Birational Superrigidity and K-stability

Ziquan Zhuang

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Abstract

We consider two different notions on Fano varieties: birational superrigidity, coming from the study of rationality, and K-stability, which is related to the existence of Kähler-Einstein metrics. In the first part, we show that birationally superrigid Fano varieties are also K-stable as long as their alpha invariants are at least $\frac{1}{2}$, partially confirming a conjecture of Odaka-Okada and Kim-Okada-Won. In the second part, we prove the folklore prediction that smooth Fano complete intersections of Fano index one are birationally superrigid and K-stable when the dimension is large. In the third part, we introduce an inductive argument to study the birational superrigidity and K-stability of singular complete intersections and in particular prove an optimal result on the birational superrigidity and K-stability of hypersurfaces of Fano index one with isolated ordinary singularities in large dimensions. Finally we provide an explicit example to show that in general birational superrigidity is not a locally closed property in families of Fano varieties, giving a negative answer to a question of Corti.

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Contents

Chapter 1

Introduction

1.1 From birational superrigidity to K-stability

One of the main goals of algebraic geometry is to understand the structures of algebraic varieties. According to the minimal model program, we may simplify the question by studying the three building blocks of algebraic varieties: Fano varieties, Calabi-Yau varieties and varieties of general type. In this thesis, we focus on Fano varieties, i.e. projective normal varieties with ample anti-canonical divisor. There are two types of structures we want to investigate on Fano varieties.

The first one is birational (super)rigidity, which concerns the Mori fiber space structure on a given Fano variety. Unlike the case of Calabi-Yau or general type varieties, a Fano variety of Picard number one can be birational to a very different Fano variety or Mori fiber space (another natural output of minimal model programs, see §2.2). It is therefore natural to identify those Fano varieties whose Mori fiber space structure is unique.

Definition 1.1.1. A Fano variety X of dimension at least 2 is said to be birationally rigid (resp. superrigid) if for all birational map $f : X \dashrightarrow Y$, where Y is the source of a Mori fiber space, we have $X \cong Y$ (resp. f is an isomorphism).

Knowing that a Fano variety is birationally (super)rigid has several interesting consequences. For example, such a variety is not birational to the projective space $\mathbb{P}^n$. Historically, this is how Iskovskih and Manin [IM71] came up with their counterexample (smooth quartic threefolds) to the 3-dimensional Lüroth problem.

The other structure we are interested in is the existence of Kähler-Einstein metrics, i.e. Kähler metrics with constant Ricci curvature (for simplicity, assume that the Fano variety is smooth so that metric is defined). Although analytic in nature, the existence of such canonical metrics is expected to be equivalent to the K-polystablity (see §2.3) of the Fano varieties and hence becomes a completely algebraic question. For Fano manifolds this is known as the Yau-Tian-Donaldson conjecture and is now a remarkable theorem of Chen-Donaldson-Sun and Tian.

Theorem 1.1.2 ([CDS15, Tia15]). *A Fano manifold admits Kähler-Einstein metric if and only if it is K-polystable.*

While the notions of birational superrigidity and K-stability have different sources of origin, recent studies in both fields suggest that both notions can be characterized by the singularities of certain anti-canonical $\mathbb{Q}$-divisors and so it becomes very natural to expect some relation between them. Indeed, the slope stability (a weaker notion of K-stability) of birationally superrigid Fano manifolds has been established by [OO13] under some mild assumptions, and it is conjectured [OO13, KOW17] that birationally rigid Fano varieties are always K-stable.

In the third chapter of the thesis (based on joint work with Charlie Stibitz [SZ18]), we study this conjecture using Tian's alpha invariants of Fano varieties:

Definition 1.1.3 ([Tia87, CS08]). The alpha invariant $\alpha(X)$ of a $\mathbb{Q}$-Fano variety (i.e. Fano variety with klt singularities) X is defined as the supremum of all $t > 0$ such that (X, tD) is log canonical for every effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$.

Theorem 1.1.4 (with C. Stibitz). *Let X be a Fano variety. If X is birationally superrigid and $\alpha(X) \geq \frac{1}{2}$, then X is K-stable.*

This gives a partial answer to the conjecture of Odaka-Okada and Kim-Okada-Won. As immediate applications, we reprove a result of K. Fujita [Fuj16a] on the K-stability of smooth hypersurfaces of degree n in $\mathbb{P}^n$.

1.2 Complete intersections

In general, it is a hard question to verify the birational superrigidity and K-stability of a given Fano variety, even for smooth Fano complete intersections. It is well known that $\mathbb{P}^n$ and smooth hyperquadrics admit Kähler-Einstein metrics and hence are K-polystable. It is also easy to see that complete intersections of Fano index (i.e. the largest integer s such that the anti-canonical divisor is divisible by s in the Picard group) at least 2 are never birationally rigid: a general pencil of hyperplane sections gives rise to a different Mori fiber space structure. When the Fano index equals 1, there is, however, some subtlety in dimension three: while the smooth quartic threefolds are birationally superrigid [IM71], a general complete intersection of a quadric and a cubic is only birationally rigid [IP96], and the complete intersections of three quadrics in $\mathbb{P}^6$ are never birationally rigid (they admit conic bundle structures by [Bea77, Chapter VI]). In the remaining cases, it is a folklore conjecture that everything is as nice as one expects.

Conjecture 1.2.1. *Every smooth Fano complete intersection $X \subseteq \mathbb{P}^n$ of degree ≥ 3 is K-stable while those of Fano index 1 and dimension ≥ 4 are birationally superrigid.*

This conjecture is known in the following cases: for smooth hypersurfaces of Fano index 1, the K-stability is proved recently by K. Fujita [Fuj16a] based on the result of [Che01] while the birational superrigidity (in dimension at least 3) is settled by de Fernex [dF16] (which in turn builds on the work of many others [IM71, Puk86, Cor95, Che00, dFEM03]) and his argument is later generalized by Suzuki [Suz17] to certain families of complete intersections; K-stability of the intersection of two hyperquadrics is treated in [AGP06] and more recently [LX17b] proves the K-stability of cubic threefolds; finally if we only consider general members of complete intersections of Fano index 1 with given degrees and dimension, then with a few exceptions, their birational superrigidity and K-stability are established by [Puk01, Puk13a, Puk14] and [Puk10, EP16, Puk17a].

In the fourth chapter of this thesis (based on the preprint [Zhu18b]), we study this conjecture for (smooth) complete intersection of Fano index 1 and large dimension. In particular, we prove the following statement.

Theorem 1.2.2. *Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Fano complete intersection of index* 1*, codimension r and dimension $n \geq 10r$, then X is birationally superrigid and K-stable.*

In the fifth chapter of the thesis (based on joint work with Yuchen Liu [LZ18] and the preprint [Zhu19]), we further study the case of singular complete intersections of Fano index 1. One of the main result of this chapter is an optimal statement (in large dimensions) on the birational superrigidity and K-stability of index 1 hypersurfaces with only isolated ordinary singularities, generalizing earlier work of [Puk02c, dF17].

Theorem 1.2.3 (with Y. Liu)**.** *Let $X \subseteq \mathbb{P}^{n+1}$ be a hypersurface of degree $n+1$ and dimension $n \geq 250$ with only isolated ordinary singularities* (*i.e. the projective tangent cones are smooth*) *of multiplicities at most m. Then*

1. *X is K-stable if $m \leq n$;*
2. *X is birationally superrigid if $m \leq n-2$;*
3. *X is birationally rigid if $m \leq n-1$. Moreover, linear projection from each point $x \in X$ of multiplicity $n-1$ induces a birational involution τ_x and the birational automorphism group* $\mathrm{Bir}(X)$ *of X is generated by* $\mathrm{Aut}(X)$ *together with these τ_x.*

1.3 Moduli problems

Since birational superrigidity and K-stability both identify certain special classes of Fano varieties, a natural and interesting question is whether the set of such Fano varieties forms a "nice" moduli. Indeed, it is well expected that the moduli functor of n-dimensional K-semistable Fano varieties is represented by an Artin stack $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ of finite type that admits a projective good moduli space $\mathcal{M}^{\mathrm{Kss}}_{n,v} \to M^{\mathrm{Kps}}_{n,v}$ (in the sense of [Alp13]), whose closed points are in bijection with n-dimensional K-polystable Fano varieties of volume v. There has been a significant amount of progress on the construction of such K-moduli [OSS16a, LWX14, LWX18, Oda15, SSY16, Jia17, CP18, BL18b, BX18]. On the other hand, very little is known about the moduli problem for birationally superrigid Fano varieties. In the last chapter of this thesis (based on [SZ18] and [Zhu19]), we investigate the existence of such moduli.

It follows from the definition that if X is birationally superrigid, then it has terminal singularities and therefore by the seminal work of Birkar [Bir16a, Bir16b] on the Borisov-Alexeev-Borisov conjecture, birationally superrigid Fano varieties belong to a bounded family. It is also not hard to see that birationally superrigid Fano varieties have finite automorphism group. Therefore, in view of the conjectural K-stability of such varieties and the K-moduli conjecture, it seems natural to expect that the moduli functor of n-dimensional birationally superrigid Fano varieties is represented by a Deligne-Mumford stack.

In order for this to be true, the first thing to check is that such moduli, if exists, is separated. We verify this by proving a valuative criterion of the moduli functor (this is obtained in joint work with Charlie Stibitz [SZ18]).

Theorem 1.3.1 (with C. Stibitz)**.** *Let $f : X \to C$, $g : Y \to C$ be two flat families of Fano varieties (i.e. all geometric fibers are integral, normal and Fano) over a smooth pointed curve $0 \in C$. Assume that the central fibers $X_0 = f^{-1}(0)$ and $Y_0 = g^{-1}(0)$ are birationally superrigid and there exists an isomorphism $\rho : X\backslash X_0 \cong Y\backslash Y_0$ over the punctured curve $C\backslash 0$. Then ρ extends to an isomorphism $X \cong Y$ over C.*

The next thing to check is that birational superrigidity is a locally closed property. Unfortunately, this is where things break down. In the last chapter of the thesis, we construct examples to show that birational superrigidity is not a locally closed property in general. We note that a similar but stronger statement (with locally closedness replaced by openness) for birational rigidity has been conjectured by Corti [Cor00, Conjecture 1.4] and disproved in dimension 3 by [CG17]; there are also 3-dimensional counterexamples [CP17, Example 6.3] to the openness of birational superrigidity, although in all these examples, birational (super)rigidity turns out to be a locally closed property in the corresponding moduli.

Our construction is based on the degeneration of hypersurfaces into double covers. Let m be a sufficiently large integer and let $x_0, \cdots, x_{n+1}, y$ be the weighted homogeneous coordinates of $\mathbb{P}(1^{n+2}, m)$. Let f_s, g_s (parameterized by $s \in \mathbb{A}^1$) be homogeneous polynomials in $x_0, \cdots, x_{n+1}$ of degree $2m$ and m respectively so that

$$\mathcal{X} = (y^2 - f_s = ty - g_s = 0) \subseteq \mathbb{P}(1^{n+2}, m) \times \mathbb{A}^2_{s,t}$$

defines a family of weighted complete intersections of dimension n parameterized by $\mathbb{A}^2$. For $t \neq 0$, it is easy to see that $\mathcal{X}_{s,t} \cong (t^2 f_s - g_s = 0) \subseteq \mathbb{P}^{n+1}$ is a hypersurface of degree $2m$ while $\mathcal{X}_{s,0}$ is the double cover of the hypersurface $G_s = (g_s = 0) \subseteq \mathbb{P}^{n+1}$ branched over the divisor $F_s \cap G_s$ where $F_s = (f_s = 0)$. We show that with suitable choices of f_s and g_s, this provides the counterexample we want in every sufficiently large odd dimension.

Theorem 1.3.2. *Notation as above. Let $x \in \mathbb{P}^{n+1}$. Assume that $n = 2m - 1$, $m \gg 0$ and the following:*

1. *F_0 and G_0 have a unique ordinary singularity at x with $\mathrm{mult}_x F_0 = 2m - 2$ and $\mathrm{mult}_x G_0 = m - 1$ and are otherwise smooth,*
2. *the projective tangent cone of $F_0 \cap G_0$ at x is a smooth complete intersection,*
3. *$\mathcal{X}_{s,t}$ is smooth when $s \neq 0$.*

Then $\mathcal{X}_{s,t}$ is birationally superrigid if and only if $s \neq 0$ or $(s,t) = (0,0)$.

We remark that it is proved by Shokurov and Choi [SC11] that birational superrigidity is a constructible condition (we will present their proof at the end of the thesis). Therefore, constructibility is the best we can hope for according to the above example.

Chapter 2

Preliminaries

2.1 Notation and conventions

Throughout, we work over the field $\mathbb{C}$ of complex numbers (or any algebraically closed field k of characteristic zero). By a *variety*, we mean an integral, separated scheme of finite type over the base field. Unless otherwise specified, all varieties are assumed to be normal and divisors are understood as $\mathbb{Q}$-divisor. A *pair* (X, D) consists of a variety X and an effective divisor $D \subseteq X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. We will follow the terminology of [KM98] concerning singularities of pairs. A variety X is said to be $\mathbb{Q}$-*Fano* (resp. weak Fano) if X is projective, $-K_X$ is $\mathbb{Q}$-Cartier and ample (resp. nef and big) and X has klt singularities. A pair (X, Δ) is *log Fano* if X is projective, $-(K_X + \Delta)$ is $\mathbb{Q}$-Cartier ample and (X, Δ) is klt. Let (X, Δ) be a pair and D a $\mathbb{Q}$-Cartier divisor on X, the *log canonical threshold*, denoted by $\mathrm{lct}(X, \Delta; D)$ (or simply $\mathrm{lct}(X; D)$ when $\Delta = 0$), of D with respect to (X, Δ) is the largest number t such that $(X, \Delta + tD)$ is log canonical. Similarly, the notation $\mathrm{lct}(X, \Delta; |D|_{\mathbb{Q}})$ (so called global log canonical threshold) stands for the infimum of $\mathrm{lct}(X, \Delta; D')$ among all $D' \sim_{\mathbb{Q}} D$ while $\mathrm{lct}(X, \Delta; Z)$ refers to the log canonical threshold of (the ideal sheaf of) a subscheme $Z \subseteq X$.

2.2 Birational (super)rigidity

Central to the definition of birational (super)rigidity is the notion of Mori fiber space.

Definition 2.2.1. Let $f : X \to Y$ be a dominant projective morphism between normal varieties. It is called a *Mori fiber space* if

1. $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\dim Y < \dim X$;
2. X has $\mathbb{Q}$-factorial terminal singularities;
3. $-K_X$ is f-ample and $\rho(X/Y) := \rho(X) - \rho(Y) = 1$.

Recall that birational (super)rigidity is defined in Definition 1.1.1. This short definition actually contains more information. If X is a Fano variety and let $\widetilde{X} \to X$ be a resolution of singularity, then $-K_{\widetilde{X}}$ is not pseudo-effective; by [BCHM10, Corollary 1.3.3], we can run a $K_{\widetilde{X}}$-MMP $\widetilde{X} \dashrightarrow X_1$ and end with a Mori fiber space $X_1 \to Y$. Therefore, if X is birationally rigid, then by definition we have $X \cong X_1$ and in particular X has $\mathbb{Q}$-factorial terminal singularities. If in addition X is birationally superrigid, one can say a bit more:

Proposition 2.2.2. *Let X be a birationally superrigid Fano variety, then X has $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$.*

Proof. By the previous discussion, X has $\mathbb{Q}$-factorial terminal singularities and admits a Mori fiber space structure $f : X \to Y$. It remains to show that X has Picard number one; in other words, Y is a point. If $\dim Y \geq 2$, let $Y' \to Y$ be the blowup of a general point on Y and let $X' = X \times_Y Y'$; then $X' \to Y'$ is also a Mori fiber space and X' is birational but not isomorphic to X, contradicting the definition of birational superrigidity. Thus $\dim Y \leq 1$. If Y is a curve, then $\rho(X) = \rho(X/Y) + \rho(Y) = 2$ and the Mori cone $\overline{\mathrm{NE}}(X)$ of X is generated by a curve in the fiber of f and another extremal ray R. Since X is terminal Fano, it is a Mori dream space by [BCHM10, Corollary 1.3.2] and we can run a K_X-MMP $X \dashrightarrow X'$ starting with the contraction of R and end with a Mori fiber space $X' \to Y'$. But since X is birationally superrigid, there are no divisorial contractions or flips in this MMP and hence the contraction of R itself induces a Mori fiber space $f' : X \to Y'$. Let F (resp. F') be a general fiber of f (resp. f'). If $\dim X \geq 3$, then as X has $\mathbb{Q}$-factorial singularities, $F \cap F'$ contains a curve C. But then by our construction, the class of C is contained in both extremal rays of $\overline{\mathrm{NE}}(X)$, a contradiction. Hence X is a smooth Fano surface and in particular is rational. But any rational variety is not birationally superrigid ($\mathbb{P}^1$ is excluded by definition, while $\mathbb{P}^n$ $(n \geq 2)$ is birational to $\mathbb{P}^k \times \mathbb{P}^{n-k}$ for any $1 \leq k \leq n-1$), again a contradiction. Thus Y must be a point and $\rho(X) = 1$. □

Remark 2.2.3. The usual definition of birational superrigidity (see e.g. [CS08, Definition 1.25]) requires the Fano variety to have $\mathbb{Q}$-factorial terminal singularities and Picard number one, thus the above proposition shows that our definition is equivalent to the usual one.

In practice, it is not straightforward to verify the birational superrigidity of a Fano variety from the definition, so we will frequently use the following criterion (known as Noether-Fano inequality).

Definition 2.2.4. Let (X, D) be a pair. A movable boundary on X is defined as an expression of the form $a\mathcal{M}$ where $a \in \mathbb{Q}$ and $\mathcal{M}$ is a movable linear system on X. Its $\mathbb{Q}$-linear equivalence class is defined in an evident way. If $M = a\mathcal{M}$ is a movable boundary, we say that the pair $(X, D + M)$ is klt (resp. canonical, lc) if for $k \gg 0$ and for general members $D_1, \cdots, D_k$ of the linear system $\mathcal{M}$, the pair $(X, D+M_k)$ (where $M_k = \frac{a}{k}\sum_{i=1}^k D_i$) is klt (resp. canonical, lc) in the usual sense (alternatively, it can also be defined via the singularity type of $(X, D; \mathfrak{b}^a)$ where $\mathfrak{b}$ is the base ideal of $\mathcal{M}$). For simplicity, we usually do not distinguish the movable boundary M and the actually divisor M_k for suitable k.

Theorem 2.2.5 ([CS08, Theorem 1.26])**.** *Let X be a Fano variety. Then it is birationally superrigid if and only if it has $\mathbb{Q}$-factorial terminal singularities, Picard number one, and for every movable boundary $M \sim_{\mathbb{Q}} -K_X$ on X, the pair (X, M) has canonical singularities.*

One consequence of this criterion is an equivalent formulation of birational superrigidity.

Proposition 2.2.6. *Let X be a Fano variety. Then X is birationally superrigid if and only if the following conditions are satisfied:*

1. *X has $\mathbb{Q}$-factorial terminal singularities and $\rho(X) = 1$;*
2. *X is not birational to any other terminal Fano variety;*
3. *there is no rational dominant map $f : X \dashrightarrow Y$ with uniruled fibers such that* $\dim X > \dim Y > 0$*;*

4. the groups Bir(X) *and* Aut(X) *coincide.*

Proof. The "if" part is obvious, so we only prove the "only if" part. Condition (1) follows from Proposition 2.2.2 while condition (4) is part of the definition. If $f : X \dashrightarrow Y$ is as in (3), let $X_1 \to X$ be a birational morphism that resolves f (i.e. it induces a morphism $f_1 : X_1 \to Y$) such that X_1 is smooth, then since f_1 has uniruled fibers, K_{X_1} is not f_1-pseudo-effective by [BDPP13], hence by [BCHM10, Corollary 1.3.3], we may run a K_{X_1}-MMP $X_1 \dashrightarrow X_2$ over Y and end with a Mori fiber space $X_2 \to W$ over Y. As $\dim W \geq \dim Y > 0$, we have $\rho(X_2) = \rho(X_2/W) + \rho(W) \geq 2$; but since $\rho(X) = 1$ and X is birational to X_2, this is impossible by the definition of birational superrigidity.

It remains to prove condition (2), which essentially uses the proof of Theorem 2.2.5. Suppose that $f : X \dashrightarrow Y$ is a birational map between terminal Fano varieties. Let m be a sufficiently large and divisible integer and let $\mathcal{M}_Y = | - mK_Y|$ be the complete linear system. Let $\mathcal{M}_X$ be its strict transform on X, then $\mathcal{M}_X$ is a movable linear system. Choose $c > 0$ such that $K_X + c\mathcal{M}_X \sim_{\mathbb{Q}} 0$. Let

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{p} & & \searrow^{q} & \\ X & & \overset{f}{\dashrightarrow} & & Y \end{array}$$

be a common log resolution of $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ and denote by $\mathcal{M}_W$ the strict transform of $\mathcal{M}_Y$. We may write

$$K_W + c\mathcal{M}_W \sim_{\mathbb{Q}} p^*(K_X + c\mathcal{M}_X) + E_X \sim_{\mathbb{Q}} q^*(K_Y + c\mathcal{M}_Y) + E_Y \qquad (2.1)$$

where E_X (resp. E_Y) is p-exceptional (resp. q-exceptional). By Theorem 2.2.5, $(X, c\mathcal{M}_X)$ has canonical singularities, thus $E_X \geq 0$ and

$$K_W + c\mathcal{M}_W \sim_{\mathbb{Q}} p^*(K_X + c\mathcal{M}_X) + E_X \sim_{\mathbb{Q}} E_X$$

has Kodaira dimension zero. Since Y has terminal singularities and $\mathcal{M}_Y$ is base point free, we also have $E_Y \geq 0$ and thus $\kappa(K_Y + c\mathcal{M}_Y) = \kappa(K_W + c\mathcal{M}_W) = 0$. Recall that

$\mathcal{M}_Y \sim -mK_Y$ and $-K_Y$ is ample, we see that $K_Y + c\mathcal{M}_Y \sim_{\mathbb{Q}} 0$ and by (2.1) we obtain $E_X \sim_{\mathbb{Q}} E_Y$. Both divisors have Kodaira dimension zero, so in fact $E_X = E_Y$. As Y has terminal singularities, every q-exceptional divisor has positive coefficient in E_Y and the above equality implies that every q-exceptional divisor is also p-exceptional. In particular, f is a birational contraction (i.e. f^{-1} has no exceptional divisor). But since $\rho(X) = 1$ and X has $\mathbb{Q}$-factorial singularities, f is a small birational map; therefore, Y has class number one and is a Mori fiber space itself. Thus f is an isomorphism by the definition of birational superrigidity. □

Also related to the Noether-Fano inequality is the higher codimensional alpha invariants of Fano varieties [Zhu18a].

Definition 2.2.7. Let (X, Δ) be a log Fano pair and let $L = -(K_X + \Delta)$. Let $1 \le k \le n = \dim X$ be an integer. We define the alpha invariant of codimension k for (X, Δ) to be

$$\alpha^{(k)}(X, \Delta) = \inf \alpha_m^{(k)}(X, \Delta) \tag{2.2}$$

where $\alpha_m^{(k)}(X, \Delta)$, the m-th alpha invariant of codimension k, is defined as the infimum of $\mathrm{lct}(X, \Delta; \frac{1}{m}\mathcal{M})$ over all integer $m > 0$ such that mL is Cartier and all linear series $\mathcal{M} \subseteq |mL|$ whose base locus has codimension at least k.

In particular, when $k = 1$ this reduces to Tian's alpha invariant [Tia87, §5] and when $\Delta = 0$ this reduces to [Zhu18a, Definition 1.1] and in this case we denote $\alpha^{(k)}(X, \Delta)$ by $\alpha^{(k)}(X)$. Note that the definition makes sense for any klt pair (X, Δ) and any ample $\mathbb{Q}$-Cartier divisor L, so we can define $\alpha^{(k)}(X, \Delta; L)$ in a similar fashion. This invariant will be further studied in the last chapter.

2.3 K-stability

We first recall the definition of K-stability, originally introduced by [Tia97, Don02].

Definition 2.3.1 ([Tia97, Don02, LX14, Li15, OS15])**.** Let (X, Δ) be an n-dimensional log Fano pair. Let L be an ample line bundle on X such that $L \sim_{\mathbb{Q}} -l(K_X + \Delta)$ for some $l \in \mathbb{Q}_{>0}$.

1. A *normal test configuration* $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})/\mathbb{A}^1$ of $(X, \Delta; L)$ consists of the following data:

   - a normal variety $\mathcal{X}$, an effective $\mathbb{Q}$-divisor Δ_{tc} on $\mathcal{X}$, together with a flat projective morphism $\pi : (\mathcal{X}, \mathrm{Supp}\ (\Delta_{\mathrm{tc}})) \to \mathbb{A}^1$;
   - a π-ample line bundle $\mathcal{L}$ on $\mathcal{X}$;
   - a $\mathbb{G}_m$-action on $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ such that π is $\mathbb{G}_m$-equivariant with respect to the standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$ via multiplication;
   - $(\mathcal{X}\backslash\mathcal{X}_0, \Delta_{\mathrm{tc}}|_{\mathcal{X}\backslash\mathcal{X}_0}; \mathcal{L}|_{\mathcal{X}\backslash\mathcal{X}_0})$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, \Delta; L)\times(\mathbb{A}^1\backslash\{0\})$ (where $\mathcal{X}_0 = \pi^{-1}(0)$).

   A normal test configuration is called a *product* test configuration if $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) \cong (X, \Delta; L) \times \mathbb{A}^1$. A product test configuration is called a *trivial* test configuration if the above isomorphism is $\mathbb{G}_m$-equivariant with respect to the trivial $\mathbb{G}_m$-action on X and the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ via multiplication.

   A normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ is called a *special test configuration* if $\mathcal{L} \sim_{\mathbb{Q}} -l(K_{\mathcal{X}/\mathbb{A}^1} + \Delta_{\mathrm{tc}})$ and $(\mathcal{X}, \mathcal{X}_0 + \Delta_{\mathrm{tc}})$ is plt. In this case, we say that (X, Δ) *specially degenerates to* $(\mathcal{X}_0, \Delta_{\mathrm{tc},0})$ which is necessarily a log Fano pair.

2. Assume $\pi : (\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) \to \mathbb{A}^1$ is a normal test configuration of $(X, \Delta; L)$. Let $\bar{\pi} : (\overline{\mathcal{X}}, \overline{\Delta}_{\mathrm{tc}}; \overline{\mathcal{L}}) \to \mathbb{P}^1$ be the natural $\mathbb{G}_m$-equivariant compactification of π. The *generalized Futaki invariant* (sometimes called Donaldson-Futaki invariant) of $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})$ is defined by the intersection formula

$$\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) := \frac{1}{(-K_X - \Delta)^n}\left(\frac{n}{n+1} \cdot \frac{(\overline{\mathcal{L}}^{n+1})}{l^{n+1}} + \frac{(\overline{\mathcal{L}}^n \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \Delta_{\mathrm{tc}}))}{l^n}\right).$$

3. The log Fano pair (X, Δ) is said to be *K-semistable* if $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})/\mathbb{A}^1$ and any $l \in \mathbb{Q}_{>0}$ for which L is Cartier. It is said to be *K-stable* (resp. *K-polystable*) if it is K-semistable and $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) = 0$ for a normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})/\mathbb{A}^1$ if and only if it is a trivial (resp. product) test configuration.

This definition of K-stability is not easy to work with, and a powerful tool in our study is a valuative criterion discovered by K. Fujita and C. Li.

Definition 2.3.2 ([Fuj16b, Definition 1.1 and 1.3])**.** Let (X, Δ) be a log Fano pair. Let n be the dimension of X and let $L = -(K_X + \Delta)$. Let F be a prime divisor over X, i.e., there exists a projective birational morphism $\pi : Y \to X$ with Y normal such that F is a prime divisor on Y.

1. For any $t \geq 0$, we define $\mathrm{vol}_X(L - tF) := \mathrm{vol}_Y(\pi^* L - tF)$.

2. The *pseudo-effective threshold* $\tau(F)$ of F with respect to L is defined as

$$\tau(F) := \sup\{\tau > 0 \,|\, \mathrm{vol}_X(L - \tau F) > 0\}.$$

3. Let $A_{X,\Delta}(F)$ be the log discrepancy of F with respect to (X, Δ). We set

$$\beta_{X,\Delta}(F) := A_{X,\Delta}(F) \cdot (L^n) - \int_0^{\tau(F)} \mathrm{vol}_X(L - tF)\mathrm{d}t.$$

4. The prime divisor F over X is said to be *dreamy* if the $\mathbb{Z}^2_{\geq 0}$-graded algebra

$$\bigoplus_{k,j \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(klL - jF))$$

is finitely generated for some $l \in \mathbb{Z}_{>0}$ for which lL is Cartier. Note that this definition does not depend on the choice of l.

The following theorem summarizes results from [Fuj16b, Theorems 1.3 and 1.4], [Li17, Theorem 3.7], and [BX18, Corollary 4.3].

Theorem 2.3.3 ([Fuj16b, Li17, BX18])**.** *Let X be a $\mathbb{Q}$-Fano variety. Then the following are equivalent:*

1. *X is K-stable (resp. K-semistable);*

2. *$\beta_{X,\Delta}(F) > 0$ (resp. $\beta(F) \geq 0$) holds for every prime divisor F over X;*

3. *$\beta_{X,\Delta}(F) > 0$ (resp. $\beta(F) \geq 0$) holds for every dreamy prime divisor F over X.*

Chapter 3

K-stability of birationally superrigid Fano varieties

3.1 Introduction

The notion of birational superrigidity was originally introduced as a generalization of Iskovskikh and Manin's work [IM71] on the non-rationality of quartic threefolds; on the other hand, the concept of K-stability emerges in the study of Kähler-Einstein metrics on Fano manifolds. While the two notions have different nature of origin, they seem to resemble each other in the following sense: it is well known that a Fano variety X of Picard number 1 is birationally superrigid if and only if (X, M) has canonical singularities for every movable boundary $M \sim_{\mathbb{Q}} -K_X$; on the other hand, by the recent work of [FO16, BJ17], the K-(semi)stability of X is (roughly speaking) characterized by the log canonicity of basis type divisors, which is the average of a basis of some pluri-anticanonical system. In other words, both notions are tied to the singularities of certain anticanonical $\mathbb{Q}$-divisors and so it is very natural to expect some relation between them. Indeed, the slope stability (a weaker notion of K-stability) of birationally superrigid Fano manifolds has been established by [OO13] under some mild assumptions, and it is conjectured [OO13, KOW17] that birationally rigid Fano varieties are always K-stable.

In this chapter, we give a partial solution to this conjecture. Here is our main result (most results in this chapter come from joint work with C. Stibitz):

Theorem 3.1.1. *Let X be a $\mathbb{Q}$-Fano variety of class number* 1 *and dimension n. Assume that for every effective divisor $D \sim_{\mathbb{Q}} -K_X$ and every movable boundary $M \sim_{\mathbb{Q}} -K_X$, the pair $(X, \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is log canonical* (*resp. klt*)*, then X is K-semistable* (*resp. K-stable*)*.*

One may compare the above result with the well known criterion of Tian:

Theorem 3.1.2 ([Tia87, DK01, OS12, Fuj16a])**.** *Let X be a $\mathbb{Q}$-Fano variety of dimension n. Assume that $(X, \frac{n}{n+1}D)$ is log canonical* (*resp. klt; or log canonical if X is smooth*) *for every effective divisor $D \sim_{\mathbb{Q}} -K_X$, then X is K-semistable* (*resp. K-stable*)*.*

Indeed, since $\frac{1}{n+1}D + \frac{n-1}{n+1}M \sim_{\mathbb{Q}} -\frac{n}{n+1}K_X$, the assumption on singularities in Theorem 3.1.1 is automatically implied by those of Theorem 3.1.2. Thus Theorem 3.1.1 can be considered as a generalization of Tian's criterion for $\mathbb{Q}$-Fano varieties of class number one. On the other hand, our assumption seems easier to satisfy as movable boundaries on a Fano variety usually have mild singularities and the most singular divisor D only gets the weight $\frac{1}{n+1}$ (as opposed to $\frac{n}{n+1}$ in Theorem 3.1.2) in our criterion. In particular, if X is birationally superrigid and hence (X, M) has canonical singularities for every movable boundary $M \sim_{\mathbb{Q}} -K_X$, then as $\frac{1}{n+1}D + \frac{n-1}{n+1}M$ is a convex combination of $\frac{1}{2}D$ and M, we obtain the following result relating birational superrigidity and K-stability:

Corollary 3.1.3 (=Theorem 1.1.4)**.** *Let X be a Fano variety. If X is birationally superrigid and $\alpha(X) \geq \frac{1}{2}$, then X is K-stable.*

It is well known that smooth Fano hypersurfaces of index 1 and dimension $n \geq 3$ are birationally superrigid [IM71, Che00, dFEM03, dF16] and their alpha invariants are at least $\frac{n}{n+1}$ by [Che01], hence we have the following immediate corollary, recovering the K-stability of Fano hypersurfaces of index one:

Corollary 3.1.4 ([Fuj16a])**.** *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+1 \geq 4$, then X is K-stable.*

Another application is to the K-stability of general index two hypersurfaces. By [Che01] and [Puk16b], such hypersurfaces have alpha invariant $\frac{1}{2}$ and (X, M) is lc for every movable boundary $M \sim_{\mathbb{Q}} -K_X$. Hence by analyzing the equality case in Theorem 3.1.1, we prove:

Corollary 3.1.5. *Let $n \geq 16$ and let $U \subseteq \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(n))$ be the parameter space of smooth index two hypersurfaces. Let $T \subseteq U$ be the set of hypersurfaces that are not K-stable. Then* $\mathrm{codim}_U T \geq \frac{1}{2}(n-11)(n-10)-10$.

Although alpha invariants are in general hard to estimate, the known examples seem to suggest that birationally (super)rigid varieties have large alpha invariants. In view of Corollary 3.1.3, it is therefore natural to ask the following question:

Question 3.1.6. Let X be a birationally superrigid Fano variety. Is it true that $\alpha(X) \geq \frac{1}{2}$?

Obviously, a positive answer to this question will confirm the K-stability of all birationally superrigid Fano varieties. At this point, we only have a weaker estimate:

Theorem 3.1.7. *Let X be a $\mathbb{Q}$-Fano variety of Picard number 1 and dimension $n \geq 3$. Assume that X is birationally superrigid (or more generally, $\alpha^{(2)}(X) \geq 1$), $-K_X$ generates the class group $\mathrm{Cl}(X)$ of X and $|-K_X|$ is base point free, then $\alpha(X) \geq \frac{1}{n+1}$.*

Note that this is in line with the conjectural K-stability of such varieties, since by [FO16, Theorem 3.5], the alpha invariant of a K-semistable Fano variety is always $\geq \frac{1}{n+1}$. We also remark that the assumptions about the index and base point freeness in the above theorem seem to be mild and they are satisfied by most known example of birationally superrigid varieties.

3.2 Proof of the criterion

In this section we give the proof of Theorem 3.1.1 and Corollary 3.1.3.

Definition 3.2.1. Let (X, Δ) be a log Fano pair and let $L = -(K_X + \Delta)$. Let F be a prime divisor over X. The *movable threshold* $\eta(F)$ of L with respects to F is defined as the supremum of all $\eta > 0$ such that every divisor in the stable base locus of $\pi^*L - \eta F$ is exceptional over X.

Note that if F is a dreamy divisor, then the supremum is indeed a maximum in the above definition.

Lemma 3.2.2. *With notation as in Definition 2.3.2 and 3.2.1 and assume that X is $\mathbb{Q}$-factorial and $\rho(X) = 1$, we have the inequality*

$$\frac{1}{(L^n)} \int_0^\infty \mathrm{vol}_X(L - xF)\mathrm{d}x \leq \frac{1}{n+1}\tau(F) + \frac{n-1}{n+1}\eta(F).$$

Remark 3.2.3. Without the Picard number one assumption, Fujita [Fuj17, Proposition 2.1] proves a weaker inequality where the right hand side becomes $\frac{n}{n+1}\tau(F)$.

Proof. The argument is a refinement of the proof of [Fuj17, Proposition 2.1]. For ease of notation, let $\eta = \eta(F)$ and $\tau = \tau(F)$. Let $\pi : Y \to X$ be a projective birational morphism such that F is a prime divisor on Y. Let

$$b = \frac{1}{(L^n)} \int_0^\infty \mathrm{vol}_X(L - xF)\mathrm{d}x.$$

As in the proof of [Fuj17, Proposition 2.1], we have

$$\int_0^\tau (x - b) \cdot \mathrm{vol}_{Y|F}(\pi^* L - xF)\mathrm{d}x = 0 \tag{3.1}$$

where $\mathrm{vol}_{Y|F}$ denotes the restricted volume of a divisor to F (see [ELM$^+$09]). For simplicity, we let $V_t = \mathrm{vol}_{Y|F}(\pi^* L - tF)$. It is clear (as in [Fuj17, Proposition 2.1]) that F is not contained in the augmented base locus $\mathbf{B}_+(\pi^* L - xF)$ when $0 \leq x < \tau$. So by [ELM$^+$09, Theorem A], the restricted volume $\mathrm{vol}_{Y|F}(\pi^* L - xF)$ is log-concave when $0 \leq x < \tau$ and in particular we have

$$(x - x_0) \cdot V_x \leq (x - x_0) \left(\frac{x}{x_0}\right)^{n-1} V_{x_0} \tag{3.2}$$

for every $0 \leq x, x_0 \leq \tau$. We may assume that $\eta < \tau$, otherwise the lemma simply follows from [Fuj17, Proposition 2.1]. By the definition of pseudo-effective threshold, there exists $D \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{ord}_F(D) > \eta$. Since X is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, we may assume that D is irreducible. Such D is necessarily unique by the definition of $\eta(F)$. In particular, there are no other divisors $D' \sim_{\mathbb{Q}} -K_X$ with $\mathrm{ord}_F(D') > \mathrm{ord}_F(D)$ and hence $\mathrm{ord}_F(D) = \tau$. Moreover, if $\eta \leq x \leq \tau$ and $D' \sim_{\mathbb{Q}} -K_X$ is such that $\mathrm{ord}_F(D') \geq x$ and we write $D' = aD + \Gamma$

where $D \not\subseteq \text{Supp}\,(\Gamma)$, then $\text{ord}_F(\Gamma) \le \eta$. As

$$\pi^*L - xF = \frac{\tau - x}{\tau - \eta}(\pi^*L - \eta F) + \frac{x-\eta}{\tau-\eta}(\pi^*L - \tau F),$$

we see that $a \ge \frac{x-\eta}{\tau-\eta}$ and therefore, if in addition $x \in \mathbb{Q}$ and m is sufficiently divisible, then the natural inclusion

$$H^0\left(Y, \frac{\tau - x}{\tau-\eta}(-m\pi^*K_X - m\eta F)\right) \hookrightarrow H^0(Y, -m\pi^*K_X - mxF)$$

given by the multiplication of $m \cdot \frac{x-\eta}{\tau-\eta}(\pi^*D - \tau F)$ is an isomorphism. By the definition and the continuity of restricted volume, this implies (note that F is not in the support of $\pi^*D - \tau F$) that

$$V_x = \left(\frac{\tau - x}{\tau - \eta}\right)^{n-1} V_\eta. \tag{3.3}$$

when $\eta \le x \le \tau$.

Now first suppose that $b \ge \eta$, then combining (3.1), (3.2) (with $x_0 = \eta$) and (3.3) we have

$$0 \le \int_0^\eta (x-b)\left(\frac{x}{\eta}\right)^{n-1} V_\eta \mathrm{d}x + \int_\eta^\tau (x-b)\left(\frac{\tau-x}{\tau-\eta}\right)^{n-1} V_\eta \mathrm{d}x,$$

which reduces to $b \le \frac{1}{n+1}\tau + \frac{n-1}{n+1}\eta$. Suppose on the other hand that $b < \eta$, then combining (3.2) (with $x_0 = b$) and (3.3) we have

$$V_x \le \left(\frac{\eta}{b}\right)^{n-1}\left(\frac{\tau-x}{\tau-\eta}\right)^{n-1} V_b$$

when $\eta \le x \le \tau$. Combining this with (3.2) (with $x_0 = b$ again) and (3.1) we have

$$0 \le \int_0^\eta (x-b)\left(\frac{x}{b}\right)^{n-1} V_b \mathrm{d}x + \int_\eta^\tau (x-b)\left(\frac{\eta}{b}\right)^{n-1}\left(\frac{\tau-x}{\tau-\eta}\right)^{n-1} V_b \mathrm{d}x,$$

which again reduces to $b \le \frac{1}{n+1}\tau + \frac{n-1}{n+1}\eta$. This proves the lemma. □

Remark 3.2.4. It is easy to see from the proof that this lemma holds for any ample $\mathbb{Q}$-Cartier divisor L on X and (X, Δ) doesn't have to be log Fano.

Comparing with the expression of $\beta_{X,\Delta}(F)$ we immediately obtain:

Corollary 3.2.5. *Assume that* $\frac{1}{n+1}\tau(F) + \frac{n-1}{n+1}\eta(F) \le A_{X,\Delta}(F)$ *(resp.* $< A_{X,\Delta}(F)$*), then* $\beta_{X,\Delta}(F) \ge 0$ *(resp.* > 0*).* ☐

Proof of Theorem 3.1.1. Let F be a dreamy divisor over X and let $\eta = \eta(F)$, $\tau = \tau(F)$. Then for $m \gg 0$, the linear system $|-mK_X - m\tau F|$ (i.e. the sub-linear system of $|-mK_X|$ consisting of divisors that vanish with order at least $m\tau$ along F) is non-empty while $|-mK_X - m\eta F|$ is movable. Let $D \in |-mK_X - m\tau F|$ and $M = \frac{1}{m}|-mK_X - m\eta F|$, then M is a movable boundary, $D \sim_{\mathbb{Q}} M \sim_{\mathbb{Q}} -K_X$, $\mathrm{ord}_F(D) = \tau$ and $\mathrm{ord}_F(M) = \eta$. Thus if $(X, \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is lc (resp. klt), then we have $\frac{1}{n+1}\tau + \frac{n-1}{n+1}\eta \le A_{X,\Delta}(F)$ (resp. $< A_{X,\Delta}(F)$). As this holds for every dreamy divisor F, X is K-semistable (resp. K-stable) by Theorem 2.3.3 and Corollary 3.2.5. ☐

Proof of Corollary 3.1.3. By assumption and Theorem 2.2.5, for every movable boundary $M \sim_{\mathbb{Q}} -K_X$, (X, M) has canonical singularities while for every effective $D \sim_{\mathbb{Q}} -K_X$, $(X, \frac{1}{2}D)$ is lc, so as $\Delta = \frac{2}{n+1} \cdot \frac{1}{2}D + (1 - \frac{2}{n+1})M$ is a convex combination of the two boundary divisors, $(X, \Delta = \frac{1}{n+1}D + \frac{n-1}{n+1}M)$ is klt, hence X is K-semistable (resp. K-stable) by Theorem 3.1.1. ☐

Remark 3.2.6. Let us also give a somewhat more conceptual proof of Corollary 3.1.3. In terms of the equation (3.1), it suffices to check that the center of mass of the interval $[0, \tau]$ with density function $f(x) = \frac{1}{(-K_X)^n}\mathrm{vol}_{Y|F}(-\pi^*K_X - xF)$ is smaller than $A := A_{X,\Delta}(F)$. The assumption that X is birationally superrigid implies that $g(x) = f(x)^{1/(n-1)}$ is linear when $x \ge A - 1$ and $g(\tau) = 0$ if $\tau > A - 1$, while $\alpha(X) \ge \frac{1}{2}$ implies that the length of the interval is at most $2A$. Since $g(x)$ is also concave, it is clear (by looking at the graph of $g(x)$) that the center of mass is smaller than A.

3.3 Applications

It is not hard to characterize the equality case from the proof of Theorem 3.1.1. For our application to K-stability of general index two hypersurfaces, we only need the following special case.

Corollary 3.3.1. *Let X be a $\mathbb{Q}$-Fano variety of class number one. Assume that $\alpha^{(2)}(X) \geq 1$ and $\alpha(X) \geq \frac{1}{2}$, then X is K-semistable. If X is not K-stable, then there exists a dreamy prime divisor F over X, a movable boundary $M \sim_{\mathbb{Q}} -K_X$ and an effective divisor $D \sim_{\mathbb{Q}} -K_X$ such that F is a log canonical place of (X, M) and $(X, \frac{1}{2}D)$.*

Proof. We keep the notation from the proof of Theorem 3.1.1. Let F be a dreamy divisor over X, then by our assumptions we have $\eta := \eta(F) \leq A := A_X(F)$ and $\tau := \tau(F) \leq 2A$. In particular, X is K-semistable by Corollary 3.2.5 and Theorem 2.3.3. If X is not K-stable, we may choose F for which $\beta_X((F) = 0$. But this forces $\eta = A$ and $\tau = 2A$. As F is dreamy, for $m \gg 0$, the linear system $|-mK_X - m\tau F|$ (i.e. the sub-linear system of $|-mK_X|$ consisting of divisors that vanish with order at least $m\tau$ along F) is non-empty while $|-mK_X - m\eta F|$ is movable. Let $D \in |-mK_X - m\tau F|$ and $M = \frac{1}{m}|-mK_X - m\eta F|$, then M is a movable boundary, $D \sim_{\mathbb{Q}} M \sim_{\mathbb{Q}} -K_X$ and F is a log canonical place of both (X, M) and $(X, \frac{1}{2}D)$. □

Proof of Corollary 3.1.5. Let $S \subseteq U$ be the set of *regular* hypersurfaces as defined in [Puk17b, §0.2]. By [Puk17b, Theorem 2], S is non-empty and the complement of S has codimension at least $\frac{1}{2}(n-11)(n-10) - 10$. Therefore, it suffices to show that every hypersurface in the set S is K-stable. Let X be such a hypersurface.

Let H be the hyperplane class and let $D \sim_{\mathbb{Q}} H \sim_{\mathbb{Q}} -\frac{1}{2}K_X$ be an effective divisor. By [Che01, Lemma 3.1], (X, D) is lc and indeed by [Puk02a, Proposition 5], we have $\mathrm{mult}_x D \leq 1$ for all but finitely many $x \in X$, hence by [Kol97, 3.14.1], (X, D) has canonical singularities outside a finite number of points. It follows that every lc center of (X, D) is either a divisor on X or an isolated point.

On the other hand, let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary, then by the main result of [Puk17b], the only possible center of maximal singularities of (X, M) is a linear section of X codimension 2. It follows that (X, M) is lc and every lc center of (X, M) is a linear section of codimension two.

Hence X is K-semistable by Corollary 3.3.1. Suppose that it is not K-stable, then by Corollary 3.3.1 there exists a movable boundary $M \sim_{\mathbb{Q}} -K_X$ and an effective divisor $D \sim_{\mathbb{Q}} H$ such that (X, M) and (X, D) have a common lc center. But by the previous analysis, this is impossible as the lc centers of (X, M) and (X, D) always have different dimension. Therefore X is K-stable and the proof is complete. $\square$

We conclude this chapter by proving Theorem 3.1.7.

Proof of Theorem 3.1.7. It suffices to show that $\mathrm{lct}(X; D) \geq \frac{1}{n+1}$ for every $D \sim_{\mathbb{Q}} -K_X$. We may assume that D is irreducible. Since $\mathrm{Cl}(X)$ is generated by $-K_X$, we have $\mathrm{mult}_\eta D \leq 1$ where η is the generic point of D. It follows that (X, D) is log canonical in codimension one [Kol97, (3.14.1)], hence the multiplier ideal $\mathcal{J}(X, (1-\epsilon)D)$ (where $0 < \epsilon \ll 1$) defines a subscheme of codimension at least two. By Nadel vanishing,

$$H^i(X, \mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-rK_X)) = 0$$

for every $i > 0$ and $r \geq 0$, therefore by Castelnuovo-Mumford regularity (see e.g. [Laz04, §1.8]), the sheaf $\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-nK_X)$ is generated by its global sections and we get a movable linear system

$$\mathcal{M} = |\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-nK_X)|.$$

Suppose that $\mathrm{lct}(X; D) < \frac{1}{n+1}$, let E be an exceptional divisor over X that computes it, let $A = A_X(E)$ and let $\pi : Y \to X$ be a projective birational morphism such that the center of E on Y is a divisor, then $A \in \mathbb{Z}$ (since $-K_X$ is Cartier by assumption) and we have $d = \mathrm{ord}_E(D) > (n+1)A$ and $\mathcal{J}(X, (1-\epsilon)D) \subseteq \pi_*\mathcal{O}_Y((A - 1 - \lfloor (1-\epsilon)d \rfloor)E) \subseteq \pi_*\mathcal{O}_Y(-(nA+1)E)$. It follows that $\mathrm{ord}_E(\mathcal{M}) \geq nA + 1$, hence for the movable boundary $M = \frac{1}{n}\mathcal{M} \sim_{\mathbb{Q}} -K_X$ we have $\mathrm{ord}_E(M) > A$ and (X, M) is not log canonical, violating our assumption. Thus $\mathrm{lct}(X; D) \geq \frac{1}{n+1}$ and we are done. $\square$

Chapter 4

Smooth Fano complete intersections of index 1

4.1 Introduction

Despite their theoretical interest, birational (super)rigidity and K-stability are not so easy to verify in general. Indeed, it's a folklore conjecture that every smooth Fano complete intersection $X \subseteq \mathbb{P}^n$ of degree at least 3 is K-stable while those of index 1 (i.e. $-K_X$ is linearly equivalent to the hyperplane class) and dimension at least 4 is birationally superrigid, and only some partial progress has been made in this direction. For birational superrigidity, the hypersurface case was settled by the work of [IM71, Puk86, Cor95, Che00, dFEM03, dF16]; in the case of higher codimensions, [Puk01, Puk13a, Puk14] proves that a *general* member of complete intersections of given degree and codimension is birationally superrigid provided they have large dimension, while [Suz17] shows the birational superrigidity of certain families of complete intersections, albeit under some assumptions on the degrees of their defining equations. As for K-stability, the intersection of two (hyper)quadric is treated in [AGP06], the case of cubic threefolds is settled recently by [LX17b] and in most remaining cases, we only have Tian's criterion (Theorem 3.1.2): if $\alpha(X) \geq \frac{n}{n+1}$ (resp. $> \frac{n}{n+1}$) where $n = \dim X$, then X is K-semistable (resp. K-stable). This has been successfully applied to smooth hypersurfaces of index 1 [Che01, Fuj16a], to *general* members of some given type of complete intersections of index 1 [Puk10, EP16, Puk17a] and to certain Fano 3-folds [CS08, Che09a, KOW17]. However, the alpha invariants of Fano varieties can still be hard to control at times, especially for *special* members of a given family.

In this chapter, we introduce a new method of estimating log canonical thresholds. Combining with the results from the previous chapter, we are able to prove the birational superrigidity and K-stability of many Fano complete intersections. As a major application, we prove the following two results:

Theorem 4.1.1 (=Theorem 1.2.2)**.** *Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Fano complete intersection of index* 1*, codimension r and dimension $n \geq 10r$, then X is birationally superrigid and K-stable.*

Theorem 4.1.2 (with C. Stibitz)**.** *Let $r, s \in \mathbb{Z}_+$, then there exists an integer $N = N(r,s)$ depending only on r and s such that every smooth Fano complete intersection of index s and codimension r in $\mathbb{P}^{n+r}$ is conditionally birationally superrigid if $n \geq N$.*

We refer to §4.3 for the definition of conditional birational superrigidity. We'd like to point our that in Theorem 4.1.2, the integer can be explicitly figured out for each given r and s; also in Theorem 4.1.1, the bound on the dimension is not optimal in general: one can usually further weaken the assumption on the dimension n for each fixed codimension r. For example, when $r = 2$, we find $n \geq 12$ is enough.

While our method is more effective in large dimensions, it also applies to some Fano threefolds of small degree. In particular, we verify the K-stability of smooth Fano threefolds of degree 6 (this is the only birationally rigid Fano threefolds whose K-stability was not known before).

Theorem 4.1.3. *Let $X = X_{2,3} \subseteq \mathbb{P}^5$ be a smooth complete intersection of a quadric and a cubic, then X is K-stable* (*hence admits Kähler-Einstein metric*)*.*

In light of Theorem 1.1.4, both theorems above rely on estimate of log canonical thresholds on the varieties in question. A key ingredient for such estimate is given by the following:

Theorem 4.1.4. *Let (X, Δ) be a pair. Let D be an effective $\mathbb{Q}$-divisor on X and L a line bundle. Let $\lambda > 0$ be a constant. Assume that*

1. *$L-(K_X+\Delta+(1-\epsilon)D)$ is nef and big and $(X,\Delta+(1-\epsilon)D)$ is klt outside a finite set T of points for all $0<\epsilon\ll 1$;*

2. *For all 0-dimensional subschemes $\Sigma\subseteq X$ supported on T such that $\ell(\mathcal{O}_\Sigma)\le h^0(X,L)$, we have $\mathrm{lct}(X,\Delta;\Sigma)\ge\lambda$.*

Then $\mathrm{lct}(X,\Delta;D)\ge\frac{\lambda}{\lambda+1}$. Moreover, when equality holds, there exists some 0-dimensional subscheme $\Sigma\subseteq X$ satisfying the assumption (2) such that every divisor that computes $\mathrm{lct}(X,\Delta;D)$ also computes $\mathrm{lct}(X,\Delta;\Sigma)=\lambda$.

For example, if $h^0(X,L)=0$, then we may choose λ to be any constant, and the theorem implies that $\mathrm{lct}(X,\Delta;D)\ge 1$. As another example, if X is smooth, $\Delta=0$ and L is the trivial line bundle, then $\lambda=n=\dim X$ satisfies the assumption (2) and we have $\mathrm{lct}(X;D)\ge\frac{n}{n+1}$, with equality if and only if $\mathrm{mult}_x(D)=n+1$ for some $x\in X$ (since $\mathrm{lct}(X;x)$ is computed exactly by the blowup of x in this case). These observations lead to a simple proof of the following well-known result:

Corollary 4.1.5 ([CP02, dFEM03])**.** *Let $X\subseteq\mathbb{P}^{n+1}$ $(n\ge 3)$ be a smooth hypersurface of degree d and let H be the hyperplane class, then*

1. *$\mathrm{lct}(X;|H|_{\mathbb{Q}})=1$ if $d\le n$;*

2. *$\mathrm{lct}(X;|H|_{\mathbb{Q}})\ge\frac{n}{d}$ if $d\ge n+1$.*

In the latter case, equality holds if and only if X has an Eckardt point (i.e. there exsits a hyperplane section with multiplicity d at the point).

More interesting applications come in when we apply Theorem 4.1.4 to the case when L has some positivity (indeed, the proof of most results in this paper consists of multiple use of Theorem 4.1.4 in this setting). In those cases, we can usually find the constant λ by the work of [dFEM04] (or its variants) and this in particular yields:

Corollary 4.1.6. *Let X,D,L be as in Theorem 4.1.4 and $\Delta=0$. Assume that X is smooth and $h^0(X,L)\le\frac{n^n}{n!}$, then $\mathrm{lct}(X;D)>\frac{1}{2}$.*

As will be clear from the proof, the number $\frac{n^n}{n!}$ can be replaced by the minimum number of lattice points in the simplex $Q_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \,|\, \mathbf{a} \cdot \mathbf{x} < 1\}$ among all possible choices of $\mathbf{a} \in \mathbb{R}^n_+$ such that $(1, 1, \cdots, 1) \in \overline{Q_{\mathbf{a}}}$. More generally, if X is singular, we may replace it by the minimal non-klt colengths (see §4.2) of the singularities. These observations will be important in the next chapter when we consider singular complete intersections.

Apart from its obvious connection to Theorem 1.1.4, Corollary 4.1.6 is also a key step in the proof of Theorem 4.1.1. The idea is that given a movable boundary $M \sim_{\mathbb{Q}} -K_X$ on a complete intersection X of index 1, one can usually show that $(X, 2M)$ is log canonical outside a set of small dimension (as in [dF16]), so after cutting down by hyperplanes, we can always reduce to the setting of Theorem 4.1.4 and it suffices to show that $\mathrm{lct}(X; 2M) \geq \frac{1}{2}$. In the hypersurface case [dFEM03, dF16], this is done by projecting X to $\mathbb{P}^n$ and then applying [dFEM03, dFEM04]. Such strategy doesn't seem to carry over to complete intersections since the projection is in general a hypersurface of large degree (compared to the dimension) and the bound on log canonical threshold given by the argument of [dFEM03, dF16] is not sufficient. To get around this issue, we estimate the log canonical threshold using local information of the multiplier ideal (instead of taking projection) and Corollary 4.1.6 plays an important role here.

4.2 Lower bounds of log canonical thresholds

In this section we prove Theorem 4.1.4 and its applications (including Theorem 4.1.1).

Proof of Theorem 4.1.4. We may assume that $\mathrm{lct}(X, \Delta; D) < 1$, otherwise there is nothing to prove. Let $0 < \epsilon \ll 1$, then by the first assumption, the multiplier ideal $\mathcal{J} = \mathcal{J}(X, \Delta + (1 - \epsilon)D)$ defines a 0-dimensional subscheme $\Sigma \subseteq X$ supported on T that does not depend on ϵ. Since $L - (K_X + \Delta + (1 - \epsilon)D)$ is nef and big, by Nadel vanishing we have $H^1(X, \mathcal{J}(X, \Delta + (1 - \epsilon)D) \otimes L) = 0$, thus the natural restriction map $H^0(X, L) \to H^0(\Sigma, L|_{\Sigma}) \cong H^0(\Sigma, \mathcal{O}_{\Sigma})$ is surjective. In particular, $\ell(\mathcal{O}_{\Sigma}) \leq h^0(X, L)$. Hence by our second assumption, $\mathrm{lct}(X, \Delta; \Sigma) \geq \lambda$. Now let E be a divisor over X that computes $\operatorname{lct}(X,\Delta;D)$, then by the definition of multiplier ideal, for every $f\in\mathcal{J}$ we have

$$\operatorname{ord}_E(f)\geq\lfloor(1-\epsilon)\operatorname{ord}_E(D)-A_{(X,\Delta)}(E)+1\rfloor>(1-\epsilon)\operatorname{ord}_E(D)-A_{(X,\Delta)}(E)$$

where $A_{(X,\Delta)}(E)$ is the log discrepancy of E with respect to (X,Δ). Letting $\epsilon\to 0$ we get

$$\operatorname{ord}_E(f)\geq\operatorname{ord}_E(D)-A_{(X,\Delta)}(E). \tag{4.1}$$

On the other hand, if f is general in $\mathcal{J}$ then we have

$$\frac{A_{(X,\Delta)}(E)}{\operatorname{ord}_E(f)}\geq\operatorname{lct}(X,\Delta;\Sigma)\geq\lambda. \tag{4.2}$$

Combining these two inequalities we obtain $\lambda^{-1}A_{(X,\Delta)}(E)\geq\operatorname{ord}_E(D)-A_{(X,\Delta)}(E)$, which reduces to $\operatorname{lct}(X,\Delta;D)=\frac{A_{(X,\Delta)}(E)}{\operatorname{ord}_E(D)}\geq\frac{\lambda}{\lambda+1}$. If equality holds, then the inequality (4.2) is an equality, hence in particular we have $\operatorname{lct}(X,\Delta;\Sigma)=\lambda$ and it is computed by E. □

Remark 4.2.1. Using the same argument we can also get a pointwise statement as follows. Keeping notation from the above proof, let $\Sigma=\cup_{i=1}^r\Sigma_i$ be the decomposition of Σ into irreducible components and let $x_i=\operatorname{Supp}(\Sigma_i)$. Then we have $\operatorname{lct}(X,\Delta;\Sigma_i)\geq\lambda$ implies $\operatorname{lct}(X,\Delta;D)\geq\frac{\lambda}{\lambda+1}$ in a neighbourhood of x_i, with equality if and only if every exceptional divisor centered at x_i that computes $\operatorname{lct}(X,\Delta;D)$ around x_i also computes $\operatorname{lct}(X,\Delta;\Sigma_i)$. This observation will be important in the proof below as well as in the last section.

Let us first apply Theorem 4.1.4 to compute log canonical thresholds on hypersurfaces.

Proof of Corollary 4.1.5. Let $D\sim_{\mathbb{Q}}H$ be an effective divisor on X. It suffices to show that $\operatorname{lct}(X;D)\geq\min\{\frac{n}{d},1\}$. By [Puk02b, Proposition 5], $\operatorname{mult}_x D\leq 1$ except at finitely many points $x\in X$, hence by [Kol97, (3.14.1)], $(X,(1-\epsilon)D)$ is klt outside a finite set of points ($0<\epsilon\ll 1$). If $d\leq n$, then we may apply Theorem 4.1.4 with $L=\mathcal{O}_X(-H)$, $\Delta=0$ and obtain $\operatorname{lct}(X;D)\geq\frac{\lambda}{\lambda+1}$ for any $\lambda>0$, thus $\operatorname{lct}(X;D)\geq 1$. If $d\geq n+1$, let $x\in X$ and let $\gamma:X\to\mathbb{P}^n$ be a general projection such that γ is étale in the neighbourhood of x and $\gamma|_D$ is injective in the neighbourhood of $\gamma(x)$, then we have $\operatorname{lct}(X;D)=\operatorname{lct}(\mathbb{P}^n,\gamma(D))$ near x and since $\frac{n+1}{d}\leq 1$, $(\mathbb{P}^n,\frac{n+1}{d}(1-\epsilon)\gamma(D))$ is klt in a punctured neighbourhood of

$\gamma(x)$. We apply Theorem 4.1.4 to the pair $(\mathbb{P}^n, \frac{n+1}{d}\gamma(D))$ with $L = 0 \sim_{\mathbb{Q}} K_{\mathbb{P}^n} + \frac{n+1}{d}\gamma(D)$, $\Delta = 0$ and $T = \{\gamma(x)\}$. Note that the only 0-dimensional subscheme Σ supported at x with $\ell(\mathcal{O}_\Sigma) \le h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 1$ is the closed point x itself, and for such point we always have $\mathrm{lct}(\mathbb{P}^n; x) = n$. Hence we may take $\lambda = n$ and obtain $\mathrm{lct}(\mathbb{P}^n; \frac{n+1}{d}\gamma(D)) \ge \frac{n}{n+1}$. It follows that $(\mathbb{P}^n, \frac{n}{d}\gamma(D))$ is log canonical at $\gamma(x)$ and hence $(X, \frac{n}{d}D)$ is log canonical at x as well. Since $x \in X$ is arbitrary, we get $\mathrm{lct}(X; D) \ge \frac{n}{d}$. Suppose that equality $\mathrm{lct}(X; |H|_{\mathbb{Q}}) = \frac{n}{d}$ holds, then by [Bir16b, Theorem 1.5], there exists $D \sim_{\mathbb{Q}} -K_X$ (which we may assume to be irreducible as X has Picard number one) with $\mathrm{lct}(X; D) = \frac{n}{d}$. Let $x \in X$ be a point where $(X, \frac{n}{d}D)$ is not klt and let $\gamma : X \to \mathbb{P}^n$ be as before. Then by the equality case of Theorem 4.1.4, every divisor that computes $\mathrm{lct}(\mathbb{P}^n; \gamma(D))$ also computes $\mathrm{lct}(\mathbb{P}^n; \gamma(x))$. It follows that $\mathrm{lct}(\mathbb{P}^n; \gamma(D))$ is computed by $\mathrm{mult}_{\gamma(x)}$ and hence $\mathrm{mult}_x D = \mathrm{mult}_{\gamma(x)}\gamma(D) = d$. If D is not a hyperplane section, then let $W = T_x X \cap X$ be the restriction of the tangent hyperplane at x, then $\mathrm{mult}_x W \ge 2$ and $d = \deg(D \cdot W) \ge \mathrm{mult}_x(D \cdot W) \ge 2d$, a contradiction. Hence D is a hyperplane section with multiplicity d at x. $\square$

Next we use Theorem 4.1.4 to give some lower bounds of log canonical thresholds on complete intersections. To this end, we introduce the following definition:

Definition 4.2.2. Let $x \in (X, D)$ be a klt singularity. The minimal non-klt (resp. non-lc) colength of $x \in (X, D)$ with coefficient λ is defined as

$$\begin{aligned} \ell_{\mathrm{nklt}}(x, X, D; \lambda) &:= \min\{\ell(\mathcal{O}_X/\mathcal{J}) \,|\, \mathrm{Supp}(\mathcal{O}_X/\mathcal{J}) = \{x\} \text{ and } (X, D; \mathcal{J}^\lambda) \text{ is not klt}\} \\ (\text{resp. } \ell_{\mathrm{nlc}}(x, X, D; \lambda) &:= \min\{\ell(\mathcal{O}_X/\mathcal{J}) \,|\, \mathrm{Supp}(\mathcal{O}_X/\mathcal{J}) = \{x\} \text{ and } (X, D; \mathcal{J}^\lambda) \text{ is not lc}\}). \end{aligned}$$

When $D = 0$, we use the abbreviation $\ell_{\mathrm{nklt}}(x, X; \lambda)$ (resp. $\ell_{\mathrm{nlc}}(x, X; \lambda)$).

We can then rephrase Theorem 4.1.4 in terms of minimal non-klt (resp. non-lc) colengths.

Theorem 4.2.3. *Let (X, Δ) be a klt pair, D an effective divisor on X and L a line bundle such that $L - (K_X + \Delta + (1-\epsilon)D)$ is big and nef for $0 < \epsilon \ll 1$. Assume that $(X, \Delta + D)$ is log canonical outside a finite set of points T and that $h^0(X, L) < \ell_{\mathrm{nklt}}(x, X, \Delta; \lambda)$ (resp. $< \ell_{\mathrm{nlc}}(x, X, \Delta; \lambda)$) for every $x \in T$, then* $\mathrm{lct}(X, \Delta; D) > \frac{\lambda}{\lambda+1}$ *(resp.* $\ge \frac{\lambda}{\lambda+1}$*).* $\square$

Hence for various applications, it suffices to find a suitable lower bound of the minimal non-klt (resp. non-lc) colengths and compare it with $h^0(X, L)$. In the smooth case, this can be given by the work of [dFEM04] (or more precisely, by the proof therein). To state the result, we need more notation: for $\mathbf{a} \in \mathbb{R}_+^n$ and $\lambda > 0$, let

$$\begin{aligned} Q_{\mathbf{a}} &= \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \,|\, \mathbf{a} \cdot \mathbf{x} < 1\}, \\ \sigma_{n,\lambda} &= \min\{\#(Q_{\mathbf{a}} \cap \mathbb{Z}^n) \,|\, \mathbf{a} \in \mathbb{R}^n_+ \text{ s.t. } (\lambda, \lambda, \cdots, \lambda) \in Q_{\mathbf{a}}\}, \\ \bar{\sigma}_{n,\lambda} &= \min\{\#(Q_{\mathbf{a}} \cap \mathbb{Z}^n) \,|\, \mathbf{a} \in \mathbb{R}^n_+ \text{ s.t. } (\lambda, \lambda, \cdots, \lambda) \in \overline{Q_{\mathbf{a}}}\}. \end{aligned}$$

Clearly $\sigma_{n,\lambda} \geq \bar{\sigma}_{n,\lambda}$.

Lemma 4.2.4. *Let X be a smooth variety of dimension n and $x \in X$. Let $\lambda > 0$, then*

$$\ell_{\mathrm{nlc}}(x, X; \lambda^{-1}) \geq \sigma_{n,\lambda}, \quad \ell_{\mathrm{nklt}}(x, X; \lambda^{-1}) \geq \bar{\sigma}_{n,\lambda}.$$

Proof. We may assume that $(X, x) = (\mathbb{A}^n, 0)$ since the statement is étale local. Moreover, as in the proof of [dFEM04, Theorem 1.1], we may assume that $\mathcal{J} \subseteq \mathcal{O}_X$ is a monomial ideal by the lower semicontinuity (see e.g. [DK01]) of log canonical thresholds. Let P be the Newton polytope of $\mathcal{J}$, defined as the convex hull in $\mathbb{R}^n_{\geq 0}$ of all the points corresponding to monomials in $\mathcal{J}$. By [How01], letting $\mu = \mathrm{lct}(\mathbb{A}^n; \mathcal{J})^{-1}$, we have

$$\mu = \min\{t > 0 \,|\, (t, t, \cdots, t) \in P\}.$$

Let W be a supporting hyperplane of P at $(\mu, \cdots, \mu) \in \partial P$. Write the equation of W as $\mathbf{a} \cdot \mathbf{x} = 1$ where $\mathbf{a} \in \mathbb{R}^n_+$, then we have $\ell(\mathcal{O}_X/\mathcal{J}) = \#((\mathbb{R}^n_{\geq 0} \backslash P) \cap \mathbb{Z}^n) \geq \#(Q_{\mathbf{a}} \cap \mathbb{Z}^n)$. If $(X; \mathcal{J}^{1/\lambda})$ is not lc (resp. not klt), then $\mu > \lambda$ (resp. $\geq \lambda$), hence $(\lambda, \cdots, \lambda) \in Q_{\mathbf{a}}$ (resp. $\in \overline{Q_{\mathbf{a}}}$) and the lemma simply follows from the definition of $\sigma_{n,\lambda}$ (resp. $\bar{\sigma}_{n,\lambda}$). □

Corollary 4.2.5. *Let X, D, L be as in Theorem 4.1.4 and $\Delta = 0$. Assume that X is smooth of dimension n and $h^0(X, L) < \bar{\sigma}_{n,\lambda}$ (resp. $< \sigma_{n,\lambda}$), then $\mathrm{lct}(X; D) > \frac{1}{\lambda+1}$ (resp. $\geq \frac{1}{\lambda+1}$).*

Proof. This is immediate from Theorem 4.2.3 and Lemma 4.2.4. □

In light of this, all subsequent estimates of log canonical thresholds essentially reduce to finding lower bounds of $\sigma_{n,\lambda}$ (or $\bar{\sigma}_{n,\lambda}$). Here are some sample applications:

Proof of Corollary 4.1.6. It is clear that $\bar{\sigma}_{n,1} > \mathrm{vol}(Q_{\mathbf{a}}) \geq \frac{n^n}{n!}$ if $(1,\cdots,1) \in \overline{Q_{\mathbf{a}}}$ and $n \geq 2$ (see the proof of [dFEM04, Theorem 1.1]), so the result follows directly from Corollary 4.2.5 with $\lambda = 1$. □

Lemma 4.2.6. *Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Fano complete intersection of codimension r and dimension $n \geq 6r$. Let H be the hyperplane class. Then* $\mathrm{lct}(X;|H|_{\mathbb{Q}}) > \frac{1}{2}$.

Proof. By [Bir16b, Theorem 1.5], it suffices to show that for every $D \sim_{\mathbb{Q}} H$ we have $\mathrm{lct}(X;D) > \frac{1}{2}$. By [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(D) \leq 1$ for every subvariety $S \subseteq X$ of dimension r, hence for all $0 < \epsilon \ll 1$, the pair $(X,(1-\epsilon)D)$ is klt outside a subset of dimension at most $r-1$ in X. Let $x \in X$ be an arbitrary point and let $Y = X \cap V \subseteq \mathbb{P}^{n+1}$ be a general linear space section containing x of codimension $r-1$. Let $D_Y = D|_Y$ and $L = (r-1)H|_Y$. Then by adjunction $L - (K_Y + (1-\epsilon)D_Y)$ is ample and the pair $(Y,(1-\epsilon)D_Y)$ is klt outside a finite set of points. So as long as

$$h^0(Y,L) \leq h^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(r-1)) = \binom{n+r}{r-1} \leq \frac{(n-r+1)^{n-r+1}}{(n-r+1)!}, \tag{4.3}$$

we have $\mathrm{lct}(Y;D_Y) > \frac{1}{2}$ by Corollary 4.1.6. Granting this for the moment, then $(Y, \frac{1}{2}D_Y)$ is klt and by inversion of adjunction (see e.g. [Kol13, Theorem 4.9]) $(X, Y + \frac{1}{2}D)$ is plt in a neighbourhood of Y. In particular, $(X, \frac{1}{2}D)$ is klt at x. Since x is arbitrary, we see that $(X, \frac{1}{2}D)$ is klt.

It remains to prove (4.3) when $n \geq 6r$. As $\frac{r^r}{r!} < e^r$, we see that $\binom{n+r}{r-1} < \binom{n+r}{r} \leq \frac{(n+r)^r}{r!} < e^r(a+1)^r$ where $a = \frac{n}{r}$; on the other hand, $\frac{(n-r+1)^{n-r+1}}{(n-r+1)!} > \frac{(n-r)^{n-r}}{(n-r)!} > 2^{n-r} = 2^{(a-1)r}$ when $n-r \geq 6$ (we may assume $r \geq 2$ by Corollary 4.1.5), so (4.3) holds as long as $2^{a-1} \geq e(a+1)$, which is trivial since $a \geq 6$. □

It is not hard to see that one can actually do slightly better if more precise value of $\sigma_{n,\lambda}$ or $\bar{\sigma}_{n,\lambda}$ is known. For example, we have

Lemma 4.2.7. *Let $X \subseteq \mathbb{P}^{n+2}$ be a smooth Fano complete intersection of codimension 2 and dimension $n \geq 4$. Let H be the hyperplane class. Then $\mathrm{lct}(X; |H|_{\mathbb{Q}}) > \frac{1}{2}$.*

Proof. Taking $r = 2$ in (4.3) and using Lemma 4.2.5 instead of Corollary 4.1.6 in the proof of Lemma 4.2.6, we see that is suffices to show that $n+2 = \binom{n+r}{r-1} < \bar{\sigma}_{n-1,1}$. For $n \geq 4$, this follows from [Kol18, Corollary 54]. □

Corollary 4.2.8. *The smooth complete intersection in $\mathbb{P}^6$ of a quadric and a quartic not containing a plane is K-stable.*

Proof. By [Che03], such varieties are birationally superrigid, so the result follows from Theorem 1.1.4 and Lemma 4.2.7. □

Using the same strategy, we now prove the birational superrigidity of Fano complete intersections in large dimension.

Lemma 4.2.9. *Let (X, D) be a pair and $x \in X$. Assume that X is smooth, (X, D) has canonical singularities outside a subset of codimension at least $m+1$ in X, but is not canonical at x. Let $V \subseteq X$ be a general complete intersection subvariety of dimension m containing x, then $(V, D|_V)$ is not log canonical at x.*

Proof. Let $n = \dim X$. By assumption, after taking at most $n-m-1$ hypersurface sections containing x, the restriction of the pair will have isolated non-canonical singularities at x; but then another hypersurface section makes the pair non-lc, and by inversion of adjunction, the restriction of the pair to any further hypersurface section is also non-lc. By choosing the right number of additional hypersurface sections, we obtain the statement of the lemma. □

Proof of Theorem 4.1.1. By Theorem 1.1.4 and Lemma 4.2.6, it suffices to show that X is birationally superrigid. Let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary on X. We need to show that (X, M) has canonical singularities. First note that by [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(M^2) \leq 1$ for every subvariety $S \subseteq X$ of dimension at least $2r$; in other words, there exists a subset $Z \subseteq X$ of dimension at most $2r-1$ such that $\mathrm{mult}_x(M^2) \leq 1$ for all $x \notin Z$. Let $x \in X \backslash Z$ and let S be a general surface section of X containing x, then by [dFEM04, Theorem 0.1], $(S, 2M|_S)$ is lc at x (note that as $M^2|_S$ is a complete intersection

0-dimensional subscheme, its multiplicity at x is the same as the Hilbert-Samuel multiplicity of its defining ideals), hence by inversion of adjunction, $(X, 2M)$ is lc at x as well. It follows that for all $0 < \epsilon \ll 1$, the pair $(X, 2(1-\epsilon)M)$ is klt outside Z. Let $x \in X$ be any point and let $Y \subseteq X$ be cut out by a general linear subspace $V \subseteq \mathbb{P}^{n+r}$ of codimension $2r-1$ containing x. Then $Y \subseteq \mathbb{P}^{n-r+1}$ is also a codimension r complete intersection and we have $K_Y \sim 2(r-1)H$ where H is the restriction of the hyperplane class. Let $D = 2M|_Y$ and $L = 2rH \sim_{\mathbb{Q}} K_Y + D$. Since V is general and $\dim Z \leq 2r-1$, $(Y, (1-\epsilon)D)$ is klt outside a finite set of points (i.e. those in $V \cap Z$). By Corollary 4.1.6 we have $\mathrm{lct}(Y; D) > \frac{1}{2}$ as long as

$$h^0(Y, L) \leq h^0(\mathbb{P}^{n-r+1}, \mathcal{O}_{\mathbb{P}^{n-r+1}}(2r)) = \binom{n+r+1}{2r} < \frac{(n-2r+1)^{n-2r+1}}{(n-2r+1)!}. \tag{4.4}$$

Assuming this inequality for the moment, then $(Y, M|_Y) = (Y, \frac{1}{2}D)$ is klt. On the other hand by [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(M) \leq 1$ for every subvariety $S \subseteq X$ of dimension at least r, so (X, M) is canonical outside a subset of dimension at most $r-1$ in X by [Kol97, (3.14.1)]. Suppose that (X, M) is not canonical at x, then since $r-1 < 2r-1 = \mathrm{codim}_X Y$, $(Y, M|_Y)$ is not lc by Lemma 4.2.9, a contradiction. Hence (X, M) is canonical and we are done.

It remains to prove (4.4) when $n \geq 10r$. Let $m = n - 2r + 1 > 8r$, then as in the proof of Lemma 4.2.6, it is easy to see that (4.4) is implied by the following weaker inequality

$$2^m \geq \left(\frac{e(m+3r)}{2r} \right)^{2r},$$

or equivalently, $2^a \geq \frac{e^2}{4}(a+3)^2$ where $a = \frac{m}{r}$. This last inequality is obviously satisfied as $a > 8$. □

4.3 Conditional birational superrigidity

In an attempt to study the birational geometry of Fano varieties, Suzuki proposed (in a paper that was later withdrawn) the following notion of conditional birational superrigidity:

Definition 4.3.1. Let X be a Fano manifold of Picard number one and let $s \geq 2$ be an integer. Consider the following condition on X:

(C_s) every birational map from X to a Mori fiber space whose undefined locus has codimension at least s is an isomorphism.

We say X is *conditionally birationally superrigid* if it satisfies condition (C_{i_X+1}) where i_X is the index of X (i.e. $-K_X = i_X H$ where H is the ample generator of $\mathrm{Pic}(X)$).

For example, when X has index one, conditional birational superrigidity is just the usual birational superrigidity. On the other hand, if X is a complete intersection of index $i_X \geq 2$, then X does not satisfy condition (C_{i_X}) due to the existence of general linear projections $X \dashrightarrow \mathbb{P}^{i_X - 1}$.

In this section, we apply the techniques from the previous section to give a short proof of the conditional birational superrigidity of Fano complete intersections in large dimension (i.e. Theorem 4.1.2). Indeed, we prove something stronger:

Theorem 4.3.2. *Let $m, r \in \mathbb{Z}_+$, then there exists an integer $N = N(r,m)$ depending only on m and r such that for every smooth Fano complete intersection of codimension r and dimension $n \geq N$ in $\mathbb{P}^{n+r}$ and every movable boundary $M \sim_{\mathbb{Q}} mH$ whose base locus has codimension at least $m+1$ (where H is the hyperplane class), the pair (X, M) is canonical.*

Corollary 4.3.3 (Theorem 4.1.2)**.** *Let $r, s \in \mathbb{Z}_+$, then there exists an integer $N = N(r,s)$ depending only on r and s such that every smooth Fano complete intersection of index s and codimension r in $\mathbb{P}^{n+r}$ is conditionally birationally superrigid if $n \geq N$.*

Remark 4.3.4. It is also conjectured that for every birational map $\phi : X \dashrightarrow X'$ from a Fano hypersurface of index s to a Mori fiber space $f : X' \to S$ that is not an isomorphism, we have $\dim S \leq s - 1$, see e.g. [Puk16a, Conjecture 1.1]. Unfortunately our result doesn't say anything about this conjecture.

As before, we need some estimate of $\sigma_{n,\lambda}$ for the proof of Theorem 4.3.2. This is given as follows:

Lemma 4.3.5. *Fix $\lambda > 0$. Then there exists a constant $c > 1$ (depending on λ) such that $\sigma_{n,\lambda} > c^n$ for $n \gg 0$.*

Proof. We may assume that $\lambda < 1$ since $\sigma_{n,\lambda}$ is non-decreasing in the variable λ. Let $\mathbf{a} \in \mathbb{R}^n_+$ be such that $(\lambda, \cdots, \lambda) \in Q = Q_{\mathbf{a}}$. We may assume that $\mathbf{a} = (a_1, \cdots, a_n)$ where $a_1 \leq \cdots \leq a_n$. We have $a_1 + \cdots + a_n < \frac{1}{\lambda}$. Let $m = \lfloor n\lambda \rfloor$, then $a_1 + \cdots + a_m \leq \frac{m}{n}(a_1 + \cdots + a_n) < \frac{m}{n\lambda} \leq 1$. It follows that every vertex of $[0,1]^m \times \{(0, \cdots, 0)\}$ is contained in Q, hence $\#(Q \cap \mathbb{Z}^n) \geq 2^m > 2^{n\lambda - 1}$ and therefore the statement of the lemma holds for any $1 < c < 2^\lambda$. □

Proof of Theorem 4.3.2. By [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(M^m) \leq m^m$ for every subvariety $S \subseteq X$ of dimension at least mr; in other words, there exists a subset $Z \subseteq X$ of dimension at most $mr - 1$ such that $\mathrm{mult}_x(M^m) \leq m^m$ for every $x \notin Z$. We first claim that (X, M) has canonical singularities outside Z. Suppose this is not the case and (X, M) is not canonical at $x \notin Z$, let $V \subseteq X$ be a general complete intersection subvariety of dimension m containing x, then since (X, M) is obviously canonical outside the base locus of M, which has codimension at least $m+1$, we see that $(V, M|_V)$ is not lc at x by Lemma 4.2.9. But since V is general we have $\mathrm{mult}_x(M|_V^m) = \mathrm{mult}_x(M^m) \leq m^m$ (see e.g. [dFEM03, Proposition 4.5]) and since $M|_V^m$ is a zero dimension complete intersection subscheme, its multiplicity is the same as the Hilbert-Samuel multiplicity of its defining ideal, so by [dFEM04, Theorem 0.1], $(V, M|_V)$ is lc at x, a contradiction. This proves the claim.

By [Suz17, Proposition 2.1] again, we have $\mathrm{mult}_S(M^{m+1}) \leq m^{m+1}$ for every subvariety $S \subseteq X$ of dimension at least $(m+1)r$, hence by a similar application of [dFEM04, Theorem 0.1] as before, the pair $(X, \frac{m+1}{m}M)$ is log canonical outside a subset of dimension at most $mr + r - 1$. Let $x \in X$ be an arbitrary point and let $Y \subseteq X$ be a general linear space section of codimension $mr + r - 1$ containing x, then the pair $(Y, \frac{m+1}{m}M|_Y)$ is log canonical outside a finite set of points. Let $L = (mr + m + r - 1)H$, then since X is Fano, $L - (K_Y + \frac{m+1}{m}M)$ is nef, hence by Corollary 4.2.5 (with $\lambda = m^{-1}$) we see that $\mathrm{lct}(Y, \frac{m+1}{m}M|_Y) \geq \frac{m}{m+1}$ as long as

$$h^0(Y, L) \leq h^0(\mathbb{P}^{n-mr+1}, \mathcal{O}_{\mathbb{P}^{n-mr+1}}(mr + m + r - 1)) = \binom{n + m + r}{mr + m + r - 1} < \sigma_{n,m^{-1}} \quad (4.5)$$

By Lemma 4.3.5, $\sigma_{n,m^{-1}}$ grows exponentially with n, hence (4.5) is always satisfied for $n \geq N$ where N is an integer depending only on m and r. It follows that $(Y, M|_Y)$ is log canonical when $n \geq N$. On the other hand, (X, M) is canonical outside Z, which has codimension at least $n - mr + 1 > \dim Y$ in X, thus by Lemma 4.2.9, (X, M) is also canonical at x. Since $x \in X$ is arbitrary, we are done. □

Remark 4.3.6. For any given m and r, we can always find an explicit $N(r, m)$ using the inequality (4.5) and the estimate $\sigma_{n,m^{-1}} > 2^{\frac{n}{m}-1}$ from the proof of Lemma 4.3.5. For example, we may take $N(1, 2) = 36$, $N(1, 4) = 200$.

Proof of Corollary 4.3.3. Let N be the number given by Theorem 4.3.2 with $m = s$. Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Fano complete intersection of index s, codimension r and dimension n. Suppose that $\phi : X \dashrightarrow X'$ is a birational map from X to a Mori fiber space X' such that ϕ is not an isomorphism and the undefined locus of ϕ has codimension at least $s + 1$. By the usual method of maximal singularities (see e.g. [Puk13b, §2]), we find a movable boundary $M \sim_{\mathbb{Q}} -K_X = sH$ whose base locus is contained in the undefined locus of ϕ (in particular, the base locus has codimension at least $s + 1$) such that the pair (X, M) is not canonical. By Theorem 4.3.2, this is impossible if $n \geq N$. □

4.4 Fano threefolds of degree 6

In this section, we make a more delicate use of Theorem 4.1.4 to prove the K-stability of $X_{2,3} \subseteq \mathbb{P}^5$ (base on Theorem 3.1.1). Again, we start with some lower bound of $\sigma_{n,\lambda}$ and $\bar{\sigma}_{n,\lambda}$. In the surface case, these numbers can be approximated quite precisely using Pick's theorem.

Lemma 4.4.1. *Let $m \in \mathbb{Z}_+$, then $\sigma_{2,m} \geq \frac{1}{2}(4m^2 + 3m + 3)$.*

Proof. Let $\mathbf{a} = (\frac{1}{s}, \frac{1}{t})$ be such that $(m, m) \in Q = Q_{\mathbf{a}}$, then we have $\frac{1}{s} + \frac{1}{t} < \frac{1}{m}$ and $s + t > 4m$. We may slightly decrease s, t and assume that $s, t \notin \mathbb{Z}$. Let $u = \lfloor s \rfloor$ and $v = \lfloor t \rfloor$, then $u + v \geq 4m - 1$. Now consider the polygon P given by the following vertices: $(0, 0)$, $(u, 0)$, (m, m) and $(0, v)$. Clearly $P \subseteq Q$, so it suffices to prove

$$\#(P \cap \mathbb{Z}^2) \geq \frac{1}{2}(4m^2 + 3m + 3). \tag{4.6}$$

On the other hand, by Pick's theorem, we have $i + \frac{1}{2}b = A + 1$ where $i = \#(P^\circ \cap \mathbb{Z}^2)$, $b = \#(\partial P \cap \mathbb{Z}^2) \geq u + v + 2$ and $A = \text{Area}(P) = \frac{1}{2}m(u+v)$, hence

$$\#(P \cap \mathbb{Z}^2) = i + b = A + \frac{1}{2}b + 1 \geq \frac{1}{2}(m+1)(u+v) + 2 \geq \frac{1}{2}(m+1)(4m-1) + 2,$$

which gives (4.6) and we are done. □

Lemma 4.4.2. *Let $m \in \mathbb{Z}_+$, then $\bar{\sigma}_{2,m} \geq m(2m+1)$.*

Proof. Let s, t, Q be as in the proof of Lemma 4.4.1. It suffices to show that $\#(Q \cap \mathbb{Z}^2) \geq m(2m+1)$. We have $\frac{1}{s} + \frac{1}{t} \leq \frac{1}{m}$ and thus $s + t \geq 4m$. Let $0 < \epsilon \ll 1$ and let $u = \lfloor s - \epsilon \rfloor$, $v = \lfloor t - \epsilon \rfloor$, then $u + v \geq s + t - 2 \geq 4m - 2$. Consider again the polygon P given by the vertices $(0,0)$, $(u,0)$, (m,m) and $(0,v)$. As before by Pick's theorem we have

$$\#(P \cap \mathbb{Z}^2) = i + b = A + \frac{1}{2}b + 1 \geq \frac{1}{2}(m+1)(u+v) + 2 \geq (m+1)(2m-1) + 2 = m(2m+1) + 1.$$

Since every lattice point of P (except possibly (m,m)) is contained in Q, we obtain $\#(Q \cap \mathbb{Z}^2) \geq m(2m+1)$ as desired. □

We are now ready to present

Proof of Theorem 4.1.3. Let $D \sim_{\mathbb{Q}} -K_X$ be an effective divisor on X and let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary, by Theorem 3.1.1 it suffices to show that $(X, \frac{1}{4}D + \frac{1}{2}M)$ is klt (note that $n = 3$). Since being klt is preserved under convex linear combination and $\rho(X) = 1$, we may assume that D is irreducible. As $\text{Pic}(X)$ is generated by $-K_X$, we have $D = \frac{1}{r}D_0$ where D_0 is integral and $D_0 \in |-rK_X|$ for some $r \in \mathbb{Z}$. Let H be the hyperplane class on X and let $\Delta = \frac{1}{4}D + \frac{1}{2}M$. Depending on the value of r, we separate into three cases.

(1) First suppose that $r \geq 3$. Then for $0 < \epsilon \ll 1$, $(X, 12(1-\epsilon)\Delta)$ is klt outside a subset of dimension at most 1 since every component of 12Δ has coefficient at most 1. Let $x \in X$ and let $S \subseteq X$ be a general hyperplane section containing x. Let $\Delta_S = 12\Delta|_S$, then $(S, (1-\epsilon)\Delta_S)$ is klt outside a finite number of points. By adjunction S is a smooth K3 surface and $K_S + \Delta_S \sim_{\mathbb{Q}} 9H|_S$. We claim that $\text{lct}(S, \Delta_S) > \frac{1}{12}$. Indeed, by Corollary 4.2.5, it suffices to show that $h^0(S, \mathcal{O}_S(9H)) < \bar{\sigma}_{2,11}$. But by Riemann-Roch, we have

$h^0(S, \mathcal{O}_S(9H)) = \frac{9^2}{2}(H|_S^2) + 2 = 245$ while by Lemma 4.4.2 with $m = 11$ we have $\bar{\sigma}_{2,11} \geq 11 \cdot 23 = 253$, proving the claim. It follows that $(S, \frac{1}{12}\Delta_S) = (S, \Delta|_S)$ is klt, hence by inversion of adjunction, (X, Δ) is also klt at x. Since $x \in X$ is arbitrary, we see that (X, Δ) is klt in this case.

(2) Next suppose that $r = 2$. Let $\Gamma = \frac{4}{3}\Delta = \frac{1}{3}D + \frac{2}{3}M$.

Claim. (X, Γ) is log canonical in dimension 1.

Proof of Claim. Suppose not, let C be a curve in the non-lc locus of (X, Γ). We first show that C is a line. Otherwise if S is a general hyperplane section then by adjunction $(S, \Gamma|_S)$ is not log canonical at at least 2 points (those in $C \cap S$), say, x_1 and x_2. Let $\Delta_S = 6\Gamma|_S$, then as before $(S, (1-\epsilon)\Delta_S)$ is klt outside a finite set of points for $0 < \epsilon \ll 1$ and we have $K_S + \Delta_S \sim_{\mathbb{Q}} 6H|_S$. As $(S, \Gamma|_S)$ is not lc at x_i $(i = 1, 2)$, we have $\mathrm{lct}(S; \Delta_S) < \frac{1}{6}$ in the neighbourhood of x_i, thus by Theorem 4.1.4 and Remark 4.2.1 we have $\mathrm{lct}(S; \Sigma_i) < \frac{1}{5}$ where Σ_i is a 0-dimensional subscheme supported at x_i and $\ell(\mathcal{O}_{\Sigma_1}) + \ell(\mathcal{O}_{\Sigma_2}) \leq h^0(S, \mathcal{O}_S(6H)) = \frac{6^2}{2}(H|_S^2) + 2 = 110$. But by Lemma 4.2.4 and 4.4.1 with $m = 5$ we also have $\ell(\mathcal{O}_{\Sigma_i}) \geq \sigma_{2,5} \geq 59$ $(i = 1, 2)$, a contradiction. Hence $\deg C \leq 1$ and C is a line.

We next prove that $\mathrm{mult}_C D \leq 1$, or equivalently, $s := \mathrm{mult}_C D_0 \leq 2$. To see this, take a general hyperplane section S containing the line C. By dimension count it is easy to see that S is smooth. We have $D_0|_S = sC + Z$ where Z is integral. As S is a K3 surface and $C \cong \mathbb{P}^1$, we have $(H|_S^2) = 6$, $(H \cdot C) = 1$, $(C^2) = -2$ and hence $(Z^2) = (2H|_S - sC)^2 = 24 - 4s - 2s^2$. On the other hand, since Z is an integral curve on a K3 surface we have $(Z^2) \geq -2$. Thus $24 - 4s - 2s^2 \geq -2$ and it follows that $s \leq 2$ as $s \in \mathbb{Z}$.

Now if (X, Γ) is not lc along C, then by [dFEM04, Theorem 2.2] we have

$$\left(\frac{2}{3}\right)^2 \mathrm{mult}_C(M^2) > 4\left(1 - \frac{1}{3}\mathrm{mult}_C D\right) \geq \frac{8}{3}$$

or $\mathrm{mult}_C(M^2) > 6$. But $\mathrm{mult}_C(M^2) = \mathrm{mult}_C(M^2) \cdot (H \cdot C) \leq (M^2 \cdot H) = 6$, a contradiction. This proves the claim. □

It follows from the claim that $(X, (1-\epsilon)\Gamma)$ is klt outside a finite set of points. Note that $K_X + \Gamma \sim_{\mathbb{Q}} 0$, we may apply Theorem 4.1.4 with $L = 0$ and $\lambda = n$ (as in the proof of Corollary 4.1.5) to conclude that $\mathrm{lct}(X;\Gamma)\ge\frac{3}{4}$, with equality if and only if $\mathrm{mult}_x\Gamma=4$ for some $x\in X$. But it is easy to see that $\mathrm{mult}_x D\le(D\cdot H^2)=6$ and $(\mathrm{mult}_x M)^2\le(M^2\cdot H)=6$, thus $\mathrm{mult}_x\Gamma=\frac{1}{3}\mathrm{mult}_x D+\frac{2}{3}\mathrm{mult}_x M\le 2+\frac{2}{3}\sqrt{6}<4$. Therefore the equality of $\mathrm{lct}(X;\Gamma)\ge\frac{3}{4}$ in never achieved and $(X,\Delta)=(X,\frac{3}{4}\Gamma)$ is klt as desired.

(3) We are left with the case $r=1$, in other words, D is a hyperplane section. Let $\Gamma=\frac{4}{3}\Delta$ be as in the previous case. Again we claim

Claim. (X,Γ) is log canonical in dimension 1.

Proof of Claim. The proof is very similar to the previous case, so we only give a sketch. Let C be a curve in the non-lc locus of (X,Γ). We have $\deg C\le 2$ (otherwise there exists 0-dimensional subscheme Σ on a general hyperplane section S such that $\ell(\mathcal{O}_\Sigma)\le\lfloor\frac{1}{3}h^0(S,\mathcal{O}_S(3H))\rfloor=9$ and $\mathrm{lct}(S;\Sigma)<\frac{1}{2}$, but the latter inequality implies $\ell(\mathcal{O}_\Sigma)\ge\sigma_{2,2}\ge 13$ by Lemma 4.2.4 and 4.4.1 with $m=2$, a contradiction). If C is a line, we simply argue as in the previous case (i.e. take a general hyperplane section containing C to prove $\mathrm{mult}_C D\le 1$ and then apply [dFEM04, Theorem 2.2] to get a contradiction). So we assume that C is a conic. We claim that $\mathrm{mult}_C D\le 2$. Suppose not, then we have $\mathrm{mult}_C D\ge 3$. Let S be a general hyperplane section containing C, then S is smooth along C: otherwise, there exists $x\in C$ such that $\mathrm{mult}_x S\ge 2$; as $\mathrm{mult}_x D\ge 3$, $D\cap S$ is a curve with degree 6 and multiplicity at least 6 at x, hence it is a union of 6 lines; but $D\cap S$ already contains the conic, a contradiction. As $6\ge\deg(D\cdot S)\ge\mathrm{mult}_C D\cdot\deg C\ge 6$, we must have $D|_S=3C$. Since D is a hyperplane section we have $(C^2)>0$ on S; but as $C\cong\mathbb{P}^1$ is in the smooth locus of the singular K3 surface S, by adjunction we have $(C^2)=-2$, a contradiction. Hence we always have $\mathrm{mult}_C D\le 2$. Now another application of [dFEM04, Theorem 2.2] gives

$$\left(\frac{2}{3}\right)^2\mathrm{mult}_C(M^2)>4\left(1-\frac{1}{3}\mathrm{mult}_C D\right)\ge\frac{4}{3}$$

or $\mathrm{mult}_C(M^2)>3$ and therefore $(M^2\cdot H)\ge\mathrm{mult}_C(M^2)\cdot\deg C>6$, a contradiction. This proves the claim. □

So we are in the same situation as in the previous case and the rest of the proof is identical to the one there. Hence in all cases (X,Δ) is klt and we conclude that X is

K-stable using Theorem 3.1.1. □

Chapter 5

Singular case

5.1 Introduction

Thanks to the work of [IM71, Puk86, Che00, dFEM03, dF16, Suz17, Fuj16a] and the results from the previous chapter, the birational superrigidity and K-stability of smooth complete intersections of index one is now better understood. However, when the Fano varieties are singular, the situation is less clear, even for hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of index one with only ordinary singularities (i.e. isolated singularities whose projective tangent cone is smooth). In general, they fail to be birationally superrigid or K-stable when they are too singular. For example, X is rational (resp. admits a birational involution) if it has a point of multiplicity n (resp. $n-1$), and therefore is not birationally superrigid. By [Liu18], we also know that a Fano variety is not K-(semi)stable if the local volume at a singular point is too small. On the other hand, it is expected that the index one hypersurfaces X are birationally superrigid and K-stable when their singularities are mild. Indeed, Pukhlikov [Puk02c] shows that a *general* hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $n+1$ having a singular point of multiplicity $\mu \leq n-2$ is birationally superrigid (see also [EP18] which treats general complete intersections whose projective tangent cones are intersections of hyperquadrics), de Fernex [dF17] establishes their birational superrigidity under certain numerical assumption (involving the dimension of singular locus and the Jacobian ideals of linear space sections of X) which in particular applies to index one hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with ordinary singularities of multiplicity at most $2\sqrt{n+1}-7$ and Suzuki [Suz17] generalizes his method to certain singular complete intersections. As for K-stability, very little is known except in small dimensions or small degrees, e.g. log del Pezzo surfaces [OSS16b], quasi-smooth 3-folds [JK01], complete intersection of two quadric hypersurfaces in all dimensions [SS17], and cubic threefolds with isolated A_k ($k \leq 4$) singularities [LX17b].

In this chapter (mostly based on joint work with Yuchen Liu), we introduce arguments that can be used to prove both birational superrigidity and K-stability for a large class of singular Fano complete intersections of Fano index one (i.e. $-K_X$ is linearly equivalent to the hyperplane class). In particular, we have the following results.

Theorem 5.1.1. *Let $\delta \geq -1$ and $m \geq 1$ be integers. Let $X \subseteq \mathbb{P}^{n+1}$ be a hypersurface of degree $n+1$. Assume that*

1. *The singular locus of X has dimension at most δ and $n \geq \frac{3}{2}\delta + 1$;*
2. *X has multiplicity $\mathrm{e}(x, X) \leq m$ at every $x \in X$ and the corresponding tangent cone is smooth in codimension $\mathrm{e}(x, X) - 1$;*
3. $\frac{1}{m^m} \cdot \frac{(n-2-\delta)^{n-2-\delta}}{(n-2-\delta)!} \geq \binom{n+2}{\delta+3}$.

Then X is birationally superrigid and K-stable.

Theorem 5.1.2. *Let $X \subseteq \mathbb{P}^{n+1}$ be a hypersurface of degree $n+1$ and dimension $n \geq 250$ with only isolated ordinary singularities (i.e. the projective tangent cones are smooth) of multiplicities at most m. Then*

1. *X is K-stable if $m \leq n$;*
2. *X is birationally superrigid if $m \leq n-2$;*
3. *X is birationally rigid if $m \leq n-1$. Moreover, linear projection from each point $x \in X$ of multiplicity $n-1$ induces a birational involution τ_x and the birational automorphism group $\mathrm{Bir}(X)$ of X is generated by $\mathrm{Aut}(X)$ together with these τ_x.*

We note that in Theorem 5.1.2, the assumptions on the multiplicities are optimal. While our results are stated for hypersurfaces here, the techniques are indeed applicable to complete intersections and even weighted complete intersections as well. We refer to §5.2 and the constructions in the last chapter for more details.

It may be interesting to look at the asymptotics of Theorem 5.1.1 as the dimension gets large. By Stirling's formula, the condition (3) can be replaced by the following weaker inequality (at least when $n-\delta \gg 0$):

$$e^{n-2-\delta} \geq m^m (n+2)^{\delta+3}.$$

After taking the logarithm, it is not hard to see that this inequality is satisfied if $\delta, m \leq Cn^{1-\epsilon}$ for some fixed constants $C, \epsilon > 0$ and $n \gg 0$. Hence our conditions are asymptotically better than those of [dF17].

Our proof of these results has close ties to some local invariants of the singularities, i.e. the minimal non-klt (resp. minimal non-lc) colengths defined as introduced in Definition 4.2.2. These local invariants govern the birational superrigidity and K-stability of index one complete intersection in a certain sense. More precisely, we prove the following criterion.

Theorem 5.1.3. *Let $\delta \geq -1$ be an integer and let $X \subseteq \mathbb{P}^{n+r}$ be a Fano complete intersection of index 1, codimension r and dimension n. Assume that*

1. *The singular locus of X has dimension at most δ and $n \geq 2r+\delta+2$;*
2. *For every $x \in X$ and every general linear space section $Y \subseteq X$ of codimension $2r+\delta$ containing x, we have $\ell_{\mathrm{nlc}}(x, Y) > \binom{n+r+1}{2r+\delta+1}$.*

Then X is birationally superrigid. If in addition,

3. *For every $x \in X$ and every general linear space section $Y \subseteq X$ of codimension $r+\delta$ containing x, we have $\ell_{\mathrm{nklt}}(x, Y) > \binom{n+r}{r+\delta}$,*

then X is also K-stable.

When the singularities are mild, the corresponding minimal non-klt (resp. non-lc) colengths tend to be large and indeed they are often exponential in n for a given singularity type. In particular, they grow faster than any polynomial in n and the above criterion automatically applies to yield the birational superrigidity and K-stability of many complete intersections in large dimension. As such, the core of our argument consists of finding suitable lower bounds of minimal non-klt (resp. non-lc) colengths for the various singularities we encounter. This can be done in several different ways by relating these local invariants to the conditional birational superrigidity (as we do for Theorem 5.1.2) or K-stability (as in the proof of Theorem 5.1.1) of the tangent cone of the singular point, which is a Fano variety of smaller dimension. Hence essentially we obtain an inductive argument for proving birational superrigidity and K-stability using lower dimensional information.

A natural lower bound of the minimal non-klt (non-lc) colength is given by the local volume and normalized colength (see §5.3 for notation and definition) of the singularity, as introduced in [Li18, BL18a]. While these numbers are hard to compute in general, we have an explicit formula for the cone over a K-semistable base as, by the work of [LL19, LX16], the local volume is computed by the blowup of the vertex in this case. By degeneration and semi-continuity argument, we then get an estimate of local volumes for certain ordinary singularities (indeed, a conjectural lower bound for every ordinary singularity as well). For general singularities, we may try to reduce to this case by taking hyperplane sections. Therefore, a key step in our approach is a comparison between the local volumes of the singularity and its hypersurface section.

Theorem 5.1.4. *Let $x \in (X, D)$ be a klt singularity of dimension n. Let H be a normal reduced Cartier divisor of X containing x. Assume that H is not an irreducible component of D. Then we have*

$$\frac{\widehat{\mathrm{vol}}(x, X, D)}{n^n} \geq \frac{\widehat{\mathrm{vol}}(x, H, D|_H)}{(n-1)^{n-1}}, \quad \frac{\widehat{\ell}(x, X, D)}{n^n/n!} \geq \frac{\widehat{\ell}(x, H, D|_H)}{(n-1)^{n-1}/(n-1)!}.$$

One may interpret this as saying that "local volume (resp. normalized colength) density" does not increase upon taking hypersurface section, reflecting the principle that singularities can only get worse after restriction to a hypersurface.

5.2 An elementary criterion

We start with the proof of Theorem 5.1.3, which relies on the following result:

Lemma 5.2.1. *Let X be a projective variety with klt singularities and L an ample line bundle on X. Let $D \sim_{\mathbb{Q}} L - K_X$ be a divisor on X such that (X, D) is log canonical outside* *a finite set T of points. Assume that $h^0(X,L) < \ell_{\mathrm{nklt}}(x,X)$ (resp. $< \ell_{\mathrm{nlc}}(x,X)$) for every $x \in T$, then* $\mathrm{lct}(X;D) > \frac{1}{2}$ *(resp.* $\geq \frac{1}{2}$*).*

Proof. This is a special case of Theorem 4.2.3 (with $\Delta = 0$ and $\lambda = 1$). □

Proof of Theorem 5.1.3. We first prove the birational superrigidity of X under the assumptions (1) and (2). As $n \geq 2r+\delta+2 \geq \delta+4$, the singular locus of X has codimension at least 4 and by [Gro68, Corollaire 3.14], X is factorial. We also have $\rho(X) = 1$ since the cone over X is also factorial by another application of [Gro68, Corollaire 3.14]. By assumption (2), for every $x \in X$, a general complete intersection of codimension $2r+\delta > \delta+1$ containing x has klt singularities, hence by inversion of adjunction, X has terminal singularities (note that if $x \in X$ is an isolated singularity and is not terminal, then a hyperplane section containing x is not klt). By Theorem 2.2.5, it remains to show that the pair (X, M) is canonical for every movable boundary $M \sim_{\mathbb{Q}} -K_X$.

Let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary. By [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(M^2) \leq 1$ for every subvariety S of dimension $\geq 2r$ in the smooth locus of X. It follows that $\mathrm{mult}_x(M^2) \leq 1$ outside a subset Z of dimension $\leq 2r+\delta$ (since a general complete intersection of codimension $\delta+1$ in X is smooth). By [dFEM04, Theorem 0.1], $(X, 2M)$ is log canonical outside Z. Let $x \in X$ and let $Y = X \cap \mathbb{P}^{n-r-\delta}$ be a general linear space section of codimension $2r+\delta$ containing x, then $(Y, 2M|_Y)$ is lc outside a finite set of points. We have $K_Y + 2M|_Y \sim_{\mathbb{Q}} (2r+\delta+1)H = L$ where H is the hyperplane class. By Lemma 5.2.1 and the assumption (2), we have $\mathrm{lct}(Y; 2M|_D) \geq \frac{1}{2}$ since $h^0(Y,L) \leq h^0(\mathbb{P}^{n-r-\delta}, \mathcal{O}_{\mathbb{P}^{n-r-\delta}}(2r+\delta+1)) = \binom{n+r+1}{2r+\delta+1}$. Therefore, (Y, M) is log canonical and by inversion of adjunction as before, (X, M) has canonical singularities and hence X is birationally superrigid.

Now assume that X also satisfies (3). By Theorem 1.1.4, X is K-stable as long as $\mathrm{lct}(X;D) > \frac{1}{2}$ for every $D \sim_{\mathbb{Q}} -K_X$. By [Suz17, Proposition 2.1], we have $\mathrm{mult}_S(D) \leq 1$ for every subvariety S of dimension $\geq r$ in the smooth locus of X hence $\mathrm{mult}_x(D) \leq 1$ and (X, D) is log canonical outside a subset Z of dimension $\leq r+\delta$. Let $x \in X$ and let Y be a general linear space section of codimension $r+\delta$ containing x, then $(Y, D|_Y)$ is lc outside a finite set of points. Let $L = (r+\delta)H \sim_{\mathbb{Q}} K_Y + D|_Y$, then as

$$h^0(Y,L) \le h^0(\mathbb{P}^{n-\delta}, \mathcal{O}_{\mathbb{P}^{n-\delta}}(r+\delta)) = \binom{n+r}{r+\delta} < \ell_{\mathrm{nklt}}(x,Y)$$

by assumption (3), we get $\mathrm{lct}(Y, D|_Y) > \frac{1}{2}$ by Lemma 5.2.1, hence by inversion of adjunction we get $\mathrm{lct}(X; D) > \frac{1}{2}$ as desired. □

Using this criterion, we prove a preliminary version of Theorem 5.1.2.

Lemma 5.2.2. *Given $m, r \in \mathbb{Z}_+$, then there exists a constant N_0 depending only on m and r such that for every ordinary canonical complete intersection singularity $x \in X$ of dimension $n \ge N_0$ and embedding codimension r whose tangent cone has Fano index at least m, we have*

$$\ell_{\mathrm{nklt}}(x,X) \ge \binom{n-1}{m-1}, \quad \ell_{\mathrm{nlc}}(x,X) \ge \binom{n-1}{m}.$$

Proof. For simplicity we only prove the inequality for minimal non-lc colength, since the other case is very similar. By [Zhu18b, Theorem A.2], there exists a constant N_0 depending only on m and r such that for every smooth Fano complete intersection (in some $\mathbb{P}^N$) of codimension r and dimension at least $N_0 - 1$ and every movable boundary $M \sim_{\mathbb{Q}} mH$ whose base locus has codimension at least $m+1$ (where H is the hyperplane class), the pair (X, M) is canonical. We will prove that the lemma holds under this choice of N_0. In other words, given $n \ge N_0$ and X as in the statement of the lemma, we need to show that $\ell(\mathcal{O}_X/\mathfrak{a}) \ge \binom{n-1}{m}$ for every $\mathfrak{a}$ co-supported at x such that $(X, \mathfrak{a})$ is not lc.

First notice that by the same degeneration argument as in §5.3, it suffices to prove this under the assumption that X is the cone over a Fano complete intersection $V \subseteq \mathbb{P}^{n-1+r}$ of codimension r and index $s \ge m$ and the ideal $\mathfrak{a}$ is homogeneous. In this case, we have $X = \mathrm{Spec}(R)$ where

$$R = \bigoplus_{i=0}^{\infty} H^0(V, \mathcal{O}_V(i))$$

and $\mathfrak{a} \subseteq R$ is given by a graded system of linear series $\mathcal{M}_\bullet = (\mathcal{M}_i)$ where $\mathcal{M}_i \subseteq H^0(V, \mathcal{O}_V(i))$. Clearly $\ell(R/\mathfrak{a}) \ge h^0(V, \mathcal{O}_V(i)) - \dim \mathcal{M}_i$ for all $i \ge 0$, hence the lemma would immediately follow once we prove the following claim:

Claim. If $n \geq N_0$, then $h^0(V, \mathcal{O}_V(m)) - \dim \mathcal{M}_m \geq \binom{n-1}{m}$.

To see this, let Z be the base locus of $\mathcal{M}_m$. Suppose that every component of Z has codimension at least $m+1$, then since $\dim V = n-1 \geq N_0-1$, the pair $(V, \mathcal{M}_m)$ is canonical by the choice of N_0. As V has Fano index at least m, $-(K_V + \mathcal{M}_m)$ is nef, hence the cone over $(V, \mathcal{M}_m)$ is lc by [Kol13, Lemma 3.1]. In particular (since $\mathcal{M}_m \subseteq \mathfrak{a}$), the pair $(X, \mathfrak{a})$ is lc, contrary to our assumption on $\mathfrak{a}$. It follows that some irreducible component, say, Z_0 of Z has codimension at most m. In other words, $\dim Z_0 \geq n-1-m$.

Let $\pi : V \dashrightarrow \mathbb{P}^{n-1-m}$ be a general linear projection whose restriction to Z_0 is generically finite. Consider $\mathcal{N} = f^*|\mathcal{O}_{\mathbb{P}^{n-1-m}}(m)| \subseteq H^0(V, \mathcal{O}_V(m))$, then it is easy to see that $\dim \mathcal{N} = \binom{n-1}{m}$, so the claim would follow if we have $\mathcal{M} \cap \mathcal{N} = \{0\}$. This last statement holds since every element of $\mathcal{M}$ vanishes along Z_0 while by construction of π, none of the elements of $\mathcal{N}$ (except zero) is identically zero along Z_0. We thus complete the proof of the claim and hence the lemma as well. □

Corollary 5.2.3. *Let $\delta \geq -1$ and $r \geq 1$ be integers, then there exists a constant N depending only on δ and r such that if $X \subseteq \mathbb{P}^{n+r}$ is a Fano complete intersection of Fano index 1, codimension r and dimension $n \geq N$ such that*

1. *The singular locus of X has dimension at most δ,*
2. *Every (projective) tangent cone of X is a Fano complete intersection of index at least $4r + 2\delta + 2$ and is smooth in dimension $r + \delta$,*

then X is birationally superrigid and K-stable.

Proof. Let $x \in X$ and let $Y \subseteq X$ be a general complete intersection of codimension $2r + \delta$ containing x, then the second assumption implies that Y has smooth tangent cone of Fano index at least $2r + \delta + 2$ at x. By Lemma 5.2.2, $\ell_{\text{nlc}}(x, Y)$ grows at least like a polynomial in n of degree $2r + \delta + 2$, hence for $n \gg 0$ we have $\ell_{\text{nlc}}(x, Y) > \binom{n+r+1}{2r+\delta+1}$ and therefore X is birationally superrigid by Theorem 5.1.3. The proof of K-stability is similar. □

Corollary 5.2.4. *There exists an absolute constant $N \leq 250$ such that every degree $n+1$ hypersurface $X \subseteq \mathbb{P}^{n+1}$ with only ordinary singularities of multiplicity at most $n-5$ is birationally superrigid and K-stable when $n \geq N$.*

Proof. The existence of N follows by taking $\delta = 0$ and $r = 1$ in the previous corollary. By Remark 4.3.6, we may take $N_0 = 200$ in Theorem 4.3.2 for $m = 4$, $r = 1$. Hence $N \leq 250$ by a careful inspection of the inequalities involved in the proof of Corollary 5.2.3 (with $\delta = 0$ and $r = 1$). $\square$

5.3 Adjunction for local volumes and normalized colengths

In this section we prove Theorem 5.1.4 and its stronger form Theorem 5.3.8, using a degeneration argument similar to [LX16]. First let us recall the definition of the local volume and normalized colength of a singularity.

Definition 5.3.1. Let $x \in (X, D)$ be a klt singularity of dimension n. We define the *local volume* $\widehat{\mathrm{vol}}(x, X, D)$ and the *normalized colength* $\widehat{\ell}(x, X, D)$ of $x \in (X, D)$ by

$$\widehat{\mathrm{vol}}(x, X, D) := \inf_{\mathfrak{a}\colon \mathfrak{m}_x\text{-primary}} \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \mathrm{e}(\mathfrak{a}),$$

$$\widehat{\ell}(x, X, D) := \inf_{\mathfrak{a}\colon \mathfrak{m}_x\text{-primary}} \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \ell(\mathcal{O}_{X,x}/\mathfrak{a}).$$

If $x \in (X, D)$ is not klt, we set $\widehat{\mathrm{vol}}(x, X, D) = \widehat{\ell}(x, X, D) = 0$. When $D = 0$, we will simply write $\widehat{\mathrm{vol}}(x, X)$ and $\widehat{\ell}(x, X)$.

Note that our definition of local volume of a singularity is equivalent to Li's original definition [Li18] in terms of valuations by [Liu18, Theorem 27]. A refined version of normalized colengths was introduced in [BL18a] in order to prove the lower semicontinuity of local volumes in families. These two invariants are related by the following inequality.

Proposition 5.3.2. *For any klt singularity $x \in (X, D)$ of dimension n, we have*

$$\frac{1}{n!\mathrm{e}(x, X)} \widehat{\mathrm{vol}}(x, X, D) \leq \widehat{\ell}(x, X, D) \leq \frac{1}{n!} \widehat{\mathrm{vol}}(x, X, D).$$

If moreover x is a smooth point on X, then $\widehat{\mathrm{vol}}(x, X, D) = n! \cdot \widehat{\ell}(x, X, D)$.

Proof. The first inequality follows from Lech's inequality [Lec60, Theorem 3]

$$n! \cdot \ell(\mathcal{O}_X/\mathfrak{a}) \cdot \mathrm{e}(x, X) \geq \mathrm{e}(\mathfrak{a}).$$

The second inequality follows from the fact that $n!\ell(\mathcal{O}_X/\mathfrak{a}^m) = \mathrm{e}(\mathfrak{a})m^n + O(m^{n-1})$. $\square$

We now proceed with the proof of Theorem 5.1.4. For simplicity, we may assume $X = \mathrm{Spec}(R)$ is affine and (X, D) is klt. In addition, we may assume that $K_X + D \sim_{\mathbb{Q}} 0$ by shrinking X if necessary. Let $\mathfrak{m}$ be the maximal ideal of R whose cosupport is x. Let $H = (h = 0)$ be a normal Cartier divisor on X with $h \in \mathfrak{m}$. Denote $A := R/(h)$ so that $H = \mathrm{Spec}\, A$. In this section, H can be an irreducible component of D. Consider the extended Rees algebra $\mathcal{R} := \oplus_{k\in\mathbb{Z}}\mathfrak{a}_k t^{-k}$ where $\mathfrak{a}_k := \mathfrak{a}_k(\mathrm{ord}_H) = (h^{\max\{k,0\}})$. Clearly, $\mathcal{R}$ is a sub $\mathbb{C}[t]$-algebra of $R[t, t^{-1}]$. From [LX16, Section 4] we know that

$$\mathcal{R} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong R[t, t^{-1}], \quad \mathcal{R} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong \oplus_{k\in\mathbb{Z}_{\geq 0}}\mathfrak{a}_k/\mathfrak{a}_{k+1} =: T.$$

For any $f \in R$, suppose $k = \mathrm{ord}_H(f)$. Then we define $\tilde{f} \in \mathcal{R}$ to be the element $ft^{-k} \in \mathcal{R}_k$. Denote by $\mathbf{in}(f) := [f]_{\mathfrak{a}_{k+1}} \in \mathfrak{a}_k/\mathfrak{a}_{k+1} = T_k$. For an ideal $\mathfrak{b}$ of R, let $\mathfrak{B}$ be the ideal of $\mathcal{R}$ generated by $\{\tilde{f} : f \in \mathfrak{b}\}$. Let $\mathbf{in}(\mathfrak{b})$ be the ideal of T generated by $\{\mathbf{in}(f) : f \in \mathfrak{b}\}$. Let $\mathfrak{m}_T := \mathbf{in}(\mathfrak{m})$, then it is clear that $\mathfrak{m}_T$ is a maximal ideal of T.

Lemma 5.3.3 ([LX16, Lemma 4.1]). *1. We have the identities:*

$$\mathcal{R}/\mathfrak{B} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong (R/\mathfrak{b})[t, t^{-1}], \quad \mathcal{R}/\mathfrak{B} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t) \cong T/\mathbf{in}(\mathfrak{b});$$

2. The $\mathbb{C}[t]$-algebra $\mathcal{R}/\mathfrak{B}$ is free and thus flat as a $\mathbb{C}[t]$-module. In particular, we have the identity of dimensions:

$$\dim_{\mathbb{C}}(R/\mathfrak{b}) = \dim_{\mathbb{C}}(T/\mathbf{in}(\mathfrak{b}));$$

3. If $\mathfrak{b}$ is $\mathfrak{m}$-primary, then $\mathbf{in}(\mathfrak{b})$ is an $\mathfrak{m}_T$-primary homogeneous ideal.

Since $\mathcal{R}$ is a $\mathbb{C}[t]$-algebra that is a flat $\mathbb{C}[t]$-module, it induces a flat morphism $\pi : \mathcal{X} = \mathrm{Spec}\,\mathcal{R} \to \mathbb{A}^1$. We know that $\mathcal{X} \setminus \mathcal{X}_0 = \pi^{-1}(\mathbb{A}^1 \setminus \{0\}) \cong X \times (\mathbb{A}^1 \setminus \{0\})$. Let $\mathcal{D}$ be the Zariski closure of $D \times (\mathbb{A}^1 \setminus \{0\})$. Then $(K_{\mathcal{X}} + \mathcal{D})|_{\mathcal{X}\setminus\mathcal{X}_0}$ corresponds to the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $(K_X + D) \times (\mathbb{A}^1 \setminus \{0\})$ on $X \times (\mathbb{A}^1 \setminus \{0\})$. Since $K_X + D \sim_{\mathbb{Q}} 0$, we know that

$K_{\mathcal{X}} + \mathcal{D}|_{\mathcal{X}\setminus\mathcal{X}_0} \sim_{\mathbb{Q}} 0$. Hence $K_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}$-linearly equivalent to a multiple of $\mathcal{X}_0$. Since $\mathcal{X}_0$ is Cartier on $\mathcal{X}$, we have that $K_{\mathcal{X}} + \mathcal{D}$ is $\mathbb{Q}$-Cartier on $\mathcal{X}$. Hence $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1$ is a $\mathbb{Q}$-Gorenstein flat family.

Next we analyze the divisor $\mathcal{D}_0$. Denote $D = cH + \sum_{i=1}^{l} c_i D_i$ where $c_i > 0$ for each $1 \leq i \leq l$. Let $\mathfrak{p}_i$ be the height 1 prime ideal in R corresponding to the generic point of D_i. Let $\mathcal{D}_i$ be the Zariski closure of $D_i \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{X}$. Then we know that the generic point $\mathcal{D}_i$ corresponds to a height 1 prime ideal $\mathfrak{q}_i$ of $\mathcal{R}$, where

$$\mathfrak{q}_i = \mathfrak{p}_i[t, t^{-1}] \cap \mathcal{R} = \oplus_{k\in\mathbb{Z}}(\mathfrak{p}_i \cap \mathfrak{a}_k)t^{-k}.$$

Hence $\mathcal{D}_{i,0}$ is the same as the divisorial part of the ideal sheaf $\mathfrak{q}_i|_{\mathcal{X}_0} = \oplus_{k\in\mathbb{Z}_{\geq 0}}(\mathfrak{a}_k \cap \mathfrak{p}_i + \mathfrak{a}_{k+1})/\mathfrak{a}_{k+1}$ on $\mathcal{X}_0 = \operatorname{Spec} T$. Since $\mathfrak{a}_k = (h^k)$ for $k \geq 0$ and $h \notin \mathfrak{p}_i$, we know that $T \cong A[s]$ with $A = R/(h)$, and $\mathfrak{a}_k \cap \mathfrak{p}_i = (h^k)\mathfrak{p}_i$. Under this isomorphism, $\mathfrak{q}_i|_{\mathcal{X}_0}$ corresponds to $(\mathfrak{p}_i + (h)/(h))[s]$. Thus $\mathcal{D}_{i,0}$ corresponds to $D_i|_H \times \mathbb{A}^1$ under the isomorphism $\mathcal{X}_0 \cong H \times \mathbb{A}^1$. Similarly, let $\mathcal{H}$ be the Zariski closure of $H \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{X}$. Let $\mathfrak{q}$ be the height 1 prime ideal of $\mathcal{R}$ corresponding to the generic point of $\mathcal{H}$. We have

$$\mathfrak{q} = (h)[t, t^{-1}] \cap \mathcal{R} = \oplus_{k\in\mathbb{Z}}((h) \cap \mathfrak{a}_k)t^{-k}.$$

Hence $\mathcal{H}_0$ is the same as the divisorial part of the ideal sheaf $\mathfrak{q}|_{\mathcal{X}_0} = \oplus_{k\in\mathbb{Z}_{\geq 0}}(\mathfrak{a}_k \cap (h) + \mathfrak{a}_{k+1})/\mathfrak{a}_{k+1}$ on $\mathcal{X}_0$. Under the isomorphism $T \cong A[s]$, it is easy to see that $\mathfrak{q}|_{\mathcal{X}_0}$ corresponds to the principal ideal (s). Thus $\mathcal{H}_0$ corresponds to $H \times \{0\}$ under the isomorphism $\mathcal{X}_0 \cong H \times \mathbb{A}^1$. To summarize, we have shown the following proposition.

Proposition 5.3.4. *With the above notation, the family $\pi : (\mathcal{X}, \mathcal{D}) \to \mathbb{A}^1$ is $\mathbb{Q}$-Gorenstein and flat. Moreover, if $D = cH + \sum_{i=1}^{l} c_i D_i$ then*

$$(\mathcal{X}_0, \mathcal{D}_0) \cong (H \times \mathbb{A}^1, c(H \times \{0\}) + \sum_{i=1}^{l} c_i(D_i|_H \times \mathbb{A}^1)).$$

Let $x_0 \in \mathcal{X}_0$ be the closed point corresponding to $\mathfrak{m}_T$. Then it is clear that π provides a special degeneration of $(x \in (X, D))$ to $(x_0 \in (\mathcal{X}_0, \mathcal{D}_0))$. By Proposition 5.3.4, there is a natural $\mathbb{G}_m$-action on $(x_0 \in (\mathcal{X}_0, \mathcal{D}_0))$ induced by the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$. We define the $\mathbb{G}_m$*-invariant local volume/normalized colength* of $(x_0 \in (\mathcal{X}_0, \mathcal{D}_0))$ as

$$\widehat{\mathrm{vol}}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0) := \inf\{\mathrm{lct}(\mathcal{X}_0, \mathcal{D}_0; \mathfrak{a})^n \cdot \mathrm{e}(\mathfrak{a}) \mid \mathfrak{a} \text{ is } \mathfrak{m}_T\text{-primary and } \mathbb{G}_m\text{-invariant}\};$$
$$\widehat{\ell}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0) := \inf\{\mathrm{lct}(\mathcal{X}_0, \mathcal{D}_0; \mathfrak{a})^n \cdot \ell(T/\mathfrak{a}) \mid \mathfrak{a} \text{ is } \mathfrak{m}_T\text{-primary and } \mathbb{G}_m\text{-invariant}\}.$$

By [LX16, Section 4], our definition of $\mathbb{G}_m$-invariant local volume is the same as the infimum of normalized volumes of all $\mathbb{G}_m$-invariant valuations.

The following lemma is similar to [LX16, Lemma 4.3 and 4.4].

Lemma 5.3.5. *With the above notation, let $\mathfrak{b}$ be an $\mathfrak{m}$-primary ideal of R. Then*

$$\mathrm{lct}(X, D; \mathfrak{b}) \geq \mathrm{lct}(\mathcal{X}_0, \mathcal{D}_0, \mathbf{in}(\mathfrak{b})), \quad \ell(R/\mathfrak{b}) = \ell(T/\mathbf{in}(\mathfrak{b})).$$

Moreover,

$$\widehat{\ell}(x, X, D) \geq \widehat{\ell}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0) = \widehat{\ell}(x_0, \mathcal{X}_0, \mathcal{D}_0),$$
$$\widehat{\mathrm{vol}}(x, X, D) \geq \widehat{\mathrm{vol}}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0) = \widehat{\mathrm{vol}}(x_0, \mathcal{X}_0, \mathcal{D}_0).$$

Proof. The inequalities on log canonical thresholds follows from the lower semi-continuity of lct [Amb16, Corollary 1.10]. Thus for any $\mathfrak{m}$-primary ideal $\mathfrak{b}$, we have

$$\mathrm{lct}(X, D; \mathfrak{b})^n \cdot \ell(R/\mathfrak{b}) \geq \mathrm{lct}(\mathcal{X}_0, \mathcal{D}_0; \mathbf{in}(\mathfrak{b}))^n \cdot \ell(T/\mathbf{in}(\mathfrak{b})).$$

Thus we have $\widehat{\ell}(x, X, D) \geq \widehat{\ell}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0)$. Apply this initial degeneration argument to ideals in T, we get $\widehat{\ell}(x_0, \mathcal{X}_0, \mathcal{D}_0) \geq \widehat{\ell}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0)$. By definition we have $\widehat{\ell}(x_0, \mathcal{X}_0, \mathcal{D}_0) \leq \widehat{\ell}^{\mathbb{G}_m}(x_0, \mathcal{X}_0, \mathcal{D}_0)$, hence the inequality on normalized colengths is proved. The last inequality on local volumes is proved in the same way as [LX16, Lemma 4.3] using graded sequence of ideals. $\square$

Next we compare the local volume and normalized colength of $x_0 \in (\mathcal{X}_0, \mathcal{D}_0)$ to $x \in (H, (D - cH)|_H)$.

Lemma 5.3.6. *With the above notation, we have*

$$\widehat{\ell}(x_0, \mathcal{X}_0, \mathcal{D}_0) \geq \left(\frac{n}{n-1}\right)^{n-1}(1-c)\widehat{\ell}(x, H, (D-cH)|_H),$$
$$\widehat{\mathrm{vol}}(x_0, \mathcal{X}_0, \mathcal{D}_0) = \frac{n^n}{(n-1)^{n-1}}(1-c)\widehat{\mathrm{vol}}(x, H, (D-cH)|_H).$$

Proof. If $(\mathcal{X}_0, \mathcal{D}_0)$ is not klt at x_0, then $(H, (D-cH)|_H)$ is not klt at x by inversion of adjunction, in which case the lemma is trivial. Thus we may assume that $(\mathcal{X}_0, \mathcal{D}_0)$ is klt at x_0, and $(H, (D-cH)|_H)$ is klt at x. By Proposition 5.3.4, we have

$$(x_0 \in (\mathcal{X}_0, \mathcal{D}_0)) \cong ((x, 0) \in (H \times \mathbb{A}^1, c(H \times \{0\}) + (D-cH)|_H \times \mathbb{A}^1))$$

We need to use a lemma on $\mathbb{G}_m$-equivariant valuations. Assume $H = \operatorname{Spec} A$ is affine. Let s be the parameter of $\mathbb{A}^1$ in $H \times \mathbb{A}^1$.

Lemma 5.3.7 ([AIP+12, Section 11] and [LX17a, Theorem 2.15])**.** *For any valuation v on $\mathbb{C}(H)$ and any $\xi \in \mathbb{R}$, we can define a valuation $\tilde{v}_\xi$ on $\mathbb{C}(H \times \mathbb{A}^1)$ as follows:*

$$\tilde{v}_\xi(f) := \min_{0 \leq i \leq m,\ f_i \neq 0} \{v(f_i) + \xi \cdot i\} \quad \text{for any } f = \sum_{i=0}^{m} f_i s^i \in A[s].$$

Conversely, any $\mathbb{G}_m$-invariant valuation on $\mathbb{C}(H \times \mathbb{A}^1)$ is of the form $\tilde{v}_\xi$ for some $v \in \mathrm{Val}_H$ and $\xi \in \mathbb{R}$. Moreover, $\tilde{v}_\xi$ is centered at $(x, 0)$ if and only if v is centered at x and $\xi > 0$.

Now assume $\mathfrak{b}$ is an ideal sheaf on $H \times \mathbb{A}^1$ cosupported at $(x, 0)$, then we may write $\mathfrak{b}$ as

$$\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 s + \cdots + \mathfrak{b}_m s^m + (s^{m+1}) \subset A[s].$$

For simplicity, we will not distinguish $(\mathcal{X}_0, \mathcal{D}_0)$ from $(H \times \mathbb{A}^1, c(H \times \{0\}) + (D-cH)|_H \times \mathbb{A}^1)$. Assume $\mathrm{lct}(\mathfrak{b}) := \mathrm{lct}(\mathcal{X}_0, \mathcal{D}_0, \mathfrak{b})$ is computed by a $\mathbb{G}_m$-invariant divisorial valuation $\tilde{v}_\xi$ (for some valuation v on $\mathbb{C}(H)$ and some $\xi \in \mathbb{R}$). By [Liu18, Lemma 26], $\tilde{v}_\xi$ is centered at $(x, 0)$. Denote $k := \tilde{v}_\xi(\mathfrak{b})$ and $a := A_{(H,(D-cH)|_H)}(v)$, then $A_{(H,(D-cH)|_H) \times \mathbb{A}^1}(\tilde{v}_\xi) = a + \xi$ (by [JM12, §5], it suffices to prove this when v is quasi-monomial, in which case $\tilde{v}_\xi$ is also quasi-monomial and the result is clear). Since $\mathcal{D}_0 = cH \times \{0\} + (D - cH)|_H \times \mathbb{A}^1$, we have

$$\operatorname{lct}(\mathfrak{b}) = \frac{A_{(\mathcal{X}_0,\mathcal{D}_0)}(\tilde{v}_\xi)}{\tilde{v}_\xi(\mathfrak{b})} = \frac{A_{(H,(D-cH)|_H)\times\mathbb{A}^1}(\tilde{v}_\xi) - \tilde{v}_\xi(cH \times \{0\})}{\tilde{v}_\xi(\mathfrak{b})} = \frac{a + (1-c)\xi}{k}.$$

From the definition of $\tilde{v}_\xi$ we see that $v(\mathfrak{b}_i) + \xi \cdot i \geq \tilde{v}_\xi(\mathfrak{b}) = k$. Hence

$$\operatorname{lct}(\mathfrak{b}_i) := \operatorname{lct}(H, (D - cH)|_H; \mathfrak{b}_i) \leq \frac{a}{v(\mathfrak{b}_i)} \leq \frac{a}{k - \xi \cdot i} \text{ for any } i < \frac{k}{\xi}.$$

We know that $\operatorname{lct}(\mathfrak{b}_i)^{n-1} \cdot \ell(A/\mathfrak{b}_i) \geq \widehat{\ell}(x, H, (D - cH)|_H)$. This implies for any $i < \frac{k}{\xi}$ we have

$$\ell(A/\mathfrak{b}_i) \geq \frac{\widehat{\ell}(x, H, (D - cH)|_H)}{\operatorname{lct}(\mathfrak{b}_i)^{n-1}} \geq \frac{\widehat{\ell}(x, H, (D - cH)|_H)}{a^{n-1}}(k - \xi \cdot i)^{n-1}.$$

Since $s^{m+1} \in \mathfrak{b}$, we have that $k = \tilde{v}_\xi(\mathfrak{b}) \leq \tilde{v}_\xi(s^{m+1}) = \xi(m+1)$, i.e. $\lceil k/\xi \rceil - 1 \leq m$. Thus

$$\begin{aligned}
\ell(A[s]/\mathfrak{b}) &\geq \sum_{i=0}^{\lceil k/\xi \rceil - 1} \ell(A/\mathfrak{b}_i) \geq \frac{\widehat{\ell}(x, H, (D - cH)|_H)}{a^{n-1}} \sum_{i=0}^{\lceil k/\xi \rceil - 1} (k - \xi \cdot i)^{n-1} \\
&\geq \frac{\widehat{\ell}(x, H, (D - cH)|_H)}{a^{n-1}} \int_0^{k/\xi} (k - \xi \cdot w)^{n-1} dw \\
&= \frac{k^n}{n a^{n-1} \xi} \widehat{\ell}(x, H, (D - cH)|_H).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{lct}(\mathfrak{b})^n \cdot \ell(A[s]/\mathfrak{b}) &\geq \left(\frac{a + (1-c)\xi}{k}\right)^n \frac{k^n}{n a^{n-1} \xi} \widehat{\ell}(x, H, (D - cH)|_H) \\
&= \frac{(a + (1-c)\xi)^n}{n a^{n-1} \xi} \widehat{\ell}(x, H, (D - cH)|_H) \\
&\geq \left(\frac{n}{n-1}\right)^{n-1} (1-c) \widehat{\ell}(x, H, (D - cH)|_H)
\end{aligned}$$

where the last inequality follows from the AM-GM inequality. This proves the inequality on normalized colengths.

For the equality on local volumes, note that (see e.g. [Li18] for the definition of log discrepancy, volume and normalized volume of a valuation)

$$A_{(\mathcal{X}_0,\mathcal{D}_0)}(\tilde{v}_\xi) = a + (1-c)\xi, \quad \operatorname{vol}(\tilde{v}_\xi) = \frac{\operatorname{vol}(v)}{\xi}.$$

Hence again by the AM-GM inequality,

$$\widehat{\mathrm{vol}}(\tilde{v}_\xi) = \frac{(a+(1-c)\xi)^n}{\xi}\mathrm{vol}(v) \geq \frac{n^n(1-c)}{(n-1)^{n-1}}a^{n-1}\mathrm{vol}(v) = \frac{n^n(1-c)}{(n-1)^{n-1}}\widehat{\mathrm{vol}}(v).$$

On the other hand, we have

$$\widehat{\mathrm{vol}}(\tilde{v}_{\frac{a}{(1-c)(n-1)}}) = \frac{n^n(1-c)}{(n-1)^{n-1}}\widehat{\mathrm{vol}}(v).$$

Thus the equality is also proved. □

Combining Lemma 5.3.5 and 5.3.6, we have the following theorem which yields Theorem 5.1.4 when $c = 0$.

Theorem 5.3.8. *Let $x \in (X, D)$ be a klt singularity of dimension n. Let H be a normal reduced Cartier divisor of X containing x. Let c be the coefficient of H in D. Then we have*

$$\frac{\widehat{\mathrm{vol}}(x, X, D)}{n^n} \geq (1-c)\frac{\widehat{\mathrm{vol}}(x, H, (D-cH)|_H)}{(n-1)^{n-1}}, \quad \frac{\widehat{\ell}(x, X, D)}{n^n/n!} \geq (1-c)\frac{\widehat{\ell}(x, H, (D-cH)|_H)}{(n-1)^{n-1}/(n-1)!}.$$

The equality of local volumes in Lemma 5.3.6 suggests a conjectural product formula for local volumes as follows.

Conjecture 5.3.9. *Let $(x_i \in (X_i, D_i))$ be klt singularities for $i = 1, 2$. Denote $n_i := \dim X_i$, then*

$$\frac{\widehat{\mathrm{vol}}((x_1, x_2), X_1 \times X_2, \pi_1^* D_1 + \pi_2^* D_2)}{(n_1+n_2)^{n_1+n_2}} = \frac{\widehat{\mathrm{vol}}(x_1, X_1, D_1)}{n_1^{n_1}} \cdot \frac{\widehat{\mathrm{vol}}(x_2, X_2, D_2)}{n_2^{n_2}}.$$

Remark 5.3.10. Let us make some remarks on Conjecture 5.3.9.

1. If each singularity $(x_i \in X_i)$ lives on a Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds, then by [LX17a, Corollary 5.7] we know that

$$\widehat{\mathrm{vol}}(x_i, X_i) = n_i^{n_i}\Theta(x_i, X_i)$$

where $\Theta(\cdot, \cdot)$ is the volume density of a singularity (see e.g. [HS17, SS17]). It is well known that $\Theta((x_1, x_2), X_1 \times X_2) = \Theta(x_1, X_1) \cdot \Theta(x_2, X_2)$, so Conjecture 5.3.9 holds in this case. More generally, similar arguments (e.g. [LX17a, Theorem 1.5]) show that Conjecture 5.3.9 is true if each singularity ($x_i \in X_i$) admits a special degeneration to a Ricci-flat Kähler cone singularity.

2. Our method in this section should be enough to prove special cases of Conjecture 5.3.9 when one of the klt singularities is toric.

3. It is also natural to expect the following statement to be true: let v_i be a $\widehat{\mathrm{vol}}$-minimizing valuations over $(x_i \in (X_i, D_i))$, then there exists a $\widehat{\mathrm{vol}}$-minimizer v over $((x_1, x_2) \in (X_1 \times X_2, \pi_1^* D_1 + \pi_2^* D_2))$ such that for any function $f_{i,j} \in \mathbb{C}(X_i)^\times$,

$$v(\sum_j f_{1,j} \otimes f_{2,j}) = \min_j \left(\frac{n_1 v_1(f_{1,j})}{A_{(X_1,D_1)}(v_1)} + \frac{n_2 v_2(f_{2,j})}{A_{(X_2,D_2)}(v_2)} \right).$$

As an application of Theorem 5.1.4, we give the proof of Theorem 5.1.1. To start with, note that the minimal non-klt (or non-lc) colength of a singularity is bounded from below by its normalized colength.

Lemma 5.3.11. *Let $x \in (X, D)$ be a klt singularity. Then*

$$\ell_{\mathrm{nklt}}(x, X, D) \geq \widehat{\ell}(x, X, D), \quad \ell_{\mathrm{nlc}}(x, X, D) > \widehat{\ell}(x, X, D).$$

Proof. The minimal non-klt (resp. non-lc) colength of $x \in (X, D)$ is achieved by some ideal $\mathfrak{a}$. By definition, we have $\mathrm{lct}(X, D; \mathfrak{a}) \leq 1$ (resp. < 1), hence $\widehat{\ell}(x, X, D) \leq \mathrm{lct}(X, D; \mathfrak{a})^n \cdot \ell(\mathcal{O}_X/\mathfrak{a}) \leq$ (resp. $<$) $\ell(\mathcal{O}_X/\mathfrak{a}) = \ell_{\mathrm{nklt}}(x, X, D)$ (resp. $\ell_{\mathrm{nlc}}(x, X, D)$). □

We then have the following criterion (in terms of normalized colengths) by combining Theorems 5.1.3 and Theorem 5.1.4.

Corollary 5.3.12. *Let $\delta \geq -1$ be an integer and let $X \subseteq \mathbb{P}^{n+r}$ be a Fano complete intersection of index 1, codimension r and dimension n. Assume that*

1. *The singular locus of X has dimension at most δ and $2n+1 \geq 3(r+\delta)$;*

2. *For every $x \in X$ and every general linear space section $Y \subseteq X$ of codimension $2r+\delta$ containing x, the normalized colength of (Y, x) satisfies $\widehat{\ell}(x, Y) \geq \binom{n+r+1}{2r+\delta+1}$.*

Then X is birationally superrigid and K-stable.

Proof. By Theorem 5.1.3 and Lemma 5.3.11, it suffices to show that

$$\widehat{\ell}(x,Y) > \binom{n+r}{r+\delta}$$

for every $x \in X$, where $Y \subseteq X$ is a general linear section of codimension $r+\delta$ containing x. Let $Y' \subseteq Y$ be a general complete intersection of codimension r in Y (in particular, Y' is a general linear section of codimension $2r+\delta$ in X). By Theorem 5.1.4, we have

$$\widehat{\ell}(x,Y) \geq \left(\frac{m}{m-1}\right)^{m-1} \widehat{\ell}(x,H) \geq 2\widehat{\ell}(x,H)$$

where $m = \dim Y$ and $H \subseteq Y$ is a general hyperplane section containing x, hence by our second assumption and a repeated use of the above inequality,

$$\begin{aligned} \widehat{\ell}(x,Y) &\geq 2^r \widehat{\ell}(x,Y') \geq 2^r \binom{n+r+1}{2r+\delta+1} \\ &> 2^r \binom{n+r}{2r+\delta} = \binom{n+r}{r+\delta} \cdot \prod_{i=0}^{r-1} \frac{2(n-i-\delta)}{2r-i+\delta} \\ &\geq \binom{n+r}{r+\delta}, \end{aligned}$$

where the last inequality holds as $2n \geq 3r+3\delta-1 \geq 2r+3\delta+i$. This completes the proof. □

Theorem 5.1.1 can now be deduced from the following lower bounds of local volumes and normalized colengths.

Lemma 5.3.13. *Let $x \in X$ be a hypersurface singularity of multiplicity $m \geq 2$ and dimension $n \geq m$. Assume that the tangent cone of $x \in X$ is smooth in codimension $m-1$, then*

$$\widehat{\mathrm{vol}}(x,X) \geq \frac{n^n}{m^{m-1}}, \quad \widehat{\ell}(x,X) \geq \frac{n^n/n!}{m^m}.$$

Proof. It suffices to prove the first inequality since the second follows from the first by Proposition 5.3.2. Let $Y = X \cap H_1 \cap H_2 \cap \cdots \cap H_{n-m}$ be a general complete intersection of dimension m containing x, then we have $\mathrm{e}(x,Y) = \mathrm{e}(x,X) = m$ and by our assumption on the tangent cone of X, we see that $x \in Y$ has a smooth tangent cone V given by a smooth hypersurface of degree m in $\mathbb{P}^m$. It is clear that there exists a $\mathbb{Q}$-Gorenstein flat family over $\mathbb{A}^1$ specially degenerating Y to the affine cone $C_p(V)$ over V (with vertex p), hence by [BL18a, Theorem 1], we have $\widehat{\mathrm{vol}}(x,Y) \geq \widehat{\mathrm{vol}}(p, C_p(V))$. On the other hand, V is K-stable by [Fuj16a, Theorem 1.2 and Example 1.3(2)], thus by [LL19, Corollary 1.5], the local volume of $p \in C_p(V)$ is computed by the blowup of p and we get $\widehat{\mathrm{vol}}(p, C_p(V)) = m$. Combining these with Theorem 5.1.4 we obtain:

$$\frac{\widehat{\mathrm{vol}}(x,X)}{n^n} \geq \frac{\widehat{\mathrm{vol}}(x, X \cap H_1)}{(n-1)^{n-1}} \geq \cdots \geq \frac{\widehat{\mathrm{vol}}(x,Y)}{m^m} \geq \frac{\widehat{\mathrm{vol}}(p, C_p(V))}{m^m} = \frac{1}{m^{m-1}},$$

giving the required statement of the lemma. □

Proof of Theorem 5.1.1. The condition (2) is preserved by taking general hyperplane sections containing a given point, hence for every $x \in X$ and every general complete intersection $Y \subseteq X$ of codimension $2+\delta$ containing x, we have

$$\widehat{\ell}(x,Y) \geq \frac{1}{m^m} \cdot \frac{(n-2-\delta)^{n-2-\delta}}{(n-2-\delta)!} \geq \binom{n+2}{\delta+3}$$

by Lemma 5.3.13 and condition (3). Hence X satisfies both assumptions of Corollary 5.3.12 (with $r = 1$) and the result follows. □

If the hypersurface $X \subseteq \mathbb{P}^{n+1}$ has isolated ordinary singularities (i.e. the tangent cone is smooth), then one can usually expect a better lower bound of its corresponding local volume. Indeed, we should have $\widehat{\mathrm{vol}}(x,X) \geq m(n+1-m)^n$ (where $m = \mathrm{e}(x,X)$) by [LL19, Corollary 1.5] and the conjectural K-stability of smooth Fano hypersurfaces. For small multiplicities, this usually give better results than Theorems 5.1.2 and Theorem 5.1.1. For instance, in the case of $m = 2$ we do have $\widehat{\mathrm{vol}}(x,X) \geq 2(n-1)^n$ (quadric hypersurfaces admit Kähler-Einstein metrics as they are homogeneous), hence:

Corollary 5.3.14. *Let $X \subseteq \mathbb{P}^{n+1}$ be a hypersurface of degree $n+1$. Assume that X has at most ordinary double points, then X is birationally superrigid and K-stable when $n \geq 11$.*

Proof. Take $\delta = 0$ and $r = 1$ in Corollary 5.3.12. By the above remark and Proposition 5.3.2, for every general complete intersection $x \in Y \subseteq X$ of codimension 2 we have

$$\widehat{\ell}(x, Y) \geq \frac{1}{2(n-2)!}\widehat{\mathrm{vol}}(x, Y) = \frac{(n-3)^{n-2}}{(n-2)!}.$$

Hence by Corollary 5.3.12, such hypersurface X is birationally superrigid and K-stable as long as $\frac{(n-3)^{n-2}}{(n-2)!} \geq \binom{n+2}{3}$, or equivalently, $n \geq 11$. □

5.4 Hypersurfaces with ordinary singularities

In this section we complete the proof of Theorem 5.1.2. The proof is a bit lengthy, so we divide it into several steps. Throughout the section, X always denotes a hypersurface of degree $n+1$ and dimension $n \geq 250$ with only isolated ordinary singularities. We first treat the (super)rigidity of X.

Lemma 5.4.1. *Let $x \in X$ be a point of multiplicity $m_0 \geq n-4$ and let $\pi : Y \to X$ be the blowup of X at x with exceptional divisor E. Let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary and let $\tilde{M}$ be it strict transform on Y. Then $(Y, \tilde{M})$ is canonical along E.*

Proof. We start with some multiplicity estimate. By assumption, $E \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree m_0. Let Z be the codimension 2 cycle $\tilde{M}^2$ in Y. Note that by [Puk02b, Proposition 5], we have $\mathrm{mult}_x M^2 \leq 1$ outside a finite union of surfaces, hence $\mathrm{mult}_y Z \leq 1$ away from E and another set of dimension at most 2. Decompose Z into $Z = Z_1 + Z_2$ such that the irreducible components of Z_1 (resp. Z_2) is contained (resp. not contained) in E. We may view Z_1 as a divisor in E and since E has Picard number one, there exists some $b > 0$ such that $Z_1 \sim_{\mathbb{Q}} b \cdot c_1(\mathcal{O}_E(1)) \cap [E] \sim_{\mathbb{Q}} -bE^2$. By [Puk02b, Proposition 5] (note that E is a smooth hypersurface), we have $\mathrm{mult}_y Z_1 \leq b$ outside a finite number of points. Let $c = \mathrm{ord}_E M$, then we have $\tilde{M} \sim_{\mathbb{Q}} \pi^* M - cE$, $Z \sim_{\mathbb{Q}} \pi^* H^2 + c^2 E^2$ and $Z_2 = Z - Z_1 \sim_{\mathbb{Q}} \pi^* H^2 + (c^2 + b)E^2$. Let $W = Z_2 \cdot E$, then $W \sim_{\mathbb{Q}} (c^2 + b)E^3$ and as a codimension 2 cycle in E we have $W \sim_{\mathbb{Q}} (c^2 + b) c_1(\mathcal{O}_E(1))^2 \cap [E]$, hence by [Puk02b, Proposition 5] again we see that $\mathrm{mult}_y W \leq c^2 + b$ outside a finite union of curves in E. On the other hand, as $\pi^*H - E$ is a nef line bundle on Y, we have

$$bm_0 = Z_1 \cdot (\pi^*H - E)^{n-2} \leq Z \cdot (\pi^*H - E)^{n-2} = \tilde{M}^2 \cdot (\pi^*H - E)^{n-2} = n + 1 - c^2 m_0,$$

hence $b \leq c^2 + b \leq \frac{n+1}{m_0}$ and as $m_0 \geq n - 4$ and $n \geq 250$ by assumption, we obtain $\text{mult}_y Z \leq (c^2 + 2b) \leq \frac{2(n+1)}{m_0} < \frac{9}{4}$ outside a subset V of dimension at most 2 in Y (at least in a neighbourhood of E; the same remark applies to the other claims about singularities in this proof). In particular, if $y \in Y \backslash V$ and S is a general surface section in Y containing y, then by [Cor00, Theorem 3.1] (or [dFEM04, Theorem 0.1]), $(S, \frac{4}{3}\tilde{M}|_S)$ is klt at y and thus by inversion of adjunction, $(Y, \frac{4}{3}\tilde{M})$ is klt away from V.

We also need to show that $(Y, \tilde{M})$ has canonical singularities away from V. Let $y \in E \backslash V$ and let $K \subseteq E$ be a center of maximal singularity containing y. If K has codimension at least 3 in Y then by adjunction $(S, \tilde{M}|_S)$ is not lc at y where S is a general surface section in Y containing y, but from the above discussion $(S, \frac{4}{3}\tilde{M}|_S)$ is klt at y, a contradiction. Thus K has codimension 2 in Y and is a divisor in E (the pair (X, M) is already canonical outside a finite union of curves, see the proof of [LZ18, Theorem 1.2]) and we have $\text{mult}_K \tilde{M} > 1$. It follows that (in the above notations) $b > 1$ and $c > 1$, but $c^2 + b \leq \frac{n+1}{m_0} < 2$, a contradiction. Thus $(Y, \tilde{M})$ has canonical singularities away from V.

The rest of the proof is similar to those in [Zhu18b]. Let $y \in E$, let Y' be a complete intersection in Y cut out by three general hypersurfaces in $|\pi^*H - E|$ containing y and let $M' = \tilde{M}|_{Y'}$. Then (Y', M') is klt away from y and it is not hard to verify that for $L = 5\pi^*H + (n - 3 - m_0)E$, $L - (K_{Y'} + \frac{4}{3}M')$ is nef and big. By Theorem 4.2.3 and Lemma 5.4.2 (applied to $\lambda = 3$), we see that (Y', M') is lc as long as

$$h^0(Y', L) \leq 2^{\frac{n-3}{3} - 1}. \tag{5.1}$$

But since $\pi(Y')$ is a complete intersection in $\mathbb{P}^{n-2}$, we have

$$h^0(Y', L) \leq h^0(\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(5)) \leq \binom{n+3}{5}$$

and then it is not hard to see that (5.1) holds when $n \geq 250$. Therefore (Y', M') is lc and since $\dim V \le 2$, $(Y, \tilde{M})$ is canonical by Lemma 4.2.9. $\square$

The following lemma is used in the above proof.

Lemma 5.4.2. *Let X be a smooth variety of dimension n and $x \in X$. Let $\lambda > 0$, then $\ell_{\mathrm{nlc}}(x, X; \lambda) > 2^{\frac{n}{\lambda}-1}$.*

Proof. This follows directly from Lemma 4.2.4 and the proof of Lemma 4.3.5. $\square$

We are ready to prove the birational (super)rigidity part of Theorem 5.1.2.

Proof of Theorem 5.1.2 (2)(3). Let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary. Suppose that (X, M) is not canonical at some point $x \in X$ with multiplicity m_0. By (the proof of) Corollary 5.2.4, (X, M) has canonical singularities outside points of multiplicity at most $n-5$. In particular, (X, M) is canonical in a punctured neighbourhood of x and we have $m_0 \ge n-4$. Let $\pi : Y \to X$ be the blowup of x, let $\tilde{M}$ be the strict transform of M and let E be the exceptional divisor. By Lemma 5.4.1, $(Y, \tilde{M})$ has canonical singularities, thus as (X, M) is not canonical, E must be a center of maximal singularity and we obtain $a(E; X, M) = n - m_0 - \mathrm{ord}_E M < 0$. As M is movable and $\pi^*H - E$ is nef (where H is the hyperplane class on X), we have $(\tilde{M}^2 \cdot (\pi^*H - E)^{n-2}) \ge 0$ and hence $(n-m_0)^2 m_0 < (\mathrm{ord}_E M)^2 m_0 \le \deg X = n+1$, which can only be true when $m_0 = n-1$. It follows that the only possible maximal singularity of X is the ordinary blowup of a point of multiplicity $n-1$. By standard argument (see e.g. [CPR00, §3]), this proves parts (2) and (3) of Theorem 5.1.2. $\square$

Next we show that X is K-stable when all points have multiplicity at most $n-1$. By Theorem 1.1.4 and Corollary 3.3.1, it suffices to show that $\alpha(X) > \frac{1}{2}$ and that in case X has multiplicity $n-1$ at some point x, (X, M) is lc at x for every movable boundary $M \sim_{\mathbb{Q}} -K_X$ (using a modification of the above argument).

Lemma 5.4.3. *Let $x \in X$ be a point of multiplicity $m_0 \le n-1$, then (X, M) is lc at x for every movable boundary $M \sim_{\mathbb{Q}} -K_X$ and $(X, \frac{1}{2}D)$ is klt at x for every effective divisor $D \sim_{\mathbb{Q}} -K_X$.*

Proof. As before let $\pi : Y \to X$ be the blowup of of x with exceptional divisor E and let $\tilde{M}$ be the strict transform of M. From the above proof of Lemma 5.4.1 and Theorem 5.1.2(2), we see that (X, M) is indeed canonical at x if $m_0 \le n-2$. Hence we may assume that $m_0 = n-1$. We have $K_Y + \tilde{M} + (c-1)E = \pi^*(K_X + M)$ where $c = \mathrm{ord}_E M$ and as before since $\tilde{M}^2 \cdot (\pi^*H - E)^{n-2} \ge 0$, we obtain $c^2 \le \frac{n+1}{m_0} = 1 + \frac{2}{n-1}$ (hence $c - 1 < \frac{1}{n-1}$) and $\mathrm{mult}_y \tilde{M}^2 < \frac{2(n+1)}{n-1}$ outside a set V of dimension at most 2. We may assume that $c > 1$, otherwise (X, M) is already canonical by Lemma 5.4.1. By [Cor00, Theorem 3.1] applied to a general surface section of $(Y, (c-1)E)$ containing some $y \in Y \backslash V$ as before (noting that $\frac{16}{9}\mathrm{mult}_y \tilde{M}^2 \le 4(1 - \frac{4}{3}(1-c)))$, we find that $(Y, \frac{4}{3}\Gamma)$ is lc away from V (where $\Gamma = (c-1)E + \tilde{M}$) and the same argument as before (i.e. using [Zhu18b, Theorem 3.3] and Lemma 5.4.2) implies that $\mathrm{lct}(Y; \frac{4}{3}\Gamma) \ge \frac{3}{4}$ along E. In particular, $(Y, \tilde{M} + (c-1)E)$ is lc along E and (X, M) is lc at x as desired.

Similarly, let $\widetilde{D}$ be the strict transform of D on Y, then $\widetilde{D} \sim_{\mathbb{Q}} \pi^*H - aE$ for some $a \le \frac{n+1}{m_0}$ (as $\widetilde{D} \cdot (\pi^*H - E)^{n-1} \ge 0$) and $\mathrm{mult}_y \widetilde{D} \le \frac{n+1}{m_0}$ outside a set V_1 of dimension at most 1 (by bounding the degree of $\widetilde{D}|_E$ and using [Puk02b, Proposition 5]). If $m_0 \le n-5$ then by the proof of [LZ18, Theorem 1.2], $(X, \frac{1}{2}D)$ is klt at x. If $m_0 \ge n-4$, then by the above multiplicity estimate $(Y, \lambda\widetilde{D})$ is klt away from V_1 for some $\lambda > \frac{3}{4}$ and a similar argument as before involving [Zhu18b, Theorem 3.3] and Lemma 5.4.2 (this time applied to $\lambda = 2$) implies that $\mathrm{lct}(Y; \lambda\widetilde{D}) \ge \frac{2}{3}$ and thus $(Y, \frac{1}{2}\widetilde{D})$ is klt along E. Since $a(E; X, \frac{1}{2}D) = n - m_0 - \frac{1}{2}a \ge 1 - \frac{1}{2}a > 0$, $(X, \frac{1}{2}D)$ is also klt at x. ☐

Finally we treat the case when some points of X has multiplicity n. For this we use the criterion [Zhu18b, Theorem 1.5] and need to analyze the singularities of a few more auxiliary pairs.

Lemma 5.4.4. *Let $D \sim_{\mathbb{Q}} -K_X$ be an effective divisor and $M \sim_{\mathbb{Q}} -K_X$ a movable boundary. Then $(X, \frac{1}{n}D + \frac{n-1}{n}M)$ is lc over the smooth locus of X.*

Proof. Let $\Gamma = \frac{1}{n}D + \frac{n-1}{n}M$ and let $c = \frac{3}{2}$. We first show that $(X, c\Gamma)$ is lc outside a set of dimension at most 2. By [Puk02b, Proposition 5], we have $\mathrm{mult}_x D \le 1$ and $\mathrm{mult}_x(M^2) \le 1$ away from a set Z of dimension at most 2. Let $S \subseteq X$ be a general surface section containing a point $x \in X \backslash Z$, then it's not hard to verify that

$$\frac{4c}{n}\mathrm{mult}_x(D|_S) + c^2\left(\frac{n-1}{n}\right)^2 \mathrm{mult}_x(M|_S^2) \leq \frac{4c}{n} + c^2\left(1-\frac{1}{n}\right)^2 < 4,$$

hence by [dFEM04, Theorem 2.2], $(S, c\Gamma|_S)$ is lc at x and therefore by inversion of adjunction we see that $(X, c\Gamma)$ is lc away from Z. Now let $y \in X$ be any smooth point and let $Y \subseteq X$ be a general linear section of codimension 2 containing y, then $(Y, c\Gamma|_Y)$ is lc in a punctured neighbourhood of y and $L - (K_Y + c\Gamma|_Y)$ is ample where $L = 3H|_Y$. By [Zhu18b, Theorem 3.3] and Lemma 5.4.2 (applied to $\lambda = 2$), $\mathrm{lct}(Y, c\Gamma|_Y) \geq \frac{2}{3} = c^{-1}$ (in a neighbourhood of y) as long as $h^0(Y, L) = \binom{n+2}{3} \leq 2^{\frac{n-2}{2}-1}$, which holds when $n \geq 250$. Therefore, $(Y, \Gamma|_Y)$ is lc at y and by inversion of adjunction (X, Γ) is also lc at y. □

Lemma 5.4.5. *Let $X \subseteq \mathbb{P}^n$ be a smooth Fano hypersurface and let $D \sim_{\mathbb{Q}} \ell H$ be an effective divisor on X for some $\ell \leq n(n-2)$ (where H is the hyperplane class). Assume that (X, D) is lc away from a finite number of points, then $(X, \frac{1}{n}D)$ is lc.*

Proof. By [Zhu18b, Theorem 3.3] and Lemma 5.4.2 (applied to $L = n(n-2)H$ and $\lambda = \frac{1}{n-1}$ respectively), we have $\mathrm{lct}(X; D) \geq \frac{1}{n}$ as long as $h^0(X, L) = \binom{n+n(n-2)}{n} \leq 2^{n(n-1)-1}$. Since $\binom{n+m}{m} \leq \frac{1}{2}2^{n+m}$ for all $n \neq m$, the result follows. □

We are now ready to finish the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2(1). Let $D \sim_{\mathbb{Q}} -K_X$ be an effective divisor on X and let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary. We claim that $(X, \Delta = \frac{n}{n+1}\Gamma)$ is klt where $\Gamma = \frac{1}{n}D + \frac{n-1}{n}M$. It then follows from Theorem 3.1.1 that X is K-stable.

As X has Picard number one, we may assume that D is irreducible (being klt is preserved under convex linear combination). Let $x \in X$ be a point of multiplicity m_0. If $m_0 \leq n-1$, then by Lemma 5.4.3, (X, M) is lc at x and $(X, \frac{1}{2}D)$ is klt at x, hence as $\Delta = \frac{2}{n+1} \cdot \frac{1}{2}D + (1 - \frac{2}{n+1})M$ is a convex linear combination of $\frac{1}{2}D$ and M, we see that (X, Δ) is klt at x. It remains to consider the case $m_0 = n$. After a change of coordinate, we may assume that $x = [0 : \cdots : 0 : 1]$ and the defining equation of X can be written as $x_{n+1}f_n(x_0, \cdots, x_n) + f_{n+1}(x_0, \cdots, x_n) = 0$ where f_i $(i = n, n+1)$ is homogeneous of degree

i. Let $D_0 = (f_n = 0) \cap X$. As before, let $\pi : Y \to X$ be the blowup of x with exceptional divisor E and strict transforms $\widetilde{D}$, $\tilde{M}$, $\widetilde{D}_0$, $\tilde{\Gamma}$ and $\tilde{\Delta}$.

Suppose first that D is not supported on D_0, then as before we have $\mathrm{ord}_E D \le 1$ since $\widetilde{D} \cdot \widetilde{D}_0 \cdot (\pi^* H - E)^{n-2} \ge 0$. Similarly $\mathrm{ord}_E M \le 1$ and thus $\mu = \mathrm{ord}_E \Gamma \le 1$. We claim that in a neighbourhood of E, $(Y, \pi^*\Gamma)$ is lc outside a finite number of point. By Lemma 5.4.4, (X, Γ) is lc in a punctured neighbourhood of x. Thus as E appears with coefficient at most 1 in $\pi^*\Gamma$, by inversion of adjunction it suffices to show that $(E, \tilde{\Gamma}|_E)$ is lc outside a finite number of points. But as E is a smooth hypersurface in $\mathbb{P}^n$ and $\tilde{\Gamma}|_E \sim_{\mathbb{Q}} -\mu E|_E \sim_{\mathbb{Q}} \mu c_1(\mathcal{O}_E(1)) \cap [E]$ where $\mu \le 1$, this follows from [Puk02b, Proposition 5] and therefore $(Y, \pi^*\Gamma)$ is lc outside a finite number of point. By Theorem 4.1.4 applied to $(Y, \pi^*\Gamma)$ with $L = 0$ (note that $K_Y + \pi^*\Gamma \sim_{\mathbb{Q}} \pi^*(K_X + \Gamma) \sim_{\mathbb{Q}} 0$ and $\pi^*\Gamma$ is nef and big) we obtain $\mathrm{lct}(Y; \pi^*\Gamma) \ge \frac{n}{n+1}$ with equality if and only if $\mathrm{mult}_y(\pi^*\Gamma) = n+1$ for some $y \in Y$. But if such y exists, then clearly $y \in E$ and since $\pi^*\Gamma = \tilde{\Gamma} + \mu E$ where $\mu \le 1$, we have $\mathrm{mult}_y(\tilde{\Gamma}|_E) \ge \mathrm{mult}_y \tilde{\Gamma} \ge n$. Recall that E is a smooth hypersurface of degree n and $\tilde{\Gamma}|_E \sim_{\mathbb{Q}} \mu c_1(\mathcal{O}_E(1))$, we also have $\mathrm{mult}_y(\tilde{\Gamma}|_E) \le n\mu \le n$. Hence we must have equality everywhere, in particular, $\mu = 1$ and $\mathrm{mult}_y \tilde{M} = \mathrm{mult}_y(\tilde{M}|_E) = n$. But as $\tilde{M}$ is movable, if we choose $n-3$ general members $P_1, \cdots, P_{n-3}$ of $|\pi^*H - E|$ passing through y and another general member Q of $|2\pi^*H - E|$ (which is very ample) containing y, then the intersection $\tilde{M}^2 \cdot P_1 \cdot \ldots \cdot P_{n-3} \cdot Q$ is zero dimensional and we get

$$n + 2 = 2(H^n) - \deg E = (\tilde{M}^2 \cdot P_1 \cdot \ldots \cdot P_{n-3} \cdot Q) \ge (\mathrm{mult}_y \tilde{M})^2 = n^2,$$

a contradiction. Thus such y doesn't exists and we indeed have $\mathrm{lct}(Y; \pi^*\Gamma) > \frac{n}{n+1}$. In other words, $(Y, \pi^*\Delta)$ (and hence also (X, Δ)) is klt.

It remain to treat the case when D is supported on D_0 (i.e. $D = \frac{1}{n} D_0$). As $\mathrm{ord}_E M \le 1$ and $\mathrm{ord}_E D = \frac{n+1}{n}$, we still have $\mathrm{ord}_E \Delta < 1$. Since $X = (x_{n+1} f_n + f_{n+1} = 0)$ has only isolated singularities, a direct computation shows that $\widetilde{D}_0|_E = (f_{n+1} = 0) \cap E$ has only isolated singularities as well. By Lemma 5.4.5, $(E, \widetilde{D}|_E = \frac{1}{n} \widetilde{D}_0|_E)$ is lc. On the other hand, as E is a smooth hypersurface of degree n in $\mathbb{P}^n$ and $\tilde{M}|_E \sim_{\mathbb{Q}} \mu c_1(\mathcal{O}_E(1))$ where $\mu \le 1$, we see that $(E, \frac{n-1}{n} \tilde{M}|_E)$ is also lc by Corollary 4.1.5. Taking convex linear combination

$\tilde{\Delta} = \frac{1}{n+1}\widetilde{D}|_E + \frac{n}{n+1} \cdot \frac{n-1}{n}\tilde{M}|_E$, it follows that $(E, \tilde{\Delta}|_E)$ is lc and thus by inversion of adjunction $(Y, E + \tilde{\Delta})$ is lc as well. As (X, Δ) is klt away from x, all the lc centers of $(Y, E + \tilde{\Delta})$ are contained in E, hence as $\mathrm{ord}_E \Delta < 1$, we deduce that $(Y, \pi^*\Delta)$ and (X, Δ) are both klt and this finishes the proof. $\square$

Chapter 6

Moduli

6.1 Introduction

Birationally (super)rigid or K-(semi)stable Fano varieties are special classes of Fano varieties and a natural question is whether they form a "nice" moduli. Recently this question has been extensively studied for K-semistable Fano varieties and it is well expected (known as the K-moduli conjecture) that the moduli functor $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ of n-dimensional K-semistable $\mathbb{Q}$-Fano varieties, which sends $S \in \mathsf{Sch}_k$ (where k is our base field) to

$$\mathcal{M}^{\mathrm{Kss}}_{n,v}(S) = \left\{ \begin{array}{c} \text{Flat proper morphisms } X \to S \text{, whose fibers are} \\ n\text{-dimensional K-semistable } \mathbb{Q}\text{-Fano varieties} \\ \text{with volume } v \text{, satisfying Kollár's condition} \end{array} \right\}$$

is represented by an Artin stack $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ of finite type and admits a projective good moduli space $\mathcal{M}^{\mathrm{Kss}}_{n,v} \to M^{\mathrm{Kps}}_{n,v}$ (in the sense of [Alp13]), whose closed points are in bijection with n-dimensional K-polystable $\mathbb{Q}$-Fano varieties of volume v. Here we say that the family $X \to S$ satisfies the Kollár condition if for any $m \in \mathbb{Z}$ the reflexive power $\omega^{[m]}_{X/S}$ commutes with arbitrary base change (see [Kol08, 24]). There has been a lot of progress on the construction of this moduli in recent years: by studying Gromov-Hausdorff limits of smooth del Pezzo surface, [OSS16a] constructs explicit good moduli spaces for K-polystable del Pezzo surfaces; a much more complete picture is provided by [LWX14, LWX18] (see also

[SSY16, Oda15]), where a proper K-moduli for smoothable Fano varieties is constructed using analytic tools; in the general singular case, [Jia17] shows that K-semistable Fano varieties with anti-canonical volume bounded from below form a bounded family, [BL18b] shows that uniform K-stability is an open property, while [BX18] proves that $M_{n,v}^{\mathrm{Kps}}$ is separated, so at least we have a separated Deligne-Mumford stack $\mathcal{M}_{n,v}^{\mathrm{uKs}}$ of finite type parametrizing uniformly K-stable Fano varieties with coarse moduli space $M_{n,v}^{\mathrm{uKs}}$; some progress on the (quasi-)projectivity of $M_{n,v}^{\mathrm{Kps}}$ and $M_{n,v}^{\mathrm{uKs}}$ is made in [CP18].

In contrast, very little is known about the moduli property of birationally superrigid Fano varieties. We have an analogous moduli functor $\mathcal{M}_{n,v}^{\mathrm{bsr}}$ of n-dimensional birationally superrigid Fano varieties sending $S \in \mathsf{Sch}_k$ to

$$\mathcal{M}_{n,v}^{\mathrm{bsr}}(S) = \left\{ \begin{array}{c} \text{Flat proper morphisms } X \to S\text{, whose fibers are} \\ n\text{-dimensional birationally superrigid Fano varieties} \\ \text{with volume } v\text{, satisfying Kollár's condition} \end{array} \right\}.$$

As birationally superrigid Fano varieties have finite automorphism groups (see the end of §6.2) and are conjectured to be K-stable, one may expect $\mathcal{M}_{n,v}^{\mathrm{bsr}}$ to be represented by a separated Deligne-Mumford stack of finite type (by comparing with the K-moduli conjecture). In this chapter, we investigate this question in detail.

By Theorem 2.2.2, birationally superrigid Fano varieties have terminal singularities, hence they form a bounded family by the seminal work of Birkar [Bir16a, Bir16b] on the Borisov-Alexeev-Borisov conjecture. With such boundedness at hand, the construction of the moduli can be reduced to proving a few more concrete statements.

The first thing to show is that the moduli space (if exists) is separated. For this we prove the following valuative criterion (joint with C. Stibitz):

Theorem 6.1.1 (=Theorem 1.3.1)**.** *Let $f : X \to C$, $g : Y \to C$ be two flat families of Fano varieties (i.e. all geometric fibers are integral, normal and Fano) over a smooth pointed curve $0 \in C$. Assume that the central fibers $X_0 = f^{-1}(0)$ and $Y_0 = g^{-1}(0)$ are birationally superrigid and there exists an isomorphism $\rho : X \backslash X_0 \cong Y \backslash Y_0$ over the punctured curve $C \backslash 0$. Then ρ induces an isomorphism $X \cong Y$ over C.*

Note that if the answer to Question 3.1.6 is positive, then Theorem 6.1.1 follows immediately from [Che09b, Theorem 1.5].

The next step is to check whether birational superrigidity is a locally closed property or not. We remark that it has been known that birational superrigidity is not open in moduli, at least in dimension 3: one such counterexample is given by the family of quasi-smooth quintic hypersurfaces in $\mathbb{P}(1^4, 2)$ [CP17, Example 6.3]. It turns out that in these examples, birational superrigidity is still a locally closed property, which suffices for moduli construction. Therefore, locally closedness is the best we can hope.

Unfortunately, this is still too good to be true and we construct counterexamples in every sufficiently large odd dimensions. Our construction is based on the degeneration of hypersurfaces into double covers. Let m be a sufficiently large integer and let $x_0, \cdots, x_{n+1}, y$ be the weighted homogeneous coordinates of $\mathbb{P}(1^{n+2}, m)$. Let f_s, g_s (parameterized by $s \in \mathbb{A}^1$) be homogeneous polynomials in $x_0, \cdots, x_{n+1}$ of degree $2m$ and m respectively so that

$$\mathcal{X} = (y^2 - f_s = ty - g_s = 0) \subseteq \mathbb{P}(1^{n+2}, m) \times \mathbb{A}^2_{s,t}$$

defines a family of weighted complete intersections of dimension n parameterized by $\mathbb{A}^2$. For $t \neq 0$, it is easy to see that $\mathcal{X}_{s,t} \cong (t^2 f_s - g_s = 0) \subseteq \mathbb{P}^{n+1}$ is a hypersurface of degree $2m$ while $\mathcal{X}_{s,0}$ is the double cover of the hypersurface $G_s = (g_s = 0) \subseteq \mathbb{P}^{n+1}$ branched over the divisor $F_s \cap G_s$ where $F_s = (f_s = 0)$. We show that with suitable choices of f_s and g_s, this provides the counterexample we want.

Theorem 6.1.2 (=Theorem 1.3.2)**.** *Notation as above. Let $x \in \mathbb{P}^{n+1}$. Assume that $n = 2m - 1$, $m \gg 0$ and the following:*

1. *F_0 and G_0 have a unique ordinary singularity at x with $\mathrm{mult}_x F_0 = 2m - 2$ and $\mathrm{mult}_x G_0 = m - 1$ and are otherwise smooth,*
2. *the projective tangent cone of $F_0 \cap G_0$ at x is a smooth complete intersection,*
3. *$\mathcal{X}_{s,t}$ is smooth when $s \neq 0$.*

Then $\mathcal{X}_{s,t}$ is birationally superrigid if and only if $s \neq 0$ or $(s, t) = (0, 0)$.

Since double covers of hypersurfaces appear as members $\mathcal{X}_{s,t}$ of this family, part of our proof is devoted to the study of the birational superrigidity of these double hypersurfaces. Indeed we prove a bit more:

Theorem 6.1.3. *Fix $r, s \in \mathbb{Z}$, then there exists an integer M such that every smooth weighted complete intersection $X \subseteq \mathbb{P}(1^m, a_1, \cdots, a_s)$ of codimension r and index one (i.e. $-K_X \sim H := c_1(\mathcal{O}_X(1))$) with a base-point-free anticanonical linear system $|-K_X|$ is birationally superrigid and K-stable when $m \geq M$.*

Despite the counterexamples in Theorem 6.1.2, a positive result is provided by [SC11, Corollary 7.8]: birational superrigidity is a constructible condition. We expect most naturally defined invariants (e.g. alpha invariants) of Fano varieties to be constructible in bounded families, as are the properties they characterize (e.g. weakly exceptional). As an evidence for this principle, we prove the constructibility of the alpha invariant function in certain families of smooth Fano complete intersections.

Theorem 6.1.4. *Let $r \in \mathbb{N}$ and let $\mathcal{X} \subseteq \mathbb{P}^{n+r} \times T$ be a smooth family of Fano complete intersections of codimension r over T. Assume that $n \geq 10r$, then the function $t \mapsto \alpha(\mathcal{X}_t)$ is constructible.*

6.2 Separatedness

In this section we prove the separatedness statement (Theorem 6.1.1). For this we recall the following criterion:

Lemma 6.2.1. *Let $f : X \to C$, $g : Y \to C$ be flat families of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$ with central fibers X_0 and Y_0. Assume that K_X and K_Y are $\mathbb{Q}$-Cartier and let $D_X \sim_{\mathbb{Q}} -K_X$, $D_Y \sim_{\mathbb{Q}} -K_Y$ be effective divisors not containing X_0 or Y_0. Assume that there exists an isomorphism*

$$\rho : (X, D_X) \times_C C^\circ \cong (Y, D_Y) \times_C C^\circ$$

over $C^\circ = C \backslash 0$, that $(X_0, D_X|_{X_0})$ is klt and $(Y_0, D_Y|_{Y_0})$ is lc. Then ρ extends to an isomorphism $(X, D_X) \cong (Y, D_Y)$.

Proof. This is well known to experts and follows from the separatedness of the moduli functor of klt log Calabi-Yau pairs (see e.g. [LWX14, Theorem 5.2] or [BX18, Proposition 3.2]). We give a proof for the convenience of reader.

Let

$$\begin{array}{ccc} & W & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & \overset{\rho}{\dashrightarrow} & Y \end{array}$$

be a common log resolution of $(X, X_0 + D_X)$ and $(Y, Y_0 + D_Y)$ and denote by X_0^W and Y_0^W the strict transform of X_0 and Y_0. By assumption, the strict transform of D_X and D_Y coincide on W and we denote it by D_W.

Note that if $X_0^W = Y_0^W$, then $\rho : X \dashrightarrow Y$ is an isomorphism in codimension one and $\rho_* K_X = K_Y$, thus it induces an isomorphism

$$X = \mathrm{Proj}_C \bigoplus_{m \geq 0} f_* \mathcal{O}_X(-mK_X) \cong \mathrm{Proj}_C \bigoplus_{m \geq 0} g_* \mathcal{O}_Y(-mK_Y) = Y,$$

in which case ρ extends to an isomorphism $(X, D_X) \cong (Y, D_Y)$.

Hence it remains to show that $X_0^W = Y_0^W$. Assume that this is not the case, we may write

$$K_W + D_W = p^*(K_X + D_X) + aY_0^W + E_X = q^*(K_Y + D_Y) + +bX_0^W + E_Y$$

where E_X and E_Y are both $p-$ and $q-$exceptional. Since $(X_0, D_X|_{X_0})$ is klt, by inversion of adjunction we see that $(X, X_0 + D_X)$ is plt and since X_0 is a Cartier divisor and the center of Y_0^W is contained in X_0, we have

$$a = a(Y_0^W; X, D_X) = a(Y_0^W; X, X_0 + D_X) + \mathrm{ord}_{Y_0^W}(X_0) > -1 + 1 = 0.$$

Similarly as $(Y_0, D_Y|_{Y_0})$ is lc we obtain $b \geq 0$. Now as

$$aY_0^W - bX_0^W + (E_X - E_Y) = q^*(K_Y + D_Y) - p^*(K_X + D_X) \sim_{\mathbb{Q}} 0,$$

by the negativity lemma (c.f. [KM98, Lemma 3.39-3.41]), there exists $c \in \mathbb{Q}$ such that $aY_0^W - bX_0^W + (E_X - E_Y) = cp^*X_0 = cq^*Y_0$. But comparing the coeficients of X_0^W and Y_0^W we obtain $0 < a = c = -b \le 0$, a contradiction. Thus $X_0^W = Y_0^W$ and we finish the proof. □

Proof of Theorem 6.1.1. Since birationally superrigid Fano varieties have terminal singularities, K_X and K_Y are $\mathbb{Q}$-Cartier by [dFH11, Proposition 3.5]. Hence the result follows from Theorem 2.2.5 and the following more general statement. □

Lemma 6.2.2. *Let $f : X \to C$, $g : Y \to C$ be flat families of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$ that are isomorphic over $C^\circ = C \backslash 0$. Let X_0 and Y_0 be their central fibers. Assume that*

1. *K_X and K_Y are $\mathbb{Q}$-Cartier.*
2. *For every movable boundary $M_X \sim_{\mathbb{Q}} -K_{X_0}$, (X_0, M_X) is klt.*
3. *For every movable boundary $M_Y \sim_{\mathbb{Q}} -K_{Y_0}$, (Y_0, M_Y) is lc.*

Then $X \cong Y$ over C.

Proof. By assumption X is birational to Y over C. Let m be a sufficiently large and divisible integer and let $D_1 \in |-mK_{X_0}|$, $D_2 \in |-mK_{Y_0}|$ be general divisors in the corresponding linear system. Choose effective divisors $D_{X,1} \sim -mK_X$, $D_{Y,2} \sim -mK_Y$ not containing X_0 or Y_0 such that $D_{X,1}|_{X_0} = D_1$ and $D_{Y,2}|_{Y_0} = D_2$. Let $D_{Y,1}$ and $D_{X,2}$ be their strict transforms to the other family. Since X and Y are isomorphic over C°, we have $D_{Y,1} \sim -mK_Y + W$ where W is supported on Y_0; but as Y_0 is irreducible, we have $W = \ell Y_0 = \ell g^*(0)$ for some integer ℓ. Since the question is local around $0 \in C$, we may shrink C so that $Y_0 \sim 0$ and thus $D_{Y,1} \sim -mK_Y$. Similarly we also have $D_{X,2} \sim -mK_X$. Let $D_1' = D_{Y,1}|_{Y_0}$ and $D_2' = D_{X,2}|_{X_0}$. Let $\mathcal{M}_X$ be the linear system spanned by $D_{X,1}$ and $D_{X,2}$ and let $M_X = \frac{1}{m}\mathcal{M}_X \sim_{\mathbb{Q}} -K_X$. Similar we have $\mathcal{M}_Y$ and $M_Y \sim_{\mathbb{Q}} -K_Y$. As D_1 and D_2 are general, D_1 and D_2' have no common components, hence the restriction of M_X to X_0 is still a movable boundary and therefore by our second assumption, $(X_0, M_X|_{X_0})$ is klt. Similarly, $(Y_0, M_Y|_{Y_0})$ is lc and we conclude by Lemma 6.2.1. □

As an easy application of Theorem 6.1.1, we show that birationally superrigid Fano varieties have finite automorphism group (for experts this is well known).

Corollary 6.2.3. *Let X be a birationally superrigid Fano variety. Then* $\mathrm{Aut}(X)$ *is finite.*

Proof. Assume that $\mathrm{Aut}(X)$ is not finite, then it contains $\mathbb{G}_m$ or $\mathbb{G}_a$. Let $\mathcal{X}_1 = \mathcal{X}_2 = X \times \mathbb{C}$. If $\mathbb{G}_m \subseteq \mathrm{Aut}(X)$, then it induces an isomorphism $\mathcal{X}_1 \times_{\mathbb{C}} \mathbb{C}^* \xrightarrow{\sim} \mathcal{X}_2 \times_{\mathbb{C}} \mathbb{C}^* = X \times \mathbb{C}^*$ through the diagonal action of $\mathbb{G}_m$. Since X is birationally superrigid, this map extends to an isomorphism of $\mathcal{X}_1$ and $\mathcal{X}_2$ over $\mathbb{C}$ by Theorem 6.1.1. Thus the map $\mathbb{G}_m \to \mathrm{Aut}(X)$ extends to $\mathbb{A}^1 \supseteq \mathbb{G}_m$, a contradiction. Similarly, if $\mathbb{G}_a \subseteq \mathrm{Aut}(X)$, then the inclusion $\mathbb{C}^* \subseteq \mathbb{G}_a$ given by $t \mapsto t^{-1}$ induces an automorphism of $X \times \mathbb{C}^*$ over $\mathbb{C}^*$ which extends to an automorphism over $\mathbb{C}$, hence $\mathbb{G}_a \to \mathrm{Aut}(X)$ extends to a map $\mathbb{P}^1 \to \mathrm{Aut}(X)$, again a contradiction. □

Using somewhat different argument, one can actually show that birationally solid Fano varieties have finite automorphism groups.

Definition 6.2.4 ([AO18, Definition 1.4])**.** A Fano variety X is said to be *birationally solid* if there is no birational map $X \dashrightarrow Y$ to the source of a strict (i.e. the base has dimension at least 1) Mori fiber space.

Proposition 6.2.5. *Let X be a Fano variety that is not birational to a conic bundle. Then* $\mathrm{Aut}(X)$ *is finite. In particular, birationally solid Fano varieties have finite automorphism groups.*

Proof. Assume that $\mathrm{Aut}(X)$ is not finite, then it contains $\mathbb{G}_m$ or $\mathbb{G}_a$. By the following Lemma 6.2.6 (with $L = -mK_X$ for some sufficiently divisible $m \in \mathbb{N}$), there exists a G-invariant (where $G = \mathbb{G}_m$ or $\mathbb{G}_a$) dense open subset $U \subseteq X$ for which the geometric quotient $\pi : U \to U/G$ exists. Thus we have rational dominant map $X \dashrightarrow U/G$ whose generic fiber is a rational curve. As in the proof of Proposition 2.2.6, this implies that X is birational to a conic bundle and in particular is not birationally solid. □

Lemma 6.2.6. *Let X be a projective variety and L an ample line bundle on X. Let $G = \mathbb{G}_m$ or $\mathbb{G}_a$ acts on (X, L). Then there exists a G-invariant dense open subset $U \subseteq X$ for which the geometric quotient $\pi : U \to U/G$ exists (i.e. the geometric fibers of π are precisely the geometric orbits of G).*

Proof. Replacing L by a multiple, we may assume that L is very ample. By assumption we have a G-equivariant embedding

$$\phi_{|L|} : X \hookrightarrow \mathbb{P}(H^0(X, L)^*).$$

If $G = \mathbb{G}_m$, let $Z \subseteq \mathbb{P}(H^0(X, L)^*)$ be the fixed points of G, then Z is a union of linear subspaces and does not contain X. It is easy to see that the complement of Z in $\mathbb{P}(H^0(X, L)^*)$ admits a geometric quotient by $\mathbb{G}_m$, hence $U = X \backslash Z$ satisfies the conclusion of the lemma. If $G = \mathbb{G}_a$, then there exists a G-invariant hyperplane $H \subseteq \mathbb{P}(H^0(X, L)^*)$ (every linear representation of a unipotent group has a nontrivial invariant subspace). Let $V = X \backslash H$, then V is a G-invariant affine subset of X. Let $A = k[V]$, then (modulo scaling) the Lie algebra Lie(G) of G may be identified with a nilpotent derivation δ of A (i.e. $\delta^m(x) = 0$ for all $f \in A$ and all $m \gg 0$). Since the action of G is not trivial, $\delta(A) \neq 0$. We may then choose $f \in A$ such that $\delta(f) \neq 0$ while $\delta^2(f) = 0$. In particular, $U = V \backslash (\delta(f) = 0)$ is a G-invariant dense affine subset of X. By [GP93, Lemma 3.1], the geometric quotient $U \to U/G$ exists (and indeed we have an G-equivariant isomorphism $U = U/G \times G$ where G acts trivially on U/G) hence satisfies the conclusion of the lemma. □

6.3 Counterexample to locally closedness

In this section we prove Theorems 6.1.3 and 6.1.2. For ease of notation, we denote by $\mathbb{P}$ the weighted projective space $\mathbb{P}(1^m, a_1, \cdots, a_s)$ and let $x_1, \cdots, x_m, y_1, \cdots, y_s$ be the weighted homogeneous coordinates. Given a weighted complete intersection $X \subseteq \mathbb{P}$, we also denote by QSing(X) the subset along which X is not quasi-smooth and let $\delta_X = \dim \mathrm{QSing}(X)$ (by convention, $\dim(\emptyset) = -1$).

Lemma 6.3.1. *Let $X \subseteq \mathbb{P}$ be a weighted complete intersection of codimension r and let $Y = X \cap (y_s = 0)$. Then $\delta_Y \le \delta_X + r$.*

Proof. Let $f_1, \cdots, f_r$ be the defining equations of X. We claim that

$$\mathrm{QSing}(Y) \cap \left(\frac{\partial f_1}{\partial y_s} = \cdots = \frac{\partial f_r}{\partial y_s} = 0 \right) \subseteq \mathrm{QSing}(X),$$

from which the lemma immediately follows. Denote the left hand side by W and let $y \in W$. Then as Y is not quasi-smooth at y, the rank of the Jacobian

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i\le r, 1\le j\le m+s-1}$$

is less than r (where $x_{m+q} = y_q$, $q = 1, \cdots, s$). Since $\frac{\partial f_1}{\partial y_s} = \cdots = \frac{\partial f_r}{\partial y_s} = 0$ at y, this rank is equal to the rank of the Jacobian for X at y. Hence X is not quasi-smooth at y either and $y \in \mathrm{QSing}(X)$ as desired. □

Lemma 6.3.2. *Let $X \subseteq \mathbb{P}$ be a quasi-smooth weighted complete intersection of codimension r and let $Z \subseteq X$ be an effective cycle of pure codimension k such that*

$$Z \equiv c_1(\mathcal{O}_X(1))^k \cap [X].$$

Then for every subvariety $S \subseteq X$ of dimension $\ge s + (2^s k + 2^s - 1)r$, we have $\mathrm{mult}_S Z \le 1$.

Proof. We prove by induction on s. When $s = 0$ the statement follows from [Puk02b, Proposition 5] or [Suz17, Proposition 2.1]. Assume that $s \ge 1$ and that the statement has been proven for smaller values of s. We may assume that $\dim X \ge s + (2^s k + 2^s - 1)r \ge 2k + r + 1$, otherwise there is no such S. Then by [Dim85, Proposition 6], X has Betti number $b_{2k}(X) = 1$, hence every irreducible component of Z is numerically proportional to Z and it suffices to show $\mathrm{mult}_S Z \le 1$ under the assumption that Z itself is irreducible. Let $Y = X \cap (y_s = 0)$, then by the previous lemma $\delta_Y \le r - 1$. Let d be a sufficiently large and divisible integer and let Y_1 be a complete intersection in Y cut out by r general weighted hypersurfaces of degree d, then Y_1 is a quasi-smooth weighted complete intersection of codimension $2r$ in $\mathbb{P}(1^m, a_1, \cdots, a_{s-1})$. Let $T = S \cap Y_1$, then $\dim T \ge \dim S - (r+1) \ge s - 1 + (2^{s-1}k + 2^{s-1} - 1) \cdot 2r$. If $Z \not\subseteq Y$, then $W = (Z \cdot Y_1)$ is a well-defined cycle of pure codimension k in Y_1 such that $W \equiv c_1(\mathcal{O}(1))^k \cap [Y_1]$. By induction hypothesis we have $\mathrm{mult}_T W \le 1$, hence $\mathrm{mult}_S Z \le 1$ in this case. If otherwise $Z \subseteq Y$, then we may view Z as a cycle of codimension $k - 1$ in Y and get a well-defined cycle $W = (Z \cdot Y_1)$ of pure codimension $k - 1$ in Y_1. By [Dim85, Proposition 6], we have $b_{2k-2}(Y_1) = 1$, hence there exists some $\lambda \in \mathbb{Q}$ such that $W \equiv \lambda \cdot c_1(\mathcal{O}(1))^{k-1} \cap [Y_1]$. Comparing the degrees of both sides we see that $\lambda = a_s^{-1} \le 1$. Therefore, using our induction hypothesis again we obtain $\mathrm{mult}_T W \le 1$ and thus $\mathrm{mult}_S Z \le 1$ as desired. □

Corollary 6.3.3. *Let $X \subseteq \mathbb{P}$ be a weighted complete intersection of codimension r and let $Z \subseteq X$ be an effective cycle of pure codimension k such that $Z \sim_{\mathbb{Q}} c_1(\mathcal{O}_X(1))^k \cap [X]$. Then for every subvariety $S \subseteq X$ of dimension $\ge s + \delta_X + 1 + (2^s k + 2^s - 1)(r + \delta_X + 1)$, we have $\mathrm{mult}_S Z \le 1$.*

Proof. Let d be a sufficiently large and divisible integer and let X_1 be a complete intersection in X cut out by $\delta_X + 1$ general weighted hypersurfaces of degree d, then X_1 is quasi-smooth and the result follows from Lemma 6.3.2 applied to X_1. □

Remark 6.3.4. The main point of Lemma 6.3.2 and Corollary 6.3.3 is that there exists an integer N depending only on the discrete data (k, r, s, δ_X) (and most importantly, not on $\dim X$ or $a_1, \cdots, a_s$) such that $\mathrm{mult}_S Z \le 1$ whenever $\dim S \ge N$. Our choice of N above is probably far from optimal (for example, for smooth cyclic covers over a hypersurface, one can just take $N = 2k + 1$ by a modification of the above argument), but it is sufficient for our need.

Proof of Theorem 6.1.3. The argument is almost identical to the proof of Theorem 4.1.1, so we only give a sketch. Let $M \sim_{\mathbb{Q}} -K_X$ be a movable boundary. By Lemma 6.3.2, there exists an integer N depending only on r and s such that $\mathrm{mult}_S M \le 1$ and $\mathrm{mult}_S(M^2) \le 1$ for every subvariety $S \subseteq X$ of dimension $\ge N$. In particular, by [Kol97, 3.14.1] and [dFEM04, Theorem 0.1], (X, M) (resp. $(X, 2M)$) has canonical (resp. lc) singularities outside a set of dimension at most $N - 1$ in X. Let $x \in X$ and let $Y \subseteq X$ be a complete intersection cut out by N general members of the linear system $|H|$ (it is base point free by assumption) containing x, then $(Y, 2M|_Y)$ is lc outside x. Let $L = K_Y + 2M \sim (N+1)H$, then $h^0(Y, L)$ is bounded by a (fixed) polynomial $P(m)$ of degree $N + 1$ in m. Choose sufficiently large $M > 0$ such that $p(m) \le \frac{c^c}{c!}$ (where $c = \dim Y = m + s - 1 - r - N$) for all $m \ge M$, then by Corollary 4.1.6, $(Y, M|_Y)$ is lc at x, hence by inversion of adjunction, (X, M) has canonical singularities and X is birationally superrigid by Theorem 2.2.5. Similarly for all effective divisor $D \sim_{\mathbb{Q}} -K_X$, we have $\mathrm{lct}(X; D) > \frac{1}{2}$, thus $\alpha(X) > \frac{1}{2}$ and by Theorem 1.1.4, X is also K-stable. $\square$

Combining ideas from the proof of Theorems 6.1.3 and 5.1.2, we now complete the proof of Theorem 6.1.2.

Proof of Theorem 6.1.2. It is easy to see that the $\mathcal{X}_{0,t}$ ($t \neq 0$) is not birationally superrigid since the assumptions imply that $\mathrm{mult}_x \mathcal{X}_{0,t} = 2m - 2$, hence a general line through x in $\mathbb{P}^{n+1}$ intersects the hypersurface $\mathcal{X}_{0,t}$ in exactly two other points and therefore induces a birational involution by interchanging these two point. By [dF16] (applied to the smooth hypersurfaces $\mathcal{X}_{s,t}$ when $st \neq 0$) and Theorem 6.1.3 (applied to $\mathcal{X}_{s,0}$, which are smooth double covers of hypersurfaces when $s \neq 0$), it is clear that $\mathcal{X}_{s,t}$ is birationally superrigid when $s \neq 0$. It remains to show that $X = \mathcal{X}_{\mathbf{0}}$ is birationally superrigid where $\mathbf{0} = (0,0) \in \mathbb{A}^2$. For ease of notation, we let $f = f_0$, $F = F_0$, etc.

We may identify $x \in F \cap G$ with its preimage in X. By assumption X is smooth outside x and $-K_X \sim H$ where H is the pullback of the hyperplane class on G. Let $M \sim_{\mathbb{Q}} -K_X$, by Theorem 2.2.5, it suffices to show that (X, M) has canonical singularities. Note that $X = (y^2 - f = g = 0) \subseteq \mathbb{P}(1^{n+2}, m)$ is a weighted complete intersection, thus by Corollary 6.3.3, there exists a constant (i.e. independent of m) $N_0 \in \mathbb{Z}$ such that $\mathrm{mult}_y(M^2) \leq 1$ away from a subset of dimension at most N_0. It then follows from the proof of Theorem 6.1.3 that there exists another constant N_1 such that (X, M) is canonical over the smooth locus of X when $n \geq N_1$. It remains to show that the pair is also canonical at x. Let $x_0, \cdots, x_{n+1}, y$ be the weighted homogeneous coordinate of $\mathbb{P}(1^{n+2}, m)$ and after a change of coordinates we may put $x = [0 : \cdots : 0 : 1 : 0]$. Let $\pi : Y \to X$ be the weighted blowup at x with associated weights $\mathrm{wt}(x_i) = 1$ ($i = 0, \cdots, n$) and $\mathrm{wt}(y) = m - 1$. Then the exceptional divisor E is isomorphic to the double cover of the projective tangent cone of G branched over the projective tangent cone of F. In particular, $E \subseteq \mathbb{P}(1^{n+1}, m-1)$ is a smooth weighted complete intersection and Y is also smooth. We can write $g = x_{n+1} g_{m-1}(x_0, \cdots, x_n) + g_m(x_0, \cdots, x_n)$ where $\deg(g_i) = i$ ($i = m-1, m$). Let $D = (g_{m-1} = 0) \subseteq X$, then $\mathrm{ord}_E D = m$. Let $c = \mathrm{ord}_E M$ and let $\tilde{M} \sim_{\mathbb{Q}} \pi^* H - cE$ (resp. $\widetilde{D} \sim (m-1)\pi^* H - mE$) be the strict transform of M (resp. D) on Y. Since $\tilde{M}$ is movable and $\pi^* H - E$ is base point free on Y, we have $0 \leq \tilde{M} \cdot \widetilde{D} \cdot (\pi^* H - E)^{n-2} =$

$(m-1)(H^n) - cm\deg E = 2m(m-1)(1-c)$, thus $c \leq 1$. We now show that $(Y, \tilde{M})$ is canonical, then as $K_Y + \tilde{M} = \pi^*(K_X + M) + (1-c)E$, (X, M) is also canonical as desired. The rest of the argument is similar to Theorem 1.2.3. Let $Z = \tilde{M}^2$ and write $Z = Z_1 + Z_2$ such that the irreducible components of Z_1 (resp. Z_2) are (resp. not) contained in E. We have $Z_1 \sim_{\mathbb{Q}} -bE^2$ for some $b \geq 0$. As in the proof of Theorem 1.2.3, we have $(b+c^2)\deg E \leq (H^n)$ (or equivalently $b + c^2 \leq \frac{m}{m-1}$. It then follows from Lemma 6.3.2 that $\mathrm{mult}_y Z_i \leq \frac{m}{m-1}$ and $\mathrm{mult}_y Z \leq \frac{2m}{m-1}$ outside a set of dimension at most N_0 (possibly by increasing the constant N_0). By [Cor00, Theorem 3.1] or [dFEM04, Theorem 0.1], there exists some $\mu > 1$ such that $(Y, \mu\tilde{M})$ is lc outside a set of dimension at most N_0. The same proof as in Theorem 1.2.3 then implies that $(Y, \tilde{M})$ is canonical when $m \gg 0$. The proof is now complete. □

6.4 Constructibility

In this section we prove Theorem 6.1.4. Indeed we will prove something more general. In [Tia12, Conjecture 5.3], Tian conjectured that in the definition (2.2) of alpha invariants, the infimum is indeed a minimum:

Conjecture 6.4.1. *Let (X, Δ) be a klt pair and L an ample line bundle on X. Then there exists an integer $m > 0$ such that*

$$\alpha(X, \Delta; L) = \alpha_m(X, \Delta; L).$$

This is confirmed by Birkar [Bir16b, Theorem 1.5] when (X, Δ) is log Fano, $L = -(K_X + \Delta)$ and $\alpha(X, \Delta; L) \leq 1$. Another special case is smooth quartic surfaces [ACS18, Theorem 1.2] where one can just take $m = 1$. The main result of this section (which implies Theorem 6.1.4) is:

Theorem 6.4.2. *Fix $r \in \mathbb{N}$ and $\epsilon > 0$. Then there exists an integer N such that for every smooth complete intersection $X \subseteq \mathbb{P}^{n+r}$ of codimension r and dimension $n \geq N$ such that $K_X \sim sH$ where $s \leq (1-\epsilon)n$ and H is the hyperplane class, we have $\alpha(X; H) = \alpha_m(X; H)$ for some integer m that only depends on n, r and s.*

The proof is based on the following criterion.

Proposition 6.4.3. *Let (X, Δ) be a klt pair of dimension n and L an ample $\mathbb{Q}$-Cartier divisor on X. Let $r > 0$, $s \geq 0$ be integers. Assume that*

1. *$sL - (K_X + \Delta)$ is nef and rL is globally generated,*
2. *the class group $\mathrm{Cl}(X)$ is generated by L,*
3. *$\alpha := \alpha(X, \Delta; L) < \alpha_2 := \alpha^{(2)}(X, \Delta; L)$.*

Then either $\alpha(X, \Delta; L) = \mathrm{lct}(X, \Delta; D)$ for some $D \sim_{\mathbb{Q}} L$ supported on an irreducible component of Δ or there exists an integer $m \leq \frac{(s+nr+\alpha_2)\alpha}{\alpha_2 - \alpha}$ such that $\alpha(X, \Delta; L) = \alpha_m(X, \Delta; L)$.

Proof. The argument is very similar to the proof of [BL18b, Theorem 5.1]. Let $D \sim_{\mathbb{Q}} L$ be an effective divisor. Since X is $\mathbb{Q}$-factorial and $\rho(X) = 1$ by assumption, each irreducible component of D is $\mathbb{Q}$-linearly equivalent to some rational multiple of L. As being lc is closed under convex linear combination, we may replace D by the suitable multiple of one of its irreducible components without increasing $\mathrm{lct}(X, \Delta; D)$. It follows that $\alpha(X, \Delta; L)$ is also the infimum of $\mathrm{lct}(X, \Delta; D)$ for all irreducible $D \sim_{\mathbb{Q}} L$.

Let $0 < \epsilon \ll 1$. Let $D \sim_{\mathbb{Q}} L$ be an irreducible divisor such that $\mathrm{lct}(X, \Delta; D) < \alpha + \epsilon$. Write $D = \lambda\Gamma$ for some irreducible and reduced divisor $\Gamma \subseteq X$ and $\lambda > 0$. Since $\mathrm{Cl}(X)$ is generated by L, we have $\Gamma \in |mL|$ for some integer m and thus $\lambda = \frac{1}{m}$ since $D \sim_{\mathbb{Q}} L$. Let E be an exceptional divisor over X that computes $\mathrm{lct}(X, \Delta; D)$ and let $A = A_{X,\Delta}(E)$ be the log discrepancy. Then we have $\mathrm{ord}_E(D) \geq \frac{A}{\alpha+\epsilon}$ and hence for sufficiently small $c > 0$ we have

$$\begin{aligned} \mathrm{ord}_E \mathcal{J}(X, \Delta + m(1-c)D) &\geq \lfloor m(1-c)\mathrm{ord}_E(D) - a(E; X, \Delta) \rfloor \\ &> m(1-c)\mathrm{ord}_E(D) - A \\ &\geq A\left(\frac{m(1-c)}{\alpha+\epsilon} - 1\right) \end{aligned}$$

where $\mathcal{J}(X, \Delta + m(1-c)D)$ denotes the multiplier ideal of the pair. Letting $c \to 0$ we get

$$\mathrm{ord}_E \mathcal{J}(X, \Delta + m(1-c)D) \geq A\left(\frac{m}{\alpha+\epsilon} - 1\right).$$

Since $rL - (K_X + \Delta)$ is ample and $D \sim_{\mathbb{Q}} L$, we have

$$H^i(X, \mathcal{J}(X, \Delta + m(1-c)D) \otimes \mathcal{O}_X((m+r)L)) = 0$$

by Nadel vanishing. As rL is globally generated, we see that $\mathcal{J}(X, \Delta + m(1-c)D) \otimes \mathcal{O}_X((m+r+nr)L)$ is also globally generated by Castelnuovo-Mumford regularity (see e.g. [Laz04, §1.8]) and gives rise to a sub linear series $\mathcal{M} \subseteq |(m+s+nr)L|$ with

$$\mathrm{lct}(X, \Delta; \frac{1}{m+s+nr} M) \leq \frac{(m+s+nr)A}{\mathrm{ord}_E \mathcal{J}(X, \Delta + m(1-c)D)} \leq \frac{m+s+nr}{m-\alpha-\epsilon}(\alpha+\epsilon). \tag{6.1}$$

Now if Γ is not a component of Δ, then $(X, \Delta + m(1-c)D)$ is klt in codimension one, thus the base locus of $\mathcal{M}$ has codimension at least two and we have $\mathrm{lct}(X, \Delta; \frac{1}{m+s+nr} M) \geq \alpha_2$ by definition. Combined with (6.1) this yields

$$\alpha_2 \leq \frac{m+s+nr}{m-\alpha-\epsilon}(\alpha+\epsilon),$$

or equivalently (using $0 < \epsilon \ll 1$), $m \leq \frac{(s+nr+\alpha_2)\alpha}{\alpha_2 - \alpha}$. In other words, we have shown that $\alpha(X, \Delta; D)$ is either computed by a component of Δ or given by the infimum of $\mathrm{lct}(X, \Delta; \frac{1}{m}\Gamma)$ over all $m \leq \frac{(s+nr+\alpha_2)\alpha}{\alpha_2-\alpha}$ and $\Gamma \in |mL|$. This concludes the proof. □

Thus in order to prove Theorem 6.4.2, we need to exhibit a gap between the codimension 2 alpha invariant and the usual alpha invariant of a smooth complete intersection. Since it is clear that $\alpha(X; H) \leq 1$, it suffices to show that $\alpha^{(2)}(X; H) > 1 + \delta$ for some absolute constant $\delta > 0$. This can be done using the same argument as in Chapter 4.

Lemma 6.4.4. *There exists a constant $\delta = \delta(n) > 0$ depending only on n such that if X is a smooth projective variety of dimension n, D an effective $\mathbb{Q}$-divisor on X and L a line bundle such that*

1. *$L - (K_X + (1-\epsilon)D)$ is nef and big for all $0 < \epsilon \ll 1$,*

2. *(X, D) is lc outside a finite number of points, and*

3. *$h^0(X, L) < \frac{1}{n}\binom{2n-2}{n-1}$,*

then $\operatorname{lct}(X;D) > \frac{1}{2} + \delta$.

Proof. Let $\bar{\sigma}_n = \min\{\#(Q_{\mathbf{a}} \cap \mathbb{Z}^n) \,|\, \mathbf{a} \in \mathbb{R}^n_+ \text{ s.t. } (1, 1, \cdots, 1) \in \overline{Q_{\mathbf{a}}}\}$ where

$$Q_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \,|\, \mathbf{a} \cdot \mathbf{x} < 1\}.$$

By the proof of Lemma 4.2.4 (or [dFEM04, Theorem 1.1]), if $\mathcal{I}$ is a monomial ideal co-supported at $0 \in \mathbb{A}^n$ such that $\ell(\mathcal{O}_{\mathbb{A}^n}/\mathcal{I}) < \bar{\sigma}_n$, then $(\mathbb{A}^n, \mathcal{I})$ is klt. Since there are only finitely many such monomial ideals, there exists a constant $\delta_0 > 0$ depending only on n such that $\operatorname{lct}(\mathbb{A}^n; \mathcal{I}) > 1 + \delta_0$ for all such $\mathcal{I}$. Hence by lower semi-continuity of lct as in the proof of Lemma 4.2.4, for any $x \in X$ and $\mathcal{I} \subseteq \mathcal{O}_X$ co-supported at x such that $\ell(\mathcal{O}_X/\mathcal{I}) < \bar{\sigma}_n$, we have $\operatorname{lct}(X; \mathcal{I}) > 1 + \delta_0$. Therefore, by Theorem 4.1.4 we have $\operatorname{lct}(X;D) > \frac{1+\delta_0}{2+\delta_0} > \frac{1}{2} + \delta$ for some constant δ depending only on n as long as $h^0(X,L) < \bar{\sigma}_n$. Thus it remains to prove that $\bar{\sigma}_n \geq \frac{1}{n}\binom{2n-2}{n-1}$. This comes from the fact that if $(1, 1, \cdots, 1) \in \overline{Q_{\mathbf{a}}}$ so that $\sum_{i=1}^n a_i \leq 1$ and $m_1, \cdots, m_n \in \mathbb{Z}$ satisfy $\sum_{i=1}^n m_i \leq n-1$, then at least one cyclic permutation of $(m_1, \cdots, m_n)$ lies in $Q_{\mathbf{a}}$ (see [Kol18, Paragraph 57]). $\square$

Lemma 6.4.5. *Fix $r \in \mathbb{N}$, $\epsilon > 0$ and let $N \gg 0$. Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth complete intersection of dimension n and codimension r. Let H be the hyperplane class. Suppose $K_X = sH$, $s \leq (1-\epsilon)n$ and $n \geq N$, then $\alpha^{(2)}(X;H) > 1 + \delta$ for some $\delta > 0$ that only depends on n and r.*

Proof. Let $M \sim_{\mathbb{Q}} H$ be a movable boundary on X. By [Suz17, Proposition 2.1], there exists a subset $Z \subseteq X$ of dimension at most $2r-1$ such that $\operatorname{mult}_x(M^2) \leq 1$ for all $x \notin Z$. Let $x \in X \backslash Z$ and let S be a general surface section of X containing x, then by [dFEM04, Theorem 0.1], $(S, 2M|_S)$ is lc at x, hence by inversion of adjunction, $(X, 2M)$ is lc at x as well. It follows that for all $0 < c \ll 1$, the pair $(X, 2(1-c)M)$ is klt outside Z. Let $x \in X$ be any point and let $Y \subseteq X$ be cut out by a general linear subspace $V \subseteq \mathbb{P}^{n+r}$ of codimension $2r-1$ containing x. Then $Y \subseteq \mathbb{P} := \mathbb{P}^{n-r+1}$ is also a codimension r complete intersection and we have $K_Y \sim (s + 2r - 1)H$. Let $D = 2M|_Y$ and $L = (s+2r+1)H \sim_{\mathbb{Q}} K_Y + D$. Since V is general and $\dim Z \leq 2r-1$, $(Y, (1-c)D)$ is klt outside a finite set of points. By

Lemma 6.4.4 we have $\mathrm{lct}(Y;D) > \frac{1}{2}+\delta$ for some absolute constant $\delta > 0$ as long as

$$h^0(Y,L) \le h^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(s+2r+1)) = \binom{n+s+r+2}{s+2r+1} < \frac{1}{n-2r+1}\binom{2(n-2r+1)}{n-2r+1} \quad (6.2)$$

which holds as $n \ge N \gg 0$. □

Proof of Theorem 6.4.2. This follows directly from Proposition 6.4.3 and Lemma 6.4.5. □

Proof of Theorem 6.1.4. Since the function $t \mapsto \alpha_m(\mathcal{X}_t)$ is constructible, by Theorem 6.4.2 it suffices to show that (6.2) holds when $s \le -1$ and $n \ge 10r$, which can be easily verified. □

Remark 6.4.6. For general polarized klt pairs $(X,\Delta;L)$, Conjecture 6.4.1 is not true: it is not hard to see that the alpha invariant only depends on the numerical equivalence class of the line bundle L, but if $D \sim_{\mathbb{Q}} L$ is a divisor whose lct computes $\alpha(X,\Delta;L)$ and N is a non-torsion numerically trivial line bundle on X, then D is no longer the support of a divisor in some $|m(L+N)|$ and it is unlikely to have another divisor that computes $\alpha(X,\Delta;L+N)$. The following example shows that in general there may not even exist a divisor D numerically equivalent to L such that $\mathrm{lct}(X,\Delta;D) = \alpha(X,\Delta;L)$, so Tian's conjecture is false even up to replacing L by a numerically equivalent one.

Example 6.4.7. Let C be a curve of genus at least 2 and $\mathcal{E}$ a stable vector bundle of degree 0 and rank 2 on C such that the line bundle $L_0 = \mathcal{O}(1)$ on $X = \mathbb{P}_C(\mathcal{E})$ is pseudo-effective but not numerically equivalent to any effective divisor (see e.g. [Laz04, Example 1.5.1]). Let $\Delta = 0$ and $L = L_0 + aF$ where F is the fiber class and $a \ge 2$ is an integer. It is clear that $\alpha(X;L) \le \frac{1}{a}$. On the other hand we claim that for any divisor $D \equiv L$ we have $\mathrm{lct}(X;D) > \frac{1}{a}$. To see this, fix a fiber F and let $D = cF + D_0$ where D_0 does not contain F in its support, then $c < a$ by our assumption on L_0. Since $(D_0 \cdot F) = (L \cdot F) = 1$, $(F, D_0|_F)$ is lc and thus $(X, F + D_0)$ is also lc by inversion of adjunction. Hence either $c \le 1$ and (X,D) is lc or $1 < c < a$ and $(X, \frac{1}{c}D)$ is lc. In both cases, $\mathrm{lct}(X;D) \ge \min\{1, \frac{1}{c}\} > \frac{1}{a}$. Hence $\alpha(X;L) = \frac{1}{a}$ but is not computed by any divisor $D \equiv L$. Note that the same argument also proves that $\alpha^{(2)}(X;L) \ge 1 > \alpha(X;L)$, hence the Picard number one assumption in Proposition 6.4.3 cannot be removed. Also note that although the alpha invariant is not computed by some $D \equiv L$, it is computed by some divisor E over X in the sense that

$$\alpha(X, \Delta; L) = \frac{A(E; X, \Delta)}{\tau(L; E)} \tag{6.3}$$

where $\tau(L; E)$ is the pseudo-effective threshold of L with respect to E (i.e. the largest $t > 0$ such that $\pi^*L - tE$ is pseudoeffective where $\pi : Y \to X$ is a birational morphism that extracts E). Therefore, instead of Conjecture 6.4.1 it seems better to ask

Question 6.4.8. Let (X, Δ) be a klt pair and L an ample line bundle. Does there always exist a divisor E over X such that (6.3) holds?

We conclude this section by presenting the proof of [SC11, Corollary 7.8] on the constructibility of birational superrigidity.

Theorem 6.4.9 ([SC11, Corollary 7.8])**.** *Let $f : \mathcal{X} \to T$ be a flat family of Fano varieties, then the set of $t \in T$ for which $\mathcal{X}_t = f^{-1}(t)$ is birationally superrigid is a constructible subset of T.*

A key ingredient in the proof is the following characterization of birational superrigidity.

Lemma 6.4.10. *Let X be a Fano variety with terminal $\mathbb{Q}$-factorial singularities. Suppose that X is not birationally superrigid and $\rho(X) = 1$, then there exists a weak Fano variety $\widetilde{X}$ with terminal $\mathbb{Q}$-factorial singularities and Picard number two such that X is obtained from $\widetilde{X}$ by running a $K_{\widetilde{X}}$-MMP.*

Proof. By Theorem 2.2.5 and our assumptions, there exists a movable boundary $M \sim_{\mathbb{Q}} -K_X$ for which (X, M) does not have canonical singularities. Let $0 < c < 1$ be the largest number such that (X, cM) is canonical. Let $\pi : Y \to X$ be the terminal modification of (X, cM) (c.f. [Kol13, Theorem 1.33]) and let M_Y be the strict transform of M. Then we have $K_Y + cM_Y = \pi^*(K_X + cM)$ and (Y, cM_Y) has terminal singularities. By perturbing M, we may assume that there exists a unique divisor E over X such that $a(E; X, cM) = 0$; in particular, we may assume that E is the unique exceptional divisor of π and $\rho(Y) = 2$.

As $-(K_X + cM) \sim_{\mathbb{Q}} -(1 - c)K_X$ is ample, Y is log Fano. By [BCHM10, Corollary 1.3.1], we may run a $(-K_Y)$-MMP $p : Y \dashrightarrow Y_1$ where $-K_{Y_1}$ is nef and big. Since $-K_Y =$

$cM_Y - \pi^*(K_X + cM)$ is movable, this MMP only consists of flips. Choose a general divisor $H \sim_{\mathbb{Q}} -(K_X + cM)$ such that $(Y, cM_Y + \pi^*H)$ is still terminal. As $K_Y + cM_Y + \pi^*H \sim_{\mathbb{Q}} 0$, the above MMP is $(K_Y + cM_Y + \pi^*H)$-crepant and thus if M_1 and H_1 are the strict transforms of M_Y and π^*H on Y_1, we have $K_{Y_1} + cM_1 + H_1 \sim_{\mathbb{Q}} 0$ and $(Y_1, cM_1 + H_1)$ is also terminal. In particular, Y_1 is a weak Fano variety with terminal $\mathbb{Q}$-factorial singularities and since Y_1 is obtained by a sequence of $(-K_Y)$-flips, Y can be obtained by a sequence of K_{Y_1}-flips. Finally, since X has terminal singularities, $K_Y = \pi^*K_X + aE$ for some $a > 0$; thus by the negativity lemma [KM98, Lemma 3.39], any K_Y-MMP over X terminates with X. In particular, X is obtained by a run of K_Y-MMP and this completes the proof. $\square$

Proof of Theorem 6.4.9. It suffices to exhibit a bounded family $f : Y \to U$ of Fano varieties such that a terminal $\mathbb{Q}$-factorial Fano varieties X of Picard number 1 is not birationally superrigid if and only if X appears as a fiber of f. The rest of the proof then follows from fairly standard Hilbert scheme arguments.

By [Bir16b, Theorem 1.1], weak Fano varieties with terminal singularities form a bounded family. Let $g : Z \to B$ be such a bounded family (i.e. every terminal weak Fano appears as a fiber). By shrinking B, we may assume that all fibers of g are integral and normal. By [Kol13, Corollary 4.10] and [dFH11, Proposition 3.5], we may assume that all fibers have terminal singularities after replacing B by a stratification. Replacing B by an open subset, we may further assume that all fibers are weak Fano by [dFH11, Proposition 3.8]. After taking a finite base change of B, we may also assume that all fibers are $\mathbb{Q}$-factorial of Picard number 2 and the natural maps

$$N^1(Z/B) \to N^1(Z_b), \quad N_1(Z_b) \to N_1(Z/B)$$

are isomorphisms for all $b \in B$ by [dFH11, Proposition 6.5 and Lemma 6.6]. We also assume that B is affine.

Now by [BCHM10, Corollary 1.3.1], we can run the K_Z-MMP over B since $-K_Z$ is g-ample. Since the relative Picard number is two, this MMP is unique once the initial extremal ray is chosen (and there are only two ways to choose the initial ray). Over a dense open subset of B, every K-MMP of the fiber is obtained by restricting such a K_Z-MMP. Thus by

Noetherian induction, it is not hard to see that there exists a bounded family $f : Y \to U$ of Fano varieties whose fibers correspond exactly to those Fano varieties obtained from some K-MMP of fibers of g (i.e. we discard outputs of those MMPs that are strict Mori fiber spaces).

As indicated at the beginning, the constructibility of birational superrigidity now follows from Lemma 6.4.10, [dFH11, Corollary 4.10, Proposition 6.5] and standard Hilbert scheme arguments. $\square$

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