# On the Models of the Fluid-Polymer Systems

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# Abstract

The purpose of this work is to study fluid-polymer systems. A fluid-polymer system is a system consisting of solvent fluids and polymers, either suspended in the bulk (polymeric fluid systems) or attached on the boundaries. Mathematically, they are coupled multi-scale systems of partial differential equations, consisting of a fluid portion modeled by the Navier-Stokes equation, and a polymer portion modeled by the Fokker-Planck equation. Key difficulties lie in the coupling of two equations.

We propose a new approach to show the well-posedness of a certain class of polymeric fluid systems. In this approach, we use "moments" to translate a multi-scale system to a fully macroscopic system (consisting of infinitely many equations), solve the macroscopic system, and recover the solution of the original multi-scale system. As an application, we obtain the large data global well-posedness of a certain class of polymeric fluid systems.

We also show the local well-posedness when a polymeric fluid system is written in Lagrangian coordinates. This approach allows us to show the uniqueness in lower regularity space and the Lipschitz dependence on initial data.

Finally, we propose a new boundary condition which describes the situation where polymers are attached on the fluid-wall interface. Using kinetic theory, we derive a dynamic boundary condition which can be interpreted as a "history-dependent slip" boundary condition, and we prove global well-posedness in 2D case. Also, we show that the inviscid limit holds for an incompressible Navier-Stokes system with this boundary condition.

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# Contents

# Chapter 1

# Introduction

Complex fluids are everywhere. Although the two most ubiquitous fluids, water and air, are generally regarded as simple fluids, most of other fluids in our life are complex fluids. One example is blood in our bodies, which consists of various cells, materials, and water, and its dynamics are different from those of Newtonian fluids due to contributions of elasticity of cells and materials. Liquid crystals are used in various display devices, and polymeric fluids are also widely utilized in the chemical industry, for applications such as adhesives. Many fluids in the food industry are also complex, mixtures of diverse materials.

Furthermore, complex fluids show various interesting behaviors, which are rarely observed in Newtonian fluids. Their unique properties are not only scientifically interesting, but also useful for real-world applications. For example, the addition of a minuscule amount of polymers to a turbulent flow in a pipe channel significantly enhances the flow, which is called the *polymer drag reduction phenomena.* This property is used to increase energy efficiency in various applications. ( [128], [132], [124], [25], and [129].)

Therefore, it is natural that complex fluids have drawn scholarly attention of numerous researchers from many disciplines, including chemical and mechanical engineering, physics, scientific computing, and mathematics. There are plenty of brilliant works which have significantly deepened our understanding of the behavior of complex fluids. However, our understanding of the dynamics of complex fluids remains incomplete due to their inherent complexity. First of all, there are many types of complex fluids, and most have very different properties. For instance, it is not likely that foams and liquid crystals are described by a common model, as they behave very differently. Even two polymeric fluid solutions with identical polymers and solvents, but varying concentrations may display very different dynamics. Therefore, in order to appropriately describe the dynamics of complex fluids, it is necessary to narrow down and specify which types of complex fluids we try to describe.

In this thesis, we are interested in a very specific class of complex fluids: *the fluid-polymer system.* In this work, the fluid-polymer system is a fluid that consists of solvent and dissolved polymers, either suspended in the bulk of the fluid (Chapter 2, 3, 4) or where one-end is attached to wall and other parts are floating freely in the solvent (Chapter 5). In addition, we are interested in a dilute fluid-polymer system, that is, a system with low enough polymer concentrations so that direct interaction between two polymers can be ignored.

In order to obtain a mathematical description of fluid-polymer systems, we have made several simplifications and assumptions. First, we have filtered out detailed information of polymers. The underlying idea, which is one of the main themes of the thesis and will appear repeatedly is the following: *we are primarily interested in a macroscopic description of the flow, in which the specific details of the microscopic system are ignored and the effect of the microscopic system is characterized as collective operators of microscopic states acting on the macroscopic system.* We simplify polymers to obtain a *coarse-grained model* of polymers: for example, we simplify flexible polymers as a spring with two beads on two ends, and rigid, rod-like polymers as a rod. Also, polymers of the same type have an identical configuration space. Such simplifications of polymers are drastic; flexible polymers have very complex shapes, and two polymers in the system may have a different number of monomers and will not be specifically identical. However, as far as we are interested in the dynamics of macroscopic flow, such simplified models still give a good agreement in qualitative behaviors.

Once we have obtained a simplified model of each polymer, we adopt the kinetic description of polymers, that is, we keep track of the statistics of the polymers in the configuration space and ignore the history of the individual polymers. There are good reasons for adopting these kinetic descriptions. First, the number of polymers is so tremendous that it is nearly impossible to keep track of each polymer. Second, most of the details of polymer states are not very useful in understanding the macroscopic flow. As previously mentioned, we only need the information related to the evolution of macroscopic flow, not specific details about microscopic polymer systems.

The final step is to understand the interaction between polymers and solvents. To understand the system completely, we need to know both the effect of fluids to the polymer distribution and the effect of polymers to the solvent flow. The former is straightforward, as one can easily see what solvent flow does to the polymers from the equation. The latter is more delicate, as polymers influence the flow by exerting stress on fluid parcels, which is less direct. The coupling of these two makes the dynamics more interesting.

In this chapter, we present background materials for the thesis. This chapter proceeds as follows. We start with the dynamics of solvents, which are assumed to be a *Newtonian fluids*, in Section 1.1. Basic languages and facts concerning fluid mechanics are provided. In Section 1.2, the coarse-graining procedure of polymers is briefly discussed. Section 1.3 focuses on the kinetic theoretic description of polymers in the simplest setting. Their interaction with solvents is discussed in Section 1.4. We also present the goal of this thesis in Section 1.5 and we review the literature in Section

1.6.

## 1.1 The Equations of Fluid Motion

In many scientific and engineering applications, it is essential to understand the motion of fluid substances. It is also the case if one wants to understand the dynamics of fluid-polymer systems. We model the fluid as a continuum; we ignore that matter consists of atoms, which is a good approximation in many situations. The description of the motion is obtained from balance laws and constitutive equations, as elaborated below.

### 1.1.1 Eulerian and Lagrangian coordinates

Before we investigate the motion of fluids, we introduce the coordinate systems. There are two important coordinate systems of observing fluid motion. In the first system, which is called the Eulerian system, an observer has a fixed position. In the second system, which is called the Lagrangian system, an observer moves with the velocity $u$ of the fluid. The two systems are related in the following manner. Let $f(x,t)$ be a quantity (field) of interest and let $\frac{D}{Dt}f$ be the time derivative of $f$ with respect to the observer moving with the velocity $u$ (which is also called as the material derivative of $f$). Also, let $X(x,t)$ be the position of the particle with respect to the Eulerian observer, at time $t$ and initially at $x$. Then we have the following initial value problem, which defines the *flow* $X$:

$$\frac{d}{dt}X(x,t) = u(X(x,t),t), \;\; X(x,0) = x. \tag{1.1}$$

Then, the Lagrangian derivative $\frac{D}{Dt}f = \frac{d}{dt}f(X(x,t),t)$ at the particle originally at $x$ can be written as

$$\frac{d}{dt}f(X(x,t),t) = \left(\partial_t f + \frac{d}{dt}X(x,t)\cdot\nabla_x f\right)(X(x,t),t) = (\partial_t + u\cdot\nabla_x)\, f(X(x,t),t), \tag{1.2}$$

in the Eulerian observer's coordinate, by the chain rule.

One important related result is the *Reynolds Transport Theorem.* Let $V(t)$ be a volume whose surface points move with the local fluid velocity $u(x,y)$ (the set of surface points of $V(t)$ will be denoted by $S(t)$.) Then

$$\frac{d}{dt}\int_{V(t)} f(x,t)dx = \int_{V(t)} \left[\frac{Df}{Dt} + f\nabla\cdot u\right] dx \tag{1.3}$$

holds.

Two coordinate systems are equivalent if the fluid velocity field $u$ is smooth enough. On the other hand, if the velocity field is not regular enough, defining unique flow $X$ becomes nontrivial and there are plenty of works addressing this issue (for example [46]). In Chapters 2, 3, and 5, we will pose our system in Eulerian coordinates. In Chapter 4, we will pose our system in Lagrangian coordinates.

### 1.1.2 Balance laws

We introduce three unknowns - the density $\rho(x,t)$, the velocity $u(x,t)$, and the pressure $p(x,t)$ of a fluid particle at position $x$ and time $t$.

**Mass balance.** First, we have a balance of mass:

$$\partial_t\rho + \nabla_x\cdot(u\rho) = 0. \tag{1.4}$$

**Linear momentum balance.** Second, we have a linear momentum balance, which states that the time rate of change of linear momentum equals the sum of all the forces acting on the material. There are two types of forces, the body forces and the surface forces.

Body forces are long range forces which act on all the material points. Examples include gravity or electric/magnetic forces. Body force per density is denoted by $f_b$. Surface forces are short range forces, and they model two types of molecular scale interactions. First, they represent intermolecular forces between nearby fluid particles, which give an effective friction in fluids. Second, they represent linear momentum exchange due to random thermal motions. We denote the surface force on the surface normal to $\hat{n}$ by $\mathbf{t}_{\hat{n}}$. The integral form of the linear momentum balance reads

$$\frac{d}{dt}\int_{V(t)} \rho u dx = \int_{V(t)} \rho f_b dx + \int_{S(t)} \mathbf{t}_{\hat{n}} dS. \tag{1.5}$$

By the Reynolds transport theorem and mass balance (1.4) this can be rewritten as

$$\int_{V(t)} \rho \frac{Du}{Dt} dx = \int_{V(t)} \rho f_b dx + \int_{S(t)} \mathbf{t}_{\hat{n}} dS. \tag{1.6}$$

We point out that $\mathbf{t}_{\hat{n}}$ has a specific form, that is, $\mathbf{t}_{\hat{n}} = \hat{n} \cdot \Sigma_F$, where $\Sigma_F$ is called the Cauchy stress tensor, which is a second-order tensor. The components $(\Sigma_F)_{ij}$ of $\Sigma_F$ represent the force per unit area in the $x_j$-direction on a surface which is normal to the $x_i$-direction. The argument is due to Cauchy, based on the order-of-magnitude analysis. We briefly review the argument.

We begin with the equation (1.6). Suppose that we choose a volume around some point $p$, whose diameter is $\ell$, and we shrink the volume as $\ell \to 0$. The surface force term will diminish to the order of $O(\ell^2)$, while the other terms diminish to the order of $O(\ell^3)$, as they are integrated over the volume. Therefore, the surface integral term is the leading term. Then, first we choose a "pill-box" shaped region: for any vector

$\hat{n}$, we take a small circle in the plane normal to $\hat{n}$ and radius $\ell$ and take a cylinder with height $\ell^2$. Again, the leading order term comes from integrals over two discs, and taking the limit we obtain $\mathbf{t}_{\hat{n}} = -\mathbf{t}_{-\hat{n}}$. Next, we take a tetrahedron volume (*Cauchy tetrahedron*), specified by a unit normal vector $\hat{n}$. For example, we take $p$ as the origin, and take our tetrahedron by the volume formed by the $x_1x_2$-plane, $x_2x_3$-plane, $x_3x_1$-plane, and $\hat{n} \cdot x = \ell$. Again, the leading order term is the surface integral term, and a the force balance, we obtain

$$\mathbf{t}_{\hat{n}} = n_1\mathbf{t}_{\hat{e}_1} + n_2\mathbf{t}_{\hat{e}_2} + n_3\mathbf{t}_{\hat{e}_3}. \tag{1.7}$$

Therefore, $\mathbf{t}_{\hat{n}} = \hat{n} \cdot \Sigma_F$, where $(\Sigma_F)_{ij} = (\mathbf{t}_{\hat{e}_i})_j$. Applying the divergence theorem to (1.6) and noting that $V(t)$ can be arbitrary, we obtain the equation for the Momentum Balance:

$$\rho\frac{Du}{Dt} = \nabla \cdot \Sigma_F + \rho f_b, \tag{1.8}$$

### 1.1.3 Incompressible Navier-Stokes equations for simple fluids

In this subsection, we find the representation for $\Sigma_F$ given that the fluid is simple, incompressible and homogeneous. We will see that for such a fluid, the stress tensor $\Sigma_F$ in (1.8) is given by:

$$\Sigma_F = -p\mathbb{I} + 2\mu D(u),\ D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \tag{1.9}$$

where $p$ is the pressure and $\mu$ is a constant, called the shear viscosity. Here incompressible and homogeneous means $\rho$ is uniform over space.

**Fluid statics.** We first consider case $u = 0$, the static case. Although fluids cannot support tangential (shear) stresses, they can support tensile/compressive or *normal* forces. Thus, the only stresses acting on a fluid in the static state are normal to any surface, and we call this stress the static (equilibrium) fluid pressure $p$. Thus,

$$\Sigma_F = -p\mathbb{I}, \tag{1.10}$$

where the minus sign indicates that the pressure is compressive.

**Fluid dynamics.** If there is no motion, the constitutive equation should reduce to the static case (1.10). Therefore, we write

$$\Sigma_F = -p\mathbb{I} + \sigma, \tag{1.11}$$

where $\sigma$ is called the *deviatoric stress*, which is the non-equilibrium stress. Our goal is to derive a relation between $\sigma$ and $u$. We can derive (1.9) for a simple fluid with the following assumptions:

1. *Galilean invariance.* We want the equation (1.8) to be invariant under Galilean transformation. Then, we want $\sigma = \sigma(\nabla u)$.

2. *No dependence on local rigid body rotation.* This implies that $\sigma = \sigma(D(u))$. There are some discussions concerning this assumption. See p. 144 of [13].

3. *Instantaneous, local response to $D(u)$.* We want $\sigma(x,t) = \sigma(D(u(x,t)))$. We do not want non-locality or history-dependence.

4. *Linear dependence.* Similar to simple constitutive equations, we want the linear relationship between the second-order tensor $\sigma$ and $D(u)$. Therefore, we want $\sigma = \mathbf{A} : D(u)$, where $A$ is a fourth-order tensor and, in the index notation this becomes $\sigma_{ij} = A_{ijk\ell} D(u)_{k\ell}$.

5. *Isotropy.* Finally, we want our constitutive relation to be *isotropic*, which means that there is no preferred direction in the material. Mathematically this means

that the fourth-order tensor $A$ is isotropic, that is, it has the same components in all rotated coordinate systems. It is well known that such $A$ is of the form

$$A_{ijk\ell} = \lambda_1 \delta_{ij}\delta_{k\ell} + \lambda_2 \delta_{ik}\delta_{j\ell} + \lambda_3 \delta_{i\ell}\delta_{jk}, \tag{1.12}$$

where $\lambda_1, \lambda_2$, and $\lambda_3$ are scalars, we conclude that

$$\sigma = \lambda_1 \nabla \cdot u \mathbb{I} + 2\lambda_2 D(u), \tag{1.13}$$

or, by rewriting,

$$\Sigma_F = -(p - \kappa \nabla \cdot u)\mathbb{I} + 2\mu \left( D(u) - \frac{1}{3} \nabla \cdot u \mathbb{I} \right) \tag{1.14}$$

where $\kappa$ is referred as the bulk viscosity, and $\mu$ is known as the shear viscosity. Fluids following such assumptions are called the *Newtonian fluids*. Here we regard $\mu$ and $\kappa$ as material constants in Newtonian fluids (in general they depend on state fields, especially on the density and the temperature). Also it can be shown that $\mu$ and $\kappa$ are nonnegative using an argument based on the second law of thermodynamics. Finally, noting that incompressibility means $\nabla \cdot u = 0$, we recover (1.9).

Using incompressibility and (1.9), we can rewrite (1.8) as

$$\begin{gathered} \frac{\partial}{\partial t} u + u \cdot \nabla_x u = -\nabla_x p + \nu \Delta_x u + f_b, \\ \nabla_x \cdot u = 0, \end{gathered} \tag{1.15}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. Also, if $\nu = 0$, then we call (1.15) as the *incompressible Euler equation.* To ask the question of existence and uniqueness of the solution for the system (1.15), we need two more complementary data: the *initial data* $u_0$ and the *boundary condition*, if boundary exists. We will investigate the second point in Section 1.1.5.

### 1.1.4 Non-dimensionalization and the Reynolds number

It is very useful to non-dimensionalize the system (1.15), especially when we are interested in the asymptotic behavior. The first equation of the system (1.15) has the dimension $\frac{\text{(length)}}{\text{(time)}^2}$. If we write $V$ the characteristic velocity scale of the system and $L$ the characteristic length-scale of the system, then the characteristic time-scale of the system becomes $T = L/V$. Also let $\tilde{x} = \frac{x}{L}$, $\tilde{t} = \frac{t}{T}$ be non-dimensional space and time variables. Also we write $\tilde{u} = \frac{u}{V}$ for non-dimensional velocity, $\tilde{p} = p\frac{T^2}{L^2}$ for non-dimensional pressure, and $\tilde{f}_b = f_b\frac{T^2}{L}$ be non-dimensional body force. Then (1.15) can be rewritten as

$$\begin{aligned} \frac{V^2}{L}\left(\partial_{\tilde{t}}\tilde{u} + \tilde{u}\right) &= \frac{V^2}{L}\left(-\nabla_{\tilde{x}}\tilde{p} + \tilde{f}_b\right) + \frac{\nu V}{L^2}\Delta_{\tilde{x}}\tilde{u}, \\ \frac{V}{L}\nabla_{\tilde{x}} \cdot \tilde{u} &= 0, \end{aligned} \tag{1.16}$$

or

$$\begin{aligned} \partial_{\tilde{t}}\tilde{u} + \tilde{u} &= -\nabla_{\tilde{x}}\tilde{p} + \tilde{f}_b + \frac{1}{\mathbf{Re}}\Delta_{\tilde{x}}\tilde{u}, \\ \nabla_{\tilde{x}} \cdot \tilde{u} &= 0, \end{aligned} \tag{1.17}$$

where **Re** is the *Reynolds number*

$$\mathbf{Re} = \frac{VL}{\nu}. \tag{1.18}$$

The Reynolds number **Re** captures the essentials of the dynamics of the system. For a given geometry of the domain and initial conditions, the effect on a flow field of changing various parameters ($V, L, \rho, \mu$ or a combination of them) can be described uniquely by the consequent change of **Re** alone. This is called *the principle of hydrodynamic similarity*; two flows in equivalent geometries and with the same Reynolds number are essentially the same.

Also, the magnitude of **Re** may be regarded as providing an estimate of the relative strength of the inertial and viscous forces acting on the fluid [13].

Various asymptotic limits can be studied in the non-dimensionalized setting. For example, in the limit $\mathbf{Re} \to \infty$, which is called the *vanishing viscosity limit*, one might expect that in the limit the system might behave as if $\frac{1}{\mathbf{Re}} = 0$, that is, the solution of the Euler equation. Indeed, this is the case if the system has no boundaries, provided a smooth solution of the latter equation exists.

**Theorem 1.1.1.** *Suppose that $u^{\boldsymbol{Re}}$ is a smooth enough solution of (1.17) for fixed $\boldsymbol{Re}$ with initial data $u_0$ (independent of $\boldsymbol{Re}$) on the domain $\mathbb{T}^d$, $d = 2$ or $3$. Also, suppose that there is a smooth solution, denoted by $u^E$, of the Euler system ((1.17) without $\frac{1}{\boldsymbol{Re}}\Delta_{\tilde{x}}\tilde{u}$ term) on $\mathbb{T}^d \times [0,T)$ with same initial data $u_0$. Then,*

$$\lim_{\boldsymbol{Re}\to\infty} \sup_{t\in[0,T)} \|u^{\boldsymbol{Re}}(t) - u^E(t)\|_{L^2(\mathbb{T}^d)} = 0. \tag{1.19}$$

*Proof.* Let $w = u^{\mathbf{Re}} - u^E$. Then by subtracting two equations, multiplying $w$ and integrating over $\mathbb{T}^d$, we obtain the following:

$$\begin{aligned}\frac{d}{2dt}\|w\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{\mathbf{Re}}\|\nabla u^{\mathbf{Re}}\|^2_{L^2(\mathbb{T}^d)} &= -\int_{\mathbb{T}^d} w \cdot \nabla u^E \cdot w dx - \frac{1}{\mathbf{Re}}\int_{\mathbb{T}^d} \nabla u^E : \nabla u^{\mathbf{Re}} dx \\ &\le \|\nabla u^E\|_{L^\infty(\mathbb{T}^d)}\|w\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{\mathbf{Re}}\|\nabla u^E\|_{L^2(\mathbb{T}^d)}\|\nabla u^{\mathbf{Re}}\|_{L^2(\mathbb{T}^d)}\end{aligned} \tag{1.20}$$

and therefore

$$\frac{d}{dt}\|w\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{\mathbf{Re}}\|\nabla u^{\mathbf{Re}}\|^2_{L^2(\mathbb{T}^d)} \le 2\|\nabla u^E\|_{L^\infty(\mathbb{T}^d)}\|w\|^2_{L^2(\mathbb{T}^d)} + \frac{1}{\mathbf{Re}}\|\nabla u^E\|^2_{L^2(\mathbb{T}^d)} \tag{1.21}$$

and applying the Grönwall inequality with $w(0) \equiv 0$, we have

$$\sup_{t\in[0,T)} \|w(t)\|^2_{L^2(\mathbb{T}^d)} \le \frac{1}{\mathbf{Re}} \exp\left(2\int_0^T \|\nabla u^E(t)\|_{L^\infty(\mathbb{T}^d)}dt\right) \int_0^T \|\nabla u^E(t)\|^2_{L^2(\mathbb{T}^d)}dt. \tag{1.22}$$

Therefore, $\sup_{t\in[0,T)} \|u^{\mathbf{Re}}(t) - u^E(t)\|_{L^2(\mathbb{T}^d)} = O\left(\frac{1}{\sqrt{\mathbf{Re}}}\right)$, as desired. □

However, in the real world, fluid systems have boundaries, and the boundary conditions impose key difficulties, which will be briefly described at the end of the next section.

### 1.1.5 The boundary conditions

To describe the system completely, the system of equations (1.15) should be complemented with appropriate boundary conditions. There are various boundaries such as fluid-fluid or fluid-vacuum, but in this thesis we only consider fluid-wall boundaries.

**Remark 1.** *We note that understanding boundary conditions for a fluid-wall interface still remains an interesting problem in fluid dynamics. A review paper [87] provides a good explanation and references.*

First of all, we have the no-penetration condition:

$$u \cdot \hat{n} = 0, \tag{1.23}$$

where $\hat{n}$ is the (outward) unit normal vector at the boundary of the domain. This boundary condition says that fluids cannot cross boundaries. For the Euler equation, describing behavior of inviscid fluids, (1.23) is a natural boundary condition to impose. Note that as the Euler equation is a first-order equation, we need one equation for its boundary condition.

For the Navier-Stokes equation describing a flow of viscous fluids, one also need boundary conditions for tangential directions of the flow. The most widely used boundary condition is the no-slip condition:

$$u = 0, \tag{1.24}$$

on the boundary. This boundary condition is physically justified by that at the fluid-wall interface, the attractive force between the fluid particle and particles comprising the wall is greater than that between the fluid particles [13]. This is an example of the Dirichlet boundary condition, which imposes values of the variables at the boundary. The Navier-slip boundary condition (introduced by Navier [110], also proposed later by Maxwell in [106]) is also frequently used (for example, [26], [100], [69]). It states that there is a slip velocity, which is tangential to the boundary. The slip velocity is proportional to the tangential component of the viscous stress tensor.

$$\begin{aligned} u \cdot \hat{n} &= 0, \\ \hat{\tau}_i \cdot (2\nu D(u) \cdot \hat{n} + \alpha u) &= 0, \ \ i = 1, 2 \end{aligned} \tag{1.25}$$

where $\alpha$ is a scalar friction function, which is positive and smooth. This boundary condition can be rigorously derived from the reflection-diffusion boundary condition ([71], [57]) in the kinetic theory of gases.

In Chapter 5, we will introduce a new boundary condition, which describes the situation where polymers are grafted on the wall. The boundary condition is the following;

$$\begin{aligned} u \cdot \hat{n} &= 0, \\ (\partial_t + 1)(\hat{\tau}_i \cdot (2D(u) \cdot \hat{n} + u)) &= -u \cdot \hat{\tau}_i, \ \ i = 1, 2. \end{aligned} \tag{1.26}$$

We conclude this section with a brief remark on the vanishing viscosity (infinite Reynolds number) limit. In many cases, it is of interest to investigate the limit behavior of the fluid systems with their Reynolds number converging to infinity. As we have seen in Theorem 1.1.1, as the Reynolds number goes to infinity, the limit system converges to the solution of the Euler equation, if there is no boundary. In contrast, the boundary condition imposes a key difficulty in studying the limit behavior when the domain has a boundary (for example [78], we discuss more in Section 1.6.3).

One expects the limit is a solution of Euler equation, naturally from the equation. However, for the no-slip boundary condition (1.24), there is an obvious boundary mismatch, and it is not known whether the system converges to the Euler system as the Reynolds number tends to infinity. Similarly, for the Navier slip boundary condition (1.25), the existence of the vanishing viscosity limit is unknown unless $\alpha \to 0$ as $\nu \to 0$. One thing to note is that the boundary condition (1.26) allows the vanishing viscosity limit, which will be discussed further in Chapter 5.

## 1.2 Coarse-grained models for Polymers

As was introduced earlier, in order to obtain a tractable model of fluid-polymer systems, we need a coarse-graining procedure for polymers. We emphasize that *flexible polymers are often simplified as a spring with two beads at both ends*; in this thesis, we always use this simplification for flexible polymers.

**Remark 2.** *In this Chapter, we only discuss flexible polymers. We exclude the derivation of rigid, rod-like polymer models, although in Chapter 3 we discuss the well-posedness of these systems. For more detailed information for rigid, rod-like polymer models, see [47].*

The goal of this section is to give a heuristic explanation of why flexible polymers can be regarded as a spring. In short, *they show elastic (spring-like) behavior due to an entropic effect:* if a spring represents the difference vector between the ends of a sub-chain consisting of many monomers or independent flexible units, then the central limit theorem implies that the distribution of end-to-end vectors is Gaussian. For these purely entropic effects, the Helmholtz free energy obtained as the logarithm of the Gaussian probability density is quadratic, and this corresponds to a linear spring force [116].

To further explain this point, we introduce a very simple model called *the freely jointed* *model*: it consists of $N$ links, each of length $b_0$ and each can point in any direction independently of each other. Its configuration is determined by the set of bond vectors $(r_1, \cdots, r_N)$, where $r_i$ is the direction of the $i$-th link. The probability distribution for $r_i$ is uniform over $b_0 S^{d-1}$. The main quantity of interest is the end-to-end vector $\mathbf{R}$ of the link:

$$\mathbf{R} = \sum_{i=1}^{N} r_n. \tag{1.27}$$

If we let $\langle \mathbf{R} \rangle$ be the expectation of $\mathbf{R}$, then one can easily see that

$$\langle |\mathbf{R}|^2 \rangle = N b_0^2. \tag{1.28}$$

Also one can see ([47]) that the probability density function $f_N(\mathbf{R})$ for the end-to-end vector can be approximated by:

$$f_N(\mathbf{R}) \propto \exp\left(-\frac{3|\mathbf{R}|^2}{2N b_0^2}\right). \tag{1.29}$$

We note that in the distribution, the local structure of the chain appears only through the bond length $b_0$. Therefore, if we are interested in the global properties of polymers, we can regard the polymer as a chain with Gaussian distribution for an end-to-end vector, which is sometimes called *the Gaussian chain*. The Gaussian chain is often represented by a mechanical model in which the two end 'beads' are connected by a harmonic spring whose potential energy is given by

$$U(\mathbf{R}) = \frac{3}{2N b_0^2} k_B \overline{T} |\mathbf{R}|^2, \tag{1.30}$$

where $k_B$ is the Boltzmann constant and $\overline{T}$ is the temperature. The equilibrium for such a model is the same as (1.29).

## 1.3 The kinetic theory

In a fluid-polymer system, we keep track of the probability density function of polymers in the configuration space, and we try to find the governing equation for the evolution of the probability density function. In the dilute regime, polymers will be subject to a drift-diffusion process due to

1. their own potential forces,

2. Brownian motion due to their collisions with solvent particles, and

3. hydrodynamic interactions, that is, the effects due to flowing solvents.

To obtain the description of the evolution of the probability density function, we will make a number of multi-scale assumptions, which will be briefly reviewed in this section. Finally, we describe the collective effect of polymers, which is called the *Kramers expression.*

### 1.3.1 The separation of scales - I

There is one point to remark concerning the multi-scale nature of the models of polymeric fluids based on the kinetic theory. In these models, there are two important assumptions, both called *separation of scales*. The first one states the following:

- The polymer length-scale is much larger than the mean free path of solvent particles.

This assumption allows us to employ various hydrodynamic approximations to calculate the effect of fluids to polymers.

**Flexible polymers.** As we have seen in Section 1.2, we can regard a flexible polymer as a spring with a bead on its each end. Within this model, the end-to-end vector $m$ is the configuration of this polymer. Due to the first and the second assumptions, we may suppose that the solvent flow surrounding the beads is in the Stokes regime. First, hydrodynamic representation of the solvent near a bead is reasonable, as a bead is larger than the solvent mean-free path length. Also, with this small length-scale, Reynolds number is small enough near the bead. Therefore, we can apply Stokes' law to obtain its friction coefficient. This fact is used in the subsequent section, and also in Chapter 5.

### 1.3.2 Fokker-Planck equations

From now on, we will denote the configuration space of polymers by $M$. Our goal is to obtain the description of $f(m)$, which denotes the probability of a particle to be in the configuration $m \in M$.

In this subsection, we assume that solvents do not flow while solvent particles are randomly jittering, exhibiting Brownian motion. Also, in this subsection we assume that polymers distribute uniformly in space, so that $f$ does not depend on $x$, and diffusion of center-of-mass positions can be ignored. At this point we would like to obtain a description of how configuration changes over time without considering its spatial distributions. In this case, polymers experience both diffusion and drift effects due to their potential forces. By diffusion we mean the diffusion in the configuration, which originates from the random collision of solvent particles and polymers. We denote the polymer (elastic) potential by $U$. The modified free energy is given by

$$\mathcal{E} = \int_M f(\log f + U)dm. \tag{1.31}$$

An equilibrium distribution is given by a minimizer of $\mathcal{E}$ and satisfies the Onsager equation

$$f = \frac{e^{-U}}{\mathcal{Z}}, \tag{1.32}$$

where $\mathcal{Z} = \int e^{-U} dm$ is the normalizer. We note that solutions to the equation (1.32) can be non-unique, if $U$ depends on $f$.

If $f$ is not in equilibrium, then it relaxes to an equilibrium following the Fokker-Planck equation:

$$\partial_t f = \epsilon \frac{\delta \mathcal{E}}{\delta f} \tag{1.33}$$

or

$$\partial_t f = \epsilon \left( \Delta_m f + \nabla_m \cdot (\nabla_m U f) \right). \tag{1.34}$$

Here, $\epsilon$ is a diffusivity, a positive constant. In Section 1.3.3, we will see how $\epsilon$ is determined from other physical coefficients.

We note that $\mathcal{E}$ is a Lyapunov functional: we have

$$\frac{d}{dt} \mathcal{E} = -\mathcal{D} \leq 0, \tag{1.35}$$

where

$$\mathcal{D} = \int_M \frac{|\nabla_m f + \nabla_m U f|^2}{f} dm. \tag{1.36}$$

It is worth noting that there is a stochastic differential equation corresponding to (1.34). For simplicity, let $M = \mathbb{R}^d$, and let $m$ be a particle subject to the potential effect and random fluctuation effect. Then the corresponding stochastic differential equation is:

$$dm_t = -\epsilon \nabla_m U(m_t) dt + \sqrt{2\epsilon} dW_t, \tag{1.37}$$

where $W_t$ is a standard Wiener process. In Chapter 5, we deal with the case where $M$ has a boundary, and in that case, we need to use a *local time* term, representing a boundary condition [98].

Another remark is that there is an interpretation of (1.34) in terms of the gradient flow structure, following Jordan, Kinderlehrer, and Otto [72]. Again, for simplicity, we assume that $M = \mathbb{R}^d$ and $U = |m|^{2k}$ with $k$ a positive integer. Then the solution

$f$ of (1.34) can be approximated by the following discrete scheme: for any $T > 0$ we divide the time interval $[0, T]$ by intervals of length $h = \frac{T}{N} > 0$. Then given $f^{(k-1)}$ at time $t_{k-1}$, we determine $f^{(k)}$ by the minimizer of

$$\frac{1}{2} d(f^{(k-1)}, f)^2 + h\mathcal{E}[f] \tag{1.38}$$

over nonnegative integrable functions $f$ with total mass 1 and finite second moments. Here $d$ is the 2-Wasserstein metric:

$$d(\mu_1, \mu_2)^2 = \inf_{p \in \mathcal{P}(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 p(dxdy), \tag{1.39}$$

where $\mathcal{P}(\mu_1, \mu_2)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal $\mu_1$ and second marginal $\mu_2$. Then $f_h(t) = f^{(k)}$ for $t \in [kh, (k+1)h)$ converges to $f$ weakly in $L^1(\mathbb{R}^d)$ for $[0, T]$.

### 1.3.3 Stokes-Einstein Relation

In this section, we establish the relation between the diffusivity constant $\epsilon$ in (1.34) and other physical coefficients, namely solvent viscosity $\nu$. This analysis is particularly related to the investigation of asymptotic limits in Chapter 5. First, we note that

$$\epsilon = \frac{k_B \overline{T}}{\zeta}, \tag{1.40}$$

where $\zeta$ is the friction coefficient of the bead (in the bead-spring model of the flexible polymers). Then, the friction coefficient of the bead, $\zeta$, is determined by the Stokes-Einstein relation:

$$\zeta = 6\pi a_b \rho \nu \tag{1.41}$$

where $\rho$ is the solvent density, $\nu$ is the solvent kinematic viscosity, and $a_b$ is the bead radius. Here we emphasize again that separation of scales has been used; (1.41) is only valid for Stokes flow past the sphere.

## 1.4 The coupled system: Navier-Stokes-Fokker-Planck equations

In this section, we complete the description of the coupled system of fluid and polymers. To achieve this, we need to understand both the effect of fluids to polymers and the collective effect of polymers to fluids. From now on, we consider the effect of macroscopic movement of solvents, and we consider $x$ the dependence of $f$ and center-of-mass diffusion of polymers. Standard references include [47], [116], and [16].

### 1.4.1 The separation of scales - II

The second *separation of scales* assumption states the following:

- The length-scale of polymers, say the diameter of $M$, is much smaller than the length-scale of the macroscopic flow.

This assumption has two important consequences. First, it provides the ground for the continuum description of polymer probability distributions. For a certain point $x$, one can think of the grid centered at $x$ with its side length $\Delta x$, where $\Delta x$ is small enough that the discretization of the Navier-Stokes equation with spatial grid size $\Delta x$ is accurate. By the separation of scales, we may assume that the diameter of $M$ is much smaller than $\Delta x$ and that the statistical distribution of configurations of the collection of polymers in $B_{\Delta x}(x)$, which is $f(x, m)$. Second, it suggests that the effect of an individual polymer to the macroscopic flow is negligible, and the effect of the polymer to the flow should be collective; all polymers at $x$ (we interpret this as polymers in a small neighborhood of $x$) exert some influence on the flow, and we sum them. Furthermore, this assumption is used in calculating the velocity gradient between two beads, which distorts the polymers and essentially contributes to the nontrivial dynamics of the polymeric fluid systems.

### 1.4.2 The effect of fluids to polymers

Flows of solvents influence the polymer distribution. First, flows transport fluid parcels, and thus, polymer distribution experiences advection. Second, polymers are rotated (or distorted, especially in the case of flexible polymers) by the flow velocity gradient. We briefly illustrate this effect in the case of the bead-spring model of flexible polymers, using the stochastic formulation of Fokker-Planck equation. Suppose that a polymer located at $x$ has its end-to-end vector $m$, with $b_1, b_2$ at its end. Then we want to calculate the difference of the solvent flow velocity field between the two end-beads. It is a subtle point since the flow velocity at $x$ is given by $u(x)$. However, we note that by the first and the second separation of scales assumptions, we have the following picture: we interpret $u(x)$ as an average velocity in a neighborhood of $x$ (say, in the grid), and there are local fluctuation of flow velocity inside the neighborhood, and for the two given points $b_1, b_2$ in the grid, with $b_2 - b_1 = m$, we choose the leading order term of the Taylor expansion

$$u(b_2) - u(b_1) = (\nabla_x u(x))m \tag{1.42}$$

as the solvent velocity difference between $b_2$ and $b_1$. Therefore, this gradient-of-velocity effect contributes to $dm_t$ in the stochastic setting by $(\nabla_x u(x))m_t$. Therefore, we have the following evolution equation for the probability density of polymers, which we will call the *Fokker-Planck equation* in the thesis.

$$\partial_t f + u \cdot \nabla_x f + (\nabla_x u) m \cdot \nabla_m f = \epsilon(\Delta_m f + \nabla_m \cdot (\nabla_m U f)) + \nu_2 \Delta_x f, \tag{1.43}$$

where $\nu_2$ is the center-of-mass diffusivity.

We remark that in some literature researchers set $\nu_2 = 0$ on the basis that $\nu_2$ is much smaller than $\epsilon$. This is a consequence of the second separation of scales assumption: $\nu_2 \propto \epsilon\alpha$, where $\alpha$ is a ratio between the polymer length-scale and macroscopic length-scale, so $\nu_2$ is very small. In Chapter 4 we set $\nu_2 = 0$. If $\nu_2 = 0$, polymer distributions are material, that is, they move along the fluid parcel. This makes analysis in Lagrangian coordinates convenient. On the other hand, in Chapter 2 we set $\nu_2 > 0$, which makes the system becomes fully parabolic and enables us to prove the global well-posedness.

### 1.4.3 Stress fields due to polymers

Polymers influence the flow by exerting stress. Thus, the stress term $\Sigma_F$ in the momentum balance equation (1.8), can be determined by

$$\Sigma_F = -p\mathbb{I} + 2\mu D(u) + K\sigma, \tag{1.44}$$

where $K$ is the parameter representing the strength of polymeric stress, and $\sigma$ is the non-dimensional polymeric stress. The polymeric stress is determined by *Kramers expression*:

$$\sigma[f] = \int_M m \otimes \nabla_m U f dm. \tag{1.45}$$

The expression (1.45) has a nice energy balance; see Chapter 2 and Chapter 5. For derivation of (1.45), see [47] or [116]; also Section 5.4.2 contains essential ideas for the derivation, although polymers are attached on the wall in this case.

### 1.4.4 The Hookean spring case: the Oldroyd-B model

Among the choices of spring potentials, the Hookean spring potential $U = \frac{|m|^2}{2}$ has a special property: we obtain a macroscopic closure for the stress evolution, which is called the *Oldroyd-B model*. To see this, we first note that in (1.45) the polymeric stress can be rewritten as

$$\sigma[f] = \int_{\mathbb{R}^d} m \otimes m f dm. \tag{1.46}$$

Thus, multiplying $m \otimes m$ to (1.43) and integrating in $m$ variable formally gives

$$\begin{aligned}(\partial_t + u \cdot \nabla_x)\, \sigma[f] = (\nabla_x u)\sigma[f] + \sigma[f](\nabla_x u)^T - 2\epsilon\sigma[f] + 2\epsilon M_{0,0}[f] + \nu_2 \Delta_x \sigma[f], \\ (\partial_t + u \cdot \nabla_X)\, M_{0,0}[f] = \nu_2 \Delta_x M_{0,0}[f], \\ M_{0,0}[f] = \int_{\mathbb{R}^d} f dm.\end{aligned} \tag{1.47}$$

The Oldroyd-B model is simpler to solve than Fokker-Planck equation, since it is fully macroscopic. The Oldroyd-B model is widely used because of this reason. We note that to derive this equation rigorously, one needs to justify the integration by parts and therefore existence of higher moments. Rigorous derivations can be found in [12] and [85].

## 1.5 Goal of the thesis

There are three goals of this thesis. First, this thesis provides a new framework to prove the global well-posedness of the systems describing polymeric fluids system, which are based on the kinetic theory. In this framework, one first translates the multi-scale system into a fully macroscopic system at the cost of having infinitely many equations. Then one solves the transformed system, and finally one recovers the solution of the original system. The description of this approach is summarized in Chapter 2 for the flexible polymer suspensions and in Chapter 3 for the rigid polymer suspensions.

Second, this thesis provides a proof of the non-diffusive Oldroyd-B system using a new framework provided by Constantin [29], [30] (also [35], together with the author) which is natural and allows lower-regularity uniqueness and Lipschitz dependence on initial data. Chapter 4 provides the proof.

Third, the thesis proposes a new boundary condition which models a situation where polymers are grafted near the wall and drag is reduced. In Chapter 5, the model is described and mathematical results are proven based on the work of the author and Drivas [50].

## 1.6 Previous works

### 1.6.1 Polymeric fluid models - existence and uniqueness of solutions

There is a large body of literature on complex fluids. This section examines this literature to situate this work in the existing scholarly debate.

**Oldroyd-B and relevant macroscopic models.** Macroscopic models for viscoelasticity, such as Oldroyd-B, have been studied extensively. First we discuss the results concerning the non-diffusive models. Guillopé and Saut proved the local existence, uniqueness of the strong solution, and the global existence of the strong solution for small initial data, in the case of the bounded domain, in [65] and in [66]. Fernández-Cara, Guillén, and Ortega extended the results of Guillopé and Saut to $L^p$ setting in [59], [60], and [61]. In addition, Hieber, Naito, and Shibata studied the system in the case of the exterior domain in [67]. Chemin and Masmoudi studied the system in critical Besov spaces, and proved the local well-posedness of the system and provided a Beale-Kato-Majda type ([14]) criterion in [24]. Other Beale-Kato-Majda type sufficient conditions were given by Kupferman, Mangoubi, and Titi in [84], and by Lei, Masmoudi, and Zhou in [89]. In addition, Lions and Masmoudi showed the global existence of a weak solution for corotational models in [96]. Hu and Lin proved in [68] the global existence of weak solution for non-corotational models, given that the initial deformation gradient is close to the identity and the initial velocity is small. In [93], Lin, Liu, and Zhang developed an approach based on the deformation tensor and Lagrangian particle dynamics. Lei and Zhou studied the system via the incompressible limit in [90] and proved the global existence for small data. Also, Lei, Liu, and Zhou studied the global existence for small data and the incompressible limit in [88]. Moreover, in [58], Fang and Zi proved the global well-posedness for initial data whose vertical velocity field can be large. Constantin and Sun proved the global existence for small data with large gradients for Oldroyd-B and considered a regularization of Oldroyd-B model in [39]. Thomases and Shelley provided numerical evidence for singularities for the Oldroyd-B system in [127]. Next, we discuss the results for the diffusive Oldroyd-B models. Barrett and Boyaval proved the global existence of weak solution in [7]. In [33], Constantin and Kliegl proved the global well-posedness of the strong solution. Also we refer to Elgindi and Rousset ([55]) and Elgindi and Liu ([54]) for the Oldroyd-B type systems where fluid viscosity is ignored.

**Multiscale models, especially FENE models.** Macro-micro models, especially FENE models and some simplifications of them have been studied by many authors. In this paragraph, we discuss results concerning non-diffusive multiscale models. Renardy proved the local existence of solution for FENE models in Sobolev space with potential $U(m) = (1 - |m|^2)^{1-k}$ for some $k > 1$, as well as infinitely extensible models, in [120]. E, Li, and Zhang considered modified models with stochastic setting in [53]. Jourdain, Leliévre, and Le Bris proved the local existence for the FENE model in [74], in the setting of the coupled system of Navier-Stokes equation and stochastic Fokker-Planck equation. Jourdain, Le Bris, Leliévre, Otto proved the exponential convergence to equilibrium in [73] using theentropy inequality method. There are also various other local existence results, for example Zhang and Zhang ([133]), Kreml and Pokorný ([83]), and Masmoudi ([102]). In [102], the author controlled the stress tensor by the $H^1$ norm in $m$ coming from diffusion in $m$, thanks to Hardy type inequalities and noted that initial data do not need to be regular in $m$ variable. Lin, Liu, and Zhang discussed near-equilibrium situations in [94]. In [105], Masmoudi, Zhang, and Zhang proved the global well-posedness for corotational case. One remarkable result, the global existence of a weak solution for the FENE model, is proven by Masmoudi in [103]. The author used the defect measure to overcome difficulties from compactness issue.

**Smoluchowski models.** Smoluchowski equations, which refer to the models whose configuration spaces $M$ are compact manifolds, are also discussed by various authors. In [32], Constantin, Fefferman, Titi, and Zarnescu studied the nonlinear Fokker-Planck equation driven by a time averaged Navier-Stokes system in 2D. Constantin ([27]), Constantin and Masmoudi ([36]), Constantin and Seregin ([38], [37]) showed the global existence of smooth solutions for large data in 2D was established.
The system (3.1) is an example of Smoluchowski equations. Otto and Tzavaras discussed Doi model (3.1) in [117]. Also, Lions and Masmoudi proved the global existence of weak solution in [97] with an important observation of dissipative nature of viscous stress. Also Zhang and Zhang proved the local and small data global well-posedness for (3.1) models for small $\eta$ in [134]. The compressible Doi model is discussed by Bae and Trivisa in [1], [2], and [3]. The relationship between rigid rod-like polymer suspension models and the Ericksen-Leslie model for nematic liquid crystal has been investigated in [130]. For more general introduction for complex fluids, there are excellent references including [104], [92], and [28]. In addition, the author also discussed

the 2-dimensional Doi model with viscous stress tensor in [86]. Chapter 3 is based on [86].

**Diffusive models and other regularized models.** There are results concerning regularized dumbbell models, for example introducing mollifiers to some terms in the equation ([135]). Especially, dumbbell models with center-of-mass diffusion are discussed by Barrett and Süli ([8], [9], [11], [10], [12]), and Barrett and Boyaval ([7]). Chapter 2 is based on [85]. Also, Schonbek discussed the regularized model, with corotational assumption in [122].

### 1.6.2 Lagrangian-Eulerian method

Many authors discussed Lagrangian formulation of fluid systems. In [93], Lin, Liu, and Zhang developed an approach based on deformation tensor and Lagrangian particle dynamics. Also, based on Lagrangian approach, Constantin and Sun proved the global existence for small data with large gradients for Oldroyd-B, and considered regularization of Oldroyd-B model in [39]. In a sequence of works ([29], [30], [35]), Constantin developed the formulation of Lagrangian-Eulerian method. The gist of the formulation is that, we use Lagrangian coordinates to represent variables, convert it to Eulerian coordinates to compute field interactions, and then come back to Lagrangian coordinates. Physically, this approach is natural: since the response of a viscoelastic fluid particle depends on the history of deformation of that particle, it is reasonable to track down the history of stress and velocity field in the Lagrangian coordinate. On the other hand, since interaction between fields are not material due to diffusion and non-local effects, it is also reasonable to compute those interactions in Eulerian coordinates. Mathematically, this approach has following advantages: first it allows us to prove uniqueness for lower regularity space, for example $\sigma \in C^{\alpha}$, while in Eulerian variable convective derivative $u \cdot \nabla_x \sigma$ does not make sense, which makes an uniqueness proof difficult in the Eulerian framework. Second, since the approach is essentially based on ODE, we can obtain Lipschitz dependence on initial data easily.

### 1.6.3 Polymer drag reduction, Boundary condition and vanishing viscosity limit

Chapter 5 is based on [50], which introduces a new boundary condition, where polymers are grafted on the wall and interact with bulk fluids, and corresponding vanishing viscosity limits. This result should be contrasted with the situation without polymer. The two most commonly used boundary conditions for neutral Navier-Stokes fluids are the so-called no-slip and Navier-friction (with viscosity dependent slip-length) conditions. No-slip, or stick, boundary conditions (1.24) correspond to the situation in which the fluid velocity matches the boundary velocity (which we here consider stationary): On the other hand, the Navier-friction boundary condition (1.25) allows the fluid to slip tangentially along the boundary for all $\nu > 0$. The (variable) slip-length is defined as $\ell_s := \nu/\alpha$. Both the no-slip and Navier-friction condition above arise rigorously from the Boltzmann equation in the hydrodynamic limit with appropriate scalings [71]. The nature of the inviscid limit for the Navier-Stokes system coupled with either of these physical boundary conditions and its connection to the Euler equations for an inviscid fluid is an outstanding open problem. We briefly review the status presently.

The main physical process which makes the behavior of fluids with small viscosity so rich and difficult to analyze is the formation of thin viscous boundary layers which may become singular in the inviscid limit, detach from the walls, and generate turbulence in the bulk. In contrast to the situation without solid boundaries, the process can occur even if a strong Euler solutions exists (which holds true globally in time, for example, in two spatial dimensions from smooth initial conditions). A fundamental result in this area is due to Kato [77], who proved that the following two conditions are equivalent: (i) the integrated energy dissipation vanishes in a very thin boundary layer of thickness $O(\nu)$ and (ii) any Navier-Stokes solution with no slip boundary conditions at the wall converges strongly in $L_t^\infty L_x^2$ to the Euler solution as $\nu \to 0$. Additionally, the above holds if and only if the *global* dissipation $\langle \varepsilon^\nu \rangle$ vanishes in the inviscid limit

$$\langle \varepsilon^\nu \rangle \to 0 \text{ as } \nu \to 0. \tag{1.48}$$

Another important equivalence condition of particular relevance to our work was established by Bardos and Titi (Theorem 4.1 of [6], Theorem 10.1 of [82]), who prove that convergence to strong Euler in the energy space is equivalent to the wall-friction velocity $u_*$ (related to the local shear stress at the wall) vanishing

$$u_*^2 := \nu(\partial_{\hat{n}} u^\nu)_{\hat{\tau}} \rightharpoonup 0 \text{ as } \nu \to 0 \tag{1.49}$$

in a weak sense on $\partial\Omega \times [0,T]$, integrating against $\varphi \in C^1([0,T] \times \partial\Omega)$ test functions. Aside from these equivalence theorems, most of the known results establish the strong inviscid limit under a variety of conditions, see for example, [78], [6], [31], [34], [80], [126], and [79]. For no-slip boundaries, some unconditional results are known in settings for which laminar flow can be controlled for short times [121], [101], [62], [81],[99], and [107]. These unconditional results hold before any boundary layer separation or other turbulent behavior occurs.

On the negative side, it has been recently shown that the Prandtl Ansatz is, in general, false for no-slip conditions and that the $L^\infty$-based Prandtl expansion fails for unsteady flows [64]. Moreover, there is a vast amount of experimental and numerical evidence for anomalous dissipation, i.e. the phenomenon of non-vanishing dissipation of energy in the limit of zero viscosity, in the presence of solid boundaries.

For example, see the experimental work of [123] and [119] from wind tunnel experiments and of [23] for more complex geometries. In two-dimensions, the works [114] and [113] convincingly show through a careful numerical study that anomalous dissipation occurs from vortex dipole initial configurations with both no-slip (1.24) and Navier-friction conditions (1.25) respectively. See extended discussion of the evidence in [51],[48]. In light of Kato's equivalence, in situations exhibiting anomalous dissipation convergence cannot be towards a strong solution of Euler. Recently progress has been made towards giving minimal conditions for the inviscid limit to weak Euler solutions to hold [40], [52], [118]. Such solutions may provide a framework to describe the anomalous dissipation in the inviscid limit as envisioned by Onsager [115]. See [51], [52], [5], and [4] for recent progress in this direction.

# Chapter 2

# Moment solution methods for flexible polymer solutions

## 2.1 Introduction

We are interested in the following system:

$$\begin{gathered}
\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \nu_1 \Delta_x u + K \nabla_x \cdot \sigma, \\
\nabla_x \cdot u = 0, \\
\sigma = \int_{\mathbb{R}^2} m \otimes (\nabla_m U(m)) f(x,t,m) dm, \\
\partial_t f + u \cdot \nabla_x f + (\nabla_x u) m \cdot \nabla_m f = \epsilon \left( \Delta_m f + \nabla_m \cdot (f \nabla_m U) \right) + \nu_2 \Delta_x f, \\
U(m) = |m|^{2q}, \\
u(0) = u_0, f(0) = \mu_0,
\end{gathered} \tag{2.1}$$

where $q \geq 1$ is a real number, and the vector of position, configuration, and time $(x, m, t)$ is in $\mathbb{R}^2 \times \mathbb{R}^2 \times (0, T)$. For the simplicity of notation, we assume that $q$ is an integer, but our method works for any real number $q \geq 1$. We may also normalize $\mu_0$ so that $\int_x \int_m \mu_0(dm) dx = 1$. The variable $u$ represents the velocity of the solvent fluid, $p$ represents the pressure, $f$ represents the distribution of the polymer, $\sigma$ represents the stress field due to polymer, and $\nu_1, K, \epsilon, \nu_2$ are positive constants. We want to investigate the existence and uniqueness of smooth solution for this system. However, we note that the regularity required for the macroscopic equation (the first equation of (2.1)) is not same as the regularity required for the microscopic equation (the fourth equation of (2.1)); for flows of the fluid to be smooth, we need the smoothness for $u$, but the only thing that we require for $f$ is the smoothness of $\sigma[f]$. In particular, smoothness in $m$ variable does not seem to be important. In addition, since $f$ contributes to flows of the whole system only by the macroscopic quantity $\sigma[f]$, it would be interesting if we can transform this microscopic-macroscopic system into a fully macroscopic system, possibly a coupled system of infinitely many variables. In this regard, we define the moment solution in section 2.2.3, which is a sense of solution for the microscopic equation that we use in this paper. In short, a moment solution is a weak solution such that all moments of $f$ are controlled. A moment of $f$ is a weighted (usually weights are monomials $m^I$) integral in $m$ variable, and thus, a macroscopic quantity, depending only on $x$ and $t$. Appropriate initial data for moment solutions are nonnegative measures on $\mathbb{R}^2_m \times \mathbb{R}^2_x$ such that norms of moments of them are controlled.

**Remark 3.** *We remark that the idea of transforming an equation to the coupled system of infinitely many variables is not new. In the context of turbulence theory, Friedmann-Keller equation ([108]) employs an infinite chain of equations for the infinite set of moments.*

Next, we state our main results.We first prove the existence and uniqueness of the moment solution, given smooth flow $u$:

**Theorem 2.1.1** (Theorem 2.3.1)**.** *Given a smooth fluid field $u$(satisfying (2.81)), and appropriate initial data $\mu_0$ (satisfying (2.82), (2.83), and (2.84)), there exists unique* *moment solution for the fourth equation of (2.1). Furthermore, various norms of moments of this moment solution are controlled solely by the initial data and flow field $u$ (estimates (2.133), (2.134), (2.135), (2.136), and (2.137)).*

Presence of the term $\epsilon \nabla_m \cdot (f \nabla_m U)$ introduces higher order terms to evolution equations of moments if $q > 1$. Another problem in the justification of this formal calculation is the potential loss of decay in $m$; in formal derivation of evolution equations of moments, we use integration by parts to deal with terms with $\nabla_m f$. We need to know the finiteness of higher moments to justify the integration by parts. In the paper, we see how to overcome this difficulty. Next, we prove that the stress field depends continuously on the flow field. For this result we require finite entropy condition for the initial data.

**Theorem 2.1.2** (Theorem 2.3.1). *Given two smooth fluid fields $u$, $v$, and appropriate initial data $\mu_0$ satisfying finite entropy condition (2.85), if we let $\sigma_1$ and $\sigma_2$ to be stress fields of the moment solutions with velocity fields $u$ and $v$, respectively, then $\sigma_1 - \sigma_2$ is controlled by $u - v$ ((2.185)).*

The main reason why we need the finite entropy condition is that we have to deal with $\nabla_m f$ term when taking difference $\sigma_1 - \sigma_2$. It will be clear in the paper that we cannot simply use integration by parts to rule out derivatives in $m$ variable in this case. Then the above theorems can be used to prove local existence and uniqueness of the solution of the system (2.1), using the contraction mapping scheme.

**Theorem 2.1.3** (Theorem 2.4.1). *Given $u_0 \in \mathbb{P}W^{2,2}$ and appropriate initial data $\mu_0$ with finite entropy condition, there is a unique solution $(u, f)$ for the system (2.1) for some time. $u$ is the strong solution for macroscopic equation, and $f$ is the moment solution for the microscopic equation with the velocity field $u$.*

In addition, this result shows that for the Hookean spring potential case ($q = 1$), the Oldroyd-B model is the exact closure of the system (2.1). This extends the result

([12]) of Barrett and Süli to a larger class of data. Next, we prove global existence and uniqueness of the system (2.1). The proof uses arguments from [33], but the first step, (2.203), needs a justification, since it involves an $L^1$ estimate for the stress field.

**Theorem 2.1.4** (Theorem 2.4.2)**.** *Given $u_0 \in \mathbb{P}W^{2,2}$, appropriate initial data $\mu_0$ with finite entropy condition, and an arbitrary $T > 0$ there exists a unique solution $(u, f)$ for $(0, T)$. In addition, there are explicit bounds ((2.203), (2.209), (2.211), (2.212), (2.213), and (2.214) ) for the norm of the solution.*

Finally, we establish a free energy estimate. Here we make an additional assumption (2.242), to guarantee that initial free energy is finite.

**Theorem 2.1.5** (Theorem 2.4.4)**.** *For the solution of the system (2.1), its free energy, which is defined as the sum of kinetic energy of the fluid ($\|u(t)\|_{L^2}^2$) and free energy of polymer distribution $\left(\int f(t) \log\left(\frac{f(t)}{\int f(t)dm \frac{e^{-U(m)}}{Z}}\right) dmdx\right)$, does not increase over time (bound (2.249).*

The main challenge for proving this theorem is to control the limit of integrals of nonlinear terms.

## 2.2 Function space and Moment solution

### 2.2.1 Preliminaries

Let $\mathcal{M}(\mathbb{R}^2)$ be the space of signed Borel measures. $\mathcal{M}(\mathbb{R}^2)$ is a Banach space, where the norm is the total variation of $\mu$, $|\mu|(\mathbb{R}^2)$. Given $\mu \in \mathcal{M}(\mathbb{R}^2)$, we denote the moment of $\mu$ as

$$M_{a,b}[\mu] = \int_{\mathbb{R}^2} m_1^a m_2^b \mu(dm), \tag{2.2}$$

where $a, b \geq 0$ are integers, the radial absolute moment of $\mu$ as

$$\bar{M}_k[\mu] = \int_{\mathbb{R}^2} |m|^k |\mu|(dm) \tag{2.3}$$

where $k \geq 0$ is an integer, the vector of moments of degree $k$ as

$$\dot{\vec{M}}_k[\mu] = (M_{k,0}[\mu], M_{k-1,1}[\mu], \cdots, M_{0,k}[\mu]) \tag{2.4}$$

and the vector of moments of degrees up to $k$ as

$$\vec{M}_k[\mu] = \left(\dot{\vec{M}}_0[\mu], \dot{\vec{M}}_1[\mu], \cdots, \dot{\vec{M}}_k[\mu]\right), \tag{2.5}$$

and the vector of moments of even degrees up to $2k$ as

$$\vec{M}^e_{2k}[\mu] = \left(\dot{\vec{M}}_0[\mu], \dot{\vec{M}}_2[\mu], \cdots, \dot{\vec{M}}_{2k}[\mu]\right). \tag{2.6}$$

In probability theory, moment problem refers to the problem of determining a probability measure when moments are given. We only briefly mention what is needed for us, and more detailed explanation can be found in [70]. We first introduce the Riesz functional and positive semidefinite sequence.

**Definition 2.2.1** (Riesz' functional)**.** *Given $m = \{m_{a,b}\}_{(a,b)\in\mathbb{Z}^2_{\geq 0}}$, we define the associated Riesz functional $L_m$ on $\mathbb{R}[x]$ by $L_m(x^I) := m^I$ for all $I = (a,b) \in \mathbb{Z}^2_{\geq 0}$.*

**Definition 2.2.2** (Positive semidefinite sequence)**.** *A sequence $m = \{m_{a,b}\}_{(a,b)\in\mathbb{Z}^2_{\geq 0}}$ of real numbers is said to be positive semidefinite if for any $k \in \mathbb{N}$, $c_1, \cdots, c_k \in \mathbb{R}$ and $(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k) \in \mathbb{Z}^2_{\geq 0}$,*

$$\sum_{i,j=1}^{k} m_{(a_i,b_i)+(a_j,b_j)} c_i c_j \geq 0 \tag{2.7}$$

*holds, or equivalently, $L_m(h^2) \geq 0$ for any $h \in \mathbb{R}[x]$.*

For moment problems for measures on $\mathbb{R}^d$, $d \geq 2$, the multivariate Carleman's condition, which is a constraint on the growth rate of moments over degree, provides a sufficient condition for uniqueness.

**Theorem 2.2.3.** *Let $\mu, \nu$ be positive measures in $\mathbb{R}^2$ where $M_{a,b}[\mu] = M_{a,b}[\mu] < \infty$. Let $m = \{M_{a,b}\}_{(a,b)\in\mathbb{Z}^2_{\geq 0}}$. If*

$$\sum_{n=1}^{\infty} L_m(x_1^{2n})^{-\frac{1}{2n}} = \sum_{n=1}^{\infty} L_m(x_2^{2n})^{-\frac{1}{2n}} = \infty \tag{2.8}$$

*then $\mu = \nu$.*

The condition (2.8) is known as the multivariate Carleman's condition.

**Theorem 2.2.4.** *Let $m = \{m_{a,b}\}_{(a,b)\in\mathbb{Z}^2_{\geq 0}}$ be a positive semidefinite sequence satisfying the multivariate Carleman's condition (2.8). Then there exists a unique non-negative Borel measure $\mu$ such that $m_{a,b} = M_{a,b}[\mu]$ for all $(a, b)$.*

Also we need the following result, which states that if a given measure is determined uniquely by its moments, and if moments of a sequence of measures converge to moments of this measure, then the sequence of measures converge to the measure weakly. We mainly refer to [17]. A sequence of (signed) Borel measures on $\mathbb{R}^2$ is uniformly tight if for every $\epsilon > 0$ there is a compact set $K_\epsilon \subset \mathbb{R}^2$ such that $|\mu_n|(\mathbb{R}^2 - K_\epsilon) < \epsilon$ for all $n$. Also we define the weak convergence of measures.

**Definition 2.2.5.** *A sequence of Borel measures on $\mathbb{R}^2$ $\{\mu_n\}$ is called weakly convergent to a Borel measure $\mu$ on $\mathbb{R}^2$ if for every bounded continuous real function $f$ on $\mathbb{R}^2$, one has*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f(m)\mu_n(dm) = \int_{\mathbb{R}^2} f(m)\mu(dm). \tag{2.9}$$

The following lemma is useful.

**Lemma 2.2.6.** *Let $\mu_n$ be a sequence of nonnegative Borel measures on $\mathbb{R}^2$ which is uniformly bounded in total variation norm and converges weakly to a Borel measure $\mu$. Then for every continuous function $f$ on $\mathbb{R}^2$ satisfying the condition*

$$\lim_{R\to\infty} \sup_n \int_{|f|\geq R} |f|\mu_n(dm) = 0, \tag{2.10}$$

*one has*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f\mu_n(dm) = \int_{\mathbb{R}^2} f\mu(dm). \tag{2.11}$$

*Proof.* First we let $f_m = \min(|f|, m)$. Then $f_m \leq |f|$, and from the assumption on $f$ there is some $R_0 > 0$ such that

$$\sup_n \int_{|f|\geq R_0} |f|\mu_n(dm) \leq 1, \tag{2.12}$$

while

$$\sup_n \int_{|f|\leq R_0} |f|\mu_n(dm) \leq R_0 \sup_n \int_{\mathbb{R}^2} \mu_n(dm) = R_0 C < \infty \tag{2.13}$$

so that

$$\sup_{n,m} \int_{\mathbb{R}^2} f_m\mu_n(dm) \leq 1 + CR_0 = M < \infty. \tag{2.14}$$

Since $f_m$ is continuous and bounded, by weak continuity we have

$$\int_{\mathbb{R}^2} f_m\mu(dm) \leq M \tag{2.15}$$

and by monotone convergence we have $f \in L^1(\mu)$. For a given $\epsilon > 0$, we can pick $R > 0$ such that there is some $N > 0$ such that for all $n \geq N$

$$\int_{|f|\geq R} |f|\mu_n(dm) + \int_{|f|\geq R} |f|\mu(dm) < \epsilon. \tag{2.16}$$

Let $g = \max(\min(f, R), -R)$ be the truncation of $f$ up to $R$: $g = f$ if $|f| < R$, $g = R$ if $f \geq R$, and $g = -R$ if $f \leq -R$. Since $g$ is continuous and bounded, there is some $N' > N$ such that for all $n \geq N'$

$$\left|\int_{\mathbb{R}^2} g\mu_n(dm) - \int_{\mathbb{R}^2} g\mu(dm)\right| < \epsilon. \tag{2.17}$$

Then for such $n$, we have

$$\left| \int_{\mathbb{R}^2} f \mu_n(dm) - \int_{\mathbb{R}^2} f \mu(dm) \right| < 3\epsilon, \tag{2.18}$$

as desired. □

Then the Prohorov's theorem states the following.

**Theorem 2.2.7** (Prohorov)**.** *The sequence $\mu_n$ of (signed) Borel measures on $\mathbb{R}^2$ contains a weakly convergent subsequence if and only if $\mu_n$ is uniformly tight and uniformly bounded in the total variation norm.*

Using Prohorov's theorem and Lemma 2.2.6, we can prove the following ([15]):

**Theorem 2.2.8.** *Suppose that $\mu_n$ is a sequence of nonnegative Borel measures on $\mathbb{R}^2$ having all moments $M_{a,b}[\mu_n] < \infty$, and $\mu$ is a nonnegative Borel measure on $\mathbb{R}^2$ with $M_{a,b}[\mu] < \infty$ too. Suppose that $\mu$ is determined by its moment: if there is a nonnegative Borel measure $\nu$ such that $M_{a,b}[\mu] = M_{a,b}[\nu]$ for all $a, b$, then $\mu = \nu$. Also suppose that $M_{a,b}[\mu_n] \to M_{a,b}[\mu]$ for all $a, b$. Then $\mu_n$ converges to $\mu$ weakly, at least for a subsequence.*

*Proof.* First note that $\bar{M}_2[\mu_n]$ is uniformly bounded, say by $C$, since it is convergent: then by Chebyshev, we have

$$\mu_n \left( \{ m \in \mathbb{R}^2 : |m| > K \} \right) \leq \frac{C}{K^2}, \tag{2.19}$$

so $\mu_n$ is uniformly tight. Also since $M_{0,0}[\mu_n]$ is also uniformly bounded, so $\mu_n$ has a weakly convergent subsequence, converging to $\nu$. Note that all $M_{a,b}[\mu_n]$ is uniformly bounded due to convergence, and note that for $a, b \geq 0$ we have that $\mu_n^{a,b,+} = \frac{(m_1^a m_2^b)^+}{1+|m|^{a+b}} \mu_n$ converges weakly to $\frac{(m_1^a m_2^b)^+}{1+|m|^{a+b}} \nu$ and $\mu_n^{a,b,-} = \frac{(m_1^a m_2^b)^-}{1+|m|^{a+b}} \mu_n$ converges weakly to $\frac{(m_1^a m_2^b)^-}{1+|m|^{a+b}} \nu$. Those measures are uniformly bounded in total variation norm,

and

$$\begin{aligned}
&\lim_{R\to\infty}\sup_n \int_{|m|^{a+b}+1\geq R^{a+b}+1} (|m|^{a+b}+1)\mu_n^{a,b,+} \\
&= \lim_{R\to\infty}\sup_n \int_{|m|^{a+b}+1\geq R^{a+b}+1} (m_1^a m_2^b)^+ \mu_n(dm) \\
&\leq \lim_{R\to\infty}\frac{1}{R}\sup_n \int_{|m|^{a+b}+1\geq R^{a+b}+1} |m|^{a+b+1}\mu_n(dm) = 0
\end{aligned} \tag{2.20}$$

and same for $\mu_n^{a,b,-}$. Therefore, by Lemma 2.2.6, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} m_1^a m_2^b \mu_n(dm) = \int_{\mathbb{R}^2} m_1^a m_2^b \nu(dm) \tag{2.21}$$

or $M_{a,b}[\mu_n] \to M_{a,b}[\nu]$. But since $\mu$ is determined by its moments, we have $\mu = \nu$. □

**Remark 4.** *If $\{M_{a,b}[\mu]\}_{(a,b)}$ satisfies the multivariate Carleman's condition (2.8), and if $\mu_n$ satisfies all the assumptions in Theorem 2.2.8, then for all $p \geq 0$ $|m|^{2p}\mu_n$ converges weakly to $|m|^{2p}\mu$ in a subsequence.*

*Proof.* First, we observe that

$$M_{2j,0}[m_1^{2p}\mu] = M_{2(j+p),0}[\mu], \tag{2.22}$$

which guarantees that $M_{a,b}[m_1^{2p}\mu]$ also satisfies the multivariate Carleman's condition. The proof of this claim is given in the last. Also, $m_1^{2p}\mu$ is also determined by its moments, and same for $m_2^{2p}\mu$. Therefore, by Theorem 2.2.8 we see that $m_1^{2p}\mu_n$ weakly converges to $m_1^{2p}\mu$ and similarly $m_2^{2p}\mu_n$ weakly converges to $m_2^{2p}\mu$. Also $\mu_n$ weakly converges to $\mu$, and we see that $\frac{|m|^{2p}}{1+m_1^{2p}+m_2^{2p}}$ is a continuous bounded function, so $|m|^{2p}\mu_n$ converges to $|m|^{2p}\mu$ weakly (in subsequence). It only remains to show that

$$\sum_{j=1}^{\infty}\left(\frac{1}{M_{2(j+p),0}}\right)^{-\frac{1}{2j}} = \infty. \tag{2.23}$$

Since $c_j = M_{2(j+p),0}$ satisfies $c_j^2 \leq c_{j-1}c_{j+1}$, by Denjoy-Carleman Theorem ([70]) it is equivalent to show

$$\int_1^\infty \frac{\log T_p(r)}{r^2} dr = \infty, \tag{2.24}$$

where $T_p(r) = \max_j \frac{r^j}{M_{2(j+p),0}}$. However,

$$T_p(r) = \max_{j \geq 1} \frac{r^j}{M_{2(j+p),0}} \geq \max_{j \geq 1+p} \frac{r^j}{M_{2j,0}} \frac{1}{r^p} = T(r) \frac{1}{r^p}. \tag{2.25}$$

But note that already we know $\int_1^\infty \frac{\log T(r)}{r^2} dr = \infty$, and $\int_1^\infty \frac{\log r}{r^2} dr < \infty$ so we are done. □

Also, we have the following Fatou-type lemma.

**Lemma 2.2.9** (Varadarajan)**.** *Suppose that the sequence of (signed) Borel measures $\mu_n$ converges weakly to a Borel measure $\mu$. Then for any functionally open ($f^{-1}((0,\infty))$ for some continuous function $f$ on $\mathbb{R}^2$) set $U$ we have*

$$\liminf_n |\mu_n|(U) \geq |\mu|(U). \tag{2.26}$$

*In this situation, the sequence $|\mu_n|$ converges weakly to $|\mu|$ precisely when $|\mu_n|(\mathbb{R}^2) \to |\mu|(\mathbb{R}^2)$.*

On the other hand, we also need the following ([44]).

**Theorem 2.2.10.** *Let $[0, T]$ be endowed with usual $\sigma$-algebra and Lebesgue measure. Let $X$ be a reflexive Banach space. For any $1 \leq p < \infty$, $(L^p(0, T; X))^* \simeq L^q(0, T; X^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

Also we use Banach-Alaoglu theorem.

**Theorem 2.2.11** (Banach-Alaoglu)**.** *Let $X$ be a normed space. Hence $X^*$ is also normed with the operator norm. Then the closed unit ball of $X^*$ is compact with respect to the weak* topology.*

We also need Rellich-Kondrachov theorem and Aubin-Lions lemma.

**Theorem 2.2.12** (Rellich-Kondrachov)**.** *Suppose that $\Omega$ is bounded domain with smooth boundary. Then the inclusion $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ and $W_0^{1,1}(\Omega) \subset L^1(\Omega)$ are compact.*

**Theorem 2.2.13** (Aubin-Lions)**.** *Let $X_0, X_1, X_2$ be three Banach spaces, $X_0 \subset X_1 \subset X_2$. Suppose that $X_0$ is compactly embedded in $X_1$ and $X_1$ is continuously embedded in $X_2$. For $1 \leq p, q, \leq \infty$, let*

$$W = \{u \in L^p([0,T]; X_0) \,:\, \partial_t u \in L^q([0,T]; X_2)\}. \tag{2.27}$$

*If $p < \infty$, the embedding $W \subseteq L^p([0,T]; X_1)$ is compact. If $p = \infty$ and $q > 1$, the embedding $W \subseteq L^p([0,T]; X_1)$ is compact.*

Also we use results from parabolic theory, especially existence, uniqueness, and estimates of Fokker-Planck-Kolmogorov equations. We mainly refer to [18]. Suppose we are given an open set $\Omega_T = \Omega \times (0,T) \subset \mathbb{R}^d \times (0,T)$, where $\Omega \subset \mathbb{R}^d$ is an open set and $T > 0$, and Borel functions $a^{ij}$, $b^i$, and $c$ on $\Omega_T$, where $i, j = 1, \cdots, d$. We suppose that the matrix $A = (a^{ij})_{ij}$ is symetric nonnegative definite. We discuss the Fokker-Planck-Kolmogorov equation of the form

$$\partial_t \mu = \partial_{x_i} \partial_{x_j} \left(a^{ij} \mu\right) - \partial_{x_i} \left(b^i \mu\right). \tag{2.28}$$

Let

$$L_{A,b}\phi = a^{ij}(x,t) \partial_{x_i} \partial_{x_j} \phi(x,t) + b^i(x,t) \partial_{x_i} \phi(x,t), \tag{2.29}$$

which is the adjoint operator of the right side of (2.28).

**Definition 2.2.14.** *A locally bounded Borel measure $\mu$ on the domain $\Omega_T$, which can be written as $\mu = \mu_t(dx)dt$ is a solution to the Cauchy problem (2.28) with $\mu|_{t=0} = \nu$*

*if* $a^{ij}, b^i \in L^1_{loc}(\mu)$, *for every function* $\phi \in C_0^\infty(\Omega_T)$ *we have*

$$\int_{\Omega_T} (\partial_t \phi + L_{A,b}\phi)\, d\mu = 0, \tag{2.30}$$

*and for every function* $f \in C_0^\infty(\Omega)$ *there is a set of full measure* $J_f \subset (0, T)$, *depending on* $f$, *such that*

$$\int_\Omega f(x)\nu(dx) = \lim_{t\to 0, t\in J_f} \int_\Omega f(x)\mu_t(dx). \tag{2.31}$$

Note that this definition is equivalent to the following: for every function $\phi \in C_0^\infty(\Omega)$ there exists a set of full measure $J_\phi \subset (0, T)$, depending on $\phi$, such that for all $t \in J_\phi$ we have

$$\int_\Omega \phi d\mu_t = \int_\Omega \phi d\nu + \lim_{\tau\to 0+, \tau\in J_\phi} \int_\tau^t \int_\Omega L_{A,b}\phi d\mu_s ds. \tag{2.32}$$

We have the following results. For the proof one can see [18], where more general statements and proof are given. Let $\Omega = \mathbb{R}^d$.

**Theorem 2.2.15** (Existence, existence of density, and uniqueness of Fokker-Planck--Kolmogorov equation)**.** *Suppose that for every ball* $U$ *in* $\mathbb{R}^d$ *the functions* $a^{ij}$, $b^i$ *are bounded in* $U \times [0, T]$ *and there exist positive numbers* $m$ *and* $M$ *such that*

$$m\mathbb{I}_d \le A(x, t) \le M\mathbb{I}_d, (x, t) \in \Omega \times [0, T]$$

*and there exist positive number* $\lambda$ *such that*

$$|a^{ij}(x, t) - a^{ij}(y, t)| \le \lambda|x - y|, x, y \in \mathbb{R}^d, t \in (0, T)$$

*holds. Then for every probability measure* $\nu$, *there is a solution to the Cauchy problem (2.28) with* $\mu|_{t=0} = \nu$, *where each* $\mu_t$ *is a nonnegative Borel measures on* $\mathbb{R}^d$, *such* *that for almost all $t \in (0,T)$ we have*

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d). \tag{2.33}$$

*Also, $\mu = \rho dxdt$ for some locally integrable function $\rho$. If $J = [T_0, T_1] \subset (0,T)$, $W$ is a neighborhood of $\bar{U} \times J$ with compact closure in $\Omega_T$, then for each $r < \frac{d+2}{d+1}$ one has*

$$\|\rho\|_{L^r(U\times J)} \leq C(d,r,\lambda,m,M,W)\left(\mu(W) + \|b\|_{L^1(W,\mu)}\right) \tag{2.34}$$

*where $C(d,r,\Lambda,m,M,W)$ depends only on $d,r,\lambda,m,M,$ and the distance from $U \times J$ to $\partial W$. In addition, suppose further that $\mu$ satisfies the following: for every ball $U \subset \mathbb{R}^d$*

$$|b| \in L^2(\mu, U \times (0,T)) \tag{2.35}$$

*and*

$$|a^{ij}| + |b^i| \in L^1\left(\mu, \mathbb{R}^d \times (0,T)\right). \tag{2.36}$$

*Then there is no solution to the Cauchy problem (2.28) with $\mu|_{t=0} = \nu$ satisfying (2.33) and (2.35) other than $\mu$. Furthermore, suppose that there is a function $V \in C^{2,1}(\Omega_T) \cap C(\mathbb{R}^d \times [0,T))$ such that for every compact interval $[\alpha,\beta] \subset (0,T)$ we have*

$$\lim_{|x|\to\infty} \min_{t\in[\alpha,\beta]} V(x,t) = +\infty \tag{2.37}$$

*and for some $K, H \in L^1((0,T))$, where $H \geq 0$, and for all $(x,t) \in \Omega_T$*

$$\partial_t V(x,t) + L_{A,b}V(x,t) \leq K(t) + H(t)V(x,t), \tag{2.38}$$

*and also $V(\cdot,0) \in L^1(\nu)$. Then for almost all $t \in (0,T)$ we have $\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d) = 1$.*

Also we have the following result for the square integrability of logarithmic gradients. First we adopt the following convention: for $\rho(x,t) \in W^{1,1}_{loc}$,

$$\frac{\nabla_x \rho(x,t)}{\rho(x,t)} := 0$$

whenever $\rho(x,t) = 0$. Also we recall that a probability measure $\nu$ on $\mathbb{R}^d$ has finite entropy if $\nu = \rho_0 dx$ and

$$\int_{\mathbb{R}^d} |\log \rho_0(x)| \, \rho_0(x) dx < \infty.$$

**Theorem 2.2.16** (Bounds on entropy production)**.** *Suppose that a measure $\mu = (\mu_t)$ is a solution to the Cauchy problem (2.28) with $\mu|_{t=0} = \nu$, each $\mu_t$ is a probability measure, and same condition for $a^{ij}$ as in Theorem 2.2.15 holds, and $|b| \in L^2(\mu, \Omega_T)$. Suppose also that the function $\Lambda(x) = \log \max(|x|, 1)$ belongs to $L^2(\mu, \Omega_T)$. If the initial distribution $\nu = \rho_0 dx$ on $\mathbb{R}^d = \Omega$ has finite entropy, then $\mu_t = \rho(\cdot, t)dx$, where $\rho(\cdot, t) \in W^{1,1}(\mathbb{R}^d)$, moreover, for every $\tau < T$ we have*

$$\int_0^\tau \int_{\mathbb{R}^d} \frac{|\nabla_x \rho(x,t)|^2}{\rho(x,t)} dx dt < \infty. \tag{2.39}$$

*If the integrals $\int_{\mathbb{R}^d} \rho(x,t)\Lambda(x) dx$ remain bounded as $t \to T$, then (2.39) is true with $\tau = T$.*

We also briefly review the proof of Theorem 2.2.16 in section 2.4, to establish the free energy estimate.

### 2.2.2 Function space based on moments

We introduce relevant function spaces and the notion of moment solution. We first define two power series based on moments: for $\mu \in L^1_{loc}(\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2))$ we let

$$\begin{aligned} F[\mu]^e(r) &= \sum_{p=0}^{\infty} \frac{\|\bar{M}_{2p}[\mu]\|_{L^2}}{(2p)!} r^{2p}, \\ F[\mu](r) &= \sum_{p=0}^{\infty} \frac{\|\bar{M}_p[\mu]\|_{L^2}}{p!} r^p. \end{aligned} \tag{2.40}$$

Note that $F[\mu](r)$ is a norm in the space

$$X^r = \{\mu \in L^1_{loc}(\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2)) \,:\, \|\mu\|_{X^r} = F[\mu](r) < \infty\}. \tag{2.41}$$

Then $F[\mu]^e(r)$ is an equivalent norm in $X^r$. Obviously, $F[\mu]^e(r) \leq F[\mu](r)$. On the other hand, by Cauchy-Schwarz inequality,

$$\frac{\|\bar{M}_{2j+1}\|_{L^2}}{(2j+1)!} r^{2j+1} \leq \frac{\|\bar{M}_{2j}\|_{L^2}}{(2j)!} r^{2j} + \frac{\|\bar{M}_{2(j+1)}\|_{L^2}}{(2(j+1))!} r^{2(j+1)} \tag{2.42}$$

and we conclude $F[\mu](r) \leq 3F[\mu]^e(r)$. We also have the following:

**Lemma 2.2.17.** *Suppose that $\{M_{a,b}\}_{a,b}$ is a sequence of functions on $\mathbb{R}^2$ such that there is a sequence of functions $\bar{M}_k$ on $\mathbb{R}^2$ where*

$$\begin{aligned} &|M_{a,b}(x)| \leq \bar{M}_{a+b}(x), \text{ for almost all } x, \\ &\sum_{p=0}^{\infty} \frac{\|\bar{M}_p\|_{L^2}}{p!} r^p < \infty \text{ for some } r > 0. \end{aligned} \tag{2.43}$$

*Then for almost every $x \in \mathbb{R}^2$, the sequence $\{M_{a,b}(x)\}_{(a,b)}$ satisfies the multivariate Carleman's condition (2.8).*

*Proof.* It suffices to show that for almost every $x$,

$$\sum_{p=0}^{\infty} \bar{M}_{2p}(x)^{-\frac{1}{2p}} = \infty. \tag{2.44}$$

By Chebyshev's inequality, we have

$$\left| \left\{ x \,:\, \bar{M}_{2p}(x) > (2p)! \left( \frac{1}{\lambda} \right)^{2(p+1)} \right\} \right| \leq 2 \left( \frac{\|\bar{M}_{2p}\|_{L^2}}{(2p)!} \lambda^{2p} \right)^2 \lambda^4. \tag{2.45}$$

Therefore, we have

$$\left| \left\{ x \,:\, \text{for some } p \geq 0,\ \bar{M}_{2p}(x) > (2p)! \left( \frac{1}{\lambda} \right)^{2(p+1)} \right\} \right| \leq 2 \sum_{p=0}^{\infty} \left( \frac{\|\bar{M}_{2p}\|_{L^2}}{(2p)!} \lambda^{2p} \right)^2 \lambda^4 \tag{2.46}$$

and by taking $\lambda \to 0$ we conclude that for almost every $x$, there exist some $\lambda = \lambda(x) \in (0, r)$ such that

$$\bar{M}_{2p}(x) \leq (2p)! \left( \frac{1}{\lambda} \right)^{2(p+1)} \quad \text{for all } p \geq 0 \tag{2.47}$$

and thus we have

$$\sum_{p=0}^{\infty} \bar{M}_{2p}(x)^{-\frac{1}{2p}} \geq \sum_{p=1}^{\infty} \left( (2p)! \left( \frac{1}{\lambda} \right)^{2(p+1)} \right)^{-\frac{1}{2p}} \geq C\lambda \sum_{p=1}^{\infty} \frac{1}{p} = \infty. \tag{2.48}$$

□

We define

$$X^{k,r} = \{ \mu \in X^r \,:\, \nabla_{m,x}^{\ell} \mu \in X^r \text{ for all } 0 \leq \ell \leq k. \} \tag{2.49}$$

We have the following:

**Lemma 2.2.18.** *$X^{k,r}$ is a Banach space for all $k \geq 0$ with norm $\|\mu\|_{X^{k,r}} = \sum_{|\ell| \leq k} \left\| \nabla_{x,m}^{\ell} \mu \right\|_{X^r}$.*

*Proof.* First note that it suffices to show that $X^r$ is Banach: for a Cauchy sequence $\mu_n$ in $X^{k,r}$ each $\nabla^\ell_{x,m}\mu_n$ is Cauchy in $X^r$, and $\mu_n \to \mu$ in $X^r$ implies $\lim_n \nabla^\ell_{x,m}\mu_n = \nabla^\ell_{x,m}\mu$. Suppose that $\mu_n$ is a Cauchy sequence in $X^r$. Then we know that all $\bar{M}_k[\mu_n]$ is a Cauchy sequence in $L^2(\mathbb{R}^2)$ and so converges to $\bar{M}_k(x) \in L^2(\mathbb{R}^2)$. Furthermore, we see that

$$F[\mu_n](r) = \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_p[\mu_n]\right\|_{L^2}}{p!} r^p \to \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_p\right\|_{L^2}}{p!} r^p \tag{2.50}$$

because $G_n(z) = F[\mu_n](z)$ is a sequence of holomorphic functions in closed $r$-ball which is Cauchy in sup norm:

$$|G_n(z) - G_m(z)| \le \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_p[\mu_n] - \bar{M}_p[\mu_m]\right\|_{L^2}}{p!} z^p \le F[\mu_n - \mu_m](z) \tag{2.51}$$

so $G_n(z)$ converges to some holomorphic function $G(z)$ uniformly in closed $r$-ball. Then we consider the power series representation of $G(z)$ near 0: its coefficients can be represented by Cauchy integral formula and we see

$$\frac{G^{(m)}(0)}{m!} = \frac{1}{2\pi i}\int_{C(0,a)} \frac{G(z)}{z^{m+1}} dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{C(0,a)} \frac{G_n(z)}{z^{m+1}} dz = \lim_{n\to\infty} \left\|\bar{M}_m[\mu_n]\right\|_{L^2}. \tag{2.52}$$

Note that $M_{a,b}[|\mu_n|](x) \le \bar{M}_{a+b}[\mu_n](x)$ and so by dominated convergence we have that

$$\begin{gathered} M_{a,b}[|\mu_n|] \to M^{tv}_{a,b} \text{ in } L^2(\mathbb{R}^2), \\ M_{a,b}[\mu_n^+] \to M^+_{a,b} \text{ in } L^2(\mathbb{R}^2), \\ M_{a,b}[\mu_n^-] \to M^-_{a,b} \text{ in } L^2(\mathbb{R}^2), \\ M^{tv}_{a,b} = M^+_{a,b} + M^-_{a,b},\ \left|M^+_{a,b}\right|, \left|M^-_{a,b}\right|, \left|M^{tv}_{a,b}\right| \le \bar{M}_{a+b}. \end{gathered} \tag{2.53}$$

where $\mu_n^+$ is the positive part (due to Jordan decomposition) of $\mu_n$ and $\mu_n^-$ is the negative part. In particular, the sequences $\{M^+_{a,b}(x)\}_{a,b}$ and $\{M^-_{a,b}(x)\}_{a,b}$ are positive semidefinite sequences for almost every $x$, because they are pointwise limit of positive semidefinite sequences. Furthermore, by Lemma 2.2.17, and Theorem 2.2.4 we see that for almost all $x$, there is a nonnegative measure $\mu^+(x)$ and $\mu^-(x)$ and subsequences $\mu^+_{n_k}, \mu^-_{n_k}$ such that

$$\begin{aligned} M^+_{a,b}(x) &= \int_{\mathbb{R}^2} m_1^a m_2^b \mu^+(x; dm),\ M^-_{a,b}(x) = \int_{\mathbb{R}^2} m_1^a m_2^b \mu^-(x; dm), \\ \lim_{k\to\infty} M_{a,b}[\mu^+_{n_k}](x) &= M_{a,b}[\mu^+](x),\ \lim_{k\to\infty} M_{a,b}[\mu^-_{n_k}](x) = M_{a,b}[\mu^-](x) \text{ a.e.}, \\ \bar{M}_p(x) &= \int_{\mathbb{R}^2} |m|^p \left(\mu^+(x; dm) + \mu^-(x; dm)\right) = \bar{M}_p[\mu^+](x) + \bar{M}_p[\mu^-](x). \end{aligned} \tag{2.54}$$

Furthermore, by putting $\mu(x; dm) = \mu^+(x; dm) - \mu^-(x; dm)$ we see that

$$F[\mu](r) = \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_p[\mu]\right\|_{L^2}}{p!} r^p = \lim_{n\to\infty} F[\mu_n](r) < \infty. \tag{2.55}$$

To show that $\mu_n$ converges to $\mu$ in $X^r$, we evaluate the equivalent norm $F[\mu-\mu_n]^e(r)$: first we know that from Theorem 2.2.8 and its remark, we see that up to subsequence $|m|^{2p}\mu_{m_k} = |m|^{2p}(\mu^+_{m_k} - \mu^-_{m_k})$ converges weakly to $|m|^{2p}\mu$. Therefore, $|m|^{2p}(\mu_n - \mu)$ is a weak limit of $|m|^{2p}(\mu_n - \mu_{m_k})$ for some subsequence $\mu_{m_k}$. Therefore, by Lemma 2.2.9, we have

$$\liminf_k \left(|m|^{2p}|\mu_n - \mu_{m_k}|\right)(\mathbb{R}^2)(x) = \liminf_k \bar{M}_{2p}[\mu_n - \mu_{m_k}](x) \geq \bar{M}_{2p}[\mu_n - \mu](x) \tag{2.56}$$

for almost all $x$. Therefore by Fatou's lemma, we have

$$\begin{aligned} F[\mu_n - \mu]^e(r) &= \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_{2p}[\mu_n - \mu]\right\|_{L^2}}{(2p)!} r^p \\ &\leq \liminf_k \sum_{p=0}^{\infty} \frac{\left\|\bar{M}_{2p}[\mu_n - \mu_{m_k}]\right\|_{L^2}}{(2p)!} r^p \end{aligned} \tag{2.57}$$

which converges to 0 as $n \to \infty$. Therefore, $\mu_n \to \mu$ in $X^r$. □

Also, we consider approximation to identity by Gaussian in the space $X^r$. Let $g_\delta$ be the Gaussian function

$$g_\delta(z) = \frac{1}{2\pi\delta^2} \exp\left(-\frac{|z|^2}{2\delta^2}\right) \tag{2.58}$$

with standard deviation $\delta$. We only have weak convergence, but this is enough for our purpose.

**Lemma 2.2.19.** *Given $\mu_0 \in X^r$ with $\mu_0(x; dm)$ nonnegative measures for all $x$, for almost every $x$ $\mu_0^\delta(x) = g_\delta *_x (g_\delta *_m \mu_0)$ converges to $\mu_0(x)$ weakly. Furthermore,*

$$\begin{gathered} M_{a,b}[\mu_0^\delta] \to M_{a,b}[\mu_0] \text{ in } W^{k,2} \text{ if } M_{a',b'}[\mu_0] \in W^{k,2} \,\text{for all}\, a' + b' \leq a + b \\ \bar{M}_p[\mu_0^\delta] \to \bar{M}_p[\mu_0] \text{ in } L^2(\text{ or } L^p, 1 \leq p < \infty), \\ \left\|\mu_0^\delta\right\|_{X^r} \leq C \left\|\mu_0\right\|_{X^r} \end{gathered} \tag{2.59}$$

*Proof.* We begin with $g_\delta *_m \mu_0$. We first show that $g_\delta *_m \mu_0 \in X^r$. We have the following basic but frequently used estimate for convolution of moments:

$$\begin{aligned} & M_{a,b}[g_\delta *_m \mu_0] \\ = & \sum_{p=0}^{a} \sum_{q=0}^{b} \begin{pmatrix} a \\ p \end{pmatrix} \begin{pmatrix} b \\ q \end{pmatrix} \int (m_1 - n_1)^p (m_2 - n_2)^q g_\delta(m-n) dm n_1^{a-p} n_2^{b-q} \mu_0(dn) \\ & = \sum_{p=0}^{a} \sum_{q=0}^{b} \begin{pmatrix} a \\ p \end{pmatrix} \begin{pmatrix} b \\ q \end{pmatrix} M_{p,q}[g_\delta(m)] M_{a-p,b-q}[\mu_0] \end{aligned} \tag{2.60}$$

and

$$\begin{aligned}\bar{M}_k[g_\delta *_m \mu_0] &\leq \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \int |m-n|^p |n|^{k-p} g_\delta(m-n) dm |\mu_0|(dn) \\ &\leq \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \delta^p 2^{\frac{p}{2}} \Gamma\left(\frac{p+2}{2}\right) \bar{M}_{k-p}[\mu_0].\end{aligned} \tag{2.61}$$

Therefore, we have

$$F[g_\delta *_m \mu_0](r) \leq CF[\mu_0](r) \tag{2.62}$$

where

$$C = \sum_{p=0}^{\infty} \frac{1}{p!} \Gamma\left(\frac{p+2}{2}\right) \left(\delta 2^{\frac{1}{2}} r\right)^p \leq Ce^{C(\delta r)^2}. \tag{2.63}$$

So we have

$$\|g_\delta *_m \mu_0\|_{X^r} \leq Ce^{C(\delta r)^2} \|\mu_0\|_{X^r}. \tag{2.64}$$

Also (2.60) and (2.61) shows that $M_{a,b}[g_\delta *_m \mu_0](x)$ and $\bar{M}_k[g_\delta *_m \mu_0](x)$ are dominated by a $L^2$ function, and $M_{a,b}[g_\delta *_m \mu_0](x)$ converges to $M_{a,b}[\mu_0]$ in $L^2$ and also almost everywhere, and $\bar{M}_p[g_\delta *_m \mu_0]$ converges to $\bar{M}_p[\mu_0]$ in $L^2$ (or other $L^p$, $p < \infty$) and almost everywhere, as $\delta \to 0$. Therefore, by Theorem 2.2.8 we note that for almost $x$ $g_\delta *_m \mu_0(x)$ converges to $\mu_0(x)$ weakly in a subsequence. Also by (2.60) if all $M_{a',b'}[\mu_0] \in W^{k,2}$ for $a' + b' \leq a + b$ then $M_{a,b}[g_\delta *_m \mu_0] \to M_{a,b}[\mu_0]$ in $W^{k,2}$. Since $\mu_0 \geq 0$, we have $M_{a,b}[\mu_0^\delta] = g_\delta *_x M_{a,b}[g_\delta *_m \mu_0]$ and $\bar{M}_k[\mu_0^\delta] = g_\delta *_x \bar{M}_k[g_\delta *_m \mu_0]$. Since convolution with $g_\delta$ is an approximate identity, all the conclusions of the lemma holds. □

Also we can prove the following:

**Lemma 2.2.20.** *Let $\mu \in X^r$ is given by a smooth density $\mu = \mu(x,m)dm$. If $\bar{M}_p[\nabla_x^k \mu] \in L^2$ for some nonnegative integer $p$, then for all $a, b \geq 0$ with $a + b = p$ we* *have* $\nabla_x^k M_{a,b}[\mu] = M_{a,b}[\nabla_x^k \mu] \in L^2(\mathbb{R}^2)$ *and*

$$\left\|\nabla_x^k M_{a,b}[\mu]\right\|_{L^2} \leq \left\|\bar{M}_p[\nabla_x^k \mu]\right\|_{L^2}. \tag{2.65}$$

*Especially, if* $\mu \in X^{k,r}$ *then* $M_{a,b}[\mu] \in W^{k,2}$ *for all* $a, b \geq 0$. *Also, if* $\mu(t) \in C^1([0,T], X^r)$ *is a continuously differentiable family, and* $\mu(t) = \mu(x,m,t)dm$ *is given by the smooth density functions, then* $\partial_t M_{a,b}[\mu](t) = M_{a,b}[\partial_t \mu] \in L^2$.

*Proof.* We prove only the first assertion; the second assertion can be proven in the same way. First, note that $\left|M_{a,b}[\nabla_x^k \mu]\right| \leq \bar{M}_{a+b}[\nabla_x^k \mu]$, so $M_{a,b}[\nabla_x^k \mu] \in L^2$ for $a+b = p$. Then we have

$$\begin{aligned}
&\partial_{x_i} M_{a,b}[\mu](x) - M_{a,b}[\partial_{x_i}\mu](x) \\
&= \lim_{h\to 0} \int_{\mathbb{R}^2_m} m_1^a m_2^b \int_0^1 \partial_{x_i}\mu(x + hse_i, m) ds dm - M_{a,b}[\partial_{x_i}\mu](x) \\
&= \lim_{h\to 0} \int_0^1 \left(M_{a,b}[\partial_{x_i}\mu](x + hse_i) - M_{a,b}[\partial_{x_i}\mu](x)\right) ds
\end{aligned} \tag{2.66}$$

by Taylor expansion and Fubini's theorem. On the other hand, since translation in space is continuous in $L^2(\mathbb{R}^2, dx)$ we have

$$\lim_{h\to 0} \left\| \int_0^1 \left(M_{a,b}[\partial_{x_i}\mu](x + hse_i) - M_{a,b}[\partial_{x_i}\mu](x)\right) ds \right\|_{L^2(\mathbb{R}^2, dx)} = 0 \tag{2.67}$$

and by Fatou we are done. □

**Remark 5.** *We conclude this section with the remark showing that the growth of moments condition is a mild constraint to polymer distributions. We consider the following probability distribution*

$$f(m,x) = \exp\left(-\left(\frac{|m|^2}{c(x)}\right)^q\right),$$

*where* $c(x) > 0$ *is a parameter representing the degree of stretch of polymer at position*

*$x$. For example, when $M_{0,0} = 1$ and $q = 1$, this corresponds to the case $\sigma = 2c(x)\mathbb{I}$. Suppose that $c \in W^{1,2}(\mathbb{R}^2)$. We can show that for some $0 < r < C\|\nabla_x c\|_{L^2}^{-\frac{1}{2}}$, $f \in X^r$. First, by a direct calculation we obtain*

$$\bar{M}_{2r}[f](x) = 2\pi\Gamma\left(\frac{r+1}{q}\right)|c(x)|^{r+1},$$

*and by Gagliardo-Nirenberg inequality ([20]) we have*

$$\|c\|_{L^{2(r+1)}}^{r+1} \leq (r+1)!\,\|\nabla_x c\|_{L^2}^{r+1}\,. \tag{2.68}$$

*Therefore,*

$$\frac{\left\|\bar{M}_{2r}[f]\right\|_{L^2}}{(2r)!}z^{2r} \leq 2\pi\left\|\nabla_x c\right\|_{L^2}\frac{\Gamma\left(\frac{r+1}{q}\right)(r+1)!}{(2r)!}\left(\|\nabla_x c\|_{L^2}^{\frac{1}{2}}z\right)^{2r}, \tag{2.69}$$

*as desired.*

**Remark 6.** *Another example is the following:*

$$f(m,x) = c(x)\frac{1}{Z}e^{-|m|} \tag{2.70}$$

*where $\int_{\mathbb{R}^2_m}\frac{1}{Z}e^{-|m|}dm = 1$ and $c(x) \geq 0 \in L^1 \cap L^2$. Then for each $k$*

$$\bar{M}_k[f](x) = \int_{\mathbb{R}^2_m} c(x)|m|^k\frac{1}{Z}e^{-|m|}dm = c(x)\frac{2\pi}{Z}\int_0^\infty r^{k+1}e^{-r}dr = \frac{2\pi(k+1)!}{Z}c(x) \tag{2.71}$$

*and therefore we have*

$$\|f\|_{X^r} = \sum_{k=0}^{\infty}\frac{2\pi(k+1)\,\|c\|_{L^2}}{Z}r^k < \infty \tag{2.72}$$

*for $0 < r < 1$.*

### 2.2.3 Moment solution and its properties

Here we define the notion of moment solution and investigate its properties.

**Definition 2.2.21** (Moment solution). *Let $\mu = \mu(x,t;dm) \in L^1_{loc}([0,T]\times\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2))$ and $u \in L^\infty(0,T;L^\infty)$ with $\nabla_x u \in L^2(0,T;L^\infty)$ be a given divergence free field. We say $\mu$ is a moment solution of the Fokker-Planck equation with velocity field $u$ if the following holds:*

1. *$\mu$ is a solution to the Cauchy problem*

$$\partial_t\mu = \epsilon\Delta_m\mu + \nu_2\Delta_x\mu - \nabla_x\cdot(u(t)\mu) - \nabla_m\cdot\left(\left(\nabla_x u(t)m - \nabla_m U\right)\mu\right) \tag{2.73}$$

   *with $\mu|_{t=0} = \mu_0$,*

2. *$\mu = \mu(x,t;dm)dxdt$ is nonnegative measures for almost all $x, t$, and for almost all $t \in (0,T)$*

$$\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}\mu(x,t;dm)dx \le \int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}\mu_0(x;dm)dx \tag{2.74}$$

   *and*

$$\int_{\mathbb{R}^2_m}\mathrm{Tr}\left(m\otimes\nabla_m U\right)\mu(x,t;dm) \in L^\infty(0,T;L^1_x) \tag{2.75}$$

   *holds;*

3. *There is a nonincreasing, positive function $r : [0,T] \to \mathbb{R}_+$, $r(0) = r < \infty$ such that*

$$\|\mu(t)\|_{X^{r(t)}} \le C(r,T,\|u\|)\,\|\mu(0)\|_{X^r} \tag{2.76}$$

   *holds; and*

4. *For all $a, b \ge 0$ $M_{a,b}[\mu](x,t) \in L^2(0,T;W^{1,2})$ and $\partial_t M_{a,b}[\mu](x,t) \in L^2(0,T;W^{-1,2})$.*

We see that for all $a, b \geq 0$ we have in fact $M_{a,b} \in C([0,T]; L^2)$. Also we see that moments of moment solutions are weak solutions for formal evolution equation of moments:

**Lemma 2.2.22.** *Let $\mu \in L^1_{loc}([0,T]\times\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2))$ be a moment solution of the Fokker-Planck equation. Then for all $a, b \geq 0$, $M_{a,b}[\mu] = M_{a,b}$ are weak solutions of the evolution equation*

$$\begin{aligned} &\partial_t M_{a,b} + u \cdot \nabla_x M_{a,b} - \nu_2 \Delta_x M_{a,b} \\ &= -2q\epsilon(a+b)M_{a,b}[|m|^{2(q-1)}\mu] + \epsilon\left(a(a-1)M_{a-2,b} + b(b-1)M_{a,b-2}\right) \\ &\quad + a\partial_1 u_1 M_{a,b} + a\partial_1 u_2 M_{a-1,b+1} + b\partial_2 u_1 M_{a+1,b-1} + b\partial_2 u_2 M_{a,b}, \end{aligned} \tag{2.77}$$

*that is, for any $\Phi \in L^2(0,T; W^{1,2})$ with $\Phi(T) = 0$ with $\partial_t \Phi \in L^2(0,T; W^{-1,2})$, we have*

$$\begin{aligned} \int_0^T \langle \partial_t M_{a,b}, \Phi\rangle_{W^{-1,2},W^{1,2}} dt + \int_0^T \langle u \cdot \nabla_x M_{a,b}, \Phi\rangle_{L^2,L^2} + \nu_2 \int_0^T \langle \nabla_x M_{a,b}, \nabla_x \Phi\rangle_{L^2,L^2} \\ = \int_0^T \langle R, \Phi\rangle_{L^2,L^2} \end{aligned} \tag{2.78}$$

*where $R$ is the collection of terms in the right side of (2.77).*

*Proof.* In (2.30), put our test functions in the form of

$$\phi = \phi_1(x,t) m_1^a m_2^b \psi_\alpha(m) \tag{2.79}$$

where $\psi$ is a smooth cutoff and $\psi_\alpha(m) = \psi(\frac{m}{\alpha})$. Then we apply dominated convergence, and then we apply integration by parts to $\partial_t \phi_1 M_{a,b}$ term and $\nu_2 \Delta_x \phi_1 M_{a,b}$ term. Then by density we are done. □

Also moment solution is unique, given initial data.

**Lemma 2.2.23.** *Suppose $\mu_1$ and $\mu_2$ are two moment solutions with same initial data.*

*Then* $\mu_1 = \mu_2$ *in* $L^1_{loc}([0,T]\times\mathbb{R}^2, \mathcal{M}(\mathbb{R}^2))$.

*Proof.* This is a immediate consequence of Theorem 2.2.15. By definition, $\mu$ is a solution to the Cauchy problem of (2.73). Then we have

$$\begin{aligned} u \in L^1(0,T;\mu(x,t;dm)dxdt),\ \nabla_m U \otimes m \in L^1(0,T;\mu(x,t;dm)dxdt), \\ |\nabla_x u(t)m| \le |\nabla_x u(t)|^2 + 1 + C|m|^{2q} \in L^1(0,T;\mu(x,t;dm)dxdt). \end{aligned} \tag{2.80}$$

Condition (2.35) is obvious. □

## 2.3 Solution scheme for Fokker-Planck equation

The purpose of this section is to prove the following theorem.

**Theorem 2.3.1.** *Given a fluid velocity field* $u$ *and initial data* $\mu_0$ *satisfying (2.81), (2.82), (2.83), (2.84), and (2.85), there exists a unique moment solution to the Fokker-Planck equation (2.73). Furthermore, it is given by nonnegative densities* $\mu(x,t;dm) = f(x,t,m)$ *and moments* $M_{a,b} = M_{a,b}[\mu]$ *satisfy bounds (2.133), (2.134), (2.135), (2.136), and (2.137). Furthermore, if the fluid velocity fields* $u$ *and* $v$ *satisfy (2.81) and if we let* $f$ *and* $g$ *be solutions to the Fokker-Planck equation (2.73) with velocity field* $u$ *and* $v$*, respectively, and if we let* $\sigma_1$ *and* $\sigma_2$ *be corresponding stress fields for* $f$ *and* $g$ *respectively, then they satisfy the estimate (2.185).*

### 2.3.1 Approximate solutions

Our goal is to find a moment solution for Fokker-Planck equation, given a fluid velocity field $u$. We establish such solution by setting up an approximation scheme. There are two main modifications in the sequence of approximate solutions: the first is to introduce smooth cutoff to the drift and potential, so that the coefficients remain finite. This modification enables us to employ integration by parts in $m$ variable rigorously, and we can investigate of the bounds on moments. The second is to mollify velocity field and initial data to guarantee higher regularities. Let $\Psi$ be a smooth, decreasing compactly supported function in the closed half-line $\{r \geq 0\}$, $0 \leq \Psi \leq 1$, with $\Psi \equiv 1$ for $r \leq 1$ and $\Psi \equiv 0$ for $r \geq 2$. Then for $\alpha > 0$, we let $\psi_\alpha(m) = \Psi\left(\frac{|m|}{\alpha}\right)$.

**Definition 2.3.2.** *Suppose that*

$$\begin{aligned} u &\in L^\infty(0,T;\mathbb{P}W^{2,2}) \cap L^2(0,T;\mathbb{P}W^{3,2}), \\ \partial_t u &\in L^\infty(0,T;\nabla_x L^1 + L^2) \cap L^2(0,T;\mathbb{P}W^{1,2}), \end{aligned} \tag{2.81}$$

$$\mu_0 \geq 0, \int\int \mu_0(dm)dx = 1 \tag{2.82}$$

$$\mu_0 \in X^r, \tag{2.83}$$

$$M_{a,b}[\mu_0] \in W^{1,2} \text{ for } a+b = 2p \leq 8q-2, \bar{M}_{4q}[\mu_0] \in L^1, \tag{2.84}$$

$$\begin{gathered} \mu_0 = f_0(x,m)dmdx, \\ \int_{\mathbb{R}^2_m \times \mathbb{R}^2_x} f_0 \log f_0 dmdx \in \mathbb{R}, \\ \int_{\mathbb{R}^2_x} |\Lambda(x)|^2 M_{0,0}[f_0](x)dx < \infty, \Lambda(x) = \log\left(\max(|x|,1)\right), \end{gathered} \tag{2.85}$$

*be given. For $\alpha > 0$, a $\alpha$-truncated Fokker-Planck solution of the Cauchy problem of (2.73) with $\mu|_{t=0} = \mu_0$ is a function $f^\alpha \in C^1([0,T]; W^{k,2}_x W^{k,2}_m \cap X^{k,r})$, $k = 20$, satisfying*

$$\begin{gathered} \partial_t f^\alpha + u^\alpha \cdot \nabla_x f^\alpha + \nabla_m \cdot ((\nabla_x u^\alpha) m \psi_\alpha f^\alpha) = \epsilon \Delta_m f^\alpha + \epsilon \nabla_m \cdot ((\nabla_m U)\psi_\alpha f^\alpha) + \nu_2 \Delta_x f^\alpha, \\ f^\alpha(x,m,0) = \mu_0^{\frac{1}{\alpha}}(x,m) =: f_0^\alpha \end{gathered} \tag{2.86}$$

*pointwise where $u^\alpha = g_{\frac{1}{\alpha}} *_x u$.*

We first start with existence and uniqueness of such $\alpha$-truncated Fokker-Planck solution. First note that

$$f_0^\alpha \in W_x^{p,2} W_m^{p,2} \cap X^{p,r} \cap W_x^{p,1} W_{m,1}^{p,1},\ \ M_{0,0}[f_0^\alpha], \bar{M}_{2q}[f_0^\alpha], \bar{M}_{4q}[f_0^\alpha] \in W_x^{p,1} \tag{2.87}$$

for any $p \geq 0$: this is because $\nabla_x^a \nabla_m^b f_0^\alpha = (\nabla_x^a g_{\frac{1}{\alpha}}) *_x (\nabla_m^b g_{\frac{1}{\alpha}}) *_m \mu_0$ so we can apply the same argument in Lemma 2.2.19 to conclude that $\nabla_x^a \nabla_m^b f_0^\alpha \in X^r$, and using Young's inequality for measure

$$\|h *_m \mu_0\|_{L^2(\mathbb{R}^2;dm)} \leq \|h\|_{L^2} \|\mu_0\| \tag{2.88}$$

we see that $\nabla_x^a \nabla_m^b f_0^\alpha \in L_x^2 L_m^2$. Also note that for all $p \geq 0$

$$\begin{aligned} u^\alpha \in L^\infty(0,T;\mathbb{P}W^{p,2}), \partial_t u^\alpha \in L^\infty(0,T;\mathbb{P}W^{p,2}), \\ u^\alpha \to u \,\text{in}\, L^\infty(0,T;\mathbb{P}W^{2,2}) \cap L^2(0,T;\mathbb{P}W^{3,2}). \end{aligned} \tag{2.89}$$

The equation (2.86) has a solution map:

$$\begin{aligned} f^\alpha(t) = e^{t(\epsilon\Delta_m + \nu_2\Delta_x)} f_0^\alpha - \int_0^t \nabla_x (e^{\tau(\epsilon\Delta_m + \nu_2 x)}) \cdot u^\alpha(t-\tau) f^\alpha(t-\tau) d\tau \\ - \int_0^t \nabla_m (e^{\tau(\epsilon\Delta_m + \nu_2 x)}) \cdot (\nabla_x u^\alpha(t-\tau) m\psi_\alpha f^\alpha(t-\tau)) d\tau \\ + \int_0^t \epsilon \nabla_m (e^{\tau(\epsilon\Delta_m + \nu_2 x)}) \cdot (f^\alpha(t-\tau) \nabla_m U \psi_\alpha) d\tau. \end{aligned} \tag{2.90}$$

Then we have

$$\nabla_x^p \nabla_m^q f^\alpha(t) = e^{t(\epsilon\Delta_m+\nu_2 x)} \nabla_x^p \nabla_m^q f_0^\alpha$$
$$- \int_0^t \nabla_x (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \sum_{p'} \begin{pmatrix} p \\ p' \end{pmatrix} \nabla_x^{p'} u^\alpha(t-\tau) \nabla_x^{p-p'} \nabla_m^q f^\alpha(t-\tau) d\tau$$
$$- \int_0^t \nabla_m (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \cdot \sum_{p',q'} \begin{pmatrix} p \\ p' \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} \nabla_x \nabla_x^{p'} u^\alpha(t-\tau) \nabla_m^{q'}(m\psi_\alpha) \nabla_x^{p-p'} \nabla_m^{q-q'} f^\alpha(t-\tau) d\tau$$
$$+ \int_0^t \nabla_m (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \cdot \sum_{q'} \begin{pmatrix} q \\ q' \end{pmatrix} \nabla_m^{q'}(\nabla_m U \psi_\alpha) \nabla_x^p \nabla_m^{q-q'} f^\alpha(t-\tau) d\tau \tag{2.91}$$

and

$$\partial_t \nabla_x^p \nabla_m^q f^\alpha(t) = e^{t(\epsilon\Delta_m+\nu_2 x)}(\epsilon\Delta_m + \nu_2\Delta_x) \nabla_x^p \nabla_m^q f_0^\alpha - e^{t(\epsilon\Delta_m+\nu_2 x)} \nabla_x \cdot \nabla_x^p (u^\alpha(0) \nabla_m^q f_0^\alpha)$$
$$- e^{t(\epsilon\Delta_m+\nu_2 x)} \nabla_x^p (\nabla_x u^\alpha(0) \nabla_m \nabla_m^q (m\psi_\alpha f_0^\alpha)) + \epsilon e^{t(\epsilon\Delta_m+\nu_2 x)} \cdot \nabla_m \nabla_m^q (\nabla_x^p f_0^\alpha \nabla_m U \psi_\alpha)$$
$$- \int_0^t \nabla_x (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \cdot \nabla_x^p (\partial_t u^\alpha \nabla_m^q f^\alpha + u^\alpha \partial_t \nabla_m^q f^\alpha)(t-\tau) d\tau$$
$$- \int_0^t \nabla_m (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \cdot \nabla_x^p (\partial_t \nabla_x u^\alpha(t-\tau) \nabla_m^q (m\psi_\alpha f^\alpha(t-\tau))$$
$$+ \nabla_x u^\alpha(t-\tau) \nabla_m^q (m\psi_\alpha \partial_t f^\alpha(t-\tau))) d\tau$$
$$+ \epsilon \int_0^t \nabla_m (e^{\tau(\epsilon\Delta_m+\nu_2 x)}) \cdot \nabla_m^q (\partial_t \nabla_x^p f^\alpha(t-\tau) \nabla_m U \psi_\alpha) d\tau. \tag{2.92}$$

From this we conclude that the solution map (2.90) is a contraction mapping in the complete metric space

$$\{f \in C^1([0,T], W_x^{k,2} W_m^{k,2} \cap X^{k,r}) \,:\, f(0) = f_0^\alpha\} \tag{2.93}$$

since all the terms in (2.90), (2.91), (2.92) are either of the form

$$e^{t(\epsilon\Delta_m+\nu_2 x)} A \nabla_x^{p'} \nabla_m^{q'} f_0^\alpha \tag{2.94}$$

where $A$ is 1 or $\nabla_x^{r'} u(0)$ and $p'$, $q'$ are derivatives higher than at most 2 degrees to the left hand side term that it occurs, or

$$\int_0^t \nabla_{x,m}(e^{\tau(\epsilon\Delta_m+\nu_2 x)})A\partial_t^{r'}\partial_x^{p'}\partial_m^{q'}f^\alpha(t-\tau)d\tau \tag{2.95}$$

where $A$ is of the form of some constant, $\nabla_m^{l'}(\nabla_m U\psi_\alpha)$, or $\nabla_x^{k'}\partial_t^{i'}u^\alpha(t-\tau)$, and $\partial_t^{r'}\partial_x^{p'}\partial_m^{q'}f^\alpha$ are terms with derivatives lower than or equal to the left hand side term that it occurs. The terms we denoted by $A$ are innocent, because $\|A\|_{L^\infty(0,T;L_x^\infty)} \leq C(\alpha) < \infty$. Therefore, the $W_{x,m}^{k,2} \cap X^{k,r}$ norm of first term can be bounded by $C\,\|f_0^\alpha\|_{W_{x,m}^{k+2,2}\cap X^{k+2,r}}$, which is finite, and the $W_{x,m}^{k,2} \cap X^{k,r}$ of the second term can be bounded by

$$C\int_0^t \frac{1}{\tau^{\frac{1}{2}}}\,\|f^\alpha\|_{C([0,T];W_{x,m}^{k,2}\cap X^{k,r})}\,d\tau = C\tau^{\frac{1}{2}}\,\|f^\alpha\|_{C([0,T];W_{x,m}^{k,2}\cap X^{k,r})}\,. \tag{2.96}$$

Furthermore, the left hand side is continuous in time since each term is either heat semigroup of some function or time integral of $L^1(0,T;W_{x,m}^{k,2}\cap X^{k,r})$ functions. Therefore, by contraction mapping principle, there is unique function $f^\alpha \in C^1([0,T];W_x^{k,2}W_m^{k,2}\cap X^{k,r})$ satisfying (2.90). One consequence is that $f^\alpha$ is a classical solution of (2.86). That is, by Sobolev embedding $f^\alpha \in C^1([0,T];C^2(x,m))$ and satisfies (2.86) pointwise. Therefore, in view of the maximum principle, we have $f^\alpha \geq 0$ for all $(x,m,t)$. Then same argument as above and $f^\alpha \geq 0$ show that $M_{0,0}[f^\alpha], \bar{M}_{2q}[f^\alpha], \bar{M}_{4q}[f^\alpha] \in C^1\left([0,T],W_x^{k,1}\right)$.

### 2.3.2 Uniform bounds on moments

In this section, we investigate bounds on moments for approximate solutions, which is uniform in $\alpha$. By Lemma 2.2.20, we conclude that

$$M_{a,b}^\alpha = M_{a,b}[f^\alpha] \in Lip(0,T;W^{2,2}), \tag{2.97}$$

and we saw $M_{0,0}[f^\alpha], \bar{M}_{2q}[f^\alpha], \bar{M}_{4q}[f^\alpha] \in C^1\left([0,T], W_x^{k,1}\right)$. Also, since $\nabla_m f^\alpha \in X^r$, by integration by parts we wee that

$$\int_{\mathbb{R}^2} m_1^a m_2^b \nabla_m (m\psi_\alpha f^\alpha) dm(x,t) = -\int_{\mathbb{R}^2} \nabla_m (m_1^a m_2^b) m\psi_\alpha f^\alpha dm \in L^\infty(0,T;L^2) \quad (2.98)$$

and similar identity holds for $\epsilon\nabla_m \cdot (f^\alpha \nabla_m U \psi_\alpha)$ term. Therefore, we see that the following equation holds for all $a, b \geq 0$ and almost every $(x,t)$:

$$\begin{aligned} \partial_t M_{a,b}^\alpha + u^\alpha \cdot \nabla_x M_{a,b}^\alpha - \nu_2 \Delta_x M_{a,b}^\alpha + 2q\epsilon(a+b) \int_{\mathbb{R}^2} m_1^a m_2^b |m|^{2(q-1)} \psi_\alpha f^\alpha dm \\ = \epsilon\left(a(a-1)M_{a-2,b}^\alpha + b(b-1)M_{a,b-2}^\alpha\right) \\ + a\partial_1 u_1^\alpha \int_{\mathbb{R}^2} m_1^a m_2^b \psi_\alpha f^\alpha dm + a\partial_1 u_2^\alpha \int_{\mathbb{R}^2} m_1^{a-1} m_2^{b+1} \psi_\alpha f^\alpha dm \\ + b\partial_2 u_1^\alpha \int_{\mathbb{R}^2} m_1^{a+1} m_2^{b-1} \psi_\alpha f^\alpha dm + b\partial_2 u_2^\alpha \int_{\mathbb{R}^2} m_1^a m_2^b \psi_\alpha f^\alpha dm \end{aligned} \quad (2.99)$$

and all the terms are in $L^\infty(0,T;L_x^2)$. Especially, for

$$\bar{M}_{2k}^\alpha = \bar{M}_{2k}[f^\alpha] \quad (2.100)$$

we have the following:

$$\begin{aligned} \partial_t \bar{M}_{2k}^\alpha + u^\alpha \cdot \nabla_x \bar{M}_{2k}^\alpha - \nu_2 \Delta_x \bar{M}_{2k}^\alpha + (2q)\epsilon(2k) \int_{\mathbb{R}^2} |m|^{2(k+q-1)} \psi_\alpha f^\alpha dm \\ = \epsilon(2k)^2 \bar{M}_{2(k-1)}^\alpha + \mathrm{Tr}\left((\nabla_x u^\alpha)(2k) \int_{\mathbb{R}^2} |m|^{2(k-1)} m \otimes m \psi_\alpha f^\alpha dm\right). \end{aligned} \quad (2.101)$$

From (2.99) and (2.101) we derive four estimates independent of $\alpha$: the first one is a set of $L^2$ estimates for all even moments, which gives us an $X^r$ estimate for the limiting object. The second one is a set of $L^\infty(0.T;L^2) \cap L^2(0,T;W^{1,2})$ bounds for all moments. The third one is a set of $L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{2,2})$ estimates for even moments up to degree $2q$, which enables us to establish regularity for the stress field $\sigma$. Finally we obtain a $L^p$ estimate, $1 \leq p \leq 2$ for $\bar{M}_{4q}$ and $M_{0,0}$, which gives us a $L^1$ bound for $\sigma$. Then we use them to bound $\partial_t M^\alpha_{a,b}$ uniformly in $\alpha$, in the space $L^2(0,T;W^{-1,2})$.

To obtain first three bounds, we need to deal with the terms coming from restoring force $\nabla_m \cdot (\nabla_m U \psi_\alpha f^\alpha)$ because it contains higher moments. However, they are harmless in $L^2$ norm due to the following simple observation:

**Lemma 2.3.3.** *Let $\mu_1(dm), \mu_2(dm)$ be nonnegative measures and $p$ be a nonnegative integer. Then*

$$\sum_{a,b\geq 0, a+b=2p} M_{a,b}[\mu_1] M_{a,b}[\mu_2] \geq 0. \tag{2.102}$$

*Proof.* This follows from Cauchy-Schwarz inequality: if $a, b$ are odd, then

$$|M_{a,b}[\mu_1]| \leq \sqrt{M_{a+1,b-1}[\mu_1]}\sqrt{M_{a-1,b+1}[\mu_1]} \tag{2.103}$$

and same for $M_{a,b}[\mu_2]$. Then the left side of the claimed inequality is bounded below by sum of perfect squares

$$\sum_{a'=0}^{p-1} \left(\sqrt{M_{2(a'+1),2(p-a'-1)}[\mu_1] M_{2(a'+1),2(p-a'-1)}[\mu_2]} - \sqrt{M_{2a',2(p-a')}[\mu_1] M_{2a',2(p-a')}[\mu_2]}\right)^2. \tag{2.104}$$

□

$L^2$ **bounds.** By multiplying $\bar{M}^\alpha_{2k}$ to (2.101) and integrating, and applying integration by parts to spatial derivatives for $\nu_2 \Delta_x \bar{M}^\alpha_{2k}$ term (which is rigorous since $\nabla^p_x \bar{M}_{2k} \in L^2$ for $p \leq 2$) and $\bar{M}^\alpha_{2k} \partial_t \bar{M}^\alpha_{2k} = \frac{1}{2}\partial_t \left(\bar{M}^\alpha_{2k}\right)^2$ (which is also rigorous since $\left(\bar{M}^\alpha_{2k}\right)^2 \in C^1([0,T];L^1)$ ), and applying Lemma 2.3.3 as $\mu_1 = |m|^{2(k+q-1)}\psi_\alpha f^\alpha$ and $\mu_2 = |m|^{2k} f^\alpha$ with $p = 0$, and bounding $\psi_\alpha f^\alpha$ by $f^\alpha$ and $m \otimes m \psi_\alpha f^\alpha$ by $|m|^2 f^\alpha$ we have

$$\frac{1}{2}\frac{d}{dt}\left\|\bar{M}_{2k}^{\alpha}\right\|_{L^2}^2+\nu_2\left\|\nabla_X\bar{M}_{2k}^{\alpha}\right\|_{L^2}^2\le\epsilon(2k)^2\left\|\bar{M}_{2(k-1)}^{\alpha}\right\|_{L^2}\left\|\bar{M}_{2k}^{\alpha}\right\|_{L^2}+2k\left\|\nabla_x u(t)\right\|_{L^\infty}\left\|\bar{M}_{2k}^{\alpha}\right\|_{L^2}^2 \tag{2.105}$$

where Young's inequality $\|\nabla_x u^\alpha(t)\|_{L^\infty}\le\|g_\alpha\|_{L^1}\|\nabla_x u(t)\|_{L^\infty}$ is used. Dividing this by $(2k)!\left\|\bar{M}_{2k}^{\alpha}\right\|_{L^2}$, multiplying $z^{2k}$ and summing those up for all $k\ge 0$ we get

$$\frac{d}{dt}\sum_{k=0}^{\infty}\frac{\left\|\bar{M}_{2k}^{\alpha}(t)\right\|_{L^2}}{(2k)!}z^{2k}\le 2\epsilon\sum_{k=1}^{\infty}\frac{\left\|\bar{M}_{2(k-1)}^{\alpha}(t)\right\|_{L^2}}{(2(k-1))!}z^{2k}+2k\left\|\nabla_x u(t)\right\|_{L^\infty}\sum_{k=0}^{\infty}\frac{\left\|\bar{M}_{2k}^{\alpha}(t)\right\|_{L^2}}{(2k)!}z^{2k}. \tag{2.106}$$

Introducing

$$F_e^{\alpha}(t;z)=\sum_{k=0}^{\infty}\frac{\left\|\bar{M}_{2k}^{\alpha}(t)\right\|_{L^2}}{(2k)!}z^{2k} \tag{2.107}$$

we get

$$\frac{d}{dt}F_e^{\alpha}(t;z)\le 2\epsilon z^2F_e^{\alpha}(t;z)+\left\|\nabla_x u(t)\right\|_{L^\infty}z\frac{d}{dz}F_e^{\alpha}(t;z). \tag{2.108}$$

Therefore, we have

$$F_e^{\alpha}(t;z)\le F_e^{\alpha}(0;ze^{\int_0^t\|\nabla_x u(\tau)\|_{L^\infty}d\tau})\exp\left(2\epsilon\int_0^t z^2e^{2\int_s^t\|\nabla_x u(\tau)\|_{L^\infty}d\tau}ds\right), \tag{2.109}$$

in other words,

$$\|f^\alpha(t)\|_{X^{\frac{r}{\int_0^t\|\nabla_x u\|_{L^\infty}d\tau}}}\le e^{2\epsilon Tr^2}\|f_0^\alpha\|_{X^r}\le C(r,T)\|\mu_0\|_{X^r} \tag{2.110}$$

where the last inequality comes from Lemma 2.2.19. We also establish $L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ estimates for all moments. For $a,b\ge 0$ with $a+b=2k\le 2p$, we multiply $M_{a,b}^{\alpha}$ to each of (2.99), sum over all such $a,b$, and integrate in $x$. Again we bound truncated terms $\psi_\alpha$ by 1 and if $m_1^a m_2^b$ by $|m|^{a+b}$. Then we get

$$\frac{1}{2}\frac{d}{dt}\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}^2+\nu_2\left\|\nabla_x\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}^2\leq C\epsilon(2k)^2\left\|\dot{\vec{M}}_{2(k-1)}^{\alpha}\right\|_{L^2}\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}+Ck\left\|\nabla_x u(t)\right\|_{L^2}\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^4}^2 \tag{2.111}$$

where

$$\dot{\vec{M}}_{2k}^{\alpha}=\left(M_{2k,0}^{\alpha},M_{2k-1,1}^{\alpha},\cdots,M_{0,2k}^{\alpha}\right). \tag{2.112}$$

Again we used Lemma 2.3.3 with $\mu_1 = |m|^{2(q-1)}\psi_\alpha f^\alpha$ and $\mu_2 = f^\alpha$. Then by Ladyzhenskaya's inequality,

$$\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^4}^2\leq C\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}\left\|\nabla_x\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}, \tag{2.113}$$

and we have the following by summing over all $k \leq p$:

$$\frac{d}{dt}\sum_{k=0}^{p}\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}^2+\nu_2\sum_{k=0}^{p}\left\|\nabla_x\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}^2\leq C(\epsilon,\nu_2)p^2(\|\nabla_x u(t)\|_{L^2}^2+1)\sum_{k=0}^{p}\left\|\dot{\vec{M}}_{2k}^{\alpha}\right\|_{L^2}^2 \tag{2.114}$$

or by introducing

$$\vec{M}_{2p}^{e,\alpha}=\left(\dot{\vec{M}}_{0}^{\alpha},\dot{\vec{M}}_{2}^{\alpha},\cdots,\dot{\vec{M}}_{2p}^{\alpha}\right) \tag{2.115}$$

we have

$$\frac{d}{dt}\left\|\vec{M}_{2p}^{e,\alpha}\right\|_{L^2}^2+\nu_2\left\|\nabla_x\vec{M}_{2p}^{e,\alpha}\right\|_{L^2}^2\leq C(\epsilon,\nu_2)p^2(\|\nabla_x u(t)\|_{L^2}^2+1)\left\|\vec{M}_{2p}^{e,\alpha}\right\|_{L^2}^2 \tag{2.116}$$

and by Grönwall we have

$$\begin{aligned}&\left\|\vec{M}_{2p}^{e,\alpha}(t)\right\|_{L^2}^2+\nu_2\int_0^t\left\|\nabla_x\vec{M}_{2p}^{e,\alpha}(s)\right\|_{L^2}^2ds\\ &\leq\exp\left(Cp^2\left(\|\nabla_x u\|_{L^\infty(0,T;L^2)}^2T+T\right)\right)C(p)\left\|\vec{M}_{2p}^{e}[\mu_0]\right\|_{L^2}^2.\end{aligned} \tag{2.117}$$

Then using this we can find a $L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ bound for $M_{a,b}$ where $a+b=2p+1$; from (2.99) we bound all terms of the form $\int_{\mathbb{R}^2} m_1^{a'}m_2^{b'}\psi_\alpha f^\alpha dm$

by $C\int_{\mathbb{R}^2}(|m|^{a'+b'-1}+|m|^{a'+b'+1})f^\alpha dm$, that is, we bound truncation $\psi_\alpha$ by 1, and moments with odd degree $m_1^{a'}m_2^{b'}$ by arithmetic mean of neighboring radial moments $|m|^{a'+b'-1}+|m|^{a'+b'+1}$. Then using all the same techniques, we obtain

$$\left\|\vec{M}^\alpha_{2p+1}(t)\right\|^2_{L^2}+\nu_2\int_0^t\left\|\nabla_x\vec{M}^\alpha_{2p+1}(s)\right\|^2_{L^2}ds\le C(p,\epsilon)^{\|\nabla_x u\|^2_{L^\infty(0,T;L^2)}T+T}\left\|\vec{M}_{2(p+1)}[\mu_0]\right\|^2_{L^2} \tag{2.118}$$

where

$$\vec{M}^\alpha_{2p+1}=\left(\dot{\vec{M}}^\alpha_0,\dot{\vec{M}}^\alpha_1,\cdots,\dot{\vec{M}}^\alpha_{2p+1}\right). \tag{2.119}$$

Note that instead of bounding $\int_0^T\|\nabla_x u(t)\|^2_{L^2}dt$ by $\|\nabla_x u\|^2_{L^\infty(0,T;L^2)}T$ we can bound it by $\|\nabla_x u\|^2_{L^2(0,T;L^2)}$ to obtain a similar estimate

$$\begin{aligned}&\left\|\vec{M}^{e,\alpha}_{2p}(t)\right\|^2_{L^2}+\nu_2\int_0^t\left\|\nabla_x\vec{M}^{e,\alpha}_{2p}(s)\right\|^2_{L^2}ds\\&\le\exp\left(Cp^2\left(\|\nabla_x u\|^2_{L^2(0,T;L^2)}+T\right)\right)C(p)\left\|\vec{M}^e_{2p}[\mu_0]\right\|^2_{L^2},\end{aligned} \tag{2.120}$$

which is crucial in global well-posedness, and

$$\left\|\vec{M}^\alpha_{2p+1}(t)\right\|^2_{L^2}+\nu_2\int_0^t\left\|\nabla_x\vec{M}^\alpha_{2p+1}(s)\right\|^2_{L^2}ds\le C(p,\epsilon)^{\|\nabla_x u\|^2_{L^2(0,T;L^2)}+T}\left\|\vec{M}_{2(p+1)}[\mu_0]\right\|^2_{L^2}. \tag{2.121}$$

$W^{1,2}$ **bounds.** Then, we consider the third estimate, $L^\infty(0,T;W^{1,2})\cap L^2(0,T;W^{2,2})$ bounds for even moments of degree up to $2k$, where $k=4q-1$. We can apply same technique for odd moments too, but we only need even moments for the proof of our result. We multiply $-\Delta_x M^\alpha_{a,b}$ to the equation (2.99) and integrate: again integration by parts are rigorous. We use previous pointwise bound for truncated moments, and we get

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}+\nu_2\left\|\Delta_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}\\ &\leq C(\epsilon)k^2\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}+C\left\|\nabla_x u(t)\right\|_{L^2}\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^4}\\ &+C(\nu_2)k^2\left\|\nabla_x u(t)\right\|^2_{L^4}\left\|\vec{M}^{e,\alpha}_{2k}\right\|^2_{L^4}+C(\epsilon,\nu_2)(kq)^2\left\|\vec{M}^{e,\alpha}_{2(k+q-1)}\right\|^2_{L^2}\end{aligned} \tag{2.122}$$

and again by Ladyzhenskaya's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}+\nu_2\left\|\Delta_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}\\ &\leq C(\epsilon,\nu_2)k^2\left(1+\left\|\nabla_x u(t)\right\|^2_{L^2}+\left\|\nabla_x u(t)\right\|_{L^2}\left\|\Delta_x u(t)\right\|_{L^2}\right)\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}\right\|^2_{L^2}\\ &+C(\epsilon,\nu_2)(kq)^2\left\|\vec{M}^{e,\alpha}_{2(k+q-1)}\right\|^2_{L^2},\end{aligned} \tag{2.123}$$

and again by Grönwall we have

$$\begin{aligned}&\left\|\nabla_x \vec{M}^{e,\alpha}_{2k}(t)\right\|^2_{L^2}+\nu_2\int_0^t\left\|\Delta_x \vec{M}^{e,\alpha}_{2k}(s)\right\|^2_{L^2}ds\\ &\leq C(\epsilon,\nu_2,k,q)^{T+\left\|\nabla_x u\right\|^2_{L^\infty(0,T;L^2)}T+\left\|\nabla_x u\right\|_{L^\infty(0,T;L^2)}\left\|\Delta_x u\right\|_{L^2(0,T;L^2)}T^{\frac{1}{2}}}\\ &\left(\left\|\nabla_x \vec{M}^{e}_{2k}[\mu_0]\right\|^2_{L^2}+C(q,\epsilon)^{\left\|\nabla_x u\right\|^2_{L^\infty(0,T;L^2)}T+T}\left\|\vec{M}^{e}_{2(k+q-1)}[\mu_0]\right\|^2_{L^2}\right).\end{aligned} \tag{2.124}$$

$L^1$ **bounds.** In addition, we have $L^1$ bound for $\bar{M}^{\alpha}_{4q}$ and $M^{\alpha}_{0,0}$: first we have

$$\partial_t M^{\alpha}_{0,0}+u^{\alpha}\cdot\nabla M^{\alpha}_{0,0}-\nu_2\Delta_x M^{\alpha}_{0,0}=0 \tag{2.125}$$

and we can integrate them rigorously to conclude

$$\left\|M^{\alpha}_{0,0}(t)\right\|_{L^1}=\left\|M_{0,0}[\mu_0]\right\|_{L^1}. \tag{2.126}$$

Also, we have, by pointwise estimate

$$\int_{\mathbb{R}^2} |m|^{2(3q-1)}\psi_\alpha f^\alpha dm \geq 0,$$
$$\left|\int_{\mathbb{R}^2} |m|^{2(2q-1)} m\otimes m\psi_\alpha f^\alpha dm\right| \leq \bar{M}^\alpha_{4q}, \tag{2.127}$$
$$\bar{M}^\alpha_{2(2q-1)} \leq C(q)(M^\alpha_{0,0} + \bar{M}^\alpha_{4q})$$

and integrating we get

$$\frac{d}{dt}\left\|\bar{M}^\alpha_{4q}\right\|_{L^1} \leq C(q,\epsilon)(\|\nabla_x u(t)\|_{L^\infty}+1)\left\|\bar{M}^\alpha_{4q}\right\|_{L^1} + C(q,\epsilon)\left\|M^\alpha_{0,0}\right\|_{L^1}, \tag{2.128}$$

and here by Agmon's inequality

$$\|\nabla_x u(t)\|_{L^\infty} \leq \|\nabla_x u(t)\|^{\frac{1}{2}}_{L^2}\|\Delta_x\nabla_x u(t)\|^{\frac{1}{2}}_{L^2} \tag{2.129}$$

and by Grönwall we have

$$\left\|\bar{M}^\alpha_{4q}(t)\right\|_{L^1} \leq C(q,\epsilon)^{\|\nabla_x u\|_{L^2(0,T;W^{2,2})}T^{\frac{1}{2}}+T}(\left\|\bar{M}_{4q}[\mu_0]\right\|_{L^1} + C(q,\epsilon)\|M_{0,0}[\mu_0]\|_{L^1}T), \tag{2.130}$$

and from this we can say that $\bar{M}^\alpha_{4q}$ (and also $\bar{M}_{2q}$ by the above pointwise estimate) is bounded in $L^\infty(0,T;L^p)$ where $1\leq p\leq 2$ uniformly in $\alpha$ due to interpolation, bounds depend only on initial data.

$W^{-1,2}$ **bounds for $\partial_t M_{a,b}$s.** Finally, due to (2.99), we notice that $\partial_t M^\alpha_{a,b}$ is uniformly bounded in $L^2(0,T;W^{-1,2})$; since $u^\alpha\in L^\infty(0,T;L^\infty)$ and $\nabla_x u^\alpha\in L^2(0,T;L^\infty)$ are uniformly bounded and all $M^\alpha_{a,b}\in L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ are uniformly bounded, terms involving $u^\alpha$ are uniformly bounded in $L^2(0,T;L^2)$. Other terms except for $\Delta_x M^\alpha_{a,b}$ are uniformly bounded in $L^\infty(0,T;L^2)$, and $\Delta_x M^\alpha_{a,b}$ is uniformly bounded in $L^2(0,T;W^{-1,2})$.

**Weak limit of moments.** Since

$$L^\infty(0,T;L^2) = \left(L^1(0,T;L^2)\right)^*, \; L^2(0,T;L^2) = \left(L^2(0,T;L^2)\right)^*,$$
$$L^\infty(0,T;L^q) = \left(L^1(0,T;L^{q'})\right)^*, 1 < q < 2, \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.131}$$
$$L^2(0,T;W^{-1,2}) = \left(L^2(0,T;W^{1,2})\right)^*,$$

by Theorem 2.2.10, and since we have bounds (2.110), (2.117), (2.118), (2.120), (2.121), (2.124), (2.130) (and $L^\infty(0,T;L^p), 1 < p < 2$ bounds due to interpolation), by Banach-Alaoglu there is a weak* limit $M_{a,b}$,

$$M^\alpha_{a,b} \to M_{a,b} \tag{2.132}$$

in the weak-* topology of $L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2})$ with the bounds

$$\sum_{p=0}^{\infty} \frac{\left\|\bar{M}_{2p}(t)\right\|_{L^2}}{(2p)!} \left(\frac{r}{\exp\left(\int_0^t \|\nabla_x u(s)\|_{L^\infty} ds\right)}\right)^{2p} \le C(r,T)\|\mu_0\|_{X^r}, \tag{2.133}$$

$$\left\|\vec{M}_k\right\|^2_{L^\infty(0,T;L^2)} + \nu_2 \left\|\vec{M}_k\right\|^2_{L^2(0,T;W^{1,2})} \le C(k)^{\|\nabla_x u\|^2_{L^\infty(0,T;L^2)}T+T} \left\|\vec{M}_{k+(k\bmod 2)}[\mu_0]\right\|^2_{L^2}, \tag{2.134}$$

$$\left\|\vec{M}_k\right\|^2_{L^\infty(0,T;L^2)} + \nu_2 \left\|\vec{M}_k\right\|^2_{L^2(0,T;W^{1,2})} \le C(k)^{T+\|\nabla_x u\|^2_{L^2(0,T;L^2)}} \left\|\vec{M}_{k+(k\bmod 2)}[\mu_0]\right\|^2_{L^2}, \tag{2.135}$$

$$\begin{aligned} &\left\|\vec{M}^e_{8q-2}\right\|^2_{L^\infty(0,T;W^{1,2})} + \nu_2 \left\|\vec{M}^e_{8q-2}\right\|^2_{L^2(0,T;W^{2,2})} \\ &\le C(\epsilon,\nu_2,q)^{T+\|u\|^2_{L^\infty(0,T;W^{1,2})}T+\|u\|_{L^\infty(0,T;W^{1,2})}\|u\|_{L^2(0,T;W^{2,2})}T^{\frac{1}{2}}} \\ &\times \left(\left\|\vec{M}^e_{8q-2}[\mu_0]\right\|^2_{W^{1,2}} + \left\|\vec{M}_{16q-6}[\mu_0]\right\|^2_{L^2}\right), \end{aligned} \tag{2.136}$$

$$\|M_{0,0}(t)\|_{L^1} = \|M_{0,0}[\mu_0]\|_{L^1},$$
$$\left\|\bar{M}_{4q}\right\|_{L^\infty(0,T;L^1)} \le C(q,\epsilon)^{\|\nabla_x u\|_{L^2(0,T;W^{2,2})}T^{\frac{1}{2}}+T}(\left\|\bar{M}_{4q}[\mu_0]\right\|_{L^1} + C(q,\epsilon)T\|M_{0,0}[\mu_0]\|_{L^1}) \tag{2.137}$$

where the last bound in (2.137) is due to bounds on $L^\infty(0,T;L^p)$, $1 < p \leq 2$, and the fact that $p \to \|f\|_{L^p}$ is continuous. Furthermore, $\partial_t M_{a,b} \in L^2(0,T;W^{-1,2})$ with bounds depending only on the initial data, due to weak* convergence. Also we have $\|M_{0,0}(t)\|_{L^1} = \|M_{0,0}[\mu_0]\|_{L^1}$ instead of $\leq$ sign by the last assertion of Theorem 2.2.15: take $V = |m|^2 + \log\max(|x|,1)$, where $\log\max(|x|,1)$ should be understood, by a slight abuse of notation, a smooth, bounded function equals it for $|x| > 2$. Then $K(t) = C + \|u(t)\|_{L^\infty}$, $H(t) = C\|\nabla_x u(t)\|_{L^\infty}$ works. We remark that (2.134) and (2.135) look similar, but in the estimate (2.135) requires only a bound on $\|\nabla_x u\|_{L^2(0,T;L^2)}$, and this fact will be used in proving global well-posedness of the coupled system.

### 2.3.3 Existence of moment solution

In this subsection, we prove the existence of moment solution using the limits $\{M_{a,b}\}_{a,b}$. There are two points to remark: first, since the convergence of $M^\alpha_{a,b}$ to $M_{a,b}$ is weak and not pointwise a priori, so we need Aubin-Lions compactness lemma to make the convergence locally pointwise. Second, since the Fokker-Planck equation we consider is fully parabolic, in fact we can rely on parabolic theory to find limit density function. First we establish positive semidefiniteness for $\{M_{a,b}\}_{a,b}$. For all $\alpha > 0$, the sequence $\{M^\alpha_{a,b}\}_{(a,b)}$ are positive semidefinite, since they are moments of nonnegative measures. Therefore,

$$\int_0^T \int_{\mathbb{R}^2} \sum_{i,j} c_i c_j M^\alpha_{a_i+a_j,b_i+b_j}(x,t)\phi(x,t)\,dxdt \geq 0 \tag{2.138}$$

for all nonnegative test functions $\phi \in L^1(0,T;L^2)$: then by the weak* limit

$$\int_0^T \int_{\mathbb{R}^2} \sum_{i,j} c_i c_j M_{a_i+a_j,b_i+b_j}(x,t)\phi(x,t)\,dxdt \geq 0 \tag{2.139}$$

and that means, $\{M_{a,b}\}_{(a,b)}$ is also positive semidefinite. Similarly,

$$\int_0^T \int_{\mathbb{R}^2} \phi(x,t) \left(\bar{M}^{\alpha}_{a+b} \pm M^{\alpha}_{a,b}\right) dxdt \geq 0 \tag{2.140}$$

so for almost all $(x,t)$ $|M_{a,b}(x,t)| \leq \bar{M}_{a+b}(x,t)$. Then, from (2.133) and Lemma 2.2.17 we see that for almost all $(x,t)$ there is a nonnegative measure $\mu = \mu(x,t;dm)$ such that $M_{a,b}(x,t) = M_{a,b}[\mu](x,t)$ for all $a,b \geq 0$. It remains to show that actually $\mu$ is a weak solution to the Fokker-Planck equation: first we show that for $\phi \in C_0^\infty([0,T] \times \mathbb{R}^2_x \times \mathbb{R}^2_m)$ with $\phi(T,x,m) = 0$ we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2_x} \int_{\mathbb{R}^2_m} & \left(\partial_t \phi + u(t) \cdot \nabla_x \phi + \left((\nabla_x u(t)) - \nabla_m U\right) m \cdot \nabla_m \phi + \epsilon \Delta_m \phi + \nu_2 \Delta_x \phi\right) \\ & \mu(x,t;dm) dxdt = -\int_{\mathbb{R}^2_x} \int_{\mathbb{R}^2_m} \phi(0,x,m) \mu_0(x;dm) dx. \end{aligned} \tag{2.141}$$

Suppose that supp $\phi \subseteq [R_1,R_2] \times B(0,R)_x \times B(0,R)_m$, which is a compact rectangle. Let $\eta$ be a $C_0^\infty([0,T] \times \mathbb{R}^2_x)$ function, $0 \leq \eta \leq 1$, $\eta = 1$ in $[R_1,R_2] \times B(0,R)_x$ and $\eta = 0$ outside $[R_1 - 1, R_2 + 1] \times B(0,2R)_x$. Then for any $a,b \geq 0$, we have

$$\begin{aligned} \eta M^{\alpha}_{a,b} \in L^2(0,T;W^{1,2}_0(\Omega)) &\to \eta M_{a,b} \text{ weak}* \text{ in } L^2(0,T;W^{1,2}_0(\Omega)) \\ \partial_t(\eta M^{\alpha}_{a,b}) \in L^2(0,T;W^{-1,2}(\Omega)) &\to \partial_t(\eta M_{a,b}) \text{ weak}* \text{ in } L^2(0,T;W^{-1,2}(\Omega)) \end{aligned} \tag{2.142}$$

where $\Omega = [R_1 - 1, R_2 + 1] \times B(0,2R)_x$. By Rellich-Kondrachov theorem $W^{1,2}_0(\Omega) \subseteq L^2(\Omega)$ is compact and $L^2(\Omega) \subseteq W^{-1,2}(\Omega)$ is continuous. Therefore, by Aubin-Lions lemma we see that there is a subsequence $\eta M^{\beta}_{a,b}$ which converges to $\eta M_{a,b}$ in $L^2(0,T;L^2(\Omega))$. By a standard diagonalization method, there is a subsequence $\eta M^{\gamma}_{a,b}$ such that all moments $\eta M^{\gamma}_{a,b}$ converges to $\eta M_{a,b}$ in the topology of $L^2(0,T;L^2(\Omega))$. Therefore, there is a subsequence, again denoted by $\eta M^{\alpha}_{a,b}$, converges to $\eta M_{a,b}$ almost everywhere, for all moments $a,b \geq 0$. Especially, $M^{\alpha}_{a,b}(x,t) \to M_{a,b}(x,t)$ for almost all $(x,t) \in [R_1,R_2] \times B(0,R)_x$. Therefore, by Theorem 2.2.8 we see that $\mu^{\alpha}(x,t;dm)$

converges weakly to $\mu(x,t;dm)$. Note that $f^\alpha$ satisfies

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}&(\partial_t\phi+u^\alpha(t)\cdot\nabla_x\phi+((\nabla_x u^\alpha(t))m-\nabla_m U)\,\psi_\alpha\cdot\nabla_m\phi\\ &+\epsilon\Delta_m\phi+\nu_2\Delta_x\phi)f^\alpha(x,t;dm)dxdt=-\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}\phi(0,x,m)f_0^\alpha(x;dm)dx.\end{aligned}\tag{2.143}$$

If $\alpha>R$ then $m\psi_\alpha\cdot\nabla_m\phi=m\nabla_m\phi$. Also for almost every $x,t$

$$\begin{aligned}&\int_{\mathbb{R}^2_m}(\partial_t\phi-\nabla_m U\cdot\nabla_m\phi+\epsilon\Delta_m\phi+\nu_2\Delta_x\phi)\,f^\alpha(x,t;dm)\\ &\to\int_{\mathbb{R}^2_m}(\partial_t\phi-\nabla_m U\cdot\nabla_m\phi+\epsilon\Delta_m\phi+\nu_2\Delta_x\phi)\,\mu(x,t;dm)\end{aligned}\tag{2.144}$$

by weak convergence. Furthermore, the left term is bounded by $C_\phi(x,t)\eta M_{0,0}[f^\alpha]$, where $\eta M_{0,0}[f^\alpha]\to\eta M_{0,0}[\mu]\in L^2(0,T;L^2(\Omega))$ and $C_\phi(x,t)\in L^2(0,T;L^2(\Omega))$ so we can apply generalized dominated convergence theorem to conclude that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}(\partial_t\phi-\nabla_m U\cdot\nabla_m\phi+\epsilon\Delta_m\phi+\nu_2\Delta_x\phi)\,f^\alpha(x,t;dm)\\ &\to\int_0^T\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m}(\partial_t\phi-\nabla_m U\cdot\nabla_m\phi+\epsilon\Delta_m\phi+\nu_2\Delta_x\phi)\,\mu(x,t;dm).\end{aligned}\tag{2.145}$$

Finally, for the term $u^\alpha(t)\cdot\nabla_x\phi+\nabla_x u^\alpha(f)m\cdot\nabla_m\phi$, we note that since $C_0^\infty([0,T]\times\mathbb{R}^2_x\times\mathbb{R}^2_m)=C_0^\infty([0,T]\times\mathbb{R}^2_x)\otimes C_0^\infty(\mathbb{R}^2_m)$ we only need to consider functions of the form $\phi(x,m,t)=\phi_1(x,t)\phi_2(m)$. Then the integral involving $u^\alpha(t)\cdot\nabla_x\phi$ becomes

$$\int_0^T\int_{\mathbb{R}^2_x}u^\alpha(t)\cdot\nabla_x\phi_1\int_{\mathbb{R}^2_m}\phi_2(m)f^\alpha(x,t;dm)dxdt\tag{2.146}$$

and we note that $u^\alpha(t)\cdot\nabla_x\phi_1\to u(t)\cdot\nabla_x\phi_1$ in $L^2(0,T;L^2(\Omega))$ and

$$\int_{\mathbb{R}^2_m}\phi_2(m)f^\alpha(x,t;dm)\to\int_{\mathbb{R}^2_m}\phi_2(m)\mu(x,t;dm)\tag{2.147}$$

in $L^2(0,T;L^2(\Omega))$ as before. We can deal with the term $\nabla_x u^\alpha(t)m \cdot \nabla_m \phi$ in the same way. Finally, by Lemma 2.2.19 we see that $\int_{\mathbb{R}^2_m} \phi(0,x,m) f_0^\alpha(x;dm)$ converges to $\int_{\mathbb{R}^2_m} \phi(0,x,m)\mu(x;dm)$ almost every $x$, and they are bounded by $C_\phi M_{0,0}[f_0^\alpha]$ which converges to $C_\phi M_{0,0}[\mu_0]$ in $L^2$ (but since $\phi(0)$ is compactly supported it converges in $L^1$ too) so by generalized dominated convergence

$$-\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m} \phi(0,x,m) f_0^\alpha(x;dm)dx \to -\int_{\mathbb{R}^2_x}\int_{\mathbb{R}^2_m} \phi(0,x,m)\mu_0(x;dm)dx. \tag{2.148}$$

Similarly, for $\phi \in C_0^\infty(\mathbb{R}^2_x \times \mathbb{R}^2_m)$ we see that

$$\int \phi f^\alpha(t) dmdx - \int \phi f_0^\alpha dmdx = A_\alpha(t) \tag{2.149}$$

and

$$A_\alpha(t) \to \int_0^t \int_{m,x} \left(u\cdot\nabla_x\phi + (\nabla_x u m - \nabla_m U)\cdot\nabla_m\phi + \epsilon\Delta_m\phi + \nu_2\Delta_x\phi\right)\mu(x,\tau;dm)dxd\tau, \tag{2.150}$$

where

$$A_\alpha(t) = \int_0^t \int_{m,x} \left(u^\alpha\cdot\nabla_x\phi + (\nabla_x u^\alpha m - \nabla_m U)\psi_\alpha\cdot\nabla_m\phi + \epsilon\Delta_m\phi + \nu_2\Delta_x\phi\right) f^\alpha dmdxd\tau. \tag{2.151}$$

Note that by

$$\begin{gathered}
\|u^\alpha\|_{L^1(\mathbb{R}^2_m\times\mathbb{R}^2_x\times[0,T];f^\alpha dmdxdt)} \le \|u\|_{L^\infty(0,T;L^\infty)} \|M_{0,0}[\mu_0]\|_{L^1},\\
\|\nabla_x u^\alpha m\psi_\alpha\|_{L^1(\mathbb{R}^2_m\times\mathbb{R}^2_x\times[0,T];f^\alpha dmdxdt)} \le \|\nabla_x u\|_{L^\infty(0,T;L^2)} C(T,u) \left\|\vec{M}_2[\mu_0]\right\|_{L^2},\\
\|\nabla_m U m\psi_\alpha\|_{L^1(\mathbb{R}^2_m\times\mathbb{R}^2_x\times[0,T];f^\alpha dmdxdt)} \le C(T,u)(\left\|\bar{M}_{2q}[\mu_0]\right\|_{L^1} + C\|M_{0,0}[\mu_0]\|_{L^1}),
\end{gathered} \tag{2.152}$$

we see that $|A_\alpha(t)| \le C(\phi)|t|$, where $C(\phi)$ depends only on $\phi$ and independent of $\alpha$. Furthermore, again $\int \phi f^\alpha(x,t,m)dm$ is pointwise bounded by $C_\phi M_{0,0}[f^\alpha]$, and note that in a ball $V \in \mathbb{R}^2_x$ containing the support of $\int \phi f^\alpha(x,t,m)dm$ and a smooth cutoff $\eta$ which is 1 in $\bar{V}$, with support contained in another ball $W$, $M_{0,0}[\mu^\alpha]\eta \in L^\infty(0,T;W^{1,2}(W))$ with $\partial_t M_{0,0}\eta \in L^2(0,T;W^{-1,2}(W))$: and $W^{1,2}(W) \subset L^2(W)$ is again compact. Therefore again by Aubin-Lions, we see that for a subsequence $M_{0,0}[f^\alpha]\eta \to M_{0,0}[\mu]\eta$ strongly in $L^\infty(0,T;L^2)$. Therefore, we conclude, by generalized dominated convergence, for almost every $t \in [0,T]$ $\int \phi f^\alpha(t)dmdx \to \int \phi\mu(t;dm)dx$, and we know that $\int \phi f_0^\alpha dmdx \to \int \phi\mu_0(dm)dx$. Therefore, we proved (2.32). Then we prove that in fact $\mu$ can be represented as a density function $f(x,t,m)$. Here we use the same argument to prove Theorem 2.2.15, used in [18]. Let $U_k = B(0,k)_x \times B(0,k)_m \subset \mathbb{R}^2_x \times \mathbb{R}^2_m$, $J_k = [\frac{T}{k}, T\left(1-\frac{1}{k}\right)]$, and $W_k$ be a neighborhood of $\bar{U}_k \times J_k$ with compact closure in $\mathbb{R}^2_x \times \mathbb{R}^2_m \times (0,T)$, for each $k > 2$. We then consider the subsequence of $f^\alpha$ that converging to $\mu(x,t;dm)$, what we used before. Since we have by Theorem 2.2.15

$$\|f^\alpha\|_{L^{\frac{7}{6}}(U_k \times J_k)} \leq C(W_k, T, u, \mu_0) \tag{2.153}$$

for each $k > 2$. Then by Banach-Alaoglu and standard diagonalization technique, we can find a subsequence of $f^\alpha$ which converges weakly to a function $f(x,m,t)$ in $L^{\frac{7}{6}}(U_k \times J_k)$ for all $k > 2$. Furthermore, $f(x,m,t)dm = \mu(x,t;dm)$ for almost every $(x,t)$.

### 2.3.4 Dependence on fluid velocity fields

In this subsection, we prove the last assertion of Theorem 2.3.1. Suppose that $u, v$ satisfies (2.81) and $f, g$ be solutions of two microscopic equations with velocity field $u$ and $v$ respectively and same initial data $\mu_0$ satisfying conditions (2.82), (2.83), (2.84), (2.85). Also $f^\alpha$ and $g^\alpha$ is defined same as before. Then we have

$$
\begin{aligned}
&\partial_t(f^\alpha - g^\alpha) + u^\alpha \cdot \nabla_x (f^\alpha - g^\alpha) + (\nabla_x u^\alpha) m\psi_\alpha \cdot \nabla_m (f^\alpha - g^\alpha) \\
&-\nabla_m \cdot (\nabla_m U \psi_\alpha (f^\alpha - g^\alpha)) - \epsilon \Delta_m (f^\alpha - g^\alpha) - \nu_2 \Delta_x (f^\alpha - g^\alpha) \\
&\quad = -(u^\alpha - v^\alpha) \cdot \nabla_x g^\alpha - \nabla_x (u^\alpha - v^\alpha) m\psi_\alpha \cdot \nabla_m g^\alpha
\end{aligned}
\tag{2.154}
$$

in the classical sense. Let $sgn_\beta$ be a smooth, increasing regularization of sign function where $sgn_\beta(s) = sign(s)$ for $|s| \geq \beta$, and $|s|_\beta = \int_0^s sgn_\beta(r)dr$. By multiplying $|m|^{2k} sgn_\beta(f^\alpha - g^\alpha)$, where $k \leq 2q - 1$, to (2.154) we have

$$
\begin{aligned}
&\partial_t \left(|m|^{2k}|f^\alpha - g^\alpha|_\beta\right) + u^\alpha \cdot \nabla_x \left(|m|^{2k}|f^\alpha - g^\alpha|_\beta\right) + \nabla_x u^\alpha m\psi_\alpha |m|^{2k} \cdot \nabla_m |f^\alpha - g^\alpha|_\beta \\
&-\nabla_m \cdot (\nabla_m U \psi_\alpha)|m|^{2k}(f^\alpha - g^\alpha) sgn_\beta(f^\alpha - g^\alpha) - \nu_2 |m|^{2k} sgn_\beta(f^\alpha - g^\alpha)\Delta_x(f^\alpha - g^\alpha) \\
&-\nabla_m U\psi_\alpha \cdot \nabla_m |f^\alpha - g^\alpha|_\beta |m|^{2k} - \epsilon |m|^{2k} sgn_\beta(f^\alpha - g^\alpha)\Delta_m(f^\alpha - g^\alpha) \\
&= -(u^\alpha - v^\alpha) \cdot \nabla_x g^\alpha |m|^{2k} sgn_\beta(f^\alpha - g^\alpha) - \nabla_x(u^\alpha - v^\alpha) m\psi_\alpha \cdot \nabla_m g^\alpha |m|^{2k} sgn_\beta(f^\alpha - g^\alpha).
\end{aligned}
\tag{2.155}
$$

Integrating in $m$ variable, we have

$$
\begin{aligned}
&(\partial_t + u^\alpha \cdot \nabla_x) \int |m|^{2k}|f^\alpha - g^\alpha|_\beta dm - (\nabla_x u^\alpha) : \int \nabla_m (m\psi_\alpha |m|^{2k}) |f^\alpha - g^\alpha|_\beta dm \\
&= I_1 + I_2 + I_3 + I_4 - \int (u^\alpha - v^\alpha) \cdot \nabla_x g^\alpha |m|^{2k} sgn_\beta(f^\alpha - g^\alpha) dm \\
&\quad - \int \nabla_x (u^\alpha - v^\alpha) m\psi_\alpha \cdot \nabla_m g^\alpha |m|^{2k} sgn_\beta(f^\alpha - g^\alpha) dm
\end{aligned}
\tag{2.156}
$$

where

$$
\begin{aligned}
I_1 &= \int \nabla_m \cdot (\nabla_m U \psi_\alpha)|m|^{2k}(f^\alpha - g^\alpha) sgn_\beta(f^\alpha - g^\alpha) dm, \\
I_2 &= -\int \nabla_m \cdot (\nabla_m U \psi_\alpha |m|^{2k}) |f^\alpha - g^\alpha|_\beta dm, \\
I_3 &= \epsilon \int |m|^{2k} sgn_\beta(f^\alpha - g^\alpha) \Delta_m (f^\alpha - g^\alpha) dm, \\
I_4 &= \nu_2 \int |m|^{2k} sgn_\beta(f^\alpha - g^\alpha) \Delta_x (f^\alpha - g^\alpha) dm.
\end{aligned}
\tag{2.157}
$$

Note that

$$\left|\int \nabla_m(m\psi_\alpha|m|^{2k})|f^\alpha - g^\alpha|_\beta dm\right| \\ \leq C\frac{1}{\alpha}\int |m|^{2k+1}|f^\alpha - g^\alpha|dm + Ck\int |m|^{2k}|f^\alpha - g^\alpha|_\beta dm, \tag{2.158}$$

and

$$I_1 + I_2 = \int \nabla_m \cdot (\nabla_m U\psi_\alpha)|m|^{2k}\left((f^\alpha - g^\alpha)sgn_\beta(f^\alpha - g^\alpha) - |f^\alpha - g^\alpha|_\beta\right)dm \\ -2k\int |m|^{2(k-1)}m \cdot \nabla_m U\psi_\alpha|f^\alpha - g^\alpha|_\beta dm \tag{2.159}$$

and the first term, denoted by $J_{\alpha,\beta}$, is bounded pointwise by

$$C\left(\frac{1}{\alpha}\bar{M}_{2(k+q)-1}[f^\alpha + g^\alpha] + \bar{M}_{2(k+q-1)}[f^\alpha + g^\alpha]\right)$$

and pointwisely converges to 0 as $\beta \to 0$. On the other hand, the second term is nonpositive. Thus $I_1 + I_2 \leq J_{\alpha,\beta}$. On the other hand,

$$I_3 = -\epsilon\int \nabla_m\left(|m|^{2k}sgn_\beta(f^\alpha - g^\alpha)\right)\cdot\nabla_m(f^\alpha - g^\alpha)dm \\ = -\epsilon\int 2k|m|^{2(k-1)}m\cdot\nabla_m|f^\alpha - g^\alpha|_\beta dm - \epsilon\int |m|^{2k}sgn'_\beta(f^\alpha - g^\alpha)|\nabla_m(f^\alpha - g^\alpha)|^2 dm \\ \leq 2k\epsilon\int \nabla_m\cdot(|m|^{2(k-1)}m)|f^\alpha - g^\alpha|_\beta dm, \tag{2.160}$$

and finally

$$I_4 = \nu_2\int \nabla_x\cdot\left(|m|^{2k}sgn_\beta(f^\alpha - g^\alpha)\nabla_x(f^\alpha - g^\alpha)\right)dm \\ -\nu_2\int |m|^{2k}sgn'_\beta(f^\alpha - g^\alpha)|\nabla_x(f^\alpha - g^\alpha)|^2 dm \\ \leq \nu_2\nabla_x\cdot\left(\int\left(\nabla_x(|m|^{2k}|f^\alpha - g^\alpha|_\beta)\right)dm\right). \tag{2.161}$$

Therefore, we have

$$
\begin{aligned}
&(\partial_t + u^\alpha \cdot \nabla_x) \left( \int |m|^{2k} |f^\alpha - g^\alpha|_\beta dm \right) \\
&\le C \|\nabla_x u(t)\|_{L^\infty_x} \left( k \int |m|^{2k} |f^\alpha - g^\alpha|_\beta dm + \frac{1}{\alpha} \int |m|^{2k+1} |f^\alpha - g^\alpha|_\beta dm \right) \\
&+ J_{\alpha,\beta} + Ck^2\epsilon \int |m|^{2(k-1)} |f^\alpha - g^\alpha|_\beta dm + \nu_2 \nabla_x \cdot \int \nabla_x (|m|^{2k} |f^\alpha - g^\alpha|_\beta) dm \\
&+ \|(u-v)(t)\|_{L^\infty_x} \int |m|^{2k} |\nabla_x g^\alpha| dm + \|\nabla_x (u-v)(t)\|_{L^\infty_x} \int |m|^{2k+1} |\nabla_m g^\alpha| dm.
\end{aligned} \tag{2.162}
$$

Then we multiply $\int |m|^{2k} |f^\alpha - g^\alpha|_\beta dm$, and integrate in $x$. Finally, we divide the both sides by $\left\| \int |m|^{2k} |f^\alpha - g^\alpha|_\beta(t) dm \right\|_{L^2_x}$ to obtain

$$
\begin{aligned}
&\frac{d}{dt} \left\| \int |m|^{2k} |f^\alpha - g^\alpha|_\beta(t) dm \right\|_{L^2_x} \\
&\le C(\|\nabla_x u(t)\|_{L^\infty_x} + 1) \left\| \int |m|^{2k} |f^\alpha - g^\alpha|_\beta(t) dm \right\|_{L^2_x} + \|J_{\alpha,\beta}\|_{L^2_x} \\
&\left\| \int |f^\alpha - g^\alpha|_\beta dm \right\|_{L^2_x} + \frac{C \|\nabla_x u(t)\|_{L^\infty_x}}{\alpha} \left( \left\| \bar{M}_{2k+1}[f^\alpha] \right\|_{L^2_x} + \left\| \bar{M}_{2k+1}[g^\alpha] \right\|_{L^2_x} \right) \\
&+ \|(u-v)(t)\|_{L^\infty_x} \left( \int \bar{M}_{4k}[g^\alpha] \left( \int \frac{|\nabla_x g^\alpha|^2}{g^\alpha} dm \right) dx \right)^{\frac{1}{2}} \\
&+ \|\nabla_x (u-v)(t)\|_{L^\infty_x} \left( \int \bar{M}_{4k+2}[g^\alpha] \left( \int \frac{|\nabla_m g^\alpha|^2}{g^\alpha} dm \right) dx \right)^{\frac{1}{2}}
\end{aligned} \tag{2.163}
$$

Since $f^\alpha(0) = g^\alpha(0)$, by Grönwall we have

$$
\left\| \int |m|^{2k} |f^\alpha - g^\alpha|_\beta(t) dm \right\|_{L^2_x} \le \exp(C(\|\nabla_x u\|_{L^1(0,T;L^\infty_x)} + 1))(I_1 + I_2 + I_3 + I_4 + I_5), \tag{2.164}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \|J_{\alpha,\beta}\|_{L^2} dx, \\
I_2 &= \frac{C}{\alpha} \|\nabla_x u\|_{L^2(0,T;L_x^\infty)} \left( \left\|\bar{M}_{2k+1}[f^\alpha]\right\|_{L^2(0,T;L_x^2)} + \left\|\bar{M}_{2k+1}[g^\alpha]\right\|_{L^2(0,T;L_x^2)} \right), \\
I_3 &= \left\| \int |f^\alpha - g^\alpha|_\beta dm \right\|_{L^1(0,T;L_x^2)}, \\
I_4 &= \int_0^T \|u - v(t)\|_{L_x^\infty} \left\|\bar{M}_{4k}[g^\alpha](t)\right\|_{L_x^\infty}^{\frac{1}{2}} \left( \int\int \frac{|\nabla_x g^\alpha|^2}{g^\alpha} dmdx \right)^{\frac{1}{2}} dt, \\
I_5 &= \int_0^T \|\nabla_x (u-v)(t)\|_{L_x^\infty} \left\|\bar{M}_{4k+2}[g^\alpha](t)\right\|_{L_x^\infty}^{\frac{1}{2}} \left( \int\int \frac{|\nabla_m g^\alpha|^2}{g^\alpha} dmdx \right)^{\frac{1}{2}} dt
\end{aligned}
\tag{2.165}
$$

and by (2.134) we have that $\left\|\bar{M}_{2k+1}[f^\alpha]\right\|_{L_t^\infty L_x^2} + \left\|\bar{M}_{2k+1}[g^\alpha]\right\|_{L_t^\infty L_x^2} \leq C$ where $C$ depends only on initial data $\mu_0$ and $\nabla_x u, \nabla_x v$, independent of $\alpha$. Also, by (2.136), and by Agmon's inequality, we have $\bar{M}_{4k}[g^\alpha], \bar{M}_{4k+2}[g^\alpha] \in L^2(0,T;L^\infty)$ with bounds depending only on initial data $\mu_0$ and velocity field $v$, again independent of $\alpha$. Also, $g^\alpha$ satisfies the conditions of Theorem 2.2.16 :

$$
\begin{aligned}
\int\int |v(x,t)|^2 g^\alpha(x,m,t) dmdx &\leq \|v\|_{L^\infty(0,T;L_x^\infty)}^2 \|M_{0,0}[g^\alpha]\|_{L^\infty(0,T;L^1)}, \\
\int_0^T \int\int |\nabla_x v(x,t) m|^2 g^\alpha(x,m,t) dmdx &\leq \|\nabla_x v\|_{L^2(0,T;L_x^\infty)}^2 \left\|\bar{M}_2[g^\alpha]\right\|_{L^\infty(0,T;L_x^1)}, \\
\int\int |\nabla_m U|^2 g^\alpha(x,m,t) dmdx &\leq C(q) \left\|\bar{M}_{4q-2}[g^\alpha]\right\|_{L^\infty(0,T;L^1)}
\end{aligned}
\tag{2.166}
$$

and since

$$
\log\left(\max(\sqrt{|x|^2+|m|^2}, 1)\right) \leq \frac{\log 2}{2} + \log\left(\max(|x|,1)\right) + \log\left(\max(|m|,1)\right),
$$

$\log\left(\max(|m|,1)\right)^2 \leq C(1+|m|^2)$ and so

$$
\int\int \log\left(\max(|m|,1)\right)^2 g^\alpha(t) dmdx \leq C\left( \|M_{0,0}[g^\alpha]\|_{L^\infty(0,T;L_x^1)} + \left\|\bar{M}_2[g^\alpha]\right\|_{L^\infty(0,T;L_x^1)} \right),
$$

which is bounded by a constant depending only on $u$ and $\mu_0$, and not in $\alpha$, it suffices to bound

$$\int\int \log\left(\max(|x|,1)\right)^2 g^\alpha dm dx = \int M_{0,0}[g^\alpha] \log\left(\max(|x|,1)\right)^2 dx.$$

Let $\Psi(x)$ be a smooth, nonnegative function in $x$ such that $\Psi \geq \log\left(\max(|x|,1)\right)^2$, $\Psi = \log\left(\max(|x|,1)\right)^2$ for $|x| \geq 2$. Since $M_{0,0}[g^\alpha]$ satisfies

$$\partial_t M_{0,0}[g^\alpha] + v^\alpha \cdot \nabla_x M_{0,0}[g^\alpha] = \nu_2 \Delta_x M_{0,0}[g^\alpha] \tag{2.167}$$

it can be easily seen that

$$\begin{aligned} &\int\int \log\left(\max(|x|,1)\right)^2 M_{0,0}[g^\alpha](t) dx \\ &\leq C\left(1 + \|v\|_{L^\infty(0,T;L^\infty_x)} T + \int \log\left(\max(|x|,1)\right)^2 M_{0,0}[\mu_0^\alpha] dx\right) \end{aligned} \tag{2.168}$$

but note that $M_{0,0}[\mu_0^\alpha](x) = (g_{\alpha^{-1}} *_x M_{0,0}[\mu_0])(x)$, and we have a following simple inequality

$$\log\left(\max(|x+y|,1)\right)^2 \leq 4 + 2\log\left(\max(|x|,1)\right)^2 + 2\log\left(\max(|y|,1)\right)^2 \tag{2.169}$$

so we have

$$\begin{aligned} \int \Lambda(x)^2 g_{\alpha^{-1}} *_x M_{0,0}[\mu_0](x) dx \leq 4\left\|M_{0,0}[\mu_0]\right\|_{L^1} + 2\int \Lambda^2 M_{0,0}[\mu_0] dx \\ +2\left\|M_{0,0}[\mu_0]\right\|_{L^1} \int g_{\alpha^{-1}}(x)\Lambda(x)^2 dx. \end{aligned} \tag{2.170}$$

However, note that

$$\int g_{\alpha^{-1}}(x)\Lambda(x)^2 dx = \int_{|x|\geq 1} g_{\alpha^{-1}}(x)\left(\log|x|\right)^2 dx \tag{2.171}$$

and if $|x| \geq 1$ and $\alpha \geq 4$, $g_{\alpha^{-1}}(x) \leq g_4(x)$ so again we can find a bound for $\int\int \log\left(\max(|x|,1)\right)^2 M_{0,0}[g^\alpha](t)dx$ which depends only in uniform data and $v$, is independent of $\alpha$ (for large enough $\alpha$), and is uniform in $[0,T]$. Also note that our initial condition implies that $\int\int \mu_0^\alpha \left|\log \mu_0^\alpha\right| dmdx < \infty$. Then by the bound obtained in the proof of Theorem 2.2.16, we conclude that

$$\begin{aligned}
&\int_0^T \int\int \frac{|\nabla_x g^\alpha|^2 + |\nabla_m g^\alpha|^2}{g^\alpha} dmdxdt \\
&\leq C(T \|v\|^2_{L^\infty(0,T;L^\infty)} \|M_{0,0}[\mu_0]\|_{L^1} + (\|\nabla_x v\|_{L^2(0,T;L^\infty)} + T) \left\|\bar{M}_2[g^\alpha]\right\|_{L^\infty(0,T;L^1)} \\
&\quad + \left\|\bar{M}_{4q-2}[g^\alpha]\right\|_{L^\infty(0,T;L^1)} T + \|v\|_{L^\infty(0,T;L^\infty)} T + 1 + \int \Lambda^2 M_{0,0}[\mu_0]dx) \\
&\quad + \int\int \mu_0^\alpha \log \mu_0^\alpha dmdx.
\end{aligned} \tag{2.172}$$

But note that by Jensen's inequality applied to $\Phi(s) = s\log s$, we have, for each $(x,m)$,

$$\begin{aligned}
&\mu_0^\alpha \log \mu_0^\alpha(x,m) \\
&= \int\int \mu_0(x-y,m-n) g_{\alpha^{-1}}(y,n) \log\left(\int \mu_0(x-y',m-n') g_{\alpha^{-1}}(y',n')\right) dndy \\
&= \Phi\left(\mathbb{E}_{g_{\alpha^{-1}}}[\mu_0(x-\cdot,m-\cdot)]\right) \leq \mathbb{E}_{g_{\alpha^{-1}}}\left[\Phi(\mu_0(x-\cdot,m-\cdot))\right] \\
&= \int\int g_{\alpha^{-1}}(y,n)\Phi(\mu_0(x-y,m-n))dndy
\end{aligned} \tag{2.173}$$

and therefore

$$\int\int \mu_0^\alpha \log \mu_0^\alpha dxdm \leq \int\int \mu_0 \log \mu_0 dxdm. \tag{2.174}$$

Therefore, by Hölder's inequality we can bound the last two terms in the (2.164) by

$$\|u-v\|_{L^4(0,T;W^{1,\infty})} C(\|v\|, \|\mu_0\|, T),$$

where $C$ is given by

$$C = \left\|\bar{M}_{4k}[g^\alpha] + \bar{M}_{4k+2}[g^\alpha]\right\|^{\frac{1}{2}}_{L^2(0,T;L^\infty_x)} \left(\int_0^T \int\int \frac{|\nabla_x g^\alpha|^2 + |\nabla_m g^\alpha|^2}{g^\alpha} dmdxdt\right)^{\frac{1}{2}} \tag{2.175}$$

where $C(\|v\|, \|\mu_0\|, T)$ depends only on those three (except for coefficients like $\nu_2, \epsilon$), is increasing in each of the variables, and does not blow up for finite $\|v\|, \|\mu_0\|$, or $T$. The term

$$\left\|\int |f^\alpha - g^\alpha|_\beta dm\right\|_{L^1(0,T;L^2)} \tag{2.176}$$

can be bounded in the same way, just plugging in $k = 0$ to (2.164) and removing the term $\left\|\int |f^\alpha - g^\alpha|_\beta dm\right\|_{L^1(0,T;L^2)}$ in the right side, and since we have $|m|^{2k}|f^\alpha - g^\alpha|,\ |m|^{2k}|f^\alpha - g^\alpha|_\beta \leq |m|^{2k}(f^\alpha + g^\alpha)$ by taking $\beta \to 0$ to apply dominated convergence and taking $\alpha \to \infty$ we have

$$\left\|\int |m|^{2k}|f(t) - g(t)|dm\right\|_{L^2_x} \leq C(\|u\|, \|v\|, \|\mu_0\|, T)\, \|u - v\|_{L^4(0,T;W^{1,\infty})} \tag{2.177}$$

where again $C(\|u\|, \|v\|, \|\mu_0\|, T)$ depends only on those four (except for coefficients like $\nu_2, \epsilon$), is increasing in each of the variables, and does not blow up for finite $\|u\|, \|v\|, \|\mu_0\|$, or $T$. Here $\|u\| = \|u\|_{L^\infty(0,T;W^{2,2})\cap L^2(0,T;W^{3,2})}$ and similar for $\|v\|$, and $\|\mu_0\|$ is a bound for (2.84) and (2.85). Let

$$\sigma_1 = \int \nabla_m U \otimes mf dm,\ \sigma_2 = \int \nabla_m U \otimes mg dm. \tag{2.178}$$

Then in the weak sense as in Lemma 2.2.22, we have

$$\partial_t(\sigma_1 - \sigma_2) + u \cdot \nabla_x(\sigma_1 - \sigma_2) - \nu_2 \Delta_x(\sigma_1 - \sigma_2) = I_1 + I_2 \tag{2.179}$$

where

$$\begin{aligned} I_1 &= -(u-v)\cdot\nabla_x\sigma_2 \\ &+4q(q-1)\int |m|^{2(q-2)}\left((\nabla_x u-\nabla_x v):m\otimes m\right)m\otimes m g dm \\ &+2q\int |m|^{2(q-1)}((\nabla_x u-\nabla_x v)m\otimes m+m\otimes(\nabla_x u-\nabla_x v)m)g dm \end{aligned} \tag{2.180}$$

and

$$\begin{aligned} I_2 &= -(2q)^3\epsilon\int |m|^{4(q-1)}m\otimes m(f-g)dm \\ &+4q(q-1)\int |m|^{2(q-2)}((\nabla_x u):m\otimes m)m\otimes m(f-g)dm \\ &+2q\int |m|^{2(q-1)}((\nabla_x u)m\otimes m+m\otimes(\nabla_x u)m)(f-g)dm \\ &+2q\epsilon\left(4q(q-1)\int |m|^{2(q-2)}m\otimes m(f-g)dm+4\int |m|^{2(q-1)}\mathbb{I}(f-g)dm\right). \end{aligned} \tag{2.181}$$

Then we see that

$$\|I_1(t)\|_{L^2}+\|I_2(t)\|_{L^2}\le C(\|\mu_0\|,\|u\|,\|v\|,T)\,\|u-v\|_{L^4(0,T;W^{1,\infty})}. \tag{2.182}$$

Therefore, by multiplying $\sigma_1-\sigma_2$ and integrating in $x$ variables, and using $\sigma_1(0)=\sigma_2(0)$ we have

$$\sup_{0\le t\le T}\|\sigma_1(t)-\sigma_2(t)\|_{L^2}^2+\nu_2\int_0^T\|\nabla_x(\sigma_1-\sigma_2)\|_{L^2}^2\,dt\le CT\,\|u-v\|_{L^4(0,T;W^{1,\infty})}^2. \tag{2.183}$$

Also, multiplying $-\Delta_x(\sigma_1-\sigma_2)$ and integrating in $x$ variable we get

$$\sup_{0\le t\le T}\|\nabla_x(\sigma_1(t)-\sigma_2(t))\|_{L^2}^2+\nu_2\int_0^T\|\Delta_x(\sigma_1-\sigma_2)\|_{L^2}^2\,dt\le CT\,\|u-v\|_{L^4(0,T;W^{1,\infty})}^2. \tag{2.184}$$

In conclusion, we have

$$\begin{aligned}
&\|\sigma_1 - \sigma_2\|_{L^\infty(0,T;W^{1,2})\cap L^2(0,T;W^{2,2})} \\
&\leq C(\nu_2, \|u\|, \|v\|, \|\mu_0\|, T)\sqrt{T}\, \|u - v\|_{L^\infty(0,T;\mathbb{P}W^{2,2})\cap L^2(0,T;\mathbb{P}W^{3,2})}
\end{aligned} \tag{2.185}$$

again $C(\nu_2, \|u\|, \|v\|, \|\mu_0\|, T)$ has the same property as before, and $C \to \infty$ as $\nu_2 \to 0$.

**Remark 7.** *If we assume the initial data $\mu_0$ for $f$, and the initial data $\nu_0$ for $g$ do not coincide, then previous arguments give the following modification of (2.185):*

$$\begin{aligned}
&\|\sigma_1 - \sigma_2\|_{L^\infty(0,T;W^{1,2})\cap L^2(0,T;W^{2,2})} \\
&\leq C\, \|\sigma_1(0) - \sigma_2(0)\|_{W^{1,2}} \\
&+C\sqrt{T}\left(\|u - v\|_{L^\infty(0,T;\mathbb{P}W^{2,2})\cap L^2(0,T;\mathbb{P}W^{3,2})} + \left\|\bar{M}_0[\mu_0 - \nu_0] + \bar{M}_{4q-2}[\mu_0 - \nu_0]\right\|_{L^2}\right).
\end{aligned} \tag{2.186}$$

*For any $k \geq 0$, the term $\bar{M}_{2k}[\mu_0 - \nu_0]$ cannot be controlled by $\bar{M}_{2k}[\mu_0] - \bar{M}_{2k}[\nu_0]$. However, this term is unavoidable; it is possible that $\bar{M}_{2k}[\mu_0] = \bar{M}_{2k}[\nu_0]$ while $\mu_0 \neq \nu_0$.*

Therefore, we have proved Theorem 2.3.1.

**Remark 8.** *As mentioned before, the condition (2.83) can be dropped in proving local and global well-posedness of the coupled system: we can only assume (2.81), (2.82), (2.84), and that $\left\|\vec{M}_{16q}[\mu_0]\right\|_{L^2_x} < \infty$ to show that there exists a unique weak solution to the Fokker-Planck equation (2.73), satisfying all the conditions fo the definition for the moment solution except for third one, and satisfying bounds (2.134), (2.135), (2.136), and (2.137). Also, note that (2.85) is used only for the estimate (2.185), which is used in proving local existence of the coupled system.*

**Remark 9.** *In the condition (2.85), the condition $\int_{\mathbb{R}^2} |\Lambda(x)|^2 M_{0,0}[f_0](x)dx < \infty$, which controls the growth of $f_0$ at infinity, is introduced in many kinetic models, for example, Boltzmann equation ([45]). Although the physical interpretation of the above* *condition is not evident, that condition guarantees us that the entropy* $\int f \log f dx$ *remains greater than* $-\infty$. *Here is an example showing that if we do not have such restriction, our solution starts with finite entropy but fall into* $-\infty$ *entropy after some time. Suppose that we are solving 1-dimensional heat equation* $\partial_t f = \partial_x^2 f$ *in the whole line, and let the initial data be*

$$f_0(x) = \sum_{n=1}^{\infty} 1_{(10n-a_n,10n+a_n)}(x) \tag{2.187}$$

*where*

$$a_n = \frac{c}{(n+1)\left(\log(n+1)\right)^2} \tag{2.188}$$

*where* $c$ *is chosen that* $\sum_{n=1}^{\infty} a_n \leq 1$ *and* $a_n < \frac{1}{2}$ *for all* $n$. *Let*

$$\Phi(s) = s\log s, \;\; g_r(x) = \frac{1}{\sqrt{4\pi r}} e^{-\frac{x^2}{4r}}. \tag{2.189}$$

*Then* $\int_{\mathbb{R}} \Phi(f_0)dx = 0$, *since* $\Phi(f_0)(x) = 0$ *for all* $x$. *Then*

$$f(x,t) = \sum_{n=1}^{\infty} g_t * 1_{(10n-a_n,10n+a_n)}(x) \tag{2.190}$$

*and we see that* $\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^1} \|g_t\|_{L^\infty} < \frac{1}{4\sqrt{\pi t}} \leq \frac{1}{2}$ *for all* $t > 1$ *and* $f(x,t) \geq 0$ *for all* $(x,t)$. *For* $t = 1$, *if* $|x - 10n| < t$, *we see that*

$$\frac{1}{2} \geq f(x,t) \geq g_t * 1_{(10n-a_n,10n+a_n)}(x) \geq \frac{a_n}{e\sqrt{\pi}} = \frac{a_n}{\sqrt{\pi t}} e^{-t}. \tag{2.191}$$

*Then since* $\Phi(s)$ *is decreasing for* $0 \leq s \leq \frac{1}{2}$, $\Phi(f(x,t)) \leq \Phi(\frac{a_n}{e\sqrt{\pi}}) = \frac{a_n}{e\sqrt{\pi}} \log a_n - a_n \log(e\sqrt{\pi})$. *Then*

$$\int_{\mathbb{R}} \Phi(f(x,t))dx \leq \sum_{n=1}^{\infty} \int_{(10n-t,10n+t)} \Phi(f(x,t))dx \leq 2\sum_{n=1}^{\infty} a_n \log a_n - 2C = -\infty. \tag{2.192}$$

*Therefore, although $f_0$ started with zero entropy, $f(t)$ has $-\infty$ entropy at $t = 1$. Same argument shows that $f(t)$ has $-\infty$ entropy for all $t > 1$.*

## 2.4 Local and global well-posedness of the coupled system

### 2.4.1 Local well-posedness

Using the results in section 2.3, we can prove the local existence of the system. We define the function space $\mathcal{X}$ as

$$\mathcal{X} = L^\infty(0,T;\mathbb{P}W^{2,2}) \cap L^2(0,T;\mathbb{P}W^{3,2}). \tag{2.193}$$

For the subspace of $\mathcal{X}$ defined by

$$\tilde{\mathcal{X}} = \{u \in \mathcal{X} \,:\, \partial_t u \in L^\infty(0,T;\nabla_x L^1 + L^2) \cap L^2(0,T;\mathbb{P}W^{1,2})\} \tag{2.194}$$

by Theorem 2.3.1 we know that there exists a unique moment solution to the Fokker-Planck equation (2.73), denoted by $\mu$. Then we define

$$\sigma[u] = \int_{\mathbb{R}^2} m \otimes \nabla_m U \mu(dm). \tag{2.195}$$

We set up a fixed point equation $u = F(u)$ in $\tilde{\mathcal{X}}$. We establish a contraction mapping in $\mathcal{X}$ and observe that if $u \in \tilde{\mathcal{X}}$ then $F(u) \in \tilde{\mathcal{X}}$ too. Following [33], our $F$ is defined as

$$F(u) = e^{\nu_1 t \Delta_x} u_0 + Q_1(u,u) + L_1(\sigma) \tag{2.196}$$

where

$$Q_1(u,v) = -\int_0^t e^{\nu_1 (t-s)\Delta_x} \mathbb{P}(u(s) \cdot \nabla_x v(s)) ds \tag{2.197}$$

and

$$L_1(\sigma) = K \int_0^t e^{\nu_!(t-s)\Delta_x} \mathbb{P}\left(\nabla_x \cdot \sigma(s)\right) ds. \tag{2.198}$$

We check that

$$\begin{aligned} \|Q_1(u,v)\|_{\mathcal{X}} &\leq \delta \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}, \\ \|L_1(\sigma)\|_{\mathcal{X}} &\leq C_1 \|\sigma\|_{L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{2,2})}, \\ \|\sigma\|_{L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{2,2})} &\leq C_2 C_3^{\delta \|u\|_{\mathcal{X}}^2}, \end{aligned} \tag{2.199}$$

where $\delta$ can be made as small as we want by making $T$ small.The first and second one can be found in [33], and the third one is a direct consequence of (2.134). Using (2.199) we can find $A$ and $\delta$ (so we adjust $T$ too) such that if $\|u\|_{\mathcal{X}} \leq A$, then $\|F(u)\|_{\mathcal{X}} \leq A$. If $\|u\|_{\mathcal{X}} \leq A$, then we have

$$\|F(u)\|_{\mathcal{X}} \leq A_0 + \delta A^2 + C_1 C_2 C_3^{\delta A^2}, \tag{2.200}$$

where $A_0$ depends only on initial data and $C_1, C_2, C_3$ are independent of $A$. For example, we can put $A = A_0 + 1 + C_1 C_2 C_3$ and choose $\delta$ small enough so that $\delta A^2 < 1$. Also, by (2.185) we have

$$\|\sigma[u] - \sigma[v]\|_{L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{2,2})} \leq C_4 \delta \|u - v\|_{\mathcal{X}} \tag{2.201}$$

where $C_4 = C_4(A, A_0)$. Then

$$\begin{aligned} &\|F(u) - F(v)\|_{\mathcal{X}} \\ &\leq \|Q_1(u, u-v)\|_{\mathcal{X}} + \|Q_1(u-v, v)\|_{\mathcal{X}} + \|L_1(\sigma[u] - \sigma[v])\|_{\mathcal{X}} \\ &\leq \delta(2A + C_1 C_4) \|u - v\|_{\mathcal{X}}. \end{aligned} \tag{2.202}$$

Therefore, by choosing $\delta$ small enough again, we see that the sequence $u^{n+1} = F(u^n)$, $u^1$ be the solution of Navier-Stokes equation with initial data $u_0$ converges exponentially to the unique fixed point. Therefore, we have proved the following.

**Theorem 2.4.1.** *Given $u_0 \in \mathbb{P}W^{2,2}$, $\mu_0$ satisfying (2.82), (2.83), (2.84), and (2.85), there is a $T_0 > 0$ such that there is a unique solution $(u, f)$ to (2.1) for $t \in (0, T_0)$ satisfying (2.81) and $f$ is the unique moment solution of the Fokker-Planck equation with velocity field $u$.*

## 2.4.2 Global well-posedness

From this point, we investigate the global existence: we need to establish the bound

$$\begin{aligned} &\frac{1}{2}\|u\|^2_{L^\infty(0,T;L^2)} + \sup_{0\le t\le T}\frac{K}{2q(2q-1)}\|\sigma(t)\|_{L^1} + \nu_1\|\nabla_x u\|^2_{L^2(0,T;L^2)} \\ &\le A(\epsilon,q)\|M_{0,0}\|[\mu_0]T + \frac{1}{2}\|u_0\|_{L^2} + \frac{K}{2q(2q-1)}\|\sigma_0\|_{L^1} = B_1(T). \end{aligned} \tag{2.203}$$

Here $B_1(T)$ depends only on initial data and $T$. For this we come back to our approximating sequence $f^\alpha$: by multiplying $u$ to the first equation of (2.1) and adding $C = \frac{K}{2q(2q-1)}$ times of (2.101), and using the pointwise estimate $|m|^{2(q-1)} \le A + |m|^{4q-2}$ then integrating we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u(t)\|^2_{L^2} + C\int \bar{M}^\alpha_{2q}(t)dx\right) + \nu_1\|\nabla_x u(t)\|^2_{L^2} \le CA\|M_{0,0}[\mu_0]\|_{L^1} \\ + \int\int |m|^{4q-2}(1-\psi_\alpha)f^\alpha dmdx \\ + \int \mathrm{Tr}\left(\nabla_x u^\alpha \int |m|^{2(q-1)} m\otimes m\psi_\alpha f^\alpha dm - \nabla_x u\int |m|^{2(q-1)} m\otimes mfdm\right)dx. \end{aligned} \tag{2.204}$$

Then we have

$$\begin{aligned} &\|u\|^2_{L^\infty(0,T;L^2)} + C\left\|\bar{M}^\alpha_{2q}\right\|_{L^\infty(0,T;L^1)} + 2\nu_1\|\nabla_x u\|^2_{L^2(0,T;L^2)} \\ &\le \|u_0\|^2_{L^2} + C\left\|\bar{M}^\alpha_{2q}(0)\right\|_{L^1} + AT\|M_{0,0}[\mu_0]\|_{L^1} \\ &+ I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{2.205}$$

where

$$\begin{aligned} I_1 &= \int_0^T \int\int |m|^{4q-2}(1-\psi_\alpha)f^\alpha dm dx dt, \\ I_2 &= \int_0^T \int \mathrm{Tr}\left( (\nabla_x u^\alpha - \nabla_x u) \int |m|^{2(q-1)} m \otimes m \psi_\alpha f^\alpha dm \right) dx dt, \\ I_3 &= \int_0^T \int \mathrm{Tr}\left( \nabla_x u \int |m|^{2(q-1)} m \otimes m (\psi_\alpha - 1) f^\alpha dm \right) dx dt, \\ I_4 &= \int_0^T \int \mathrm{Tr}\left( \nabla_x u \left( \int |m|^{2(q-1)} m \otimes m f^\alpha dm - \int |m|^{2(q-1)} m \otimes m f dm \right)\right) dx dt. \end{aligned} \tag{2.206}$$

First we note that $\lim_{\alpha\to\infty} \left\|\bar{M}_{2q}^\alpha\right\|_{L^\infty(0,T;L^1)} \geq \left\|\bar{M}_{2q}[f]\right\|_{L^\infty(0,T;L^1)}$. Then we note that for $k < 2q$

$$\int |m|^{2k}(1-\psi_\alpha)f^\alpha dm \leq \int_{|m|\geq\alpha} |m|^{2k}\left(\frac{|m|}{\alpha}\right)^{4q-2k} f^\alpha dm \leq \frac{1}{\alpha}\int |m|^{4q} f^\alpha dm. \tag{2.207}$$

Then we also note that $\int |m|^{4q} f^\alpha dm$ is uniformly bounded, say by $C$, in $L^\infty(0,T;L^1_x)$ by (2.130). Therefore, we have $\lim_\alpha I_1 = \lim_\alpha I_3 = 0$. Then we note that $M_{a,b}^\alpha$ converges to $M_{a,b}[f]$ in weak* topology of $L^2(0,T;L^2)$. Since $\nabla_x u \in L^2(0,T;L^2)$, we see that $I_4 \to 0$ as $\alpha\to\infty$. Finally, we note that $\int |m|^{2q}\psi_\alpha f^\alpha dm$ is uniformly bounded in $L^\infty(0,T;L^2)$. Also, for each $t$, $\|\nabla_x u^\alpha(t) - \nabla_x u(t)\|_{L^2} \to 0$ as $\alpha\to\infty$, so by dominated convergence in $t$ variable, we conclude that $\|\nabla_x u^\alpha - \nabla_x u\|_{L^1(0,T;L^2)} \to 0$. Therefore, $\lim_\alpha I_2 = 0$. In conclusion, we have

$$\|u\|^2_{L^\infty(0,T;L^2)} + C\,\|\mathrm{Tr}\sigma\|_{L^\infty(0,T;L^1)} + 2\nu_1\,\|\nabla_x u\|_{L^2(0,T;L^2)} \leq AT\,\|M_{0,0}[\mu_0]\|_{L^1}\,, \tag{2.208}$$

and since $|\sigma_{12}| \leq \frac{1}{2}\mathrm{Tr}(\sigma)$ we obtain (2.203). From (2.135) we see that

$$\|\sigma\|^2_{L^\infty(0,T;L^2)} + \nu_2\,\|\nabla_x\sigma\|^2_{L^2(0,T;L^2)} \leq B_2(T) \tag{2.209}$$

where again $B_2(T) = C(q)^{T+B_1(T)} \left\| \bar{M}[\mu_0]_{2q} \right\|_{L^2}$ depends only on initial data and $T$. Then we take curl to the first equation of the (2.1) to get vorticity equation: for $\omega = \nabla_x^\perp \cdot u$

$$\partial_t \omega + u \cdot \nabla_x \omega = \nu_1 \Delta_x \omega + K \nabla_x^\perp \cdot \nabla_x \cdot \sigma. \tag{2.210}$$

Multiplying $\omega$ to (2.210) and integrating, we obtain

$$\|\omega\|_{L^\infty(0,T;L^2)}^2 + \nu_1 \|\nabla_x \omega\|_{L^2(0,T;L^2)} \le C(\nu_1) \|\nabla_x \sigma\|_{L^2(0,T;L^2)}^2 = CB_2(T). \tag{2.211}$$

Then by (2.136) we have

$$\|\sigma\|_{L^\infty(0,T;W^{1,2})}^2 + \nu_2 \|\sigma\|_{L^2(0,T;W^{2,2})}^2 \le B_3(T) \tag{2.212}$$

where $B_3(T) = C(\epsilon, \nu_2, q, K)^{T + CB_2(T)T + B_2(T)\sqrt{T}}$ again depends only on initial data and $T$. Finally, by multiplying $-\Delta_x \omega$ to (2.210) and integrating, we have

$$\begin{aligned} \|\nabla_x \omega\|_{L^\infty(0,T;L^2)}^2 + \|\Delta_x \omega\|_{L^2(0,T;L^2)}^2 &\le \exp\left( C \int_0^T \|u(t)\|_{L^2}^2 \|\omega(t)\|_{L^2}^2 \, dt \right) \\ &\quad \left( \|\nabla_x \omega(0)\|_{L^2}^2 + C(K, \nu_1) \|\Delta_x \sigma\|_{L^2(0,T;L^2)}^2 \right) \\ &\le \exp\left( CB_1(T)B_2(T) \right) \left( \|\nabla_x \omega(0)\|_{L^2}^2 + C(K, \nu_1) B_3(T) \right) = B_4(T). \end{aligned} \tag{2.213}$$

Therefore, we see that

$$\|u\|_{\mathcal{X}} \le B_1 + CB_2 + B_4 = B_5, \tag{2.214}$$

which only depends on initial data and $T$. Thus, we have the global existence, following the proof of [33]. Theorem 2.4.1 guarantees that there is $T_0 > 0$ such that the solution exists for $[0, T_0]$. We consider the maximal interval of existence: $T_1 = \sup T_0 \le T$ such that the solution exists for $[0, T_0]$. Then it must be that $T_1 = T$, because otherwise we could extend the solution beyond $T_1$.

**Theorem 2.4.2.** *Given $u_0 \in \mathbb{P}W^{2,2}$, $\mu_0$ satisfying (2.82), (2.83), (2.84), and (2.85),* *and arbitrary $T > 0$, there is a unique solution $(u, f)$ to (2.1) for $t \in (0, T)$ satisfying (2.81) and $f$ is the unique moment solution of the Fokker-Planck equation with velocity field $u$. In addition, the bounds (2.203), (2.209), (2.211), (2.212), (2.213), and (2.214) are satisfied.*

**Remark 10.** *In fact, local Lipschitz dependence of solution on the initial data can be proved with similar standard energy estimates in this subsection, together with (2.186). That is, if $u_0, v_0 \in \mathbb{P}W^{2,2}$ and $\mu_0, \nu_0$ satisfy (2.82), (2.83), (2.84), and (2.85), then*

$$\|u - v\|_{\mathcal{X}} \leq C\left( \|u_0 - v_0\|_{\mathbb{P}W^{2,2}} + \|\sigma[\mu_0] - \sigma[\nu_0]\|_{W^{1,2}} + \sum_{k=0}^{2q-1} \left\|\bar{M}_{2k}[\mu_0 - \nu_0]\right\|_{L^2} \right), \tag{2.215}$$

*where $C(u_0, v_0, \mu_0, \nu_0, T)$.*

**Corollary 2.4.3.** *Suppose that $q = 1$ in the system (2.1), in other words, $U(m) = |m|^2$. Suppose that the initial data $u_0, \mu_0$ satisfies conditions $u_0 \in \mathbb{P}W^{2,2}$, (2.82), (2.83), (2.84), and (2.85), and (2.242). Then*

$$(u, \sigma, \rho) = (u, \int m \otimes \nabla_m U f dm, M_{0,0}[f])$$

*is the unique strong solution for the diffusive Oldroyd-B equation*

$$\begin{gathered}
\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \nu_1 \Delta_x u + K \nabla_x \cdot \sigma, \\
\nabla_x \cdot u = 0, \\
\partial_t \sigma + u \cdot \nabla_x \sigma = (\nabla_x u)\sigma + \sigma(\nabla_x u)^T - 2\epsilon\sigma + 2\epsilon\rho\mathbb{I} + \nu_2 \Delta_x \sigma, \\
\partial_t \rho + u \cdot \nabla_x \rho = \nu_2 \Delta_x \rho, \\
u(0) = u_0, \sigma(0) = \int m \otimes \nabla_m U \mu_0 dm, \rho(0) = M_{0,0}[\mu_0].
\end{gathered} \tag{2.216}$$

*Proof.* It is a consequence of Lemma 2.2.22 and Theorem 2.4.2. Although $\sigma$ is a weak solution of the corresponding equation of (2.77), it has enough regularity to perform integration by parts, so in fact it is a strong solution. By the uniqueness of diffusive Oldroyd-B system ([33]), it is the unique solution. □

### 2.4.3 Free energy bound

In this section, we prove the estimates (2.241) and (2.249). For this purpose, we briefly review the proof of Theorem 2.2.16. We follow the proof in [18].

*proof of Theorem 2.2.16 .* For simplicity, we assume that $a^{ij}(x,t) = a^{ij}$ for some constant, positive definite matrix $(a^{ij})_{ij}$. We use the following simple observation: given two nonnegative functions $f_1, f_2 \in L^1(\mathbb{R}^d)$, for every measurable function $\psi$ with the property that $|\psi|^2 f_1 \in L^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \frac{|(\psi f_1) * f_2|^2}{f_1 * f_2} dx \leq \int_{\mathbb{R}^d} |\psi|^2 f_1 dx \int_{\mathbb{R}^d} f_2 dx, \tag{2.217}$$

where $\frac{|(\psi f_1) * f_2(x)|^2}{f_1 * f_2(x)} := 0$ if $f_1 * f_2(x) = 0$. Also we set

$$\rho * \omega_\epsilon(x,t) := \int_{\mathbb{R}^d} \omega_\epsilon(x-y)\rho(y,t)dy,$$

where $\omega_\epsilon(x) = \epsilon^{-d} g\left(\frac{x}{\epsilon}\right)$ where $g$ is the standard Gaussian and $\epsilon \in (0,T)$. Then $\mu = \rho dxdt$ and in the Sobolev sense

$$\partial_t(\rho * \omega_\epsilon) = (a^{ij}\rho) * (\partial_{x_i}\partial_{x_j}\omega_\epsilon) - (b^i\rho) * \partial_{x_i}\omega_\epsilon. \tag{2.218}$$

We have the following version of $\rho * \omega_\epsilon$ defined by the formula

$$\rho * \omega_\epsilon(x,t) := \rho * \omega_\epsilon(x,0) + \int_0^t v(x,s)ds \tag{2.219}$$

where $v$ is the right side of (2.218). One can readily check that this version is absolutely continuous in $t$ on $[0,T]$ and belongs to the class $C_b^\infty(\mathbb{R}^d)$ in $x$, and for almost every $t$, including $t = 0$, this version coincides for all $x$ with the original version defined by convolution. This version is bounded pointwise by $\epsilon^{-d}$, for all $(x,t) \in \mathbb{R}^d \times [0,T]$. We set

$$\rho_\epsilon := \rho * \omega_\epsilon, \;\; f_\epsilon(x,t) := \rho_\epsilon(x,t) + \epsilon \max(1,|x|)^{-(d+1)}, \tag{2.220}$$

where $\rho * \omega_\epsilon$ should be understood as the version (2.219) and by $\max(1,|x|)^{-(d+1)}$ we mean, again by a slight abuse of notation, a smooth, bounded function equals it for $|x| > 2$. Since the function $\rho\Lambda$ is integrable, there is $\tau$ as close to $T$ as we wish such that

$$\int_{\mathbb{R}^d} \rho(x,\tau)\Lambda(x)dx < \infty, \tag{2.221}$$

and for every $\epsilon = \frac{1}{n}$ our version of $\rho_\epsilon(x,\tau)$ coincides with the function $\rho(\cdot,\tau) * \omega_\epsilon(x)$ for all $x$. Then by inequality

$$\log \max(|x+y|,1) \leq \log \max(|x|,1) + |y|$$

gives the estimate

$$\int_{\mathbb{R}^d} f_\epsilon(x,\tau)\Lambda(x)dx \leq \int_{\mathbb{R}^d} \rho(x,\tau)\Lambda(x)dx + \epsilon M_1, \tag{2.222}$$

where $M_1$ is a constant independent of $\epsilon$. Then by (2.218), we have

$$\int_0^\tau \int_{\mathbb{R}^d} \partial_t(\rho * \omega_\epsilon) \log f_\epsilon dxdt = \int_0^\tau \int_{\mathbb{R}^d} \left(a^{ij}\left(\rho * \partial_{x_i}\partial_{x_j}\omega_\epsilon\right) - \left(b^i\rho\right) * \partial_{x_i}\omega_\epsilon\right) \log f_\epsilon dxdt, \tag{2.223}$$

and by $|\log f_\epsilon| \leq C\left(\log\frac{1}{\epsilon} + 1 + \Lambda\right)$, (2.217), $|b| \in L^2(\mu)$, and the estimate

$$|\log \max(|x+y|,1)|^2 \leq 4 + 2|\log \max(|x|,1)|^2 + 2|\log \max(|y|,1)|^2$$

the integrand of the right side of (2.223) is integrable in $\mathbb{R}^d \times (0,T)$. Furthermore, one can observe that one can integrate by parts of the right side of (2.223) using the similar argument: therefore we get

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{R}^d} \partial_t \rho_\epsilon \log f_\epsilon dx dt \\ &= -\int_0^\tau \int_{\mathbb{R}^d} \frac{\partial_{x_i} f_\epsilon}{f_\epsilon} \left(a^{ij}\partial_{x_j}(\rho * \omega_\epsilon) - (b^i\rho) * \omega_\epsilon\right) dx dt.\end{aligned} \tag{2.224}$$

The integrand of left side of (2.224) can be written as $\partial_t(f_\epsilon \log f_\epsilon) - \partial_t \rho_\epsilon$, and since $\rho_t$ are probability measures, the left side of (2.224) equals

$$L_\epsilon := \int_{\mathbb{R}^d} \left(f_\epsilon(x,\tau)\log f_\epsilon(x,\tau) - f_\epsilon(x,0)\log f_\epsilon(x,0)\right) dx. \tag{2.225}$$

By (2.222) and $|\log f_\epsilon| \le C\left(\log\left(\frac{1}{\epsilon}\right) + 1 + \Lambda\right)$ we have $f_\epsilon(\cdot,\tau)\log f_\epsilon(\cdot,\tau) \in L^1(\mathbb{R}^d)$ and similarly $f_\epsilon(\cdot,0)\log f_\epsilon(\cdot,0) \in L^1(\mathbb{R}^d)$. By Jensen's inequality applied to $\Phi(s) = s\log s$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} f_\epsilon(x,0)\log f_\epsilon(x,0) dx \\ &\le \lambda \int \Phi\left(\frac{\rho_\epsilon}{\lambda}\right) dx + (1-\lambda)\int \Phi\left(\frac{\epsilon}{1-\lambda}\max(|x|,1)^{-(d+1)}\right) dx \\ &\le \int \rho_0 \log \rho_0 dx + \log\frac{1}{\lambda} + \epsilon \int \frac{1}{1-\lambda}\max(|x|,1)^{-(d+1)} dx\end{aligned} \tag{2.226}$$

for any $\lambda \in (0,1)$. On the other hand, we have Csiszár-Kullback-Pinsker inequality ([75])

$$\begin{aligned}&\int f\log f - f\log g - f + g dx \ge \frac{1}{2}\|f-g\|_{L^1}^2, \\ &\text{where } f, g \in L^1, f \ge 0, g > 0, \int f = \int g = 1\end{aligned} \tag{2.227}$$

with $f = \frac{1}{\|f_\epsilon\|_{L^1}} f_\epsilon = \frac{1}{1+\epsilon C} f_\epsilon$ and

$$g = \frac{1}{\|\max(|x|,1)^{-(d+1)}\|_{L^1}} \max(|x|,1)^{-(d+1)} = C\max(|x|,1)^{-(d+1)}.$$

Thus, we obtain

$$\begin{aligned}\int f_\epsilon(x,\tau)\log f_\epsilon(x,\tau)dx &\geq (1+C\epsilon)\log C(1+C\epsilon)-(d+1)\int f_\epsilon(x,\tau)\Lambda(x)dx\\ &\geq -(d+1)\int \rho(x,\tau)\Lambda(x)dx+o(\epsilon).\end{aligned} \tag{2.228}$$

From (2.224) we obtain

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^d} a^{ij}\frac{\partial_{x_i}f_\epsilon}{f_\epsilon}\partial_{x_j}f_\epsilon dxdt\\ =&\int_0^\tau\int_{\mathbb{R}^d}\frac{\partial_{x_i}f_\epsilon}{f_\epsilon}\left((b^i\rho)*\omega_\epsilon+\epsilon a^{ij}\partial_{x_j}\max(|x|,1)^{-(d+1)}\right)dxdt-L_\epsilon\end{aligned} \tag{2.229}$$

and the right side in this inequality is bounded by

$$\begin{aligned}&\left(\int_0^\tau\int_{\mathbb{R}^d}\frac{|\nabla f_\epsilon|^2}{f_\epsilon}dxdt\right)^{\frac{1}{2}}\left(\|b\|_{L^2(\mu)}+o(\epsilon)\right)\\ &+(d+1)\int\rho(x,\tau)\Lambda(x)dx+o(\epsilon)+\int\rho_0\log\rho_0 dx-\log\lambda+\frac{o(\epsilon)}{1-\lambda}.\end{aligned} \tag{2.230}$$

Using $A\geq m\mathbb{I}$, taking $\epsilon\to 0$, using Fatou's lemma, and putting $\lambda\to 1$ we get

$$m^2\int_0^\tau\int_{\mathbb{R}^d}\frac{|\nabla\rho|^2}{\rho}dxdt\leq\left(\|b\|^2_{L^2(\mu)}+(d+1)\int\rho(x,\tau)\Lambda(x)dx+\int\rho_0\log\rho_0 dx\right) \tag{2.231}$$

as desired. □

To prove entropy estimate, we start from (2.224) and $L_\epsilon$: first we prove that as $\epsilon\to 0$,

$$\int_0^\tau\int_{\mathbb{R}^d}\frac{\partial_{x_i}f_\epsilon}{f_\epsilon}\partial_{x_j}(\rho*\omega_\epsilon)dxdt\to\int_0^\tau\int\frac{\partial_{x_i}\rho\partial_{x_j}\rho}{\rho}dxdt.$$

We begin with observing that $\frac{\partial_{x_i}f_\epsilon}{f_\epsilon}\partial_{x_j}(\rho*\omega_\epsilon)$ is bounded by

$$q_\epsilon=\frac{|\nabla\rho*\omega_\epsilon|^2}{\rho_\epsilon}+C\epsilon\frac{1}{f_\epsilon^{\frac{1}{2}}}\max(|x|,1)^{-(d+2)}\frac{|\nabla\rho*\omega_\epsilon|}{(\rho*\omega_\epsilon)^{\frac{1}{2}}}. \tag{2.232}$$

For almost all $t \in (0,\tau)$, $\int_{\mathbb{R}^d} \frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}dx < \infty$. Therefore, for such $t$, by (2.217) we see that

$$\int \frac{|\nabla\rho * \omega_\epsilon(t)|^2}{\rho * \omega_\epsilon(t)}dx \leq \int_{\mathbb{R}^d} \frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}dx,$$

and so for integral over $t$ too. Then the integral over $\mathbb{R}^d \times [0,\tau)$ of the second term is also bounded by $\sqrt{\epsilon}C$, where $C$ is independent of $\epsilon$. Therefore, we have

$$\limsup_{\epsilon\to 0} \int_0^\tau \int q_\epsilon(x,t)dxdt \leq \int_0^\tau \int_{\mathbb{R}^d} \frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}dxdt.$$

On the other hand, note that $q_\epsilon(x,t) \to \frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}$ for almost all $(x,t) \in \mathbb{R}^d \times [0,\tau)$, at least for a subsequence of $\epsilon = \frac{1}{n}$ because we have $L^1(x,t)$ convergence of approximate identity in $x$ variable. Therefore, by Fatou's lemma we have

$$\int_0^\tau \int_{\mathbb{R}^d} \frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}dx \leq \liminf \int_0^\tau \int q_\epsilon(x,t)dxdt.$$

Therefore, we see that $\frac{\partial_{x_i} f_\epsilon}{f_\epsilon}\partial_{x_j}(\rho * \omega_\epsilon)$ is bounded by $q_\epsilon(x,t)$ pointwise, which is integrable and converges to $\frac{|\nabla\rho(x,t)|^2}{\rho(x,t)}$ pointwise, and its integral also converges to the integral of the limit. Therefore, by generalized dominated convergence, we prove the claim. In a similar manner, we see that

$$\int_0^\tau \int_{\mathbb{R}^d} \frac{|(b^i\rho) * \omega_\epsilon|^2}{\rho * \omega_\epsilon}dxdt \to \int_0^\tau \int_{\mathbb{R}^d} \frac{|(b^i\rho)|^2}{\rho}dxdt.$$

Then again $\frac{\partial_{x_i} f_\epsilon}{f_\epsilon}(b^i\rho) * \omega_\epsilon$ is bounded by

$$q'_\epsilon = \left(\frac{|\nabla\rho * \omega_\epsilon|}{\sqrt{\rho_\epsilon}} + C\epsilon\frac{1}{f_\epsilon^{\frac{1}{2}}}\max(|x|,1)^{-(d+2)}\right)\frac{|(b^i\rho) * \omega_\epsilon|}{\sqrt{\rho_\epsilon}}, \tag{2.233}$$

and we can again use generalized dominated convergence to conclude that

$$\int_0^\tau \int_{\mathbb{R}^d} \frac{\partial_{x_i} f_\epsilon}{f_\epsilon}(b^i\rho) * \omega_\epsilon dxdt \to \int_0^\tau \int_{\mathbb{R}^d} b^i\partial_{x_i}\rho dxdt. \tag{2.234}$$

On the other hand, to control $L_\epsilon$ term we observe that $\Psi(x) = x\log x - x + 1 \geq 0$ for all $x \geq 0$: then for $g = \max(|x|, 1)^{-(d+1)}$ by Fatou we have

$$
\begin{aligned}
&\int \rho \log \rho(\tau) dx + (d+1) \int \rho(\tau)\Lambda dx - 1 + \int g dx = \int \Psi\left(\frac{\rho(\tau)}{g}\right) g dx \\
\leq &\liminf_{\epsilon \to 0} \int \Psi\left(\frac{f_\epsilon}{g}\right) g dx = \liminf \int f_\epsilon \log f_\epsilon(\tau) dx + (d+1) \int \rho(\tau)\Lambda dx - 1 + \int g dx.
\end{aligned}
\tag{2.235}
$$

Here we used that $\int f_\epsilon(\tau)\Lambda dx \to \int \rho(\tau)\Lambda dx$, which comes from (2.222) and Fatou. Therefore, by taking $\epsilon \to 0$ to (2.224) we get

$$
\int \rho \log \rho(\tau) dx + \int_0^\tau \int_{\mathbb{R}^d} \frac{a^{ij}\partial_{x_i}\rho \partial_{x_j}\rho}{\rho} dx dt - \int_0^\tau \int_{\mathbb{R}^d} b^i \partial_{x_i}\rho dx dt \leq \int \rho_0 \log \rho_0 dx.
\tag{2.236}
$$

Applying (2.236) to our equation, and applying integration by parts to $b^i\partial_{x_i}\rho$, which is possible since $b)\rho, (\partial_{x_i}b^i)\rho \in L^1$, we get

$$
\begin{aligned}
\int f(\tau)\log f(\tau) dm dx + \int_0^\tau \int \nu_2 \frac{|\nabla_x f|^2}{f} + \epsilon \frac{|\nabla_m f|^2}{f} dm dx dt \\
-\epsilon \int_0^\tau \int \Delta_m U f dm dx dt \leq \int f_0 \log f_0 dm dx.
\end{aligned}
\tag{2.237}
$$

On the other hand, applying similar argument as (2.203), we have

$$
\begin{aligned}
&\int \bar{M}^\alpha_{2q}(\tau) dx + \epsilon(2q)^2 \int_0^\tau \int \bar{M}^\alpha_{4q-2} dx dt \\
= &\int \bar{M}^\alpha_{2q}(0) dx + \int_0^\tau \int \mathrm{Tr}((\nabla_x u)\sigma) dx dt + \epsilon(2q)^2 \int_0^\tau \int \bar{M}^\alpha_{2(q-1)} dx dt + I_\alpha
\end{aligned}
\tag{2.238}
$$

where $I_\alpha \to 0$ as $\alpha \to \infty$. Note that we know, by weak convergence,

$$
\int \bar{M}_{2q}[f](\tau) dx + \epsilon(2q)^2 \int_0^\tau \int \bar{M}_{4q-2}[f] dx dt
$$

does not exceed the limit inferior of the left side of (2.238). On the other hand, we need

$$\int_0^\tau \int \bar{M}^\alpha_{2(q-1)} dxdt \to \int_0^\tau \int \bar{M}_{2(q-1)}[f] dxdt,$$

which can be obtained by the following: since

$$\int \Lambda(x)|m|^{2(q-1)} f^\alpha dmdx \leq \int \left(|\Lambda|^2 + |m|^{4(q-1)}\right) f^\alpha dmdx$$

we see, from bounds in section 2.3.2 and section 2.3.4 we note that $\int_0^\tau \int \Lambda(x)\bar{M}^\alpha_{2(q-1)} dxdt$ is bounded by some constant $C$ depending on initial data $\mu_0$ and $u$, uniform in $\alpha$. Therefore, for any $R > 1$, we have

$$\int_0^\tau \int_{|x|>R} \bar{M}^\alpha_{2(q-1)} dxdt \leq \frac{C}{\log R}.$$

On the other hand, we note that

$$|\nabla_x \bar{M}^\alpha_{2(q-1)}| \leq \bar{M}^\alpha_{4(q-1)} + \int \frac{|\nabla_x f^\alpha|^2}{f^\alpha} dm,$$

and bounds in section 2.3.2 and section 2.3.4 gives that $\left\|\nabla_x \bar{M}^\alpha_{2(q-1)}\right\|_{L^1(0,T;L^1)}$ is uniformly bounded in $\alpha$. Also, by (2.99) we can see that $\partial_t \bar{M}^\alpha_{2(q-1)} \in L^1(0,T;W^{-1,1})$ is uniformly bounded in $\alpha$: for terms involving velocity fields, one can use $L^2$ bounds on moments, and for plain moment terms one note that the highest moment in that equation has degree $4(q-1)$, and it has bound in $L^\infty(0,T;L^1)$, which is uniform in $\alpha$. Then for any $B(0,R)$, $W^{1,1}(B(0,R)) \subset L^1(B(0,R))$ compactly by Rellich-Kondrachov, and $L^1(B(0,R)) \subset W^{-1,1}(B(0,R))$ by Morrey-Sobolev embedding $W^{1,q'} \subset L^\infty$ for $q < \frac{d}{d-1}$. Therefore, by Aubin-Lions, by applying some cutoff function if necessary, we have

$$\lim_{\alpha\to\infty} \int_0^\tau \int_{B(0,R)} \bar{M}^\alpha_{2(q-1)} dxdt = \int_0^\tau \int_{B(0,R)} \bar{M}_{2(q-1)}[f] dxdt.$$

To summarize, we have

$$\begin{aligned}&\int \bar{M}_{2q}[f](\tau)dx + \epsilon(2q)^2\int_0^\tau\int \bar{M}_{4q-2}[f]dxdt\\ \leq &\int \bar{M}_{2q}[f_0]dx + \int_0^\tau\int \mathrm{Tr}((\nabla_x u)\sigma)dxdt + \epsilon(2q)^2\int_0^\tau\int \bar{M}_{2(q-1)}[f]dxdt,\end{aligned} \tag{2.239}$$

or, in other words,

$$\begin{aligned}&-\int \log\left(e^{-U(m)}\right)f(\tau)dmdx + \epsilon\int_0^\tau\int |\nabla_m U|^2 f dmdxdt\\ -\epsilon&\int_0^\tau\int \Delta_m U f dmdxdt \leq -\int \log\left(e^{-U(m)}\right)f_0 dmdx + \int_0^\tau\int \mathrm{Tr}((\nabla_x u)\sigma)dxdt.\end{aligned} \tag{2.240}$$

Note that we can apply integration by parts to the term $\int_0^\tau\int \Delta_m f dmdxdt$: since $|\nabla_m U\nabla_m f| \leq \frac{|\nabla_m f|^2}{f} + |\nabla_m U|^2 f$ so it is integrable in $L^1([0,T]\times\mathbb{R}^{2+2})$, and $\nabla_m U f$ is also integrable. Therefore, by adding (2.237) and (2.240), and adding the velocity part we get

$$\begin{aligned}&\int f(\tau)\log\frac{f(\tau)}{e^{-U}/Z}dmdx + \epsilon\int_0^\tau\int f\left|\nabla_m\log\left(\frac{f}{e^{-U}/Z}\right)\right|^2 dmdxdt\\ +\nu_2&\int_0^\tau\int f\left|\nabla_x\log\left(\frac{f}{e^{-U}/Z}\right)\right|^2 dmdxdt + \frac{1}{K}\|u(\tau)\|_{L^2}^2 + \frac{\nu_1}{K}\int_0^\tau \|\nabla_x u\|_{L^2}^2 dxdt\\ &\leq \|u_0\|_{L^2}^2 + \int f_0\log\frac{f_0}{e^{-U}/Z}dmdx\end{aligned} \tag{2.241}$$

where $Z = \int e^{-U}dm$. On the other hand, suppose that

$$\int M_{0,0}[f_0]\log\left(M_{0,0}[f_0]\right)dx < \infty. \tag{2.242}$$

Using the same technique as before, we can show that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^\tau\int \nu_2\frac{|\nabla_x M_{0,0}^\epsilon|^2}{M_{0,0}^\epsilon}dxdt = \int_0^\tau\int \nu_2\frac{|\nabla_x M_{0,0}|^2}{M_{0,0}}dxdt,\\ &\lim_{\epsilon\to 0}\int M_{0,0}^\epsilon(x,0)\log M_{0,0}^\epsilon(x,0)dx = \int M_{0,0}(x,0)\log M_{0,0}(x,0)dx.\end{aligned} \tag{2.243}$$

where $M_{0,0}^\epsilon = M_{0,0} * \omega_\epsilon + \epsilon \max(|x|, 1)^{-3}$ and the remaining task is to show

$$\liminf_{\epsilon \to 0} \int M_{0,0}^\epsilon(x,\tau) \log M_{0,0}^\epsilon(x,\tau) dx = \int M_{0,0}(x,\tau) \log M_{0,0}(x,\tau) dx.$$

For this we recall the following fact about Fatou from [91], which comes from Brézis-Lieb inequality ([21]) : if $\{h_n\}$ is a sequence of nonnegative functions, converging almost everywhere to $h$, and $\int h_n$ is uniformly bounded, then

$$\liminf_n \int |h_n - h| + \int h = \liminf_n \int h_n. \tag{2.244}$$

We apply this to $\Psi\left(\frac{M_{0,0}^\epsilon}{g}\right) g \geq 0$, where as before $\Psi(s) = s \log s - s + 1$ and $g(x) = \max(|x|, 1)^{-3}$. We know that for $f \geq 0, f \in L^1 \cap L^2, \int f\Lambda < \infty$, we have a pointwise estimate

$$f|\log f| \leq C f\Lambda + Cg + |f|^2, \tag{2.245}$$

where the first term corresponds to the case $g(x)^2 \leq f(x) \leq 1$, the second term corresponds to the case $0 \leq f \leq g(x)^2$, and the last term corresponds to the case $f(x) > 1$. Therefore,

$$\int_{\mathbb{R}^2} \Psi\left(\frac{M_{0,0}^\epsilon}{g}\right) g dx = \int M_{0,0}^\epsilon \log M_{0,0}^\epsilon + 3M_{0,0}^\epsilon \Lambda - M_{0,0}^\epsilon + g dx \tag{2.246}$$

so by (2.245) and (2.222) they are uniformly bounded in $\epsilon$. Thus it suffices to show

$$\int \left| \Psi\left(\frac{M_{0,0}^\epsilon}{g}\right) - \Psi\left(\frac{M_{0,0}}{g}\right) \right| g dx \to 0.$$

But this term is bounded by

$$\int |M_{0,0}^\epsilon \log M_{0,0}^\epsilon - M_{0,0} \log M_{0,0}| + |M_{0,0}^\epsilon - M_{0,0}|(\Lambda + 1) dx, \tag{2.247}$$

which converges to 0 by the pointwise estimate (2.245) and generalized dominated convergence theorem. Therefore, we have

$$\int M_{0,0}(\tau)\log M_{0,0}(\tau)dx + \nu_2 \int_0^\tau \int \frac{|\nabla_x M_{0,0}|^2}{M_{0,0}} dxdt = \int M_{0,0}[f_0]\log M_{0,0}[f_0]dx. \tag{2.248}$$

Noting that

$$\frac{|\nabla_x M_{0,0}|^2}{M_{0,0}} = M_{0,0}\left|\nabla_x \left(\log M_{0,0}\right)\right|^2 = \int f \left|\nabla_x \left(\log M_{0,0}\right)\right|^2 dm$$

and by subtracting (2.248) to (2.241) we get

$$\begin{aligned} &\int f(\tau)\log \frac{f(\tau)}{M_{0,0}[f(\tau)]e^{-U}/Z} dmdx + \epsilon \int_0^\tau \int f \left|\nabla_m \log\left(\frac{f}{M_{0,0}[f]e^{-U}/Z}\right)\right|^2 dmdxdt \\ &+\nu_2 \int_0^\tau \int f \left|\nabla_x \log\left(\frac{f}{M_{0,0}[f]e^{-U}/Z}\right)\right|^2 dmdxdt + \frac{1}{K}\left\|u(\tau)\right\|_{L^2}^2 + \frac{\nu_1}{K}\int_0^\tau \left\|\nabla_x u\right\|_{L^2}^2 dxdt \\ &\le \left\|u_0\right\|_{L^2}^2 + \int f_0 \log \frac{f_0}{M_{0,0}[f_0]e^{-U}/Z} dmdx. \end{aligned} \tag{2.249}$$

Therefore, we have proved the following.

**Theorem 2.4.4.** *If the system (2.1) has initial data satisfying* $u_0 \in \mathbb{P}W^{2,2}$, *(2.82), (2.83), (2.84), and (2.85), then for almost all* $\tau \in (0, +\infty)$ *(2.241) holds. If in addition (2.242) holds, then (2.249) also holds for almost all* $\tau \in (0, +\infty)$.

# Chapter 3

# Moment solution methods for rigid polymer solutions

## 3.1 Introduction

In this chapter, we are interested in a dilute suspension of rigid rod-like polymers, in dimension 2. In particular, we investigate the Doi model:

$$\begin{aligned}
&\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \Delta_x u + \nabla_x \cdot \sigma, \\
&\nabla_x \cdot u = 0, \\
&\partial_t f + u \cdot \nabla_x f = k\Delta_m f + \nu \Delta_x f - \nabla_m \cdot \left(P_{m^\perp}\left((\nabla_x u) m f\right)\right), \\
&\sigma = 2\int_{\mathbb{S}^1} (m \otimes m - \frac{1}{2}\mathbb{I}_2) f dm + \eta \int_{\mathbb{S}^1} \left((\nabla_x u) : m \otimes m\right) m \otimes m f dm, \\
&(x, m, t) \in \mathbb{T}^2 \times \mathbb{S}^1 \times (0, T), \\
&u(x, 0) = u_0(x), f(x, m, 0) = f_0(x, m),
\end{aligned} \tag{3.1}$$

where $u$ is the velocity field of the fluid, $p$ is the pressure, $\sigma$ is the added stress field due to the presence of polymer, $f = f(x, t, m)$ is the polymer distribution, and $u_0, f_0$ are initial data. Also constant parameters $k, \nu > 0$ represents configurational and spatial diffusivity of polymers, respectively, and $\eta > 0$ is a constant parameter representing the concentration of the polymers. We prove global well-posedness of strong solution of (3.1). The term

$$P_{m^\perp}(g\vec{v}) = (m^\perp \cdot \vec{v})gm^\perp$$

is the projection to the tangent space of $\mathbb{S}^1$ at $m$, and $\nabla_m = \partial_\theta$ in local coordinates. The polymer stress tensor $\sigma$ can be decomposed into two terms: $\sigma = \sigma_E + \sigma_V$, where

$$\sigma_E(f) = 2\int_{\mathbb{S}^1} (m \otimes m - \frac{1}{2}\mathbb{I}_2) f dm, \tag{3.2}$$

and

$$\sigma_V(f) = \eta \int_{\mathbb{S}^1} ((\nabla_x u) : m \otimes m) m \otimes m f dm. \tag{3.3}$$

The presence of viscous stress tensor is the main difficulty for the well-posedness of the Doi model. Viscous stress tensors arise from rigidity constraint of the polymer( [47]), and mathematically $\sigma_V(f)$ is not elliptic in $u$, which makes the momentum equation of (3.1) non-parabolic for large $\eta$. This difficulty can be clearly illustrated in the approximate Doi model:

$$\begin{gathered}
\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \Delta_x u + \nabla \cdot \sigma, \\
\nabla_x \cdot u = 0, \\
\sigma = \eta(\nabla_x u : A)A, \\
\partial_t A + u \cdot \nabla_x A = (\nabla_x u)A + A(\nabla_x u)^T - 2(\nabla_x u : A)A - 2k(2A - \mathbb{I}_2) + \nu \Delta_x A, \\
u(x,0) = u_0(x), A(x,0) = A_0(x), \\
(x,t) \in \mathbb{T}^2 \times (0,T).
\end{gathered} \tag{3.4}$$

The model (3.4) is an approximate closure of Doi model (3.1) obtained by letting $A = \int_{S^1} n \otimes n f dn$ and adopting the decoupling approximation $\sigma \simeq \eta(\nabla_x u : A)A$ and ignoring elastic stress part. We establish the energy estimate:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \eta \int |(\nabla_x u) : A|^2 dx = 0, \tag{3.5}$$

and we see that in fact viscous stress is another dissipative structure for $u$. Based on this remarkable property, which holds in (3.1) also, Lions and Masmoudi proved global existence of weak solution of (3.1) in [97]. However, when we apply the vorticity estimate, at first point we can only obtain

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla_x \omega\|_{L^2}^2 \leq \|\nabla_x \cdot \sigma\|_{L^2}^2 \leq \eta^2 \left(\|\nabla\nabla u\|_{L^2}^2 + (\text{Error})\right) \tag{3.6}$$

which makes the right hand side for the second inequality intractable if $\eta > \eta_c$ for some threshold $\eta_c$.

Recently, in [109], the authors numerically discovered that, when $\eta$ exceeds some threshold $\eta_c$, the flow governed by (3.4) becomes chaotic. It was hence unclear this phenomenon supports the claim that the systems (3.1) and (3.4) lack structure to control higher regularity of $u$. However, in this work we find that actually the viscous stress tensor adds dissipation for higher derivatives of $u$ also, modulo derivatives in polymer variables ( (3.74), (3.80)). This observation is crucial in proving global well-posedness of (3.1) and (3.4) in diffusive systems $\nu > 0$.

**Notion of the solution.** For the notion of solution, we follow the argument in [85]. By focusing on the evolution of macroscopic variables (trigonometric moments in this case), we can set up well-posedness of strong solutions for large class of initial data. In particular, higher regularity of Fokker-Planck equation is not necessary, and weak solution for Fokker-Planck equation is sufficient. On the other hand, since the effect of polymer to the flow are characterized by stresses, which are moments in (3.1), requiring spatial regularity for appropriate moments is necessary. In this regard, we introduce a terminology: for any $n \in \mathbb{Z}_{>0}$, we let

$$M_n(x,t) := \left(M_n^I(x,t)\right)_{I:|I|=n} := \left(\int_{\mathbb{S}^1} m^I f(x,t,m)dm\right)_{I:|I|=n} \tag{3.7}$$

be the vector of all moments of $f$ of order $n$. Also, we define the weak solution as following ([18]):

**Definition 3.1.1.** *Given a divergence-free vector field $v \in L^\infty(0,T;W^{2,2}) \cap L^2(0,T;W^{3,2})$, $\mu$ is a weak solution to the Cauchy problem*

$$\partial_t \mu + v \cdot \nabla_x \mu = k\Delta_m \mu + \nu \Delta_x \mu - \nabla_m \cdot (P_{m^\perp}((\nabla_x v)m\mu)), \mu(t=0) = \nu$$

*if for almost every $t \in (0,T)$,*

$$\begin{aligned} &\int_{\mathbb{T}^2\times\mathbb{S}^1} \phi_x(x)\phi_m(m)d\mu(x,t;dm)dx - \int_{\mathbb{T}^2\times\mathbb{S}^1} \phi_x(x)\phi_m(m)d\nu(x;dm)dx \\ &= \lim_{\tau\to 0}\int_\tau^t \int_{\mathbb{T}^2\times\mathbb{S}^1} \left[v\cdot\nabla_x\phi + k\Delta_m\phi + \nu\Delta_x\phi + \nabla_m\phi \cdot P_{m^\perp}((\nabla_x v)m)\right]\mu(x;s;dm)dxds \end{aligned} \tag{3.8}$$

*for every $\phi = \phi_x\phi_m$, where $\phi_x \in C^\infty(\mathbb{T}^2)$, $\phi_m \in C^\infty(\mathbb{S}^1)$.*

Our main result is the following:

**Theorem 3.1.2.** *Suppose that $u_0 \in \mathbb{P}W^{2,2}(\mathbb{T}^2)$, $f_0 \geq 0 \in L^1(\mathbb{T}^2 \times \mathbb{S}^1)$ with $\sigma_E(f_0) \in W^{1,2}(\mathbb{T}^2)$, $\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}(f_0 \log f_0 - f_0 + 1)dmdx < \infty$, $M_0 \in L^\infty(\mathbb{T}^2)$, $M_4(f_0) \in W^{2,2}(\mathbb{T}^2)$, and $M_6(f_0) \in W^{1,2}(\mathbb{T}^2)$. Then there is a unique solution $(u,f)$ to (3.1), where $u \in L^\infty(0,T;W^{2,2}) \cap L^2(0,T;W^{3,2})$ is the strong solution of the evolution equation of $u$ for (3.1), $\sigma_E(f), M_6(t) \in L^\infty(0,T;W^{1,2}) \cap L^2(0,T;W^{2,2})$, and $M_4(f) \in L^\infty(0,T;W^{2,2}) \cap L^2(0,T:W^{3,2})$. Also $f$ is given by a density $f(x,t,m)$, and $f$ is a weak solution to the Cauchy problem of the Fokker-Planck equation of (3.1). Furthermore, the estimates (3.52), (3.61), (3.64) for $n=4$, (3.71), (3.77), (3.78), and (3.83) hold. In addition,*

*$f(t) \in W^{1,1}(\mathbb{T}^2 \times \mathbb{S}^1)$ holds.*

## 3.2 Global well-posedness of the strong solution of (3.4)

In this section, we prove the following theorem.

**Theorem 3.2.1.** *Given $(u_0, A_0) \in \mathbb{P}W^{2,2}(\mathbb{T}^2) \times W^{2,2}(\mathbb{T}^2)$, where $A_0$ is a $2 \times 2$ positive definite matrix valued function with $\mathrm{Tr}\, A_0 \equiv 1$, then for any $T > 0$ there is a unique strong solution $(u, A) \in (L^\infty(0, T; \mathbb{P}W^{2,2}(\mathbb{T}^2)) \cap L^2(0, T; \mathbb{P}W^{3,2}(\mathbb{T}^2))) \times (L^\infty(0, T; W^{2,2}(\mathbb{T}^2)) \cap L^2(0, T; W^{3,2}(\mathbb{T}^2)))$ satisfying (3.4) and $\mathrm{Tr}\, A \equiv 1$ and $A$ remains positive definite. Furthermore, the solution satisfies the estimates (3.12), (3.15), (3.18), (3.24), (3.26), and (3.32).*

### 3.2.1 A priori estimates

First we have the propagation of positive-definiteness and Trace 1 for $A$.

**Proposition 3.2.2.** *Suppose that $(\nabla_x u) : A \in L^1(0, T; L^\infty)$ and $\mathrm{Tr}\, A(0) \equiv 1$ and $A(0)$ is positive definite. Then $A(t)$ remains positive definite with $\mathrm{Tr}\, A \equiv 1$.*

*Proof.* We need to check $\det A > 0$ and $\mathrm{Tr}\, A \equiv 1$. For $\mathrm{Tr}\, A \equiv 1$, we take the trace of the third equation of (3.4) to get

$$(\partial_t + u \cdot \nabla_x)\mathrm{Tr}\, A = 2\left((\nabla_x u : A) + 2k\right)(1 - \mathrm{Tr}\, A) + \nu \Delta_x \mathrm{Tr}\, A \tag{3.9}$$

and by the maximum principle we are done. For $\det A > 0$, we have

$$
\begin{aligned}
&(\partial_t + u\cdot\nabla_x)\det A \\
&= -4(((\nabla_x u):A) + 2k)\det A + 2k\mathrm{Tr}\,A + \nu\Delta_x \det A - 2\nu\nabla_x A_{11}\cdot\nabla_x A_{22} + 2\nu|\nabla_x A_{12}|^2 \\
&= -4(((\nabla_x u):A) + 2k)\det A + \nu\Delta_x\det A + (2k + 2\nu|\nabla_x A_{11}|^2 + 2\nu|\nabla_x A_{12}|^2)
\end{aligned}
\tag{3.10}
$$

where we used $\mathrm{Tr}\,A \equiv 1$. Then by the maximum principle we are done again. $\square$

We investigate a priori estimates. First, usual energy estimates give us

$$
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 = -\int \nabla_x u : \sigma = -\eta\int |(\nabla_x u):A|^2 dx, \tag{3.11}
$$

that is,

$$
\frac{1}{2}\|u\|_{L^\infty(0,T;L^2)}^2 + \|\nabla_x u\|_{L^2(0,T;L^2)}^2 + \eta\|(\nabla_x u):A\|_{L^2(0,T;L^2)}^2 \le \frac{1}{2}\|u_0\|_{L^2}^2. \tag{3.12}
$$

For $\|A\|_{L^\infty(0,T;L^1\cap L^\infty)}$, we know that from $\mathrm{Tr}\,A \equiv 1$ and since $A$ is positive definite, $\|A(t)\|_{L^\infty} \le 1$ for all $t$. Also $0 < \det A(x,t) \le \frac{1}{4}$ is obtained. On the other hand, using (3.10), we can obtain an estimate for $\|\nabla_x A\|_{L^2(0,T;L^2)}$. Integrating (3.10) with respect to $x$, it can be written as

$$
\frac{d}{dt}\int \det A dx + 4\int((\nabla_x u):A)dx + \int 8k\det A dx = \nu\|\nabla_x A\|_{L^2}^2 + 2k|\mathbb{T}^2|. \tag{3.13}
$$

However, using that $\det A \le \frac{1}{4}$, that $|A|^2 = \sum_{ij} A_{ij}^2 = 1 - 2\det A$ by $\mathrm{Tr}\,A \equiv 1$, and Cauchy-Schwarz inequality we have

$$
\frac{1}{2}\frac{d}{dt}\|A\|_{L^2}^2 + \nu\|\nabla_x A(t)\|_{L^2}^2 \le 4\|(\nabla_x u):A(t)\|_{L^2}. \tag{3.14}
$$

Integrating over time, we have

$$\frac{1}{2}\|A\|^2_{L^\infty(0,T;L^2)} + \nu\|\nabla_x A\|^2_{L^2(0,T;L^2)} \le \frac{1}{2}\|A(0)\|^2_{L^2} + C\sqrt{T}\min\left(\frac{1}{\sqrt{\eta}},1\right). \quad (3.15)$$

Also, by multiplying $-\Delta_x A$ to the fourth equation of (3.4) and integrating we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x A\|^2_{L^2} + 4k\|\nabla_x A\|^2_{L^2} + \nu\|\Delta_x A\|^2_{L^2} \le \|\Delta_x A\|_{L^2}\left(\|u\|_{L^4}\|\nabla_x A\|_{L^4} + 4\|\nabla_x u\|_{L^2}\right), \quad (3.16)$$

and by Ladyzhenskaya's inequality

$$\|u\|^2_{L^4} \le C\|u\|_{L^2}\|\nabla_x u\|_{L^2}$$

applied to $\|u\|_{L^4}$ and $\|\nabla_x A\|_{L^4}$ and Young's inequality we have

$$\frac{d}{dt}\|\nabla_x A\|^2_{L^2} + 8k\|\nabla_x A\|^2_{L^2} + \nu\|\Delta_x A\|^2_{L^2} \le \frac{C}{\nu^3}\|u\|^2_{L^2}\|\nabla_x u\|^2_{L^2}\|\nabla_x A\|^2_{L^2} + \frac{C}{\nu}\|\nabla_x u\|^2_{L^2}, \quad (3.17)$$

so we have, by Grönwall,

$$\begin{aligned}&\|\nabla_x A\|^2_{L^\infty(0,T;L^2)} + 8k\|\nabla_x A\|^2_{L^2(0,T;L^2)} + \nu\|\Delta_x A\|^2_{L^2(0,T:L^2)}\\ &\le \exp\left(\frac{C}{\nu^3}\|u\|^2_{L^\infty(0,T;L^2)}\|\nabla_x u\|^2_{L^2(0,T;L^2)}\right)\left(\|\nabla_x A(0)\|^2_{L^2} + \frac{C}{\nu}\|\nabla_x u\|^2_{L^2(0,T:L^2)}\right) \le C_1,\end{aligned} \quad (3.18)$$

where $C_1$ depends only on the norm of initial data. Then we take the curl $(-\partial_2, \partial_1)\cdot$ to the velocity equation of (3.4), multiply $\omega = \partial_1 u_2 - \partial_2 u_1$, and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2} + \|\nabla_x \omega\|^2_{L^2} = \int \omega \nabla_x^\perp \cdot (\nabla_x \cdot \sigma)dx. \quad (3.19)$$

**Controlling $\int \omega\nabla_x^\perp \cdot (\nabla_x \cdot \sigma)dx$.** If we try to estimate the term $\int \omega\nabla_x^\perp \cdot (\nabla_x \cdot \sigma)dx$ using Cauchy-Schwarz inequality, the term $\|\nabla_x \omega\|^2_{L^2}$ becomes uncontrollable.

However, a closer look at the term allows us a better estimate. Note that

$$\begin{aligned}
\int \omega \nabla_x^\perp \cdot (\nabla_x \cdot \sigma) dx &= \int \omega \left( (\partial_1^2 - \partial_2^2)\sigma_{12} + \partial_1\partial_2(\sigma_{22} - \sigma_{11}) \right) dx \\
&= \int (\partial_1^2 - \partial_2^2)\omega\sigma_{12} + \partial_1\partial_2\omega(\sigma_{22} - \sigma_{11}) dx \\
&= \eta \int \left( (\partial_1^2 - \partial_2^2)\omega A_{12} + \partial_1\partial_2\omega(A_{22} - A_{11}) \right) (\nabla_x u) : A dx.
\end{aligned} \tag{3.20}$$

Also, note that

$$(\nabla_x u) : A = \partial_1 u_1 A_{11} + \partial_1 u_2 A_{12} + \partial_2 u_1 A_{12} + \partial_2 u_2 A_{22}$$

and we introduce the stream function $\psi$, that is, $u = \nabla^\perp \psi = (-\partial_2\psi, \partial_1\psi)$. Then we have

$$\omega = \Delta_x \psi, \ -\partial_1 u_1 = \partial_2 u_2 = \partial_1\partial_2\psi, \ \partial_1 u_2 = \partial_1^2\psi, \ \partial_2 u_1 = -\partial_2^2\psi.$$

Therefore, we have

$$\begin{aligned}
\int \omega \nabla_x^\perp \cdot (\nabla_x \cdot \sigma) dx &= \eta \int \left( (\partial_1^2 - \partial_2^2)\omega A_{12} + \partial_1\partial_2\omega(A_{22} - A_{11}) \right) (\nabla_x u) : A dx \\
&= \eta \int \left( \Delta_x(\partial_1^2 - \partial_2^2)\psi A_{12} + \Delta_x\partial_1\partial_2\psi(A_{22} - A_{11}) \right) \\
&\quad \times \left( \partial_1\partial_2\psi(A_{22} - A_{11}) + (\partial_1^2 - \partial_2^2)\psi A_{12} \right) dx \\
&= -\eta \int |A_{12}\nabla_x(\partial_1^2 - \partial_2^2)\psi + (A_{22} - A_{11})\nabla_x(\partial_1\partial_2)\psi|^2 dx + I
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
|I| &\leq C\eta \int |\nabla_x(\Delta_x)\psi||A||\nabla_x A| dx \leq C\eta \left\| \nabla_x \omega \right\|_{L^2} \left\| \nabla_x A \right\|_{L^2} \\
&\leq \frac{1}{2} \left\| \nabla_x \omega \right\|_{L^2}^2 + C\eta^2 \left\| \nabla_x A \right\|_{L^2}^2
\end{aligned} \tag{3.22}$$

Applying this to (3.19), we obtain

$$\frac{d}{dt} \left\| \omega \right\|_{L^2}^2 + \left\| \nabla_x \omega \right\|_{L^2}^2 + 2\eta \left\| (\nabla(\partial_k u) : A)_k \right\|_{L^2}^2 \leq C\eta^2 \left\| \nabla_x A \right\|_{L^2}^2, \tag{3.23}$$

and by Grönwall we obtain

$$\begin{aligned}&\|\omega\|^2_{L^\infty(0,T;L^2)} + \|\nabla_x\omega\|^2_{L^2(0,T;L^2)} + 2\eta \|(\nabla(\partial_k u) : A)_k\|^2_{L^2(0,T;L^2)} \\ &\leq \|\omega(0)\|^2_{L^2} + C\eta^2 \|\nabla_x A\|^2_{L^2(0,T;L^2)} = C_2,\end{aligned} \tag{3.24}$$

where again $C_2$ depends only on the initial data. Then we multiply $(\Delta)^2 A$ to the fourth equation of (3.4) and integrate to obtain

$$\begin{aligned}&\frac{d}{dt}\|\Delta_x A\|^2_{L^2} + 4k\|\Delta_x A\|^2_{L^2} + \nu\|\nabla_x\Delta_x A\|^2_{L^2} \\ &= \int (\Delta_x^2 A)\left(-u\cdot\nabla_x A + (\nabla_x u)A + A(\nabla_x u)^T - 2(\nabla_x u : A)A\right) dx.\end{aligned} \tag{3.25}$$

The first term in the left-hand side is controlled by

$$\begin{aligned}&\|\nabla_x\Delta_x A\|_{L^2}\left(\|\nabla_x u\|_{L^4}\|\nabla_x A\|_{L^4} + \|u\|_{L^4}\|\Delta_x A\|_{L^4}\right) \\ &\leq \frac{\nu}{4}\|\nabla_x\Delta_x A\|^2_{L^2} + \|\nabla_x A\|^2_{L^2}\|\Delta_x A\|^2_{L^2} + \frac{C}{\nu^2}\|\nabla_x u\|^2_{L^2}\|\nabla_x\omega\|^2_{L^2} \\ &\quad + \frac{C}{\nu^3}\|u\|^2_{L^2}\|\nabla_x u\|^2_{L^2}\|\Delta_x A\|^2_{L^2}.\end{aligned}$$

The second and the third term is controlled by

$$\begin{aligned}&\|\nabla_x\Delta_x A\|_{L^2}\left(\|\nabla_x\omega\|_{L^2}\|A\|_{L^\infty} + \|\nabla_x u\|_{L^4}\|\nabla_x A\|_{L^4}\right) \\ &\leq \frac{\nu}{4}\|\nabla_x\Delta_x A\|^2_{L^2} + \frac{C}{\nu}\|\nabla_x\omega\|^2_{L^2} + \frac{C}{\nu}\left(\|\nabla_x u\|^2_{L^2} + \|\Delta_x u\|^2_{L^2}\right) \\ &\quad + \frac{C}{k\nu^2}\|\nabla_x A\|^2_{L^2} + k\|\Delta_x A\|^2_{L^2}\end{aligned}$$

and the last term is controlled by the same term, by $\|A\|_{L^\infty} \leq 1$. Therefore, we have

$$\|\Delta_x A\|^2_{L^\infty(0,T;L^2)} + \nu\|\nabla_x\Delta_x A\|^2_{L^2(0,T;L^2)} \leq C\exp(C_1 + C\eta^2)(\|\Delta_x A(0)\|^2_{L^2} + C(1+\eta^4)) = C_3 \tag{3.26}$$

where $C_3$ depends only on the initial data, using (3.18) instead of (3.15) when controlling $\|\nabla_x A\|^2_{L^2(0,T:L^2)}$. FInally, we multiply $-\Delta_x\omega$ to the vorticity equation and integrating to obtain

$$\frac{d}{dt}\left\|\nabla_x\omega\right\|_{L^2}^2+\left\|\Delta_x\omega\right\|_{L^2}^2=\int\Delta_x\omega u\cdot\nabla_x\omega+\int(-\Delta_x\omega)\nabla_x^{\perp}\cdot(\nabla_x\cdot\sigma)dx.$$

However, by similar calculation to (3.21), we have

$$\int(-\Delta_x\omega)\nabla_x^{\perp}\cdot(\nabla_x\cdot\sigma)dx=-\eta\int|(\nabla_x\Delta_x u):A|^2dx+I', \tag{3.27}$$

where

$$\begin{aligned}|I'|&\leq C\eta\left\|\Delta_x\omega\right\|_{L^2}\left(\left\|\nabla_x\omega\right\|_{L^4}\left\|\nabla_xA\right\|_{L^4}+\left\|\omega\right\|_{L^4}\left\|\Delta_xA\right\|_{L^4}+\left\|\omega\right\|_{L^2}\left\|\nabla_xA\right\|_{L^\infty}^2\right)\\&\leq C\eta\left\|\Delta_x\omega\right\|_{L^2}^{\frac{3}{2}}\left\|\nabla_xA\right\|_{L^2}^{\frac{1}{2}}\left\|\Delta_xA\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla_x\omega\right\|_{L^2}^{\frac{1}{2}}\\&+C\eta\left\|\Delta_x\omega\right\|_{L^2}\left\|\Delta_xA\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla_x\Delta_xA\right\|_{L^2}^{\frac{1}{2}}\left\|\omega\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla_x\omega\right\|_{L^2}^{\frac{1}{2}}\\&+C\eta\left\|\Delta_x\omega\right\|_{L^2}\left\|\nabla_xA\right\|_{L^2}\left\|\nabla_x\Delta_xA\right\|_{L^2}\left\|\omega\right\|_{L^2}\\&\leq\frac{1}{4}\left\|\Delta_x\omega\right\|_{L^2}^2+C\eta^3\left\|\nabla_xA\right\|_{L^2}^2\left\|\Delta_xA\right\|_{L^2}^2\left\|\nabla_x\omega\right\|_{L^2}^2\\&+C\eta^2\left(\left\|\nabla_xA\right\|_{L^2}^2\left\|\nabla_x\omega\right\|_{L^2}^2+\left\|\nabla_x\Delta_xA\right\|_{L^2}^2\left\|\omega\right\|_{L^2}^2\right)\\&+C\eta^2\left\|\nabla_xA\right\|_{L^2}^2\left\|\nabla_x\Delta_xA\right\|_{L^2}^2\left\|\omega\right\|_{L^2}^2,\end{aligned} \tag{3.28}$$

and

$$\int\Delta_x\omega u\nabla_x\omega dx=-\int u\cdot\nabla_x\left(|\nabla_x\omega|^2\right)dx-\int(\nabla_xu\nabla_x\omega)\nabla_x\omega dx, \tag{3.29}$$

with Gagliardo-Nirenberg applied to conclude that

$$\left|\int(\nabla_xu\nabla_x\omega)\nabla_x\omega dx\right|\leq\left\|\nabla_xu\right\|_{L^3}\left\|\nabla_x\omega\right\|_{L^3}^2\leq\frac{1}{4}\left\|\Delta_x\omega\right\|_{L^2}^2+C\left\|\omega\right\|_{L^2}^2\left\|u\right\|_{L^2}\left\|\nabla_x\omega\right\|_{L^2}^2 \tag{3.30}$$

To sum up, we have

$$\frac{d}{dt}\|\nabla_x\omega\|_{L^2}^2+\frac{1}{2}\|\Delta_x\omega\|_{L^2}^2+\eta\|(\nabla_x\Delta_x u):A\|_{L^2}^2$$
$$\leq C(\|u_0\|_{L^2}\|\omega\|_{L^2}^2+\eta^3C_1\|\Delta_x A\|_{L^2}^2+\eta^2\|\nabla_x A\|_{L^2}^2)\|\nabla_x\omega\|_{L^2}^2+C\eta^2C_1C_2\|\nabla_x\Delta_x A\|_{L^2}^2 \tag{3.31}$$

and by Grönwall we have

$$\begin{aligned}&\|\nabla_x\omega\|_{L^\infty(0,T;L^2)}^2+\frac{1}{2}\|\Delta_x\omega\|_{L^2(0,T;L^2)}+\eta\|(\nabla_x\Delta_x u):A\|_{L^2(0,T;L^2)}^2\\&\leq \exp\left(C_4(1+\eta^3C_3)\right)\left(\|\nabla_x\omega(0)\|_{L^2}^2+C_5(1+\eta^4)\right)\end{aligned} \tag{3.32}$$

where $C_4, C_5$ depend only on the initial data (and parameters except for $\eta$).

**Remark 11.** *Same cancellation argument works for the original Doi model (3.1), so we can prove global well-posedness of diffusive Doi model for any $\eta > 0$. In the presence of an external forcing (applied to the fluid field), the global well-posedness can still be proved by similar estimates.*

### 3.2.2 Local well-posedness

In this section we prove the local well-posedness. Before we start, we briefly check the difficulty in the usual contraction mapping scheme. We define the Banach space $B = X \times Y$, where

$$X = L^\infty(0,T_0;\mathbb{P}W^{2,2}(\mathbb{T}^2))\cap L^2(0,T_0;\mathbb{P}W^{3,2}(\mathbb{T}^2))$$

and

$$Y = L^\infty(0,T_0;W^{2,2}(\mathbb{T}^2))\cap L^2(0,T_0;W^{3,2}(\mathbb{T}^2)).$$

We set up a fixed point equation $U = F(U)$ in $B$ for $U = (u, A)$, where $F(U) = (u^{new}, A^{new})$ given by

$$\begin{aligned} u^{new}(t) &= e^{t\Delta_x} u_0 + Q_1(u, u) + L(u, A), \\ A^{new}(t) &= e^{(\nu\Delta_x - 4k)t} A_0 + Q_2(u, A) + \frac{1}{2}\left(1 - e^{-4kt}\right)\mathbb{I}_2, \end{aligned} \tag{3.33}$$

where

$$Q_1(u, v) = -\int_0^t e^{(t-s)\Delta_x}\mathbb{P}(u(s)\cdot\nabla_x v(s))ds, \tag{3.34}$$

$$L(u, A) = \int_0^t e^{(t-s)\Delta_x}\eta\mathbb{P}(\mathrm{div}_x((\nabla_x u(s) : A(s))A(s)))ds, \tag{3.35}$$

and

$$\begin{aligned} Q_2(u, A) &= \int_0^t e^{(t-s)(\nu\Delta_x - 4k)} B(u, A)(s)ds, \\ B &= (-u(s)\cdot\nabla_x A(s) + (\nabla_x u(s))A(s) + A(s)(\nabla_x u(s))^T - 2(\nabla_x u(s) : A(s))A(s)). \end{aligned} \tag{3.36}$$

We can easily check that

$$\begin{aligned} \|Q_1(u, v)\|_X &\leq C\sqrt{T_0}\,\|u\|_X\,\|v\|_X\,, \\ \|L(u, A)\|_X &\leq C\,\|u\|_X\,\|A\|_Y\,, \\ \|Q_2(u, A)\|_Y &\leq C\sqrt{T_0}\,\|u\|_X\left(\|A\|_Y + \|A\|_Y^2\right). \end{aligned} \tag{3.37}$$

For example, if we let $q = Q_2(u, A)$, then $q$ is the solution of the equation

$$\partial_t q - \nu\Delta_x q + 4kq = R,\ \ q(0) = 0, \tag{3.38}$$

where

$$Q_2(u, A) = \int_0^t e^{(t-s)(\nu\Delta_x - 4k)} R(s)ds.$$

Then by the standard estimate we obtain

$$\|q\|^2_{L^\infty(0,T_0;W^{2,2})\cap L^2(0,T_0;W^{3,2})} \le C \|R\|^2_{L^2(0,T;W^{1,2})},$$

and in this case

$$\begin{aligned} C\|R(s)\|_{W^{1,2}} &\le \|u(s)\|_{L^\infty}\|\nabla_x A(s)\|_{L^2} + \|\nabla_x u(s)\|_{L^4}\|\nabla_x A(s)\|_{L^4} \\ &+ \|u(s)\|_{L^\infty}\|\Delta_x A(s)\|_{L^2} + \|\nabla_x u(s)\|_{L^2}\|A(s)\|_{L^\infty} \\ &+ \|\Delta u(s)\|_{L^2}\|A(s)\|_{L^\infty} + \|\nabla_x u(s)\|_{L^2}\|A(s)\|^2_{L^\infty} \\ &+ \|\Delta_x u(s)\|_{L^2}\|A(s)\|^2_{L^\infty} + \|\nabla_x u(s)\|_{L^4}\|\nabla_x A(s)\|_{L^4}\|A\|_{L^\infty} \\ &\le C\|u(s)\|_{W^{2,2}}\|A(s)\|_{W^{2,2}}(1+\|A(s)\|_{W^{2,2}}) \le C\|u\|_X\|A\|_Y(1+\|A\|_Y). \end{aligned} \tag{3.39}$$

The problem is in $L(u,A)$: to find a contraction mapping we need to guarantee that $F$ is a mapping from a ball $B(0,R) \subset B$ to itself: however, the bounds for $\|u^{new}(t)\|_X$ that we can obtain from this method is $U_0 + \|u\|_X (C_1 T_0 \|u\|_X + C_2\|A\|_Y)$, and if $\|A\|_Y \ge \frac{1}{C_2}$ then this method fails to bound which holds for both $\|u\|_X$ and $\|u^{new}\|_X$. Therefore, instead of contraction mapping principle, we use an approximation scheme for $u$ equation and go with contraction mapping principle for $A$ equation.

**Approximation scheme.** Suppose that $u_n \in X$ is given with $\|u_n\|_X < \infty$ and $u_n(0) = u_0$. We solve

$$\begin{cases} \partial_t A_n + u_n \cdot \nabla_x A_n \\ = (\nabla_x u_n)A_n + A_n(\nabla_x u^T) - 2(\nabla_x u_n : A_n)A_n + \nu\Delta_x A_n + 2k(2A_n - \mathbb{I}_2), \\ A_n(0) = A_0. \end{cases} \tag{3.40}$$

For this equation contraction mapping works well, and local well-posedness is guaranteed, and proposition 3.2.2, a priori estimates (3.15), (3.18), (3.26) are satisfied except that all the estimates concerning $u$ are replaced by $u_n$. This means that $A_n$ is guaranteed to exist until the time of existence of $u_n$. Our approximation scheme for $u_{n+1}$ is the following.

$$\begin{cases} \partial_t u_{n+1} + u_{n+1} \cdot \nabla_x u_{n+1} \\ = -\nabla_x p_{n+1} + \Delta_x u_{n+1} + \nabla_x \cdot \mathcal{J}_{n+1} \left( (\eta(\mathcal{J}_{n+1} \left( \nabla_x u_{n+1} \right) : A_n) A_n) \right), \\ \nabla_x \cdot u_{n+1} = 0, \ u_{n+1}(0) = u_0, \end{cases} \tag{3.41}$$

where $\mathcal{J}_{n+1} f$ is the orthogonal projection of $f$ into space spanned by eigenvectors corresponding to first $(n+1)$-th eigenvalues. Therefore, $\mathcal{J}_{n+1}$ s are symmetric (in fact self-adjoint) and they commute with differentiation. Then we can prove the local well-posedness of the system (3.41) via contraction mapping, since for the modified polymer-induced nonlinear structure

$$L^{n+1}(u_{n+1}, A_n) = \int_0^t e^{(t-s)\Delta_x} \eta \mathbb{P}(\nabla_x \cdot \mathcal{J}_{n+1}((\mathcal{J}_{n+1}(\nabla_x u_{n+1}) : A_n) A_n))(s) ds$$

has the estimate

$$\left\| L^{n+1}(u_{n+1}, A_n) \right\|_X \leq \eta(n+1) C \sqrt{T_0} \left\| u_{n+1} \right\|_X \left\| A_n \right\|_Y^2 .$$

We then find an estimate of $\|u_{n+1}\|_X$ independent of $n$, which allow us to guarantee existence of the solution $u_{n+1}$ until the time of existence of $u_n$, and also the existence of a weak limit of the sequence $\{u_n\}_n$. This estimate can be obtained in the same manner as (3.12), (3.24), and (3.32), which is essentially the usual energy method together with the cancellation structures (3.21), (3.27), and those estimates hold with the bound depending only on initial data and $T_0$, independent of $n$. Then we have the uniform bounds

$$\|u_n\|_X \leq D_1, \|A_n\|_Y \leq D_2,$$

so by compactness we have weak limits $u \in X, A \in Y$, and we can check that for some subsequence of $(u_n, A_n)$, again denoted by $(u_n, A_n)$

$$\begin{aligned}(\nabla_x u_n)A_n &\to (\nabla_x u)A \text{ in } L^2(0,T;L^2)\\ (\nabla_x u_n : A_n)A_n &\to (\nabla_x u : A)A \text{ in } L^2(0,T;L^2)\\ u_n \cdot \nabla_x u_n &\to u \cdot \nabla_x u \text{ in } L^2(0,T;L^2)\end{aligned} \tag{3.42}$$

and

$$\nabla_x \cdot \mathcal{J}_n((\mathcal{J}_n(\nabla_x u_n) : A_n)A_n) \to \nabla_x \cdot ((\nabla_x u : A)A) \text{ in } L^2(0,T;W^{-1,2}). \tag{3.43}$$

Note that $u_n, A_n \in L^\infty(0,T_0;W^{2,2})$ are uniformly bounded and $\partial_t u_n, \partial_t A_n \in L^2(0,T;W^{1,2})$ are also uniformly bounded, by Aubin-Lions there is a subsequence of $A_n$ converging to $A$ and $u_n$ converging to $u$ strongly in $C([0,T];W^{2-\epsilon,2})$ for small $\epsilon > 0$. For the first convergence, note that

$$\begin{aligned}\|(\nabla_x u_n)A_n - (\nabla_x u)A\|_{L^2} &\le \|\nabla_x u_n\|_{L^\infty}\|A_n - A\|_{L^2} + \|\nabla_x(u_n - u)\|_{L^2}\|A\|_{L^\infty}\\ &\le D_1\|A_n - A\|_{L^2} + \|u_n - u\|_{W^{1,2}} D_2\end{aligned} \tag{3.44}$$

and by Aubin-Lions we are done. Other two can be shown similarly. The last convergence is also straightforward:

$$\begin{aligned}&\mathcal{J}_n\left(\left(\mathcal{J}_n\left(\nabla_x u_n\right) : A_n\right)A_n\right) - (\nabla_x u : A)A\\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned} \tag{3.45}$$

where

$$
\begin{gathered}
I_1 = \mathcal{J}_n\left(\left(\mathcal{J}_n\left(\nabla_x u_n\right) : A_n\right)\left(A_n - A\right)\right), \; I_2 = \mathcal{J}_n\left(\left(\mathcal{J}_n\left(\nabla_x u_n\right) : A_n - A\right) A\right), \\
I_3 = \mathcal{J}_n\left(\left(\mathcal{J}_n\left(\nabla_x u_n - \nabla_x u\right) : A\right) A\right), \; I_4 = \mathcal{J}_n\left(\left(\left(\mathcal{J}_n\left(\nabla_x u\right) - \nabla_x u\right) : A\right) A\right), \\
I_5 = \mathcal{J}_n\left(\left(\left(\nabla_x u\right) : A\right) A\right) - (\nabla_x u : A)A.
\end{gathered} \tag{3.46}
$$

We have

$$
\left\|I_1\right\|_{L^2}, \left\|I_2\right\|_{L^2} \leq D_2 \left\|\nabla_x u_n\right\|_{L^\infty} \left\|(A_n - A)\right\|_{L^2} \leq D_2 \left\|(A_n - A)\right\|_{L^\infty(0,T;W^{1,2})} \left\|u_n\right\|_{W^{3,2}}. \tag{3.47}
$$

By Aubin-Lions lemma, $\|I_1\|_{L^2(0,T_0;L^2)} + \|I_2\|_{L^2(0,T_0;L^2)} \to 0$ as $n \to \infty$. $I_3$ can be similarly treated by Aubin-Lions lemma, and $I_4$, $I_5$ can be treated by the property of $\mathcal{J}_n$.

**Uniqueness of the solution.** Suppose that $(u, A) \in B$ and $(v, B) \in B$ are two solutions to the initial value problem (3.4). Then we have

$$
\begin{gathered}
\partial_t(u - v) + u \cdot \nabla_x(u - v) + (u - v) \cdot \nabla_x v = -\nabla_x(p_u - p_v) + \Delta_x(u - v) \\
+\eta \nabla_x \cdot (((\nabla_x u - \nabla_x v) : A)A + ((\nabla_x v) : (A - B))A + ((\nabla_x v) : B)(A - B)), \\
\partial_t(A - B) + u \cdot \nabla_x(A - B) + (u - v) \cdot \nabla_x B = (\nabla_x(u - v))A + (\nabla_x v)(A - B) \\
+A(\nabla_x(u - v))^T + (A - B)(\nabla_x v)^T - 4k(A - B) + \nu \Delta_x(A - B) \\
-2(((\nabla_x u - \nabla_x v) : A)A + ((\nabla_x v) : (A - B))A + ((\nabla_x v) : B)(A - B))
\end{gathered} \tag{3.48}
$$

and standard relative energy estimate gives

$$\begin{aligned}
&\frac{d}{dt}\|u-v\|_{L^2}^2+\|\nabla_x(u-v)\|_{L^2}^2+2\eta\|\nabla_x(u-v):A\|_{L^2}^2\\
&\quad\le C\|\nabla_x v\|_{L^2}\|A-B\|_{L^2}\|u-v\|_{L^2},\\
&\frac{1}{2}\frac{d}{dt}\|A-B\|_{L^2}^2+\nu\|\nabla_x(A-B)\|_{L^2}^2+4k\|A-B\|_{L^2}^2\\
&\quad\le C\|A-B\|_{L^2}I,\\
&I=\|u-v\|_{L^2}\|\nabla_x A\|_{L^2}+\|A-B\|_{L^2}\|\nabla_x v\|_{L^2}\\
&\quad+\|(u-v)\|_{L^2}\|A\|_{W^{1,2}}+\|A-B\|_{L^2}\|\nabla_x v\|_{L^2}
\end{aligned}\tag{3.49}$$

and by a priori estimates on $u, v, A, B$ we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|u-v\|_{L^2}^2+\|A-B\|_{L^2}^2\right)+\|\nabla_x(u-v)\|_{L^2}+\nu\|A-B\|_{L^2}^2\\
&\quad\le C\left(\|u-v\|_{L^2}^2+\|A-B\|_{L^2}^2\right)
\end{aligned}\tag{3.50}$$

and by Grönwall inequality $u=v$, $A=B$.

## 3.3 A priori estimate for (3.1)

In this section, we establish a priori estimates for (3.1). More precisely, we prove the following theorem:

**Theorem 3.3.1.** *Let $(u, f)$ be a stroong solution of (3.1) on $[0,T]$ with initial data satisfying $M_0(0)\in L^\infty, \sigma_E(0)\in W^{1,2}$, $M_4(0)\in W^{2,2}, M_6(0)\in W^{1,2}$, $u_0\in\mathbb{P}W^{2,2}$, and $\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}(f\log f-f+1)(0)dmdx<\infty$. Then $(u,f)$ satisfies the bounds (3.52), (3.61), (3.64) for $n=4$, (3.71), (3.77), (3.78), and (3.83).*

### 3.3.1 Free energy estimate

The first one is the well-known free energy estimate.

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|u\|_{L^2}^2+\int_{\mathbb{T}^2}\int_{\mathbb{S}^1} f\log f-f+1dmdx\right)+k\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}\frac{|\nabla_m f|^2}{f}dmdx\\ +\nu&\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}\frac{|\nabla_x f|^2}{f}dmdx+\eta\int((\nabla_x u):m\otimes m)^2 fdmdx+\|\nabla_x u\|_{L^2}^2\,dx=0.\end{aligned} \tag{3.51}$$

From this we can obtain the bound

$$\begin{aligned}&\|u\|_{L^\infty(0,T;L^2)}^2+\sup_{t\in[0,T]}\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}(f\log f-f+1)(t)dmdx\\ &+\|\nabla_x u\|_{L^2(0,T;L^2)}^2+\left\|\nabla_{m,x}\sqrt{f}\right\|_{L^2(0,T;L^2(\mathbb{T}^2\times\mathbb{S}^1))}^2\leq B_1\end{aligned} \tag{3.52}$$

where

$$B_1=C\|u_0\|_{L^2}^2+\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}(f\log f-f+1)(0)dmdx$$

with $C$ a constant depending only on parameters $k,\nu$ and $\eta$.

### 3.3.2 Estimate on moments

In this section, we investigate bounds on moments, which are useful in establishing bounds of elastic and viscous stresses.

**Local coordinates.** To study the evolution of moments and elastic tensors, it is useful to write the Fokker-Planck equation of (3.1) in the local expression. The configuration space $\mathbb{S}^1$ can be represented by $m(\theta)=(\cos\theta,\sin\theta)$, and the Fokker-Planck equation of (3.1) is

$$\partial_t f+u\cdot\nabla_x f=k\partial_\theta^2 f+\nu\Delta_x f-\partial_\theta\left(m(\theta)^\perp\cdot((\nabla_x u)m(\theta))\,f\right) \tag{3.53}$$

where

$$m(\theta)^{\perp} \cdot ((\nabla_x u) m(\theta)) = \frac{1}{2}\cos 2\theta(\partial_1 u_2 + \partial_2 u_1) + \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1) - \frac{1}{2}\sin 2\theta(\partial_1 u_1 - \partial_2 u_2).$$

Also, the expression for elastic stress can be rewritten as:

$$\sigma_E = \int_0^{2\pi} \frac{1}{2} f \left( \cos 2\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sin 2\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) d\theta \tag{3.54}$$

and

$$\sigma_V = \int_0^{2\pi} \frac{\eta f}{4} \left(\cos 2\theta(\partial_1 u_1 - \partial_2 u_2) + \sin 2\theta(\partial_2 u_1 + \partial_1 u_2)\right) \left(\mathbb{I}_2 + \cos 2\theta \mathbb{J}_1 + \sin 2\theta \mathbb{J}_2\right) d\theta. \tag{3.55}$$

where

$$\begin{aligned} \mathbb{J}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \mathbb{J}_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.56}$$

**Evolution of moments.** The evolution equation for $M_n$, $n > 0$ is derived from (3.1):

$$\partial_t M_n + u \cdot \nabla_x M_n = T_{1,n} M_n + \nu \Delta_x M_n + T_{2,n}(\nabla_x u, M_{n+2}) \tag{3.57}$$

where $T_{1,n}$ is a constant-coefficient (depending on $n$) matrix and $T_{2,n}(A, B)$ is a constant-coefficient (also depending on $n$) bilinear tensor on $A$ and $B$. On the other hand, when $n = 0$, the evolution equation for $M_0$ is given by

$$\partial_t M_0 + u \cdot \nabla_x M_0 = \nu \Delta_x M_0 \tag{3.58}$$

and from this we obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\left\|M_0\right\|_{L^2}^2+\nu\left\|\nabla_x M_0\right\|_{L^2}^2=0, \\ \frac{d}{dt}\left\|\nabla_x M_0\right\|_{L^2}^2+\nu\left\|\Delta_x M_0\right\|_{L^2}^2\leq C\left\|u\right\|_{L^2}^2\left\|\nabla_x u\right\|_{L^2}^2\left\|\nabla_x M_0\right\|_{L^2}^2 \\ \leq CB_1\left\|\nabla_x u\right\|_{L^2}^2\left\|\nabla_x M_0\right\|_{L^2}^2 \end{aligned} \tag{3.59}$$

and so

$$\begin{aligned} \left\|M_0\right\|_{L^\infty(0,T;L^2)}^2+\left\|\nabla_x M_0\right\|_{L^2(0,T;L^2)}^2\leq B_2, \\ \left\|\nabla_x M_0\right\|_{L^\infty(0,T;L^2)}^2+\left\|\Delta_x M_0\right\|_{L^2(0,T;L^2)}^2\leq B_3, \end{aligned} \tag{3.60}$$

and

$$\left\|M_0\right\|_{L^\infty(0,T;L^\infty)}\leq B_4, \tag{3.61}$$

where

$$B_2=C\left\|M_0(0)\right\|_{L^2}^2,\ B_3=C\exp(CB_1^2)\left\|\nabla_x M_0(0)\right\|_{L^2}^2,\ B_4=\left\|M_0(0)\right\|_{L^\infty}$$

where again $C$ depends only on parameters, and estimate (3.61) follows from the maximum principle. One simple but important observation is the following:

$$|M_n^I(x,t)|\leq M_0(x,t) \tag{3.62}$$

due to positivity of $f$ and compactness of $\mathbb{S}^1$. By (3.62), we obtain estimates for $M_n$, $n>0$; from

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|M_n\right\|_{L^2}^2+\nu\left\|\nabla_x M_n\right\|_{L^2}^2\leq C_n\left(\left\|M_n\right\|_{L^2}^2+\left\|\nabla_x u\right\|_{L^2}\left\|M_n\right\|_{L^2}B_4\right), \\ \frac{1}{2}\frac{d}{dt}\left\|\nabla_x M_n\right\|_{L^2}^2+\nu\left\|\Delta_x M_n\right\|_{L^2}^2 \\ \leq\left(\left\|u\right\|_{L^4}\left\|\nabla_x M_n\right\|_{L^4}+C_n\left\|\nabla_x u\right\|_{L^2}\right)\left\|\Delta_x M_n\right\|_{L^2}+C_n\left\|\nabla_x M_n\right\|_{L^2}^2, \end{aligned} \tag{3.63}$$

where $C_n$ depends only on $n$ and parameters ($k$ in these cases), we have

$$\begin{aligned} \|M_n\|^2_{L^\infty(0,T;L^2)} + \|\nabla_x M_n\|^2_{L^2(0,T;L^2)} &\le B_{5,n}, \\ \|\nabla_x M_n\|^2_{L^\infty(0,T;L^2)} + \|\Delta_x M_n\|^2_{L^2(0,T;L^2)} &\le B_{6,n} \end{aligned} \tag{3.64}$$

where

$$B_{5,n} = C\exp(C_n T)\left(\|M_n(0)\|^2_{L^2} + B_4^2 B_1\right),$$

$$B_{6,n} = C\exp(C_n T + B_1^2)\left(\|\nabla_x M_n(0)\|^2_{L^2} + C_n B_1\right).$$

Also, similar to estimate (3.26), we have

$$\begin{aligned} \frac{d}{dt}\|\Delta_x M_n\|^2_{L^2} + \nu\|\nabla_x\Delta_x M_n\|^2_{L^2} &\le C_n\|\Delta_x M_n\|^2_{L^2} \\ &+C_n\|\nabla_x\Delta_x M_n\|_{L^2} I, \\ I = \|u\|_{L^4}\|\Delta_x M_n\|_{L^4} &+ \|\nabla_x u\|_{L^4}\|\nabla_x M_n\|_{L^4} \\ +\|\Delta_x u\|_{L^2}\|M_{n+2}\|_{L^\infty} &+ \|\nabla_x u\|_{L^4}\|\nabla_x M_{n+2}\|_{L^4} \end{aligned} \tag{3.65}$$

After burying $\|\nabla_x\Delta_x M_n\|_{L^2}$ term using Cauchy-Schwarz inequality, the first term is bounded by

$$C_n\|u\|^2_{L^2}\|\nabla_x u\|^2_{L^2}\|\Delta_x M_n\|^2_{L^2} \le C_n B_1\|\nabla_x u\|^2_{L^2}\|\Delta_x M_n\|^2_{L^2},$$

the second term is bounded by

$$\begin{aligned} &C_n\left(\|\Delta_x u\|^2_{L^2}\|\nabla_x M_n\|^2_{L^2} + \|\nabla_x u\|^2_{L^2}\|\Delta_x M_n\|^2_{L^2}\right) \\ &\le C_n\|\nabla_x u\|^2_{L^2}\|\Delta_x M_n\|^2_{L^2} + C_n B_{6,n}\|\Delta_x u\|^2_{L^2}, \end{aligned} \tag{3.66}$$

the third term is bounded by $C_n B_4^2\|\Delta_x u\|^2_{L^2}$, and the fourth term is bounded by

$$\begin{aligned} &C_n\left(\|\nabla_x u\|^2_{L^2}\|\Delta_x M_{n+2}\|^2_{L^2} + \|\nabla_x M_{n+2}\|^2_{L^2}\|\Delta_x u\|^2_{L^2}\right) \\ &\le C_n B_{6,n+2}\|\Delta_x u\|^2_{L^2} + C_n\|\nabla_x u\|^2_{L^2}\|\Delta_x M_{n+2}\|^2_{L^2}. \end{aligned} \tag{3.67}$$

To sum up, we have

$$\frac{d}{dt}\left\|\Delta_x M_n\right\|_{L^2}^2+\nu\left\|\nabla_x\Delta_x M_n\right\|_{L^2}^2\leq C_n\left(1+(B_1+1)\left\|\nabla_x u\right\|_{L^2}^2\right)\left\|\Delta_x M_n\right\|_{L^2}^2$$
$$+C_n\left((B_{6,n}+B_4^2+B_{6,n+2})\left\|\nabla_x u\right\|_{L^2}^2+\left\|\nabla_x u\right\|_{L^2}^2\left\|\Delta_x M_{n+2}\right\|_{L^2}^2\right) \tag{3.68}$$

and therefore

$$\left\|\Delta_x M_n\right\|_{L^\infty(0,T;L^2)}^2+\left\|\nabla_x\Delta_x M_n\right\|_{L^2(0,T;L^2)}^2\leq B_{7,n}\left(B_{8,n}+\left\|\nabla_x u\right\|_{L^\infty(0,T;L^2)}^2 B_{6,n+2}\right) \tag{3.69}$$

where

$$B_{7,n}=C_n\exp\left(T+(B_1+1)B_1\right),\ B_{8,n}=B_1(B_{6,n}+B_4^2+B_{6,n+2})+\left\|\Delta_x M_n(0)\right\|_{L^2}^2.$$

### 3.3.3 Control of elastic stress

Elastic stress can be bounded by bounds on $M_2$, since each component of $\sigma_E$ is a component of $M_2$, but we can get better estimates:

$$\partial_t\sigma_E+u\cdot\nabla_x\sigma_E=-4k\sigma_E+\nu\Delta_x\sigma_E+T'_{2,2}(\nabla_x u,M_4) \tag{3.70}$$

where $T'_{2,2}$ is another constant-coefficient bilinear tensor. Then

$$\left\|\sigma_E\right\|_{L^\infty\cap L^2(0,T;L^2)}+\left\|\nabla_x\sigma_E\right\|_{L^2(0,T;L^2)}^2\leq C(\left\|\sigma_E(0)\right\|_{L^2}^2+B_1)=B_9,$$
$$\left\|\nabla_x\sigma_E\right\|_{L^\infty\cap L^2(0,T;L^2)}+\left\|\Delta_x\sigma_E\right\|_{L^2(0,T;L^2)}^2\leq C\exp(B_1^2)\left(\left\|\nabla_x\sigma_E(0)\right\|_{L^2}^2+B_4^2B_1\right)=B_{10}. \tag{3.71}$$

### 3.3.4 Control of higher derivatives of $u$ and viscous stress

We take curl to the Navier-Stokes equation to obtain

$$\partial_t \omega + u \cdot \nabla_x \omega = \Delta_x \omega + \nabla_x^\perp \cdot \nabla_x \cdot \sigma_E + \eta \nabla_x^\perp \cdot \nabla_x \cdot \int_{\mathbb{S}^1} ((\nabla_x u) : m \otimes m) m \otimes m f dm \quad (3.72)$$

and

$$\begin{gathered} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla_x \omega\|_{L^2}^2 = - \int_{\mathbb{T}^2} \nabla_x^\perp \omega \cdot (\nabla_x \cdot \sigma_E) dx \\ + \eta \int_{\mathbb{T}^2} \omega \nabla_x^\perp \cdot \nabla_x \cdot \int_{\mathbb{S}^1} ((\nabla_x u) : m \otimes m) m \otimes m f dm dx. \end{gathered} \quad (3.73)$$

We investigate the last term: note that $\omega = \epsilon_{ij} \partial_i u_j$ where $\epsilon_{ij}$ is the Levi-Civita symbol (in this case just $\epsilon_{12} = 1$ and $\epsilon_{21} = -1$) and the last term can be written as:

$$\begin{gathered} \eta \int_{\mathbb{T}^2} \epsilon_{ij} \partial_i u_j \left( \epsilon_{k\ell} \partial_k \partial_p \int_{\mathbb{S}^1} ((\nabla_x u) : m \otimes m) m_p m_\ell f dm \right) dx \\ = \eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} (\epsilon_{ij} \epsilon_{k\ell} \partial_k \partial_p \partial_i u_j m_p m_\ell) ((\nabla_x u) : m \otimes m) f dm dx \\ = \eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} ((\partial_i \partial_p \partial_i u_\ell - \partial_j \partial_k \partial_\ell u_j) m_p m_\ell) ((\nabla_x u) : m \otimes m) f dm dx \\ = \eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} \Delta_x ((\nabla_x u) : m \otimes m) ((\nabla_x u) : m \otimes m) f dm dx \end{gathered} \quad (3.74)$$

and

$$\begin{gathered} \eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} \Delta_x ((\nabla_x u) : m \otimes m) ((\nabla_x u) : m \otimes m) f dm dx \\ = -\eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} |\nabla_x ((\nabla_x u) : m \otimes m)|^2 f dm dx - \eta \int_{\mathbb{T}^2} T_3(\nabla_x \nabla_x u, \nabla_x u, \nabla_x M_4) dx \end{gathered} \quad (3.75)$$

where $T_3$ is a constant-coefficient trilinear form. Therefore,

$$\begin{gathered} \frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla_x \omega\|_{L^2}^2 + \eta \int_{\mathbb{T}^2} \int_{\mathbb{S}^1} |\nabla_x ((\nabla_x u) : m \otimes m)|^2 f dm dx \\ \leq C \|\nabla_x \sigma_E\|_{L^2}^2 + C\eta \|\nabla_x \nabla_x u\|_{L^2} \|\nabla_x u\|_{L^4} \|\nabla_x M_4\|_{L^4} \\ \leq C \|\nabla_x \sigma_E\|_{L^2}^2 + \frac{1}{2} \|\nabla_x \omega\|_{L^2}^2 + C \|\omega\|_{L^2}^2 \|\nabla_x M_4\|_{L^2}^2 \|\Delta_x M_4\|_{L^2}^2 \end{gathered} \quad (3.76)$$

where $C$ depends only on parameters (the last $C$ is proportional to $\eta^4$) and

$$\|\omega\|_{L^\infty(0,T;L^2)}^2 + \|\nabla_x\omega\|_{L^2(0,T;L^2)}^2 \le C\exp\left(B_{5,4}B_{6,4}\right)\left(\|\omega(0)\|_{L^2}^2 + B_9\right) = B_{11}. \quad (3.77)$$

Then, by (3.77) and (3.69) with $n = 4$ we have

$$\|\Delta_x M_4\|_{L^\infty(0,T;L^2)}^2 + \|\nabla_x\Delta_x M_4\|_{L^2(0,T;L^2)}^2 \le B_{7,4}\left(B_{8,4} + B_{11}B_{6,6}\right) = B_{12}. \quad (3.78)$$

Finally, by multiplying $-\Delta_x\omega$ to (3.72) and integrating, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla_x\omega\|_{L^2}^2 + \|\Delta_x\omega\|_{L^2}^2 = \int_{\mathbb{T}^2} u\cdot\nabla_x\omega\Delta_x\omega dx + \int_{\mathbb{T}^2}\Delta_x\omega(\nabla_x^\perp\cdot\nabla_x\cdot\sigma_E)dx \\ -\eta\int_{\mathbb{T}^2}\Delta_x\omega\nabla_x^\perp\cdot\nabla_x\cdot\int_{\mathbb{S}^1}((\nabla_x u):m\otimes m)m\otimes mfdmdx.\end{aligned} \quad (3.79)$$

Again the last term can be rewritten as, by the same calculation to (3.74),

$$\begin{aligned}&\eta\int_{\mathbb{T}^2}\Delta_x\omega\nabla_x^\perp\cdot\nabla_x\cdot\int_{\mathbb{S}^1}((\nabla_x u):m\otimes m)m\otimes mfdmdx \\ &= \eta\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}\Delta_x^2\left((\nabla_x u):m\otimes m\right)\left((\nabla_x u):m\otimes m\right)fdmdx\end{aligned} \quad (3.80)$$

and

$$\begin{aligned}&\eta\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}\Delta_x^2\left((\nabla_x u):m\otimes m\right)\left((\nabla_x u):m\otimes m\right)fdmdx \\ &= \eta\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}\left(\Delta_x\left((\nabla_x u):m\otimes m\right)\right)^2 fdmdx \\ &+\eta\int_{\mathbb{T}^2}T_4\left(\nabla_x\Delta_x u,\nabla_x\nabla_x u,\nabla_x M_4\right)dx + \eta\int_{\mathbb{T}^2}T_5\left(\nabla_x\Delta_x u,\nabla_x u,\Delta_x M_4\right)dx\end{aligned} \quad (3.81)$$

again $T_4$ and $T_5$ are constant-coefficient trilinear tensors. Thus,

$$\begin{aligned}&\frac{d}{dt}\|\nabla_x\omega\|_{L^2}^2+\|\Delta_x\omega\|_{L^2}^2+\eta\int_{\mathbb{T}^2}\int_{\mathbb{S}^1}(\Delta_x((\nabla_x u):m\otimes m))^2 f dmdx\\ &\le C(\|u\|_{L^2}^2\|\nabla_x u\|_{L^2}^2+\|\nabla_x M_4\|_{L^2}^2\|\Delta_x M_4\|_{L^2}^2+\|\nabla_x\Delta_x M_4\|_{L^2}^2)\|\nabla_x\omega\|_{L^2}^2\\ &\quad+C(\|\Delta_x\sigma_E\|_{L^2}^2+\|\nabla_x u\|_{L^2}^2\|\Delta_x M_4\|_{L^2}^2)\end{aligned} \tag{3.82}$$

and

$$\|\nabla_x\omega\|_{L^\infty(0,T;L^2)}^2+\|\Delta_x\omega\|_{L^2(0,T;L^2)}^2\le B_{!3} \tag{3.83}$$

where

$$B_{13}=C\exp\left(C(B_1^2+B_{6,4}^2+B_{12})\right)\left(\|\nabla_x\omega(0)\|_{L^2}^2+B_{10}+B_{11}B_{6,4}\right).$$

**Remark 12.** *The cancellation structures (3.74) and (3.80) hold for 3D case also. To illustrate, we have*

$$\begin{aligned}&\int dmdx\nabla_x\wedge u\cdot\nabla_x\wedge(\nabla_x\cdot(((\nabla_x u):m\otimes m)m\otimes mf))\\ &=\int dmdx\epsilon_{ij'k'}\partial_{j'}u_{k'}\epsilon_{ijk}\partial_j\partial_\ell(((\nabla_x u):m\otimes m)m_\ell m_k f)\\ &=\int dmdx\epsilon_{ij'k'}\epsilon_{ijk}(\partial_j\partial_j'\partial_\ell u_{k'})m_\ell m_k((\nabla_x u):m\otimes m)f\\ &=\int dmdx(\partial_j^2\partial_\ell u_k-\partial_j\partial_k\partial_\ell u_j)m_\ell m_k((\nabla_x u):m\otimes m)f\\ &=\int dmdx(\Delta_x((\nabla_x u):m\otimes m))((\nabla_x u):m\otimes m)f.\end{aligned} \tag{3.84}$$

## 3.4 Local well-posedness of (3.1)

In this section, we prove local well-posedness of (3.1). Once local well-posedness is established, global well-posedness follows from the a priori estimates established in the previous section.

### 3.4.1 Local existence of the solution

We follow the method presented in Constantin and Seregin: the existence of the system follows from uniform bounds on the approximate system

$$
\begin{gathered}
\partial_t u + u \cdot \nabla_x u = -\nabla_x p + \Delta_x u + \nabla_x \cdot \mathcal{J}_\ell(\sigma_E) \\
+\nabla_x \cdot \mathcal{J}_\ell\left(\eta \int_{\mathbb{S}^1} (\mathcal{J}_\ell(\nabla_x u) : m \otimes m) m \otimes m f dm\right), \\
\nabla_x \cdot u = 0, \\
\partial_t f + \mathcal{J}_\ell(u) \cdot \nabla_x f = k\Delta_m f + \nu \Delta_x f - \nabla_m \cdot (P_{m^\perp}(\mathcal{J}_\ell(\nabla_x u) m f))
\end{gathered}
\tag{3.85}
$$

which satisfies the same bounds (3.52), (3.61), (3.64) for $n = 4$, (3.71), (3.77), (3.78), and (3.83), and solutions of these systems are obtained by an implicit iteration scheme, using linear equations in each step of the approximation:

$$
\begin{gathered}
\partial_t u_{n+1} + u_n \cdot \nabla_x u_{n+1} = -\nabla_x p_{n+1} + \Delta_x u_{n+1} \\
+\nabla_x \cdot \mathcal{J}_\ell(\sigma_E(f_n)) + \nabla_x \cdot \mathcal{J}_\ell\left(\eta \int_{\mathbb{S}^1} ((\mathcal{J}_\ell(\nabla_x u_{n+1})) : m \otimes m)\, m \otimes m f_n dm\right), \\
\nabla_x \cdot u_{n+1} = 0, \\
\partial_t f_{n+1} + \mathcal{J}_\ell(u_n) \cdot \nabla_x f_{n+1} = k\Delta_m f_{n+1} + \nu \Delta_x f_{n+1} - \nabla_m \cdot (P_{m^\perp}(\mathcal{J}_\ell \nabla_x u_n) m f_{n+1})\,.
\end{gathered}
\tag{3.86}
$$

Existence of (3.85) follows from standard arguments in Fokker-Planck equation: first each system in (3.86) has smooth solution (same regularity as in the a priori estimate, uniform bounds in $n$), and therefore we have weakly convergent subsequence $u_n$ converging to $u$ in $L^\infty(0,T;W^{2,2}) \cap L^2(0,T;W^{3,2})$, and by Aubin-Lions and Rellich-Kondrachov we have $u_n \to u \in L^2(0,T;W^{2-\epsilon,2})$, which is a strong convergence. Also we establish similar strong convergence in moments, and we establish convergence of evolution equation of $u_n$ to that of $u$, which proves that the limit $u$ is a weak solution of (3.85), and since $u$ has enough regularity it is a strong solution. We also find the limit $f$ of $f_n$, using the results from the trigonometric moment problem. We see that

$f$ is a weak solution of (3.85), and that $f$ is given by the density, and the standard theory gives the free energy estimate (3.52).

**Uniform bounds on solutions of (3.86).** Suppose that

$$\|u_q\|^2_{L^\infty(0,T;W^{j,2})\cap L^2(0,T;W^{j+1,2})} \leq B^j_{app},$$

and $\|M^q_2\|^2_{L^\infty(0,T;W^{j,2})\cap L^2(0,T;W^{j+1,2})} + \|M^q_4\|^2_{L^\infty(0,T;W^{j,2})\cap L^2(0,T;W^{j+1,2})} \leq F^j_{app}$ for $j = 0,1,2,$ and $\|M^q_6\|^2_{L^\infty(0,T:W^{j,2})\cap L^2(0,T;W^{j+1,2})} \leq F^j_{app}$ for $j = 0,1,$ and for all $q \leq n$. We will determined the exact values of $B^j_{app}$ and $F^j_{app}$ in the subsequent estimates. Then we have

$$\begin{aligned} \frac{d}{dt}\|u_{n+1}\|^2_{L^2} + \|\nabla_x u_{n+1}\|^2_{L^2} + \eta\int\left(\mathcal{J}_\ell(\nabla_x u_{n+1}) : m\otimes m\right)^2 f_n dm dx \\ \leq C\|\sigma_E(f_n)\|^2_{L^2} \leq CB^2_4, \end{aligned} \tag{3.87}$$

from the energy estimate, and from this we obtain

$$\|u_{n+1}\|^2_{L^\infty(0,T;L^2)} + \|\nabla_x u_{n+1}\|^2_{L^2(0,T;L^2)} \leq \|u(0)\|^2_{L^2} + CB^2_4 T = B^0_{app}. \tag{3.88}$$

The vorticity equation becomes

$$\begin{aligned} \partial_t\omega_{n+1} + u_n\cdot\nabla_x\omega_{n+1} = -\epsilon_{ik}\partial_i u^n_j \partial_j u^{n+1}_k + \Delta_x\omega_{n+1} + \nabla^\perp_x\cdot\nabla_x\cdot\mathcal{J}_\ell\left(\sigma_E(f_n)\right) \\ +\nabla^\perp_x\cdot\nabla_x\cdot\mathcal{J}_\ell\left(\eta\left(\left(\mathcal{J}_\ell(\nabla_x u_{n+1})\right) : m\otimes m\right) m\otimes m f_n dm\right) \end{aligned} \tag{3.89}$$

which leads to the estimate

$$\begin{aligned}
&\frac{d}{dt}\left\|\omega_{n+1}\right\|_{L^2}^2+\left\|\nabla_x \omega_{n+1}\right\|_{L^2}^2+\eta \int\left(\mathcal{J}_{\ell}(\nabla_x \nabla_x u_{n+1}): m \otimes m\right)^2 f_n dm dx \\
&\leq C\left(\left\|\nabla_x \sigma_E(f_n)\right\|_{L^2}^2+\left\|\omega_{n+1}\right\|_{L^2}^2\left(\left\|\nabla_x M_4^n\right\|_{L^2}^2\left\|\Delta_x M_4^n\right\|_{L^2}^2+1\right)+\left\|\omega_n\right\|_{L^2}^2\right), \\
&\frac{d}{dt}\left\|\nabla_x \omega_{n+1}\right\|_{L^2}^2+\left\|\Delta_x \omega\right\|_{L^2}^2+\eta\left(\Delta_x((\nabla_x u): m \otimes m)^2 f_n dm dx\right) \\
&\leq C\left\|\nabla_x \omega_{n+1}\right\|_{L^2}^2 I_1+C I_2, \\
&I_1=\left(\left\|u_n\right\|_{L^2}^2\left\|\nabla_x u_n\right\|_{L^2}^2+\left\|\nabla_x M_4^n\right\|_{L^2}^2\left\|\Delta_x M_4^n\right\|_{L^2}^2+\left\|\nabla_x \Delta_x M_4^n\right\|_{L^2}^2+\left\|\omega_n\right\|_{L^2}^2\right), \\
&I_2=\left(\left\|\Delta_x \sigma_E(f_n)\right\|_{L^2}^2+\left\|\nabla_x u_{n+1}\right\|_{L^2}^2\left\|\Delta_x M_4^n\right\|_{L^2}^2+\left\|\nabla_x \omega_n\right\|_{L^2}^2\left\|\omega_{n+1}\right\|_{L^2}^2\right).
\end{aligned} \tag{3.90}$$

Also we have

$$\begin{aligned}
&\left\|M_4^n\right\|_{L^{\infty}(0,T;L^2)}^2+\left\|\nabla_x M_4^n\right\|_{L^2(0,T;L^2)}^2 \leq e^{CT}(\left\|M_4^n(0)\right\|_{L^2}^2+B_4 B_{app}^1) \\
&\leq e^{CT}(\left\|M_4(0)\right\|_{L^2}^2+B_4 B_{app}^0)=F_{app}^0, \\
&\left\|\nabla_x M_4^n\right\|_{L^{\infty}(0,T;L^2)}^2+\left\|\Delta_x M_4^n\right\|_{L^2(0,T;L^2)}^2 \leq e^{(B_{app}^0)^2+CT}\left(\left\|\nabla_x M_4(0)\right\|_{L^2}^2+B_{app}^0\right)=F_{app}^1,
\end{aligned} \tag{3.91}$$

and the same bound for $M_2^n$ (or $\sigma_E(f_n)$) and $M_6^n$ in place of $M_4^n$. From this we conclude that

$$\left\|\omega_{n+1}\right\|_{L^{\infty}(0,T;L^2)}^2+\left\|\nabla_x \omega_{n+1}\right\|_{L^2}^2 \leq e^{CT+C(F_{app}^1)^2}\left(\left\|\omega(0)\right\|_{L^2}^2+C F_{app}^0+B_{app}^0\right)=B_{app}^1. \tag{3.92}$$

Then

$$\begin{aligned}
&\left\|\Delta_x M_4^n\right\|_{L^{\infty}(0,T;L^2)}^2+\left\|\nabla_x \Delta_x M_4^n\right\|_{L^2(0,T;L^2)}^2 \\
&\leq e^{C(B_{app}^0+1)B_{app}^0}\left(\left\|\Delta_x M_4(0)\right\|_{L^2}^2+C(F_{app}^1+B_4^2)B_{app}^1\right)=F_{app}^2,
\end{aligned} \tag{3.93}$$

and finally

$$\begin{aligned}
&\left\|\nabla_x \omega_{n+1}\right\|_{L^{\infty}(0,T;L^2)}^2+\left\|\Delta_x \omega_{n+1}\right\|_{L^2(0,T;L^2)}^2 \\
&\leq e^{C((B_{app}^0)^2+(F_{app}^1)^2+F_{app}^2+B_{app}^0)}(\left\|\nabla_x \omega(0)\right\|_{L^2}^2+F_{app}^1+F_{app}^1 B_{app}^1+(B_{app}^1)^2)=B_{app}^2.
\end{aligned} \tag{3.94}$$

This verifies that $\|u_n\|_{L^\infty(0,T:W^{2,2})\cap L^2(0,T;W^{3,2})}$ is uniformly bounded. Furthermore, $\partial_t u_n$ is also uniformly bounded in $L^2(0,T;L^2)$.

**Convergence of $u_n$ to $u$ and existence of solution for $u$ equation of (3.85).**

By Banach-Alaoglu, we have a subsequence of $u_n$, weakly converging to $u$ in $L^\infty(0,T;W^{2,2})\cap L^2(0,T;W^{3,2})$, and by Aubin-Lions, in fact

$$u_n \to u \in C([0,T];W^{2-\epsilon,2}) \text{ strongly}$$

for small enough $\epsilon > 0$, for a further subsequence. We extract further subsequence that $u_n \to u$, $\nabla_x u_n \to \nabla_x u$ almost everywhere. Moreover, we can find a further subsequence such that there is $\sigma_E$ and $M_4$ such that

$$\sigma_E(f_n) \to \sigma_E, M_4(f_n) = M_4^n \to M_4 \in C([0,T];W^{2-\epsilon,2}) \text{ strongly} \tag{3.95}$$

also. To show that $u$ is a solution of (3.85), we first recall that $W^{2-\epsilon,2}(\mathbb{T}^2)$ is a Banach algebra for $\epsilon < 1$, and also a refined version of Agmon inequality:

$$\|u\|_{L^\infty(\mathbb{T}^2)} \le C\,\|u\|_{W^{2-\epsilon,2}(\mathbb{T}^2)} \tag{3.96}$$

Now the evolution equation for $u_{n+1}$ of (3.86) can be rewritten as the following:

$$\begin{gathered}\partial_t u_{n+1} = \mathbb{P}I,\\ I = (-u_n\cdot\nabla_x u_{n+1} + \Delta_x u_{n+1} + \nabla_x\cdot\mathcal{J}_\ell(\sigma_E(f_n)) + \nabla_x\cdot\mathcal{J}_\ell T(\mathcal{J}_\ell(\nabla_x u_{n+1}), M_4(f_n)))\end{gathered} \tag{3.97}$$

where $\mathbb{P}$ is the orthogonal projection to divergence-free vector field and $T$ is a constant-coefficient bilinear tensor between two arguments. We first control $u_n\cdot\nabla_x u_{n+1}$. Since $u_n \to u \in C([0,T];L^\infty)$ by Agmon, $\nabla_x u_{n+1} \to \nabla_x u \in C([0,T];L^2)$, and $\mathbb{P}(u_n\cdot\nabla_x u_{n+1}) \to \mathbb{P}(u\cdot\nabla_x u) \in C([0,T];L^2)$. Also, $\mathbb{P}\Delta_x u_{n+1} \to \mathbb{P}\Delta_x u \in C([0,T];W^{-1,2})$,

$\mathbb{P}\nabla_x \cdot \mathcal{J}_\ell(\sigma_E(f_n)) \to \mathbb{P}\nabla_x \cdot \mathcal{J}_\ell(\sigma_E) \in C([0,T]; W^{1-\epsilon,2})$ (uniformly in $\ell$), and finally since $M_4(f_n) \to M_4 \in C([0,T]; L^\infty)$ by Agmon and $\nabla_x u_{n+1} \to \nabla_x u \in C([0,T]; L^2)$, $T(\mathcal{J}_\ell(\nabla_x u_{n+1}), M_4(f_n)) \to T(\mathcal{J}_\ell(\nabla_x u), M_4) \in C([0,T]; L^2)$ and therefore we have $\nabla_x \cdot \mathcal{J}_\ell T(\mathcal{J}_\ell(\nabla_x u_{n+1}), M_4(f_n)) \to \nabla_x \cdot \mathcal{J}_\ell T(\mathcal{J}_\ell(\nabla_x u), M_4) \in C([0,T]; W^{-1,2})$ uniformly in $\ell$. Finally, since $\partial_t u_n$ is weakly convergent to $\partial_t u$ in $L^2(0,T;L^2)$, we see that $u$ is a weak solution of $u$-part of (3.85).

**Convergence of $f_n$ to $f$.** To deal with this issue, we recall the result from the trigonometric moment problem:

**Theorem 3.4.1** (Carathéodory-Toeplitz)**.** *For a complex sequence $s = (s_j) \in \mathbb{N} \cup \{0\}$ the following are equivalent:*

1. *There exists a (nonnegative) radon measure $\mu$ on $\mathbb{T}^1$ such that*

$$s_j = \int_{\mathbb{S}^1} e^{-ij\theta} d\mu(\theta)$$

   *for all $j \in \mathbb{Z}$. Here $s_{-j} := \bar{s}_j$ for $n \geq 1$.*

2. *$\sum_{j,k=0}^{\infty} s_{j-k} c_k \bar{c}_j \geq 0$ for all finite complex sequences $(c_j)_{j \in \mathbb{N} \cup \{0\}}$.*

*The measure $\mu$ is uniquely determined by determined by $s_j$.*

In this regard, we define the trigonometric moments

$$s_j(f_n) = \int_{\mathbb{S}^1} e^{-ij\theta} f_n(\theta) d\theta.$$

By the very similar argument as before, we obtain

$$\|s_j(f_n)\|^2_{L^\infty(0,T;L^2)} + \|\nabla_x s_j(f_n)\|^2_{L^2} \leq B_4^2(1 + j^2 B^0_{app}),$$

which is uniform in $n$, and $\|\partial_t s_j(f_n)\|_{W^{-1,2}} \leq (\sqrt{B^2_{app}} + C\nu) \|\nabla_x s_j(f_n)\|_{L^2} + CjB_4 \|\nabla_x u\|_{L^2}$

so $\partial_t s_j(f_n)$ is uniformly bounded in $L^2(0,T;W^{-1,2})$. Therefore, again by Aubin-Lions and diagonalization argument, we can find a further subsequence of $f_n$ such that $s_j(f_n)$ converges to some $s_j$ strongly in $L^2(0,T;L^2)$, and therefore almost everywhere, for all $j$. Also, we see that

$$\sum_{j,k=0}^{\infty} s_{j-k}(f_n)(x,t)c_k\bar{c}_j \geq 0$$

for all $x, t$ for all finite complex sequences $(c_j)_{j\in\mathbb{N}\cup\{0\}}$, and by almost everywhere convergence $\sum_{j,k=0}^{\infty} s_{j-k}(x,t)c_k\bar{c}_j \geq 0$ for almost every $(x,t)$. Therefore, we see that there exists a (nonnegative) radon measure $\mu$ such that $s_j = \int_{\mathbb{S}^1} e^{-ij\theta} d\mu(\theta)$ for all $j \in \mathbb{Z}$. Next we show that for almost every $(x,t)$ $f_n(x,t,m)dm$ converges to $\mu(x,t;dm)$ weakly. The argument is analogous to the method of moment: since $\int_{\mathbb{S}^1} f_n(x,t,m)dm \leq B_4$, $(f_n(x,t)dm)$ is uniformly bounded and obviously uniformly tight. Therefore, by Prokhorov's theorem $f_n(x,t,m)dm$ converges weakly to some Radon measure $\nu(x,t;dm)$. Thus, $s_j(f_n)(x,t,m)dm$ converges to $\int e^{-ij\theta} d\nu(x,t;dm)$ for each $j$, but this equals to $s_j = \int_{\mathbb{S}^1} e^{-ij\theta} d\mu(\theta)$. Since $\mu$ is determined by the trigonometric moments, $\nu = \mu$. Note that $M_k[\mu] = M_k$ also holds.

**$f$ is a weak solution of (3.85).** To show that $\mu$ is the weak solution of (3.85), we write the Fokker-Planck equation of (3.86) in the weak form as in (3.8), and check the convergence. First, we note that

$$\begin{aligned} &|E| = 0, \\ &E = \{t \in (0,T) : \\ &\big|\{x \in \mathbb{T}^2 : f_n(x,t,dm) \text{ does not converge weakly to } \mu(x,t;dm)\}\big| > 0\}, \end{aligned} \tag{3.98}$$

For $t \in (0,T) - E$, for almost all $x$, $\int_{\mathbb{S}^1} \phi_m(m) f_{n+1}(x,t,m)dm \to \int_{\mathbb{S}^1} \phi_m(m)\mu(x,t;dm)$ by weak convergence, and $\left|\int_{\mathbb{S}^1} \phi_m(m) f_{n+1}(x,t,m)dm\right| \leq C_{\phi_m} B_4$ for almost every $x$, so by Dominated convergence theorem,

$$\int_{\mathbb{T}^2\times\mathbb{S}^1} \phi_x(x)\phi_m(m)f_{n+1}(x,t;m)dmdx \to \int_{\mathbb{T}^2\times\mathbb{S}^1} \phi_x(x)\phi_m(m)\mu(x,t;dm)dx.$$

The second term is easy since the initial data of $f_n$ are just mollified ones of $f(0)$. We can show convergence for other terms except for the ones involving velocity field $u$, using the very same argument: by weak convergence we have almost everywhere convergence for $m$ integral part first, and for that term we have uniform bound (depending on $\phi_m$), then we apply dominated convergence theorem. For the terms involving velocity fields $u$, we apply the generalized dominated convergence theorem instead. Then standard parabolic regularity theory guarantees that actually $\mu$ is given by density $f(x,t,m)dmdx$, and if initial entropy is finite, then it remains finite, with $f(t) \in W^{1,1}(\mathbb{T}^2 \times \mathbb{S}^1)$, and $\int_0^T \int_{\mathbb{T}^2\times\mathbb{S}^1} \frac{|\nabla_{x,m}f|^2}{f} dmdt < \infty$.

**Solution of (3.1).** Existence of a solution of (3.1) is just a repetition of arguments for establishing solutions of (3.85). In this case, we set up $\ell \to \infty$.

### 3.4.2 Uniqueness of the solution

Uniqueness of the solution follows from relative energy method. Suppose that $(u, f)$ and $(v, g)$ are two solutions of (3.1) with same initial data $(u_0, f_0)$ satisfying our assumptions.

**Control of $u - v$.** By taking $L^2$ estimates, vorticity estimate, and $W^{1,2}$ norm estimates for $u - v$, we have

$$\begin{aligned} &\frac{d}{dt}\|u-v\|^2_{W^{2,2}} + \|(u-v)\|^2_{W^{3,2}} \le C_1(t)\|u-v\|^2_{W^{2,2}} \\ &+C\|\sigma_E(f)-\sigma_E(g)\|^2_{W^{2,2}} + C_2(t)\|M_4(f)-M_4(g)\|^2_{W^{2,2}} \end{aligned} \tag{3.99}$$

where $C_1(t), C_2(t) \in L^1(0,T)$ coming from norms of $v$ and $C$ is a constant independent of time.

**Control of $\int_{\mathbb{S}^1} |f-g|dm$.** The key quantity of control is $\int_{\mathbb{S}^1} |f-g|dm$. Let $\mathrm{sgn}_\beta$ be a smooth, increasing regularization of the sign function such that $\mathrm{sgn}_\beta(s) = \mathrm{sign}(s)$ for $|s| \geq \beta$, and let $|s|_\beta = \int_0^s \mathrm{sgn}_\beta(r)dr$. Then as $\beta \to 0$, we have $|s|_\beta \to |s|$. Then by subtracting two Fokker-Planck equations of (3.1) for $f$ and $g$, then by replacing $\phi_x(x)\phi_m(m)$ in (3.8) by $\mathrm{sgn}_\beta(f-g)\int |f-g|_\beta dm$ (we can do this since $C^\infty(\mathbb{T}^2)\otimes C^\infty(\mathbb{S}^1)$ is dense in $L^p(\mathbb{T}^2 \times \mathbb{S}^1, f(x,t,m)dxdm(g(x,t,m)dxdm))$ for any $p \geq 1$), then by checking that terms from diffusion are positive, and finally taking the limit $\beta \to 0$ (and dividing by $\left\| \int_{\mathbb{S}^1} |f-g|dm \right\|_{L^2}$), we obtain

$$\begin{aligned} &\left\| \int_{\mathbb{S}^1} |f-g|dm \right\|_{L^2}(t) \\ \leq &\int_0^t \left( \|u-v\|_{W^{1,\infty}} \left\| \int_{\mathbb{S}^1} |\nabla_{x,m} g|dm \right\|_{L^2} + CB_4 \left\| \nabla_x(u-v) \right\|_{L^2} \right) ds. \end{aligned} \tag{3.100}$$

Noting that

$$\left\| \int_{\mathbb{S}^1} |\nabla g|dm \right\|_{L^2} = \left( \int_{\mathbb{T}^2} \left( \int_{\mathbb{S}^1} |\nabla g|dm \right)^2 dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{T}^2} B_4 \int_{\mathbb{S}^1} \frac{|\nabla g|^2}{g} dmdx \right)^{\frac{1}{2}}$$

we obtain

$$\begin{aligned} &\left\| \int_{\mathbb{S}^1} |f-g|dm \right\|_{L^2}(t) \\ \leq &\int_0^t \left( B_4^{\frac{1}{2}} \left( \int_{\mathbb{T}^2\times\mathbb{S}^1} \frac{|\nabla_{x,m} g|^2}{g}(s)dmdx \right)^{\frac{1}{2}} + CB_4 \right) \|(u-v)(s)\|_{W^{3,2}} ds \\ \leq &C \|u-v\|_{L^2(0,t;W^{3,2})} \left( \sqrt{t} + \left( \int_0^t \int_{\mathbb{T}^2\times\mathbb{S}^1} \frac{|\nabla_{x,m} g|^2}{g}(s)dmdxds \right)^{\frac{1}{2}} \right) \\ \leq &C(\sqrt{B_1} + \sqrt{t}) \|u-v\|_{L^2(0,t;W^{3,2})} \end{aligned} \tag{3.101}$$

thanks to the free energy estimate (3.52). Therefore,

$$\left\| \int_{\mathbb{S}^1} |f-g| dm(t) \right\|_{L^2}^2 \leq C(1+t) \|u-v\|_L^2 (0,t;W^{3,2})^2. \tag{3.102}$$

**Control of moments.** Finally, we apply the relative energy estimates for evolution equation of moments (3.57), and apply (3.102) in closing the effect of higher moments to obtain the following:

$$\begin{aligned}
&\frac{d}{dt} \|M_n(f)-M_n(g)\|_{W^{1,2}}^2 + \nu \|M_n(f)-M_n(g)\|_{W^{2,2}}^2 \\
&\leq C^n \|M_n(f)-M_n(g)\|_{W^{1,2}}^2 + C_n \|u-v\|_{W^{1,2}}^2 + C_3(t) \left\| \int_{\mathbb{S}^1} |f-g| dm \right\|_{L^2}^2, \\
&\frac{d}{dt} \|M_n(f)-M_n(g)\|_{W^{2,2}}^2 + \nu \|M_n(f)-M_n(g)\|_{W^{3,2}}^2 \\
&\leq C^n \|M_n(f)-M_n(g)\|_{W^{2,2}}^2 + C^n \|u-v\|_{W^{2,2}}^2 + \frac{\nu}{2} \|\Delta_x(M_{n+2}(f)-M_{n+2}(g))\|_{L^2}^2 \\
&\quad + \frac{C^n}{\nu} \left( \left\| \int_{\mathbb{S}^1} |f-g| dm \right\|_{L^2}^2 + \|\nabla_x (M_{n+2}(f)-M_{n+2}(g)\|_{L^2}^2 \right),
\end{aligned} \tag{3.103}$$

where $C_3 \in L^1(0,T)$ depending on the norms of $v$ and $g$, and $C^n$ are constants depending only on $n$ and norms of $v$ and $g$. Summing up (3.99), (3.102), and (3.103), we finally obtain

$$\begin{aligned}
&\frac{d}{dt} F(t) + G(t) \leq \phi(t) \left( F(t) + \int_0^t G(s) ds \right), \phi \in L^1(0,T), G(t) = \|(u-v)(t)\|_{W^{3,2}}^2, \\
&F(t) = \|u-v\|_{W^{2,2}}^2 + \|\sigma_E(f)-\sigma_E(g)\|_{W^{2,2}}^2 \\
&\quad + \|M_4(f)-M_4(g)\|_{W^{2,2}}^2 + \|M_6(f)-M_6(g)\|_{W^{1,2}}^2 (t)
\end{aligned} \tag{3.104}$$

as desired. Noting that $F(0)=0$ and applying Grönwall's inequality, we see that $F(t) = \int_0^t G(s) ds = 0$. This proves the uniqueness of the solution, and completes the proof of Theorem 3.1.2.

# Chapter 4

# Lagrangian-Eulerian method

## 4.1 Introduction

In this Chapter, we consider the non-diffusive Oldroyd-B system coupled with Navier-Stokes system:

$$\begin{cases} \partial_t u - \nu \Delta u = \mathbb{H} \left( \text{div } (\sigma - u \otimes u) \right), \\ \nabla \cdot u = 0, \\ \partial_t \sigma + u \cdot \nabla \sigma = (\nabla u)\sigma + \sigma(\nabla u)^T - 2k\sigma + 2\rho K((\nabla u) + (\nabla u)^T), \\ u(x,0) = u_0(x), \sigma(x,0) = \sigma_0(x). \end{cases} \tag{4.1}$$

on $(x,t) \in \mathbb{R}^2 \times [0,T)$, where $\mathbb{H} = (\mathbb{I} + R \otimes R)$ with $R = (R_1, R_2)$ being Riesz transforms and $\nu, \rho K, k$ are positive constants. We recall that $u$ is the velocity field of the solvent and $\sigma$ is the stress field due to the presence of polymer. This system exhibits viscoelasticity, which means that it can show both viscous and elastic behavior, and one of the consequence is that due to elastic effect, the behavior of the solution depends on the history of its deformation.

**Main differences between existing literatures and this work** There are two main distinctive features in this work compared to other works on Lagrangian-Eulerian formulation. First, in [30], Constantin generalized and formalized the Lagrangian-Eulerian formulation to various incompressible flow system, including 3D Euler equation, Surface-Quasigeostrophic equation(SQG), Porous medium equation, Boussinesq system, and Oldroyd-B system coupled with Stokes system. Formally, all these systems can be written as the following:

$$\begin{cases} \partial_t X = \mathcal{U}(X, \tau), \\ \partial_t \tau = \mathcal{T}(X, \tau). \end{cases} \tag{4.2}$$

Here $X(\cdot, t) : \mathbb{R}^d \to \mathbb{R}^d$ is a volume preserving diffeomorphism, which represents the particle path, and $\tau$ is a Lagrangian (material) field variable coupled with the evolution of the fluid. The corresponding Eulerian variables are

$$u = \partial_t X \circ X^{-1}, \ \sigma = \tau \circ X^{-1}. \tag{4.3}$$

The evolution of $\tau$ follows the following ODE:

$$\frac{d}{dt}\tau = F(g, \tau) \tag{4.4}$$

where $g = \nabla_x u \circ X$ and $F$ is a polynomial (depending on the model). The Eulerian velocity is determined by $\sigma$ in terms of a fixed (model-dependent), time-independent operator

$$u = \mathbb{U}(\sigma) \tag{4.5}$$

and the Eulerian velocity gradient is also determined by $\sigma$ again in terms of a fixed (model-dependent), time-independent operator

$$\nabla_x u = \mathbb{G}(\sigma). \tag{4.6}$$

The nonlinearities $\mathcal{U}, \mathcal{T}$ are given by

$$\begin{aligned} \mathcal{U}(X, \tau) &= \mathbb{U}(\tau \circ X^{-1}) \circ X, \\ \mathcal{T}(X, \tau) &= F(\mathbb{G}(\tau \circ X^{-1}) \circ X, \tau). \end{aligned} \tag{4.7}$$

This setting does not include the Oldroyd-B model coupled with Navier-Stokes system. In Section 4.2, we will see that $\mathbb{U}$ and $\mathbb{G}$ are time dependent. This seems natural since Oldroyd-B model describes behavior of viscoelastic liquids, and they usually have memory effect; the behavior of fluid particle depends on the history of its deformation. The time-dependence of $\mathbb{U}$ and $\mathbb{G}$ imposes mathematical difficulties and demands us to use the full capacity of Lipschitz-in-time norm: in [30] we only needed $\|\cdot\|_{L^\infty(0,T;C^{\alpha,p})}$ norm to be small to perform contraction mapping; in this work we need $\|\cdot\|_{Lip(0,T;C^{\alpha,p})}$ norm to be small.

## 4.2 The Lagrangian-Eulerian formulation

The solution map for $u(x,t)$ of (4.1) is

$$u(x,t) = \mathbb{L}_\nu(u_0)(x,t) + \int_0^t g_{\nu(t-s)} * (\mathbb{H}(\text{div } (\sigma - u \otimes u)))(x,s)ds. \tag{4.8}$$

where

$$\mathbb{L}_\nu(u_0)(x,t) = g_{\nu t} * u_0(x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4\nu t}} u_0(y)dy. \tag{4.9}$$

Also, note that the gradient of the velocity satisfies

$$(\nabla u)(x,t) = \mathbb{L}_\nu(\nabla u_0)(x,t) + \int_0^t \left(g_{\nu(t-s)} * (\mathbb{H}\nabla \text{div } (\sigma - u \otimes u))\right)(x,s)ds. \quad (4.10)$$

We will denote

$$\begin{cases} \mathbb{U}(f)(x,t) = \displaystyle\int_0^t (g_{\nu(t-s)} * \mathbb{H}\text{div } f)(x,s)ds, \\ \mathbb{G}(f)(x,t) = \displaystyle\int_0^t (g_{\nu(t-s)} * \mathbb{H}\nabla\text{div } f)(x,s)ds. \end{cases} \quad (4.11)$$

Note that for a second order tensor $f$, $\mathbb{G}(f) = \nabla_x \mathbb{U}(f) = R \otimes R\left(\mathbb{U}(\nabla_x f)\right)$. Now we introduce Lagrangian variables. Let $X$ be the Lagrangian path,

$$\begin{gathered} v = \frac{\partial X}{\partial t} = u \circ X, \\ \tau = \sigma \circ X. \end{gathered} \quad (4.12)$$

Also we let

$$\begin{gathered} g(a,t) = (\nabla u)(X(a,t),t) = \mathbb{L}_\nu(\nabla u_0) \circ X(a,t) \\ +\mathbb{G}\left(\tau \circ X^{-1}\right) \circ X(a,t) - \mathbb{U}\left(\nabla_x\left((v \otimes v) \circ X^{-1}\right)\right) \circ X(a,t). \end{gathered} \quad (4.13)$$

Then the solution map for $X$ is

$$\begin{gathered} X(a,t) = a + \int_0^t u(X(a,s),s)ds \\ = a + \int_0^t \mathbb{L}_\nu(u_0)(X(a,s),s) + \mathbb{U}(\sigma - u \otimes u)(X(a,s),s)ds \end{gathered} \quad (4.14)$$

and the third equation of the system (4.1) is written as, by composing $X$,

$$\partial_t \tau = g\tau + \tau g^T - 2k\tau + 2\rho K(g + g^T). \quad (4.15)$$

Therefore, in Lagrangian variables the system is

$$\begin{cases} X(a,t) = a + \displaystyle\int_0^t \mathcal{V}(X,\tau,a,s)ds, \\ \tau(a,t) = \sigma_0(a) + \displaystyle\int_0^t \mathcal{T}(X,\tau,a,s)ds, \\ v(a,t) = \mathcal{V}(X,\tau,t) \end{cases} \tag{4.16}$$

where the Lagrangian nonlinearities $\mathcal{V}, \mathcal{T}$ are

$$\begin{cases} \mathcal{V}(X,\tau,a,s) = \mathbb{L}_\nu(u_0) \circ X(a,s) + (\mathbb{U}\left(\left(\tau - v \otimes v\right) \circ X^{-1}\right)) \circ X(a,s), \\ \mathcal{T}(X,\tau,a,s) = \left(g\tau + \tau g^T - 2k\tau + 2\rho K(g + g^T)\right)(a,s), \end{cases} \tag{4.17}$$

where $g$ is defined as above. Also, by chain rule the following is straightforward:

$$\nabla_a \mathcal{V} = (\nabla_a X)\, g. \tag{4.18}$$

We then consider variations of Lagrangian variables. We take a family $(X_\epsilon, \tau_\epsilon)$ of flow maps depending smoothly on a parameter $\epsilon \in [1,2]$, with initial data $u_{\epsilon,0}$ and $\sigma_{\epsilon,0}$. Note that $v_\epsilon = \partial_t X_\epsilon$. Also we denote

$$\begin{cases} u_\epsilon = \partial_t X_\epsilon \circ X_\epsilon^{-1}, g'_\epsilon = \dfrac{d}{d\epsilon} g_\epsilon \\ X'_\epsilon = \dfrac{d}{d\epsilon} X_\epsilon,\ \eta_\epsilon = X'_\epsilon \circ X_\epsilon^{-1}, \\ v'_\epsilon = \dfrac{d}{d\epsilon} v_\epsilon, \\ \sigma_\epsilon = \tau_\epsilon \circ X_\epsilon^{-1}, \\ \tau'_\epsilon = \dfrac{d}{d\epsilon} \tau_\epsilon, \delta_\epsilon = \tau'_\epsilon \circ X_\epsilon^{-1}, \end{cases} \tag{4.19}$$

and

$$u'_{\epsilon,0} = \frac{d}{d\epsilon} u_\epsilon(0), \sigma'_{\epsilon,0} = \frac{d}{d\epsilon} \sigma_\epsilon(0). \tag{4.20}$$

We can represent

$$
\begin{cases}
X_2(a,t) - X_1(a,t) = \displaystyle\int_1^2 \mathcal{X}'_\epsilon d\epsilon, \\
\tau_2(a,t) - \tau_1(a,t) = \displaystyle\int_1^2 \pi_\epsilon d\epsilon, \\
v_2(a,t) - v_1(a,t) = \displaystyle\int_1^2 \frac{d}{d\epsilon}\mathcal{V}_\epsilon d\epsilon,
\end{cases} \tag{4.21}
$$

where

$$
\begin{gathered}
\mathcal{X}'_\epsilon = \int_0^t \frac{d}{d\epsilon}\mathcal{V}_\epsilon ds, \ \pi_\epsilon = \int_0^t \frac{d}{d\epsilon}\mathcal{T}_\epsilon ds + \sigma'_{\epsilon,0}, \\
\mathcal{V}_\epsilon = \mathcal{V}(X_\epsilon, \tau_\epsilon), \ \mathcal{T}_\epsilon = \mathcal{T}(X_\epsilon, \tau_\epsilon).
\end{gathered} \tag{4.22}
$$

The following is a detailed calculation of $\frac{d}{d\epsilon}\mathcal{V}_\epsilon$: first, for the term $\mathbb{L}_\nu(u_{\epsilon,0}) \circ X_\epsilon(a,s)$

$$
\begin{aligned}
\frac{d}{d\epsilon}\left(\mathbb{L}_\nu(u_{\epsilon,0}) \circ X_\epsilon(a,s)\right) &= \frac{d}{d\epsilon}\int_{\mathbb{R}^d} g_{\nu s}(X_\epsilon(a,s)-y)u_{\epsilon,0}(y)dy \\
= \int_{\mathbb{R}^d} \frac{d}{d\epsilon} g_{\nu s}(X_\epsilon(a,s)-y)u_{\epsilon,0}(y)dy &+ \int_{\mathbb{R}^d} g_{\nu s}(X_\epsilon(a,s)-y)\frac{d}{d\epsilon}u_{\epsilon,0}(y)dy \\
= \int_{\mathbb{R}^d} X'_\epsilon(a,s) \cdot (\nabla_x g_{\nu s})(X_\epsilon(a,s)-y)u_{\epsilon,0}(y)dy &+ \mathbb{L}_\nu(u'_{\epsilon,0}) \circ X_\epsilon(a,s) \\
= \left(X'_\epsilon \cdot \left(\mathbb{L}_\nu(\nabla u_{\epsilon,0}) \circ X_\epsilon\right)\right)(a,s) &+ \mathbb{L}_\nu(u'_{\epsilon,0}) \circ X_\epsilon(a,s),
\end{aligned} \tag{4.23}
$$

or by composing $X_\epsilon^{-1}$ we have

$$
\left(\frac{d}{d\epsilon}\left(\mathbb{L}_\nu(u_{\epsilon,0}) \circ X_\epsilon\right)\right) \circ X_\epsilon^{-1} = \eta_\epsilon \cdot \mathbb{L}_\nu\left(\nabla_x u_{\epsilon,0}\right) + \mathbb{L}_\nu\left(u'_{\epsilon,0}\right). \tag{4.24}
$$

We split the second term into two terms $\mathbb{U}(\tau_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon$ and $-\mathbb{U}(v_\epsilon \otimes v_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon$. The $\epsilon$-derivative of the first one is

$$
\begin{aligned}
&\frac{d}{d\epsilon}\left(\mathbb{U}(\tau_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon(a,s)\right) \\
&= \frac{d}{d\epsilon}\int_0^s \int_{\mathbb{R}^d} g_{\nu(s-r)}(X_\epsilon(a,s)-y)\mathbb{H}\mathrm{div}\,(\tau_\epsilon \circ X_\epsilon^{-1})(y,r)dydr \\
&= \left(\frac{d}{d\epsilon}X_\epsilon(a,s)\right)\cdot \int_0^s \left(\nabla_x\left(g_{\nu(s-r)} * \mathbb{H}\mathrm{div}\,(\tau_\epsilon \circ X_\epsilon^{-1})\right)\right)(X_\epsilon(a,s),r)dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} g_{\nu(s-r)}(X_\epsilon(a,s)-y)\frac{d}{d\epsilon}\mathbb{H}\mathrm{div}\,(\tau_\epsilon \circ X_\epsilon^{-1})(y,r)dydr \\
&= X_\epsilon'(a,s)\cdot(\nabla_x \mathbb{U}(\sigma_\epsilon))(X_\epsilon(a,s),s) + \mathbb{U}\left(\frac{d}{d\epsilon}\left(\tau_\epsilon \circ X_\epsilon^{-1}\right)\right)(X_\epsilon(a,s),s)
\end{aligned} \tag{4.25}
$$

since $\mathbb{H}\mathrm{div}$ and differentiation with respect to $\epsilon$ commute. Also we note that

$$
\frac{d}{d\epsilon}\tau_\epsilon(X_\epsilon^{-1}(z,r),r) = \left(\frac{d\tau_\epsilon}{d\epsilon}\circ X_\epsilon^{-1}\right)(z,r) + \sum_k \left(\frac{dX_{\epsilon,k}^{-1}}{d\epsilon}(\partial_{a_k}\tau_\epsilon)\circ X_\epsilon^{-1}\right)(z,r). \tag{4.26}
$$

Since

$$
\frac{dX_\epsilon^{-1}}{d\epsilon} = -\left(X_\epsilon' \circ X_\epsilon^{-1}\right)\left(\nabla_x X_\epsilon^{-1}\right) \tag{4.27}
$$

we have

$$
\frac{d}{d\epsilon}\tau_\epsilon \circ X_\epsilon^{-1} = \tau_\epsilon' \circ X_\epsilon^{-1} - \eta_\epsilon \cdot \nabla_x(\sigma_\epsilon) = \delta_\epsilon - \eta_\epsilon \cdot \nabla_x \sigma_\epsilon. \tag{4.28}
$$

Therefore, we have

$$
\frac{d}{d\epsilon}\left(\mathbb{U}(\tau_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon\right) = X_\epsilon' \cdot ((\nabla_x \mathbb{U})(\sigma_\epsilon)\circ X_\epsilon) + \mathbb{U}(\delta_\epsilon)\circ X_\epsilon - \mathbb{U}(\eta_\epsilon \cdot \nabla_x \sigma_\epsilon)\circ X_\epsilon \tag{4.29}
$$

or by composing $X_\epsilon^{-1}$

$$
\left(\frac{d}{d\epsilon}\left(\mathbb{U}(\tau_\epsilon \circ X_\epsilon^{-1})\circ X_\epsilon\right)\right)\circ X_\epsilon^{-1} = [\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon) + \mathbb{U}(\delta_\epsilon) \tag{4.30}
$$

where

$$
[\eta \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon) = \eta_\epsilon \cdot \nabla_x(\mathbb{U}(\sigma_\epsilon)) - \mathbb{U}(\eta_\epsilon \cdot \nabla_x \sigma_\epsilon) \tag{4.31}
$$

and applying the same calculation with replacing $\tau_\epsilon$ with $v_\epsilon \otimes v_\epsilon$, we have

$$\begin{aligned} &\left(\frac{d}{d\epsilon}\mathbb{U}(v_\epsilon \otimes v_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon\right) \circ X_\epsilon^{-1} \\ &= [\eta_\epsilon \cdot \nabla_x, \mathbb{U}](u_\epsilon \otimes u_\epsilon) + \mathbb{U}((v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}). \end{aligned} \tag{4.32}$$

**Remark 13.** *Here the commutator structure essentially comes from the fact that the original variable was Lagrangian, it composed with back to label map to become Eulerian, and then finally it got back to Lagrangian variable.*

To sum up, we have

$$\begin{aligned} &\left(\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right) \circ X_\epsilon^{-1} = \eta_\epsilon \cdot (\mathbb{L}_\nu(\nabla_x u_{\epsilon,0})) + \mathbb{L}_\nu(u'_{\epsilon,0}) \\ &+[\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon - u_\epsilon \otimes u_\epsilon) + \mathbb{U}(\delta_\epsilon - (v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}). \end{aligned} \tag{4.33}$$

Due to a straightforward calculation $\frac{d}{d\epsilon}\mathcal{T}_\epsilon$ reads

$$\frac{d}{d\epsilon}\mathcal{T}_\epsilon = g'_\epsilon \tau_\epsilon + g_\epsilon \tau'_\epsilon + \tau'_\epsilon g_\epsilon^T + \tau_\epsilon (g'_\epsilon)^T - 2k\tau'_\epsilon + 2\rho K(g'_\epsilon + (g'_\epsilon)^T). \tag{4.34}$$

To find a bound for $\frac{d}{d\epsilon}\mathcal{T}_\epsilon$, we need to calculate $g'_\epsilon$. The calculation is performed in the same way as the calculation of $\frac{d}{d\epsilon}\mathcal{V}_\epsilon$.

$$\begin{aligned} g'_\epsilon \circ X_\epsilon^{-1} &= \eta_\epsilon \cdot \mathbb{L}_\nu(\nabla_x \nabla_x u_{x,0}) + \mathbb{L}_\nu(\nabla_x u'_{\epsilon,0}) + [\eta_\epsilon \cdot \nabla_x, \mathbb{G}](\sigma_\epsilon) + \mathbb{G}(\delta_\epsilon) \\ &-[\eta_\epsilon \cdot \nabla_x, \mathbb{U}]\left(\nabla_x(u_\epsilon \otimes u_\epsilon)\right) - \mathbb{U}\left(\nabla_x\left((v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}\right)\right) \end{aligned} \tag{4.35}$$

where

$$[\eta_\epsilon \cdot \nabla_x, \mathbb{G}](\sigma_\epsilon) = \eta_\epsilon \cdot \nabla_x(\mathbb{G}(\sigma_\epsilon)) - \mathbb{G}(\eta_\epsilon \cdot \nabla_x \sigma_\epsilon). \tag{4.36}$$

Also note that from (4.18) and that differentiation in $a$ and $\epsilon$ commute, we have

$$\frac{d}{d\epsilon}(\nabla_a \mathcal{V}_\epsilon) = (\nabla_a X'_\epsilon) g_\epsilon + (\nabla_a X_\epsilon) g'_\epsilon. \tag{4.37}$$

Summarizing, we have

$$
\begin{cases}
\left(\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right) \circ X_\epsilon^{-1} = \eta_\epsilon \cdot (\mathbb{L}_\nu(\nabla_x u_{\epsilon,0})) + \mathbb{L}_\nu(u'_{\epsilon,0}) \\
+[\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon - u_\epsilon \otimes u_\epsilon) + \mathbb{U}(\delta_\epsilon - (v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}), \\
g_\epsilon = \mathbb{L}(\nabla_x u_{\epsilon,0}) \circ X_\epsilon + \mathbb{G}(\sigma_\epsilon) \circ X_\epsilon - \mathbb{U}(\nabla_x(u_\epsilon \otimes u_\epsilon)) \circ X_\epsilon, \\
g'_\epsilon \circ X_\epsilon^{-1} = \eta_\epsilon \cdot \mathbb{L}_\nu(\nabla_x \nabla_x u_{\epsilon,0}) + \mathbb{L}_\nu(\nabla_x u'_{\epsilon,0}) + [\eta_\epsilon \cdot \nabla_x, \mathbb{G}](\sigma_\epsilon) + \mathbb{G}(\delta_\epsilon) \\
-[\eta_\epsilon \cdot \nabla_x, \mathbb{U}]\left(\nabla_x(u_\epsilon \otimes u_\epsilon)\right) - \mathbb{U}\left(\nabla_x\left((v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}\right)\right), \\
\frac{d}{d\epsilon}(\nabla_a \mathcal{V}_\epsilon) = (\nabla_a X'_\epsilon) g_\epsilon + (\nabla_a X_\epsilon) g'_\epsilon, \\
\frac{d}{d\epsilon}\mathcal{T}_\epsilon = g'_\epsilon \tau_\epsilon + g_\epsilon \tau'_\epsilon + \tau'_\epsilon g_\epsilon^T + \tau_\epsilon (g'_\epsilon)^T - 2k\tau'_\epsilon + 2\rho K(g'_\epsilon + (g'_\epsilon)^T),
\end{cases}
\tag{4.38}
$$

## 4.3 Functions, operators, and commutators

We consider function spaces

$$C^{\alpha,p} = C^\alpha(\mathbb{R}^2) \cap L^p(\mathbb{R}^2) \tag{4.39}$$

with norm

$$\|f\|_{\alpha,p} = \|f\|_{C^\alpha(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)} \tag{4.40}$$

for $\alpha \in (0,1), p \in (1,\infty)$, $C^{1+\alpha}(\mathbb{R}^2)$ with norm

$$\|f\|_{C^{1+\alpha}(\mathbb{R}^2)} = \|f\|_{L^\infty(\mathbb{R}^2)} + \|\nabla f\|_{C^\alpha(\mathbb{R}^2)} \tag{4.41}$$

and

$$C^{1+\alpha,p} = C^{1+\alpha}(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2) \tag{4.42}$$

with norm

$$\|f\|_{1+\alpha,p} = \|f\|_{C^{1+\alpha}(\mathbb{R}^2)} + \|f\|_{W^{1,p}(\mathbb{R}^2)}. \tag{4.43}$$

We also use spaces of paths, $L^\infty(0,T;Y)$ with the usual norm,

$$\|f\|_{L^\infty(0,T;Y)} = \sup_{t\in[0,T]} \|f(t)\|_Y\,, \tag{4.44}$$

spaces $Lip(0,T;Y)$ with norm

$$\|f\|_{Lip(0,T;Y)} = \sup_{t\neq s, t,s\in[0,T]} \frac{\|f(t)-f(s)\|_Y}{|t-s|} + \|f\|_{L^\infty(0,T;Y)} \tag{4.45}$$

where $Y$ is $C^{\alpha,p}$ or $C^{1+\alpha,p}$ in the following. We use the following lemmas.

**Lemma 4.3.1** ( [30]). *Let $0<\alpha<1$, $1<p<\infty$. Let $\eta\in C^{1+\alpha}(\mathbb{R}^d)$ and let*

$$(\mathbb{K}\sigma)(x) = P.V.\int_{\mathbb{R}^d} k(x-y) \tag{4.46}$$

*be a classical Calderon-Zygmund operator with kernel $k$ which is smooth away from the origin, homogeneous of degree $-d$ and with mean zero on spheres about the origin. Then the commutator $[\eta\cdot\nabla,\mathbb{K}]$ can be defined as a bounded linear operator in $C^{\alpha,p}$ and*

$$\|[\eta\cdot\nabla,\mathbb{K}]\sigma\|_{C^{\alpha,p}} \le C\,\|\eta\|_{C^{1+\alpha}(\mathbb{R}^d)}\,\|\sigma\|_{C^{\alpha,p}}\,. \tag{4.47}$$

**Lemma 4.3.2** (Generalized Young's inequality). *Let $1\le q\le\infty$ and $C>0$. Suppose $K$ is a measurable function on $\mathbb{R}^d\times\mathbb{R}^d$ such that*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|dy \le C,\ \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|dx \le C. \tag{4.48}$$

*If $f\in L^q(\mathbb{R}^d)$, the function $Tf$ defined by*

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy \tag{4.49}$$

*is well-defined almost everywhere and is in $L^q$, and $\|Tf\|_{L^q}\le C\,\|f\|_{L^q}$.*

First we try to find the bounds for Eulerian path variables, which are composed with $X^{-1}$s, in terms of Lagrangian path variables. For simplicity of notation, let us denote

$$M_X = 1 + \|X - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}. \tag{4.50}$$

**Theorem 4.3.3.** *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. Also let $X$ be a path for incompressible flow such that $X - \mathrm{Id} \in Lip(0,T;C^{1+\alpha})$. Then*

$$\left\|\sigma \circ X^{-1}\right\|_{L^\infty(0,T;C^{\alpha,p})} \leq \|\sigma\|_{L^\infty(0,T;C^{\alpha,p})} M_X^\alpha. \tag{4.51}$$

*If $X' \in Lip(0,T;C^{1+\alpha})$, then*

$$\left\|X' \circ X^{-1}\right\|_{L^\infty(0,T;C^{1+\alpha})} \leq \|X'\|_{L^\infty(0,T;C^{1+\alpha})} M_X^{1+2\alpha}. \tag{4.52}$$

*If $v \in Lip(0,T;W^{1,p})$, then*

$$\left\|v \circ X^{-1}\right\|_{L^\infty(0,T;W^{1,p})} \leq \|v\|_{L^\infty(0,T;W^{1,p})} M_X. \tag{4.53}$$

*If in addition $\partial_t X'$, $\partial_t X$ exist in $L^\infty(0,T;C^{1+\alpha})$, then*

$$\left\|X' \circ X^{-1}\right\|_{Lip(0,T;C^\alpha)} \leq \|X'\|_{Lip(0,T;C^{1+\alpha})} \|X - \mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})} M_X^{1+3\alpha}. \tag{4.54}$$

*Proof.*

$$\left\|\sigma \circ X^{-1}\right\|_{L^p \cap L^\infty} = \|\sigma\|_{L^p \cap L^\infty}, \tag{4.55}$$

and

$$\begin{aligned} \left[\sigma \circ X^{-1}(t)\right]_\alpha &= \sup_{a\neq b, a,b\in\mathbb{R}^2} \frac{|\sigma(X^{-1}(a,t),t) - \sigma(X^{-1}(b,t),t)|}{|a-b|^\alpha} \\ = \sup_{a\neq b, a,b\in\mathbb{R}^2} & \frac{|\sigma(X^{-1}(a,t),t) - \sigma(X^{-1}(b,t),t)|}{|X^{-1}(a,t) - X^{-1}(b,t)|^\alpha} \left(\frac{|X^{-1}(a,t) - X^{-1}(b,t)|}{|a-b|}\right)^\alpha \end{aligned} \tag{4.56}$$

so

$$\left[\sigma \circ X^{-1}(t)\right]_\alpha \leq [\sigma(t)]_\alpha \left\|\nabla_x X^{-1}(t)\right\|_{L^\infty}^\alpha \leq [\sigma(t)]_\alpha \left(1 + \|X - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^\alpha. \tag{4.57}$$

Note that this shows that the same bound holds when we replace $X^{-1}$ by $X$. For the second and third part, it suffices to remark that

$$\nabla_x(X' \circ X^{-1}) = \left((\nabla_a X) \circ X^{-1}\right)^{-1} \left((\nabla_a X') \circ X^{-1}\right) \tag{4.58}$$

and the previous part gives the bound in terms of Lagrangian variables. For the last part, we note that

$$\begin{aligned} &\frac{1}{t-s}\left(X'\left(X^{-1}(x,t),t\right) - X'\left(X^{-1}(x,s),s\right)\right) \\ &= \int_0^1 \left((\partial_t X')\left(X^{-1}(x,\beta_\tau),\beta_\tau\right) + \left(\partial_t X^{-1}\right)(x,\beta_\tau)(\nabla_a X')\left(X^{-1}(x,\beta_\tau),\beta_\tau\right)\right) d\tau, \end{aligned} \tag{4.59}$$

where

$$\beta_\tau = \tau t + (1-\tau)s. \tag{4.60}$$

Now noting that

$$\partial_t X^{-1} = -\left((\partial_t X) \circ X^{-1}\right)\left((\nabla_a X)^{-1} \circ X^{-1}\right) \tag{4.61}$$

we have

$$
\begin{aligned}
&\frac{1}{t-s}\left\|X'\circ X^{-1}(t)-X'\circ X^{-1}(s)\right\|_{C^{\alpha}}\\
\leq&\left(\|\partial_t X'\|_{L^\infty(0,T;C^\alpha)}+\|\partial_t X\|_{L^\infty(0,T;C^\alpha)}\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\right)\left(1+\|X-\mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^{1+3\alpha}
\end{aligned}
\tag{4.62}
$$

so that

$$
\begin{aligned}
&\left\|X'\circ X^{-1}\right\|_{Lip(0,T;C^\alpha)}\\
\leq&\|X'\|_{Lip(0,T;C^{1+\alpha})}\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\left(1+\|X-\mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^{1+3\alpha}.
\end{aligned}
\tag{4.63}
$$

□

Now we try to bound $\mathbb{L}_\nu$, $\mathbb{U}$ and $\mathbb{G}$. From now on, we assume that $X-\mathrm{Id}\in Lip(0,T;C^{1+\alpha})$ with $\partial_t X\in L^\infty(0,T;C^{1+\alpha})$.

**Theorem 4.3.4.** *Let $0<\alpha<1, 1<p<\infty$ and let $T>0$. There exists a constant $C$ independent of $T$ and $\nu$ such that for any $0<t<T$*

$$
\begin{aligned}
\|\mathbb{L}_\nu(u_0)\|_{L^\infty(0,T;C^{\alpha,p})}&\leq C\|u_0\|_{\alpha,p},\\
\|\mathbb{L}_\nu(u_0)\|_{L^\infty(0,T;C^{1+\alpha,p})}&\leq C\|u_0\|_{1+\alpha,p},\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p}&\leq\frac{C}{(\nu t)^{\frac{1}{2}}}\|u_0\|_{\alpha,p},\\
\|\mathbb{L}_\nu(\nabla u_0)\|_{L^\infty(0,T;C^{\alpha,p})}&\leq C\|u_0\|_{1+\alpha,p}.
\end{aligned}
\tag{4.64}
$$

*Proof.*

$$
\begin{aligned}
\|\mathbb{L}_\nu(u_0)(t)\|_{\alpha,p}&\leq\|g_{\nu t}\|_{L^1}\|u_0\|_{\alpha,p}=\|u_0\|_{\alpha,p},\\
\|\mathbb{L}_\nu(u_0)(t)\|_{1+\alpha,p}&\leq\|g_{\nu t}\|_{L^1}\|u_0\|_{1+\alpha,p}=\|u_0\|_{1+\alpha,p},\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p}&\leq\|\nabla g_{\nu t}\|_{L^1}\|u_0\|_{1+\alpha,p}=\frac{C}{(\nu t)^{\frac{1}{2}}}\|u_0\|_{\alpha,p},\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p}&\leq\|g_{\nu t}\|_{L^1}\|\nabla u_0\|_{\alpha,p}\leq\|u_0\|_{1+\alpha,p}.
\end{aligned}
\tag{4.65}
$$

□

**Theorem 4.3.5.** *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. There exists a constant $C$ such that*

$$\|\mathbb{U}(\sigma)\|_{L^\infty(0,T;C^{\alpha,p})} \leq C \left(\frac{T}{\nu}\right)^{\frac{1}{2}} \|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{4.66}$$

*Proof.*

$$\begin{aligned} \|\mathbb{U}(\sigma)(t)\|_{C^{\alpha,p}} &\leq C \int_0^t \left\|\nabla g_{\nu(t-s)}\right\|_{L^1} \|\sigma(s)\|_{\alpha,p}\, ds \\ \leq \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds\, \|\sigma\|_{L^\infty(0,T;C^{\alpha,p})} &\leq \frac{C}{\nu^{\frac{1}{2}}} \sqrt{T}\, \|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}. \end{aligned} \tag{4.67}$$

□

**Theorem 4.3.6.** *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. There exists a constant $C_1, C_2$ depending only on $\alpha$ and $\nu$, and $C_3(T,X), C_4(T,X)$ such that*

$$\begin{aligned} \left\|\mathbb{G}(\tau \circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})} \leq C_1 \|X - \mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \|\tau(0)\|_{\alpha,p} (1 + C_3(T,X)) \\ + C_2 \|\tau\|_{Lip(0,T;C^{\alpha,p})} C_4(T,X) \end{aligned} \tag{4.68}$$

*where $C_3(T,X)$ and $C_4(T,X)$ are of the form*

$$CT^{\frac{1}{2}} \left(\|X - \mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} + \|X - \mathrm{Id}\|^4_{Lip(0,T;C^{1+\alpha})}\right).$$

*Proof.* Since $\mathbb{G} = (R \otimes R)\mathbb{H}\Gamma$ where

$$\Gamma(\tau \circ X^{-1}) = \int_0^t \Delta g_{\nu(t-s)} * (\tau \circ X^{-1}(s)) ds, \tag{4.69}$$

we can replace $\mathbb{G}$ by $\Gamma$. Then $\Gamma(\tau \circ X^{-1})$ can be written as

$$\begin{aligned}\Gamma(\tau \circ X^{-1})(t) = &\int_0^t \Delta g_{\nu(t-s)} * \left(\left(\tau \circ X^{-1}\right)(s) - \left(\tau \circ X^{-1}\right)(t)\right) ds \\ &+ \int_0^t \Delta g_{\nu(t-s)} * \left(\tau \circ X^{-1}\right)(t) ds.\end{aligned} \tag{4.70}$$

But

$$\int_0^t \Delta g_{\nu(t-s)} * (\tau \circ X^{-1})(t) ds = \tau \circ X^{-1}(t) - g_{\nu t} * (\tau \circ X^{-1})(t) \tag{4.71}$$

so the second term is bounded by $2\left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})} M_X^\alpha$ by Theorem 4.3.3. Now we let

$$\tau \circ X^{-1}(x,s) - \tau \circ X^{-1}(x,t) = \Delta_1 \tau(x,s,t) + \Delta_2 \tau(x,s,t), \tag{4.72}$$

where

$$\begin{aligned}\Delta_1 \tau(x,s,t) &= \tau(X^{-1}(x,s),s) - \tau(X^{-1}(x,s),t), \\ \Delta_2 \tau(x,s,t) &= \tau(X^{-1}(x,s),t) - \tau(X^{-1}(x,t),t).\end{aligned} \tag{4.73}$$

But since

$$\left\|\Delta_1 \tau(s,t)\right\|_{C^{\alpha,p}} \leq |t-s| M_X^\alpha \left\|\tau\right\|_{Lip(0,T;C^{\alpha,p})}, \tag{4.74}$$

by the proof of Theorem 4.3.3 we get

$$\left\|\int_0^t \Delta g_{\nu(t-s)} * \Delta_1 \tau(s,t) ds\right\|_{\alpha,p} \leq \frac{Ct}{\nu} \left\|\tau\right\|_{Lip(0,T;C^{\alpha,p})} M_X^\alpha, \tag{4.75}$$

On the other hand,

$$\int_0^t \Delta g_{\nu(t-s)} * \Delta_2 \tau(s,t) ds = \int_0^t \int_{\mathbb{R}^2} K(x,z,t,s) \tau(z,t) dz ds, \tag{4.76}$$

where

$$K(x,z,t,s) = \Delta g_{\nu(t-s)}(x - X(z,s)) - \Delta g_{\nu(t-s)}(x - X(z,t)). \tag{4.77}$$

We show the following lemma.

**Lemma 4.3.7.** *$K(x,z,t,s)$ is $L^1$ in both $x$ variable and $z$ variable, and*

$$\sup_z \|K(\cdot,z,t,s)\|_{L^1}, \sup_x \|K(x,\cdot,t,s)\|_{L^1} \leq \frac{C\,\|X-\mathrm{Id}\|_{Lip(0,T;L^\infty)}}{|t-s|^{\frac{1}{2}}\nu^{\frac{3}{2}}}. \tag{4.78}$$

*Proof.* We define

$$S(x) = 4\pi e^{-|x|^2}\left(|x|^2 - \frac{d}{2}\right) \tag{4.79}$$

so that

$$(\Delta g_{\nu(t-s)}) = (4\pi\nu(t-s))^{-(\frac{d}{2}+1)} S\left(\frac{x}{(4(t-s))^{\frac{1}{2}}}\right). \tag{4.80}$$

Then

$$\begin{aligned}\int |K(x,z,t,s)|dz &= \int (4\pi\nu(t-s))^{-\left(\frac{d}{2}+1\right)}\left|S\left(\frac{x-X(z,s)}{(4\nu(t-s))^{\frac{1}{2}}}\right) - S\left(\frac{x-X(z,t)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| dz \\ &= \int (4\pi\nu(t-s))^{-\left(\frac{d}{2}+1\right)}\left|S\left(\frac{x-y}{(4\nu(t-s))^{\frac{1}{2}}}\right) - S\left(\frac{x-X(y,t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| dy \\ &= (4\pi\nu(t-s))^{-1}\pi^{-\left(\frac{d}{2}+1\right)}\int \left|S(u) - S\left(u - \frac{(X-\mathrm{Id})(x-(4(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| du.\end{aligned} \tag{4.81}$$

However, for each $u$

$$\begin{aligned}&\left|S(u) - S\left(u - \frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| \\ &\leq \left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right| \\ &\times \sup\left\{|\nabla S(u-z)| : |z| \leq \left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right|\right\}\end{aligned} \tag{4.82}$$

and we have

$$\left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right| \leq \|(X-\mathrm{Id})\|_{Lip(0,T;L^\infty)}\frac{|t-s|^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \leq CT^{\frac{1}{2}} \tag{4.83}$$

and obviously

$$\tilde{S}(u) = \sup_{z \leq CT^{\frac{1}{2}}} |(\nabla S)(u - z)| \tag{4.84}$$

is integrable in $\mathbb{R}^d$; because $\nabla S$ is Schwartz,

$$|(\nabla S)(x)| \leq \frac{C_d}{(1 + 2C^2 T + |x|^2)^d} \tag{4.85}$$

for some constant $C_d$, but if $|z| \leq CT^{\frac{1}{2}}$, then $|u - z|^2 \geq |u|^2 - C^2 T$ and

$$|(\nabla S)(u - z)| \leq \frac{C_d}{(1 + C^2 T + |u|^2)^d} \tag{4.86}$$

and the right side of above is clearly integrable with bound depending only on $d = 2$ and $T$. Therefore, we have

$$\int |K(x, z, t, s)| dz \leq |t - s|^{-\frac{1}{2}} \nu^{-\frac{3}{2}} \|(X - \mathrm{Id})\|_{Lip(0,T;L^\infty)} C(d, T). \tag{4.87}$$

Similarly,

$$\begin{aligned}\int |K(x, z, t, s)| dx &= \int (4\pi\nu(t - s))^{-(\frac{d}{2}+1)} \left| S(\frac{x - X(z, s)}{(4\nu(t - s))^{\frac{1}{2}}}) - S(\frac{x - X(z, t)}{(4\nu(t - s))^{\frac{1}{2}}}) \right| dx \\ &= \int (4\pi\nu(t - s))^{-1} \pi^{-(\frac{d}{2}+1)} \left| S(y) - S(y + \frac{X(z, s) - X(z, t)}{(4\nu(t - s))^{\frac{1}{2}}}) \right| dy\end{aligned} \tag{4.88}$$

and again we have

$$\left| \frac{X(z, s) - X(z, t)}{(4\nu(t - s))^{\frac{1}{2}}} \right| \leq \|(X - \mathrm{Id})\|_{Lip(0,T;L^\infty)} |t - s|^{\frac{1}{2}} \nu^{-\frac{1}{2}} \leq CT^{\frac{1}{2}}. \tag{4.89}$$

Therefore, we have the bound

$$\int |K(x, z)| dx \leq |t - s|^{-\frac{1}{2}} \nu^{-\frac{3}{2}} \|(X - \mathrm{Id})\|_{Lip(0,T;L^\infty)} C(d, T). \tag{4.90}$$

$\square$

From Lemma 4.3.7 and generalized Young's inequality, we have

$$\begin{aligned} &\left\|\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds\right\|_{L^p\cap L^\infty} \\ \leq\ & \frac{C}{\nu}\left(\left(\frac{t}{\nu}\right)^{\frac{1}{2}} \|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)\|\tau\|_{L^\infty(0,T;L^p\cap L^\infty)}. \end{aligned} \tag{4.91}$$

For the Holder seminorm, we measure the finite difference. Let us denote $\delta_h f(x,t) = f(x+h,t) - f(x,t)$. If $|h| < t$, then

$$\delta_h\left(\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds\right) = \int_0^t \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2\tau(s,t)ds. \tag{4.92}$$

If $0 < t-s < |h|$, then $\left\|\delta_h \Delta g_{\nu(t-s)}\right\|_{L^1} \leq 2\left\|\Delta g_{\nu(t-s)}\right\|_{L^1} \leq \frac{C}{\nu(t-s)}$ and since

$$\|\Delta_2\tau(s,t)\|_{L^\infty} \leq |t-s|^\alpha \|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})} \tag{4.93}$$

we have

$$\left\|\int_{t-|h|}^t \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2\tau(s,t)ds\right\|_{L^\infty} \leq \frac{C}{\nu\alpha}|h|^\alpha \|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{4.94}$$

If $|h| < t-s < t$, then following lines of Lemma 4.3.7 $\delta_h(\Delta g_{\nu(t-s)})$ is a $L^1$ function with

$$\left\|\delta_h(\Delta g_{\nu(t-s)})\right\|_{L^1} \leq \frac{C|h|}{(\nu(t-s))^{\frac{3}{2}}} \tag{4.95}$$

we have

$$\begin{aligned} &\left\|\int_0^{t-|h|} \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2\tau(s,t)ds\right\|_{L^\infty} \\ \leq\ & \begin{cases} \frac{C}{\nu^{\frac{3}{2}}}\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}|h|^{\frac{1}{2}}\frac{t^\alpha}{\alpha} & \alpha \leq \frac{1}{2}, \\ \frac{C}{\nu^{\frac{3}{2}}}\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}|h|\frac{t^{\alpha-\frac{1}{2}}}{\alpha-\frac{1}{2}} & \alpha > \frac{1}{2}. \end{cases} \end{aligned} \tag{4.96}$$

If $|h| \geq t$, then we only have the first term. Therefore, we have

$$\frac{1}{|h|^\alpha} \left\| \delta_h \left( \int_0^t \Delta g_{\nu(t-s)} * \Delta_2 \tau(s,t) ds \right) \right\|_{L^\infty} \leq \frac{C(\alpha)}{\nu} \|X - \mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \|\tau\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{4.97}$$

We note that

$$\|\tau(t)\|_{\alpha,p} \leq \|\tau(0)\|_{\alpha,p} + t \|\tau\|_{Lip(0,T;C^{\alpha,p})}. \tag{4.98}$$

To summarize, we have

$$\begin{aligned} &\left\|\Gamma(\tau \circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\leq C(\alpha)\left(1+\frac{1}{\nu}\right)\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau(0)\|_{\alpha,p} \\ &+C(\alpha)\left(1+\frac{1}{\nu}\right)\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} T \|\tau\|_{Lip(0,T;C^{\alpha,p})} \\ &+\frac{C(\alpha)}{\nu}\left(\frac{T}{\nu}\right)^{\frac{1}{2}} \max\{\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}, \|X-\mathrm{Id}\|^4_{Lip(0,T;C^{1+\alpha})}\} \\ &\times(\|\tau(0)\|_{\alpha,p} + T\|\tau\|_{Lip(0,T;C^{\alpha,p})}). \end{aligned} \tag{4.99}$$

□

**Theorem 4.3.8.** *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. Let $X' \in Lip(0,T;C^{1+\alpha})$ with $\partial_t X' \in L^\infty(0,T;C^{1+\alpha})$. There exists a constant $C$ such that*

$$\begin{aligned} &\left\|\left[X' \circ X^{-1} \cdot \nabla, \mathbb{U}\right](\sigma)\right\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\leq C\left(\left(\frac{T}{\nu}\right)^{\frac{1}{2}} + \frac{T}{\nu}\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right) M_X^{1+3\alpha}\|X'\|_{Lip(0,T;C^{1+\alpha})}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})} \end{aligned} \tag{4.100}$$

*Proof.* First, we denote

$$\eta = X' \circ X^{-1}. \tag{4.101}$$

Then we have

$$
\begin{aligned}
&[\eta \cdot \nabla, \mathbb{U}]\,(\sigma)(t) \\
&= \eta(t) \cdot \nabla \int_0^t g_{\nu(t-s)} * \mathbb{H}\mathrm{div}\,\sigma(s) ds - \int_0^t g_{\nu(t-s)} * \mathbb{H}\mathrm{div}\,(\eta(s) \cdot \nabla \sigma(s)) ds \\
&= [\eta(t) \cdot \nabla, \mathbb{H}] \int_0^t g_{\nu(t-s)} * \mathrm{div}\,\sigma(s) ds + \mathbb{H} \int_0^t (\nabla g_{\nu(t-s)}) * (\nabla \cdot \eta(s)\sigma(s))\, ds \\
&\quad - \mathbb{H} \int_0^t (\nabla\nabla g_{\nu(t-s)}) * (\eta(s) - \eta(t))\, \sigma(s) ds \\
&\quad + \mathbb{H} \int_0^t \big(\eta(t) \cdot (\nabla\nabla g_{\nu(t-s)}) * \sigma(s) - (\nabla\nabla g_{\nu(t-s)}) * (\eta(t)\sigma(s))\big)\, ds,
\end{aligned}
\tag{4.102}
$$

where $(\nabla\nabla g_{\nu(t-s)}) * (\eta(s) - \eta(t))\sigma(s)$, $\eta(t) \cdot (\nabla\nabla g_{\nu(t-s)}) * \sigma(s)$, and $(\nabla\nabla g_{\nu(t-s)}) * (\eta(s)\sigma(s))$ represent

$$
\begin{aligned}
&\sum_{i,j} (\partial_i \partial_j g_{\nu(t-s)} *)(\eta_i(s) - \eta_i(t))\sigma_{jk}(s), \\
&\sum_{i,j} \eta_i(t) \left(\partial_i \partial_j g_{\nu(t-s)}\right) * \sigma_{jk}(s), \text{ and } \sum_{i,j} \left(\partial_i \partial_j g_{\nu(t-s)}\right) * (\eta_i(s)\sigma_{jk}(s))
\end{aligned}
\tag{4.103}
$$

respectively. The first term, by Lemma 4.3.1, and the second term, by standard estimate, are bounded by

$$
\begin{aligned}
&\left\| [\eta(t) \cdot \nabla, \mathbb{H}] \int_0^t g_{\nu(t-s)} * \mathrm{div}\sigma(s) ds \right\|_{\alpha,p} \le C \left\| \eta(t) \right\|_{C^{1+\alpha}} \left(\frac{t}{\nu}\right)^{\frac{1}{2}} \left\| \sigma \right\|_{L^\infty(0,T;C^{\alpha,p})}, \\
&\left\| \mathbb{H} \int_0^t (\nabla g_{\nu(t-s)}) * (\nabla \cdot \eta(s)\sigma(s)) ds \right\|_{\alpha,p} \le C \left(\frac{t}{\nu}\right)^{\frac{1}{2}} \left\| \eta \right\|_{L^\infty(0,T;C^{1+\alpha})} \left\| \sigma \right\|_{L^\infty(0,T;C^{\alpha,p})}.
\end{aligned}
\tag{4.104}
$$

The third term is bounded by

$$
\frac{Ct}{\nu} \left\| \eta \right\|_{Lip(0,T;C^\alpha)} \left\| \sigma \right\|_{L^\infty(0,T;C^{\alpha,p})}
\tag{4.105}
$$

by the virtue of Theorem 4.3.3. For the last term, note that

$$\begin{aligned} &\left(\eta(t)\cdot(\nabla\nabla g_{\nu(t-s)})*\sigma(s)-(\nabla\nabla g_{\nu(t-s)})*(\eta(t)\sigma(s))\right)(x)\\ &=\int_{\mathbb{R}^2}\nabla\nabla g_{\nu(t-s)}(z)z\cdot\left(\int_0^1\nabla\eta(x-(1-\lambda)z,t)d\lambda\right)\sigma(x-z,s)dz \end{aligned} \tag{4.106}$$

and note that $\nabla\nabla g_{\nu(t-s)}(z)z$ is a $L^1$ function with

$$\left\|\nabla\nabla g_{\nu(t-s)}(z)z\right\|_{L^1}\le\frac{C}{(\nu(t-s))^{\frac{1}{2}}}. \tag{4.107}$$

Therefore,

$$\begin{aligned} &\left\|\left(\eta(t)\cdot(\nabla\nabla g_{\nu(t-s)})*\sigma(s)-(\nabla\nabla g_{\nu(t-s)})*(\eta(t)\sigma(s))\right)\right\|_{\alpha,p}\\ &\le\frac{C}{(\nu(t-s))^{\frac{1}{2}}}\left\|\eta(t)\right\|_{C^{1+\alpha}}\left\|\sigma(s)\right\|_{\alpha,p} \end{aligned} \tag{4.108}$$

so that the last term is bounded by

$$C\left(\frac{t}{\nu}\right)^{\frac{1}{2}}\left\|\eta(t)\right\|_{C^{1+\alpha}}\left\|\sigma\right\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{4.109}$$

We finish the proof by replacing $\eta$ by $X'$ using Theorem 4.3.3. □

**Theorem 4.3.9.** *Let $0<\alpha<1$, $1<p<\infty$ and let $T>0$. Let $X'\in Lip(0,T;C^{1+\alpha})$ with $\partial_t X'\in L^\infty(0,T;C^{1+\alpha})$. There exists a constant $C(\alpha)$ depending only on $\alpha$ such that*

$$\begin{aligned} &\left\|\left[X'\circ X^{-1}\cdot\nabla,\mathbb{G}\right](\tau\circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})}\\ &\le(\left\|X'\right\|_{L^\infty(0,T;C^{1+\alpha})}+\left\|X'\right\|_{Lip(0,T;C^{1+\alpha})}T^{\frac{1}{2}})R \end{aligned} \tag{4.110}$$

*where $R$ is a polynomial function on $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$, $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$, whose coefficients depend on $\alpha$, $\nu$, and $T$, and in particular it grows polynomially in $T$ and bounded below.*

*Proof.* Again we denote $\eta = X' \circ X^{-1}$. Also it suffices to bound

$$[\eta \cdot \nabla, \Gamma]\left(\tau \circ X^{-1}\right) = \eta(t) \cdot \nabla \Gamma\left(\tau \circ X^{-1}\right) - \Gamma\left(\eta \cdot \nabla\left(\tau \circ X^{-1}\right)\right) \tag{4.111}$$

where $\Gamma$ is as defined in (4.69), since

$$[\eta \cdot \nabla, \mathbb{G}] = (R \otimes R)\mathbb{H}\,[\eta \cdot \nabla, \Gamma] + [\eta(t) \cdot \nabla, (R \otimes R)\mathbb{H}]\,\Gamma \tag{4.112}$$

and the second term is bounded by Lemma 4.3.1. For the first term, we have

$$[\eta \cdot \nabla, \Gamma]\,(\tau \circ X^{-1})(t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{4.113}$$

where

$$\begin{aligned}
I_1 = \int_0^t \eta(t) \cdot \left(\nabla \Delta g_{\nu(t-s)} * \left(\tau \circ X^{-1}(t)\right)\right) - \nabla \Delta g_{\nu(t-s)} * \left(\eta(t) \tau \circ X^{-1}(t)\right) ds,& \\
I_2 = \int_0^t \eta(t) \cdot \left(\nabla \Delta g_{\nu(t-s)} * \left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right)& \\
-\nabla \Delta g_{\nu(t-s)} * \left(\eta(t)\left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right) ds,& \\
I_3 = -\int_0^t \nabla \Delta g_{\nu(t-s)} * \left((\eta(s) - \eta(t))\left(\tau \circ X^{-1}(s)\right)\right) ds,& \\
I_4 = \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot (\eta(s) - \eta(t))\, \tau \circ X^{-1}(s)\right) ds,& \\
I_5 = \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot \eta(t)\left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right) ds,& \\
I_6 = -\frac{1}{\nu}\left(\nabla \cdot \eta(t) \tau \circ X^{-1}(t) - g_{\nu t} * \left(\nabla \cdot \eta(t) \tau \circ X^{-1}(t)\right)\right).&
\end{aligned} \tag{4.114}$$

We denote these 6 terms on the right hand side of above by $I_1, I_2, I_3, I_4, I_5$ and $I_6$. First, $I_1 + I_6$ can be easily bounded:

$$\begin{aligned}
I_1 + I_6 = \frac{1}{\nu}\left(\eta(t) \cdot \nabla\left(g_{\nu t} * \left(\tau \circ X^{-1}(t)\right)\right) - \nabla\left(g_{\nu t} * \left(\eta(t) \tau \circ X^{-1}(t)\right)\right)\right)& \\
-\frac{1}{\nu} g_{\nu t} * \left(\nabla \cdot \eta(t)\left(\tau \circ X^{-1}(t)\right)\right)&
\end{aligned} \tag{4.115}$$

and the first term is treated in the same way as (4.106). Since the first term is

$$\frac{1}{\nu}\left(\int_{\mathbb{R}^2} \nabla g_{\nu t}(y)y \cdot \int_0^1 \nabla\eta(x-(1-\lambda)y,t)d\lambda\left(\tau\circ X^{-1}\right)(x-y,t)dy\right) \tag{4.116}$$

and

$$\|\nabla g_{\nu t}(y)y\|_{L^1} \leq C \tag{4.117}$$

the $C^{\alpha,p}$-norm of the first term is bounded by

$$\frac{C}{\nu}\|\eta(t)\|_{C^{1+\alpha}}\left\|\tau\circ X^{-1}(t)\right\|_{\alpha,p}. \tag{4.118}$$

However, the $C^{\alpha,p}$-norm of the second term is also bounded by the same bound. Therefore,

$$\|I_1+I_6\|_{L^\infty(0,T;C^{\alpha,p})} \leq \frac{C}{\nu}M_X^{1+3\alpha}\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})} \tag{4.119}$$

The term $I_3$ is bounded due to Theorem 4.3.3. Since $\eta \in Lip(0,T;C^\alpha)$ we have

$$\begin{aligned}\|I_3\|_{L^\infty(0,T;C^{\alpha,p})} \leq \frac{C}{\nu}\left(\frac{T}{\nu}\right)^{\frac{1}{2}} M_X^{1+4\alpha}\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\\ \|X'\|_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}\end{aligned} \tag{4.120}$$

The terms $I_4$, and $I_5$ are treated in the spirit of Theorem 4.3.6. We treat $L^p \cap L^\infty$ norm and Holder seminorm separately. For the term $I_5$, we have

$$I_5 = \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla\cdot\eta(t)\left(\Delta_1\tau(s,t)+\Delta_2\tau(s,t)\right)\right)ds \tag{4.121}$$

where $\Delta_1\tau$ and $\Delta_2\tau$ are the same as (4.73). From the same arguments from the above,

$$\begin{aligned}&\left\|\int_0^t \Delta g_{\nu(t-s)} * (\nabla\cdot\eta(t)\Delta_1\tau(s,t))\,ds\right\|_{\alpha,p}\\&\leq \frac{Ct}{\nu}\left\|\eta\right\|_{L^\infty(0,T;C^{1+\alpha})}\left\|\tau\right\|_{Lip(0,T;C^{\alpha,p})}M_X^\alpha.\end{aligned}\tag{4.122}$$

On the other hand,

$$\begin{aligned}\Delta g_{\nu(t-s)} * (\nabla\cdot\eta(t)\Delta_2\tau(s,t))\,(x) &= \int_{\mathbb{R}^2}(K(x,z,t,s)\,(\nabla\cdot\eta)\,(X(z,t),t)\\+\Delta g_{\nu(t-s)}(x-X(z,t))\,((\nabla\cdot\eta)\,(X(z,s),t)&-(\nabla\cdot\eta)\,(X(z,t),t)))dz\end{aligned}\tag{4.123}$$

where $K$ is as in (4.77). Then as in the proof of Lemma 4.3.7, by the Generalized Young's inequality we have

$$\begin{aligned}&\left\|\int_0^t \Delta g_{\nu(t-s)} * (\nabla\cdot\eta(t)\Delta_2\tau(s,t))\,ds\right\|_{L^p\cap L^\infty} \leq C\left\|\tau(t)\right\|_{L^p\cap L^\infty}\left\|\eta\right\|_{L^\infty(0,T;C^{1+\alpha})}\\&\left\|X-\mathrm{Id}\right\|_{Lip(0,T;C^{1+\alpha})}\left(\frac{t^\alpha}{\nu\alpha}+\left(\frac{t}{\nu^3}\right)^{\frac{1}{2}}+\frac{t^2}{\nu^3}\left\|X-\mathrm{Id}\right\|^3_{Lip(0,T;C^{1+\alpha})}\right).\end{aligned}\tag{4.124}$$

For the Holder seminorm, we repeat the same argument in the proof of Theorem 4.3.6, using the bound (4.95). Then we obtain

$$\begin{aligned}&\frac{1}{|h|^\alpha}\left\|\delta_h\left(\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds\right)\right\|_{L^\infty}\\&\leq \frac{C(\alpha)}{\nu}\left(1+\left(\frac{t}{\nu}\right)^{\frac{1}{2}}+\left(\frac{t}{\nu}\right)^2\right)\left\|X-\mathrm{Id}\right\|^\alpha_{Lip(0,T;C^{1+\alpha})}\left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})}\left\|\eta\right\|_{L^\infty(0,T;C^{1+\alpha})}.\end{aligned}\tag{4.125}$$

Therefore,

$$\begin{aligned}\left\|I_5\right\|_{L^\infty(0,T;C^{\alpha,p})} &\leq \frac{C(\alpha)}{\nu}\left(1+t+\left(\frac{t}{\nu}\right)^2\right)\left(1+\left\|X-\mathrm{Id}\right\|_{Lip(0,T;C^{1+\alpha})}\right)^3 M_X^{1+2\alpha}\\&\left\|X'\right\|_{L^\infty(0,T;C^{1+\alpha})}\left\|\tau\right\|_{Lip(0,T;C^{\alpha,p})}.\end{aligned}\tag{4.126}$$

The term $I_4(t)$ is treated in the exactly same way, by noting that

$$\nabla\cdot(\eta(s)-\eta(t)) = \nabla_x X^{-1}(s) : (\Delta_1\nabla_a X'(s,t)) + \nabla_x X^{-1}(s) : (\Delta_2\nabla_a X'(s,t)) \\ + \left(\nabla_x X^{-1}(s) - \nabla_x X^{-1}(t)\right) : \left(\nabla_a X' \circ X^{-1}\right)(t), \tag{4.127}$$

where as in (4.73)

$$\begin{aligned} \Delta_1\nabla_a X'(x,s,t) &= \nabla_a X'(X^{-1}(x,s),s) - \nabla_a X'(X^{-1}(x,s),t), \\ \Delta_2\nabla_a X'(x,s,t) &= \nabla_a X'(X^{-1}(x,s),t) - \nabla_a X'(X^{-1}(x,t),t), \end{aligned} \tag{4.128}$$

and

$$\nabla_x\left(X^{-1}(x,s) - X^{-1}(x,t)\right) = \left(\nabla_a X \circ X^{-1}\right)(x,t)\left(\nabla_a\left(X-\mathrm{Id}\right)\right)\left(X^{-1}(x,t),t-s\right) \tag{4.129}$$

so that

$$\left\|\nabla_x X^{-1}(s) - \nabla_x X^{-1}(t)\right\|_{C^\alpha} \le |t-s|\,\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})} M_X^{1+2\alpha}. \tag{4.130}$$

Also note that

$$\|\Delta_2\nabla_a X'(s,t)\|_{L^\infty} \le \|\nabla_a X'(t)\|_{C^\alpha}\,\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;L^\infty)}\,|t-s|^\alpha \tag{4.131}$$

so that

$$\left\|\int_0^t \Delta g_{\nu(t-s)} * \left(\nabla_x X^{-1}(s) : (\Delta_2\nabla_a X'(s,t))\tau\circ X^{-1}(s)\right) ds\right\|_{C^{\alpha,p}} \\ \le \frac{C(\alpha)}{\nu}\left(1+t^\alpha+\left(\frac{t}{\nu}\right)^2\right) M_X^{1+2\alpha}\,\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \\ \|X'\|_{L^\infty(0,T;C^{1+\alpha})}\,\|\tau\|_{L^\infty(0,T;C^{\alpha,p})} \tag{4.132}$$

The final result is

$$\|I_4(t)\|_{\alpha,p} \leq \frac{C(\alpha)}{\nu}\left(1+t+\left(\frac{t}{\nu}\right)^2\right) M_X^{2+4\alpha} \|X'\|_{L^\infty(0,T;C^{1+\alpha})} \|\tau\|_{L^\infty(0,T;C^{\alpha,p})} \\ +C\frac{t}{\nu}M_X^{1+3\alpha} \|X'\|_{Lip(0,T;C^{1+\alpha})} \|\tau\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{4.133}$$

Finally, $I_2$ can be bounded using the combination of the technique in Theorem 4.3.6 and Theorem 4.3.8. First, we have

$$I_2(x,t) = \\ \int_0^t \int_{\mathbb{R}^2} \nabla\Delta g_{\nu(t-s)}(y)\cdot y\cdot\left(\int_0^1 \nabla\eta(x-(1-\lambda)y,t)d\lambda\,(\Delta_1\tau(x-y,s,t))\right)dyds \\ +\int_0^t \int_{\mathbb{R}^2} \nabla\Delta g_{\nu(t-s)}(x-z)\cdot(x-z)\cdot\left(\int_0^1 \nabla\eta(\lambda x+(1-\lambda)z,t)d\lambda\,(\Delta_2\tau(z,s,t))\right)dzds. \tag{4.134}$$

Then applying the argument of the proof of Theorem 4.3.8, the first term is bounded by

$$\frac{C}{\nu}tM_X^{\alpha}\|\eta\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{Lip(0,T;C^{\alpha,p})}. \tag{4.135}$$

The second term is treated using the method used in Theorem 4.3.6. By changing variables to form a kernel similar to (4.77), and applying generalized Young's inequality, the $L^p \cap L^\infty$ norm of the second term is bounded by

$$\frac{C(\alpha)}{\nu}\left(t^\alpha+\left(\frac{t}{\nu}\right)^{\frac{1}{2}}+\left(\frac{t}{\nu}\right)^2\right)\left(1+\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)^4\|\eta\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;L^p\cap L^\infty)}. \tag{4.136}$$

Finally, the Holder seminorm of the second term is bounded by the same method as Theorem 4.3.6. The only additional point is the finite difference of $\nabla\eta$ term, but this term is bounded by a trivial estimate. The bound for the Holder seminorm of the second term is

$$\frac{C(\alpha)}{\nu}\left(1+t^{\alpha}+\left(\frac{t}{\nu}\right)^{\frac{1}{2}}+\left(\frac{t}{\nu}\right)^{2}\right)\|X-\mathrm{Id}\|^{\alpha}_{Lip(0,T;C^{1+\alpha})}\|\eta\|_{L^{\infty}(0,T;C^{1+\alpha})}\|\tau\|_{L^{\infty}(0,T;C^{\alpha,p})}. \tag{4.137}$$

To sum up, we have

$$\begin{aligned}\|I_2(t)\|_{\alpha,p}\leq \frac{C(\alpha)}{\nu}\left(1+t+\left(\frac{t}{\nu}\right)^{2}\right)\left(1+\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)^{4}M_X^{1+3\alpha}\\ \|X'\|_{L^{\infty}(0,T;C^{1+\alpha})}\|\tau\|_{Lip(0,T;C^{\alpha,p})}.\end{aligned} \tag{4.138}$$

If we put this together,

$$\begin{aligned}&\left\|\left[X'\circ X^{-1}\cdot\nabla,\mathbb{G}\right](\tau\circ X^{-1})\right\|_{L^{\infty}(0,T;C^{\alpha,p})}\\ &\leq C\|X'\|_{L^{\infty}(0,T;C^{1+\alpha})}M_X^{1+2\alpha}\left\|\Gamma(\tau\circ X^{-1})\right\|_{L^{\infty}(0,T;C^{\alpha,p})}\\ &+(\|X'\|_{L^{\infty}(0,T;C^{1+\alpha})}+\|X'\|_{Lip(0,T;C^{1+\alpha})}T^{\frac{1}{2}})F_1(\nu,\alpha,X,\|\tau\|_{Lip(0,T;C^{\alpha,p})},T)\end{aligned} \tag{4.139}$$

where $F_1$ depends on the written variables and grows like polynomial in $T$, $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$, and $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$. The bound on $\Gamma(\tau\circ X^{-1})$ is given by Theorem 4.3.6. □

## 4.4 Bounds on variations and variables

Using the results from the previous section, we will find bound for variations and variables. For simplicity, we adopt the notation

$$M_\epsilon = 1+\|X_\epsilon-\mathrm{Id}\|_{L^{\infty}(0,T;C^{1+\alpha})}. \tag{4.140}$$

First, we will find bound for $\frac{d}{d\epsilon}\mathcal{V}_\epsilon$. Note that $X_\epsilon(0) = \mathrm{Id}$, so $X'_\epsilon(0) = 0$ and by Theorem 4.3.3 and since $X'_\epsilon \in Lip(0,T;C^{1+\alpha,p})$ we have

$$\begin{gathered} \|X'_\epsilon\|_{L^\infty(0,T;C^{1+\alpha})} \leq T\,\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})}\,, \\ \|\eta_\epsilon(t)\|_{C^\alpha} \leq t\,\|X'\|_{Lip(0,T;C^{1+\alpha,p})}\,M_\epsilon^\alpha. \end{gathered} \tag{4.141}$$

Then by the Theorem 4.3.4, we have

$$\begin{gathered} \|\eta_\epsilon\cdot\mathbb{L}_\nu(\nabla_x u_{\epsilon,0})\|_{L^\infty(0,T;C^{\alpha,p})} \leq C\left(\frac{T}{\nu}\right)^{\frac{1}{2}} M_\epsilon^\alpha\,\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})}\,\|u_{\epsilon,0}\|_{1+\alpha,p}\,, \\ \left\|\mathbb{L}_\nu(u'_{\epsilon,0})\right\|_{L^\infty(0,T;C^{\alpha,p})} \leq C\left\|u'_{\epsilon,0}\right\|_{\alpha,p}. \end{gathered} \tag{4.142}$$

By the Theorem 4.3.8, we have

$$\begin{gathered} \|[\eta_\epsilon\cdot\nabla_x,\mathbb{U}]\,(\sigma_\epsilon - u_\epsilon\otimes u_\epsilon)\|_{L^\infty(0,T;C^{\alpha,p})} \leq C\left(\left(\frac{T}{\nu}\right)^{\frac{1}{2}} + \left(\frac{T}{\nu}\right)\right) M_\epsilon^{2+4\alpha} \\ \|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha})}\,\|\tau_\epsilon - v_\epsilon\otimes v_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} \end{gathered} \tag{4.143}$$

and by the Theorem 4.3.5, we have

$$\begin{gathered} \left\|\mathbb{U}\left(\delta_\epsilon - (v'_\epsilon\otimes v_\epsilon + v_\epsilon\otimes v'_\epsilon)\circ X_\epsilon^{-1}\right)\right\|_{L^\infty(0,T;C^{\alpha,p})} \leq C\left(\frac{T}{\nu}\right)^{\frac{1}{2}} M_\epsilon^\alpha \\ \|\tau'_\epsilon - (v'_\epsilon\otimes v_\epsilon + v_\epsilon\otimes v'_\epsilon)\|_{L^\infty(0,T;C^{\alpha,p})}\,. \end{gathered} \tag{4.144}$$

Therefore,

$$\begin{gathered} \left\|\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})} \leq C\left\|u'_{\epsilon,0}\right\|_{\alpha,p} \\ +S_1(T)(\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})} + \|v'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} + \left\|\sigma'_{\epsilon,0}\right\|_{\alpha,p} + \|\tau'_\epsilon\|_{Lip(0,T;C^{\alpha,p})})Q_1 \end{gathered} \tag{4.145}$$

where $S_1(T)$ vanishes as $T^{\frac{1}{2}}$ as $T\to 0$ and $Q_1$ is a polynomial in variables $\|u_{\epsilon,0}\|_{1+\alpha,p}$, $\|X_\epsilon - \mathrm{Id}\|_{Lip(0,T;C^{1+\alpha,p})}$, $\|\tau_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}$, and $\|v_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}$, whose coefficients depend on $\nu$. Similarly,

$$\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} \le M_X^\alpha \|u_0\|_{1+\alpha,p} + C_1 \|X - \mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \|\sigma_{\epsilon,0}\|_{\alpha,p} + S_2(T)Q_2 \tag{4.146}$$

where $S_2(T)$ vanishes as $T^{\frac{1}{2}}$ as $T \to 0$ and $Q_2$ is polynomial in variables $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$ and $\|X - \mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$, whose coefficients depend on $\alpha$ and $\nu$. Also

$$\begin{aligned} \|g'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} &\le C(\left\|u'_{\epsilon,0}\right\|_{1+\alpha,p} + \|X - \mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \left\|\tau'_{\epsilon,0}\right\|_{\alpha,p}) \\ &+ S_3(T)(\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})} + \left\|\sigma'_{\epsilon,0}\right\|_{\alpha,p} + \|\tau'_\epsilon\|_{Lip(0,T;C^{\alpha,p})} + \|v'_\epsilon\|_{L^\infty(0,T;C^{1+\alpha,p})})Q_3 \end{aligned} \tag{4.147}$$

where $S_3(T)$ vanishes as $T^{\frac{1}{2}}$ as $T \to 0$ and $Q_3$ is polynomial in variables $\|u_{\epsilon,0}\|_{1+\alpha,p}$, $\|X - \mathrm{Id}\|_{Lip(0,T;C^{1+\alpha,p}}$, $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$, and $\|v_\epsilon\|_{L^\infty(0,T;C^{1+\alpha,p})}$, whose coefficients depend on $\nu$ and $\alpha$. Then we have

$$\left\|\nabla_a \frac{d}{d\epsilon}\mathcal{V}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})} \le T\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha})}\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} + M_\epsilon \|g'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} \tag{4.148}$$

and

$$\begin{aligned} \left\|\frac{d}{d\epsilon}\mathcal{T}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})} &\le 2\|g'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}\left(\|\tau_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} + 2\rho K\right) \\ &+ \|\tau'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}\left(\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})} + 2k\right). \end{aligned} \tag{4.149}$$

## 4.5 Local existence of solution

We define the function space $\mathcal{P}_1$ and the set $\mathcal{I}$.

$$\begin{aligned} \mathcal{P}_1 &= Lip(0,T;C^{1+\alpha,p}) \times Lip(0,T;C^{\alpha,p}) \times L^\infty(0,T;C^{1+\alpha,p}) \\ \mathcal{I} &= \{(X,\tau,v) : \|(X - \mathrm{Id},\tau,v)\|_{\mathcal{P}_1} \le \Gamma, v = \frac{dX}{dt}\}, \end{aligned} \tag{4.150}$$

where $\Gamma > 0$ and $T > 0$ are to be determined. Now for given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$ we define the map

$$(X, \tau, v) \to \mathcal{S}(X, \tau, v) = (X^{new}, \tau^{new}, v^{new}) \tag{4.151}$$

where

$$\begin{cases} X^{new}(t) = \mathrm{Id} + \int_0^t \mathcal{V}(X(s), \tau(s), v(s))ds, \\ \tau^{new}(t) = \sigma_0 + \int_0^t \mathcal{T}(X(s), \tau(s), v(s))ds, \\ \qquad v^{new}(t) = \mathcal{V}(X, \tau, v). \end{cases} \tag{4.152}$$

If $(X - \mathrm{Id}, \tau, v) \in \mathcal{P}_1$, then $(X^{new} - \mathrm{Id}, \tau^{new}, v^{new}) \in \mathcal{P}_1$ for any choice of $T > 0$. Moreover, we have the following:

**Theorem 4.5.1.** *For given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$, there is a $\Gamma > 0$ and $T > 0$ such that the map $\mathcal{S}$ of (4.152) maps $\mathcal{I}$ to itself.*

*Proof.* It is obvious that $\frac{d}{dt}X^{new} = v^{new}$. For the size of $\mathcal{S}(X, \tau, v)$, first note that if $(X - \mathrm{Id}, \tau, v)_{\mathcal{P}_1} \leq \Gamma$, then

$$M_X = 1 + \|X - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})} \leq 1 + T\Gamma. \tag{4.153}$$

Now applying Theorem 4.3.4 and Theorem 4.3.5, we know that

$$\|\mathcal{V}\|_{L^\infty(0,T;C^{\alpha,p})} \leq \|u_0\|_{\alpha,p} + A_1(T)B_1(\Gamma, \|u_0\|_{\alpha,p}, \|\sigma_0\|_{\alpha,p}) \tag{4.154}$$

where $A_1(T)$ vanishes like $T^{\frac{1}{2}}$ for small $T > 0$ and $B_1$ is a polynomial in its arguments, and some coefficients depend on $\nu$. Now

$$\|g\|_{L^\infty(0,T;C^{\alpha,p})} \leq \|u_0\|_{1+\alpha,p} + C_1\Gamma^\alpha \|\sigma_0\|_{\alpha,p} + A_2(T)B_2(\Gamma, \|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}) \tag{4.155}$$

where $C_1$ is as in Theorem 4.3.6, depending only on $\alpha$ and $\nu$, $A_2(T)$ vanishes in the same order as $A_1(T)$ as $T \to 0$, and $B_2$ is a polynomial in its arguments, and some coefficients depend on $\nu$ and $\alpha$. Now from (4.18) we conclude

$$\|\mathcal{V}\|_{L^\infty(0,T;C^{1+\alpha,p})} \leq K_1(\|u_0\|_{1+\alpha,p} + \Gamma^\alpha \|\sigma_0\|_{\alpha,p}) + A_3(T)B_3(\Gamma, \|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}) \tag{4.156}$$

where $K_1$ is a constant depending only on $\nu$ and $\alpha$, and $A_3$ and $B_3$ have the same properties as previous $A_i$s and $B_i$s. Now we measure $\mathcal{T}$. From (4.98) and previous estimate on $g$ we have

$$\|\mathcal{T}\|_{L^\infty(0,T;C^{\alpha,p})} \leq K_2(\|u_0\|_{1+\alpha,p}(\rho K + \|\sigma_0\|_{\alpha,p}) + \|\sigma_0\|_{\alpha,p}(\Gamma^\alpha \|\sigma_0\|_{\alpha,p} + \rho K \Gamma^\alpha + k)) \\ + A_4 B_4 \tag{4.157}$$

where $K_2$ is a constant depending on $\nu$ and $\alpha$, and $A_4$ and $B_4$ are as before. Since $\alpha < 1$, we can appropriately choose large $\Gamma > \|\sigma_0\|_{\alpha,p} + \|u_0\|_{1+\alpha,p}$ and correspondingly small $\frac{1}{6} > T > 0$ so that the right side of (4.156) and (4.157) are bounded by $\frac{\Gamma}{6}$. Then $\|(X^{new} - \mathrm{Id}, \tau^{new}, \upsilon^{new})\|_{\mathcal{P}_1} \leq \Gamma$. □

Now we can prove that $\mathcal{S}$ is a contraction mapping on $\mathcal{I}$ for a short time.

**Theorem 4.5.2.** *For given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$, there is a $\Gamma$ and $T > 0$, depending only on $\|u_0\|_{1+\alpha,p}$ and $\|\sigma_0\|_{\alpha,p}$, such that the map $\mathcal{S}$ is a contraction mapping on $\mathcal{I} = \mathcal{I}(\Gamma, T)$, that is*

$$\|\mathcal{S}(X_2, \tau_2, \upsilon_2) - \mathcal{S}(X_1, \tau_1, \upsilon_1)\|_{\mathcal{P}_1} \leq \frac{1}{2}\|(X_2 - X_1, \tau_2 - \tau_1, \upsilon_2 - \upsilon_1)\|_{\mathcal{P}_1}. \tag{4.158}$$

*Proof.* First from Theorem 4.5.1 we can find a $\Gamma$ and $T_0 > 0$, depending only on the size of initial data, say

$$N = \max\{\|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}\} \tag{4.159}$$

which guarantees that $\mathcal{S}$ maps $\mathcal{I}$ to itself. This property still holds if we replace $T_0$ by any smaller $T > 0$. Reminding that $\mathcal{I}$ is convex, we may put

$$\begin{aligned} X_\epsilon &= (2-\epsilon)X_1 + (\epsilon-1)X_2, \\ \tau_\epsilon &= (2-\epsilon)\tau_1 + (\epsilon-1)\tau_2, 1 \le \epsilon \le 2. \end{aligned} \tag{4.160}$$

Then $(X_\epsilon, \tau_\epsilon, v_\epsilon) \in \mathcal{I}$, $v_\epsilon = (2-\epsilon)v_1 + (\epsilon-1)v_2$, $u_{\epsilon,0} = u_0$, and $\sigma_{\epsilon,0} = \sigma_0$. This means that

$$X'_\epsilon = X_2 - X_1, v'_\epsilon = v_2 - v_1, u'_{\epsilon,0} = 0, \sigma'_{\epsilon,0} = 0. \tag{4.161}$$

Then from the results of Section 4.4, we see that

$$\begin{aligned} \left\|\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right\|_{L^\infty(0,T;C^{1+\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})})S'_1(T)Q'_1(\Gamma) \\ \|\mathcal{X}'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})})S'_2(T)Q'_2(\Gamma), \end{aligned} \tag{4.162}$$

$$\begin{aligned} \|\pi_\epsilon\|_{Lip(0,T;C^{\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})})S'_3(T)Q'_3(\Gamma). \end{aligned}$$

where $\mathcal{X}'_\epsilon$ and $\pi_\epsilon$ are defined in (4.22), $S'_1(T), S'_2(T), S'_3(T)$ vanish to 0 in the rate of $T^{\frac{1}{2}}$ as $T \to 0$, and $Q'_1(\Gamma), Q'_2(\Gamma), Q'_3(\Gamma)$ are polynomials in $\Gamma$, whose coefficients depend only on $\nu$ and $\alpha$. By choosing $0 < T < T_0$ small enough, depending on the size of $Q'_i(\Gamma)$s, we are done. □

Therefore, we have a solution to the system (4.1) in the function space $\mathcal{P}_1$ for a short time, that is, $(X, \tau, v)$ satisfying $v = \frac{dX}{dt}$ and satisfying (4.16). Now we have the uniqueness.

**Theorem 4.5.3.** *Assume that for given* $\Gamma > 0$ *and* $T > 0$, $(X_1, \tau_1, v_1), (X_2, \tau_2, v_2) \in \mathcal{I}$, *with initial data* $(u_1(0), \sigma_1(0)), (u_2(0), \sigma_2(0))$ *respectively, satisfies (4.16), that is, they solve the system (4.1). Then there exists* $T_0 > 0$ *such that*

$$\begin{aligned} &\|X_2 - X_1\|_{Lip(0,T_0;C^{1+\alpha,p})} + \|\tau_2 - \tau_1\|_{Lip(0,T_0;C^{\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T_0,C^{1+\alpha,p})} \\ &\qquad \le C(\|u_2(0) - u_1(0)\|_{1+\alpha,p} + \|\tau_2(0) - \tau_1(0)\|_{\alpha,p}) \end{aligned} \tag{4.163}$$

*Proof.* We repeat the calculation of the Theorem 4.5.2, but this time $u'_{\epsilon,0} = u_1(0) - u_2(0)$ and $\sigma'_{\epsilon,0} = \sigma_1(0) - \sigma_2(0)$. Then we choose $T_0$ small enough that $S'_i(T_0)Q'_1(\Gamma) < \frac{1}{2}$. □

# Chapter 5

# Polymer drag reduction

## 5.1 Navier-Stokes – End-Functionalized Polymer System

Here, we provide a *formal* (non-rigorous) derivation of a system of equations and boundary conditions to describe the setting of a neutral fluid confined to a domain with end-functionalized polymer along the solid walls. Our assumptions, $(A_1)$–$(A_8)$, are detailed below.

### 5.1.1 Kinetic Theoretic Derivation

We consider general bounded domains $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$. At the end of the section, we will discuss the interpretation for two-dimensional case. Our models are based on the following set of assumptions.

- $(A_1)$ *One-end anchored.* The layer consists of polymers floating in the solvent with one end anchored to the wall (e.g. chemically bound or strongly adsorbed).
- $(A_2)$ *Wall coating.* The grafted polymers covers the boundary surface, and the thickness of this covering layer is the order of characteristic length-scale,

denoted by $R$, of polymers. We can think of $R$ as the gyration radius of the tethered polymer.

- $(A_3)$ *Multi-scale assumption.* We assume that at the scale of the polymer, the surrounding fluid can be described as a continuum and also that the polymer appears 'infinitesimal' from the perspective of the macroscopic fluid, i.e. we assume scale seperation

$$\lambda_{mf} \ll R \ll \lambda_{\nabla}, \tag{5.1}$$

where $\lambda_{mf}$ is the mean-free path of the molecules making up the solvent and $\lambda_{\nabla}$ is the gradient length of the continuum description of the fluid (i.e. typical variation scale of the macroscopic flow). In particular, the polymer should fit well within the near-wall viscous sublayer of the flow. Additionally, in the case of domains with curvilinear boundary, we assume that the typical scale of the polymer $R$ is much small relative to the radius of curvature of the boundary

$$R \ll (\text{minimum radius of boundary curvature}), \tag{5.2}$$

say $\frac{1}{R} > 4\kappa$, where $\kappa$ is the boundary curvature defined by (5.26). Therefore, the configuration space for polymers at $x \in \partial\Omega$ with its outward normal vector $\hat{n} = \hat{n}(x)$ is given by a flat half-space,

$$M(x) := \{m \in \mathbb{R}^d : m \cdot (-\hat{n}(x)) > 0\}. \tag{5.3}$$

In the case where finite extend mode is employed (e.g. FENE), then this domain is intersected with a ball $B_r(0)$, thereby building in the finite stretching range $r$ of the polymer.

The above assumptions are concerned with small-scale polymer structure and allow us to determine how they are effectively 'seen' by the large scale fluid solvent. We now make an assumption on the structure of the near-wall velocity at those scales of $O(R)$, which determines how the fluid interacts with them. This "microscopic" structure assumption will be forgotten in our continuum model, within which it translates simply to a tangential slip velocity along the boundary.

- $(A_4)$ *Velocity field of the flow inside the layer.* Microscopically (at the scale of the polymer $R$), we approximate the velocity of the flow inside the layer by a linear shear. Specifically, the velocity linearly interpolates between the wall side where it vanishes (assuming no-slip on the polymer scale) and its value at near the boundary of the polymer layer which is $u$ and which is tangent to the boundary. This "outer" velocity $u$ becomes the velocity *at the boundary* in our macroscopic closure.

Because of assumptions $(A_1)$ – $(A_4)$, we impose the following boundary condition: since the thickness of the layer is far less than the macroscopic length-scale, we only care about the response of the layer for the flow at wall. We do not incorporate the thickness or shape of the layer in our model. We do not have stress balance condition for normal stress $\hat{n} \cdot \Sigma_F \cdot \hat{n}$. One can ask whether or not the normal stresses also balance, i.e. whether $\hat{n} \cdot \Sigma_L \cdot \hat{n} = \hat{n} \cdot \Sigma_F \cdot \hat{n}$. In our work, we work in a regime in which the layer does not appreciably move or deform in the normal direction. Consequently, the net force (per unit area) in the normal direction acting on the layer is zero, that is, $\Sigma_L \cdot \hat{n} + \vec{N} = \Sigma_F \cdot \hat{n}$, where $\vec{N}$ is the normal force (per unit area) that the wall exerts to the polymer layer. That is, the fluid parcels adjacent to the wall feel the presence of the wall in the normal direction. To explain further, we note that along the fluid-layer boundary the force (per unit area) $(\Sigma_F - \Sigma_L) \cdot \hat{n}$ is applied to the layer. On the other hand, along the layer-wall boundary the normal force (per unit area) $\vec{N}$ is applied to the layer. Then we have balance of two forces, as the layer is steady in the normal direction. On the other hand we have stress balance condition for the shear stress since the layer, which is a mixture of solvent and polymer, covers the wall. We formalize this as an assumption:

- $(A_5)$ *Tangential stress balance.* The layer along (impermeable) wall exerts elastic stress due to the restoring force of the fluid-polymer layer which balances the viscous stress of the bulk fluid.

This assumption gives the following: given a point $x$ on the boundary, let $\hat{n}$ be the outward normal vector and $u$ be the fluid velocity at $x$. Let $\Sigma_L$ be the stress exerted by the layer (normalized by $\rho$), and $\Sigma_F$ be the stress exerted by the bulk fluid. By impermeability and $(A_5)$ we have

$$\begin{aligned} u \cdot \hat{n} = 0, \text{ on } \partial\Omega, \\ \hat{\tau}_i \cdot \Sigma_L \cdot \hat{n} = \hat{\tau}_i \cdot \Sigma_F \cdot \hat{n}, \text{ on } \partial\Omega, \ i = 1, \ldots, d-1, \end{aligned} \tag{5.4}$$

where, for every $x \in \partial\Omega$, the vectors $\{\hat{\tau}_i(x)\}_{i=1}^{d-1}$ form an orthogonal basis of the tangent space of $\partial\Omega$ at $x$. The stress that the layer exerts is understood as a combination of the polymer stress $\Sigma_P$ and the fluid solvent in the layer $\Sigma_S$,

$$\Sigma_L = \Sigma_S + \Sigma_P. \tag{5.5}$$

The stress associated to the solvent in the layer is determined from assumption $(A_4)$. In particular, it is set by the relative velocity near the wall (as it is in for, e.g. Navier-friction boundary condition) so that $\hat{n} \cdot \Sigma_S = -\frac{\nu}{2R} u + \vec{N}$, where $\vec{N}$ is the wall normal force. The corresponding stress balance (5.4) then reads

$$\hat{n} \cdot \Sigma_F \cdot \hat{\tau}_i = \hat{n} \cdot \Sigma_P \cdot \hat{\tau}_i - \frac{\nu}{2R} u \cdot \hat{\tau}_i, \text{ on } \partial\Omega, \ i = 1, \ldots, d-1. \tag{5.6}$$

Without polymer, this stress-balance argument yields the Navier-friction boundary condition (1.25). Specifically, under the assumption $(A_4)$, we consider a fluid parcel of thickness $\lambda$, which is much smaller than the flow length-scale $L$, which is in contact with the wall. As in our case, we set up an effective boundary condition on top of this fluid parcel. Again we assume there is no inflow from the rest of the fluid domain to this fluid parcel. Then, its normal stress $\Sigma_L \cdot \hat{n}$ can be similarly approximated by $-\frac{\nu}{2\lambda}u$ and by the continuity of stress for a Navier-Stokes fluid we obtain

$$2\,(D(u)\hat{n}) \cdot \hat{\tau}_i + \frac{1}{2\lambda/L} u \cdot \hat{\tau}_i = 0. \tag{5.7}$$

The natural regime of validity for the above assumptions to hold in a viscous fluid without polymer additives forces $\lambda = O(\nu)$ so that the layer lies within the viscous sublayer. In this way, (5.7) recovers the Navier-friction boundary condition (1.25).

The final ingredient for our model is then $\Sigma_P$, the polymer layer stress. To obtain this, we need to say something about the structure and dynamics of the polymer additives. Based on $(A_1) - (A_4)$, we assume

- $(A_6)$ *Bead-Spring approximation.* Polymers are modeled as an elastic dumbbell with spring potential $k_B\overline{T}U(m)$, where $k_B$ is the Boltzmann constant, $\overline{T}$ is the temperature, $U(m)$ is non-dimensional spring potential, with one end anchored to the wall. Its configuration is characterized by its end-to-end vectors, $m$.

- $(A_7)$ *Reflecting condition.* We adopt the reflecting boundary condition for beads: the bead reflects in the direction of the inward normal vector if ever it hits the wall.

- $(A_8)$ *Single-Chain approximation at the wall.* For simplicity, we ignore the interaction between polymers anchored at the wall. We calculate the dynamics of each polymer as if there is only single chain anchored at the wall, and add them. This puts us in the so-called *mushroom regime.*

We remark that to be in the "mushroom regime" in which the polymers do not interact, one requires that the polymer number density $N_P$ defined by (5.9) satisfy

$N_P < N^*$ where $N^* \sim a_0^{-2} N^{-6/5}$ where $N$ is the polymerization index [42] and $a_0$ is the monomer size (see Chp. 13 of [112]).

From assumptions $(A_1) - (A_8)$, we may describe the dynamics of polymers anchored at the wall, and derive Fokker-Planck equation for the polymer probability distribution, denoted by $f_P(x, m, t)$. The final ingredient of the model, required for (5.4), is the expression for the stress, and we use Kramers formula [116] (we already introduceed Kramers formula (1.45) in Chapter 1, but we re-introduced since we care of what is the coefficient in this Chapter) :

$$\Sigma_P = \frac{k_B \overline{T}}{\rho} \int_{M(x)} m \otimes \nabla_m U f_P dm. \tag{5.8}$$

Although the expression (5.8) is standard in theoretical polymer physics, we provide a short derivation in Section 5.4.2 as it is crucial for the derivation of our model. We make a brief remark now about dimensions. We note that $\rho$, the solvent mass-density, is taken constant and has units of $M/L^d$. Then $k_B \overline{T}/\rho$ has units $L^{2+d}/T^2$. Also we assume that polymers are uniformly grafted over the wall. Specifically, the polymer number density $N_P$ at every $x \in \partial\Omega$ (which is preserved in time by the dynamics for each $x$), is taken to be constant on the boundary, i.e.

$$N_P := \int_{M(x)} f_P dm = (\text{const.}). \tag{5.9}$$

The units of $N_P$ is taken as $1/L^d$. The dimension of $k_B \overline{T} N_P/\rho$ is $(L/T)^2$, the same as that of stress $\Sigma_L$.

**Remark 14.** *Examples for potential choices of configuration spaces and spring potentials are:*

1. *Hookean-type dumbbell: we set* $r$ *in* $(A_4)$ *to be* $r = \infty$ *and*

$$U(m) = H\left(\frac{|m|}{R}\right)^{2k}, \qquad k \geq 1, \tag{5.10}$$

*where* $H$ *is the non-dimensionalized spring constant. Note that, compared with the standard (dimensional) spring constant* $H_{st}$ *where* $k = 1$*, we have the relation* $H_{st} = Hk_B\overline{T}/R^2$*.*

2. *FENE (Finitely Extensible Nonlinear Elastic) models: we have a finite* $r < \infty$ *in* $(A_4)$ *and take*

$$U(m) = -H\log\left(1 - \frac{|m|^2}{R^2}\right). \tag{5.11}$$

To derive a governing equation for the end-functionalized polymers, we follow Ottinger [116]. For the polymer of configuration $m$, anchored at the wall of position $x$ and initially in configuration $m_0$, the evolution of $m := m_t(m_0)$ is determined by the deterministic forces (drift velocity and elastic restoring force) and random fluctuation. Since the length-scale of the polymer $R$ is assumed small relative to the minimum radius of curvature at the boundary across the domain, a polymer pinned at any given $x \in \partial\Omega$ on the boundary is assumed to wander around the half-space $M(x)$ defined by the normal $\hat{n}(x)$ at that point. Moreover, we assume that if the polymer end is simply reflected in the direction of the wall-normal $\hat{n}(x_0)$ in the event that it randomly hits the boundary. Specifically, under the bead-spring approximation $(A_6)$, drift velocity from the near-wall linear shear $(A_4)$ on the polymer is given by

$$\text{(drift by fluid experienced by polymer)} = \left(\frac{m}{R}\cdot(-\hat{n})\right)u. \tag{5.12}$$

The elastic restoring force is simply $\frac{k_B\overline{T}}{\zeta}\nabla_m U$ and also contributes to the drift on the bead. The noise is assumed to be of additive Brownian type with strength $\sqrt{\frac{2k_B\overline{T}}{\zeta}}$. Therefore, for each $x \in \partial\Omega$, the polymer end-to-end extension $m_t(m_0) := m_t(m_0; x) \in$

$M(x)$ is a stochastic process described by a reflecting drift-diffusion process on the half-plane $M(x)$:

$$\begin{aligned} dm_t(m_0) &= \left( \frac{u(x,t)}{R} m_t(m_0) \cdot (-\hat{n}(x)) - \frac{k_B \overline{T}}{\zeta} \nabla_m U(m_t(m_0)) \right) dt \\ &\quad + \sqrt{\frac{2k_B \overline{T}}{\zeta}} dW_t + \hat{n}(x)\ d\ell_t(m_0), \\ & m_t(m_0)|_{t=0} = m_0 \in M(x) \end{aligned} \tag{5.13}$$

where $W_t$ is a $d$-dimensional standard Brownian motion, and $\ell_t(m_0)$ is the boundary local time density which, for a stochastic polymer end located at some $m \in M(x)$ at time $t$ is the time within the interval $[0,t]$ which is spent near the boundary $\partial M(x)$ per unit distance [125], [98]. It is formally defined by

$$\ell_t(m_0) = \int_0^t \delta\left(\mathrm{dist}(m_s(m_0), \partial M(x)\right) ds. \tag{5.14}$$

See Theorem 2.6 of [22]. We remark that Lions & Sznitman [98] proved existence and uniqueness of stochastic processes as strong solutions to this "Skorohod problem" with Lipschitz drifts and sufficient smooth boundaries with regular normal vectors $\hat{n}$. For an extended discussion, see §2 of [49]. The Fokker-Planck equation associated to the stochastic differential equation (5.13) reads

$$\begin{aligned} \partial_t f_P + \nabla_m \cdot \left( \left( \frac{u(x,t)}{R} (m \cdot (-\hat{n})) - \frac{k_B \overline{T}}{\zeta} \nabla_m U \right) f_P \right) &= \frac{k_B \overline{T}}{\zeta} \Delta_m f_P \\ \text{in} \quad [0,T] \times M(x)&, \\ \hat{n}(x) \cdot \nabla_m f_P = 0 \qquad\qquad \text{on} \quad [0,T] \times \partial M(x)&, \end{aligned} \tag{5.15}$$

for each $x \in \partial\Omega$. To sum up, we arrive at the micro-macro system

$$\begin{aligned} \partial_t u = \nabla_x \cdot \Sigma_F + f_b, \text{ in } \Omega \times (0,T), \\ u|_{t=0} = u_0 \text{ on } \Omega \times \{t = 0\}, \\ \nabla \cdot u = 0 \text{ in } \Omega \times [0,T), \\ u \cdot \hat{n} = 0 \text{ on } \partial\Omega \times [0,T), \\ \hat{\tau}_i \cdot \Sigma_F \cdot \hat{n} = \hat{\tau}_i \cdot \Sigma_L \cdot \hat{n} \text{ on } \partial\Omega \times (0,T), \; i = 1, \ldots, d-1 \end{aligned} \tag{5.16}$$

where $f_b$ is a body forcing, the $\Sigma_F$ is the fluid stress tensor, which for a simple Navier-Stokes fluid reads

$$\Sigma_F := -u^\nu \otimes u^\nu - p^\nu \mathbb{I} + 2\nu D(u^\nu), \tag{5.17}$$

recalling that $D(u) = 1/2(\nabla_x u + (\nabla_x u)^t)$ is the symmetric part of the velocity gradient tensor and

$$\hat{\tau}_i \cdot \Sigma_L \cdot \hat{n} = \hat{n} \cdot \Sigma_P \cdot \hat{\tau}_i - \frac{\nu}{2R} u \cdot \hat{\tau}_i, \tag{5.18}$$

where the polymer stress $\Sigma_P$ is given by the Kramers expression (5.8), which is closed by the Fokker-Planck equation (5.15) for the polymer distribution at the boundary, $f_P$ which is supplied with initial conditions $f_P(0)$. The system (5.15) – (5.16) comprises our proposed microscopic-macroscopic system to describe the Navier-Stokes-fluid/end-functionalized polymer interaction. Note that due to the impermeability condition $u \cdot \hat{n} = 0$ on the boundary the stress that the fluid exerts on the wall is entirely due to viscosity

$$\hat{\tau}_i \cdot \Sigma_F \cdot \hat{n} = 2\nu \; \hat{\tau}_i \cdot D(u) \cdot \hat{n}. \tag{5.19}$$

**Remark 15** (On the validity of assumptions)**.** *In our opinion, the most subtle of our assumptions are* $(A_4)$ *and* $(A_8)$*. First, one may question whether* $(A_8)$ *(single-chain* *approximation so that the polymers do not interact with eachother) can be compatible with* $(A_2)$ *(that, from the macroscopic point of view, the polymer forms a continuous carpet along the boundary). We believe there is a regime of validity where these assumptions coexist, however, even if it is not the case, we interpret* $(A_8)$ *as a first-hand approximation of the regime in which polymers are close enough to effectively cover the wall but their interactions are not too strong. This interpretation naturally asks a more realistic assumption to replace* $(A_8)$*. Perhaps the most natural thing to consider is the "polymer brush" regime, in which the polymers are spaced close together on the boundary and may strongly interact with each other [112, 111]. It is unclear to us whether or not a fully macroscopic description for this regime will be possible. If not, a coupled microscopic-macroscopic system must be studied to understand the behavior in this regime.*

*For* $(A_4)$*, the central issue is the range of parameters which makes linear shear approximation valid. For large enough* $\boldsymbol{\alpha}$ *and small enough* $\boldsymbol{Re}$*, the flow will be laminar near the walls and assumption* $(A_4)$ *should valid. On the other hand, for large* $\boldsymbol{Re}$ *the flow will develop small-scale possible invalidating the aforementioned justification of* $(A_4)$*. If this boundary condition regularizes the macroscopic (outside the polymer layer) near-wall flow and it resembles a linear shear, it provides a supporting evidence for* $(A_4)$*. Therefore, it would be interesting to test whether the boundary condition allows us to maintain a shear-like near-wall flow for certain* $\boldsymbol{\alpha}$ *and* $\boldsymbol{Re}$*. It would be also interesting to test the validity of* $(A_4)$ *by more microscopic method, for example, molecular dynamics simulations [43, 131].*

### 5.1.2 Energetics: microscopic/macroscopic balance

**Proposition 5.1.1.** *Suitably smooth solutions of* (5.16) *satisfy the following global energy balance*

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u|^2 dx + \frac{k_B\overline{T}}{\rho} R\mathcal{E}\right) = -\int_\Omega \nabla_x u : \Sigma_F dx \\ -&\frac{\nu}{2R}\int_{\partial\Omega} |u|^2 dS - \frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int f_P \left|\nabla_m(\log f_P + U)\right|^2 dm dS. \end{aligned} \tag{5.20}$$

*Proof.* We set the body force $f_b \equiv 0$ for simplicity. The kinetic energy for (5.16) satisfies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx = -\int_\Omega \nabla_x u : \Sigma_F dx + \int_{\partial\Omega} u \cdot \Sigma_F \cdot \hat{n} dS \\ = &-\int_\Omega \nabla_x u : \Sigma_F dx + \sum_{i=1}^{d-1}\int_{\partial\Omega} u_{\tau_i}\hat{\tau}_i \cdot \Sigma_P \cdot \hat{n} dS - \sum_{i=1}^{d-1}\frac{\nu}{2R}\int_{\partial\Omega} |u_{\tau_i}|^2 dS \end{aligned} \tag{5.21}$$

where $u_{\tau_i} = u \cdot \hat{\tau}_i$ and the last identity comes from the no-flow condition of (5.16). Now we calculate the free energy of $f_L$:

$$\begin{aligned} \mathcal{E} &= \int_{\partial\Omega}\int_M f_P \log\left(\frac{f_P}{N_P e^{-U}}\right) dm dS \\ &= \int_{\partial\Omega}\int_M f_P \log f_P dm dS - N_P \log N_P |\partial\Omega| + \int_{\partial\Omega}\int_{M(x)} U f_P dm dS. \end{aligned} \tag{5.22}$$

A straightforward computation gives the evolution

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E} = \int_{\partial\Omega}\int_{M} \nabla_m f_P \cdot \left(\left(\frac{u(x,t)}{R}\,(m\cdot(-\hat{n})) - \frac{k_B\overline{T}}{\zeta}\nabla_m U\right)\right) dmdS \\
-\frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int \frac{|\nabla_m f_P|^2}{f_P} dmdS + \frac{d}{dt}\int_{\partial\Omega}\int_{M(x)} U f_P dmdS \\
= \frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int \Delta_m U f_P dmdS - \frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int \frac{|\nabla_m f_P|^2}{f_P} dmdS \\
+\sum_{i=1}^{d-1}\int_{\partial\Omega}\int \partial_{m_{\tau_i}} f_P(m\cdot\hat{n})dm\frac{u_{\tau_i}}{R}dS + \frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int \Delta_m U f_P dmdS \\
-\int_{\partial\Omega}\int \frac{k_B\overline{T}}{\zeta}|\nabla_m U|^2 f_P dmdS + \sum_{i=1}^{d-1}\frac{\rho}{k_B\overline{T}}\int_{\partial\Omega}\frac{u_{\tau_i}}{R}\hat{\tau}_i\cdot\Sigma_P\cdot(-\hat{n})dS \\
= \sum_{i=1}^{d-1}\frac{\rho}{k_B\overline{T}}\int_{\partial\Omega}\frac{u_{\tau_i}}{R}\hat{\tau}_i\cdot\Sigma_P\cdot(-\hat{n})dS - \frac{k_B\overline{T}}{\zeta}\int_{\partial\Omega}\int f_P\left|\nabla_m(\log f_P + U)\right|^2 dmdS.
\end{aligned}
\tag{5.23}
$$

The tangential polymer boundary stress appears in the evolution (5.23) of the free energy. Therefore, we find that the total energy of the system (kinetic energy of the bulk flow together with the free energy of the polymer layer) satisfies the balance (5.20). □

Note that for fluid models satisfying the following energy condition,

$$
\int_{\Omega} \nabla_x u : \Sigma_F dx \geq 0, \tag{5.24}
$$

the total energy (5.20) is non-increasing in time. This condition holds for a simple Navier-Stokes fluid for which $\Sigma_F$ is given by (5.17), provided that the domain has non-positive boundary curvatures. To see this, note that by incompressibility and the no-flow condition of (5.16) we have

$$\begin{aligned}\int_\Omega \nabla_x u^\nu : \Sigma_F dx &= \nu \int_\Omega |\nabla_x u^\nu|^2 dx + \nu \int_\Omega \nabla_x u^\nu : (\nabla_x u^\nu)^t dx \\ &= \nu \int_\Omega |\nabla_x u^\nu|^2 dx + \nu \sum_{i=1}^{d-1} \int_{\partial\Omega} (u^\nu \cdot \hat{\tau}_i) \partial_{\tau_i} u^\nu \cdot \hat{n} \; dS \\ &= \nu \int_\Omega |\nabla_x u^\nu|^2 dx - \sum_{i,j=1}^{d-1} \nu \int_{\partial\Omega} (u^\nu \cdot \hat{\tau}_i) \kappa_{ij} (u^\nu \cdot \hat{\tau}_j) dS\end{aligned} \tag{5.25}$$

where the boundary curvatures were introduced

$$\kappa_{ij} = \hat{\tau}_i \cdot \nabla \hat{n} \cdot \hat{\tau}_j. \tag{5.26}$$

If $\kappa \leq 0$ (negative semidefinite) at all points on the boundary, then energy condition (5.24) is automatically satisfied (this is true, for example, the canonical setting of flow on a channel with periodic side-walls for which $\kappa \equiv 0$, or in pipe flow for which the curvature is constant and negative). Otherwise, because of the condition $(A_3)$, if $1/R > 4 \sup_{x \in \partial\Omega} \kappa$ then we have the control of the curvature term.

### 5.1.3 Macroscopic closure: Navier-Stokes fluid and Hookean dumbbell polymer

If the solvent is taken to be a incompressible Navier-Stokes fluid and the polymer model is taken to be Hookean, that is, the radius $r$ in (5.3) is given by $r = \infty$ and the potential $U$ is chosen to be (5.10) with $k = 1$, i.e. $U(m) = H \left(\frac{|m|}{R}\right)^2$, we arrive at the closed system under some additional mild assumptions detailed below

$$
\begin{gathered}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \nu \Delta u^\nu + f_b \text{ in } \Omega \times (0,T), \\
u^\nu|_{t=0} = u_0 \text{ on } \Omega \times \{t=0\}, \\
\nabla \cdot u^\nu = 0 \text{ in } \Omega \times [0,T), \\
u^\nu \cdot \hat{n} = 0 \text{ on } \partial\Omega \times [0,T), \\
\left(\partial_t + \frac{4Hk_B\overline{T}}{R\zeta}\right)\left(2(D(u^\nu)\hat{n}) \cdot \hat{\tau}_i + \frac{1}{2R} u^\nu \cdot \hat{\tau}_i\right) = -\frac{k_B\overline{T}N_P}{\rho\nu R} u^\nu \cdot \hat{\tau}_i \\
\text{on } \partial\Omega \times (0,T),\ i = 1,\ldots,d-1.
\end{gathered}
\tag{5.27}
$$

To derive this fully macroscopic closure (5.27), first note that the Kramers formula (5.8) for the Hookean dumbbell becomes simply

$$
\Sigma_P = 2H\frac{k_B\overline{T}}{\rho}\int_M \frac{m}{R} \otimes \frac{m}{R} f_P dm. \tag{5.28}
$$

From the Fokker-Planck equation (5.15), the evolution of $\Sigma_P$ is derived

$$
\begin{aligned}
\partial_t \left(\Sigma_P\right)_{ij} &= 2H\frac{k_B\overline{T}}{\rho}\int_M \partial_{m_k}\left(\frac{m_i m_j}{R^2}\right)\frac{u_k^\nu}{R}(m \cdot (-\hat{n}))f_P dm \\
&-2H\frac{k_B\overline{T}}{\rho}\frac{k_B\overline{T}}{\zeta}\int_M \partial_{m_k}\left(\frac{m_i m_j}{R^2}\right)2H\frac{m_k}{R^2}f_P dm + 2H\frac{k_B\overline{T}}{\rho}\frac{k_B\overline{T}}{\zeta}\int_M \Delta_m\left(\frac{m_i m_j}{R^2}\right)f_P dm \\
&= \left(\sum_{\ell=1}^{d-1}\frac{u^\nu \cdot \hat{\tau}_\ell}{R}\left(\hat{\tau}_\ell \otimes (-\hat{n})\Sigma_P + \Sigma_P(-\hat{n}) \otimes \hat{\tau}_\ell\right) - \frac{4H}{R^2}\frac{k_B\overline{T}}{\zeta}\Sigma_P + \frac{4Hk_B\overline{T}}{R^2\rho}\frac{k_B\overline{T}}{\zeta}N_P\mathbb{I}\right)_{ij}
\end{aligned}
\tag{5.29}
$$

since $u^\nu \cdot \hat{n} = 0$. Then, contracting with the appropriate boundary normal and tangent vectors, we have

$$
\partial_t\left(\hat{\tau}_i \cdot \Sigma_P \cdot (-\hat{n})\right) = \frac{u^\nu \cdot \hat{\tau}_i}{R}\left((-\hat{n}) \cdot \Sigma_P \cdot (-\hat{n})\right) - \frac{4H}{R^2}\frac{k_B\overline{T}}{\zeta}\left(\hat{\tau}_i \cdot \Sigma_P \cdot (-\hat{n})\right), \tag{5.30}
$$

$$
\partial_t\left((-\hat{n}) \cdot \Sigma_P \cdot (-\hat{n})\right) = -\frac{4H}{R^2}\frac{k_B\overline{T}}{\zeta}\left((-\hat{n}) \cdot \Sigma_P \cdot (-\hat{n})\right) + \frac{4Hk_B\overline{T}}{R^2\rho}\frac{k_B\overline{T}}{\zeta}N_P. \tag{5.31}
$$

Note that the evolution of $((-\hat{n})\cdot\Sigma_P\cdot(-\hat{n}))$ completely decouples and does not depend on the tangential velocity. Further, equation (5.31) shows that at long times it converges to its equilibrium configuration,

$$((-\hat{n})\cdot\Sigma_P\cdot(-\hat{n}))_{eq} = \frac{k_B\overline{T}}{\rho}N_P. \tag{5.32}$$

For simplicity, we assume that $((-\hat{n})\cdot\Sigma_P\cdot(-\hat{n}))$ already reached at the equilibrium and therefore can be identified with the constant (5.32). This is non-essential for the macroscopic closure. If so, (5.30) becomes

$$\partial_t\left(\hat{\tau}_i\cdot\Sigma_P\cdot(-\hat{n})\right) = \frac{k_B\overline{T}N_P}{R\rho}(u^\nu\cdot\hat{\tau}_i) - \frac{4H}{R^2}\frac{k_B\overline{T}}{\zeta}\left(\hat{\tau}_i\cdot\Sigma_P\cdot(-\hat{n})\right). \tag{5.33}$$

By (1.41), (5.16) and (5.17), the above is equivalent to the stated boundary condition of (5.27).

### 5.1.4 Non-dimensionalization

Defining a characteristic length, $L$ (say the diameter of the domain $L = \mathrm{diam}(\Omega)$), characteristic velocity $V$ and convective time scale $T = L/V$, we write introduce dimensionless variables by taking $u = V\tilde{u}, t = T\tilde{t}, x = L\tilde{x}$. Note that the polymer relaxation time is $\lambda = \zeta R^2/4Hk_B\overline{T}$. We may now introduce the non-dimensional Reynolds number $\mathbf{Re}$, Weissenberg number $\mathbf{Wi}$, the relative stress strength $\boldsymbol{\tau}$ and the ratio of polymer to domain size $\boldsymbol{\alpha}$ as follows

$$\mathbf{Re} = \frac{VL}{\nu}, \mathbf{Wi} = \frac{\lambda}{T}, \boldsymbol{\tau} = \frac{\rho V^2}{k_B\overline{T}N_P}, \boldsymbol{\alpha} = \frac{L}{R}. \tag{5.34}$$

For definitions of the physical constants, see the introduction. Also we note that $(A_3)$ translates to $\boldsymbol{\alpha} > 4\kappa$. With these convensions, the equations for the non-dimensional variables in the bulk become

$$\begin{aligned} \partial_{\tilde{t}}\tilde{u}^{\nu} + \tilde{u}^{\nu}\cdot\nabla_{\tilde{x}}\tilde{u}^{\nu} &= -\nabla_{\tilde{x}}\tilde{p}^{\nu} + \frac{1}{\mathbf{Re}}\Delta_{\tilde{x}}\tilde{u}^{\nu} + \tilde{f}_b, \\ \nabla_{\tilde{x}}\cdot\tilde{u}^{\nu} &= 0, \end{aligned} \tag{5.35}$$

and, on the boundary, the following non-dimensionalized boundary condition holds

$$\left(\partial_{\tilde{t}} + \frac{1}{\mathbf{Wi}}\right)\left(2\tilde{D}(\tilde{u}^{\nu})\hat{n}\cdot\hat{\tau}_i + \frac{\boldsymbol{\alpha}}{2}\tilde{u}^{\nu}\cdot\hat{\tau}_i\right) = -\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}}\tilde{u}^{\nu}\cdot\hat{\tau}_i,\ i = 1,\ldots,d-1, \tag{5.36}$$

thereby reproducing the system (5.27). Note that, an alternative interpretation of the ratio $\boldsymbol{\alpha}\mathbf{Re}/\boldsymbol{\tau}$ appearing in the boundary condition is

$$\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} = \frac{\boldsymbol{\alpha}}{\mathbf{Wi}}\frac{\mu_p}{\mu_s},\ \mu_s = \rho\nu,\ \mu_p = N_P\lambda k_B\overline{T}, \tag{5.37}$$

where involving dynamic viscosities of the solvent $\mu_s$ and polymer $\mu_p$. The polymer viscosity $\mu_p$ is determined from kinetic theory as (number density)$\times$ (polymer relaxation time)$\times k_B\overline{T}$. The benefit of the non-dimensionalization (5.37) is that it allows one to base a Reynolds number on the total viscosity instead of accounting for the change in $\mathbf{Re}$ due to presence of polymers. Occasionally, a fourth parameter known as the elasticity $\mathbf{E} := \mathbf{Wi}/\mathbf{Re}$, is sometimes used. It is the ratio of polymer time scale to viscous time scale; it is thought to be more relevant in many cases. See Figure 4 of [63] for discussion about parameter regimes for drag reduction for dilute polymers added to the bulk.

For notational simplicity, we hereon drop the tildes and understand all variables to be dimensionless. That is, we write the system as

$$
\begin{gathered}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p^\nu + \frac{1}{\mathbf{Re}}\Delta u^\nu + f_b \text{ in } \Omega \times (0,T), \\
u^\nu|_{t=0} = u_0 \text{ on } \Omega \times \{t=0\}, \\
\nabla \cdot u^\nu = 0 \quad \text{in } \Omega \times [0,T), \\
u^\nu \cdot \hat{n} = 0 \text{ on } \partial\Omega \times [0,T), \\
\left(\partial_t + \frac{1}{\mathbf{Wi}}\right)\left(2(D(u^\nu)\hat{n}) \cdot \hat{\tau}_i + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}_i\right) = -\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} u^\nu \cdot \hat{\tau}_i \text{ on } \partial\Omega \times (0,T), \; i = 1,\dots,d-1
\end{gathered}
\tag{5.38}
$$

**Proposition 5.1.2.** *Suitably smooth solutions of* (5.27) *satisfy the following global energy balance*

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|u^\nu(x,t)|^2 dx + \sum_{i=1}^{d-1} \frac{\boldsymbol{\tau}}{2\boldsymbol{Re}^2\boldsymbol{\alpha}} \int_{\partial\Omega} |(2D(u^\nu)\hat{n} + \frac{\boldsymbol{\alpha}}{2}u^\nu) \cdot \hat{\tau}_i|^2 dS\right) \\
&\quad = -\frac{1}{\boldsymbol{Re}} \int_\Omega |\nabla u^\nu(x,t)|^2 dx + \int_\Omega u^\nu \cdot f_b dx - \sum_{i=1}^{d-1} \frac{\boldsymbol{\alpha}}{2\boldsymbol{Re}} \int_{\partial\Omega} |u^\nu \cdot \hat{\tau}_i|^2 dS \\
&- \sum_{i=1}^{d-1} \frac{\boldsymbol{\tau}}{\boldsymbol{Re}^2\boldsymbol{\alpha}\,\boldsymbol{Wi}} \int_{\partial\Omega} |(2D(u^\nu)\hat{n} + \frac{\boldsymbol{\alpha}}{2}u^\nu) \cdot \hat{\tau}_i|^2 dS + \sum_{i,j=1}^{d-1} \frac{1}{\boldsymbol{Re}} \int_{\partial\Omega} (u^\nu \cdot \hat{\tau}_i)\kappa_{ij}(u^\nu \cdot \hat{\tau}_j) dS.
\end{aligned}
\tag{5.39}
$$

*where* $\kappa_{ij} := \hat{\tau}_i \cdot \nabla\hat{n} \cdot \hat{\tau}_j$ *are the boundary curvatures.*

*Proof.* The balance (5.39) follows from (5.21) together with (5.25) and and from (5.33) in the form

$$
\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} (\hat{\tau}_i \cdot \Sigma_P \cdot \hat{n})^2\, dS = -\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} \int_{\partial\Omega} (u^\nu \cdot \hat{\tau}_i)(\hat{\tau}_i \cdot \Sigma_P \cdot \hat{n}) dS - \frac{1}{\mathbf{Wi}} \int_{\partial\Omega} (\hat{\tau}_i \cdot \Sigma_P \cdot \hat{n})^2\, dS.
\tag{5.40}
$$

Substituting and noting that $\hat{\tau}_i \cdot \Sigma_F \cdot \hat{n} = \mathbf{Re}^{-1}(2D(u^\nu)\hat{n}) \cdot \hat{\tau}_i$ completes the proof. □

**Remark 16** (Navier-Stokes – End-Functionalized Polymer system in two-dimensions)**.** *Of course, one may always regard the system (5.27) in* 2d *as simply a mathematical analogue of the* 3d *situation. However, there are physical regimes in which* *the two-dimensional equations should appear as the correct effective dynamics. On immediate difficulty in doing so is, as discussed in Footnote 1 of the introduction, the validity of Stokes-Einstein relation (1.41) in two dimensions is not well established. On the other hand, we argue now that, if the spring potential is Hookean, then we may regard the system (5.27) in* 2d *as a representation of the fluid-polymer system in* 3d *which is either confined in a large aspect ratio domain or homogeneous in one direction. To understand this, note that although we think of two-dimensional flow, physically fluids occupy three-dimensional space. If the domain is taken to be* $\Omega = \{(x_1, x_2, x_3) \in \Omega_P \times I\}$, *then we argue that the flow is well described by two dimensional dynamics if either (i)* $|I|$ *is much smaller than the scale of* $\Omega_P$, *or (ii)* $I = \mathbb{T}^1$ *and the flow is homogeneous in* $x_3$ *direction. In the case (i), the multi-scale assumption (5.1) should be interpreted as that* $R$ *is also much smaller than the scale of* $|I|$. *In both cases, (5.15) can be formally rewritten in terms of*

$$f_P^*(x^*, t, m^*) = \int f_P dm_3, \tag{5.41}$$

*where* $x^* = (x_1, x_2)$ *and* $m^* = (m_1, m_2)$. *Note that* $f_L^*$ *is independent of* $x_3$ *since (i) the system already ignores* $x_3$ *dependence or (ii) the system is homogeneous in* $x_3$ *direction, by the following:*

$$\begin{gathered}\partial_t f_P^* + \nabla_{m^*} \cdot \left( \frac{u(x,t)}{R} (m^* \cdot (-\hat{n})) f_P^* - \frac{k_B \overline{T}}{\zeta} \int \nabla_{m^*} U f_P dm_3 \right) = \frac{k_B \overline{T}}{\zeta} \Delta_{m^*} f_P^* \\ \textit{in } [0,T] \times M^*(x), \\ \hat{n}(x) \cdot \nabla_m f_P^* = 0 \qquad\qquad \textit{on} \quad [0,T] \times \partial M^*(x),\end{gathered} \tag{5.42}$$

*where* $M^*(x) = \{(m_1, m_2) : (m_1, m_2, m_3) \in M(x)\}$, *since* $u_3 = 0$ *and* $\hat{n} = (n_1, n_2, 0)$. *Crucially, in the Hookean dumbbell case, we have* $\nabla_{m^*} U = H m^*$ *which is manifestly* *independent of* $m_3$*. Thus,*

$$\int \nabla_{m^*} U f_P dm_3 = \nabla_{m^*} U f_P^* \tag{5.43}$$

*and consequently we can replace the boundary equation (5.15) with the above effective 2d ones.*

**Remark 17** (Recovery of no-slip boundary conditions)**.** *Note that the tangential boundary condition of* (5.27) *can be expressed as*

$$2\left(\partial_t + \frac{1}{\boldsymbol{Wi}}\right)(2D(u^\nu)\cdot\hat{n})\cdot\hat{\tau}_i + \boldsymbol{\alpha}\left(\partial_t u^\nu + \frac{1}{\boldsymbol{Wi}}\left(1+\frac{2\mu_p}{\mu_s}\right)u^\nu\right)\cdot\hat{\tau}_i = 0. \tag{5.44}$$

*If the polymer is taken much smaller than the domain so that the parameter* $\boldsymbol{\alpha} = L/R$ *is taken to infinity with* $\boldsymbol{Wi}$ *and* $\frac{\mu_p}{\mu_s}$ *fixed, then the formal* $\boldsymbol{\alpha} \to \infty$ *limit shows that* $u^\nu$ *converges to the no-slip boundary conditions (if* $u_0|_{\partial\Omega} = 0$*, otherwise they converge exponentially fast (in time) to no-slip).*

## 5.2 Global existence of strong solutions in $2d$

It is convenient for our analysis to express (5.27) in terms of the vorticity $\omega = \nabla^\perp \cdot u$ where $\nabla^\perp = (-\partial_2, \partial_1)$. By Lemma 2.1 of [26], provided that $u \in H^2(\Omega)$ and $u \cdot \hat{n} = 0$ on $\partial\Omega$, then

$$\omega|_{\partial\Omega} = 2(D(u)\hat{n})\cdot\hat{\tau}|_{\partial\Omega} + 2\kappa(u\cdot\hat{\tau})|_{\partial\Omega}. \tag{5.45}$$

Thus, the vorticity satisfies the following closed system

$$\begin{gathered}
\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \frac{1}{\mathbf{Re}} \Delta \omega^\nu + \nabla^\perp \cdot f_b \text{ in } \Omega \times (0,T), \\
\omega^\nu|_{t=0} = \omega_0 \ \text{, on } \Omega \times \{0\}, \\
\left(\partial_t + \frac{1}{\mathbf{Wi}}\right) \omega^\nu = \left(2\kappa - \frac{\boldsymbol{\alpha}}{2}\right) \partial_t (u^\nu \cdot \hat{\tau}) - \left(\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} - \frac{2\kappa - \frac{\boldsymbol{\alpha}}{2}}{\mathbf{Wi}}\right) u^\nu \cdot \hat{\tau} \\
\text{on } \partial\Omega \times (0,T),
\end{gathered} \tag{5.46}$$

where, for each fixed time, the velocity $u^\nu$ is recovered from the vorticity using the Biot-Savart law:

$$u^\nu = K_\Omega[\omega^\nu]. \tag{5.47}$$

Here, $K_\Omega$ is an integral operator of order $-1$ with a kernel given by $\nabla^\perp G_\Omega$, where $G_\Omega$ is the Green's function for Laplacian on $\Omega$ with Dirichlet boundary conditions. More specifically, for any $v \in W^{-1,p}(\Omega)$, the Biot-Savart law says $K_\Omega[v] = \nabla^\perp \psi$, where $\psi$ is the unique solution of

$$\begin{gathered}
\Delta \psi = v, \ \text{ in } \Omega, \\
\psi = 0 \text{ on } \partial\Omega.
\end{gathered} \tag{5.48}$$

By standard elliptic regularity, it follows that for $k \geq 0$ and $p \in (1,\infty)$ if $v \in W^{k,p}(\Omega)$, then $K_\Omega[f]$ satisfies

$$\|K_\Omega[v]\|_{W^{k,p}(\Omega)} \leq C \|v\|_{W^{k-1,p}(\Omega)}. \tag{5.49}$$

For details see e.g. Chapter III §4 of [19] and Theorem 1 of [56].

We now prove the following theorem.

**Theorem 5.2.1** (Global Well-Posedness)**.** *Suppose* $\omega_0 \in H^2(\Omega) \cap C(\bar{\Omega})$*. For any* $T > 0$*, there exists a unique*

$$\omega^\nu \in C(0,T;H^1(\Omega)) \cap C([0,T] \times \bar{\Omega}) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{5.50}$$

*solving the system (5.46) where the boundary condition is understood in the sense of*

$$\begin{aligned}\omega^\nu(t) = \left(2\kappa - \frac{\boldsymbol{\alpha}}{2}\right) u^\nu(t)\cdot\hat{\tau} + e^{-\frac{1}{\boldsymbol{Wi}}t}\left(\omega_0 - \left(2\kappa - \frac{\boldsymbol{\alpha}}{2}\right) u_0\cdot\hat{\tau}\right) \\ -\frac{\boldsymbol{\alpha Re}}{\boldsymbol{\tau}}\int_0^t e^{-\frac{1}{\boldsymbol{Wi}}(t-s)} u^\nu(s)\cdot\hat{\tau}\,ds\end{aligned} \tag{5.51}$$

*holding pointwise in* $(t,x)\in[0,T]\times\partial\Omega$.

For simplicity of notation, we denote $\beta = 2\kappa - \frac{\boldsymbol{\alpha}}{2}$.

### 5.2.1 A priori estimates

First, the energy balance for the Navier-Stokes – End-Functionalized system immediately gives some apriori control on the kinetic energy and viscous energy dissipation. We note that this control does not depend on the particular model of the spring potential $U$ used in the model.

**Lemma 5.2.2** (Energy Bounds). *For any* $T>0$*, we have*

$$\begin{aligned}&\|u^\nu\|^2_{L^\infty(0.T;L^2(\Omega))} + \frac{1}{\boldsymbol{Re}}\|u^\nu\|^2_{L^2(0;T;H^1(\Omega))} + \frac{\boldsymbol{\alpha}}{4\boldsymbol{Re}}\|u^\nu\|^2_{L^2(0.T;L^2(\partial\Omega))} \\ &\le e^T\left(\|u_0\|^2_{L^2(\Omega)} + \|f_b\|^2_{L^2(0,T;L^2(\Omega)} + \frac{\boldsymbol{\tau}}{\boldsymbol{Re}^2\boldsymbol{\alpha}}\left(\|2D(u_0^\nu)\hat{n}|^2_{L^2(\partial\Omega)} + \frac{\boldsymbol{\alpha}^2}{4}\|u_0\|^2_{L^2(\partial\Omega)}\right)\right).\end{aligned} \tag{5.52}$$

*Proof.* Recall the balance (5.39) with $\boldsymbol{\alpha} > 4\max_{x\in\partial\Omega}\kappa$, which is consistent with our assumption $(A_3)$. For general spring potential $U$, we start from (5.20). □

The system (5.27) also admits an apriori estimate for the vorticity in $L^\infty$ space-time, at least within the Hookean dumbbell closure. The proof of this fact follows essentially from the argument to prove Lemma 3 of [100] which holds for Navier-friction boundary conditions. Remarkably, the $L^\infty$ bound on vorticity is insensitive to high Reynolds number – this is a consequence of the Stokes-Einstein relation (1.41) for the bead-friction coefficient of the polymer which is reflected in the ratio $\boldsymbol{\alpha}\mathbf{ReWi}/\boldsymbol{\tau}$

being independent of Reynolds **Re** if the latter is varied either by changing solvent viscosity $\nu$ or characteristic velocity $V$. This will be discussed at length in Remark 18.

**Lemma 5.2.3** (Vorticity Bound)**.** *For any $T > 0$, there exists $C_2 > 0$ defined by (5.57) such that*

$$\|\omega^\nu\|_{C([0,T]\times\bar{\Omega})} \le C_2. \tag{5.53}$$

*Proof.* Let $C_1$ be the right side of (5.52). For any $p > 2$, from the embedding and Sobolev interpolation between $W^{1,p}$ and $L^2$ we have

$$\begin{aligned}
\|u^\nu(t)\cdot\hat{\tau}\|_{L^\infty(\partial\Omega)} &\le \|u^\nu(t)\|_{C(\bar{\Omega})} \le \|u^\nu(t)\|^{\theta}_{L^2(\Omega)}\|u^\nu\|^{1-\theta}_{W^{1,p}(\Omega)} \le C\|u^\nu(t)\|^{\theta}_{L^2(\Omega)}\|\omega^\nu(t)\|^{1-\theta}_{L^p(\Omega)} \\
&\le C^{\frac{1}{\theta}}\epsilon^{-\frac{1-\theta}{\theta}} \sup_{t\in[0,T]}\|u^\nu(t)\|_{L^2(\Omega)} + \epsilon \sup_{t\in[0,T]}\|\omega^\nu(t)\|_{L^p(\Omega)} \\
&\le C\sqrt{C_1}\epsilon^{-1} + \epsilon\|\omega^\nu\|_{L^\infty(0,T;L^\infty(\Omega))},
\end{aligned} \tag{5.54}$$

where $\theta = \frac{p-2}{2(p-1)}$, we used the energy bound from Lemma 5.2.2 and Young's inequality introduced the arbitrarily small $\epsilon$ and taking the limit $p \to \infty$. On the other hand, from Duhamel's formula and the boundary condition of (5.46) we obtain (5.51). Also note that $|\beta| \le \boldsymbol{\alpha}$. Therefore, we have the following

$$\begin{aligned}
\|\omega^\nu(t)\|_{L^\infty(\partial\Omega)} &\le 2\boldsymbol{\alpha}\|u^\nu\cdot\hat{\tau}\|_{L^\infty((0,T)\times\partial\Omega)} + \|\omega_0\|_{L^\infty(\partial\Omega)} \\
&\quad + \frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}}\int_0^t e^{-\frac{1}{\mathbf{Wi}}(t-s)}\|u^\nu\cdot\hat{\tau}\|_{L^\infty((0,T)\times\partial\Omega)}ds \\
&\le \|\omega_0\|_{L^\infty(\partial\Omega)} + \left(2\boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}\right)\|u^\nu\cdot\hat{\tau}\|_{L^\infty((0,T)\times\partial\Omega)} \\
&\le \left(2\boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}\right)\left(C\sqrt{C_1}\epsilon^{-1} + \epsilon\|\omega^\nu\|_{L^\infty(0,T;L^\infty(\Omega))}\right) + \|\omega_0\|_{L^\infty(\partial\Omega)}.
\end{aligned} \tag{5.55}$$

On the other hand, from maximum principle we have

$$\|\omega^\nu\|_{C([0,T]\times\bar{\Omega})} \le \|\omega_0\|_{L^\infty(\Omega)} + \|\omega^\nu\|_{L^\infty((0,T)\times\partial\Omega)} + T\|\nabla^\perp\cdot f_b\|_{L^\infty([0,T]\times\bar{\Omega})}. \tag{5.56}$$

By taking $\epsilon$ small enough,

$$\epsilon = \frac{1}{2}\left(2\boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}\mathbf{Re}\mathbf{Wi}}{\boldsymbol{\tau}}\right)^{-1},$$
$$C_2 = 4\left(2\boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}\mathbf{Re}\mathbf{Wi}}{\boldsymbol{\tau}}\right)^{2} C\sqrt{C_1} + 4\|\omega_0\|_{C(\bar{\Omega})} + 2T\|\nabla^{\perp}\cdot f_b\|_{L^{\infty}([0,T]\times\bar{\Omega})}, \tag{5.57}$$

we may conclude the claimed bound (5.53). □

**Lemma 5.2.4** (Higher Regularity)**.** *For any $T > 0$, there exists*

$$C := C(\boldsymbol{Re}, \boldsymbol{Wi}, \boldsymbol{\tau}, \boldsymbol{\alpha}, u_0, \Omega, T)$$

*such that*

$$\|\omega^{\nu}\|_{C(0,T;H^1(\Omega))} \leq C, \qquad \|\Delta\omega^{\nu}\|_{L^2(0,T;L^2(\Omega))} \leq C, \qquad \|\omega^{\nu}\|_{H^1(0,T;L^2(\Omega))} \leq C. \tag{5.58}$$

*Proof.* By multiplying $(-\Delta)\omega^{\nu}$ to (5.46) and integrating we have

$$\int_{\Omega}(-\Delta\omega^{\nu})\partial_t\omega^{\nu}dx + \frac{1}{\mathbf{Re}}\int_{\Omega}|\Delta\omega^{\nu}|^2dx = \int_{\Omega}\Delta\omega^{\nu}u^{\nu}\cdot\nabla\omega^{\nu}dx - \int_{\Omega}\Delta\omega^{\nu}\nabla^{\perp}\cdot f_b dx. \tag{5.59}$$

Note now that the first term of the left hand side of (5.59) can be rewritten as

$$\begin{aligned} &-\int_{\Omega}\nabla\cdot(\nabla\omega^{\nu}\partial_t\omega^{\nu})\,dx + \int_{\Omega}\nabla\omega^{\nu}\cdot\partial_t\nabla\omega^{\nu}dx \\ &= -\int_{\partial\Omega}\hat{n}\cdot\nabla\omega^{\nu}\partial_t\omega^{\nu}dS + \frac{1}{2}\frac{d}{dt}\|\nabla\omega^{\nu}\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.60}$$

Thus we obtain the following evolution

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\omega^{\nu}\|^2_{L^2(\Omega)} + \frac{1}{\mathbf{Re}}\|\Delta\omega^{\nu}\|^2_{L^2(\Omega)}dx &= \int_{\partial\Omega}\hat{n}\cdot\nabla\omega^{\nu}\partial_t\omega^{\nu}dS \\ &+ \int_{\Omega}\Delta\omega^{\nu}u^{\nu}\cdot\nabla\omega^{\nu}dx - \int_{\Omega}\Delta\omega^{\nu}\nabla^{\perp}\cdot f_b dx. \end{aligned} \tag{5.61}$$

Using the boundary condition of (5.46) the first term in the right hand side of (5.60) reads

$$\begin{aligned}&\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu \partial_t\omega^\nu dS \\ &= \int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu \left(\beta\partial_t u\cdot\hat{\tau} - \frac{1}{\mathbf{Wi}}\omega^\nu - \left(\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} - \frac{\beta}{\mathbf{Wi}}\right)u^\nu\cdot\hat{\tau}\right)dS.\end{aligned} \tag{5.62}$$

The second term on the right-hand-side can be written as a bulk term

$$\frac{1}{\mathbf{Wi}}\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu\omega^\nu dS = \frac{1}{\mathbf{Wi}}\int_\Omega \nabla\cdot(\nabla\omega^\nu\omega^\nu)dx = \frac{1}{\mathbf{Wi}}\int_\Omega \Delta\omega^\nu\omega^\nu dx + \frac{1}{\mathbf{Wi}}\|\nabla\omega^\nu\|^2_{L^2(\Omega)}. \tag{5.63}$$

Therefore, we find that the boundary term becomes

$$\begin{aligned}&\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu\partial_t\omega^\nu dS = -\frac{1}{\mathbf{Wi}}\|\nabla\omega^\nu\|^2_{L^2(\Omega)} - \frac{1}{\mathbf{Wi}}\int_\Omega \omega^\nu\Delta\omega^\nu dx \\ &-\int_{\partial\Omega}\hat{n}\cdot\nabla\omega^\nu\left(\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} - \frac{\beta}{\mathbf{Wi}}\right)u^\nu\cdot\hat{\tau}\ dS + \int_{\partial\Omega}\hat{n}\cdot\nabla\omega^\nu\beta\partial_t(u^\nu\cdot\hat{\tau})dS.\end{aligned} \tag{5.64}$$

The second term of (5.64) is controlled by

$$\left|\frac{1}{\mathbf{Wi}}\int_\Omega \omega^\nu\Delta\omega^\nu dx\right| \le \frac{1}{\mathbf{Wi}}\|\Delta\omega^\nu\|_{L^2(\Omega)}\|\omega^\nu\|_{L^2(\Omega)}. \tag{5.65}$$

To deal with the third term of (5.64), we introduce a thin enough tubular neighborhood of $\partial\Omega$, smoothly extend the vector field $\left(\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} - \frac{\beta}{\mathbf{Wi}}\right)\tau$ on $\partial\Omega$ whose support is compactly embedded in this neighborhood, and we denote this vector field as $\Phi$. Then we have

$$\begin{aligned}&\int_{\partial\Omega}\hat{n}\cdot\nabla\omega^\nu\left(\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} - \frac{\beta}{\mathbf{Wi}}\right)u^\nu\cdot\hat{\tau}\ dS = \int_{\partial\Omega}\hat{n}\cdot\nabla\omega^\nu u^\nu\cdot\Phi dS \\ &= \int_{\partial\Omega}\hat{n}\cdot\nabla(\omega^\nu u^\nu\cdot\Phi)dS - \int_{\partial\Omega}\hat{n}\cdot\nabla(u^\nu\cdot\Phi)\omega^\nu dS \\ &= \int_\Omega \nabla\cdot(\nabla(\omega^\nu u^\nu\cdot\Phi))dx - \int_{\partial\Omega}\hat{n}\cdot\nabla(u^\nu\cdot\Phi)\omega^\nu dS.\end{aligned} \tag{5.66}$$

The first term of (5.66) is controlled by

$$\begin{aligned}
&\left|\int_\Omega \nabla\cdot(\nabla(\omega^\nu u^\nu\cdot\Phi))dx\right| \\
&\le \left(\|\Delta\omega^\nu\|_{L^2(\Omega)}\|u^\nu\|_{L^2(\Omega)}\|\Phi\|_{L^\infty(\Omega)} + \|\omega^\nu\|_{H^1(\Omega)}\|u^\nu\|_{H^2(\Omega)}\|\Phi\|_{W^{1,\infty}(\Omega)}\right) \\
&\le c\|\Delta\omega^\nu\|_{L^2(\Omega)}\|u^\nu\|_{L^2(\Omega)} + C\|\omega^\nu\|^2_{H^1(\Omega)},
\end{aligned} \tag{5.67}$$

since $\Phi$ depends only on $\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}}, \mathbf{Wi}, \boldsymbol{\alpha}$, and $\Omega$ (in particular, on $\kappa$). The second term of (5.66) is controlled by

$$\begin{aligned}
\left|\int_{\partial\Omega} \hat{n}\cdot\nabla(u^\nu\cdot\Phi)\omega^\nu dS\right| &\le \|\nabla(u^\nu\cdot\Phi)\|_{L^2(\partial\Omega)}\|\omega^\nu\|_{L^2(\partial\Omega)} \\
&\le \|u^\nu\cdot\Phi\|_{H^{\frac{3}{2}}(\Omega)}\|\omega^\nu\|_{H^1(\Omega)} \le C\|\omega^\nu\|^2_{H^1(\Omega)}
\end{aligned} \tag{5.68}$$

by the Sobolev trace inequality. It suffices to treat the term

$$\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu(2\kappa)\partial_t u^\nu\cdot\hat{\tau}dS. \tag{5.69}$$

First note that, from the vorticity equation and the Biot-Savart law, we may express

$$\partial_t u^\nu = K_\Omega[\partial_t\omega^\nu] = K_\Omega\left[-\nabla\cdot(u^\nu\omega^\nu) + \frac{1}{\mathbf{Re}}\Delta\omega^\nu\right]. \tag{5.70}$$

Using this correspondence, we have

$$\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu\beta\partial_t u^\nu\cdot\hat{\tau}dS = \int_\Omega \nabla\cdot\left(\nabla\omega^\nu\Psi\cdot\left(K_\Omega[-\nabla\cdot(u^\nu\omega^\nu)] + \frac{1}{\mathbf{Re}}K_\Omega[\Delta\omega^\nu]\right)\right)dx, \tag{5.71}$$

where $T_{\partial\Omega}\Psi = \beta\hat{\tau}$. We now note that

$$\begin{aligned}
&\left|\int_\Omega \nabla\cdot\left(\nabla\omega^\nu\Psi\cdot\frac{1}{\mathbf{Re}}K_\Omega[\Delta\omega^\nu]\right)dx\right| \\
&\leq \left|\int_\Omega \Delta\omega^\nu\Psi\cdot\frac{1}{\mathbf{Re}}K_\Omega[\Delta\omega^\nu]dx\right| + \left|\int_\Omega \nabla\omega^\nu\nabla\Psi\cdot\frac{1}{\mathbf{Re}}K_\Omega[\Delta\omega^\nu]dx\right| \\
&\quad + \left|\int_\Omega \nabla\omega^\nu\Psi\cdot\nabla\frac{1}{\mathbf{Re}}K_\Omega[\Delta\omega^\nu]dx\right| \\
&\leq \frac{1}{\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2}\|\Psi\|_{L^\infty}\|K_\Omega[\Delta\omega^\nu]\|_{L^2} + \frac{1}{\mathbf{Re}}\|\nabla\omega^\nu\|_{L^2}\|\Psi\|_{W^{1,\infty}}\|K_\Omega[\Delta\omega^\nu]\|_{H^1} \\
&\leq \frac{C(\Psi)}{\mathbf{Re}}\left(\|\Delta\omega^\nu\|_{L^2}\|\Delta\omega^\nu\|_{H^{-1}} + \|\nabla\omega^\nu\|_{L^2}\|\Delta\omega^\nu\|_{L^2}\right) \leq \frac{C(\Psi)}{\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2}\|\nabla\omega^\nu\|_{L^2},
\end{aligned} \tag{5.72}$$

where we used $\|\Delta\omega^\nu\|_{H^{-1}} \leq \|\nabla\omega^\nu\|_{L^2}$. Note that this estimate involves no boundary terms since $H^{-1}(\Omega)$ is the dual of $H^1_0(\Omega)$. Now, the first term becomes

$$\begin{aligned}
&\left|\int_\Omega \nabla\cdot\left(\nabla\omega^\nu\Psi\cdot K_\Omega[-\nabla\cdot(u^\nu\omega^\nu)]\right)dx\right| \\
&\leq \left|\int_\Omega \Delta\omega^\nu\Psi\cdot K_\Omega[-\nabla\cdot(u^\nu\omega^\nu)]dx\right| + \left|\int_\Omega \nabla\omega^\nu\cdot\nabla(\Psi\cdot K_\Omega[-\nabla\cdot(u^\nu\omega^\nu)])dx\right| \\
&\leq \|\Delta\omega^\nu\|_{L^2}\|\Psi\|_{L^\infty}\|\nabla\cdot(u^\nu\omega^\nu)\|_{H^{-1}} + \|\nabla\omega^\nu\|_{L^2}\|\Psi\|_{W^{1,\infty}}\|\nabla\cdot(u^\nu\omega^\nu)\|_{L^2} \\
&\leq C\|\Delta\omega^\nu\|_{L^2} + C'\|\nabla\omega^\nu\|^2_{L^2}
\end{aligned} \tag{5.73}$$

for some constants $C, C' > 0$. To obtain the above, we noted that we used the bounds on $\|u^\nu\|_C, \|\omega^\nu\|_C$ and $\|\omega^\nu\|_{L^2}$ and therefore $\|u^\nu\omega^\nu\|_{H^1} \leq \|u^\nu\|_{H^1\cap C}\|\omega^\nu\|_{H^1\cap C}$. Thus we obtained

$$\begin{aligned}
\left|\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu(2\kappa)\partial_t u^\nu\cdot\hat{\tau}dS\right| &\leq \frac{C(\Psi)}{\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2}\|\nabla\omega^\nu\|_{L^2} + C\|\Delta\omega^\nu\|_{L^2} + C'\|\nabla\omega^\nu\|^2_{L^2} \\
&\leq C + \frac{1}{2\mathbf{Re}}\|\Delta\omega^\nu\|^2_{L^2} + C\|\omega^\nu\|^2_{H^1}.
\end{aligned} \tag{5.74}$$

Finally, combining (5.65), (5.67), (5.68) and (5.74), we bound the terms on the right-hand-side of Eqn. (5.61) by

$$\left|\int_{\partial\Omega} \hat{n}\cdot\nabla\omega^\nu \partial_t \omega^\nu dS\right| \leq C + \frac{1}{2\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2}^2 + C\|\omega^\nu\|_{H^1}^2 + \mathbf{Re}^2\|\omega^\nu\|_{L^2}^2 + \mathbf{Re}^2\|u^\nu\|_{L^2}^2,$$

$$\begin{aligned}\left|\int_\Omega \Delta\omega^\nu u^\nu\cdot\nabla\omega^\nu dx - \int_\Omega \Delta\omega^\nu \nabla^\perp\cdot f_b dx\right| \\ \leq \|\Delta\omega^\nu\|_{L^2}\left(\|u^\nu\|_{L^\infty}\|\nabla\omega^\nu\|_{L^2} + \|\nabla^\perp\cdot f_b\|_{L^2}\right) \\ \leq \frac{1}{2\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2}^2 + C\mathbf{Re}\|u^\nu\|_{L^\infty(\Omega)}^2\|\omega^\nu\|_{H^1}^2 + \|\nabla^\perp\cdot f_b\|_{L^2}.\end{aligned} \tag{5.75}$$

Noting that by Poincare inequality $\|\omega^\nu\|_{H^1(\Omega)}$ and $\|\nabla\omega^\nu\|_{L^2(\Omega)}$ are comparable, and using Cauchy-Schwarz inequality to bury all $\|\Delta\omega^\nu\|_{L^2(\Omega)}$ terms, we end up with

$$\begin{aligned}\frac{d}{dt}\|\nabla\omega^\nu\|_{L^2(\Omega)}^2 + \frac{1}{\mathbf{Re}}\|\Delta\omega^\nu\|_{L^2(\Omega)}^2 + \frac{2}{\mathbf{Wi}}\|\nabla\omega^\nu\|_{L^2(\Omega)}^2 \\ \leq C\left(\left(\|u^\nu\|_{L^\infty(\Omega)}^2 + 1\right)\|\nabla\omega^\nu\|_{L^2(\Omega)}^2\left(\|\nabla^\perp\cdot f_b\|_{L^2(\Omega)}^2 + \|\omega^\nu\|_{L^2(\Omega)}^2 + \|u^\nu\|_{L^2(\Omega)}^2\right)\right), \\ C = C(\mathbf{Re}, \mathbf{Wi}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \Omega)\end{aligned} \tag{5.76}$$

Note finally that from the a priori estimate $\omega^\nu \in C([0,T]\times\overline{\Omega})$ of Lemma 5.2.3, we have $u^\nu = K_\Omega[\omega^\nu] \in L^\infty(0,T;W^{1,p}(\Omega))$ for all $1 \leq p < \infty$. In particular, combining this with (5.54) we find $u^\nu \in C([0,T]\times\overline{\Omega})$. Whence, by Lemma 5.2.3, the above estimate allows us to conclude that $\omega^\nu \in C(0,T;H^1(\Omega))$ and consequently $u^\nu \in C(0,T;H^2(\Omega))$. Moreover, from the vorticity equation we have

$$\|\partial_t\omega^\nu\|_{L^2} \leq \|u^\nu\|_{L^\infty}\|\nabla\omega^\nu\|_{L^2} + \|\Delta\omega^\nu\|_{L^2}, \tag{5.77}$$

which implies that $\omega^\nu \in H^1(0,T;L^2(\Omega))$. □

### 5.2.2 Proof of Theorem 5.2.1: Global Strong Solutions

To construct the solution for the system (5.46), we first propose the function space for the solution;

$$\begin{gathered}\mathcal{X} = \{\omega \in C_t H^1(\Omega) \cap C_t C(\bar{\Omega}) \cap H_t^1 L^2(\Omega) \mid \omega(0) \in H^1(\Omega) \cap C(\bar{\Omega}), \Delta\omega(0) \in L^2(\Omega)\}, \\ \mathcal{X}' = \{\omega \in C_t H^1(\Omega) \cap H_t^1 L^2(\Omega) \mid \omega(0) \in H^1(\Omega), \Delta\omega(0) \in L^2(\Omega)\},\end{gathered} \tag{5.78}$$

with the natural norm $\|\omega\|_{\mathcal{X}} = \|\omega\|_{C_t H^1(\Omega)} + \|\omega\|_{C_t C(\bar{\Omega})} + \|\omega\|_{H_t^1 L^2(\Omega)}$ and $\|\omega\|_{\mathcal{X}'} = \|\omega\|_{C_t H^1(\Omega)} + \|\omega\|_{H_t^1 L^2(\Omega)}$. Here $C_t, H_t^1, L_t^2$ are shorthand for time interval $[0, T]$. To prove Theorem 5.2.1, we will:

1. Establish a contraction mapping $F$ in $\mathcal{X}'$, so that for $\omega(0) \in H^1(\Omega) \cap \{\Delta\omega(0) \in L^2(\Omega)\}$, there is unique $\omega \in \mathcal{X}'$ such that $\omega = F(\omega)$ for a short time $T$.

2. Check that if $\omega(0) \in C(\bar{\Omega})$ then $\omega \in \mathcal{X}$ in fact. Then Lemma 5.2.3 and consequently Lemma 5.2.4 become valid, establishing a priori estimates on $\mathcal{X}$.

3. Noting that $\Delta\omega(t) \in L^2(\Omega)$ for almost every $t \in [0, T]$, so we can continue a point close to $T$, thereby obtaining global well-posedness.

*Proof.* For the description of boundary behavior, we define the following operator:

$$N_\Omega[\omega] := N_\Omega^1[\omega] + N_\Omega^2[\omega] + N_\Omega^3[\omega], \tag{5.79}$$

where

$$\begin{aligned} N_\Omega^1[\omega](t) &= \Psi_1\Psi_2 \cdot K_\Omega[\omega(t)], \\ N_\Omega^2[\omega](t) &= e^{-\frac{1}{\mathbf{Wi}}t}\left(\omega(0) - \Psi_1\Psi_2 \cdot K_\Omega[\omega(0)]\right), \\ N_\Omega^3[\omega](t) &= -\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}}\int_0^t e^{-\frac{1}{\mathbf{Wi}}(t-s)}\Psi_2 \cdot K_\Omega[\omega(s)]ds, \end{aligned} \tag{5.80}$$

where and $\Psi_1$ and $\Psi_2$ are smooth extensions of $\beta$ and $\hat{\tau}$, respectively satisfying that the boundary traces $T_{\partial\Omega}\Psi_1 = \beta$, and $T_{\partial\Omega}\Psi_2 = \hat{\tau}$ together with the support condition

(with $\rho$ to be specified later in the proof)

$$\operatorname{supp}(\Psi_i) \subset E_\rho(\partial\Omega) := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq \rho\},\ i = 1, 2, \tag{5.81}$$

together with the estimate

$$\|D^k \Psi_i\|_{L^\infty(\Omega)} \leq \frac{C}{\rho^k},\ i = 1, 2,\ k = 0, 1, 2. \tag{5.82}$$

Note that

$$\|N_\Omega^1[\omega]\|_{C_t H^2(\Omega) \cap H_t^1 H^1(\Omega)} + \|N_\Omega^3[\omega]\|_{C_t H^2(\Omega) \cap H_t^1 H^1(\Omega)} \leq C\left(1 + \frac{1}{\rho^2}\right)\|\omega\|_{\mathcal{X}'}, \tag{5.83}$$

$$\|\Delta N_\Omega^2[\omega]\|_{C_t L^2(\Omega)} + \|N_\Omega^2[\omega]\|_{C_t H^1(\Omega) \cap H_t^1 H^1(\Omega)} \leq C\left(1 + \frac{1}{\rho^2}\right)\left(\|\omega(0)\|_{H^1(\Omega)} + \|\Delta\omega(0)\|_{L^2}\right). \tag{5.84}$$

Furthermore, by the Sobolev embedding $\|\omega(t)\|_{C(\bar{\Omega})} \leq C\|\omega(t)\|_{H^2(\Omega)}$ and $\omega(0) \in C(\bar{\Omega})$, we have

$$\|N_\Omega[\omega]\|_{C_t C(\bar{\Omega})} \leq C\left(1 + \frac{1}{\rho^2}\right)\left(\|\omega\|_{\mathcal{X}'} + \|\omega(0)\|_{H^1(\Omega) \cap C(\bar{\Omega})}\right). \tag{5.85}$$

**Step 1: (Solution Scheme)** Let $F$ be an operator on $\mathcal{X}$ defined by $F(\omega) = \omega^n$, where $\omega^n$ is the solution of

$$\begin{gathered}\partial_t \omega^n = \frac{1}{\mathbf{Re}}\Delta\omega^n - K_\Omega[\omega] \cdot \nabla\omega^n + \nabla^\perp \cdot f_b,\ \text{in } \Omega \times (0, T),\\ \omega^n(0) = \omega(0),\ \text{on } \Omega \times \{t = 0\},\\ T_{\partial\Omega}[\omega^r] = T_{\partial\Omega}[N_\Omega[\omega]] \text{ on } \partial\Omega \times (0, T).\end{gathered} \tag{5.86}$$

Let $\omega^r = \omega^n - N_\Omega[\omega]$. Then $\omega^r$ solves

$$\begin{aligned}
\partial_t \omega^r &= \frac{1}{\mathbf{Re}}\Delta\omega^r - K_\Omega[\omega]\cdot\nabla\omega^r + R, \text{ in } \Omega\times(0,T),\\
R &= \nabla^\perp \cdot f_b - \left(\partial_t + K_\Omega[\omega]\cdot\nabla - \frac{1}{\mathbf{Re}}\Delta\right) N_\Omega[\omega],\\
&\omega^r(0) = 0, \text{ on } \Omega\times\{t=0\},\\
&T_{\partial\Omega}[\omega^r] = 0 \text{ on } \partial\Omega\times(0,T).
\end{aligned} \tag{5.87}$$

Since $R \in L^2_t L^2(\Omega)$ from previous calculations, there is a unique $\omega^r$ solving them, satisfying

$$\omega^r \in C_t H^1_0(\Omega) \cap L^2_t (H^2 \cap H^1_0)(\Omega) \cap H^1_t L^2(\Omega). \tag{5.88}$$

As a consequence, we have

$$\omega^n = \omega^r + N_\Omega[\omega] \in C_t H^1(\Omega) \cap H^1_t L^2(\Omega), \tag{5.89}$$

with $\omega^n(0) = \omega(0)$ and solves the system (5.86). In addition, since $N_\Omega[\omega] \in C_t C(\bar{\Omega})$ by the maximum principle $\omega^n \in C_t C(\bar{\Omega})$. Note that we only used $\omega \in \mathcal{X}'$ and $\omega(0) \in C(\bar{\Omega})$ to obtain $F(\omega) = \omega^n \in \mathcal{X}$, and we do not need $\omega \in \mathcal{X}$. Finally, we note that $\Delta\omega^n = \Delta\omega^r + \Delta N_\Omega[\omega] \in L^2_t L^2(\Omega)$.

**Step 2: (Contraction Mapping)** Next, we show that for a given $\omega_0 \in H^1(\Omega) \cap C(\bar{\Omega})$ with $\Delta\omega_0 \in L^2(\Omega)$, $F$ is in fact a contraction mapping in

$$\mathcal{Y} = \{\omega \in \mathcal{X}' \mid \|\omega\|_{\mathcal{X}'} \leq B,\ \omega(0) = \omega_0\}, \tag{5.90}$$

for a suitable $B > 0$, and small enough time $T$. Since we have enough regularity, we can rigorously perform the following calculation: for $\omega \in \mathcal{Y}$, let $v = F(\omega)$. Then

$$\begin{aligned} \partial_t v &= \frac{1}{\mathbf{Re}}\Delta v - K_\Omega[\omega]\cdot\nabla v + \nabla^\perp\cdot f_b, \text{ in } \Omega\times(0,T), \\ v(0) &= \omega_0, \text{ on } \Omega\times\{t=0\}, \\ T_{\partial\Omega}v &= T_{\partial\Omega}[N_\Omega[\omega]] \text{ on } \partial\Omega\times(0,T). \end{aligned} \tag{5.91}$$

Since $\Delta v \in L^2_t L^2(\Omega)$ we have

$$\int_\Omega(-\Delta v)\partial_t v dx + \frac{1}{\mathbf{Re}}\int_\Omega|\Delta v|^2dx = \int_\Omega K_\Omega[\omega]\cdot\nabla v(\Delta v)dx - \int_\Omega(\Delta v)(\nabla^\perp\cdot f_b)dx. \tag{5.92}$$

The first term of (5.92) becomes

$$\begin{aligned} &-\int_\Omega \nabla\cdot(\nabla v\partial_t v)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2dx \\ &= -\int_{\partial\Omega}\hat{n}\cdot T_{\partial\Omega}(\nabla v)T_{\partial\Omega}(\partial_t v)dS + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega)} \\ &= -\int_{\partial\Omega}\hat{n}\cdot T_{\partial\Omega}(\nabla v)T_{\partial\Omega}(\partial_t N_\Omega[\omega])dS + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega)} \\ &= -\int_\Omega \nabla\cdot(\nabla v\partial_t N_\Omega[\omega])dx + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega)} \\ &= -\int_\Omega \Delta v\partial_t N_\Omega[\omega]dx - \int_\Omega\nabla v\cdot\nabla\partial_t N_\Omega[\omega]dx + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.93}$$

The issue is control of $\|\partial_t N_\Omega[\omega]\|_{L^2_t L^2(\Omega)}$ and $\|\nabla\partial_t N_\Omega[\omega]\|_{L^2_t L^2(\Omega)}$. Here we use two tricks.

1. We have a freedom in choosing $\rho$, and for small enough, fixed $T$ we choose $\rho = T^\beta$ accordingly.

2. When controlling the term $\int_\Omega \nabla v \cdot \nabla \partial_t N_\Omega[\omega] dx$, we use

$$\begin{aligned}\int_\Omega \nabla v \cdot \nabla \partial_t N_\Omega[\omega] dx &\leq \|\nabla v\|_{L^2(\Omega)} \|\nabla \partial_t N_\Omega[\omega]\|_{L^2(\Omega)} \\ &\leq t^{-\alpha} \|\nabla v\|^2_{L^2(\Omega)} + t^\alpha \|\nabla \partial_t N_\Omega[\omega]\|^2_{L^2(\Omega)},\end{aligned} \tag{5.94}$$

which enables the control of $\|\nabla \partial_t N_\Omega[\omega]\|^2_{L^2(\Omega)}$ term for a short time.

For $\|\partial_t N_\Omega[\omega]\|_{L^2_t L^2(\Omega)}$, we have

$$\begin{aligned}&\|\partial_t N_\Omega[\omega](t)\|_{L^2(\Omega)} \leq C\|\omega_0\|_{L^2(\Omega)} \\ &+C\|\Psi_2\|_{L^{p'}(\Omega)} \left( \|K_\Omega[\partial_t \omega(t)]\|_{L^p(\Omega)} + \|K_\Omega[\omega(t)]\|_{L^p(\Omega)} + \int_0^t \|K_\Omega[\omega(s)]\|_{L^p(\Omega)} ds \right) \\ &\leq C\|\omega_0\|_{H^1(\Omega)} + CT^{\frac{\beta}{p'}} \left( \|\partial_t \omega(t)\|_{L^2(\Omega)} + \|\omega(t)\|_{L^2(\Omega)} + \sqrt{t}\|\omega\|_{L^2_t L^2(\Omega)} \right),\end{aligned} \tag{5.95}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$, $2 < p < \infty$ and $p' > 2$, by Sobolev embedding $H^1(\Omega) \subset L^p(\Omega)$ and the bound $\|\Psi_i\|_{L^{p'}(\Omega)} \leq CT^{\frac{\beta}{p'}}$. Similarly for $\|\nabla \partial_t N_\Omega[\omega]\|_{L^2_t L^2(\Omega)}$,

$$\begin{aligned}&\|\nabla \partial_t N_\Omega[\omega](t)\|_{L^2(\Omega)} \leq \|\nabla(\Psi_1 \Psi_2)\|_{L^{p'}(\Omega)} \|K_\Omega[\partial_t \omega(t)]\|_{L^p(\Omega)} \\ &+\|\Psi_1\Psi_2\|_{L^\infty(\Omega)} \|\nabla K_\Omega[\partial_t \omega(t)\|_{L^2(\Omega)} \\ &+\frac{1}{\mathbf{Wi}} \left( \|\omega_0\|_{H^1(\Omega)} + \|\nabla(\Psi_1\Psi_2)\|_{L^{p'}(\Omega)} \|K_\Omega[\omega_0]\|_{L^p(\Omega)} + \|\Psi_1\Psi_2\|_{L^\infty(\Omega)} \|\nabla K_\Omega[\omega_0\|_{L^2(\Omega)} \right) \\ &+\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} \int_0^t \left( \|\nabla \Psi_2\|_{L^2(\Omega)} \|K_\Omega[\omega(s)]\|_{L^\infty(\Omega)} + \|\Psi_2\|_{L^\infty(\Omega)} \|\nabla K_\Omega[\omega(s)]\|_{L^2(\Omega)} \right) ds \\ &+\frac{\boldsymbol{\alpha}\mathbf{Re}}{\boldsymbol{\tau}} \left( \|\nabla \Psi_2\|_{L^2(\Omega)} \|K_\Omega[\omega(t)]\|_{L^\infty(\Omega)} + \|\Psi_2\|_{L^\infty(\Omega)} \|\nabla K_\Omega[\omega(t)]\|_{L^2(\Omega)} \right) \\ &\leq C(1 + T^{\beta(\frac{1}{p'}-1)}) \left( \|\omega_0\|_{H^1(\Omega)} + \|\partial_t \omega(t)\|_{L^2(\Omega)} + (1+t)\|\omega\|_{C_t H^1(\Omega)} \right),\end{aligned} \tag{5.96}$$

by Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$ and the bounds $\|\nabla \Psi_2\|_{L^2(\Omega)} \leq \|\nabla \Psi_2\|_{L^\infty(\Omega)} T^{\frac{\beta}{2}}$

together with $a^{\frac{1}{p'}-1} > a^{-\frac{1}{2}}$ for $p' > 2$ and $0 < a < 1$. Therefore, we have

$$\begin{aligned}\frac{d}{dt}\|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{\mathbf{Re}}\|\Delta v\|^2_{L^2(\Omega)} &\leq C(\|\omega\|^2_{C_tH^1(\Omega)} + t^{-\alpha})\|\nabla v\|^2_{L^2(\Omega)} \\ &+C\left(\|\omega_0\|^2_{H^1(\Omega)} + \|f\|^2_{L^\infty_t H^1(\Omega)} + \|\omega_0\|^2_{H^1(\Omega)}\right) \\ &+CT^{\frac{\beta}{p'}}\left(\|\partial_t\omega(t)\|^2_{L^2(\Omega)} + \|\omega(t)\|^2_{L^2(\Omega)} + t\|\omega\|^2_{L^2_tL^2(\Omega)})\right) \\ &+Ct^\alpha(1+T^{\beta(\frac{1}{p'}-1)})^2\left(\|\omega_0\|^2_{H^1(\Omega)} + \|\partial_t\omega(t)\|^2_{L^2(\Omega)} + (1+t^2)\|\omega\|^2_{C_tH^1(\Omega)}\right).\end{aligned} \tag{5.97}$$

Noting from (5.91) that $\|\partial_t v\|^2_{L^2(\Omega)} \leq \mathbf{Re}^{-2}\|\Delta v\|^2_{L^2} + \|\omega\|^2_{C_tH^1(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|f\|^2_{L^\infty_t H^1(\Omega)}$, and by Grönwall's inequality we have

$$\begin{aligned}\|v\|^2_{\mathcal{X}'} &= \|v\|^2_{C_tH^1(\Omega)} + \|\partial_t v\|^2_{L^2_tL^2(\Omega)} \\ &\leq C\exp\left(T\|\omega\|^2_{C_tH^1(\Omega)} + T^{1-\alpha}\right) \\ &\times\left(T\|\omega_0\|^2_{H^1(\Omega)} + T\|f\|^2_{L^\infty_t H^1(\Omega)} + T(1+T^\alpha(1+T^{\beta(\frac{1}{p'}-1)})^2)\|\omega_0\|^2_{H^1(\Omega)}\right) \\ &+C\exp\left(T\|\omega\|^2_{C_tH^1(\Omega)} + T^{1-\alpha}\right) \\ &\times\left((T^{\frac{\beta}{p'}}(1+T^2) + T^\alpha(1+T^{\beta(\frac{1}{p'}-1)})^2)\|\omega\|^2_{H^1_tL^2(\Omega)} + T(1+T^2)\|\omega\|^2_{C_tH^1(\Omega)}\right) \\ &\leq Ce^{B^2T+T^{1-\alpha}}O(T^q)\left(\|\omega_0\|^2_{H^1(\Omega)} + \|f\|^2_{L^\infty_t H^1(\Omega)} + B^2\right),\end{aligned} \tag{5.98}$$

where we choose $\alpha + 2\beta(1-\frac{1}{p'}) > 0$. Then for any $B > 0$, for sufficiently small $T$ we have $\|v\|_{\mathcal{X}'} \leq B$. The same calculation shows that $F$ is a contraction mapping on $\mathcal{Y}$ for a sufficiently small $T$. Let $\omega_1, \omega_2 \in \mathcal{Y}$ with $y = \omega_1 - \omega_2$, and let $z = F(\omega_1) - F(\omega_2)$. Then $z$ solves

$$\begin{aligned}\partial_t z = \frac{1}{\mathbf{Re}}\Delta z - K_\Omega[\omega_1]\cdot\nabla z - K_\Omega[y]\cdot\nabla F(\omega_2),\ \text{in } \Omega\times(0,T), \\ z(0) = 0,\ \text{on } \Omega\times\{t=0\}, \\ T_{\partial\Omega}[z] = T_{\partial\Omega}[N_\Omega[y]] \text{ on } \partial\Omega\times(0,T).\end{aligned} \tag{5.99}$$

Then, the same computations as above gives the following bound on $z$ in $\mathcal{X}'$:

$$\|z\|_{\mathcal{X}'}^2 \leq C \exp\left(2TB + T^{1-\alpha}\right) O(T^q) \|y\|_{\mathcal{X}'}^2 (1 + B^2), \tag{5.100}$$

which follows from the estimate

$$\|K_\Omega[y] \cdot \nabla F(\omega_2)\|_{L_t^\infty L^2(\Omega)}^2 \leq C \|K_\Omega[y]\|_{C_t H^2(\Omega)}^2 \|F(\omega_2)\|_{C_t H^1(\Omega)}^2 \leq C \|y\|_{\mathcal{X}'}^2 B^2. \tag{5.101}$$

Consequently, there is unique $\omega \in \mathcal{X}'$ such that $F(\omega) = \omega$, and since $F(\omega) \in \mathcal{X}$ we have $\omega \in \mathcal{X}$. Then by Lemma 5.2.3 and Lemma 5.2.4 we have a bound

$$\|\omega\|_{\mathcal{X}} \leq C(\omega_0, T), \tag{5.102}$$

which does not blow up for finite $T > 0$ or $\|\omega\|$. Also, $\Delta\omega(t) \in L^2(\Omega)$ for a.e. $t \in [0, T]$, which means that we can continue the solution. Finally, this proves global well-posedness of the system in $\mathcal{X}$. □

**Corollary 5.2.5.** *If $\omega_0 \in H^2(\Omega)$, then $\omega \in L^2(0, T; H^2(\Omega))$.*

*Proof.* Note that $N_\Omega[\omega] \in C_t H^2(\Omega)$ if $\omega_0 \in H^2(\Omega)$ by estimates (5.83) and definition of $N_\Omega^2[\omega]$. Note that $\omega = \omega^r + N_\Omega[\omega]$, where $\omega^r$ solves the system (5.87), and therefore $\omega^r \in L^2(0, T; H^2(\Omega))$. □

## 5.3 Inviscid limit and quantitative drag reduction

Consider a smooth solution $u$ of the Euler equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p + f_b \text{ in } \Omega \times (0,T), \\ \nabla \cdot u &= 0 \text{ in } \Omega \times (0,T), \\ u \cdot \hat{n} &= 0 \text{ on } \partial\Omega \times (0,T), \\ u|_{t=0} &= u_0 \text{ on } \Omega \times \{t=0\}. \end{aligned} \tag{5.103}$$

Strong Euler solutions are guaranteed to exist globally starting from regular initial data in two-dimensions on domains with smooth boundaries [76]. The nature of the inviscid limit (high-Reynolds number) of solutions of the Navier-Stokes–End-Functionalized polymer system (5.27) is a natural question; do solutions with infinitesimal viscosity behave approximately as strong solutions of the inviscid equations? We answer this question in the affirmative below, and provide a rate of convergence as Reynolds number tends to infinity.

**Theorem 5.3.1** (Inviscid Limit and Drag Reduction)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^2$ boundary. Fix $T > 0$ and let $u^\nu$ be a strong solution of* (5.27) *with initial data $u_0$ on $[0,T] \times \Omega$ and mean-zero forcing provided by Theorem 5.2.1. Let $u$ be the global strong Euler solution* (5.103) *with initial data $u_0$. Then*

$$\sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|_{L^2(\Omega)} = O(\boldsymbol{Re}^{-1/2}). \tag{5.104}$$

*Furthermore, the wall friction factor $\langle f \rangle$ (global momentum defect) vanishes as*

$$\langle f \rangle := \frac{1}{\boldsymbol{Re}} \int_0^T \int_{\partial\Omega} \hat{n} \cdot \nabla u^\nu(x,t) dS dt = O(\boldsymbol{Re}^{-1}), \tag{5.105}$$

*and the global energy dissipation tends to zero as*

$$\langle \varepsilon^\nu \rangle := \frac{1}{\boldsymbol{Re}} \int_0^T \int_\Omega |\nabla u^\nu(x,t)|^2 dxdt = O(\boldsymbol{Re}^{-1}). \qquad (5.106)$$

**Remark 18** (Scaling Limits)**.** *The Navier-Stokes – End-Functionalized polymer system has four non-dimensional parameters,* $\boldsymbol{Re}$*,* $\boldsymbol{Wi}$*,* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\tau}$*. Our argument below shows the the key dimensionless quantities for passage to Euler in the inviscid limit and obtaining drag reduction are the following two ratios*

$$\boldsymbol{\alpha} = \frac{L}{R}, \qquad \frac{\boldsymbol{\alpha}\boldsymbol{Re}\boldsymbol{Wi}}{\boldsymbol{\tau}} = \boldsymbol{\alpha}\frac{\mu_p}{\mu_s}, \qquad (5.107)$$

*where, recall,* $\mu_s = \rho\nu$ *is the dynamic solvent viscosity,* $\mu_p = N_P \lambda k_B \overline{T}$ *is the polymer viscosity,* $\lambda = \zeta R^2/4Hk_B\overline{T}$ *is the polymer relaxation time and* $\zeta = 6\pi\rho\nu a$ *is the bead friction coefficient. If the quantities* (5.107) *behave well, say they are* $O(\boldsymbol{Re}^\gamma)$ *for some* $\gamma < 1$*, then an inspection of our proof shows that the high-Reynolds number limit holds as* $\boldsymbol{Re} \to \infty$*, albeit with a slower rate of* $\boldsymbol{Re}^{(\gamma-1)/2}$*. In taking the high–*$\boldsymbol{Re}$ *limit, we imagine we accomplish this either by taking viscosity small, taking the characteristic velocity* $V$ *large, taking large characteristic scales* $L$*, or some combination thereof. Thinking of applications such as pipe of channel flow, one might think of* $L$ *as fixed (the pipes may be long in extend, but turbulent scales are set by the cross-sectional width which is not necessarily large.) and vary Reynolds number be either reducing the viscosity of the solvent of driving the fluid faster through the pipe by increasing the pressure head.*
*Let us analyze a few situations of varying Reynolds number* $\boldsymbol{Re}$*, paying attention to the ratio* (5.107).

1. *Perhaps the most physical of the potential limits is to hold* $\nu$ *and* $L$ *fixed and*

*vary V. In this case,*

$$\boldsymbol{\alpha}, \frac{\boldsymbol{\alpha Re Wi}}{\boldsymbol{\tau}} = O_V(1) = O(\boldsymbol{Re}^0) \text{ with } \nu, L \text{ fixed,} \tag{5.108}$$

*since neither* $\boldsymbol{\alpha}$ *nor* $\mu_p/\mu_s$ *depend at all on the characteristic velocity* $V$*.*

2. *First, if* $L$ *and* $U$ *are held fixed and* $\nu$ *is varied, recalling the Stokes–Einstein relation* $\zeta = 6\pi\rho\nu a$ *we find* $\mu_p/\mu_s$ *is independent of viscosity* $\nu$*. Consequently,*

$$\boldsymbol{\alpha}, \frac{\boldsymbol{\alpha Re Wi}}{\boldsymbol{\tau}} = O_\nu(1) = O(\boldsymbol{Re}^0) \text{ with } V, L \text{ fixed.} \tag{5.109}$$

*However, as remarked before, these limits should physically be interpreted as intermediate asymptotics. In particular, decreasing viscosity will decrease the viscous sublayer of the flow near the wall, which is order* $O(\nu)$*. Our tacit assumption is that the typical polymer length should be smaller than the gradient length of the flow which, near the wall, should be on the order of the sublayer. Therefore, varying* $\nu$ *and keeping* $R$ *fixed is liable to break down when* $R$ *and the sublayer become of comparable sizes.*

*In order to maintain our effective continuum model description, one might consider performing a sequence of experiments where* $R$ *is decreased together with* $\nu$ *as* $R = O(\boldsymbol{Re}^{-\gamma})$ *for* $\gamma \in [0, 1]$*, while maintaining a sufficiently dense coating. This requires, in particular, that the number density be taken of the order* $N_P \sim R^{-(d-1)}$ *where* $d$ *is the spatial dimension so that the continuous carpet approximation and mushroom regime remain valid. For consistency, since polymer length-scale itself is shrinking, the effective bead scale* $a$ *should be taken of order* $O(R^\beta)$ *for some* $\beta \geq 1$*. In that case,* $\boldsymbol{\alpha} = L/R = O(\boldsymbol{Re}^\gamma)$ *and if* $R$ *is* *taken $O(\nu)$, then the ratio (5.107) is order*

$$\boldsymbol{\alpha} = \boldsymbol{Re}^{\gamma}, \ \frac{\boldsymbol{\alpha Re Wi}}{\boldsymbol{\tau}} = O(\boldsymbol{Re}^{(d-2-\beta)\gamma}) \text{ with } V, L \text{ fixed.} \tag{5.110}$$

*Thus, provided that $\beta > 0$ and $\gamma < 1$, we again obtain inviscid limit while maintaining our continuum description for all viscosity. The borderline case $\gamma = 1$ is exactly parallel to the critical Navier-slip boundary conditions, see discussion in [52].*

3. *In order to obtain our inviscid limit results, we cannot fix $V$ and $\nu$, and take $L$ large to increase Reynolds number. This would result in $\boldsymbol{\alpha} = O(\boldsymbol{Re})$ while the ratio $\mu_p/\mu_s$ remains fixed, which is again critical.*

*In summary, taking the limit $\boldsymbol{Re} \to \infty$ either by modifying the viscosities of the fluids or their characteristic speeds, our Theorem 5.3.1 says that $u^{\nu} \to u$ the strong Euler solution and the wall-drag/ dissipation vanishes.*

**Remark 19.** *The conclusions of Theorem 5.3.1 extend in a straightforward manner for dimensions $d \geq 3$ on any time interval over which strong solutions $u^{\nu}$ of the Navier-Stokes–end-functionalized polymer system and strong Euler solutions $u$ exist. Moreover, the initial conditions and forces need not be taken identical, strong convergence in $L^2$ suffices to pass to Euler in the inviscid limit.*

*Proof.*

**Step 1: (Convergence to Euler)** Let $w = u^\nu - u$ be the difference of the two solutions. Then

$$\begin{gathered}
\partial_t w + w \cdot \nabla u + u^\nu \cdot \nabla w = -\nabla q + \frac{1}{\mathbf{Re}} \Delta u^\nu \text{ in } \Omega \times (0, T), \\
\nabla \cdot w = 0 \text{ in } \Omega \times (0, T), \\
w \cdot \hat{n} = 0 \text{ on } \partial\Omega \times (0, T), \\
w|_{t=0} = 0 \text{ on } \Omega \times \{t = 0\}.
\end{gathered} \tag{5.111}$$

The energy in the difference field satisfies

$$\partial_t \left( \frac{1}{2} |w|^2 \right) + w \cdot \nabla u \cdot w + \nabla \cdot \left( \frac{1}{2} |w|^2 u^\nu + qw \right) = \frac{1}{\mathbf{Re}} w \cdot \Delta u^\nu. \tag{5.112}$$

Integrating and using the boundary conditions $u^\nu \cdot \hat{n}$ and $w \cdot \hat{n}$, we find

$$\frac{1}{2} \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)}^2 + \frac{1}{\mathbf{Re}} \int_\Omega w \cdot \Delta u^\nu dx. \tag{5.113}$$

Now first note that

$$\begin{aligned}
\int_\Omega w \cdot \Delta u^\nu dx &= -\|\nabla u^\nu\|_{L^2(\Omega)}^2 + \int_\Omega \nabla u : \nabla u^\nu dx + \int_{\partial\Omega} w \cdot (\hat{n} \cdot \nabla) u^\nu dS \\
&\leq -\frac{1}{2} \|\nabla u^\nu\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} w \cdot (\hat{n} \cdot \nabla) u^\nu dS.
\end{aligned} \tag{5.114}$$

Now note that for any tangential vector field to the boundary $v$ satisfying $v \cdot \hat{n} = 0$ we have

$$\begin{aligned}
&\int_{\partial\Omega} v \cdot (\hat{n} \cdot \nabla) u^\nu dS = \int_{\partial\Omega} (v \cdot \hat{\tau})((\hat{n} \cdot \nabla) u^\nu \cdot \hat{\tau}) dS \\
&= \int_{\partial\Omega} (v \cdot \hat{\tau})(2(D(u^\nu)n) \cdot \hat{\tau}) dS - \int_{\partial\Omega} (v \cdot \hat{\tau})(\hat{\tau} \cdot \nabla n) \cdot u^\nu dS \\
&= \int_{\partial\Omega} (v \cdot \hat{\tau})(2(D(u^\nu)n) \cdot \hat{\tau}) dS - \int_{\partial\Omega} 2\kappa (v \cdot \hat{\tau})(u^\nu \cdot \hat{\tau}) dS
\end{aligned} \tag{5.115}$$

where we recall that $\kappa = \hat{\tau} \cdot \nabla \hat{n} \cdot \hat{\tau}$ is the boundary curvature. Combining with the boundary condition on Navier-Stokes

$$u^\nu \cdot \hat{\tau} = -\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{Re}} \left( \partial_t + \frac{1}{\mathbf{Wi}} \right) \left( 2(D(u^\nu)\hat{n}) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau} \right), \tag{5.116}$$

we have the following equality

$$\begin{aligned} &\int_{\partial\Omega} u^\nu \cdot (\hat{n} \cdot \nabla) u^\nu dS \\ &= \int_{\partial\Omega} (u^\nu \cdot \hat{\tau}) \left( 2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau} \right) dS - \int_{\partial\Omega} \left( \frac{\boldsymbol{\alpha}}{2} + 2\kappa \right) (u^\nu \cdot \hat{\tau})^2 dS \\ &= -\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{Re}} \frac{d}{dt} \int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS \\ &-\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{ReWi}} \int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS - \int_{\partial\Omega} \left( \frac{\boldsymbol{\alpha}}{2} + 2\kappa \right) (u^\nu \cdot \hat{\tau})^2 dS. \end{aligned} \tag{5.117}$$

Consequently

$$\begin{aligned} \int_{\partial\Omega} w \cdot (\hat{n} \cdot \nabla) u^\nu dS &= -\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{Re}} \frac{d}{dt} \int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS \\ -\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{ReWi}} &\int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS - \int_{\partial\Omega} \left( \frac{\boldsymbol{\alpha}}{2} + 2\kappa \right) (u^\nu \cdot \hat{\tau})^2 dS \\ - \int_{\partial\Omega} (u \cdot \hat{\tau}) &\left( 2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau} \right) dS + \int_{\partial\Omega} \left( \frac{\boldsymbol{\alpha}}{2} + 2\kappa \right) (u \cdot \hat{\tau})(u^\nu \cdot \hat{\tau}) dS. \end{aligned} \tag{5.118}$$

The Euler/Navier-Stokes cross-terms are handled as follows. First,

$$\begin{aligned} &\left| \int_{\partial\Omega} (u \cdot \hat{\tau}) \left( 2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau} \right) dS \right| \\ &\leq \sqrt{\int_{\partial\Omega} (u \cdot \hat{\tau})^2 dS \int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS} \\ &\leq \frac{2\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}} \int_{\partial\Omega} (u \cdot \hat{\tau})^2 dS + \frac{\boldsymbol{\tau}}{2\boldsymbol{\alpha}\mathbf{ReWi}} \int_{\partial\Omega} |2(D(u^\nu)n) \cdot \hat{\tau} + \frac{\boldsymbol{\alpha}}{2} u^\nu \cdot \hat{\tau}|^2 dS. \end{aligned} \tag{5.119}$$

The inequality (5.119) allows us to hide the first cross-terms above. As for the other cross-term, we note first that if $\boldsymbol{\alpha} > 4 \max_{x \in \partial\Omega} \kappa$ (which is consistent with our assumption ($A_3$)), then this term is negative and can be dropped. Otherwise, more generally we assume $\boldsymbol{\alpha} \neq 4\kappa$ and we have

$$\begin{aligned}\left|\int_{\partial\Omega}\left(\frac{\boldsymbol{\alpha}}{2}+2\kappa\right)(u^\nu\cdot\hat{\tau})(u\cdot\hat{\tau})dS\right| &\leq \frac{1}{2}\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}(u^\nu\cdot\hat{\tau})^2dS \\ &+\frac{1}{2}\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}(u\cdot\hat{\tau})^2dS.\end{aligned} \tag{5.120}$$

We estimate the boundary term by trace inequality and embedding as follows

$$\int_{\partial\Omega}(u^\nu\cdot\hat{\tau})^2dS \leq 4\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}\|u^\nu\|^2_{L^2(\Omega)}+\frac{1}{4\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}}\|\nabla u^\nu\|^2_{L^2(\Omega)}. \tag{5.121}$$

Thus, putting this together with (5.118) and (5.119) we find

$$\begin{aligned}\int_{\partial\Omega}w\cdot(\hat{n}\cdot\nabla)u^\nu dS &\leq -\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{Re}}\frac{d}{dt}\int_{\partial\Omega}|2(D(u^\nu)n)\cdot\hat{\tau}+\frac{\boldsymbol{\alpha}}{2}u^\nu\cdot\hat{\tau}|^2dS \\ &-\frac{\boldsymbol{\tau}}{2\boldsymbol{\alpha}\mathbf{ReWi}}\int_{\partial\Omega}|2(D(u^\nu)n)\cdot\hat{\tau}+\frac{\boldsymbol{\alpha}}{2}u^\nu\cdot\hat{\tau}|^2dS \\ &+2\|\boldsymbol{\alpha}/2+2\kappa\|^2_{L^\infty(\partial\Omega)}\|u^\nu\|^2_{L^2(\Omega)}+\frac{1}{4}\|\nabla u^\nu\|^2_{L^2(\Omega)} \\ &+\left(\frac{1}{2}\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}+\frac{2\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}\right)\int_{\partial\Omega}(u\cdot\hat{\tau})^2dS.\end{aligned} \tag{5.122}$$

Finally, we obtain the following relative energy inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|w(t)\|^2_{L^2(\Omega)}+\frac{\boldsymbol{\tau}}{\boldsymbol{\alpha}\mathbf{Re}^2}\int_{\partial\Omega}|2(D(u^\nu)\hat{n})\cdot\hat{\tau}|^2dS\right)+\frac{1}{4\mathbf{Re}}\|\nabla u^\nu\|^2_{L^2(\Omega)} \\ &+\frac{\boldsymbol{\tau}}{2\boldsymbol{\alpha}\mathbf{Re}^2\mathbf{Wi}}\int_{\partial\Omega}|2(D(u^\nu)\hat{n})\cdot\hat{\tau}|^2dS \leq \|\nabla u\|_{L^\infty(\Omega)}\|w(t)\|^2_{L^2(\Omega)}+\frac{\mathcal{E}(t)}{\mathbf{Re}} \\ &\|w(0)\|^2_{L^2(\Omega)}=0\end{aligned} \tag{5.123}$$

where

$$\begin{aligned}\mathcal{E}(t) := &\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}+2\|\boldsymbol{\alpha}/2+2\kappa\|^2_{L^\infty(\partial\Omega)}\|u^\nu\|^2_{L^2(\Omega)} \\ &+\left(\frac{1}{2}\|\boldsymbol{\alpha}/2+2\kappa\|_{L^\infty(\partial\Omega)}+\frac{2\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}\right)\int_{\partial\Omega}(u\cdot\hat{\tau})^2dS.\end{aligned} \tag{5.124}$$

Recalling Lemma 5.2.2 for the bound on kinetic energy and working in the settings of (1) or (2) detailed in Remark 18, we have $\frac{\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}} = O(\mathbf{Re}^0)$ and $\boldsymbol{\alpha} = O(\mathbf{Re}^0)$ and thus

$$\sup_{t\in[0,T]} \mathcal{E}(t) = O_{\mathbf{Re}}(1). \tag{5.125}$$

Integrating the above, using Grönwall's inequality and the fact that $A > 0$ we find for any $T > 0$

$$\sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|_{L^2(\Omega)} = O(\mathbf{Re}^{-1/2}). \tag{5.126}$$

Thus, we have convergence $u^\nu \to u$ strongly in $L^\infty(0,T;L^2(\Omega))$.

**Step 2: (Vanishing of Wall Drag)** The global momentum balance for Navier-Stokes reads

$$\frac{d}{dt}\int_\Omega u^\nu dx = -\int_{\partial\Omega} \hat{n} p^\nu dS + \frac{1}{\mathbf{Re}}\int_{\partial\Omega} \partial_n u^\nu dS. \tag{5.127}$$

The last term is the viscosity induced wall-friction, which we aim to show vanishes. Indeed, using the divergence-free condition $\nabla \cdot u^\nu = 0$ we have

$$\hat{n}\cdot\partial_n u^\nu|_{\partial\Omega} = -\hat{\tau}\cdot\partial_\tau u^\nu|_{\partial\Omega}. \tag{5.128}$$

To see this, we extend $\hat{n}(x)$ and $\hat{\tau}(x)$ smoothly into a tubular neighborhood of $\partial\Omega$ and such that they remain an orthonormal basis of $\mathbb{R}^2$. Then expressing $\nabla = \hat{n}\partial_n + \hat{\tau}\partial_\tau$, forming $\nabla\cdot u = \hat{n}\partial_n u + \hat{\tau}\partial_\tau u$ and tracing on the boundary $\partial\Omega$ (recalling that $u \in L^\infty(0,T;H^2(\Omega))$, so that the trace makes sense), we obtain (5.128). Recalling also the identity (5.45) for vorticity along the walls

$$\omega^\nu|_{\partial\Omega} = 2(D(u^\nu)\hat{n})\cdot\hat{\tau}|_{\partial\Omega} + 2\kappa(u^\nu\cdot\hat{\tau})|_{\partial\Omega}, \tag{5.129}$$

and returning to the wall-friction in (5.127), we have

$$
\begin{aligned}
\frac{1}{\mathbf{Re}}\int_{\partial\Omega}\partial_n u^\nu dS &= \frac{1}{\mathbf{Re}}\int_{\partial\Omega}\hat{n}\cdot\partial_n u^\nu\ \hat{n}dS + \frac{1}{\mathbf{Re}}\int_{\partial\Omega}\hat{\tau}\cdot\partial_n u^\nu\ \hat{\tau}dS \\
&= -\frac{1}{\mathbf{Re}}\int_{\partial\Omega}\hat{\tau}\cdot\partial_\tau u^\nu\ \hat{n}dS + \frac{1}{\mathbf{Re}}\int_{\partial\Omega}2(D(u^\nu)\hat{n})\cdot\hat{\tau}\ \hat{\tau}dS - \frac{1}{\mathbf{Re}}\int_{\partial\Omega}\hat{n}\cdot\partial_\tau u^\nu\ \hat{\tau}dS \\
&= \frac{1}{\mathbf{Re}}\int_{\partial\Omega}(u^\nu\cdot\hat{\tau})\left[\hat{\tau}\cdot\partial_\tau(\hat{\tau}\otimes\hat{n}+\hat{n}\otimes\hat{\tau})\right]dS + \frac{1}{\mathbf{Re}}\int_{\partial\Omega}2(D(u^\nu)\hat{n})\cdot\hat{\tau}\ \hat{\tau}dS
\end{aligned}
\tag{5.130}
$$

Then, by trace theorem and the energy equality (5.39), we find for some $C := C(\Omega, T, \frac{2\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}})$ such that

$$
\begin{aligned}
&\frac{1}{\mathbf{Re}}\left|\int_0^T\int_{\partial\Omega}\partial_n u^\nu dSdt\right| \\
&\leq \frac{C}{\mathbf{Re}}\|u^\nu\|_{L^\infty(0,T;L^2(\Omega))}\|\nabla u^\nu(t)\|_{L^2(0,T;L^2(\Omega))} + \frac{C}{\mathbf{Re}}\|(2D(u^\nu)\hat{n})\cdot\hat{\tau}\|_{L^2(0,T;L^2(\partial\Omega))} \\
&\leq \frac{\mathbf{Re}\boldsymbol{\alpha}\mathbf{Wi}}{\boldsymbol{\tau}}\times\frac{C}{\mathbf{Re}} = O(\mathbf{Re}^{-1}),
\end{aligned}
\tag{5.131}
$$

where we used the bound (5.52) and (5.123). Note that the $L^\infty(0,T;L^2(\Omega))$ convergence established above implies that the pressure integrals must likewise converge. Consequently, the limiting global momentum balance reads: for any $0 \leq t' \leq t \leq T$

$$
\int_\Omega u(t)dx = \int_\Omega u(t')dx - \int_{t'}^t\int_{\partial\Omega}\hat{n}p(s)dSds. \tag{5.132}
$$

**Step 3: (Vanishing of Energy Dissipation)** Finally we note that, directly from (5.126) and (5.123) upon integration,

$$
\frac{1}{\mathbf{Re}}\int_0^T\int_\Omega|\nabla u^\nu(x,t)|^2dxdt \leq \frac{C(\frac{\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}, u_0, \Omega)}{\mathbf{Re}}. \tag{5.133}
$$

This bound would hold also in higher dimensions, provided smooth Navier-Stokes-End-Functionalized polymer solution and Euler solutions exists on the a common time interval. In two dimensions, the result follows again directly from the apriori bound on vorticity found in Lemma 5.2.3. Specifically, using (5.49) we have

$$\frac{1}{\mathbf{Re}}\int_0^T\int_\Omega |\nabla u^\nu(x,t)|^2 dxdt \lesssim \frac{1}{\mathbf{Re}}\int_0^T\int_\Omega |\omega^\nu(x,t)|^2 dxdt \leq \frac{C(\frac{\boldsymbol{\alpha}\mathbf{ReWi}}{\boldsymbol{\tau}}, u_0, \Omega)}{\mathbf{Re}}. \tag{5.134}$$

□

## 5.4 Appendices.

### 5.4.1 Appendix A. Well-posedness theory of parabolic equations

We recall some standard results on parabolic equations that we have used. Consider the problem

$$\begin{aligned} \partial_t u + v\cdot\nabla u - \nu\Delta u &= f \text{ in } \Omega\times[0,T], \\ u &= 0 \text{ on } \partial\Omega\times[0,T], \\ u|_{t=0} &= u_0 \text{ on } \Omega\times\{t=0\}, \end{aligned} \tag{5.135}$$

where $v \in C([0,T];C(\Omega))$ with $\operatorname{div} v = 0$, and $\Omega$ is bounded with $C^2$ boundary. If $f \in L^2(\Omega\times[0,T])$ and $u_0 \in H^1_0(\Omega)$, then there is a unique solution of (5.135) satisfying

$$\begin{aligned} &u \in C([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega)), \\ &\partial_t u \in L^2(0,T;L^2(\Omega)). \end{aligned} \tag{5.136}$$

For $v = 0$ one can find this from Lions and Magenes [95] or Brezis [20]. For general $v$, one may follow the standard argument summarized below; for a full argument (see [95] or [41]).

**Lemma 5.4.1** (Lions Projection Lemma). *Let $H$ be a Hilbert space and $\Phi \subset H$ a dense space.*

*Let $a : H \times \Phi \to \mathbb{R}$ be a bilinear form with the following two properties:*

1. *for all $\phi \in \Phi$, the linear form $u \to a(u, \phi)$ is continuous on $H$,*

2. *there is $\alpha > 0$ such that*

$$a(\phi, \phi) \geq \alpha \|\phi\|_H^2 \text{ for all } \phi \in \Phi. \tag{5.137}$$

*Then, for each continuous linear form $f \in H'$, there exists $u \in H$ such that*

$$\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'} \tag{5.138}$$

*and*

$$a(u, \phi) = \langle f, \phi \rangle \text{ for all } \phi \in \Phi. \tag{5.139}$$

To solve the system (5.135), we set

$$\begin{aligned} H &= L^2(0, T; H_0^1(\Omega)), \\ \Phi &= \{\phi = v|_{(0,T)\times\Omega} \mid v \in C_0^\infty((-\infty, T) \times \Omega)\}, \\ a(u, \phi) &= \int_{(0,T)\times\Omega} (\nabla u \cdot \nabla\phi - u\partial_t\phi - uv \cdot \nabla\phi)\, dxdt. \end{aligned} \tag{5.140}$$

Then, Lemma 5.4.1 implies existence of solution of (5.135) in the weak sense and, together with

$$\int_{(0,T)\times\Omega} (\partial_t uv + u\partial_t v)\, dxdt = \int_{\Omega\times\{t=T\}} uvdx - \int_{\Omega\times\{t=0\}} uvdx \tag{5.141}$$

and a standard density argument gives uniqueness. Finally, higher regularity follows from $v = 0$ case with $f$ replaced by $f - v \cdot \nabla u \in L^2(\Omega \times [0, T])$.

### 5.4.2 Appendix B. Derivation of Kramers expression for polymer stresses

Due to its central nature to our work, we here provide a short derivation of Kramers expression (Eqn (5.8)) for the polymer stresses for completeness. The derivation is standard and can be found, for example in the textbook of Ottinger [116] on pages 158–159. We will calculate here only the components $(-\hat{n}) \cdot \Sigma_P$, which are the force acting on the fluid in the direction normal to the wall. This is the only component of the stress tensor used in our physical derivation and it has the most intuitive interpretation.

First note that, within the bead-spring approximation, a polymer can exert force on a fluid parcel if and only if its end bead is contained in that fluid parcel. Thus, we set up a cut-off between polymer layer and fluid parcel. In other words, we imagine a tubular neighborhood along the wall of size $\ell$. The thickness (in the wall-normal direction) of the near-wall fluid parcel acted upon by the polymer has characteristic size on the order of $r$, the maximal extent of the polymer defined in assumption $(A_4)$. Its length-scale in the wall-tangential direction is taken larger than that of the typical polymers. As a consequence, the bead does not belong to the fluid parcel only if $(-\hat{n}) \cdot m < \ell$. The thickness scale is justified since we are interested in the fluid parcel directly communicating with polymer. Let $(-\hat{n}) \cdot \Sigma_P^\ell$ be the (spring) force per surface, divided by solvent density. This is the force that polymers exert on the near-wall fluid parcel sitting at distance $\ell$ uniformly from the wall. Fixing $\ell$, this force is mathematically expressed as

$$(-\hat{n}) \cdot \Sigma_P^\ell = r \int_{M(x)} \chi_{\{(-\hat{n})\cdot m \geq \ell\}}(m) \frac{k_B \overline{T}}{\rho} \nabla_m U(m) f_L(m) dm. \tag{5.142}$$

However, we note the following: there is no obvious choice for cut-off distance $\ell$ for polymer layer and fluid particles. Thus, to obtain the cumulative force $(-\hat{n}) \cdot \Sigma_L$, we average over possible scales $\ell$ and obtain

$$(-\hat{n})\cdot\Sigma_P = \frac{1}{r}\int_0^r (-\hat{n})\cdot\Sigma_P^\ell d\ell. \tag{5.143}$$

(In the case of the Hookean dumbbell model for which $r=\infty$ which can be understood in suitable limiting sense. We do not detail this here.) Therefore,

$$\begin{aligned}(-\hat{n})\cdot\Sigma_P &= \frac{k_B\overline{T}}{\rho}\int_0^r\int_{M(x)}\chi_{\{(-\hat{n})\cdot m\geq\ell\}}\nabla_m U f_L(m)dmd\ell \\ = \frac{k_B\overline{T}}{\rho}\int_{M(x)}\int_0^{(-\hat{n})\cdot m}&\nabla_m U f_L(m)d\ell dm = \frac{k_B\overline{T}}{\rho}\int_{M(x)}(-\hat{n})\cdot m\nabla_m U f_P dm.\end{aligned} \tag{5.144}$$

We thereby recover the Kramer formula (5.8) for the normal component of polymer stress along the wall.

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