Topics in Fano Varieties and Singularities

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Abstract

In this thesis, we look at several problems in two areas of algebraic geometry: singularities and Fano varieties. From the singularities side, we examine the relationship between local fundamental groups and étale covers of the regular locus of a normal scheme. Here we are able to classify the obstructions to the map $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ being an isomorphism, and show that if they are finite there exists an étale cover of the regular locus of X where the maps are an isomorphism.

In the area of Fano varieties, we study the notion of birational superrigidity. We show that under some extra conditions it implies K-stability a notion originating in the study of nice metrics on the Fano varieties. Moreover we show that hypersurfaces of sufficiently high dimension with respect to their index must satisfy some sort of rigidity assumption, restricting the base locus of any birational map to a Mori fiber space.

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Contents

Chapter 1

Motivation and Results

This thesis will focus on some topics in two fields of algebraic geometry: singularities and Fano varieties.

1.1 Topology of Singularities

From the singularities point of view, our concern is the local topology of singularities on a variety and its corresponding impact on the geometry of the variety reflected via étale covers of the regular locus. Mumford proved that for a normal complex surface S a point $x \in S$ is smooth if and only if its local fundamental group is trivial [25]. This allows us to ask two questions in higher dimensions. One is whether there are certain nice classes of singularities where we can control the local fundamental groups. In this direction, Xu proved that it is possible to say that the étale local fundamental groups are finite for klt singularities over $\mathbb{C}$ [39]. A similar statement was then shown to hold true for strongly F-regular schemes in characteristic p [6], which are a class of singularities corresponding closely to klt singularities in characteristic 0. The work of Bhatt, Gabber and Olsson [3] furthermore shows that it is also possible to use the finite characteristic result to reprove the result of Xu in characteristic 0.

The other question we can ask is whether if all the local fundamental groups (or étale local fundamental groups) of a normal variety X are finite does this allow us to control the geometry of X. The first guess might be that the kernel of the map of fundamental groups $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ would be finite. Some of the first examples show that this is in fact too strong to be true. However as first shown in [19] for klt singularities, it is true that the map is an isomorphism for the étale fundamental groups after some étale cover of the regular locus. This was extended to strongly F-regular singularities in characteristic p by [2].

We show that such results are in fact independent of the singularities and are only dependent on the finiteness of the local fundamental groups. Moreover they hold in a great generality, essentially only relying on the existence of alterations for the class of normal schemes we are interested in. Finally we show that the existence of these covers of the regular locus automatically imply that the images of the local fundamental group in $\pi_1(X_{\text{reg}})$ are finite, so these results can be represented as the following equivalent statements.

Theorem 1.1.1. *[32] Suppose that X is a normal noetherian scheme of finite type over an excellent base B of dimension ≤ 2. Let $Z = \operatorname{Sing}(X) \subseteq X$. Then the following are equivalent.*

(i) For every geometric point $x \in Z$ the image $G_x := \operatorname{im}[\pi_1(X_x \backslash Z_x) \to \pi_1(X \backslash Z)]$ is finite.

(ii) There exists a finite index closed normal subgroup $H \subseteq \pi_1(X \backslash Z)$ such that $G_x \cap H$ is trivial for every geometric point $x \in X$.

(iii) For every tower of quasi-étale Galois covers of X

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

$X_{i+1} \to X_i$ are étale for i sufficiently large.

(iv) There exists a finite, Galois, quasi-étale cover $Y \to X$ by a normal scheme Y such that any étale cover of Y_{reg} extends to an étale cover of Y.

The idea behind these results is that we can kill the image of any finite local fundamental group in $\pi_1(X_{\mathrm{reg}})$ after an étale cover of X_{reg}. The problem is then that we may have to do this for infinitely many points. To solve this, it is possible to show that there exists a stratification of X into locally closed subsets such that if we deal with any point in a single stratum then we have dealt with all points. In [19] they showed that the Whitney stratification of X solves this problem. In [2] they showed by stratifying the size of local fundamental groups they could achieve similar results assuming they had a bound on the size of the local fundamental groups. We show here that such a stratification exists only depending on the existence of a regular alteration X. Moreover the stratification is easy to compute from our alteration. The main property of this stratification is summed up in the following.

Theorem 1.1.2. *[32] Suppose that X is a normal Noetherian scheme and $\pi : \tilde{X} \to X$ a regular alteration. Then there exists a stratification $X = \bigcup_{i \in I} Z_i$ into locally closed subsets for some index set I such that for any finite map $f : Y \to X$ étale over the regular locus, with Y a normal Noetherian scheme,* $\mathrm{Branch}(f) = \bigcup_{i \in J \subset I} Z_i$*, where $J \subseteq I$ is some subset.*

These results can be applied to give a uniform proof of the main theorems of [19] and [2], as well as many of their corollaries.

1.2 Birational Superrigidity

In this section of the thesis we take a look at the notion of birational superrigidity. The notion of birational superrigidity started with attempts to generalize the proof of [21] of the nonrationality of a quartic threefold. It was noticed that the Mori fiber space structures on many Fano varieties of Picard number one were very limited. This leads to the definition of birational rigidity and superrigidity.

Definition 1.2.1. *A Fano variety X is birationally rigid if for any birational map $X \dashrightarrow X'$ where X' is a Mori fiber space we must have $X' \cong X$. It is birationally superrigid if the birational map is always an isomorphism.*

In particular there exists no birational map to $\mathbb{P}^n$ for these varieties, and there exists no rational conic bundle structure on them either.

The method of proving such results always follows a similar route in theory. Given a birational map $X \dashrightarrow X'$ where X' is a Mori fiber space, by looking at the singularities of the strict transform on X of a general hyperplane section from X' it turns out it must be singular in a measurable since. Such estimates are known as Noether-Fano inequalities, dating back to the work of the Noether and the Italian algebraic geometers. Then the method works by showing that no sufficiently singular divisors can exist on X.

The descriptions of the singularities we need to exclude work very well with the singularities of the minimal model program. Corti noticed that the Noether-Fano inequalities can be described as the singularities of the pair (X, M) being canonical where $M \sim_{\mathbb{Q}} -K_X$ is a movable linear system [9]. In particular the techniques for studying these singularities in birational geometry can apply.

One of the problem studied in this section is the relation of birational superrigidity to K-stability. K-stability, which originated in the study of nice metrics on Fano varieties, has recently been found to be equivalent to a condition on singularities of pluricanonical linear systems (cf. [4]). In particular both birational superrigidity and K-stability say that certain pluricanonical linear systems are not too singular in certain measures. The first theorem (joint with Zhuang) of this section says that under extra conditions, these two concepts are related.

Theorem 1.2.2. *[33] Suppose that X is a $\mathbb{Q}$-Fano variety of Picard number* 1 *with*

$\alpha(X) > \frac{1}{2}$. *Then if X is birationally superrigid it is also K-stable.*

The next result (also joint with Zhuang), looks at rigidity type results of higher index Fano varieties. So far the method of using the Noether-Fano inequalities has proven useful mostly in the case where the anticanonical divisor generates the Picard group. In higher indices, it is possible to be birational to another Mori fiber space, but it still expected that there exists restrictions on what type of Mori fiber spaces they can be birational to. Although we did not restrict these Mori fiber spaces, we can at least show for the example of complete intersections that there exists certain restriction on the base locus of a birational map to another Mori fiber space.

Theorem 1.2.3. *([41],appendix) Suppose that X is a Fano complete intersection in $\mathbb{P}^N$. Then for N sufficiently large compared to the codimension and the index, it follows that the base locus of any birational map $X \dashrightarrow X'$ to a Mori fiber space must have codimension less than or equal to the index of the Fano variety.*

Chapter 2

Local Fundamental Groups and Étale Covers

2.1 Background

Throughout this chapter X will denote a normal Noetherian scheme. We will make no assumption that X is over a field except for examples and applications. After fixing a geometric base point $x \mapsto X$, we will be interested in the algebraic fundamental group $\pi_1(X,x)$ (we will use $\pi_1^{\top}(X,x)$ to denote the topological fundamental groups if X is over the complex numbers). If no confusion will occur we will suppress the basepoint and just write $\pi_1(X)$. We are interested in the following question.

Question 2.1.1. *Suppose that X is a normal scheme and U is an open subset. Suppose that the complement $Z = X \backslash U$ has codimension ≥ 2. Then when is the map $\pi_1(U) \to \pi_1(X)$ an isomorphism?*

An open subset U as above such that $X \backslash U$ has codimension ≥ 2 will be called *big*. In one situation this result is classical. Namely if X is over a field it was proved by Zariski in 1958 that this map is always an isomorphism [40]. This was extended by Nagata to the case of an arbitrary regular local rings. In particular the following was proved

Lemma 2.1.2. *(Purity of the branch locus [27]) Suppose that $f : Y \to X$ is a quasi-finite morphism of Noetherian schemes, with X regular and Y normal. If the map is unramified at every codimension 1 points of Y then is is étale.*

We can apply this to fundamental groups as follows. If $g : V \to U$ is an étale cover of U, then by taking the normalization of X in $k(V)$, it will extend to a finite map $f : Y \to X$. By our assumption on the codimension of $Z = X \backslash U$ this will be étale in codimension 1 and hence étale. Hence every étale cover of U extends to one of X which will give us that the above map is an isomorphism. It will be useful to give a name to the type of maps we need to consider.

Definition 2.1.3. *A map $f : Y \to X$ of Noetherian normal schemes is called quasi-étale if it is étale over an open subset $U \subseteq X$ where $Z = X \backslash U$ has codimension ≥ 2 in X.*

It is easy to see that all the assumptions of the purity lemma are necessary. For instance if X is an isolated quotient singularity we get covers that are étale away from the singularity but will not be étale at the singular point. Similarly if Y is two planes meeting at a point, this maps to a single plane, and is étale in codimension one. However it will not be étale where the two planes meet, which shows that the normality assumption is necessary as well.

Purity also allows us to reduce our question to the case when $U = X_{\text{reg}}$. We can see this, as if we have an étale cover of U it extends by purity of the branch locus to an étale cover of $U \cup X_{\text{reg}}$. Hence if every étale cover of X_{reg} extends to an étale cover of X, then the same will be true for any U. In particular the most interesting case of the question is whether the map $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ is an isomorphism. We will use this reduction throughout this chapter.

2.2 Obstructions

We will now look at the obstructions to the above map on fundamental groups being an isomorphism. We will get one of these obstructions for each geometric point $x \in Z$. For a quasi-étale finite cover $f : Y \to X$ we need to check that it is étale over each $x \in Z$. To do this we will show that extends to an étale morphism in a local neighborhood of x. If we were in the complex case a contractible neighborhood would suffice. In general we will take the following replacement.

Definition 2.2.1 (Local Neighborhoods)**.** *Suppose that $x \in Z$ is a geometric point of X. Then we define $X_x = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x})$ where this denotes the strict Henselization of the local ring. This comes equipped with a map $\iota : X_x \to X$. We then define $Z_x = \iota^{-1}(Z)$ and $U_x = \iota^{-1}(U)$.*

The importance of using Henselizations is that a local ring is Henselian if and only if every finite extension is a product of local rings. In particular a local extension of a strictly Henselian ring is étale and only if it is an isomorphism. In particular we can prove the following

Lemma 2.2.2. *([2] Claim 3.5) Suppose that $f : Y \to X$ is a quasi-étale morphism of normal schemes, that is étale away from some subset Z of codimension ≥ 2. Then f is étale over a geometric point $x \in X$ if and only if the pull back of the map to $U_x := X_x \backslash Z_x$ is trivial.*

Proof. It is enough to prove that the map is étale once we pull back to the strict Henselization of the local ring. Now in this case if the map f is étale then it induces a trivial cover of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ and hence of the open set V. On the other hand if the cover of U_x is trivial, then so is the cover of $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$ since the varieties are normal. Hence the morphism is étale. □

This shows we need to know when étale covers of U_x extend to X_x. This is again determined by a map of fundamental groups. This leads to the following definition.

Definition 2.2.3. *Suppose that $x \in X$ is a geometric point lying in Z. We define the local fundamental group at x to be $\pi_1(U_x)$. We will define the obstruction group at x to be $G_x := \mathrm{im}[\pi_1(U_x) \to \pi_1(U)]$.*

Remark 2.2.4. This definition varies from the standard definition considered for example in [39] and [19] where they use the fundamental groups $\pi_1(X_x \backslash \{x\})$. We will see that this difference is important though will not matter for the case of klt singularities. Moreover we also see this definition is dependent on a base choice (x is not the base point as $x \notin U$). In particular the groups G_x are only defined up to conjugacy.

The importance of the obstruction groups are then summed up in the following lemma.

Lemma 2.2.5. *Suppose that X is a Noetherian normal scheme and $U \subseteq X$ is a big open subset. Then $\pi_1(U) \to \pi_1(X)$ is an isomorphism if and only if $G_x = 1$ for every geometric point $x \in X$*

Proof. First suppose that $G_x = 1$ for every $x \in X$. Let $f : Y \to X$ be a finite cover that is étale over U. Consider any point $x \in X$ a geometric point. Then if we restrict to the X_x, we will get a finite cover $Y_x \to X_x$ that is étale over U_x. In particular by our assumption $Y_x \to X_x$ is étale. Hence $Y \to X$ will be étale in a neighborhood of x.

On the other hand suppose $\pi_1(U) \to \pi_1(X)$ is an isomorphism. Suppose that $x \in X$ is a geometric point. Then we get a commutative diagram

$$\begin{array}{ccc} \pi_1(U_x) & \longrightarrow & \pi_1(X_x) \\ \downarrow & & \downarrow \\ \pi_1(U) & \longrightarrow & \pi_1(X) \end{array}$$

Then $\pi_1(X_x)$ is trivial as this is the spectrum of a Henselian ring. Hence the image of $\pi_1(U_x)$ in $\pi_1(X) \cong \pi_1(U)$ is trivial. This means that G_x is trivial. □

Hence the G_x do in fact provide the obstructions we want to consider to answer for our question. We will want to know when they are at least finite. To show finiteness, note that it is enough to show that the local fundamental group $\pi_1(U_x)$ is finite. If we restrict the singularities to certain nice classes, the results of [39] and [6] show that this will be true.

Theorem 2.2.6. *[39] Suppose that $x \in X$ is a complex klt singularity of dimension ≥ 2. Then $\pi_1(U_x)$ is finite.*

Proof. This does not follow exactly from [39]. There it was only proven that $\pi_1(X_x \backslash \{x\})$ is finite. However the same proof holds as in the last step we cut down through a surface through x and hence the two fundamental groups will agree. Alternatively we can use the techniques of [3]. □

A similar result holds in positive characteristic. In this case no modification of the proof is required.

Theorem 2.2.7. *[6] Suppose that $x \in X$ is a strongly F-regular singularity, F-finite normal scheme of dimension ≤ 2. Then $\pi_1(U_x)$ is finite.*

In particular in either of these cases we see that every G_x is at least finite.

2.3 Main Theorem and Examples

In this section we will state the main theorem as well as give some relevant examples. If we are in the case where the map $\pi_1(U) \to \pi_1(X)$ is not an isomorphism we can still hope to find something useful. If we have that some G_x is finite then we know that after some finite quasi-étale cover that we can kill this obstruction. Then the question is whether it is possible to do this for all points. The following theorem of [19] was the first result along this line.

Theorem 2.3.1. *[19] Suppose that X is a normal complex variety and Δ is a $\mathbb{Q}$-Weil divisor on X such that (X, Δ) is klt. Then for any tower*

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

of quasi-étale finite morphisms that are Galois over X. The morphisms in the tower will eventually become étale.

They used this result to obtain the following corollary.

Corollary 2.3.2. *[19] Suppose that (X, Δ) is a klt pair. Then there exists a quasi-étale cover $\tilde{X} \to X$ such that $\pi_1(\tilde{X}_{\text{reg}}) \to \pi_1(\tilde{X})$ is an isomorphism.*

The following simple two dimensional example shows how this theorem works in practice.

Example 2.3.3. Consider the quotient $\pi : A \to A/\pm = X$, where A is an abelian surface and X is a singular Kummer surface over $\mathbb{C}$. Away from the 16 2-torsion points this map is étale, but at the 2-torsion points it ramifies. Each of these 2-torsion points gives rise to a nontrivial $\mathbb{Z}/2\mathbb{Z}$ obstruction. In particular any étale cover of A will give a cover of X not satisfying purity of the branch locus, and in particular there are infinitely many such covers. On the other hand the fundamental group of X is trivial, which can be seen as X will be diffeomorphic to a standard Kummer surface given as a singular nodal quartic in $\mathbb{P}^3$ with the maximum number of nodes. In particular its étale fundamental group is trivial, and there are no étale covers of X. Note that on the other hand as A is smooth we obtain purity on a finite cover of X that is étale away from a set of codimension 2 as in (iv) of the above theorem. In this sense, although the kernel of the map $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ is large it is not far away from satisfying purity.

We see a few things from this example. For one it does not tell us that if all the G_x are finite then the kernel of $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ must be finite as well. Even in the case when the singularities are all isolated and all $G_x \cong \mathbb{Z}/2\mathbb{Z}$ we see that this kernel can be large. We also get a sense for how the proof should go in general: we find a quasi-étale cover that kills each of the G_x and keep iterating until all the G_x are trivial. The problem with this is that if the singularities are not isolated then we could have infinitely many points to check. We will deal with this issue by stratifying the singular set of X

The above theorem was subsequently extended to a similar class of varieties in characteristic p.

Theorem 2.3.4. *[2] Suppose that X is an F-finite Noetherian integral strongly F-regular scheme. Suppose there exists a tower*

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

of finite quasi-étale morphisms Galois over X. Then all but finitely many of the morphisms are étale.

This allows for the following corollary.

Corollary 2.3.5. *[2] Suppose that X is an F-finite strongly F-regular Noetherian integral scheme. Then there exists a quasi-étale Galois cover $\tilde{X} \to X$ such that $\pi_1(\tilde{X}_{\text{reg}}) \to \pi_1(\tilde{X})$ is an isomorphism*

Moreover they were able to obtain a purely group theoretic statement as well.

Corollary 2.3.6. *[2] Suppose we are in the situation above. Then there exists an open normal subgroup $H \subseteq \pi_1(X_{\text{reg}})$ such that $H \cap G_x = 1$ for each x.*

The idea of the main theorem of this section is to put these two theorems and their corollaries into a uniform setting. Moreover we show that it can be phrased completely in terms of the groups G_x and does not require any additional assumption on the singularities of the scheme X. Finally the theorem will hold in a very high generality including mixed characteristic.

Theorem 2.3.7. *[32] Suppose that X is a normal noetherian scheme of finite type over an excellent base B of dimension ≤ 2. Let $Z = \mathrm{Sing}(X) \subseteq X$. Then the following are equivalent.*

(i) For every geometric point $x \in Z$ the image $G_x = \mathrm{im}[\pi_1(X_x \backslash Z_x) \to \pi_1(X \backslash Z)]$ is finite.

(ii) There exists a finite index closed normal subgroup $H \subseteq \pi_1(X \backslash Z)$ such that $G_x \cap H$ is trivial for every geometric point $x \in X$.

(iii) For every tower of quasi-étale Galois covers of X

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

$X_{i+1} \to X_i$ are étale for i sufficiently large.

(iv) There exists a finite, Galois, quasi-étale cover $\tilde{X} \to X$ by a normal scheme $\tilde{X}$ such that any étale cover of $\tilde{X}_{\mathrm{reg}}$ extends to an étale cover of $\tilde{X}$.

Here all fundamental groups denote the algebraic fundamental groups, which are the profinite completion of their analogues over $\mathbb{C}$. In particular we see that all the different conditions proved in the theorems and corollaries of the previous works are in fact equivalent to having finite obstructions.

2.4 Quasi-Étale Covers and Alterations

In this section we will explain how to tell whether a finite morphism is étale by pulling back to an alteration. We will use alterations as a replacement for a resolution. First we give the definition of alterations as appearing in [13].

Definition 2.4.1. *A map* $\pi : \tilde{X} \to X$ *is an alteration if it is proper dominant and generically finite. If* $\tilde{X}$ *is regular the map is called a regular alteration.*

In particular a resolution is a regular alteration, and alterations should be thought of a generalization of a resolution. Although the existence of resolutions is not known outside characteristic 0, alterations are known to exist for much larger classes of schemes.

Theorem 2.4.2. *[13] Suppose that* X *is a Noetherian scheme of finite type over an excellent base of dimension* ≤ 2*. Then there exists a regular alteration* $\pi : \tilde{X} \to X$*.*

As in the case of resolutions we will need to understand the divisors that are contracted by π

Definition 2.4.3. *Suppose that* $\pi : \tilde{X} \to X$ *is an alteration. Then a divisor* $E \subseteq \tilde{X}$ *is called exceptional if* $\pi(E)$ *has codimension* ≥ 2 *in* X*.*

Our goal is to now examine whether a given finite map $f : Y \to X$ is étale by looking at an alteration $\pi : \tilde{X} \to X$. The most easy to see claim is

Claim 2.4.4. *Denote by* $Y' = (Y \times_X \tilde{X})$*. Then* f *is étale if and only if* $Y' \to \tilde{X}$ *is étale.*

Proof. One direction is clear as we are pulling back an étale morphism so we end up with something étale. On the other hand if $Y' \to \tilde{X}$ is étale, then we can tell by the fibers (which are the same as those for f) that the map f is étale as well. □

Although this gives a seemingly easy way to tell whether f is étale, in practice this is not that useful. This is due to the fact that while X, Y, and $\tilde{X}$ all be normal, the fiber product Y' can still fail to be normal in general. In particular it will be hard to use geometric techniques on Y' itself, and we will need to go to normalization. Note that if the map were étale then Y' would already be regular, but for a general quasi-étale map we cannot rule out that Y' is going to not be normal. In our proof we will need to normalize Y' so that we can apply purity of the branch locus to the morphism $(Y')^n \to \tilde{X}$.

Now let $\tilde{Y} = (Y \times_X \tilde{X})^n$ be the normalized fiber product which comes with a finite morphism $\tilde{f} : \tilde{Y} \to \tilde{X}$. Then the following lemma adjusts the claim to the case when we consider $\tilde{Y}$ instead of Y'.

Lemma 2.4.5. *Suppose that X is normal Noetherian scheme, with a regular alteration $\pi : \tilde{X} \to X$. Let $f : Y \to X$ be a finite morphism of Noetherian schemes, and denote by $\tilde{f} : \tilde{Y} \to \tilde{X}$ the normalized fiber product of the maps. Then f is étale if and only if $\tilde{f}$ is étale and for any geometric point $x \in X$, $\tilde{f}$ induces a trivial cover of $\pi^{-1}(x)$.*

Proof. First suppose that $f : Y \to X$ is étale. Then the base change $Y \times_X \tilde{X} \to \tilde{X}$ is étale, so that the fiber product was already normal. Hence it follows $\tilde{f}$ is étale. Then since $\tilde{Y}$ is just the fiber product and f is étale, for any of the d points $q \in Y$ mapping to $p \in X$ we see that $\sigma^{-1}(q) \cong \pi^{-1}(p) \times_{k(p)} k(q)$. Therefore $\tilde{f}$ is just the trivial degree d cover on every fiber $\pi^{-1}(p)$.

Now suppose that $\tilde{f}$ is étale and induces a trivial cover on every fiber of π. Then in particular for any $x \in X$, $f^{-1}(x)$ will have $\deg(f)$ geometric connected components. In particular it must be étale at x ([26], V.7). □

Using the above lemma we can check whether f is étale based on a single regular alteration independent of f. Note that versus the case of the regular fiber product we have two conditions to check about the map $\tilde{f}$. The next example shows that both these conditions can fail.

Example 2.4.6. First take as an example $X = \mathbb{A}^2/\pm$. This has a quasi-étale double cover $f : \mathbb{A}^2 \to X$. Blowing up the origins gives a map of the normalized fiber-products $\mathrm{BL}_0\, \mathbb{A}^2 \to \mathrm{BL}_0\, X$, that will ramify along the exceptional divisors.

On the other hand we can take the X to be the cone over an elliptic curve E. Take an étale cover $E' \to E$, which will induce a quasi-étale cover $X' \to X$ of their cones. After blowing up the origins we obtain the map of normalized fiber products which is étale, but induces a nontrivial cover of the exceptional divisors.

Now we note that f is quasi-étale then the map $\tilde{f} : \tilde{Y} \to \tilde{X}$ will be étale over U. In particular using purity of the branch locus on $\tilde{X}$ will show that the only place that $\tilde{f}$ can fail to be étale is over the exceptional divisors. In particular we see that the branch locus of $\tilde{f}$ is limited to a finite number of possible divisors. In the next section we will use this limitation on the branch locus of $\tilde{f}$ to also limit the branch locus of f.

2.5 Stratification of the Branch Locus

In this section we will show that if X is a normal scheme with a regular alteration $\pi : \tilde{X} \to X$, then the branch locus of any quasi-étale finite morphism $f : Y \to X$ can only be one of finitely many options. In more detail we will show that there exists a stratification of X (or $\mathrm{Sing}(X)$) into locally closed subsets such that the branch locus is a union of strata. This stratification need to satisfy the condition that if one point in a stratum satisfies our conditions to tell whether f is étale via $\tilde{f}$, then all points in the stratum likewise will satisfy these conditions. Hence our first goal will be to identify conditions that guarantee each of the properties hold on a stratum regardless of which f we choose. Above we saw that the images of exceptional divisors already gives some of the pieces of the stratification we desire. The other part will need the following condition.

Condition 2.5.1 (Condition $*$)**.** *Suppose that $g : Z \to S$ is a proper morphism of Noetherian schemes with S integral. Then g satisfies this condition if there exists a purely inseparable morphism $i : S' \to S$ such that if $Z' = Z \times_S S' \to S'$ is the base* *change, then $Z'_{red} \to S'$ is flat with geometrically reduced fibers.*

Roughly speaking this says that the reduced fibers fit together in a flat family. In particular it will tell us that the fibers are not changing topologically. Our first step will be to show that any morphism can be stratified into locally closed subsets such that this condition $*$ is satisfied over each stratum. The proof of this fact is similar to the fact that there exists a stratification of a morphism where the morphism is flat over each stratum.

Lemma 2.5.2. *Let $\pi : \tilde{X} \to X$ be a morphism of Noetherian schemes. Then there exists a stratification $X = \bigcup S_i$, where the S_i are irreducible locally closed subsets, such that $\pi^{-1}(S_i) \to S_i$ satisfies condition $(*)$ for all i.*

Proof. We will proceed by Noetherian induction on X. Take an irreducible component S of X. Consider the map $\pi^{-1}(S)_{\text{red}} \to S$. Taking an irreducible component W of $\pi^{-1}(S)_{\text{red}}$ if $W \to S$ is not separable, we can take some high enough power of the Frobenius so that the pullback by the map is separable. Doing this for every irreducible component of $\pi^{-1}(S)_{\text{red}}$ we may assume that the general fiber is reduced. Then taking an open subset U of S we may assume that every fiber of $\pi^{-1}(U)_{\text{red}} \to U$ is reduced and that this morphism is flat. Continuing on will give the desired stratification. □

Example 2.5.3. Let us apply this to the resolution of the Whitney umbrella $X = \{zx^2 = y^2\}$. Let $\pi : \tilde{X} \to X$ denote the blow up along the z-axis given by $\mathbb{A}^2_{s,t} \to X$ with $(s,t) \mapsto (s, st, t^2)$. Then for the point $P = \{x = y = z = 0\}$ we see that the fiber is given by $(s = t^2 = 0)$, which is nonreduced of length 2. Then if we look at the line $L = \{x = y = 0\}$ we see that for any $Q \in S = L \setminus \{P\}$ that the fiber over Q is of the form $\{s = t^2 - b = 0\}$, which consists of two points in characteristic $\neq 2$. Finally if we take a point in $U = \mathbb{A}^2 \setminus L$ the fiber will be of the form $\{s - a = t - b/a = 0\}$, and in particular is reduced of length 1. Therefore we see that the stratification becomes $\mathbb{A}^2 = U \cup S \cup \{P\}$. Although the resolution is flat over L, the point P is different from the other points on the line which is represented by it appearing in a different stratum.

The next lemma explains the important property of the condition $*$ that we will need to use.

Lemma 2.5.4. *(e.g. [20] 7.8.6) Suppose that $g : Z \to S$ is a morphism of Noetherian schemes satisfying condition $(*)$ with S integral. Then the number of connected components of the geometric fibers is constant.*

Proof. We have a purely inseparable morphism $S' \to S$ such that $Z' \to S'$ is flat with geometrically reduced fibers. Since $S' \to S$ is a universal homeomorphism, it follows that $Z' \to Z$ is a homeomorphism. Hence the number of connected components remains the same, so we can assume from the beginning that $Z \to S$ is flat with geometrically reduced fibers.

Now in this case we will show that the Stein factorization of $g : Z \to S$ factors as $Z \to \hat{S} \to S$ where $\hat{S} \to S$ is étale. Taking the strict Henselization of the local ring at any point we can reduce to the case where S is the spectrum of a strictly Henselian local ring. In this case $\hat{S}$ is a product of finitely many local rings. Our goal is to show that these are isomorphic to S. Now consider a connected component W of Z, so that the map $g : W \to S$ is flat and proper, with geometrically reduced fibers. Now since W is connected and S is the spectrum of strictly Henselian ring, the special fiber W_0 is also connected. But then since W_0 is reduced $H^0(W_0, \mathcal{O}_{W_0}) = k(0)$. Hence we see by the theorem of Grauert that $\mathcal{O}_S \to g_*\mathcal{O}_W$ is an isomorphism. This implies that $\hat{S} \to S$ is étale, so in particular the number of connected components of the geometric fibers are constant. □

We now know enough that we can prove the main use of our stratification to branch loci of quasi-étale maps.

Theorem 2.5.5. *Suppose that X is a normal Noetherian scheme and $\pi : \tilde{X} \to X$ a regular alteration. Then there exists a stratification $X = \bigcup_{i \in I} Z_i$ into locally closed subsets such that for any $f : Y \to X$ quasi-étale, with Y a normal Noetherian scheme,* $\operatorname{Branch}(f) = \bigcup_{i \in J \subset I} Z_i$.

Proof. The above lemma gives a stratification $X = \bigcup_i S_i$ such that $\pi^{-1}(S_i) \to S_i$ satisfies condition $*$. Moreover we a finite number of exceptional divisors E_i giving closed subsets $\pi(E_i)$ on X. Taking all possible intersections gives a stratification of X. Our goal is then to show that any branch locus of a quasi-étale morphism is a union of these strata.

Consider $\tilde{Y} = (\tilde{X} \times_X Y)^n$ the normalized fiber product which comes with a morphism $\tilde{f} : \tilde{Y} \to \tilde{X}$ that is étale away from the exceptional locus. Now by purity of the branch locus $\operatorname{Branch}(\tilde{f}) = \bigcup_i E_i$ where the E_i are some subset of the exceptional divisors. In particular the branch locus of f will include $B = \bigcup_i \pi(E_i)$, which will be a union of some strata. Now looking on the complement of B, and replacing X by $X \backslash B$ we can assume that $\tilde{f}$ is in fact étale. In particular $\tilde{f}^{-1}(\pi^{-1}(S_i)) \to \pi^{-1}(S_i) \to S_i$ satisfies condition $*$ as $\tilde{f}$ is étale. Hence the number of connected components of the fibers are constant. This implies that for any point $s \in S_i$ that if the cover of $\pi^{-1}(s)$ is geometrically trivial, then the corresponding cover for other point in S_i is also trivial. Hence we see that the branch locus must be a union of the strata. □

The key part of this proof is that the above stratification only depends on the alteration and not the finite morphism $f : Y \to X$. This will allow us to work with only finitely many sets even if we are dealing with an infinite number of maps. On the other hand we unfortunately in the proof of the main theorem will need to apply this not only to X, but also to some quasi-étale covers of X.

2.6 Proof of the Main Theorem

In this section we will prove the main theorem of this chapter. With the use of the stratification from the previous section, most implications are straight forward.

(i) $\Rightarrow$ (ii).

Proof. Consider a regular alteration $\pi : \hat{X} \to X$. This will give us a stratification $X = \bigcup_i Z_i$. Now for each of the finitely many generic points η_i of the different strata consider the finitely many finite groups $G_i = G_{\eta_i}$. Then as $\pi_1^{\text{ét}}(U)$ is profinite there exists some finite index closed normal subgroup H intersecting all of these G_i trivially. This corresponds to a quasi-étale cover $\gamma : Y \to X$ that is étale over U. Moreover by our choice of stratification for any geometric point x we will also have that $G_x \cap H$ is trivial. Hence such a finite index normal subgroup H can be taken uniformly for all $x \in X$. □

(ii) $\Rightarrow$ (i).

Proof. Our assumption (ii) gives a closed finite index normal subgroup $H \subseteq \pi_1^{\text{ét}}(U)$ such that $G_x \cap H = \{1\}$. Then in particular $G_x \cong G_x / G_x \cap H \subseteq \pi_1^{\text{ét}}(U)/H$ which is finite. Hence G_x is finite as well. □

(i) $\Rightarrow$ (iii).

Proof. We proceed by Noetherian induction. Consider our tower of finite morphisms denoted by $\gamma_k : X_{k+1} \to X_k$, and consider the collection $\mathcal{U}$ of open sets $U \subseteq X$ such that when we restrict the tower over U the morphisms are eventually étale. The assumption that all the morphisms are quasi-étale implies that $X_{\text{reg}} \in \mathcal{U}$. Since X is assumed to be Noetherian this collection has a maximal element and our goal is to show that this must be all of X.

Therefore we need to show that if $U \in \mathcal{U}$ and $U \neq X$ then we can find a larger

$U' \in \mathcal{U}$. To do this take any x a generic point of an irreducible component of $X\backslash U$. Consider $X_x = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\eta})$ and restrict the tower of X_i over X_x to get a tower

$$\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\eta}) = X_{x,0} \leftarrow X_{x,1} \leftarrow X_{x,2} \leftarrow X_{x,3} \leftarrow \cdots$$

Now using the assumption (i) applied to the point x, it follows that eventually the covers will be trivial when restricted over the regular locus and hence will be étale. This then shows that there exists some $N >> 0$ such that γ_n is étale over η for $n \geq N$ and they are étale over the open set U coming from Noetherian induction.

Now take a regular alteration $\pi : \hat{X}_N \to X_N$. Then using π we construct a stratification $X_n = \bigcup_i Z_i$ as before. Then any of the maps $X_{N+k} \to X_N$ must be étale over U and η. But because the branch locus must be a union of strata it follows that these are all étale over some open set $U' \supset U$ with $U' \ni \eta$. Hence such a larger $U' \in \mathcal{U}$ exists and by Noetherian induction we see that $X \in \mathcal{U}$. This proves property (iii). □

(iii) ⇒ (iv).

Proof. Assuming that no such cover exists, we inductively construct a tower $X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$ as in (iii) of the main theorem using Galois closures, such that none of the $X_{i+1} \to X_i$ are étale. This will contradict our assumption, so eventually every étale cover of one of the $X_{i,\mathrm{reg}}$ will extend to an étale cover of X_i. This gives the desired cover satisfying purity. □

(iv) ⇒ (i).

Proof. Consider a geometric point x of X. Take a cover $f : Y \to X$ as in (iv), and a geometric point y of Y mapping to x. Denote by U the regular locus of X and $Z = X\backslash U$ the singular locus. This gives rise to the following commutative diagram of fundamental groups.

$$\begin{array}{ccc} \hat{\pi}_1^{\mathrm{loc}}(Y_y \backslash f^{-1}(Z)_y) & \longrightarrow & \hat{\pi}_1(f^{-1}(U)) \\ \big\downarrow & & \big\downarrow \\ \hat{\pi}_1^{\mathrm{loc}}(X_x \backslash Z_x) & \longrightarrow & \hat{\pi}_1(U) \end{array}$$

Now the assumption on Y implies that the top map is zero. On the other hand, the image of the map on the left is a finite index normal subgroup. Hence looking at the images in $\hat{\pi}_1(U)$, we see that G_x has a trivial finite index subgroup and hence must be finite. □

2.7 Corollaries

The most straightforward corollary of the theorem is the following.

Corollary 2.7.1. *Suppose that X is a Noetherian normal scheme of finite type over an excellent base of dimension ≤ 2. Suppose moreover that $\pi_1(U_x)$ if finite for every geometric point $x \in X$ where $U = X_{\mathrm{reg}}$. Then the following hold*

(i) There exists a finite, Galois, quasi-étale cover $f : \hat{X} \to X$ such that $\pi_1(f^{-1}(U)) \to \pi_1(\hat{X})$ is an isomorphism.

(ii) There exists a normal closed subgroup of finite index $H \subseteq \pi_1(X_{\mathrm{reg}})$ such that $H \cap G_x = 1$ for every geometric point $x \in X$.

In particular using the results from [39] and [6] we can simultaneously prove the results of [19] and [2].

Corollary 2.7.2. *Suppose that X is a klt variety in characteristic 0 or a strongly F-regular variety in characteristic p. Then (i) and (ii) from the above corollary hold.*

Originally in the paper of [39] and [19] the more classical local fundamental group $\pi_1(X_x \backslash \{x\})$ was used instead of the one occurring in our theorem. Note that these fundamental groups are smaller than the one we use, and thus can be trivial while $\pi_1(U_x)$ may not be. We can still ask though if some form of the theorem holds when we replace finiteness of G_x by finiteness of the local fundamental groups $\pi_1(X_x \backslash \{x\})$. The following example shows that this is not the case in general.

Example 2.7.3. Consider $X = CS$ the cone over a Kummer surface $S = A/\pm$ where A is an abelian surface. Then there are three types of singular points:

First consider the case where x is the generic point of the cone over one of the nodes. Then X_x has a regular double quasi-étale cover ramifying at x (note that since we have localized there is no difference between the two possible fundamental groups). This shows $\pi_1(X_x \backslash \{x\}) = \pi_1(X_x \backslash Z_x) \cong \mathbb{Z}/2\mathbb{Z}$, and there is no ambiguity in which definition we choose. In general this will work for the generic point of any irreducible component of the singular locus.

The next type of point where we start to see a difference is when x is a closed point in the cone over a node of S. Then in this case $\pi_1(X_x \backslash \{x\})$ is trivial while $\pi_1^{\text{loc}}(X_x \backslash \operatorname{Sing}(X)_x) \cong \mathbb{Z}/2\mathbb{Z}$. Although they are different they are at least both finite. On the other hand if we desire for these groups to behave well under specialization it is clear that $\pi_1(X_x \backslash \operatorname{Sing}(X)_x)$ is the better choice.

The last type of point, where the real problem occurs, is the cone point $x \in CS$. First consider the fundamental group $\pi_1(X_x \backslash \{x\})$. Then this will be isomorphic to $\pi_1(S) \cong 0$ by the Lefschetz hyperplane theorem. In particular all the local fundamental groups defined in this sense are finite. So if this version of the theorem were true then any tower as above would stabilize. On the other hand $\pi_1(S_{\text{reg}})$ is infinite, giving an infinite tower of cones $X = CS \leftarrow CS_1 \leftarrow CS_2 \leftarrow \cdots$ Galois over X and quasi-étale. In particular finiteness of all the local fundamental groups $\pi_1(X_x \backslash \{x\})$ does not imply finiteness of the local fundamental groups $\pi_1(X_x \backslash \operatorname{Sing}(X)_x)$. Note that in this case the singularity at the origin is not klt.

On the other hand the proof does work for klt singularities as is shown in [19]. The key to this working is that when we take a quasi-étale cover we will still have finite local fundamental groups in the classical sense. This is summed up in the following remark.

Remark 2.7.4. Suppose that $\mathcal{R}$ is a class of normal varieties such that for every $X \in \mathcal{R}$ and every quasi-étale cover $Y \to X$, $Y \in \mathcal{R}$ as well. Suppose that for every $X \in \mathcal{R}$ and every geometric point $x \in X$ that $\pi_1(X_x \backslash \{x\})$ is finite. Then the conclusions (iii) and (iv) of the theorem still hold.

The proof is essentially the same as the proof of (i) implies (iii) in the main theorem, except now we use the fact that on each cover the local fundamental groups are still finite. In particular this applies to the case of klt singularities.

Now we will discuss some of the applications of these types of theorems that have occurred in the literature since [19]. Most of these arguments involve extending some classification that holds for smooth varieties to one involving singular varieties by using a cover as in (iii) of the theorem. All of these results will be over $\mathbb{C}$ and the varieties will be projective.

Theorem 2.7.5. *(Characterization of singular torus quotients, [19]) Suppose that X is a klt variety over $\mathbb{C}$ with $K_X \equiv 0$. Assume that X is regular in codimension 2 and that there exists a resolution $\pi : \tilde{X} \to X$ and ample divisors $H_1, \ldots, H_{n-2}$ on X such that $c_2(\mathcal{T}_{\tilde{X}}) \cdot \pi^* H_1 \cdots \pi^* H_{n-2} = 0$. Then there exists a Galois cover $f : A \to X$ that is étale in codimension 2, where A is an abelian variety.*

In another paper they are able to use these covers to obtain a characterization of singular ball quotients as well.

Theorem 2.7.6. *[17] Suppose that X is a klt variety of general type with K_X ample. Suppose moreover that*

$$\big(2(n+1) \cdot \hat{c}_2(\mathcal{T}_X) - n \cdot \hat{c}_1(\mathcal{T}_X)^2\big) \cdot [K_X]^{n-2} = 0$$

where the orbifold Chern classes $\hat{c}_i$ are defined in [17]. Then X is regular in codimension 2 and there exists a Galois quasi-étale cover $Y \to X$ where Y is a ball quotient.

Finally we note that they are also able to obtain some form of the Beauville-Bogomolov decomposition theorem for singular varieties with this technique.

Theorem 2.7.7. *[18] Suppose that X is a projective klt variety with $K_X \equiv 0$. Then there exists a quasi-étale cover $f : A \times Z \to X$ where A is abelian and $\dim(A) = \hat{q}(X)$ and Z is a canonical variety with $K_Z \sim 0$ and $\hat{q}(Z) = 0$. Here $\hat{q}$ denotes the augmented irregularity, which is the maximal irregularity of any quasi-étale cover.*

We now will discuss some interesting open questions in this area. One of the first to be asked is the following.

Question 2.7.8. *Suppose that X is a klt variety. Then are the topological local fundamental groups $\pi_1^{top}(X_x \backslash \{x\})$ and $\pi_1^{top}(U_x)$ finite?*

So far this is known up to dimension 3 by the work of [37]. In fact they show this is equivalent to the orbifold fundamental groups of log Fano pairs being finite in one lower dimension.

Another point of view comes from mixed characteristic. As we know the local to global statement holds even in mixed characteristic it is natural to ask whether we can apply it anywhere. This leads to the following question.

Question 2.7.9. *What classes of singularities in mixed characteristic satisfy the assumption that $\pi_1(U_x)$ is finite for all $x \in X$? In particular is there an analog of strongly F-regular singularities whose local fundamental groups are all finite?*

As of now there are multiple definitions of the right analog of strongly F-regular singularities, that are not known to be equivalent. Moreover there is no analog of the F-signature which allows us to bound the sizes of local fundamental groups.

Chapter 3

Birational Superrigidity

3.1 Introduction

In this chapter we will look at the topic of birational superrigidity of Fano varieties. Unlike in the case of when the canonical bundle is ample there is the potential for many different birational maps between different Fano varieties. This include the case when the Fano varieties can have large birational automorphism group. On the other hand there is a certain class of Fano varieties whose birational behavior is much more similar to a variety with ample canonical bundle then to a rational variety. These varieties are called birationally rigid and form an interesting but mysterious class of Fano varieties.

Definition 3.1.1. *A variety X is birationally rigid if for any birational map $X \dashrightarrow Y$, where Y is a Mori fiber space, we must have $Y \cong X$. X is called birationally superrigid if moreover the above birational map is an isomorphism.*

All results on birationally rigidity and superrigidity proceed by use of the Noether-Fano inequalities. These roughly state that any birational map between Mori fiber spaces requires that the base locus of the map to be sufficiently singular enough. The general approach is then to describe what linear systems can be singular enough, and in some cases prove that they simply cannot exist on our variety.

The history of the Noether-Fano inequalities goes back to the early history of birational geometry. Some forms of these equalities originated in the work of Noether who used them to study the Cremona group in dimension 2. He showed there were inequalities in the multiplicities of base loci of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. These were applied to find generators of the Cremona group. After this attempts were made by the Italian school of algebraic geometers to generalize these results to threefolds, though there methods were not always rigorous.

From the modern point of view, the Noether-Fano inequalities found their importance again in the work of Iskovskih and Manin [21]. In their work they were able to essentially prove the superrigidity of quartic threefolds. This provided one of the first negative solutions to the Lüroth problem, as some were known to be unirational. These results were later generalized to many other Fano varieties of index 1. Moreover it was noted by Corti [9] that the Noether-Fano inequalities can be phrased very simply in terms of the singularities seen in the minimal model program.

The first problem we will look at is the relationship of birational superrigidity to K-stability in section 3.5. There we prove under some extra assumptions that birational superrigidity in fact implies K-stability. Second we look at the base locus of birational maps from a higher index Fano hypersurface to a Mori fiber space. Although these will not be superrigid, we show that there still is some restriction on the base locus of the birational map that can occur.

3.2 Rationality

In this section we review some of the notions from the theory of rationality. We will work in characteristic 0 throughout this section though many definitions carry through to characteristic p (some requiring a slight variation to deal with inseparable maps). The basic problem is to tell how far a given variety is from projective space $\mathbb{P}^n$. In attempting to solve this problems many different notions of being similar to $\mathbb{P}^n$ have been introduced, with some being harder to show in practice than others. In this chapter we will pay particular attention to the case when X is a hypersurface, which is the simplest case of the problem, yet already proves to be hard. The first definition though is the following.

Definition 3.2.1. *A variety X is rational if it is birational to $\mathbb{P}^n$. It is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$.*

For a hypersurface X_d of degree d in $\mathbb{P}^n$, we can easily see that if $d = 1$ we have $X_d \cong \mathbb{P}^{n-1}$ and by projecting from a point if $d = 2$ then X_d is birational to $\mathbb{P}^{n-1}$. Even in degree 3 the result becomes incredibly difficult. Instead of trying to find explicit ways to prove rationality of X_d we will instead look for restrictions that rationality puts on X.

The first thing we notice about $\mathbb{P}^n$ is that it is covered by lines. In particular if we have a birational map $\mathbb{P}^n \dashrightarrow X$ there exists a rational curve through most points (in fact all). This leads to the weakest notion of a variety being similar to $\mathbb{P}^n$.

Definition 3.2.2. *A variety X is uniruled if there exists a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ where $\dim(Y) = \dim(X) - 1$. A variety X is ruled if there exists a map as above which is birational.*

Although this is a very weak notion from the point of view of rationality, it turns out uniruledness is at least in principle easy to detect. This is due to the fact that any ruled variety $Y \times \mathbb{P}^1$ has few pluricanonical forms. Hence any uniruled variety should also have few pluricanonical forms, or we could pull these back to $Y \times \mathbb{P}^1$. This leads to the following theorem.

Proposition 3.2.3. *Suppose that X is a uniruled variety. Then $H^0(X, \omega_X^k) = 0$ for all k.*

It is conjectured that the converse of this is also true, and in particular we should be able to tell whether a variety is uniruled by looking only at pluricanonical forms. A special case should be that if $-K_X$ is positive then X should be uniruled. Using his bend and break technique, Mori showed that this is in fact true.

Theorem 3.2.4. *[24] Suppose that X is a Fano variety. Then X is uniruled.*

Looking at the example of $X_d \subseteq \mathbb{P}^n$ a hypersurface of degree d, uniruledness already tells us that for many degrees we can rule out the possibility of rationality. In particular as $\omega_X \cong \mathcal{O}_X(d-n-1)$, we can conclude that if $d \geq n+1$, then X cannot be uniruled while for $d \leq n$ X is uniruled. Hence if we want to look at rationality we should look in the region where the degree satisfies $1 \leq d \leq n$.

Although the conjecture characterizing uniruledness is not known as stated above, it is known that uniruledness is equivalent to some numerical condition on the canonical bundle.

Theorem 3.2.5. *[5] A variety X is uniruled if and only if K_X is not pseudoeffective.*

Although this is easy to check in theory, uniruled varieties can still be very different from rational varieties. For instance $Y \times \mathbb{P}^1$ is ruled by definition, but we can take Y as far away from $\mathbb{P}^{n-1}$ as we want. For example Y could contain no rational curves, and hence we would only have one family of rational curves on $Y \times \mathbb{P}^1$. The next notion we can consider fixes this by looking at the existence of rational curves between two general points.

Definition 3.2.6. *A variety X is rationally connected if there exists a rational curve through any two general points on X.*

In fact it was proven by Kollár, Miyaoka, and Mori that Fano varieties satisfy this stronger condition on rational curves.

Theorem 3.2.7. *[23]. Suppose that X is a smooth Fano variety. Then X is rationally connected.*

In particular any hypersurface of degree $d \leq n$ in $\mathbb{P}^n$ is rationally connected. It is clear that any unirational variety is still rationally connected. However we do not have a single example of a rationally connected variety which is not unirational. This is due to the fact that there is no easy numerical condition to check to tell whether a variety is unirational.

One question left to ask is whether there is a difference between rationality and unirationality. This is known as the Lüroth problem and was unsolved until 3 different techniques appeared in the 1970s to tell whether a variety was not rational. The Lüroth problem can be stated completely in terms of fields, asking whether if K is a field of transcendence degree n over k such that $K \subseteq k(x_1, \ldots, x_n)$ is it true that $K \cong k(x_1, \ldots, x_n)$. The case of dimension 1 is easy and was proved by Lr̈oth in the 1800s, while the case of dimension 2 follows from the classification of surfaces and was proved by Castelnuovo in 1893. In dimension 3 and higher though it turns out the result is false.

We will now discuss the three techniques that came up in the 1970s to show that a Fano variety is not rational. To this day most nonrationality results follow one of these techniques. The exception to this are two more recent techniques: the first being a result of Kollár using a degeneration to characteristic p [22]. The second more recent technique of Voisin [38] uses the decomposition of the diagonal of X in the chow ring $A^n(X \times X)$ (though it is in the same spirit of the method of Artin and Mumford).

The first technique to be published was that of Iskovskih and Manin [21]. They looked at birational maps $X \dashrightarrow X'$ where X and X' are both quartic threefolds. By examining the singularities of the strict transform of a general hyperplane on one of the quartics, they were able to show that any such birational map must have already been an isomorphism. In particular this implies that $\mathrm{Bir}(X) \cong \mathrm{Bir}(X')$. All of the automorphisms of X on the other hand must come from projective linear transformations, and in particular there can only be finitely many. Hence these could not be rational. In the other direction, Segre had shown in 1960 that a particular quartic threefold was unirational, hence showing that the Lüroth problem does not hold in dimension 3.

Next in the same year Clemens and Griffith showed that a smooth cubic threefold is in fact also not rational [8]. To do this they looked at the intermediate Jacobian, a complex torus defined via the Hodge structure on the cohomology of the cubic threefold. Although the intermediate Jacobian is not a birational invariant it behaves well under blow ups and in particular it is possible to restrict the intermediate Jacobian of any rational variety. On the other hand it is known that cubic threefolds are unirational, so this gives another solution to the Lüroth problem.

Finally in the same year, Artin and Mumford gave a third example of a variety which is unirational but not rational [1]. In more detail they showed that the torsion in the third cohomology of X was a birational invariant for any smooth variety. Hence by looking for varieties where this was nontrivial, they showed that certain quartic double solids were irrational but still were unirational. In fact their method also shows that they are not stably rational, meaning they will remain irrational even after taking a product with any $\mathbb{P}^k$.

Of these three techniques introduced around the same time, each has its strength and weaknesses. The Clemens and Griffith method is very easy to apply to threefolds, but does not seem have a clear generalization to varieties of higher dimension. Also it is computable mostly in the case when X has a conic bundle structure. On the other hand the Artin-Mumford example applies in any dimension and can detect stable rationality as well. The problem is showing the invariant is non-trivial, which usually requires a degeneration to a very general variety. In particular it fails to give concrete examples in many of the cases where it can be applied.

The first technique of Iskovskih and Manin is what we will look at in the remainder of this chapter. It has the advantage of, at least theoretically, generalizing to higher dimensions. It also can be applied to specific varieties and even has been applied to varieties with singularities [21]. Finally it gives much more structure on the variety then simply irrationality. It implies very strict conditions on the birational geometry of the variety. On the other hand, the generalizations to higher dimension usually require harder estimates on singularities of linear systems in higher dimension. Moreover it has proven hard to apply the techniques in the case when $-K_X$ does not generate the Picard group. Yet of the methods above, Iskovskih and Manin's method can be framed most closely to the modern techniques developed in the minimal model program, where large achievements have been made in recent years.

3.3 Birational Rigidity and Noether Fano Inequalities

In this section we discuss some of the basics of birational rigidity and the technique that is used to prove it: the Noether-Fano inequalities. We first recall the definition of Mori fiber space that will be used throughout this chapter.

Definition 3.3.1. *A Mori fiber space $Y \to S$ is a morphism with connected fibers such that*

- *Y is terminal,*
- *$\rho(Y/S) = 1$,*
- *$-K_{Y/S}$ is ample.*

In particular Fano varieties of Picard number 1 and conic bundles of relative Picard number 1 are both examples of Mori fiber spaces. Starting with a uniruled variety, these are the end products of the minimal model program. Note on the other hand in the case that X is of general type we can find a canonical model X_{can}. These have the following nice property.

Proposition 3.3.2. *Suppose that $X \dashrightarrow Y$ is a birational morphism of varieties with ample canonical bundles and canonical singularities. Then f is an isomorphism.*

On the other hand for Mori fiber spaces there is no corresponding result. For instance even in dimension 2, $\mathbb{P}^2$ has both birational automorphisms as well as birational maps to other Mori fiber spaces, e.g. $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Even though these maps exist, they still put some limits on the singularities of the linear systems of the varieties. This is summed up by Noether-Fano inequalities, which in the general form that we present here are due to Corti.

Theorem 3.3.3. *[9] Suppose that X is a terminal Fano variety of Picard number 1 and that $f : X \dashrightarrow Y$ is a birational morphism to a Mori fiber space $Y \to S$. Suppose that H is a very ample divisor on Y. Then if f is not an isomorphism $\mathcal{M} = f_*^{-1}|H|$ will be such that $(X, \frac{1}{r}\mathcal{M})$ is not canonical where $\mathcal{M} \sim_{\mathbb{Q}} -rK_X$.*

Although the Noether-Fano inequalities are stated here using the singularities of the minimal model program, they go back much further than any such definitions of singularities. For instance Noether knew that there were restrictions on the multiplicities of the base loci of birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. He used this to find generators for the Cremona group in two variables. Later the Italian algebraic geometers attempted (not always rigorously) to the classification of threefolds. Essentially there results only take into consideration the discrepancies on the blow-ups of subvarieties, and not discrepancies over other exceptional divisors. Moreover these attempts were well before a proof of resolution of singularities so access to the full generality of the Noether-Fano inequalities would have been difficult.

The first modern attempt to use the Noether-Fano varieties was by Iskovskih and Manin. They applied these to quartic threefolds to look at birational maps $X_4 \dashrightarrow X_4'$

and found that these exist if and only if they are an isomorphism, This lead to the following definition of rigidity.

Definition 3.3.4. *A Fano variety X with terminal singularities is called birationally rigid if for any birational map $f : X \dashrightarrow Y$, where $Y \to S$ is a Mori fiber space, implies that $Y \cong X$. A Fano variety is birationally superrigid if moreover the above map is an isomorphism, or equivalently* $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

We can rephrase this as saying there exists no other Mori fiber space structures on X. In other words rigidity implies that the varieties behave much more similar to varieties with ample canonical bundles. With the form of the Noether-Fano varieties above, there is a clear way to prove birational superrigidity: namely we need to show that for every $\mathcal{M}$ a movable linear system with $\mathcal{M} \sim -rK_X$ that $(X, \frac{1}{r}\mathcal{M})$. This says that any movable linear system is not too badly singular in the sense measured by the canonical threshold. Although it was not stated in this form, what Iskovskih and Manin had essentially shown that a smooth quartic threefold is birationally superrigid.

Remark 3.3.5. Since birationally superrigidity has a clear strategy of proof via the Noether-Fano inequalities it would be nice if every birationally rigid variety was superrigid, and we wouldn't have to worry about the difference. This is in fact false: a complete intersection $X_{2,3} \subseteq \mathbb{P}^5$ is birationally rigid, but it has a birational involution. The technique to prove birational rigidity is then to identify some birational automorphisms $X \dashrightarrow X$ on the Fano variety, and show that after suitably twisting by these automorphisms we get closer to being able to apply the Noether-Fano inequalities.

Not only do the Noether-Fano inequalities give a way to prove birational superrigidity, but we can show now show using the minimal model program that they are in fact sharp.

Theorem 3.3.6. *[7] Suppose that X is a terminal Fano variety of Picard number* 1 *and there exists a movable linear system $\mathcal{M} \sim -rK_X$, with $(X, \frac{1}{r}\mathcal{M})$ not canonical.*

Then X has a birational map $X \dashrightarrow Y$, where $Y \to S$ is a Mori fiber space, that is not an isomorphism.

Hence if we want to prove birational superrigidity we really need to prove the Noether-Fano varieties always hold on our Fano variety. Although the basic technique of showing superrigidity always follows the Noether-Fano inequalities, the details of applying this to different varieties has a long history. The expected behavior is summed up in the following conjecture.

Conjecture 3.3.7. *[29] Suppose that X is a Fano manifold of Picard number 1 and index 1. Then X is birationally superrigid if $\dim(X) \geq 5$.*

One of the most studied cases of this conjecture is when X is a hypersurface of degree n in $\mathbb{P}^n$. In the case when $n = 4$, the work of Iskovskih and Manin show essentially that X is birationally superrigid [21], though they only considered maps to other quartic threefolds and not arbitrary Mori fiber spaces. Many names have been involved in the generalization to higher dimensions, for instance [11],[31], which led to the first complete proof in all dimensions by de Fernex [10]. This has been recently simplified and extended to complete intersections by Zhuang [41].

3.4 K-stability

In this section we will give a short introduction to the concept of K-stability of complex Fano varieties. Although this subject originates in differential geometry, we will look here at them from the perspective of birational geometry, where they can be defined in terms of the singularities of pluricanonical divisors on the Fano variety. Although the definition of K-stability requires a discussion of Donaldson-Futaki invariants [36], we will omit the definition here as recent theorems provide a much more simple characterization from the point of view of the thesis. The importance of K-stability though comes from the problem of finding a constant Kähler curvature metric, where the equivalence of the two is known as the Yau-Tian-Donaldson conjecture. Recently this conjecture has been known in greater generality. For instance one proven form of this conjecture that will be useful for this section is the following.

Theorem 3.4.1. [] *Suppose that X is a Fano manifold with finite automorphism group. Then X is uniformly K-stable if and only if X admits a Kähler-Einstein metric.*

Here the uniform K-stability refers to boundedness of the Donaldson-Futaki invariants from below. The relation to singularities of linear systems starts with the work of Tian. He studied the following invariant defined via pluricanonical linear systems on our Fano variety in questions.

Definition 3.4.2. *[35] Suppose that X is a Fano variety. Define the alpha invariant of X to be the following.*

$$\alpha(X) := \sup_c \{c \in \mathbb{Q} : (X, cD) \text{ is log canonical for } D \sim_{\mathbb{Q}} -K_X\}$$

In other words the alpha invariant measures the most singular pluricanonical divisor measured via the log canonical threshold. Although this may seem like a natural invariant from the viewpoint of algebraic geometry its real interest comes from the following theorem of Tian.

Theorem 3.4.3. *[35] Suppose that X is a Fano manifold, with $\alpha(X) > \frac{n}{n+1}$ (resp. $\geq \frac{n}{n+1}$). Then X is K-stable (X is K-semistable).*

This gives a computational way to prove that a given Fano manifold is K-(semi)stable using the techniques of algebraic geometry. Although it gives a criterion for proving K-(semi)stability it will not detect K-(semi)stability in general. For example $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$, yet $\mathbb{P}^n$ is K-semistable. More recently, finer invariants of linear systems have been introduced that can detect K-stability. For example the δ-invariant of

Fujita and Odaka measures not the most singular divisor, but the most singular basis of $H^0(X, -mK_X)$ for $m >> 0$. This leads to the following definition.

Definition 3.4.4. *[16] [4] Suppose that X is a Fano variety. Define δ_m for X as following*

$$\delta_m(X) = \inf(\{\text{lct}(X, D) : D \textit{ is of } m \textit{ basis type}\}$$

where here we say D is of m basis type if it is of the form

$$D = \frac{1}{N_m}(D_1 + \cdots + D_{N_m})$$

where the D_i form a basis of $H^0(X, -mK_X)$. The the δ invariant of X is

$$\delta(X) = \lim_{n \to \infty} \delta_m(X)$$

Although this invariant is harder to calculate due to the fact that we must know a whole basis for $m >> 0$ and not just a single divisor, its importance is due to the following.

Theorem 3.4.5. *[16] [4] Suppose that X is a $\mathbb{Q}$-Fano variety. Then*

- *X is K-semistable if and only if $\delta(X) \geq 1$,*
- *X is uniformly K-stable if and only if $\delta(X) > 1$.*

In particular this gives a characterization of K-stability completely in terms of singularities of linear systems. In particular it looks very similar to the type of bounds on the singularities seen on birationally superrigid varieties via the Noether-Fano inequality. Note also that the two invariants above, the alpha and delta invariant, must at least satisfy the following two relations.

Theorem 3.4.6. *[4] Suppose that X is a $\mathbb{Q}$-Fano variety. Then*

$$\alpha(X) \leq \frac{n}{n+1}\delta(X) \leq (n+1)\alpha(X)$$

In particular the δ invariant together with the above bounds on the invariants from above imply the result of Tian as well.

3.5 Birational Superrigidity and K-Stability

In this section we look at the following conjecture.

Question 3.5.1. *[28] Let X be a birationally (super)rigid Fano manifold. Then is it true that X is K-stable?*

This question is motivated by the fact that both of these properties are measured by the singularities of pluri-anticanonical linear systems. From the superrigidity point of view we are interested in the most singular movable linear system. On the other hand, at least asymptotically, K-stability is measured by the most singular basis for a vector space $H^0(-mK_X)$ for $m >> 0$. This question is was supported by the following result of Odaka and Okada.

Theorem 3.5.2. *[28] Suppose that X is a birationally superrigid Fano manifold of index 1. If $|-K_X|$ is base point free, then $(X, \mathcal{O}_X(-K_X))$ is slope stable*

In fact they proved the following stronger result holds where the assumption that X is birationally superrigid is weakened to a log version.

Theorem 3.5.3. *[28] Suppose that X is a Fano manifold of Picard number 1 and $|-K_X|$ base point free. Suppose that X has no log maximal singularities.Then $(X, \mathcal{O}_X(-K_X))$ is slope stable.*

In these theorems the conclusion is not K-stability, but instead slope stability, which is weaker than K-stability. In particular this requires the positivity of Donaldson-Futaki invariants only for special test configurations. The method of proof in [33] is different from the approach taken there.

In this work we show that if we have superrigidity and a reasonable bound on the α invariant from below, then we can also conclude K-(semi)stability.

Theorem 3.5.4. *[33] Suppose that X is a $\mathbb{Q}$-Fano variety of Picard number 1. If X is birationally superrigid (or more generally log maximal singularity free) and $\alpha(X) \geq \frac{1}{2}$ (resp, $> \frac{1}{2}$), then X is K-semistable (resp. K-stable).*

For smooth hypersurfaces of degree n in $\mathbb{P}^n$ it is known both that they are birationally superrigid and that there α-invariant is $\geq \frac{n}{n+1}$ [15]. Hence we can reprove the following corollary

Corollary 3.5.5. *[33] Suppose that $X \subseteq \mathbb{P}^n$ is a hypersurface of degree $n \geq 4$. Then X is K-stable.*

Before we prove the theorem we will fix some notations and definitions.

Definition 3.5.6. *Suppose that X is a $\mathbb{Q}$-Fano variety and E is an exceptional divisor over X. Suppose that $\pi : Y \to X$ is a projective birational morphism such that E is a prime divisor on Y. Then the pseudo-effective threshold $\tau(E)$ of F with respect to $-K_X$ to be*

$$\tau(E) := \sup\{\tau > 0 |\, \mathrm{vol}_Y(-\pi^* K_X - \tau F) > 0\}$$

We will also need the notion of restricted volume and its properties.

Definition 3.5.7. *[14] Let $V \subseteq X$ be an irreducible subvariety of a variety X. The restricted volume of D along V is defined to be*

$$\mathrm{vol}_{X|V}(D) = \limsup_{m\to\infty} \frac{\dim(\mathrm{im}(H^0(X, \mathcal{O}_X(mD) \to H^0(V, \mathcal{O}_V(mD)}{m^d/d!}$$

Proposition 3.5.8. *[14] Suppose that $V \subseteq X$ as above. Then the following properties of the restricted volume hold*

1. *If $V \not\subseteq \mathbf{B}_+(D)$ then $\mathrm{vol}_{X|V}(D) > 0$*

2. *For real divisor classes ξ_1, ξ_2 whose augmented base locus does not contain V*

$$\mathrm{vol}_{X|V}(\xi_1 + \xi_2)^{1/d} \geq \mathrm{vol}_{X|V}(\xi_1)^{1/d} + \mathrm{vol}_{X|V}(\xi_2)^{1/d}$$

3. *Suppose that ξ is a big $\mathbb{R}$-divisor class. Then the function $\mathrm{vol}_X(\xi + tE)$ is continuously differentiable and*

$$\frac{d}{dt}\left(\mathrm{vol}_X(\xi + tE)\right)_{t=0} = n \cdot \mathrm{vol}_{X|E}(\xi)$$

Proof of Theorem Our assumption will say that $\mathrm{lct}(X; D) \geq \frac{1}{2}$ (resp. $> \frac{1}{2}$) for every $D \sim_{\mathbb{Q}} -K_X$. Now consider a divisor E over X, that occurs as a prime divisor of Y where $\pi : Y \to X$ is a projective birational morphism. Denote by $\tau = \tau(E)$ the pseudoeffective threshold of E. Let $A = A_X(E)$ be the log discrepancy of E, and assume for sake of contradiction that $S(\mathrm{ord}_E) > A_X(E)$ (resp. $S(\mathrm{ord}_E) \geq A_X(E)$)). This is equivalent to saying that $\delta(-K_X) < 1$ (resp. $\delta(-K_X) \leq 1$).

Now we isolate the most singular divisor on X with respect to the valuation ord_E. Using the assumption that $(X, \mathcal{M})$ is log canonical for every movable $\mathbb{Q}$-linear system with $\mathcal{M} \sim_{\mathbb{Q}} -K_X$ we see that there exists at most one irreducible divisor $D \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{ord}_E(D) > A$. Note that it follows from the definition of pseudoeffective threshold that $\mathrm{ord}_E(D) = \tau$. Fix this divisor from now on.

Our next goal is to break up the restricted volume function $\mathrm{vol}_{T|E}(-\pi^* K_X - xE)$ into two regions. After we reach the log canonical threshold it follows that all sections must be very divisible by D as D is the only divisor singular enough with respect to E. Before the log canonical threshold we can use log-concavity of the restricted volume to relate it to that at the log canonical threshold. This is summed up in the following two lemmas.

Lemma 3.5.9. *Suppose that $x > A$. Then*

$$\mathrm{vol}_{Y|E}(-\pi^*K_X - xE) = \left(\frac{\tau - x}{\tau - A}\right)^{n-1} \mathrm{vol}_{X|E}(-\pi^*K_X - AE)$$

Proof. Note that we can write

$$-\pi^*K_X - xE = \frac{\tau - x}{\tau - A}(-\pi^*K_X - AE) + \frac{x - A}{\tau - A}(-\pi^*K_X - \tau F)$$

In particular we see that D must appear at least with multiplicity $\frac{x-A}{\tau-A}$ in the stable base locus of $-\pi^*K_X - xE$. Hence by homogeneity of the restricted volume we obtain the desired inequality. □

In the region where $0 \leq x \leq A$ we can get a different bound for the restricted volume in terms of $\mathrm{vol}(-\pi^*K_X - AE)$.

Lemma 3.5.10. *Suppose that $x \in [0, A]$. Then*

$$\mathrm{vol}_{X|E}(-\pi^*K_X - xE) \geq \left(\frac{x}{A}\right)^{n-1} \mathrm{vol}_{X|E}(-\pi^*K_X - AE)$$

Proof. In this region we write

$$-\pi^*K_X - xE = \frac{x}{A}(-\pi^*K_X - AE) - \left(1 - \frac{x}{A}\right)\pi^*K_X$$

Then as $-K_X$ is ample we can use log-concavity of the restricted volume to obtain the following

$$\mathrm{vol}_{Y|E}(-\pi^*K_X - xF)^{\frac{1}{n-1}} \geq \left(\frac{x}{A}\right)\mathrm{vol}_{Y|E}(-\pi^*K_X - AE)^{\frac{1}{n-1}} + \mathrm{vol}_{Y|E}(-\pi^*K_X)$$

from which the result follows. ∎

Finally using integration by parts we can obtain the following equality.

Lemma 3.5.11.

$$\int_0^\tau (x-S)\cdot \mathrm{vol}_{Y|E}(-\pi^*K_X - xF)dx = 0$$

Proof. We will denote by $G(x) = \mathrm{vol}_{Y|E}(-\pi^*K_X - xE)$ and $V(x) = \mathrm{vol}_Y(-\pi^*K_X - xE)$. Note that

$$S = \frac{1}{V(0)}\int_0^\tau V(x)dx$$

and $V'(x) = -G(x)$. Then using integration by parts we can conclude

$$\begin{aligned}
&\int_0^\tau (x-S)\cdot \mathrm{vol}_{Y|E}(-\pi^*K_X - xF)dx = \int_0^\tau xG(x)dx - \int_0^\tau SG(x)dx \\
&= -xV(x)|_0^\tau + \int_0^\tau V(x)dx - S\int_0^\tau G(x)dx \\
&= -\tau V(\tau) + \int_0^\tau V(x)dx - S\int_0^\tau G(x)dx \\
&= S\cdot V(0) + S\cdot V(\tau) - S\cdot V(0) = 0
\end{aligned}$$

∎

Putting this equality together with our two estimates of the restricted volume gives.

$$0 = \int_0^A (x-S)\cdot \mathrm{vol}_{X|E}(-\pi^*K_X - xE)dx + \int_A^\tau (x-S)\cdot \mathrm{vol}_{X|E}(-\pi^*K_X - xE)$$

$$\leq \int_0^A (x-S)\left(\frac{x}{A}\right)^{n-1}dx + \int_A^\tau (x-S)\left(\frac{\tau - x}{\tau - A}\right)^{n-1}dx$$

which after computing allows us to conclude

$$\frac{\tau - 2A}{n(n+1)} + \frac{A - S}{n} \geq 0$$

But we assumed $b > A$, so we must have $\tau > 2A$. Hence we can conclude that $\operatorname{lct}(X; D) < \frac{1}{2}$. This contradicts our assumption on the log canonical threshold of D. Hence we can conclude that X is K-semistable (resp. K-stable). $\square$

The theorem requires the assumption that $\alpha(X) > \frac{1}{2}$ to conclude K-stability. In practice this simplifies the methods of proving that $\alpha(X) > \frac{n}{n+1}$ to prove that a birationally superrigid Fano varieties is K stable. For many of these examples of birationally superrigid Fano varieties it is already known though that there α invariants are large. In particular this leads to the following question.

Question 3.5.12. *Suppose that X is a birationally superrigid Fano variety. Is it always true that $\alpha(X) > \frac{1}{2}$?*

In a large class of Fano varieties, we can at least obtain a lower bound of the alpha invariant from Castelnuovo-Mumford regularity.

Theorem 3.5.13. *[33] Suppose that X is a $\mathbb{Q}$-Fano variety of Picard number 1 and dimension $n \geq 3$. Assume moreover that $(X, \mathcal{M})$ is log canonical for every movable $\mathbb{Q}$-divisor with $\mathcal{M} \sim_{\mathbb{Q}} -K_X$. Finally assume that $|-K_X|$ is base point free and $-K_X$ generates the class group of X. Then $\alpha(X) \geq \frac{1}{n+1}$.*

Proof. Let $D \sim_{\mathbb{Q}} -K_X$ be an irreducible $\mathbb{Q}$-divisor on X. Then we need to check that $\operatorname{lct}(X; D) \geq \frac{1}{n+1}$. Hence suppose this is not the case. Now using the fact that $\operatorname{Cl}(X)$ is generated by X we can at least conclude that $\operatorname{mult}_\eta(D) \leq 1$ where η denotes the generic point of D. Hence at least in codimension 1, the pair (X, D) is log canonical so that the multiplier ideal $\mathcal{J}(X, (1-\epsilon)D)$ must define a subscheme Σ

whose codimension is ≥ 2 for $0 < \epsilon << 1$. Hence by Nadel vanishing the cohomology groups

$$H^i(X, \mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-rK_X)) = 0$$

for any $i > 0$ and $r \geq 0$. In particular this implies that $\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}(-nK_X)$ is 0-regular in the sense of Castelnuovo-Mumford regularity, so that it is in particular generated by global sections. Hence we can look at the linear system $\mathcal{M} = |\mathcal{J}(X, (1-\epsilon)D) \otimes \mathcal{O}_X(-mK_X)|$ which will be movable bu the above argument.

Now take E an exceptional divisor over X computing the lct of D and suppose that E is a prime divisor on Y where $\pi : Y \to X$ is a birational morphism. Let $A = A_X(E)$ be the log discrepancy. Hence we have $\mathrm{ord}_E(D) \geq (n+1)A$ so that $\mathcal{J}(X, (1-\epsilon)D) \subseteq \pi_* \mathcal{O}_Y(-(nA+1)E)$. This implies $\mathrm{ord}_E(\mathcal{M}) \geq nA + 1$, which contradicts our assumption on the singularities of a movable boundary. Hence $\alpha(X) \geq \frac{1}{n+1}$. □

3.6 Conditional Rigidity for Higher Index Hypersurfaces

In this section we will look at some rigidity like results for higher index hypersurfaces. The general situation cannot be so simple as for index one hypersurfaces though. This is due to the existence of general linear projections $X \dashrightarrow \mathbb{P}^{i_X - 1}$, which will induce non-trivial Mori fiber space structures on X. On the other hand in higher dimension we do expect there to be some constraints on the conic bundle structures on X. For instance Kollár proved the following about very general hypersurfaces.

Theorem 3.6.1. *[22] Suppose that $X_d \subseteq \mathbb{P}^{n+1}$ is a very general hypersurface over $\mathbb{C}$. If*

$$d \geq 3\lceil (n+3)/4 \rceil$$

then X_d is not birational to a conic bundle.

Although the proof of this theorem does not use the Noether-Fano inequalities, it is expected that the Noether-Fano inequalities will give constraints on higher index Fano varieties in large dimension as well. Moreover since the Noether-Fano inequalities apply to every smooth hypersurface, it is expected that we can weaken the assumption that the hypersurface is very general. One of the most ambitious conjectures along these lines is the following.

Conjecture 3.6.2. *([30] Conjecture 1.) Suppose that $V_d \subseteq \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d \geq \lceil (n+5)/2 \rceil$ (so that the index is $i = n+2-d$). Suppose that $f : V \to Y$ is a birational map to a rationally connected fiber space $p : Y \to S$. Then $\sim (S) \leq r-1$ and if $\dim(S) = r-1$ then the map $Y \to S$ is birational to a projection $V \to \mathbb{P}^{r-1}$ (meaning there exists a birational map $\mathbb{P}^{r-1} \dashrightarrow S$ making the appropriate diagram commute).*

So far the only case of this that is partially known is the case when the index $i = 2$. In this case Pukhlikov claims the following.

Theorem 3.6.3. *[30] Suppose that $n \geq 14$ and V is a general hypersurface of index 2 in $\mathbb{P}^{n+1}$. Suppose that $f : V \dashrightarrow Y$ is a birational map to a rationally connected fiber space $p : Y \to S$. Then $S = \mathbb{P}^1$ and there exists a projection $\pi : Y \to \mathbb{P}^1$ such that $V \to \mathbb{P}^1$ is birational to $Y \to S$.*

Although this gives an answer for index 2, the proof relies on a detailed analysis of multiplicities that can occur for the base locus of such a birational map. Hence there is a need to restrict the hypersurfaces allowed, and the theorem is hard to generalize. On the other hand we can still prove something about the base loci of such birational maps. The following definition comes from a paper of Suzuki (later withdrawn due to an error).

Definition 3.6.4. *Suppose that X is a Fano variety of Picard number 1 and s is an integer. Then X satisfies condition C_s if the following holds*

(C_s): every birational map from X to a Mori fiber space whose base locus has codimension at least s is an isomorphism

Then we say X is conditionally birationally superrigid if it satisfies condition (C_{i_X+1}) where i_x is the index of X.

In other words conditional birational superrigidity says the base locus of any birational map to a Mori fiber space must be large compared to the index. The purpose of the next result is to show that if we assume that the dimension of a complete intersection in $\mathbb{P}^{n+r}$ is sufficiently large, then it must satisfy conditional birational superrigidity.

Theorem 3.6.5. *Suppose that i and r are positive integers. Then there exists an integer $N = N(i, s)$ such that every Fano complete intersection of codimension r and index i in $\mathbb{P}^{n+r}$ is conditionally birationally superrigid if $n \geq N$.*

This result relies on the techniques of [41]. These are an improvement on the work of [10]. It should be noted that attempting to prove conditional birational superrigidity for hypersurfaces of higher index is possible using only the methods of [10]. On the other hand, the bounds presented here are better and apply to complete intersections as well. As with all results relying on the Noether-Fano inequalities we can rephrase the problem into a question about singularities of linear systems. Then it takes the form of the following.

Theorem 3.6.6. *([41],appendix) Suppose that r, m are integers. Then there exists an integer $N = N(r, m)$ such that for every smooth Fano complete intersection of codimension r and dimension $n \geq N$ in $\mathbb{P}^{n+r}$, any movable boundary $M \sim_{\mathbb{Q}} mH$ whose base locus has codimension at least $m+1$ must be such that (X, M) is canonical.*

The first step towards these results is the following bounds for multiplicities of complete intersections in projective space.

Proposition 3.6.7. *[34] Suppose that X us a complete intersection in $\mathbb{P}^{n+r}$ of codimension r. Let α be an effective cycle on X of pure codimension k such that $\alpha \sim m[H^]k$ where $[H]$ denotes the hyperplane class. Assume that $kr + \dim(\mathrm{Sing}(X)) + 1 < n + r$. Then $e_S(\alpha) \leq m$ for every closed subvariety $S \subseteq X$ of dimension $\geq kr$ not meeting the singular locus of X.*

Here $e_S(\alpha)$ refers to the Samuel multiplicity of α which for a closed point and a variety measures the top coefficient on the Hilbert-Samuel function of the local ring. For a subvariety S, $e_S(\alpha)$ will be the minimum of $e_P(\alpha)$ where $P \in S$. Finally for a cycle we can extends the definition linearly.

Proof

Proof. Using [34] we have $e_S(M^m) \leq m^m$ for every subvariety $S \subseteq X$ whose dimension is at least mr. Hence there exists a set Z of dimension $\leq mr - 1$ such that for any $x \notin Z$ $e_x(M^m) \leq m^m$. Outside of Z, the singularities of (X, M) are at least canonical. To see this note that if V is a general complete intersection variety of dimension m containing x, then $(V, M|_V)$ is not log canonical at x by inversion of adjunction. Now as V was chosen to be general $e_x(M|_V^m) = e_x(M^m)$, where now we are in the 0 dimensional case. Hence by [12] it must be that $(V, M|_V)$ is log canonical at x which contradicts our claim. Hence we must have that at least (X, M) has canonical singularities outside Z.

Now we use the result of [34] in one codimension more. This implies that $e_S(M^{m+1}) \leq m^{m+1}$ for every subvariety $S \subseteq X$ whose dimension is at least $(m + 1)r$. Hence by the same application of [12] as above the pair $(X, \frac{m+1}{m}M)$ is log canonical outside a subset of dimension at most $mr + r - 1$.

Take an arbitrary $x \in X$ and through x take a general linear section of codimension $mr + r - 1$. The the pair $(Y, \frac{m+1}{m}M|_Y)$ is log canonical outside finitely many points. Define the line bundle $L = (mr + m - +r - 1)H$, which satisfies that $L - (K_Y + \frac{m+1}{m}M|_Y)$ is nef. Hence by [41] it follows that $\mathrm{lct}(Y, \frac{m+1}{m}M|_Y) \geq \frac{m}{m+1}$ as long as

$$h^0(Y, L) \leq h^0(\mathbb{P}^{n-mr+1}, \mathcal{O}_{\mathbb{P}^{n-mr+1}}(mr + m + r - 1)) = \binom{n + m + r}{mr + m + r - 1}$$

The right hand side grows exponentially with n, and hence is satisfied for all $n \geq N(m, r)$. Hence it follows that $(Y, M|_Y)$ is log canonical for $n \geq N$. But (X, M) is canonical outside Z, so again by an application of inversion of adjunction (X, M) is also canonical at x. Since we chose $x \in X$ to be arbitrary the result follows. □

From this point the theorem on conditional superrigidity follows straight forward from the result on linear systems. If we have any map $X \dashrightarrow X'$ whose undefined locus is at least $s+1$, then by the Noether-Fano inequalities there exists a movable boundary $M \sim_{\mathbb{Q}} -K_X = sH$, whose base locus is contained in the locus of indeterminacy of the above map on X. Moreover such a pair (X, M) is not canonical, which cannot happen for $n \geq N$.

Bibliography

[1] Michael Artin and David Mumford. Some elementary examples of unirational varieties which are not rational. *Proceedings of the London Mathematical Society*, 3(1):75–95, 1972.

[2] Bhargav Bhatt, Javier Carvajal-Rojas, Patrick Graf, Karl Schwede, and Kevin Tucker. Étale fundamental groups of strongly F-regular schemes. *arXiv preprint arXiv:1611.03884*, 2016.

[3] Bhargav Bhatt, Ofer Gabber, and Martin Olsson. Finiteness of étale fundamental groups by reduction modulo p. *arXiv preprint arXiv:1705.07303*, 2017.

[4] Harold Blum and Mattias Jonsson. Thresholds, valuations, and K-stability. *arXiv preprint arXiv:arXiv:1706.04548*, 2017.

[5] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell. The pseudo-effective cone of a compact k\" ahler manifold and varieties of negative kodaira dimension. *arXiv preprint math/0405285*, 2004.

[6] Javier Carvajal-Rojas, Karl Schwede, and Kevin Tucker. Fundamental groups of F-regular singularities via F-signature. *arXiv preprint arXiv:1606.04088*, 2016.

[7] I. A. Cheltsov and K. A. Shramov. Log-canonical thresholds for nonsingular Fano threefolds. *Uspekhi Mat. Nauk*, 63(5(383)):73–180, 2008.

[8] C Herbert Clemens and Phillip A Griffiths. The intermediate jacobian of the cubic threefold. *Annals of Mathematics*, pages 281–356, 1972.

[9] Alessio Corti. Factoring birational maps of 3-folds after sarkisov. *J. Algebraic Geom.*, 4:223–254, 1995.

[10] Tommaso de Fernex. Erratum to: Birationally rigid hypersurfaces. *Invent. Math.*, 203(2):675–680, 2016.

[11] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. Bounds for log canonical thresholds with applications to birational rigidity. *Math. Res. Lett.*, 10(2-3):219–236, 2003.

[12] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaţă. Multiplicities and log canonical threshold. *J. Algebraic Geom.*, 13(3):603–615, 2004.

[13] A Johan de Jong. Smoothness, semi-stability and alterations. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 83(1):51–93, 1996.

[14] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Restricted volumes and base loci of linear series. *Amer. J. Math.*, 131(3):607–651, 2009.

[15] Kento Fujita. K-stability of Fano manifolds with not small alpha invariants. *arXiv preprint arXiv:1606.08261*, 2016. To appear in J. Inst. Math. Jussieu.

[16] Kento Fujita and Yuji Odaka. On the K-stability of Fano varieties and anticanonical divisors. *arXiv preprint arXiv:1602.01305*, 2016. To appear in Tohoku Math. J.

[17] D. Greb, S. Kebekus, T. Peternell, and B. Taji. The miyaoka-yau inequality and uniformisation of canonical models. *arXiv preprint arXiv:1511.08822*, 2015.

[18] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class-holonomy, differential forms, and fundamental groups. *arXiv preprint arXiv:1704.01408*, 2017.

[19] Daniel Greb, Stefan Kebekus, Thomas Peternell, et al. Étale fundamental groups of kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties. *Duke Mathematical Journal*, 165(10):1965–2004, 2016.

[20] Alexander Grothendieck. Eléments de géométrie algébrique (rédigés avec la collaboration de jean dieudonné): III. Etude cohomologique des faisceaux cohérents, premiere partie. *Publications Mathématiques de l'IHES*, 11:5–167, 1961.

[21] V. A. Iskovskih and Ju. I. Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. Sb. (N.S.)*, 86(128):140–166, 1971.

[22] János Kollár. Nonrational hypersurfaces. *Journal of the American Mathematical Society*, 8(1):241–249, 1995.

[23] János Kollár, Yoichi Miyaoka, Shigefumi Mori, et al. Rational connectedness and boundedness of fano manifolds. *Journal of Differential Geometry*, 36(3):765–779, 1992.

[24] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Annals of Mathematics*, 110(3):593–606, 1979.

[25] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Publications mathématiques de l'IHÉS*, 9:5–22, 1961.

[26] David Mumford and Tadao Oda. Algebraic geometry: II, 2015.

[27] Masayoshi Nagata et al. On the purity of branch loci in regular local rings. *Illinois Journal of Mathematics*, 3(3):328–333, 1959.

[28] Yuji Odaka and Takuzo Okada. Birational superrigidity and slope stability of Fano manifolds. *Math. Z.*, 275(3-4):1109–1119, 2013.

[29] Aleksandr Pukhlikov. *Birationally rigid varieties*, volume 190 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[30] Aleksandr Pukhlikov. Birational geometry of Fano hypersurfaces of index two. *Math. Ann.*, 366(1-2):721–782, 2016.

[31] Aleksandr Valentinovich Pukhlikov. Birationally rigid fano hypersurfaces. *Izvestiya: Mathematics*, 66(6):1243, 2002.

[32] Charlie Stibitz. Étale covers and local algebraic fundamental groups. *arXiv preprint arXiv:1707.08611*, 2017.

[33] Charlie Stibitz and Ziquan Zhuang. K-stability of birationally superrigid fano varieties. *arXiv preprint arXiv:1802.08381*, 2018.

[34] Fumiaki Suzuki. Birational rigidity of complete intersections. *Math. Z.*, 285(1-2):479–492, 2017.

[35] Gang Tian. On kähler-einstein metrics on certain kähler manifolds withc 1 (m)¿ 0. *Inventiones mathematicae*, 89(2):225–246, 1987.

[36] Gang Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.

[37] Zhiyu Tian and Chenyang Xu. Finiteness of fundamental groups. *Compositio Mathematica*, 153(2):257–273, 2017.

[38] Claire Voisin. Unirational threefolds with no universal codimension 2 2 cycle. *Inventiones mathematicae*, 201(1):207–237, 2015.

[39] Chenyang Xu. Finiteness of algebraic fundamental groups. *Compositio Mathematica*, 150(03):409–414, 2014.

[40] Oscar Zariski. On the purity of the branch locus of algebraic functions. *Proceedings of the National Academy of Sciences*, 44(8):791–796, 1958.

[41] Ziquan Zhuang. Birational superrigidity and k-stability of fano complete intersections of index one (with an appendix written jointly with charlie stibitz). *arXiv preprint arXiv:1802.08389*, 2018.