). When $d = 1$, Kobylanski [55] proved the existence of a unique solution in the case where f does not grow faster than quadratically in z and ξ is bounded. BSDEs with drivers of quadratic growth in z and unbounded terminal conditions have been studied by Briand and Hu [12, 13] as well as Delbaen et al. [29]. Delbaen et al. [28] showed that if the driver f only depends on z, is convex and has superquadratic growth, there exist bounded terminal conditions such that the BSDE (1.1.1) has no solution with bounded Y, and if the BSDE admits a solution with bounded Y, it has infinitely many of them. Moreover, they proved the existence of solution for Markovian BSDEs when the terminal value is a bounded continuous function of the terminal value of a forward process. Richou [74] derived the existence of solutions to more general Markovian BSDEs in the case where f and ξ satisfy a local Lipschitz condition with respect to the underlying forward process. In Cheridito and Stadje [18], it is shown that BSDEs whose drivers are convex in z have unique solutions with bounded Z if f and ξ are Lipschitz continuous functionals of the path of the underlying Brownian motion.

However, when $d > 1$, the well-posedness of BSDE has not progressed much since Pardoux and Peng [65]. Bahlali [4] verified it in the case where the Lipschitz constant L_N in the ball of radius N satisfies $L_N \sim \sqrt{\log N}$ and then further generalized the well-posedness result to subquadratic monotone driver in [5] by the same author.

Recently, Malliavin calculus has also been applied to the study of one-dimensional BSDEs by Hu et al. [46] and Briand and Elie [11] for Lipschitz BSDEs and one-dimensional quadratic BSDEs.

Since BSDE is a generalization of martingale representation, one can expect a BSDE version of the Clark-Ocone formula. Indeed, when the terminal condition and the driver are standard parameters which are Malliavin differentiable, El Karoui et al. [34] proved that this is the case. Their proposition allows one to express Z as Malliavin derivative of Y, that is, $Z_t = D_t Y_t$. Since Malliavin derivative of the solution satisfies a differentiated BSDE, one can find uniform bound on Z. Then, the usual cutoff argument can be used to further study the case where f may grow arbitrarily fast in z without assuming Markov or convexity assumptions. To be more precise, we use the following steps.

1. Cut off the driver so that the localized driver is a standard parameter.

2. Solve a BSDE with localized driver and denote the solution (Y, Z).

3. Show that $|Z|$ is smaller than the cutoff parameter.

4. Conclude (Y, Z) is the solution for the original BSDE.

In particular, the driver $f(s, y, z)$ is assumed to be Lipschitz continuous in y but only locally Lipschitz continuous in z. If ξ is bounded in addition to having a bounded Malliavin derivative, the driver needs only be locally Lipschitz continuous in y. When $d > 1$, we have a unique solution for small time T. If $d = 1$, using the comparison theorem, we have a unique global solution such that Z is bounded. In the special case where $d = 1$ and the BSDE is Markovian, we obtain existence and uniqueness results for semilinear parabolic PDEs with non-Lipschitz nonlinearities. We discuss the case where there is no lateral boundary as well as those in which there is lateral boundary condition of Dirichlet or Neumann type.

We present preliminary results based on El Karoui et al. [34] in Section 4.1. The result will be used in Section 4.2 and 4.3. In Section 4.2, we show that if ξ has a bounded Malliavin derivative and T is small enough, (1.1.1) has a unique solution for drivers f that are Lipschitz in y and locally Lipschitz in z. In Section 4.3, we will show the global existence and uniqueness of global solution under analogous conditions Moreover, if ξ is also bounded, f only needs to be locally Lipschitz in y. In Sections 4.4, we generalize the results on the relation between Markovian BSDEs and semilinear parabolic PDEs to the case of non-Lipschitz nonlinearities. In Section 4.4.1 we study Markovian BSDEs based on forward processes following standard diffusion dynamics and related PDEs for functions $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$. The general results of Section 4.3 allow us to extend findings of Amour and Ben-Artzi [1] and Gilding et al. [37] on the existence of solutions to nonlinear heat equations without lateral boundaries. Section 4.4.2 is devoted to BSDEs with random terminal times and parabolic PDEs with lateral boundary conditions of Dirichlet type. Finally, Section 4.4.3 discusses BSDEs based on reflected forward processes and their relation to parabolic PDEs with lateral boundary conditions of Neumann type.

4.1 Preliminary Results

Let us briefly review Malliavin calculus. For given $l \in \mathbb{N}$, let us denote

$$\begin{aligned}
\mathcal{H} &:= L^2([0,T];\mathbb{R}^n) \\
C_p^\infty(\mathbb{R}^k;\mathbb{R}^l) &:= \left\{ \text{the set of smooth functions } f : \mathbb{R}^k \to \mathbb{R}^l \text{ with polynomial growth} \right\} \\
W(h) &:= \int_0^T h(t)dW_t, h \in \mathcal{H} \\
\mathcal{S} &:= \left\{ f(W(h_1), W(h_2), \cdots, W(h_k)) : f \in C_p^\infty(\mathbb{R}^k;\mathbb{R}^l), \{h_1, \cdots, h_k\} \subset \mathcal{H}, k \in \mathbb{N} \right\}.
\end{aligned}$$

Then we can define Malliavin derivative on $\mathcal{S}$.

Definition 4.1.1. *For $\xi \in \mathcal{S}$, the Malliavin derivative $D\xi$ is $\mathcal{H}^l$-valued random variable*

$$D\xi = (\nabla f)(W(h_1), \cdots, W(h_k))(h_1, \cdots, h_k)^T$$

Then, it is known that D is closable and we denote $\mathbb{D}^{1,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|\xi\|_{\mathbb{D}^{1,p}}^p := \mathbb{E}|\xi|^p + \mathbb{E}\left(\int_0^T |D_t\xi|^2 dt\right)^{p/2}.$$

We define Malliavin derivative D with domain $\mathbb{D}^{1,p}$ as the closure of the Malliavin derivative on $\mathcal{S}$.

Extensive details about Malliavin calculus can be found in Nualart [63].

As a special case, if the random variable ξ is Lipschitz with respect to underlying Brownian motion, ξ has bounded Malliavin derivative.

Definition 4.1.2. *We denote the space of all continuous functions from $[0,T]$ to $\mathbb{R}^n$ starting from 0 by $C_0^n[0,T]$ and call a random variable ξ Lipschitz continuous in the Brownian motion W with constants $A_1,\ldots,A_n \in \mathbb{R}_+$ if $\xi = \varphi(W)$ for a function $\varphi : C_0^n[0,T] \to \mathbb{R}$ satisfying*

$$|\varphi(v) - \varphi(w)| \leq \sum_{i=1}^{n} A_i \sup_{0\leq t\leq T} |v^i(t) - w^i(t)|. \tag{4.1.1}$$

Note that this Lipschitzness is different from L-Lipschitzness defined in Definition 2.3.4.

Proposition 4.1.3. *Let ξ be Lipschitz continuous in W with constants $A_1,\ldots,A_n \in \mathbb{R}_+$. Then $\xi \in \mathbb{D}^{1,2}$ and $|D_t^i\xi| \leq A_i \; dt \otimes d\mathbb{P}$-a.e. for all $i = 1,\ldots,n$.*

Boundedness of Malliavin derivative does not imply the Lipschitzness. The proof and the counterexample will be provided in Section B. Another result we need is that bounded Malliavin derivative implies the random variable is in $\mathbb{L}^p$ for all p.

Lemma 4.1.4. *If $\xi \in \mathbb{D}^{1,2}$ and $D\xi$ is bounded, then $\mathbb{E}|\xi|^p < \infty$ for all $p \in [1,\infty)$.*

Proof. Since $\xi \in \mathbb{D}^{1,2}$, it is square-integrable. By the Clark–Ocone formula, one can represent ξ as $\xi = \mathbb{E}[\xi] + \int_0^T \mathbb{E}[D_t\xi|\mathcal{F}_t]\, dW_t$. Applying the Burkholder–Davis–Gundy inequality to the martingale $M_t = \int_0^t \mathbb{E}[D_t\xi|\mathcal{F}_s]\, dW_s$, one obtains a constant $c_p \in \mathbb{R}_+$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|^p\right] \leq c_p \mathbb{E}\left[\left(\int_0^T |\mathbb{E}[D_t\xi|\mathcal{F}_t]|^2 dt\right)^{p/2}\right] < \infty,$$

which proves the lemma. □

The next proposition is Proposition 5.3 of El Karoui et al. [34] and it is a starting point of this chapter. Note that the proposition is about multidimensional BSDEs with standard parameter (ξ, f).

Proposition 4.1.5. *Assume the following conditions*

- $\xi \in \mathbb{D}^{1,2} \cap \mathbb{L}^4(\mathcal{F}_T)$
- $f(\omega,s,y,z) \in \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \mapsto f(\omega,s,y,z) \in \mathbb{R}^d$ *is continuously differentiable in (y,z) and there exist constants $B,\rho \in \mathbb{R}_+$ such that $|\partial_y f| \leq B$ and $|\partial_z f| \leq \rho$*
- $f(.,0,0) \in \mathbb{H}^4(\mathbb{R}^d)$, $f(\cdot,y,z) \in \mathbb{L}_a^{1,2}$ *for all (y,z), and*

$$\int_0^T \mathbb{E}|(D_r f)(t,Y,Z)|^2 dr < \infty$$

where (Y, Z) is a solution of BSDE(ξ, f)

- *For a.e. $r \in [0,T]$, there exists a non-negative process $K_{r\cdot}$ in $\mathbb{H}^4(\mathbb{R})$ such that*

$$\int_0^T ||K_{r,\cdot}||^4_{\mathbb{H}^4} dr < \infty \quad \text{and} \quad |D_r f(t, y_1, z_1) - D_r f(t, y_2, z_2)| \le K_{rt}(|y_1 - y_2| + |z_1 - z_2|),$$

for all $t \in [0,T]$ and $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$.

Then, $(Y, Z) \in \mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^4(\mathbb{R}^{d\times n})$ and for fixed $i = 1, \dots, n$,

$$(D^i_r Y_t, D^i_r Z_t) = (U^r_t, V^r_t) \;\; dr \otimes dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{and} \quad Z^i_t = U^t_t \;\; dt \otimes d\mathbb{P}\text{-a.e.},$$

where

$$U^r_t = 0, \; V^r_t = 0, \quad 0 \le t < r \le T,$$

and for each fixed r, $(U^r_t, V^r_t)_{r\le t\le T}$ is the unique pair in $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$ solving the BSDE

$$U^r_t = D^i_r \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U^r_s + \partial_z f(s, Y_s, Z_s) V^r_s + D^i_r f(s, Y_s, Z_s)] ds - \int_t^T V^r_s dW_s. \quad \textbf{(4.1.2)}$$

El Karoui et al. [34] also stated the following remark.

Remark 4.1.6. *If $K_{r\cdot}$ is bounded for almost all $r \in [0,T]$, above proposition still holds if the conditions $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ and $\xi \in \mathbb{L}^4$ are dropped except that then, (Y, Z) is in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and not necessarily in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$.*

The proof of Proposition 4.1.5 will be provided in Section B. In order to study BSDEs with Lipschitz drivers which is not continuously differentiable, we need the following stability result.

Proposition 4.1.7 (Proposition 5.1 of El Karoui et al. [34])**.** *For every $L \in \mathbb{R}_+$ there exist constants $\mu, \nu > 0$ satisfying the following: If $T \le \mu$, then for all 2-standard parameters (f^i, ξ^i), $i = 1, 2$, such that f^1 fulfills*

$$\left|f^1(t, y, z) - f^1(t, y', z')\right| \le L(|y - y'| + |z - z'|)$$

for all $(y, z), (y', z') \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$, the BSDE solutions (Y^i, Z^i) corresponding to (f^i, ξ^i) satisfy

$$\left\|Y^1 - Y^2\right\|^2_{\mathbb{S}^2} + \left\|Z^1 - Z^2\right\|^2_{\mathbb{H}^2} \le \nu\, \mathbb{E}\left[\left|\xi^1 - \xi^2\right|^2 + \int_0^T \left|f^1(t, Y^2_t, Z^2_t) - f^2(t, Y^2_t, Z^2_t)\right|^2 dt\right].$$

The proof will be provided in Section B as well.

4.2 Local Solution for Multidimensional BSDEs

We assume the following conditions in this section.

(H1) The terminal condition ξ is a $\mathcal{F}_T$-measurable random variable in $(\mathbb{D}^{1,2})^d$. For each $i = 1, 2 \cdots, n$, there exists a function $A_i : [0,T] \to [0, C_i]$ such that $\sqrt{\mathbb{E}_t|D_r^i \xi|^2} \leq A_i(t)$ for almost all $(r, \omega) \in [0,T] \times \Omega$. Let us denote $A := \sqrt{A_1^2 + \cdots + A_n^2}$ and $C := \sqrt{C_1^2 + \cdots + C_n^2}$.

(H2) There exist function $\rho : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $B : [0,T] \to \mathbb{R}$ such that

$$f \text{ is Lipschitz in } y \quad \text{and} \quad (y - y')^T \left(f(s,y,z) - f(s,y',z)\right) \leq B(s) \left|y - y'\right|^2 \tag{4.2.1}$$

$$|f(s,y,z) - f(s,y,z')| \leq \rho(s,|z|)|z - z'| \tag{4.2.2}$$

for all $(y,z), (y',z') \in \mathbb{R}^d \times \mathbb{D}^{d\times n}$, and $s \in [0,T]$. Moreover, $\rho(s,x)$ is assumed to be uniformly bounded in s if x is bounded and nondecreasing in x for each fixed s.

(H3) $f(.,0,0) \in \mathbb{H}^4(\mathbb{R}^d)$ and there exist Borel measurable functions $q_i : [0,T] \to \mathbb{R}_+$ satisfying $D_i^2 := \int_0^T q_i^2(t)dt < \infty$ such that for every pair $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$ and for $i = 1, 2 \cdots, n$, one has $f(\cdot, y, z) \in \mathbb{L}_a^{1,2}(\mathbb{R}^d)$ and $|D_r^i f(t,y,z)| \leq q_i(t) \quad dr \otimes d\mathbb{P}$-a.e. We denote $D := \sqrt{D_1^2 + \cdots + D_n^2}$.

(H4) For a.e. $r \in [0,T]$, there exists a non-negative process $K_{r\cdot}$ in $\mathbb{H}^4(\mathbb{R})$ such that

$$\int_0^T ||K_{r,\cdot}||_{\mathbb{H}^4}^4 dr < \infty \quad \text{and} \quad |D_r f(t,y_1,z_1) - D_r f(t,y_2,z_2)| \leq K_{rt}(|y_1 - y_2| + |z_1 - z_2|),$$

for all $t \in [0,T]$ and $(y_1,z_1), (y_2,z_2) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$.

(H5) There exists a bounded function $r : [0,T] \to \mathbb{R}_+$ such that

$$\sqrt{(A(t))^2 + D^2} \exp\left(\frac{1}{2}\int_t^T (2B(s) + \rho^2(s, r(s)) + 1)ds\right) \leq r(t) \tag{4.2.3}$$

for all $t \in [0,T]$

The main result of this section is the following theorem. Its proof is provided in the next subsection.

Theorem 4.2.1. *If* (H1)–(H5) *hold, then the BSDE* (1.1.1) *has a unique solution* $(Y, Z) \in \mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^4(\mathbb{R}^{d\times n})$ *such that* Z_s *is bounded. In particular,* $|Z_s| \le r(s)$.

The condition (H5) is essential in our proof. In the case where A and B are constant and ρ is a function of x, (H5) requires T to be sufficiently small. Now we give two examples of sufficient conditions the existence of desirable $r(t)$ in (H5). The first example is the case where $f(s, y, z)$ grows polynomially in z.

Proposition 4.2.2. *For* $m > 0$*, let* $\rho(s, x) := b(s)|x|^m + c(s)$ *for some bounded functions* $b, c : [0, T] \to \mathbb{R}_+$*. If there exist* $R > 0$ *such that*

$$\int_0^T b^2(s) \exp\left(m \int_s^T \left(2B(u) + (1 + R^{-1})|c(u)|^2 + 1\right) du \right) ds < \frac{1}{2m(1 + R)\,(C^2 + D^2)^m},$$

then (4.2.3) *has a bounded solution* $r(t)$ *in* $[0, T]$.

Proof. For $R > 0$, let us solve

$$-2r' = (2B + (1 + R)b^2 r^{2m} + (1 + R^{-1})c^2 + 1)r \text{ with } r(T) = \sqrt{C^2 + D^2}$$

If there exists a bounded solution r in time $[0, T]$, (4.2.3) is satisfied by such r because

$$(2B + (br^m + c)^2 + 1)r \le (2B + (1 + R)b^2 r^{2m} + (1 + R^{-1})c^2 + 1)r.$$

Let $y(t) := d(t)r^m(t)$ where

$$d(t) = \exp\left(-\frac{m}{2} \int_t^T (2B(s) + (1 + R^{-1})|c(s)|^2 + 1)ds \right)$$

and $y(T) = \left(C^2 + D^2\right)^{m/2}$. Then,

$$y' = \frac{mr'y}{r} + m\left(B + \frac{(1 + R^{-1})c^2 + 1}{2}\right) y = my\left(\frac{r'}{r} + \left(B + \frac{(1 + R^{-1})c^2 + 1}{2}\right)\right) \tag{4.2.4}$$

$$= -my\left(\frac{1 + R}{2} b^2 r^{2m}\right) = -\frac{m(1 + R)b^2}{d^2} y^3 \tag{4.2.5}$$

Therefore

$$r(t) = d(t)^{-1/m}\left((C^2 + D^2)^{-m} - 2m(1 + R) \int_t^T \frac{b^2(s)}{d^2(s)} ds \right)^{-\frac{1}{2m}}$$

Since

$$(C^2 + D^2)^{-m} - 2m(1+R)\int_0^T \frac{b^2(s)}{d^2(s)}ds > 0$$

for some $q > 0$, $r(t)$ is bounded in $[0, T]$. $\square$

4.2.1 Proof of main theorem

We first prove Theorem 4.2.1 under more restrictive condition (H2').

(H2') $f(t, y, z)$ is continuously differentiable in (y, z) and there exist function $\rho : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $B : [0, T] \to \mathbb{R}$ such that

$$f \text{ is Lipschitz in } y \quad \text{and} \quad v^T \partial_y f(s, y, z) v \le B(s) \text{ for all unit vector } v \in \mathbb{R}^d \tag{4.2.6}$$

$$|\partial_z f(s, y, z)| \le \rho(s, |z|) \tag{4.2.7}$$

for all $(s, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n}$. Moreover, $\rho(s, x)$ is assumed to be bounded if x is bounded and nondecreasing in x for each fixed s.

Proposition 4.2.3. *Assume* (H1), (H2'), (H3), and (H4) *with* $\rho(s, x) = \rho(s)$. *Then the BSDE* (1.1.1) *has a unique solution* $(Y, Z) \in \mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^4(\mathbb{R}^{d\times n})$ *such that*

$$|Z_t| \le \sqrt{(A(t))^2 + D^2}\ \exp\left(\frac{1}{2}\int_t^T (2B(s) + \rho^2(s) + 1)ds\right) \tag{4.2.8}$$

for almost all $t \in [0, T]$.

Proof. By Lemma 4.1.4, $\xi \in \mathbb{L}^p$ for all $p < \infty$. So it follows from Proposition 4.1.5 that BSDE (1.1.1) has a unique solution (Y, Z) in $\mathbb{S}^4 \times \mathbb{H}^4$. Moreover, $(Y, Z) \in \mathbb{L}_a^{1,2}(\mathbb{R}^{d+d\times n})$, and, for fixed $i = 1, \ldots, n$,

$$(D_r^i Y_t, D_r^i Z_t) = (U_t^r, V_t^r)\ \ dr \otimes dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{and} \quad Z_t^i = U_t^t\ \ dt \otimes d\mathbb{P}\text{-a.e.},$$

where

$$U_t^r = 0,\ V_t^r = 0, \quad 0 \le t < r \le T,$$

and for each fixed r, $(U_t^r, V_t^r)_{r\le t\le T}$ is the unique pair in $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$ solving the BSDE

$$U_t^r = D_r^i \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U_s^r + \partial_z f(s, Y_s, Z_s) V_s^r + D_r^i f(s, Y_s, Z_s)]ds - \int_t^T V_s^r dW_s. \tag{4.2.9}$$

Using Itô formula on $e^{h(t)}|U_t^r|^2$ for some differentiable function h, one obtains

$$\begin{aligned} e^{h(t)}|U_t^r|^2 = e^{h(T)}|D_r^i\xi|^2 - h'(t)\int_t^T e^{h(s)}|U_s^r|^2 ds \\ + 2\int_t^T e^{h(s)}(U_s^r)^T\left(\partial_y f(s,Y_s,Z_s)U_s^r + \partial_z f(s,Y_s,Z_s)V_s + D_r^i f(s,Y_s,Z_s)\right)ds \\ - 2\int_t^T e^{h(s)}(U_s^r)^T V_s^r dW_s - \int_t^T e^{h(s)}|V_s^r|^2 ds. \end{aligned}$$

It follows from the assumptions that for all $p, q > 0$,

$$\begin{aligned} &(U_s^r)^T\left(\partial_y f(s,Y_s,Z_s)U_s^r + \partial_z f(s,Y_s,Z_s)V_s + D_r^i f(s,Y_s,Z_s)\right) \\ &\le B(s)|U_s^r|^2 + \rho(s)|U_s^r||V_s^r| + |U_s^r||D_s^i f(s,Y_s,Z_s)| \\ &\le B(s)|U_s^r|^2 + \frac{\rho^2(s)}{2}|U_s^r|^2 + \frac{1}{2}|V_s^r|^2 + \frac{1}{2}|U_s^r|^2 + \frac{1}{2}q_i^2(s) \\ &\le \left(B(s) + \frac{\rho^2(s)}{2} + \frac{1}{2}\right)|U_s^r|^2 + \frac{1}{2}|V_s^r|^2 + \frac{1}{2}q_i^2(s). \end{aligned}$$

So for $h(t) = \int_0^t (2B(s) + \rho^2(s) + 1)ds$, one has

$$e^{h(t)}|U_t^r|^2 \le e^{h(T)}|D_r^i\xi|^2 + \int_t^T e^{h(s)}q_i^2(s)ds - 2\int_t^T e^{h(s)}\langle U_s^r, V_s^r dW_s\rangle. \tag{4.2.10}$$

Since $(U^r, V^r) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$, the process

$$M_t := \int_0^t e^{h(s)}\langle U_s^r, V_s^r dW_s\rangle$$

is a local martingale satisfying

$$\begin{aligned} \mathbb{E}\sqrt{\langle M\rangle_T} &\le \mathbb{E}\left[\left(\int_0^T e^{2h(s)}|U_s^r|^2|V_s^r|^2 ds\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{2}\exp\left(\max_{s\in[0,T]} h(s)\right)\left(\mathbb{E}\sup_{0\le s\le T}|U_s^r|^2 + \mathbb{E}\int_0^T |V_s^r|^2 ds\right) < \infty. \end{aligned}$$

So it follows from the Burkholder–Davis–Gundy inequality that M is a martingale. By applying $\mathbb{E}_t$ to (4.2.10) one obtains

$$e^{h(t)}|U_t^r|^2 \le e^{h(T)}\left((A_i(t))^2 + \int_t^T q_i^2(s)ds\right),$$

If we summing over i and let $r = t$, we have

$$|Z_t|^2 \leq e^{h(T)-h(t)} \left((A(t))^2 + D^2\right),$$

$\square$

Proof of Theorem 4.2.1

Define

$$\hat{f}(t,y,z) = \begin{cases} f(t,y,z) & \text{if } |z| \leq r(t) \\ f(t,y,r(t)z/|z|) & \text{if } |z| > r(t) \end{cases}.$$

Then, $(\hat{f},\xi)$ are 4-standard parameters. So the corresponding BSDE has a unique solution (Y,Z) in $\mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^4(\mathbb{R}^{d\times n})$. Denote $x = (y,z) \in \mathbb{R}^{d+d\times n}$ and let $\beta \in C_c^\infty(\mathbb{R}^{d+d\times n})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases},$$

where the constant $\lambda \in \mathbb{R}_+$ is chosen so that $\int_{\mathbb{R}^{d+dn}} \beta(x)dx = 1$. Set $\beta^m(x) := m^{n+1}\beta(mx)$, $m \in \mathbb{N} \setminus \{0\}$, and define

$$f^m(t,\omega,x) := \int_{\mathbb{R}^{d+d\times n}} \hat{f}(t,\omega,x')\beta^m(x-x')dx'.$$

Then all f^m satisfy (H2'), (H3), and (H4). Therefore, one obtains from Proposition 4.2.3 that there exist unique solutions (Y^m,Z^m) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R})$ to the BSDEs corresponding to (f^m,ξ), and

$$|Z_t^m| \leq a(t) := \sqrt{(A(t))^2 + D^2} \exp\left(\frac{1}{2}\int_t^T (2B(s) + \rho^2(s,r(s)) + 1)ds\right).$$

Since $\hat{f}$ is Lipschitz with some coefficient $L \in \mathbb{R}_+$, one can choose constants $\mu,\nu > 0$ such that the statement of Proposition 4.1.7 holds. This gives

$$\|Y - Y^m\|^2_{\mathbb{S}^2,[T-\mu,T]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-\mu,T]} \leq \nu\,\mathbb{E}\left[\int_{T-\mu}^T \left(\hat{f}(t,Y_t^m,Z_t^m) - f^m(t,Y_t^m,Z_t^m)\right)^2 dt\right].$$

Since $\left|\hat{f} - f^m\right| \to 0$ uniformly in (t, ω, y, z) as $m \to \infty$, one obtains $\mathbb{E}\left[(Y_{T-\mu} - Y^m_{T-\mu})^2\right] \to 0$ and $|Z_t| \le a(t)$ for $T - \mu \le t \le T$. Proposition 4.1.7 applied on the interval $[T - 2\mu, T - \mu]$ yields

$$\begin{aligned} & \|Y - Y^m\|^2_{\mathbb{S}^2,[T-2\mu,T-\mu]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-2\mu,T-\mu]} \\ \le \quad & \nu\, \mathbb{E}\left[(Y_{T-\mu} - Y^m_{T-\mu})^2 + \int_{T-2\mu}^{T-\mu} \left(\hat{f}(t, Y^m_t, Z^m_t) - f^m(t, Y^m_t, Z^m_t)\right)^2 dt\right]. \end{aligned}$$

So $\mathbb{E}\left[(Y_{T-2\mu} - Y^m_{T-2\mu})^2\right] \to 0$ and $|Z_t| \le a(t)$ for $T - 2\mu \le t \le T - \mu$. By repeating this argument, one gets $|Z_t| \le a(t) \le r(t)$ for almost all $t \in [0, T]$. It follows that (Y, Z) is also a solution of the BSDE (1.1.1) with parameters (f, ξ).

Finally, if $(\tilde{Y}, \tilde{Z})$ is another solution in $\mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^\infty(\mathbb{R}^{d\times n})$ corresponding to (f, ξ), it must be equal to (Y, Z) since both solve the BSDE (1.1.1) with a 4-standard driver $\tilde{f}$ that coincides with f for $|z| \le \tilde{R}$, where $\tilde{R} \in \mathbb{R}_+$ is a bound on Z and $\tilde{Z}$. □

4.3 Global Solution for One-Dimensional BSDEs

The similar argument using Malliavin calculus can be implemented to the case where $d = 1$. Since we have comparison theorem for one dimensional BSDEs, we have global estimate for $|Z|$. Therefore, we can guarantee the existence and uniqueness of solution for arbitrary $T < \infty$. The following Theorem 4.3.1 and Corollary 4.3.5 are the main results. Now consider the conditions:

(A1) The terminal condition ξ is in $\mathbb{D}^{1,2}$ and there exist constants $A_i \in \mathbb{R}_+$ such that $|D^i_t \xi| \le A_i$ $dt \otimes d\mathbb{P}$-a.e. for all $i = 1, \dots, n$.

(A2) There exist a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y, z) - f(t, y', z)| \le B|y - y'| \quad \text{and} \quad |f(t, y, z) - f(t, y, z')| \le \rho(|z| \vee |z'|)|z - z'|$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

(A3) $f(., 0, 0) \in \mathbb{H}^4(\mathbb{R})$ and there exist Borel-measurable functions $q_i : [0, T] \to \mathbb{R}_+$ satisfying $\int_0^T q_i^2(t)dt < \infty$ such that for every pair $(y, z) \in \mathbb{R} \times \mathbb{R}^n$ with

$$|z| \le Q := \sqrt{\sum_{i=1}^n \left(A_i + \int_0^T q_i(t) e^{-B(T-t)} dt\right)^2 e^{BT}},$$

one has $f(\cdot, y, z) \in \mathbb{L}^{1,2}_a(\mathbb{R})$ and $|D^i_r f(t, y, z)| \le q_i(t)$ $dr \otimes d\mathbb{P}$-a.e. for all $i = 1, \dots, n$.

(A4) For a.a. $r \in [0,T]$, there exists a non-negative process $K_{r.}$ in $\mathbb{H}^4(\mathbb{R})$ such that

$$\int_0^T \|K_{r.}\|_{\mathbb{H}^4}^4 dr < \infty \quad \text{and} \quad |D_r f(t,y,z) - D_r f(t,y',z')| \leq K_{rt}(|y-y'| + |z-z'|)$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ satisfying $|z|, |z'| \leq Q$.

Then one has the following

Theorem 4.3.1. *If* (A1)–(A4) *hold, then the BSDE* (1.1.1) *has a unique solution* (Y,Z) *in* $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$*, and for all* $i = 1, \ldots, n$,

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.}$$

Remark 4.3.2. If for a.a. $r \in [0,T]$, the process $K_{r.}$ in (A4) is bounded, the condition $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ can be dropped from (A3). Then the statement of Theorem 4.3.1 still holds, except that Y is in $\mathbb{S}^2(\mathbb{R})$ instead of $\mathbb{S}^4(\mathbb{R})$. This is due to the fact that in this case, $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ is not needed in Proposition 4.3.3 below; see Remark 4.3.4.

We first prove Theorem 4.3.1 under the following stronger versions of conditions (A2)–(A4):

(A2') $f(t,y,z)$ is continuously differentiable in (y,z) and there exist constants $B, \rho \in \mathbb{R}_+$ such that

$$|\partial_y f(t,y,z)| \leq B, \quad |\partial_z f(t,y,z)| \leq \rho$$

for all $t \in [0,T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$.

(A3') Condition (A3) holds for all $(y,z) \in \mathbb{R} \times \mathbb{R}^n$.

(A4') Condition (A4) holds for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

Proposition 4.3.3. *If* (A1), (A2'), (A3'), (A4') *hold, then the BSDE* (1.1.1) *has a unique solution* (Y,Z) *in* $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$*, and*

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.} \tag{4.3.11}$$

Proof. By Lemma 4.1.4 above, condition (A1) implies $\mathbb{E}|\xi|^p < \infty$ for all $p \in \mathbb{R}_+$. So it follows from Theorem 5.1 and Proposition 5.3 of El Karoui et al. [34] that BSDE (1.1.1) has a unique solution (Y,Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. Moreover, $(Y,Z) \in \mathbb{L}_a^{1,2}(\mathbb{R}^{n+1})$, and for fixed $i = 1, \ldots, n$,

$$(D_r^i Y_t, D_r^i Z_t) = (U_t^r, V_t^r) \;\; dr \otimes dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{and} \quad Z_t^i = U_t^t \;\; dt \otimes d\mathbb{P}\text{-a.e.},$$

where

$$U_t^r = 0,\; V_t^r = 0, \quad 0 \le t < r \le T,$$

and for each fixed r, $(U_t^r, V_t^r)_{r \le t \le T}$ is the unique pair in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ solving the BSDE

$$U_t^r = D_r^i \xi + \int_t^T [\partial_y f(s, Y_s, Z_s) U_s^r + \partial_z f(s, Y_s, Z_s) V_s^r + D_r^i f(s, Y_s, Z_s)] ds - \int_t^T V_s^r dW_s. \quad \textbf{(4.3.12)}$$

Since (4.3.12) and the two BSDEs

$$\overline{U}_t = A_i + \int_t^T \left(B|\overline{U}_s| + \rho|\overline{V}_s| + q_i(s)\right) ds - \int_t^T \overline{V}_s dW_s \quad \textbf{(4.3.13)}$$

$$\underline{U}_t = -A_i - \int_t^T \left(B|\underline{U}_s| + \rho|\underline{V}_s| + q_i(s)\right) ds - \int_t^T \underline{V}_s dW_s \quad \textbf{(4.3.14)}$$

have 2-standard parameters, one obtains from the comparison result, Theorem 2.2 in El Karoui et al. [34], that $\underline{U}_t \le U_t^r \le \overline{U}_t$ for all $t \in [0,T]$. But the solutions to (4.3.13) and (4.3.14) are given by

$$\overline{U}_t = -\underline{U}_t = \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds\right) e^{B(T-t)}, \quad \overline{V}_t = \underline{V}_t = 0.$$

This shows (4.3.11). □

Remark 4.3.4. If for a.a. $r \in [0,T]$, the process $K_{r.}$ in (A4') is bounded, Proposition 4.3.3 still holds if the condition $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ is dropped from (A3') except that then, (Y,Z) is in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and not necessarily in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. This is true because in this case, the proof of Proposition 5.3 in El Karoui et al. [34] still works without the assumption $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ with the difference that it yields a solution (Y,Z) of the BSDE (1.1.1) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ instead of $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$.

To derive Theorem 4.3.1 from Proposition 4.3.3, we need the following result, which is Proposition 5.1 of El Karoui et al. [34] in the special case of a Brownian filtration and $p = 2$.

Proof of Theorem 4.3.1

Define

$$\hat{f}(t,y,z) = \begin{cases} f(t,y,z) & \text{if } |z| \le Q \\ f(t,y,Qz/|z|) & \text{if } |z| > Q \end{cases}.$$

Then $(\hat{f}, \xi)$ are 4-standard parameters. So the corresponding BSDE has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R}^n)$. Denote $x = (y, z) \in \mathbb{R}^{n+1}$ and let $\beta \in C_c^\infty(\mathbb{R}^{n+1})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases},$$

where the constant $\lambda \in \mathbb{R}_+$ is chosen so that $\int_{\mathbb{R}^{n+1}} \beta(x) dx = 1$. Set $\beta^m(x) := m^{n+1}\beta(mx)$, $m \in \mathbb{N} \setminus \{0\}$, and define

$$f^m(t, \omega, x) := \int_{\mathbb{R}^{n+1}} \hat{f}(t, \omega, x')\beta^m(x - x')dx'.$$

Then all f^m satisfy (A2')–(A4'). Therefore, one obtains from Proposition 4.3.3 that there exist unique solutions (Y^m, Z^m) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^4(\mathbb{R})$ to the BSDEs corresponding to (f^m, ξ), and $|Z_t^{m,i}| \leq a_i(t) := (A_i + \int_t^T q_i(s)e^{-B(T-s)}ds)e^{B(T-t)}$. Since $\hat{f}$ satisfies the Lipschitz condition (S2) for some constant $L \in \mathbb{R}_+$, one can choose constants $\mu, \nu > 0$ such that the statement of Proposition 4.1.7 holds. This gives

$$\|Y - Y^m\|^2_{\mathbb{S}^2,[T-\mu,T]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-\mu,T]} \leq \nu\, \mathbb{E}\left[\int_{T-\mu}^T \left(\hat{f}(t, Y_t^m, Z_t^m) - f^m(t, Y_t^m, Z_t^m)\right)^2 dt\right].$$

Since $\left|\hat{f} - f^m\right| \to 0$ uniformly in (t, ω, y, z) as $m \to \infty$, one obtains $\mathbb{E}\left[(Y_{T-\mu} - Y_{T-\mu}^m)^2\right] \to 0$ and $|Z_t^i| \leq a_i(t)$ for $T - \mu \leq t \leq T$. Proposition 4.1.7 applied on the interval $[T - 2\mu, T - \mu]$ yields

$$\begin{aligned} &\|Y - Y^m\|^2_{\mathbb{S}^2,[T-2\mu,T-\mu]} + \|Z - Z^m\|^2_{\mathbb{H}^2,[T-2\mu,T-\mu]} \\ \leq\;& \nu\, \mathbb{E}\left[(Y_{T-\mu} - Y_{T-\mu}^m)^2 + \int_{T-2\mu}^{T-\mu} \left(\hat{f}(t, Y_t^m, Z_t^m) - f^m(t, Y_t^m, Z_t^m)\right)^2 dt\right]. \end{aligned}$$

So $\mathbb{E}\left[(Y_{T-2\mu} - Y_{T-2\mu}^m)^2\right] \to 0$ and $|Z_t^i| \leq a_i(t)$ for $T - 2\mu \leq t \leq T - \mu$. By repeating this argument, one gets $|Z^i(t)| \leq a_i(t)$ for all $t \in [0, T]$. It follows that (Y, Z) is also a solution of the BSDE (1.1.1) with parameters (f, ξ).

Finally, if $(\tilde{Y}, \tilde{Z})$ is another solution in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ corresponding to (f, ξ), it must be equal to (Y, Z) since both solve the BSDE (1.1.1) with a 4-standard driver $\tilde{f}$ that coincides with f for $|z| \leq \tilde{Q}$, where $\tilde{Q} \in \mathbb{R}_+$ is a bound on Z and $\tilde{Z}$. □

In the following corollary, we assume that the terminal condition ξ is bounded and has bounded Malliavin derivative. This allows us to relax some of the assumptions of Theorem

4.3.1 on the driver f. The precise conditions we need are the following:

(B1) ξ satisfies (A1) and there exists a constant $C \in \mathbb{R}_+$ such that $|\xi| \le C$.

(B2) There exist constants $B, D \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t,y,z) - f(t,y',z)| \le B|y-y'|$$
$$|f(t,y,z) - f(t,y,z')| \le \rho(|z| \vee |z'|)|z-z'|$$
$$|f(t,y,z)| \le D(1+|y|) + \rho(|z|)|z|$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ with $|y|, |y'| \le R := (C+1)e^{DT} - 1$ and all $z, z' \in \mathbb{R}^n$.

(B3) Condition (A3) holds for all $(y,z) \in \mathbb{R} \times \mathbb{R}^n$ such that $|y| \le R$ and

$$|z| \le Q := \sqrt{\sum_{i=1}^n \left(A_i + \int_0^T q_i(t) e^{-B(T-t)} dt \right)^2} e^{BT}.$$

(B4) Condition (A4) holds for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ such that $|y|, |y'| \le R$ and $|z|, |z'| \le Q$.

Corollary 4.3.5. *Assume* (B1)–(B4). *Then the BSDE* (1.1.1) *has a unique solution* (Y,Z) *in* $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$*, and*

$$|Y_t| \le (C+1)e^{D(T-t)} - 1 \quad \textit{for all } t \in [0,T]$$
$$|Z^i_t| \le \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\textit{-a.e.} \quad \textit{for all } i = 1, \dots, n.$$

Proof. Consider the following three BSDEs

$$Y_t = \xi + \int_t^T \hat{f}(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \tag{4.3.15}$$
$$\overline{Y}_t = C + \int_t^T \overline{f}(s, \overline{Y}_s, \overline{Z}_s) ds - \int_t^T \overline{Z}_s dW_s \tag{4.3.16}$$
$$\underline{Y}_t = -C + \int_t^T \underline{f}(s, \underline{Y}_s, \underline{Z}_s) ds - \int_t^T \underline{Z}_s dW_s, \tag{4.3.17}$$

where $\hat{f}(t,y,z) := f(t, \tilde{y}, \tilde{z})$ for

$$\tilde{y} := \begin{cases} y & \text{if } |y| \le R \\ Ry/|y| & \text{if } |y| > R \end{cases} \quad \text{and} \quad \tilde{z} := \begin{cases} z & \text{if } |z| \le Q \\ Qz/|z| & \text{if } |z| > Q \end{cases},$$

$\overline{f}(t,y,z) := D(1+|y|)+\rho(Q)|z|$ and $\underline{f}(t,y,z) := -\overline{f}(t,y,z)$. $\hat{f}$ satisfies (A2)–(A4) and has the following two properties:

1) $\hat{f}(t,y,z) = f(t,y,z)$ for all (t,y,z) such that $|y| \leq R$ and $|z| \leq Q$

2) $\underline{f}(t,y,z) \leq \hat{f}(t,y,z) \leq \overline{f}(t,y,z)$ for all (t,y,z).

It follows from Theorem 4.3.1 that (4.3.15) has a unique solution (Y,Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and

$$|Z^i_t| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)}.$$

Moreover, one obtains from Theorem 2.2 in El Karoui et al. [34] that

$$\underline{Y}_t \leq Y_t \leq \overline{Y}_t, \quad 0 \leq t \leq T,$$

and it can easily be checked that

$$\overline{Y}_t = -\underline{Y}_t = (C+1)e^{D(T-t)} - 1, \quad \overline{Z}_t = \underline{Z}_t = 0.$$

This gives $|Y_t| \leq (C+1)e^{D(T-t)} - 1 \leq R$. So (Y,Z) solves the BSDE (1.1.1) with parameters (f,ξ).

To conclude the proof, assume that $(\tilde{Y},\tilde{Z})$ is another solution in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$. Let $\tilde{Q} \in \mathbb{R}_+$ be a bound on $\tilde{Z}$ and assume

$$t^* := \sup \left\{ s \in [0,T] : \mathbb{P}[|\tilde{Y}_s| \geq R] > 0 \right\} > 0.$$

On $[t^*,T]$, $\tilde{Y}$ is bounded by R, and hence, $(\tilde{Y},\tilde{Z})$ is equal to (Y,Z) since both solve the BSDE (1.1.1) with a 4-standard driver $\tilde{f}$ that coincides with f for $|y| \leq R$ and $|z| \leq Q \vee \tilde{Q}$. In particular, $|\tilde{Y}_{t^*}| \leq (C+1)e^{D(T-t^*)} - 1 < R$. It follows that there exists an $\varepsilon > 0$ such that

$$|\tilde{Y}_t| = |\mathbb{E}_t \tilde{Y}_{t^*} + \int_t^{t^*} \mathbb{E}_t f(s,\tilde{Y}_s,\tilde{Z}_s) ds| \leq (C+1)e^{D(T-t^*)} - 1 + (t^*-t)[D(1+R)+\rho(\tilde{Q})\tilde{Q}] < R$$

for all $t \in [t^*-\varepsilon, t^*]$, a contradiction to the definition of t^*. This shows that $t^* = 0$ and $(\tilde{Y},\tilde{Z}) = (Y,Z)$.

□

4.4 Relationship with Semilinear Parabolic PDEs

When f is Lipschitz, there is Feynman-Kac formula which relate Markovian BSDE with semilinear parabolic PDE; see [34]. If there is $W^{1,2}$ solution for semilinear parabolic PDE where $W^{1,2}$ is Sobolev space, we can apply Itô formula to find a solution of the corresponding BSDE. On the other hand, when $d = 1$, the solution of BSDE is known to be a viscosity solution of the corresponding PDE. See Appendix B for the definition of viscosity solution.

In this section, we will extend the result to the case where the driver has superquadratic growth in z using the result from previous section. Since we bound Z process, most of results directly followed if we can bound the Malliavin derivative of the terminal condition and the driver. We will restrict ourselves to $d = 1$ case.

4.4.1 Markovian BSDEs and semilinear parabolic PDEs

For $(t, x) \in [0, T] \times \mathbb{R}^m$, consider an SDE of the form

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r)dW_r \quad t \leq s \leq T, \tag{4.4.1}$$

where $b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0, T] \to \mathbb{R}^{m \times n}$ are Borel measurable functions for which there exist constants $E, F \in \mathbb{R}_+$ such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}^m$ and i, j,

$$|\sigma_{ij}(t)| \leq E \tag{4.4.2}$$

$$|b_i(t, x)| \leq F(1 + \max_k |x_k|) \tag{4.4.3}$$

$$|b_i(t, x) - b_i(t, x')| \leq F \max_k |x_k - x'_k|. \tag{4.4.4}$$

Denote $W_s^t := W_s - W_t$, $s \in [t, T]$, and let $(\mathcal{F}_s^t)_{s \in [t,T]}$ be the filtration generated by W^t. By $\mathbb{S}_t^p(\mathbb{R}^d)$ we denote the space of all $\mathbb{R}^d$-valued continuous $(\mathcal{F}_s^t)$-adapted processes with finite $\mathbb{S}^p$-norm on $[t, T]$, and by $\mathbb{H}_t^p(\mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued $(\mathcal{F}_s^t)$-predictable processes with finite $\mathbb{H}^p$-norm on $[t, T]$. Analogously, we denote by $\mathbb{D}_t^{1,2}$ and $\mathbb{L}_{a,t}^{1,2}$ the spaces $\mathbb{D}^{1,2}$ and $\mathbb{L}_a^{1,2}$ with respect to $(W_s^t)_{s \in [t,T]}$.

Under (4.4.2)–(4.4.4) the SDE (4.4.1) has a unique strong solution in $\mathbb{S}_t^2(\mathbb{R}^m)$; see for instance, Karatzas and Shreve [52]. A Markovian BSDE based on $X^{t,x}$ is of the form

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r \tag{4.4.5}$$

for measurable functions $g : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$.

It is well-known that if g is sufficiently regular in (r,x) and Lipschitz in (y,z), $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the parabolic PDE with terminal condition

$$u_t(t,x) + \mathcal{L}_{(t,x)} u(t,x) + g(t,x,u(t,x),(\nabla u \sigma)(t,x)) = 0, \quad u(T,x) = h(x), \tag{4.4.6}$$

where

$$\mathcal{L}_{(t,x)} := \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(t) \partial_{x_i} \partial_{x_j} + \sum_i b_i(t,x) \partial_{x_i};$$

see El Karoui et al. [34]. Since Theorem 4.3.1 and Corollary 4.3.5 give bounds on solutions of BSDEs, we can generalize this relationship between BSDEs and PDEs to the case where g is non-Lipschitz in (y,z). To do that we require g and h to satisfy the following conditions:

(C1) There exists a constant $A \in \mathbb{R}_+$ such that $|h(x) - h(x')| \le A \max_i |x_i - x'_i|$ for all $x, x' \in \mathbb{R}^m$.

(C2) There exist a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,x,y,z) - g(t,x,y',z)| \le B|y-y'| \text{ and } |g(t,x,y,z) - g(t,x,y,z')| \le \rho\left(|z| \vee |z'|\right)|z-z'|$$

for all $t \in [0,T]$, $x \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$.

(C3) $\int_0^T g(t,0,0,0)^2 dt < \infty$ and there exists a constant $G \in \mathbb{R}_+$ such that

$$|g(t,x,y,z) - g(t,x',y,z)| \le G \max_i |x_i - x'_i|$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$ with

$$|z| \le N := \sqrt{n}\left(A + \frac{1-e^{-BT}}{B} G\right) E e^{(B+F)T}.$$

(C4) There exists a constant $H \in \mathbb{R}_+$ such that

$$|g(t,x,y,z) - g(t,x',y,z) - g(t,x,y',z') + g(t,x',y',z')| \le H \max_i |x_i - x'_i| \left(|y-y'| + |z-z'|\right)$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ with $|z|, |z'| \le N$.

Proposition 4.4.1. *Assume* (C1)–(C4). *Then for every* $(t,x) \in [0,T] \times \mathbb{R}^m$*, the Markovian BSDE*

(4.4.5) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^2_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$*, and*

$$|Z^{t,x,i}_s| \le \left(A + \frac{1 - e^{-B(T-s)}}{B} G \right) E e^{B(T-s)} e^{F(T-t)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n.$$

Proof. If we can show that the BSDE (4.4.5) satisfies (A1) with $A_i = AEe^{F(T-t)}$, (A2), (A3) with $q_i \equiv GEe^{F(T-t)}$ but without $f(.,0,0) \in \mathbb{H}^4(\mathbb{R})$ and (A4) with a constant K, then the proposition follows from Theorem 4.3.1 and Remark 4.3.2.

(A2) is a direct consequence of (C2). By Lemma 4.4.2 below, $X^{t,x}_s$ is in $(\mathbb{D}^{1,2}_t)^m$ for all $t \le s \le T$ and $|D^i_r X^{t,x,j}_s| \le Ee^{F(T-t)}$ $dr \otimes d\mathbb{P}$-a.e. for all i and j. It follows from the Lipschitz condition (C1) and Proposition 1.2.4 of Nualart [63] that $h(X^{t,x}_T)$ is in $\mathbb{D}^{1,2}_t$ and for all $i = 1, \dots, n$, there exists an m-dimensional random vector Λ satisfying

$$D^i_r h(X^{t,x}_T) = \sum_{j=1}^m \Lambda^j D^i_r X^{t,x,j}_T \quad \text{and} \quad \sum_{j=1}^m |\Lambda^j| \le A.$$

This shows that the terminal condition $\xi = h(X^{t,x}_T)$ satisfies (A1) with $A_i = AEe^{F(T-t)}$. Analogously, it follows from (C3) that for every pair (y,z) such that $|z| \le N$, $g(., X^{t,x}_., y, z)$ belongs to $\mathbb{L}^{1,2}_{a,t}$ and $|D^i_r g(s, X^{t,x}_s, y, z)| \le GEe^{F(T-t)}$. So (A3) holds with $q_i \equiv GEe^{F(T-t)}$. The same argument applied to

$$\tilde{g}(s,x,y,y',z,z') = g(s,x,y,z) - g(s,x,y',z')$$

gives $|D^i_r g(s, X^{t,x}_s, y, z) - D^i_r g(s, X^{t,x}_s, y', z')| \le HEe^{F(T-t)}(|y - y'| + |z - z'|)$ for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ with $|z|, |z'| \le N$. This shows that (A4) holds with a constant K. □

Lemma 4.4.2. *For all* $0 \le t \le s \le T$ *and* $x \in \mathbb{R}^m$*,* $X^{t,x}_s$ *is in* $(\mathbb{D}^{1,2}_t)^m$ *and*

$$|D^i_r X^{t,x,j}_s| \le Ee^{F(T-t)} \quad dr \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

Proof. It follows from Theorem 2.2.1 of Nualart [63] that $X^{t,x}_s$ is in $(\mathbb{D}^{1,2}_t)^m$. Moreover, one obtains from the Lipschitz condition (4.4.4) and Proposition 1.2.4 of Nualart [63] that there exists an $\mathbb{R}^{m \times m}$-valued process Λ such that

$$D^i b_j(s, X^{t,x}_s) = \sum_{l=1}^m \Lambda^{jl}_s D^i X^{t,x,l}_s \quad \text{and} \quad \sum_{l=1}^m |\Lambda^{jl}_s| \le F.$$

It follows that $\int_t^s b_j(u, X_u^{t,x})du \in \mathbb{D}_t^{1,2}$ with

$$\left| D_r^i \int_t^s b_j(u, X_u^{t,x})du \right| \leq \int_t^s \left| D_r^i b_j(u, X_u^{t,x}) \right| du \leq F \int_t^s \max_l |D_r^i X_u^{t,x,l}| du.$$

Moreover, $|D^i \int_t^s \sum_{l=1}^n \sigma_{jl}(u) dW_u^l| = |\sigma_{ji} 1_{[t,s]}| \leq E$. Therefore,

$$\max_j |D_r^i X_s^{t,x,j}| \leq E + F \int_t^s \max_j |D_r^i X_u^{t,x,j}| du,$$

and one obtains from Gronwall's lemma that $|D_r^i X_s^{t,x,j}| \leq E e^{F(s-t)}$ $dr \otimes d\mathbb{P}$-a.e. □

If the function h is bounded, one can relax some of the assumptions of Proposition 4.4.1 on g as follows:

(D1) The function h satisfies (C1) and is bounded by a constant $C \in \mathbb{R}_+$.

(D2) There exist constants $B, D \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t,x,y,z) - g(t,x,y',z)| \leq B|y - y'|$$
$$|g(t,x,y,z) - g(t,x,y,z')| \leq \rho(|z| \vee |z'|)|z - z'|$$
$$|g(t,x,y,z)| \leq D(1 + |y|) + \rho(|z|)|z|$$

for all $t \in [0,T]$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}$ with $|y|, |y'| \leq R := (C+1)e^{DT} - 1$ and all $z, z' \in \mathbb{R}^n$.

(D3) Condition (C3) holds for all for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$ such that $|y| \leq R$ and $|z| \leq N$.

(D4) Condition (C4) holds for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^n$ such that $|y|, |y'| \leq R$ and $|z|, |z'| \leq N$.

Proposition 4.4.3. *Assume* (D1)–(D4). *Then for all* $(t,x) \in [0,T] \times \mathbb{R}^m$*, the Markovian BSDE* (4.4.5) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}_t^\infty(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$*, and*

$$|Y_s^{t,x}| \leq (C+1)e^{D(T-s)} - 1 \quad \textit{for all } s \in [t,T]$$
$$|Z_s^{t,x,i}| \leq \left(A + \frac{1 - e^{-B(T-s)}}{B} G \right) E e^{B(T-s)} e^{F(T-t)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \textit{for all } i = 1, \ldots, n.$$

Proof. (D1)–(D4) imply (B1)–(B4). Therefore, the proposition follows from Corollary 4.3.5 like Proposition 4.4.1 follows from Theorem 4.3.1. □

Before we proceed, let us define a viscosity solution of parabolic PDE (4.4.6).

Definition 4.4.4. *A continuous function $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ is called a viscosity subsolution of* (4.4.6) *if $u(T,x) = h(x)$ and for each $(t,x) \in [0,T] \times \mathbb{R}^m$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R}^m; \mathbb{R})$ such that $\phi(t,x) = u(t,x)$ and (t,x) is a minimum of $\phi - u$,*

$$\partial_t \phi(t,x) + \mathcal{L}_{(t,x)} \phi(t,x) + f(t,x,\phi(t,x),(\nabla\phi\sigma)(t,x)) \geq 0$$

On the other hand, a continuous function $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ is called a viscosity supersolution of (4.4.6) *if $u(T,x) = h(x)$ and for each $(t,x) \in [0,T] \times \mathbb{R}^m$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R}^m; \mathbb{R})$ such that $\phi(t,x) = u(t,x)$ and (t,x) is a maximum of $\phi - u$,*

$$\partial_t \phi(t,x) + \mathcal{L}_{(t,x)} \phi(t,x) + f(t,x,\phi(t,x),(\nabla\phi\sigma)(t,x)) \leq 0$$

If u is called a viscosity solution if it is both a viscosity subsolution and viscosity supersolution.

Extensive details about viscosity solution of second order elliptic and parabolic PDE can be found in [21].

Corollary 4.4.5. *If the assumptions of Proposition 4.4.1 or Proposition 4.4.3 hold, then the PDE* (4.4.6) *has a viscosity solution u such that for all $(t,x) \in [0,T] \times \mathbb{R}^m$, $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$, where $X^{t,x}$ and $Y^{t,x}$ are solutions of* (4.4.1) *and* (4.4.5)*, respectively.*

Proof. If the assumptions of Proposition 4.4.1 hold, the BSDE (4.4.5) has for all $(t,x) \in [0,T] \times \mathbb{R}^m$ a solution $(Y^{t,x}, Z^{t,x})$ such that $Z^{t,x}$ is bounded by N. So $(Y^{t,x}, Z^{t,x})$ also solves (4.4.5) if g is replaced by a function $\tilde{g}$ that agrees with g for $|z| \leq N$ and is Lipschitz in (x,y,z). It follows from Theorem 4.3 of Pardoux and Peng [66] that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (4.4.6) such that $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$.

Under the assumptions of Proposition 4.4.3, the BSDE (4.4.5) has a solution $(Y^{t,x}, Z^{t,x})$ such that $Y^{t,x}$ is bounded by $(C+1)e^{DT} - 1$ and $Z^{t,x}$ by N. Then $(Y^{t,x}, Z^{t,x})$ still solves (4.4.5) if g is replaced by a function $\tilde{g}$ that is Lipschitz in (x,y,z) and agrees with g for $|y| \leq (C+1)e^{DT} - 1$ and $|z| \leq N$. As above it follows that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (4.4.6) such that $u(s, X_s^{t,x}) = Y_s^{t,x}$, $t \leq s \leq T$. □

Corollary 4.4.6. *Assume the conditions of Proposition 4.4.3 hold and set $u(t,x) := Y_t^{t,x}$. If for every $L \in \mathbb{R}_+$, there exists a constant $\gamma_L \in \mathbb{R}$ and a continuous function $\delta_L : \mathbb{R}_+ \to \mathbb{R}_+$ with*

$\delta_L(0) = 0$ *such that*

$$
\begin{aligned}
g(t,x,y',v\sigma(t)) - g(t,x,y,v\sigma(t)) &\geq \gamma_L(y-y') \\
|g(t,x,y,v\sigma(t)) - g(t,x',y,v\sigma(t))| &\leq \delta_L(|x-x'|(1+|v|))
\end{aligned}
\tag{4.4.7}
$$

for all $(t,x,x') \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m$*,* $-L \leq y' \leq y \leq L$ *and* $v \in \mathbb{R}^m$*, then* u *is the unique bounded viscosity solution of the PDE* (4.4.6)*.*

Proof. This follows from Section 4.2 of Ishii and Lions [48]. □

Under appropriate assumptions on the coefficients b, σ, g and h, the PDE (4.4.6) has a unique classical solution.

Corollary 4.4.7. *Assume* $\int_0^T g^2(t,0,0,0)dt < \infty$*,* b *only depends on* x*,* σ *is a constant and* b, g, h *are all* C^3 *in* (x,y,z)*. Then one has the following:*

a) *If* (C1)–(C2) *hold and there exists a constant* $G \in \mathbb{R}_+$ *such that* $|\frac{\partial}{\partial x_i} g(t,x,y,z)| \leq G$ *for all* i*,* t, x, y *and* z *with*

$$|z| \leq N := \sqrt{n}\left(A + \frac{1-e^{-BT}}{B}G\right) E e^{(B+F)T},$$

and b, g, h *have bounded derivatives of first, second and third order in* (x,y,z) *on the set* $\{(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq N\}$*, then the PDE* (4.4.6) *has a unique solution* u *of class* $C^{1,2}$ *such that* $\nabla u \sigma$ *is bounded, and*

$$|\nabla u\sigma(t,x)| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right) E e^{(B+F)(T-t)} \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^m.$$

b) *If* (D1)–(D2) *hold and there exists a constant* $G \in \mathbb{R}_+$ *such that* $|\frac{\partial}{\partial x_i} g(t,x,y,z)| \leq G$ *for all* i*,* t, x, y *and* z *with*

$$|y| \leq (C+1)e^{DT} - 1 \quad \textit{and} \quad |z| \leq N := \sqrt{n}\left(A + \frac{1-e^{-BT}}{B}G\right) E e^{(B+F)T},$$

and b, g, h *have bounded derivatives of first, second and third order in* (x,y,z) *on the set* $\left\{(t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq (C+1)e^{DT} - 1, |z| \leq N\right\}$*, then* (4.4.6) *has a unique solution* u *of class* $C^{1,2}$ *such that* u *and* $\nabla u\sigma$ *are bounded. Moreover, one has*

$$|u(t,x)| \leq (C+1)e^{D(T-t)} - 1 \quad \textit{and} \quad |\nabla u\sigma(t,x)| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B}G\right) E e^{(B+F)(T-t)}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^m$.

Proof. It follows from the assumptions by the mean value theorem that in case a), (C3)–(C4) are satisfied and in case b), (D3)–(D4) hold. So one obtains from Propositions 4.4.1 and 4.4.3 that in both cases, the BSDE (4.4.5) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^2_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$. Moreover,

$$|Z^{t,x}| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B} G\right) E e^{(B+F)(T-t)},$$

and in case b), $|Y^{t,x}| \leq (C+1)e^{D(T-t)} - 1$. By modifying g for pairs (y,z) that are not attained by $(Y^{t,x}, Z^{t,x})$, one can assume that it is Lipschitz in (y,z). Then it follows from Theorem 3.2 of Pardoux and Peng [66] that $u(t,x) := Y^{t,x}_t$ defines a $C^{1,2}$ solution of the PDE (4.4.6). By Corollary 4.1 of El Karoui et al. [34], one has

$$|(\nabla u \sigma)(t,x)| = |Z^{t,x}_t| \leq \sqrt{n}\left(A + \frac{1-e^{-B(T-t)}}{B} G\right) E e^{(B+F)(T-t)},$$

and in case b), $|u(t,x)| = |Y^{t,x}_t| \leq (C+1)e^{D(T-t)} - 1$.

Finally, let us prove uniqueness. In case a), if the PDE (4.4.6) has another solution v of class $C^{1,2}$ such that $\nabla v \sigma$ is bounded, it follows from Itô's lemma that

$$(\tilde{Y}^{t,x}_s, \tilde{Z}^{t,x}_s) = (v(s, X^{t,x}_s), (\nabla v \sigma)(s, X^{t,x}_s))$$

solves the BSDE (4.4.5). Boundedness of $\tilde{Z}^{t,x}$ implies that $\tilde{Y}^{t,x}$ is in $\mathbb{S}^2_t(\mathbb{R})$. By the uniqueness result of Propositions 4.4.1, one has $(Y^{t,x}, Z^{t,x}) = (\tilde{Y}^{t,x}, \tilde{Z}^{t,x})$, and therefore, $u = v$. In case b), uniqueness follows from the same argument. □

As a consequence of the results in this section, one obtains the following corollary for PDEs with initial conditions of the form

$$u_t = \triangle u + g(u, \nabla u), \quad u(0,x) = h(x), \tag{4.4.8}$$

where $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$.

Corollary 4.4.8. *Consider the following conditions:*

(i) g *and* h *satisfy* (C1)–(C2).

(ii) g *and* h *satisfy* (D1)–(D2).

(iii) *For every* $L \in \mathbb{R}_+$ *there exists a constant* $\gamma_L \in \mathbb{R}$ *such that* $g(y', z) - g(y, z) \geq \gamma_L(y - y')$ *for all* $-L \leq y' \leq y \leq L$ *and* $z \in \mathbb{R}^n$.

(iv) g *and* h *have bounded derivatives of first, second and third order on the set*

$$\left\{(x, y, z) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |z| \leq \sqrt{n} A e^{BT}\right\}.$$

(v) g *and* h *have bounded derivatives of first, second and third order on the set*

$$\left\{(x, y, z) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n : |y| \leq (C+1)e^{DT} - 1, |z| \leq \sqrt{n} A e^{BT}\right\}.$$

Then the following hold:

a) *If* (i) *is satisfied, the PDE* (4.4.8) *has a viscosity solution* u.

b) *If* (ii) *is satisfied, the PDE* (4.4.8) *has a viscosity solution* u *satisfying* $|u(t,x)| \leq (C+1)e^{Dt} - 1$.

c) *If* (ii) *and* (iii) *are satisfied, the PDE* (4.4.8) *has a unique bounded viscosity solution.*

d) *If* (i) *and* (iv) *are satisfied, the PDE* (4.4.8) *has a unique* $C^{1,2}$*-solution with bounded gradient* ∇u*, and* $|\nabla u(t,x)| \leq \sqrt{n} A e^{Bt}$.

e) *If* (ii) *and* (v) *are satisfied, the PDE* (4.4.8) *has a unique bounded* $C^{1,2}$*-solution with bounded gradient* ∇u*, and one has* $|u(t,x)| \leq (C+1)e^{Dt} - 1$ *as well as* $|\nabla u(t,x)| \leq \sqrt{n} A e^{Bt}$.

Proof. Set $m = n$, $b \equiv 0$ and $\sigma \equiv \sqrt{2}\, Id$. Corollary 4.4.5 applied to $\tilde{g}(y,z) = g(y, z/\sqrt{2})$ yields that under (i) or (ii) the PDE with terminal condition,

$$v_t + \Delta v + g(v, \nabla v) = 0, \quad v(T, x) = h(x), \tag{4.4.9}$$

has a viscosity solution $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. Moreover, if (ii) holds, one obtains from Proposition 4.4.3 that $|v(t,x)| \leq (C+1)e^{D(T-t)} - 1$. It follows that under both conditions, (i) and (ii), $u(t,x) := v(T-t, x)$ is a viscosity solution of (4.4.8), which in case (ii) satisfies $u(t,x) \leq (C+1)e^{Dt} - 1$. This shows a) and b). If (ii) and (iii) hold, one obtains from Corollary 4.4.6 that v is the unique bounded viscosity solution of (4.4.9). Therefore, u is the unique bounded viscosity solution of (4.4.8). This proves c). Finally, d) and e) follow from Corollary 4.4.7. □

Remark 4.4.9. In the special case $g(y, z) = \mu|z|^p$ the PDE (4.4.8) was studied by Amour and Ben-Artzi [1] as well as Gilding et al. [37]. Amour and Ben-Artzi [1] proved the existence and uniqueness of a classical solution for $\mu \neq 0$, $p > 1$ and h a bounded C^2 function with bounded derivatives of first and second order. Gilding et al. [37] proved the existence and uniqueness of a classical solution for $\mu = 1$, $p > 0$ and h a continuous bounded function. Equation (4.4.8) is more general, but for the existence of a viscosity solution we need g to be locally Lipschitz in z. To obtain a classical solution we have to assume that g and h are C^3.

4.4.2 BSDEs with random terminal times and parabolic PDEs with lateral Dirichlet boundary conditions

BSDEs with random terminal times

Let $\tau \leq T$ be a stopping time and ξ an $\mathcal{F}_\tau$-measurable random variable.

Definition 4.4.10. *We say an $\mathbb{R} \times \mathbb{R}^n$-valued predictable process (Y, Z) solves the BSDE with random terminal time,*

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} Z_s dW_s, \tag{4.4.10}$$

if $\int_0^\tau (|f(t, Y_t, Z_t)|dt + |Z_t|^2)dt < \infty$, $Z_t = 0$ for $t > \tau$ and (4.4.10) *is satisfied for all $0 \leq t \leq T$.*

Suppose that for every $\omega \in \Omega$, the ODE

$$y_t(\omega) = \xi(\omega) - \int_{\tau(\omega)}^{t} f(s, \omega, y_s(\omega), 0)ds, \quad t \in [\tau(\omega), T], \tag{4.4.11}$$

has a unique solution $y(\omega)$, and set $\hat{\xi}(\omega) := y_T(\omega)$. Note that $1_{\{\tau\leq t\}}y_t$ is adapted, and in the special case $f(t, y, 0) = 0$, $t > \tau$, one has $\xi = \hat{\xi}$.

Proposition 4.4.11. *Assume $\hat{\xi}$ satisfies* (A1) *and f fulfills* (A2)–(A4)*. Then the BSDE* (4.4.10) *has a unique solution (Y, Z) in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and*

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s)e^{-B(T-s)}ds\right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, ..., n. \tag{4.4.12}$$

Proof. If $\hat{\xi}$ satisfies (A1) and f fulfills (A2)–(A4), it follows from Theorem 4.3.1 that the BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T f(s, \hat{Y}_s, \hat{Z}_s)ds - \int_t^T \hat{Z}_s dW_s$$

has a unique solution $(\hat{Y}, \hat{Z})$ in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and $\hat{Z}$ satisfies the bound (4.4.12). Let

$$Q := \sqrt{\sum_{i=1}^n \left(A_i + \int_0^T q_i(t) e^{-B(T-t)} dt \right)^2} e^{BT},$$

and notice that $(\hat{Y}, \hat{Z})$ also solves the BSDE

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f}(s, \hat{Y}_s, \hat{Z}_s) ds - \int_t^T \hat{Z}_s dW_s, \tag{4.4.13}$$

where $\hat{f}$ is the 4-standard driver

$$\hat{f}(t, y, z) = \begin{cases} f(t, y, z) & \text{if } |z| \leq Q \\ f(t, y, Qz/|z|) & \text{if } |z| > Q \end{cases}.$$

By Theorem 3.4 of Darling and Pardoux [81], the BSDE with random terminal time,

$$Y_t = \xi + \int_{t \wedge \tau}^{\tau} \hat{f}(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{\tau} Z_s dW_s, \tag{4.4.14}$$

has a unique solution (Y, Z) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$. Now, note that the pair $(\tilde{Y}, \tilde{Z})$ given by $\tilde{Y}_t := Y_t 1_{\{t \leq \tau\}} + y_t 1_{\{\tau < t\}}$ and $\tilde{Z}_t := Z_t 1_{\{t \leq \tau\}}$ is in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ and solves the BSDE (4.4.13). But since (4.4.13) can only have one solution in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$, one has $(\tilde{Y}, \tilde{Z}) = (\hat{Y}, \hat{Z})$. In particular, (Y, Z) belongs to $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and Z satisfies the bound (4.4.12). It follows that (Y, Z) solves the BSDE (4.4.10).

Finally, if (Y', Z') is another solution in $\mathbb{S}^4(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ it must be equal to (Y, Z) since both solve the BSDE (4.4.14) for a 4-standard driver f' that coincides with f for $|z| \leq Q'$, where $Q' \in \mathbb{R}_+$ is a bound for Z and Z'. □

Proposition 4.4.12. *If ξ is bounded by a constant $C \in \mathbb{R}_+$, $\hat{\xi}$ satisfies* (A1) *and f fulfills* (B2)–(B4) *with $R = (C+1)e^{2DT} - 1$ instead of $R = (C+1)e^{DT} - 1$, then the BSDE* (4.4.10) *has a unique solution (Y, Z) in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$, and*

$$|Y_t| \leq (C+1)e^{D(T-t)} - 1 \quad \textit{for all } t \in [0, T] \tag{4.4.15}$$

$$|Z_t^i| \leq \left(A_i + \int_t^T q_i(s) e^{-B(T-s)} ds \right) e^{B(T-t)} \quad dt \otimes d\mathbb{P}\text{-}\textit{a.e.} \quad \textit{for all } i = 1, \ldots, n. \tag{4.4.16}$$

Proof. By condition (B2), one has $|y_t(\omega)| \leq C + \int_{\tau(\omega)}^t D(1 + |y_s(\omega)|) ds$. So one obtains from

Gronwall's lemma that $|\hat{\xi}| \leq (C+1)e^{DT} - 1$. Now it follows from Corollary 4.3.5 by the same arguments as in the proof of Proposition 4.4.11 that the BSDE (4.4.10) has a unique solution (Y, Z) in $\mathbb{S}^\infty(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ and the bound (4.4.16) is satisfied. To complete the proof, notice that since one has $Y_t = \xi$ and $Z_t = 0$ for $t > \tau$, (Y, Z) satisfies the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) 1_{\{s \leq \tau\}} ds - \int_t^T Z_s dW_s.$$

So it follows from the comparison argument in the proof of Corollary 4.3.5 that (4.4.15) holds.

□

Semilinear parabolic PDEs with lateral Dirichlet boundary conditions

Let $\mathcal{O}$ be an open connected subset of $\mathbb{R}^m$. For every pair $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, consider the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r) dW_r,$$

where b and σ fulfill the conditions (4.4.2)–(4.4.4). Define the stopping time

$$\tau^{t,x} := \inf \left\{ s \geq t : X_s^{t,x} \notin \mathcal{O} \right\} \wedge T,$$

and consider the BSDE with random terminal time

$$Y_s^{t,x} = h(X_{\tau^{t,x}}^{t,x}) + \int_{s \wedge \tau^{t,x}}^{\tau^{t,x}} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_{s \wedge \tau^{t,x}}^{\tau^{t,x}} Z_r^{t,x} dW_r, \quad t \leq s \leq T, \tag{4.4.17}$$

where $h : \bar{\mathcal{O}} \to \mathbb{R}$ and $g : [0, T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Let $\bar{g} : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be an extension of g such that for every ω, the ODE

$$y_s^{t,x}(\omega) = h(X_{\tau^{t,x}}^{t,x}(\omega)) - \int_{\tau^{t,x}(\omega)}^s \bar{g}(r, X_r^{t,x}(\omega), y_r^{t,x}(\omega), 0) dr, \quad \tau^{t,x}(\omega) \leq s \leq T,$$

has a unique solution $y^{t,x}(\omega)$, and set $\hat{\xi}^{t,x}(\omega) := y_T^{t,x}(\omega)$. For the following results we need the following condition:

(E) there exist constants $A_i \in \mathbb{R}_+$ such that for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $\hat{\xi}^{t,x} \in \mathbb{D}^{1,2}$ and $|D_r^i \hat{\xi}^{t,x}| \leq A_i \; dr \otimes d\mathbb{P}$-a.e. for all i.

Proposition 4.4.13. *Assume g has an extension $\bar{g} : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (E) *and*

(C2)–(C4) *with*

$$N = \sqrt{\sum_i \left(A_i + \frac{GEe^{FT}(1-e^{-BT})}{B}\right)^2} e^{BT}$$

instead of $N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right)Ee^{(B+F)T}$*. Then, for each pair* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$*, the BSDE* (4.4.17) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^2_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$*, and*

$$|Z^{t,x,i}_s| \le \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-s)})}{B}\right) e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \ldots, n. \quad (4.4.18)$$

Proof. Fix (t,x), and set $\xi^{t,x} := h(X^{t,x}_{\tau^{t,x}})$. By assumption (E), $\hat{\xi}^{t,x}$ satisfies condition (A1), and it follows from the other assumptions like in the proof of Proposition 4.4.1 that $\bar{g}(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)$ fulfills (A2), (A3) with $q_i \equiv GEe^{F(T-t)}$ but without $\bar{g}(s, X^{t,x}_s, 0, 0) \in \mathbb{H}^4(\mathbb{R})$ and (A4) with a constant K. Now the proposition follows from Theorem 4.3.1 and Remark 4.3.2 like Proposition 4.4.13 followed from Theorem 4.3.1. □

Proposition 4.4.14. *Assume* h *is bounded by a constant* $C \in \mathbb{R}_+$ *and* g *has an extension* $\bar{g}$: $[0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *satisfying* (E) *and* (D2)–(D4) *with*

$$N = \sqrt{\sum_i \left(A_i + \frac{GEe^{FT}(1-e^{-BT})}{B}\right)^2} e^{BT}$$

instead of $N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B}G\right)Ee^{(B+F)T}$ *and* $R = (C+1)e^{2DT} - 1$ *instead of* $R = (C+1)e^{DT} - 1$*. Then, for each* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$*, the BSDE* (4.4.17) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$*, and*

$$|Y^{t,x}_s| \le (C+1)e^{D(T-s)} - 1 \text{ for all } s \in [t,T]$$
$$|Z^{t,x,i}_s| \le \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-s)})}{B}\right) e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \text{for all } i = 1, \ldots, n.$$

Proof. The result follows from Corollary 4.3.5 like Proposition 4.4.13 follows from Theorem 4.3.1 and Remark 4.3.2. □

Under appropriate assumptions, a solution to the BSDE (4.4.17) yields a solution to the following parabolic PDE with Dirichlet boundary conditions:

$$\begin{aligned} &u_t(t,x) + \mathcal{L}_{(t,x)}u(t,x) + g(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0 \quad \text{for } (t,x) \in [0,T) \times \mathcal{O} \\ &u(t,x) = h(x) \quad \text{for } (t,x) \in [0,T] \times \partial\mathcal{O} \text{ and } (t,x) \in \{T\} \times \mathcal{O}, \end{aligned} \quad (4.4.19)$$

where

$$\mathcal{L}_{(t,x)} := \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(t)\partial_{x_i}\partial_{x_j} + \sum_i b_i(t,x)\partial_{x_i}.$$

The next result is a consequence of Theorem 2.2, Lemma 3.1 and Theorem 3.2 of Peng [69].

Theorem 4.4.15. *Assume the following conditions hold:*

(F1) $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *is in* $C^{1,2}([0,T] \times \bar{\mathcal{O}})$, $\sigma : [0,T] \to \mathbb{R}^{m\times n}$ *is in* $C^1[0,T]$*, and there exists a constant* $\varepsilon > 0$ *such that* $\sum_{i,j} \left(\sigma\sigma^T\right)_{ij}(t)v_iv_j \geq \varepsilon|v|^2$ *for all* $(t,v) \in [0,T] \times \mathbb{R}^m$

(F2) $\mathcal{O}$ *is bounded and* $\partial\mathcal{O}$ *is* C^3

(F3) h *is* C^3 *and* $\mathcal{L}_{(t,x)}h(x) + g(T,x,h(x),\nabla h(x)\sigma(T)) = 0$ *for* $x \in \partial\mathcal{O}$

(F4) $g(t,x,y,z)$ *is continuously differentiable in* $(t,x,y,z) \in [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n$ *with bounded derivatives.*

Then the BSDE (4.4.17) *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *in* $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$ *and* $u(t,x) := Y^{t,x}_t$ *is the unique* $C^{1,2}$*-solution of the PDE* (4.4.19)*.*

By applying Proposition 4.4.14, one can weaken condition (F4) in Theorem 4.4.15.

Corollary 4.4.16. *Assume* (F1)–(F3) *are satisfied,* g *is continuously differentiable in* $(t,x,y,z) \in [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n$ *and the assumptions of Proposition 4.4.14 hold. Let* $(Y^{t,x}, Z^{t,x})$ *be the unique solution of the BSDE* (4.4.17) *in* $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$*. Then* $u(t,x) := Y^{t,x}_t$ *is the unique* $C^{1,2}$*-solution of the PDE* (4.4.19)*, and one has*

$$|u(t,x)| \leq (C+1)e^{D(T-t)} - 1, \quad |\nabla u(t,x)| \leq \frac{1}{\sqrt{\varepsilon}}\sqrt{\sum_i \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-t)})}{B}\right)^2 e^{B(T-t)}}. \tag{4.4.20}$$

Proof. It follows from Proposition 4.4.14 that the BSDE (4.4.17) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty(\mathbb{R}^n)$ with $|Y^{t,x}_s| \leq (C+1)e^{D(T-s)} - 1$ and

$$|Z^{t,x}_s| \leq \sqrt{\sum_i \left(A_i + \frac{GEe^{F(T-t)}(1-e^{-B(T-t)})}{B}\right)^2 e^{B(T-t)}} \quad ds \otimes d\mathbb{P}\text{-a.e.}$$

By modifying g for pairs (y,z) that are not attained by $(Y^{t,x}, Z^{t,x})$, one can assume that it has bounded derivatives. Then one obtains from Theorem 4.4.15 that $u(t,x) := Y^{t,x}_t$ is a $C^{1,2}$-solution of the PDE (4.4.19). It can be seen in the proof of Theorem 3.2 of Peng [69] that $Z^{t,x}_t = \nabla u(t,x)\sigma(t)$. So the bounds (4.4.20) follow from condition (F1).

If v is another $C^{1,2}$-solution of (4.4.19), v and ∇v are bounded. Moreover, it follows from Itô's formula that $\tilde{Y}^{t,x}_s := v(s \wedge \tau^{t,x}, X^{t,x}_{s\wedge\tau^{t,x}})$, $\tilde{Z}^{t,x}_s := \nabla v(s, X^{t,x}_{s\wedge\tau^{t,x}})\sigma(s) 1_{\{s\le\tau\}}$ solve the BSDE (4.4.17). So one obtains from the uniqueness result of Proposition 4.4.14 that $u(t,x) = v(t,x)$. □

An important assumption of Propositions 4.4.13 and 4.4.14 as well as Corollary 4.4.16 is that g has an extension $\bar{g}$ such that condition (E) holds. $X^{t,x}_{\tau^{t,x}}$ is typically not Malliavin differentiable. For instance, if τ is a stopping time such that $W_\tau \in \mathbb{D}^{1,2}$, then τ must be a constant. Indeed, for $W_\tau = \int_0^\infty 1_{\{s<\tau\}} dW_s \in \mathbb{D}^{1,2}$, one obtains from Proposition 5.3 of El Karoui et al. [34] that $1_{\{s<\tau\}} \in \mathbb{D}^{1,2}$ for almost all s, and therefore, by Proposition 1.2.6 of Nualart [63], $\mathbb{P}[s < \tau] = 0$ or 1. So to ensure that condition (E) holds, one has to require the functions g and h to be regular enough. The following lemma gives sufficient conditions for (E).

Lemma 4.4.17. *Assume g has an extension $\bar{g} : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying $\bar{g}(t,x,y,0) = 0$ for all t, x and y. Then* (E) *holds if $h : \bar{\mathcal{O}} \to \mathbb{R}$ is C^2 with bounded gradient and there exist measurable functions $\alpha : [0,T] \times h(\bar{\mathcal{O}}) \to \mathbb{R}$ and $\beta : h(\bar{\mathcal{O}}) \to \mathbb{R}^n$ such that*

(i) $\mathcal{L}_{(t,x)} h(x) = \alpha(t, h(x))$, $(\nabla h\sigma)(x) = \beta(h(x))$ *for* $(t,x) \in [0,T] \times \mathcal{O}$

(ii) $\alpha(t,h(x)) = 0$ *and* $\beta(h(x)) = 0$ *for* $x \in \partial\mathcal{O}$,

(iii) $|\alpha(t,x) - \alpha(t,y)| \le \kappa|x-y|$, $\alpha(t,0) \le \kappa$ *and* $|\beta(x) - \beta(y)| \le \kappa|x-y|$ *for some constant* $\kappa \in \mathbb{R}_+$

(iv) *there exist countably many values* $h_1, h_2, \dots$ *in* $\mathbb{R}$ *such that* $\partial\mathcal{O} = \bigcup_i \{x \in \partial\mathcal{O} : h(x) = h_i\}$.

Proof. Fix $(t,x) \in [0,T] \times \mathcal{O}$. Since $g(s,x,y,0) = 0$, one has $\hat{\xi}^{t,x} = h(X^{t,x}_{\tau^{t,x}})$, and by Itô's Lemma,

$$
\begin{aligned}
h(X^{t,x}_{s\wedge\tau^{t,x}}) &= h(x) + \int_t^{s\wedge\tau^{t,x}} \mathcal{L}_{(t,x)} h(X^{t,x}_r) dr + \int_t^{s\wedge\tau^{t,x}} (\nabla h\sigma)(X^{t,x}_r) dW_r \\
&= h(x) + \int_t^s \alpha(r, h(X^{t,x}_{r\wedge\tau^{t,x}})) dr + \int_t^s \beta(h(X^{t,x}_{r\wedge\tau^{t,x}})) dW_r.
\end{aligned}
$$

It follows from Theorem 2.2.1 of Nualart [63] that $h(X^{t,x}_{\tau^{t,x}})$ belongs to $\mathbb{D}^{1,2}$. Moreover, since the Malliavin derivative is local (see Proposition 1.3.16 in [63]), one has

$$
D\hat{\xi}^{t,x} = Dh(X^{t,x}_{\tau^{t,x}}) = 1_{\{\tau^{t,x}=T\}} Dh(X^{t,x}_T) + \sum_i 1_{\left\{\tau^{t,x}<T,\, h(X^{t,x}_{\tau^{t,x}})=h_i\right\}} Dh_i = 1_{\{\tau^{t,x}=T\}} Dh(X^{t,x}_T),
$$

and by the chain rule, $Dh(X^{t,x}_T) = \sum_j \partial_j h(X^{t,x}_T) DX^{t,x,j}_T$. So since ∇h is bounded, it follows from Lemma 4.4.2 that $D\hat{\xi}^{t,x}$ is uniformly bounded in t and x. □

The following example describes a more concrete situation in which condition (E) holds.

Example 4.4.18. Assume $X = W$ for a one-dimensional Brownian motion W and $\bar{\mathcal{O}} = [a, b]$. If g has an extension $\bar{g} : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ such that $g(t, x, y, 0) = 0$ for all t, x, y, and $h : [a, b] \to \mathbb{R}$ is C^2 such that $h'(x) = \beta(h(x))$ for a C^2-function $\beta : h[a, b] \to \mathbb{R}$ satisfying $\beta(a) = \beta(b) = 0$, then $\mathcal{L}_{(t,x)} h(x) = h''(x)/2 = G'(h(x))G(h(x))/2$. So the conditions of Lemma 4.4.17 are fulfilled, and (E) holds.

4.4.3 Markovian BSDEs based on reflected SDEs and parabolic PDEs with lateral Neumann boundary conditions

In this whole section, $\mathcal{O} \subset \mathbb{R}^n$ is an open connected domain and $b : \bar{\mathcal{O}} \to \mathbb{R}^n$, $\sigma : \bar{\mathcal{O}} \to \mathbb{R}^{n \times n}$ are bounded Lipschitz functions. We assume that $\mathcal{O}$ satisfies the uniform exterior sphere condition and uniform interior cone condition introduced by Saisho [76]. They are defined as follows: For $y \in \partial\mathcal{O}$ and $r > 0$, define $\mathcal{N}_{y,r} := \{v \in \mathbb{R}^n : |v| = 1, B_r(y - rv) \cap \mathcal{O} = \emptyset\}$ and $\mathcal{N}_y := \cup_{r>0} \mathcal{N}_{y,r}$ where $B_r(y)$ denotes the open ball around y with radius r.

Uniform exterior sphere condition

There exists a constant $r_0 > 0$ such that $\mathcal{N}_y = \mathcal{N}_{y,r_0} \neq \emptyset$ for all $y \in \partial\mathcal{O}$.

Uniform interior cone condition

There exist constants $\delta > 0$ and $\varepsilon \in [0, 1)$ with the following property: for every $y \in \partial\mathcal{O}$, there exists a unit vector $v \in \mathbb{R}^n$ such that

$$\{z \in B_\delta(y) : \langle z - x, v \rangle \geq \varepsilon |z - x|\} \subset \bar{\mathcal{O}} \quad \text{for all } x \in B_\delta(y) \cap \partial\mathcal{O}.$$

Reflected SDEs and Markovian BSDEs

For every pair $(t, x) \in [0, T] \times \bar{\mathcal{O}}$ we define a diffusion $X^{t,x}$ that is reflected at the boundary of $\mathcal{O}$. Let $v(y) \in \mathcal{N}_y$ be a vector field on $\partial\mathcal{O}$. Note that if $\partial\mathcal{O}$ is smooth, then $v(y)$ is the unit inward normal vector at y. It is shown in Saisho [76] that for all (t, x), there exists a unique pair $(X^{t,x}, L^{t,x})$ of continuous adapted processes with values in $\bar{\mathcal{O}} \times \mathbb{R}_+$ such that for all $s \in [t, T]$,

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r + \int_t^s v(X_r^{t,x}) dL_r^{t,x} \\ L_s^{t,x} &= \int_t^s 1_{\{X_r^{t,x} \in \partial\mathcal{O}\}} dL_r^{t,x} \quad \text{and} \quad L^{t,x} \text{ is nondecreasing.} \end{aligned} \tag{4.4.21}$$

Let $g : [0,T] \times \bar{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $h : \bar{\mathcal{O}} \to \mathbb{R}$ be measurable functions and consider the BSDE

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, \quad t \le s \le T. \tag{4.4.22}$$

Proposition 4.4.19. *Assume there exists a constant $M \in \mathbb{R}_+$ such that for all $0 \le t \le s \le T$ and $x \in \bar{\mathcal{O}}$,*

$$X_s^{t,x} \in \mathbb{D}^{1,2} \quad \textit{and} \quad |D_r X_s^{t,x}| \le M \quad dr \otimes d\mathbb{P}\text{-a.e.} \tag{4.4.23}$$

If g and h satisfy (C1)–(C4) *with*

$$N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B} G\right) M e^{BT} \quad \textit{instead of} \quad N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B} G\right) E e^{(B+F)T},$$

then (4.4.22) *has for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}_t^2(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$, and*

$$|Z_s^{t,x,i}| \le \left(A + \frac{1-e^{-B(T-s)}}{B} G\right) M e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \textit{for all } i = 1, \dots, n.$$

If g and h satisfy (D1)–(D4) *with*

$$N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B} G\right) M e^{BT} \quad \textit{instead of} \quad N = \sqrt{n}\left(A + \frac{1-e^{BT}}{B} G\right) E e^{(B+F)T},$$

then (4.4.22) *has a unique solution $(Y^{t,x}, Z^{t,x}) \in \mathbb{S}_t^\infty(\mathbb{R}) \times \mathbb{H}_t^\infty(\mathbb{R}^n)$, and*

$$\begin{aligned} |Y_s^{t,x}| &\le (C+1)e^{D(T-s)} - 1 \quad \textit{for all } s \in [t,T] \ a.s. \\ |Z_s^{t,x,i}| &\le \left(A + \frac{1-e^{-B(T-s)}}{B} G\right) M e^{B(T-s)} \quad ds \otimes d\mathbb{P}\text{-a.e.} \quad \textit{for all } i = 1, \dots, n. \end{aligned}$$

Proof. If g and h satisfy (C1)–(C4), the proposition follows like Proposition 4.4.1, and if g and h fulfill (D1)–(D4), it follows like Proposition 4.4.3. □

(4.4.23) is a crucial assumption of Proposition 4.4.19. The following lemma gives a sufficient condition for it.

Lemma 4.4.20. *Assume that $\mathcal{O}$ is a convex polyhedron with nonempty interior in $\mathbb{R}^n$, $b = 0$, and $\sigma = c\,Id$ for a constant $c \in \mathbb{R}_+$. Then condition* (4.4.23) *holds.*

Proof. It follows from Theorems 2.1 and 2.2 in Dupuis and Ishii [31] that $X_s^{t,x}$ is Lipschitz continuous in W with constants $A_1, \dots, A_n$ independent of t, s and x. So the statement follows from Proposition B.1.2. □

Semilinear parabolic PDEs with lateral Neumann boundary conditions

Assume that $\mathcal{O} \subset \mathbb{R}^n$ is bounded and there exists a function $w \in C^2(\mathbb{R}^n)$ with bounded derivatives of first and second order such that $\mathcal{O} = \{w > 0\}, \partial\mathcal{O} = \{w = 0\}, \mathbb{R}^n \setminus \bar{\mathcal{O}} = \{w < 0\}$, and $|\nabla w(x)| = 1$ for $x \in \partial\mathcal{O}$. Then $\mathcal{O}$ satisfies the uniform exterior sphere condition and uniform interior cone condition. So for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, there exists a unique pair of continuous adapted processes $(X^{t,x}, L^{t,x})$ with values in $\bar{\mathcal{O}} \times \mathbb{R}_+$ such that

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r + \int_t^s \nabla w(X_r^{t,x})dL_r^{t,x} \\ L_s^{t,x} &= \int_t^s 1_{\{X_r^{t,x} \in \partial\mathcal{O}\}} dL_r^{t,x} \quad \text{and} \quad L^{t,x} \text{ is nondecreasing.} \end{aligned}$$

If the forward process is of this form, the Markovian BSDE (4.4.22) is related to the following parabolic PDE with lateral Neumann boundary conditions:

$$\begin{aligned} &u_t(t,x) + \mathcal{L}_x u(t,x) + g\left(t,x,u(t,x),(\nabla u\sigma)(t,x)\right) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathcal{O} \\ &\frac{\partial u}{\partial n}(t,x) = 0 \quad \text{for } (t,x) \in (0,T) \times \partial\mathcal{O} \quad \text{and} \quad u(T,x) = h(x) \quad \text{for } x \in \bar{\mathcal{O}}, \end{aligned} \tag{4.4.24}$$

where

$$\frac{\partial}{\partial n} := \sum_{i=1}^n \frac{\partial w}{\partial x_i}(x)\frac{\partial}{\partial x_i}, \quad \text{and} \quad \mathcal{L}_x := \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_i b_i(x)\partial_{x_i}.$$

Definition 4.4.21. *A continuous function $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ is called a viscosity solution of* (4.4.24) *if $u(T,x) = h(x)$ and for each $(t,x) \in [0,T] \times \mathbb{R}^m$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R}^m;\mathbb{R})$ such that $\phi(t,x) = u(t,x)$ and (t,x) is a local minimum of $\phi - u$,*

$$\begin{aligned} &\partial_t\phi(t,x) + \mathcal{L}_x\phi(t,x) + f(t,x,\phi(t,x),(\nabla\phi\sigma)(t,x)) \geq 0 && \text{if } x \in \mathcal{O} \\ &\max\left\{\partial_t\phi(t,x) + \mathcal{L}_{(t,x)}\phi(t,x) + f(t,x,\phi(t,x),(\nabla\phi\sigma)(t,x)), \frac{\partial\phi}{\partial n}(t,x)\right\} \geq 0 && \text{if } x \in \partial\mathcal{O} \end{aligned}$$

and for each $(t',x') \in [0,T] \times \mathbb{R}^m$ and $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m;\mathbb{R})$ such that $\varphi(t',x') = u(t',x')$ and (t',x') is a local maximum of $\varphi - u$,

$$\begin{aligned} &\partial_t\varphi(t',x') + \mathcal{L}_x\varphi(t',x') + f(t',x',\varphi(t',x'),(\nabla\varphi\sigma)(t',x')) \leq 0 && \text{if } x' \in \mathcal{O} \\ &\min\left\{\partial_t\varphi(t',x') + \mathcal{L}_x\varphi(t',x') + f(t',x',\varphi(t',x'),(\nabla\varphi\sigma)(t',x')), \frac{\partial\varphi}{\partial n}(t',xx)\right\} \leq 0 && \text{if } x' \in \partial\mathcal{O}. \end{aligned}$$

Proposition 4.4.22. *Assume condition* (4.4.23) *holds and* g, h *satisfy* (D1)–(D4) *with*

$$N = \sqrt{n}\left(A + \frac{1 - e^{BT}}{B}G\right) Me^{BT} \quad \textit{instead of} \quad N = \sqrt{n}\left(A + \frac{1 - e^{BT}}{B}G\right) Ee^{(B+F)T}.$$

Let $(Y^{t,x}, Z^{t,x})$ *be the solution of the BSDE* (4.4.22)*. Then,* $u(t,x) := Y_t^{t,x}$ *is a viscosity solution of the PDE* (4.4.24) *satisfying* $|u(t,x)| \leq (C+1)e^{D(T-t)} - 1$ *for all* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$*.*

Proof. One can assume that g is Lipschitz in (x,y,z) by modifying it for large (x,y,z). Then the results of Pardoux and Zhang [68] apply, and one obtains that $u(t,x) := Y_t^{t,x}$ is a viscosity solution of the PDE (4.4.24). By Proposition 4.4.19, it is bounded by $(C+1)e^{D(T-t)} - 1$. □

If one makes stronger assumptions on $\mathcal{O}, b, \sigma$ and g, the viscosity solution u of Proposition 4.4.22 is unique. We denote by $\mathcal{S}^n$ the set of all symmetric $n \times n$-matrices and define the function $F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ by

$$F(t,x,y,v,S) := -\frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(x)S_{ij} - \sum_i b_i(x)v_i - g(T-t,x,y,v\sigma(x)).$$

Proposition 4.4.23. *Assume the boundary function* w *is* C^3 *with bounded derivatives of first, second and third order,* g *is continuous in* (t,x,y,z) *and the conditions of Proposition 4.4.22 hold. Moreover, suppose that for all* $L, L' \in \mathbb{R}_+$*, there exist a constant* $\gamma_L \in \mathbb{R}$ *and a function* $\delta_{L,L'} : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\lim_{x \downarrow 0} \delta_{L,L'}(x) = 0$ *such that the following two conditions hold:*

(i) $g(t,x,y',v\sigma(x)) - g(t,x,y,v\sigma(x)) \geq \gamma_L(y - y')$ *for all* $(t,x) \in [0,T] \times \bar{\mathcal{O}}$*,* $-L \leq y' \leq y \leq L$ *and* $v \in \mathbb{R}^n$*.*

(ii)

$$F(t,x',y,v',S') - F(t,x,y,v,S) \leq \delta_{L,L'}\left(\eta + |x - x'|(1 + |v| \vee |v'|) + \frac{|x-x'|^2}{\varepsilon^2}\right)$$

for all $\eta, \varepsilon \in (0,1]$*,* $t \in [0,T]$*,* $x, x' \in \bar{\mathcal{O}}$*,* $|y| \leq L$*,* $v, v' \in \mathbb{R}^n$ *and* $S, S' \in \mathcal{S}^n$ *satisfying the following three properties:*

$$-\frac{L'}{\varepsilon^2} Id \leq \begin{pmatrix} S & 0 \\ 0 & -S' \end{pmatrix} \leq \frac{L'}{\varepsilon^2}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + L'\eta Id$$

$$|v - v'| \leq L'\eta\varepsilon(1 + |v| \wedge |v'|)$$

$$|x - x'| \leq L'\eta\varepsilon.$$

Let $(Y^{t,x}, Z^{t,x})$ *be the solution of the BSDE* (4.4.22). *Then* $u(t,x) := Y_t^{t,x}$ *is the unique viscosity solution of the PDE* (4.4.24).

Proof. By Proposition 4.4.22, $u(t,x) := Y_t^{t,x}$ is a viscosity solution of (4.4.24). Uniqueness follows from Theorem 3.1 of Barles [7]. □

The unique viscosity solution of Proposition 4.4.23 is actually of class $C^{1,2}$ if one strengthens the assumptions.

Proposition 4.4.24. *Assume the conditions of Proposition 6.4 are satisfied and the following hold:*

(i) σ *is* $C^2(\bar{\mathcal{O}})$ *with bounded derivatives of first and second order and there exists a constant* $\varepsilon > 0$ *such that* $\sum_{i,j} \left(\sigma\sigma^T\right)_{ij}(x) v_i v_j \geq \varepsilon |v|^2$ *for all* $x, v \in \mathbb{R}^n$.

(ii) $b(x)v + g(t,x,y,v\sigma(x))$ *is continuously differentiable in* (t,x,y,v)

(iii) $h \equiv 0$.

Then the PDE (4.4.24) *has a unique* $C^{1,2}$*-solution* u*, and*

$$|u(t,x)| \leq e^{D(T-t)} - 1 \tag{4.4.25}$$

$$|\nabla u(t,x)| \leq \sqrt{\frac{n}{\varepsilon}} \left(A + \frac{1 - e^{-B(T-t)}}{B} G \right) M e^{B(T-t)}. \tag{4.4.26}$$

Proof. We can assume that g is Lipschitz in (x,y,z) by modifying it for large (x,y,x). Then it follows from Theorem V.7.4 of Ladyženskaja et al. [57] that there exists a $C^{1,2}$ solution. So the unique viscosity solution u of Proposition 4.4.23 is $C^{1,2}$. From Pardoux and Zhang [68], we know that $Y_s^{t,x} = u(s, X_s^{t,x})$. Since $h \equiv 0$, one obtains from Proposition 4.4.22 that u satisfies (4.4.25). Now fix $(t,x) \in (0,T) \times \mathcal{O}$ and let $\alpha > 0$ be a constant such that $\{y \in \mathbb{R} : |y - x| \leq \alpha\} \subset \mathcal{O}$. Define the stopping time $\tau^{t,x} := \inf\{s \geq t : |X_s^{t,x} - x| \geq \alpha\} \wedge (t+\alpha)$. Then $(Y_{s\wedge\tau^{t,x}}^{t,x}, Z_s^{t,x} 1_{\{s \leq \tau^{t,x}\}})$ and $(u(s \wedge \tau^{t,x}, X_{s\wedge\tau^{t,x}}^{t,x}), (\nabla u\sigma)(s, X_s^{t,x}) 1_{\{s \leq \tau^{t,x}\}})$ are bounded solutions of the BSDE

$$\tilde{Y}_s = u(\tau^{t,x}, X_{\tau^{t,x}}^{t,x}) + \int_s^{t+\alpha} g(r, X_r^{t,x}, \tilde{Y}_r^{t,x}, \tilde{Z}_r^{t,x}) 1_{\{s \leq \tau^{t,x}\}} dr - \int_s^{t+\alpha} \tilde{Z}_r^{t,x} dW_r, \tag{4.4.27}$$

on $[t, t+\alpha]$. By modifying g for large (x,y,z), one can assume that it is Lipschitz in (x,y,z). Then (4.4.27) is a standard BSDE and has a unique solution. Therefore, one obtains from

Proposition 4.4.19 that

$$|(\nabla u\sigma)(s,X_s^{t,x})1_{\{s\le\tau^{t,x}\}}| = |Z_s^{t,x}1_{\{s\le\tau^{t,x}\}}| \le \sqrt{n}\left(A+\frac{1-e^{-B(T-s)}}{B}G\right)Me^{B(T-s)} \quad ds\otimes d\mathbb{P}\text{-a.e.}$$

on $[t,t+\alpha]$, and in particular,

$$|(\nabla u\sigma)(t,x)| \le \sqrt{n}\left(A+\frac{1-e^{-B(T-t)}}{B}G\right)Me^{B(T-t)},$$

which by condition (i), gives the bound (4.4.26). □

As a consequence of Propositions 4.4.19, 4.4.22, 4.4.23 and 4.4.24 one obtains the following result for PDEs of the form:

$$\begin{aligned} &u_t = u_{xx} + g(u,u_x) \quad \text{on } [0,T]\times(c,d)\\ &u_x = 0 \quad \text{on } \mathbb{R}_+\times\{c,d\} \quad \text{and} \quad u(0,x)=h(x) \quad \text{for } x\in(c,d), \end{aligned} \tag{4.4.28}$$

where $u:[0,T]\times[c,d]\to\mathbb{R}$.

Corollary 4.4.25. *Assume h satisfies* (C1) *and g fulfills* (D2). *Then* (4.4.28) *has a viscosity solution u satisfying*

$$|u(t,x)| \le \left(\sup_{c<x<d}|h(x)|+1\right)e^{Dt}-1, \quad (t,x)\in[0,T]\times[c,d].$$

Moreover, if g is continuous in y, for every $L\in\mathbb{R}_+$, there exists a constant $\gamma_L\in\mathbb{R}$ such that for all $-L\le y'\le y\le L$ and $z\in\mathbb{R}^n$, one has

$$g(y',z)-g(y,z)\ge\gamma_L(y-y') \tag{4.4.29}$$

and $F(t,x,y,v,S)=\sum_{i,j}S_{ij}-g(y,v)$ satisfies condition (ii) *of Proposition 4.4.23, then u is the unique viscosity solution. If in addition, $h\equiv 0$ and g is C^1, then u is $C^{1,2}$ and satisfies*

$$|u_x(t,x)|\le 3Ae^{Bt} \quad \textit{for all } (t,x)\in[0,T]\times[c,d].$$

Proof. Set $b\equiv 0$, $\sigma\equiv\sqrt{2}\,Id$ and $\tilde g(y,z):=g(y,z/\sqrt{2})$. Since h is Lipschitz continuous and $[c,d]$ is compact, h is bounded. Therefore, $\tilde g$ and h satisfy (D1)–(D4) with $C=\sup_{c<x<d}|h(x)|$ and $G=H=0$. So one obtains from Proposition 4.4.19 and Lemma 4.4.20 that the BSDE

(4.4.22) has a unique solution $(Y^{t,x}, Z^{t,x})$ in $\mathbb{S}^\infty_t(\mathbb{R}) \times \mathbb{H}^\infty_t(\mathbb{R}^n)$ with $|Y^{t,x}_s| \leq (C+1)e^{D(T-s)} - 1$. It can be seen from Theorems 2.1 and 2.2 of Dupuis and Ishii [31] together with Proposition B.1.2 that condition (4.4.23) is satisfied with $M = 3\sqrt{2}$. Therefore, Proposition 4.4.19 yields $|Z^{t,x}_s| \leq 3\sqrt{2}Ae^{B(T-s)}$. By Proposition 4.4.22, $v(t,x) := Y^{t,x}_t$ is a viscosity solution of the PDE

$$
\begin{aligned}
&v_t + v_{xx} + g(v, v_x) = 0 \quad \text{on } [0,T] \times (c,d) \\
&v_x = 0 \quad \text{on } [0,T] \times \{c,d\} \quad \text{and} \quad v(T,x) = h(x) \quad \text{for } x \in (c,d),
\end{aligned}
$$

satisfying $|v(t,x)| \leq (C+1)e^{D(T-t)} - 1$. So $u(t,x) := v(T-t,x)$ is a viscosity solution of (4.4.28) with $|u(t,x)| \leq (C+1)e^{Dt} - 1$. If g is continuous in y, (4.4.29) holds and $F(t,x,y,v,S) = \sum_{i,j} S_{ij} - g(y,v)$ fulfills condition (ii) of Proposition 4.4.23, then the conditions of Proposition 4.4.23 are satisfied. So u is the unique viscosity solution. If in addition, $h \equiv 0$ and g is of class C^1, one obtains from Proposition 4.4.24 that u is of class $C^{1,2}$ and $|u_x(t,x)| \leq 3Ae^{Bt}$. □

Appendix A

Sobolev Space of Random Variables

In this appendix, we will review Da Prato [23] regarding Chapter 2. We will define Sobolev space of measurable functions on separable Hilbert space $\mathcal{H}$ with respect to Gaussian measure μ and prove the general version of Propositions 2.3.5, 2.3.6, and 2.3.8. If we let $\mathcal{H} := L^2([0,T];\mathbb{R}^n)$ and $\mathbb{P} = \mu$, the results corresponds to our propositions in Chapter 2.

A.1 Introduction to Sobolev Space

Without losing generality, we will assume $d = 1$. We denote $\mathcal{H}$ to be the separable Hilbert space of functions endowed with inner product $\langle \cdot, \cdot \rangle$. On $\mathcal{H}$, we endow Borel algebra $\mathcal{F}$. Let a continuous linear operator $Q : \mathcal{H} \to \mathcal{H}$ be self-adjoint, positive, and $\operatorname{Tr}(Q) < \infty$ (trace class). For such Q, we define $\lambda := \sup_k \lambda_k$ where $\{\lambda_k : k \in \mathbb{N}\}$ is a set of eigenvalues of Q. We define e_k be the eigenvector corresponding to λ_k.

Theorem A.1.1. *N_Q is a Gaussian measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ with mean zero and covariance operator Q if*

$$\int_{\mathcal{H}} e^{i\langle h,x\rangle} N_Q(dx) = \exp\left(-\frac{1}{2}\langle Qh, h\rangle\right).$$

We call N_Q is non-degenerate if $\ker Q = \{0\}$. Now, let us define Sobolev space $W^{1,2}$ in $\mathcal{H}$.

Definition A.1.2. *Let $\mathcal{E}(\mathcal{H})$ be the linear span of all real and imaginary parts of functions*

$\varphi_h, h \in \mathcal{H}$ in $C_b(\mathcal{H};\mathbb{R})$, where

$$\varphi_h(\omega) = e^{i\langle h,\omega\rangle}.$$

and $C_b(\mathcal{H};\mathbb{R})$ is the set of uniformly continuous and bounded mapping. For any $\varphi \in \mathcal{E}(\mathcal{H})$ and any $k \in \mathbb{N}$, we denote $\mathbf{D_k}\varphi$ to be the derivative of φ in the direction of e_k, namely

$$\mathbf{D}_k\varphi(\omega) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\varphi(\omega+\varepsilon e_k) - \varphi(\omega)\right), \quad \omega \in \mathcal{H}.$$

Then, the mapping

$$\mathbf{D} : \mathcal{E}(\mathcal{H}) \subset L^2(\mathcal{H},\mu) \to L^2(\mathcal{H},\mu;\mathcal{H}), \varphi \mapsto \mathbf{D}\varphi$$

is closable by Lemma A.1.6. We will maintain the notation $\mathbf{D}$ for the closure of $\mathbf{D}$. We shall denote the domain of $\mathbf{D}$ by $W^{1,2}(\mathcal{H},\mu)$. $W^{1,2}(\mathcal{H},\mu)$, endowed with the inner product

$$\langle\varphi,\psi\rangle_{W^{1,2}(\mathcal{H},\mu)} := \int_{\mathcal{H}} \left(\varphi\psi + \langle \mathbf{D}\varphi, \mathbf{D}\psi\rangle\right) d\mu$$

is a Hilbert space.

We will need integration by parts formula (Corollary A.1.5) to prove Lemma A.1.6. We need the following lemmas.

Lemma A.1.3. *For all $\varphi \in C_b(\mathcal{H};\mathbb{R})$, there exists a two-index sequence $(\varphi_{k,n}) \subset \mathcal{E}(\mathcal{H})$ such that*

$$\lim_{k\to\infty}\lim_{n\to\infty}\varphi_{k,n}(x) = \varphi(x) \qquad \forall x \in \mathcal{H}$$
$$\sup_x |\varphi_{k,n}(x)| \le \sup_x |\varphi(x)| + \frac{1}{n} \qquad \forall k,n \in \mathbb{N}.$$

Proof. Assume that $\mathcal{H}$ is finite dimensional and let $\varphi \in C_b(\mathcal{H};\mathbb{R})$. By the Stone-Weierstrass theorem, for any $R \in \mathbb{N}$, there exists a sequence $(\psi_{R,n}) \subset \mathcal{E}(\mathcal{H})$ such that

$$\sup_{|x|\le R}|\varphi(x) - \psi_{R,n}(x)| \le \frac{1}{n}$$
$$\sup_x |\psi_{R,n}(x)| \le \sup_x |\varphi(x)| + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

The claim is now can be proved by a standard diagonalization argument.

Now consider the case where $\mathcal{H}$ is infinite dimensional. For $\varphi \in C_b(\mathcal{H};\mathbb{R})$ and $k \in \mathbb{N}$, set

$\varphi_k := \varphi \circ P_k$ where $P_k x := \sum_{h=1}^{k} \langle x, e_h \rangle e_h$. There exists a sequence $(\varphi_{k,n}) \subset \mathcal{E}(\mathcal{H})$ such that

$$\lim_{n\to\infty} \varphi_{k,n}(x) = \varphi_k(x) \quad \forall x \in \mathcal{H}$$
$$\sup_x |\varphi_{k,n}(x)| \leq \sup_x |\varphi_k(x)| + \frac{1}{n} \leq \sup_x |\varphi(x)| + \frac{1}{n}$$

Therefore,

$$\lim_{k\to\infty} \lim_{n\to\infty} \varphi_{k,n}(x) = \varphi(x) \quad \forall x \in \mathcal{H}.$$

□

Lemma A.1.4. *For $\varphi, \psi \in \mathcal{E}(\mathcal{H})$,*

$$\int_{\mathcal{H}} \mathbf{D}_k \varphi \psi d\mu = -\int_{\mathcal{H}} \varphi \mathbf{D}_k \psi d\mu + \frac{1}{\lambda_k} \int_{\mathcal{H}} x_k \varphi \psi d\mu$$

Proof. By dominated convergence theorem and previous lemma, it is enough to prove the identity for $\varphi(x) = e^{i\langle f,x \rangle}$ and $\psi(x) = e^{i\langle g,x \rangle}$ for $f, g \in \mathcal{H}$. Note that

$$\int_{\mathcal{H}} \mathbf{D}_k \varphi \psi d\mu + \int_{\mathcal{H}} \varphi \mathbf{D}_k \psi d\mu = i(f_k + g_k) \exp\left(-\frac{1}{2} \langle Q(f+g), f+g \rangle\right)$$
$$\begin{aligned} \int_{\mathcal{H}} x_k \varphi \psi d\mu &= \int_{\mathcal{H}} x_k e^{i\langle f+g,x \rangle} d\mu = -i \frac{d}{dt} \int_{\mathcal{H}} e^{i\langle f+g+te_k, x \rangle} d\mu \Big|_{t=0} \\ &= -i \frac{d}{dt} \exp\left(-\frac{1}{2} \langle Q(f+g+te_k), f+g+te_k \rangle\right) \Big|_{t=0} \\ &= i \langle Q(f+g), e_k \rangle \exp\left(-\frac{1}{2} \langle Q(f+g), f+g \rangle\right) \\ &= i\lambda_k (f_k + g_k) \exp\left(-\frac{1}{2} \langle Q(f+g), f+g \rangle\right). \end{aligned}$$

Therefore, the claim is proved. □

The following corollary is the integration by parts formular we need.

Corollary A.1.5. *For $\varphi, \psi \in \mathcal{E}(\mathcal{H})$ and $z \in Q^{1/2}(\mathcal{H})$,*

$$\int_{\mathcal{H}} \langle \mathbf{D}\varphi, z \rangle \psi d\mu = -\int_{\mathcal{H}} \langle \mathbf{D}\psi, z \rangle \varphi d\mu + \int_{\mathcal{H}} W_{Q^{-1/2}z} \varphi \psi d\mu$$

where W is the white noise mapping defined by Definition 2.3.1.

Proof. For $n \in \mathbb{N}$ and $P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$,

$$\begin{aligned}\int_{\mathcal{H}} \langle \mathbf{D}\varphi, P_n z \rangle \psi d\mu &= \sum_{k=1}^{n} z_k \int_{\mathcal{H}} \mathbf{D}_k \varphi \psi d\mu \\ &= \sum_{k=1}^{n} z_k \left(- \int_{\mathcal{H}} \varphi \mathbf{D}_k \psi d\mu + \frac{1}{\lambda_k} \int_{\mathcal{H}} x_k \varphi \psi d\mu \right) \\ &= - \int_{\mathcal{H}} \langle \mathbf{D}\psi, P_n z \rangle \varphi d\mu + \int_{\mathcal{H}} \sum_{k=1}^{n} \frac{z_k x_k}{\lambda_k} \varphi \psi d\mu.\end{aligned}$$

Note that

$$W_{Q^{-1/2} P_n z}(x) = \left\langle x, Q^{-1} P_n z \right\rangle = \sum_{k=1}^{n} \frac{x_k z_k}{\lambda_k}.$$

By letting $n \to \infty$, the claim is proved. □

Now, we can prove the closability of $\mathbf{D}$.

Lemma A.1.6. *The mapping*

$$\mathbf{D} : \mathcal{E}(\mathcal{H}) \subset L^2(\mathcal{H}, \mu) \to L^2(\mathcal{H}, \mu; \mathcal{H})$$

is closable.

Proof. Let $(\varphi_n) \subset \mathcal{E}(\mathcal{H})$ be a sequence such that

$$\varphi_n \to 0 \text{ in } L^2(\mathcal{H}, \mu)$$

$$D\varphi_n \to F \text{ in } L^2(\mathcal{H}, \mu; \mathcal{H})$$

as $n \to \infty$. If $F = 0$, then the claim is proved. Let $\psi \in \mathcal{E}(\mathcal{H})$ and $z \in Q^{1/2}(\mathcal{H})$. Then, by Corollary A.1.5,

$$\int_{\mathcal{H}} \langle \mathbf{D}\varphi_n, z \rangle \psi d\mu = - \int_{\mathcal{H}} \langle \mathbf{D}\psi, z \rangle \varphi_n d\mu + \int_{\mathcal{H}} W_{Q^{-1/2} z} \varphi_n \psi d\mu.$$

If we let $n \to \infty$,

$$\int_{\mathcal{H}} \langle F(x), z \rangle \psi \mu(dx) = 0.$$

Since ψ and z are arbitrary, $F = 0$. □

We will state chain rule and product rule without proof: see Da Prato [23] for the proofs.

Proposition A.1.7. *Let $\varphi \in W^{1,2}(\mathcal{H},\mu)$ and $g \in C_b^1(\mathbb{R};\mathbb{R})$ with bounded derivative. Then, $g(\varphi) \in W^{1,2}(\mathcal{H},\mu)$ and*

$$\mathbf{D}g(\varphi) = g'(\varphi)\mathbf{D}\varphi.$$

Proposition A.1.8. *Let $\varphi, \psi \in W^{1,2}(\mathcal{H},\mu)$ and suppose that ψ and $\mathbf{D}\psi$ are bounded. Then $\varphi\psi \in W^{1,2}(\mathcal{H},\mu)$ and*

$$\mathbf{D}(\varphi\psi) = (\mathbf{D}\varphi)\psi + \varphi(\mathbf{D}\psi).$$

A.2 Relationship between Da Prato's Derivative D and Malliavin Derivative D

For simplicity, we will only consider one-dimensional random variables. Let Γ be the set of all mappings

$$\gamma : \mathbb{N} \to \{0\} \cup \mathbb{N}, n \mapsto \gamma_n$$

such that

$$|\gamma| := \sum_{k=1}^{\infty} \gamma_k < \infty.$$

That is, we can represent γ as a sequence $(\gamma_1, \gamma_2, \cdots)$. We define $\gamma_n^{(h)} := \gamma_n$ if $n \neq h$ and $\gamma_n^{(n)} := \gamma_n - 1$. Also, define

$$H_\gamma(x) := \prod_{k=1}^{\infty} H_{\gamma_k}(W_{e_k}(x))$$

where W_h is the white noise mapping defined in Definition 2.3.1 except we use a general Hilbert space $\mathcal{H}$ instead of Ω and H_n is Hermite polynomial defined by

$$H_n(\xi) := \frac{(-1)^n}{\sqrt{n!}} e^{\xi^2/2}(\partial_\xi)^n(e^{-\xi^2/2}).$$

Since $(H_\gamma)_{\gamma\in\Gamma}$ is orthonormal and complete, we can use it as a basis of $L^2(\mathcal{H},\mu)$. The representation of φ with respect to this basis is called Wiener chaos decomposition. Consider a random variable $\varphi \in L^2(\mathcal{H},\mu)$ with Wiener chaos

$$\varphi = \sum_{\gamma\in\Gamma} \varphi_\gamma H_\gamma.$$

Note that

$$\mathbf{D}W_{e_k} = \frac{1}{\sqrt{\lambda_k}} e_k \quad \text{and} \quad DW_{e_k} = e_k.$$

By applying the chain rules, Da Prato's derivative $\mathbf{D}$ is

$$\mathbf{D}\varphi = \sum_h e_h \left(\sum_{\gamma\in\Gamma} \sqrt{\frac{\gamma_h}{\lambda_h}} \varphi_\gamma H_{\gamma^{(h)}} \right),$$

while the Malliavin derivative D is

$$\begin{aligned} D\varphi &= D\left(\sum_{\gamma\in\Gamma} \varphi_\gamma H_\gamma\right) = \sum_{\gamma\in\Gamma} \varphi_\gamma DH_\gamma = \sum_{\gamma\in\Gamma} \varphi_\gamma \sum_{h=1}^{\infty} \left(\prod_{i=1, i\neq h}^{\infty} \sqrt{\gamma_h} H_{\gamma_i}(W_{e_i}) H_{\gamma_h - 1}(W_{e_h}) e_h \right) \\ &= \sum_{\gamma\in\Gamma} \varphi_\gamma \sum_{h=1}^{\infty} \left(\sqrt{\gamma_h} H_{\gamma^{(h)}} e_h \right) \\ &= \sum_{h=1}^{\infty} \left(\sum_{\gamma\in\Gamma} \sqrt{\gamma_h} \varphi_\gamma H_{\gamma^{(h)}} \right) e_h \end{aligned}$$

Since $\sum_k \lambda_k < \infty$, we know $\lambda_k < \infty$ and $\lim_{k\to\infty} \lambda_k = 0$. If $\varphi \in \mathbb{W}^{1,2}$,

$$\int_{\mathcal{H}} \langle \mathbf{D}\varphi, \mathbf{D}\varphi \rangle \, d\mu = \sum_{\gamma\in\Gamma} \sum_{h=1}^{\infty} \frac{\gamma_h |\varphi_\gamma|^2}{\lambda_h},$$

and this implies that at most finitely many $\sqrt{\gamma_h}\varphi_\gamma$ are nonzero. Then, automatically, $\varphi \in \mathbb{D}^{1,2}$.

A.3 Proof of Compact Embedding Theorem 2.3.6

We will use the notations in Appendices A.1 and A.2.

Proposition A.3.1. *$W^{1,2}(\mathcal{H}, \mu)$ is compactly embedded to $L^2(\mathcal{H}, \mu)$. That is, any bounded sequence in $W^{1,2}(\mathcal{H}, \mu)$ has a subsequence which is convergent in $L^2(\mathcal{H}, \mu)$.*

Proof. Let $(\varphi^{(n)})$ be a sequence in $W^{1,2}(\mathcal{H}, \mu)$ such that $\langle \varphi^{(n)}, \varphi^{(n)} \rangle_{W^{1,2}} \leq K$ for some $K > 0$. Since $L^2(\mathcal{H}, \mu)$ is reflexive, there exists a subsequence $(\varphi^{(n_k)}) \subset (\varphi^{(n)})$ such that weakly convergent to some $\varphi \in L^2(\mathcal{H}, \mu)$. We will show this subsequence converge strongly and then our claim is proved.

For $N \in \mathbb{N}$, define $\Gamma_N := \left\{\gamma \in \Gamma : \left\langle \gamma, \lambda^{-1} \right\rangle \leq N\right\}$ where $\left\langle \gamma, \lambda^{-1} \right\rangle := \sum_h \gamma_h \lambda_h^{-1}$. Note that

$$\begin{aligned}\int_{\mathcal{H}} \left|\varphi - \varphi^{(n_k)}\right|^2 d\mu &= \sum_{\gamma \in \Gamma_N} \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 + \sum_{\gamma \in (\Gamma_N)^c} \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 \\ &\leq \sum_{\gamma \in \Gamma_N} \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 + \frac{1}{N} \sum_{\gamma \in \Gamma} \left\langle \gamma, \lambda^{-1} \right\rangle \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 .\end{aligned}$$

For any $k \in \mathbb{N}$, since we have

$$\int_{\mathcal{H}} \langle \mathbf{D}\psi, \mathbf{D}\psi \rangle \, d\mu = \sum_{\gamma \in \Gamma} \left\langle \gamma, \lambda^{-1} \right\rangle |\psi_\gamma|^2$$

for any $\psi \in W^{1,2}(\mathcal{H}, \mu)$, the following inequality holds.

$$\sum_{\gamma \in \Gamma} \left\langle \gamma, \lambda^{-1} \right\rangle |\varphi_\gamma^{(n_k)}|^2 \leq K$$

This implies

$$\sum_{\gamma \in \Gamma} \left\langle \gamma, \lambda^{-1} \right\rangle |\varphi_\gamma|^2 \leq K$$

Therefore,

$$\int_{\mathcal{H}} \left|\varphi - \varphi^{(n_k)}\right|^2 d\mu \leq \sum_{\gamma \in \Gamma_N} \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 + \frac{2K}{N}$$

Since Γ_N is finite and $(\varphi^{(n_k)})$ converges weakly to φ,

$$\lim_{k \to \infty} \sum_{\gamma \in \Gamma_N} \left|\varphi_\gamma - \varphi_\gamma^{(n_k)}\right|^2 = 0.$$

The conclusion follows since N is arbitrary. $\square$

Appendix B

Appendix for Chapter 4

B.1 Malliavin Derivative of Lipschitz Random Variables

Let us show that terminal conditions ξ which are Lipschitz continuous in the underlying Brownian motion W are Malliavin differentiable with bounded Malliavin derivative. On the other hand, we give an example of a terminal condition with bounded Malliavin derivative that is not Lipschitz continuous in W. This shows that condition (A1) in Chapter 4 is weaker than Lipschitz continuity in W.

Definition B.1.1. *We denote the space of all continuous functions from $[0,T]$ to $\mathbb{R}^n$ starting from 0 by $C_0^n[0,T]$ and call a random variable ξ Lipschitz continuous in the Brownian motion W with constants $A_1,\dots,A_n \in \mathbb{R}_+$ if $\xi = \varphi(W)$ for a function $\varphi : C_0^n[0,T] \to \mathbb{R}$ satisfying*

$$|\varphi(v) - \varphi(w)| \leq \sum_{i=1}^{n} A_i \sup_{0 \leq t \leq T} |v^i(t) - w^i(t)|. \tag{B.1.1}$$

Proposition B.1.2. *Let ξ be Lipschitz continuous in W with constants $A_1,\dots,A_n \in \mathbb{R}_+$. Then $\xi \in \mathbb{D}^{1,2}$ and $|D_t^i \xi| \leq A_i \; dt \otimes d\mathbb{P}$-a.e. for all $i = 1,\dots,n$.*

Proof. Assume ξ is of the form $\varphi(W)$ for a function φ satisfying (B.1.1). For $m \in \mathbb{N}$, set $t_j^m := jT/m$, $j = 0,\dots,m$, and define the mapping $l^m : \left\{x = (x_j)_{j=1}^m : x_j \in \mathbb{R}^n\right\} \to C_0^n[0,T]$ by

$$l_0^m(x) := 0 \quad \text{and} \quad l_t^m(x) := x_1 + \dots + x_{j-1} + \frac{t - t_{j-1}^m}{T/m} x_j \quad \text{for } t_{j-1}^m < t \leq t_j^m.$$

Set $\xi^m := \varphi \circ l^m(\Delta W_{t_1^m},\dots,\Delta W_{t_m^m})$. For every $p \in [2,\infty)$, there exists a constant $b_p \in \mathbb{R}_+$ such that

$$
\begin{aligned}
\mathbb{E}|\xi - \xi^m|^p \leq b_p\, \mathbb{E} \sup_{0\leq t\leq T} |W_t^1 - l_t^{m,1}(\Delta W_{t_1^m}, \ldots, \Delta W_{t_m^m})|^p \\
\leq \quad b_p\, \mathbb{E} \max_{j=1,\ldots,m} \sup_{t_{j-1}^m < t \leq t_j^m} \left| W_t^1 - W_{t_{j-1}^m}^1 - \frac{t - t_{j-1}^m}{T/m} \Delta W_{t_j^m}^1 \right|^p \\
\leq \quad b_p m \mathbb{E} \sup_{0<t\leq T/m} \left| W_t^1 - \frac{t W_{T/m}^1}{T/m} \right|^p,
\end{aligned}
$$

where for the last inequality, we used that W has stationary increments. It follows that

$$
\begin{aligned}
\|\xi - \xi^m\|_p &\leq (b_p m)^{1/p} \left\| \sup_{0<t\leq T/m} \left| W_t^1 - \frac{t W_{T/m}^1}{T/m} \right| \right\|_p \\
&\leq (b_p m)^{1/p} \left(\left\| \sup_{0<t\leq T/m} |W_t^1| \right\|_p + \left\| W_{T/m}^1 \right\|_p \right) \\
&\leq (b_p m)^{1/p} c_p \left\| W_{T/m}^1 \right\|_p \leq (b_p m)^{1/p} d_p \sqrt{T/m},
\end{aligned}
$$

where c_p and d_p are constants depending on p, and the third inequality follows from Doob's maximal inequality. For $p > 2$ the last term goes to 0 as $m \to \infty$. This shows that $\xi^m \to \xi$ in $\mathbb{L}^p$ for all $p \in (2, \infty)$ and therefore also in $\mathbb{L}^2$.

Note that for $x, y \in \mathbb{R}^{mn}$,

$$
|\varphi \circ l^m(x) - \varphi \circ l^m(y)| \leq \sum_{i,j} A_i |x_j^i - y_j^i|. \tag{B.1.2}
$$

Let $\beta \in C_c^\infty(\mathbb{R}^{mn})$ be the mollifier

$$
\beta(x) := \begin{cases} \lambda \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases},
$$

where λ is a constant so that $\int_{\mathbb{R}^{mn}} \beta(x)dx = 1$. Set $\beta^m(x) := m^{mn}\beta(mx)$ and define

$$
\varphi^m(x) := \int_{\mathbb{R}^{mn}} \varphi \circ l^m(y) \beta^m(x - y) dy, \quad \tilde{\xi}^m := \varphi^m(\Delta W_{t_1^m}, \ldots, \Delta W_{t_m^m}).
$$

By Proposition 1.2.3 of Nualart [63], one has

$$
D^i \tilde{\xi}^m = \sum_{j=1}^m \frac{\partial}{\partial x_j^i} \varphi^m(\Delta W_{t_1^m}, \ldots, \Delta W_{t_m^m}) 1_{(t_{j-1}^m, t_j^m]}.
$$

But it follows from (B.1.2) that $\left|\frac{\partial}{\partial x_j^i}\varphi^m(x)\right| \leq A_i$ for all i, j. So $|D_t^i \tilde{\xi}^m| \leq A_i \; dt \otimes d\mathbb{P}$-a.e. Moreover $\tilde{\xi}^m \to \xi$ in $\mathbb{L}^2$. Hence, one obtains from Lemma 1.2.3 of Nualart [63] that ξ is in $\mathbb{D}^{1,2}$ and $D\tilde{\xi}^m \to D\xi$ in the weak topology of $L^2(\Omega; \mathcal{H})$. This implies that $|D_t^i \xi| \leq A_i \; dt \otimes d\mathbb{P}$-a.e. □

In the following example we construct a random variable with bounded Malliavin derivative that is not Lipschitz in the underlying Brownian motion.

Example B.1.3. Assume $T = n = 1$. Define

$$g(t) := \sum_{k=1}^{\infty} (-1)^{k-1} 2^k 1_{\{1-2^{1-k} < t \leq 1-2^{-k}\}}, \quad h(t) := \int_0^t g(s) ds,$$

and set

$$\xi := \int_0^1 h(t) dW_t.$$

Then $\xi \in \mathbb{D}^{1,2}$ and $D\xi = h$ is bounded by 1.

On the other hand, it follows from integration by parts that

$$\int_0^{1-2^{-2k}} h(t) dW_t = -\int_0^{1-2^{-2k}} g(t) W_t dt \quad \text{for all } k \geq 1.$$

Therefore,

$$\xi = -\lim_{k\to\infty} \int_0^{1-2^{-2k}} g(t) W_t dt,$$

which shows that ξ cannot be of the form $\xi = \varphi(W)$ for a Lipschitz continuous function $\varphi : C_0[0,1] \to \mathbb{R}$.

B.2 Proof for Proposition 4.1.5

Proposition B.2.1. *Assume the following conditions*

- $\xi \in \mathbb{D}^{1,2} \cap \mathbb{L}^4(\mathcal{F}_T)$
- $f(\omega, s, y, z) \in \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \mapsto f(\omega, s, y, z) \in \mathbb{R}^d$ *is continuously differentiable in* (y, z) *and there exist constants* $B, \rho \in \mathbb{R}_+$ *such that* $|\partial_y f| \leq B$ *and* $|\partial_z f| \leq \rho$
- $f(.,0,0) \in \mathbb{H}^4(\mathbb{R}^d)$, $f(\cdot, y, z) \in \mathbb{L}_a^{1,2}$ *for all* (y, z)*, and*

$$\int_0^T \mathbb{E}|(D_r f)(t, Y, Z)|^2 dr < \infty$$

where (Y,Z) is a solution of **BSDE**(ξ, f)

- *For a.e. $r \in [0,T]$, there exists a non-negative process $K_{r\cdot}$ in $\mathbb{H}^4(\mathbb{R})$ such that*

$$\int_0^T ||K_{r,\cdot}||^4_{\mathbb{H}^4} dr < \infty \quad \text{and} \quad |D_r f(t,y_1,z_1) - D_r f(t,y_2,z_2)| \leq K_{rt}(|y_1-y_2| + |z_1-z_2|),$$

for all $t \in [0,T]$ and $(y_1,z_1),(y_2,z_2) \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$.

Then, $(Y,Z) \in \mathbb{S}^4(\mathbb{R}^d) \times \mathbb{H}^4(\mathbb{R}^{d\times n})$ and for fixed $i = 1,\dots,n$,

$$(D^i_r Y_t, D^i_r Z_t) = (U^r_t, V^r_t) \;\; dr \otimes dt \otimes d\mathbb{P}\text{-a.e.} \quad \text{and} \quad Z^i_t = U^t_t \;\; dt \otimes d\mathbb{P}\text{-a.e.},$$

where

$$U^r_t = 0,\; V^r_t = 0, \quad 0 \leq t < r \leq T,$$

and for each fixed r, $(U^r_t, V^r_t)_{r\leq t\leq T}$ is the unique pair in $\mathbb{S}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times n})$ solving the BSDE

$$U^r_t = D^i_r \xi + \int_t^T [\partial_y f(s,Y_s,Z_s)U^r_s + \partial_z f(s,Y_s,Z_s)V^r_s + D^i_r f(s,Y_s,Z_s)]ds - \int_t^T V^r_s dW_s. \quad \text{(B.2.3)}$$

Before we proceed, let us state the following lemma which is proved in Lemma 2.3 of Pardoux and Peng [66].

Lemma B.2.2. *Assume that $\int_t^T Z_s dW_s \in \mathbb{D}^{1,2}$. Then, $Z \in L^2([t,T];\mathbb{D}^{1,2})$ and*

$$D^i_r\left(\int_t^T Z_s dW_s\right) = \int_t^T D^i_r Z_s dW_s \quad \text{if} \quad r \leq t,$$
$$D^i_r\left(\int_t^T Z_s dW_s\right) = Z^i_r + \int_r^T D^i_r Z_s dW_s \quad \text{if} \quad r > t$$

$dr \otimes d\mathbb{P}$-a.s.

Proof of Proposition 4.1.5. We follow the proof in El Karoui et al. [34]. Without losing generality, we can assume $d = n = 1$. Note that if we can prove the proposition for small enough T, we can partition $[0,T]$ and iterate the argument. Therefore, we only need to prove for small enough T. Let $(Y^0, Z^0) = (0,0)$ and define (Y^k, Z^k) iteratively by solving

$$Y^{k+1}_t = \xi + \int_t^T f(s, Y^k_s, Z^k_s)ds - \int_t^T Z^{k+1}_s dW_s$$

Note that, for small T, (Y^k, Z^k) converges in $\mathbb{S}^4 \times \mathbb{H}^4$ to the solution (Y,Z) of BSDE as one can see in the proof of Proposition 4.1.7. For any finite T, one can partition $[0,T]$ to small intervals such that $(Y^k, Z^k) \xrightarrow{\mathbb{S}^4\times\mathbb{H}^4} (Y,Z)$ in those intervals. Therefore, (Y^k, Z^k) converges in $\mathbb{S}^4 \times \mathbb{H}^4$ to the solution (Y,Z).

We will use mathematical induction to show $(Y^k, Z^k) \in L^2([0,T]; \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ for all $k \in \mathbb{N}$. Assume that $(Y^k, Z^k) \in L^2([0,T]; \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$. Then,

$$Y_t^{k+1} = \mathbb{E}_t\left(\xi + \int_t^T f(s, Y_s^k, Z_s^k)ds\right) \in \mathbb{D}^{1,2}$$
$$\int_0^T Z_s^{k+1} dW_s = \xi + \int_0^T f(s, Y_s^k, Z_s^k)ds - Y_0^{k+1} \in \mathbb{D}^{1,2},$$

and by previous lemma, we have $Z^{k+1} \in L^2([0,T]; \mathbb{D}^{1,2})$. Moreover, for $0 \le r \le t$, we have

$$\begin{aligned} D_r Y_t^{k+1} = D_r\xi &+ \int_t^T \left[\partial_y f(s, Y_s^k, Z_s^k) D_r Y_s^k + \partial_z f(s, Y_s^k, Z_s^k) D_r Z_s^k + (D_r f)(s, Y_s^k, Z_s^k)\right] ds \\ &- \int_t^T D_r Z_s^{k+1} dW_s. \end{aligned}$$

Now, using Theorem 2.2.3 for the case where $\mathbb{F} = \mathbb{F}^W$, define $(U^r, V^r) \in \mathbb{S}^2 \times \mathbb{H}^2$ by the solution of the following BSDE.

$$\begin{aligned} U_t^r = D_r\xi &+ \int_t^T \left[\partial_y f(s, Y_s, Z_s) U_s^r + \partial_z f(s, Y_s, Z_s) V_s^r + +(D_r f)(s, Y_s, Z_s)\right] ds \\ &- \int_t^T V_s^r dW_s. \end{aligned}$$

By the estimate in Proposition 4.1.7, there exists a constant C such that

$$\left\|D_r Y^{k+1} - U^r\right\|_{\mathbb{S}^2}^2 + \left\|D_r Z^{k+1} - V^r\right\|_{\mathbb{H}^2}^2 \le C\mathbb{E}\left(\int_0^T |\delta_s^k| ds\right)^2$$

holds for almost all $r \in [0,T]$, where

$$\begin{aligned} \delta_s^k := &D_r f(s, Y_s, Z_s) - D_r f(s, Y_s^k, Z_s^k) \\ &+ \partial_y f(s, Y_s, Z_s) U_s^r - \partial_y f(s, Y_s^k, Z_s^k) D_r Y_s^k \\ &+ \partial_z f(s, Y_s, Z_s) V_s^r - \partial_z f(s, Y_s^k, Z_s^k) D_r Z_s^k. \end{aligned}$$

Then, there exists a constant C' such that $\mathbb{E}\left(\int_0^T |\delta_s^k| ds\right)^2 \le C'(A_k^r + B_k^r + C_k^r)$, where

$$
\begin{aligned}
A_k^r &:= \mathbb{E}\left(\int_r^T \left|D_r f(s, Y_s, Z_s) - D_r f(s, Y_s^k, Z_s^k)\right| ds\right)^2 \\
B_k^r &:= \mathbb{E}\left(\int_r^T \left|\partial_y f(s, Y_s^k, Z_s^k)(U_s^r - D_r Y_s^k)\right| ds\right)^2 + \mathbb{E}\left(\int_r^T \left|\partial_z f(s, Y_s^k, Z_s^k)(V_s^r - D_r Z_s^k)\right| ds\right)^2 \\
C_k^r &:= \mathbb{E}\left(\int_r^T \left|\left(\partial_y f(s, Y_s, Z_s) - \partial_y f(s, Y_s^k, Z_s^k)\right) U_s^r\right| ds\right)^2 \\
&\quad + \mathbb{E}\left(\int_r^T \left|\left(\partial_z f(s, Y_s, Z_s) - \partial_z f(s, Y_s^k, Z_s^k)\right) V_s^r\right| ds\right)^2 .
\end{aligned}
$$

Note that, by Cauchy Schwartz inequality,

$$
\begin{aligned}
A_k^r &\le \mathbb{E}\left(\int_r^T K_{rs}\left(|Y_s - Y_s^k| + |Z_s - Z_s^k|\right) ds\right)^2 \\
&\le \mathbb{E}\left(\int_r^T K_{rs}^2 ds \int_r^T \left(|Y_s - Y_s^k| + |Z_s - Z_s^k|\right)^2 ds\right) \\
&\le \left(\mathbb{E}\left(\int_r^T K_{rs}^2 ds\right)^2\right)^{1/2} \left(\mathbb{E}\left(\int_r^T \left(|Y_s - Y_s^k| + |Z_s - Z_s^k|\right)^2 ds\right)^2\right)^{1/2} \\
&\le 4\,\|K_{r\cdot}\|_{\mathbb{H}^4}^2 \left(\left\|Y - Y^k\right\|_{\mathbb{H}^4}^2 + \left\|Z - Z^k\right\|_{\mathbb{H}^4}^2\right)
\end{aligned}
$$

Since $(Y^k, Z^k) \xrightarrow{\mathbb{S}^4 \times \mathbb{H}^4} (Y, Z)$,

$$
\lim_{k \to \infty} \int_0^T A_k^r dr = 0.
$$

On the other hand, note that

$$
C_k^r \le T\mathbb{E} \int_r^T \left(\left|\left(\partial_y f(s, Y_s, Z_s) - \partial_y f(s, Y_s^k, Z_s^k)\right) U_s^r\right|^2 + \left|\left(\partial_z f(s, Y_s, Z_s) - \partial_z f(s, Y_s^k, Z_s^k)\right) V_s^r\right|^2\right) ds.
$$

Since $\partial_y f$ and $\partial_z f$ are bounded and continuous with respect to y and z, and $\int_0^T \left(\|U^r\|_{\mathbb{S}^2}^2 + \|V^r\|_{\mathbb{H}^2}^2\right) dr$ is finite, by dominated convergence theorem,

$$
\lim_{k \to \infty} \int_0^T C_k^r dr = 0.
$$

Next, note that

$$B_k^r \le (B^2+\rho^2)\left(T^2\left\|U^r - D_rY^k\right\|_{\mathbb{S}^2}^2 + T\left\|V^r - D_rZ_s^k\right\|_{\mathbb{H}^2}^2\right)$$
$$\le (B^2+\rho^2)\left(T^2+T\right)\left(\left\|U^r - D_rY^k\right\|_{\mathbb{S}^2}^2 + \left\|V^r - D_rZ_s^k\right\|_{\mathbb{H}^2}^2\right).$$

Let T small enough so that $\alpha := (B^2+\rho^2)(T^2+T) < 1$ and fix $\varepsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that for all $k \ge N$,

$$\int_0^T \left(\left\|D_rY^{k+1} - U^r\right\|_{\mathbb{S}^2}^2 + \left\|D_rZ^{k+1} - V^r\right\|_{\mathbb{H}^2}^2\right) dr$$
$$\le C\int_0^T \mathbb{E}\left(\int_0^T |\delta_s^k| ds\right)^2 dr \le CC'\int_0^T (A_k^r + B_k^r + C_k^r)dr$$
$$\le \varepsilon + \alpha\int_0^T \left(\left\|U^r - D_rY^k\right\|_{\mathbb{S}^2}^2 + \left\|V^r - D_rZ_s^k\right\|_{\mathbb{H}^2}^2\right) dr.$$

If we define

$$H_k := \int_0^T \left(\left\|D_rY^k - U^r\right\|_{\mathbb{S}^2}^2 + \left\|D_rZ^k - V^r\right\|_{\mathbb{H}^2}^2\right) dr,$$

above inequality implies that

$$H_{k+1} - \frac{\varepsilon}{1-\alpha} \le \alpha\left(H_k - \frac{\varepsilon}{1-\alpha}\right)$$

for all $k \ge N$, and therefore,

$$H_k \le \frac{\varepsilon}{1-\alpha} + \alpha^{k-N}\left(H_N - \frac{\varepsilon}{1-\alpha}\right)$$

for $k \ge N$. Since $\varepsilon > 0$ is arbitrary and $\alpha^{k-N}\left(H_N - \varepsilon(1-\alpha)^{-1}\right) \to 0$ as $k \to \infty$, we have

$$H_k = \int_0^T \left(\left\|D_rY^k - U^r\right\|_{\mathbb{S}^2}^2 + \left\|D_rZ^k - V^r\right\|_{\mathbb{H}^2}^2\right) dr \to 0$$

as $k \to \infty$. Since $L([0,T];\mathbb{D}^{1,2}\times\mathbb{D}^{1,2})$ is closed under its norm, it follows that the limit $(Y,Z) \in L([0,T];\mathbb{D}^{1,2}\times\mathbb{D}^{1,2})$ and a version of (D_rY, D_rZ) is given by (U^r, V^r).

Note that for $t \le s$,

$$Y_s = Y_t - \int_t^s f(u, Y_u, Z_u)du + \int_t^s Z_u dW_u.$$

Then, by the previous lemma, it follows that, for $t < r \leq s$,

$$D_rY_s = Z_r - \int_r^s [\partial_y f(u, Y_u, Z_u) D_r Y_u + \partial_z f(u, Y_u, Z_u) D_r Z_u + (D_r f)(u, Y_u, Z_u)]\, du + \int_r^s D_r Z_u dW_u.$$

By taking $s = r$, we get $D_s Y_s = Z_s$ a.s. □

B.3 Proof for Proposition 4.1.7

We prove the more general version of Proposition 4.1.7.

Proposition B.3.1. *For every $L \in \mathbb{R}_+$ there exist constants $\mu, \nu > 0$ satisfying the following: If $T \leq \mu$, then for all p-standard parameters (f^i, ξ^i), $i = 1, 2$, such that f^1 fulfills*

$$\left| f^1(t, y, z) - f^1(t, y', z') \right| \leq L(|y - y'| + |z - z'|)$$

for all $(y, z), (y', z') \in \mathbb{R}^d \times \mathbb{R}^{d\times n}$, the BSDE solutions (Y^i, Z^i) corresponding to (f^i, ξ^i) satisfy

$$\left\| Y^1 - Y^2 \right\|_{\mathbb{S}^p}^p + \left\| Z^1 - Z^2 \right\|_{\mathbb{H}^p}^p \leq \nu\, \mathbb{E}\left[\left| \xi^1 - \xi^2 \right|^p + \left(\int_0^T \left| f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2) \right|^2 dt \right)^{p/2} \right].$$

Proof. We follow the proof by El Karoui et al. [34]. We will use C for a constant that changes line to line and does not depend on T or L. Let us denote $\delta Y := Y^1 - Y^2, \delta Z := Z^1 - Z^2$, and $\delta f := f^1(\cdot, Y^2, Z^2) - f^2(\cdot, Y^2, Z^2)$. Then,

$$\delta Y_t = \delta Y_T + \int_t^T (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y^2, Z^2))ds - \int_t^T \delta Z_s dW_s$$

Note that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \left| f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) \right|^2 ds \right)^{p/2} \\
&\leq \mathbb{E}\left(\int_0^T (L\, |\delta Y_s| + L\, |\delta Z_s| + |\delta f_s|)^2\, ds \right)^{p/2} \\
&\leq CL^p \mathbb{E}\left(\int_0^T |\delta Y_s|^2\, ds \right)^{p/2} + CL^p \mathbb{E}\left(\int_0^T |\delta Z_s|^2\, ds \right)^{p/2} + C\mathbb{E}\left(\int_0^T |\delta f_s|^2\, ds \right)^{p/2} \\
&\leq CL^p T^{p/2} \left\| \delta Y \right\|_{\mathbb{S}^p}^p + CL^p \left\| \delta Z \right\|_{\mathbb{H}^p}^p + C\mathbb{E}\left(\int_0^T |\delta f_s|^2\, ds \right)^{p/2}.
\end{aligned}$$

Since

$$\delta Y_t = \mathbb{E}_t\left[\delta Y_T + \int_t^T (f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2))ds\right],$$

by Doob's maximal inequality, we get

$$\begin{aligned}\|\delta Y\|_{\mathbb{S}^p}^p &\leq C\mathbb{E}\left[|\delta Y_T|^p + T^{p/2}\left(\int_0^T \left|f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\right|^2 ds\right)^{p/2}\right] \\ &\leq C\left(\mathbb{E}|\delta Y_T|^p + T^{p/2}\mathbb{E}\left(\int_0^T |\delta f_t|^2 ds\right)^{p/2}\right) + CL^pT^p\,\|\delta Y\|_{\mathbb{S}^p}^p + CL^pT^{p/2}\,\|\delta Z\|_{\mathbb{H}^p}^p \quad \text{(B.3.4)}\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^T \delta Z_s dW_s &= \delta Y_T + \int_0^T \left(f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\right) ds - \delta Y_0 \\ &= \delta Y_T + \int_0^T \left(f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\right) ds \\ &\qquad - \mathbb{E}\left[\delta Y_T + \int_0^T (f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2))ds\right],\end{aligned}$$

and therefore,

$$\begin{aligned}\|\delta Z\|_{\mathbb{H}^p}^p &\leq C\mathbb{E}\left(|\delta Y_T|^p + T^{p/2}\left(\int_0^T \left|f^1(s,Y_s^1,Z_s^1) - f^2(s,Y_s^2,Z_s^2)\right|^2 ds\right)^{p/2}\right) \\ &\leq C\mathbb{E}|\delta Y_T|^p + CL^pT^p\,\|\delta Y\|_{\mathbb{S}^2}^2 + CL^pT^{p/2}\,\|\delta Z\|_{\mathbb{H}^2}^2 + CT^{p/2}\mathbb{E}\int_0^T |\delta f_t|^2 dt \qquad \text{(B.3.5)}\end{aligned}$$

Let T small enough so that $\max\left\{CL^pT^p, CL^pT^{p/2}, T^{p/2}\right\} \leq 1/4$. Then, if we sum (B.3.4) and (B.3.5), then

$$\|\delta Y\|_{\mathbb{S}^p}^p + \|\delta Z\|_{\mathbb{H}^p}^p \leq 2C\left(\mathbb{E}|\delta Y_T|^p + \mathbb{E}\left(\int_0^T |\delta f_t|^2 ds\right)^{p/2}\right) + \frac{1}{2}\left(\|\delta Y\|_{\mathbb{S}^p}^p + \|\delta Z\|_{\mathbb{H}^p}^p\right),$$

and the claim is proved with $\nu = 4C$. □

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