On the Burau representation of the braid group B_4

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Abstract

In this thesis, we establish strong constraints on the kernel of the (reduced) Burau representation $\beta_4 : B_4 \to \mathrm{GL}_3\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$ of the braid group B_4, addressing a conjecture originally posed in the 1930s. The strategy of the proof is a concrete interpretation of $\beta_4(\sigma)$ in terms of the Garside normal form for $\sigma \in B_4$. More specifically, if σ is a positive braid in B_4 satisfying certain constraints, then we show that $\beta_4(\sigma)$ is not a diagonal matrix by considering a new decomposition of positive braids and combinatorially interpreting $\beta_4(\sigma)$ in terms of this decomposition.

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Contents

Chapter 1

Introduction

1.1 The main result

The (reduced) Burau representation $\beta_n : B_n \to \mathrm{GL}_{n-1}\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$, defined by Burau [4] in the 1930s, is a classical representation of the braid group B_n. Note that B_n can be defined as the mapping class group of the n-times punctured disk $\mathbb{D}_n$ and β_n can be defined as the action of B_n on the first homology of a certain infinite cyclic covering space $\widetilde{\mathbb{D}}_n$ of $\mathbb{D}_n$ viewed as a $\mathbb{Z}\left[q^{\pm 1}\right]$-module.

Indeed, the invariant subspace $\langle \gamma \rangle$ of the action of B_n on the first homology of the n-times punctured disk $H_1(\mathbb{D}_n)$ is primitive and infinite cyclic. We can consider the kernel of the composition $\pi_1(\mathbb{D}_n) \to H_1(\mathbb{D}_n) \to \langle \gamma \rangle$ (this depends on the second map which is a projection; the first map is abelianization) and $\widetilde{\mathbb{D}_n}$ is defined to be the covering space corresponding to this subgroup of $\pi_1(\mathbb{D}_n)$. The action of B_n on $\mathbb{D}_n$ lifts to an action of B_n on $\widetilde{\mathbb{D}}_n$. Moreover, $H_1\left(\widetilde{\mathbb{D}}_n\right)$ is a $\mathbb{Z}\left[q^{\pm 1}\right]$-module of rank $n-1$ where the action of q is induced by a generator of the deck transformation group of the covering map $\widetilde{\mathbb{D}}_n \to \mathbb{D}_n$. Of course, one can check that the isomorphism class of β_n is independent of the choice of projection $H_1(\mathbb{D}_n) \to \langle \gamma \rangle$.

Note that β_n can also be defined in terms of invertible $(n-1) \times (n-1)$ matrices with entries in $\mathbb{Z}\left[q^{\pm 1}\right]$ assigned to the generators in the Artin presentation of B_n, satisfying the relations in this presentation. In this thesis, we will use this interpretation of the Burau representation. We define β_4 this way in Definition 2.1.1.

The values of n for which β_n is faithful is a longstanding open question. Note that β_3 is known to be faithful [10]. However, in 1991, Moody [11] proved that β_n is not faithful for $n \geq 9$, and subsequently Long and Paton [9] in 1993 and Bigelow [1] in 1999 proved that β_n is not faithful for $n \geq 6$ and $n = 5$, respectively.

Nonetheless, the question as to whether or not β_4 is faithful remains an open problem. The goal of this thesis is to establish Theorem 1.1.2 constraining the kernel of β_4. We briefly review notation and terminology. Let us consider the Artin presentation of B_4:

$$\langle \sigma_1, \sigma_2, \sigma_3 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle .$$

We recall that the Garside element of B_4 is $\Delta = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$, and a positive braid in B_4 is a braid in the submonoid of B_4 generated by $\sigma_1, \sigma_2, \sigma_3$. In [5], Garside proved that every braid $\tau \in B_4$ can be expressed in the form $\tau = \Delta^k \sigma$ for $k \in \mathbb{Z}$ and σ a positive braid in B_4. If $k \in \mathbb{Z}$ is chosen to be as large as possible, then this expression is referred to as the Garside normal form of τ. (In Chapter 2, we discuss this notation and terminology in more detail.)

We introduce the following condition for positive braids in B_4 before formally stating Theorem 1.1.2, for ease of exposition.

Definition 1.1.1. Let σ be a positive braid in B_4. We write $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ for the unique product expansion of σ in the submonoid of positive braids (i.e., $a_i, b_i, c_i \geq 0$ for $1 \leq i \leq n$), with the property that the sequence of indices of the generators (read from left to right) is the smallest integer among all product expansions of σ in the submonoid of positive braids. We will assume without loss of generality that $a_i + b_i > 0$ for $2 \leq i \leq n$ and $c_i > 0$ for $1 \leq i \leq n-1$. We define the positive braid σ to be **normal** if this product expansion satisfies the following properties for each $p < n$.

(i) If $a_p > 1$, $b_p > 0$ and $c_p > 1$, then either $b_p > 1$ or $b_{p+1} > 0$.

(ii) If $a_p = 1$, $b_p > 0$ and $c_p > 1$, then $a_{p+1} > 0$, and either $b_p > 1$ or $b_{p+1} > 0$.

(iii) If $a_p = 0$, $b_p = 1$ and $c_p > 1$, then $b_{p+1} > 0$.

(iv) If $a_p > 0$, $b_p > 1$ and $c_p = 1$, then $a_p > 1$.

We are prepared to state Theorem 1.1.2.

Theorem 1.1.2. *Let $\tau \in B_4$ be a non-identity braid and write $\tau = \Delta^k \sigma$ for the Garside normal form of τ. If either $k \geq 0$ or σ is a normal braid (see Definition 1.1.1), then $\tau \notin ker(\beta_4)$.*

In this thesis, we refer to the unique product expansion of the positive braid σ in the statement of Definition 1.1.1 as the "minimal form of σ" (see Definition 2.2.1 and Lemma 2.2.3 in Chapter 2 for further discussion; it is not a new concept, as it was originally introduced by Garside in [5] under a different name). We view the constraint that a positive braid σ is normal as a set of "local constraints" on the minimal form of σ. In particular, conditions **(ii)** - **(iv)** in Definition 1.1.1 are local constraints when "isolated generators" (i.e., generators with exponent one) occur.

Theorem 1.1.2 is closely related to the open problem of whether or not the Jones polynomial of a knot detects the unknot. Indeed, in [2], it is conjectured that the Jones polynomial of a knot would *not* detect the unknot if the Jones representation of B_4 (defined by Jones in [6]) were not faithful. The Burau representation of B_4 is an irreducible summand of the Jones representation of B_4, and in particular, the kernel of the Jones representation of B_4 is contained in the kernel of the Burau representation of B_4 [6]. Thus, Theorem 1.1.2 also gives constraints on the kernel of the Jones representation of B_4.

Note that the linearity of the braid groups (i.e., the existence of *some* faithful finite dimensional linear representation) has already been established. Indeed, in [8], Lawrence introduced the Lawrence-Krammer-Bigelow representations of the braid groups. Subsequently, Bigelow [3] by topological methods, and Krammer [7] by algebraic methods, independently and concurrently established the faithfulness of these representations. The Lawrence-Krammer-Bigelow representation of B_4 is 6-dimensional and is the only known irreducible faithful finite-dimensional linear representation of B_4.

1.2 Outline of the proof

The outline of the proof of Theorem 1.1.2 is as follows. **Henceforth**, in this outline, we will only consider positive braids that are normal. (We use the phrase "positive braid" to implicitly refer to a normal positive braid.)

In Chapter 2, we will review background and terminology that we will use in the rest of the thesis and establish some elementary statements in connection to Theorem 1.1.2. In Section 2.1, we will recall the definition of the (reduced) Burau representation $\beta_4 : B_4 \to \mathrm{GL}_3\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$ explicitly as matrices assigned to the generators in the Artin presentation of B_4 (Definition 2.1.1). In Section 2.2, we will review the Garside normal form for B_4, which furnishes a solution to the word problem in B_4. The Garside normal form for the braid groups was established by Garside in his foundational paper [5]. A consequence of the results of [5] is that every braid is a product of a positive braid and an even negative power of the Garside element. (Note that the even powers of the Garside element constitute the center of the braid group.) We will consider the minimal form of a positive braid in B_4, a notion originally introduced in [5] under different terminology (in [5], it is referred to as the base of the diagram of the positive braid). The minimal form of a positive braid is a uniquely defined product expansion of the braid and we will establish a characterization of this product expansion (Lemma 2.2.3). We will use the Garside normal form to reduce proving Theorem 1.1.2 to proving Theorem 2.3.2, which states that the minimal form of the positive braid σ is equal to the minimal form of Δ^{2n} if $\beta_4(\sigma) = \beta_4(\Delta^{2n})$, where Δ is the Garside element of B_4 and $n \geq 0$. The basic approach for proving Theorem 2.3.2 is to show that there are multiple non-zero entries in a row of $\beta_4(\sigma)$ if σ is not a power of the Garside element, and that there are off-diagonal entries in $\beta_4(\sigma)$ if σ is not an even power of the Garside element. Since $\beta_4(\Delta^{2n})$ is a diagonal matrix (Lemma 2.2.6), this will complete the proof of Theorem 2.3.2, and consequently the proof of Theorem 1.1.2.

In Chapter 3, in order to execute this approach, we will discuss an interpretation of $\beta_4(\sigma)$ for a positive braid σ in terms of *admissible σ-paths*. More specifically, a σ-path is a path in the three vertex straight-line graph that is compatible with the minimal form of σ in a certain sense (Definition 3.1.1), and with length (number of edges) equal to the length of σ (as a positive braid). We will assign a weight to each σ-path (a signed monomial in q of nonnegative degree) and note that the entries of $\beta_4(\sigma)$ are weighted numbers of σ-paths (Proposition 3.1.2). The rest of the proof will be devoted to constructing σ-paths with weights contributing non-cancelling terms in multiple entries in a row of $\beta_4(\sigma)$ if the minimal form of σ does not satisfy certain constraints, corresponding to σ equalling a power of the Garside element. We will introduce a local obstruction to a σ-path admitting this property (belonging to a *distinguished pair* of σ-paths with cancelling weights in the sense of Definition 3.2.1) and refer to admissible σ-paths as those σ-paths which are not locally obstructed in this manner (see also Proposition 3.2.3).

Finally, in Chapter 4, we will establish Theorem 1.1.2 by constructing σ-paths with non-cancelling weights in terms of the minimal form of the positive braid σ. More precisely, we will construct admissible σ-paths in terms of a certain decomposition of the minimal form of σ into *blocks* and *roads*. Roughly, the blocks are "small" subproducts containing one side of a braid relation (either $\sigma_1\sigma_2\sigma_1$ or $\sigma_2\sigma_3\sigma_2$) and the roads are subproducts, maximal with respect to the property of not overlapping with a block

(Definition 4.1.1 of blocks and roads is precise). If $\sigma = \prod_{i=1}^{k} B_i R_i$ is the product decomposition of the minimal form of σ into blocks and roads, then we will show in Section 4.1 that there are multiple non-zero entries in a row of $\beta_4(\sigma)$ unless $k \leq 2$ (Corollary 4.1.19).

Indeed, we will partition P-paths into *good* and *bad* P-paths, if P is a subproduct of the minimal form of σ. We will describe how to extend P-paths along blocks and roads (Corollary 4.1.11 and Corollary 4.1.14), considering the decomposition of a road into *isolated* and *non-isolated* σ_2 subproducts (see Definition 2.2.4). We will use this partition of P-paths to ultimately produce σ-paths with weights contributing to multiple non-zero entries in a row of $\beta_4(\sigma)$ unless $k \leq 2$. (More precisely, we will show that there is a property of P phrased in terms of the weighted numbers of good and bad P-paths, which we refer to as regularity (Definition 4.1.4). We will show that $\prod_{i=1}^{2} B_i R_i$ is regular if $k \geq 3$ and that this implies there are multiple non-zero entries in a row of $\beta_4(\sigma)$, by inductively showing that $\prod_{i=1}^{l} B_i R_i$ is regular for each $3 \leq l \leq k$.) Finally, we will constrain the B_is and R_is uniquely in terms of n if $\beta_4(\sigma) = \beta_4\left(\Delta^{2n}\right)$ for some $n \geq 0$ in the proof of Theorem 2.3.2 in Section 4.2 of Chapter 4. Of course, this concludes the proof of Theorem 1.1.2.

We remark that the techniques in this thesis are not (explicitly) related in any way to those used to establish that the Burau representation $\beta_n : B_n \to \mathrm{GL}_{n-1}\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$ is not faithful for $n \geq 5$ in [1], [9], and [11]. In these works, a topological characterization of the faithfulness of the Burau representation for $n \geq 3$ is used to show that β_n is not faithful for $n \geq 5$. An interesting question would be to independently prove Theorem 1.1.2 using this characterization.

On the other hand, the Garside normal form was also used by Krammer in [7] to establish the faithfulness of the Lawrence-Krammer-Bigelow representations, although the details are quite different to those in this thesis. Indeed, in [7], the Lawrence-Krammer-Bigelow representations are studied with respect to the greedy normal form of a positive braid, whereas in the present work, the Burau representation is studied with respect to the product decomposition of the minimal form of a positive braid into blocks and roads.

Chapter 2

The Burau representation β_4 and the Garside normal form

In this chapter, we will recall background and record elementary definitions and lemmas that we will use in the rest of the thesis. In Section 2.1, we recall a definition of the (reduced) Burau representation of the braid group B_4 that we will use in this thesis. In Section 2.2, we will briefly summarize Garside's fundamental theorem for the braid groups in the case of B_4. Definition 2.2.1 and Theorem 2.2.5 are adopted from Garside's foundational paper [5] in the case of B_4. Lemma 2.2.3 is a characterization of the minimal form of a positive braid ("base of the diagram of the positive braid" in [5]) and Definition 2.2.4 is a certain product decomposition of the minimal form of a positive braid that will be important in the sequel. Finally, in Section 2.3, we will reduce proving Theorem 1.1.2 to proving Theorem 2.3.2. The subsequent chapters of the thesis will be devoted to proving Theorem 2.3.2.

2.1 Definition of β_4

Definition 2.1.1. We recall the **Artin presentation** of B_4:

$$\langle \sigma_1, \sigma_2, \sigma_3 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle .$$

The **Burau representation** $\beta_4 : B_4 \to \mathrm{GL}_3\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$ is defined in terms of the generators in this presentation as follows:

$$\sigma_1 \to \begin{pmatrix} q & -q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_2 \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & q & -q \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_3 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & q \end{pmatrix}$$

Of course, one can check that the matrices assigned to the generators are invertible and satisfy the corresponding relations in the presentation. Note that other definitions of β_4 in the literature are equivalent to the one above by a change of basis.

In this thesis, an **important convention** is that words in the generators are read from left to right and matrix multiplication is performed from right to left (i.e., $\beta_4(\sigma\tau) = \beta_4(\tau)\,\beta_4(\sigma)$, where the right hand side is a matrix product).

2.2 The minimal form of a positive braid and the Garside normal form

We adopt the following terminology from [5].

Definition 2.2.1. A **positive braid** is a braid in the submonoid of B_4 generated by $\sigma_1, \sigma_2, \sigma_3$. We will denote the submonoid of positive braids by B_4^+. A **product expansion** of a positive braid is an expression of the braid as a product of positive powers of the generators. The **length** of a positive braid is the number of generators in a product expansion of the braid. In [5], it is proven that the inclusion $B_4^+ \to B_4$ is an embedding of monoids. In particular, the length of a positive braid is well-defined, since the number of generators match on both sides of each of the relations in the Artin presentation.

The set of product expansions of a positive braid is a finite set, since all have the same length. We can totally order this set using the dictionary order with respect to the index sequence and we define the **minimal form** of the positive braid to be the product expansion minimal with respect to this order. (We remark that in [5], the terminology "base of the diagram of a positive braid" is used to refer to the minimal form of a positive braid.)

For example, the minimal form of $\sigma_2\sigma_1\sigma_2$ is $\sigma_1\sigma_2\sigma_1$ and the minimal form of $\sigma_3\sigma_1$ is $\sigma_1\sigma_3$ (consistent with our convention that words in the generators are read from left to right).

The following characterization of the minimal form of a positive braid in Lemma 2.2.3 will be important in the sequel. We include Example 2.2.2 to visually clarify the statement of Lemma 2.2.3 for ease of readability, although the reader may technically skip it.

Example 2.2.2. Lemma 2.2.3 **(i)** considers the existence of an isolated σ_1 and states that $\sigma_2\sigma_1\sigma_2$ does not occur in the minimal form of σ. Lemma 2.2.3 **(ii)** considers the existence of an isolated σ_2 and states that $\sigma_3\sigma_2\sigma_1^{a_{p+1}}\sigma_3^{b_{p+1}}$ does not occur if $b_{p+1} > 0$ in any case and $\sigma_1^{a_p}\sigma_2\sigma_1^{a_{p+1}}\sigma_3^{b_{p+1}}$ does not occur if $a_{p+1} > 0$ unless it is the beginning of the minimal form of σ. Lemma 2.2.3 **(iii)** considers the existence of an isolated σ_3 and states that neither $\sigma_3\sigma_2^{c_{p-1}}\sigma_3\sigma_2^{c_p}$ nor $\sigma_3\sigma_2\sigma_1^{a_{p-1}}\sigma_2^{c_{p-1}}\sigma_3\sigma_2^{c_p}$ occurs. Finally, Lemma 2.2.3 **(iv)** states that $\sigma_2^{c_{p-1}}\sigma_1^{a_p}\sigma_3\sigma_2$ with $c_p = 1$ does not occur ($a_p = 0$ is possible) unless it is either the ending of the minimal form of σ or there is no σ_3 preceding it in the minimal form.

Lemma 2.2.3. *Let* $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ *be a product expansion of the positive braid* σ *where* a_i+b_i *is a positive integer for* $2 \le i \le n$ *and* c_i *is a positive integer for* $1 \le i \le n-1$. *If the product expansion is in minimal form, then the following conditions are satisfied:*

(i) *If* $a_p = 1$ *and* $p > 1$, *then* $b_p > 0$, *except possibly if* $p = n$ *and* $c_p = 0$.

(ii) *If* $c_p = 1$, *then either* $b_p = 0$ *or* $b_{p+1} = 0$. *If, in addition,* $p > 1$ *and* $b_p = 0$, *then* $a_{p+1} = 0$.

(iii) *If* $a_p = 0$, $b_p = 1$ *and* $p > 1$, *then* $b_{p-1} = 0$, *except possibly if* $p = n$ *and* $c_p = 0$. *If, in addition,* $p > 2$ *and* $c_{p-2} = 1$, *then* $p = 3$ *and* $b_1 = 0$, *except possibly if* $p = n$ *and* $c_p = 0$.

(iv) *If* $b_p = 1 = c_p$, *then* $b_i = 0$ *for* $i < p$, *except possibly if* $p = n$.

Proof. In **(i)**, a single application of the braid relation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$ shows that the product expansion is not in minimal form if $b_p = 0$ and $c_{p-1}, c_p > 0$.

In **(ii)**, a_{p+1} applications of the commutativity relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ and a single application of the braid relation $\sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2$ shows that the product expansion is not in minimal form if $b_p, b_{p+1} > 0$. If $p > 1$ and $b_p = 0$, then a_p applications of the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ shows that the product expansion is not in minimal form since $c_{p-1} > 0$.

In **(iii)**, c_{p-1} applications of the braid relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ shows that the product expansion is not in minimal form if $b_{p-1}, c_p > 0$. If $p > 2$ and $c_{p-2} = 1$, then c_{p-1} applications of the braid relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$, a_{p-1} applications of the commutativity relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$, and a single application of the braid relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ shows that the product expansion is not in minimal form if $b_{p-2}, c_p > 0$. Finally, **(ii)** implies that $p = 3$ and $b_1 = 0$.

In **(iv)**, if $p < n$, then $a_{p+1} > 0$ by **(ii)**. Let us consider a general product expansion $\prod_{i=1}^{n'} \sigma_1^{a'_i}\sigma_3^{b'_i}\sigma_2^{c'_i}$ of σ with $b'_r = 1 = c'_r$ and $a'_{r+1} > 0$. If $a'_r > 0$, then consider a'_r applications of the commutativity relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ and a'_r applications of the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ to obtain a new product expansion $\prod_{i=1}^{n'+1} \sigma_1^{a''_i}\sigma_3^{b''_i}\sigma_2^{c''_i}$ of σ with $b''_r = 1 = c''_r$, $a''_r = 0$, $a''_{r+1} = 1 > 0$, $a''_i = a'_i$, $b''_i = b'_i$, and $c''_i = c'_i$ for $i \leq r-1$. If $r > 1$, $a'_r = 0$ and $b'_{r-1} = 0$, then consider c'_{r-1} applications of the braid relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ to obtain a new product expansion $\prod_{i=1}^{n'-1} \sigma_1^{a''_i}\sigma_3^{b''_i}\sigma_2^{c''_i}$ of σ with $b''_{r-1} = 1 = c''_{r-1}$, $a''_{r-1} = a'_{r-1} > 0$, $a''_r = a'_{r+1} > 0$, $a''_i = a'_i$, $b''_i = b'_i$, and $c''_i = c'_i$ for $i \leq r-2$.

Let us now consider the original product expansion of σ. Let $j < p$ be maximal with respect to the property that $b_j > 0$, if it exists. A finite number of iterations of the process in the previous paragraph (each iteration depending on whether $a'_r > 0$ or $a'_r = 0$) shows that there is a product expansion $\prod_{i=1}^{n'} \sigma_1^{a'_i}\sigma_3^{b'_i}\sigma_2^{c'_i}$ of σ with $b'_{j+1} = 1 = c'_{j+1}$, $a'_{j+1} = 0$, $a'_i = a_i$, $b'_i = b_i$, and $c'_i = c_i$ for $i \leq j$. Finally, a finite number of applications of the braid relation $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$ shows that the original product expansion of σ is not in minimal form. $\square$

In Chapter 3, we will discuss an interpretation of $\beta_4(\sigma)$ in terms of a product expansion of the positive braid σ and later in this chapter, we will note (Lemma 2.3.1) that the faithfulness of $\beta_4 : B_4 \to \mathrm{GL}_3\left(\mathbb{Z}\left[q^{\pm 1}\right]\right)$ is equivalent to the faithfulness of the restriction of β_4 to the submonoid of positive braids B_4^+. More precisely, there is a one-to-one correspondence between $\ker(\beta_4)$ and the submonoid of $\sigma \in B_4^+$ with the property that $\beta_4(\sigma) = q^{2n}I$ for some $n \geq 0$ (I denotes the 3×3 identity matrix).

Since the minimal form of a positive braid is unique, the characterization of minimal product expansions in Lemma 2.2.3 will allow us to characterize this submonoid of B_4^+ (Lemma 2.2.3 furnishes a solution to the word problem in the monoid B_4^+ with respect to the Artin presentation).

Definition 2.2.4. Let $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ be the minimal form of the positive braid σ where $a_i + b_i$ is a positive integer for $2 \leq i \leq n$ and c_i is a positive integer for $1 \leq i \leq n-1$. Let us consider a subproduct $\left(\prod_{i=p}^{p'-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_{p'}}\sigma_3^{b_{p'}}$ of the minimal form of σ, maximal with respect to the property that $c_i = 1$ for all $p \leq i \leq p'-1$.

Lemma 2.2.3 **(ii)** implies that this subproduct is one of two possible forms if $p > 1$:

$$\sigma_1^{a_p}\sigma_3^{b_p}\sigma_2\sigma_1^{a_{p+1}}\sigma_2\sigma_3^{b_{p+2}}\sigma_2\cdots \text{ or}$$

$$\sigma_1^{a_p}\sigma_2\sigma_3^{b_{p+1}}\sigma_2\sigma_1^{a_{p+2}}\sigma_2\cdots$$

(the existence or nonexistence of a σ_1 or a σ_3 between consecutive σ_2s alternates after the first σ_2) with $b_p > 0$ in the first case. Furthermore, Lemma 2.2.3 **(i)** implies that $a_i \geq 2$ for $p < i \leq p'$ in the first case and $a_i \geq 2$ for $p \leq i \leq p'$ in the second case, except possibly if $i = p' = n$ and $c_i = 0$. Similarly, Lemma 2.2.3 **(iii)** implies that $b_i \geq 2$ for $p < i \leq p'$, except possibly if $i = p' = n$ and $c_i = 0$ in either case, or $i = p+1$ in the second case. We refer to these subproducts as **isolated σ_2 subproducts of types I and II**, respectively.

If an isolated σ_2 subproduct occurs at the beginning of the minimal form of σ (i.e., if $c_1 = 1$), then we refer to it as the **initial isolated σ_2 subproduct**. An initial isolated σ_2 subproduct can be of **type III**

$$\sigma_1^{a_1}\sigma_2\sigma_1^{a_2}\sigma_3^{b_2}\sigma_2\sigma_1^{a_3}\sigma_2\sigma_3^{b_4}\sigma_2\cdots$$

(the existence or nonexistence of a σ_1 or a σ_3 between consecutive σ_2s alternates after the *second* σ_2).

Finally, we refer to a subproduct of the minimal form of σ, maximal with respect to the property that it does not overlap with an isolated σ_2 subproduct, as a **non-isolated** σ_2 **subproduct**.

We recall **Garside's theorem** [5] in the special case of B_4.

Theorem 2.2.5 ([5]). *The **Garside element** of* B_4 *is* $\Delta = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$. *A braid in* B_4 *can be expressed uniquely as the product* $\Delta^{2n}\sigma$ *where* $n \in \mathbb{Z}$ *and* σ *is a positive braid with "no* Δ^2 *factor" (i.e.,* $\sigma \neq \Delta^2\tau$ *for any positive braid* τ*). We refer to this expression as the **Garside normal form** of the braid.*

Note that the Garside normal form furnishes a solution to the word problem in the braid groups with respect to the Artin presentation. Indeed, there is an algorithm in [5] to determine the Garside normal form of a braid as the output with a product expansion of the braid in the Artin generators as the input. However, we will not refer to this algorithm in the sequel. The main relevance of Theorem 2.2.5 for this thesis is that, combined with the notion of the minimal form of a positive braid, there is a canonical product expansion of every braid.

We will also require the following statement concerning the images of the even powers of the Garside element under the Burau representation.

Lemma 2.2.6. *If* n *is even, then*

$$\beta_4(\Delta^n) = \begin{pmatrix} q^{2n} & 0 & 0 \\ 0 & q^{2n} & 0 \\ 0 & 0 & q^{2n} \end{pmatrix}$$

Proof. The statement follows by induction and the computation of $\beta_4(\Delta^2)$ (see the definition of β_4 in Definition 2.1.1 and the definition of Δ in Theorem 2.2.5). $\square$

2.3 Reduction of the main result to Theorem 2.3.2

We will now reduce proving Theorem 1.1.2 to proving Theorem 2.3.2 below. The following chapters of the thesis will be devoted to proving Theorem 2.3.2. The reduction is a straightforward corollary of Garside's theorem.

Lemma 2.3.1. *If $\Delta^{-2n}\sigma$ is the Garside normal form of a braid in B_4 (as in Definition 2.2.5), then $\Delta^{-2n}\sigma \in ker(\beta_4)$ if and only if $\beta_4(\sigma) = \beta_4(\Delta^{2n})$ and $n \geq 0$.*

Proof. We can assume $n \geq 0$ since the exponents of the polynomials in the entries of $\beta_4(\Delta^{2n})$ are negative for $n < 0$ by Lemma 2.2.6. □

Theorem 2.3.2. *If σ is a normal positive braid (see Definition 1.1.1) and if $\beta_4(\sigma) = \beta_4(\Delta^{2n})$ (for $n \geq 0$), then σ admits the following form:*

(i) *If $n = 0$, then σ is the identity. If $n = 1$, then $\sigma = \sigma_1^2\sigma_2\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3$.*

(ii) *If $n > 1$, then*

$$\sigma = \left(\sigma_1^{2n}\sigma_2\sigma_1^2\right)\left(\sigma_2^2\sigma_1^2\sigma_2^2\cdots\sigma_2^2\sigma_1^2\sigma_2^2\right)\left(\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\cdots\sigma_2\sigma_3\right)$$

is a product of a type III initial isolated σ_2 subproduct, a non-isolated σ_2 subproduct, and a type II initial isolated σ_2 subproduct (see Definition 2.2.4 for terminology). The number of σ_2^2s in the non-isolated σ_2 subproduct is $n-1$ and the number of σ_2s in the isolated σ_2 subproduct is $2n+1$.

Note that the statement of Theorem 2.3.2 determines the minimal form of Δ^{2n} uniquely for all $n \geq 0$. We prove Theorem 1.1.2 of the Introduction conditional on Theorem 2.3.2.

Proof of Theorem 1.1.2. Theorem 2.3.2 implies that if σ is a normal positive braid and if $\beta_4(\sigma) = \beta_4(\Delta^{2n})$ for $n \geq 0$, then $\sigma = \Delta^{2n}$, since it characterizes the minimal form of σ uniquely in this case. Furthermore, one can verify by induction that the minimal form of Δ^{2n} is given by the statement of Lemma 2.3.1. Theorem 1.1.2 now follows from Lemma 2.3.1. □

The remainder of the thesis will be devoted to proving Theorem 2.3.2. The outline of the proof is as follows (see Section 1.2 of the Introduction for a more detailed outline). In Chapter 3, we will introduce an interpretation of the entries of $\beta_4(\sigma)$ for a positive braid σ as weighted numbers of admissible σ-paths (Proposition 3.2.3). Note that admissible σ-paths are defined concretely in terms of the minimal form of σ. In Chapter 4, we will use Lemma 2.2.6 together with this interpretation to obtain the constraints on the minimal form of σ in the statement of Theorem 2.3.2.

Chapter 3

An interpretation of $\beta_4(\sigma)$ in terms of the minimal form of a positive braid σ

Let $\sigma \in B_4$ be a positive braid. In this chapter, we will interpret the entries in the matrix $\beta_4(\sigma)$ in terms of the minimal form of σ. In doing so, we will introduce terminology (in boldface) that we will freely use in Chapter 4. The key observation is Proposition 3.2.3, which states the entries of $\beta_4(\sigma)$ are weighted numbers of admissible σ-paths. In Section 3.1, we will define σ-paths and in Section 3.2, we will define the notion of an admissible σ-path. In Section 3.3, we will introduce terminology important in the construction of admissible σ-paths in Section 4.

3.1 σ-paths

Definition 3.1.1. Let G be the three vertex straight-line graph with vertices labelled as $\{1, 2, 3\}$ and the middle (valence two) vertex labelled as 2. We will denote a path in G by its sequence of vertices $v_0, \ldots, v_l$. We define an (r, s)**-type** σ**-path** to be a path in G, $v_0, \dots, v_l$, with the following conditions:

(i) l is the length of the positive braid σ.

(ii) $v_0 = r$ and $v_l = s$ (as vertices of G).

(iii) If the kth generator in the minimal form of σ is σ_i, then either $v_k = v_{k-1}$ or $v_k = i$ is adjacent to v_{k-1} in G; we refer to the latter as a **vertex change** at σ_i.

We define the **weight of a σ-path** to be the signed monomial in q determined as follows:

(i) The q-degree (which we also refer to as the **q-resistance of the σ-path**) is the number of k for which either $v_k = i = v_{k-1}$ and the kth generator (in the minimal form of σ) is σ_i, or there is a $i + 1 \to i$ vertex change at the kth generator.

(iii) The sign is $(-1)^e$ where e is the number of k for which there is a $i + 1 \to i$ vertex change at the kth generator.

Note that we can replace the minimal form of σ by any product expansion of σ in Definition 3.1.1 and Proposition 3.1.2 below is still true. However, since we will consider the minimal form of a positive braid as its canonical product expansion in the sequel, we will not require this generality. Furthermore, if context makes σ clear, then we will sometimes omit reference to σ and refer to σ-paths as simply paths. Also, we will often consider τ-paths for τ a subproduct of σ and construct σ-paths by extending τ-paths (see Chapter 4). Figure 3.1 illustrates Definition 3.1.1.

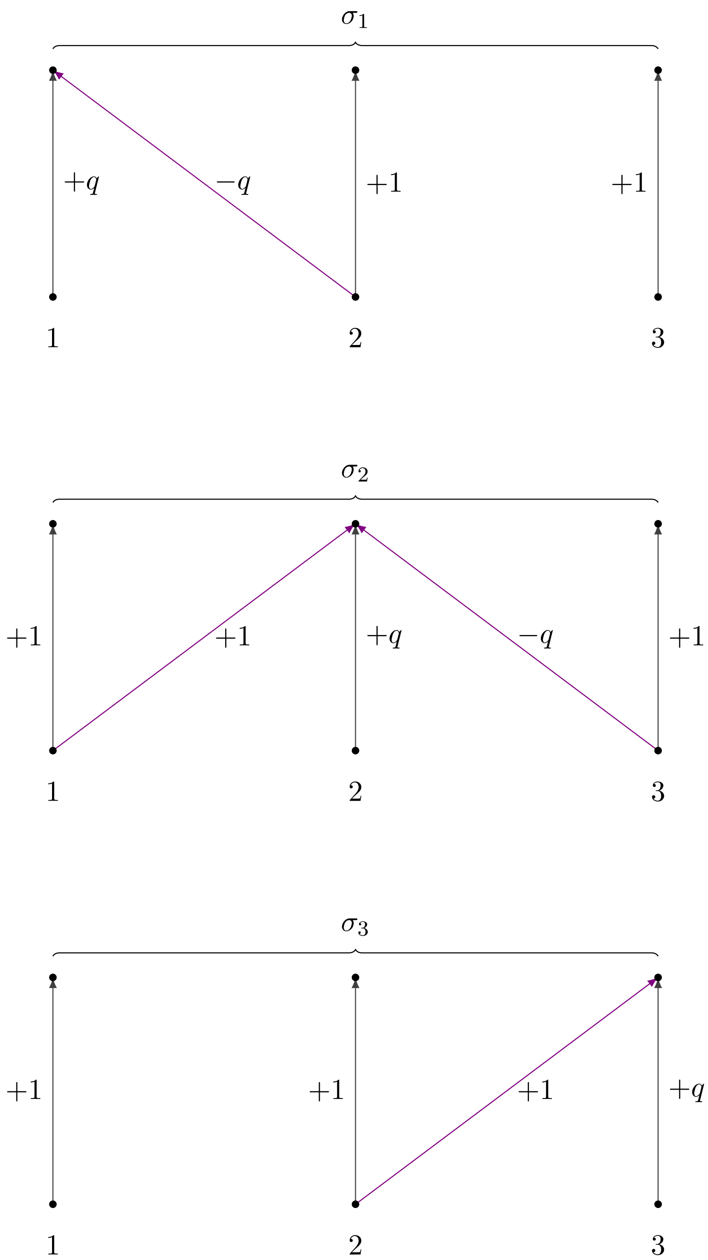

Figure 3.1: We have indicated three pictures corresponding to the three generators σ_1, σ_2, σ_3, labelled at the top of each picture. The vertices of the graph G are labelled at the bottom of each picture. The pictures indicate the possible edges of a σ-path from v_{k-1} to v_k according to which of σ_1, σ_2, σ_3 is the kth generator in the minimal form of σ. (Of course, a specific σ-path can only follow one edge in the picture corresponding to the kth generator in the minimal form of σ.) The vertical dark gray edges denote the cases $v_{k-1} = v_k$ and the diagonal violet edges denote vertex changes ($v_{k-1} \neq v_k$). The labels on the edges denote the multiplicative factor by which the weight of the σ-path changes, if the σ-path follows that edge.

Note that in the figures below, we will draw paths by composing the pictures for the generators in Figure 3.1 from bottom to top (also recall that words in the generators are read from left to right). We justify the introduction of σ-paths in relevance to the Burau representation β_4.

Proposition 3.1.2. *The* (r,s)*-entry of* $\beta_4(\sigma)$ *is the weighted number of the* (r,s)*-type* σ*-paths.*

Proof. The statement follows from the definition of the matrices $\beta_4(\sigma_i)$ (Definition 2.1.1) and Definition 3.1.1 (recalling the convention that matrix multiplication is performed from right to left). Indeed, in Figure 3.1, the (u,v) entry of $\beta_4(\sigma_i)$ is either equal to the label on the arrow from $v \to u$ in the picture corresponding to σ_i if it exists, or equal to zero if the arrow does not exist. $\square$

3.2 Admissible σ-paths

An important point in the statement of Proposition 3.1.2 is that some summands cancel in the weighted sum but what remains is the (r,s)-entry of $\beta_4(\sigma)$. In other words, there is not a one-to-one correspondence between the (non-cancelling) terms in the (r,s)-entry of $\beta_4(\sigma)$ and the (r,s)-type σ-paths.

Indeed, the weights of certain σ-paths **cancel in pairs**.

Definition 3.2.1. Let $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ be the minimal form of σ and let the last σ_2 in $\sigma_2^{c_p}$ be the kth generator in this product expansion (read from left to right). We define the following **distinguished pairs** of σ-paths and observe that the weights of the σ-paths in each distinguished pair cancel. See also the figures for accompanying illustrations of special cases of distinguished pairs (the exponents of the generators are specialized to small values to permit an illustration).

If $a_{p+1} > 0$ and $v_{k-1} = 1 = v_{k+1}$, then the paths $\delta_{k,1}$ and $\delta_{k,2}$ with $v_k = 1$ and $v_k = 2$, respectively, constitute a distinguished pair, unless $c_p = 1$ and $v_{k-b_p-2} = 2$. The other vertices match for $\delta_{k,1}$ and $\delta_{k,2}$, but are arbitrary, subject only to the conditions of a

σ-path. Note that the weights of $\delta_{k,1}$ and $\delta_{k,2}$ have opposite sign. Figure 3.2 illustrates the local picture of this distinguished pair in the case $c_p \geq 2$. Figure 3.3 illustrates the local picture of the non-example where all the conditions of being a distinguished pair in this paragraph are satisfied, except it is the case that $c_p = 1$ and $v_{k-b_p-2} = 2$.

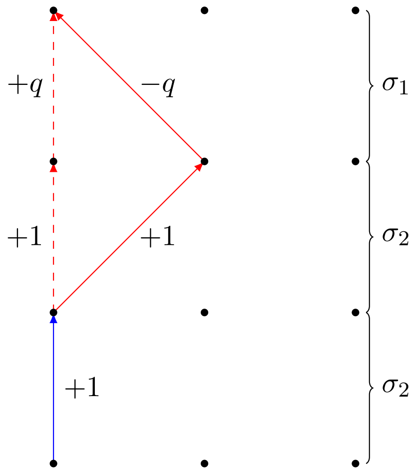

Figure 3.2: The solid and broken red paths (augmented by the blue edge) constitute the local pictures of a distinguished pair $\{\delta_{k,1}, \delta_{k,2}\}$. The weights of the local pictures are $\pm q$. In relevance to the text directly above this figure, this is the case $c_p \geq 2$.

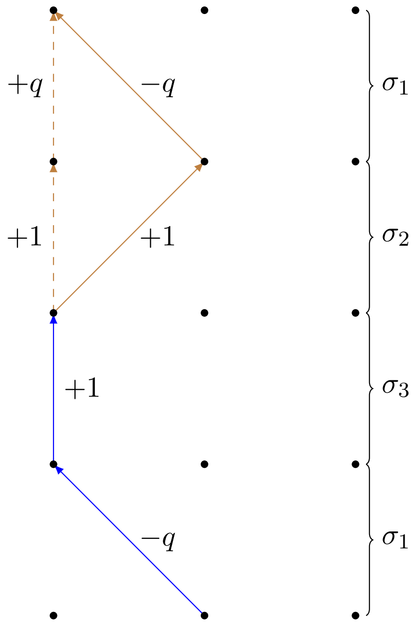

Figure 3.3: The solid and broken brown paths (augmented by the blue edges) *do not* constitute the local pictures of a distinguished pair, although the weights of the local pictures are $\pm q$. In relevance to the text directly above Figure, this is the case $b_p = 1 = c_p$.

If $b_{p+1} > 0$ and $v_{k-1} = 3 = v_{k+a_{p+1}+1}$, then the paths $\epsilon_{k,2}$ and $\epsilon_{k,3}$ with $v_k = 2$ and $v_k = 3$, respectively, constitute a distinguished pair (note that $c_p = 1$ and $b_p > 0$ is not possible by Lemma 2.2.3 **(ii)**). Figure 3.4 illustrates the local picture of this distinguished pair in the case $a_{p+1} = 1$.

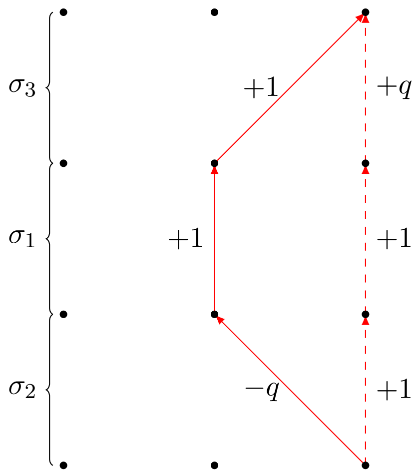

Figure 3.4: The solid and broken red paths constitute the local pictures of a distinguished pair $\{\epsilon_{k,2}, \epsilon_{k,3}\}$. In relevance to the text directly above this figure, this is the case $a_{p+1} = 1$. Note that the weights of the local pictures are $\pm q$ (for all values of $a_{p+1} \geq 0$).

Let us henceforth consider the case $c_{p+1} > 0$ and $v_k = 2$. If $a_{p+1} = 1$, $v_{k+a_{p+1}+b_{p+1}+1} = 2$, and there is a $1 \to 2$ vertex change at the kth generator, then the paths $\zeta_{k+a_{p+1}+b_{p+1},2}$ and $\zeta_{k+a_{p+1}+b_{p+1},3}$ with $v_{k+a_{p+1}+b_{p+1}} = 2$ and $v_{k+a_{p+1}+b_{p+1}} = 3$, respectively, constitute a distinguished pair (note that $b_{p+1} > 0$ by Lemma 2.2.3 **(i)**). Figure 3.5 illustrates the local picture of this distinguished pair in the case $b_{p+1} = 2$.

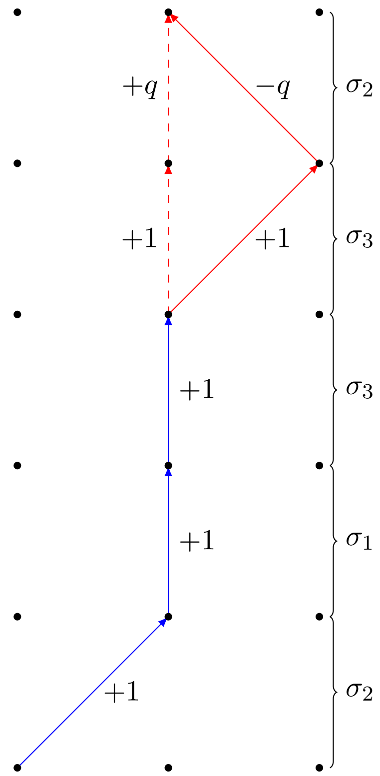

Figure 3.5: The solid and broken red paths (augmented by the blue edges) constitute the local pictures of a distinguished pair $\{\zeta_{k+a_{p+1}+b_{p+1},2}, \zeta_{k+a_{p+1}+b_{p+1},3}\}$. The weights of the local (red) pictures are $\pm q$. In relevance to the text directly above this figure, this is the case $b_{p+1} = 2$.

Let us assume that $v_{k+a_{p+1}+b_{p+1}+1} = 2$. If either $a_{p+1} = 1$ and there is no $1 \to 2$ vertex change at the kth generator, or $a_{p+1} > 1$, then the paths $\eta_{k+a_{p+1},1}$ and $\eta_{k+a_{p+1},2}$ with $v_{k+a_{p+1}} = 1$ and $v_{k+a_{p+1}} = 2$, respectively, constitute a distinguished pair. Figure 3.6 illustrates the local picture of this distinguished pair in the case $a_{p+1} = 1 = b_{p+1}$ (and there is no $1 \to 2$ vertex change at the kth generator).

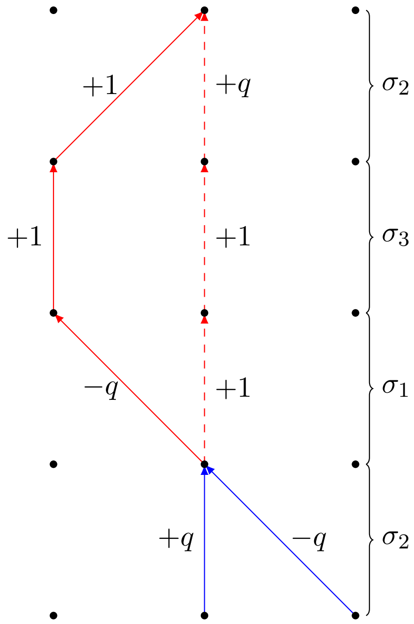

Figure 3.6: The solid and broken red paths (augmented by either of the blue edges) constitute the local pictures of a distinguished pair $\{\eta_{k+a_{p+1},1}, \eta_{k+a_{p+1},2}\}$. The weights of the local (red) pictures are $\pm q$. In relevance to the text directly above this figure, this is the case $a_{p+1} = 1 = b_{p+1}$ (and there is no $1 \to 2$ vertex change at the kth generator).

Let us assume that $v_{k+a_{p+1}+b_{p+1}+1} = 2$ and $a_{p+1} = 0$. If either $b_{p+1} = 1$ and there is no $3 \to 2$ vertex change at the kth generator, or $b_{p+1} > 1$, then the paths $\theta_{k,2}$ and $\theta_{k,3}$ with $v_{k+a_{p+1}+b_{p+1}} = 2$ and $v_{k+a_{p+1}+b_{p+1}} = 3$, respectively, constitute a distinguished pair. Figure 3.7 illustrates the local picture of this distinguished pair in the case $b_{p+1} = 1$ (and there is no $3 \to 2$ vertex change at the kth generator).

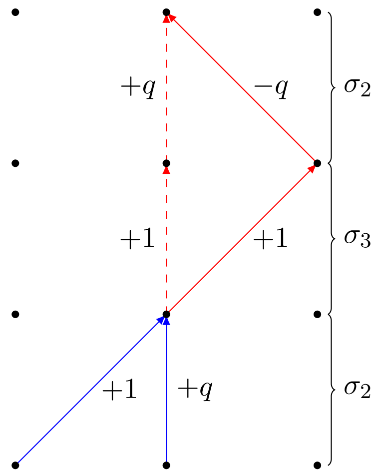

Figure 3.7: The solid and broken red paths (augmented by either of the blue edges) constitute the local pictures of a distinguished pair $\{\theta_{k+a_{p+1},1}, \theta_{k+a_{p+1},2}\}$. The weights of the local (red) pictures are $\pm q$. In relevance to the text directly above this figure, this is the case $b_{p+1} = 1$ (and there is no $3 \to 2$ vertex change at the kth generator).

We will refer to σ-paths that do not belong to a distinguished pair of the form described as **admissible σ-paths**.

The following example elucidates the notion of admissibility.

Example 3.2.2. Figure 3.8 is an illustration of the case where $a_{p+1} = 1 = b_{p+1}$ and there is no vertex change at the kth generator.

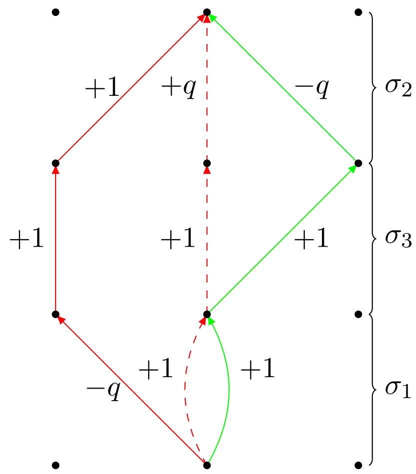

Figure 3.8: The solid and broken red paths constitute the local pictures of a distinguished pair, provided there is no $1 \to 2$ vertex change just beforehand. The local picture of the green path does not belong to a distinguished pair (roughly speaking, it is "locally admissible" in this case). Of course, if there is a $3 \to 2$ vertex change just beforehand, then the extension of the local picture of the green path by one previous step will belong to a distinguished pair (see Figure 3.4).

Let us consider $\sigma = \sigma_1\sigma_2\sigma_1$ and Figure 3.9 below. The $(1,1)$-type σ-paths $1,1,1,1$ and $1,1,2,1$ (not depicted in Figure 3.9) constitute a distinguished pair (see Figure 3.2) and their weights cancel. Furthermore, the $(2,1)$-type σ-paths $2,1,2,1$ and $2,2,2,1$ constitute a distinguished pair and their weights cancel (a variation of Figure 3.6). On the other hand, $2,1,1,1$ is an admissible σ-path although its weight has the opposite sign to the weight of $2,1,2,1$ (Figure 3.2 does not apply since there is a $2 \to 1$ vertex change just beforehand; see also Figure 3.3). In this case, the weight of $2,1,1,1$ contributes the (unique) non-zero term to the $(2,1)$ entry of $\beta_4(\sigma)$. Indeed, it is instructive to verify Proposition 3.2.3 in this case.

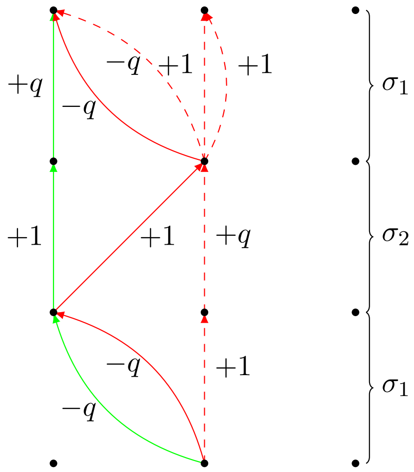

Figure 3.9: In this figure (see Example 3.2.2), there are two distinguished pairs, each consisting of a solid and a broken red path. The weights in the distinguished pair of $(2,1)$-type paths are $\pm q^2$ and the weights in the distinguished pair of $(2,2)$-type paths are $\pm q$. The green $(2,1)$-type path is admissible and its weight is $-q^2$. However, note that the constant $(1,1)$-type path (not depicted in this figure) would *not* be admissible (see Figure 3.2).

Note that a figure identical to Figure 3.9, except translated one unit to the right is applicable for $\sigma_2\sigma_3\sigma_2$. While $\sigma_1\sigma_2\sigma_1$ cannot occur in the minimal form of a positive braid except possibly at the beginning (Lemma 2.2.3 **(ii)**), $\sigma_2\sigma_3\sigma_2$ could occur in multiple places in the minimal form of a positive braid. We will consider these occurrences in Chapter 4 when we introduce the notion of a 3-block (see Definition 4.1.1).

We justify the introduction of admissible σ-paths in relevance to the Burau representation β_4.

Proposition 3.2.3. *The (r,s)-entry of $\beta_4(\sigma)$ is the weighted number of the admissible (r,s)-type σ-paths.*

Proof. Let $\mathcal{P}$ be the set of all σ-paths belonging to a distinguished pair. We will define an involution $\iota : \mathcal{P} \to \mathcal{P}$ with the property that σ-paths which correspond via ι have weights of the opposite sign. The statement follows from the existence of this involution and Proposition 3.1.2.

We define the *local picture* of a distinguished pair as the symmetric difference of the σ-paths in the distinguished pair (in each of Figures 3.2 - 3.7 above, the local picture is the union of the solid and dashed red subpaths). Note that the local picture of a distinguished pair is the union of two subpaths with the same initial and final vertex. We will establish the fact that if a σ-path belongs to two different distinguished pairs, then the corresponding local pictures of the two distinguished pairs do not overlap in an edge (they do not overlap in a vertex due to Lemma 2.2.3). If $X \in \mathcal{P}$, then we define $\iota(X)$ to be the σ-path with the property that $\{X, \iota(X)\}$ is the earliest distinguished pair containing X.

In Definition 3.2.1, there are different types of distinguished pairs, labelled by the Greek letters $\delta, \epsilon, \zeta, \eta, \theta$. Of course, the local pictures of different distinguished pairs in the same type do not overlap in an edge, since the corresponding subproducts of generators do not overlap. We will show that the local pictures of distinguished pairs in different types do not overlap in an edge.

Firstly, observe that the edges in the local picture of a δ-pair are $1 \to 1$, $1 \to 2$ and $2 \to 1$, and the edges in the local picture of an ϵ-pair are $3 \to 3$, $3 \to 2$, $2 \to 2$, and $2 \to 3$. In particular, the local pictures of a δ-pair and an ϵ-pair do not overlap in an edge.

Secondly, observe that the local pictures of the ζ-pairs, ν-pairs, and θ-pairs do not overlap in an edge, since there is an incompatibility in either the corresponding subproducts of generators in the minimal form of σ, or the conditions on the edge of the σ-path immediately preceding the local picture. Indeed, the subproduct of generators corresponding to the local picture of a ζ-pair is $\sigma_2\sigma_1\sigma_3^{b_{p+1}}\sigma_2^{c_{p+1}}$, and in this case there is a $1 \to 2$ vertex change at the first σ_2. The subproduct of generators corresponding to the local picture of an η-pair is $\sigma_2\sigma_1^{a_{p+1}}\sigma_3^{b_{p+1}}\sigma_2^{c_{p+1}}$, and either $a_{p+1} > 1$, or $a_{p+1} = 1$ and there is no $1 \to 2$ vertex change at the first σ_2. The subproduct of generators corresponding to the local picture of a θ-pair is $\sigma_2\sigma_3^{b_{p+1}}\sigma_2$.

Finally, observe that the local picture of either a δ-pair or an ϵ-pair does not overlap in an edge with the local picture of either a ζ-pair, ν-pair, or a θ-pair, due to the conditions on the nature of the vertex change immediately preceding these local pictures in Definition 3.2.1. □

In spite of Proposition 3.1.2 and Proposition 3.2.3, the weight of an admissible σ-path *does not* necessarily contribute a non-zero term to an entry of $\beta_4(\sigma)$. The failure of admissibility may be thought of as a *local* obstruction to the weight contributing a non-zero term to an entry of $\beta_4(\sigma)$.

In light of Proposition 3.2.3, **in Chapter 4, we will only consider admissible σ-paths** and we will generally omit the adjective "admissible" preceding "σ-path", unless we wish to emphasize the admissibility (e.g., if it is a subtle point).

3.3 Good and bad paths

In Chapter 4, we will construct admissible σ-paths inductively by extending admissible τ-paths for τ a subproduct of the minimal form of σ. In view of the definition of admissibility and distinguished pairs (Definition 3.2.1), the construction depends on the nature of the final vertex change of the τ-path. We introduce the notion of *good* and *bad* τ-paths to capture this idea precisely. Firstly, we introduce terminology to characterize the subproducts we will consider.

Definition 3.3.1. Let $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ be the minimal form of σ where $a_i + b_i$ is a positive integer for $2 \leq i \leq n$ and c_i is a positive integer for $1 \leq i \leq n-1$. We define an s-**subproduct** of σ as follows. If $s = 2$, then it is of the form $\prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ for some $1 \leq p \leq n+1$ ($p = n+1$ only if $c_n > 0$), and if $s \in \{1, 3\}$, then it is of the form $\left(\prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_p}\sigma_3^{b_p}$ for some $1 \leq p \leq n$. (A 1-subproduct is the same as a 3-subproduct. In particular, if P is an s-subproduct for $s \in \{1, 3\}$, then an (r, s)-type P-path refers to either an $(r, 1)$-type P-path or an $(r, 3)$-type P-path.)

In the next chapter, we will inductively construct admissible σ-paths with weights contributing non-zero terms to the entries of $\beta_4(\sigma)$, in terms of the minimal form of

σ, by considering the following partition of P-paths into two different types for P an s-subproduct of σ.

Definition 3.3.2. If P is an s-subproduct of σ, then an admissible (r,s)-type P-path is **good** if one of the following conditions is satisfied:

(i) Let $s \in \{1,3\}$ and $P = \left(\prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_p}\sigma_3^{b_p}$. If $s=1$, then the P-path does not have a $2 \to 1$ vertex change at the last σ_1 in $\sigma_1^{a_p}$ (if $a_p = 0$, then this is vacuously true). If $s=3$, then either the P-path does not have a $2 \to 3$ vertex change at the last σ_3 in P if $a_p = 0$, or the P-path does not have a $1 \to 2$ vertex change at the last σ_2 in P and a $2 \to 3$ vertex change at the last σ_3 in $\sigma_3^{b_p}$ if $a_p = 1$ (if either $b_p = 0$ or $a_p > 1$, then this is vacuously true).

(ii) Let $s = 2$ and $P = \prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$. The P-path does not have a $1 \to 2$ vertex change at the last σ_2 in P if $a_p > 0$, and the P-path does not have a $3 \to 2$ vertex change at the last σ_2 in P if $b_p > 0$.

Otherwise, the P-path is **bad**. If we wish to be more specific in the case $s=2$, then we will write that a *bad* P-path is 1-**bad** (resp. 3-**bad**) if it admits a $1 \to 2$ (resp. $3 \to 2$) vertex change at the last σ_2 in P. (If P is the empty product, then we note that the P-path is vacuously good.)

The following elementary observation concerning extensions of admissible P-paths elucidates Definition 3.3.2.

Lemma 3.3.3. *Let P be an s-subproduct of σ and let P' be the s'-subproduct for some $s' \in \{1,2,3\}$, minimal with respect to the property that $P \subsetneq P'$ (in particular, $s' \neq s$). The following statements concerning extensions of P-paths to P'-paths are true:*

(i) *An admissible (r,s)-type P-path is good if and only if its extension to an (r,s')-type P'-path by any vertex change is admissible.*

(ii) *If* $s \in \{1,3\}$*, then an extension of an admissible* (r,s)*-type* P*-path by a vertex change at the second or later* σ_2 *in* P' *after* P *is admissible. If* $s = 2$*, then an extension of an* $(r,2)$*-type* P*-path by a vertex change at either the second or later* σ_1 *or* σ_3 *in* P' *after* P *is admissible.*

(iii) *An extension of a* 1*-bad* $(r,2)$*-type* P*-path by any vertex change at a* σ_3 *in* P' *after* P*, or an extension of a* 3*-bad* $(r,2)$*-type* P*-path by any vertex change at a* σ_1 *in* P' *after* P*, is admissible.*

(iv) *Let* $s \in \{1,3\}$*, and let* P'' *be a* 2*-subproduct such that* $P \subseteq P''$*. An extension of an admissible* (r,s)*-type* P*-path to an* (r,s)*-type* P''*-path with no further vertex change is admissible if and only if there is no* σ_s *in* P'' *after* P*.*

Proof. Note that if $s \in \{1,3\}$, then $s' = 2$, and if $s = 2$, then $s' \in \{1,3\}$. We consider cases according to the value of $s \in \{1,2,3\}$. If $s \in \{1,3\}$, then we write $P = \left(\prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_p}\sigma_3^{b_p}$. If $s = 2$, then we write $P = \prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$. Let X denote an admissible (r,s)-type P-path.

If $s = 1$, then an extension of X by a vertex change at the first σ_2 in $\sigma_2^{c_p}$ is admissible if and only if X does not belong to the relevant η-pair. Of course, this is the case if and only if X does not have a vertex change at the last σ_1 in $\sigma_1^{a_p}$. The extension of X by a vertex change at the second or later σ_2 in $\sigma_2^{c_p}$ is always admissible since it does not belong to the relevant η-pair.

If $s = 3$, then an extension of X by a vertex change at the first σ_2 in $\sigma_2^{c_p}$ is admissible if and only if X does not belong to the relevant ζ-pair or θ-pair. Of course, X does not belong to a ζ-pair if and only if either $a_p > 1$, or $a_p = 1$ and X does not have a $1 \to 2$ vertex change at the last σ_2 in P, and a $2 \to 3$ vertex change at the last σ_3 in $\sigma_3^{b_p}$. Furthermore, X does not belong to a θ-pair if and only if $a_p = 0$ and X does not have a $2 \to 3$ vertex change at the last σ_3 in $\sigma_3^{b_p}$.

If $s = 2$, then an extension of X by a vertex change at the first σ_1 in P' is admissible if and only if X does not belong to the relevant δ-pair. Of course, this is the case if and only if X does not have a $1 \to 2$ vertex change at the last σ_2 in P. Similarly, an extension of X by a vertex change at the first σ_3 in P' is admissible if and only if X does not belong to the relevant ϵ-pair. Of course, this is the case if and only if X does not have a $3 \to 2$ vertex change at the last σ_2 in P. Furthermore, an extension of X by a vertex change at either the second or later σ_1 in P', or the second or later σ_3 in P' is admissible, since neither belongs to the relevant δ-pair or ϵ-pair, respectively.

Finally, the last statement follows since the extension of the (r,s)-type P-path X with $s \in \{1,3\}$ to an (r,s)-type P''-path with no further vertex change is admissible if and only if it does not belong to either a δ-pair or an ϵ-pair. $\square$

An important point is that the weight of a P-path might not contribute a non-cancelling term to $\beta_4(P)$ (e.g., the signed number of P-paths with the same q-resistance as this P-path might be zero), but an extension of the P-path to a σ-path might contribute a non-cancelling term to $\beta_4(\sigma)$. The partition of P-paths into good and bad P-paths is relevant to this point. Indeed, Lemma 3.3.3 implies that there is a one-to-one correspondence between the sets of extensions to a σ-path of a pair of either simultaneously good or simultaneously bad P-paths, such that corresponding σ-paths have cancelling weights (of opposite sign).

The following Notation 3.3.4 for the weighted numbers of good and bad P-paths will be important in Definition 4.1.4 in Section 4.1 of Chapter 4.

Notation 3.3.4. We fix $r \in \{1,2,3\}$. Let $P = \prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ be a 2-subproduct of σ.

- We write $w_\lambda^P = \sum_{i=0}^{\infty} \lambda_i^P q^{d_\lambda^P - i}$ for the weighted number of good $(r,2)$-type P-paths.
- We write $w_{\mu,1}^P = \sum_{i=0}^{\infty} \mu_{1,i}^P q^{d_{\mu,1}^P - i}$ for the weighted number of $(r,2)$-type P-paths with a $1 \to 2$ vertex change at the last σ_2 in P, $w_{\mu,3}^P = \sum_{i=0}^{\infty} \mu_{3,i}^P q^{d_{\mu,3}^P - i}$ for the weighted number of $(r,2)$-type P-paths with a $3 \to 2$ vertex change at the last

σ_2 in P, and $w^P_{\lambda,2}$ for the weighted number of $(r,2)$-type P-paths with no vertex change at the last σ_2 in P.

- If either $a_p = 0$ or $b_p = 0$, then we write $w^P_\nu = \sum_{i=0}^{\infty} \nu_i q^{d^P_\nu - i}$ for either the weighted number of $(r,1)$-type P-paths or the weighted number of $(r,3)$-type P-paths, respectively. We write $w^P_\mu = \sum_{i=0}^{\infty} \mu_i q^{d^P_\mu - i}$ for the weighted number of bad $(r,2)$-type P-paths.

Let $P = \left(\prod_{i=1}^{p-1} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}\right) \sigma_1^{a_p} \sigma_3^{b_p}$ be a 1-subproduct/3-subproduct of σ.

- If $a_p, b_p > 0$, then we write $w^P_{\lambda,1} = \sum_{i=0}^{\infty} \lambda^P_{1,i} q^{d^P_{\lambda,1} - i}$ for the weighted number of good $(r,1)$-type P-paths and $w^P_{\lambda,3} = \sum_{i=0}^{\infty} \lambda^P_{3,i} q^{d^P_{\lambda,3} - i}$ for the weighted number of good $(r,3)$-type P-paths.

- If $a_p, b_p > 0$, then we write $w^P_{\mu,1} = \sum_{i=0}^{\infty} \mu^P_{1,i} q^{d^P_{\mu,1} - i}$ for the weighted number of bad $(r,1)$-type P-paths and $w^P_{\mu,3} = \sum_{i=0}^{\infty} \mu^P_{3,i} q^{d^P_{\mu,3} - i}$ for the weighted number of bad $(r,3)$-type P-paths. (Note that if $w^P_{\mu,3} \neq 0$, then $a_p = 1$, in which case $w^P_{\lambda,1} = 0$.)

- If either $a_p = 0$ or $b_p = 0$, then we write $\sum_{i=0}^{\infty} \lambda^P_i q^{d^P_\lambda - i}$ for either the weighted number of good $(r,3)$-type P-paths or the weighted number of good $(r,1)$-type P-paths, respectively.

- If either $a_p = 0$ or $b_p = 0$, then we write $\sum_{i=0}^{\infty} \mu^P_i q^{d^P_\mu - i}$ for either the weighted number of bad $(r,3)$-type P-paths or the weighted number of bad $(r,1)$-type P-paths, respectively.

- If either $a_p = 0$ or $b_p = 0$, then we write $\sum_{i=0}^{\infty} \kappa^P_i q^{d^P_\kappa - i}$ for either the weighted number of $(r,3)$-type P-paths or the weighted number of $(r,1)$-type P-paths. (Proposition 3.2.3 implies that this is either the $(r,3)$ or the $(r,1)$-entry of $\beta_4(P)$, respectively.)

In Notation 3.3.4, Proposition 3.2.3 and the fact that admissible P-paths are partitioned into good and bad P-paths according to Definition 3.3.2, implies that the entries of $\beta_4(P)$ can be defined as sums of the relevant λ-weights, μ-weights, and ν-weights. Furthermore, if $P = \prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ is a 2-subproduct, then $w_\lambda^P = w_{\lambda,2}^P + w_{\mu,1}^P$ and $w_\mu^P = w_{\mu,3}^P$ if $a_p = 0$, and $w_\lambda^P = w_{\lambda,2}^P + w_{\mu,3}^P$ and $w_\mu^P = w_{\mu,1}^P$ if $b_p = 0$.

We can use Lemma 3.3.3 to track the changes in the λ-weights, μ-weights, and ν-weights as one passes from P-paths to P'-paths (in the notation of Lemma 3.3.3). In Chapter 4, this will be an important point for understanding the proofs.

Chapter 4

Proof of Theorem 2.3.2

In this chapter, we will prove Theorem 2.3.2 of Section 2.3 in Chapter 2, thus establishing Theorem 1.1.2. The strategy is to construct admissible σ-paths with weights contributing to multiple non-zero entries in a row of $\beta_4(\sigma)$ for a positive braid $\sigma \in B_4$, unless there are certain constraints on the minimal form of σ (corresponding to those in the statement of Theorem 2.3.2).

In Section 4.1, we will introduce the product decomposition of the minimal form of σ into *blocks* and *roads* (Definition 4.1.1). If P is an s-subproduct of σ, then we will construct admissible σ-paths by extending good and bad P-paths with respect to this product decomposition.

We will introduce a classification of the blocks in the minimal form of σ (Definition 4.1.2). A block in the minimal form of σ is either *initial*, *singular*, or *generic*. The constraints on the minimal form of σ in Lemma 2.2.3 imply that there is at most one initial block and one singular block, for any positive braid σ. We will define a property of blocks, *regularity* (Definition 4.1.4), such that the existence of a generic, regular block implies the existence of multiple non-zero entries in a row of $\beta_4(\sigma)$ (Corollary 4.1.15). Furthermore, we will show that the first generic block is regular (Corollary 4.1.17 and Lemma 4.1.18).

We conclude with Corollary 4.1.19, which implies that there are no generic blocks in the minimal form of σ, if $\beta_4(\sigma) = \beta_4\left(\Delta^{2n}\right)$ for some $n \geq 1$. Furthermore, Corollary 4.1.19 constrains a singular block in this case, if it exists. (A motivation for Corollary 4.1.19 as a key step in the proof is that there are precisely two blocks in the minimal form of Δ^{2n} (indicated in Theorem 2.3.2) for any $n \geq 1$; the first block is initial and the second block is singular.)

In Section 4.2, we will show that, in fact, there exists one initial block and one singular block in the minimal form of σ, if $\beta_4(\sigma) = \beta_4\left(\Delta^{2n}\right)$ for some $n \geq 1$. The precise formulation is Lemma 4.2.1, which will be a strengthening of Corollary 4.1.19 of Section 4.1. Finally, we will conclude the proof of Theorem 2.3.2 by constraining the roads in the minimal form of σ, and showing that the constraints on the decomposition of the minimal form of σ into blocks and roads correspond to those in the statement of Theorem 2.3.2.

In this chapter, $\sigma \in B_4$ will **always** denote a normal positive braid. Also, we will freely use the terminology introduced in Chapter 3. In particular, recall that all paths under consideration are assumed to be admissible, unless otherwise stated. Let $\sigma = \prod_{i=1}^{n} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ be the minimal form of the positive braid σ where $a_i + b_i$ is a positive integer for $2 \leq i \leq n$ and c_i is a positive integer for $1 \leq i \leq n-1$.

4.1 The product decomposition of a positive braid into blocks and roads

Definition 4.1.1. A **block** in the minimal form of σ is an expression either of the form $\sigma_1^{a_p}\sigma_3^{b_p}\sigma_2\sigma_1^{a_{p+1}}$ (a 2-**block**) or of the form $\sigma_2^{c_{p-1}}\sigma_3\sigma_2^{c_p}$ (a 3-**block**). A **subroad** is a subproduct of the minimal form of σ that does not overlap with a block. A **road** is a subproduct of the minimal form of σ, maximal with respect to the property of being a subroad.

The roads in the minimal form of σ further admit product expansions into isolated and non-isolated σ_2 subproducts (see Definition 2.2.4) and this will be important in the sequel. In these product expansions, the isolated σ_2 subproducts are either of type I with $a_p = 0$ in Definition 2.2.4 or of type II (since a road does not overlap with a 2-block). Furthermore, the non-isolated σ_2 subproducts do not contain isolated σ_3s, except possibly at the ending of the minimal form of σ (since a road does not overlap with a 3-block).

We introduce the following classification of blocks in the minimal form of σ.

Definition 4.1.2. The **initial block** in the minimal form of σ is $\sigma_1^{a_1}\sigma_2\sigma_1^{a_2}$ $(b_2 = 0)$, if it exists. A **singular block** in the minimal form of σ is either a 2-block of the form $\sigma_1^{a_p}\sigma_3\sigma_2\sigma_1^{a_{p+1}}$ or a 3-block of the form $\sigma_2^{c_{p-1}}\sigma_3\sigma_2$. A block that is neither initial nor singular is a **generic block**.

Lemma 2.2.3 **(iv)** implies that a σ_3 cannot precede a singular block in the minimal form of σ. In particular, if a singular block in the minimal form of σ exists, then it is unique and is the first non-initial block.

The key step in the proof of Theorem 2.3.2 is the constraint on the product decomposition of the minimal form of σ into blocks and roads in Lemma 4.2.1 of Section 4.2. In particular, Lemma 4.2.1 implies that there is one initial block, one singular block, and no generic blocks in the minimal form of σ, if $\beta_4(\sigma) = \beta_4\left(\Delta^{2n}\right)$ for some $n \geq 1$. Note that this is true for Δ^{2n} with $n \geq 1$ (see the statement of Theorem 2.3.2 for the minimal form of Δ^{2n}).

In this section, we will fix the following notation in the definitions, lemmas and proofs, in order to avoid repetition. In general, P will denote an s-subproduct of σ. If $s = 2$, then we write $P = \prod_{i=1}^{p-1}\sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$ and if $s \in \{1,3\}$, then we write $P = \left(\prod_{i=1}^{p-1}\sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_p}\sigma_3^{b_p}$.

We will freely use the notation introduced in Notation 3.3.4 (for weighted numbers of P-paths) at the end of Chapter 3. Note that a statement concerning the polynomials in

Notation 3.3.4 can be interpreted as a set of three statements, one for each $r \in \{1, 2, 3\}$ for which we consider the relevant weighted numbers of paths with initial vertex equal to r. We will often make such statements for a subset of $r \in \{1, 2, 3\}$ for which a certain property is true. In these cases, we will clearly indicate the relevant subset. For example, this will be the case in the following Definition 4.1.4 and the subsequent proofs.

Let us briefly motivate the following Definition 4.1.4 before stating it precisely.

Motivation 4.1.3. Let P' denote the minimal s'-subproduct containing P for $s' \neq s$. The entries in a row of $\beta_4(P')$ can be determined as sums of relevant weighted numbers of P'-paths (Notation 3.3.4). In turn, these weighted numbers of P'-paths can be determined from the weighted numbers of P-paths (Lemma 3.3.3). The first observation is that unless the degrees of these polynomials are sufficiently close or some of these polynomials are equal to zero, then there are multiple non-zero entries in a row of $\beta_4(P')$.

In fact, there are situations where the degrees are sufficiently close and without further conditions, it cannot a priori be ruled out that there is a single non-zero entry in a row of $\beta_4(P')$. However, if there are certain constraints on the leading coefficients of the polynomials when the degrees of these polynomials are sufficiently close, one can still conclude that there are multiple non-zero entries in a row of $\beta_4(P')$. The constraints are stated in Definition 4.1.4 below. The constraints constitute an inductive property that allows one to ultimately show that there are multiple non-zero entries in a row of $\beta_4(\sigma)$. In other words, we will show that there are certain conditions on the entries of $\beta_4(P)$ for an s-subproduct P of σ, that inductively imply that there are multiple non-zero entries in a row of $\beta_4(\sigma)$. In particular, these conditions are satisfied if P contains a generic block in the minimal form of σ. The induction will be with respect to the product decomposition of the minimal form of σ into blocks and roads.

The proof works only for a restricted class of positive braids, due to the nature of the inductive property. However, we expect that the argument can be generalized to a proof that the Burau representation of B_4 is faithful, and one will need to find a more general inductive property of the polynomials (rather than simply their degrees and leading coefficients, which are weak invariants of the polynomials). The main advantage of working with the leading coefficients is, as we shall see in the forthcoming proofs, it is possible in some sense to track how the leading coefficients change as we pass from P to P'.

Definition 4.1.4. If P is an s-subproduct of σ, then we write that P is **regular** if one of the following conditions **(a)** (if $s = 2$) or **(b)** (if $s \in \{1, 3\}$) on the weighted numbers of P-paths is satisfied.

(a) Let P be a 2-subproduct. In **(i)**, we will state the conditions in the case that $c_{p-1} \geq 2$, and in **(ii)**, we will state the conditions in the case that $c_{p-1} = 1$. (The reason for the partition is that Notation 3.3.4 is different in each case.)

(i) Let us assume that $c_{p-1} \geq 2$ and consider those $r \in \{1, 2, 3\}$ for which $d^P_{\lambda,2} \leq \max\{d^P_{\mu,1}, d^P_{\mu,3}\}$. The following conditions are satisfied.

Firstly, either $w^P_{\mu,1} \neq 0$ for every r or $w^P_{\mu,3} \neq 0$ for every r. Secondly, if $d^P_{\mu,1} = d^P_{\mu,3}$, then $\mu^P_{1,0}$ and $\mu^P_{3,0}$ are non-zero integers with the opposite sign. Thirdly, if either $d^P_{\mu,1} = d^P_{\mu,3} + 1$ and $a_p = 0$, or $d^P_{\mu,3} = d^P_{\mu,1} + 1$ and $b_p = 0$, then $\mu^P_{1,0}$ and $\mu^P_{3,0}$ are non-zero integers with the same sign.

(ii) Let us assume that $c_{p-1} = 1$ and consider those $r \in \{1, 2, 3\}$ for which $d^P_{\lambda} \leq d^P_{\mu}$. The following conditions are satisfied.

Firstly, either $w^P_{\mu} \neq 0$ for every r or $w^P_{\nu} \neq 0$ for every r. We will state the remaining conditions separately in the cases $a_p = 0$ and $b_p = 0$. (Lemma 2.2.3 **(ii)** implies that precisely one of these cases occurs.)

If $a_p = 0$, then let us further consider those r for which $d^P_{\lambda} + 1 \leq d^P_{\nu}$. If $d^P_{\nu} = d^P_{\mu} + 1$, then ν^P_0 and μ^P_0 are non-zero integers with the same sign. If $d^P_{\nu} = d^P_{\mu}$, then ν^P_0 and μ^P_0 are non-zero integers with the opposite sign.

If $b_p = 0$, then let us further consider those r for which $d_\lambda^P \le d_\nu^P$. If $d_\nu^P = d_\mu^P$, then ν_0^P and μ_0^P are non-zero integers with the opposite sign. If $d_\nu^P + 1 = d_\mu^P$, then ν_0^P and μ_0^P are non-zero integers with the same sign.

(b) Let P be an s-subproduct for $s \in \{1, 3\}$. In **(i)**, we will state the conditions in the case that either $a_p = 0$ or $b_p = 0$, and in **(ii)**, we will state the conditions in the case that $a_p, b_p > 0$.

(i) Let us assume that either $a_p = 0$ or $b_p = 0$ and consider those $r \in \{1, 2, 3\}$ for which $d_\lambda^P \le d_\mu^P$. The following conditions are satisfied.

Firstly, either $w_\mu^P \ne 0$ for every r or $w_\nu^P \ne 0$ for every r. We will state the remaining conditions separately in the cases $a_p = 0$ and $b_p = 0$.

Firstly, we consider the case $a_p = 0$. If $d_\nu^P = d_\mu^P + 1$, then ν_0^P and μ_0^P are non-zero integers with the same sign. If $d_\nu^P = d_\mu^P$, then ν_0^P and μ_0^P are non-zero integers with the opposite sign.

Secondly, we consider the case $b_p = 0$. If $d_\nu^P + 1 = d_\mu^P$, then ν_0^P and μ_0^P are non-zero integers with the same sign. If $d_\nu^P + 2 = d_\mu^P$, then ν_0^P and μ_0^P are non-zero integers with the opposite sign.

(ii) Let us assume that $a_p, b_p > 0$ throughout. (Note that if $a_p > 1$, then $w_{\mu,3}^P = 0$, and if $a_p = 1$, then $w_{\lambda,1}^P = 0$; see Notation 3.3.4.) The following conditions are satisfied. We will state the conditions separately in the cases $a_p > 1$ and $a_p = 1$.

Firstly, we consider the case $a_p > 1$ and consider those $r \in \{1, 2, 3\}$ for which $d_{\lambda,1}^P \le d_{\mu,1}^P$. In this case, $w_{\mu,1}^P, w_{\lambda,3}^P \ne 0$ for every r. Furthermore, $d_{\mu,1}^P \le d_{\lambda,3}^P + 1$. If $d_{\mu,1}^P = d_{\lambda,3}^P + 1$, then either $\mu_{1,0}^P + \lambda_{3,0}^P$ and $\lambda_{3,0}^P$ are non-zero integers with the same sign, or $b_p = 1$ and $\mu_{1,0}^P + \lambda_{3,0}^P = 0$.

Secondly, we consider the case $a_p = 1$ and consider those $r \in \{1, 2, 3\}$ for which $d_{\lambda,3}^P < \max\{d_{\mu,1}^P - 1, d_{\mu,3}^P\}$. In this case, either $w_{\mu,1}^P \ne 0$, $d_{\lambda,3}^P < d_{\mu,3}^P$, or $\lambda_{3,0}^P + \mu_{3,0}^P \neq 0$. Furthermore, if $d_{\mu,1}^P = d_{\mu,3}^P + 1$, then $\mu_{1,0}^P$ and $\mu_{3,0}^P$ are non-zero integers with the same sign. Finally, if $b_p > 1$, then $d_{\mu,1}^P \neq d_{\mu,3}^P$.

If B is a block in the minimal form of σ and if P is the s-subproduct immediately preceding B, then we will write that B is regular if PB (which is an s'-subproduct for $s' \neq s$) is regular.

We will establish that there is no regular generic block in the minimal form of σ (Corollary 4.1.15), if $\beta_4(\sigma)$ is a diagonal matrix. Subsequently, we will show that the first generic block in the minimal form of σ (if it exists) is regular (Corollary 4.1.17 and Lemma 4.1.18). Of course, this will allow us to obtain strong constraints on the minimal form of σ if $\beta_4(\sigma)$ is a diagonal matrix, since there can be at most two non-generic blocks in the minimal form of σ (at most one initial block and at most one singular block).

We will establish Corollary 4.1.15 with respect to the decomposition of the minimal form of σ into blocks and roads. Firstly, we will establish Corollary 4.1.11, which states that if P is regular and $P' \setminus P$ is a road, then P' is regular and there are multiple non-zero entries in a row of $\beta_4(P')$. Secondly, we will establish Corollary 4.1.14, which states that if P is regular and $P' \setminus P$ is a generic block, then P' is regular and there are multiple non-zero entries in a row of $\beta_4(P')$.

The first step is to establish Corollary 4.1.11. We will split the proof into several individual lemmas and corollaries, for ease of readability. Indeed, Corollary 4.1.11 is equivalent to the combination of Corollary 4.1.7 and Corollary 4.1.10. In turn, Corollary 4.1.7 is equivalent to the combination of Lemma 4.1.5 and Lemma 4.1.6. Similarly, Corollary 4.1.10 is equivalent to the combination of Lemma 4.1.8 and Lemma 4.1.9.

Lemma 4.1.5. *Let P be a 2-subproduct of σ and let P' be the minimal s'-subproduct containing P for $s' \in \{1,3\}$. If $P' \setminus P$ is a subroad, and if P is regular, then $\beta_4(P')$ is not a diagonal matrix.*

Proof. We recall that we write $P = \prod_{i=1}^{p-1} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}$. Firstly, we will establish the statement if either $d^P_{\lambda,2} > \max\{d^P_{\mu,1}, d^P_{\mu,3}\}$ or $d^P_\lambda > d^P_\mu$, for some $r \in \{1,2,3\}$. In these cases, the $(r,2)$-entry of $\beta_4(P')$ is non-zero. Indeed, the $(r,2)$-entry of $\beta_4(P')$ is equal to $w^P_{\lambda,2} + w^P_{\mu,1} + w^P_{\mu,3}$ if $a_p, b_p > 0$, and is equal to $w^P_\lambda + w^P_\mu$ if either $a_p = 0$ or $b_p = 0$. Furthermore, Lemma 3.3.3 implies that the $(r,1)$-entry of $\beta_4(P')$ is non-zero if $a_p > 0$ and the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $b_p > 0$. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ if either $d^P_{\lambda,2} > \max\{d^P_{\mu,1}, d^P_{\mu,3}\}$ or $d^P_\lambda > d^P_\mu$ for some $r \in \{1,2,3\}$, and $\beta_4(P')$ is not a diagonal matrix in this case.

Henceforth, we will assume that either $d^P_{\lambda,2} \leq \max\{d^P_{\mu,1}, d^P_{\mu,3}\}$ or $d^P_\lambda \leq d^P_\mu$, for every $r \in \{1,2,3\}$. Under this assumption, either $w^P_{\mu,1} \neq 0$ for every $r \in \{1,2,3\}$, or $w^P_{\mu,3} \neq 0$ for every $r \in \{1,2,3\}$, if $a_p, b_p > 0$, since P is regular. Also under this assumption, either $w^P_\mu \neq 0$ for every $r \in \{1,2,3\}$ or $w^P_\nu \neq 0$ for every $r \in \{1,2,3\}$, if either $a_p = 0$ or $b_p = 0$, since P is regular.

We will split the remainder of the proof into two cases. In **Case 1**, we will assume that either $a_p = 0$ or $b_p = 0$, and in **Case 2**, we will assume that $a_p, b_p > 0$.

Case 1 (either $a_p = 0$ or $b_p = 0$)

Note that either $a_p > 1$ or $b_p > 1$ by Lemma 2.2.3 **(i)** and since $P' \setminus P$ is a subroad (and, in particular, $P' \setminus P$ does not overlap with a 3-block). If $c_{p-1} = 1$, then $w^{P'}_\nu = w^P_\nu$. If $c_{p-1} \geq 2$, then $w^{P'}_\nu = w^P_{\mu,1}$ if $a_p = 0$, and $w^{P'}_\nu = -q^{-1} w^P_{\mu,3}$ if $b_p = 0$.

If $a_p = 0$, then the $(r,1)$-entry of $\beta_4(P')$ is $w^{P'}_\nu$, and if $b_p = 0$, then the $(r,3)$-entry of $\beta_4(P')$ is $w^{P'}_\nu$. In particular, if $w^P_\nu \neq 0$ for every $r \in \{1,2,3\}$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero for every $r \in \{1,2,3\}$ if $a_p = 0$, and the $(r,3)$-entry of $\beta_4(P')$ is non-zero for every $r \in \{1,2,3\}$ if $b_p = 0$. Therefore, $\beta_4(P')$ is not a diagonal matrix, if $w^P_\nu \neq 0$ for every $r \in \{1,2,3\}$. If $d^P_\lambda > d^P_\mu$ for some $r \in \{1,2,3\}$, then we have already observed in the beginning of the proof that $\beta_4(P')$ is not a diagonal matrix.

Since P is regular, we may assume that $w^P_\mu \neq 0$ and $d^P_\lambda \leq d^P_\mu$ for every $r \in \{1,2,3\}$. We will split the remainder of the argument according to the relative values of d^P_λ and d^P_μ. If

$d^P_\lambda = d^P_\mu$, then Lemma 3.3.3 implies that the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $a_p = 0$, and the $(r,1)$-entry of $\beta_4(P')$ is non-zero if $b_p = 0$. If $d^P_\lambda < d^P_\mu$, then the $(r,2)$-entry of $\beta_4(P')$ is non-zero. Therefore, $\beta_4(P')$ is not a diagonal matrix.

Case 2 $(a_p, b_p > 0)$

Note that $c_{p-1} \geq 2$ since $P' \setminus P$ is a subroad (and, in particular, $P' \setminus P$ does not overlap with either an initial block or a 2-block). We will establish the statement by considering subcases according to the relative values of $d^P_{\lambda,2}$, $d^P_{\mu,1}$, and $d^P_{\mu,3}$. (Of course, we are assuming that $d^P_{\lambda,2} \leq \max\{d^P_{\mu,1}, d^P_{\mu,3}\}$.) We recall that the $(r,2)$-entry of $\beta_4(P')$ is $w^P_{\lambda,2} + w^P_{\mu,1} + w^P_{\mu,3}$.

Subcase 1 (either $d^P_{\mu,1} > \max\{d^P_{\mu,3}, d^P_{\lambda,2}\}$ or $d^P_{\mu,3} > \max\{d^P_{\mu,1}, d^P_{\lambda,2}\}$)

The $(r,2)$-entry of $\beta_4(P')$ is non-zero. Furthermore, Lemma 3.3.3 **(iii)** implies that either the $(r,3)$-entry of $\beta_4(P')$ is non-zero if the first assumption is satisfied, or the $(r,1)$-entry of $\beta_4(P')$ is non-zero if the second assumption is satisfied. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ in this subcase.

Subcase 2 $(d^P_{\mu,1} = d^P_{\mu,3} > d^P_{\lambda,2})$

Lemma 3.3.3 **(iii)** implies that the $(r,1)$-entry and the $(r,3)$-entry of $\beta_4(P')$ are non-zero. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ in this subcase.

Subcase 3 $(d^P_{\lambda,2} = d^P_{\mu,1} > d^P_{\mu,3})$

Lemma 3.3.3 **(i)** and **(iii)** imply that the $(r,1)$-entry of $\beta_4(P')$ is non-zero.

Subcase 4 $(d^P_{\lambda,2} = d^P_{\mu,3} > d^P_{\mu,1})$

Lemma 3.3.3 **(i)** and **(iii)** imply that the $(r,3)$-entry of $\beta_4(P')$ is non-zero.

Subcase 5 $(d^P_{\lambda,2} = d^P_{\mu,1} = d^P_{\mu,3})$

Note that $\mu^P_{1,0}$ and $\mu^P_{3,0}$ are non-zero integers with the opposite sign, since P is regular. Furthermore, Lemma 3.3.3 **(i)** and **(iii)** imply that the $(r,1)$-entry of $\beta_4(P')$ is non-zero if $\lambda_{2,0}^{P}+\mu_{3,0}^{P}\neq 0$, and the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $\lambda_{2,0}^{P}+\mu_{1,0}^{P}\neq 0$. We deduce that either the $(r,1)$-entry or the $(r,3)$-entry of $\beta_4(P')$ is non-zero in this subcase.

We have considered all subcases and we have established that either the $(r,1)$-entry or the $(r,3)$-entry of $\beta_4(P')$ is non-zero for every $r\in\{1,2,3\}$. Therefore, $\beta_4(P')$ is not a diagonal matrix. □

We now establish that regularity is an inductive property in the context of Lemma 4.1.5.

Lemma 4.1.6. *Let P be a 2-subproduct of σ and let P' be the minimal s'-subproduct containing P for $s'\in\{1,3\}$. If $P'\setminus P$ is a subroad, and if P is regular, then P' is regular.*

Proof. We recall that we write $P=\prod_{i=1}^{p-1}\sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$. We will split the remainder of the proof into two cases. In **Case 1**, we will assume that either $a_p=0$ or $b_p=0$, and in **Case 2**, we will assume that $a_p,b_p>0$.

Case 1 (either $a_p=0$ or $b_p=0$)

We have already noted, in the proof of Lemma 4.1.5, that either $a_p>1$ or $b_p>1$ since $P'\setminus P$ is a subroad. Let us consider those $r\in\{1,2,3\}$ for which $d_{\lambda}^{P'}\leq d_{\mu}^{P'}$. Lemma 3.3.3 implies that $d_{\lambda}^{P}+1\leq d_{\mu}^{P}$. A further application of Lemma 3.3.3 implies that $d_{\mu}^{P'}=d_{\mu}^{P}$ and $\mu_0^{P'}=\mu_0^{P}$ if $a_p=0$, and $d_{\mu}^{P'}=d_{\mu}^{P}+1$ and $\mu_0^{P'}=-\mu_0^{P}$ if $b_p=0$.

We will split the remainder of the argument into two subcases. In **Subcase 1**, we will assume that $c_{p-1}\geq 2$, and in **Subcase 2**, we will assume that $c_{p-1}=1$.

Subcase 1 ($c_{p-1}\geq 2$)

Note that $w_{\nu}^{P'}=w_{\mu,1}^{P}$ if $a_p=0$, and $w_{\nu}^{P'}=-q^{-1}w_{\mu,3}^{P}$ if $b_p=0$. In particular, either $w_{\mu}^{P'}\neq 0$ for every r, or $w_{\nu}^{P'}\neq 0$ for every r. Furthermore, the conditions on $w_{\mu}^{P'}$ and $w_{\nu}^{P'}$ in Definition 4.1.4 **(b)** **(i)** (the definition of regularity for P') correspond to the conditions on $w_{\mu,1}^{P}$ and $w_{\mu,3}^{P}$ in Definition 4.1.4 **(a)** **(i)** (the definition of regularity for P). Therefore, P' is regular in this subcase.

Subcase 2 ($c_{p-1} = 1$)

Note that $w_\nu^{P'} = w_\nu^P$. In particular, either $w_\mu^{P'} \neq 0$ for every r, or $w_\nu^{P'} \neq 0$ for every r. Furthermore, the conditions on $w_\mu^{P'}$ and $w_\nu^{P'}$ in Definition 4.1.4 **(b) (i)** (the definition of regularity for P') correspond to the conditions on w_μ^P and w_ν^P in Definition 4.1.4 **(a) (ii)** (the definition of regularity for P). Moreover, if either $d_\lambda^P + 1 > d_\nu^P$ and $a_p = 0$, or $d_\lambda^P > d_\nu^P$ and $b_p = 0$, then the conditions on $w_\mu^{P'}$ and $w_\nu^{P'}$ in Definition 4.1.4 **(b) (i)** are vacuous, since $d_\lambda^P + 1 \leq d_\mu^P$. Therefore, P' is regular in this subcase.

Case 2 ($a_p, b_p > 0$)

We have already noted, in the proof of Lemma 4.1.5, that $c_{p-1} \geq 2$ since $P' \setminus P$ is a subroad. We will establish the statement by considering subcases according to the relative values of $d_{\lambda,2}^P$, $d_{\mu,1}^P$, and $d_{\mu,3}^P$. In each subcase, we will establish the statement by considering the possibilities $a_p > 1$ and $a_p = 1$, separately. (The reason for the partition is that Definition 4.1.4 **(b) (ii)** (the definition of regularity of P') is different for each possibility.)

Subcase 1 ($d_{\lambda,2}^P > \max\{d_{\mu,1}^P, d_{\mu,3}^P\}$)

If $a_p > 1$, then Lemma 3.3.3 **(i)** implies that $d_{\lambda,1}^{P'} > d_{\mu,1}^{P'}$, and the statement is vacuous.

If $a_p = 1$, then Lemma 3.3.3 **(i)** implies that $d_{\lambda,3}^{P'} > d_{\mu,3}^{P'}$ and $d_{\lambda,3}^{P'} + 1 \geq d_{\mu,1}^{P'}$, and the statement is vacuous.

Subcase 2 ($d_{\mu,1}^P > \max\{d_{\lambda,2}^P, d_{\mu,3}^P\}$)

If $a_p > 1$, then Lemma 3.3.3 **(iii)** implies that $w_{\mu,1}^{P'}, w_{\lambda,3}^{P'} \neq 0$, $d_{\mu,1}^{P'} + b_p - 2 = d_{\lambda,3}^{P'}$, $\mu_{1,0}^{P'} = -\mu_{1,0}^P$, and $\lambda_{3,0}^{P'} = \mu_{1,0}^P$. If $b_p > 1$, then $d_{\mu,1}^{P'} < d_{\lambda,3}^{P'} + 1$. Therefore, the statement is established if $a_p > 1$.

If $a_p = 1$, then Lemma 3.3.3 **(iii)** implies that $d_{\mu,1}^{P'} \leq d_{\mu,3}^{P'}$. If $b_p > 1$, then $d_{\lambda,3}^{P'} > d_{\mu,3}^{P'}$. If $b_p = 1$, then $d_{\lambda,3}^{P'} < d_{\mu,3}^{P'}$. Therefore, the statement is established if $a_p = 1$.

Subcase 3 ($d_{\mu,3}^P > \max\{d_{\lambda,2}^P, d_{\mu,1}^P\}$)

If $a_p > 1$, then Lemma 3.3.3 **(iii)** implies that $d^{P'}_{\lambda,1} > d^{P'}_{\mu,1}$, and the statement is vacuous. If $a_p = 1$, then Lemma 3.3.3 **(iii)** implies that $w^{P'}_{\mu,1} \neq 0$ and $d^{P'}_{\mu,1} \geq d^{P'}_{\mu,3} + 2$, and the statement is established.

Subcase 4 $(d^P_{\mu,1} = d^P_{\mu,3} > d^P_{\lambda,2})$

If $a_p > 1$, then Lemma 3.3.3 **(iii)** implies that $d^{P'}_{\lambda,1} > d^{P'}_{\mu,1}$, and the statement is vacuous. If $a_p = 1$, then Lemma 3.3.3 **(iii)** implies that $w^{P'}_{\mu,1}, w^{P'}_{\mu,3} \neq 0$, $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}+1$, $\mu^{P'}_{1,0} = -\mu^P_{3,0}$, and $\mu^{P'}_{3,0} = \mu^P_{1,0}$. If $b_p > 1$, then $d^{P'}_{\lambda,3} > d^{P'}_{\mu,3}$. If $b_p = 1$, then $d^{P'}_{\lambda,3} < d^{P'}_{\mu,3}$. Note that $\mu^P_{1,0}$ and $\mu^P_{3,0}$ are non-zero integers with the opposite sign, since P is regular. Therefore, the statement is established if $a_p = 1$.

Subcase 5 $(d^P_{\lambda,2} = d^P_{\mu,1} > d^P_{\mu,3})$

If $a_p > 1$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $d^{P'}_{\lambda,1} > d^{P'}_{\mu,1}$, and the statement is vacuous.

If $a_p = 1$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $w^{P'}_{\mu,1}, w^{P'}_{\mu,3} \neq 0$, $d^{P'}_{\mu,1} = d^{P'}_{\mu,3} + 1$, $\mu^{P'}_{1,0} = -\lambda^P_{2,0}$, and $\mu^{P'}_{3,0} = \mu^P_{1,0}$. If $d^{P'}_{\lambda,3} < d^{P'}_{\mu,3}$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $b_p > 1$ and $\lambda^P_{2,0} + \mu^P_{1,0} = 0$. Therefore, $\mu^{P'}_{1,0}$ and $\mu^{P'}_{3,0}$ are non-zero integers with the same sign and the statement is established.

Subcase 6 $(d^P_{\lambda,2} = d^P_{\mu,3} > d^P_{\mu,1})$

If $a_p > 1$ and $d^{P'}_{\lambda,1} \leq d^{P'}_{\mu,1}$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $\lambda^P_{2,0} + \mu^P_{3,0} = 0$. A further application of Lemma 3.3.3 **(i)** and **(iii)** imply that $w^{P'}_{\lambda,3} \neq 0$ and $d^{P'}_{\lambda,3} \geq d^P_{\lambda,2} \geq d^{P'}_{\mu,1}$. Therefore, the statement is vacuous if $a_p > 1$.

If $a_p = 1$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $d^{P'}_{\lambda,3} > d^{P'}_{\mu,3}$ and $d^{P'}_{\lambda,3} + 1 \geq d^{P'}_{\mu,1}$. Therefore, the statement is vacuous if $a_p = 1$.

Subcase 7 $(d^P_{\lambda,2} = d^P_{\mu,1} = d^P_{\mu,3})$

Note that $\mu^P_{1,0}$ and $\mu^P_{3,0}$ are non-zero integers with the opposite sign in this subcase, since P is regular.

If $a_p > 1$ and $d^{P'}_{\lambda,1} \leq d^{P'}_{\mu,1}$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $\lambda^P_{2,0} + \mu^P_{3,0} = 0$.

In particular, $\lambda_{2,0}^{P}+\mu_{1,0}^{P}\neq 0$. A further application of Lemma 3.3.3 **(i)** and **(iii)** imply that $w_{\mu,1}^{P'}, w_{\lambda,3}^{P'}\neq 0$, $d_{\lambda,3}^{P'}+1\geq d_{\mu,1}^{P'}$, $\mu_{1,0}^{P'}=-\mu_{1,0}^{P}$, and $\lambda_{3,0}^{P'}=\mu_{1,0}^{P}-\mu_{3,0}^{P}$. In particular, $\mu_{1,0}^{P'}+\lambda_{3,0}^{P'}$ and $\lambda_{3,0}^{P'}$ are non-zero integers with the same sign. Therefore, the statement is established if $a_p>1$.

If $a_p=1$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $d_{\mu,1}^{P'}\leq d_{\mu,3}^{P'}+1$ (the inequality is strict if $\lambda_{2,0}^{P}+\mu_{3,0}^{P}=0$). If $d_{\lambda,3}^{P'}<d_{\mu,3}^{P'}$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $b_p>1$ and $\lambda_{2,0}^{P}+\mu_{1,0}^{P}=0$. In this case, Lemma 3.3.3 **(i)** and **(iii)** imply that $w_{\mu,1}^{P'}, w_{\mu,3}^{P'}\neq 0$, $d_{\mu,1}^{P'}=d_{\mu,3}^{P'}+1$, and $\mu_{1,0}^{P'}=-\lambda_{2,0}^{P}-\mu_{3,0}^{P}=\mu_{1,0}^{P}-\mu_{3,0}^{P}$ and $\mu_{3,0}^{P'}=\mu_{1,0}^{P}$ are non-zero integers with the same sign, since P is regular.

On the other hand, if $w_{\mu,1}^{P'}=0$, then Lemma 3.3.3 **(i)** and **(iii)** imply that $\lambda_{2,0}^{P}+\mu_{3,0}^{P}=0$. In particular, if $d_{\lambda,3}^{P'}\leq d_{\mu,3}^{P'}$, then $b_p=1$ since P is regular, $\lambda_{3,0}^{P}=\lambda_{2,0}^{P}=-\mu_{3,0}^{P}$, and $\mu_{3,0}^{P}=\mu_{1,0}^{P}$. We conclude that $\lambda_{3,0}^{P}+\mu_{3,0}^{P}\neq 0$ in this case, since P is regular. Therefore, the statement is established if $a_p=1$.

We have considered all possibilities for the relative values of $d_{\lambda,2}^{P}$, $d_{\mu,1}^{P}$, and $d_{\mu,3}^{P}$, and the statement is established in general. □

We summarize Lemma 4.1.5 and Lemma 4.1.6.

Corollary 4.1.7. *Let P be a 2-subproduct of σ and let P' be the minimal s'-subproduct containing P for $s'\in\{1,3\}$. If $P'\setminus P$ is a subroad, and if P is regular, then P' is regular and $\beta_4(P')$ is not a diagonal matrix.*

Proof. The statement is equivalent to the combination of Lemma 4.1.5 and Lemma 4.1.6. □

We now establish an analogue of Lemma 4.1.5, and subsequently an analogue of Lemma 4.1.6, where P is an s-subproduct for $s\in\{1,3\}$, instead of a 2-subproduct.

Lemma 4.1.8. *Let P be an s-subproduct of σ for $s \in \{1,3\}$ and let P' be the minimal 2-subproduct containing P. If $P' \setminus P$ is a subroad, and if P is regular, then $\beta_4(P')$ is not a diagonal matrix.*

Proof. We recall that we write $P = \left(\prod_{i=1}^{p-1} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}\right)\sigma_1^{a_p}\sigma_3^{b_p}$. We will split the proof into two cases. In **Case 1**, we will assume that either $a_p = 0$ or $b_p = 0$, and in **Case 2**, we will assume that $a_p, b_p > 0$. (The reason for the partition is that Definition 4.1.4 **(b)** (the definition of regularity for P) is different in each case.)

Case 1 (either $a_p = 0$ or $b_p = 0$)

If $a_p = 0$, then the $(r,1)$-entry of $\beta_4(P')$ is equal to w_ν^P, and the $(r,3)$-entry of $\beta_4(P')$ is equal to $w_\lambda^P + w_\mu^P$. If $b_p = 0$, then the $(r,1)$-entry of $\beta_4(P')$ is equal to $w_\lambda^P + w_\mu^P$, and the $(r,3)$-entry of $\beta_4(P')$ is equal to w_ν^P. We will split the remainder of the argument in this case according to the relative values of d_λ^P and d_μ^P.

Firstly, we will show that there are multiple non-zero entries in the rth row of $\beta_4(P')$ if $d_\lambda^P > d_\mu^P$ for some $r \in \{1,2,3\}$. If $a_p = 0$, then the $(r,3)$-entry of $\beta_4(P')$ is non-zero, and if $b_p = 0$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero. If $w_\nu^P = 0$, then Lemma 3.3.3 **(i)** implies that the $(r,2)$-entry of $\beta_4(P')$ is non-zero, since $w_\lambda^P \neq 0$. If $w_\nu^P \neq 0$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero if $a_p = 0$, and the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $b_p = 0$. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ if $d_\lambda^P > d_\mu^P$ for some $r \in \{1,2,3\}$. In this case, $\beta_4(P')$ is not a diagonal matrix.

On the other hand, if $w_\nu^P \neq 0$ for every $r \in \{1,2,3\}$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero for every $r \in \{1,2,3\}$ if $a_p = 0$, and the $(r,3)$-entry of $\beta_4(P')$ is non-zero for every $r \in \{1,2,3\}$ if $b_p = 0$. In this case, $\beta_4(P')$ is not a diagonal matrix.

Since P is regular, we may assume that $d_\lambda^P \leq d_\mu^P$ and $w_\mu^P \neq 0$ for every $r \in \{1,2,3\}$. If $w_\nu^P = 0$ for every $r \in \{1,2,3\}$, then $\beta_4(P')$ is not an invertible matrix. Let us further assume that $w_\nu^P \neq 0$ for some $r \in \{1,2,3\}$. In this case, we have $d_\lambda^P = d_\mu^P$ and $\lambda_0^P + \mu_0^P = 0$, if either $a_p = 0$ and the $(r,3)$-entry of $\beta_4(P')$ is zero, or $b_p = 0$ and the

$(r,1)$-entry of $\beta_4(P')$ is zero. Lemma 3.3.3 **(i)** and **(iv)** imply that $\lambda_0^P + \nu_0^P = 0$ if the $(r,2)$-entry of $\beta_4(P')$ is zero, and either $d_\lambda^P + 1 = d_\nu^P$ if $a_p = 0$, or $d_\lambda^P = d_\nu^P + 1$ if $b_p = 0$. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ in this case, since P is regular. We have established that $\beta_4(P')$ is not a diagonal matrix in all cases.

Case 2 $(a_p, b_p > 0)$

We will split the argument into two subcases. In **Subcase 1**, we will assume that $a_p > 1$, and in **Subcase 2**, we will assume that $a_p = 1$. (The reason for the partition is that Definition 4.1.4 **(b) (ii)** (the definition of regularity of P) is different in each case.)

Subcase 1 $(a_p > 1)$

The $(r,1)$-entry of $\beta_4(P')$ is $w_{\lambda,1}^P + w_{\mu,1}^P$ and the $(r,3)$-entry of $\beta_4(P')$ is $w_{\lambda,3}^P$. We will split the argument in this subcase according to the relative values of $d_{\lambda,1}^P$ and $d_{\mu,1}^P$. (The reasoning is similar to that of **Case 1** with $b_p = 0$, but simpler.)

If $d_{\lambda,1}^P > d_{\mu,1}^P$ for some $r \in \{1,2,3\}$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero. Furthermore, if the $(r,3)$-entry of $\beta_4(P')$ is zero, then Lemma 3.3.3 **(i)** implies that the $(r,2)$-entry of $\beta_4(P')$ is non-zero. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ if $d_{\lambda,1}^P > d_{\mu,1}^P$ for some $r \in \{1,2,3\}$. In this case, $\beta_4(P')$ is not a diagonal matrix.

Since P is regular, we may assume that $d_{\lambda,1}^P \leq d_{\mu,1}^P$ and $w_{\mu,1}^P, w_{\lambda,3}^P \neq 0$ for every $r \in \{1,2,3\}$. However, in this case, the $(r,3)$-entry of $\beta_4(P')$ is non-zero for every $r \in \{1,2,3\}$. We have established that $\beta_4(P')$ is not a diagonal matrix in all cases.

Subcase 2 $(a_p = 1)$

The $(r,1)$-entry of $\beta_4(P')$ is $w_{\mu,1}^P$ and the $(r,3)$-entry of $\beta_4(P')$ is $w_{\lambda,3}^P + w_{\mu,3}^P$. We will split the argument in this subcase according to the relative values of $d_{\lambda,3}^P$ and $d_{\mu,3}^P$. (The reasoning is similar to that of **Case 1** with $a_p = 0$, but simpler.)

If $d_{\lambda,3}^P > d_{\mu,3}^P$ for some $r \in \{1,2,3\}$, then the $(r,3)$-entry of $\beta_4(P')$ is non-zero. Furthermore, Lemma 3.3.3 **(i)** and **(iv)** imply that either the $(r,1)$-entry or the $(r,2)$-entry of

$\beta_4(P')$, is non-zero. In this case, $\beta_4(P')$ is not a diagonal matrix.

We may assume that $d^P_{\lambda,3} \leq d^P_{\mu,3}$ for every $r \in \{1,2,3\}$. In this case, either the $(r,1)$-entry or the $(r,3)$-entry of $\beta_4(P')$ is non-zero, since P is regular. We have established that $\beta_4(P')$ is not a diagonal matrix in all cases.

We have considered all possibilities and the statement is established in general. □

We now establish an analogue of Lemma 4.1.6, where P is an s-subproduct for $s \in \{1,3\}$, instead of a 2-subproduct. Lemma 4.1.6 and the following Lemma 4.1.9 imply that regularity is an inductive property along roads (see Corollary 4.1.11).

Lemma 4.1.9. *Let P be an s-subproduct of σ for $s \in \{1,3\}$ and let P' be the minimal 2-subproduct containing P. If $P' \setminus P$ is a subroad, and if P is regular, then P' is regular.*

Proof. We recall that we write $P = \prod_{i=1}^{p-1} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}$. We will split the proof into two cases. In **Case 1**, we will assume that either $a_p = 0$ or $b_p = 0$, and in **Case 2**, we will assume that $a_p, b_p > 0$. (The reason for the partition is that Definition 4.1.4 **(b)** (the definition of regularity of P) is different in each case.)

Case 1 (either $a_p = 0$ or $b_p = 0$)

If $c_p \geq 2$, then Lemma 3.3.3 **(iv)** implies that $w^{P'}_{\mu,1} = w^P_\nu$ if $a_p = 0$, and $w^{P'}_{\mu,3} = -qw^P_\nu$ if $b_p = 0$. If $c_p = 1$, then $w^{P'}_\mu = w^P_\nu$ if $a_p = 0$, and $w^{P'}_\mu = -qw^P_\nu$ if $b_p = 0$. We will split the argument according to the relative values of d^P_λ and d^P_μ (the partition is indicated by boldface text in three instances).

Firstly, let us assume that $d^P_\lambda > d^P_\mu$ for some $r \in \{1,2,3\}$. We will further partition the argument in this subcase according to the value of c_p.

If $c_p = 1$, then $d^{P'}_\nu = d^P_\lambda$ in this subcase. Lemma 3.3.3 **(i)** implies that $d^{P'}_\lambda = d^{P'}_\nu + 1$ if $a_p = 0$, and $d^{P'}_\lambda = d^{P'}_\nu$ if $b_p = 0$. Therefore, the statement is vacuous if $c_p = 1$, since the inequality comparing $d^{P'}_\lambda$ and $d^{P'}_\nu$ is not satisfied.

Let us assume that $c_p \geq 2$. Lemma 3.3.3 **(ii)** implies that $d^{P'}_{\mu,3} = d^P_\lambda + 1$ and $\mu^{P'}_{3,0} = -\lambda^P_0$ if $a_p = 0$, and $d^{P'}_{\mu,1} = d^P_\lambda$ and $\mu^{P'}_{1,0} = \lambda^P_0$ if $b_p = 0$. If $d^{P'}_{\lambda,2} \leq \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$, then Lemma 3.3.3 **(i)** and **(iv)** imply that $w^{P'}_{\mu,1}, w^{P'}_{\mu,3} \neq 0$, $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}$ and $\mu^{P'}_{1,0} + \mu^{P'}_{3,0} = 0$. Therefore, the statement is established if $c_p \geq 2$.

Secondly, let us assume that $d^P_\lambda \leq d^P_\mu$ for some $r \in \{1, 2, 3\}$. Again, we will further partition the argument in this subcase according to the value of c_p. Firstly, note that if $d^P_\lambda < d^P_\mu$, then the conditions on $w^{P'}_\mu$ and $w^{P'}_\nu$ in Definition 4.1.4 **(a)** (the definition of regularity for P') correspond to the conditions on w^P_μ and w^P_ν in Definition 4.1.4 **(b) (i)** (the definition of regularity for P). Indeed, if $c_p = 1$, then $d^{P'}_\nu = d^P_\mu$ and $\nu^{P'}_0 = \mu^{P'}_0$. If $c_p \geq 2$, then Lemma 3.3.3 **(ii)** implies that $d^{P'}_{\mu,3} = d^P_\mu + 1$ and $\mu^{P'}_{3,0} = -\mu^P_0$ if $a_p = 0$, and $d^{P'}_{\mu,1} = d^P_\mu$ and $\mu^{P'}_0 = \mu^P_0$ if $b_p = 0$. Moreover, $d^{P'}_{\lambda,2} > \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$ if either $d^P_\nu \geq d^P_\mu + 1$ and $a_p = 0$, or $d^P_\nu + 1 \geq d^P_\mu$ and $b_p = 0$.

Finally, let us assume that $d^P_\lambda = d^P_\mu$ for some $r \in \{1, 2, 3\}$, and we will partition the argument according to the value of c_p.

If $c_p = 1$, then $d^{P'}_\nu \leq d^P_\lambda$ (the inequality is strict if $\lambda^P_0 + \mu^P_0 = 0$). Lemma 3.3.3 **(i)** implies that $d^{P'}_\lambda \geq d^{P'}_\nu + 1$ if $a_p = 0$, and $d^{P'}_\lambda \geq d^{P'}_\nu$ if $b_p = 0$ (compare to the assumption $d^P_\lambda > d^P_\mu$ considered before, where the inequalities are equalities). Therefore, the statement is vacuous if $c_p = 1$, since the inequality comparing $d^{P'}_\lambda$ and $d^{P'}_\nu$ is not satisfied.

If $c_p \geq 2$ and $d^{P'}_{\lambda,2} \leq \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$, then Lemma 3.3.3 **(i)** and **(iv)** imply that $d^P_\nu + 1 = d^P_\lambda$ if $a_p = 0$, and $d^P_\nu = d^P_\lambda$ if $b_p = 0$. Moreover, in this case, Lemma 3.3.3 **(i)** and **(iv)** also imply that $\lambda^P_0 = \nu^P_0$. We conclude that $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}$, and $\mu^{P'}_{1,0}$ and $\mu^{P'}_{3,0}$ are non-zero integers with the opposite sign. Indeed, if $a_p = 0$, then Lemma 3.3.3 **(ii)** and **(iv)** imply that $\mu^{P'}_{1,0} = \nu^P_0$ and $\mu^{P'}_{3,0} = -\nu^P_0 - \mu^P_0$. If $b_p = 0$, then Lemma 3.3.3 **(ii)** and **(iv)** imply that $\mu^{P'}_{1,0} = \nu^P_0 + \mu^P_0$ and $\mu^{P'}_{3,0} = -\nu^P_0$. Since P is regular, μ^P_0 and ν^P_0 are non-zero integers with the opposite sign, and the statement is established if $d^P_\lambda = d^P_\mu$.

Case 2 $(a_p, b_p > 0)$

We will split the argument into two subcases, according to the value of a_p. (The reason for this partition is that the definition of regularity for P (Definition 4.1.4 **(b)** **(ii)**) is different in each subcase.)

Subcase 1 $(a_p > 1)$

We will split the argument according to the relative values of $d^P_{\lambda,1}$ and $d^P_{\mu,1}$ (the partition is indicated by boldface text in two instances). The argument is similar to that of **Case 1** (with $b_p = 0$). However, we will use constraint **(i)** on the positive braid σ in the statement of Theorem 1.1.2, which states that either $b_p > 1$ or $b_{p+1} > 0$. Lemma 3.3.3 **(ii)** implies that $w^{P'}_{\mu,3} = -qw^P_{\lambda,3}$. We will split the argument according to the relative values of $d^P_{\lambda,1}$ and $d^P_{\mu,1}$ (the partition is indicated by boldface text in three instances).

Firstly, let us assume that $d^P_{\lambda,1} > d^P_{\mu,1}$ for some $r \in \{1,2,3\}$. Lemma 3.3.3 **(ii)** implies that $d^{P'}_{\mu,1} = d^P_{\lambda,1}$ and $\mu^{P'}_{1,0} = \lambda^{P'}_{1,0}$. If $d^{P'}_{\lambda,2} \leq \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$, then Lemma 3.3.3 **(i)** implies that $d^P_{\lambda,1} = d^P_{\lambda,3} + 1$ and $\lambda^P_{1,0} = \lambda^P_{3,0}$. In particular, $w^{P'}_{\mu,1}, w^{P'}_{\mu,3} \neq 0$, $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}$, and $\mu^{P'}_{1,0} + \mu^{P'}_{3,0} = 0$. Therefore, the statement is established if $d^P_{\lambda,1} > d^P_{\mu,1}$.

Secondly, if $d^P_{\lambda,1} < d^P_{\mu,1}$ for some $r \in \{1,2,3\}$, then Lemma 3.3.3 **(i)** implies that $d^{P'}_{\lambda,2} > \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$ since $d^P_{\mu,1} \leq d^P_{\lambda,3} + 1$. Therefore, the statement is vacuous if $d^P_{\lambda,1} < d^P_{\mu,1}$.

Finally, let us assume that $d^P_{\lambda,1} = d^P_{\mu,1}$ for some $r \in \{1,2,3\}$. If $d^{P'}_{\lambda,2} \leq \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$, then as before, Lemma 3.3.3 **(i)** implies that $d^P_{\lambda,1} = d^P_{\lambda,3} + 1$ and $\lambda^P_{1,0} = \lambda^P_{3,0}$. We conclude that either $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}$, and $\mu^{P'}_{1,0}$ and $\mu^{P'}_{3,0}$ are non-zero integers with the opposite sign, or $d^{P'}_{\mu,3} \geq d^{P'}_{\mu,1} + 1$ and $b_{p+1} > 0$. Indeed, Lemma 3.3.3 **(ii)** implies that either $\mu^{P'}_{1,0} = \mu^P_{1,0} + \lambda^P_{3,0}$ and $\mu^{P'}_{3,0} = -\lambda^P_{3,0}$, if the former is non-zero, or $d^{P'}_{\mu,3} \geq d^{P'}_{\mu,1} + 1$. Since P is regular, either $\mu^P_{1,0} + \lambda^P_{3,0}$ and $\lambda^P_{3,0}$ are non-zero integers with the same sign, or $\mu^P_{1,0} + \lambda^P_{3,0} = 0$ and $b_p = 1$. If the latter, then $d^{P'}_{\mu,3} \geq d^{P'}_{\mu,1} + 1$ and $b_{p+1} > 0$. Therefore, the statement is established if $d^P_{\lambda,1} = d^P_{\mu,1}$.

Subcase 2 $(a_p = 1)$

We will split the argument according to the relative values of $d^{P}_{\lambda,3}$ and $d^{P}_{\mu,3}$ (the partition is indicated by boldface text in the two instances). The argument is somewhat different to that of **Case 1** (with $a_p = 0$). We will use constraint **(ii)** on the positive braid σ in the statement of Theorem 1.1.2, which states that $a_{p+1} > 0$, and either $b_p = 1$ or $b_{p+1} > 0$. Lemma 3.3.3 **(ii)** implies that $w^{P'}_{\mu,1} = w^{P}_{\mu,1}$. We will split the argument according to the relative values of $d^{P}_{\lambda,3}$ and $d^{P}_{\mu,3}$ (the partition is indicated by boldface text in two instances).

Firstly, let us assume that $d^{P}_{\lambda,3} \geq d^{P}_{\mu,3}$ for some $r \in \{1,2,3\}$. Lemma 3.3.3 **(ii)** implies that $d^{P'}_{\mu,3} \leq d^{P}_{\lambda,3} + 1$ (the inequality is strict if $d^{P}_{\lambda,3} = d^{P}_{\mu,3}$ and $\lambda^{P}_{3,0} + \mu^{P}_{3,0} = 0$). If $d^{P'}_{\lambda,2} \leq \max\{d^{P'}_{\mu,1}, d^{P'}_{\mu,3}\}$, then Lemma 3.3.3 **(i)** and **(ii)** imply that $d^{P}_{\lambda,3} + 2 \leq d^{P}_{\mu,1}$. In particular, $d^{P'}_{\mu,3} + 1 \leq d^{P'}_{\mu,1}$. Therefore, the statement is established if $d^{P}_{\lambda,3} \geq d^{P}_{\mu,3}$, since $a_{p+1} > 0$.

Secondly, let us assume $d^{P}_{\lambda,3} < d^{P}_{\mu,3}$ for some $r \in \{1,2,3\}$. Lemma 3.3.3 **(ii)** implies that $d^{P'}_{\mu,3} = d^{P}_{\mu,3} + 1$ and $\mu^{P'}_{3,0} = -\mu^{P}_{3,0}$. If $d^{P'}_{\mu,1} = d^{P'}_{\mu,3}$, then $d^{P}_{\mu,1} = d^{P}_{\mu,3} + 1$. Note that $\mu^{P}_{1,0}$ and $\mu^{P}_{3,0}$ are non-zero integers with the same sign, since P is regular. We conclude that $\mu^{P'}_{1,0} = \mu^{P}_{1,0}$ and $\mu^{P'}_{3,0} = -\mu^{P}_{3,0}$ are non-zero integers with the opposite sign.

Of course, if $d^{P'}_{\mu,1} = d^{P'}_{\mu,3} + 1$, then the statement is vacuous since $a_{p+1} > 0$. On the other hand, if $d^{P'}_{\mu,3} = d^{P'}_{\mu,1} + 1$, then $d^{P}_{\mu,1} = d^{P}_{\mu,3}$. In particular, $b_p = 1$, since P is regular. Again, the statement is vacuous since $b_{p+1} > 0$ in this case. Therefore, the statement is established if $d^{P}_{\lambda,3} < d^{P}_{\mu,3}$.

We have considered all possibilities and the statement is established in general. $\square$

We summarize Lemma 4.1.8 and Lemma 4.1.9.

Corollary 4.1.10. *Let P be an s-subproduct of σ for $s \in \{1,3\}$ and let P' be the minimal 2-subproduct containing P. If $P' \setminus P$ is a subroad and P is regular, then P' is regular and $\beta_4(P')$ is not a diagonal matrix.*

Proof. The statement is equivalent to the combination of Lemma 4.1.8 and Lemma 4.1.9. □

Let us summarize the results established thus far (Corollary 4.1.7 and Corollary 4.1.10). In particular, we conclude that regularity is inductively preserved along roads (see Motivation 4.1.3 for further discussion).

Corollary 4.1.11. *Let P be an s-subproduct of σ and let R be a subroad of σ directly succeeding P, such that $P' = PR$ is an s'-subproduct for some $s' \in \{1, 2, 3\}$. If P is regular, then P' is regular and $\beta_4(P')$ is not a diagonal matrix. In particular, if $P' = \sigma$, then $\beta_4(\sigma)$ is not a diagonal matrix.*

Proof. The statement follows from Corollary 4.1.7 and Corollary 4.1.10. Indeed, there is a finite chain $P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_k = P'$ with the property that for $s_0 = s$ and $1 \leq i \leq k$, P_i is the minimal s_i-subproduct containing P_{i-1}, for some $s_i \in \{1, 2, 3\}$ with $s_i \neq s_{i-1}$, and $P_i \setminus P_{i-1}$ is a subroad. □

The next goal is to establish Corollary 4.1.14. Corollary 4.1.14 is an analogue of Corollary 4.1.11 where $P' \setminus P$ is a generic block, instead of a road. We will establish Corollary 4.1.14 as a consequence of Lemma 4.1.12 ($P' \setminus P$ is a generic 2-block) and Corollary 4.1.13 ($P' \setminus P$ is a generic 3-block). (Subsequently, we will address the case that $P' \setminus P$ is a singular block in Lemma 4.1.18.)

Lemma 4.1.12. *Let B be a generic 2-block and let P be the 2-subproduct immediately preceding B. If P is regular, then B is regular (i.e., PB is regular) and $\beta_4(PB)$ is not a diagonal matrix.*

Proof. Let us fix the notation that we will use in this proof. We write $B = \sigma_1^{a_p}\sigma_3^{b_p}\sigma_2\sigma_1^{a_{p+1}}$, $P_2 = P\sigma_1^{a_p}\sigma_3^{b_p}\sigma_2$, and $P' = PB$. Note that $b_p > 1$ since B is a generic 2-block. We will use constraint **(iii)** on the positive braid σ in the statement of Theorem 1.1.2, which states that $a_p > 1$.

Let $w_{\lambda,1}^{P_2} = \sum_{i=0}^{\infty} \lambda_{1,i}^{P_2} q^{d_{\lambda,1}^{P_2}-i}$ be the weighted number of $(r,1)$-type P_2-paths that have a $2 \to 1$ vertex change at the last σ_1 in $\sigma_1^{a_p}$. Note that an admissible $(r,1)$-type P_2-path with a $2 \to 1$ vertex change at the last σ_1 in $\sigma_1^{a_p}$ extends (uniquely) to an admissible $(r,1)$-type P'-path (with no further vertex changes). Indeed, the extension does not belong to a δ-pair (see Figure 3.3 in Chapter 3).

Firstly, we will establish the following **Claim** and secondly, we will establish that the **Claim** implies the statement.

Claim: If either $d_{\lambda}^{P_2} > d_{\lambda,1}^{P_2}$ or $d_{\mu}^{P_2} > d_{\lambda,1}^{P_2}$, then P' is regular and $\beta_4(P')$ is not a diagonal matrix.

Step 1 (Proof of the **Claim**)

We will use the assumption $a_p > 1$ on the positive braid σ. We observe that $w_{\lambda}^{P_2} = -qw_{\nu}^{P'}$. Indeed, $w_{\nu}^{P'}$ is the weighted number of $(r,3)$-type $P\sigma_1^{a_p}\sigma_3^{b_p}$-paths by definition, and all $(r,3)$-type $P\sigma_1^{a_p}\sigma_3^{b_p}$-paths are good since $a_p > 1$. Furthermore, we can obtain a good $(r,2)$-type P_2-path by extending an $(r,3)$-type $P\sigma_1^{a_p}\sigma_3^{b_p}$-path by a single $3 \to 2$ vertex change, and this furnishes a one-to-one correspondence between good $(r,2)$-type P_2-paths and $(r,3)$-type $P\sigma_1^{a_p}\sigma_3^{b_p}$-paths. We conclude that $w_{\lambda}^{P_2} = -qw_{\nu}^{P'}$. Also, note that the $(r,2)$-entry of $\beta_4(P')$ is $w_{\lambda}^{P_2} + w_{\mu}^{P_2}$.

Firstly, we will establish that P' is regular if the hypothesis of the **Claim** is satisfied. Lemma 3.3.3 implies that if $d_{\lambda}^{P'} \leq d_{\mu}^{P'}$, then either $\max\{d_{\lambda}^{P_2}, d_{\lambda,1}^{P_2}\}+1 \leq d_{\mu}^{P_2}$ or $d_{\lambda}^{P_2} = d_{\lambda,1}^{P_2}$. We will deduce that P' is regular as a consequence of this statement.

Indeed, if $d_{\lambda}^{P'} \leq d_{\mu}^{P'}$, and if either $d_{\lambda}^{P_2} > d_{\lambda,1}^{P_2}$ or $d_{\mu}^{P_2} > d_{\lambda,1}^{P_2}$, then $d_{\lambda}^{P_2}+1 \leq d_{\mu}^{P_2}$. A further application of Lemma 3.3.3 shows that $d_{\mu}^{P'} = d_{\mu}^{P_2} + 1$, and consequently $d_{\nu}^{P'} + 3 \leq d_{\mu}^{P'}$. Therefore, P' vacuously satisfies Definition 4.1.4 **(b) (i)** (the definition of regularity) if the hypothesis of the **Claim** is satisfied.

Secondly, we will establish that $\beta_4(P')$ is not a diagonal matrix if the hypothesis of the **Claim** is satisfied. Indeed, if the $(r,2)$-entry of $\beta_4(P')$ is zero, then $d_{\lambda}^{P_2} = d_{\mu}^{P_2}$. The hypothesis of the **Claim** implies that $d_\lambda^{P_2} = d_\mu^{P_2} > d_{\lambda,1}^{P_2}$ and the $(r,3)$-entry of $\beta_4(P')$ is non-zero. Lemma 3.3.3 implies that $d_\lambda^{P'} > d_\mu^{P'}$, and consequently the $(r,1)$-entry of $\beta_4(P')$ is non-zero. Therefore, there are multiple non-zero entries in the rth row of $\beta_4(P')$ if the $(r,2)$-entry of $\beta_4(P')$ is zero. Of course, if the $(r,2)$-entry of $\beta_4(P')$ is non-zero for each $r \in \{1,2,3\}$, then $\beta_4(P')$ is not a diagonal matrix.

Step 2 (The **Claim** implies the statement)

We will establish that the hypothesis of the **Claim** is satisfied if P is regular. We will split the argument into cases according to the relative values of $d_{\lambda,2}^P$, $d_{\mu,1}^P$, and $d_{\mu,3}^P$. We recall $b_p > 1$ (since B is a generic 2-block), but we will use the assumption $a_p > 1$ on the positive braid σ.

Case 1 ($d_{\lambda,2}^P \neq d_{\mu,1}^P$ and $\max\{d_{\lambda,2}^P, d_{\mu,1}^P\} \geq d_{\mu,3}^P$)

We observe that $d_\lambda^{P_2} > d_{\lambda,1}^{P_2}$. Indeed, Lemma 3.3.3 implies that $d_{\lambda,1}^{P_2} \leq \max\{d_{\lambda,2}^P, d_{\mu,1}^P\}+1$. Lemma 3.3.3 also implies that $d_\lambda^{P_2} > \max\{d_{\lambda,2}^P, d_{\mu,1}^P\} + 1$, since $b_p > 1$. Therefore, the hypothesis of the **Claim** is satisfied in this case.

Case 2 ($d_{\mu,3}^P > \max\{d_{\lambda,2}^P, d_{\mu,1}^P\}$ or $d_{\lambda,2}^P = d_{\mu,1}^P > d_{\mu,3}^P$)

Lemma 3.3.3 implies that $d_\mu^{P_2} > d_{\lambda,1}^{P_2}$, since $a_p > 1$. Therefore, the hypothesis of the **Claim** is satisfied in this case.

Case 3 ($d_{\lambda,2}^P = d_{\mu,1}^P = d_{\mu,3}^P$)

Since P is regular, $\mu_{1,0}^P$ and $\mu_{3,0}^P$ are non-zero integers with the opposite sign. In particular, either $\lambda_{2,0}^P + \mu_{1,0}^P \neq 0$ or $\lambda_{2,0}^P + \mu_{3,0}^P \neq 0$.

If $\lambda_{2,0}^P + \mu_{1,0}^P \neq 0$, then similar reasoning to that of **Case 1** shows that $d_\lambda^{P_2} > d_{\lambda,1}^{P_2}$. If $\lambda_{2,0}^P + \mu_{3,0}^P \neq 0$, then similar reasoning to that of **Case 2** shows that $d_\mu^{P_2} > d_\lambda^{P_2}$. Therefore, the hypothesis of the **Claim** is satisfied in this case.

We have established that the hypothesis of the **Claim** is satisfied in all cases. Therefore, **Step 1** implies the statement. □

We now establish the analogue of Lemma 4.1.12 where $P' \setminus P$ is a generic 3-block, instead of a generic 2-block.

Lemma 4.1.13. *Let B be a generic 3-block and let P be the 1-subproduct immediately preceding B. If P is regular, then B is regular (i.e., PB is regular) and $\beta_4(PB)$ is not a diagonal matrix.*

Proof. Let us write $B = \sigma_2^{c_{p-1}}\sigma_3\sigma_2^{c_p}$, $P_3 = P\sigma_2^{c_{p-1}}\sigma_3$, and $P' = PB$. Note that $c_p > 1$ since B is a generic 3-block. We will use constraint **(ii)** on the positive braid σ in the statement of Theorem 1.1.2, which states that $b_{p+1} > 0$ if $p < n$.

The proof is in two steps. In **Step 1**, we will introduce notation to denote (admissible) extensions of a P-path to a P'-path. In **Step 2**, we will establish the statement by considering cases according to the relative values of d_λ^P, d_μ^P, and d_ν^P, and the values of c_{p-1} and c_p.

Step 1 (A notation for extensions of P-paths to P'-paths.)

Let X be a P-path. We will introduce notation to denote the extensions of X to a P'-path, as this will be convenient in **Step 2**. We will also use the notation we introduce here in the proof of Lemma 4.1.18, where B is a singular 3-block, instead of a generic 3-block. In particular, we will also consider the possibility $c_p = 1$ in this **Step 1**, although we will assume that $c_p > 1$ in **Step 2** of the proof since B is a generic 3-block.

The possible extensions of X will depend on the (r, s)-type of X, and whether X is good or bad. Note that X does not extend admissibly to a P'-path unless X is either an $(r, 1)$-type or an $(r, 3)$-type P-path. Indeed, if X is an $(r, 2)$-type P-path, then any extension of X to a P'-path constitutes part of a ν-pair (see Chapter 3). We will partition the notation into two cases, according to whether X is an $(r, 1)$-type P-path or an $(r, 3)$-type P-path. The possible extensions of X to P'-paths and the notation is similar in each case, but there is an important difference, and so we consider separate cases for ease of readability.

Case 1 (X is an $(r, 1)$-type P-path)

We define the extension $X^{\alpha,\beta}$ of X to an $(r, 2)$-type P'-path by a first vertex change at the αth σ_2 in $\sigma_2^{c_{p-1}}$, a second vertex change at the σ_3 in B, and a third vertex change at the βth σ_2 in $\sigma_2^{c_p}$ if $2 \leq \beta \leq c_p$. We denote the $(r, 3)$-type P_3-path, where there is no third vertex change in this extension, by $X^{\alpha,1}$. Note that the extension of X to an $(r, 2)$-type P'-path with a third vertex change at the first σ_2 in $\sigma_2^{c_p}$ is not admissible since it belongs to a ζ-pair (see Chapter 3). Furthermore, the extension $X^{\alpha,\beta}$ is admissible if and only if either X is good, or X is bad and $2 \leq \alpha \leq c_{p-1}$. Indeed, if X is bad and $\alpha = 1$, then $X^{\alpha,\beta}$ constitutes part of a ν-pair (see Chapter 3).

Let us define the extension of X to an $(r, 2)$-type P'-path by a single vertex change at the γth σ_2 in $\sigma_2^{c_p}$ by X^γ.

We remark that $X^{\alpha,\beta}$ is a good $(r, 2)$-type P'-path if and only if $1 < \beta < c_p$. Indeed, $b_{p+1} > 0$ and $X^{\alpha,\beta}$ admits a $3 \to 2$ vertex change at the last σ_2 in P' if and only if $\beta = c_p$. Also, X^γ is a good $(r, 2)$-type P'-path if and only if either $\gamma < c_p$ or $a_{p+1} = 0$. Indeed, X^γ admits a $1 \to 2$ vertex change at the last σ_2 in P' if and only if $\gamma = c_p$.

Case 2 (X is an $(r, 3)$-type P-path)

We define the extension $X^{\alpha,\beta}$ of X to an $(r, 2)$-type P'-path by a first vertex change at the αth σ_2 in $\sigma_2^{c_{p-1}}$, a second vertex change at the σ_3 in B if $\alpha < c_{p-1}$, and a third vertex change at the βth σ_2 in $\sigma_2^{c_p}$ if $\alpha < c_{p-1}$ and $2 \leq \beta \leq c_p$. We denote the $(r, 3)$-type P_3-path, where $\alpha < c_{p-1}$ and there is no third vertex change in this extension, by $X^{\alpha,1}$. If $\alpha = c_{p-1}$, then there is no second or third vertex change and we denote the resulting $(r, 2)$-type P' path by $X^{\alpha,1}$. Indeed, there is no second or third vertex change if $\alpha = c_{p-1}$ since such a P'-path would belong to either an ϵ-pair or a ζ-pair (see Chapter 3). Furthermore, the extension $X^{c_{p-1},1}$ is admissible since it does not belong to an ϵ-pair. Indeed, this is a variation of Figure 3.3 in Chapter 3.

Note that an analogue of the extension X^γ of an $(r, 1)$-type P-path X, for an $(r, 3)$-type P-path X is not admissible. Indeed, it would constitute part of an ϵ-pair (see Chapter 3).

We remark that $X^{\alpha,\beta}$ is a good P'-path if and only if either $\alpha < c_{p-1}$ and $\beta < c_p$, or $\alpha = c_{p-1}$ and $\beta = 1$. The reasoning is similar to that of **Case 1** if $\alpha < c_{p-1}$. On the other hand, $X^{c_{p-1},1}$ is a good $(r,2)$-type P'-path since it does not admit a vertex change at any σ_2 in $\sigma_2^{c_p}$.

We have parametrized all possible (admissible) extensions of a P-path to a P'-path, and we have indicated which extensions are good and which extensions are bad. We will use this parametrization in **Step 2**.

Step 2 (The case-by-case proof of the statement in terms of the sizes of the exponents in the block B and the relative values of d_λ^P, d_μ^P, and d_ν^P.)

We will split the argument into two cases, according to the relative values of d_λ^P and d_μ^P. **Henceforth**, in this **Step 2**, we fix the following notation. We denote a good $(r,1)$-type P-path of maximal q-resistance (i.e., q-resistance equal to $d_{\lambda,1}^P$) by X, a bad $(r,1)$-type P-path of maximal q-resistance (i.e., q-resistance equal to $d_{\mu,1}^P$) by Y, an $(r,3)$-type P-path of maximal q-resistance (i.e., q-resistance equal to d_ν^P) by Z, and an $(r,3)$-type P_3-path of maximal q-resistance (i.e., q-resistance equal to the degree of the $(r,3)$-entry of $\beta_4(P_3)$) by W.

We remark that either $d_{\lambda,2}^{P'} \geq d_{\mu,1}^{P'}+1$ or $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'}+1$ implies that P' is regular. Indeed, if $d_{\lambda,2}^{P'} \leq \max\{d_{\mu,1}^{P'}, d_{\mu,3}^{P'}\}$, then either of these assumptions implies that $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'}+1$. Since $b_{p+1} > 0$, P' vacuously satisfies Definition 4.1.4 **(a) (i)** (the definition of regularity). We will use this remark repeatedly in the proof.

Case 1 ($d_\lambda^P \geq d_\mu^P$)

Note that $d_{\mu,1}^{P'} \leq d_\lambda^P$ in this case. Indeed, the $(r,2)$-type P'-paths of maximal q-resistance with respect to the property of having a $1 \to 2$ vertex change corresponding to the last generator in $\sigma_2^{c_p}$ have q-resistance at most the q-resistance of X^{c_p}. We will split the argument into subcases according to the value of c_{p-1}.

Subcase 1 $(c_{p-1} \geq 2)$

We will partition the argument according to the relative values of d_λ^P and d_ν^P. We will use boldface to indicate the partition into the three possibilities $d_\lambda^P > d_\nu^P + 1$, $d_\lambda^P = d_\nu^P + 1$ and $d_\lambda^P < d_\nu^P + 1$, for ease of readability.

Firstly, let us assume that $d_\lambda^P > d_\nu^P + 1$. We observe that the $(r,2)$-type P'-paths of maximal q-resistance are of the form $X^{1,2}$ and the $(r,3)$-type P_3-paths of maximal q-resistance are of the form $X^{1,1}$, since $d_\lambda^P \geq d_\mu^P$ and $c_{p-1} \geq 2$. Indeed, the q-resistance of $X^{1,2}$ is certainly greater than the q-resistance of $X^{\alpha,\beta}$ if either $\alpha > 1$ or $\beta > 2$. The q-resistance of $X^{1,2}$ is also greater than the q-resistance of X^γ for $1 \leq \gamma \leq c_p$ since $c_{p-1} \geq 2$. Finally, the q-resistance of $X^{1,1}$ is greater than the q-resistance of $Z^{1,1}$ since we are assuming that $d_\lambda^P > d_\nu^P + 1$. We conclude that the $(r,2)$-entry and the $(r,3)$-entry of $\beta_4(P')$ are non-zero and $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'} + 1$. Therefore, the statement is established if $d_\lambda^P > d_\nu^P + 1$.

Secondly, let us assume that $d_\lambda^P = d_\nu^P + 1$. We observe that the $(r,2)$-type P'-paths of maximal q-resistance with respect to the property of not being an extension of an $(r,3)$-type P_3-path by a $3 \to 2$ vertex change corresponding to a generator within $\sigma_2^{c_p}$, are of the form $Z^{c_{p-1},1}$. Indeed, $d_\lambda^P \geq d_\mu^P$ by assumption and the q-resistance of $Z^{c_{p-1},1}$ is greater than the q-resistance of X^γ for $1 \leq \gamma \leq c_p$ since $d_\lambda^P = d_\nu^P + 1$. We will now show that this observation implies the statement.

Firstly, we show that the observation implies that P' is regular. If $c_p = 2$, then the good $(r,2)$-type P'-paths of maximal q-resistance are all of the form $Z^{c_{p-1},1}$. We conclude that $d_{\lambda,2}^{P'} \geq d_{\mu,1}^{P'} + 1$. Therefore, P' is regular if $c_p = 2$. On the other hand, if $c_p \geq 3$ and $d_{\lambda,2}^{P'} \leq \max\{d_{\mu,1}^{P'}, d_{\mu,3}^{P'}\}$, then the q-resistance of W (an $(r,3)$-type P_3-path of maximal q-resistance) is exactly one less than the q-resistance of $Z^{c_{p-1},1}$. We deduce that $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'} + 1$. Therefore, P' is regular if $c_p \geq 3$.

Secondly, we show that the observation implies that there are multiple non-zero entries in the rth row of $\beta_4(P')$. Of course, the observation implies that either the $(r,2)$-entry or the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $d_\lambda^P \geq d_\mu^P$. Indeed, an $(r,2)$ P'-path either has the property indicated in the observation or it does not. In particular, there are multiple non-zero entries in the rth row of $\beta_4(P')$ if $d_\lambda^P > d_\mu^P$, since the $(r,1)$-entry of $\beta_4(P')$ is $w_\lambda^P + w_\mu^P$. On the other hand, if $d_\lambda^P = d_\mu^P$, then μ_0^P and ν_0^P are non-zero integers with the same sign, since P is regular and $d_\lambda^P = d_\nu^P + 1$ by assumption.

Let us assume that the $(r,1)$-entry of $\beta_4(P')$ is zero. In this case, $\lambda_0^P + \mu_0^P = 0$, and thus $\lambda_0^P - \nu_0^P \neq 0$. We conclude that the $(r,3)$-type P_3-paths of maximal q-resistance are all either of the form $X^{1,1}$ or $Z^{1,1}$. Similarly, the $(r,2)$-type P'-paths of maximal q-resistance are all either of the form $X^{1,2}$ or $Z^{1,2}$ if $c_{p-1} > 2$. Indeed, if $c_{p-1} > 2$, then the q-resistance of $Z^{1,2}$ is greater than the q-resistance of $Z^{c_{p-1},1}$. On the other hand, if $c_{p-1} = 2$, then the weights of $Z^{1,2}$ and $Z^{2,1}$ cancel, and the $(r,2)$-type P'-paths of maximal q-resistance are all of the form $X^{1,2}$. Therefore, the $(r,2)$-entry and the $(r,3)$-entry of $\beta_4(P')$ are non-zero, and the statement is established if $d_\lambda^P = d_\nu^P + 1$.

Finally, let us assume that $d_\lambda^P < d_\nu^P + 1$. Note that $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'} + 1$ and the $(r,3)$-entry of $\beta_4(P')$ is non-zero. Indeed, the maximal q-resistance $(r,3)$-type P_3-paths are of the form $Z^{1,1}$, since $d_\lambda^P < d_\nu^P + 1$ and $c_{p-1} \geq 2$. Therefore, P' is regular. However, in order to show that there are multiple non-zero entries in a row of $\beta_4(P')$, we need to consider **Case 2** ($d_\mu^P > d_\lambda^P$) for the relative values of d_λ^P and d_μ^P, while fixing $c_{p-1} \geq 2$. Indeed, if $d_\mu^P > d_\lambda^P$, then the $(r,1)$-entry of $\beta_4(P')$ is non-zero. We have shown so far in **Case 1** that there are either multiple non-zero entries in the rth row of $\beta_4(P')$, or the $(r,3)$-entry of $\beta_4(P')$ is non-zero if $d_\lambda^P < d_\nu^P + 1$. If $c_{p-1} \geq 2$, then this shows that either the $(r,1)$-entry or the $(r,3)$-entry of $\beta_4(P')$ is non-zero for each $r \in \{1,2,3\}$. Therefore, we have established that $\beta_4(P')$ is not a diagonal matrix if $c_{p-1} \geq 2$.

Subcase 2 ($c_{p-1} = 1$)

Firstly, note that either $w_\lambda^P \neq 0$ for every $r \in \{1,2,3\}$ or $w_\nu^P \neq 0$ for every $r \in \{1,2,3\}$.

Indeed, this is implicit in the assumption $d_\lambda^P \geq d_\mu^P$ of this **Case 1**, and the hypothesis that either $w_\mu^P \neq 0$ for every r or $w_\nu^P \neq 0$ for every r if $d_\lambda^P \leq d_\mu^P$ (P is regular). If $w_\lambda^P \neq 0$, then the $(r,3)$-entry of $\beta_4\,(P')$ is non-zero and $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'} + 1$. Indeed, the $(r,3)$-type P_3-paths of maximal q-resistance are of the form $X^{1,1}$ if $w_\lambda^P \neq 0$.

Let us establish that there are multiple non-zero entries in the rth row of $\beta_4\,(P')$ if $w_\lambda^P \neq 0$. If the $(r,1)$-entry of $\beta_4\,(P')$ is zero, then $d_\lambda^P = d_\mu^P$. Note that $X^{1,\beta}$ and $X^{\beta-1}$ have cancelling weights for $2 \leq \beta \leq c_p$. In particular, the $(r,2)$-type P'-paths of maximal q-resistance are either of the form Y^1 if $d_\mu^P > d_\nu^P + 2$, or of the form $Z^{1,1}$ if $d_\mu^P < d_\nu^P + 2$. If $d_\mu^P = d_\nu^P + 2$, then the $(r,2)$-type P'-paths of maximal q-resistance are either of the form Y^1 or $Z^{1,1}$, since μ_0^P and ν_0^P are non-zero integers with the opposite sign (P is regular). We conclude that the $(r,2)$-entry of $\beta_4\,(P')$ is non-zero. Therefore, the statement is established.

On the other hand, if $w_\lambda^P = 0$, then the $(r,2)$-entry of $\beta_4\,(P')$ is non-zero and $d_{\lambda,2}^{P'} > \max\{d_{\mu,1}^{P'}, d_{\mu,3}^{P'}\}$. Indeed, the $(r,2)$-type P'-paths of maximal q-resistance are of the form $Z^{1,1}$ since $w_\mu^P = 0$ and $w_\nu^P \neq 0$ if $w_\lambda^P = 0$. Therefore, P' is regular. On the other hand, we have shown so far in **Case 1** that either there are multiple non-zero entries in a row of $\beta_4\,(P')$, or the $(r,2)$-entry of $\beta_4\,(P')$ is non-zero for each $r \in \{1,2,3\}$, if $c_{p-1} = 1$. In **Case 2** ($d_\lambda^P < d_\mu^P$), the $(r,1)$-entry of $\beta_4\,(P')$ is non-zero. If $c_{p-1} = 1$, then this shows that either the $(r,1)$-entry or the $(r,2)$-entry of $\beta_4\,(P')$ is non-zero for each $r \in \{1,2,3\}$. Therefore, we have established that $\beta_4\,(P')$ is not a diagonal matrix if $c_{p-1} = 1$.

Case 2 ($d_\lambda^P < d_\mu^P$)

The $(r,1)$-entry of $\beta_4\,(P')$ is non-zero in this case, since it is equal to $w_\lambda^P + w_\mu^P$. Note also that $d_{\mu,1}^{P'} = d_\mu^P$ since the $(r,2)$-type P'-paths of maximal q-resistance with respect to the property of having a $1 \to 2$ vertex change corresponding to the last generator in $\sigma_2^{c_p}$ are of the form Y^{c_p}. As in **Case 1**, we will split the argument into subcases according to the value of c_{p-1}.

Subcase 1 $(c_{p-1} \geq 2)$

Let us make some preliminary observations, in order to establish the statement. Let us consider the $(r,2)$-type P'-paths of maximal q-resistance with respect to the property of not being an extension of an $(r,3)$-type P_3-path by a $3 \to 2$ vertex change corresponding to a generator within $\sigma_2^{c_p}$. Since $d_\lambda^P < d_\mu^P$, such $(r,2)$-type P'-paths are either of the form Y^1 if $d_\mu^P > d_\nu^P + 2$ or of the form $Z^{1,1}$ if $d_\mu^P < d_\nu^P + 2$. If $d_\mu^P = d_\nu^P + 2$, then such $(r,2)$-type P'-paths are either of the form Y^1 or $Z^{1,1}$, since μ_0^P and ν_0^P are non-zero integers with the opposite sign (P is regular). We will now show that these observations establish the statement.

We have already observed in **Case 1** that $\beta_4(P')$ is not a diagonal matrix if $c_{p-1} \geq 2$. Let us establish that P' is regular. If $c_p = 2$, then the good $(r,2)$-type P'-paths all have the indicated property. We deduce that $d_{\lambda,2}^{P'} \geq d_{\mu,1}^{P'}+1$ and P' is regular if $c_p = 2$. On the other hand, if $c_p \geq 3$ and $d_{\lambda,2}^{P'} \leq \max\{d_{\mu,1}^{P'}, d_{\mu,3}^{P'}\}$, then the q-resistance of W (an $(r,3)$-type P_3-path of maximal q-resistance) is at least $d_{\mu,1}^{P'}$. We deduce that $d_{\mu,3}^{P'} \geq d_{\mu,1}^{P'} + 1$ and P' is regular if $c_p \geq 3$.

Subcase 2 $(c_{p-1} = 1)$

We have already observed in **Case 1** that $\beta_4(P')$ is not a diagonal matrix if $c_{p-1} = 1$. We claim that $d_{\lambda,2}^{P'} > \max\{d_{\mu,1}^{P'}, d_{\mu,3}^{P'}\}$. Of course, the claim implies that P' vacuously satisfies Definition 4.1.4 **(a) (i)** (the definition of regularity).

Let us prove the claim. Since $d_\lambda^P < d_\mu^P$ and $c_{p-1} = 1$, the $(r,2)$-type P'-paths of maximal q-resistance are either of the form Y^1 if $d_\mu^P > d_\nu^P + 2$, or of the form $Z^{1,1}$ if $d_\nu^P + 2 > d_\mu^P$. If $d_\mu^P = d_\nu^P + 2$, then the $(r,2)$-type P'-paths of maximal q-resistance are either of the form Y^1 or $Z^{1,1}$, since μ_0^P and ν_0^P are non-zero integers with the opposite sign (P is regular).

Finally, the $(r,2)$-type P'-paths of maximal q-resistance with a $1 \to 2$ vertex change corresponding to the last generator in $\sigma_2^{c_p}$ are of the form Y^{c_p} since $d_\lambda^P < d_\mu^P$. Similarly, the $(r,2)$-type P'-paths of maximal q-resistance with a $3 \to 2$ vertex change corresponding to the last generator in $\sigma_2^{c_p}$ are of the form X^{1,c_p}. The claim follows and the statement is established.

We have considered all possibilities and the statement is established in general. $\square$

We combine Lemma 4.1.12 and Lemma 4.1.13. In particular, we conclude that regularity is an inductive property past blocks.

Corollary 4.1.14. *Let B be a generic block and let P be the s-subproduct immediately preceding B. If P is regular, then B is regular (i.e., PB is regular) and $\beta_4(PB)$ is not a diagonal matrix.*

Proof. The statement is equivalent to the combination of Lemma 4.1.12 and Lemma 4.1.13. $\square$

Let us summarize the results thus far.

Corollary 4.1.15. *If $\beta_4(\sigma)$ is a diagonal matrix, then there does not exist a regular generic block in the minimal form of σ.*

Proof. Let B be a regular generic block in the minimal form of σ and let P be the s-subproduct immediately preceding B. The statement follows from repeated applications of Corollary 4.1.11 and Corollary 4.1.14. Indeed, there is a finite chain $PB = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_k = \sigma$ with the property that $P_i \setminus P_{i-1}$ is a road for i odd and $P_i \setminus P_{i-1}$ is a generic block for i even. (An initial block or a singular block cannot succeed a generic block.) $\square$

The next goal is to show that the first generic block in the minimal form of σ (if it exists) is regular (Lemma 4.1.18). Of course, in light of Corollary 4.1.15, this will imply that there is no generic block in the minimal form of σ, if $\beta_4(\sigma)$ is a diagonal matrix. Firstly, we will establish this statement if there is no singular block in the minimal form of σ (Corollary 4.1.17). We begin with the following important observation concerning

P-paths, where P is the s-subproduct preceding the first non-initial block in the minimal form of σ.

Lemma 4.1.16. *Let P be the s-subproduct preceding the first non-initial block in the minimal form of σ (if it exists). Let us consider (r,s)-type P-paths. The following statements are true.*

(i) *If $r \in \{1,3\}$, then either $d_{\lambda}^{P} > d_{\mu}^{P}$, or there is a unique (r,s)-type P-path and it is bad. If the latter, then $w_{\nu}^{P} = 0$.*

(ii) *If $r = 2$ and $d_{\lambda}^{P} \leq d_{\mu}^{P}$, then $w_{\nu}^{P} = 0$. Furthermore, if $w_{\lambda}^{P} \neq 0$ in this case, then there is an initial block in the minimal form of σ.*

In particular, P is regular.

Proof. The statement is a consequence of repeated applications of Lemma 3.3.3. □

We now establish that the first non-initial block in the minimal form of σ (if it exists) is a singular block, if $\beta_4(\sigma)$ is a diagonal matrix. Of course, this is a necessary step in showing that $\sigma = \Delta^{2n}$ since the statement is true for Δ^{2n}. In fact, there are no generic blocks in Δ^{2n} and shortly we will show that this is also the case for σ, if $\beta_4(\sigma)$ is a diagonal matrix (Corollary 4.1.19).

Corollary 4.1.17. *If B is the first non-initial block in the minimal form of σ and if B is not singular, then B is regular. In particular, if $\beta_4(\sigma)$ is not a diagonal matrix, then the first non-initial block in the minimal form of σ is singular (if it exists).*

Proof. Of course, if B is neither initial nor singular, then B is generic. The first statement follows from Lemma 4.1.14 and Lemma 4.1.16. The second statement follows from the first statement and Corollary 4.1.15. □

Note that a singular block is not necessarily regular. (In fact, Δ^{2n} has a singular block, and this is not regular in light of Corollary 4.1.11, since $\beta_4\left(\Delta^{2n}\right)$ is a diagonal matrix.) However, in the following Lemma 4.1.18, we will show that the first generic block in the minimal form of σ is regular. Note that it is crucial that a singular block occurs at most once in the minimal form of σ, as we can exploit the strong constraints on P-paths in the statement of Lemma 4.1.16. Indeed, the regularity of P alone is insufficient.

Lemma 4.1.18. *Let B be a singular block in the minimal form of σ. In particular, B is either a 2-block of the form $\sigma_1^{a_p}\sigma_3\sigma_2\sigma_1^{a_{p+1}}$ or a 3-block of the form $\sigma_2^{c_{p-1}}\sigma_3\sigma_2$. Let P be the s-subproduct preceding B. If B' is the first generic block succeeding B, then B' is regular. On the other hand, if B is the final block in the minimal form of σ, then the following statements are true:*

(i) *If $r = 1$ and either $d_\lambda^P > d_\mu^P$, or $a_p > 1$ (resp. $c_{p-1} > 1$) if B is a 2-block (resp. 3-block), then there are multiple non-zero entries in the first row of $\beta_4(\sigma)$.*

(ii) *If $r = 2$, then there are multiple non-zero entries in the second row of $\beta_4(\sigma)$ if $d_\lambda^P \neq d_\mu^P$.*

Proof. Lemma 2.2.3 **(iv)** implies that $b_i = 0$ for $i < p$. If B is not the final block in the minimal form of σ, then Lemma 2.2.3 also implies that there exists a minimal $p' > p$ with $c_{p'} \geq 2$, such that B' succeeds $P' = \prod_{i=1}^{p'} \sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$. In this case, P' is regular and Corollary 4.1.11 implies that B' is regular.

On the other hand, if B is the final block in the minimal form of σ, then Lemma 3.3.3 implies statements **(i)** and **(ii)**. □

We summarize the results of this section, with the following constraints on the product decomposition of σ into blocks and roads, if $\beta_4(\sigma)$ is a diagonal matrix.

Corollary 4.1.19. *If $\beta_4(\sigma)$ is a diagonal matrix, then there are at most two blocks in the minimal form of σ. Moreover, the first block (if it exists) is initial and the second* *block (if it exists) is singular. Finally, the singular block is either a 2-block of the form $\sigma_1\sigma_3\sigma_2\sigma_1^{a_{p+1}}$ or a 3-block of the form $\sigma_2\sigma_3\sigma_2$.*

Proof. The statements follow from Corollary 4.1.15, Lemma 4.1.16, Corollary 4.1.17, and Lemma 4.1.18. □

4.2 Conclusion of the proof

Finally, we prove Theorem 2.3.2, thus establishing Theorem 1.1.2 of the Introduction. The first step is to improve on Corollary 4.1.19 at the end of Section 4.1 (the previous section) and determine more precise information on the product decomposition of σ into blocks and roads if $\beta_4(\sigma)$ is a diagonal matrix.

Lemma 4.2.1. *If $\beta_4(\sigma)$ is a diagonal matrix and $\sigma \neq e, \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$ (e denotes the identity element of B_4), then the minimal form of σ admits the product decomposition $\sigma = B_0R_0B_1R_1$, where $B_0 = \sigma_1^{a_1}\sigma_2\sigma_1^2$ is the initial block, $R_0 = \sigma_2^2\sigma_1^2\cdots$ (the σ_2^2s and σ_1^2s alternate) is the first road, and B_1 is either a singular 2-block of the form $\sigma_1\sigma_3\sigma_2\sigma_1^{a_{p_1+1}}$ or a singular 3-block of the form $\sigma_2\sigma_3\sigma_2$.*

Proof. If the first non-initial block B_1 in the minimal form of σ exists, then Corollary 4.1.19 implies the constraints on the blocks in the statement. We will establish the existence of a singular block in the minimal form of σ and constrain the first road R_0 with the same reasoning. Indeed, there is precisely one non-zero entry in the first row of $\beta_4(\sigma)$. In particular, there is a unique $(1, s)$-type σ-path if there is no non-initial block in the minimal form of σ. On the other hand, if there is a non-initial block B_1 in the minimal form of σ, and if P_1 is the s_1-subproduct immediately preceding B_1, then there is a unique $(1, s_1)$-type P_1-path by Lemma 4.1.18.

Let us consider the beginning $\sigma_1^{a_1}\sigma_3^{b_1}\sigma_2^{c_1}\sigma_1^{a_2}\sigma_3^{b_2}$ of the minimal form of σ. The $(1, 2)$-type $\sigma_1^{a_1}\sigma_3^{b_1}\sigma_2^{c_1}$-path of maximal q-resistance is good if either $c_1 \geq 2$ or $a_2 = 0$. Of course, this contradicts the statement that there is a unique $(1, s_1)$-type P_1-path/$(1, s)$-type σ-path. We deduce that $c_1 = 1$ and $a_2 > 0$. Moreover, if $b_1 > 0$, then $b_2 = 0$ by Lemma 2.2.3 **(ii)**, there is no block in the minimal form of σ by Lemma 2.2.3 **(iv)** (a σ_3 cannot precede a singular block in the minimal form of σ), and the $(3, 2)$-type $\sigma_1^{a_1}\sigma_3^{b_1}\sigma_2^{c_1}$-path of maximal q-resistance is good. In this case, there are multiple non-zero entries in the third row of $\beta_4(\sigma)$. Therefore, $b_1 = 0$ and the beginning of the minimal form of σ is $\sigma_1^{a_1}\sigma_2\sigma_1^{a_2}\sigma_3^{b_2}$.

Similarly, if $b_2 > 0$, then there is no block in the minimal form of σ by Lemma 2.2.3 **(iv)**. Moreover, $a_2 = 1 = b_2$ since there is a unique $(1, s)$-type σ-path. We now consider cases according to either $c_2 \geq 2$ or $c_2 = 1$.

If $c_2 \geq 2$, then $c_2 = 2$ and the $(1, 2)$-type $\prod_{i=1}^{2}\sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$-path of maximal q-resistance must admit a $3 \to 2$ vertex change corresponding to the last σ_2 in $\sigma_2^{c_2}$. Similarly, the $(3, 2)$-type $\prod_{i=1}^{2}\sigma_1^{a_i}\sigma_3^{b_i}\sigma_2^{c_i}$-path of maximal q-resistance must admit a $1 \to 2$ vertex change corresponding to the last σ_2 in $\sigma_2^{c_2}$. In particular, unless $\sigma = \sigma_1^{a_1}\sigma_2\sigma_1\sigma_3\sigma_2^2\sigma_1\sigma_3\sigma_2^2\cdots$, there are multiple non-zero entries in either the first row or the third row of $\beta_4(\sigma)$. However, in this case, there are also multiple non-zero entries in either the first row or the third row of $\beta_4(\sigma)$ by examination of the possibilities for the ending of the minimal form of σ.

Indeed, if the minimal form of σ ends in σ_2^2 or $\sigma_1\sigma_3$, then there are multiple non-zero entries in each of the first and the third rows of $\beta_4(\sigma)$. If the minimal form of σ ends in either σ_1 or σ_3, then there are multiple non-zero entries in precisely one of the first row or the third row of $\beta_4(\sigma)$.

If $c_2 = 1$, then consider the good $(3, 1)$-type $\sigma_1^{a_1}\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$-path defined by a first vertex change corresponding to the first σ_2 and a second vertex change corresponding to $\sigma_1 = \sigma_1^{a_2}$ (the path is admissible by a variant of Figure 3.3 in Chapter 3). In particular, there are multiple non-zero entries in the third row of $\beta_4(\sigma)$, unless $\sigma = \sigma_1^{a_1}\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$. However, in this case, there are multiple non-zero entries in the second row of σ, unless

$a_1 = 1$. Since $\sigma \neq \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$ by hypothesis, we conclude that $b_2 = 0$ and the beginning of the minimal form of σ is $\sigma_1^{a_1}\sigma_2\sigma_1^{a_2}$.

We observe that the $(3,2)$-type $\sigma_1^{a_1}\sigma_2$ path defined by a $3 \to 2$ vertex change corresponding to the σ_2 is good. Since the third row of $\beta_4(\sigma)$ has precisely one non-zero entry, we conclude that there exists a non-initial block B_1 in the minimal form of σ. Finally, since there is a unique $(1,s_1)$-type P_1-path (P_1 is the s_1-subproduct immediately preceding the non-initial block B_1), we deduce that $a_2 = 2$ and the first road R_0 in the minimal form of σ is $\sigma_2^2\sigma_1^2\cdots$. □

We now conclude the proof of Theorem 2.3.2 by obtaining constraints on the product decomposition of the minimal form of σ in the statement of Lemma 4.2.1, that correspond to those in the statement of Theorem 2.3.2.

Proof of Theorem 2.3.2. We appeal to Lemma 4.2.1. Firstly, note that $\sigma \neq \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$ since the Burau matrix of the latter braid is not diagonal (the unique admissible $(1,3)$-type $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$-path is $1,1,2,2,3,3,3$). In the statement of Lemma 4.2.1, it suffices to determine the length of R_0, the precise form of B_1 (out of the two possibilities), and R_1. Let P_1 be the s_1-subproduct immediately preceding B_1. Let us write either $B_1 = \sigma_1\sigma_3\sigma_2\sigma_1^{a_{p_1+1}}$ if it is a 2-block, or $B_1 = \sigma_2\sigma_3\sigma_2^{c_{p_1}}$ with $c_{p_1} = 1$ if it is a 3-block.

Firstly, we determine the form of R_1. Of course, there is no block after B_1 in the minimal form of σ. Since the first row of $\beta_4(\sigma)$ has precisely one non-zero entry, this implies that there is a unique extension of the $(1,s_1)$-type P_1-path to a σ-path. Furthermore, Lemma 4.1.18 implies that the q-resistance of the good $(2,s_1)$-type P_1-path equals the q-resistance of the maximal q-resistance bad $(2,s_1)$-type P_1-path. Since the second row of $\beta_4(\sigma)$ has precisely one non-zero entry, this implies that there is a unique extension of the good $(2,s_1)$-type P_1-path to a σ-path.

We deduce that $a_p \leq 2$ and $b_p \leq 2$ for all $p > p_1$. Moreover, $c_p = 1$ for all $p > p_1$. If not and $c_p \geq 2$ for some $p > p_1$, then a maximal q-resistance extension to a $\prod_{i=1}^{p} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}$-path of one of either the $(1, s_1)$-type P_1-path or the good $(2, s_1)$-type P_1-path would have a vertex change corresponding to the first σ_2 in $\sigma_2^{c_p}$. In particular, either a $(1,2)$-type or a $(2,2)$-type $\prod_{i=1}^{p} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}$-path of maximal q-resistance is good if $c_p \geq 2$. Of course, this contradicts the assumption that there is precisely one non-zero entry in each of the first and the second rows of $\beta_4(\sigma)$.

Therefore, R_1 is an isolated σ_2 subproduct with $a_p \leq 2$ and $b_p \leq 2$ for all $p > p_1$ (note that only $a_n = 1$ or $b_n = 1$ is possible, i.e., only a σ_1 or a σ_3 at the end of the minimal form of σ can have exponent one).

Finally, we will determine the value of a_1, the length of R_0, the length of R_1, and the precise form of B_1. Thus far, we have only used the assumption that there is a single non-zero entry in each row of $\beta_4(\sigma)$. Now, we use the full strength of the hypothesis that $\beta_4(\sigma) = \beta_4(\Delta^{2n})$ by considering the precise values of the entries in $\beta_4(\sigma)$ determined by Lemma 2.2.6.

Firstly, since the $(1,3)$-entry of $\beta_4(\sigma)$ is zero, it follows that R_1 ends in $\cdots \sigma_1^2 \sigma_2 \sigma_3$ and $B_1 = \sigma_2 \sigma_3 \sigma_2$. Let k be the number of σ_2^2s in R_0. We recall that the q-resistance of the maximal q-resistance good $(2, s_1)$-type P_1-path equals the q-resistance of the maximal q-resistance bad $(2, s_1)$-type P_1-path. The q-resistance of the former is $3 + 3k$ and the q-resistance of the latter is $a_1 + k + 1$. Therefore, $a_1 = 2 + 2k$.

Let $2l + 1$ be the number of σ_2s in R_1. The q-resistance of the good $(3,1)$-type $\left(\prod_{i=1}^{n-2} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}\right) \sigma_1^2$-path equals the q-resistance of the maximal q-resistance bad $(3,1)$-type $\left(\prod_{i=1}^{n-2} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}\right) \sigma_1^2$-path, since the $(3,1)$-entry of $\beta_4(\sigma)$ is zero. The former equals to $4 + 4l$ (the good $(3,1)$-type $\left(\prod_{i=1}^{n-2} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}\right) \sigma_1^2$-path is defined by a vertex change corresponding to the first σ_2 in B_1, and subsequently by a vertex change at every possible opportunity). The latter equals $3k + l + 4$ (the maximal q-resistance bad $(3,1)$-type $\left(\prod_{i=1}^{n-2} \sigma_1^{a_i} \sigma_3^{b_i} \sigma_2^{c_i}\right) \sigma_1^2$-path is defined in two stages with a vertex change corresponding to the second σ_2 in B_1). We conclude that $k = l$.

Finally, the $(1,1)$-entry of $\beta_4(\sigma)$ is q^{4n}. The q-resistance of the $(1,1)$-type σ-path is $a_1 + k + l + 2$. We have already established that $a_1 = 2 + 2k$ and we deduce that $4n = 4 + 3k + l$. Since $k = l$, it follows that $4n = 4l + 4$. Therefore, $2n = 2l + 2 = a_1$, the number of σ_2s in R_1 is $n - 1$ (since $2l + 1$ is the number of σ_2s in R_1 by definition), and the number of σ_2^2s in R_0 is $n - 1$. Note that these correspond to the constraints on the minimal form of σ in the statement. $\square$

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