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**A Comment on Consumer Demand Systems with
Binding Non-Negativity Constraints***

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Abstract

Wales and Woodland (1983) have proposed an econometric model to deal with non-negativity constraints in systems of demand equations. This paper points out the relationship between the Wales-Woodland model and the "simultaneous Tobit" model of Amemiya (1974). This relationship is important because Amemiya has proposed a simple estimation procedure that can be utilized for some cases of the Wales-Woodland model. The issue of internal consistency for models of this type is discussed. I point out that internal consistency for the Wales-Woodland model is equivalent to the second order condition for systems without quantity constraints.

1. Introduction

In a recent article Wales and Woodland (1983, hereafter noted W-W) propose an econometric model to deal with non-negativity constraints in a system of demand or expenditure equations. Previous empirical studies of demand system have not dealt with this problem, although it was recognized long ago. In fact, Houthakker (1954) discusses the non-negativity constraint as a special case of the "rationing" problem that was extensively studied in the immediate post-war period.¹ Kniessner (1976) makes a similar observation regarding the family labor supply model. Ransom (1983) develops essentially the same model as Wales and Woodland and discusses its application to a variety of quantity constrained demand problems. The purpose of this comment is to point out the relationship between the Kuhn-Tucker model of W-W and the "simultaneous equation" Tobit model of Amemiya (1974) and to comment on the relationship between the usual second-order conditions in demand models and the internal consistency condition that is required in simultaneous equation-limited dependent variable models of demand systems with quantity constraints.

2. Development

In the application presented in their paper, W-W estimate a system of Engel curves from Australian data on meat expenditures. Because the data lack price variability, W-W base their analysis on an additive quadratic utility function, although they develop a more general model. My comments here are restricted to this special case. However, since the large majority of survey data on consumption has no price variability, these comments should be of some interest.

Consider an individual who maximizes utility, $G(\mathbf{x})$, over a set of non-negative quantities, $\mathbf{x} = (x_1, \dots, x_M)$ which satisfy a budget constraint $\mathbf{p} \cdot \mathbf{x} \leq I$ where I is total expenditure (or income) and $\mathbf{p}$ is a vector of prices. Without variation in prices it is convenient to choose units such that the prices are all unity, so $\mathbf{x}$ becomes a vector of expenditures. If G is a quasiconcave, increasing utility function, the necessary and sufficient Kuhn-Tucker conditions for a constrained maximum are

$$\begin{aligned} G_i(\mathbf{x}) - \lambda &\leq 0 \leq x_i \quad i = 1, \dots, M; \\ \sum_{i=1}^M x_i - I &\leq 0 \leq \lambda. \end{aligned} \quad (1)$$

Since $G(\mathbf{x})$ is increasing, at least one good (say M) will be consumed, so (1) may be rewritten as

$$\begin{aligned} G_i(\mathbf{x}) - G_M(\mathbf{x}) &\leq 0 \leq x_i, \quad i=1, \dots, M-1; \\ \sum_{i=1}^M x_i &= I. \end{aligned} \quad (2)$$

A convenient formulation for G , suggested by W-W is the additive quadratic utility function

$$G(\mathbf{x}) = \sum_{i=1}^M (a_{i0} + u_i)x_i + \frac{1}{2} \sum_{i=1}^M a_{ii} x_i^2, \quad (3)$$

where u_i is a random variable representing the unobservable differences in tastes among consumers. Concavity requires that a_{ii} be negative for all i . For this specification, the Kuhn-Tucker conditions corresponding to (2) can be written as

$$(a_{i0} - a_{M0}) + a_{ii}x_i + a_{MM} \sum_{j=1}^{M-1} x_j - a_{MM}I + (u_i - u_M) = 0 \text{ if } x_i > 0 ,$$

$$(a_{i0} - a_{M0}) + a_{ii}x_i + a_{MM} \sum_{j=1}^{M-1} x_j - a_{MM}I + (u_i - u_M) < 0 \text{ if } x_i = 0 ,$$

for $i=1, \dots, M-1$.

It is possible to generalize the model by allowing $(a_{i0} - a_{M0})$ to vary with characteristics of the individual, as W-W on page 275. The conditions can then be written as

$$\sum_{j=1}^{M-1} \gamma_{ij} x_j + \sum_{k=1}^K \beta_{ik} Z_k + \epsilon_i = 0 \text{ if } x_i > 0 , i=1, \dots, M-1 ; \quad (4)$$

$$\sum_{j=1}^{M-1} \gamma_{ij} x_j + \sum_{k=1}^K \beta_{ik} Z_k + \epsilon_i < 0 \text{ if } x_i = 0 , i=1, \dots, M-1 ;$$

where the Z_k are individual characteristics, and $Z_1=I$, $\beta_{i1} = -a_{MM}$.

Also note that $\gamma_{ij} = a_{MM}$ if $i \neq j$, and $\gamma_{ii} = a_{MM} + a_{ii}$. This condition can also be written as

$$\begin{aligned} -\gamma_{ii}x_i &= \sum_{\substack{j=1 \\ j \neq i}}^{M-1} \gamma_{ij}x_j + \sum_{k=1}^K \beta_{ik} Z_k + \epsilon_i \text{ if RHS } > 0 \\ -\gamma_{ii}x_i &= 0 \text{ if RHS } \leq 0 . \end{aligned} \quad (5)$$

Except for the different normalization, (5) is identical to the simultaneous equation model of Amemiya (1974). This identity is

important because Amemiya proposes an estimation strategy that is more practical than maximum likelihood in some cases. This is especially true when the number of goods is large and a large fraction of the sample consumes some of each good.

3. Internal Consistency

The system of equations described by (4) is only piecewise linear in the dependent variables, so the system need not have a unique solution. That is, for given value of z and ε , there may be no value for x that satisfies (4). On the other hand, there may be many values that satisfy the conditions.

A model with a unique solution is said to be "internally consistent" or "coherent."¹ Amemiya (1974) derived conditions under which the simultaneous equation system with limited dependent variables is internally consistent. Schmidt (1981) presents a more detailed discussion of the issue.

In order to show that this model is coherent, I adopt the formulation of Amemiya (1974, p. 1006). Let each individual be subscripted by t , $t=1, \dots, T$. The model can be written in vector form as

$$\begin{aligned} \Gamma^* x_t &\geq \beta Z_t + \varepsilon_t, \\ x_t &\geq 0_1 \quad \text{and} \\ (\Gamma^* x_t)_i &= (\beta Z_t + \varepsilon_t)_i \quad \text{whenever } x_{it} > 0, \end{aligned} \tag{6}$$

where t subscripts individuals, $(\gamma_{ij}^*) = (-\gamma_{ij})$, and $()_i$ denotes the i th element of the vector in the parenthesis. β is an $(M-1) \times K$

matrix. Γ^* can be expressed as the sum of two $(M-1) \times (M-1)$ matrices

$$\Gamma^* = G + H ,$$

where all elements of G are identical positive constants,

$$\{g_{ij}\} = \{-a_{MM}\} ,$$

and H is a diagonal matrix,

$$\{h_{ij}\} = 0 \text{ if } i \neq j ,$$

$$\{h_{ij}\} = -a_{ii} \text{ if } i=j .$$

Proposition 1: The model described in equations (4), (5) and (6) above is internally consistent.

Proof: By Amemiya (1974, p. 1006) Theorem 3, the model is internally consistent if and only if all principal minors of Γ^* are positive. This will be true only if Γ^* is a positive definite matrix. To show that this is the case consider the quadratic form

$$Q = w' \Gamma^* w ,$$

where w is an arbitrary nonzero column vector. Since $\Gamma^* = (G+H)$, Q can be expressed as

$$Q = w' (G+H) w = w' G w + w' H w .$$

Γ^* will be positive definite if and only if Q is positive for any nonzero value for w . Since G is positive semidefinite, $w' G w$ will always be nonnegative. Since H is positive definite, $w' H w$ will be positive. Both of these follow from the negativity of a_{ii} .

Therefore, Q will always be positive, which implies that Γ^* is positive definite. Q.E.D.

The quasiconcavity of the utility function ensures a unique solu-

tion to the Kuhn-Tucker conditions at the outset. Nevertheless, it is interesting to make clear the nature of the internal consistency condition for this model. Schmidt (1981) has questioned the value of simultaneous Tobit models because of the apparently arbitrary constraints implied by coherency. Here I have presented a class of models for which these parameter constraints are natural.

Footnote

¹The term coherency, comes from the disequilibrium literature. For a discussion in that context see Gourieroux, Laffont and Montfort (1980). Schmidt (1981) uses the term "internal consistency."

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