Nonlinear wave equations on time dependent inhomogeneous backgrounds

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Abstract

In this thesis, I study the nonlinear wave equations on a class of asymptotically flat Lorentzian manifolds $(\mathbb{R}^{3+1}, g)$ with **time dependent** inhomogeneous metric g. Based on a new approach for proving the decay of solutions of linear wave equations, I give several applications to nonlinear problems. In particular, I show the small data global existence result for quasilinear wave equations satisfying the null condition on a class of time dependent inhomogeneous backgrounds which do not settle to any particular stationary metric.

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To my fiancee

who requests the first copy

Contents

Chapter 1

Introduction

In this thesis, I study the Cauchy problem for the nonlinear wave equations

$$\Box_g \phi + C^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\phi = Q(\phi, \partial\phi)$$

on a Lorentzian manifold $(\mathbb{R}^{n+1}, g)$, where $\Box_g$ is the covariant wave operator for the metric g and Q is at least quadratic in ϕ, $\partial\phi$.

In higher dimensions ($n \geq 4$), the decay rate of the solution to a linear wave equation is sufficient to obtain the small data global existence result [19], [21], [23], [36]. However, in $3+1$ dimensions, one can only show the almost global existence result [16], [21]. In fact, in [15], F. John showed that any nontrivial C^3 solution of

$$\Box\phi = (\partial_t\phi)^2$$

with compactly supported initial data blows up in finite time. Nevertheless, a sufficient condition on the quadratic nonlinearities, which guarantees the small data global existence result, is the celebrated null condition introduced by S. Klainerman [20]. Under this condition, D. Christodoulou [6] and S. Klainerman [22] independently proved the small data global existence result.

The approach of [6] used the conformal mapping method, which relies on the conformal embedding of Minkowski space to the Einstein cylinder $R\times S^3$. S. Klainerman used the vector field method [21], which connects the symmetries of the flat $\mathbb{R}^{n+1}$ with the quantitative decay properties of solutions of linear wave equations. The vector fields, used as commutators or multipliers, are the killing and conformal killing vector fields in $\mathbb{R}^{n+1}$ and can be given explicitly

$$\Gamma = \{\Omega_{ij} = x_i\partial_j - x_j\partial_i, L_i = t\partial_i + x_i\partial_t, \partial_\alpha, K = (t^2+r^2)\partial_t + 2tr\partial_r, S = t\partial_t + r\partial_r\}. \quad (1.1)$$

Based on this vector field method, there have been an extensive literature on generalizations and variants of D. Christodoulou and S. Klainerman's work, in particular on the multiple speed problems ([24], [37], [38]) and obstacle problems ([31], [17], [18], [33]). All these works used the scaling vector field S.

Another application of the vector field method is to the wave equations on a Lorentzian manifold $(\mathcal{M}, g)$ with metric g, which may also depend on the solution of the equation. The motivation for studying such problems arises from studying the problem of global nonlinear stability of Minkowski space in wave coordinates. The stability of Minkowski space was first established by Christodoulou-Klainerman by recasting the problem as a system of Bianchi equations for the curvature tensor [7]. Later, Lindblad-Rodnianski [28] obtained a different proof in wave coordinates, in which the problem was ted as a system of quasilinear wave equations for the components of the metric perturbation. This is an example that the background metric $g(\phi)$ depends on the solution ϕ of the wave equation, where $g(0) = m_0$ (the Minkowski metric). The quasilinear part of such equations $g^{\alpha\beta}(\phi)\partial_{\alpha\beta}\phi$ never satisfies the null condition defined in the original work of S. Klainerman [20], [22]. Nevertheless, besides the global nonlinear stability of Minkowski space aforementioned, the nonlinear wave equations

$$g^{\alpha\beta}(\phi)\partial_{\alpha\beta}\phi = 0$$

on $\mathbb{R}^{3+1}$, which was studied by H. Lindblad in [25], [26], also admits small data global solutions. A particular case

$$\partial_{tt}\phi - (1+\phi)^2\Delta\phi = 0$$

has been investigated previously by S. Alinhac [1].

The linear and nonlinear wave equations on a Lorentzian manifold with given metric g have also received considerable attention, in particular on black hole spacetimes. For the linear wave equation on $\mathbb{R}^{3+1}$, S. Alinhac [2] showed that the solution has the decay properties similar to those of a solution of a linear wave equation on Minkowski space provided that the metric g approaches the Minkowski metric suitably as $t \to \infty$. In [40], D. Tataru proved the local decay of the solution but with the assumption that the background metric is stationary or time independent. For the decay of solution of linear wave equations on Kerr spacetimes (including Schwarzschild spacetimes), we refer the readers to [9], [29], [8], [4] for references. For the nonlinear equations, J. Luk [30] proved the small data global existence result for semilinear wave equations with derivatives on slowly rotating Kerr spacetimes. In a recent work [41], Wang-Yu proved the small data global existence result for quasilinear wave equations on static spacetimes, which are more restrictive than stationary ones.

A common feature of these problems is that the background metric g settles down to a stationary metric either by the assumptions (Kerr spacetimes are stationary) or, for the case when the metric $g(\phi)$ depends on the solution, by the assumption $g(0) = m_0$ and the expected convergence $\phi(t, x) \to 0$ as $t \to \infty$. The need for such convergence, or at least convergence of the time derivative of the metric to 0, is dictated by the vector field method. All applications of the vector field method require commutations with some generators of the conformal symmetries of Minkowski space. In particular, we note that all the applications have used the scaling vector field $S = t\partial_t + r\partial_r$ or the conformal killing vector field $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ as commutators. For the problem

$$\Box_g \phi = F,$$

the error term coming from the commutation with S or K would be of the form $t\partial_t g^{\alpha\beta}\partial_{\alpha\beta}\phi$ or $t^2\partial_t g^{\alpha\beta}\partial_{\alpha\beta}\phi$ which leads to the requirement that $t\partial_t g$ is at least bounded and thus the time derivative of the metric decays to 0 as $t \to \infty$.

In [10], Dafermos-Rodnianski introduced a new method for proving decay of solutions of linear wave equations. This new method avoids the use of any vector field growing in t, in particular the scaling vector field $S = t\partial_t + r\partial_r$. The aim of this thesis is to adapt this new method to several nonlinear problems on time dependent inhomogeneous backgrounds which do not settle to any particular stationary metric.

The first problem is to study the small data global existence for nonlinear wave equations. My first result concerns the case when the metric is a small perturbation of the Minkowski metric inside some cylinder $\{(t, x), |x| \leq R\}$ and is flat outside.

Theorem 1.0.1 ([42])**.** *Consider the semilinear wave equations $\Box_g \phi = Q(\phi, \phi)$ satisfying the null condition on $(\mathbb{R}^{3+1}, g)$. Assume the metric g is smooth and is C^1 close to the Minkowski metric inside some large cylinder. Then for sufficiently small initial data, the unique solution is global in time.*

In higher dimensions, a similar small data global existence result can be obtained without imposing the null condition. Although the perturbation of the metric is assumed to be small in C^1, the curvature of the manifold can be large for all time. I also established a more general result, in which the smallness assumption on the metric perturbation can be replaced by two integrated local energy decay estimates for solutions of linear wave equations (see Chapter 4).

Later I studied the more general quasilinear wave equations on a larger class of time dependent inhomogeneous backgrounds. The metric can also be a perturbation of the Minkowski metric outside the cylinder. I proved

Theorem 1.0.2. *Assume the metric g is smooth and is C^1 close to the Minkowski metric inside some large cylinder. Outside the cylinder, assume the metric approaches the Minkowski metric with some weak rates. Then the quasilinear wave equation $\Box_g\phi + C^{\alpha\beta}(\phi)\partial_\alpha\partial_\beta\phi = 0$ satisfying the null condition on $(\mathbb{R}^{3+1}, g)$ has a unique global solution for all sufficiently small initial data.*

A special case of the theorem is that the metric approaches the Minkowski metric in the spatial directions with rate $|x|^{-1-\epsilon}$ for some $\epsilon > 0$. This implies a recent result obtained independently by Wang-Yu [41], but there the metric is assumed to be time independent.

On the other hand, very little is known about the global dynamics of solutions with large data. S. Alinhac [3] initiated the study of global stability of large solutions of nonlinear wave equations. I improved a result of S. Alinhac by imposing weaker assumptions on the given large solution and showed that it is stable under perturbations of initial data.

Theorem 1.0.3 ([43]). *Let Φ be a solution of the quasilinear wave equation*

$$\Box\phi + g^{\alpha\beta\gamma}\partial_\gamma\phi\partial_\alpha\partial_\beta\phi = Q(\partial\phi, \partial\phi) \tag{1.2}$$

satisfying the null condition in $\mathbb{R}^{3+1}$. Assume Φ obeys some weak decay estimates (much weaker than that of solutions of linear wave equations). Then the quasilinear wave equation admits a unique global solution for all initial data sufficiently close to the initial data of Φ.

In particular, it is not necessary to require that the given large solution decays uniformly in time as it was in [3].

The thesis is organized as follows: in Chapter 2, we define the basic notations and recall the energy method for wave equations; in Chapter 3, we introduce the new method of Dafermos-Rodnianski and extend the necessary estimates to solutions of linear wave equations on time dependent inhomogeneous background; then in Chapter 4, we apply this new method to semilinear wave equations and derive the small data global existence result; in Chapter 5 we extend the results for semilinear wave equations to quasilinear wave equations on more general asymptotically flat Lorentzian spacetimes; in Chapter 6, we consider the problem of global stability of solutions of nonlinear wave equations; in the last chapter, we discuss the general initial data and system of nonlinear wave equations.

Chapter 2

Preliminaries

2.1 Coordinates and null frames

In $\mathbb{R}^{3+1}$, we use the coordinate system $(t, x) = (t, x_1, x_2, x_3)$. We may also use the standard polar local coordinate system (t, r, ω) and the null coordinates (u, v, ω)

$$u = \frac{t-r}{2}, \quad v = \frac{t+r}{2}.$$

Let $\nabla\!\!\!/$ denote the induced covariant derivative, $\Delta\!\!\!/$ the induced Laplacian on the spheres of constant r, Ω the angular momentum with components $\Omega_{ij} = x_i\partial_j - x_j\partial_i$. Here ∂_i is the partial derivative ∂/∂_{x_i}. We may use ∂ to abbreviate $(\partial_t, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$. The vector fields, used as commutators, are

$$Z = \{\Omega_{ij}, \partial_t\}.$$

Following the setup in [27], we now introduce a null frame $\{L, \underline{L}, S_1, S_2\}$, which is locally a basis of the tangent space at any point (t, x) of the Minkowski space for

$r = |x| > 0$. We let

$$L = \partial_v = \partial_t + \partial_r, \quad \underline{L} = \partial_u = \partial_t - \partial_r.$$

We then let S_1, S_2 be an orthonormal basis of the tangent space of the spheres with constant radius r. We use $\overline{\partial_v}$ to denote the "good" derivatives

$$\overline{\partial_v} = (L, S_1, S_2) = (\partial_v, \nabla\!\!\!/).$$

For any symmetric two tensor $k^{\mu\nu}$, relative to the null frame $\{L, \underline{L}, S_1, S_2\}$, we have

$$k^{\underline{LL}} = \frac{1}{4} k^{\mu\nu} \underline{L}_\mu \underline{L}_\nu, \quad \underline{L}_0 = 1, \quad \underline{L}_i = -\frac{x_i}{r}.$$

2.2 The foliations

Instead of using the classical spacelike foliations $\{t = const\}$, one novel aspect of the new approach introduced by Dafermos-Rodnianski [10] is the use of the foliation Σ_τ of the Minkowski space, defined as follows

$$S_\tau := \{u = \frac{\tau - R}{2}, v \geq \frac{\tau + R}{2}\},$$
$$\Sigma_\tau := \{t = \tau, r \leq R\} \cup S_\tau,$$

where the radius R is a to-be-fixed constant. The corresponding energy flux in Minkowski space is

$$E[\phi](\tau) := \int_{\{r \leq R\} \cap \Sigma_\tau} |\partial\phi|^2 dx + \int_{S_\tau} |L\phi|^2 + |\nabla\!\!\!/\phi|^2 \quad r^2 dv d\omega.$$

Denote the incoming null hypersurface

$$\bar{C}(\tau_1, \tau_2, v) := \{(t, r, \omega)|\tau_1 \leq t - r + R \leq \tau_2, \quad t + r = 2v\}.$$

We simply use $\bar{C}(\tau_1, \tau_2)$ to denote the future null infinity (part of) where $v = \infty$. Define the energy flux through the null infinity as the limit infimum of the energy flux through $\bar{C}(\tau_1, \tau_2, v)$ as $v \to \infty$, that is,

$$E^N[\phi]_{\tau_1}^{\tau_2} := \liminf_{v \to \infty} \int_{\bar{C}(\tau_1, \tau_2, v)} (\partial_u \phi)^2 + |\nabla\!\!\!/ \phi|^2 \quad r^2 du d\omega.$$

We define the modified energy flux

$$\tilde{E}[\phi](\tau) := E[\phi](\tau) + E^N[\phi]_0^\tau.$$

2.3 Notations

Except some notations defined above, we would like to make some conventions and define the following notations which may be used throughout this thesis.

- R is a large positive constant denoting the radius of the foliation Σ_τ.
- $g = g_{\mu\nu} dx^\mu dx^\nu$, $x^0 = t$, $x^i = x_i$ is a Lorentzian metric on $\mathbb{R}^{3+1}$. Relative to the coordinates (t, x), assume

$$g^{\mu\nu} = m_0^{\mu\nu} + h^{\mu\nu}, \quad m_0 \text{ the Minkowski metric.}$$

  We always assume that the metric uniformly hyperbolic, i.e., there exists a positive constant λ such that in (t, x) coordinates

$$g^{00} \leq -\lambda, \quad \lambda|\xi|^2 \leq g^{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

- $d\text{vol}$ is the volume form with respect to the metric g. In local coordinates, $d\text{vol} = \sqrt{-G}dxdt = -\sqrt{-G}dtdx$. Here we always choose t to be the time orientation. G is the determinant of the metric component $g_{\mu\nu}$.

- $d\omega$ is the standard surface measure on the unit sphere $\mathbb{S}^2$.

- The capital letters A, B will be any vector fields in the null frame $\{L, \underline{L}, S_1, S_2\}$. Relative to the null frame, the metric components are g_{AB}. The inverse is g^{AB}. We also denote

$$\partial^\mu = g^{\mu\nu}\partial_\nu, \quad \partial^A = g^{AB}B.$$

- The Greek indices run from 0 to 3 while the Latin indices run from 1 to 3.

- At any fixed point (t, x) ($|x| > 0$), we may choose the orthonormal basis S_1, S_2 such that

$$[L, S_i] = LS_i - S_iL = -\frac{1}{r}S_i, \quad [\underline{L}, S_i] = \frac{1}{r}S, \quad [S_1, S_2]|_{(t,x)} = 0, \quad i = 1, 2.$$

  This helps to compute those geometric quantities which are independent of the choice of the local coordinates.

- For smooth function ϕ, $\bar{\partial}\phi = (\partial\phi, \frac{\phi}{1+|x|})$.

- $r_+ = 1 + r$, $\tau_+ = 1 + \tau$. Here $r = |x|$, τ is the parameter for the foliation Σ_τ.

- For real numbers α, p, β, we denote

$$
\begin{aligned}
&I^{\alpha}[\phi]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}}dxd\tau, \quad S^{\alpha}[\phi](\tau) := \int_{S_\tau} \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} r^2 dvd\omega,\\
&D^{\alpha}[F]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} (1+r)^{1+\alpha}|F|^2 dxd\tau, \quad g^p[\phi](\tau) := \int_{S_\tau} r^p|\partial_v\psi|^2 dvd\omega,\\
&G^{p,\beta}[\phi]_{\tau_2}^{\tau_2} := \int_{\tau_1}^{\tau_2} \tau_+^{-\beta}\int_{S_\tau} r^p|\partial_v\psi|^2 dvd\omega d\tau, \quad E^{\beta}[\phi]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} \tau_+^{-\beta} E[\phi](\tau)d\tau\\
&D_+^{\alpha}[F]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2}\int_{S_\tau} (1+r)^{1+\alpha}|F|^2 dxd\tau, \quad \bar{g}^p[\phi](\tau) := \int_{S_\tau} r^p|\overline{\partial_v}\psi|^2 dvd\omega,\\
&\bar{G}^{p,\beta}[\phi]_{\tau_2}^{\tau_2} := \int_{\tau_1}^{\tau_2} \tau_+^{-\beta}\int_{S_\tau} r^p|\overline{\partial_v}\psi|^2 dvd\omega d\tau,
\end{aligned}
$$

  where $\psi = r\phi$. Similarly, we have the notation for $\tilde{E}^{\beta}[\phi]_{\tau_1}^{\tau_2}$. We remark here that this notation is different from $E^N[\phi]_{\tau_1}^{\tau_2}$ which is the energy flux through the null infinity.

2.4 Several useful lemmas

We present here several lemmas which will be used frequently. These lemmas are easy to prove and the proofs could be found in other sources. To to self contained, we outline the proofs here.

Lemma 2.4.1. *Let ϕ be a smooth function on $\mathbb{R}^{3+1}$. Assume*

$$\lim_{v\to\infty} \phi(u_0, v, \omega) = 0, \quad u_0 = -\frac{1}{2}R.$$

Then in the polar coordinates (t, r, ω), we have

$$r\int_{\omega} |\phi(r+\tau-R, r, \omega)|^2 d\omega \leq 4\tilde{E}[\phi](\tau).$$

Moreover, if $E^N[\phi]_0^\tau < \infty$*, then*

$$r\int_\omega |\phi(r+\tau-R,r,\omega)|^2 d\omega \le 4E[\phi](\tau).$$

Proof. It suffices to prove the lemma when $E^N[\phi]_0^\tau$ is finite. On the incoming null hypersurface $\bar{C}(0,\tau,v)$, since $\lim_{v\to\infty}\phi(u_0,v,\omega)=0$, we can estimate

$$\int_\omega |\phi(u_\tau,v,\omega)|^2 d\omega \le \int_\omega \left(\int_{-\frac{R}{2}}^{u_\tau} |\partial_u\phi| du\right)^2 d\omega \le (\frac{1}{r}-\frac{1}{r+\tau})\int_{\bar{C}(0,\tau,v)} (\partial_u\phi)^2 r^2 du d\omega,$$

where $r=v-u_\tau$, $u_\tau=\frac{\tau-R}{2}$ and R is the radius for the foliation Σ_τ. Fix τ and let $v\to\infty$. When $\tilde{E}[\phi](\tau)$ is finite, in particular $E^N[\phi]_0^\tau$ is finite, we can conclude that

$$\liminf_{v\to\infty}\int_\omega |\phi(u_\tau,v,\omega)|^2 d\omega \le \lim_{v\to\infty}(\frac{1}{r}-\frac{1}{r+\tau})E^N[\phi]_0^\tau = 0, \quad \tau\ge 0.$$

Now, for the point (u_τ,v,ω) on S_τ, i.e., $v\ge\frac{\tau+R}{2}$, for any $v_1>v$, we have

$$\int_\omega |\phi(u_\tau,v,\omega)|^2 d\omega \le 2\int_\omega |\phi(u_\tau,v_1,\omega)|^2 d\omega + \frac{2}{r}\int_{S_\tau} |\partial_v\phi|^2 r^2 dv d\omega, \quad r=v-u_\tau.$$

Take the infimum as $v_1\to\infty$, we obtain that

$$\int_\omega |\phi(u_\tau,v,\omega)|^2 d\omega \le \frac{2}{r}\int_{S_\tau} |\partial_v\phi|^2 r^2 dv d\omega \le \frac{2}{r}E[\phi](\tau).$$

Finally when $r\le R$ (for this case, $t=\tau$), we have

$$\int_\omega |\phi(t,r,\omega)|^2 d\omega \le 2\int_\omega |\phi(t,R,\omega)|^2 d\omega + 2\int_\omega \left(\int_r^R |\partial_r\phi(t,s,\omega)| ds\right)^2 d\omega$$
$$\le \frac{4}{R}E[\phi](t) + 2(\frac{1}{r}-\frac{1}{R})\int_\omega\int_r^R |\partial_r\phi|^2 s^2 ds d\omega \le \frac{4}{r}E[\phi](\tau).$$

□

We also need the following analogue of Hardy's inequality to control ϕ by the energy. We borrow the proof from [9].

Lemma 2.4.2. *Let ϕ satisfy the same conditions as in the previous lemma. Then*

$$\int_{\{r\leq R\}\cap\Sigma_\tau}\left(\frac{\phi}{1+r}\right)^2dx+\int_{S_\tau}\left(\frac{\phi}{1+r}\right)^2r^2dvd\omega\leq 12\tilde{E}[\phi](\tau).$$

In particular

$$\int_{r\leq R}\phi^2dx\leq 12(1+R)^2\tilde{E}[\phi](\tau).$$

Here we simply use $r\leq R$ to denote the integral region $\{r\leq R\}\cap\Sigma_\tau$.

Proof. Take a function η as follows

$$\eta(r)=r-2\ln(1+r)+\frac{r}{1+r}.$$

Then

$$\eta'(r)=\frac{r^2}{(1+r)^2},\quad \eta(0)=0,\quad |\eta(r)|\leq r.$$

Denote $d\sigma$ as dx when $r\leq R$ and $r^2dvd\omega$ when $r\geq R$. By Lemma 2.4.1, we can show that

$$\begin{aligned}\int_{\Sigma_\tau}\left(\frac{\phi}{1+r}\right)^2d\sigma&=\int_\omega\int_0^R\phi^2d\eta d\omega+\int_\omega\int_{v_\tau}^\infty\phi^2d\eta d\omega\\&=\int_\omega\phi^2\eta d\omega\Big|_0^\infty-2\int_{r\leq R}\eta\phi\cdot\phi_rdrd\omega-2\int_{S_\tau}\eta\phi\cdot\phi_vdvd\omega\\&\leq 4\tilde{E}[\phi](\tau)+\frac{1}{2}\int_{\Sigma_\tau}\eta^2r^{-4}\phi^2d\sigma+2\int_{r\leq R}\phi_r^2dx+2\int_{S_\tau}\phi_v^2d\sigma\\&\leq 6\tilde{E}[\phi](\tau)+\frac{1}{2}\int_{\Sigma_\tau}\eta^2r^{-4}\phi^2d\sigma,\end{aligned}\tag{2.1}$$

where $r=v-u$ and on S_τ, u is constant.

Notice that η is nonnegative and $\ln(1+r) \geq \frac{r}{1+r}$. We conclude that

$$\frac{\eta}{r} = \frac{r}{1+r} - 2\left(\frac{\ln(1+r)}{r} - \frac{1}{1+r}\right) \leq \frac{r}{1+r}.$$

Therefore we have

$$\int_{\Sigma_\tau} \eta^2 r^{-4} \phi^2 d\sigma \leq \int_{\Sigma_\tau} \left(\frac{\phi}{1+r}\right)^2 d\sigma.$$

Then the lemma follows from the estimate (2.1). □

For solutions of linear wave equations, the good derivative ∂_v of the solution decays better. In that case, we have

Lemma 2.4.3. *Let $\alpha_1, \alpha_2 > 0$. Assume ϕ satisfies the condition in Lemma 2.4.1. Then we have*

$$\int_{S_\tau} r^{1-\alpha_1} \phi^2 dv d\omega \leq C_0 R^{1-\alpha_1} \tilde{E}[\phi](\tau) + C_0 \int_{S_\tau} r^{1+\alpha_2} |\partial_v (r\phi)|^2 dv d\omega,$$

where C_0 is a constant depending only on α_1, α_2.

Proof. Let $\psi = r\phi$. By Lemma 2.4.1, we have

$$\begin{aligned}
\int_\omega |\psi|^2(\tau, v, \omega) d\omega &\leq C_0 \int_\omega |\psi|^2(\tau, v_\tau, \omega) d\omega + C_0 \left(\int_{v_\tau}^v \int_\omega |\partial_v \psi| d\omega dv\right)^2 \\
&\leq C_0 R \tilde{E}[\phi](\tau) + C_0 \int_{v_\tau}^v \int_\omega r^{1+\alpha_2} |\partial_v \psi|^2 d\omega dv \int_{v_\tau}^v r^{-1-\alpha_2} dv \\
&\leq C_0 R \tilde{E}[\phi](\tau) + C_0 \frac{R^{-\alpha_2} - r^{-\alpha_2}}{\alpha_2} \int_{S_\tau} r^{1+\alpha_2} |\partial_v \psi|^2 d\omega dv
\end{aligned}$$

Multiply the above inequality by $r^{-1-\alpha_1}$ and then integrate from $v_\tau = \frac{\tau+R}{2}$ to infinity.

We obtain

$$\begin{aligned}\int_{S_\tau} r^{1-\alpha_1}\phi^2 dv d\omega &= \int_{v_\tau}^{\infty} r^{-1-\alpha_1}\int_\omega |\psi|^2 d\omega dv\\ &\le C_0 R^{1-\alpha_1}\tilde{E}[\phi](\tau) + C_0\frac{R^{-\alpha_1-\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)}\int_{S_\tau} r^{1+\alpha_2}|\partial_v\psi|^2 d\omega dv\\ &\le C_0 R^{1-\alpha_1}\tilde{E}[\phi](\tau) + C_0\int_{S_\tau} r^{1+\alpha_2}|\partial_v\psi|^2 d\omega dv.\end{aligned}$$

□

We will also frequently use the following simple lemma.

Lemma 2.4.4. *Suppose $f(\tau)$ is smooth. Then we have the identity*

$$\int_{\tau_1}^{\tau_2}(1+s)^\beta f(s)ds = \beta\int_{\tau_1}^{\tau_2}(1+\tau)^{\beta-1}\int_\tau^{\tau_2} f(s)dsd\tau + (1+\tau_1)^\beta\int_{\tau_1}^{\tau_2} f(s)ds$$

for $\forall\beta\in\mathbb{R}$.

Proof. Let

$$F(\tau) = \int_\tau^{\tau_2} f(s)ds.$$

Integration by parts gives the lemma. □

Finally, we state the well known Gronwall's inequality.

Lemma 2.4.5 (Gronwall's Inequality)**.** *Suppose $A(\tau)$, $E(\tau)$ are nonnegative functions on $[\tau_1,\tau_2]$. Assume that $E(\tau)$ is nondecreasing on this interval and β is a positive number. If*

$$A(\tau) \le E[\tau] + C\int_{\tau_1}^{\tau}(1+s)^{-1-\beta}A(s)ds, \qquad \forall\tau\in[\tau_1,\tau_2],$$

then

$$A(\tau) \le \exp\left(C\beta^{-1}(1+\tau_1)^{-\beta}\right)E(\tau), \qquad \forall\tau\in[\tau_1,\tau_2].$$

Proof. See [39]. □

2.5 Energy method

We review the energy method for wave equations. Recall the energy-momentum tensor of the scalar field ϕ on the Lorentzian space $(\mathbb{R}^{3+1}, g)$ with metric g

$$\mathbb{T}_{\mu\nu}[\phi] = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\gamma\phi\partial_\gamma\phi.$$

We raise and lower indices of any tensor relative to the given metric g, e.g., $\partial^\gamma = g^{\gamma\mu}\partial_\mu$. Given a vector field X, we define the currents

$$J_\mu^X[\phi] = \mathbb{T}_{\mu\nu}[\phi]X^\nu, \qquad K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi]\pi_{\mu\nu}^X,$$

where $\pi_{\mu\nu}^X = \frac{1}{2}\mathcal{L}_X g_{\mu\nu}$ is the deformation tensor of the vector field X. We denote $J^X[\phi]$ as the vector field

$$J^X[\phi] = J_\mu^X[\phi]g^{\mu\nu}\partial_\nu.$$

Recall that

$$D^\mu J_\mu^X[\phi] = X(\phi)\Box_g\phi + K^X[\phi],$$

where $\Box_g$ is the covariant wave operator and D is the covariant derivative of the Lorentz metric g. Take any function χ. We have the following identity

$$D^\mu\left(-\frac{1}{2}\partial_\mu\chi\cdot\phi^2 + \frac{1}{2}\chi\partial_\mu\phi^2\right) = \chi\phi\Box_g\phi + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box_g\chi\cdot\phi^2.$$

Modify the vector field $J^X[\phi]$ to be

$$\tilde{J}^X[\phi] = \tilde{J}_\mu^X[\phi]g^{\mu\nu}\partial_\nu = \left(J_\mu^X[\phi] - \frac{1}{2}\partial_\mu\chi\cdot\phi^2 + \frac{1}{2}\chi\partial_\mu\phi^2\right)g^{\mu\nu}\partial_\nu. \tag{2.2}$$

We then have the identity

$$D^\mu \tilde{J}^X_\mu[\phi] = (X(\phi) + \chi\phi)\Box_g\phi + K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box_g\chi \cdot \phi^2.$$

For any bounded region $\mathcal{D}$ in $\mathbb{R}^{3+1}$, using Stoke's formula, we have the following energy identity

$$\begin{aligned}\iint_{\mathcal{D}} D^\mu \tilde{J}^X_\mu[\phi] d\text{vol} &= \iint_{\mathcal{D}} \Box_g\phi(\chi\phi + X(\phi)) + K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box_g\chi \cdot \phi^2 d\text{vol} \\ &= \int_{\partial\mathcal{D}} i_{\tilde{J}^X[\phi]} d\text{vol},\end{aligned} \tag{2.3}$$

$\partial\mathcal{D}$ denotes the boundary of the domain $\mathcal{D}$ and $i_Y d\text{vol}$ denotes the contraction of the volume form $d\text{vol}$ with the vector field Y which gives the surface measure of the boundary. For example, for any basis $\{e_1, e_2, \ldots, e_n\}$, we have $i_{e_1}(de_1 \wedge de_2 \wedge \ldots de_k) = de_2 \wedge de_3 \wedge \ldots \wedge de_k$. Here we have chosen t to be the time orientation. For more details on this formula, we refer to the appendix of [5].

Throughout this thesis, the domain $\mathcal{D}$ will be regular regions bounded by the t-constant slices, the outgoing null hypersurfaces S_τ or the incoming null hypersurfaces $\bar{C}(\tau_1, \tau_2, v)$. We now compute $i_{\tilde{J}^X[\phi]} d\text{vol}$ on these three kinds of hypersurfaces.

On the t-constant slices, the surface measure is a function times dx. Recall the volume form

$$d\text{vol} = \sqrt{-G}dxdt = -\sqrt{-G}dtdx.$$

Here note that dx is a 3-form. We thus can compute

$$\begin{aligned} i_{\tilde{J}^X[\phi]} d\text{vol} &= i_{(\tilde{J}^X[\phi])^\mu\partial_\mu} d\text{vol} = -(\tilde{J}^X[\phi])^0\sqrt{-G}dx \\ &= -(\partial^t\phi X(\phi) - \frac{1}{2}X^0\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\partial^t\chi \cdot \phi^2 + \chi\partial^t\phi \cdot \phi)\sqrt{-G}dx.\end{aligned} \tag{2.4}$$

On the outgoing null hypersurface S_τ, we can write the volume form

$$d\text{vol} = \sqrt{-G}dxdt = \sqrt{-G}r^2 drdtd\omega = 2\sqrt{-G}r^2 dvdud\omega = -2\sqrt{-G}dudvd\omega.$$

Here $u = \frac{t-r}{2}$, $v = \frac{t+r}{2}$ are the null coordinates. Notice that $\underline{L} = \partial_u$. We can compute

$$i_{\tilde{J}^X[\phi]}d\text{vol} = -2\sqrt{-G}r^2(\partial^{\underline{L}}\phi X(\phi) - \frac{1}{2}X^{\underline{L}}\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\partial^{\underline{L}}\chi\phi^2 + \chi\partial^{\underline{L}}\phi \cdot \phi)dvd\omega. \quad (2.5)$$

Similarly, on the incoming null hypersurfaces $\{v = constant\}$, we have

$$i_{\tilde{J}^X[\phi]}d\text{vol} = 2\sqrt{-G}r^2(\partial^L\phi X(\phi) - \frac{1}{2}X^L\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\partial^L\chi\phi^2 + \chi\partial^L\phi \cdot \phi)dud\omega. \quad (2.6)$$

We remark here that the above three formulae hold for any vector field X and any function χ.

Chapter 3

Dafermos-Rodnianski's new approach

The long time behavior of solutions of nonlinear wave equations relies on the decay properties of solutions of linear wave equations. In the early works, e.g., [16], [23], [36], ones used the classical method (through the fundamental solution or using Fourier analysis) to obtain the decay of solutions of linear wave equations. The robust vector field method introduced by Klainerman [21] which has also been briefly mentioned in the introduction connects the invariant vector fields (1.1) of Minkowski space with the decay properties of solutions of linear wave equations. All the previous applications of the vector field method used at least one of the vector fields $S = t\partial_t + r\partial_r$, $K = (t^2 + r^2)\partial_t + 2tr\partial_r$ growing in time t, which is responsible for the requirement that the background metric settles to a stationary one suitably.

The method of Dafermos-Rodnianski we would like to introduce is a new approach to prove the decay properties of solutions of linear wave equations without using any vector fields growing in time t, in particular the scaling vector field S. This new method has a wide range of applications, not only to the study of nonlinear wave equations on perturbations of Minkowski spaces here but can also to be used to investigate the linear and nonlinear wave equations on black hole spacetimes [11], [12], [35].

This new method shares the following three basic estimates: an integrated local energy estimate, uniform boundedness of the energy $E[\phi](\tau)$ which is usually obtained through the classical energy estimates adapted to the foliations Σ_τ and a hierarchy of r-weighted energy inequalities (*p-weighted energy inequalities*) in a neighborhood of the null infinity. The first two ingredients have been studied extensively in various cases. The key ingredient— the third one can be obtained by using the vector field $r^p\partial_v$, $0 \leq p \leq 2$ as multipliers. In the following, we derive the first two estimates for solutions of linear wave equations on $(\mathbb{R}^{3+1}, g)$ adapted to the foliations Σ_τ. Next, we introduce the p-weighted energy inequalities on Minkowski spaces. In the subsequent chapters, we apply this new method to several nonlinear problems on time dependent inhomogeneous backgrounds.

3.1 The integrated energy estimates and the energy estimates

We derive the integrated local energy estimates and the classical energy estimates simultaneously. The integrated energy estimates have been well studied and was first proven by C. Morawetz [34]. We will follow the method developed in [9] which uses the vector field $f\partial_r$ as multipliers. The energy estimates can be derived by using ∂_t as multipliers.

In this section, we prove a general integrated energy inequality for solutions of the linear wave equations

$$\Box_g\phi + N(\phi) = F \tag{3.1}$$

on the Lorentzian manifold $(\mathbb{R}^{3+1}, g)$. Here $N(\phi) = N^\mu\partial_\mu\phi$ is a linear term and N is a vector field on $\mathbb{R}^{3+1}$ with components N^μ.

Fix a large constant $R > 8$ so that we can determine the foliation Σ_τ with radius R. Recall that $g = h + m_0$. In addition suppose $h^{\mu\nu}$, N^μ satisfy the following weak decay estimates

$$
\begin{aligned}
&|h^{\mu\nu}| + |\partial h^{\mu\nu}| + |N^\mu| \le \delta_0 r_+^{-1-2\alpha}, \quad r = |x| \le R,\\
&|\partial h^{\mu\nu}| + |h^{\mu\nu}| + |N^\mu| \le \delta_1 (r_+^{-\frac{1}{2}-2\alpha}\tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha} + r_+^{-1-2\alpha}), \quad (t,x) \in S_\tau,\\
&|\overline{\partial_v} h^{\mu\nu}| + |\partial h^{\underline{LL}}| + |h^{\underline{LL}}| + |N^{\underline{L}}| \le \delta_1 r_+^{-1-2\alpha}, \quad (t,x) \in S_\tau,
\end{aligned}
\tag{3.2}
$$

where δ_0, δ_1, α are positive constants and $\alpha < \frac{1}{10}$. Here we recall that $r_+ = 1 + r$, $\tau_+ = 1 + \tau$; $h^{\underline{LL}}$ is the component of h with respect to the null frame $\{L, \underline{L}, S_1, S_2\}$.

We will assume δ_1 is sufficiently small. For the metric perturbation inside the cylinder with radius R, we can either assume the perturbation is small, i.e., δ_0 is small or alternatively, for more general metric g which can be large in $C^1(\{r \le R\})$, we assume the integrated local energy estimates. More precisely, we may assume the metric g satisfies

$$
\int_{\tau_1}^{\tau_2}\int_{r\le R} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \le C(\tilde{E}[\phi](\tau) + \delta_1(S^\alpha[\phi](\tau_1) + S^\alpha[\phi](\tau_2)) + D^\alpha[\Box_g \phi]_{\tau_1}^{\tau_2}) \tag{3.3}
$$

for all smooth function ϕ and some constant C. We consider these two cases together.

Proposition 3.1.1. *Assume that the given metric g satisfy the above estimates* (3.2) *for some positive constant α, δ_0, δ_1. Let ϕ be a smooth solution of the linear wave equation* (3.1) *and satisfy the conditions in Lemma 2.4.1. If δ_0, δ_1 is sufficiently small depending only on α, or if δ_1 is small depending on α, δ_0 and in addition the metric g satisfies the integrated local energy estimate* (3.3) *and $N = 0$, then for $\forall \tau_1 \le \tau_2$ we have the boundedness of the integrated energy*

$$
I^\alpha[\phi]_0^\infty \lesssim \tilde{E}[\phi](0) + \delta_1 S^\alpha[\phi](0) + D^\alpha[F]_0^\infty. \tag{3.4}
$$

If in addition we have

$$I^\alpha[\phi]_0^\infty = \int_0^\infty \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau < \infty,$$

then

(1) Integrated energy estimate

$$I^\alpha[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\nabla\!\!\!/ \phi|^2}{1+r} dx d\tau \le C_{\alpha,\delta_0}(\tilde{E}[\phi](\tau_1) + \delta_1 S^\alpha[\phi](\tau_i) + D^\alpha[F]_{\tau_1}^{\tau_2}); \quad (3.5)$$

(2) Energy bound

$$\tilde{E}[\phi](\tau_2) + E^N[\phi]_{\tau_1}^{\tau_2} \le C_{\alpha,\delta_0}(\tilde{E}[\phi](\tau_1) + \delta_1 S^\alpha[\phi](\tau_i) + D^\alpha[F]_{\tau_1}^{\tau_2}), \quad (3.6)$$

where $S^\alpha[\phi](\tau_i) = S^\alpha[\phi](\tau_1) + S^\alpha[\phi](\tau_2)$. *The constant* C_{α,δ_0} *depends only on* α, δ_0. *The definitions for* $I^\alpha[\phi]_{\tau_1}^{\tau_2}$, $S^\alpha[\phi](\tau)$, $D^\alpha[F]_{\tau_1}^{\tau_2}$ *can be found in Sections 2.2 and 2.3.*

To prove (3.5) and (3.6), we need a priori asymptotical estimate for the solution, i.e., in this proposition, we assume the integrated energy $I^\alpha[\phi]_0^\infty$ is finite. The inequality (3.4) will be used to verify this condition with appropriate initial condition and some boundedness of the inhomogeneous term F.

Remark 3.1.2. *We mention here that variants and generalizations of estimate* (3.5) *can also be found in [32], [33] and reference therein. However, the conditions on the given metric g here is more general and the foliations used here are different.*

We use the vector field method to prove the above proposition. More precisely, we construct vector fields $X = f\partial_r$, f is a function of r. Using the energy identities (2.3) applied to the region bounded by Σ_{τ_1}, Σ_{τ_2}, we are able to derive the integrated energy estimates as well as the energy estimates.

3.1.1 The vector field $f\partial_r$

Let $v > \frac{\tau_1+R}{2}$. Consider the region $\mathcal{D}$ bounded by Σ_{τ_1}, Σ_{τ_2} and the incoming null hypersurface $\bar{C}(\tau_1, \tau_2, v)$. Let

$$\Sigma_\tau^v := \Sigma_\tau \cap \{t + r \le 2v\}.$$

Take the vector field X as follows

$$X = f\partial_r$$

for some function f of r such that $f(0) = 0$. Hence X is a well defined vector filed on $\mathbb{R}^{3+1}$. Thus in the energy inequality (2.3), we can compute the current $K^X[\phi]$

$$\begin{aligned} K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi]\pi^X_{\mu\nu} = \partial_j(f\frac{x_i}{r})\partial^j\phi\cdot\partial_i\phi - (\frac{1}{2}f' + r^{-1}f)\partial^\gamma\phi\partial_\gamma\phi \\ - \frac{1}{2}f\partial_r g^{\mu\nu}\cdot\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}f\partial_r g^{\mu\nu}\cdot g_{\mu\nu}\partial^\gamma\phi\partial_\gamma\phi, \end{aligned}$$

where we denote f' as $\partial_r f$.

Next we choose the function χ in the modified vector field (2.2) to be

$$\chi = r^{-1}f.$$

Then from the energy identity (2.3), we obtain

$$\begin{aligned} &\int_{\Sigma^v_{\tau_1}} i_{\tilde{J}^X[\phi]}d\text{vol} - \int_{\Sigma^v_{\tau_2}} i_{\tilde{J}^X[\phi]}d\text{vol} + \int_{\bar{C}(\tau_1,\tau_2,v)} i_{\tilde{J}^X[\phi]}d\text{vol} \\ &= \int_{\tau_1}^{\tau_2}\int_{\Sigma^v_\tau} \Box_g\phi(X(\phi) + \phi\chi) + \frac{1}{2}f'(|\partial_r\phi|^2 + |\partial_t\phi|^2) \\ &\qquad + (\chi - \frac{1}{2}f')|\nabla\!\!\!/\phi|^2 - \frac{1}{2}\Box_g\chi\cdot\phi^2 + E(X)d\text{vol}, \end{aligned} \tag{3.7}$$

where the error term $E(X)$ is given as follows

$$\begin{aligned} E(X) =&\partial_j(f\frac{x_i}{r})h^{j\mu}\partial_\mu\phi\cdot\partial_i\phi-\frac{1}{2}f'h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &-\frac{1}{2}f\partial_r g^{\mu\nu}\cdot\partial_\mu\phi\partial_\nu\phi+\frac{1}{4}f\partial_r g^{\mu\nu}\cdot g_{\mu\nu}\partial^\gamma\phi\partial_\gamma\phi. \end{aligned} \tag{3.8}$$

We now explicitly construct the function f as follows

$$f=2\alpha^{-1}-\frac{2\alpha^{-1}}{(1+r)^\alpha},\quad \chi=r^{-1}f.$$

We may use the following calculations

$$\begin{aligned} &\frac{1}{2}f'=\frac{1}{(1+r)^{1+\alpha}},\quad \chi'=-2\alpha^{-1}\frac{(1+r)^{1+\alpha}-(1+\alpha)r-1}{r^2(1+r)^{1+\alpha}}, \\ &\chi''=4\alpha^{-1}r^{-3}-\frac{4\alpha^{-1}}{r^3(1+r)^\alpha}-\frac{4}{r^2(1+r)^{1+\alpha}}-\frac{2(1+\alpha)}{r(1+r)^{2+\alpha}}, \\ &\partial_{ij}\chi=-\frac{2(1+\alpha)}{r(1+r)^{2+\alpha}}\cdot\frac{x_ix_j}{r^2}+\chi'\left(\frac{\delta_{ij}}{r}-\frac{3x_ix_j}{r^3}\right), \end{aligned}$$

where χ'' denotes the second order derivative of χ as a function of r and ∂_{ij} is the second order derivative on $\mathbb{R}^3$, i.e., $\partial_{ij}=\partial_i\partial_j$. Note that

$$\frac{\alpha}{1+r}\le\frac{(1+r)^\alpha-1}{r}\le(1+r)^{\alpha-1}.$$

We can show that

$$\begin{aligned} &\chi=\frac{2\alpha^{-1}}{1+r}\frac{1+r}{r}\frac{(1+r)^\alpha-1}{(1+r)^\alpha}\le\frac{2\alpha^{-1}}{1+r},\quad -\Delta\chi=-\chi''-\frac{2}{r}\chi'=\frac{2(1+\alpha)}{r(1+r)^{2+\alpha}}, \\ &\chi-r^{-1}f+\frac{1}{2}f'=r^{-1}f+\frac{1}{2}f'-\chi=\frac{1}{(1+r)^{\alpha+1}},\quad |\chi'|\le\frac{2\alpha^{-1}}{(1+r)^2}, \\ &\chi-\frac{1}{2}f'=\frac{2\alpha^{-1}\left((1+r)^\alpha-1\right)}{r(1+r)^\alpha}-\frac{1}{(1+r)^{1+\alpha}}\ge\frac{1}{(1+r)^{1+\alpha}}, \end{aligned}$$

where Δ is the standard Laplacian operator in $\mathbb{R}^3$. In particular, when $r\ge R>8$, we have the following improved estimate for $\chi - \frac{1}{2}f'$

$$\chi - \frac{1}{2}f' \geq \frac{\beta}{r} - \frac{1+\beta}{r(1+r)^{\alpha}} \geq \frac{1}{r}, \quad r \geq R. \tag{3.9}$$

This improved estimate will be used to show the improved integrated energy estimate (3.5) for the angular derivative of the solution.

In this section, we make the convention that $A \lesssim B$ means $A \leq CB$ for some constant C depending only on α, δ_0. We can estimate

$$|(\Box_g - \Delta)\chi| = |h^{ij}\partial_{ij}\chi + (\partial_\mu g^{\mu i} + \frac{1}{2}g^{\mu i}\partial_\mu g_{\nu\gamma} \cdot g^{\nu\gamma})\partial_i \chi| \lesssim \frac{|h| + |\partial h|}{r(1+r)}.$$

From the assumption (3.2), we have

$$|(\Box_g - \Delta)\chi| \lesssim \begin{cases} \frac{\delta_0}{r(1+r)^{2+\alpha}}, & r \leq R; \\ \frac{\delta_1 r_+^{-\frac{1}{2}-\alpha}\tau_+^{-\frac{1}{2}-\alpha}}{(1+r)^2} \leq \frac{\delta_1}{(1+r)^2}(r_+^{-1-\alpha} + \tau_+^{-1-\alpha}), & (t,x) \in S_\tau. \end{cases}$$

Using Lemma 2.4.2 to control the integral of $\frac{\phi^2}{(1+r)^2}$, we can show that

$$\begin{aligned} \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau^v} |(\Box_g - \Delta)\chi|\phi^2 d\mathrm{vol} &\lesssim \delta_0 \int_{\tau_1}^{\tau_2}\int_{r\leq R} \frac{|\phi|^2}{r(1+r)^{2+\alpha}} dx d\tau \\ &+ \delta_1 \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} dx d\tau + \delta_1 \int_{\tau_1}^{\tau_2} \tau_+^{-1-\alpha}\tilde{E}[\phi](\tau) d\tau. \end{aligned}$$

Recall that $-\Delta\chi = \frac{2(1+\alpha)}{r(1+r)^{2+\alpha}}$. Then from the above energy inequality (3.7), we obtain

$$\begin{aligned} \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau^v} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} + E(X) dx d\tau &\lesssim |\int_{\Sigma_{\tau_1}^v} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{\bar{C}(0,\tau_1,v)} i_{\tilde{J}^X[\phi]} d\mathrm{vol}| \\ &+ |\int_{\Sigma_{\tau_2}^v} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{\bar{C}(0,\tau_2,v)} i_{\tilde{J}^X[\phi]} d\mathrm{vol}| + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau^v} |\Box_g\phi||X(\phi) + \chi\phi| dx d\tau \\ &+ \delta_1 \int_{\tau_1}^{\tau_2} \tau_+^{-1-\alpha}\tilde{E}[\phi](\tau) d\tau + \int_{\tau_1}^{\tau_2}\int_{r\leq R} \frac{\delta_0|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{\delta_1|\partial\phi|^2}{(1+r)^{1+\alpha}} dx d\tau. \end{aligned} \tag{3.10}$$

After taking the limit $v \to \infty$, for sufficiently small δ_1 the last term in the last line will be absorbed. The second last term will be estimates either by the assumption that δ_0 is small or by the condition that the integrated local energy estimate (3.3) holds. We will later show that integral of the error term $E(X)$ can be absorbed. We first demonstrate that the integral on the boundary can be bounded by the energy $E[\phi](\tau)$.

Lemma 3.1.3. *We have*

$$\liminf_{v\to\infty} \left| \int_{\Sigma_\tau^v} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{\bar{C}(0,\tau,v)} i_{\tilde{J}^X[\phi]} d\mathrm{vol} \right| \lesssim \tilde{E}[\phi](\tau) + \delta_1 S^\alpha[\phi](\tau).$$

Proof. The boundary $\Sigma_\tau^v \cup \bar{C}(0,\tau,v)$ consists of three parts: the spacelike t-constant slice $\{r \le R\}$, the outgoing null hypersurface S_τ and the incoming null hypersurface $\bar{C}(0,\tau,v)$. On the t-constant slice restricted to the region $\{r \le R\}$, we use the formula (2.4). Recall that

$$|\chi| \lesssim \frac{1}{1+r}, \quad |f| \lesssim 1, \quad |\chi'| \lesssim \frac{1}{(1+r)^2}.$$

We can show that

$$\left| \int_{\Sigma_\tau \cap \{r \le R\}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} \right| \lesssim \int_{\Sigma_\tau \cap \{r \le R\}} |\bar{\partial}\phi|^2 dx \lesssim \tilde{E}[\phi](\tau).$$

On S_τ, using the formula (2.5), we have

$$i_{\tilde{J}^X[\phi]} d\mathrm{vol} = -2\sqrt{-G} r^2 (\partial^{\underline{L}}\phi X(\phi) - \frac{1}{2} X^{\underline{L}} \partial^\gamma \phi \partial_\gamma \phi - \frac{1}{2} \partial^{\underline{L}} \chi \cdot \phi^2 + \chi \partial^{\underline{L}} \phi \cdot \phi) dv d\omega.$$

We first estimate the last two terms in the above expression. Recall that

$$|\chi| \lesssim (1+r)^{-1}, \quad |\partial^{\underline{L}}\chi| \lesssim (1+r)^{-2}$$

and note that

$$|\partial^{\underline{L}}\phi| \leq |h||\partial\phi| + |\overline{\partial_v}\phi|,$$

where $\overline{\partial_v} = (L, S_1, S_2)$. By the assumption (3.2), we can bound

$$|-\frac{1}{2}\partial^{\underline{L}}\chi\cdot\phi^2 + \chi\partial^{\underline{L}}\phi\cdot\phi| \lesssim \frac{\phi^2}{(1+r)^2} + |\overline{\partial_v}\phi|^2 + \delta_1\frac{|\partial\phi|^2}{(1+r)^{1+\alpha}}.$$

The integral of the first two terms on the right hand side of the above inequality can be bounded by the energy flux through S_τ by using Lemma 2.4.2. The integral of the third term, by the definition, is exactly $S^\alpha[\phi](\tau)$, which will be absorbed for sufficiently small δ_1.

Now to estimate $\int_{S_\tau\cap\{t+r\leq 2v\}} i_{\tilde{J}^X[\phi]}d\text{vol}$, it remains to estimate the integral of the first two terms on the right hand side of the expression for $i_{\tilde{J}^X[\phi]}d\text{vol}$. Recall that $X = f\partial_r$, $\partial_r = \frac{1}{2}(L-\underline{L})$, $|f|\lesssim 1$. In particular, we have $X = \frac{1}{2}f(L-\underline{L})$, $X^{\underline{L}} = -\frac{1}{2}f$. Hence we have

$$\begin{aligned}|\partial^{\underline{L}}\phi X(\phi) - \frac{1}{2}X^{\underline{L}}\partial^\gamma\phi\partial_\gamma\phi| &\lesssim |\partial^{\underline{L}}\phi L(\phi)| + |\frac{1}{2}\partial^\gamma\phi\partial_\gamma\phi - \partial^{\underline{L}}\phi\underline{L}(\phi)| \\ &\lesssim |\partial^{\underline{L}}\phi L(\phi)| + \frac{1}{2}|g^{\bar{A}\bar{B}}\bar{A}(\phi)\bar{B}(\phi) - g^{\underline{L}\underline{L}}\underline{L}(\phi)\underline{L}(\phi)| \\ &\lesssim \delta_1(1+r)^{-1-\alpha}|\partial\phi|^2 + |\overline{\partial_v}\phi|^2,\end{aligned}$$

where $\bar{A}, \bar{B} \in \{L, S_1, S_2\}$. Therefore we can estimate

$$\left|\int_{S_\tau\cap\{t+r\leq 2v\}} i_{\tilde{J}^X[\phi]}d\text{vol}\right| \lesssim \tilde{E}[\phi](\tau) + \int_{S_\tau}\frac{\delta_1|\partial\phi|^2}{(1+r)^{1+\alpha}}r^2 dv d\omega \lesssim \tilde{E}[\phi](\tau) + \delta_1 S^\alpha[\phi](\tau).$$

Finally, we estimate the integral on the incoming null hypersurface $\bar{C}(0,\tau,v)$. Since the metric is asymptotically flat, using the formula (2.6), we can split the expression of $i_{\tilde{J}^X[\phi]}d\text{vol}$ according to the decomposition of the metric $g = h + m_0$

$$\begin{aligned}
&\partial^L\phi X(\phi) - \frac{1}{2}X^L\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\partial^L\chi\phi^2 + \chi\partial^L\phi\cdot\phi \\
&= h^{LA}A(\phi)X(\phi) - \frac{1}{2}X^L h^{\mu\nu}\partial_\nu\phi\partial_\mu\phi - \frac{1}{2}h^{LA}A(\chi)\phi^2 + \chi h^{LA}A(\phi)\phi \qquad (3.11)\\
&\quad + \frac{f}{4}(|\partial_u\phi|^2 - |\nabla\!\!\!\!/\phi|^2) - \frac{1}{4}\chi'\phi^2 - \frac{1}{2}\chi\partial_u\phi\phi.
\end{aligned}$$

Recall that

$$|f| \lesssim 1, \quad |\chi'| \lesssim \frac{1}{(1+r)^2}, \quad 2|\chi\partial_u\phi\cdot\phi| \leq \chi^2\phi^2 + |\partial_u\phi|^2.$$

On $\bar{C}(0,\tau,v)$, we can use a similar version of Lemma 2.4.2 to control the integral of $|\chi'|\phi^2$, $|\chi\phi|^2$ by the energy flux through $\bar{C}(0,\tau,v)$. That is we can estimate

$$\begin{aligned}
&\liminf_{v\to\infty} \Big|\int_{\bar{C}(0,\tau,v)} 2\big(\frac{f}{4}(|\partial_u\phi|^2 - |\nabla\!\!\!\!/\phi|^2) - \frac{1}{4}\chi'\phi^2 - \frac{1}{2}\chi\partial_u\phi\phi\big)r^2\sqrt{-G}dud\omega\Big| \\
&\lesssim \liminf_{v\to\infty}\int_{\bar{C}(0,\tau,v)} (|\partial_u\phi|^2 + |\nabla\!\!\!\!/\phi|^2)r^2dud\omega = E^N[\phi]_0^\tau.
\end{aligned}$$

Next we need to control the error terms which consist of the second line of the decomposition (3.11). Since we assumed that

$$I^\alpha[\phi]_0^\infty = \int_0^\infty\int_{\Sigma_\tau}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}}dxd\tau$$

is finite. In particular, we can choose a sequence $v_n \to \infty$ so that

$$\int_{\bar{C}(0,\tau,v_n)}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}}r^2dud\omega \leq Mv_n^{-1}$$

for some constant M. Therefore by the assumption on the metric (3.2), we have

$$
\begin{aligned}
&|\int_{\bar{C}(0,\tau,v_n)}(h^{LA}A(\phi)X(\phi)-\frac{1}{2}X^L h^{\mu\nu}\partial_\nu\phi\partial_\mu\phi-\frac{1}{2}h^{LA}A(\chi)\phi^2+\chi h^{LA}A(\phi)\phi)r^2dud\omega| \\
&\lesssim \int_{\bar{C}(0,\tau,v_n)}|\bar{\partial}\phi|^2 r_+^{-\frac{1}{2}-\alpha}r^2dud\omega \lesssim v_n^{\frac{1}{2}}\int_{\bar{C}(0,\tau,v_n)}|\bar{\partial}\phi|^2 r_+^{-1-\alpha}r^2dud\omega \lesssim Mv_n^{-\frac{1}{2}}.
\end{aligned}
$$

Hence from the formula (2.6), we have shown that

$$
\liminf_{v\to\infty}|\int_{\bar{C}(0,\tau,v)} i_{\tilde{J}^X[\phi]}d\mathrm{vol}| \lesssim E^N[\phi]_0^\tau + \lim_{n\to} Mv_n^{-\frac{1}{2}} = E^N[\phi]_0^\tau.
$$

The Lemma then follows as $E^N[\phi]_0^\tau \le \tilde{E}[\phi](\tau)$. □

This lemma implies that the boundary terms on the right hand side of the integrated energy estimate (3.10) can be bounded by the energy flux plus an error. Next we estimate the main error term $E(X)$ defined in line (3.8). We can compute

$$
|\partial_j(r^{-1}fx_i)| = |\partial_j(\chi x_i)| \lesssim \frac{1}{1+r}.
$$

Thus the first term in line (3.8) can be controlled by

$$
|\partial_j(r^{-1}fx_i)h^{j\mu}\partial_\mu\phi\partial_i\phi| \lesssim (\delta_0\chi_{\{|x|\le R\}}+\delta_1\chi_{\{|x|>R\}})(1+r)^{-1-\alpha}|\partial\phi|^2,
$$

where χ_A is the characteristic function for the set A. For the other terms, the idea is that if the good derivative $\overline{\partial_v}$ hits on ϕ we can use Cauchy-Schwartz inequality to bound it by $\tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2$ plus $r_+^{-1-\alpha}|\partial\phi|^2$. For the bad term $\underline{L}(\phi)\underline{L}(\phi)$, we rely on the better decay of the metric component $g^{\underline{LL}}$. First for any vector field Y such that

$Y(\underline{L}_\mu) = 0$, $\|Y\| \le 2$, we can write

$$Y(g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi = Y(g^{\mu\nu})(\not\partial_\mu + \frac{1}{2}\underline{L}_\mu\underline{L})\phi(\not\partial_\nu + \frac{1}{2}\underline{L}_\nu\underline{L})\phi$$
$$= Y(g^{\mu\nu})\not\partial_\mu\phi\not\partial_\nu\phi + Y(g^{\mu\nu})\underline{L}_\mu\not\partial_\mu\phi\underline{L}(\phi) + \frac{1}{4}Y(g^{\mu\nu}\underline{L}_\mu\underline{L}_\nu)\underline{L}(\phi)\underline{L}(\phi).$$

Here $\not\partial_\nu = \partial_\nu - \frac{1}{2}\underline{L}_\nu\underline{L}$. From the assumption on the metric (3.2), we can estimate

$$|Y(g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi| \lesssim \begin{cases} \delta_0(1+r)^{-1-\alpha}|\partial\phi|^2, & |x| \le R, \\ \delta_1(1+r)^{-1-\alpha}|\partial\phi|^2 + \delta_1\tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2, & (t,x) \in S_\tau. \end{cases}$$

Similarly, we have

$$|h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi| \lesssim \begin{cases} \delta_0(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2, & |x| \le R, \\ \delta_1(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2 + \delta_1\tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2, & (t,x) \in S_\tau. \end{cases} \tag{3.12}$$

Here $\bar{\phi} = (\partial\phi, r_+^{-1}\phi)$. In particular, we conclude that

$$\frac{1}{2}|f'h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|, \quad |\partial_r g^{\mu\nu}\cdot g_{\mu\nu}\partial^\gamma\phi\partial_\gamma\phi|$$

verify the same estimates as in (3.12). Since $|f| \lesssim 1$, $\partial_r(\underline{L}_\mu) = 0$, the integral of the error term $E(X)$ obeys the following estimates

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau}|E(X)|dxd\tau \lesssim \delta_0\int_{\tau_1}^{\tau_2}\int_{r\le R}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}}dxd\tau + \delta_1\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}}dxd\tau$$
$$+ \delta_1\int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}\tilde{E}[\phi](\tau)d\tau. \tag{3.13}$$

Finally, we discuss the inhomogeneous term F and the linear term $N(\phi)$. For the linear term $N(\phi)$, note that

$$|X(\phi) + \chi\phi| \lesssim |\bar{\partial}\phi|.$$

Therefore $|N(\phi)(X(\phi)+\chi\phi)|$ also satisfy the estimate (3.12). The inhomogeneous term F can be bounded as follows

$$|F(X(\phi)+\chi\phi)| \lesssim \epsilon_0^{-1}(1+r)^{1+\alpha}|F|^2 + \epsilon_0(1+r)^{-1-\alpha}|\bar{\partial}\phi|^2, \quad \epsilon_0 > 0.$$

Now using Lemma 3.1.3 to control the boundary terms on Σ_τ, for sufficiently small δ_1, ϵ_0, depending only on α, the integrated energy inequality (3.10) leads to

$$\begin{aligned} I^\alpha[\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_1}\int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \lesssim \delta_0 \int_{\tau_1}^{\tau_2}\int_{r\le R} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \\ + \tilde{E}[\phi](\tau_i) + \delta_1 S^\alpha[\phi](\tau_i) + \delta_1 \int_{\tau_1}^{\tau_2} \tau_+^{-1-\alpha}\tilde{E}[\phi](\tau)d\tau + D^\alpha[F]_{\tau_1}^{\tau_2}. \end{aligned}$$

If δ_0 is also small depending only on α, then the first term on the RHS of the above estimate could be absorbed. For the case when the integrated local energy estimate (3.3) holds, it can be bounded directly (note that $N=0$, in this case $\Box_g\phi = F$). Therefore, in any case, we have

$$I^\alpha[\phi]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_i) + \delta_1 S^\alpha[\phi](\tau_i) + \delta_1 \int_{\tau_1}^{\tau_2} \tau_+^{-1-\alpha}\tilde{E}[\phi](\tau)d\tau + D^\alpha[F]_{\tau_1}^{\tau_2}. \tag{3.14}$$

Here for simplicity, $\tilde{E}[\phi](\tau_i)$ denotes $\tilde{E}[\phi](\tau_1)+\tilde{E}[\phi](\tau_2)$, similarly for $S^\alpha[\phi](\tau_i)$.

To prove (3.4), we choose the domain to be the finite region bounded by Σ_0 and $\{t=t_1\}$ and we do the same estimates as above. Denote

$$\begin{aligned} &v(t_1) = 2t_1 - \tau, \quad C_0 = \tilde{E}[\phi](0) + S^\alpha[\phi](0). \\ &E(t) = \int_{r\le t+R} |\partial\phi|^2 dx\Big|_{t=t}, \quad D(t) = \int_0^t \int_{S_\tau\cap\{v\le v(\tau)\}} \tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2 r^2 dv d\omega d\tau. \end{aligned}$$

We can show that

$$\int_0^{t_1}\int_{\Sigma_\tau^{v(t_1)}} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \lesssim C_0 + E(t_1) + \delta_1 D(t_1) + D^\alpha[F]_0^{t_1}. \tag{3.15}$$

The energy flux $E(t_1)$ plays the same role as $\tilde{E}[\phi](\tau_2)+\delta_1 S^\alpha[\phi](\tau_2)$. The only difference is that $D(t)$ does not contain the part from the cylinder $\{r \le R\}$. The reason is that in fact the error term from this part could be absorbed for sufficiently small δ_0 or can be controlled by the integrated local energy estimate (3.3). Thus (3.15) holds for all $t_1 \ge 0$.

3.1.2 The vector field ∂_t

We have taken the vector field $f\partial_r$ as multipliers to obtain the above integrated energy inequality (3.14) which will imply (3.5) in Proposition 3.1.1 if we can further control the energy flux $\tilde{E}[\phi](\tau)$. Next, we take ∂_t as multipliers to obtain the classical energy estimates.

Similar to the case when $X = f\partial_r$ discussed above, we apply the energy identity (2.3) to the region bounded by Σ_{τ_1}, Σ_{τ_2} and the incoming null hypersurface $\bar{C}(\tau_1, \tau_2, v)$ and the vector field $X = \partial_t$, the function $\chi = 0$. We obtain

$$\begin{aligned}\int_{\Sigma^v_{\tau_1}} i_{J^X[\phi]} d\mathrm{vol} - \int_{\Sigma^v_{\tau_2}} i_{J^X[\phi]} d\mathrm{vol} + \int_{\bar{C}(\tau_1,\tau_2,v)} i_{J^X[\phi]} d\mathrm{vol} \\ = \int_{\tau_1}^{\tau_2}\int_{\Sigma^v_\tau} \Box_g \phi X(\phi) + K^X[\phi] d\mathrm{vol},\end{aligned} \tag{3.16}$$

The estimates of the current $K^X[\phi]$ and the inhomogeneous term $\Box_g\phi$ are quite similar to the case when $X = f\partial_r$. And we can show that

$$\begin{aligned}|K^{\partial_t}[\phi]| + |\Box_g \phi X(\phi)| &= |-\partial_t g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\partial_t g^{\mu\nu}\cdot g_{\mu\nu}\partial^\gamma\phi\partial_\gamma\phi| + |(F - N(\phi))\partial_t\phi| \\ &\lesssim \delta_0 \chi_{\{|x|\le R\}}\frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} + \frac{\delta_1|\partial\phi|^2}{(1+r)^{1+\alpha}} + \delta_1\chi_{\{|x|>R\}}\tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2 + \delta_1^{-1}(1+r)^{1+\alpha}|F|^2.\end{aligned}$$

Next we estimate the boundary terms. On $\Sigma^v_\tau \cap \{r \le R\}$, using the formula (2.4), we have

$$i_{J^X[\phi]} d\mathrm{vol} = (-\partial^t\phi\partial_t\phi + \frac{1}{2}\partial^\gamma\phi\partial_\gamma\phi)\sqrt{-G}dx.$$

Since ∂_t is uniformly timelike, we can conclude that there is a positive constant λ_1 depending only λ, δ_0 such that

$$\lambda_1 \int_{r\leq R} |\partial\phi|^2 dx \leq \int_{r\leq R} i_{J^X[\phi]} d\text{vol} \leq \lambda_1^{-1} \int_{r\leq R} |\partial\phi|^2 dx.$$

On S_τ, the formula (2.5) implies that

$$\begin{aligned} i_{J^X[\phi]} d\text{vol} &= -2(g^{\underline{L}A} A(\phi)\partial_t\phi - \frac{1}{4}\partial^\gamma\phi\partial_\gamma\phi)\sqrt{-G} r^2 dv d\omega \\ &= \frac{1}{2}(|\overline{\partial_v}\phi|^2 + Err_1)\sqrt{-G} r^2 dv d\omega, \end{aligned}$$

where the error Err_1 obeys

$$|Err_1| = |4h^{\underline{L}A} A(\phi)\partial_t\phi - h^{\gamma\nu}\partial_\gamma\phi\partial_\nu\phi| \lesssim \delta_1 \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} + \delta_1|\overline{\partial_v}\phi|^2.$$

For sufficiently small δ_1, depending only on α, we can conclude that for some positive constants $C_1 > 4$, C_2, depending on α, λ, we have

$$C_1^{-1}(\int_{\Sigma_\tau^v} |\overline{\partial_v}\phi|^2 d\sigma - C_2\delta_1 S[\phi](\tau)) \leq \int_{\Sigma_\tau^v} i_{J_\mu^{\partial_t}[\phi]} d\text{vol} \leq C_1(E[\phi](\tau) + \delta_1 S^\alpha[\phi](\tau)),$$

where $d\sigma = dx$, $r \leq R$; $d\sigma = r^2 dv d\omega$, $r > R$.

Next we estimate the boundary term on $\bar{C}(\tau_1, \tau_2, v)$. On such incoming null hypersurfaces, we have

$$i_{J^{\partial_t}[\phi]} d\text{vol} = \left(-\frac{1}{2}(|\partial_u\phi|^2 + |\nabla\!\!\!/\phi|^2) + 2h^{LA} A(\phi)\partial_t\phi - \frac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) r^2 du d\omega.$$

Since $I^\alpha[\phi]_0^\infty$ is finite, we can choose a sequence $v_n \to \infty$ such that

$$\int_{\bar{C}(\tau_1,\tau_2,v_n)} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} r^2 du d\omega \leq M v_n^{-1}$$

for some constant M. In particular, we have

$$\liminf_{n\to\infty}(-\int_{\bar{C}(\tau_1,\tau_2,v_n)} i_{J^{\partial_t}[\phi]}d\text{vol}) = \liminf_{n\to\infty}\int_{\bar{C}(\tau_1,\tau_2,v_n)} \frac{1}{2}(|\partial_u\phi|^2+|\nabla\!\!\!/\phi|^2)r^2dud\omega$$
$$= \frac{1}{2}E^N[\phi]_{\tau_1}^{\tau_2}.$$

Now from the energy identity (3.16) and all the above estimates above, we can derive

$$\tilde{E}[\phi](\tau_2) + E^N[\phi]_{\tau_1}^{\tau_2} \lesssim E[\phi](\tau_1) + \delta_1 S^\alpha[\phi](\tau_i) + \delta_1 I^\alpha[\phi]_{\tau_1}^{\tau_2} + \delta_1\int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}E[\phi](\tau)d\tau$$
$$+ \delta_0\int_{\tau_1}^{\tau_2}\int_{r\le R}\frac{|\partial\phi|^2}{(1+r)^{1+\alpha}}dxd\tau + \delta_1^{-1}D^\alpha[F]_{\tau_1}^{\tau_2}.$$

Again $S^\alpha[\phi](\tau_i) = S^\alpha[\phi](\tau_1) + S^\alpha[\phi](\tau_2)$. Since either δ_0 is small which we can take it to be δ_0 or the metric satisfies the integrated local energy estimate (3.3), using the integrated energy estimates (3.14), for sufficiently small δ_1, depending only on α, δ_0, we have

$$\tilde{E}[\phi](\tau_2) \lesssim \tilde{E}[\phi](\tau_1) + \delta_1 S^\alpha[\phi](\tau_i) + \delta_1\int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}E[\phi](\tau)d\tau + D^\alpha[F]_{\tau_1}^{\tau_2}. \tag{3.17}$$

Now the problem is how to estimate the integral of the energy with negative weights in τ_+, which can usually be bounded by using Gronwall's inequality. However, due to the presence of $S^\alpha[\phi](\tau_2)$ on the right hand side, we are not able to use Gronwall's inequality directly. Instead, let $\tau_2 = \tau$ and then integrate the above energy inequality with respect to τ from τ_1 to τ_2. Use the integrated energy inequality (3.14) to bound the integral of $S^\alpha[\phi](\tau)$. We can show that

$$\int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}\tilde{E}[\phi](\tau)d\tau \lesssim \tilde{E}[\phi](\tau_1) + \delta_1\tilde{E}[\phi](\tau_i) + \delta_1 S^\alpha[\phi](\tau_i)$$
$$+ \delta_1\int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}E[\phi](\tau)d\tau + D^\alpha[F]_{\tau_1}^{\tau_2}.$$

For small δ_1, the above estimate leads to

$$\int_{\tau_1}^{\tau_2} \tau_+^{-1-a} \tilde{E}[\phi](\tau) d\tau \lesssim \tilde{E}[\phi](\tau_1) + \delta_1 \tilde{E}[\phi](\tau_i) + \delta_1 S^\alpha[\phi](\tau_i) + D^\alpha[F]_{\tau_1}^{\tau_2}.$$

Then the energy estimate (3.6) of Proposition 3.1.1 follows from (3.17) if δ_1 is sufficiently small, depending only on α, δ_0. This energy estimate together with (3.14) implies the integrated energy estimate (3.5) of Proposition 3.1.1. For the improved integrated energy estimate for the angular derivative of ϕ, we note that we in fact have the improved decay estimate

$$\chi - \frac{1}{2} f' \geq \frac{1}{2} \frac{1}{1+r}.$$

We thus have shown the integrated energy estimate (3.5) and the energy estimate (3.6). Next we prove the boundedness of the integrated energy (3.4). We do the energy estimate on the region bounded by Σ_0 and $\{t = t_1\}$, that is, as above we apply the energy identity (2.3) to such compact region with $X = \partial_t$, $\chi = 0$. Similar to the above discussion, we can obtain

$$E(t_1) \lesssim C_0 + \delta_1 D(t_1) + D^\alpha[F]_0^{t_1}.$$

Here $E(t)$, $D(t)$, C_0 have been defined in the end of Section 3.1.1. Thus by the estimate (3.15) there, we have

$$\int_0^{t_1} \int_{\Sigma_\tau^{v(t_1)}} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \lesssim C_0 + \delta_1 D(t_1) + D^\alpha[F]_0^{t_1}.$$

Now to estimate $D(t_1)$, we do the energy estimate on the region bounded by Σ_{τ_1} and

$\{t = t_1\}$ for $0 \le \tau_1 \le t_1$. We can show that

$$\begin{aligned}\int_{S_{\tau_1}\cap\{t\le t_1\}} |\overline{\partial_v}\phi|^2 r^2 dv d\omega \lesssim & E(t_1) + \delta_1 \int_{\tau_1}^{t_1} \int_{S_\tau \cap \{t\le t_1\}} \tau_+^{-1-\alpha} |\overline{\partial_v}\phi|^2 r^2 dv d\omega + \delta_1 D(t_1) \\ & + \delta_1 \int_{S_{\tau_1}} \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} r^2 dv d\omega + C_0 + D^\alpha[F]_0^{t_1} \\ & \lesssim C_0 + \delta_1 D(t_1) + \delta_1 \int_{S_{\tau_1}} \frac{|\partial\phi|^2}{(1+r)^{1+\alpha}} r^2 dv d\omega + D^\alpha[F]_0^{t_1}.\end{aligned}$$

Multiply the above inequality by $(1+\tau_1)^{-1-a}$ and then integrated with respect to τ_1 from 0 to t_1. We can show that

$$D(t_1) \lesssim C_0 + D^a[F]_0^{t_1} + \delta_1 D(t_1) + \delta_1(C_0 + \delta_1 D(t_1) + D^\alpha[F]_0^{t_1}).$$

Let δ_1 to be sufficiently small. We conclude that

$$D(t_1) \lesssim C_0 + D^\alpha[F]_0^{t_1}.$$

This implies that

$$\int_0^{t_1} \int_{\Sigma_\tau^{v(t_1)}} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dx d\tau \lesssim C_0 + \delta_1 D(t_1) + D^\alpha[F]_0^{t_1} \lesssim C_0 + D^\alpha[F]_0^{t_1}.$$

Since the implicit constant is independent of t_1 and $C_0 = \tilde{E}[\phi](0) + S^\alpha[\phi](0)$, let $t_1 \to \infty$. We then show (3.4).

3.2 The p-weighted energy inequalities

In this section we discuss the key estimate of the new approach, the p-weighted energy inequalities in a neighborhood of null infinity. We will first recall this estimate in Minkowski space, i.e., the background metric g is flat. Then we will discuss this estimate on asymptotically flat spacetimes under very weak conditions on the background metric.

3.2.1 The p-weighted energy inequality in Minkowski space

In this section, we revisit the p-weighted energy inequality in Minkowski space originally introduced by Dafermos-Rodnianski in [10].

Consider the linear wave equation

$$\Box\phi = F,$$

on $(\mathbb{R}^{3+1}, g)$ where g is flat outside the cylinder $\{|x| \leq R\}$ with radius R. In particular the metric is flat in a neighborhood of null infinity. When $|x| \geq R$, we can write the equation in null coordinates

$$-\partial_u\partial_v\psi + \Delta\!\!\!/\,\psi = rF, \quad \psi = r\phi. \tag{3.18}$$

We have

Proposition 3.2.1. *Let ϕ be a smooth solution of the above linear wave equation. Assume ϕ satisfies the conditions in Proposition 3.1.1. Denote $\psi = r\phi$. Then for any $0 \leq p \leq 2$, we have*

$$\begin{aligned}
&\int_{S_{\tau_2}} r^p(\partial_v\psi)^2 dv d\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1}(p|\partial_v\psi|^2 + (2-p)|\nabla\!\!\!/\,\psi|^2) dv d\omega d\tau \\
&\leq C_{\alpha,R}(\tilde{E}[\phi](\tau_1) + \int_{S_{\tau_1}} r^p|\partial_v\psi|^2 dv d\omega + \int_{\tau_1}^{\tau_2} \tau_+^\alpha D_+^{p-1}[F]_\tau^{\tau_2} d\tau \\
&\quad + (\tau_1)_+^{1+\alpha} D_+^{p-1}[F]_{\tau_1}^{\tau_2} + D^\alpha[F]_{\tau_1}^{\tau_2}),
\end{aligned} \tag{3.19}$$

where the constant $C_{\alpha,R}$ depends on α, R and R is the radius of the foliation Σ_τ.

In the work of Dafermos-Rodnianski [10], this estimate has been derived for solutions of linear wave equations without the inhomogeneous term F. To apply this new approach to nonlinear problems, we also need a version with the inhomogeneous term. The proof here is essentially the same as that in [10]. The only difference is to handle the inhomogeneous term F.

Proof. Multiply the equation by $r^p\partial_v\psi$ and then integrate by parts on the region bounded by the two null hypersurfaces S_{τ_1}, S_{τ_2} and the hypersurface $\{(t,x)||x|=R\}$ and the incoming null hypersurface $\bar{C}(\tau_1,\tau_2,v)$. Denote $S_\tau^v = S_\tau \cap \{t+r\le 2v\}$. For all $v > \frac{R+\tau_2}{2}$, we obtain

$$\begin{aligned}
&\int_{S_{\tau_2}^v} r^p(\partial_v\psi)^2 dvd\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau^v} 2r^{p+1}F\cdot\partial_v\psi dvd\tau d\omega \\
&+ \int_{\tau_1}^{\tau_2}\int_{S_\tau^v} r^{p-1}\left(p(\partial_v\psi)^2 + (2-p)|\nabla\!\!\!/\psi|^2\right) dvd\tau d\omega + \int_{\bar{C}(\tau_1,\tau_2,v)} r^p|\nabla\!\!\!/\psi|^2 dud\omega \\
=& \int_{S_{\tau_1}^v} r^p(\partial_v\psi)^2 dvd\omega + \int_{\tau_1}^{\tau_2} r^p\left(|\nabla\!\!\!/\psi|^2 - (\partial_v\psi)^2\right) d\omega d\tau|_{r=R}.
\end{aligned} \tag{3.20}$$

Note that the boundary term on $\{r=R\}$ is proportional to R^p. Hence we can simply take $p=0$ to estimate it. First for any τ, we have

$$\int_{S_\tau^v} (\partial_v\psi)^2 dvd\omega = \int_{S_\tau^v} |\partial_v\phi|^2 r^2 dvd\omega + \int_{S_\tau^v} \partial_v(r\phi^2) dvd\omega.$$

By Lemma 2.4.1, we can estimate

$$\int_{S_\tau^v} (\partial_v\psi)^2 dvd\omega \le 5\tilde{E}[\phi](\tau).$$

For the inhomogeneous term $F\partial_v\psi$, we use Cauchy-Schwartz inequality together with the integrated energy inequality (3.5) to and we can show that

$$\begin{aligned}
|\int_{\tau_1}^{\tau_2}\int_{S_\tau^v} rF\cdot\partial_v\psi dvd\tau d\omega| &\lesssim D^\alpha[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\alpha}} dxd\tau \\
&\lesssim \tilde{E}[\phi](\tau_1) + D^\alpha[F]_{\tau_1}^{\tau_2}.
\end{aligned}$$

Here we note that since the metric is flat when $|x| > R$, we have $\delta_1 = 0$ in the integrated energy estimate (3.5). For the integral on $\bar{C}(\tau_1, \tau_2, v)$, we have

$$\liminf_{v\to\infty} \int_{\bar{C}(\tau_1,\tau_2,v)} |\nabla\!\!\!/\psi|^2 du d\omega \leq \liminf_{v\to\infty} \int_{\bar{C}(\tau_1,\tau_2,v)} (|\partial_u\phi|^2 + |\nabla\!\!\!/\phi|^2) r^2 du d\omega \leq E^N[\phi]_{\tau_1}^{\tau_2}.$$

There from the above p-weighted energy identity when $p = 0$, we can conclude that (other terms can be estimated by using the integrated energy estimate (3.5))

$$\left| \int_{\tau_1}^{\tau_2} r^p \left(|\nabla\!\!\!/\psi|^2 - (\partial_v\psi)^2 \right) d\omega d\tau |_{r=R} \right| \lesssim R^p (D^\alpha[F]_{\tau_1}^{\tau_2} + \tilde{E}[\phi](\tau_1)).$$

Now for general $0 < p \leq 2$, the boundary term on the incoming null hypersurface $\bar{C}(\tau_1, \tau_2, v)$ has a good sign. It suffices to estimate the integral of the inhomogeneous term $r^{p+1} F \partial_v \psi$ in the above p-weighted energy identity (3.20). On S_τ, we control it as follows

$$2r^{p+1}|F\partial_v\psi| \leq r^p |\partial_v\psi|^2 \tau_+^{-1-\alpha} + r^{p+2}|F|^2 \tau_+^{1+\alpha}.$$

The integral of the first term $r^p|\partial_v\psi|^2\tau_+^{-1-\alpha}$ can be bounded by using Gronwall's inequality. We can use Lemma 2.4.4 to control the integral of the second term. Then from the p-weighted energy identity (3.20), we have (3.19). Thus the proposition holds. □

3.2.2 Generalizations to asymptotically flat spacetimes

In this section we establish the p-weighted energy inequalities on asymptotically flat spacetimes in a neighborhood of null infinity. We still consider solutions of the linear wave equations (3.1) on $(\mathbb{R}^{3+1}, g)$ with the metric g, the vector field N satisfying the estimates (3.2) and the following additional estimate

$$|\nabla\!\!\!/ h^{\underline{L}\underline{L}}| \leq \delta_1 (r_+^{-\frac{3}{2}-2\alpha} + r_+^{-1-\alpha} \tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha}), \quad r \geq R. \tag{3.21}$$

We assume δ_1 is sufficiently small, depending only on α. Suppose either δ_0 is small or in addition the metric g verify the integrated local energy condition (3.3). Assume α is small positive constant. Then we have

Proposition 3.2.2. *Let ϵ, α_1, α_2 be positive constant such that*

$$0 < \epsilon < \frac{\alpha^2}{4} < \alpha < \frac{2\alpha + \alpha\epsilon}{2 - \alpha} \leq \alpha_1 < \alpha_2 \leq \frac{7}{3}\alpha - \alpha_1 - \epsilon.$$

Let ϕ be the solution of the linear wave equation (3.1). *Assume ϕ satisfies the conditions in Lemma 2.4.1 and $I^\epsilon[\phi]_0^\infty$ is finite. Then*

(1) : p-weighted energy inequality with weights $r^{1+\alpha_1}$

$$g^{1+\alpha_1}[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{\alpha_1} |\overline{\partial_v}\psi|^2 dv d\omega d\tau \lesssim g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau_+^\epsilon D^{\alpha_1}[F]_\tau^{\tau_2} d\tau$$
$$+ R^{1+\alpha_1}((\tau_1)_+^{1-\alpha} \tilde{E}[\phi](\tau_1) + \delta_1 (\tau_i)_+^{1-\alpha} S^\epsilon[\phi](\tau_i) + (\tau_1)_+^{1+\epsilon} D^{\alpha_1}[F]_{\tau_1}^{\tau_2}), \tag{3.22}$$

(2) : p-weighted energy inequality with weights r

$$g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau \lesssim g^1[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_\tau^{\tau_2} d\tau$$
$$+ R^{1+\epsilon}((\tau_1)_+^{1-\alpha} \tilde{E}[\phi](\tau_1) + (\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \delta_1 (\tau_i)_+^{1-\alpha} S^\epsilon[\phi](\tau_i)), \tag{3.23}$$

where $S^\epsilon[\phi](\tau_i) = S^\epsilon[\phi](\tau_1) + S^\epsilon[\phi](\tau_2)$. The implicit constants also depend on ϵ, α_1, α_2. The notations are defined in Section 2.3.

Remark 3.2.3. *ϵ is much smaller than α and can be taken to be, for example $\epsilon = \frac{\alpha}{10000}$. This small constant will appear in the integrated energy estimate* (3.5)*(since ϵ is smaller than α,* (3.5) *also holds for ϵ). $p = 1 + \alpha_1$ is the maximal p we can take in the p-weighted energy inequality.*

In the previous section, the p-weighted energy inequalities on flat spacetimes are established by multiplying the equation in null coordinates with $r^p\partial_v(r\phi)$ and then integrating by parts. On asymptotically flat spacetimes, a more robust way, as also mentioned in [10], to prove the p-weighted energy inequalities is to use the vector field method. When the metric is flat, we can in fact prove (3.19) by using the vector field $r^p\partial_v$ as multipliers. For the general metrics with very weak (α is small) decay properties, we need to construct the corresponding vector fields as multipliers.

We use the vector field method to establish the above two p-weighted energy inequalities. Let f be a smooth compactly supported nonnegative function of r defined on $[R, \infty)$. Choose the corresponding vector field as follows

$$X = fY = f(-2g^{\underline{L}A}A + g^{\underline{L}\underline{L}}\underline{L}) = f(-2\partial^{\underline{L}} + g^{\underline{L}\underline{L}}\underline{L}), \tag{3.24}$$

where A runs over the null frame $\{\underline{L}, L, S_1, S_2\}$.

Although the null frame is merely defined locally and depends on the choice of S_1 and S_2, the above X is in fact a well defined vector field when $r \geq R$. Notice that the vector $g^{\underline{L}A}A$ can be viewed as the unique vector field which is orthogonal to the hypersurface S_τ such that the inner product with $\underline{L}$ relative to the metric g is 1. Since $\underline{L}$ is a global well defined vector field when $r \geq R$, we conclude that X is also a well defined vector field on $\{r \geq R\}$.

We rely on the energy identity (2.3). We choose the vector field X as above (3.24). Let the function $\chi = r^{-1}f$. The integral region $\mathcal{D}$ is bounded by S_{τ_1}, S_{τ_2} and

$$C_R = \{r = R, \tau_1 \leq t \leq \tau_2\}.$$

Since f has compact support, the energy identity (2.3) implies that

$$\int_{S_{\tau_1}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{S_{\tau_2}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{C_R} i_{\tilde{J}^X[\phi]} d\mathrm{vol}$$
$$= \int_{\tau_1}^{\tau_2} \int_{S_\tau} (F - N(\phi))(X(\phi) + \chi\phi) + K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box_g\chi\phi^2 d\mathrm{vol}. \quad (3.25)$$

In this subsection, we define two functions on S_τ

$$H = \delta_1 \tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha} r_+^{-\frac{1}{2}-2\alpha}, \quad \bar{H} = \delta_1 r_+^{-1-2\alpha}.$$

We now estimate term by term in the above energy identity (3.25). First for the linear term $N(\phi)(X(\phi)+\chi\phi)$, note that

$$|N(\phi)| \lesssim \bar{H}|\partial\phi| + H|\overline{\partial_v}\phi|.$$

We can write

$$X(\phi) + \chi\phi = \chi L(\psi) - 2fh^{\underline{L}A}A(\phi) + fg^{\underline{LL}}\underline{L}(\phi), \quad \psi = r\phi.$$

Therefore we have

$$|N(\phi)(X\phi + \chi\phi)| \lesssim (\bar{H}|\partial\phi| + H|\overline{\partial_v}\phi|)(\chi|L(\psi)| + f\bar{H}|\partial\phi| + fH|\overline{\partial_v}\phi|). \quad (3.26)$$

Next we estimate the main term $K^X[\phi]$. Relative to the null frame $\{L, \underline{L}, S_1, S_2\}$, we can calculate the deformation tensor of the vector field X

$$\pi^X_{AB} = \frac{1}{2}(X(g_{AB}) + g([A,X],B) + g([B,X],A)),$$

where $[A,X]$ denotes the commutator of the two vector fields A, X. We remark here that π^X_{AB} is defined locally. However, the current $K^X[\phi]$ is independent of the choice of local coordinates. We thus can compute it relative to the null frame $\{L, \underline{L}, S_1, S_2\}$. We can compute

$$K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi = -\frac{1}{2}X(g^{AB})A(\phi)B(\phi) + X^C[A,C](\phi)\partial^A\phi \\ + f\partial^A Y^C C(\phi)A(\phi) + \partial^A f A(\phi)Y(\phi) - (\frac{1}{2}div(X) - \chi)\partial^\gamma\phi\partial_\gamma\phi, \tag{3.27}$$

where $div(X)$ is the divergence of the vector field X with respect to the metric g (also see the definition below). Since the metric is asymptotically flat, we decompose the above expression according to the metric decomposition $g = h + m_0$.

We first consider $div(X)$. Recall that $X = fY$. We have

$$div(X) = Y(f) + f div(Y).$$

Recall that (see Section 2.3) at a fixed point we can require that $[S_1, S_2] = 0$. We thus can compute

$$\begin{aligned}\frac{1}{2}div(Y) &= \frac{1}{2}A(Y^A) + \frac{1}{4}Y(g_{AB})g^{AB} + \frac{1}{2}Y^C g([A,C],B)g^{AB} \\ &= \frac{1}{2}A(Y^A) - \frac{1}{4}Y(g^{AB})g_{AB} + r^{-1}(Y^L - Y^{\underline{L}}) \\ &= r^{-1} - 2r^{-1}h^{L\underline{L}} + r^{-1}g^{\underline{L}\underline{L}} - L(g^{L\underline{L}}) - S(g^{S\underline{L}}) - \frac{1}{2}\underline{L}(g^{\underline{L}\underline{L}}),\end{aligned}$$

where $h^{AB} = g^{AB} - m_0^{AB}$, $S \in \{S_1, S_2\}$. Note that

$$Y(f) = -2m_0^{L\underline{L}}L(f) - 2h^{\underline{L}A}A(f) + g^{\underline{L}\underline{L}}\underline{L}(f) = f' - 2h^{\underline{L}A}A(f) + g^{\underline{L}\underline{L}}\underline{L}(f).$$

Using the estimates (3.12) to control $\partial^\gamma\phi\partial_\gamma\phi$, we then can write the last term in the expression (3.27) as

$$
\begin{aligned}
&(\frac{1}{2}div(X) - \chi)\partial^\gamma\phi\partial_\gamma\phi = \frac{1}{2}f'(-L(\phi)\underline{L}(\phi) + |\nabla\!\!\!/\phi|^2) + Er_1, \\
&|Er_1| \lesssim (\chi + |f'|)\left(H|\overline{\partial_v\phi}||\partial\phi| + \bar{H}|\partial\phi|^2\right) + \bar{H}f(|L(\phi)||\partial\phi| + |\nabla\!\!\!/\phi|^2),
\end{aligned}
\tag{3.28}
$$

where recall that $\chi = r^{-1}f$.

Similarly, we can write

$$
\begin{aligned}
X^C[A,C](\phi)\partial^A\phi &= \chi(1 - 2h^{L\underline{L}} + h^{\underline{L}\underline{L}})S(\phi)\partial^S\phi + \chi h^{\underline{L}S}S(\phi)(\partial^{\underline{L}}\phi - \partial^L\phi) \\
&= \chi|\nabla\!\!\!/\phi|^2 + Er_2, \quad |Er_2| \lesssim \chi H|\nabla\!\!\!/\phi||\partial\phi|.
\end{aligned}
\tag{3.29}
$$

Next for $\partial^A fA(f)Y(\phi)$, recall that f is a function of r. We have

$$
\begin{aligned}
&\partial^A fA(\phi) = f'(-\frac{1}{2}\underline{L}(\phi) + \frac{1}{2}L(\phi) + h^{LB}B(\phi) - h^{\underline{L}B}B(\phi)), \\
&Y(\phi) = -2\partial^{\underline{L}}\phi + g^{\underline{L}\underline{L}}\underline{L}(\phi) = L(\phi) - 2h^{\underline{L}B}B(\phi) + h^{\underline{L}\underline{L}}\underline{L}(\phi).
\end{aligned}
$$

Since

$$
|-2h^{\underline{L}B}B(\phi) + h^{\underline{L}\underline{L}}\underline{L}(\phi)| \lesssim \bar{H}|\partial\phi| + H|\overline{\partial_v}\phi|,
$$

we can write

$$
\begin{aligned}
\partial^A fA(\phi)Y(\phi) &= \frac{1}{2}f'(L(\phi) - \underline{L}(\phi))L(\phi) + Er_3, \\
|Er_3| &\lesssim H|f'||\partial\phi||\overline{\partial_v}\phi| + |f'|\bar{H}|\partial\phi|^2.
\end{aligned}
\tag{3.30}
$$

We finally estimate the main error term in (3.25)

$$
-\frac{1}{2}X(g^{AB})A(\phi)B(\phi) + f\partial^A Y^C C(\phi)A(\phi).
$$

Since this term is linear in f, it suffices to consider the quadratic form

$$(-\frac{1}{2}Y(g^{AB}) + \partial^A Y^B)A(\phi)B(\phi), \quad Y = -2\partial^{\underline{L}} + g^{\underline{LL}}\underline{L}.$$

of $\underline{L}(\phi)$, $L(\phi)$, $S(\phi)$. The coefficient of $\underline{L}(\phi)\underline{L}(\phi)$ satisfies

$$\left|-\frac{1}{2}(-2\partial^{\underline{L}} + g^{\underline{LL}}\underline{L})(g^{\underline{LL}}) - \partial^{\underline{L}} g^{\underline{LL}}\right| = \left|-\frac{1}{2}g^{\underline{LL}}\underline{L}(g^{\underline{LL}})\right| \lesssim \bar{H}^2.$$

Using the improved decay assumption (3.21) on $\nabla\!\!\!/ h^{\underline{LL}}$, we can bound the coefficients for $\underline{L}(\phi)S(\phi)$, $S \in \{S_1, S_2\}$ as follows

$$|(2\partial^{\underline{L}} - g^{\underline{LL}}\underline{L})(g^{\underline{L}S}) - 2\partial^{\underline{L}} g^{\underline{L}S} - \partial^S g^{\underline{LL}}| = |g^{\underline{LL}}\underline{L}(g^{\underline{L}S}) + \partial^S(g^{\underline{LL}})| \lesssim r^{-\frac{1}{2}}(H + \bar{H}).$$

Similarly, the coefficient for $L(\phi)\underline{L}(\phi)$ can be bounded by $C_0\bar{H}$. For $A(\phi)B(\phi)$, A, $B \in \{S_1, S_2\}$, we rely on the better decay in r of $\overline{\partial_v} g$ to control the coefficient

$$|\partial^{\underline{L}} g^{AB} - \frac{1}{2}g^{\underline{LL}}\underline{L}(g^{AB}) - \partial^A g^{\underline{L}B} - \partial^B g^{\underline{L}A}| \lesssim \bar{H} + H^2 \lesssim \bar{H}.$$

Finally for $L(\phi)\overline{\partial_v}\phi$, the coefficient can simply be bounded by $C_0 H$ for some constant C_0 depending only on α. Summarizing, we have shown

$$\left|-\frac{1}{2}X(g^{AB})A(\phi)B(\phi) + f\partial^A Y^C C(\phi)A(\phi)\right| \lesssim f\left(\bar{H}^2|\underline{L}(\phi)|^2 + \bar{H}|\underline{L}(\phi)L(\phi)|\right.$$
$$\left. + (\bar{H} + H)r^{-\frac{1}{2}}|\underline{L}(\phi)||\nabla\!\!\!/\phi| + \bar{H}|\nabla\!\!\!/\phi|^2 + H|L(\phi)||\overline{\partial_v}\phi|\right).$$

Combine this estimate with estimates (3.28), (3.29), (3.30). The equation (3.27) gives the estimate for $K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi$. Then from (3.25) and (3.26), we derive

$$\begin{aligned}&\int_{S_{\tau_1}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{S_{\tau_2}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} - \int_{C_R} i_{\tilde{J}^X[\phi]} d\mathrm{vol} \qquad (3.31)\\ &= \int_{\tau_1}^{\tau_2}\int_{S_\tau} F(X(\phi)+\chi\phi) + \frac{1}{2}f'|L(\phi)|^2 + (\chi - \frac{1}{2}f')|\nabla\!\!\!/\phi|^2 - \frac{1}{2}\Delta\chi\cdot\phi^2 + E(X) d\mathrm{vol},\end{aligned}$$

where the error term $E(X)$ can be bounded as follows

$$\begin{aligned}|E(X)| \lesssim &(\chi + |f'|)\bar{H}|\partial\phi|^2 + (|f'|H + fr^{-\frac{1}{2}}(\bar{H}+H))|\partial\phi||\overline{\partial_v}\phi| + \chi\bar{H}|L\psi||\partial\phi|\\ &+ fH^2|\overline{\partial_v}\phi|^2 + f\bar{H}(|L(\phi)||\partial\phi| + |\nabla\!\!\!/\phi|^2) + fH|L(\phi)||\overline{\partial_v}\phi|\\ &+ |(\Box_g - \Delta)\chi|\phi^2 + \chi H|L\psi||\overline{\partial_v}\phi|,\end{aligned}$$

where the Laplacian Δ is with respect to the flat metric on $\mathbb{R}^3$. We further estimate $E(X)$ by choosing the function f explicitly. Let κ be a smooth positive cutoff function on $[0,\infty)$ such that

$$\kappa(x) = 1, \quad x \le 1; \quad \kappa(x) = 0, \quad x \ge 2; \quad |\kappa'(x)| \le 2.$$

Let M be a large constant and then let $f = r^p\kappa_M = r^p\kappa(\frac{|x|-R}{M})$. Recall that $\chi = r^{-1}f$. We can show that

$$|(\Box_g - \Delta)\chi| \lesssim Hr^{p-2}, \quad (t,x) \in S_\tau.$$

Similar to the case when the metric is flat (see Proposition 3.2.1), to estimate the boundary term on C_R, we can take $p = 0$ in the p-weighted energy inequality. Hence we also need to estimate the error term $E(X)$ when $p = 0$. For this case we have

$$|f| \le 1, \quad |f'| = |\frac{\kappa'(r-R)}{M}| \lesssim r^{-1}, \quad |\chi| \le r^{-1}.$$

And we can estimate the error term $E(X)$ as follows

$$|E(X)| \lesssim \delta_1(r^{-1-\alpha}|\bar{\partial}\phi|^2 + \tau_+^{-1-\alpha}|\overline{\partial_v}\phi|^2 + \tau_+^{-1-\alpha}r^{-2}\phi^2). \tag{3.32}$$

We now estimate the error term $E(X)$ for general $p \in [0, 1+\alpha_1]$. Relative to $\psi = r\phi$, from the above estimate for $E(X)$, we can estimate $r^2E(X)$ as follows

$$\begin{aligned} r^2|E(X)| \lesssim\ & \delta_1 r^{-1-\epsilon}|\bar{\partial}\psi|^2 + r^{p-\frac{1}{2}}(\bar{H}+H)(|\bar{\partial}\psi||\overline{\partial_v}\psi| + |\bar{\partial}\psi||\phi|) + fH^2|\overline{\partial_v}\psi|^2 + fH\phi^2 \\ & + f\bar{H}(|L\psi||\bar{\partial}\psi| + |\nabla\!\!\!/\psi|^2 + |\bar{\partial}\psi||\phi|) + fH(|L\psi||\overline{\partial_v}\psi| + |\phi||\overline{\partial_v}\psi|) \\ \lesssim\ & \delta_1(r^{-1-\epsilon}|\bar{\partial}\psi|^2 + r^{p-1}|\overline{\partial_v}\psi|^2 + r^{p-1-2\alpha}|L(\psi)||\bar{\partial}\psi| + r^p\tau_+^{-1-\alpha}|L\psi|^2 \\ & + r^{p-1-2\alpha}|\phi||\bar{\partial}\psi| + r^{p-\alpha_2}\tau_+^{-1-\alpha}|\phi|^2), \end{aligned}$$

where $\bar{\partial}\psi = (\partial\psi, \frac{\psi}{1+r})$. Here we have used the assumption (3.2) and Cauchy-Schwartz inequality to obtain the above estimates.

We need to further control $r^{p-1-2\alpha}|\bar{\partial}\psi|(|\phi| + |L\psi|)$. Note that

$$p \le 1+\alpha_1, \quad \alpha_1 + \alpha_2 + \epsilon < 2\alpha + \frac{1}{3}\alpha.$$

Using Jensen's inequality, we can show that for all $p \in [0, 1+\alpha_1]$

$$\begin{aligned} r^{p-1-2\alpha}|\bar{\partial}\psi|(|\phi| + |L\psi|) &\lesssim r_+^{-1-\epsilon}\tau_+^{1-\alpha}|\bar{\partial}\psi|^2 + r^{2p-1-4\alpha+\epsilon}\tau_+^{-1+\alpha}(|L\psi|^2 + |\phi|^2) \\ &\lesssim r_+^{-1-\epsilon}\tau_+^{1-\alpha}|\bar{\partial}\psi|^2 + \left(1 + \tau_+^{-1-\frac{1}{2}\alpha}r^{p-\alpha_2}\right)(|L\psi|^2 + |\phi|^2). \end{aligned}$$

In particular, we can estimate $r^2E(X)$ as follows

$$\begin{aligned} r^2|E(X)| \lesssim\ & \delta_1(\tau_+^{1-\alpha}r^{-1-\epsilon}|\bar{\partial}\psi|^2 + r^{p-1}|\overline{\partial_v}\psi|^2 + r^p\tau_+^{-1-\frac{1}{2}\alpha}|L\psi|^2 \\ & + |L\psi|^2 + r^{p-\alpha_2}\tau_+^{-1-\frac{1}{2}\alpha}|\phi|^2 + |\phi|^2), \end{aligned}$$

As δ_1 is assumed to be small, the integral of the second term will be absorbed. The integral of all the other terms except the first one will be bounded by using Gronwall's inequality. The first term can not be bounded by using the integrated energy inequality directly due to the positive weights in τ_+. However since the integrated energy is expected to decay like $(1+\tau_1)^{-1-\alpha}$, we can use Lemma 2.4.4 to estimate it. In Lemma 2.4.4, take

$$f(\tau) = \int_{S_\tau} (1+r)^{-1-\epsilon} |\bar{\partial}\psi|^2 dv d\omega.$$

Using the integrated energy inequality (3.5) (also holds for $\alpha = \epsilon$), we can show that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{S_\tau} \tau_+^{1-\alpha} r_+^{-1-\epsilon} r^{-2} |\bar{\partial}\psi|^2 d\text{vol} \lesssim \int_{\tau_1}^{\tau_2}\int_{S_\tau} \tau_+^{1-\alpha} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dx d\tau \\
&\lesssim \tilde{E}^\alpha[\phi]_{\tau_1}^{\tau_2} + (\tau_1)_+^{1-\alpha} \tilde{E}[\phi](\tau_1) + (\tau_1)_+^{1-\alpha} D^\epsilon[F]_{\tau_1}^{\tau_2} \\
&\quad + \int_{\tau_1}^{\tau_2} \tau_+^{-\alpha} D^\epsilon[F]_\tau^{\tau_2} d\tau + \delta_1 (\tau_i)_+^{1-\alpha} S^\epsilon[\phi](\tau_i).
\end{aligned}$$

Now in the above estimate for $r^2 E(X)$, we use Lemma 2.4.3 to control the integral of $r^{p-\alpha_2}\phi^2$ and use Lemma 2.4.2 to control the integral of ϕ^2. We end up with

$$\begin{aligned}
&|\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^2 E(X) d\text{vol}| \\
&\lesssim \delta_1 \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1}|\overline{\partial_v}\psi|^2 + (1 + r^p \tau_+^{-1-\frac{1}{2}\alpha})|L\psi|^2 + (1 + r^{p-\alpha_2}\tau_+^{-1-\frac{1}{2}\alpha})|\phi|^2 dv d\omega d\tau \\
&\quad + \delta_1 \int_{\tau_1}^{\tau_2}\int_{S_\tau} \tau_+^{1-\alpha} r_+^{-1-\epsilon} r^{-2} |\bar{\partial}\psi|^2 d\text{vol} \\
&\lesssim \delta_1 \Big(\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1}|\overline{\partial_v}\psi|^2 dv d\omega d\tau + G^{p,1+\frac{1}{2}\alpha}[\phi]_{\tau_1}^{\tau_2} + R^{p-\alpha_2}\tilde{E}^{1+\frac{\alpha}{2}}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^0[\phi]_{\tau_1}^{\tau_2} \\
&+ (\tau_1)_+^{1-\alpha}(\tilde{E}[\phi](\tau_1) + D^\epsilon[F]_{\tau_1}^{\tau_2}) + \int_{\tau_1}^{\tau_2} \tau_+^{-\alpha} D^\epsilon[F]_\tau^{\tau_2} d\tau + (\tau_i)_+^{1-\alpha} S^\epsilon[\phi](\tau_i)\Big).
\end{aligned} \quad (3.33)$$

This gives the estimate for the error term $E(X)$ in the above energy identity (3.31). We use a similar idea to treat the inhomogeneous term $F(X(\phi) + \chi\phi)$. Note that

$$|X(\phi) + \chi\phi| = |fL(\phi) + \chi\phi - 2h^{\underline{L}A} A(\phi) + g^{\underline{LL}} \underline{L}(\phi)| \lesssim \chi|L\psi| + f\bar{H}|\partial\phi| + fH|\overline{\partial_v}\phi|.$$

For $p \leq 1+\alpha_1 < 1+2\alpha$, we can estimate

$$|Ff(\bar{H}|\partial\phi| + H|\overline{\partial_v}\phi|)| \lesssim \delta_1((1+r)^{1+\epsilon}|F|^2 + (1+r)^{-1-\epsilon}|\partial\phi|^2 + |\overline{\partial_v}\phi|^2).$$

For the main term $|F|\chi|L\psi|$, we use Cauchy-Schwartz's inequality to show that

$$\begin{aligned} r^2|F|\chi|L\psi| &\lesssim \delta_1^{-1}\tau_+^{1+(p-1)\alpha_1^{-1}\epsilon}|F|^2 r^{3+\alpha_1} + \delta_1|L\psi|^2 r^{2p-1-\alpha_1}\tau_+^{-1-(p-1)\alpha_1^{-1}\epsilon} \\ &\lesssim \delta_1^{-1}\tau_+^{1+(p-1)\alpha_1^{-1}\epsilon}|F|^2 r^{3+\alpha_1} + \delta_1|L\psi|^2(r^p\tau_+^{-1-\epsilon}+1), \quad p = 1 \text{ or } 1+\alpha_1. \end{aligned}$$

Similarly, we use Lemma 2.4.4 to estimate the first term on the right hand side of the above inequality. And we can show that

$$\begin{aligned} |\int_{\tau_1}^{\tau_2}\int_{S_\tau} F(X(\phi)+\chi\phi)d\text{vol}| &\lesssim \delta_1(\int_{\tau_1}^{\tau_2}\tilde{E}[\phi](\tau)d\tau + G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2}) \\ &+ \int_{\tau_1}^{\tau_2}\tau_+^{(p-1)\alpha_1^{-1}\epsilon}D^{\alpha_1}[F]_\tau^{\tau_2}d\tau + (\tau_1)_+^{(p-1)\alpha_1^{-1}\epsilon}D^{\alpha_1}[F]_{\tau_1}^{\tau_2}. \end{aligned} \tag{3.34}$$

Next we estimate the boundary terms in the energy identity (3.31).

Lemma 3.2.4. *Let X be the vector field defined in line (3.24) on the region $\{r \geq R\}$. Let $f = r^p\kappa_M(x) = r^p\kappa(\frac{|x|-R}{M})$ for the cutoff function κ. Let $\chi = r^{-1}f$. Then we have*

$$\left|\int_{S_\tau} i_{\tilde{J}^X[\phi]}d\text{vol} - \int_{S_\tau} f(\partial_v\psi)^2\sqrt{-G}dvd\omega\right| \lesssim \delta_1(S^\epsilon[\phi](\tau) + R^{p-\alpha_2}\tilde{E}[\phi](\tau) + g^p[\phi](\tau))$$

for all $p \in [1+\alpha_1]$. For the special case when $p=0$, we have

$$\left|\int_{S_\tau} i_{\tilde{J}^X[\phi]}d\text{vol}\right| \lesssim \tilde{E}[\phi](\tau) + \delta_1 S^\epsilon[\phi](\tau).$$

Here the implicit constants are independent of M.

Proof. Recall vector field $\tilde{J}^X[\phi]$ defined in line (2.2). On S_τ, we use the formula (2.5) and we can compute

$$\int_{S_\tau} i_{\tilde{J}^X[\phi]} d\text{vol} = \int_{S_\tau} (-2\partial^{\underline{L}}\phi X(\phi) + X^{\underline{L}}\partial^\gamma\phi\partial_\gamma\phi + \partial^{\underline{L}}\chi\cdot\phi^2 - \chi\partial^{\underline{L}}\phi^2)\sqrt{-G}r^2 dv d\omega.$$

For the special case when $p = 0$, note that

$$|\partial^{\underline{L}}\phi| \lesssim |\overline{\partial_v}\phi| + \bar{H}|\partial\phi|, \quad |X(\phi)| = |-2\partial^{\underline{L}}\phi + g^{\underline{L}\underline{L}}\underline{L}(\phi)| \lesssim |\overline{\partial_v}\phi| + \bar{H}|\partial\phi|, \quad |\partial^{\underline{L}}\chi| \lesssim r^{-2}.$$

Thus we have

$$|\int_{S_\tau} i_{\tilde{J}^X[\phi]} d\text{vol}| \lesssim \int_{S_\tau} |\overline{\partial_v}\phi|^2 + \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} \quad r^2 dv d\omega \lesssim \tilde{E}[\phi](\tau) + \delta_1 S^\epsilon[\phi](\tau).$$

For general p, we expand the integral on S_τ according to the metric decomposition $g = h + m_0$. Recall that $X = f(-2\partial^{\underline{L}} + g^{\underline{L}\underline{L}}\underline{L})$. In particular, $X^{\underline{L}} = -fg^{\underline{L}\underline{L}}$ and the main part of the vector field X is $-2fm_0^{L\underline{L}}L = fL$. Since $\chi = r^{-1}f$, we can write

$$\partial^{\underline{L}}\phi L(\phi) + \frac{1}{2}\chi\partial^{\underline{L}}\phi^2 = \chi\partial^{\underline{L}}\phi\cdot rL(\phi) + \chi\phi\partial^{\underline{L}}\phi = \chi\partial^{\underline{L}}\phi\cdot L(\psi), \quad \psi = r\phi.$$

Therefore we have

$$\begin{aligned}
&r^2(-2\partial^{\underline{L}}\phi X\phi + \partial^{\underline{L}}\chi\cdot\phi^2 - \chi\partial^{\underline{L}}\phi^2) \\
&= -2rf\partial^{\underline{L}}\phi L\psi - \frac{1}{2}\psi^2 L\chi - 2r^2\partial^{\underline{L}}\phi(X - fL)\phi + h^{\underline{L}A}A(\chi)\psi^2 \\
&= rfL(\phi)L(\psi) - \frac{1}{2}\psi^2 L\chi - 2rfh^{\underline{L}A}A(\phi)L(\psi) - 2r^2\partial^{\underline{L}}\phi(X - fL)\phi + h^{\underline{L}A}A(\chi)\psi^2 \\
&= f|L\psi|^2 - \frac{1}{2}L(\chi\psi^2) + Er_4,
\end{aligned}$$

where the error Er_4 can be bounded as follows

$$
\begin{aligned}
|Er_4| \lesssim\ & f|L\psi|(H|\overline{\partial_v}\psi| + H|\phi| + \bar{H}|\partial\psi|) + H|\chi'||\psi|^2 \\
& + f(|L\psi| + |\phi| + \bar{H}|\partial\psi| + H|\nabla\!\!\!/\psi|)(H|\overline{\partial_v}\psi| + H|\phi| + \bar{H}|\partial\psi|) \\
\lesssim\ & fH|L\psi||\overline{\partial_v}\psi| + f\bar{H}|L\psi||\partial\psi| + fH\bar{H}|\partial\psi||\overline{\partial_v}\psi| + fH^2|\overline{\partial_v}\psi| \\
& + fH|\overline{\partial_v}\psi||\phi| + fH\bar{H}|\partial\psi||\phi| + fH|\phi|^2 + f\bar{H}^2|\partial\psi|^2 \\
\lesssim\ & \delta_1(r^{-1-\epsilon}|\partial\psi|^2 + r^2|\overline{\partial_v}\phi|^2 + r^p|L\psi|^2 + r^{p-\alpha_2}|\phi|^2).
\end{aligned}
$$

Similarly, using the estimate (3.12) to control the null form $\partial^\gamma\phi\partial_\gamma\phi$, we have

$$
\begin{aligned}
|r^2X^{\underline{L}}\partial^\gamma\phi\partial_\gamma\phi| &\lesssim f\bar{H}(|L\psi||\partial\psi| + |\phi||\partial\psi| + |\nabla\!\!\!/\psi|^2 + |\phi|^2 + \bar{H}|\partial\psi|^2 + H|\overline{\partial_v}\psi||\partial\psi|) \\
&\lesssim \delta_1(r^{-1-\epsilon}|\partial\psi|^2 + r^2|\overline{\partial_v}\phi|^2 + r^p|L\psi|^2 + r^{p-\alpha_2}|\phi|^2).
\end{aligned}
$$

After integrating over S_τ with measure $dvd\omega$, the first term on the right hand side of the above two inequalities can be controlled by $S^\epsilon[\phi](\tau)$. The integral of the second term gives the energy flux through S_τ. The last term can be controlled by using Lemma 2.4.3. Summarizing, we have shown that

$$
\begin{aligned}
&\left|\int_{S_\tau} i_{\tilde{J}^X[\phi]}d\text{vol} - \int_{S_\tau} f|L\psi|^2\sqrt{-G}dvd\omega + \frac{1}{2}\int_{S_\tau} L(\chi\psi^2)\sqrt{-G}dvd\omega\right| \\
&\lesssim \delta_1(S^\epsilon[\phi](\tau) + R^{p-\alpha_2}\tilde{E}[\phi](\tau) + g^p[\phi](\tau)).
\end{aligned}
$$

The Lemma then follows if we can control the integral of the term $-\frac{1}{2}L(\chi\psi^2)$. We use integration by parts to pass the derivative $L = \partial_v$ to $\sqrt{-G}$. By the assumption (3.2), $L(\sqrt{-G})$ decays better in r. More precisely, using Lemma 2.4.1 and the fact that χ has compact support, we have

$$\left|\int_{S_\tau} L(\chi\psi^2)\sqrt{-G}dvd\omega\right| = \left|\int_{S_\tau} \chi\psi^2 L(\sqrt{-G})dvd\omega - \int_\omega \chi\psi^2\sqrt{-G}d\omega\Big|_{r=R}\right|$$
$$\lesssim \delta_1 \int_{S_\tau} r^{p-\alpha_2}\phi^2 dvd\omega + R^p\tilde{E}[\phi](\tau)$$

Then again using Lemma 2.4.3, we can conclude the lemma. □

The above lemma shows that the energy flux through S_τ is almost equal to $g^p[\phi](\tau)$. We proceed to estimate the other terms in the p-weighted energy identity (3.31). We will estimate the boundary term on C_R later and now we rewrite the energy terms on the right hand side of (3.31) in terms of $\psi = r\phi$. Since $\chi = r^{-1}f$, we have the identity

$$\frac{1}{2}r^2 f'|\partial_v\phi|^2 - \frac{1}{2}r^2\Delta\chi\cdot\phi^2 = \frac{1}{2}f'|\partial_v\psi|^2 - \frac{1}{2}f'\partial_v(r\phi^2) - \frac{1}{2}\partial_v f'\cdot r\phi^2$$
$$= \frac{1}{2}f'|\partial_v\psi|^2 - \frac{1}{2}\partial_v(f'r\phi^2).$$

Here Δ is the Laplacian operator on flat $\mathbb{R}^3$. The first term on the RHS of the above identity is what we want. We use integration by parts to control the integral of the second term. As f has compact support, we can show that

$$-\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{S_\tau}\partial_v(f'r\phi^2)\sqrt{-G}dvd\omega d\tau \tag{3.35}$$
$$= \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{S_\tau} f'r\phi^2\partial_v\sqrt{-G}dvd\omega d\tau + \frac{1}{2}\int_{C_R} f'r\phi^2\sqrt{-G}d\omega d\tau.$$

The first term on the RHS is an error term and we will estimate it later. We now move the second term to the left hand side of the p-weighted energy identity (3.31) and combine it with the original boundary term on C_R. The new boundary term on

C_R can be written as

$$
\begin{aligned}
&-\int_{C_R} i_{\tilde{J}^X[\phi]} d\text{vol} - \frac{1}{2}\int_{C_R} f' r\phi^2 \sqrt{-G} d\omega d\tau \\
&= -f(R)\int_{C_R} i_{J^Y[\phi]} d\text{vol} + \frac{1}{2}\int_{C_R} \left(r^2\partial^r\chi\cdot\phi^2 - fr\partial^r(\phi^2) - f'r\phi^2\right)\sqrt{-G} d\omega d\tau \\
&= -R^p(\int_{C_R} i_{J^Y[\phi]} d\text{vol} + \frac{1}{2}\int_{C_R} \partial^r(r\phi^2)\sqrt{-G} d\tau d\omega) + \frac{1}{2}\int_{C_R} (\partial^r f - f') r\phi^2 \sqrt{-G} d\omega d\tau \\
&= R^p B(C_R) + \frac{1}{2}\int_{C_R} (\partial^r f - f') r\phi^2 \sqrt{-G} d\omega d\tau,
\end{aligned}
$$

where $\partial^r = \partial^L - \partial^{\underline{L}}$ and we have used $B(C_R)$ to denote the integral. We see that $B(C_R)$ is independent of the power p. Hence to control the boundary term $B(C_R)$, it suffices to take $p = 0$, which is essentially the energy estimates we have done in the previous section. We will estimate the boundary term $B(C_R)$ later. We now group the error term on the boundary C_R in the above inequality with the error term in (3.35) and denote

$$
Er_5 = \frac{1}{2}\int_{C_R} (\partial^r f - f') r\phi^2 \sqrt{-G} d\omega d\tau - \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{S_\tau} f' r\phi^2 \partial_v \sqrt{-G} dv d\omega d\tau.
$$

Since $|\partial_v\sqrt{-G}| \lesssim \bar{H}$, using Lemma 2.4.1 and Lemma 2.4.2, we can estimate Er_5

$$
|Er_5| \lesssim \int_{\tau_1}^{\tau_2} (\int_\omega \bar{H} R^p \phi^2 d\omega + \int_{S_\tau} f\bar{H}\phi^2 dv d\omega) d\tau \lesssim \delta_1 R^{p-1-2\alpha} \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau. \quad (3.36)
$$

Now from the p-weighted energy identity (3.31), the above discussion leads to the following energy estimate

$$
\begin{aligned}
&\int_{S_{\tau_1}} i_{\tilde{J}^X[\phi]} d\text{vol} - \int_{S_{\tau_2}} i_{\tilde{J}^X[\phi]} d\text{vol} + R^p B(C_R) + Er_5 \qquad (3.37) \\
&= \int_{\tau_1}^{\tau_2}\int_{S_\tau} F(X(\phi) + \chi\phi) + \frac{1}{2} f' r^{-2} |\partial_v \psi|^2 + r^{-2}(\chi - \frac{1}{2} f')|\nabla\!\!\!/\psi|^2 + E(X) d\text{vol}.
\end{aligned}
$$

The boundary term on S_τ is almost equal to $g^p[\phi](\tau)$ by Lemma 3.2.4. If $f = r^p$, the energy term on the right hand side will give us a positive sign. This will be made to be rigorous by taking the limit $M \to \infty$. Here we recall that M is the parameter in the cutoff function κ_M.

Finally in the above energy identity (3.37), we estimate the boundary term $B(C_R)$ by taking $p = 0$. For this case $f \leq 1$, $|\chi| \leq r^{-1}$. The inhomogeneous term $F(X(\phi) + \chi\phi)$ can be bounded by

$$|F(X(\phi) + \chi\phi)| \lesssim |F||\bar{\partial}\phi| \lesssim |F|^2(1+r)^{1+\epsilon} + (1+r)^{-1-\epsilon}|\bar{\partial}\phi|^2.$$

Now if

$$\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau < \infty,$$

then we have

$$\lim_{M\to\infty} |\int_{\tau_1}^{\tau_2}\int_{S_\tau}(\kappa_M)' r^{-2}(|\partial_v\psi|^2 - |\nabla\!\!\!/\psi|^2)d\text{vol}| \lesssim \lim_{M\to\infty}\frac{1}{M}\int_{\tau_1}^{\tau_2}\tilde{E}[\phi](\tau)d\tau = 0.$$

Therefore let $M \to \infty$ in the above energy identity (3.37) with $p = 0$. Using estimate (3.32) to control the error term $E(X)$ and estimate (3.36) to bound Er_5 and Lemma 3.2.4 to control the boundary terms on S_{τ_i}, $i = 1, 2$, we then have the estimate for the boundary term $B(C_R)$

$$\begin{aligned}|B(C_R)| \lesssim \delta_1 R^{-1-2\alpha}\int_{\tau_1}^{\tau_2}\tilde{E}[\phi](\tau)d\tau + \tilde{E}[\phi](\tau_i) + \delta_1 S^\epsilon[\phi](\tau_i) + \int_{\tau_1}^{\tau_2}\tau_+^{-1-\alpha}\tilde{E}[\phi](\tau)d\tau \\ + D^\epsilon[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} + \frac{|\nabla\!\!\!/\phi|^2}{1+r}dxd\tau.\end{aligned}$$

If $\int_{\tau_1}^{\tau_2}\tilde{E}[\phi](\tau)d\tau = \infty$, then the above estimate for $B(C_R)$ holds automatically. Now we use the integrated energy inequality (3.5) and the energy inequality (3.6) to improve the above estimate for the boundary term $B(C_R)$. We have

$$|B(C_R)| \lesssim \delta_1 R^{-1-2\alpha} \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau + \tilde{E}[\phi](\tau_1) + \delta_1 S^\epsilon[\phi](\tau_i) + D^\epsilon[F]_{\tau_1}^{\tau_2}, \qquad (3.38)$$

where $S^\epsilon[\phi](\tau_i) = S^\epsilon[\phi](\tau_1) + S^\epsilon[\phi](\tau_2)$. This gives the estimate for the boundary term $B(C_R)$ in the above energy identity (3.37).

Now in the above energy identity (3.37), we have estimate (3.33) for the error term $E(X)$, estimate (3.34) for the inhomogeneous term $F(X\phi + \chi\phi)$, estimate (3.36) for the error term Er_5 and the above estimate (3.38) for the boundary term $B(C_R)$. The boundary term on S_τ has been discussed in Lemma 3.2.4. As the function f, χ depends on the parameter M in the cutoff function κ_M, we now argue that we can push the parameter M to infinity and conclude the p-weighted energy inequalities (3.23), (3.22).

Without loss of generality, we can assume that κ is decreasing. We find that

$$\frac{1}{2}f' = \frac{1}{2}pr^{p-1}\kappa_M + \frac{1}{2}M^{-1}r^p\kappa'(\frac{r-R}{M}),$$
$$\chi - \frac{1}{2}f' = (1-\frac{p}{2})r^{p-1}\kappa_M - \frac{1}{2}M^{-1}r^p\kappa'(\frac{r-R}{M}) \geq (1-\frac{p}{2})r^{p-1}\kappa_M.$$

Note that κ' is supported on $[1, 2]$. We conclude that if

$$G^{p-1,0}[\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1}|L\psi|^2 dv d\omega d\tau$$

is finite, then

$$\lim_{M\to\infty} |\int_{\tau_1}^{\tau_2}\int_{S_\tau} M^{-1}r^p\kappa'(\frac{r-R}{M})|L\psi|^2 r^{-2} dv d\omega d\tau$$
$$\leq \lim_{M\to\infty}\int_{\tau_1}^{\tau_2}\int_{r\geq M} r^{p-1}|L\psi|^2 dv d\omega d\tau = 0.$$

We first consider the p-weighted energy inequality when $p = 1$. Note that

$$G^{0,0}[\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^2 |L\phi|^2 - L(r\phi^2) dv d\omega d\tau \lesssim \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau.$$

For fixed $\tau_1 < \tau_2$, it suffices to prove Proposition 3.2.2 when

$$\tilde{E}[\phi](\tau_1) + S^\epsilon[\phi](\tau_1) + S^\epsilon[\phi](\tau_2) + D^\epsilon[F]_{\tau_1}^{\tau_2} < \infty.$$

Otherwise, all the estimates in Proposition 3.2.2 hold automatically for $\tau_1 < \tau_2$. In this case using the energy estimate (3.6), we have

$$\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau \leq C(\tau_1, \tau_2)(\tilde{E}[\phi](\tau_1) + S^\epsilon[\phi](\tau_1) + S^\epsilon[\phi](\tau_2) + D^\epsilon[F]_{\tau_1}^{\tau_2}) < \infty,$$

where $C(\tau_1, \tau_2)$ is constant. This further implies that $G^{0,0}[\phi]_{\tau_1}^{\tau_2}$ is finite. Therefore by the argument above, we can let M go to infinity in the p-weighted energy inequality (3.37) with $p = 1$ and we can conclude from the estimates (3.33), (3.34), (3.36), (3.38) together with Lemma 3.2.4 that

$$\begin{aligned} &g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\overline{\partial_v}\psi|^2 dv d\omega d\tau \lesssim \delta_1 (g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\overline{\partial_v}\psi|^2 dv d\omega d\tau) \\ &+ \delta_1 (\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau + G^{1,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}) + R(\tau_1)_+^{1-\alpha} \tilde{E}[\phi](\tau_1) + \delta_1 R(\tau_i)_+^{1-\alpha} S^\epsilon[\phi](\tau_i) \quad (3.39) \\ &+ g^1[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_\tau^{\tau_2} d\tau + R(\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2}, \end{aligned}$$

where we used the energy inequality (3.6) to estimate $\tilde{E}^{1+\frac{1}{2}\alpha}[\phi]_{\tau_1}^{\tau_2}$. For small δ_1 the first two terms can be absorbed. Usually $G^{1,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}$ can be bounded by using Gronwall's inequality. However, due to the presence of $S^\epsilon[\phi](\tau_i)$ on the right hand side, we can not use Gronwall's inequality directly. However we can take $\tau_2 = \tau$ in the above inequality. Multiply both side by $\tau_+^{-1-\epsilon}$ and then integrate it with respect to τ from τ_1 to τ_2. We retrieve $G^{1,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}$ on the left hand side. The same term will appear on the right hand side but with the small coefficient δ_1. We thus can estimate it.

Now to prove the p-weighted energy inequality (3.23) with $p = 1$, we need to recover $\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau$ on the left hand side of (3.39). Note that

$$\int_{S_\tau} |\overline{\partial_v}\psi|^2 dv d\omega = \int_{S_\tau} |\overline{\partial_v}\phi|^2 r^2 + L(r\phi^2) dv d\omega = \int_{S_\tau} |\overline{\partial_v}\phi|^2 r^2 dv d\omega - \int_\omega r\phi^2 d\omega\Big|_{r=R}.$$

For $R \geq 1$, we have

$$R^3 \int_\omega \phi^2(\tau, R, \omega) d\omega = \int_0^R \int_\omega \partial_r(r^3\phi^2) d\omega dr \leq 3 \int_{r\leq R} \phi^2 dx + \int_{r\leq R} R^2 |\partial_r\phi|^2 + \phi^2 dx.$$

Hence we have

$$R \int_\omega \phi^2(\tau, R, \omega) d\omega \leq 8 \int_{r\leq R} |\partial\phi|^2 + (1+R)^{-2}\phi^2 dx \leq 8 \int_{r\leq R} |\bar{\partial}\phi|^2 dx.$$

Now add both side of the above p-weighted energy inequality (3.39) when $p = 1$ with

$$\int_{\tau_1}^{\tau_2} \int_{r\leq R} |\partial\phi|^2 dx d\tau + R \int_{\tau_1}^{\tau_2} \int_\omega \phi^2(\tau, R, \omega) d\omega d\tau \lesssim (1+R)^{1+\epsilon} I^\epsilon[\phi]_{\tau_1}^{\tau_2}.$$

Then the left hand side of (3.39) becomes

$$g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau.$$

For small δ_1 the first term in the second line of (3.39) can be absorbed. Then using the integrated energy inequality (3.5) to control $I^\epsilon[\phi]_{\tau_1}^{\tau_2}$, we can conclude from (3.39) the p-weighted energy inequality (3.23) for $p = 1$.

Finally we prove the p-weighted energy inequality (3.22) when $p = 1 + \alpha_1$. Having the p-weighted energy inequality when $p = 1$, which in particular gives the bound for $g^1[\phi](\tau)$ (we may assume the right hand side of (3.22) is finite), we conclude that

$\int_{\tau_1}^{\tau_2} g^1[\phi](\tau)d\tau$ is finite. In particular, we have

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\alpha_1}|L\psi|^2 dvd\omega d\tau \leq \int_{\tau_1}^{\tau_2} g(1,\tau)d\tau < \infty.$$

Therefore in the p-weighted energy inequality (3.37) we can set $p = 1+\alpha_1$ and then let the parameter M in the cutoff function go to infinity. Similar to the above p-weighted energy inequality when $p = 1$, for small δ_1, we can show that

$$\begin{aligned} g^{1+\alpha_1}[\phi](\tau_2) + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\alpha_1}|\overline{\partial_v}\psi|^2 dvd\omega d\tau &\lesssim g^{1+\alpha_1}[\phi](\tau_1) + \delta_1\int_{\tau_1}^{\tau_2}\tilde{E}[\phi](\tau)d\tau \\ + \delta_1 G^{1+\alpha_1,1+\epsilon}[\phi]_{\tau_1}^{\tau_2} + R^{1+\alpha_1}(\tau_1)_+^{1-\alpha}\tilde{E}[\phi](\tau_1) &+ \delta_1 R^{1+\alpha_1}(\tau_i)_+^{1-\alpha}S^\epsilon[\phi](\tau_i) \\ + \int_{\tau_1}^{\tau_2}(\tau)_+^\epsilon D^{\alpha_1}[F]_\tau^{\tau_2}d\tau + R^{1+\alpha_1}(\tau_1)_+^{1+\epsilon}D^{\alpha_1}[F]_{\tau_1}^{\tau_2}. \end{aligned}$$

The integral of the energy flux $\tilde{E}[\phi](\tau)$ from τ_1 to τ_2 can be controlled by the p-weighted energy inequality (3.23) with $p = 1$. To estimate $G^{1+\alpha_1,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}$, we set $\tau_2 = \tau$ in the above inequality and then integrate both side with weights $\tau_+^{-1-\epsilon}$ from τ_1 to τ_2. And then we can conclude (3.22) from the above inequality. We thus finished the proof of Proposition 3.2.2.

3.3 Energy decay estimates

We have shown in the previous two sections the integrated energy estimates (Proposition 3.1.1) and the p-weighted energy inequalities (Proposition 3.2.1 and 3.2.2) without using any vector fields with positive weights in t. We now argue that under appropriate assumptions on the inhomogeneous term F as well as the data on the initial hypersurface Σ_0 the energy flux $E[\phi](\tau)$ decays in τ.

We still consider the linear wave equation (3.1) on $(\mathbb{R}^{3+1}, g)$ with metric g satisfies the conditions in Proposition 3.2.2. Let E_0 denote the size of the data on Σ_0

$$E_0 := \tilde{E}[\phi](0) + S^\epsilon[\phi](0) + g^{p_m}[\phi](0),$$

where

$$p_m = 2, \text{ if } \delta_1 = 0; \quad p_m = 1 + \alpha_1, \text{ otherwise .}$$

Here the small constant ϵ, α_1 are the ones satisfy the condition in Proposition 3.2.2. We always assume that E_0 is finite.

We first recall the derivation of the decay of the energy flux in Minkowski space and then we will show that the integrated energy $I^\epsilon[\phi]_{\tau_1}^{\tau_2}$ decays in τ_1 on asymptotically flat spacetimes.

3.3.1 Energy flux decay in Minkowski space

In this subsection we assume in addition that the metric g is flat when $|x| \geq R$. In particular, we have $\delta_1 = 0$ in the assumption (3.2). We have

Proposition 3.3.1. *Assume F satisfies the following condition*

$$D_+^1[F]_{\tau_1}^{\tau_2} \leq C_1(\tau_1)_+^{-1-2\epsilon}, \quad D^\epsilon[F]_{\tau_1}^{\tau_2} \leq C_1(\tau_1)_+^{-2} + C_0 E[\phi](\tau_1)$$

for some fixed constant C_1 and some constant C_0 depending on α, R. Then we have the energy flux decay

$$E[\phi](\tau) \leq C_{\alpha,R}(E_0 + C_1)(\tau)_+^{-2}$$

for some constant $C_{\alpha,R}$ depending on α, R.

The proof below is similar to that in [10] for the case when $F = 0$.

Proof. The assumption above in particular implies that $D^\epsilon[F]_0^\infty$ is finite. Therefore by the boundedness of the integrate energy estimate (3.4) we have $I^\epsilon[\phi]_0^\infty$ is finite.

Then by the energy estimate (3.6) we conclude that the energy flux $\tilde{E}[\phi](\tau)$ is finite for all $\tau \geq 0$. Now from Lemma 2.4.1, we infer that all the previous estimates holds if we replace $\tilde{E}[\phi](\tau)$ with $E[\phi](\tau)$. Below in this proof, we only need to consider $E[\phi](\tau)$.

Let $p = 2$ in the p-weighted energy inequality (3.19) (take $\alpha = \epsilon$ there). We obtain

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r|\partial_v \psi|^2 d\omega dv d\tau \leq \int_0^{\tau_2} \int_{S_\tau} r|\partial_v \psi|^2 d\omega dv d\tau \lesssim E_0 + C_1. \tag{3.40}$$

Here in this section we also allow the implicit constant to depend on R. We can extract a dyadic sequence $\{\tau_n\}_{n=3}^{\infty}$ such that

$$\int_{S_{\tau_n}} r|\partial_v \psi|^2 dv d\omega \leq (\tau_n)_+^{-1} \left(\epsilon^2 E_0 + C_1\right), \tag{3.41}$$

where τ_n satisfies the inequality

$$\gamma^{-2} \tau_n \leq \tau_{n-1} \leq \gamma^2 \tau_n$$

for some large constant γ depending on R, α. It suffices to show that there exists $\tau_n \in [\gamma^n, \gamma^{n+1}]$ such that (3.41) holds. Otherwise we have

$$\int_{\gamma^k}^{\gamma^{k+1}} \int_{S_\tau} r|\partial_v \psi|^2 dv d\omega d\tau \geq \ln \gamma \left(\epsilon^2 E_0 + C_1\right),$$

which contradicts to (3.40) if γ is sufficiently large.

Next set $\tau_2 = \tau \in [\tau_{n-1}, \tau_n]$, $\tau_1 = \tau_{n-1}$, $p = 1$ in the p-weighted energy inequality

(3.19). We use the estimate (3.41) to control the boundary term on $S_{\tau_{n-1}}$. We have

$$\begin{aligned}
&\int_{S_\tau} r|\partial_v\psi|^2 d\omega dv + \int_{\tau_{n-1}}^{\tau}\int_{S_t} |\partial_v\psi|^2 + |\nabla\!\!\!/\psi|^2 d\omega dv dt \\
&\lesssim \int_{S_{\tau_{n-1}}} r|\partial_v\psi|^2 d\omega dv + E[\phi](\tau_{n-1}) + (\tau_{n-1})_+^{-1} C_1 \\
&\lesssim (\tau_{n-1})_+^{-1}(E_0 + C_1) + E[\phi](\tau_{n-1}) \\
&\lesssim (\tau)_+^{-1}(E_0 + C_1) + E[\phi](\tau_{n-1}).
\end{aligned} \tag{3.42}$$

We need to recover the full energy flux $E[\phi](\tau)$ through Σ_τ. Note that

$$\int_{S_\tau} |\partial_v\psi|^2 + |\nabla\!\!\!/\psi|^2 d\omega dv \geq \int_{S_\tau} (|\partial_v\phi|^2 + |\nabla\!\!\!/\phi|^2) r^2 d\omega dv - R\int_\omega \phi^2(\tau, R, \omega) d\omega$$

For the integral on the sphere of constant R, we have

$$R^3\phi^2(\tau, R, \omega) \leq 3\int_0^R r^2\phi^2 dr + R\int_0^R (\phi^2 + |\partial_r\phi|^2) r^2 dr.$$

Then from (3.42) and the integrated local energy estimate (3.5), we can show that

$$\begin{aligned}
\int_{\tau_{n-1}}^{\tau_n} E[\phi](\tau) d\tau &= \int_{\tau_{n-1}}^{\tau_n}\int_{r\leq R} |\partial\phi|^2 dx d\tau + \int_{\tau_{n-1}}^{\tau_n}\int_{S_\tau} (|\partial_v\phi|^2 + |\nabla\!\!\!/\phi|^2) r^2 dv d\omega d\tau \\
&\lesssim \int_{\tau_{n-1}}^{\tau_n}\int_{r\leq R} |\partial\phi|^2 + \phi^2 dx d\tau + \int_{\tau_{n-1}}^{\tau_n}\int_{S_\tau} |\partial_v\psi|^2 + |\nabla\!\!\!/\psi|^2 dv d\omega d\tau \\
&\lesssim E[\phi](\tau_{n-1}) + (\tau_{n-1})_+^{-1+\alpha}(E_0 + C_1).
\end{aligned}$$

Now the energy estimate (3.6) shows that for all $\tau \leq \tau_n$, we have

$$E[\phi](\tau_n) \lesssim E[\phi](\tau) + D^\epsilon[F]_\tau^{\tau_n} \lesssim E[\phi](\tau) + (\tau)_+^{-2}(E_0 + C_1).$$

Hence the previous inequality implies that

$$(\tau_n - \tau_{n-1})E[\phi](\tau_n) \lesssim E[\phi](\tau_{n-1}) + (\tau_{n-1})_+^{-1+\alpha}(E_0 + C_1).$$

Since the sequence $\{\tau_n\}$ is dyadic, we get

$$E[\phi](\tau_n) \lesssim \tau_n^{-1}E[\phi](\tau_{n-1}) + (\tau_n)_+^{-2}(E_0 + C_1).$$

Now note that the energy inequality (3.6) quickly implies the boundedness of the energy flux $E[\phi](\tau)$. That is

$$E[\phi](\tau) \lesssim E_0 + C_1, \quad \forall \tau \geq 0.$$

Thus the previous estimate implies the improved decay for $\tau = \tau_n$

$$E[\phi](\tau_n) \lesssim \tau_n^{-1}E[\phi](\tau_{n-1}) + (\tau_n)_+^{-2}(E_0 + C_1) \lesssim \tau_n^{-1}(E_0 + C_1).$$

This in turn shows that

$$E[\phi](\tau_n) \lesssim (\tau_n)_+^{-2}(E_0 + C_1) + \tau_n^{-1}E[\phi](\tau_{n-1}) \lesssim (\tau_n)_+^{-2}(E_0 + C_1).$$

For general $\tau \in [\tau_{n-1}, \tau_n]$, we use the energy estimate (3.6) and the fact that τ_n is dyadic. We have

$$E[\phi](\tau) \lesssim E[\phi](\tau_{n-1}) + (\tau)_+^{-2}(E_0 + C_1) \lesssim (\tau)_+^{-2}(E_0 + C_1).$$

Here the implicit constants depend on α, R. □

This proposition immediately implies that for solutions of linear wave equations with compactly supported initial data the energy flux $E[\phi](\tau)$ decays in τ.

Theorem 3.3.2 (Dafermos-Rodnianski [10]). *Consider $\Box\phi = 0$ in Minkowski space with compactly support initial data. Then we have*

$$E[\phi](\tau) \leq C\tau_+^2, \quad \forall \tau \geq 0.$$

After commuting the equation with the angular momentum, the pointwise decay of the solution then follows from Lemma (2.4.1). This is the framework of the new approach in obtaining the pointwise decay of the solution without using any vector fields growing in time t.

3.3.2 Integrated energy decay on asymptotically flat spacetimes

We now consider the case when $\delta_1 \neq 0$. We are not able to show the decay of the energy flux $E[\phi](\tau)$. However, we can show that integrated energy $I^\epsilon[\phi]_\tau^\infty$ decays.

Proposition 3.3.3. *Assume that the inhomogeneous term F satisfies*

$$D^{\alpha_1}[F]_{\tau_1}^{\tau_2} \leq C_1(\tau_1)_+^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1.$$

Then we have the integrated energy decay

$$I^\epsilon[\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dx d\tau \leq C_{\alpha,R}(E_0 + C_1)(\tau_1)_+^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1.$$

Here the constant $C_{\alpha,R}$ depends on α, R. And we recall here that the small positive constants ϵ, α_1 satisfy the relations in Proposition 3.2.2.

Compared to the case when the metric is flat outside the cylinder with radius R, the main difficulty is the presence of $S^\epsilon[\phi](\tau_2)$ on the right hand side of estimates in Proposition 3.1.1, 3.2.2. The idea is that we first show that $I^\epsilon[\phi]_0^\infty$ is finite from the estimate (3.4). And then we can extract a sequence $\{\tau_n\}$ such that $S^\epsilon[\phi](\tau_n)$ decays. This will lead to the decay of the integrate energy.

Proof. We already argued in the beginning of the proof of Proposition 3.3.1 that $I^\epsilon[\phi]_0^\infty$ is finite. In particular, we can conclude that there is a sequence $\tau_n \to \infty$ such that $S^\epsilon[\phi](\tau_n)$ is finite. Then from the energy inequality (3.6), we infer that $\tilde{E}[\phi](\tau_n)$ is finite. In particular $E^N[\phi]_0^{\tau_n}$ is finite for all n. Since $\tau_n \to \infty$, we have $E^N[\phi]_0^\tau$ is finite for all τ. By Lemma 2.4.1 all the previous estimates hold if we replace $\tilde{E}[\phi](\tau)$ with $E[\phi](\tau)$.

Denote $M = E_0 + C_1$. Without loss of generality we may assume $M > 1$. For some big constant C_2 depending only on R and α, M assume

$$I^\epsilon[\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dx d\tau \leq C_2 M (1+\tau_1)^{-\beta}, \quad \forall \tau_2 \geq \tau_1 \geq 0 \tag{3.43}$$

for some $\beta \in [0, 1+\alpha]$. Since $I^\epsilon[\phi]_0^\infty$ is finite, the above estimate holds for $\beta = 0$. The proposition claims that it holds for $\beta = 1+\alpha$. We define a nonempty set

$$T = \{\tau \left| S^\epsilon[\phi](\tau) \leq \int_{\Sigma_\tau} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dx \leq 10 C_2 M \tau_+^{-1-\beta}\right.\}. \tag{3.44}$$

Let $\tau_1 = 0$ in the p-weighted energy inequality (3.22) with weights $r^{1+\alpha_1}$. We obtain

$$\int_{S_\tau} r^{1+\alpha_1} (\partial_v \psi)^2 dv d\omega \lesssim M, \quad \forall \tau \in T.$$

By the definition of T, we have

$$\int_{S_\tau} \frac{(\partial_v \psi)^2}{(1+r)^{1+\epsilon}} dv d\omega \lesssim M \tau_+^{-1-\beta}, \quad \forall \tau \in T.$$

Here recall that $\psi = r\phi$. Interpolate between the above two inequalities. We get

$$\int_{S_\tau} r(\partial_v \psi)^2 dv d\omega \lesssim M(1+\tau)^{-\theta\alpha}, \quad \forall \tau \in T,$$

where

$$\theta = \min\{1, \frac{(1+\beta)\alpha_1}{(2+\alpha_1+\epsilon)\alpha}\}.$$

Then the p-weighted energy inequality (3.23) when $p = 1$ implies that

$$\int_{\tau_1}^{\tau_2} E[\phi](\tau) d\tau \lesssim M(\tau_1)_+^{-\theta\alpha} + (\tau_1)_+^{1-\alpha} E[\phi](\tau_1), \quad \forall \tau_1, \tau_2 \in T.$$

Now the energy inequality (3.6) shows

$$E[\phi](\tau_2) \lesssim E[\phi](\tau) + S^\epsilon[\phi](\tau) + (\tau_2)_+^{-1-\alpha} + M(1+\tau_2)^{-1-\beta}, \forall \tau \leq \tau_2, \tau_2 \in T.$$

In particular, we have

$$E[\phi](\tau_2) \lesssim M, \quad \forall \tau_2 \in T.$$

Combine this with the previous two estimates. We can show that

$$\begin{aligned}(\tau_2 - \tau_1) E[\phi](\tau_2) \lesssim & M(\tau_1)_+^{-\theta\alpha} + (\tau_1)_+^{1-\alpha} E[\phi](\tau_1) \\ & + (\tau_2 - \tau_1)(\tau_2)_+^{-1-\alpha} + M(\tau_2 - \tau_1)(\tau_2)_+^{-1-\beta} + M(\tau_1)_+^{-\beta}\end{aligned} \tag{3.45}$$

for all τ_1, $\tau_2 \in T$, $\tau_1 \leq \tau_2$. Since $0 \in T$, in particular, let $\tau_1 = 0$. We get

$$E[\phi](\tau_2) \lesssim M(\tau_2)_+^{-1}, \quad \forall \tau_2 \in T.$$

Now, fix $\tau_1 \in T$, $\tau_1 \geq 1$. We can always choose $\tau_2 \in T$ such that

$$2\tau_1 \leq \tau_2 \leq 4\tau_1.$$

Otherwise by the definition of τ, we have

$$10C_2M\int_{2\tau_1}^{4\tau_1}\tau_+^{-1-\beta}d\tau=\frac{6C_2M}{\beta}((1+2\tau_1)^{-\beta}-(1+4\tau_1)^{-\beta})<C_2M(1+2\tau_1)^{-\beta}.$$

This is impossible as $\beta\leq 1+\alpha<2$, $\tau_1\geq 1$. For such τ_1 and τ_2, the estimate (3.45) then implies that

$$E[\phi](\tau_2)\lesssim C_1(\tau_2)_+^{-1-\alpha}+M(\tau_2)_+^{-1-\beta}+M(\tau_2)_+^{-1-\theta\alpha},\quad \forall\tau_2\in T. \tag{3.46}$$

Therefore from the integrated energy estimate (3.5), we can improve the integrated energy

$$I^{\epsilon}[\phi]_{\tau_1}^{\tau_2}\lesssim M(\tau_1)_+^{-1-\alpha}+M(\tau_1)_+^{-1-\beta}+M(\tau_1)_+^{-1-\theta\alpha},\quad \forall\tau_1,\tau_2\in T.$$

As the set T contains arbitrarily large τ, the above estimate holds for all $\tau_2\geq\tau_1$, $\tau_1\in T$. For general $\tau_1\geq 4$, note that we can choose $\tilde{\tau}_1\in T$ such that

$$\frac{1}{2}\tau_1\leq\tilde{\tau}_1\leq\tau_1.$$

Hence the above improved integrated energy inequality holds for all $0\leq\tau_1\leq\tau_2$. In particular, as estimate (3.43) holds for $\beta=0$, we conclude that

$$I^{\epsilon}[\phi]_{\tau_1}^{\tau_2}\lesssim M(\tau_1)_+^{-1},\quad \forall\tau_1\leq\tau_2.$$

That is the estimate (3.43) holds for $\beta=1$. Now from the definition of θ and estimate (3.46) we again can show that (3.43) holds for

$$\beta=1+\min\{\alpha,\frac{2\alpha_1}{2+\alpha_1+\epsilon}\}.$$

Recall that

$$\frac{2\alpha + \alpha\epsilon}{2 - \alpha} \leq \alpha_1.$$

Therefore estimate (3.46) holds for

$$\beta = 1 + \min\{\alpha, \frac{2\alpha_1}{2 + \alpha_1 + \epsilon}\} = 1 + \alpha.$$

We thus finished the proof for the proposition. $\square$

Chapter 4

Semilinear wave equations

4.1 Statement of Theorem 1.0.1

In this chapter, we consider the Cauchy problem to the semilinear wave equations

$$\begin{cases} \Box_g \phi = \frac{1}{\sqrt{-G}} \partial_\alpha \left(g^{\alpha\beta} \sqrt{-G} \partial_\beta \phi \right) = F(\phi, \partial\phi), \\ \phi(0,x) = \phi_0(x), \partial_t \phi(0,x) = \phi_1(x) \end{cases} \tag{4.1}$$

on a Lorentzian manifold $(\mathbb{R}^{3+1}, g)$ with initial data $\phi_0(x)$, $\phi_1(x) \in C_0^\infty(\mathbb{R}^3)$.

We can write the metric g as $g = h + m_0$, where m_0 is the Minkowski metric on $\mathbb{R}^{3+1}$. For some large constant R, we assume that h is smooth and is supported on $\{|x| \leq \frac{1}{2}R\}$. In addition we assume the metric g satisfies the following two integrated local energy estimates

$$\int_{\tau_1}^{\tau_2} \int_{r \leq R} |\partial\phi|^2 + \frac{\phi^2}{r} dx dt \leq C_0(\tilde{E}[\phi](\tau_1) + D^\epsilon[\Box_g \phi]_{\tau_1}^{\tau_2}), \tag{4.2}$$

$$\int_{\tau_1}^{\tau_2} \int_{r \leq \frac{1}{2}R} |\partial\partial_t \phi|^2 dx dt \leq C_0(\tilde{E}[\partial_t \phi](\tau_1) + D^\epsilon[\partial_t \Box_g \phi]_{\tau_1}^{\tau_2} + \tilde{E}[\phi](\tau_1) \tag{4.3}$$

$$+ D^\epsilon[\Box_g \phi]_{\tau_1}^{\tau_2}).$$

for some small positive constant ϵ and for all smooth functions ϕ.

We assume the nonlinearity F satisfies the null condition outside the large cylinder $\{(t,x)||x|\leq R\}$, that is,

$$F(\phi,\partial\phi)=A^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+O(|\phi|^3+|\partial\phi|^3),\quad |x|\geq R,$$

where $A^{\alpha\beta}$ constants such that $A^{\alpha\beta}\xi_\alpha\xi_\beta=0$ whenever $\xi_0^2=\xi_1^2+\xi_2^2+\xi_3^2$. Inside the cylinder F is at least quadratic in terms of ϕ, $\partial\phi$.

We assume the initial data ϕ_0, ϕ_1 are supported on $\{|x|\leq R\}$. Let

$$E_0:=\sum_{k+j\leq 8,k\leq 5}\int_{\mathbb{R}^3}|\partial\Omega^k T^j\phi|^2(0,x)dx.$$

Here in this chapter we may use T to denote the vector field ∂_t. We remark here that E_0 is uniquely determined by ϕ_0, ϕ_1 together with the equation (4.1).

Our main results are:

Theorem 4.1.1. *Consider the Cauchy problem for the semilinear wave equations* (4.1) *on* $(\mathbb{R}^{3+1},g)$. *There exists a positive constant* ϵ_0, *depending on* R, ϵ, E_0, C_0, $h^{\mu\nu}$, *such that if the initial data* $E_0<\epsilon_0$, *then the equation* (4.1) *admits a unique global smooth solution* ϕ *with the following properties:*

(1) Energy decay

$$E[\phi](\tau)\leq CE_0\tau_+^{-2};$$

(2) Pointwise decay: for any $0<\delta$

$$|\phi|\leq C_\delta\sqrt{E_0}(1+r)^{-1}(1+|t-r+R|)^{-\frac{1}{2}+\delta},$$
$$\sum_{|\beta|\leq 2}|\partial^\beta\phi|\leq C\sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+|t-r+R|)^{-1},$$

where the constants C, C_δ *depend* R, ϵ, E_0, C_0, $h^{\mu\nu}$ *and* C_δ *also depends on* δ.

Remark 4.1.2. *Similar results hold for the corresponding problems in higher dimensions without null condition or systems of semilinear wave equations satisfying the null condition outside a large cylinder.*

Remark 4.1.3. *The same conclusion holds if the metric g satisfies the integrated local energy inequality* (4.2) *and the deformation tensor $\pi^T_{\mu\nu} = \frac{1}{2}\partial_t g_{\mu\nu}$ is small, independent of the initial data. We remark that this is consistent with our attempt to investigate nonlinear wave equations on backgrounds far from Minkowski space.*

Remark 4.1.4. *Since we only require the smallness of the C^1 norm of the metric perturbation h, the curvature of the background spacetime $(\mathbb{R}^{3+1}, g)$ can be arbitrarily large as long as the curvature is uniformly bounded.*

Remark 4.1.5. *It is not necessary to require that the initial data have compact support. The general assumption on the initial data can be that the following quantity*

$$\sum_{k+j\leq 8, k\leq 5} \iint_{\mathbb{R}^3} r^2 |\partial \Omega^k T^j \phi(0,x)|^2 dx$$

is sufficiently small. For more details, we will discuss it in Chapter 7.

The above theorem claims that the small data global existence result can be reduced to the two integrated local energy inequalities. Below, we describe particular conditions for which the assumption these two estimates can be explicitly verified. We will show that under the assumption that g is merely $C^1(\mathbb{R}^{3+1})$ close to the Minkowski metric, then we have the two integrated local energy estimates (4.2), (4.3). More precisely, denote

$$H_0 = \|g - m_0\|_{C^1(\mathbb{R}^{3+1})} = \|h\|_{C^1(\mathbb{R}^{3+1})}.$$

Then we have

Theorem 4.1.6. *Suppose h is supported on the cylinder $\{(t,x) \| x| \leq \frac{R}{2}\}$. Then there* *exists a positive constant ϵ_1, depending only on R, such that if $H_0 < \epsilon_1$, then the metric g satisfies the estimates* (4.2), (4.3). *Hence Theorem 4.1.1 holds.*

Remark 4.1.7. *From Theorem 4.1.6, we retrieve the classical result proved by S. Klainerman [22] and D. Christodoulou [6] in Minkowski space where $H_0 = 0$.*

4.2 Pointwise Decay of the Solution

A key ingredient for proving the global existence result for nonlinear wave equations is to derive a pointwise decay of the solution. We have already shown in Section 3.3 that the energy flux $E[\phi](\tau)$ (as the metric is flat outside the cylinder with radius R) decays in τ. To obtain the pointwise decay of the solution, we need to commute the equations with higher order derivatives. In Minkowski space, this is a direct consequence of the existence of global symmetries. On our curved background, no such global symmetry exists.

In this section, we commute the equation with the vector fields Ω, $T = \partial_t$ defined in Minkowski space. To show the decay of energy for $\Omega\phi$ or $T\phi$, we need to control the error terms arising from the commutation with Ω, T. Since those error terms are supported in the cylinder $\{(t,x)||x| \leq \frac{1}{2}R\}$ (as h is support there). We control them by using elliptic estimates.

For simplicity, in this chapter, the notation $A \lesssim B$ stands for $A \leq CB$ for some constants C depending on R, ϵ, h.

Lemma 4.2.1. *The wave operator $\Box_g$ has the following properties:*

$$g^{ij}\partial_{ij} = \Box_g - g^{00}\partial_{tt} - 2g^{0i}\partial_{ti} - \frac{1}{\sqrt{-G}}\partial_\mu(g^{\mu\nu}\sqrt{-G})\partial_\nu, \tag{4.4}$$

$$[\Box_g, \Omega] = f^{\mu\nu}\partial_{\mu\nu} + f^\mu\partial_\mu, \tag{4.5}$$

$$[\Box_g, T] = -\partial_t g^{\mu\nu}\cdot\partial_{\mu\nu} - \partial_t\left(\frac{1}{\sqrt{-G}}\partial_\mu(g^{\mu\nu}\sqrt{-G})\right)\partial_\nu, \tag{4.6}$$

where $f^{\mu\nu}$, f^{μ} are smooth functions supported on the cylinder $\{(t,x)||x| \le \frac{1}{2}R\}$ and satisfy

$$|f^{\mu\nu}| \le C_R H_0, \qquad |f^{\mu}| \le C_{h,R} H_0$$

for some constant C_R depending only on R and some constant $C_{h,R}$ depending only on R and h.

Proof. We only have to recall the definition of the covariant wave operator

$$\Box_g \phi = \frac{1}{\sqrt{-G}} \partial_\mu (g^{\mu\nu} \sqrt{-G} \partial_\nu \phi)$$

and the fact that the metric g is a perturbation of the Minkowski metric inside the cylinder $\{r \le \frac{1}{2}R\}$. $\Box$

4.2.1 Elliptic estimates and Proof of Theorem 4.1.6

We first prove Theorem 4.1.6 that if H_0 is sufficiently small depending only on R then the metric g verifies the two integrated local energy estimates (4.2), (4.3).

In fact we can assume

$$H_0 \le \delta_0 (1 + \frac{1}{2}R)^{-1-\epsilon}$$

for some positive constant $\epsilon < 1$, δ_0. We assume δ_0 is sufficiently small, depending only on ϵ so that Proposition 3.2.2 holds (in this case the metric g satisfies the estimates (3.2) with $\delta_1 = 0$). In particular, from the integrated energy estimate (3.5) we conclude that the metric g satisfies the integrated local energy estimates (4.2). Then to prove Theorem 4.1.6 it remains to verify (4.3).

Consider the equation for $\partial_t \phi = T\phi$

$$\Box_g T\phi = T\Box_g \phi + [\Box_g, T]\phi.$$

We need to control $[\Box_g, T]\phi$. Using the commutator identity (4.6) and the integrated energy estimate (3.5) we can show that

$$
\begin{aligned}
D^{\epsilon}[[\Box_g, T]\phi]_{\tau_1}^{\tau_2} &\leq \int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R} H_0^2(1+r)^{1+\epsilon}|\partial^2\phi|^2 + C_{h,R}\frac{|\partial\phi|^2}{(1+r)^{1+\epsilon}}dxdt \\
&\leq \delta_0 H_0 \int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial^2\phi|^2 dxdt + C_{h,R,\epsilon}(\tilde{E}[\phi](\tau_1) + D^{\epsilon}[\Box_g\phi]_{\tau_1}^{\tau_2})
\end{aligned}
\tag{4.7}
$$

Here $C_{\cdot,\cdot,\ldots}$ are constants depending on the subscripts.

Next we use elliptic estimates to control the integral of $|\nabla^2\phi|^2$. In the ball with radius R in $\mathbb{R}^3$, we have

$$
\int_{r\leq\frac{1}{2}R}|\nabla^2\phi|^2 dx \leq 2\int_{r\leq R}|\sum g^{ij}\partial_{ij}\phi|^2 dx + C_R\int_{r\leq R}|\phi|^2 + |\nabla\phi|^2 dx,
$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$ in $\mathbb{R}^3$. For the elliptic estimates, we refer to the book [13]. Now by the equation (4.4), we have

$$
|\sum g^{ij}\partial_{ij}\phi| \leq 2|\partial T\phi| + |\Box_g\phi| + 2|\partial\phi|.
$$

Therefore from the previous estimates together with the integrated energy estimates (3.5) we derive

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial^2\phi|^2 dxd\tau \\
&\leq 10\int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial T\phi|^2 dxd\tau + C_{R,\epsilon}\left(\tilde{E}[\phi](\tau_1) + D^{\epsilon}[\Box_g\phi]_{\tau_1}^{\tau_2}\right),
\end{aligned}
\tag{4.8}
$$

Now from estimates (4.7), (4.8) and the integrated energy estimate (3.5), we can show that

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial T\phi|^2 dxd\tau &\leq (1+\frac{1}{2}R)^{1+\epsilon}I^{\epsilon}[\phi]_{\tau_1}^{\tau_2}\\ &\leq (1+\frac{1}{2}R)^{1+\epsilon}C_{\epsilon}(\tilde{E}[T\phi](\tau_1)+D^{\epsilon}[T\Box_g\phi]_{\tau_1}^{\tau_2}+D^{\epsilon}[[\Box_g,T]\phi]_{\tau_1}^{\tau_2})\\ &\leq C_{\epsilon}\delta_0^2\int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial T\phi|^2 dxd\tau + C_{\epsilon,R,h}(\tilde{E}[T\phi](\tau_1)\\ &\quad + D^{\epsilon}[T\Box_g\phi]_{\tau_1}^{\tau_2}+\tilde{E}[\phi](\tau_1)+D^{\epsilon}[\Box_g\phi]_{\tau_1}^{\tau_2}).\end{aligned}$$

If δ_0 is sufficiently small, that is, H_0 is sufficiently small depending only on R, then we can conclude from the above estimate the integrated local energy estimates (4.3). This gives the proof for Theorem 4.1.6.

In particular, from now on in this chapter, we can assume the metric g satisfies the two integrated local energy estimates (4.2), (4.3). As a corollary of the above discussion, we have the following useful lemma.

Lemma 4.2.2. *Suppose g satisfies the two integrated local energy estimates* (4.2), (4.3). *Let ϕ satisfies the conditions in Proposition 3.2.2. Then*

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R}|\partial^2\phi|^2 dxd\tau \leq &C_{\epsilon,R,h}(\tilde{E}[T\phi](\tau_1)+\tilde{E}[\phi](\tau_1)\\ &+D^{\epsilon}[T\Box_g\phi]_{\tau_1}^{\tau_2}+D^{\epsilon}[\Box_g\phi]_{\tau_1}^{\tau_2}),\end{aligned}\tag{4.9}$$

$$\begin{aligned}\int_{\{r\leq R\}\cap\Sigma_\tau}|\partial^2\phi|^2 dx \leq &C_{\epsilon,R,h}(\tilde{E}[T\phi](\tau^+)+D^{\epsilon}[T\Box_g\phi]_{\tau^+}^{\tau+R}+\tilde{E}[\phi](\tau^+)\\ &+D^{\epsilon}[\Box_g\phi]_{\tau^+}^{\tau+R}),\end{aligned}\tag{4.10}$$

where $\tau^+=\max\{\tau-R,0\}$.

Estimate (4.9) will be used when we commute the equation with Ω or T in order to obtain the energy decay for $\Omega\phi$ or $T\phi$. Estimate (4.10) is useful for deriving the pointwise decay of the solution inside the cylinder $\{r\leq R\}$.

Proof. Estimate (4.9) follows from the elliptic estimates (4.8) (the constant 10 should be replaced by some constant C_h depending on h) and the integrated local energy estimates (4.3).

Next we use the estimate (4.9) to show (4.10). Similarly by elliptic estimates, we have

$$\begin{aligned}\int_{r\leq R}|\partial^2\phi|^2dx &\lesssim \int_{r\leq R}|\partial T\phi|^2dx + \int_{r\leq 2R}|\sum g^{ij}\partial_{ij}\phi|^2 + |\phi|^2dx \\ &\lesssim \tilde{E}[T\phi](\tau) + \tilde{E}[\phi](\tau) + \int_{r\leq 2R}|\Box_g\phi|^2 + |T^2\phi|^2 + |\phi|^2dx. \qquad (4.11)\end{aligned}$$

We need to estimate the integral of $|\Box_g\phi|^2 + |T^2\phi|^2 + |\phi|^2$ on the ball with radius $2R$. We first consider the case when $\tau \geq R$. Take $\tau_1 = \tau - R$ and $\tau_2 = \tau + R$ in the integrated energy estimate (3.5). We have

$$\int_{\tau}^{\tau+R}\int_{r\leq 2R}|T\phi|^2 + \phi^2dxdt \lesssim I^{\epsilon}[\phi]_{\tau-R}^{\tau+R} \lesssim \tilde{E}[\phi](\tau-R) + D^{\epsilon}[\Box_g\phi]_{\tau-R}^{\tau+R}.$$

In particular, we have

$$\int_{r\leq 2R}\phi^2(\tau,x)dx \lesssim \tilde{E}[\phi](\tau-R) + D^{\epsilon}[\Box_g\phi]_{\tau-R}^{\tau+R}.$$

Similarly we have

$$\int_{r\leq 2R}|\Box_g\phi|^2(\tau,x)dx \lesssim D^{\epsilon}[\Box_g\phi]_{\tau-R}^{\tau+R} + D^{\epsilon}[T\Box_g\phi]_{\tau-R}^{\tau+R}.$$

Estimate (4.10) then follows from (4.11) if we can show that

$$\int_{r\leq 2R}|T^2\phi|^2dx \lesssim \tilde{E}[T\phi](\tau-R) + D^{\epsilon}[T\Box_g\phi]_{\tau-R}^{\tau} + \tilde{E}[\phi](\tau-R) + D^{\epsilon}[\Box_g\phi]_{\tau-R}^{\tau}. \quad (4.12)$$

To show this estimate, we do the energy estimate on the region bounded by $\Sigma_{\tau-R}$

and the t-constant surface $\{t = \tau\}$. Let the vector field $X = T$ in the energy identity (2.3) and the integral region be the compact one described here. We can get

$$\begin{aligned} \int_{\{r \le 2R\} \cap \{t=\tau\}} i_{J^T[T\phi]} d\text{vol} = & \int_{\Sigma_{\tau-R} \cap \{t \le \tau\}} i_{J^T[T\phi]} d\text{vol} \\ & - \int_{\tau-R}^{\tau} \int_{r \le R+t-\tau} \Box_g T\phi \cdot \partial_t T\phi + K^T[T\phi] d\text{vol}. \end{aligned} \tag{4.13}$$

Since that the metric is flat when $r \ge \frac{1}{2}R$, by the integrated local energy estimate (4.3), we have

$$\begin{aligned} \int_{\tau-R}^{\tau} \int_{r \le R+t-\tau} |K^T[T\phi]| d\text{vol} &\lesssim \int_{\tau-R}^{\tau} \int_{r \le R} |\partial T\phi|^2 dx dt \\ &\lesssim \tilde{E}[T\phi](\tau - R) + D^{\epsilon}[\Box_g T\phi]_{\tau-R}^{\tau}. \end{aligned}$$

Using Cauchy-Schwartz inequality, we can estimate

$$\begin{aligned} \int_{\tau-R}^{\tau} \int_{r \le R+t-\tau} |\Box_g T\phi \cdot \partial_t T\phi| d\text{vol} &\lesssim \int_{\tau-R}^{\tau} \int_{\Sigma_t} |\Box_g T\phi|^2 r_+^{1+\epsilon} + \frac{|\partial_t T\phi|^2}{(1+r)^{1+\epsilon}} dx dt \\ &\lesssim \tilde{E}[T\phi](\tau - R) + D^{\epsilon}[\Box_g T\phi]_{\tau-R}^{\tau}. \end{aligned}$$

For the boundary terms in the identity (4.13), we have the positivity of the energy current on $\{t = constant\}$

$$\begin{aligned} \int_{r \le 2R} |\partial_t T\phi|^2 dx &\lesssim \int_{r \le 2R} i_{J^T[\phi]} d\text{vol}, \\ \int_{\Sigma_{\tau-R} \cap \{t \le \tau\}} i_{J^T[T\phi]} d\text{vol} &\lesssim \int_{\Sigma_{\tau-R}} i_{J^T[T\phi]} d\text{vol} \lesssim E[T\phi](\tau - R). \end{aligned}$$

Thus from the identity (4.13) and the estimate (4.9) we have already established, we can derive

$$\begin{aligned}
&\int_{r\leq 2R}|\partial_t T\phi|^2dx \lesssim \tilde{E}[T\phi](\tau-R)+D^{\epsilon}[\Box_g T\phi]_{\tau-R}^{\tau}\\
&\lesssim \tilde{E}[T\phi](\tau-R)+D^{\epsilon}[T\Box_g\phi]_{\tau-R}^{\tau}+\int_{\tau-R}^{\tau}\int_{r\leq\frac{1}{2}R}|\partial^2\phi|^2+|\partial\phi|^2dxdt\\
&\lesssim \tilde{E}[T\phi](\tau-R)+D^{\epsilon}[T\Box_g\phi]_{\tau-R}^{\tau}+\tilde{E}[\phi](\tau-R)+D^{\epsilon}[\Box_g\phi]_{\tau-R}^{\tau}.
\end{aligned}$$

Here we have used the commutator identity (4.6). Hence the estimate (4.12) holds. We thus have shown estimate (4.10) for the case when $\tau\geq R$.

When $\tau\leq R$, without loss of generality we may assume ϕ is supported on $\{r\leq t+R\}$. Thus the above estimates still hold if we simply replace $\tau-R$ with 0. □

4.2.2 Energy decay for higher order derivatives

In this section, we consider the pointwise decay of the solution ϕ of the semilinear wave equation (4.1).

Since the data have compact support, we conclude from the finite speed of propagation for solutions of wave equations (see [39]) that ϕ vanishes in the region $\{t+R\leq r\}$.

Proposition 4.2.3. *Let ϕ be the solution of the semilinear wave equation* (4.1). *Suppose the nonlinear term F satisfies*

$$D_{+}^{1}[F]_{\tau_1}^{\tau_2}\leq C_1(\tau_1)_{+}^{-1-2\epsilon},\quad D^{\epsilon}[F]_{\tau_1}^{\tau_2}\leq C_1(\tau_1)_{+}^{-2} \tag{4.14}$$

for some fixed constant C_1. Then on Σ_τ, for all $0<\delta\leq 1$, we have

$$r^2\int_{\omega}|\phi|^2d\omega\leq C_{\epsilon,R,h,\delta}\tau_{+}^{-1+\delta}\left(E_0+C_1\right),\qquad r\geq R, \tag{4.15}$$

$$r\int_{\omega}|\phi|^2d\omega\leq C_{\epsilon,R,h}\tau_{+}^{-2}\left(E_0+C_1\right),\qquad r\geq R. \tag{4.16}$$

If TF also satisfies the above decay estimate (4.14) *and the energy of $T\phi$ decays, i.e.,*

$$E[T\phi](\tau) \leq C_{\epsilon,R}\left(E_0 + C_1\right)\tau_+^{-2},$$

then we have the pointwise decay of the solution inside the cylinder $\{r \leq R\}$

$$|\phi|^2 \leq C_{\epsilon,R,h}\tau_+^{-2}\left(E_0 + C_1\right), \qquad r \leq R. \tag{4.17}$$

Proof. For the same argument in the proof of Proposition 3.3.1, the energy flux $\tilde{E}[\phi](\tau)$ is finite for all $\tau \geq 0$. Thus it suffices to consider $E[\phi](\tau)$ instead of $\tilde{E}[\phi]$. Proposition 3.3.1 implies that the energy for $T\phi$ decays

$$E[\phi](\tau) \lesssim (E_0 + C_1)\tau_+^{-2}.$$

Here as we have mentioned that the implicit constant depends on ϵ, R, h. Hence the estimate (4.16) follows directly from Lemma 2.4.1.

For the improved decay estimate (4.15) in the spatial direction, we use the p-weighted energy inequality (3.19). First note that

$$\int_{S_\tau}(\partial_v\psi)^2 d\omega dv \lesssim E[\phi](\tau) \lesssim \tau_+^{-2}\left(E_0 + C_1\right), \quad \forall\tau \geq 0, \quad \psi = r\phi.$$

Take $p = 2$ in the p-weighted energy inequality (3.19). We obtain

$$\int_{S_\tau} r^2|\partial_v\psi|^2 dv d\omega \lesssim E_0 + C_1.$$

We now interpolate between the above two estimates. We get

$$\int_{S_\tau} r^{1+\delta}(\partial_v\psi)^2 d\omega dv \lesssim \tau_+^{-1+\delta}\left(E_0 + C_1\right), \quad 0 < \delta \leq 1. \tag{4.18}$$

Hence from estimate (4.16), we can show that

$$\int_\omega r^2|\phi|^2(\tau,v,\omega)d\omega \lesssim \int_\omega r^2|\phi|^2(\tau,v_\tau,\omega)d\omega + \left(\int_{v_\tau}^v\int_\omega |\partial_v\psi|d\omega dv\right)^2$$
$$\lesssim \tau_+^{-2}\left(E_0+C_1\right) + \int_{v_\tau}^v\int_\omega r^{1+\delta}|\partial_v\psi|^2 d\omega dv \int_{v_\tau}^v r^{-1-\delta}dv$$
$$\lesssim \tau_+^{-1+\delta}\left(E_0+C_1\right).$$

Here $v_\tau = \frac{R+\tau}{2}$. This proves estimate (4.15).

Finally, we use Lemma 4.2.2, in particular estimate (4.10), to show the pointwise decay of the solution when $r \le R$. Using Sobolev embedding and Lemma 2.4.2, 4.2.2, we can show that

$$|\phi|^2 \lesssim \int_{r\le R} |\nabla^2\phi|^2 + \phi^2 dx \lesssim E[T\phi](\tau^+) + D^\epsilon[TF]_{\tau^+}^{\tau+R} + E[\phi](\tau^+) + D^\epsilon[F]_{\tau^+}^{\tau+R}$$
$$\lesssim (1+\tau^+)^{-2}\left(E_0+C_1\right) \lesssim \tau_+^{-2}\left(E_0+C_1\right), \quad r\le R,$$

where $\tau^+ = \max\{\tau - R, 0\}$. □

The above Proposition shows that in order to obtain the pointwise decay of the solution, we need to show the decay of the energy for $\Omega\phi$, $T\phi$.

Proposition 4.2.4. *Let the vector field X be T or Ω. Assume that F, XF satisfy the estimate* (4.14) *for some constant C_1.*

(1) If $X = T$, then

$$E[T\phi](\tau) \lesssim \tau_+^{-2}\left(E_0+C_1\right). \tag{4.19}$$

(2) If $X = \Omega$ and

$$E[T\phi](\tau) \lesssim \tau_+^{-2}\left(E_0+C_1\right),$$

then

$$E[\Omega\phi](\tau) \lesssim \tau_+^{-2+\alpha}\left(E_0+C_1\right). \tag{4.20}$$

Moreover, in any case, we have $\Box_g X(\phi)$ satisfies the condition in Proposition 3.3.1, that is,

$$D_+^1[\Box_g X(\phi)]_{\tau_1}^{\tau_2} \lesssim (\tau_1)_+^{-1-2\epsilon}, \quad D^\epsilon[\Box_g X\phi]_{\tau_1}^{\tau_2} \lesssim (C_1 + E_0)(\tau_1)_+^{-2} + E[T\phi](\tau_1).$$

Proof. By the commutator identities (4.5), (4.6), we can write the equation for $X(\phi)$ as follows

$$\Box_g X\phi = X(F) + f^{\mu\nu}\partial_{\mu\nu}\phi + f^\mu \partial_\mu \phi.$$

To show the energy decay of $X\phi$, it suffices to show that $\Box_g X\phi$ verifies the condition in Proposition 3.3.1. Since the functions $f^{\mu\nu}$, f^μ are supported on $\{r \geq \frac{1}{2}R\}$, we have $\Box_g X(\phi) = XF$ when $|x| \geq R$. In particular, we have

$$D_+^1[\Box_g X(\phi)]_{\tau_1}^{\tau_2} = D_+^1[XF]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-2\epsilon}.$$

Next we check the condition for $D^\epsilon[\Box_g X(\phi)]_{\tau_1}^{\tau_2}$. Using the integrated energy estimate (4.9) for the second order derivative of ϕ, we can show that

$$\begin{aligned} D^\epsilon[\Box_g X\phi]_{\tau_1}^{\tau_2} &\lesssim D^\epsilon[X\Box_g\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{r\leq\frac{1}{2}R} |\partial^2\phi|^2 + |\partial\phi|^2 dx d\tau \\ &\lesssim D^\epsilon[XF]_{\tau_1}^{\tau_2} + D^\epsilon[TF]_{\tau_1}^{\tau_2} + D^\epsilon[F]_{\tau_1}^{\tau_2} + E[T\phi](\tau_1) + E[\phi](\tau_1) \\ &\lesssim (C_1 + E_0)(\tau_1)_+^{-2} + E[T\phi](\tau_1). \end{aligned}$$

Here note that we already have the decay for $E[\phi](\tau)$. Therefore if $X = T$, the above estimate implies that $\Box_g X(\phi)$ satisfies conditions in Proposition 3.3.1. Hence we have the decay of the energy flux for $T\phi$, that is, the estimate (4.19) holds. If $X = \Omega$, we have the extra decay assumption of $E[T\phi](\tau)$. In any case, we have shown that $\Box_g X(\phi)$ satisfies the condition in Proposition 3.3.1. Thus the proposition here holds. □

The above proposition shows if we can control the nonlinear term F, we then can obtain the energy decay for $T\phi$. However, to show the energy decay of $\Omega\phi$, we need to show the energy decay of $T\phi$ first. The idea is that we first commute the equation with T. Then we pass the T derivatives to Ω derivatives.

We now define two sets

$$A := \{(k,j)|k+j \leq 8, k \leq 5\}, \quad B := \{(k,j)|(k,j+2) \in A\},$$

where k, j are always nonnegative integers.

Corollary 4.2.5. *Suppose $\Omega^k T^j F$ satisfies the following estimates*

$$D^1_+[\Omega^k T^j F]^{\tau_2}_{\tau_1} \leq C_1(\tau_1)_+^{-1-2\epsilon}, \quad D^\epsilon[\Omega^k T^j F]^{\tau_2}_{\tau_1} \leq C_1(\tau_1)_+^{-2}, \quad \forall (k,j) \in A.$$

Then we have the energy decay

$$E[\Omega^k T^j \phi](\tau) \lesssim (E_0 + C_1)\, \tau_+^{-2}.$$

Proof. We prove by induction on k. When $k = 0$, $j \leq 8$, the energy decay for $T^j\phi$ follows from the above Proposition 4.2.4. The energy decay of $T^j\phi$, $j \leq 8$ then implies the energy decay of $\Omega^k T^j$, $k \leq 1$, $j \leq 7$. Repeat this argument until we have the case when $k = 5$ and $j \leq 3$. That covers all pairs $(k,j) \in A$. Thus the corollary follows. □

This corollary together with Proposition 4.2.3 leads to the pointwise decay of the solution after using Sobolev embedding on the unit sphere as long as the nonlinear term F decays appropriately. In particular, we have the decay of solutions of linear wave equations without using any vector fields growing in time t.

Remark 4.2.6. *In Minkowski space, we only have to commute the equation with T for 3 times and with Ω for 5 times.*

4.3 Bootstrap Argument

We use bootstrap argument to prove the main theorem in this chapter.

Proposition 4.3.1. *Suppose the nonlinear term F in the semilinear wave equation (4.1) satisfies the following estimates*

(a) $\sum_{(k,j)\in A} \int_{\{r\le R\}\cap\Sigma_\tau} |\Omega^k T^j F|^2 dx \le 2E_0 \tau_+^{-3}$;

(b) $\sum_{(k,j)\in B} \int_{\{r\le R\}\cap\Sigma_\tau} |\nabla\Omega^k T^j F|^2 dx \le 2E_0 \tau_+^{-3}$;

(c) $\sum_{(k,j)\in A} \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\Omega^k T^j F|^2 r^2 \quad r^2 dv d\omega d\tau \le 2E_0 (\tau_1)_+^{-2}$

for all $\tau_2 \ge \tau_1 \ge 0$. Then

$$\sum_{(k,j)\in A} \int_{\{r\le R\}\cap\Sigma_\tau} |\Omega^k T^j F|^2 dx \lesssim E_0^2 \tau_+^{-3}, \tag{4.21}$$

$$\sum_{(k,j)\in B} \int_{\{r\le R\}\cap\Sigma_\tau} |\nabla\Omega^k T^j F|^2 dx \lesssim E_0^2 \tau_+^{-3}, \tag{4.22}$$

$$\sum_{(k,j)\in A} \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\Omega^k T^j F|^2 r^{3-\epsilon} r^2 dv d\omega d\tau \lesssim E_0^2 (\tau_1)_+^{-2}, \tag{4.23}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$ and the implicit constant depends only on R, $\epsilon > 0$, h. The sets A, B are defined in the end of the previous section.

We note that estimate (4.23) is more robust than the corresponding assumption (c). Estimates (4.21) and (4.23) are sufficient to conclude our main theorems. The extra bootstrap assumption (b) is used to prove the estimate (4.21). We will use elliptic estimates to show (4.21), which will lead to the estimate (4.22). For (4.23), we rely on the p-weighted energy inequality and the null structure of the quadratic nonlinearity of F.

First under the bootstrap assumptions on the nonlinear term F, Proposition 3.3.1

and Proposition 4.2.4 imply the decay of the energy of $\Omega^k T^j \phi$, that is,

$$E[\Omega^k T^j \phi](\tau) \lesssim E_0(1+\tau)^{-2}, \quad \forall (k,j) \in A.$$

Since the cubic or higher order nonlinearities of F behave better, in the sequel we will simply assume that $F = A^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ with constants $A^{\mu\nu}$ satisfying the null condition when $r \geq R$. When $r \leq R$, we may assume that F is at least quadratic in ϕ, $\partial\phi$. For all $(k,j) \in A$, we have

$$\Omega^k T^j F = \Omega^k T^j (A^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) = \sum A^{\mu\nu}\partial_\mu \Omega^{k_1} T^{j_1}\phi \cdot \partial_\nu \Omega^{k_2} T^{j_2}\phi, \tag{4.24}$$

where we have used the fact that $[T, \partial_\alpha] = 0$, $[\Omega, \partial_\alpha] = 0$ or ∂_β up to a constant and the constant $A^{\mu\nu}$ satisfies the null condition. The above sum is taken over all pairs (k_1, j_1), (k_2, j_2) such that $k_1 + k_2 \leq k$, $j_1 + j_2 \leq j$.

In this section we denote

$$\phi_1 = \Omega^{k_1} T^{j_1}\phi, \quad \phi_2 = \Omega^{k_2} T^{j_2}\phi$$

for pairs (k_1, j_1), (k_2, j_2) in the set A. Before we prove Proposition 4.3.1, we show a simple lemma.

Lemma 4.3.2. *Assume* $(k_1 + k_2, j_1 + j_2) \in A$. *Then*

(1) $(k_i + 2, j_i + 1) \in A$ *for at least one* $i \in \{1, 2\}$;

(2) $(k_i, j_i) \in B$ *for at least one* $i \in \{1, 2\}$;

(3) If $(k, j) \in A$ *or* B *then* $(k', j') \in A$ *or* B *for any* $k' \leq k$, $j' \leq j$.

Proof. For the first property, since $k_1 + j_1 + k_2 + j_2 \leq 8$, without loss of generality, we assume $k_1 + j_1 \leq 4$. If $k_1 \leq 3$, then $(k_1 + 2, j_1 + 1) \in A$ by definition; If $k_1 \geq 4$, then $k_2 \leq 1$ and $k_2 + 2 + j_2 + 1 \leq 4 + 3 \leq 8$. Thus $(k_2 + 2, j_2 + 1) \in A$.

For the second property, without loss of generality, we assume $j_1 \leq j_2$. Then $k_1 + j_1 + 2 \leq k_1 + j_1 + k_2 + j_2 \leq 8$, which shows $(k_1, j_1 + 2) \in A$. By definition, $(k_1, j_1) \in B$. The third property holds by the definition. $\square$

4.3.1 Proof of estimate (4.21) and estimate (4.22)

We first prove (4.21). Note that

$$\sum_{(k,j)\in A} |\Omega^k T^j F|^2 \lesssim |\bar{\partial}\phi_1|^2 |\bar{\partial}\phi_2|^2, \tag{4.25}$$

where $\bar{\partial}\phi = (\partial\phi, \frac{\phi}{1+r})$. We divide the integral region $\{r \leq R\} \cap \Sigma_\tau$ into two parts: $\{r \leq \frac{1}{2}R\} \cap \Sigma_\tau$, $\{\frac{1}{2}R \leq r \leq R\} \cap \Sigma_\tau$. First we consider the integral on the ball with radius $\frac{1}{2}R$. Since $k_1 + k_2 \leq k$, $j_1 + j_2 \leq j$, without loss of generality, assume $(k_1, j_1) \in B$ according to Lemma 4.3.2, that is, by definition $(k_1, j_1 + 2) \in A$. We claim that

$$|\bar{\partial}\phi_1|^2 \lesssim E_0 \tau_+^{-2}, \quad r \leq \frac{1}{2}R, \quad (t,x) \in \Sigma_\tau. \tag{4.26}$$

For $\partial_t\phi_1$, since $(k_1, j_1 + 2) \in A$, Proposition 4.2.4 shows that

$$E[T\phi](\tau) + E[T\partial_t\phi_1] \lesssim E_0 \tau_+^{-2}$$

and $T\Box_g(\partial_t\phi_1)$, $T\Box_g\phi_1$ satisfy the conditions in Proposition 3.3.1. Therefore using Proposition 4.2.3, we can obtain the pointwise decay for $\partial_t\phi_1$ and ϕ_1. That is

$$|\partial_t\phi_1| + |\phi_1| \lesssim E_0 \tau_+^{-2}, \quad r \leq R.$$

Next we estimate $\nabla\phi_1$. Using the elliptic estimates and the second order integrated local energy estimate (4.10) , we can show that

$$\begin{aligned}
&\int_{r\le\frac{1}{2}R} |\nabla\partial^2\phi_1|^2 dx \lesssim \int_{r\le R} |g^{ij}\partial_{ij}\nabla\phi_1|^2 + |\nabla\partial T\phi_1|^2 + |\partial^2\phi_1|^2 + |\partial\phi_1|^2 dx \\
&\lesssim \int_{r\le R} \left|\left(\Box_g - g^{00}\partial_{tt} - 2g^{0i}\partial_{ti} - \frac{1}{\sqrt{-G}}\partial_\alpha(g^{\alpha\beta}\sqrt{-G})\partial_\beta\right)\nabla\phi_1\right|^2 dx \\
&\quad + \int_{r\le R} |\nabla\partial T\phi_1|^2 + |\partial^2\phi_1|^2 + |\partial\phi_1|^2 dx \\
&\lesssim \int_{r\le R} |\nabla\Box_g\phi_1|^2 + |\nabla T^2\phi_1|^2 + |\nabla\partial T\phi_1|^2 + |\partial^2\phi_1|^2 dx + |\partial\phi_1|^2 dx \\
&\lesssim \int_{r\le R} |\nabla\Box_g\phi_1|^2 dx + E_0(\tau^+)_+^{-2},
\end{aligned}$$

where $\tau^+ = \max\{\tau - R, 0\}$. To estimate the integral of $\nabla\Box_g\phi_1$, we first write

$$\begin{aligned}
\nabla\Box_g\phi_1 = \nabla\Omega^{k_1}T^{j_1}F &+ \sum_{\substack{k\le k_1, j\le j_1\\ k+j<k_1+j_1}} f_{kj}^{\mu\nu}\nabla\partial_{\mu\nu}\Omega^k T^j\phi + f_{kj}^{\mu}\nabla\partial_\mu\Omega^k T^j\phi \\
&+ \sum_{k\le k_1, j\le j_1} F_{kj}^{\mu\nu}\partial_{\mu\nu}\Omega^k T^j\phi + F_{kj}^{\mu}\partial_\mu\Omega^k T^j\phi,
\end{aligned}$$

where $f_{kj}^{\mu\nu}, f_{kj}^{\mu}$, $F_{kj}^{\mu\nu}, F_{kj}^{\mu}$ are smooth functions depending on g and are supported on $\{r \le \frac{1}{2}R\}$. Therefore, using the bootstrap assumptions (b) in Proposition 4.3.1 and the second order integrated local energy estimate (4.10), we can show that

$$\begin{aligned}
\int_{r\le\frac{1}{2}R} |\nabla\partial^2\phi_1|^2 dx &\lesssim \int_{r\le R} |\nabla\Box_g\phi_1|^2 dx + E_0(\tau^+)_+^{-2} \\
&\lesssim E_0\tau_+^{-2} + \sum_{1'<1}\int_{r\le\frac{1}{2}R} |\nabla\partial^2\phi_{1'}|^2 dx,
\end{aligned}$$

where $1' < 1$ means $k_{1'} \le k_1$, $j_{1'} \le j_1$ and $k_{1'} + j_{1'} < k_1 + j_1$. By induction, we can conclude that

$$\int_{r\le\frac{1}{2}R} |\nabla\partial^2\phi_1|^2 dx \lesssim E_0\tau_+^{-2}.$$

Then Sobolev embedding implies that

$$\|\nabla\phi_1\|^2_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})} \lesssim E_0\tau_+^{-2}, \tag{4.27}$$

where we use B_r to denote the ball with radius r on $\mathbb{R}^3$ and C^γ, $\gamma < 1$ is the Hölder space. Since we already shown the pointwise estimate for $\partial_t\phi_1$, in particular, we have the pointwise estimate (4.26) for $\bar{\partial}\phi_1$. Therefore we can estimate

$$\begin{aligned}\sum_{(k,j)\in A}\int_{r\le\frac{1}{2}R}|\Omega^k T^j F|^2 dx &\lesssim \tau_+^{-2}E_0\int_{r\le\frac{1}{2}R}|\bar{\partial}\phi_2|^2 dx\\ &\lesssim \tau_+^{-2}E_0 E[\phi_2](\tau) \lesssim E_0^2\tau_+^{-3}.\end{aligned}$$

Next we estimate the integral of $|\Omega^k T^j F|^2$ on $\{\frac{1}{2}R \le r \le R\} \cap \Sigma_\tau$. We make use of the angular momentum Ω. Observe that $k_1 + k_2 \le 5$. Using Sobolev embedding on the unit sphere, we always have

$$\int_\omega |\bar{\partial}\phi_1|^2 \cdot |\bar{\partial}\phi_2|^2 d\omega \lesssim \sum \int_\omega |\bar{\partial}\phi_{1'}|^2 d\omega \cdot \int_\omega |\bar{\partial}\phi_{2'}|^2 d\omega, \tag{4.28}$$

where the sum is taken over all pairs $(k_{1'}, j_1)$, $(k_{2'}, j_2)$ such that $k_{1'} \le k_1+2$, $k_{2'} = k_2$ if $k_1 \le k_2$ or $k_{1'} = k_1$, $k_{2'} \le k_2+2$ if $k_2 < k_1$. In any case, we have $(k_{1'}, j_1), (k_{2'}, j_2) \in A$. Since $j_1 + j_2 = j$, without loss of generality, we may assume $(k_{1'}, j_1 + 1) \in A$ by Lemma 4.3.2. Therefore by Lemma 4.2.2, we have

$$\begin{aligned}&\int_\omega |\bar{\partial}\phi_{1'}|^2 d\omega \lesssim \int_{r\le R} |\bar{\partial}\phi_{1'}|^2 + |\nabla\bar{\partial}\phi_{1'}|^2 dx\\ &\lesssim E[\phi_{1'}](\tau) + E[T\phi_{1'}](\tau^+) + D^\epsilon[TF]_{\tau^+}^{\tau+R} + E[\phi_{1'}](\tau^+) + D^\epsilon[F]_{\tau^+}^{\tau+R}\\ &\lesssim \tau_+^{-2}E_0, \quad \frac{1}{2}R \le r \le R,\end{aligned} \tag{4.29}$$

where $\tau^+ = \max\{\tau - R, 0\}$. Thus we can show that

$$\sum_{(k,j)\in A} \int_{\frac{1}{2}R\leq r\leq R} |\Omega^k T^j F|^2 dx \lesssim \tau_+^{-2} E_0 \int_{r\leq R} |\partial\phi_{2'}|^2 dx$$
$$\lesssim \tau_+^{-2} E_0 E[\phi_{2'}](\tau) \lesssim \tau_+^{-3} E_0^2.$$

We hence proved the estimate (4.21).

Remark 4.3.3. *We remark here that estimate* (4.29) *holds only when r is bigger than a constant. The angular momentum Ω vanishes at the space origin $r = 0$. Thus we are not able to obtain estimate for the solution through angular momentum Ω for small r. Instead, for small r, we use the vector field T and the elliptic estimates.*

Next we use the estimate (4.21) we have just established to prove the estimate (4.22). For all $(k, j) \in B$, using Sobolev embedding, we can show that

$$\int_\omega |\nabla\Omega^k T^j F|^2 d\omega \lesssim \int_\omega |\bar{\partial}\phi_1|^2 \cdot |\bar{\partial}\partial\phi_2|^2 d\omega \lesssim \sum \int_\omega |\bar{\partial}\phi_{1'}|^2 d\omega \cdot \int_\omega |\bar{\partial}\partial\phi_{2'}|^2 d\omega,$$

where the sum is taken over all pairs $(k_{1'}, j_1)$, $(k_{2'}, j_2)$ such that $k_{1'} \leq k_1 + 2$, $k_{2'} = k_2$ if $k_1 \leq k_2$ or $k_{1'} = k_1$, $k_{2'} \leq k_2 + 2$ if $k_2 < k_1$. In any case, $(k_{1'}, j_1), (k_{2'}, j_2) \in B$. Thus using the pointwise estimate (4.26) when $r \leq \frac{1}{2}$ and the estimate (4.29) when $\frac{1}{2}R \leq r \leq R$, we can show that

$$\int_\omega |\bar{\partial}\phi_{1'}|^2 d\omega \lesssim \tau_+^{-2} E_0, \quad r \leq R, \quad (t, x) \in \Sigma_\tau.$$

Therefore using the second order integrated local energy estimate (4.10), we have

$$\sum_{(k,j)\in B} \int_{r\leq R} |\nabla\Omega^k T^j F|^2 dx \lesssim \tau_+^{-2} E_0 \int_{r\leq R} |\partial^2\phi_{2'}|^2 dx \lesssim E_0^2 \tau_+^{-3}.$$

We thus have shown the estimate (4.22).

4.3.2 Proof of estimate (4.23)

For this part, we rely on the p-weighted energy inequality and the null structure of the nonlinearity. Recall that the p-weighted energy inequality (3.19) is an estimate in terms of $\psi = r\phi$ instead of ϕ. We need to expand $\Omega^k T^j F$ with respect to ψ.

Lemma 4.3.4. *Assume F is quadratic in $\partial\phi$ and satisfies the null condition when $r \geq R$. Then we have*

$$r^4|\Omega^k T^j F|^2 \lesssim \sum_{1,2} r^2 \phi_1^2 |\bar{\partial}\phi_2|^2 + |\nabla\!\!\!/ \psi_1|^2 ||\partial\psi_2|^2 + |\partial_v \psi_1|^2 |\partial_u \psi_2|^2, \tag{4.30}$$

where $\psi_1 = r\phi_1$, $\psi_2 = r\phi_1$. The sum is taken over all pairs (k_1, j_1), (k_2, j_2) such that $k_1 + k_2 \leq k$, $j_1 + j_2 \leq j$, $\phi_1 = \Omega^{k_1} T^{j_1}\phi$, $\Omega^{k_2} T^{j_2}\phi$.

Proof. The lemma follows from the identity (4.24) and direct computations. □

Remark 4.3.5. *If ϕ is a scalar function, we in fact can get better estimate for the quadratic form satisfying the null condition. The above estimates for the general null form hold for vector valued functions. In particular, the argument in this chapter is also valid for system of nonlinear wave equations.*

This lemma shows that in order to estimate the integral of $r^{3-\epsilon}|\Omega^k T^j F|^2$, it suffices to consider the three kinds of terms on the right hand side of (4.30). Note that

$$|\partial\psi| \lesssim |\partial_t \psi| + |\partial_v \psi| + |\nabla\!\!\!/ \psi|$$

We estimate the following terms

$$r^2\phi_1^2|\bar{\partial}\phi_2|^2; \quad |\nabla\!\!\!/ \psi_1|^2|\nabla\!\!\!/ \psi_2|^2; \quad |\nabla\!\!\!/ \psi_1|^2|\partial_v\psi_2|^2, k_1 < 5; \quad |\nabla\!\!\!/ \psi_1|^2|\partial_t\psi_2|^2 \tag{4.31}$$

in a uniform way. Without loss of generality, for the term $|\nabla\!\!\!/ \psi_1|^2|\nabla\!\!\!/ \psi_2|^2$, we may assume $k_1 \leq k_2$. For the first three terms, we let Φ_1 be ϕ_1 or $\nabla\!\!\!/ \psi_1$; Φ_2 be $r\bar{\partial}\phi_2$, $\nabla\!\!\!/ \psi_2$

or $\partial_v\psi_2$ correspondingly. For the last term $|\nabla\!\!\!/\psi_1|^2|\partial_t\psi_2|^2$, we let $\Phi_1 = \nabla\!\!\!/\psi_1$, $\Phi_2 = \partial_t\psi_2$ if $k_1 \leq k_2$; otherwise $\Phi_2 = \nabla\!\!\!/\psi_1$, $\Phi_1 = \partial_t\psi_2$.

As $k_1 + k_2 \leq 5$, using Sobolev embedding on the unit sphere, we always have

$$\int_\omega |\Phi_1|^2|\Phi_2|^2 d\omega \lesssim \sum \int_\omega |\Phi_{1'}|^2 d\omega \cdot \int_\omega |\Phi_{2'}|^2 d\omega, \tag{4.32}$$

where the sum is taken over all pairs $(k_{1'}, j_1)$, $(k_{2'}, j_2)$ such that $k_{1'} \leq k_1 + 2$, $k_{2'} = k_2$ if $k_1 \leq k_2$ or $k_{1'} = k_1$, $k_{2'} \leq k_2 + 2$ if $k_2 < k_1$. Note that $\nabla\!\!\!/\psi_1 = \Omega\phi_1$. We thus can always write $\Phi_{1'}$ as $\Omega^k T^j \phi$ for some $(k, j) \in A$. In particular, estimate (4.15) implies that

$$r^2 \int_\omega |\Phi_{1'}|^2 d\omega \lesssim \tau_+^{-1+\delta} E_0, \quad \forall 1 > \delta > 0.$$

Therefore, by the integrated energy estimate (3.5), we can the terms in line (4.31) as follows

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{1-\epsilon}\Phi_1^2\Phi_2^2 dv d\omega d\tau &\lesssim \int_{\tau_1}^{\tau_2} \int_{v_\tau}^{\infty} r^2 \int_\omega |\Phi_{1'}|^2 d\omega \cdot r^{-1-\epsilon} \int_\omega |\Phi_{2'}|^2 d\omega dv d\tau \\
&\lesssim (\tau_1)_+^{-1+\delta} E_0 \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\Phi_{2'}|^2}{r^{3+\epsilon}} r^2 dv d\omega d\tau \\
&\lesssim (\tau_1)_+^{-1+\delta} E_0 (\tau_1)_+^{-2} E_0 \lesssim (\tau_1)_+^{-2} E_0^2.
\end{aligned} \tag{4.33}$$

To prove estimate (4.23), it remains to consider the main terms $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$ on the right hand side of (4.30) and the special case $|\nabla\!\!\!/\psi_2|^2|\partial_v\psi_1|^2$, $k_2 = 5$ (we change the subscript 1 and 2). The special term will be estimated together with the main terms. For the main terms, first recall that $\psi_1 = r\Omega^{k_1}T^{j_1}\phi$, $\psi_2 = \Omega^{k_2}T^{j_2}\phi$ for pairs (k_1, j_1), (k_2, j_2) such that $(k_1 + k_2, j_1 + j_2) \in A$. We have two cases for $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$.

We first consider the case when $k_1 \leq k_2$ for $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$ as well as the special term $|\nabla\!\!\!/\psi_2|^2|\partial_v\psi_1|^2$, $k_2 = 5$. Since $k_1 + k_2 \leq 5$, we have $k_1 \leq 2$. The idea is that we bound $|\partial_v\psi_1|$ uniformly and then control $|\partial_u\psi_2|^2 + |\nabla\!\!\!/\psi_2|^2$ by the energy flux through the v-constant null hypersurface. We first establish a lemma that shows that the energy flux through the v-constant null hypersurface is bounded.

Lemma 4.3.6. *Consider the region* $\mathcal{D} = [u_1, u_2] \times [v_1, \infty) \subset S_\tau \times [\tau_1, \tau_2]$. *Then*

$$\int_{u_1}^{u_2} \int_\omega (\partial_u \psi_2)^2 + |\nabla\!\!\!/ \psi_2|^2 du d\omega \lesssim (\tau_1)_+^{-2} E_0.$$

Proof. We apply the energy identity (2.3) with the vector field $X = T$ on the region $\mathcal{D}$. We have

$$\begin{aligned}\int_{S_{\tau_1} \cap \{v \geq v_1\}} i_{J^T[\phi_2]} d\text{vol} + \int_{\bar{C}(\tau_1, \tau_2)} i_{J^T[\phi_2]} d\text{vol} &= \int_{S_{\tau_2} \cap \{v \geq v_1\}} i_{J^T[\phi_2]} d\text{vol} \\ &+ \int_{v = v_1, u_1 \leq u \leq u_2} i_{J^T[\phi_2]} d\text{vol} + \int_{\mathcal{D}} \Box_g \phi_2 \cdot \partial_t \phi_2 d\text{vol}.\end{aligned}$$

On S_τ, we can compute that

$$i_{J^T[\phi_2]} d\text{vol} = \frac{1}{2} (|\partial_v \phi_2|^2 + |\nabla\!\!\!/ \phi_2|^2) r^2 dv d\omega.$$

On the incoming null hypersurface $\{u_1 \leq u \leq u_2, v = constant\}$, we have

$$i_{J^T[\phi_2]} d\text{vol} = -\frac{1}{2} (|\partial_u \phi_2|^2 + |\nabla\!\!\!/ \phi_2|^2) r^2 du d\omega.$$

Hence from the above energy identity on the region $\mathcal{D}$, we can show that

$$\begin{aligned}\int_{u_1}^{u_2} \int_\omega (\partial_u \psi_2)^2 + |\nabla\!\!\!/ \psi_2|^2 d\omega du &= \int_{u_1}^{u_2} \int_\omega r^2 |\partial_u \phi_2|^2 + \partial_u (r \phi_2^2) + |\nabla\!\!\!/ \psi_2|^2 d\omega du \\ &\leq -2 \int_{u_1}^{u_2} \int_\omega i_{J^T[\phi_2]} d\text{vol} + \int_\omega r \phi_2^2 (v_1, u_2, \omega) d\omega \\ &\leq 2 \tilde{E}[\phi_2](\tau_2) + 2 \int_M |\Box_g \phi_2| |\partial_t \phi_2| d\text{vol} \\ &\lesssim E[\phi_2](\tau_1) + D^\epsilon [\Box_g \phi_2]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_{S_\tau} \frac{|\partial \phi_2|^2}{(1+r)^{1+\epsilon}} dx dt \\ &\lesssim (\tau_1)_+^{-2} E_0,\end{aligned}$$

where we have used the energy estimate (3.6) and the integrated energy estimate (3.5). The decay for $D^{\epsilon}[\Box_g \phi_2]_{\tau_1}^{\tau_2}$ follows from Proposition 4.2.4 and our bootstrap assumptions. $\Box$

We continue our proof for the estimate (4.23) when $k_1 \leq 2$. The above lemma shows that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{-1-\epsilon}|\partial_v\psi_1|^2(|\partial_u\psi_2|^2+|\nabla\!\!\!/\psi_2|^2)r^2 dv d\omega d\tau \\
&=\int_{v_{\tau_1}}^{\infty}\int_{u_{\tau_1}}^{u(v)}\int_{\omega}(|\partial_u\psi_2|^2+|\nabla\!\!\!/\psi_2|^2)r^{1-\epsilon}|\partial_v\psi_1|^2 du d\omega dv \\
&\leq \int_{v_{\tau_1}}^{\infty}\int_{u_{\tau_1}}^{u(v)}\int_{\omega}|\partial_u\psi_2|^2+|\nabla\!\!\!/\psi_2|^2 d\omega du \cdot \sup_{u,\omega} r^{1-\epsilon}|\partial_v\psi_1|^2 dv \\
&\lesssim (1+\tau_1)^{-2}E_0\int_{v_{\tau_1}}^{\infty}\sup_{u,\omega} r^{1-\epsilon}|\partial_v\psi_1|^2 dv.
\end{aligned} \tag{4.34}$$

Here $u\in[u_{\tau_1},u(v)]$, $u(v)=u_{\tau_2}$ if $v\geq v_{\tau_2}$; $u(v)=v-2R$ if $v\in[v_{\tau_1},v_{\tau_2}]$. And we recall that $u_\tau=\frac{\tau-R}{2}$, $v_\tau=\frac{\tau+R}{2}$. Next we estimate $\sup r^{1-\epsilon}|\partial_v\psi_1|^2$. First, using the Sobolev embedding on the unit sphere, we have

$$|\partial_v\psi_1|^2 \lesssim \sum_{a\leq 2}\int_\omega |\Omega^a\partial_v\psi_1|^2 d\omega \lesssim \sum_{a\leq 2}\int_\omega |\partial_v\Omega^a\psi_1|^2\omega = \int_\omega |\partial_v\psi_{1'}|^2 d\omega,$$

where $k_{1'}\leq k_1+2$ and we omitted the summation sign. Then on the interval $[u_1,u_2]=[u_{\tau_1},u(v)]$, we have

$$\begin{aligned}
r^{1-\alpha}|\partial_v\psi_{1'}|^2 &\lesssim r^{1-\epsilon}|\partial_v\psi_{1'}|^2\big|_{u=u_1}+\int_{u_1}^{u_2} r^{1-\epsilon}|\partial_v\psi_{1'}|^2 du \\
&+\int_{u_1}^{u_2} r^{1-\epsilon}|\partial_u\partial_v\psi_{1'}|^2 du+\int_{u_1}^{u_2} r^{-\epsilon}|\partial_v\psi_{1'}|^2 du \\
&\lesssim r^{1-\epsilon}|\partial_v\psi_{1'}|^2\big|_{u=u_1}+\int_{u_1}^{u_2} r^{1-\epsilon}|\partial_v\psi_{1'}|^2 du \\
&+\int_{u_1}^{u_2} r^{1-\epsilon}|\Delta\!\!\!/\psi_{1'}|^2+r^{3-\epsilon}|F_{1'}|^2 du,
\end{aligned}$$

where we have used the wave equation (3.18) in null coordinates for $\phi_{1'}$ and $F_{1'} = \Box_g \phi_{1'}$. Integrate both side of the above inequality over the unit sphere. We can show that

$$\begin{aligned}\int_{v_{\tau_1}}^{\infty} \sup_{u,\omega} r^{1-\epsilon}|\partial_v\psi_1|^2 dv \lesssim & \int_{S_{\tau_1}} r^{1-\epsilon}|\partial_v\psi_{1'}|^2 dv d\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}|\partial_v\psi_{1'}|^2 dv d\omega d\tau \\ & + \int_{\tau_1}^{\tau_2}\int_{S_\tau} |\nabla\!\!\!/ \Omega\phi_{1'}|^2 r^{1-\epsilon} + r^{3-\epsilon}|F_{1'}|^2 dv d\omega d\tau \\ \lesssim & E_0 + \int_{\tau_1}^{\tau_2}\int_{S_\tau} \frac{|\nabla\!\!\!/ \Omega\phi_{1'}|^2}{(1+r)^{1+\epsilon}} r^2 dv d\omega d\tau \lesssim E_0. \end{aligned} \tag{4.35}$$

Here we have used the p-weighted energy inequality (3.19) and the integrated energy inequality (3.5) as well as the fact $(k_{1'}+1, j_1) \in A$. The estimate for $F_{1'} = \Box_g \phi_1$ follows from Proposition 4.2.4. In particular, from (4.34), we derive

$$\int_{\tau_1}^{\tau_2} r^{1-\epsilon}\int_{S_\tau} |\partial_v\psi_1|^2(|\partial_u\psi_2|^2 + |\nabla\!\!\!/ \psi_2|^2) dv d\omega d\tau \lesssim (1+\tau_1)^{-2}E_0^2.$$

This is the desired estimate for the case $k_1 \le k_2$ of the term $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$ as well as the special term $|\nabla\!\!\!/ \psi_2|^2|\partial_v\psi_1|^2$, $k_2 = 5$.

Next we consider the case when $k_2 \le k_1$ for $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$, in particular we have $k_2 \le 2$. For $|\partial_v\psi_1|^2|\partial_u\psi_2|^2$, we bound $\partial_u\psi_2$ uniformly and estimate the integral of $|\partial_v\psi_1|^2$ by the p-weighted energy inequality. First we have

$$|\partial_u\psi_2|^2 \lesssim \sum_{a\le 2}\int_\omega |\Omega^a\partial_u\ \psi_2|^2 d\omega \lesssim \sum_{a\le 2}\int_\omega |\partial_u\Omega^a\psi_1|^2 d\omega = \int_\omega |\partial_u\psi_{2'}|^2 d\omega,$$

where $k_{2'} \le k_2 + 2$ and we omitted the summation sign. Now using the p-weighted energy inequality (3.19) with $p = 1$, that is, estimate (4.18) with $\delta = \epsilon$, we can show that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}|\partial_v\psi_1|^2|\partial_u\psi_2|^2 dvd\omega d\tau \\
&\lesssim \int_{\tau_1}^{\tau_2}\int_{S_\tau} r|\partial_v\psi_1|^2\cdot\int_\omega r^{-\epsilon}|\partial_u\psi_{2'}|^2 d\omega \quad dvd\omega d\tau \qquad (4.36)\\
&\lesssim (1+\tau_1)^{-1}E_0\int_{\tau_1}^{\tau_2}\sup_v r^{-\epsilon}\int_\omega |\partial_u\psi_{2'}|^2 d\omega d\tau.
\end{aligned}$$

For all $v\in[v_\tau,\infty)$, we have

$$\begin{aligned}
&r^{-\epsilon}(\partial_u\psi_{2'})^2\\
\lesssim\ & r^{-\epsilon}(\partial_u\psi_{2'})^2\big|_{v=v_{\tau_2}} + \left|\int_v^{v_{\tau_2}} r^{-1-\epsilon}|\partial_u\psi_{2'}|^2 dv\right| + \left|\int_v^{v_{\tau_2}} r^{-\epsilon}|\partial_u\psi_{2'}\cdot\partial_v\partial_u\psi_{2'}|dv\right|\\
\lesssim\ & r^{-\epsilon}(\partial_u\psi_{2'})^2\big|_{v=v_{\tau_2}} + \int_{v_\tau}^{\infty} r^{-1-\epsilon}|\partial_u\psi_{2'}|^2 dv + \int_{v_\tau}^{\infty} r^{1-\epsilon}(\partial_v\partial_u\psi_{2'})^2 dv\\
\lesssim\ & r^{-\epsilon}(\partial_u\psi_{2'})^2\big|_{v=v_{\tau_2}} + \int_{v_\tau}^{\infty}\frac{|\partial_u\psi_{2'}|^2+|\nabla\!\!\!/\Omega\psi_{2'}|^2}{(1+r)^{1+\epsilon}}dv + \int_{v_\tau}^{\infty} r^{3-\epsilon}|F_{2'}|^2 dv.
\end{aligned}$$

Here again we have used the wave equation for $\phi_{2'}$ in null coordinates and $F_{2'}=\Box_g\phi_{2'}$. Integrate both side of the above inequality on the unit sphere. We derive

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\sup_v r^{-\epsilon}\int_\omega|\partial_u\psi_{2'}|^2 d\omega d\tau \lesssim \int_{u_{\tau_1}}^{u_{\tau_2}}\int_\omega(\partial_u\psi_{2'})^2(u,v_{\tau_2},\omega)dud\omega\\
&\quad+\int_{\tau_1}^{\tau_2}\int_{S_\tau}\frac{|\partial\psi_{2'}|^2+|\nabla\!\!\!/\Omega\psi_{2'}|^2}{r^{1+\epsilon}}dvd\omega d\tau + \int_{\tau_1}^{\tau_2}\int_{S_\tau}(1+r)^{1+\epsilon}|F_{2'}|^2r^2dvd\omega d\tau\\
&\quad\lesssim E_0(1+\tau_1)^{-2}. \qquad (4.37)
\end{aligned}$$

Here we have used Lemma 4.3.6 to control the energy flux through the incoming null hypersurface. The first term on the second line of (4.37) can be controlled by using the integrated energy estimate (3.5) together with the fact that $k_{2'}\le k_2+2\le 4$ (which implies that $(k_{2'}+1,j_2)\in A$). The decay of the integral of $(1+r)^{1+\epsilon}|F_{2'}|^2$

follows from Proposition 4.2.4. Hence from (4.36), we derive

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}|\partial_v\psi_1|^2|\partial_u\psi_2|^2 dv d\omega d\tau \lesssim (1+\tau_1)^{-2}E_0^2.$$

We thus finished the proof of estimate (4.23). In particular, Proposition 4.3.1 holds.

4.4 Proof of the Main Theorem

We used the foliation Σ_τ, part of which is null, in the our argument. However, we do not have a local existence result with respect to the foliation Σ_τ. We thus use the standard Picard iteration process. Take $\phi_{-1}(t,x)=0$. We solve the following linear wave equation recursively

$$\begin{cases} \Box_{g(t,x)}\phi_{n+1} = F(\phi_n, \partial\phi_n), \\ \phi_{n+1}(0,x) = \phi_0(x), \partial_t\phi_{n+1}(0,x) = \phi_1(x). \end{cases} \tag{4.38}$$

Now suppose the implicit constant in Proposition 4.3.1 is C_1, which, according to our notation, depends only on R, ϵ, λ, h and C_0. Set

$$\epsilon_0 = \frac{1}{\sqrt{C_1}}.$$

Then for all $E_0 \le \epsilon_0$, we have

$$C_1E_0^2 \le E_0.$$

Thus by the continuity of $F(\phi_n, \partial\phi_n)$, we in fact have shown that the nonlinear term F satisfies

$$\sum_{(k,j)\in A} \int_{\{r\leq R\}\cap\Sigma_\tau} |\Omega^k T^j F(\phi_n, \partial\phi_n)|^2 dx \leq C_1 E_0^2 \tau_+^{-3} \leq E_0 \tau_+^{-3},$$
$$\sum_{(k,j)\in A} \int_{\tau_1}^{\tau_2} \int_{S_\tau} |\Omega^k T^j F(\phi_n, \partial\phi_n)|^2 r^{3-\epsilon} dx dt \leq E_0(1+\tau_1)^{-2}, \quad \forall n.$$

In particular, Corollary 4.2.5 implies that

$$E[\Omega^k T^j \phi_n](\tau) \lesssim E_0 \tau_+^{-2}, \quad \forall (k,j) \in A.$$

We remark here that all the implicit constants are independent of n.

Then Proposition 4.2.3 together with the Sobolev embedding on the unit sphere implies that

$$\sum_{k\leq 2, j\leq 2} |\Omega^k T^j \phi_n| \lesssim \sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+|t-r+R|)^{-1},$$
$$|\phi_n| \lesssim_\delta \sqrt{E_0}(1+r)^{-1}(1+|t-r+R|)^{-\frac{1}{2}+\frac{1}{2}\delta}, \quad \forall \delta > 0.$$

We next show that solutions are bounded in C^2. The first step is to show that ϕ_n is uniformly bounded in C^1. When $r \leq \frac{1}{2}R$, we use estimate (4.26) to show that

$$|\partial\Omega^k T^j \phi_n| \lesssim \sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+\tau)^{-1}, \quad \forall k \leq 1, j \leq 2,$$

where τ is the parameter of the foliation Σ_τ and we always evaluate the solutions at points on Σ_τ. When $\frac{1}{2}R \leq r \leq R$, we use estimate (4.29) and we can show that $|\partial\Omega^k T^j|$, $k \leq 1, j \leq 2$ obeys the same estimate as for the case $r \leq \frac{1}{2}R$. We now show that ϕ_n is C^1 when $r \geq R$. Since $\nabla\!\!\!/ = r^{-1}\Omega$, it suffices to estimate $\partial_r \phi_n$. From the inequality (4.37), we have that

$$\int_{\tau_1}^{\tau_2} \sup_v r^{-\epsilon} \int_\omega |\partial_u \psi|^2 + |\partial_t \partial_u \psi|^2 d\omega d\tau \lesssim (1+\tau_1)^{-2} E_0,$$

where $\psi = r\Omega^k T^j \phi_n$, $k \le 3$, $j \le 2$. Using Sobolev embedding on $S^2 \times [\tau_1, \tau_2]$, we have

$$|r\partial_u \Omega^k T^j \phi_n|^2 \lesssim r^\epsilon |\Omega^k T^j \phi_n|^2 + r^\epsilon \tau_+^{-2} E_0, \quad \forall k \le 1, j \le 2.$$

Since $\partial_u = \partial_t - \partial_r$ and we have shown that

$$|\Omega^k T^j \phi_n|^2 + |T\Omega^k T^j \phi_n|^2 \lesssim (1+r)^{-1} \tau_+^{-2} E_0$$

We thus conclude that

$$|\partial_r \Omega^k T^j \phi_n| \lesssim (1+r)^{-\frac{1}{2}} \tau_+^{-1} \sqrt{E_0}, \quad \forall k \le 1, j \le 2, \quad r \ge R.$$

Combine with the above discussion for $r \le R$. We in fact have shown that

$$|\partial \Omega^k T^j \phi_n| \lesssim (1+r)^{-\frac{1}{2}} \tau_+^{-1} \sqrt{E_0}, \quad \forall k \le 1, j \le 2.$$

Finally, we show that ϕ_n is bounded in C^2. When $\{r \ge \frac{1}{4}R\}$, we use the equation (4.38) of ϕ_n. We can write

$$g^{rr} \partial_{rr} \phi_n = F(\phi_{n-1}, \partial\phi_{n-1}) - g^{\mu\nu} \partial_{\mu\nu} \phi_n + g^{rr} \partial_{rr} \phi_n - \frac{1}{\sqrt{-G}} \partial_\mu \left(g^{\mu\nu} \sqrt{-G} \right) \partial_\nu \phi_n.$$

As we have shown that

$$|\partial \phi_n| + |\Omega^2 \phi_n| + |\partial_{tt} \phi_n| + |\phi_n| + |\partial \Omega \phi_n| + |\partial T \phi_n| \lesssim r_+^{-\frac{1}{2}} (\tau_1)_+^{-1} \sqrt{E_0}, \quad \forall n,$$

we conclude that

$$|\partial^2\phi_n| \lesssim \sum_{k+j\leq 1} |\partial\Omega^k T^j \phi_n| + |\partial_{rr}\phi_n| \lesssim \sqrt{E_0} r_+^{-\frac{1}{2}} \tau_+^{-1}.$$

As the angular momentum Ω vanishes at the space origin, the above argument for the C^2 estimate of ϕ_n fails when $r \leq \frac{1}{4}R$. In this region, we instead use elliptic theory to obtain the C^2 estimate of ϕ_n. First, we have the elliptic equation for ϕ_{n+1}

$$\begin{aligned} g^{ij}\partial_{ij}\phi_{n+1} =& F(\phi_n, \partial\phi_n) - \frac{1}{\sqrt{-G}}\partial_\mu \left(g^{\mu\nu}\sqrt{-G}\right)\partial_\nu\phi_{n+1} \\ & - g^{00}T^2\phi_{n+1} - 2g^{0i}\partial_i T\phi_{n+1}. \end{aligned}$$

We have shown from estimates (4.27) that

$$\begin{aligned} \|\partial\Omega^k T^j\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})} &\leq \|\nabla\Omega^k T^j\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})} + \|\Omega^k T^{j+1}\phi_n\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})} \\ &\lesssim \sqrt{E_0} r_+^{-\frac{1}{2}} \tau_+^{-1} \end{aligned}$$

for all $k \leq 1, j \leq 2$. In particular, the right hand side of the above elliptic equation is uniformly bounded in $C^{\frac{1}{2}}(B_{\frac{1}{2}R})$, that is,

$$\|g^{ij}\partial_{ij}\phi_{n+1}\|_{C^{\frac{1}{2}}(B_{\frac{1}{2}R})} \lesssim \sqrt{E_0} r_+^{-\frac{1}{2}} \tau_+^{-1}, \quad \forall n.$$

Then Schauder estimates [13] imply that

$$\|\phi_{n+1}\|_{C^{2,\frac{1}{2}}(B_{\frac{1}{4}R})} \lesssim \sqrt{E_0} r_+^{-\frac{1}{2}} \tau_+^{-1}.$$

Therefore, we have show that the solutions ϕ_n is bounded in C^2 and satisfy the estimate

$$\sum_{|\beta|\leq 2} |\partial^\beta \phi_n| \lesssim \sqrt{E_0} r_+^{-\frac{1}{2}} \tau_+^{-1}. \tag{4.39}$$

Finally, the classical local existence theory for wave equations shows that there exists a time $t^* > 0$ and a unique smooth solution $\phi(t, x) \in C^\infty([0, t^*) \times \mathbb{R}^3)$ of the equation (4.1). Moreover, the proof of the local existence theory indicates that

$$\phi_n(t, x) \to \phi(t, x)$$

in $C^\infty([0, t^*) \times \mathbb{R}^3)$. Therefore by (4.39), we have pointwise bound for the solution ϕ

$$\sum_{|\beta|\leq 2} |\partial^\beta \phi| \lesssim \sqrt{E_0}(1 + r)^{-\frac{1}{2}}(1 + \tau)^{-1}, \quad \forall (t, x) \in [0, t^*) \times \mathbb{R}^3.$$

By a theorem of Hörmander [14] that as long as the solution is bounded up to the second order derivatives, the solution exists globally. That is there exists a unique global solution $\phi(t, x) \in C^\infty(\mathbb{R}^{3+1})$ which solves the nonlinear wave equation (4.1). Moreover since

$$\phi_n(t, x) \to \phi(t, x), \quad (t, x) \in \mathbb{R}^{3+1},$$

we conclude that ϕ admits all the estimates of ϕ_n obtained above. This proves the main Theorem 4.1.1 in this chapter.

Chapter 5

Quasilinear wave equations

5.1 Statement of Theorem 1.0.2

In this chapter, we consider the more general quasilinear wave equations

$$\begin{cases} \Box_g \phi + C^{\mu\nu}(\phi)\partial_{\mu\nu}\phi = F(\phi, \partial\phi), \\ \phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x) \end{cases} \tag{5.1}$$

on a Lorentzian manifold $(\mathbb{R}^{3+1}, g)$, where $F(\phi, \partial\phi)$ denotes the semilinear terms of the equation, $C^{\mu\nu}(\phi)$ is the coefficient of the quasilinear part.

As having discussed in the previous chapter, F is assumed to be at least quadratic in terms of ϕ, $\partial\phi$ and the quadratic part satisfies the null condition outside a cylinder with radius R. For the quasilinear term considered here, we need similar conditions. We assume that the quasilinear terms satisfy the null condition. More precisely, assume $C^{\alpha\beta}$ is smooth and $C^{\alpha\beta}(0) = 0$. For some large constant R

$$C^{\alpha\beta}(\phi) = g^{\alpha\beta\gamma}\partial_\gamma \phi + O(|\phi|^2 + |\partial\phi|^2), \quad |x| \geq R \tag{5.2}$$

for small ϕ, $\partial\phi$. And $g^{\alpha\beta\gamma}$ are constants such that $g^{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma = 0$ whenever $\xi_0^2 =$

$\xi_1^2 + \xi_1^2 + \xi_3^2$.

The background metric $g = h + m_0$ is assumed to be perturbations of Minkowski metric inside some large cylinder with radius R (that is h is supported in $\{|x| \leq R\}$). In this chapter we consider asymptotically flat inhomogeneous background. The conditions on h has in fact already been given in Section 3.1 (see conditions (3.2)). However, for nonlinear problem, we also need estimates for higher order derivatives of the metric components. We now describe the conditions on the metric for the quasilinear problem here

$$
\begin{aligned}
&|h^{\mu\nu}| + |\partial h^{\mu\nu}| \leq \delta_0 r_+^{-1-2\alpha}, \quad r = |x| \leq R, \\
&|\partial h^{\mu\nu}| + |h^{\mu\nu}| + |Z^k h^\mu \nu| \leq \delta_0 (r_+^{-\frac{1}{2}-2\alpha} \tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha} + r_+^{-1-2\alpha}), \quad (t,x) \in S_\tau, \\
&|\overline{\partial_v} h^{\mu\nu}| + |\partial h^{\underline{LL}}| + |Z^k h^{\underline{LL}}| \leq \delta_0 r_+^{-1-2\alpha}, \quad (t,x) \in S_\tau,
\end{aligned}
\tag{5.3}
$$

for some small positive constant $\alpha < \frac{1}{10}$. δ_0 will be required to be small positive constant depending only on α. We remark here that such metric is still time dependent and the spacetime may have arbitrarily large curvature.

For the initial data ϕ_0, ϕ_1, we simply (but without loss of generality, see discussions in Chapter (7)) are supported on $\{|x| \leq R\}$. Let

$$E_0 := \sum_{k \leq 6} \int_{\mathbb{R}^3} |\partial Z^k \phi|^2 (0, x) dx.$$

Here recall that $Z = \{\Omega, \partial_t\}$. We remark here that E_0 is uniquely determined by ϕ_0, ϕ_1 together with the equation (5.1).

For such time dependent backgrounds, we have the following small data global existence result for quasilinear wave equations.

Theorem 5.1.1. *Suppose the nonlinearities in the quasilinear wave equation* (5.1) *satisfy the null condition. Assume the initial data* (ϕ_0, ϕ_1) *are smooth and are support on* $\{|x| \leq R\}$ *for some large constant* R. *Assume that the asymptotically flat back-*

ground metric g satisfy the above decay estimates (5.3) *for small positive constants δ_0, α. Then there exist $\delta_m > 0$, depending only on α, and $\epsilon_0 > 0$, depending on α, R, g, such that for all $\delta_0 < \delta_m$, $E_0 < \epsilon_0$, there exists a unique global smooth solution ϕ of equation* (5.1) *with the following properties*

(1) Energy decay

$$E[Z^k\phi](\tau) \leq CE_0(1+\tau)^{-1-\alpha}, \quad |k| \leq 5.$$

(2) Pointwise decay:

$$\sum_{|k|\leq 3} |Z^k\phi| \leq C\sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+|t-r+R|)^{-\frac{1}{2}-\frac{1}{2}\alpha};$$

$$\sum_{|k|\leq 2} |\underline{L}Z^k\phi| + \sum_{|k|\leq 1} |\partial\underline{L}Z^k\phi| \leq C_\epsilon\sqrt{E_0}(1+r)^{-1+\epsilon}(1+|t-r+R|)^{-\frac{1}{2}-\frac{\alpha}{2}};$$

$$\sum_{|k|\leq 2} |\overline{\partial_v}Z^k\phi| + \sum_{|k|\leq 1} |\partial\overline{\partial_v}Z^k\phi| \leq C_\epsilon\sqrt{E_0}(1+r)^{-\frac{3}{2}+\epsilon}, \quad \epsilon > 0.$$

where the constant C depends only on R, α, g and C_ϵ depends also depends on ϵ

We give several remarks

Remark 5.1.2. *A similar result can be obtained in higher dimensions without null condition.*

Remark 5.1.3. *It is not necessary to require that the initial data have compact support. The general assumption on the initial data can be that the following quantity*

$$\sum_{|k|\leq 6} \int_{\mathbb{R}^3} r^{1+\alpha}|\partial Z^k\phi(0,x)|^2 dx$$

is sufficiently small. In particular the constant R in the assumptions on the metric g can be different from the radius of the support of the initial data.

Remark 5.1.4. *We remark here that the special case when the metric g approaches the Minkowski metric in the spatial directions with a rate $(1+r)^{-1-\epsilon}$ has been discussed* *in the recent work [41]. But in that work there is an extra condition that the metric is static and is independent of time t.*

5.2 Bootstrap assumptions

The semilinear term F in the equation (5.1) has already been discussed in Chapter 4. The quasilinear part $C^{\mu\nu}(\phi)\partial_\mu\nu$ satisfies the null condition given in line (5.2). Cubic or higher order terms are always easy to handle for nonlinear wave equations. To simplify the proof of Theorem 5.1.1 but without loss of generality, instead of the general equation (5.1), we consider the following simple model of quasilinear wave equations

$$\begin{cases} \Box_g \phi + g^{\mu\nu\gamma}\partial_\gamma\phi \cdot \partial_{\mu\nu}\phi = 0, \\ \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x), \end{cases} \tag{5.4}$$

on the time dependent inhomogeneous background $(\mathbb{R}^{3+1}, g)$ where the metric g satisfies the estimates (5.3) and $g^{\mu\nu\gamma}$ are constants satisfy the null condition. We have to point out here that although we write the quasilinear part as $g^{\mu\nu\gamma}\partial_\gamma\phi\partial_{\mu\nu}\phi$, the null structure will never be used in the region $\{|x| \leq R\}$. In this region the nonlinear term can be any quadratic form of ϕ, $\partial\phi$.

We assume δ_0 is sufficiently small, depending only on α, such that Proposition 3.1.1, Proposition 3.2.2 and Proposition 3.3.3 hold. Recall that the initial data (ϕ_0, ϕ_1) are smooth and are supported on $\{|x| \leq R\}$. We use bootstrap argument to prove the main Theorem 5.1.1 in this chapter.

First we fix the foliation Σ_τ by choosing the radius R to be one appeared in the assumption (5.3) for the background metric g. We start with the following bootstrap assumptions on the solution ϕ. On S_τ $(r \geq R)$, we assume

$$\begin{aligned} &\sum_{|k|\leq 4} \int_\omega |\underline{L} Z^k \phi|^2 d\omega + \sum_{|k|\leq 3} \int_\omega |\partial \underline{L} Z^k \phi|^2 d\omega \leq 2H^2, \\ &\sum_{|k|\leq 4} \int_\omega |\overline{\partial_v} Z^k \phi|^2 d\omega + \sum_{|k|\leq 3} \int_\omega |\partial \overline{\partial_v} Z^k \phi|^2 d\omega \leq 2\bar{H}^2, \end{aligned} \tag{5.5}$$

where

$$\bar{H} = \delta_0 (1+|x|)^{-1-2\alpha}, \quad H = \bar{H} + \delta_0 (1+|x|)^{-\frac{1}{2}-2\alpha}(1+\tau)^{-\frac{1}{2}-\frac{1}{2}\alpha}$$

and τ is the parameter of the foliation Σ_τ. When $r \leq R$, we assume

$$\sum_{|k|\leq 4} \int_\omega |\partial Z^k \phi|^2 d\omega + \sum_{|k|\leq 3} \int_\omega |\partial^2 Z^k \phi|^2 d\omega \leq 2\bar{H}^2, \quad 1 \leq r \leq R, \tag{5.6}$$

$$\sum_{|k|\leq 3} |\partial Z^k \phi|^2 + \sum_{|k|\leq 2} |\partial^2 Z^k \phi|^2 \leq 2\delta_0^2, \quad |x| \leq 1. \tag{5.7}$$

To close the above bootstrap assumptions, we commute the equation (5.4) with Z for k times and show the decay of the integrated energy of $Z^k\phi$. We then use Sobolev embedding when $r \geq 1$ and elliptic estimates when $r \leq 1$ to improve the above bootstrap assumptions. That is we will show that the above bootstrap assumptions hold if we replace 2 with $E_0 C$ for some constant C depending only on R, α. Therefore if E_0 is sufficiently small, we can improve the bootstrap assumptions and conclude the main theorem.

Before we go to the estimates for the integrated energy decay for $Z^k\phi$, we prove several lemmas which will be used in this chapter. First of all, we choose the small positive constant ϵ, α_1, α_2 satisfying the conditions in Proposition 3.2.2, where ϵ is much smaller than α and $0 < \epsilon < \alpha < \alpha_1 < \alpha_2$. All the implicit constants appeared in the sequel may depend on these small constants. However, since the choice of ϵ, α_1, α_2 depends on α, we can let α to be the representative for the dependence of the implicit constants in the sequel. The only point we need to emphasize is that since we want to show that the smallness of δ_0 is independent of R, we may have to keep track of the dependence of R. From now on, unless we point it out, the implicit constant $A \lesssim B$ depends only on α.

We consider the solutions of the linear wave equation

$$\Box_g \phi + N(\phi) = F$$

with the metric g and N satisfying the estimates (3.2) (see equation (3.1) in section 3.1). The first lemma is an analogue of Lemma 4.2.2 and will be used to show the pointwise decay of the solution when $r \leq 1$.

Lemma 5.2.1. *Let ϕ be the solution of the linear wave equation* (3.1)*. Then we have*

$$\int_{\tau_1}^{\tau_2} \int_{r\leq 2} |\partial^2\phi|^2 dx d\tau \leq C_\alpha (D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2})$$

for some constant C_α depending only on α.

Proof. Note that g^{ij} is uniformly elliptic for sufficiently small δ_0. From the equation (3.1), we derive by using elliptic estimates that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \int_{r\leq 1} |\partial^2\phi|^2 dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{r\leq 2} \sum_{i,j=1}^{3} |\partial_{ij}\phi|^2 dx d\tau + \int_{\tau_1}^{\tau_2} \int_{r\leq 2} |\partial\partial_t\phi|^2 dx \\
&\lesssim \int_{\tau_1}^{\tau_2} \int_{r\leq 4} |\sum_{i,j=1}^{3} g^{ij}\partial_{ij}\phi|^2 + |\partial\partial_t\phi|^2 + |\phi|^2 dx d\tau \\
&\lesssim \int_{\tau_1}^{\tau_2} \int_{r\leq 4} (1+r)^{1+\alpha_1} |F|^2 dx d\tau + \int_{\tau_1}^{\tau_2} \int_{r\leq 4} \frac{|\partial\partial_t\phi|^2 + |\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dx d\tau \\
&\lesssim D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2}.
\end{aligned}$$

This proves the Lemma. □

For a symmetric two tensor $k^{\mu\nu}$, we may need to decompose the differential operator $k^{\mu\nu}\partial_{\mu\nu}$ with respect to the null frame $\{L, \underline{L}, S_1, S_2\}$.

Lemma 5.2.2. *Assume $k^{\mu\nu} = k^{\nu\mu}$. Then we have*

$$|k^{\mu\nu}\partial_{\mu\nu}\phi| \le |k^{\underline{L}\underline{L}}||\underline{L}Z\phi| + |k|(|\overline{\partial_v}Z\phi| + |L\underline{L}\phi| + r^{-1}|\partial\phi|), \quad r = |x| \ge 1.$$

where $k^{\underline{L}\underline{L}} = k^{\mu\nu}\frac{1}{2}\underline{L}_\mu\frac{1}{2}\underline{L}_\nu$ and $|k| = \sum_{\mu,\nu}|k^{\mu\nu}|$.

Proof. We decompose the derivative ∂_μ relative to the null frame $\{L, \underline{L}, S_1, S_2\}$

$$\partial_\mu = \nabla\!\!\!/_\mu + \frac{1}{2}\underline{L}_\mu\underline{L} + \frac{1}{2}L_\mu L, \quad L_0 = 1, \quad L_i = \frac{x_i}{r},$$

where $\nabla\!\!\!/_\mu$ is a linear combination of S_1 and S_2. Note that $L(\underline{L}_\mu) = L(L_\mu) = \underline{L}(\underline{L}_\mu) = \underline{L}(L_\mu) = 0$. We can compute

$$\begin{aligned}
\partial_{\mu\nu} &= (\frac{1}{2}\underline{L}_\mu\underline{L} + \frac{1}{2}L_\mu L + \nabla\!\!\!/_\mu)(\frac{1}{2}\underline{L}_\nu\underline{L} + \frac{1}{2}L_\nu L + \nabla\!\!\!/_\nu) \\
&= \frac{1}{2}\underline{L}_\mu\frac{1}{2}\underline{L}_\nu\underline{L}\underline{L} + \frac{1}{2}\underline{L}_\mu L_\nu L\underline{L} + \frac{1}{2}\underline{L}_\mu\underline{L}\nabla\!\!\!/_\nu + \frac{1}{2}L_\mu\frac{1}{2}\underline{L}_\nu L\underline{L} \\
&\quad + \frac{1}{2}L_\mu\frac{1}{2}L_\nu LL + \frac{1}{2}L_\mu L\nabla\!\!\!/_\nu + \nabla\!\!\!/_\mu(\frac{1}{2}\underline{L}_\nu\underline{L}) + \nabla\!\!\!/_\mu(\frac{1}{2}L_\nu L) + \nabla\!\!\!/_\mu\nabla\!\!\!/_\nu.
\end{aligned}$$

Recall that $L = 2\partial_t - \underline{L}$. We have

$$\begin{aligned}
k^{\mu\nu}\partial_{\mu\nu} &= k^{\underline{L}\underline{L}}\underline{L}\underline{L} + k^{LL}LL + 2k^{\underline{L}L}L\underline{L} + k^{\mu\nu}\nabla\!\!\!/_\mu\nabla\!\!\!/_\nu + k^{\mu\nu}\underline{L}_\mu\underline{L}\nabla\!\!\!/_\nu + k^{\mu\nu}L_\mu L\nabla\!\!\!/_\nu \\
&\quad + \frac{1}{2}k^{\mu\nu}(\nabla\!\!\!/_\mu\underline{L}_\nu \cdot \underline{L} + \nabla\!\!\!/_\mu L_\nu \cdot L + \underline{L}_\nu[\nabla\!\!\!/_\mu, \underline{L}] + L_\nu[\nabla\!\!\!/_\mu, L]) \\
&= 2k^{\underline{L}\underline{L}}\underline{L}\partial_t + 2k^{LL}L\partial_t + (2k^{\underline{L}L} - k^{LL} - k^{\underline{L}\underline{L}})L\underline{L} + k^{\mu\nu}\nabla\!\!\!/_\mu\nabla\!\!\!/_\nu + 2k^{\mu\nu}\underline{L}_\mu\nabla\!\!\!/_\nu\partial_t \\
&\quad + k^{\mu\nu}(L_\mu - \underline{L}_\mu)L\nabla\!\!\!/_\nu + \frac{1}{2}k^{\mu\nu}(\nabla\!\!\!/_\mu\underline{L}_\nu \cdot \underline{L} + \nabla\!\!\!/_\mu L_\nu \cdot L + \underline{L}_\nu[\nabla\!\!\!/_\mu, \underline{L}] + L_\nu[\nabla\!\!\!/_\mu, L]).
\end{aligned}$$

Note that

$$|\nabla\!\!\!/_\mu\underline{L}_\nu| \le r^{-1}, \quad |[\nabla\!\!\!/_\mu, \underline{L}]\phi| \le r^{-1}|\partial\phi|, \quad \nabla\!\!\!/ = r^{-1}\Omega, \quad r \ge 1.$$

The Lemma then follows. □

The following lemma gives the estimate for $L\underline{L}\phi$.

Lemma 5.2.3. *We have*

$$|L\underline{L}\phi| \leq C_\alpha(\frac{|\partial\phi| + |\partial Z\phi|}{r} + \tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha}(|\overline{\partial_v}Z\phi| + |\overline{\partial_v}\phi|) + |F|), \quad (t,x) \in S_\tau;$$
$$|L\underline{L}\phi| \leq C_\alpha(\frac{|\partial\phi| + |\partial Z\phi|}{r} + |F|), \quad 1 \leq r \leq R.$$

Proof. Write the equation (3.1) in null coordinates

$$-L\underline{L}\phi + \frac{2}{r}\partial_r\phi + \Delta\!\!\!/\phi + h^{\mu\nu}\partial_{\mu\nu}\phi + \tilde{N}(\phi) = F,$$

where

$$\tilde{N}^\mu = N^\mu + \frac{1}{\sqrt{-G}}\partial_v(g^{\mu\nu}\sqrt{-G}).$$

When $r \geq R$, we can show that

$$|\tilde{N}(\phi)| \lesssim r^{-1}|\partial\phi| + \tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha}|\overline{\partial_v}\phi|.$$

Using Lemma 5.2.2 to control $h^{\mu\nu}\partial_{\mu\nu}\phi$, we have pointwise bound for $L\underline{L}\phi$

$$|L\underline{L}\phi| \lesssim \frac{|\partial\phi| + |\nabla\!\!\!/\Omega\phi|}{r} + |h^{\underline{LL}}||\partial Z\phi| + |h|(|\overline{\partial_v}Z\phi| + |L\underline{L}\phi|) + |\tilde{N}(\phi)| + |F|.$$

Since $|h| \leq H + \bar{H} \leq \delta_0$, $|h^{\underline{LL}}| \leq \bar{H} \leq r^{-1}$ and δ_0 is small, the above inequality implies that

$$|L\underline{L}\phi| \lesssim \frac{|\partial\phi| + |\partial Z\phi|}{r} + |h||\overline{\partial_v}Z\phi| + |\tilde{N}(\phi)| + |F|.$$

The Lemma then follows from the assumption (3.2). □

For any two functions Φ, ϕ, we denote the null form

$$Q(\Phi, \phi) = g^{\mu\nu\gamma}\partial_\gamma \Phi \cdot \partial_{\mu\nu}\phi$$

for constants $g^{\mu\nu\gamma}$ satisfying the null condition. To simplify the notation, for another set of constants $\tilde{g}^{\mu\nu\gamma}$ satisfying the null condition, we still use $Q(\Phi, \phi)$ to denote $\tilde{g}^{\mu\nu\gamma}\partial_\gamma \Phi \cdot \partial_{\mu\nu}\phi$.

Lemma 5.2.4. *Let $g^{\mu\nu\gamma}$ be constants satisfying the null condition. Then for any two smooth functions Φ, ϕ, we have*

$$|Q(\Phi, \phi)| \lesssim |\overline{\partial_v}\Phi||\partial Z\phi| + |\partial\Phi|(|\overline{\partial_v}Z\phi| + |L\underline{L}\phi| + r^{-1}|\partial\phi|), \quad |x| \geq 1,$$
$$ZQ(\Phi, \phi) = Q(Z\Phi, \phi) + Q(\Phi, Z\phi) + Q(\Phi, \phi).$$

The last term $Q(\Phi, \phi)$ should be interpreted as $\tilde{g}^{\mu\nu\gamma}\partial_\gamma\Phi\partial_{\mu\nu}\phi$ for new constants $\tilde{g}^{\mu\nu\gamma}$ satisfying the null condition.

Proof. The null condition $g^{\mu\nu\gamma}\xi_\gamma\xi_\mu\xi_\nu = 0$ whenever $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ implies that $\underline{L}\Phi \cdot \underline{L}\underline{L}\phi$ will not appear in the decomposition of the null from $Q(\Phi, \phi)$ relative to the null frame $\{L, \underline{L}, S_1, S_2\}$. Using Lemma 5.2.2, we can get the first inequality. For the second inequality, we note that $Zr = 0$, $[Z, \underline{L}] = [Z, L] = 0$. □

5.3 Integrated energy decay for $Z^k\phi$

Since the initial data for the simplified quasilinear wave equation (5.4) have compact support, from Proposition 3.3.3 we conclude that under the bootstrap assumptions (5.7), (5.6), (5.5) the integrated energy $I^\epsilon[\phi]_{\tau_1}^{\tau_2}$ for ϕ decays in τ_1. As having discussed in the previous section, to close the bootstrap assumptions, we need to show the decay of the integrated energy for higher order derivatives of the solution. We thus can commute the equation with the vector fields $Z = \{\Omega, \partial_t\}$. However, after commuting the equation with Z, the resulting equation is not of the form in Proposition 3.3.3 (an addition second order derivative term $k^{\mu\nu}\partial_{\mu\nu}$ appears). That is we are not able to apply Proposition 3.3.3 directly to obtain the decay of the integrated energy for $Z\phi$. Below we consider the equations for $Z\phi$, and show that the integrated energy for $Z\phi$ also decays.

Let ϕ be the solution of the following linear wave equation

$$\Box\phi + h^{\mu\nu}\partial_{\mu\nu}\phi + N^{\mu}\partial_{\mu}\phi = F \tag{5.8}$$

where $h^{\mu\nu}$, N^{μ} satisfy the estimates (3.2) (but with $\delta_1 = \delta_0$). We have the equation for $Z\phi$

$$\Box Z\phi + h^{\mu\nu}\partial_{\mu\nu}Z\phi + \tilde{h}^{\mu\nu}\partial_{\mu\nu}\phi + N^{\mu}\partial_{\mu}(Z\phi) + \tilde{N}^{\mu}\partial_{\mu}\phi = ZF, \tag{5.9}$$

where

$$\tilde{h}^{\mu\nu}\partial_{\mu\nu}\phi = Z(h^{\mu\nu}\partial_{\mu\nu}\phi) - h^{\mu\nu}\partial_{\mu\nu}Z\phi, \quad \tilde{N} = [Z, N].$$

We assume that $\tilde{h}$, $\tilde{N}^{\mu}$ satisfy the following estimates

$$\begin{aligned} &|\tilde{N}^{\underline{L}}| + |\tilde{h}^{\underline{L}\underline{L}}| \leq \bar{H}, \quad |\tilde{N}^{\mu}| + |\tilde{h}^{\mu\nu}| \leq H, \quad (t,x) \in S_{\tau}; \\ &|\tilde{N}^{\mu}| + |\tilde{h}^{\mu\nu}| \leq \bar{H}, \quad |x| \leq R. \end{aligned} \tag{5.10}$$

Here we recall that

$$\tilde{h}^{\underline{L}\underline{L}} = \tilde{h}^{\mu\nu}\frac{1}{2}\underline{L}_{\mu}\frac{1}{2}\underline{L}_{\nu}, \quad \tilde{N}^{\underline{L}} = \tilde{N}^{\mu}\frac{1}{2}\underline{L}_{\mu}.$$

Denote

$$E_0 = \tilde{E}[\phi](0) + \tilde{E}[Z\phi](0) + S^{\epsilon}[\phi](0) + S^{\epsilon}[Z\phi](0).$$

Proposition 5.3.1. *Assume $h^{\mu\nu}$, N^{μ} satisfy the estimates* (3.2)*. $\tilde{h}^{\mu\nu}$, $\tilde{N}^{\mu}$ are defined as above and satisfy the similar estimates* (5.10)*. Let ϕ be the solution of the linear* *wave equation* (5.8). *Assume*

$$D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + D^{\alpha_1}[ZF]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1$$

for some constant C_1*. If* δ_0 *is sufficiently small, depending only on* α*, then*

$$\begin{aligned} &I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2} + D^{\alpha_1}[\tilde{h}^{\mu\nu}\partial_{\mu\nu}\phi]_{\tau_1}^{\tau_2} + D^{\alpha_1}[\tilde{N}^{\mu}\partial_{\mu}Z\phi]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} \\ &\leq C_{\alpha,R}(C_1+E_0)(\tau_1)_+^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \end{aligned}$$

for some constants $C_{\alpha,R}$ *depending on* α*,* R*.*

Proof. For simplicity, in the proof we denote

$$E_1 = C_1 + E_0, \quad \bar{N} = N^{\mu}\partial_{\mu}Z\phi, \quad F_h = \tilde{h}^{\mu\nu}\partial_{\mu\nu}\phi, \quad F_1 = ZF - \tilde{N}(\phi).$$

We move $\bar{N} + F_h$ to the right hand side of the equation (5.9) and treat it as inhomogeneous term. Using the smallness of δ_0 we will absorb F_h and $\tilde{N}$. And then the decay of $I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2}$ follows from the same argument for proving the decay of $I^{\epsilon}[\phi]_{\tau_1}^{\tau_2}$ in Proposition 3.3.3. The main difficulty is that we need to show that the smallness of δ_0 depends only on α. Note that the implicit constants in the integrated energy estimate (3.5) and the energy estimate (3.6) depend only on α. We mainly rely on these two estimates to control F_h and $\bar{N}$.

First using the estimates (5.10) and Proposition 3.3.3 we can show that

$$\begin{aligned} D^{\alpha_1}[\tilde{N}(\phi)]_{\tau_1}^{\tau_2} &\leq C_{\alpha}(I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + E^{1+\alpha}[\phi]_{\tau_1}^{\tau_2}) \leq E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}(1 + \int_0^{\tau_2} E[\phi](\tau)d\tau) \\ &\leq E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}. \end{aligned}$$

We have used the p-weighted energy inequality (3.23) in the last step. In particular, we have

$$D^{\alpha_1}[F_1]_{\tau_1}^{\tau_2} = D^{\alpha_1}[ZF - \tilde{N}(\phi)]_{\tau_1}^{\tau_2} \le E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}. \tag{5.11}$$

Using Lemma 5.2.2 and the assumption (5.10), we can estimate

$$|F_h| + |\tilde{N}| \lesssim \begin{cases} \bar{H}|\partial Z\phi| + H(|\overline{\partial_v} Z\phi| + |L\underline{L}\phi| + r^{-1}|\partial\phi|), \quad |x| \ge R; \\ \bar{H}(|\partial Z\phi| + |L\underline{L}\phi| + r^{-1}|\partial\phi|), \quad 1 \le r < R; \\ \bar{H}(|\partial^2\phi| + |\partial\phi|). \end{cases} \tag{5.12}$$

Then using Lemma 5.2.3 to bound $L\underline{L}\phi$, we can show that

$$\begin{aligned} D^{\alpha_1}[F_h]_{\tau_1}^{\tau_2} + D^{\alpha_1}[\bar{N}]_{\tau_1}^{\tau_2} &\lesssim \delta_0^2 \Big(\int_{\tau_1}^{\tau_2}\int_{r\le 1} |\partial^2\phi|^2 dx d\tau + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau \cap\{r\ge 1\}} |L\underline{L}\phi|^2 dx d\tau \\ &\quad + E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2} \Big) \\ &\lesssim \delta_0^2 (D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2}). \end{aligned}$$

Here we have used Lemma 5.2.1 to estimate $\partial^2\phi$ in $\{|x| \le 1\}$. To estimate $E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2}$ (see the definition in Section 2.2), set $\tau_2 = \tau$ in the energy inequality (3.6) and multiply both side with $\tau_+^{-1-\alpha}$ and then integrate with respect to τ from τ_1 to τ_2. We can derive

$$E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[Z\phi](\tau_1) + S^\epsilon[Z\phi](\tau_1) + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + D^\epsilon[F_1 - F_h - \bar{N}]_{\tau_1}^{\tau_2}.$$

Here recall that $Z\phi$ satisfies the above linear wave equation (5.9). Now from the previous estimate we obtain

$$D^{\alpha_1}[F_1 - F_h - \bar{N}]_{\tau_1}^{\tau_2} \lesssim D^{\alpha_1}[F_1]_{\tau_1}^{\tau_2} + \delta_0^2(D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} + I^\epsilon[\phi]_{\tau_1}^{\tau_2}).$$

Let δ_0 be sufficiently small depending only on α (the implicit constant depends only on α). We can absorb $E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2}$ and thus to derive

$$E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[Z\phi](\tau_1) + S^{\epsilon}[Z\phi](\tau_1) + I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2} + \tilde{D}_{\tau_1}^{\tau_2}, \tag{5.13}$$

$$D^{\alpha_1}[F_1 - F_h - \bar{N}]_{\tau_1}^{\tau_2} \lesssim \delta_0^2(\tilde{E}[Z\phi](\tau_1) + S^{\epsilon}[Z\phi](\tau_1) + I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2}) + \tilde{D}_{\tau_1}^{\tau_2}, \tag{5.14}$$

where we denote

$$\tilde{D}_{\tau_1}^{\tau_2} = D^{\alpha_1}[F_1]_{\tau_1}^{\tau_2} + D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + I^{\epsilon}[\phi]_{\tau_1}^{\tau_2}.$$

From estimate (5.11) and Proposition 3.3.3, we have

$$\tilde{D}_{\tau_1}^{\tau_2} \leq E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}. \tag{5.15}$$

We now use the above estimates (5.13), (5.14) to simplify the integrated energy estimate (3.5), the energy estimate (3.6) as well as the p-weighted energy inequalities (3.22), (3.23) for $Z\phi$. For the integrated energy estimate and the energy estimate, from (5.14), for sufficiently small δ_0, we have

$$E[Z\phi](\tau_2) + I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau} \frac{|\nabla\!\!\!/ Z\phi|^2}{r} dx d\tau \lesssim E[Z\phi](\tau_1) + \delta_0 S^{\epsilon}[Z\phi](\tau_i) + \tilde{D}_{\tau_1}^{\tau_2}.$$

Here since $E^N[Z\phi]_0^{\infty}$ is finite, all the estimates hold if we replace $\tilde{E}[Z\phi](\tau)$ with $E[Z\phi](\tau)$. Now from the p-weighted energy inequalities (3.23), we obtain the p-weighted energy estimate when $p = 1$ for $Z\phi$

$$\begin{aligned} g^1[Z\phi](\tau_2) + \int_{\tau_1}^{\tau_2} E[Z\phi](\tau) d\tau &\lesssim g^1[Z\phi](\tau_1) + \delta_0^2 \int_{\tau_1}^{\tau_2} (\tau)_+^{-\alpha} E[Z\phi](\tau) d\tau d\tau \\ &+ C_R((\tau_1)_+^{1-\alpha} E[Z\phi](\tau_1) + (\tau_1)_+^{-\alpha} + \delta_0(\tau_i)_+^{1-\alpha} S^{\epsilon}[Z\phi](\tau_i)), \end{aligned}$$

where we use the estimate (5.14) to bound the inhomogeneous term $F_1 - F_h - \bar{N}$. Let δ_0 to be sufficiently small, depending only on α (as the implicit constant depends only on α). We conclude from the above estimate that

$$\begin{aligned}&\int_{\tau_1}^{\tau_2} E[Z\phi](\tau)d\tau \\ &\lesssim g^1[Z\phi](\tau_1) + C_R((\tau_1)_+^{1-\alpha}E[Z\phi](\tau_1) + (\tau_1)_+^{-\alpha} + \delta_0(\tau_i)_+^{1-\alpha}S^\epsilon[Z\phi](\tau_i)).\end{aligned} \tag{5.16}$$

Similarly, we obtain the p-weighted energy inequality when $p = 1+\alpha_1$ for $Z\phi$

$$\begin{aligned}g^{1+\alpha_1}[Z\phi](\tau_2)dvd\omega d\tau &\lesssim g^{1+\alpha_1}[Z\phi](\tau_1) + \int_{\tau_1}^{\tau_2} E[Z\phi](\tau)d\tau \\ &+ C_R((\tau_1)_+E[Z\phi](\tau_1) + \delta_0(\tau_i)_+S^\epsilon[Z\phi](\tau_i) + E_1).\end{aligned}$$

Let $\tau_1 = 0$. From the previous estimate for the integral of the energy flux, we derive

$$g^{1+\alpha_1}[Z\phi](\tau)dvd\omega d\tau \lesssim C_R(E_1 + \tau_+S^\epsilon[Z\phi](\tau)).$$

By Proposition 3.1.1 we have

$$I^\epsilon[Z\phi]_0^\infty \leq C_{\alpha,R}E_1.$$

Then the above two p-weighted energy estimates for $Z\phi$ are sufficiently to prove the decay of the integrated energy for $Z\phi$ (the proof is then the same as the proof in Proposition 3.3.3). That is

$$I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} \leq E_1C_{\alpha,R}(\tau_1)_+^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

To finish the proof for Proposition 5.3.1, it suffices to prove the decay of $E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2}$. Note that

$$E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \frac{E[Z\phi](\tau)}{(1+\tau)^{1+\alpha}}d\tau \leq (\tau_1)_+^{-1-\alpha}\int_{\tau_1}^{\tau_2} E[Z\phi](\tau)d\tau.$$

Since we have shown

$$\int_{\tau_1}^{\tau_2} S^\epsilon[Z\phi](\tau)d\tau \le I^\epsilon[Z\phi]_{\tau_1}^{\tau_2} \le E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha},$$

we can choose τ_2 arbitrarily large such that

$$S^\epsilon[Z\phi](\tau_2) \le E_1 C_{\alpha,R}(\tau_2)_+^{-1-\alpha}.$$

for some arbitrarily large τ_2. Then in the p-weighted energy inequality (5.16) set $\tau_1 = 0$, we derive

$$\int_0^{\tau_2} E[Z\phi](\tau)d\tau \le E_1 C_{\alpha,R}.$$

This constant is independent of τ_2. In particular, we have

$$\int_{\tau_1}^{\tau_2} E[Z\phi](\tau)d\tau \le E_1 C_{\alpha,R}.$$

Therefore we have

$$E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} \le (\tau_1)_+^{-1-\alpha} \int_{\tau_1}^{\tau_2} E[Z\phi](\tau)d\tau \le E_1 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}.$$

This finishes the proof for Proposition 5.3.1. □

We now consider the solution of the quasilinear wave equation (5.4) under the bootstrap assumptions (5.5), (5.6), (5.7). We show that the integrated energy for $Z^k\phi$, $k \le 6$ decays.

Proposition 5.3.2. *Let ϕ be the solution of* (5.4) *with compactly supported initial data ϕ_0, ϕ_1 described in Theorem 5.1.1. Then*

$$I^\epsilon[Z^k\phi]_{\tau_1}^{\tau_2} \le E_0 C_{\alpha,R}(1+\tau_1)^{-1-\alpha}, \quad \forall k \le 6$$

for some constant $C_{\alpha,R}$ depending on R, α. Here E_0 is defined before Theorem 5.1.1.

To show the integrated energy decay for $Z^k\phi$, we consider the equation of $Z^k\phi$ obtained by commuting the equation (5.4) with Z^k. Let N be the vector field with components

$$N^\mu = \frac{1}{\sqrt{-G}}\partial_\nu(g^{\mu\nu}\sqrt{-G}).$$

Then we can write the equation (5.4) as

$$\Box\phi + Q(\phi,\phi) + h^{\mu\nu}\partial_{\mu\nu}\phi + N(\phi) = 0$$

Commute this equation with Z^k. We obtain the equation for $Z^k\phi$

$$\Box Z^k\phi + Q(\phi, Z^k\phi) + h^{\mu\nu}\partial_{\mu\nu}Z^k\phi + Q_1^k + H_1^k + Q(Z^k\phi,\phi) + N(Z^k\phi) = F_2^k. \quad (5.17)$$

with the following definitions for Q_1^k, H_1^k, F_2^k. Using Lemma 5.2.4, we let Q_1^k be the collection of all those terms containing $\partial_{\mu\nu}Z^{k-1}\phi$ in the expansion of $Z^kQ(\phi,\phi)$. More precisely,

$$Q_1^k = Q(Z\phi, Z^{k-1}\phi) + Q(\phi, Z^{k-1}\phi).$$

We remark here that Q denotes a general null form for constants $\tilde{g}^{\mu\nu\gamma}$ satisfying the null condition. It may be different from $g^{\mu\nu\gamma}$ appeared in the equation (5.4). For example we in fact have

$$Q(Z\phi, Z^{k-1}\phi) = kg^{\mu\nu\gamma}\partial_\gamma Z\phi \cdot \partial_{\mu\nu}Z^{k-1}\phi.$$

Similarly, we let H_1^k be the collection of all those terms in the expansion of $Z(h^{\mu\nu}\partial_{\mu\nu}\phi)$ containing $\partial_{\mu\nu}Z^{k-1}\phi$, which can be given as follows

$$H_1^k = k(Z(h^{\mu\nu}\partial_{\mu\nu}Z^{k-1}\phi) - h^{\mu\nu}\partial_{\mu\nu}Z^k\phi).$$

Finally, we denote

$$Q_2^k = -\sum_{k_2 \le k-2, k_1+k_2 \le k, k_1 < k} Q(Z^{k_1}\phi, Z^{k_2}\phi), \quad H_2^k = -Z^k(h^{\mu\nu}\partial_{\mu\nu}\phi) + h^{\mu\nu}\partial_{\mu\nu}Z^k\phi + H_1^k$$

$$Q^k = Q(Z^k\phi, \phi), N^k = N(Z^k\phi), F_2^k = Q_2^k + H_2^k - [Z^k, N]\phi, \quad F^k = F_2^k - Q_1^k - H_1^k.$$

We first check that $g^{\mu\nu\gamma}\partial_\gamma\phi$ satisfies the same estimates (3.2), (3.21) as $h^{\mu\nu}$. Note that

$$g^{\underline{L}\underline{L}\gamma}\partial_\gamma\phi = g^{\mu\nu\gamma}\frac{1}{2}\underline{L}_\mu\frac{1}{2}\underline{L}_\nu\partial_\gamma\phi, \quad g^{\mu\nu\gamma}\underline{L}_\mu\underline{L}_\nu\underline{L}_\gamma = 0.$$

The bootstrap assumption (5.5) together with the Sobolev embedding on the unit sphere shows that

$$|g^{\underline{L}\underline{L}\gamma}\partial_\gamma\phi| + |\partial g^{\underline{L}\underline{L}\gamma}\partial_\gamma\phi| \le 2\bar{H}, \quad |\nabla\!\!\!/ g^{\underline{L}\underline{L}\gamma}\partial_\gamma\phi| \le |r^{-1}g^{\underline{L}\underline{L}\gamma}\partial_\gamma Z\phi| \le 2r^{-1}\bar{H}, \quad r \ge R.$$

The other estimates in (3.2), (3.21) follow directly from the bootstrap assumptions (5.5), (5.6), (5.7) after using Sobolev embedding.

To apply Proposition 5.3.1, we can write the equation for $Z^{k-1}\phi$ as

$$\Box Z^{k-1}\phi + (g^{\mu\nu\gamma}\partial_\gamma\phi + h^{\mu\nu})\partial_{\mu\nu}Z^{k-1}\phi + (g^{\mu\nu\gamma}\partial_{\mu\nu}\phi + N^\gamma)\partial_\gamma(Z^{k-1}\phi) = F^{k-1}.$$

Then the equation for $Z^k\phi$ will be of the form (5.9) if we denote $\tilde{h}^{\mu\nu}$, be functions such that

$$\tilde{h}^{\mu\nu}\partial_{\mu\nu}\phi = Q_1^k + H_1^k.$$

The vector field N^μ there corresponds to $g^{\mu\nu\gamma}\partial_{\mu\nu}\phi + N^\gamma$ here. And $\tilde{N}$ is the Z derivative of N. We can check that $\tilde{h}^{\mu\nu}$, $\tilde{N}^\mu$, N satisfy condition (5.10). In fact for $g^{\mu\nu\gamma}\partial_\gamma Z\phi$ or $g^{\nu\gamma\mu}\partial_{\nu\gamma}\phi$ contributed by the null form Q_1^k, we can show that the condition (5.10) is satisfied by using Lemma 5.2.4 together with the bootstrap assumptions. For the part from H_1^k, we have the assumption (5.3). This implies that we can apply

Proposition 3.3.3 and 5.3.1 to show the integrated energy decay of $Z^k\phi$.

In particular, when $k \leq 1$, we have $F_2^k = 0$. Thus Proposition 3.3.3 and 5.3.1 imply that

$$I^{\epsilon}[Z\phi]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Q_1^1 + H_1^1]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Q^1 + N^1]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z\phi]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}, \quad k \leq 1.$$

Here we recall that $Q^k = Q(Z^k\phi, \phi)$, $N^k = N(Z^k\phi)$. We now use induction argument to show Proposition 5.3.2. We assume that for some fixed $k \leq 6$

$$\begin{aligned} &I^{\epsilon}[Z^l\phi]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Q_1^l + H_1^l]_{\tau_1}^{\tau_2} + D^{\alpha_1}[F_2^l]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Q^l + N^l]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z^l\phi]_{\tau_1}^{\tau_2} \\ &\leq E_0 C_{\alpha,R,k-1}(\tau_1)_+^{-1-\alpha}, \quad \forall l \leq k-1. \end{aligned} \tag{5.18}$$

We have shown that this is true when $k = 2$. Now we want to show that the above estimate also holds for $l = k$.

First note that the induction assumption in particular implies that

$$D^{\alpha_1}[F^l]_{\tau_1}^{\tau_2} = D^{\alpha_1}[F_2^l - Q_1^l - H_1^l]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R,k-1}(\tau_1)_+^{-1-\alpha}, \quad \forall l \leq k-1.$$

Therefore by Proposition 5.3.1, the estimate (5.18) holds for k if we can show that

$$D^{\alpha_1}[F_2^k]_{\tau_1}^{\tau_2} \leq D^{\alpha_1}[Q_2^k]_{\tau_1}^{\tau_2} + D^{\alpha_1}[H_2^k]_{\tau_1}^{\tau_2} + D^{\alpha_1}[[Z^k, N]\phi]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R,k}(\tau_1)_+^{-1-\alpha},$$

which follows from the following two lemmas.

Lemma 5.3.3. *Under the induction assumption* (5.18)*, we have*

$$D^{\alpha_1}[H_2^k]_{\tau_1}^{\tau_2} + D^{\alpha_1}[[Z^k, N]\phi]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R,k}(\tau_1)_+^{-1-\alpha}.$$

Proof. We use condition (5.3) and Lemma 5.2.2. We can show that

$$|H_2^k| + |[Z^k, N]\phi| \lesssim \begin{cases} \sum_{l\leq k-1} \bar{H}|\partial Z^l\phi| + H(|\overline{\partial_v} Z^l\phi| + r^{-1}|\partial Z^l\phi| + |L\underline{L}Z^{l-1}\phi|), r \geq 1; \\ \bar{H}\sum_{l\leq k-2} |\partial^2 Z^l\phi| + |\partial Z^{l+1}\phi|, \quad r < 1. \end{cases} \tag{5.19}$$

Thus we have

$$D^{\alpha_1}[H_2^k]_{\tau_1}^{\tau_2} + D^{\alpha_1}[[Z^k, N]\phi]_{\tau_1}^{\tau_2} \lesssim \sum_{l\leq k-1}\int_{\tau_1}^{\tau_2}\int_{r\leq 1}|\partial^2 Z^{l-1}\phi|^2 dx d\tau + I^{\epsilon}[Z^l\phi]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z^l\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}}|L\underline{L}Z^{l-1}\phi|^2 dx d\tau.$$

Here $Z^{-1}\phi = 0$. Now using Lemma 5.2.1 and the induction assumption (5.18), we can estimate

$$\int_{\tau_1}^{\tau_2}\int_{r\leq 1}|\partial^2 Z^l\phi|^2 dx d\tau \lesssim D^{\alpha_1}[F^l]_{\tau_1}^{\tau_2} + \sum_{l_1\leq l+1} I^{\epsilon}[Z^{l_1}\phi]_{\tau_1}^{\tau_2} \leq C_{\alpha,R,k}E_0(\tau_1)_+^{-1-\alpha}, \forall l \leq k-2.$$

Similarly using Lemma 5.2.3 to control $L\underline{L}Z^{k-2}\phi$, we get

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}}|L\underline{L}Z^l\phi|^2 dx d\tau &\lesssim \sum_{l_1\leq l+1} I^{\epsilon}[Z^{l_1}\phi]_{\tau_1}^{\tau_2} + E^{1+\alpha}[Z^{l_1}]_{\tau_1}^{\tau_2} + D^{\alpha_1}[F^l]_{\tau_1}^{\tau_2} \\ &\leq C_{\alpha,R,k}E_0(\tau_1)_+^{-1-\alpha}, \quad \forall l \leq k-2.\end{aligned}$$

The proves the lemma. □

For $D^{\alpha_1}[Q_2^k]_{\tau_1}^{\tau_2}$, we have

Lemma 5.3.4. *We have*

$$D^{\alpha_1}[Q_2^k]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R,k}(\tau_1)_+^{-1-\alpha}.$$

Proof. The proof will be the same to that of the previous lemma once we can estimate the null form. Recall the definition of Q_2^k after equation (5.17). It suffices to consider $Q(Z^{k_1}\phi, Z^{k_2}\phi)$ for some pair (k_1, k_2) such that $k_1+k_2 \leq k$, $k_2 \leq k-2$, $k_1 \leq k-1$. We need a Sobolev embedding to estimate the null form. We claim that for such pair (k_1, k_2), we always have

$$\begin{aligned}\int_\omega |\partial Z^{k_1}\phi|^2 |\partial^2 Z^{k_2}\phi|^2 d\omega \lesssim & \sum_{l\leq 3}\int_\omega |\partial^2 Z^l\phi|^2 d\omega \cdot \sum_{l\leq k-1}\int_\omega |\partial Z^l \phi| d\omega \\ & + \sum_{l\leq 4} |\partial Z^l\phi|^2 d\omega \cdot \sum_{l\leq k-2}\int_\omega |\partial^2 Z^l\phi|^2 d\omega.\end{aligned}$$

We only prove the above claim for the case $k_1+k_2 = k = 6$, $k_1 \leq 5$, $k_2 \leq 4$. If $k_2 \leq 1$ or $k_1 \leq 3$, the above inequality follows from Sobolev embedding on the unit sphere. If $k_1 = 4$, $k_2 = 2$, we use

$$\|\partial Z^{k_1}\phi \partial^2 Z^{k_2}\phi\|_{L^2(\mathbb{S}^2)} \lesssim \|\partial Z^{k_1}\phi\|_{L^4(\mathbb{S}^2)}\|\partial^2 Z^{k_2}\phi\|_{L^4(\mathbb{S}^2)} \lesssim \|\partial Z^{k_1}\phi\|_{H^1(\mathbb{S}^2)}\|\partial^2 Z^{k_2}\phi\|_{H^1(\mathbb{S}^2)}.$$

Thus the above Sobolev embedding holds. Now using Lemma 5.2.4 and the bootstrap assumptions (5.5), (5.6), we can show that when $r \geq 1$

$$\begin{aligned}&\int_\omega |Q(Z^{k_1}\phi, Z^{k_2}\phi)|^2 d\omega \\ &\lesssim \sum_{l\leq k-1}\int_\omega H^2|\overline{\partial_v} Z^l\phi|^2 + \bar{H}^2|\partial Z^l\phi|^2 + H^2|L\underline{L}Z^{l-1}\phi|^2 + H^2 r^{-2}|\partial Z^l\phi|^2 d\omega.\end{aligned} \tag{5.20}$$

When $r \leq 1$, note that $k_1+k_2 \leq k \leq 6$. In particular, $k_1 \leq 3$ or $k_2 \leq 2$. Thus by the bootstrap assumption (5.7), we can get

$$|Q(Z^{k_1}\phi, Z^{k_2}\phi)| \lesssim \bar{H}\sum_{l\leq k-2}|\partial^2 Z^l\phi| + |\partial Z^{l+1}\phi|.$$

Then the Lemma follows from the same argument for proving Lemma 5.3.3. □

The above two lemmas together with Proposition 5.3.1 implies that (5.18) holds for $l = 6$. In particular we have Proposition 5.3.2. From the proof, we in fact can prove an important integrated energy inequality for $L\underline{L}Z^k\phi$ with positive weights in r which will be used to derive the pointwise decay of the derivative of the solution.

Corollary 5.3.5. *We have*

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{1-\epsilon}|L\underline{L}Z^k\phi|^2 dx d\tau \leq E_0 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}, \quad \forall k\leq 5.$$

Proof. Using Lemma 5.2.3, we can show that for all $k \leq 5$

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{1-\epsilon}|L\underline{L}Z^k\phi|^2 dx d\tau &\lesssim \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{|\partial Z^k\phi|^2 + |\partial Z^{k+1}\phi|^2}{(1+r)^{1+\epsilon}} dx d\tau + D^{\alpha_1}[F^k]_{\tau_1}^{\tau_2} \\ &\lesssim \sum_{l\leq 6} I^\epsilon[Z^l\phi]_{\tau_1}^{\tau_2} + D^{\alpha_1}[F^k]_{\tau_1}^{\tau_2} \leq E_0 C_{\alpha,R}(\tau_1)_+^{-1-\alpha}.\end{aligned}$$

□

5.4 Pointwise estimates

The corollary in the end of the last section plays an important role in showing the pointwise estimates for the solution when $\{r \geq 1\}$. Next we use this integrated energy decay estimate together with the p-weighted energy estimates proven in Proposition 3.1.1, 3.2.2 to obtain the pointwise estimates for the solution ϕ and hence to close the bootstrap assumptions (5.5), (5.7), (5.6). We divide our argument into several steps.

In the argument below, the notation $A \lesssim B$ means $A \leq C_{\alpha,R}B$ for some constant $C_{\alpha,R}$ depending only on α, R.

First we estimate $Z^k\phi$ for $k \leq 5$. Since Z can be ∂_t or Ω, from Proposition 5.3.2, we can bound $S^\epsilon[Z^k\phi](\tau)$ as follows

$$S^\epsilon[Z^k\phi](\tau) \leq I^\epsilon[Z^k\phi]_\tau^{\tau+1} + I^\epsilon[Z^{k+1}\phi]_\tau^{\tau+1} \lesssim E_0\tau_+^{-1-\alpha}, \quad \forall k \leq 5.$$

In particular, the set (3.44) defined in the proof of Proposition 3.3.3 is $[0,\infty)$ for all $Z^k\phi$, $k \le 5$. Hence the proof there implies that the energy decays for all $\tau \ge 0$. That is

$$E[Z^k\phi](\tau) \lesssim E_0\tau_+^{-1-\alpha}, \quad \forall k \le 5, \quad \forall \tau \ge 0. \tag{5.21}$$

Here we have used the estimate (5.18) which holds for all $l \le 6$. Now let $\tau_1 = 0$ in the p-weighted energy inequality (3.22) when $p = 1 + \alpha_1$, we have

$$g^{1+\alpha_1}[Z^k\phi](\tau) = \int_{S_\tau} r^{1+\alpha_1}|L(rZ^k\phi)|^2 dvd\omega \lesssim E_0, \quad \forall k \le 5.$$

From Lemma 2.4.3, we can show

$$\int_{S_\tau} r^{1-\epsilon}|Z^k\phi|^2 dvd\omega \lesssim E[Z^k\phi](\tau) + g^{1+\alpha_1}[Z^k\phi](\tau) \lesssim E_0, \quad \forall k \le 5.$$

The good derivative of the solution decays better. Quantitatively, from the previous estimate, we derive

$$\int_{S_\tau} r^{3-\epsilon}|LZ^k\phi|^2 dvd\omega \lesssim g^{1-\epsilon}[Z^k\phi](\tau) + \int_{S_\tau} r^{1-\epsilon}|Z^k\phi|^2 dvd\omega \lesssim E_0, \quad \forall k \le 5. \tag{5.22}$$

Using the decay estimates for the energy $E[Z^k\phi](\tau)$, $k \le 5$, Lemma 2.4.1 quickly yields the spherical average estimate for $Z^k\phi$

$$\int_\omega |Z^k\phi|^2(\tau, r, \omega)d\omega \lesssim E[Z^k\phi](\tau) \lesssim E_0 r^{-1}\tau_+^{-1-\alpha}, \quad \forall k \le 5. \tag{5.23}$$

Recall that $\nabla\!\!\!/ = r^{-1}\Omega$. The above estimate in particular implies the improved decay estimates for the angular derivative of the solution

$$\int_\omega |\nabla\!\!\!/ Z^k\phi|^2 d\omega \le r^{-2}\int_\omega |\Omega Z^k\phi|^2 d\omega \lesssim E_0 r^{-3}\tau_+^{-1-\alpha}, \quad \forall k \le 4. \tag{5.24}$$

Next, we estimate $LZ^k\phi$ and $\underline{L}Z^k\phi$ which we rely on Corollary 5.3.5. Corollary 5.3.5 implies that

$$\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{3-\epsilon}|L\underline{L}Z^k\phi|^2 dvd\omega \leq \int_\tau^{\tau+1}\int_{\Sigma_\tau\cap\{r\geq 1\}} r^{1-\epsilon}(|L\underline{L}Z^k\phi|^2 + |L\underline{L}\partial_t Z^k\phi|^2)dxd\tau$$
$$\lesssim E_0(1+\tau)^{-1-\alpha}, \quad \forall k\leq 4. \tag{5.25}$$

Recall that $L = 2\partial_t - \underline{L}$ and Z can be ∂_t. We can also obtain estimates for $LLZ^k\phi$. In fact, from estimate (5.22), we can show that

$$\int_{S_\tau} r^{3-\epsilon}|LLZ^k\phi|^2 dvd\omega$$
$$\leq \int_{S_\tau} r^{3-\epsilon}|L\underline{L}Z^k\phi|^2 dvd\omega + 2\int_{S_\tau} r^{3-\epsilon}|\partial_v\partial_t Z^k\phi|^2 dvd\omega \lesssim E_0, \quad \forall k\leq 4.$$

This estimate together with estimate (5.22) implies that

$$\int_\omega |LZ^k\phi|^2 d\omega \lesssim E_0 r^{-3+\epsilon}, \quad r\geq R, \quad \forall k\leq 4. \tag{5.26}$$

Here note that $\partial_v = L = \partial_t + \partial_r$. This estimate together with (5.24) gives the estimate for $\overline{\partial_v}Z^k\phi$ when $r\geq R$, $k\leq 4$. To close the bootstrap assumptions, we also need to estimate $\underline{L}Z^k\phi$.

We consider $\underline{L}Z^k\phi$ on the larger domain $\Sigma_\tau\cap\{r\geq 1\}$. We first argue that

$$\liminf_{v\to\infty}\int_\omega |\underline{L}(Z^k\phi)|^2(u,v,\omega)d\omega = 0, \quad k\leq 5. \tag{5.27}$$

This follows immediately from the fact that

$$\int_{S_\tau}\frac{|\partial Z^k\phi|^2}{(1+r)^{1+\epsilon}} r^2 dvd\omega \lesssim E_0\tau_+^{-1-\alpha}, \quad k\leq 5.$$

We can also see this as solutions of linear wave equations decays at null infinity. To solve our nonlinear equations, we see from the last section of the previous chapter that we use Picard iteration process and approximate the solution by linear solutions. As linear solution decays at null infinity, we have (5.27). Hence on $\Sigma_\tau \cap \{r \geq 1\}$, from the estimates (5.25), we can show that

$$\begin{aligned}\int_\omega |\underline{L}Z^k\phi|^2(\tau, v, \omega)d\omega &\leq \int_v^\infty \int_\omega |L\underline{L}Z^k\phi|^2 r^{3-\epsilon} dv d\omega \cdot \int_v^\infty \int_\omega r^{-3+\epsilon} dv d\omega \\ &\lesssim (2v - \tau + R)^{-2+\epsilon} \int_{\Sigma_\tau \cap \{r \geq 1\}} |L\underline{L}Z^k\phi|^2 r^{1-\epsilon} dx d\omega \\ &\lesssim E_0(1+\tau)^{-1-\alpha} r^{-2+\epsilon}, \quad \forall k \leq 4,\end{aligned}$$

where we recall that $v = \frac{t+r}{2} = r + \frac{\tau - R}{2}$. We must remark here that the above argument holds for $(\tau, v, \omega) \in S_\tau$. When $1 \leq r \leq R$, splitting the integral into two parts: integral on S_τ and integral on $r \leq R$, we can get the same estimates. This gives the estimate for $\underline{L}Z^k\phi$ when $r \geq 1$. Since $\underline{L}$, $Z = \{\partial_t, \Omega\}$ can form a basis of the tangent space at any point where $r \geq 1$, a weaker estimate for $\partial Z^k\phi$ can be that

$$\begin{aligned}\int_\omega |\partial Z^k\phi|^2 d\omega &\lesssim \int_\omega |\partial_t Z^k\phi|^2 + |\underline{L}Z^k\phi|^2 d\omega \\ &\lesssim E_0 r^{-1}(1+\tau)^{-1-\alpha}, \quad r \geq 1, \quad \forall k \leq 4.\end{aligned} \tag{5.28}$$

This can be used to estimate $\partial Z^k\phi$ when $1 \leq r \leq R$ (as $r \leq R$ we can improve the decay in r to be $r^{-3+\epsilon}$ up to a constant depending only on R).

The above discussion gives us the pointwise estimates (after using Sobolev embedding on the unit sphere) for the first order derivatives of the solution. To close our bootstrap assumptions, we also need to estimate the second order derivative of the solution. We first consider the case when $r \geq 1$. Note that

$$|\partial^2 Z^k\phi| \lesssim |L\underline{L}Z^k\phi| + |\partial Z^k\phi| + r^{-1}|Z^k\phi|, \quad r \geq 1.$$

Thus to estimate the full second order derivative of the solution, it suffices to estimate $L\underline{L}Z^k\phi$. We rely on Lemma 5.2.3, which shows that it suffices to estimate F^k (see the equation (5.17) for $Z^k\phi$) and notations thereafter. We see that F^k consists of $H_2^k - [Z^k, N]\phi$ satisfying the estimate (5.19) in the proof of Lemma 5.3.3, Q_2^k satisfying estimate (5.20) in the proof of Lemma 5.3.4, $Q_1^k + H_1^k$ satisfying estimates given in the line (5.12). Therefore, we can estimate F^k as follows

$$\int_\omega |F^k|^2 d\omega \lesssim \sum_{l\le k} \int_\omega r^{-2}|\partial Z^l\phi|^2 + |L\underline{L}Z^{l-1}\phi|^2 + H^2|\overline{\partial_v}Z^l\phi|^2 d\omega, \quad r \ge R;$$
$$\int_\omega |F^k|^2 d\omega \lesssim \sum_{l\le k} \int_\omega r^{-2}|\partial Z^l\phi|^2 + |L\underline{L}Z^{l-1}\phi|^2 d\omega, \quad 1 \le r \le R.$$

Note that we already have estimates for $\overline{\partial_v}Z^k\phi$ (see (5.26), (5.24)) and estimate for $\partial Z^k\phi$ (inequality (5.28)). Then from Lemma 5.2.3, we can bound

$$\int_\omega |L\underline{L}Z^k\phi|^2 d\omega \lesssim E_0 r^{-3}\tau_+^{-1-\alpha} + \sum_{l\le k-1} \int_\omega |L\underline{L}Z^l\phi|^2 d\omega, \quad r \ge 1, \quad \forall k \le 3.$$

As $F^0 = 0$, $Z^{-1}\phi = 0$, we conclude from the above inequality (simply by an induction argument) that

$$\int_\omega |L\underline{L}Z^k\phi|^2 d\omega \lesssim E_0 r^{-3}\tau_+^{-1-\alpha}, \quad r \ge 1, \quad \forall k \le 3.$$

These estimates are sufficient to obtain all the necessary C^2 estimates of the solution when $\{r \ge 1\}$. In fact, from the above discussions, we have shown that

$$\int_\omega |\partial\underline{L}Z^k\phi|^2 d\omega \lesssim \int_\omega |\underline{L}Z^{k+1}\phi|^2 d\omega + \int_\omega |L\underline{L}Z^k\phi|^2 d\omega \lesssim E_0 r^{-2+\epsilon}(1+\tau)^{-1-\alpha},$$
$$\int_\omega |\partial\overline{\partial_v}Z^k\phi|^2 d\omega \lesssim \int_\omega |\overline{\partial_v}Z^{k+1}\phi|^2 + |L\underline{L}Z^k\phi|^2 + |\underline{L}\nabla\!\!\!\!/ Z^k\phi|^2 d\omega \lesssim E_0 r^{-3+\epsilon}, \quad k \le 3.$$

Summarizing, we have shown that when $r \geq 1$

$$\begin{aligned} &\sum_{|k|\leq 4} \int_\omega |\underline{L} Z^k \phi|^2 d\omega + \sum_{|k|\leq 3} \int_\omega |\partial \underline{L} Z^k \phi|^2 d\omega \leq C_{\alpha,R} E_0 (1+r)^{-2+\epsilon} \tau_+^{-1-\alpha}, \\ &\sum_{|k|\leq 4} \int_\omega |\overline{\partial_v} Z^k \phi|^2 d\omega + \sum_{|k|\leq 3} \int_\omega |\partial \overline{\partial_v} Z^k \phi|^2 d\omega \leq C_{\alpha,R} E_0 (1+r)^{-3+\epsilon} \end{aligned} \tag{5.29}$$

for some constant $C_{\alpha,R}$ depending only on R and α. Here τ is the parameter for the foliation Σ_τ. We can let $\alpha < \frac{1}{10}$, $\epsilon < \frac{\alpha}{4}$. If we let

$$\epsilon_0 = \frac{\delta_0}{\sqrt{C_{\alpha,R}}}, \quad E_0 \leq \epsilon_0,$$

then the bootstrap assumptions (5.5), (5.6) can be improved.

Finally we need to close the bootstrap assumption (5.7) when $r \leq 1$. Since the angular momentum Ω vanishes when $r = 0$, we can not get much information of the solution by commuting the equation with the angular momentum for small r. We instead rely on the vector field ∂_t. Since we have estimates for $\partial_t \phi$, $\partial_{tt}\phi$, for fixed time t, we can consider the elliptic equation for ϕ and use the robust elliptic theory to obtain the C^2 estimates for the solution on the compact region $r \leq 1$ (when t is fixed). We will use Schauder's estimates to show that the solution is bounded in C^2. We first use Sobolev embedding and L^p elliptic theory to show the C^1 estimate for the solution.

Fix τ. Lemma 5.2.1 and Proposition 5.3.2 imply that

$$\|Z^k \phi\|^2_{H^2(B_2)} \lesssim \int_\tau^{\tau+1} \|\partial_t Z^k \phi\|^2_{H^2(B_2)} d\tau \lesssim E_0 (1+\tau)^{-1-\alpha}, \quad \forall k \leq 4. \tag{5.30}$$

In particular using Sobolev embedding, we have

$$\|Z^k \phi\|^2_{C^{\frac{1}{2}}(B_2)} \lesssim E_0 (1+\tau)^{-1-\alpha}, \quad \forall k \leq 4. \tag{5.31}$$

Here B_r stands for the ball in $\mathbb{R}^3$ with radius r. Next we show the C^1 estimates. Let $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. Commute the equation (5.17) with ∇. We have the elliptic equation for $\nabla Z^k\phi$, $k \leq 3$

$$\begin{aligned}\Delta(\nabla Z^k\phi) + (g^{ij\gamma}\partial_\gamma\phi + h^{ij}) \cdot \partial_{ij}(\nabla Z^k\phi) &= \nabla(F^k - Q^k - N^k) + \partial_{tt}(\nabla Z^k\phi) \\ &- Q(\nabla\phi, Z^k\phi) - \nabla h^{\mu\nu}\partial_{\mu\nu}Z^k\phi - 2h^{i0}\partial_i\nabla\partial_t Z^k\phi - h^{00}\partial_t\partial_t Z^k\phi \\ &- 2g^{0i\gamma}\partial_\gamma\phi \cdot \partial_i\nabla\partial_t Z^k\phi - g^{00\gamma}\partial_\gamma\phi \cdot \nabla\partial_t\partial_t Z^k\phi. \qquad (5.32)\end{aligned}$$

Here Δ is the Laplacian operator in $\mathbb{R}^3$. The bootstrap assumptions (5.7), (5.6) on ϕ as well as the assumption (5.3) on $h^{\mu\nu}$ imply that

$$m_0^{kl} + g^{ij\gamma}\partial_\gamma\phi + h^{ij}$$

is uniformly elliptic. We want to show that the right hand side of the above elliptic equation is bounded in $L^2(B_{r_k})$ for some $r_k \in (1, 2)$. For $\nabla(F^k - Q^k - N^k)$, by the definitions, it consists of two parts: null form which is quadratic in $Z^l\phi$ and $H_1^k + H_2^k - N^k$ contributed by the metric perturbation $h^{\mu\nu}$. The later one is easy to estimate as the bound on $Z^l h^{\mu\nu}$ is given. Since we have the $H^2(B_2)$ bound for $Z^k\phi$, $k \leq 4$ (estimate (5.30)), we can show that

$$\|\nabla(H_1^k + H_2^k - N^k)\|^2_{L^2(B_{r_k})} \lesssim E_0(1+\tau)^{-1-\alpha} + \sum_{l<k} \|\nabla Z^l\phi\|^2_{H^2(B_{r_k})}, \quad k \leq 3.$$

For the quadratic terms from the null form, using Lemma 5.2.4, it suffices to consider $Q(Z^{k_1}\phi, Z^{k_2}\phi)$, $k_1 + k_2 \leq k \leq 3$, $k_2 \leq k-1$ (this also includes $Q^k = Q(Z^k\phi, \phi)$). We first have

$$|\nabla Q(Z^{k_1}\phi, Z^{k_2}\phi)| \lesssim |\partial\nabla Z^{k_1}\phi||\partial^2 Z^{k_2}\phi| + |\partial Z^{k_1}\phi||\partial^2\nabla Z^{k_2}\phi|.$$

Since $k_1 + k_2 \leq 3$, without loss of generality, we may assume that $k_1 \leq 1$. Then the bootstrap assumptions (5.7), (5.6) show that $|\partial^2 Z^{k_1}| \lesssim 1$. As $k_2 \leq k - 1 \leq 2$, we conclude that $\partial^2 Z^{k_2}$ is bounded in $L^2(B_2)$. That is the first term is bounded in $L^2(B_2)$. For the second term, we always have

$$\int_\omega |\partial Z^{k_1}\phi|^2 |\partial^2 \nabla Z^{k_2}\phi|^2 d\omega \lesssim \sum_{l\leq 3} \int_\omega |\partial Z^l \phi|^2 d\omega \cdot \sum_{l \leq k-1} \int_\omega |\partial^2 \nabla Z^l \phi|^2 d\omega.$$

In any case, we can show that

$$\|\nabla Q(Z^{k_1}\phi, Z^{k_2}\phi)\|_{L^2(B_{r_k})}^2 \lesssim E_0(1+\tau)^{-1-\alpha} + \sum_{l\leq k-1} \|\nabla Z^l \phi\|_{H^2(B_{r_k})}$$

This gives the estimate for $\nabla(F^k - Q^k - N^k)$. For the other terms on the right hand side of the above elliptic equation (5.32) for $\nabla Z^k\phi$, their $L^2(B_{r_k})$ norm can be bounded by $\sqrt{E_0}\tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha}$ (up to a constant depending only on α, R) as $Z^k\phi$ is bounded in $H^2(B_2)$. Therefore the elliptic theory shows that

$$\begin{aligned} \|\nabla Z^k \phi\|^2_{C^{\frac{1}{2}}(B_{r'_k})} &\lesssim \|\nabla Z^k\phi\|^2_{H^2(B'_{r_k})} \\ &\lesssim \sum_{l\leq k-1} \|\nabla Z^l\phi\|^2_{H^2(B_{r_k})} + E_0 \tau_+^{-1-\alpha}, \quad 1 < r'_k < r_k < 2. \end{aligned}$$

If we choose $1 < r'_k < r_k < r'_{k-1} \leq 2$, $r_0 = 2$ then the above estimate implies that

$$\|\partial Z^k\phi\|^2_{C^{\frac{1}{2}}(B_{r'_3})} \lesssim E_0(1+\tau)^{-1-\alpha}, \quad \forall k \leq 3, \quad r'_3 > 1.$$

In particular, this gives the C^1 estimates for the solution when $\{r \leq 1\}$.

Finally, we use the $C^{1,\frac{1}{2}}(B_{r'_3})$ estimates for $Z^k\phi$, $k \leq 3$ to show the C^2 estimates of the solution. We now consider the elliptic equation for $Z^k\phi$, $k \le 2$

$$\Delta Z^k\phi + (g^{ij\gamma}\partial_\gamma\phi + h^{ij}) \cdot \partial_{ij}Z^k\phi = F^k - Q^k - N^k + \partial_{tt}Z^k\phi$$
$$- 2(g^{0i\gamma}\partial_\gamma\phi + h^{0i}) \cdot \partial_i\partial_t Z^k\phi - (g^{00\gamma}\partial_\gamma\phi + h^{00}) \cdot \partial_{tt}Z^k\phi.$$

Similarly, by the definition of F^k, Q^k, N^k, we can estimate their $C^{\frac{1}{2}}$ norm as follows

$$\|F^k - Q^k - N^k\|^2_{C^{\frac{1}{2}}(B_{s_k})} \lesssim E_0\tau_+^{-1-\alpha} + \sum_{l \le k-1} \|Z^l\phi\|^2_{C^{2,\frac{1}{2}}(B_{s_k})}, \quad 1 < s_k < r_3'.$$

For the other terms on the right hand side of the above elliptic equation for $Z^k\phi$, we already have estimates of $Z^k\phi$, $k \le 3$ in $C^{1,\frac{1}{2}}(B_{r_3'})$ and estimates of $Z^k\phi$, $k \le 4$ in $C^{\frac{1}{2}}(B_{r_3'})$. Hence Schauder's estimates imply that for all $s_k' < s_k$

$$\|Z^k\phi\|^2_{C^{2,\frac{1}{2}}(B_{s_k'})} \lesssim \sum_{l \le k+2} \|Z^k\phi\|^2_{C^{\frac{1}{2}}(B_{s_k})} + \sum_{l \le k-1} \|Z^l\phi\|_{C^{2,\frac{1}{2}}(B_{s_k})} + \sqrt{E_0}\tau_+^{-\frac{1}{2}-\frac{1}{2}\alpha}.$$

If we choose $s_0 = r_3'$, $1 < s_k' < s_k < s_{k-1}' \le r_3'$, then we have

$$\|Z^k\phi\|^2_{C^{2,\frac{1}{2}}(B_{s_2'})} \lesssim E_0\tau_+^{-1-\alpha}, \quad \forall k \le 2.$$

In particular, this yields the C^2 estimates for $Z^k\phi$, $k \le 2$ when $r \le 1 < s_2'$. To summarize, we have shown that

$$\sum_{|k| \le 3} |\partial Z^k\phi|^2 + \sum_{|k| \le 2} |\partial^2 Z^k\phi|^2 \le C_{\alpha,R}E_0\tau_+^{-1-\alpha}, \quad r \le 1 \tag{5.33}$$

for some constant $C_{\alpha,R}$ depending only on R and α. Without loss of generality, we may assume this constant is the same as the one in (5.29). If

$$\epsilon_0 \le \frac{\delta_0}{\sqrt{C_{\alpha,R}}}, \quad E_0 \le \epsilon_0,$$

then the bootstrap assumptions, (5.7), (5.6), (5.7) are improved. We thus closed all the bootstrap assumptions.

5.5 Proof of the main theorem

Since the initial data have compact support, the finite speed of propagation for solutions of wave equations implies that the solution of the quasilinear wave equation (5.1) vanishes when $r \geq t + R$. Like what we did in the end of Chapter 4, we can run the same Picard iteration process and the above argument shows that limiting solution (may be local in time) ϕ of the quasilinear wave equation (5.1) is bounded in C^2. Then by using a theorem of Hörmander [14] that as long as the solution is bounded in C^2, the solution exists globally. We thus proved the small data global existence result for quasilinear wave equations. Moreover, the solution ϕ satisfies the estimates (5.29), (5.33). Using Sobolev embedding, we conclude that the solution ϕ satisfies the estimates as claimed in Theorem 5.1.1. We thus proved Theorem 5.1.1.

Chapter 6

Global stability of solutions of nonlinear wave equations

6.1 Introduction and statements of the main theorems

In [3], S. Alinhac studied the stability of large solutions to the quasilinear wave equations

$$\Box w + g^{\alpha\beta\gamma}\partial_\gamma w \cdot \partial_{\alpha\beta} w = 0$$

in Minkowski space, where $g^{\alpha\beta\gamma}$ are constants satisfying the null condition (see (5.2) in the previous chapter). More specifically, starting with a global solution $\Phi(t,x) \in C^\infty(\mathbb{R}^{3+1})$, consider the Cauchy problem of the above quasilinear wave equation with perturbed initial data

$$w(0,x) = \Phi(0,x) + \phi_0(x), \quad \partial_t w(0,x) = \partial_t \Phi(0,x) + \phi_1(x).$$

He showed that if Φ satisfies the decay estimates

$$|g^{ij\gamma}\partial_\gamma \Phi \cdot \xi_i \xi_j| \leq \alpha_0 \sum_{i=1}^{3} |\xi_i|^2, \quad \sum_{|k|\leq 7} |\Gamma^k \partial \Phi| \leq C_0 (1+t)^{-1}(1+|r-t|)^{-\frac{1}{2}} \tag{6.1}$$

for some positive constants $\alpha_0 < 1$ and C_0, then the solution of the quasilinear wave equation with perturbed initial data exists globally and is close to the given solution Φ. Here Γ denotes the collection of Lorentz vector fields, see [21]. The problem of global stability of Φ can be reduced to the following small data Cauchy problem

$$\begin{cases} \Box\phi + g^{\alpha\beta\gamma}\partial_\gamma\phi \cdot \partial_{\alpha\beta}\phi + g^{\alpha\beta\gamma}\partial_\alpha\Phi\partial_{\beta\gamma}\phi + g^{\alpha\beta\gamma}\partial_{\beta\gamma}\Phi\partial_\alpha\phi = 0, \\ \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x) \end{cases} \tag{6.2}$$

with given function Φ satisfying condition (6.1). The approach in [3] relies on the vector field method. In particular, Alinhac used the scaling vector field $S = t\partial_t + r\partial_r$ with weights growing in t as commutators. The use of such weighted vector fields requires one to make the rather strong assumption that the given solution $\Phi(t,x)$ decays uniformly in time t as it was in (6.1).

In this chapter, we study the problem of global stability of solutions to nonlinear wave equations by using the approach developed in the previous three chapters. This new method is sufficient robust to show the small data global existence result for quasilinear wave equations on time dependent inhomogeneous backgrounds. We have avoided using any vector fields growing in t, in particular the scaling vector field $S = t\partial_t + r\partial_r$. For the stability problem in this chapter, this in fact allows us to propose much weaker assumptions on the given solution such that it is global stable under perturbations of initial data. In particular, we do not require the given solution to decay uniformly in time t.

We consider the following nonlinear wave equation

$$
\begin{cases}
\Box w + C^{\mu\nu}(\phi)\partial_{\mu\nu}\phi = \mathcal{N}(\partial w), \\
w(0,x) = \Phi_0(x) + \phi_0(x), \quad \partial_t w(0,x) = \Phi_1(x) + \phi_1(x)
\end{cases}
\tag{6.3}
$$

in Minkowski space. The data $\Phi_i(x)$, $\phi_i(x)$ are smooth and have compact support. We assume Φ is a given smooth solution of the above nonlinear wave equation with initial data (Φ_0, Φ_1). (ϕ_0, ϕ_1) denotes the perturbation of the initial data. We want to show that under appropriate conditions on the given solution Φ the above nonlinear wave equation still admits a unique global solution provided that the perturbation of the initial data (ϕ_0, ϕ_1) is sufficiently small in certain weighted Sobolev norm.

We assume the semilinear term $\mathcal{N}(w)$ satisfies the following expansion

$$
\mathcal{N}(\partial\Phi + \partial\phi) = \mathcal{N}(\partial\Phi) + Q_0(\Phi,\phi) + \mathcal{N}^{\mu}(\partial\Phi)\partial_\mu\phi + \mathcal{N}^{\mu\nu}(\partial\Phi)\partial_\mu\phi\partial_\nu\phi + O(|\partial\phi|^3)
$$

when $\partial\phi$ is small. Q_0 is a quadratic null form, i.e.,

$$
Q_0(\Phi,\phi) = A^{\mu\nu}\partial_\mu\Phi\partial_\nu\phi, \tag{6.4}
$$

where $A^{\mu\nu}$ are constants such that $A^{\mu\nu}\xi_\mu\xi_\nu = 0$ whenever $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. The coefficients $\mathcal{N}^{\mu}(\partial\Phi)$, $\mathcal{N}^{\mu\nu}(\partial\Phi)$ obey the estimates

$$
\begin{aligned}
|Z^\beta \mathcal{N}^{\mu}(\partial\Phi)| &\le C(\partial\Phi)\sum_{\beta'\le\beta}|Z^{\beta'}\partial\Phi|^{2+\alpha_0}, \quad \forall\beta\le 4, \\
|Z^\beta \mathcal{N}^{\mu\nu}(\partial\Phi)| &\le C(\partial\Phi)\sum_{\beta'\le\beta}|Z^{\beta'}\partial\Phi|^{\alpha_0}, \quad \forall\beta\le 4
\end{aligned}
\tag{6.5}
$$

for some positive constant α_0. The constant $C(\partial\Phi)$ depends only on $\sum_{\beta\le 4}\|Z^\beta\partial\Phi\|_{C^0}$. Here we recall that $Z = \{\Omega, \partial_t\}$.

Definition 6.1.1. *We call $\Phi \in C^\infty(\mathbb{R}^{3+1})$ a $(\delta, \alpha, t_0, R_1, C_0)$-**weak wave** if*

(i) : $|\partial\Phi(t,x)| \leq C_0, \quad t \leq t_0,$

(ii) : $|\partial\Phi(t,x)| \leq C_0(1+r)^{-\frac{1}{2}}(1+(t-|x|)_+)^{-\frac{1}{2}-2\alpha}, \quad |x| \geq R_1, \quad t \geq t_0,$

(iii) : $|L\Phi(t,x)| \leq C_0(1+r)^{-1-2\alpha}, \quad |x| \geq R_1, \quad t \geq t_0,$

(iv) : $|\partial\Phi(t,x)| \leq \delta_0(1+r)^{-1-\alpha}, \quad |x| \leq R_1, \quad t \geq t_0$

for some positive constants $\delta_0, \alpha, t_0, R_1, C_0$. Here

$$L = \partial_t + \partial_r, (t-|x|)_+ = \max\{0, t-|x|\}.$$

Without loss of generality, we can assume $\alpha \leq \frac{1}{10}$ and $R_1 \leq t_0$.

Remark 6.1.2. *Solution of a free wave equation $\Box\Phi = 0$ with compactly supported initial data decays uniformly in time t and is always a $(\delta_0, \alpha, t_0, R_1, C_0)$-weak wave for some constants $\delta_0, \alpha, t_0, R_1, C_0$. We remark here that a weak wave does not have to decay uniformly in time t in the cylinder $\{(t,x)||x| \leq R_1\}$.*

We denote

$$E_0 = \sum_{k\leq 4} \int_{\mathbb{R}^3} |\nabla Z^k \phi_0|^2 + |Z^k\phi_1|^2 dx, \tag{6.6}$$

where ∇ is the full derivative in $\mathbb{R}^3$.

Our first result is for the semilinear wave equation, that is, $C^{\mu\nu} = 0$.

Theorem 6.1.3. *Consider the semilinear wave equation* (6.3) *with $C^{\mu\nu} = 0$. Assume the given solution $Z^k\Phi$ is $(\delta, \alpha, t_0, R_1, C_0)$-weak wave for all $k \leq 4$. Suppose the initial data $\phi_0(x), \phi_1(x)$ are supported in $\{|x| \leq R_0\}$. Then there exists $\delta_m > 0$, depending only on α, and $\epsilon_0 > 0$, depending on R_0, α_0, α, t_0, R_1, C_0, such that for any $\delta_0 < \delta_m$, $E_0 < \epsilon_0$, there exists a unique global smooth solution w of equation* (6.3) *with the property that $\exists$ positive constant R, depending on t_0, α, α_0, R_1, C_0, R_0, such that for the foliation Σ_τ with radius R, the difference $\phi = w - \Phi$ satisfies*

(1) Energy decay

$$E[\phi](\tau) \leq CE_0(1+\tau)^{-1-\frac{1}{2}\alpha'}, \quad \alpha' = \min\{\frac{\alpha_0}{6}, \alpha\}.$$

(2) Pointwise decay:

$$|\phi| \leq C\sqrt{E_0}(1+r)^{-1},$$
$$\sum_{|\beta|\leq 2} |\partial^\beta \phi| \leq C\sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+|t-r+R|)^{-\frac{1}{2}-\frac{\alpha'}{4}}, \quad \alpha' = \min\{\frac{\alpha_0}{6}, \alpha\},$$

where C depends on R, α_0, α, t_0, R_1, C_0.

Remark 6.1.4. *The weak decay of $\partial\Phi$ in the spatial direction ($(1+|x|)^{-\frac{1}{2}}$) excludes general cubic nonlinearities of $\mathcal{N}(\partial\Phi)$ (cubic nonlinearities satisfying the null condition are allowed). However if condition (ii) in the definition of* ***weak wave*** *Φ is improved to*

$$\sum_{|\beta|\leq 4} |\partial Z^\beta \Phi| \leq C_0(1+r)^{-\frac{1}{2}-\alpha}(1+(t-|x|)_+)^{-\frac{1}{2}-4\alpha},$$

then we can allow $\alpha_0 = 0$ in the assumption (6.5). In particular, $\mathcal{N}(\partial w)$ can be any cubic (or higher) nonlinearity of ∂w.

Since Φ solves (6.3) when $\phi_0 = 0$, $\phi_1 = 0$, the problem of global stability of solutions of semilinear wave equations (equation (6.3) with $C^{\mu\nu} = 0$) is then reduced to the following small data Cauchy problem

$$\begin{cases} \Box\phi + Q_0(\Phi, \phi) + N(\phi) = F(\partial\phi), \\ \phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \phi_1(x), \end{cases} \tag{6.7}$$

where $Q_0(\Phi, \phi)$ is the quadratic null form defined in (6.4), N is a vector field with components N^μ and $N(\phi) = N^\mu \partial_\mu \phi$. The nonlinearity $F(\partial\phi)$ is of the form

$$F(\partial\phi) = Q_0(\phi, \phi) + k^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \text{cubic and higher order terms of } \partial\phi,$$

Here N^μ, $k^{\mu\nu}$ are given functions in $\mathbb{R}^{3+1}$.

We study the small data Cauchy problem of the above semilinear wave equations but with linear terms $Q_0(\Phi, \phi)$, $N^\mu\partial_\mu\phi$ and quadratic terms $k^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ where the functions $\Phi(t, x)$, $N^\mu(t, x)$, $k^{\mu\nu}(t, x)$ decay rather weakly, given as follows:

For positive constants δ_0, α, t_0, R_1, C_0, we assume $Z^k\Phi$ is $(\delta_0, \alpha, t_0, R_1, C_0)$-weak wave for all $k \le 4$. In addition, we assume the following weak decay estimates

$$\begin{aligned}
&|\nabla^2 Z^\beta \Phi| + |\nabla Z^\beta L^\mu| \le C_0, \quad |x| \le 2, \quad \forall \beta \le 2;\\
&|Z^\beta N^\mu| + |Z^\beta k^{\mu\nu}| \le C_0, \quad t \le t_0, \quad \beta \le 4; \qquad (6.8)\\
&|Z^\beta k^{\mu\nu}| \le C_0(1 + |x|)^{-\alpha}, \quad t \ge t_0, \quad \beta \le 4.
\end{aligned}$$

When $t \ge t_0$, $\beta \le 4$, we assume N^μ satisfies **one** of the following two decay estimates

$$|Z^\beta N^\mu(t, x)| \le \delta_0(1 + |x|)^{-1-2\alpha}, \qquad (6.9)$$

$$|Z^\beta N^\mu(t, x)| \le C_0(1 + |x|)^{-1-3\alpha}(1 + (t - |x|)_+)^{-\alpha}. \qquad (6.10)$$

We see that when t is sufficiently large (6.10) implies (6.9). In fact eventually we will work under the condition (6.9) where δ_0 is small positive constant.

Theorem 6.1.3 follows from:

Theorem 6.1.5. *Let $\Phi(t, x)$, $N^\mu(t, x)$, $k^{\mu\nu}(t, x)$ be given smooth functions satisfying the above conditions. Assume the initial data $\phi_0(x)$, $\phi_1(x)$ are smooth and supported in $\{|x| \le R_0\}$. Then there exists $\delta_m > 0$, depending only on α, and $\epsilon_0 > 0$, depending on R_0, α, t_0, R_1, C_0, such that for any $\delta_0 < \delta_m$, $E_0 < \epsilon_0$, there exists a unique global smooth solution ϕ of the equation* (6.7) *with the property that there exists a positive* *constant R, depending on t_0, α, R_1, C_0, R_0, such that for the foliation Σ_τ with radius R, the solution ϕ satisfies*

(1) Energy decay

$$E[Z^k\phi](\tau) \leq CE_0(1+\tau)^{-1-\frac{1}{2}\alpha}, \quad k \leq 4.$$

(2) Pointwise decay

$$|\phi| \leq C\sqrt{E_0}(1+r)^{-1},$$
$$\sum_{|\beta|\leq 2} |\partial^\beta \phi| \leq C\sqrt{E_0}(1+r)^{-\frac{1}{2}}(1+|t-r+R|)^{-\frac{1}{2}-\frac{\alpha}{4}}$$

for some constant C depending on R, α, t_0, R_1, C_0.

Remark 6.1.6. *Notice that α can be arbitrarily small. The decay assumptions on N^μ (condition* (6.9)*) and $k^{\mu\nu}$ are sharp as there exists soliton solution (independent of time t) to the linear wave equation if N^μ decays only like $(1+|x|)^{-1}$ and any nontrivial C^3 solution of the equation $\Box\phi = \phi_t^2$ with compactly supported initial data blows up in finite time [15].*

Remark 6.1.7. *We can also consider equation* (6.7) *with zeroth order linear term $f_0\phi$, leading to the same conclusion provided that f_0 decays like $(1+|x|)^{-3-a}$. Hence for the stability problem of large solution (Theorem 6.1.3), specific dependence on w of the nonlinearity $\mathcal{N}(w)$ is also allowed.*

Remark 6.1.8. *As in the previous two chapters, the null structure of the equation here ($Q_0(\Phi,\phi)$ and the quadratic part of $F(\partial\phi)$) is only needed outside a large cylinder with radius R. Inside the cylinder, they can be any quadratic form.*

The main difficulty of considering nonlinear wave equation (6.7) with linear terms $Q_0(\Phi,\phi)$, $N^\mu\partial_\mu\phi$ and quadratic terms $k^{\mu\nu}(t,x)\partial_\mu\phi\partial_\nu\phi$ is the rather weak decay of the functions $\Phi(t,x)$, N^μ, $k^{\mu\nu}$. Although we have considered the linear terms $N(\phi)$ in the previous two chapters (see equation (3.1)), we note that the decay assumption on $\partial\Phi$ is weaker than that in (for example, (5.3)) in Chapter 5. More precisely, we note that $\partial\Phi$ decays only like $(1+r)^{-\frac{1}{2}}$ in the spatial direction while in the pervious chapters the corresponding coefficients N^μ decays at least $(1+r)^{-\frac{1}{2}-\alpha}$, $\alpha > 0$ in the spatial direction. A logarithmic growth may occur if we teat $Q_0(\Phi, \phi)$ as before. The idea is to use the p-weighted energy estimates to control Q_0.

We will establish an integrated energy estimate and the necessary p-weighted energy inequalities for solutions of (6.7) based on those we have obtained in Proposition 3.1.1 and Proposition 3.2.2. The framework to obtain these estimates has been given in Chapter 3. For the situation here, it suffices to treat the linear term $Q_0(\Phi, \phi)$ and the quadratic term $k^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

For the quasilinear wave equations, based on Theorem 5.1.1 in Chapter 5, we have a similar stability result. For implicity, we consider the quasilinear wave equation (6.3) with $C^{\mu\nu} = g^{\mu\nu\gamma}\partial_\gamma\phi$, $\mathcal{N} = 0$, where $g^{\mu\nu\gamma}$ are constants satisfying the null condition (see definition (5.2)).

Again, assume Φ is a given smooth solution with initial data (Φ_0, Φ_1). We now consider the same equation with initial data $(\Phi_0 + \phi_0, \Phi_1 + \phi_1)$. Before some large time t_0, we assume the metric $m_0 + g^{\mu\nu\gamma}\partial_\gamma\Phi$ is hyperbolic and

$$|\partial^2\Phi| + |Z^k\partial\Phi| \leq C_1, \quad t \leq t_0, \quad k \leq 6 \tag{6.11}$$

for some constant C_1. After time t_0, we assume Φ satisfies the following weak decay estimates

$$\begin{aligned}
&|\partial^2\Phi| + |Z^k\partial\Phi| \leq \delta_0(r_+^{-\frac{1}{2}-2\alpha}\tau_+^{-\frac{1}{2}-2\alpha} + r_+^{-1-2\alpha}), \quad (t,x) \in S_\tau, \quad t \geq t_0;\\
&|\partial\overline{\partial_v}\Phi| + |Z^k\overline{\partial_v}\Phi| \leq \delta_0 r_+^{-1-2\alpha}, \quad (t,x) \in S_\tau, \quad t \geq t_0;\\
&|\partial^2\Phi| + |\partial Z^k\phi| \leq \delta_0(1+r)^{-1-2\alpha}, \quad r \leq R, \quad t \geq t_0, \quad \forall k \leq 6,
\end{aligned} \tag{6.12}$$

where the radius of the foliation $R = R_0 + t_0$ and R_0 is the radius of the support of the initial data.

We have the following global stability of solutions to quasilinear wave equations.

Theorem 6.1.9. *Consider the quasilinear wave equation* (6.3) *with* $C^{\mu\nu} = g^{\mu\nu\gamma}\partial_\gamma\phi$, $\mathcal{N} = 0$ *satisfying the null condition. Let* Φ *be a smooth solution with initial data* (Φ_0, Φ_1) *and satisfy the above two estimates* (6.11), (6.12). *Then there exist two small positive constants* $\delta_m > 0$, *depending on* α, *and* $\epsilon_0 > 0$, *depending on* α, R_0, t_0, C_1 *such that for all* $\delta_0 < \delta_m$, $E_0 < \epsilon_0$, *there exists a unique global smooth solution* w *with initial data* $(\Phi_0 + \phi_0, \Phi_1 + \phi_1)$. *Moreover, the difference* $\phi = w - \Phi$ *satisfies the same estimates as given in Theorem 5.1.1 but with the constant* C *depending on* α, R_0, t_0, C_1.

This theorem follows from Theorem 5.1.1 once we consider it as a small data problem for the quasilinear equation (6.2).

6.2 The radius of the foliation

In this section, we fix the radius R of the foliation and improve the estimates for $LZ^k\phi$ when $|x|$ is large. The idea is to choose another set of constants α', t_0', R_1' such that

$$|LZ^k\Phi| \le \delta_0(1+|x|)^{-1-2\alpha'}, \quad t \ge t_0', \quad |x| \ge R_1', \quad k \le 4, \tag{6.13}$$

$$|\partial Z^k\Phi| \le 2\delta_0(1+|x|)^{-1-\alpha'}, \quad t \ge t_0', \quad |x| \le R_1', \quad |k| \le 4, \tag{6.14}$$

$$(1+t_0')^{\alpha'}\delta_0 \ge C_0. \tag{6.15}$$

In fact, since $Z^k\Phi$ is $(\delta_0, \alpha, t_0, R_1, C_0)$-weak wave for all $k \le 4$, choose $\alpha' = \frac{1}{2}\alpha$ and R_1' large enough such that (6.13) holds. Then (6.14) and (6.15) are satisfied if t_0' is sufficiently large. In particular, we have $Z^k\Phi$ is $(\delta_0, \alpha', t_0', R_1', C_0)$-weak wave but with the estimate

$$|LZ^k\phi| \leq C_0(1+r)^{-1-2\alpha}, \quad t \geq t_0', \quad |x| \geq R_1'$$

replaced with (6.13) where we gain a small constant δ_0.

Moreover by choosing these new constants we can simply assume (6.9) for N^μ even if N^μ satisfies (6.10) as by (6.15) when $t \geq t_0'$, we can show that

$$C_0(1+|x|)^{-\alpha}(1+(t-|x|)_+)^{-\alpha} \leq C_0(1+\tau)^{-\alpha} \leq C_0(1+t_0')^{-\alpha} \leq \delta_0.$$

Therefore we can assume $LZ^k\Phi$ satisfies (6.13) and N^μ satisfies (6.9) but with new constants α', R_1', t_0'.

Then the radius R for the foliation can be fixed as

$$R = t_0' + R_0,$$

where R_0 is the radius of the support of the initial data. However, to avoid too many constants, we still use the constants α, R_1, t_0 to denote α', R_1', t_0' respectively in the sequel.

Finally, we choose the small positive constants ϵ, α_1, α_2 as in Proposition 3.2.2, depending on α. We only have to keep in mind that ϵ appears in the integrated energy $I^\epsilon[\phi]_{\tau_1}^{\tau_2}$. $1+\alpha_1$ is the maximal weights in the p-weighted energy inequality. α_2 is slightly larger than α_1.

6.3 Integrated energy inequalities

We consider solution ϕ of the equation (6.7) but treat $F(\partial\phi)$ as inhomogeneous term. We establish the integrated energy inequality and the p-weighted energy inequalities for ϕ, based on those we have obtained in Chapter 3.

We make a convention in this section that $A \lesssim B$ means $A \leq C_R B$ for some constant C_R depending on α, R, t_0, R_1, R_0, C_0. The constant C_α denotes a constant depending only on α.

Proposition 6.3.1. *Let ϕ be a solution of (6.7) with initial data (ϕ_0, ϕ_1) which are supported on $\{|x| \le R_0 < R\}$. Then for sufficiently small δ_0 , depending only on α, we have*

(1) Integrated local energy estimate

$$\tilde{E}[\phi](\tau_2) + I^\epsilon[\phi]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + (\tau_1)_+^{-1-\alpha}\left(g^{1+\alpha_1}[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \tau_+^\epsilon D^{\alpha_1}[F]_\tau^{\tau_2} d\tau\right). \tag{6.16}$$

(3) p-weighted energy inequalities in a neighborhood of null infinity

$$g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau)d\tau \lesssim g^1[\phi](\tau_1) + (\tau_1)_+^{1-\alpha}\tilde{E}[\phi](\tau_1) + (\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_\tau^{\tau_2} d\tau + (\tau_1)_+^{-\alpha}(g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau_+^\epsilon D^{\alpha_1}[F]_\tau^{\tau_2} d\tau); \tag{6.17}$$

$$g^{1+\alpha_1}[\phi](\tau_2) + \bar{G}^{\alpha_1,0}[\phi]_{\tau_1}^{\tau_2} \lesssim g^{1+\alpha_1}[\phi](\tau_1) + (\tau_1)_+^{1-\alpha}\tilde{E}[\phi](\tau_1) + (\tau_1)_+^{1+\epsilon} D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tau_+^\epsilon D^{\alpha_1}[F]_\tau^{\tau_2} d\tau. \tag{6.18}$$

The linear term $N(\phi)$ has already been discussed in Chapter 3 (as having argued in the previous section, we can assume N^μ satisfies the condition (6.9)). It suffices to estimate $Q_0(\Phi, \phi)$.

Lemma 6.3.2. *For all $\tau_2 \ge \tau_1 \ge t_0$, we have*

$$D^\epsilon[|Q_0| + |N|]_{\tau_1}^{\tau_2} \le C_\alpha \delta_0^2 I^\epsilon[\phi]_{\tau_1}^{\tau_2} + C_R(\bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2}).$$

Here N is short for the linear term $N(\phi)$ in the equation (6.7).

Proof. Since $Q_0(\Phi, \phi)$ is a null form, we have the estimate (also see Lemma 4.3.4)

$$r|Q_0(\Phi, \phi)| \leq C_\alpha(|\partial\Phi||\overline{\partial_v}\psi| + |\partial_v\Phi||\bar{\partial}\psi| + |\partial\Phi||\phi|), \quad \psi = r\phi, r = |x| > 1. \quad (6.19)$$

When $r \leq R_1$, using the fact that $\partial\Phi$ is small in this region (see Definition 6.1.1), we can simply bound

$$|Q_0| \leq C_\alpha \delta_0 (1+r)^{-1-2\alpha}|\partial\phi|, \quad r \leq R_1.$$

On S_τ, in particular $r \geq R > R_1$, we have

$$r^2|Q_0|^2 \leq C_R r^{-1}\tau_+^{-1-4\alpha}(|\overline{\partial_v}\psi|^2 + |\phi|^2) + C_\alpha \delta_0^2 (1+r)^{-2-2\alpha}|\bar{\partial}\psi|^2.$$

We use Lemma 2.4.2 and Lemma 2.4.3 to bound ϕ^2. From Lemma 2.4.2, we have

$$\int_{S_\tau} \phi^2 dv d\omega \leq 12\tilde{E}[\phi](\tau).$$

Apply Lemma 2.4.3 by taking $\alpha_1 = \alpha_2 = \epsilon$ (in that lemma). We get

$$\int_{S_\tau} r^{1-\epsilon}\phi^2 dv d\omega \leq C_\alpha(R^{1-\epsilon}\tilde{E}[\phi](\tau) + g^{1+\epsilon}[\phi](\tau)).$$

Interpolate between the previous two estimates. We obtain the estimates for ϕ^2 with weights r^ϵ

$$\int_{S_\tau} r^\epsilon \phi^2 dv d\omega \leq C_\alpha(R^\epsilon \tilde{E}[\phi](\tau) + (\tilde{E}[\phi](\tau))^{1-\gamma}(g^{1+\epsilon}[\phi](\tau))^\gamma), \quad \gamma = \frac{\epsilon}{1-\epsilon}.$$

For $N(\phi)$, by condition (6.9) (see the argument in the previous section), we have

$$|N(\phi)| \leq C_\alpha \delta_0^2 (1+r)^{-1-2\alpha}|\partial\phi|.$$

Therefore we can estimate

$$
\begin{aligned}
D^{\epsilon}[|Q_0|+|N|]_{\tau_1}^{\tau_2} \le C_{\alpha}\delta_0^2 I^{2\alpha-\epsilon}[\phi]_{\tau_1}^{\tau_2} + C_R(&\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\epsilon}\tau_+^{-1-2\alpha}|\overline{\partial_v}\psi|^2 dv d\omega d\tau \\
&+ \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{R_1\le r\le R} \tau_+^{-1-2\alpha} r^{\epsilon}|\partial\phi|^2 dx d\tau \\
&+ \int_{\tau_1}^{\tau_2} \tau_+^{-1-4\alpha}\tilde{E}[\phi](\tau)^{1-\gamma}(g^{1+\epsilon}[\phi](\tau))^{\gamma} d\tau) \\
&\le C_{\alpha}\delta_0^2 I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + C_R(\bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2}).
\end{aligned}
$$

Here see the notations defined in Section 2.2 and we have the relation

$$
\epsilon + \alpha_1 < 2\alpha, \quad \gamma(2+2\alpha) + (1-\gamma)(1+2\alpha) \le 1 + 4\alpha.
$$

□

6.3.1 The integrated energy inequality

We now use the above lemma to prove Proposition 6.3.1. We first consider the case when $\tau_2 \ge \tau_1 \ge t_0$. By the integrated energy inequality (3.5), we derive (the boundary term $S^{\epsilon}[\phi](\tau_i)$ does not appear as the background metric here is flat)

$$
\begin{aligned}
I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} &\le C_{\alpha}(\tilde{E}[\phi](\tau_1) + D^{\epsilon}[F]_{\tau_1}^{\tau_2} + D^{\epsilon}[Q_0]_{\tau_1}^{\tau_2}) \\
&\le C_{\alpha}\delta_0^2 I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + C_R(\tilde{E}[\phi](\tau) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} \\
&\quad + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[F]_{\tau_1}^{\tau_2}).
\end{aligned}
$$

For sufficiently small δ_0, depending only on α, we can conclude that

$$
I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_1) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[F]_{\tau_1}^{\tau_2}.
$$

Then by the energy estimate (3.6), we can also obtain

$$\tilde{E}[\phi](\tau_2) \lesssim \tilde{E}[\phi](\tau_1) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[F]_{\tau_1}^{\tau_2}.$$

Here by our convention, the implicit constant depends on R, α, t_0, C_0, R_1, R_0. Now using Gronwall's inequality (Lemma 2.4.5), we can absorb the term $\tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2}$ on the right hand side of the previous two estimates. In particular, we obtain

$$\begin{aligned} & I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}[\phi](\tau_2) + \tilde{E}^{1+\alpha}[\phi]_{\tau_1}^{\tau_2} \\ & \lesssim \tilde{E}[\phi](\tau_1) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[F]_{\tau_1}^{\tau_2}. \end{aligned} \tag{6.20}$$

We have shown the above estimate holds for all $\tau_2 \geq \tau_1 \geq t_0$. This estimate is in fact valid for $\tau_2 \geq \tau_1 \geq 0$. When $\tau_1 \leq \tau_2 \leq t_0$, the finite speed of propagation for wave equation [39] shows that ϕ vanishes when $r \geq t + R_0$. Hence we can simply bound

$$D^{\alpha_1}[Q_0]_{\tau_1}^{\tau_2} \lesssim \int_{\tau_1}^{\tau_2} \int_{r \leq R} |\bar{\partial}\phi|^2 dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau,$$

When considering the energy inequality (3.6), $\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau) d\tau$ can be bounded by using Gronwall's inequality as τ_1, τ_2 are bounded above by t_0. Hence we have (6.20) for all $0 \leq \tau_1 \leq \tau_2 \leq t_0$. For the case $\tau_1 \leq t_0 \leq \tau_2$, split the interval $[\tau_1, \tau_2]$ into $[\tau_1, t_0]$ and $[t_0, \tau_2]$, on which we have two separate inequalities. Combining them together, we get (6.20). Therefore (6.20) holds for all $0 \leq \tau_1 \leq \tau_2$.

6.3.2 The p-weighted energy inequality

For the p-weighted energy inequalities, we instead rely on the derivation in Section 3.2.1, that is the p-weighted energy inequality in Minkowski space obtained in Proposition 3.2.1. The proof there in fact implies that

$$
\begin{aligned}
&g^p[\phi](\tau_2) + \bar{G}^{p-1,0}[\phi]_{\tau_1}^{\tau_2} \\
&\lesssim \tilde{E}[\phi](\tau_1) + g^p[\phi](\tau_1) + D^\epsilon[\Box\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p+1}|\Box\phi||\partial_v\psi| dv d\omega d\tau
\end{aligned}
\tag{6.21}
$$

for all $p \in (0,2)$. The implicit constant also depends on p. However, only the case when $p = 1$ and $p = 1 + \alpha_1$ will be considered. For solution of (6.7), we have $\Box\phi = F - Q_0 - N$. Again, we consider the case $\tau_2 \geq \tau_1 \geq t_0$ first. By Lemma 6.3.2 and estimate (6.20), we can bound

$$
D^\epsilon[F - Q_0 - N]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_1) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} + D^\epsilon[F]_{\tau_1}^{\tau_2}. \tag{6.22}
$$

Next we estimate the integral of $r^{p+1}|F - Q_0 - N||\partial_v\psi|$. We use the null structure of Q_0 (inequality (6.19)) and the fact that Φ is a $(\delta_0, \alpha, t_0, R_1, C_0)$-weak wave to bound $|Q_0|$. For the linear term $N(\phi)$, we use the condition (6.9) (although N^μ may satisfy condition (6.10), by the argument in previous section, it suffices to assume (6.9)). The inhomogeneous term $r^{p+1}F\partial_v\psi$ will be estimated similarly as in Proposition 3.2.1. First on S_τ we can estimate

$$
\begin{aligned}
|r^{p+1}(Q_0 + N)\partial_v\psi| &\lesssim r^p(|\partial\Phi||\overline{\partial_v}\phi| + |\partial\Phi||\phi| + |\partial_v\Phi||\partial\psi| + r|N^\mu||\partial_\mu\phi|)|\partial_v\psi| \\
&\lesssim r^p\left(r^{-\frac{1}{2}}\tau_+^{-\frac{1}{2}-2\alpha}(|\overline{\partial_v}\psi| + |\phi|) + r^{-1-2\alpha}|\bar{\partial}\psi|\right)|\partial_v\psi| \\
&\lesssim \epsilon_1 r^{p-1}|\overline{\partial_v}\psi|^2 + \epsilon_1^{-1}r^p\tau_+^{-1-\alpha}|\partial_v\psi|^2 + r^{p-1-2\alpha}|\bar{\partial}\psi||\partial_v\psi| \\
&\quad + r^{1-\epsilon}\tau_+^{-1-\alpha}|\phi|^2 + (r^p\tau_+^{-1-\epsilon} + \tau_+^{-\epsilon})|\partial_v\psi|^2, \quad \forall\epsilon_1 > 0
\end{aligned}
$$

for $p = 1$ or $p = 1 + \alpha_1$. The last step follows from the relation

$$
r^{2p-2+\epsilon}\tau_+^{-3\alpha} \lesssim r^p\tau_+^{-1-\epsilon} + \tau_+^{-\epsilon}, \quad p = 1, 1 + \alpha_1.
$$

Therefore, we can estimate

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p+1}|Q_0+N||\partial_v\psi|dvd\omega d\tau \lesssim \epsilon_1 \bar{G}^{p-1,0}[\phi]_{\tau_1}^{\tau_2} + (1+\epsilon_1^{-1})G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}$$
$$+ (\tau_1)_+^{-\epsilon}G^{0,0}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau}(r^{p-1-2\alpha}|\bar{\partial}\psi||\partial_v\psi| + r^{1-\epsilon}\tau_+^{-1-\alpha}|\phi|^2)dvd\omega d\tau.$$

We use Lemma 2.4.3 and estimate (6.20) to further estimate ϕ^2

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}\tau_+^{-1-\alpha}|\phi|^2 dvd\omega d\tau \lesssim \tilde{E}^{1+\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,1+\alpha}[\phi]_{\tau_1}^{\tau_2}$$
$$\lesssim \tilde{E}[\phi](\tau_1) + \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,1+\alpha}[\phi]_{\tau_1}^{\tau_2} + D^{\epsilon}[F]_{\tau_1}^{\tau_2}.$$

For the integral of $|\bar{\partial}\psi||\partial_v\psi|$, we use Lemma 2.4.4 together with the integrated energy estimate (6.20) to show that for $p = 1$ or $1+\alpha_1$

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{p-1-2\alpha}|\bar{\partial}\psi||\partial_v\psi|dvd\omega d\tau \\
&\lesssim \left(\int_{\tau_1}^{\tau_2}\tau_+^{1-\alpha}\int_{S_\tau}\frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}}dxd\tau\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{2p-1-4\alpha+\epsilon}\tau_+^{-1+\alpha}|\partial_v\psi|^2 dvd\omega d\tau\right)^{\frac{1}{2}} \\
&\lesssim \epsilon_2^{-1}\int_{\tau_1}^{\tau_2}\tau_+^{1-\alpha}S^{\epsilon}[\phi](\tau)d\tau + \epsilon_2\left(G^{0,0}[\phi]_{\tau_1}^{\tau_2}\right)^{\frac{1+4\alpha-\epsilon-p}{p}}\left(G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}\right)^{\frac{2p-1-4\alpha+\epsilon}{p}} \\
&\lesssim \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\tau_+^{-\alpha}I^{\epsilon}[\phi]_{\tau}^{\tau_2}d\tau) + \epsilon_2\left(G^{0,0}[\phi]_{\tau_1}^{\tau_2} + G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}\right) \\
&\lesssim \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}(\tilde{E}[\phi](\tau_1) + D^{\epsilon}[F]_{\tau_1}^{\tau_2}) + \bar{G}^{\epsilon,3\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,1+3\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{\alpha}[\phi]_{\tau_1}^{\tau_2} \\
&\quad + \int_{\tau_1}^{\tau_2}\tau_+^{-\alpha}D^{\epsilon}[\phi]_{\tau}^{\tau_2}d\tau) + \epsilon_2\left(G^{0,0}[\phi]_{\tau_1}^{\tau_2} + G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2}\right), \quad \forall \epsilon_2 > 0.
\end{aligned}$$

Since p will be taken as 1 or $1+\alpha_1$, to control $\bar{G}^{\epsilon,3\alpha}[\phi]_{\tau_1}^{\tau_2}$, we interpolate

$$\bar{G}^{\epsilon,3\alpha}[\phi]_{\tau_1}^{\tau_2} \lesssim \epsilon_2\epsilon_3\bar{G}^{\alpha_1,2\alpha} + (\epsilon_2\epsilon_3)^{-1}(\tau_1)_+^{-2\alpha}\bar{G}^{0,0}[\phi]_{\tau_1}^{\tau_2}, \quad \forall \epsilon_3 > 0.$$

Similarly we have

$$G^{1+\epsilon,1+\alpha}[\phi]_{\tau_1}^{\tau_2} \lesssim \epsilon_2\epsilon_3 G^{1+\alpha_1,1+\alpha}[\phi]_{\tau_1}^{\tau_2} + (\epsilon_2\epsilon_3)^{-1}(\tau_1)_+^{-2\alpha}\bar{G}^{0,0}[\phi]_{\tau_1}^{\tau_2}.$$

Here the positive constant α_2 is the one appeared above. Our ultimate goal is to obtain estimate for the energy flux $\tilde{E}[\phi](\tau)$. Note that we have the following inequality

$$\int_{\tau_1}^{\tau_2} E[\phi](\tau)d\tau \le \bar{G}^{0,0}[\phi]_{\tau_1}^{\tau_2} + 2\int_{\tau_1}^{\tau_2}\int_{r\le R} |\partial\phi|^2 + |\phi|^2 dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau.$$

See the proof in the end of Section 3.2.2.

The above discussion give us the estimates for $r^{p+1}|Q_0 + N||\partial_v\psi|$. Finally for the inhomogeneous term F, we have for $p = 1$ or $p = 1 + \alpha_1$

$$\begin{aligned} r^{p+1}|F||\partial_v\psi| &\lesssim \epsilon_4^{-1}\tau_+^{1+(p-1)\alpha_1^{-1}\epsilon}|F|^2 r^{3+\alpha_1} + \epsilon_4|\partial_v\psi|^2 r^{2p-1-\alpha_1}\tau_+^{-1-(p-1)\alpha_1^{-1}\epsilon} \\ &\lesssim \epsilon_4^{-1}\tau_+^{1+(p-1)\alpha_1^{-1}\epsilon}|F|^2 r^{3+\alpha_1} + \epsilon_4|\partial_v\psi|^2(r^p\tau_+^{-1-\epsilon} + 1), \quad \forall \epsilon_4 > 0. \end{aligned} \tag{6.23}$$

(Also see the proof in Section 3.2.2 and the small constants ϵ, α, α_1 satisfy the conditions in Section 6.2). The second term can be further estimated by using Lemma 2.4.4. Now combine all the above estimates (from estimate (6.22) to estimate (6.23)) and take the positive constants $\epsilon_1 = \epsilon_3 < 1$, $\epsilon_2 = \epsilon_4 < 1$. Then the inequality (6.21) leads to

$$\begin{aligned} &g^p[\phi](\tau_2) + \bar{G}^{p-1,0}[\phi]_{\tau_1}^{\tau_2} \lesssim g^p[\phi](\tau_1) + \epsilon_1(\bar{G}^{\alpha_1,2\alpha}[\phi]_{\tau_1}^{\tau_2} + \bar{G}^{p-1,0}[\phi]_{\tau_1}^{\tau_2} + G^{1+\alpha_1,1+\alpha}[\phi]_{\tau_1}^{\tau_2}) \\ &+ \left(\frac{(\tau_1)_+^{-2\alpha}}{\epsilon_2^2\epsilon_1} + \epsilon_2 + \frac{(\tau_1)_+^{-\epsilon}}{\epsilon_2}\right)\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau + \epsilon_1^{-1}G^{p,1+\epsilon}[\phi]_{\tau_1}^{\tau_2} \\ &+ \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}(\tilde{E}[\phi](\tau_1) + (\tau_1)_+^{1+(p-1)\alpha_1^{-1}\epsilon}D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tau_+^{(p-1)\alpha_1^{-1}\epsilon}D^{\alpha_1}[F]_{\tau}^{\tau_2}d\tau). \end{aligned}$$

6.3.3 Proof of Proposition 6.3.1

In the above p-weighted energy inequality, we first take $p = 1+\alpha_1$ and the constant ϵ_1 to be sufficiently small so that the terms $\bar{G}^{\alpha_1,0}[\phi]_{\tau_1}^{\tau_2}$ can be absorbed. Using Gronwall's inequality Lemma 2.4.5, we obtain

$$g^{1+\alpha_1}[\phi](\tau_2) + \bar{G}^{\alpha_1,0}[\phi]_{\tau_1}^{\tau_2} \lesssim g^{1+\alpha_1}[\phi](\tau_1) + \left(\frac{(\tau_1)_+^{-2\alpha}}{\epsilon_2^2} + \epsilon_2 + \frac{(\tau_1)_+^{-\epsilon}}{\epsilon_2}\right)\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau$$
$$+ \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}\tilde{E}[\phi](\tau_1) + (\tau_1)_+^{1+\epsilon}D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tau_+^{\epsilon}D^{\alpha_1}[F]_{\tau}^{\tau_2}d\tau). \quad (6.24)$$

Next we take $p = 1$ in the previous general p-weighted energy inequality. Add

$$2\int_{\tau_1}^{\tau_2} |\phi|^2 + |\phi|^2 dx d\tau$$

to both sides and using the integrated energy estimate (6.20) to bound this newly added term. Similarly taking ϵ_1 to be sufficiently small and using Gronwall's inequality, we obtain

$$g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau)d\tau \lesssim g^1[\phi](\tau_1) + \bar{G}^{\alpha_1,2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\alpha_1,1+\alpha}[\phi]_{\tau_1}^{\tau_2}$$
$$+ \left(\frac{(\tau_1)_+^{-2\alpha}}{\epsilon_2^2} + \epsilon_2 + \frac{(\tau_1)_+^{-\epsilon}}{\epsilon_2}\right)\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau$$
$$+ \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}(\tilde{E}[\phi](\tau_1) + (\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_{\tau}^{\tau_2}d\tau).$$

We now use the above p-weighted energy inequality (6.24) to bound $\bar{G}^{\alpha_1,2\alpha}[\phi]_{\tau_1}^{\tau_2}$ and $G^{1+\alpha_1,1+\alpha}[\phi]_{\tau_1}^{\tau_2}$. We can improve the above p-weighted energy inequality when $p = 1$

$$g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau)d\tau \lesssim g^1[\phi](\tau_1) + \left(\frac{(\tau_1)_+^{-2\alpha}}{\epsilon_2^2} + \epsilon_2 + \frac{(\tau_1)_+^{-\epsilon}}{\epsilon_2}\right)\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau$$
$$+ \epsilon_2^{-1}((\tau_1)_+^{1-\alpha}(\tilde{E}[\phi](\tau_1) + (\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_{\tau}^{\tau_2}d\tau) \quad (6.25)$$
$$+ (\tau_1)_+^{-\alpha}g^{1+\alpha_1}[\phi](\tau_1) + \epsilon_2^{-1}(\tau_1)_+^{-\alpha}\int_{\tau_1}^{\tau_2} \tau_+^{\epsilon}D^{\alpha_1}[F]_{\tau}^{\tau_2}d\tau.$$

Without loss of generality, we may assume $\tilde{E}[\phi](\tau)$ is finite (as it suffices to assume the right hand side of estimates (6.16), (6.18), (6.17) is finite). Hence by Lemma 2.4.1, all the above estimates hold if we replace $\tilde{E}[\phi](\tau)$ with $E[\phi](\tau)$. Now, in the above inequality, we first take ϵ_2 to be sufficiently small. Then we choose T_0 large enough such that

$$\frac{T_0^{-2\alpha}}{\epsilon_2^2} + \epsilon_2 + \frac{T_0^{-\epsilon}}{\epsilon_2}$$

is sufficiently small. Therefore for all $\tau_2 \geq \tau_1 \geq T_0$, we can show that

$$\begin{aligned} g^1[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} E[\phi](\tau)d\tau &\lesssim g^1[\phi](\tau_1) + (\tau_1)_+^{1-\alpha}\tilde{E}[\phi](\tau_1) + (\tau_1)_+ D^{\alpha_1}[F]_{\tau_1}^{\tau_2} \\ &+ \int_{\tau_1}^{\tau_2} D^{\alpha_1}[F]_{\tau}^{\tau_2} d\tau + (\tau_1)_+^{-\alpha} g^{1+\alpha_1}[\phi](\tau_1) + (\tau_1)_+^{-\alpha}\int_{\tau_1}^{\tau_2} \tau_+^{\epsilon} D^{\alpha_1}[F]_{\tau}^{\tau_2} d\tau. \end{aligned}$$

In particular, we have shown (6.17) for all $\tau_2 \geq \tau_1 \geq T_0$. Once we have estimate for the integral of the energy flux, the p-weighted energy inequality (6.24) implies (6.18) for all $\tau_2 \geq \tau_1 \geq T_0$.

For $t_0 \leq \tau_1 \leq \tau_2 \leq T_0$, we make use of the boundedness of τ. Let $\epsilon_2 = 1$ in inequality (6.24). We get

$$g^{1+\alpha_1}[\phi](\tau_2) + \bar{G}^{\alpha_1,0}[\phi]_{\tau_1}^{\tau_2} \lesssim g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau + \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2}.$$

In particular, we have

$$\begin{aligned} \bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} &\lesssim \int_{\tau_1}^{\tau_2} g^{1+\alpha_1}[\phi](\tau) + \bar{G}^{\alpha_1,0}[\phi]_{\tau}^{\tau_2} d\tau \\ &\lesssim g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau + \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2}. \end{aligned}$$

Here as $\tau_1 \leq \tau_2 \leq T_0$. Hence from the integrated energy estimate (6.20), we conclude

$$\tilde{E}[\phi](\tau_2) \lesssim g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau + \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2}.$$

Thus Gronwall's inequality indicates that

$$\int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau \lesssim g^{1+\alpha_1}[\phi](\tau_1) + \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2}.$$

Hence (6.17) and (6.18) follow from (6.24) and (6.25) respectively.

For $\tau_1 \leq \tau_2 \leq t_0$, the finite speed of propagation for wave equation shows that $g^p[\phi](\tau)$ vanishes. Thus the p-weighted energy inequalities (6.17) and (6.18) hold. For general $\tau_2 \geq \tau_1 \geq 0$, divide the interval $[\tau_1, \tau_2]$ into three(possibly two) such intervals: $[\tau_1, t_0]$, $[t_0, T_0]$ and $[T_0, \tau_2]$. Then (6.17) and (6.18) follow by combining those three (or two) inequalities together. This completes the proof for the p-weighted energy inequalities (6.17) and (6.18).

Finally, we use these two estimates to prove the integrated energy estimate (6.16) and hence finish the proof for Proposition 6.3.1. From estimate (6.20) it suffices to estimate

$$\bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2}.$$

Using interpolation, we can bound

$$\bar{G}^{\epsilon,1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\epsilon,2+2\alpha}[\phi]_{\tau_1}^{\tau_2} \lesssim (\tau_1)_+^{-1-\alpha}\bar{G}^{\alpha_1,0}[\phi]_{\tau_1}^{\tau_2} + G^{1+\alpha_1,2+\alpha}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \tilde{E}[\phi](\tau)d\tau$$

Therefore from the p-weighted energy inequalities (6.17), (6.18), we have

$$\begin{aligned} &I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}[\phi](\tau_2) + \tilde{E}^{1+\alpha}[\phi]_{\tau_1}^{\tau_2} \\ &\lesssim \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + (\tau_1)_+^{-1-\alpha}(g^{1+\alpha_1}[\phi](\tau_1) + \int_{\tau_1}^{\tau_2} \tau_+^{\epsilon} D^{\alpha_1}[F]_{\tau}^{\tau_2} d\tau). \end{aligned}$$

This proves (6.16). Thus Proposition (6.3.1) holds.

We end this section by proving a corollary of Proposition (6.3.1) which improves the estimate in Lemma 6.22.

Corollary 6.3.3. *For all $\tau_2 \geq \tau_1 \geq t_0$, we have*

$$D^{\alpha_1}[|Q_0| + |N|]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + (\tau_1)_+^{-1-\alpha}\left(g^{1+\alpha_1}[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \tau_+^{\epsilon} D^{\alpha_1}[F]_{\tau}^{\tau_2} d\tau\right).$$

Proof. Similar to the proof of Lemma 6.22, we can show that

$$\begin{aligned} D^{\alpha_1}[|Q_0| + |N|]_{\tau_1}^{\tau_2} \lesssim\ & I^{2\alpha-\alpha_1}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\alpha_1}\tau_+^{-1-2\alpha}|\overline{\partial_v}\psi|^2 dv d\omega d\tau \\ & + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2}\int_{R_1\leq r\leq R} \tau_+^{-1-2\alpha} r^{\alpha_1}|\partial\phi|^2 dx d\tau \\ & + \int_{\tau_1}^{\tau_2} \tau_+^{-1-4\alpha}\tilde{E}[\phi](\tau)^{1-\gamma}(g^{1+\epsilon}[\phi](\tau))^{\gamma} d\tau) \\ \lesssim\ & I^{\epsilon}[\phi]_{\tau_1}^{\tau_2} + \bar{G}^{\alpha_1, 1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + \tilde{E}^{1+2\alpha}[\phi]_{\tau_1}^{\tau_2} + G^{1+\alpha_1, 2+2\alpha}[\phi]_{\tau_1}^{\tau_2}. \end{aligned}$$

Then by Proposition 6.3.1, we have

$$D^{\alpha_1}[|Q_0| + |N|]_{\tau_1}^{\tau_2} \lesssim \tilde{E}[\phi](\tau_1) + D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + (\tau_1)_+^{-1-\alpha}\left(g^{1+\alpha_1}[\phi](\tau_2) + \int_{\tau_1}^{\tau_2} \tau_+^{\epsilon} D^{\alpha_1}[F]_{\tau}^{\tau_2} d\tau\right).$$

□

6.4 Decay of the solution

In the framework of the previous chapters, we use Proposition 6.3.1 to show the decay of the energy flux $E[\phi](\tau)$ under appropriate assumptions on the inhomogeneous term F. After commuting the equation with the vector fields Z, we obtain the pointwise decay of the solution outside the cylinder $\{(t, x)||x| \leq R\}$ by using Sobolev embedding and inside the cylinder by using elliptic estimates.

Proposition 6.4.1. *Suppose there is a constant C_1 such that*

$$D^{\alpha_1}[F]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \qquad \forall \tau_2 \geq \tau_1 \geq 0.$$

Then for solution ϕ of the wave equation (6.7)*, we have energy flux decay*

$$E[\phi](\tau) \lesssim (E_0 + C_1)\tau_+^{-1-\alpha}.$$

The proof is similar to that of Proposition 3.3.1. We outline the proof here.

Proof. Since the initial data are supported in the region $\{|x| \leq R_0 \leq R\}$, the finite speed of propagation shows that $g^{1+\alpha_1}[\phi](0)$ vanishes. Take $\tau_1 = 0$ in the p-weighted energy inequality (6.18). We get

$$\int_{S_\tau} r^{1+\alpha_1}(\partial_v\psi)^2 dv d\omega + \int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{\alpha_1}(\partial_v\psi)^2 dv d\omega d\tau \lesssim C_1 + E_0.$$

Then we can extract a dyadic sequence $\{\tau_n \to \infty\}$ such that

$$\int_{S_{\tau_n}} r^{\alpha_1}(\partial_v\psi)^2 dv d\omega \leq (1+\tau_n)^{-1}(C_1 + E_0).$$

Interpolation implies that

$$\int_{S_{\tau_n}} r(\partial_v\psi)^2 dv d\omega \lesssim (1+\tau_n)^{-\alpha_1}(E_0 + C_1).$$

Then the p-weighted energy inequality (6.17) implies that for $\tau \geq \tau_n$

$$\int_{\tau_n}^{\tau} E[\phi](s) ds \lesssim (\tau_n)_+^{-\alpha}(E_0 + C_1) + \tau_n^{1-\alpha} E[\phi](\tau_n).$$

On the other hand the energy inequality (6.16) shows that for all $s \leq \tau$

$$E[\phi](\tau) \lesssim E[\phi](s) + (1+s)^{-1-\alpha} \left(E_0 + C_1\right).$$

In particular

$$E[\phi](\tau_1) \lesssim E[\phi](0) + E_0 + C_1 \lesssim E_0 + C_1.$$

Therefore we can show that

$$(\tau - \tau_n)E[\phi](\tau) - \int_{\tau_n}^{\tau} s_+^{-1-\alpha} \left(E_0 + C_1\right) ds \lesssim (\tau_n)_+^{-\alpha} \left(E_0 + C_1\right) + \tau_n^{1-\alpha} E[\phi](\tau_n).$$

In particular for $n = 1$

$$E[\phi](\tau) \lesssim \tau_+^{-1} \left(E_0 + C_1\right).$$

Then let $\tau = \tau_{n+1}$ in the previous estimate. We obtain

$$(\tau_{n+1} - \tau_n)E[\phi](\tau_{n+1}) \lesssim (1+\tau_n)^{-\alpha} \left(E_0 + C_1\right).$$

Since τ_n are dyadic, we have

$$E[\phi](\tau_n) \lesssim \tau_n^{-1-\alpha} \left(E_0 + C_1\right), \quad \forall n.$$

Finally, for $\tau \in [\tau_n, \tau_{n+1}]$, we use the energy estimate (6.16) and we can show that

$$E[\phi](\tau) \lesssim E[\phi](\tau_n) + (\tau_n)_+^{-1-\alpha} \left(E_0 + C_1\right) \lesssim (\tau_n)_+^{-1-\alpha} \left(E_0 + C_1\right) \lesssim \tau_+^{-1-\alpha} \left(E_0 + C_1\right).$$

$\square$

With the energy flux decay, we can obtain the decay of the spherical average of the solution.

Corollary 6.4.2. *Assume that there is a constant C_1 such that*

$$D^{\alpha_1}[F]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \qquad \forall \tau_2 \geq \tau_1 \geq 0.$$

Then on the hypersurface S_τ, we have

$$\int_\omega |r\phi|^2 d\omega \lesssim (1+\tau)^{-\alpha+\delta}(E_0 + C_1), \quad \forall \delta > 0, \quad r \geq R,$$
$$\int_\omega r|\phi|^2 d\omega \lesssim (1+\tau)^{-1-\alpha}\left(E_0 + C_1\right), \qquad r \geq R.$$

Proof. By Proposition 6.4.1, the second estimate follows from Lemma 2.4.1. For the first inequality, first by the p-weighted energy inequalities (6.17), (6.18), we have

$$g^{1+\alpha_1}[\phi](\tau) \lesssim E_0 + C_1, \quad g^1[\phi](\tau) \lesssim \tau_+^{-\alpha}(E_0 + C_1).$$

Thus for all $\alpha > \delta > 0$, we have the estimate

$$g^{1+\delta}[\phi](\tau) \lesssim (1+\tau)^{-\alpha+\delta}(E_0 + C_1).$$

Therefore similarly to the proof of Proposition 4.2.3, we can show that

$$\int_\omega r^2|\phi|^2(\tau, v, \omega) d\omega \lesssim E[\phi](\tau) + g^{1+\delta}[\phi](\tau) \lesssim \tau_+^{-\alpha+\delta}\left(E_0 + C_1\right), \quad \forall \delta > 0.$$

Here $v_\tau = \frac{R+\tau}{2}$. □

In order to obtain the pointwise decay of the solution which is usually a consequence of Sobolev embedding, we need energy estimates for the derivative of the solution. For this purpose, we commute the equation with the vector fields Z. Under appropriate assumptions on the inhomogeneous term F, we want to derive the energy decay for $Z^k\phi$. Recall the null form $Q_0(\Phi, \phi)$. We have the identity

$$Z^\beta Q_0(\Phi, \phi) = \sum_{\beta_1+\beta_2=\beta} Q_0(Z^{\beta_1}\Phi, Z^{\beta_2}\phi)$$

for $Z = \Omega$ or ∂_t.

Based on the Corollary 6.3.3, we are able to prove the decay of the energy flux of $Z^\beta\phi$ after commuting the equation (6.7) with Z^β.

Proposition 6.4.3. *Assume that there is a constant C_1 such that the inhomogeneous term F in (6.7) satisfies*

$$D^{\alpha_1}[Z^\beta F]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \quad \forall \tau_2 \geq \tau_1 \geq 0, \quad \forall \beta \leq \beta_0$$

for some $\beta_0 \leq 4$. Then we have

$$E[Z^\beta\phi](\tau) \lesssim (C_1 + E_0)\,(1+\tau_1)^{-1-\alpha}, \tag{6.26}$$

$$D^{\alpha_1}[Q_0(Z^{\beta_1}\Phi, Z^\beta\phi)]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Z^{\beta_1}N^\mu \cdot Z^\beta\partial_\mu\phi]_{\tau_1}^{\tau_2} \lesssim (C_1 + E_0)\,(1+\tau_1)^{-1-\alpha} \tag{6.27}$$

for $\forall\beta \leq \beta_0$, $\beta_1 \leq 4$.

Proof. We prove the proposition by induction. When $\beta = 0$, (6.26) follows from Proposition 6.4.1. Since $Z^{\beta_1}\Phi$ is $(\delta, \alpha, t_0, R_1, C_0)$-weak wave, $\forall\beta_1 \leq 4$, Corollary 6.3.3 implies that

$$D^{\alpha_1}[Q_0(Z^{\beta_1}\Phi, \phi)]_{\tau_1}^{\tau_2} + D^{\alpha_1}[Z^{\beta_1}N^\mu \cdot \partial_\mu\phi]_{\tau_1}^{\tau_2} \lesssim (C_1 + E_0)(1+\tau_1)^{-1-\alpha}, \quad \forall\beta_1 \leq 4.$$

Assume that the estimates (6.26), (6.27) hold for all $\beta' < \beta$. Commute the equation

(6.7) with Z^β. We have the equation for $Z^\beta\phi$

$$\Box(Z^\beta\phi) + Q_0(\Phi, Z^\beta\phi) + N(Z^\beta\phi) \tag{6.28}$$
$$= Z^\beta F - \sum_{\beta_1+\beta_2\leq\beta, \beta_2<\beta} Q_0(Z^{\beta_1}\Phi, Z^{\beta_2}\phi) + Z^{\beta_1}N^\mu \cdot Z^{\beta_2}\partial_\mu\phi.$$

Since $\beta_2 < \beta$, by the induction assumptions, we get

$$D^{\alpha_1}\left[Z^\beta F - \sum_{\beta_2<\beta} Q_0(Z^{\beta_1}\Phi, Z^{\beta_2}\phi) + Z^{\beta_1}N^\mu \cdot Z^{\beta_2}\partial_\mu\phi\right]_{\tau_1}^{\tau_2} \lesssim (C_1 + E_0)\,(1+\tau_1)^{-1-\alpha}.$$

Hence for $Z^\beta\phi$, inequality (6.26) follows from Proposition 6.4.1 and inequality (6.27) follows from Corollary 6.3.3. □

Since the angular momentum Ω is vanishing for $r = 0$, we are not able to obtain the pointwise bound of the solution in the cylinder $\{|x| \leq R\}$ by commuting the equation with Ω. We instead rely on elliptic estimates and the vector ∂_t as commutators.

Lemma 6.4.4. *Assume that there is a constant C_1 such that*

$$D^{\alpha_1}[F]_{\tau_1}^{\tau_2} + D^{\alpha_1}[\partial_t F]_{\tau_1}^{\tau_2} \leq C_1(1+\tau_1)^{-1-\alpha}, \qquad \forall \tau_2 \geq \tau_1 \geq 0.$$

Then for solution of the wave equation (6.7)*, we have*

$$\int_{r\leq R} |\partial^2\phi|^2 dx \lesssim (E_0 + C_1)\,(1+\tau)^{-1-\alpha}.$$

Proof. The previous Proposition implies that

$$E[\partial_t^j\phi](\tau_1) + D^{\alpha_1}[\partial_t^j Q_0(\Phi,\phi) + \partial_t^j N(\phi)]_{\tau_1}^{\tau_2} \lesssim (E_0 + C_1)\,(\tau_1)_+^{-1-\alpha}, j \leq 1.$$

Then from the estimate (4.10) in Lemma 4.2.2, we can show that

$$\begin{aligned}\int_{r\leq R} |\partial^2\phi|^2 dx &\lesssim E[\partial_t\phi](\tau^+) + D^{\epsilon}[\partial_t\Box\phi]_{\tau^+}^{\tau+R} + E[\phi](\tau^+) + D^{\epsilon}[\Box\phi]_{\tau^+}^{\tau+R} \\ &\lesssim \sum_{j\leq 1} E[\partial_t^j\phi](\tau^+) + D^{\alpha_1}[\partial_t^j Q_0(\Phi,\phi) + \partial_t^j N(\phi)]_{\tau_1}^{\tau_2} + D^{\alpha_1}[\partial_t^j F]_{\tau^+}^{\tau+R} \\ &\lesssim (E_0 + C_1)(1+\tau)^{-1-\alpha}, \quad \tau^+ = \max\{0, \tau - R\}.\end{aligned}$$

□

Having the above lemma, we can show the pointwise estimate of the solution inside the cylinder with radius R.

Corollary 6.4.5. *Let F satisfies the same condition in the previous Lemma. Then*

$$|\phi|^2 \lesssim (C_1 + E_0)\,(1+t)^{-1-\alpha}, \qquad r \leq R.$$

Proof. This estimate follows from the previous Lemma and Sobolev embedding. □

6.5 Proof of the main theorems

We use the same framework in Chapter 4 to prove the main theorem Theorem 6.1.5 in this chapter. We use bootstrap argument to show control the nonlinearity F in the equation (6.7).

Proposition 6.5.1. *If the nonlinearity F in* (6.7) *satisfies*

$$\begin{aligned}&D^{\alpha_1}[Z^\beta F]_{\tau_1}^{\tau_2} \leq 2E_0(1+\tau_1)^{-1-\alpha}, \quad \forall\beta\leq 4, \quad \forall\tau_2\geq\tau_1\geq 0, \\ &\int_{r\leq R} |\nabla Z^\beta F|^2 dx \leq 2E_0(1+\tau)^{-1-\alpha}, \quad \forall\beta\leq 2, \quad \forall\tau\geq 0,\end{aligned} \tag{6.29}$$

then we can show that

$$D^{\alpha_1}[Z^\beta F]_{\tau_1}^{\tau_2} \lesssim E_0^2(1+\tau_1)^{-1-\alpha}, \quad \forall \beta \leq 4, \quad \forall \tau_2 \geq \tau_1 \geq 0, \tag{6.30}$$

$$\int_{r\leq R} |\nabla Z^\beta F|^2 dx \lesssim E_0^2(1+\tau)^{-1-\alpha}, \quad \forall \beta \leq 2, \quad \forall \tau \geq 0. \tag{6.31}$$

Proof. The proof of this Proposition is similar to that of Proposition 4.3.1 in Chapter 4, considering all the necessary ingredients we have obtained for solutions of the equation (6.7). The extra assumption (6.29) is used to show the pointwise estimate of the solution inside the cylinder with radius R, which is, as we have seen in the previous chapters, obtained through elliptic estimates. The only difference between this Proposition and Proposition 4.3.1 is that F contains the quadratic term $k^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (the decay rates here are weaker, but the estimates we want to show are also weaker). To prove this proposition, we only have to treat this quadratic term with coefficients $k^{\mu\nu}$ satisfying the decay estimates (6.8).

Since inside the cylinder with radius R, the null condition is not necessary (which has been proved in Proposition 4.3.1). It hence suffices to show that

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1+\alpha_1}|Z^k(k^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)|^2 dx d\tau \lesssim E_0^2(\tau_1)_+^{-1-\alpha}, \quad \forall k \leq 4.$$

Recall from the condition (6.8) that

$$|Z^k k^{\mu\nu}| \lesssim (1+r)^{-\alpha}, \quad \forall k \leq 4.$$

Denote

$$\phi_1 = Z^{k_1}\phi, \quad \phi_2 = Z^{k_2}\phi, \quad k_1 + k_2 \leq 4.$$

Then it is sufficient to prove that

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}|\partial\phi_1|^2|\partial\phi_2|^2 dx d\tau \lesssim E_0^2(\tau_1)_+^{-1-\alpha}, \quad \forall k_1+k_2\le 4.$$

Here we note that $\alpha_1 - 2\alpha < \epsilon$. We now decompose the full derivative ∂ with respect to the null frame $\{\partial_v, \partial_u, \nabla\!\!\!\!/\}$ as those quadratic terms satisfying the null condition have already been bounded in Proposition 4.3.1. We can show that

$$\sum_{k_1+k_2\le 4} r^2|\partial(Z^{k_1}\phi)||\partial(Z^{k_2}\phi)| \lesssim \sum_{k_1+k_2\le 4} r|\phi_1||\bar\partial\phi_2| + |\nabla\!\!\!\!/\psi_1||\partial\psi_2|$$
$$+ |\partial_u\psi_1||\partial_v\psi_2| + |\partial_t\psi_1||\partial\psi_2|,$$

where $\psi_1 = r\phi_1 = rZ^{k_1}\phi$. The first three terms on the right hand side

$$r|\phi_1||\bar\partial\phi_2|, \quad |\nabla\!\!\!\!/\psi_1||\partial\psi_2|, \quad |\partial_u\psi_1||\partial_v\psi_2|$$

are those appeared on the right hand side of estimate (4.30) in Lemma 4.3.4. By the proof in that section, we conclude that the above three terms can be bounded as expected. Therefore to prove the proposition here, it remains to control $|\partial_t\psi_1||\partial\psi_2|$.

If $k_1 \le 3$, then we always have

$$\int_\omega |\partial_t\psi_1|^2|\partial\psi_2|^2 d\omega \lesssim \sum_{k_{1'}\le 3, k_{2'}\le 4}\int_\omega |\partial_t\psi_{1'}|^2 d\omega\cdot\int_\omega|\partial\psi_{2'}|^2 d\omega.$$

It suffices to check this estimate for the case $k_1 = k_2 = 2$ (otherwise Sobolev embedding on the unit sphere leads to the inequality). We have

$$\int_\omega |\partial_t Z^2\phi|^2|\partial Z^2\phi|^2 d\omega \lesssim \|\partial_t Z^2\phi\|^2_{H^1(S^2)}\|\partial Z^2\phi\|^2_{H^1(S^2)}.$$

Therefore the previous estimate holds. Since $k_{1'} \leq 3$, from Corollary 6.4.2, we have

$$\int_\omega |\partial_t \psi_{1'}|^2 d\omega \lesssim E_0 \tau_+^{-\alpha+\delta}.$$

Thus by the integrated energy estimate (6.16) we can estimate

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{1-\epsilon} r^{-4} |\partial_t \psi_1|^2 |\partial \psi_2|^2 r^2 dv d\omega d\tau \lesssim E_0 (\tau_1)_+^{-\alpha+\delta} \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{-1-\epsilon} |\partial \psi_{2'}|^2 dv d\omega d\tau$$
$$\lesssim E_0 (\tau_1)_+^{-\alpha+\delta} I^\epsilon [Z^{k_{2'}} \phi]_{\tau_1}^{\tau_2} \lesssim E_0^2 (\tau_1)_+^{-1-2\alpha+\delta}$$

for all $\delta > 0$.

Now we only have to consider the case when $k_1 = 4$, $k_2 = 0$ (as $k_1 + k_2 \leq 4$). That is the term $|\partial_t (r Z^4 \phi)||\partial \psi|$. We use the estimate (4.35) proven in Section 4.3.2 to show the pointwise bound for $\partial \psi$. In fact we conclude from (4.35) that

$$\int_{t_2}^{t_1} \int_\omega |Z^\beta \partial_v \psi| d\omega dt \lesssim E_0, \quad \psi = r\phi, \quad \beta \leq 2.$$

Then using Sobolev embedding on $[t_1, t_2] \times S^2$, we obtain

$$r^2 |\partial_v \phi|^2 \leq |\partial_v \psi|^2 + |\phi|^2 \lesssim E_0.$$

In particular we have

$$|\partial \psi|^2 \lesssim r^2 (|Z\phi|^2 + |\partial_v \phi|^2) \lesssim E_0.$$

Therefore we can estimate

$$\int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{1-\epsilon} r^{-4} |\partial_t (r Z^4 \phi)|^2 |\partial \psi|^2 r^2 dv d\omega d\tau \lesssim E_0 \int_{\tau_1}^{\tau_2} \int_{S_\tau} r^{-1-\epsilon} |\partial (r Z^4 \phi)|^2 dv d\omega d\tau$$
$$\lesssim E_0 I^\epsilon [Z^4 \phi]_{\tau_1}^{\tau_2} \lesssim E_0^2 (\tau_1)_+^{-1-\alpha}.$$

This finished the proof for Proposition 6.5.1. □

Having this proposition, our main theorem Theorem 6.1.5 then follows from the same argument in Section 4.4.

To prove the second main Theorem 6.1.3, it suffices to check that the functions Φ, $\mathcal{N}^\mu(\partial\Phi)$, $\mathcal{N}^{\mu\nu}(\partial\Phi)$ satisfy the conditions in Theorem 6.1.3 that Φ, N^μ, $k^{\mu\nu}$ satisfy correspondingly. In fact, notice that

$$|\partial Z^\beta \mathcal{N}^\mu(\partial\Phi)| \leq C(\mathcal{N})|\partial^2 Z^\beta \Phi|, \quad \forall \beta \leq 2.$$

The boundedness of $\partial^2 Z^\beta\Phi$, for all $\beta \leq 2$, follows from the equation for $Z^\beta\Phi$ (similarly, express the only unknown term $\partial_{rr} Z^\beta\Phi$ as a combination of terms with known L^∞ norm). Hence $|\partial Z^\beta \mathcal{N}(\partial\Phi)| + |\partial^2 Z^\beta\Phi|$ is bounded by a constant depending on C_0 and the nonlinearity $\mathcal{N}$. For the other conditions, when $t \leq t_0$, we have

$$|Z^\beta \mathcal{N}^\mu(\partial\Phi)| + |Z^\beta \mathcal{N}^{\mu\nu}(\partial\Phi)| \leq C(\mathcal{N}, C_0), \quad \forall \beta \leq 4.$$

When $t \geq t_0$, we can show

$$\begin{aligned} |Z^\beta \mathcal{N}^\mu| &\leq C(\mathcal{N}, C_0)(1+|x|)^{-1-\frac{1}{2}\alpha_0}(1+(t-|x|)_+)^{-1}, \\ |Z^\beta \mathcal{N}^{\mu\nu}(\Phi)| &\leq C(\mathcal{N}, C_0)(1+|x|)^{-\frac{1}{2}\alpha_0}, \quad \beta \leq 4. \end{aligned}$$

Replace α with $\min\{\frac{\alpha_0}{6}, \alpha\}$. Then the functions $\mathcal{N}^\mu(\partial\Phi)$, $\mathcal{N}^{\mu\nu}(\partial\Phi)$ satisfy the conditions in Theorem 6.1.5. We thus have the stability result of Theorem 6.1.3.

Chapter 7

Further discussions

7.1 General initial data

The initial data for the nonlinear wave equations we have considered in the previous chapters were assumed to have compact support (supported on the ball with radius R in $\mathbb{R}^3$). The finite speed of propagation of solutions of wave equations implies that the solution vanishes on $\{|x| \geq t + R\}$. As claimed in Remark 5.1.3, as long as

$$E_0 = \sum_{|k|\leq 6} \int_{\mathbb{R}^3} r^{1+\alpha} |\partial Z^k \phi(0, x)|^2 dx \tag{7.1}$$

is sufficiently small for some positive constant α, the solution of the quasilinear wave equation satisfying the null condition (see equation (5.1) in Chapter 5) exists globally and satisfies the estimates described in Theorem 5.1.1. To verify this claim, it suffices to consider the equation on the region $\{|x| \geq t + R\}$. We still use this new approach that we have introduced in the previous chapters.

For simplicity, we will take the semilinear wave equation

$$\Box\phi = F(\partial\phi)$$

for example to outline the proof for that if initially the quantity (7.1) is sufficiently small, then the solution exists globally.

Fix a large constant $R > 0$. We do the integrated energy estimates and the p-weighted energy estimates on the region $\{|x| \geq t + R\}$. We define some notations here. For $R \leq r_1 \leq r_2$, we use S_{r_1,r_2} to denote the following outgoing null hypersurface emanating from the sphere with radius r_1

$$S_{r_1,r_2} := \{u = -\frac{r_1}{2}, \quad r_1 \leq r \leq r_2\}$$

Similarly define $\bar{C}_{r_1,r_2}$ to be the following incoming null hypersurface emanating from the sphere with radius r_2

$$\bar{C}_{r_1,r_2} := \{v = \frac{r_2}{2}, \quad r_1 \leq r \leq r_2\}.$$

On the initial hypersurface $\mathbb{R}^3$, the annulus with radii r_1, r_2 is

$$B_{r_1,r_2} := \{t = 0, \quad r_1 \leq r \leq r_2\}.$$

We use S_r to be short for $S_{r,\infty}$. Similarly we have $\bar{C}_r$ and B_r.

We use $\mathcal{D}_{r_1,r_2}$ to denote the region bounded by S_{r_1,r_2}, B_{r_1,r_2}, $\bar{C}_{r_1,r_2}$. Let $E[\phi](\Sigma)$ to be the energy flux for ϕ through the hypersurface Σ in the Minkowski space. In particular,

$$E[\phi](S_{r_1,r_2}) = \int_{S_{r_1,r_2}} |\overline{\partial_v}\phi|^2 r^2 dv d\omega, \quad E[\phi](\bar{C}_{r_1,r_2}) = \int_{\bar{C}_{r_1,r_2}} |\overline{\partial_u}\phi|^2 r^2 du d\omega,$$

where $\overline{\partial_u} = (\partial_u, \nabla\!\!\!/)$. On the initial hypersurface

$$E[\phi](B_{r_1,r_2}) = \int_{B_{r_1,r_2}} |\partial\phi|^2 dx.$$

In the energy identity (2.3), take the region $\mathcal{D}$ to be $\mathcal{D}_{r_1,r_2}$, the vector field $X = \partial_t$ and the function $\chi = 0$. We obtain the classical energy estimate

$$2\iint_{D_{R_1,R_2}} \Box\phi \cdot \partial_t\phi d\text{vol} + E[\phi](C_{R_1,R_2}) + E[\phi](\bar{C}_{R_1,R_2}) = E[\phi](B_{R_1,R_2}). \tag{7.2}$$

We also need an integrated energy estimate adapted to the region $\mathcal{D}_{r_1,r_2}$. For some small positive constant ϵ, we construct the same vector field $X = f(r)\partial_r$, the functions f, χ as those in the proof of Proposition 3.1.1 (see Section 3.1 but with the constant α there replaced with ϵ here). We have already done all the necessary estimates in Section 3.1.1. And thus we can derive that

$$I^\epsilon[\phi]_{r_1}^{r_2} \leq C_\epsilon(E[\phi](B_{r_1}) + E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) + D^\epsilon[\Box\phi]_{r_1}^{r_2}), \tag{7.3}$$

where we denote

$$I^\epsilon[\phi]_{r_1}^{r_2} := \iint_{\mathcal{D}_{r_1,r_2}} \frac{|\bar{\partial}\phi|^2}{(1+r)^{1+\epsilon}} dxdt, \quad D^\epsilon[F]_{r_1}^{r_2} := \iint_{\mathcal{D}_{r_1,r_2}} (1+r)^{1+\epsilon}|F|^2 dxdt.$$

These notations should not be confused with those we previously used as the solutions in this chapter live only on the region $\{r \geq t + R\}$. The constant in (7.3) depends only on ϵ and is independent of r_1, r_2. For the derivation of estimate (7.3), it is almost the same as estimate (3.14) in Section 3.1.1 (note that $\delta_1 = 0$ here). The only place that we have to point out here is that we use the fact that the solution ϕ goes to zero as $r \to \infty$ on the initial hypersurface together with a similar version of Lemma 2.4.2 to control $\frac{|\phi|^2}{(1+r)^2}$. This is also the reason that we have $E[\phi](B_{r_1})$ (using our notations above this is $E[\phi](B_{r_1,\infty})$) instead of $E[\phi](B_{r_1,r_2})$ in the estimate (7.3).

Combine the above two estimates (7.2), (7.3). We derive the following integrated energy estimates

$$E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) + I^{\epsilon}[\phi]_{r_1}^{r_2} \leq C_{\epsilon}(E[\phi](B_{r_1}) + D^{\epsilon}[\Box\phi]_{r_1}^{r_2}) \qquad (7.4)$$

for some constant C_ϵ depending only on ϵ. The energy decay (energy flux through the outgoing null hypersurface S_{r_1}) and the integrated energy decay can be derived directly from the above inequality under appropriate assumptions on $\Box\phi$. In fact since initially E_0 (see (7.1)) is sufficiently small, we can conclude that

$$E[\phi](B_{r_1}) \leq E_0 r_1^{-1-\alpha}.$$

Therefore if we assume that

$$D^{\alpha}[\phi]_{r_1}^{r_2} \leq 2E_0 r_1^{-1-\alpha} \qquad (7.5)$$

then we can show the decay of the energy flux $E[\phi](S_{r_1})$ as well as the integrated energy $I^{\epsilon}[\phi]_{r_1}^{r_2}$. Here the small constant $\epsilon < \alpha$. And the above assumption (7.5) can be viewed as the bootstrap assumption on the nonlinearity. We note that the decay of the energy flux through S_r, $r \geq R$ does not rely on the p-weighted energy inequality.

However, to close the bootstrap assumption (7.5), we still need the p-weighted energy inequalities. In the energy identity (2.3), we take

$$X = f\partial_v, \quad \chi = r^{p-1}, \quad f = r^p.$$

We can compute

$$\int_{B_{r_1,r_2}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} = \frac{1}{2}\int_{B_{r_1,r_2}} f(|\partial_v\psi|^2 + |\nabla\!\!\!/\psi|^2) - \partial_r(fr\phi^2) + f'r\phi^2 drd\omega,$$

$$\int_{S_{r_1,r_2}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} = \int_{S_{r_1,r_2}} f|\partial_v\psi|^2 - \frac{1}{2}\partial_v(fr\phi^2)dvd\omega,$$

$$\int_{\bar{C}_{r_1,r_2}} i_{\tilde{J}^X[\phi]} d\mathrm{vol} = -\int_{\bar{C}_{r_1,r_2}} f|\nabla\!\!\!/\psi|^2 + f'r\phi^2 + \frac{1}{2}\partial_u(fr\phi^2)dud\omega,$$

$$\iint_{\mathcal{D}_{r_1,r_2}} K^X[\phi] + \chi\partial^\gamma\phi\partial_\gamma\phi - \frac{1}{2}\Box\chi\cdot\phi^2 d\mathrm{vol}$$
$$= \iint_{\mathcal{D}_{r_1,r_2}} \frac{1}{2}f'|\partial_v\psi|^2 + (\chi - \frac{1}{2}f')|\nabla\!\!\!/\psi|^2 - \frac{1}{2}\partial_v(f'r\phi^2)drdt.$$

Here $\psi = r\phi$. We can do integration by parts on $\mathcal{D}_{r_1,r_2}$ to estimate the integral of $\partial_v(f'r\phi^2)$. We can also modify the current vector field $\tilde{J}^X[\phi]$ defined in line (2.2) to

$$\hat{J}^X[\phi] = \tilde{J}^X[\phi] + \frac{1}{2}f'r\phi^2\partial_v.$$

Notice that

$$-\int_{B_{r_1,r_2}} \partial_r(fr\phi^2)drd\omega - \int_{\bar{C}_{r_1,r_2}} \partial_u(fr\phi^2)dud\omega + \int_{S_{r_1,r_2}} \partial_v(fr\phi^2)dvd\omega = 0.$$

Then from the energy identity (2.3) and the above calculations, we obtain

$$\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}(|\partial_v\psi|^2 + (2-p)|\nabla\!\!\!/\psi|^2)drdtd\omega + \int_{\bar{C}_{r_1,r_2}} r^p|\nabla\!\!\!/\psi|^2 dud\omega$$
$$+ \int_{S_{r_1,r_2}} r^p|\partial_v\psi|^2 dvd\omega = \int_{B_{r_1,r_2}} r^p|\overline{\partial_v}\psi|^2 drd\omega - 2\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}\Box\phi\partial_v\psi dxdt.$$

We can take $p = 1+\alpha$. And we can estimate the inhomogeneous term $\Box\phi$ as follows

$$|2\iint_{\mathcal{D}_{r_1,r_2}} r^{p-1}\Box\phi\partial_v\psi dxdt| \le \iint_{\mathcal{D}_{r_1,r_2}} \tau_+^{1+\epsilon}r^{1+\alpha}|\Box\phi|^2 dxdt$$
$$+ \iint_{\mathcal{D}_{r_1,r_2}} \tau_+^{-1-\epsilon}r^{1+\alpha}|\partial_v\psi|^2 dvd\omega dt,$$

where $\tau_+ = \max\{r - t, 1\}$ and the role that τ_+ plays here is the same as τ (the parameter of the foliation Σ_τ in the previous chapters). The first term on the right hand side of the above inequality can be bounded by using the bootstrap assumption (7.5). The second term can be controlled by using Gronwall's inequality. In particular, we can obtain the p-weighted energy inequality for solutions of linear wave equation on the region $\{|x| \geq t + R\}$. Having this p-weighted energy inequality as well as the integrated energy estimate (7.4), the similar argument in Section 4.3 then implies that we can close the bootstrap assumption (7.5) if E_0 is sufficiently small. Therefore we can show the existence of the solution on the region $\{|x| \geq R + t\}$. Finally to show that there is a global solution on the whole spacetime, we only have to note that $g^{1+\alpha}[\phi](0)$ which is

$$\int_{S_R} r^{1+\alpha} |\partial_v \psi|^2 dv d\omega$$

in this section is small. This is sufficient to make the argument in the previous chapters work.

7.2 Systems of quasilinear wave equations

In this section, we discuss the system of quasilinear wave equations satisfying the null condition. We briefly discuss here how to adapt our method developed in the previous chapters to system of quasilinear wave equations. We take the equations

$$\Box u^I + h^{IJ,\mu\nu} \partial_{\mu\nu} u^J = 0, \quad I = 1, 2, \ldots, D \tag{7.6}$$

for example, where D is a positive integer. For fixed μ, ν, $h^{IJ,\mu\nu}$ is a symmetric $D \times D$ matrix. We can simply assume

$$h^{IJ,\mu\nu} = B_K^{IJ,\mu\nu\gamma} \partial_\gamma u^K,$$

where $B_K^{IJ,\mu\nu\gamma}$ are constants satisfying the null condition for any triple (I, J, K) and are symmetric in I, J. Here the capital letters I, J, K run from 1 to D, the Greek letters μ, ν, γ run from 0 to 3.

To use our new method, we have to generalize the integrated energy estimates and the p-weighted energy inequalities to solutions of system of linear wave equations. The integrated energy estimates (Proposition 3.1.1 established in Section 3.1) can be easily extended to solutions of system as there the solution can be vector valued. For the p-weighted energy inequality, the only relative place that we need the equation to be a scalar equation is that the vector field X defined in line (3.24) depends on the background metric $g^{\mu\nu} = (m_0)^{\mu\nu} + h^{\mu\nu}$. For system, this vector field is different for different wave component. In particular, it is not possible to make the coefficient of the "bad" term $\underline{L}(u^I)\underline{L}(u^J)$, $I \neq J$ to decay sufficiently fast in r (see the proof after line (3.30)). And the argument there fails for solutions of system of quasilinear wave equations.

However, we note that the choice of the vector field X (defined in line (3.24)) was used to estimate the term $\underline{L}(\phi)\underline{L}(\phi)$ in the current $K^X[\phi]$. For those other terms which contain at least one "good" derivative of the solution (for example $L(\phi)\underline{L}(\phi)$), the estimates there (proof of Proposition 3.2.2) hold for solutions of system of linear wave equations. Therefore for system, to derive a useful p-weighted energy inequality, it suffices to treat the terms $h^{IJ,\underline{L}\underline{L}}\underline{L}\underline{L}(u^J)$, $I \neq J$ in the equation (7.6). We view these terms as inhomogeneous term. As the vector field X is close to $r^p L$ (error term from the vector field X is easier to estimate), we only have to bound the integral

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} h^{IJ,\underline{L}\underline{L}}\underline{L}\underline{L}(u^J) r^p L(u^I) dx d\tau.$$

Note that

$$h^{IJ,\underline{L}\underline{L}}\underline{L}\underline{L}(u^J)r^pL(u^I) = \underline{L}(h^{IJ,\underline{L}\underline{L}}\underline{L}(u^J)r^pL(u^I)) - \underline{L}(h^{IJ,\underline{L}\underline{L}})\underline{L}(u^J)r^pL(u^I)$$
$$- h^{IJ,\underline{L}\underline{L}}\underline{L}(u^J)r^p\underline{L}L(u^I) - h^{IJ,\underline{L}\underline{L}}\underline{L}(u^J)pr^{p-1}L(u^I).$$

Using the bootstrap assumptions (similar as (5.5)), we have the decay

$$|h^{IJ,\underline{L}\underline{L}}| + |\underline{L}(h^{IJ,\underline{L}\underline{L}})| \leq C\delta_1(1+r)^{-1-2\alpha}.$$

In the previous identity, the first term can be estimated by using integration by parts. The second and the last term contains the "good" term $L(u^I)$. Thus it suffices to estimate the third term which contains $\underline{L}L(u^I)$. We can use the equation of u^I. In particular, if we assume that

$$I^\epsilon[Zu^I]_{\tau_1}^{\tau_2} + I^\epsilon[u^I]_{\tau_1}^{\tau_2} \leq C, \quad \forall I \in \{1, 2, \ldots, D\}, \tag{7.7}$$

then from Lemma 5.2.3 or Corollary 5.3.5 we can conclude that

$$\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{1-\epsilon}|\underline{L}Lu^I|^2 dx d\tau \leq C.$$

Here $Z = \{\partial_t, \Omega\}$ and we have pointed out that the integrated energy estimate (3.5) and the energy estimate (3.6) hold for solutions of system of linear wave equations. For $p = 1 + \alpha_1$, $\alpha + \epsilon < \alpha_1 + 2\epsilon < 2\alpha$, we then can show that

$$|\int_{\tau_1}^{\tau_2}\int_{S_\tau} h^{IJ,\underline{L}\underline{L}}\underline{L}(u^J)r^p\underline{L}L(u^I)dxd\tau| \leq C\int_{\tau_1}^{\tau_2}\int_{S_\tau} r^{-1-2\alpha+1+\alpha_1-1+\epsilon}|\underline{L}(u^J)|^2 dxd\tau$$
$$\leq C\delta_1 I^\epsilon[u^J]_{\tau_1}^{\tau_2}.$$

Having this estimate, we can obtain the same p-weighted energy inequality as those in

Proposition 3.2.2. Therefore if we have the boundedness of $I^{\epsilon}[Zu^I]_{\tau_1}^{\tau_2}$, $I^{\epsilon}[u^I]_{\tau_1}^{\tau_2}$, then we can use the same method of Proposition 3.3.3 to derive the integrated energy decay

$$I^{\epsilon}[u^I]_{\tau_1}^{\tau_2} \leq C(1+\tau_1)^{-1-\alpha}.$$

However, we note that the boundedness of the integrated energy follows from the estimate (3.4) in Proposition 3.1.1 which can be generalized to solutions of system of linear wave equations. Thus we can commute the system of equations with Z for 8 times to obtain the boundedness of $I^{\epsilon}[Z^k u^I]_{\tau_1}^{\tau_2}$, $k \leq 8$. Then above argument then implies that

$$I^{\epsilon}[Z^k u^I]_{\tau_1}^{\tau_2} \leq C(1+\tau_1)^{-1-\alpha}, \quad k \leq 7.$$

Similar to the proof of Theorem 5.1.1, this estimate is sufficient to prove the pointwise decay of the solution u^I. We thus can show the small data global existence for system of quasilinear wave equations satisfying the null condition.

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