Semigroups for One-Dimensional Schrödinger Operators with Multiplicative Gaussian Noise

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Abstract

A problem of fundamental interest in mathematical physics is that of understanding the structure of the spectrum of random Schrödinger operators. An important tool in such investigations is the Feynman-Kac formula, which provides a simple probabilistic representation of the semigroup of Schrödinger operators, thus making exponential functionals of the eigenvalues amenable to explicit computation. Classical results in the theory of random Schrödinger semigroups concern discrete operators (i.e., acting on a lattice) or continuous operators with a smooth random noise. In many applications, however, it is more natural to consider continuous operators with very irregular random noises modelled by Schwartz distributions (such as Gaussian white noise).

In this thesis, we take the first steps in developing a general semigroup theory for one-dimensional random Schrödinger operators whose noise is given by the formal derivative of a continuous Gaussian process with stationary increments. The main source of inspiration for these results is the stochastic semigroup theory pioneered by Gorin and Shkolnikov in the random matrix theory literature. Our main result consists of a Feynman-Kac formula for such operators, which naturally extends the classical Feynman-Kac formula for random Schrödinger operators with smooth Gaussian noise. As a consequence of our main result, we obtain an explicit representation of the Laplace transforms of the eigenvalue point process of one-dimensional Schrödinger operators with generalized Gaussian noise in terms of exponential functionals of Brownian local time.

We present two applications of this new semigroup theory. Firstly, we use our new Feynman-Kac formulas to provide the first method capable of proving the occurrence of a property called number rigidity in the spectrum of general random Schrödinger operators with irregular noise. Secondly, we study the convergence of discrete approximations of our Feynman-Kac formulas and their applications in proving limit laws for the extremal eigenvalues of random matrices.
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Chapter 1

Introduction

This introductory chapter is organized as follows: In Section 1.1, we provide some general background and motivation on the spectral and semigroup theories of random Schrödinger operators. In Section 1.2, we state the main problem studied in this thesis, namely, the semigroup theory of one-dimensional continuous random Schrödinger operators with generalized Gaussian noise. In Section 1.3, we state our main results regarding the latter. Then, in Sections 1.4 and 1.5, we discuss applications of our main results to the study of rigidity in the spectrum random Schrödinger operators and random matrix theory. Finally, we discuss in Section 1.6 a few research questions that are of immediate interest given the results in this thesis.

1.1 Background and Motivation

1.1.1 Schrödinger Operators

Let \(d \in \mathbb{N}\), and let \(I \subset \mathbb{R}^d\) be some subset of Euclidean space. A Schrödinger operator on \(I\), denoted by \(H\), is defined as an operator of the form

\[
Hf(x) := -c_H \Delta f(x) + V(x)f(x),
\] (1.1.1)
where

1. $H$ acts on a dense subset $D(H) \subset L^2(I)$ or $\ell^2(I)$ of square integrable or summable functions $f : I \to \mathbb{R}$, which is called the domain of $H$;

2. $V : I \to \mathbb{R}$ is a function called the potential; and

3. $c_H > 0$ is a positive constant and $\Delta$ is an appropriate Laplacian operator on $I$.

Throughout this thesis, every Schrödinger operator that we consider is assumed to be self-adjoint on its domain (e.g., [Sim15, Chapter 7]).

**Example 1.1.1.** If $I$ is continuous (for instance, a simply connected open set in $\mathbb{R}^d$), then the Laplacian operator is given by

$$
\Delta f(x) := \sum_{k=1}^{d} \partial^2_{x_k} f(x), \quad x = (x_1, \ldots, x_d).
$$

If $I$ is the vertex set of some lattice graph (such as a subset of the integer lattice $\mathbb{Z}^d$), then it is customary to take

$$
\Delta f(x) := \sum_{y \in I: y \sim x} (f(x) - f(y)), \quad (1.1.2)
$$

where $y \sim x$ indicates that $x$ and $y$ are adjacent in the lattice graph that induces $I$.

Schrödinger operators are fundamental objects of study in mathematics and physics, due in large part to their appearance in the Schrödinger equation and heat-type diffusion equations. As explained below, we note that in both of these applications, the spectral theory of $H$ (namely, the study of its eigenvalues and eigenfunctions) is of particular interest.
Application – Schrödinger Equation

(We refer to, e.g., [Tak08, Chapters 2 and 3] for more details on the physical concepts and terminology discussed here.) Let $i$ denote the imaginary unit. Up to minor changes of variables (to account for particles with different masses), the Schrödinger equation postulates that the wave function $w(t, x)$ ($t \in \mathbb{R}, \ x \in I$) of a system of nonrelativistic particles that are subjected to the potential $V$ satisfies

$$i \frac{\partial}{\partial t} w(t, x) = H w(t, x), \quad w(0, x) = w_0(x), \quad (1.1.3)$$

where $w_0$ is the initial condition. In this context, $-c_H \Delta$ is the kinetic energy operator and $V$ is the potential energy operator (the latter may include “external” contributions to the system depending only on the position $x \in I$, as well as “internal” contributions coming from interactions between particles); thus $H$ represents the Hamiltonian of the particle system.

Given that (1.1.3) can be reformulated as $w(t, x) = e^{-itH} w_0(x)$, studying the solutions of the Schrödinger equation is equivalent to the study of the unitary group of $H$, that is, the one-parameter family of operators $(e^{-itH})_{t \in \mathbb{R}}$. In this context, the solutions $(\lambda, \psi)$ of $H$’s eigenvalue problem

$$H \psi(x) = \lambda \psi(x), \quad x \in I, \quad (1.1.4)$$

are of physical significance, as the eigenvalues $\lambda$ in (1.1.4) represent the possible energy levels of the system, and the eigenfunctions $\psi$ represent the stationary states (the eigenvalue equation (1.1.4) implies that $e^{-itH} \psi = e^{-it\lambda} \psi$, which is oscillatory in $t$ with period $2\pi/\lambda$).
The diffusion equation with potential $V$ and initial condition $u_0$ is given by

$$\partial_t u(t, x) = -Hu(t, x), \quad u(0, x) = u_0(x), \tag{1.1.5}$$

where $t \geq 0$ and $x \in I$. In this case, the contribution of $c_H \Delta$ to $-H$ reflects the tendency of the solution $x \mapsto u(t, x)$ of (1.1.5) as a function of the “space variable” $x$ to smooth out its irregularities as the “time variable” $t$ increases. The pointwise multiplication with $-V$ allows for the quantity $u(t, x)$ (for every fixed $x \in I$) to increase or decrease in $t$ at a rate proportional to itself with constant $-V(x)$.

In the special case where $V = 0$, (1.1.5) corresponds to the classical heat equation due to Fourier (see, e.g., [Wid75, Chapter I] for a derivation from elementary principles), wherein $x \mapsto u(t, x)$ represents the distribution of heat at time $t$ in some physical body $I$. If $V \neq 0$, then (1.1.5) is often thought of as describing the evolution of particle diffusions with reproduction/kill rate $-V(x)$ depending on the environment (reproduction rate if $-V(x) > 0$ and kill rate if $-V(x) < 0$; e.g., [GM90]), though there are many more interesting applications for these types of equations.

Somewhat analogously to the Schrödinger equation, (1.1.5) can be reformulated as $u(t, x) = e^{-tH}u_0(x)$; hence the study of diffusion equations of the form (1.1.5) is intimately connected to understanding the semigroup of $H$, that is, the one-parameter family of operators $(e^{-tH})_{t>0}$. In particular, if $H$ is bounded below and has a compact resolvent (e.g., [RS78, Section XIII.14]), meaning that $H$’s eigenvalues $-\infty < \lambda_1(H) \leq \lambda_2(H) \leq \cdots \uparrow \infty$ are bounded below and without accumulation point, and that the matching eigenfunctions $\psi_1(H), \psi_2(H), \ldots$ form an orthonormal basis of $L^2(I)$ or $\ell^2(I)$, then the solution of (1.1.5) satisfies the spectral expansion

$$u(t, \cdot) = \sum_{k=1}^{\infty} e^{-t\lambda_k(H)} \langle \psi_k(H), u_0 \rangle \psi_k(H), \tag{1.1.6}$$
where \(\langle \cdot, \cdot \rangle\) denotes the standard inner product in \(L^2(I)\) or \(\ell^2(I)\). Therefore, a good understanding of the magnitude of \(H\)'s eigenvalues and the “shape” of its eigenfunctions may produce insights into the geometry of the solutions of the associated diffusion equation.

1.1.2 Schrödinger Semigroups

As alluded to in the previous section, the semigroup of a Schrödinger operator \(H\) is the family of operators \((e^{-tH})_{t>0}\). One of the most fundamental results in the semigroup theory of Schrödinger operators is the Feynman-Kac formula, which asserts that under very general assumptions on \(V\) and \(D(H)\), several features of the operator \(e^{-tH}\) admit explicit probabilistic representations that only involve classical stochastic processes. Omitting for the moment any specific mention of the assumptions required for the Feynman-Kac formula to hold, the following two examples illustrate the kinds of probabilistic representations involved in the descriptions of \(H\)'s semigroup:

**Example 1.1.2** (e.g., [Sim82, (A26)]). If \(I = \mathbb{R}^d\) and \(c_H = 1/2\) in (1.1.1), then for \(f \in L^2(\mathbb{R}^d)\), one has

\[
e^{-tH} f(x) = \mathbb{E}^x \left[ \exp \left( - \int_0^t V(B(s)) \, ds \right) f(B(t)) \right], \quad x \in \mathbb{R}^d,
\]

where \(B\) is a \(d\)-dimensional Brownian motion and \(\mathbb{E}^x\) signifies that we are taking the expected value with respect to \(B\) conditioned on the starting point \(B(0) = x\).

**Example 1.1.3** (e.g., [GM90, (2.1)]). If \(I = \mathbb{Z}^d\), \(c_H = 1/2d\), and we use the lattice Laplacian (1.1.2), then for \(f \in \ell^2(\mathbb{Z}^d)\), one has

\[
e^{-tH} f(x) = \mathbb{E}^x \left[ \exp \left( - \int_0^t V(B(s)) \, ds \right) f(B(t)) \right], \quad x \in \mathbb{Z}^d,
\]
where $B$ is a simple symmetric random walk on $\mathbb{Z}^d$ with i.i.d. exponential jump times, and $\mathbb{E}^x$ denotes the expected value with respect to $(B|B(0) = x)$.

There are many ways in which such probabilistic representations are fruitful: On the one hand, certain features of $H$’s spectrum can be accessed using tools from classical stochastic analysis. For example, if $H$’s semigroup is trace class (e.g., [RS80, Page 207]), then by combining the Feynman-Kac formula with the spectral expansion

$$\text{Tr}[e^{-tH}] = \sum_{k=1}^{\infty} e^{-\lambda_k(H)} , \quad t > 0,$$

we obtain that the Laplace transform $L(t) := \sum_k e^{-\lambda_k(H)}$ of $H$’s eigenvalues (which completely characterizes the spectrum) can be computed by probabilistic means. On the other hand, the Feynman-Kac formula allows for the explicit construction of solutions of diffusion equations of the form (1.1.5) using exponential functionals of Brownian motions, random walks, etc. We refer to [Sim82] and [CZ95] for more details on these kinds of applications.

**Remark 1.1.4.** In closing this section, we note that, apart from the benefit of making the spectral theory of Schrödinger operators amenable to probabilistic methods, the Feynman-Kac formula can in fact form the basis of the definition of $H$ itself, as done, for instance, in [McK77].

### 1.1.3 Multiplicative Noise

In many applications, it is natural to consider random perturbations of (1.1.1):

$$\hat{H}f(x) := Hf(x) + \xi(x)f(x), \quad (1.1.9)$$

where $\xi : I \to \mathbb{R}$ is a random function, which we call the noise. Indeed, $\xi$ allows to model disorder or impurities in the potential of the quantum models (1.1.3) or
diffusion equations (1.1.5). Operators of the form (1.1.9) are usually referred to as random Schrödinger operators with “multiplicative noise” to emphasize that $\xi$ multiplies the function on which $\hat{H}$ is acting (and thus distinguishes the present model from an “additive noise” perturbation $H f(x) + \xi(x)$). We refer to [AW15, CL90] for comprehensive expositions of the theory of such random Schrödinger operators in discrete and continuous space.

Following up on Sections 1.1.1 and 1.1.2, a problem of great interest in mathematical physics is to understand the structure of $\hat{H}$’s random spectrum, and the random Feynman-Kac formulas describing the semigroup $(e^{-t\hat{H}})_{t>0}$ are often useful tools in such investigations. Applications of the spectral theory of these kinds of random operators include the study of the dynamics of disordered quantum systems (e.g., [AW15, Chapter 2]) and the geometry of diffusion in stochastic heat-type equations (e.g., [Kön16, Sections 2.1 and 2.2]).

**Example 1.1.5.** Continuing on Examples 1.1.2 and 1.1.3 (where $I = \mathbb{R}^d$ and $\mathbb{Z}^d$, respectively), in the present setting of the random operator $\hat{H}$, the Feynman-Kac formula showcased in (1.1.7) and (1.1.8) can be extended to

$$e^{-t\hat{H}} f(x) = \mathbf{E}^x \left[ \exp \left( - \int_0^t V(Z(s)) + \xi(Z(s)) \, ds \right) f(B(t)) \right]$$

(1.1.10)

for all $x \in I$, where

1. $Z$ is the Brownian motion $B$ or the random walk $\mathcal{B}$, depending on context;

2. the noise $\xi$ is independent of the process $Z$; and

3. $\mathbf{E}^x$ now denotes the expectation with respect to the process $(Z|Z(0) = x)$ conditioned on $\xi$.  

7
1.2 Outline of Main Results

Our main purpose in this thesis is to lay out the foundations of a general semigroup theory (or Feynman-Kac formulas) for random Schrödinger operators of the form \((1.1.9)\) in the special case where

1. \(c_H = 1/2\) in \((1.1.1)\), and the set \(I\) on which \(H\) acts is a (possibly unbounded) one-dimensional open interval in \(\mathbb{R}\) of the form \(I = (a, b)\) with \(a \in [-\infty, \infty)\) and \(b \in (a, \infty]\); and

2. the noise \(\xi\) is a generalized stationary Gaussian noise on \(\mathbb{R}\).

Informally, we think of \(\xi\) as a centered Gaussian process on \(\mathbb{R}\) with a covariance of the form \(\mathbb{E}[\xi(x)\xi(y)] = \gamma(x - y)\), where \(\gamma\) is an even almost-everywhere-defined function or Schwartz distribution. In many cases that we consider, \(\gamma\) is not an actual function, and thus \(\xi\) cannot be defined as a random function on \(\mathbb{R}\). In such generality \(\xi\) is instead defined rigorously as a random Schwartz distribution, i.e., a centered Gaussian process on an appropriate function space with covariance

\[
\mathbb{E}[\xi(f)\xi(g)] = \int_{\mathbb{R}^2} f(x)\gamma(x - y)g(y) \, dx \, dy, \quad f, g : \mathbb{R} \to \mathbb{R}.
\]

Example 1.2.1. If \(\gamma = \sigma^2\delta_0\), where \(\sigma > 0\) and \(\delta_0\) denotes the delta Dirac distribution, then \(\xi\) is a Gaussian white noise with variance \(\sigma^2\). Informally, we think of white noise as the derivative of a Brownian motion \(W\) with variance \(\sigma^2\), i.e., \(\xi = W'\).

Since we consider very irregular noises (i.e., in general \(\xi\) is not a proper function that can be evaluated at points in \(\mathbb{R}\)), the semigroup theory of \(\hat{H}\) falls outside the purview of the classical theory, which relies crucially on the ability to evaluate the noise at every point. As a first step in this program, we show that a variety of tools and ideas recently developed in the random matrix theory literature (e.g., [BV13, GLS19, GS18b, KRV16, Min15, RRV11]) can be suitably extended to provide
Feynman-Kac formulas for $e^{-t\hat{H}}$ under fairly general conditions. The main restriction of our assumptions is that we consider cases where the semigroup $e^{-t\hat{H}}$ is trace class, which implies in particular that $\hat{H}$ must have a purely discrete spectrum.

Our main results regarding the semigroup theory of $\hat{H}$ are discussed in Section 1.3 and proved in details in Chapter 2. Then, in Sections 1.4 and 1.5, we present applications of this new semigroup theory in the study of rigidity properties in the spectrum of random Schrödinger operators, and the spectral convergence of random matrices. These applications are proved in detail Chapters 3 and 4 respectively.

1.3 Preview of Chapter 2. Semigroup Theory

The results and proofs in Chapter 2 are based on the paper [GL19b].

1.3.1 Problem of Irregular Noise

As mentioned in the previous section, much of the challenge involved in our program comes from the fact that, in general, Gaussian noises are Schwartz distributions. This creates two main technical obstacles.

The first obstacle is that it is not immediately obvious how to define the operator $\hat{H}$. Indeed, if we interpret $\xi$ as being part of the potential of $\hat{H}$, then the action

$$\hat{H} f(x) \doteq -\frac{1}{2} f''(x) + (V(x) + \xi(x)) f(x)$$

of the operator on a function $f$ includes the “pointwise product” $\xi(x)f(x)$, which is not well defined if $\xi$ cannot be evaluated at single points in $\mathbb{R}$. The second obstacle comes from the definition of $e^{-t\hat{H}}$. Arguably, the most natural guess for this semigroup
would be to use the Feynman-Kac formula (1.1.10), which yields
\[
e^{-t\hat{H}}f(x) = \mathbb{E}^x \left[ \exp \left(-\int_0^t V(B(s)) + \xi(B(s)) \; ds \right) f(B(t)) \right]. \tag{1.3.1}
\]
However, this again requires the ability to evaluate $\xi$ at every point.

### 1.3.2 Definition of the Operator

The key to overcoming these obstacles is to interpret $\xi$ as the distributional derivative of an actual Gaussian process. More precisely, let $\Xi$ be the Gaussian process on $\mathbb{R}$ defined as
\[
\Xi(x) := \begin{cases} 
\xi(1_{[0,x)}), & x \geq 0 \\
\xi(-1_{[x,0)}), & x \leq 0.
\end{cases} \tag{1.3.2}
\]
Assuming $\Xi$ has a version with continuous sample paths (and we neglect boundary values for simplicity), a formal integration by parts yields
\[
\xi(f) = \langle f, \Xi' \rangle := -\langle f', \Xi \rangle.
\]
Following this line of thought, we may then settle on a “weak” definition of $\hat{H}$ through a sesquilinear form: Letting $I = \mathbb{R}$ for simplicity, we expect that
\[
\langle f, \hat{H}g \rangle := \langle f, Hg \rangle + \xi(fg) = \langle f, Hg \rangle - \langle f'g + fg', \Xi \rangle. \tag{1.3.3}
\]
We note that this type of definition for $\hat{H}$ has previously appeared in the literature (e.g., [BV13, FN77, Min15, RRV11]) for various potentials $V$ on the half line $I = (0, \infty)$ as well as $V = 0$ on a bounded interval $I = (0, L)$ ($L > 0$). We also note an alternative approach outlined by Bloemendal in [Blo11, Appendix A] that allows one
(in principle) to recast $\hat{H}$ as the classical Sturm-Liouville operator

$$S f = -w^{-1}(\frac{w}{2}f')' + (V - 2\Xi^2)f, \quad \text{where } w(x) := \exp \left(4 \int_0^x \Xi(y) \, dy \right) \quad (1.3.4)$$

through a suitable Hilbert space isomorphism. Our first result is an extension of these statements:

**Proposition 1.3.1** (Informal Statement). Suppose that $V$ is bounded below, locally integrable, and grows faster than $\log |x|$ at infinity if $I \subset \mathbb{R}$ is unbounded. Suppose that $\xi = \Xi'$ in the sense of Schwartz distributions, where $\Xi$ is a continuous centered Gaussian process with stationary increments. Almost surely, there exists a unique self-adjoint operator $\hat{H}$ on $L^2(I)$ such that $\langle f, \hat{H} g \rangle$ is of the form (1.3.3). Our result holds for $I = \mathbb{R}$, as well as (possibly half-infinite) intervals $I \neq \mathbb{R}$ with a variety of boundary conditions.

A formal statement of this result is given in Proposition 2.1.6; we refer to Definition 2.1.3 for a definition of the sesquilinear form $\langle f, \hat{H} g \rangle$ in cases where $I \neq \mathbb{R}$.

### 1.3.3 Feynman-Kac Formula

With $\hat{H}$ defined by Proposition 1.3.1, we can then define $e^{-t\hat{H}}$ by functional calculus. As it turns out, the interpretation $\xi = \Xi'$ also leads to a natural candidate for a Feynman-Kac formula for the latter: Let $L^a_t(B)$ $(a \in \mathbb{R}, \ t \geq 0)$ be the local time process of the Brownian motion $B$ so that for any measurable function $f$, we have

$$\int_0^t f(B(s)) \, ds = \int_{\mathbb{R}} L^a_t(B) f(a) \, da.$$
Assuming a stochastic integral with respect to $\Xi$ can be meaningfully defined, we may then interpret the problematic term in $e^{-t\hat{H}}$’s intuitive derivation (1.3.1) thusly:

$$\int_0^t V(B(s)) + \xi(B(s)) \, ds := \int_\mathbb{R} L_t^\omega(B) \, dQ(a),$$

where $Q$ is the process $dQ(x) = V(x) \, dx + d\Xi(x)$, which we assume to be independent of $B$. In the case where $I = \mathbb{R}$, for example, this suggests that

$$e^{-t\hat{H}} f(x) = \mathbf{E}^\Xi \left[ \exp \left( - \int_\mathbb{R} L_t^\omega(B) \, dQ(a) \right) f(B(t)) \right], \quad (1.3.5)$$

where $\mathbf{E}^\Xi$ denotes the conditional expectation of $(B|B(0) = x)$ given $\Xi$. This type of Feynman-Kac formula has appeared in [GLS19, GS18b] in the special case where $I$ is the positive half line $(0, \infty)$, $V(x) = x$, and $\Xi$ is a Brownian motion (so that $\xi$ is a Gaussian white noise). Our second and main result is as follows:

**Theorem 1.3.2** (Informal Statement). *Let $V$ and $\xi = \Xi'$ be as in Proposition 1.3.1, with an additional technical assumption to ensure that a stochastic integral with respect to $\Xi$ can be meaningfully defined. Almost surely, $e^{-t\hat{H}}$ is a Hilbert-Schmidt/trace class operator for all $t > 0$. Moreover, for every $t > 0$, a Feynman-Kac formula of the form (1.3.5) holds with probability one.*

We provide a formal statement of this result in Theorem 2.1.19; we refer to (2.1.11) for a statement of our Feynman-Kac formula when $I \neq \mathbb{R}$. We provide in Section 2.2.2 a detailed exposition of our method of proof. In short, the argument consists of the following steps.

1. We introduce smooth approximations $\xi_\varepsilon$ ($\varepsilon > 0$) of the noise such that $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$ in the space of Schwartz distributions.
2. Since $\xi$ have smooth sample paths, we can use classical semigroup theory to show that the semigroup of the operator $\hat{H}_\varepsilon := H + \xi_\varepsilon$ admits a probabilistic representation given by the Feynman-Kac formula (e.g., (1.1.10)).

3. We show that $\hat{H}_\varepsilon$ converges to the operator $\hat{H}$ as defined through the sesquilinear form (1.3.3) and that the probabilistic representation of $e^{-t\hat{H}_\varepsilon}$ converges to the stochastic-integral-type Feynman-Kac formula on the right-hand side of (1.3.5), thus providing a probabilistic representation of $e^{-t\hat{H}}$.

1.3.4 Application - Laplace Transforms via Local Times

One interesting consequence of Theorem 2.1.19 is the following connection between the random functional (1.3.5) and the spectrum of $\hat{H}$: Let $\lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \cdots$ be the eigenvalues of $\hat{H}$ and $\psi_1(\hat{H}), \psi_2(\hat{H}), \ldots$ be the associated eigenfunctions, which are defined by the variational principle (i.e., Courant-Fischer) associated with the form (1.3.3). By Theorem 2.1.19, in many cases the spectral expansion

$$e^{-t\hat{H}}f = \sum_{k=1}^{\infty} e^{-t\lambda_k(\hat{H})}\langle \psi_k(\hat{H}), f \rangle \psi_k(\hat{H}), \quad f \in L^2(\mathbb{R})$$

admits an explicit probabilistic representation of the form (1.3.5). In particular, our Feynman-Kac formula provides a means of computing the “Laplace transforms”

$$E \left[ \prod_{i=1}^{\ell} \sum_{k=1}^{\infty} e^{-t_i \lambda_k(\hat{H})} \right] = E \left[ \prod_{i=1}^{\ell} \text{Tr}[e^{-t_i \hat{H}}] \right], \quad t_1, \ldots, t_\ell > 0, \quad (1.3.6)$$

which characterize the joint distribution of $\hat{H}$’s eigenvalues.

To give a specific example, given that

$$\int_{\mathbb{R}} f(x) \, d\Xi(x) \sim N \left( 0, \int_{\mathbb{R}^2} f(x) \gamma(x-y) f(y) \, dx dy \right)$$
for every deterministic function $f : \mathbb{R} \to \mathbb{R}$, straightforward computations using the Feynman-Kac formula (1.3.5) suggest that, in the special case $I = \mathbb{R}$,

$$
E \left[ \sum_{k=1}^{\infty} e^{-t\lambda_k(B)} \right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} E_{t}^{x,x} \left[ e^{-\int_\mathbb{R} L_t^y(B)\gamma da + \frac{1}{2} \int_\mathbb{R}^2 L_t^y(B)\gamma^2(\gamma - b)L_t^y(B)da db} \right] \, dx,
$$

where $E_{t}^{x,x}$ denotes the expected value with respect to the law of the Brownian bridge $(B|B(0) = x = B(t))$. The study of (1.3.6) is thus reduced to the computation of exponential functionals of Brownian local time.

In Sections 1.4 and 1.5, we discuss applications of the ability to compute (1.3.6) in the study of operator limits of random matrices and the occurrence of number rigidity in the spectrum of general random Schrödinger operators.

### 1.4 Preview of Chapter 3. Number Rigidity

The results and proofs in Chapter 3 are based on the paper [GLGY19], which is joint work with Promit Ghosal and Yuchen Liao.

#### 1.4.1 Spatial Conditioning and Number Rigidity

Let $\Lambda$ be a point process on Euclidean space $\mathbb{R}^d$, that is, a random locally finite counting measure on $\mathbb{R}^d$, which we think of as a random configuration of point particles in space. Point processes are well-studied in probability and mathematical physics [DVJ08, Kal17], due in large part to their many applications in varied disciplines (e.g., [BGM+06]).

Among the simplest examples of point processes is the Poisson process, which is such that $\Lambda(A)$ and $\Lambda(E)$ are independent whenever $A \cap E = \emptyset$. Thus the number of point particles inside of some region $A \subset \mathbb{R}^d$ is independent of the configuration of particles outside of $A$. In contrast, in many applications, one is instead interested in
point processes with interactions between distinct particles. In such situations, the notion of spatial conditioning, that is, the distribution of points inside a bounded set conditional on the point configuration outside the set, is of basic interest. Pioneering work on this subject includes the Dobrushin-Lanford-Ruelle (DLR) formalism (e.g., [Der18, Sections 1.4-2.4]), which describes spacial conditioning for Gibbs point processes. In Chapter 3, we are interested in a type of structure related to spacial conditioning called number rigidity.

**Definition 1.4.1** ([GP17a]). For any Borel set $A \subset \mathbb{R}$, we let $$F_\Lambda(A) := \sigma\{\Lambda(E) : E \subset A\}$$ denote the $\sigma$-algebra generated by the configuration of particles inside of $A$. $\Lambda$ is called number rigid if $\Lambda(A)$ is $F_\Lambda(\mathbb{R} \setminus A)$-measurable for every bounded Borel set $A \subset \mathbb{R}$.

In other words, $\Lambda$ is number rigid if for every bounded Borel set $A \subset \mathbb{R}^d$, the number of point particles inside of $A$ is completely determined by the configuration of particles outside of $A$. The earliest proof of number rigidity appears to be the work of Aizenman and Martin in [AM81] (see also [HS13]). More recently, there has been a notable increase of interest in this property stemming from the work of Ghosh and Peres [GP17b]. Therein, it is proved that the zero set of the planar Gaussian analytic function and the Ginibre process are number rigid. Since then, number rigidity has been shown to be connected to several other interesting properties of point processes (e.g., [Buf16a, BQ17a, BQ17b, Gho15, Gho16, GL17a, GL17b, GL18, PS14]), and understanding conditions under which number rigidity occurs has developed into an active field of research. In this context, the general problem in which we are interested in Chapter 3 is the following.

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Problem 1.4.2. Suppose that we define a random Schrödinger operator \( \hat{H} \) in such a way that it has compact resolvent, so that its eigenvalues

\[
-\infty < \lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \cdots \nearrow \infty
\]

form a point process. Under what conditions (if any) are \( \hat{H} \)’s eigenvalues number rigid?

Among other things, if \( \hat{H} \)’s eigenvalues are number rigid, then for every two energies \( E_1 < E_2 \), the number of energy levels of \( \hat{H} \) in the interval \([E_1, E_2] \) is completely determined by the configuration of energy levels outside of \([E_1, E_2] \). Consequently, in such a situation, the configuration of \( \hat{H} \)’s energy levels exhibits very nontrivial structure despite its randomness.

1.4.2 The Ghosh-Peres Criterion

Although several tools have been successfully used to prove number rigidity in different point processes (such as DLR equations [DHLM18] or deletion tolerance [PS14]), the most commonly used method consists of a simple sufficient condition by Ghosh and Peres based on the computation of linear statistics (e.g., [Buf16b, BNQ18, Gho15, GL17b, GP17a, RN18]).

Definition 1.4.3. Let \( f : \mathbb{R}^d \to \mathbb{R} \). The linear functional associated with \( f \), denoted \( \Lambda(f) \), is defined as

\[
\Lambda(f) := \int_{\mathbb{R}^d} f(x) \, d\Lambda(x).
\]

Proposition 1.4.4 ([GP17a]). Let \( A \subset \mathbb{R} \) be a bounded Borel set. Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence of functions satisfying the following conditions.

1. Almost surely, \( \Lambda(f_n \mathbf{1}_E) < \infty \) for every \( n \in \mathbb{N} \) and \( E \subset \mathbb{R} \).
2. \(|f_n - 1| \to 0 \) as \( n \to \infty \) uniformly on \( A \).
3. \( \text{Var}[\Lambda(f_n)] \to 0 \) as \( n \to \infty \).

Then, \( \Lambda(A) \) is \( \mathcal{F}_A(\mathbb{R} \setminus A) \)-measurable.

Thanks to this criterion, proving number rigidity can be reduced to controlling the variance of linear statistics that converge to 1 uniformly on compacts.

### 1.4.3 Rigidity for Airy-2

**Definition 1.4.5.** For every \( \beta > 0 \), let \( W_\beta \) be a Brownian motion with variance \( 4/\beta \).

We define the stochastic Airy operator with parameter \( \beta \) as

\[
\text{SAO}_\beta f(x) := -\Delta f(x) + xf(x) + W'_\beta(x)f(x),
\]

acting on functions of the interval \((0, \infty)\) with Dirichlet or Robin boundary condition at the origin.

To the best of our knowledge, until \[\text{GLGY19}\], there was only one random Schrödinger operator whose eigenvalues were known to be number rigid, namely:

**Theorem 1.4.6 ([Buf16b]).** The eigenvalues of \( \text{SAO}_2 \) with Dirichlet boundary condition are number rigid.

Indeed, \( \text{SAO}_2 \) with Dirichlet boundary condition has the property that its eigenvalues generate a determinantal point process known as the Airy-2 process. Thanks to this special algebraic structure, Bufetov had at his disposal a simple explicit formula for the variance of linear statistics, which he used to construct sequences of test functions that satisfy the Ghosh-Peres criterion in Proposition 1.4.4.

In light of Theorem 1.4.6, it is natural to wonder if rigidity occurs in more general random Schrödinger operators. However, the method used to prove Theorem 1.4.6 relies crucially on very special algebraic structure that is only present in \( \text{SAO}_2 \) with Dirichlet boundary conditions, and is thus ill-suited to answer this question. There
is therefore an interest in the development of methods capable of proving number rigidity in general random Schrödinger operators (i.e., methods that do not rely on special algebraic structure that is only present in one or a handful of examples).

### 1.4.4 Rigidity via Feynman-Kac

In Chapter 3, we introduce a method capable of proving number rigidity in a large class of random Schrödinger operators. Our method is based on a combination of the Ghosh-Peres criterion (Proposition 1.4.4) and the Feynman-Kac formula: Let \((t_n)_{n \in \mathbb{N}}\) be a vanishing sequence (as \(n \to \infty\)) of positive numbers. For every \(n \in \mathbb{N}\), define the test function \(f_n(x) := e^{-tnx}\). Given that \(f_n \to 1\) uniformly on every compact set, if \(\Lambda(f_n) < \infty\) and \(\text{Var}[\Lambda(f_n)] \to 0\) as \(n \to \infty\), then it follows from Proposition 1.4.4 that \(\Lambda\) is number rigid. If \(\Lambda\) is the eigenvalue point process of a random Schrödinger operator \(\hat{H}\), we note that \(\Lambda(f_n) = \text{Tr}[e^{-tn\hat{H}}]\). Consequently, we have the following sufficient condition for number rigidity in random Schrödinger operators:

**Corollary 1.4.7.** Let \(\hat{H}\) be a random Schrödinger operator with compact resolvent. If it holds that

\[
\lim_{t \to 0} \text{Var}[\text{Tr}[e^{-t\hat{H}}]] = 0, \quad (1.4.1)
\]

then \(\hat{H}\)'s eigenvalues are number rigid.

Then, by using the Feynman-Kac formula, we have an explicit probabilistic representation of \(\text{Var}[\text{Tr}[e^{-t\hat{H}}]]\) with which we can potentially prove (1.4.1).

Inspired by the fact that the one random Schrödinger operator previously known to be number rigid is SAO\(_2\), in Chapter 3 we use a combination of Theorem 1.3.2 and Corollary 1.4.7 to prove the number rigidity of the eigenvalues of a large class of one-dimensional continuous random Schrödinger operators with Gaussian noise:
Theorem 1.4.8 (Informal Statement). Let \( \hat{H}, V, \) and \( \xi \) be as in Theorem 1.3.2. When \( \hat{H} \) acts on a bounded interval, its eigenvalues are always number rigid. When \( \hat{H} \) acts on an unbounded interval, there is a constant \( c_\gamma > 0 \) that only depends on the covariance \( \gamma \) of the noise such that if

\[
\lim_{|x| \to \infty} V(x)/|x|^\gamma = \infty,
\]

then \( \hat{H} \)'s eigenvalues are number rigid.

We refer to Theorem 3.1.2 for a formal statement of this result, and to Theorem 3.1.4 for explicit lower bounds on the constant \( c_\gamma \) in several examples.

Remark 1.4.9. While Theorem 1.4.8 shows that using Corollary 1.4.7 allows to prove number rigidity for very general random Schrödinger operators, it is known that the sufficient condition (1.4.1) is not necessary for number rigidity (e.g., Proposition 3.1.6) in the spectrum of random Schrödinger operators. In particular, we do not recover the rigidity of SAO\(_2\) with Dirichlet boundary condition, since it can be proved that the limit (1.4.1) does not hold in that case. We refer to Section 3.1.2 for more details on the optimality of Theorem 1.4.8.

1.5 Preview of Chapter 4. Random Matrices

The results and proofs in Chapter 4 are based on the paper [GL19a].

1.5.1 Operator Limits of Random Matrices

The \( \beta \)-ensembles are a class of finite point processes \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) described by joint densities proportional to

\[
\left( \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^\beta \right) \left( \prod_{i=1}^n w_n(\lambda_i)^\beta \right),
\] (1.5.1)
where $\beta > 0$ represents the inverse temperature of the point particles $\lambda_i$, and $w_n$ is a so-called weight function. The $\beta$-ensembles have been of longstanding interest in mathematical physics due to them being simple examples of particle systems exhibiting repulsion (through the interaction term $(\lambda_j - \lambda_i)^\beta$); and for $\beta = 1, 2, 4$ and special choices of $w_n$, (1.5.1) is the joint eigenvalue density of random matrices that arise naturally in physics and statistics (see, e.g., the $\beta$-Hermite and $\beta$-Laguerre ensembles in [DE02, For10] and references therein).

In the last two decades, it was discovered that the fluctuations (as $n \to \infty$) of the largest/smallest particles of many $\beta$-ensembles are described by the eigenvalues of one-dimensional random Schrödinger operators of the form $-\frac{1}{2} \Delta + V + \xi$. More precisely, it was observed that:

1. ([DE02, KRV16]) for many choices of $w_n$, one can construct a $n \times n$ random tridiagonal matrix $T_n$ whose joint eigenvalue density is (1.5.1); and

2. ([ES07]) there exists a rescaling $H_n := \sigma_n (\mu_n - T_n)$ (the quantities $\sigma_n, \mu_n$ capture the size/location of the fluctuations of $T_n$'s extremal eigenvalues) for which we can naturally write a representation of the form

$$H_n = "-\frac{1}{2} \text{discrete Laplacian} + \text{deterministic potential} + \text{discrete noise}".$$ (1.5.2)

The question of convergence of random matrices of the form (1.5.2) to continuum Schrödinger operators was then systematically investigated in the seminal papers [BV13, RRV11]:

**Theorem 1.5.1** (Informal Statement; [BV13, RRV11]). Assume that

1. the “deterministic potential” part of $H_n$ converges to a function $V$;

2. the “discrete noise” part of $H_n$ converges to a Schwartz distribution $\xi = \Xi'$; and
3. additional technical conditions.

Let $\hat{H} = -\frac{1}{2} \Delta + V + \xi$, with $V$ and $\xi = \Xi'$ as in the above assumptions. Then, $H_n \to \hat{H}$ as $n \to \infty$, in the sense that $H_n$’s quadratic form converges to the quadratic form of $\hat{H}$ (i.e., (1.3.3)).

Moreover, this work identified the spectrum of the stochastic Airy operator (see Definition 1.4.5) as a universal limit describing the fluctuations of a large class of $\beta$-ensembles of natural interest. There is thus a strong incentive to develop tools that could contribute to a deeper understanding of the spectral properties of $\hat{H}$ (with an emphasis on the special case $\hat{H} = \frac{1}{2} \text{SAO}_\beta$), as well as questions of universality regarding their random matrix approximations.

1.5.2 Stochastic Semigroups

The first results regarding the semigroup theory of $\hat{H}$ (as discussed in Section 1.3) came from random matrix considerations in [GS18b]. Let us henceforth refer to the $\beta$-ensemble with weight function $w_n(\lambda) := e^{-\lambda^2/2}$ as $\beta$-Hermite.

**Theorem 1.5.2** (Informal Statement; [GS18b]). For every $n \in \mathbb{N}$, let $T_n^{(\beta)}$ be the random $n \times n$ tridiagonal matrix associated with $\beta$-Hermite. For every $t > 0$, let

$$M_n^{(\beta)}(t) := \frac{1}{2}(T_n^{(\beta)}/2\sqrt{n})^{\lfloor tn^{2/3} \rfloor} + \frac{1}{2}(T_n^{(\beta)}/2\sqrt{n})^{\lfloor tn^{2/3} \rfloor+1}. \quad (1.5.3)$$

As $n \to \infty$, it holds that $M_n^{(\beta)}(t) \to e^{-t\text{SAO}_\beta/2}$ ($t > 0$), where $\text{SAO}_\beta$ has a Dirichlet boundary condition at zero. The semigroup $(e^{-t\text{SAO}_\beta/2})_{t>0}$ is Hilbert-Schmidt/trace class and admits a random Feynman-Kac formula of the form (1.3.5).

The main technical innovation of [GS18b] consists of the development of a combinatorial method to analyze growing powers (i.e., on the order $n^{2/3}$) of the $\beta$-Hermite
matrices $T_{n}^{(\beta)}$. For this, the chief insight is to relate the convergence of these high powers with strong invariance principles for random walks and their local time. On a more conceptual level, Theorem 1.5.2 provides a new tool with which $\text{SAO}_{\beta}$ can be studied, namely, the random Feynman-Kac formula describing its semigroup $e^{-t\text{SAO}_{\beta}/2}$. Using this tool, Gorin and Shkolnikov uncovered a new property of $\text{SAO}_{\beta}$’s eigenvalue process that sheds some light on the special integrable structure that arises in $\beta$-Hermite for the value $\beta = 2$ [GS18b, Proposition 2.7 and Corollary 2.3]. These result were later extended in [GLS19] to rank-one additive perturbations of $\beta$-Hermite tridiagonal random matrices; the semigroup of $\text{SAO}_{\beta}$ with Robin boundary condition at zero arises as the limit of the latter.

### 1.5.3 General Convergence Result

In Chapter 4, we introduce a modification of the stochastic semigroup formalism developed in [GLS19, GS18b]. Our main results establish the convergence of high powers of a large class of random tridiagonal matrices to the semigroups of continuum Schrödinger operators with Gaussian white noise potential. Informally stated, our result is as follows.

**Theorem 1.5.3** (Informal Statement). With $H_n$ of the form (1.5.2), assume that

1. the “deterministic potential” part of $H_n$ converges to a function $V$;

2. the “discrete noise” part of $H_n$ converges to a white noise $\xi = W'$; and

3. additional technical conditions.

Define the random matrices

$$K_n(t) := (1 - H_n/3\kappa_n)^{[3\kappa_n t]}, \quad t > 0,$$

(1.5.4)
where \( \kappa_n \nearrow \infty \) is a suitably chosen diverging sequence. Let \( \hat{H} = -\frac{1}{2} \Delta + V + \xi \), with \( V \) and \( \xi = W' \) as in the above assumptions. Then, \( K_n(t) \to e^{-t\hat{H}} \) as \( n \to \infty \).

We refer to Theorems 4.1.9 and 4.1.10 for formal statements. Our results improve on Theorem 1.5.2/[GS18b] (and its extension in [GLS19]) in two significant ways.

Firstly, a notable feature of the combinatorial analysis in [GS18b] is that it requires the tridiagonal matrices under consideration to have diagonal entries of smaller order than their super/sub-diagonal entries (see Section 4.3.3 for more details). In particular, the argument in question is not directly applicable to the tridiagonal matrices of several classical \( \beta \)-ensembles, such as \( \beta \)-Laguerre (e.g., [DE02, For10]). In contrast, Theorem 1.5.1/[BV13, RRV11] applies to rather general random tridiagonal matrices, including \( \beta \)-Laguerre. In this context, one contribution of this chapter is to develop an improved version of the stochastic semigroup formalism that does not have restrictions on the relative size of the diagonal and off-diagonal entries (hence applicable to a much wider class of tridiagonal random matrices). As a demonstration of this greater generality, we prove that our main results apply to every matrix model considered in [GLS19, GS18b], as well as generalized \( \beta \)-Laguerre ensembles (see Section 4.2.1).

Secondly, a notable feature of our convergence results is that they apply to non-symmetric matrices. As a consequence, we can prove new limit laws for the eigenvalues of certain non-symmetric random tridiagonal matrices (Theorem 4.2.5). In particular, we identify a new matrix model whose edge fluctuations are in the stochastic Airy operator/Tracy-Widom universality class (Corollary 4.2.17). These results complement previous investigations regarding the spectrum of non-symmetric random tridiagonal matrices, such as [GS18a, GK00, GK03, GK05].

**Remark 1.5.4.** Although our informal statements of the hypotheses of Theorems 1.5.1 and 1.5.3 (numbered 1–3 therein) are essentially the same, the details of the actual statements are substantially different. In fact, there are several classes of random
tridiagonal matrices that can be treated by Theorem 1.5.1 and not Theorem 1.5.3, and vice versa. Thus, an interesting feature of our main result is the demonstration that, if one is interested in convergence of the random matrices (1.5.2) to Schrödinger operators with white noise, for certain classes of $H_n$, the semigroup approach is more natural. In particular, due to its crucial reliance on the Courant-Fischer variational principle, the convergence results of Theorem 1.5.1/[BV13, RRV11] can only be applied to symmetric matrices; Theorem 1.5.3 has no such restriction.

In light of Theorem 1.5.1, the hypotheses of Theorem 1.5.3 suggest that $H_n \to \hat{H}$ in some sense. Since $(1 - H_n/3\kappa_n)^{[3\kappa_nt]} \approx e^{-3tH_n/3} = e^{-tH_n}$, the conclusion of Theorem 1.5.3 is therefore expected. In fact, if we are given any sequence of functions $(F_{n,t})_n$ such that $F_{n,t}(x) \to e^{-tx}$ in a strong enough sense, then we also expect that $F_{n,t}(H_n) \to e^{-t\hat{H}}$. In this context, the main contribution of Theorem 1.5.3 is the identification of $F_{n,t}(x) = (1 - x/3\kappa_n)^{[3\kappa_nt]}$ as a particularly good choice of such a function, in that (1.5.4) is both amenable to tractable combinatorial analysis and applicable to very general tridiagonal matrices (which is not the case for arguably more “obvious” choices; see Remark 4.1.4 for more details).

Several features of the strategy of proof in [GLS19, GS18b] for analyzing the combinatorics of large powers of tridiagonal matrices carry over to this chapter. For instance, strong invariance principles for occupation measures of random walks also play a fundamental role in our proofs. That being said, the differences are significant enough that many nontrivial modifications and new ideas need to be introduced. Most notably, several results in the literature concerning strong approximations of Brownian local time that are used without modification in [GLS19, GS18b] require significant work to be applicable to our setting (Sections 4.4 and 4.5).
1.6 Some Future Work

1.6.1 Semigroup Theory in Higher Dimensions

In light of the results discussed in Section 1.3, it is natural to wonder if a similar theory can be developed for random Schrödinger operators with generalized Gaussian noise acting on domains in $\mathbb{R}^d$ for $d \geq 2$. However, the technical problems involved in the construction of $\hat{H}$ and $e^{-t\hat{H}}$ in those cases are significantly more challenging.

Looking for example at white noise in higher dimensions, while it is still possible to define quadratic forms similar to (1.3.3) for well-behaved test functions, the forms in question cannot be closed on a dense subspace of $L^2$ (see, for instance, the introduction of [Lab18]), and thus they do not correspond to self-adjoint operators. Furthermore, the nonexistence of local times for higher dimensional Brownian motions (at least as a process with enough regularity to be integrated with respect to a Brownian sheet) makes the corresponding generalizations of (1.3.5) non-obvious.

In some cases, random operators/PDEs with irregular noise in higher dimensions can be constructed by using more advanced techniques, such as regularity structures (e.g., [Hai14, Lab18]), paracontrolled calculus (e.g., [AC15, GIP15]), or other means (e.g., [HL15]). (These constructions typically involve the renormalization of smooth approximations of the operators/PDEs in question with diverging quantities.) Thus, while it is conceivable that a useful semigroup theory for such objects can be developed, it is expected that significantly new ideas will need to be introduced.

1.6.2 Rigidity Beyond 1D Continuous Operators

The sufficient condition for number rigidity stated in Corollary 1.4.7 is by no means unique to one-dimensional continuous random Schrödinger operators. Therefore, we believe that the results in Chapter 3 have the potential for substantial generalization. In future work, we intend to study the usefulness of Corollary 1.4.7 as a means to prove
eigenvalue rigidity for random Schrödinger operators acting on a variety of discrete lattices or continuous domains in $\mathbb{R}^d$ with $d \geq 2$. We expect that controlling the variance $\text{Var}[\text{Tr}[e^{-tH}]]$ in those cases may uncover interesting connections with the theory of random walk self-intersections, and the renormalization theory of singular operators/PDEs in higher dimensions.
Chapter 2

Semigroup Theory

2.1 Main Results

In this section, we provide detailed statements of our main results. Throughout this thesis, we make the following assumption regarding the interval $I$ on which the operator is defined and its boundary conditions.

**Assumption DB.** (Domain/Boundary) We consider three different types of domains: The full space $I = \mathbb{R}$ (Case 1), the positive half line $I = (0, \infty)$ (Case 2), and the bounded interval $I = (0, b)$ for some $b > 0$ (Case 3).

In Case 2, we consider Dirichlet and Robin boundary conditions at the origin:

\[
\begin{align*}
    f(0) &= 0 \quad \text{(Case 2-D)} \\
    f'(0) + \alpha f(0) &= 0 \quad \text{(Case 2-R)}
\end{align*}
\tag{2.1.1}
\]

where $\alpha \in \mathbb{R}$ is fixed.
In Case 3, we consider the Dirichlet, Robin, and mixed boundary conditions at the endpoints 0 and \( b \):

\[
\begin{align*}
&f(0) = f(b) = 0 \quad \text{(Case 3-D)} \\
&f'(0) + \alpha f(0) = -f'(b) + \beta f(b) = 0 \quad \text{(Case 3-R)} \\
&f'(0) + \alpha f(0) = f(b) = 0 \quad \text{(Case 3-M)}
\end{align*}
\] (2.1.2)

where \( \alpha, \beta \in \mathbb{R} \) are fixed.

**Remark 2.1.1.** Case 3-M should technically also include the mixed boundary conditions of the form \( f(0) = -f'(b) + \beta f(b) = 0 \). However, the latter can easily be obtained from Case 3-M by considering the transformation \( x \mapsto f(b - x) \).

Throughout the thesis, we make the following assumption on the potential \( V \).

**Assumption PG.** (Potential Growth) Suppose that \( V : I \mapsto \mathbb{R} \) is nonnegative and locally integrable on \( I \)'s closure. If \( I \) is unbounded, then we also assume that

\[
\liminf_{x \to \pm \infty} \frac{V(x)}{\log |x|} = \infty.
\] (2.1.3)

**Remark 2.1.2.** As is usual in the theory of Schrödinger operators and semigroups, the assumption that \( V \geq 0 \) is made for technical ease, and all of our results also apply in the case where \( V \) is merely bounded from below on \( I \).

### 2.1.1 Self-Adjoint Operator

Our first result concerns the realization of \( \hat{H} \) as a self-adjoint operator. As explained in the passage following equation (1.3.3), this is done through a sesquilinear form.

We assume that the Gaussian process \( \Xi \) driving the noise is as follows:

**Assumption FN.** (Form Noise) \( \xi = \Xi' \) in the sense of Schwartz distributions, where \( \Xi : \mathbb{R} \to \mathbb{R} \) is a centered Gaussian process such that
1. almost surely, $\Xi(0) = 0$ and $\Xi$ has continuous sample paths; and

2. $\Xi$ has stationary increments.

We now define the sesquilinear form for $\hat{H}$.

**Definition 2.1.3.** Let $L^2 = L^2(I)$ denote the set of square integrable functions on $I$, with its usual inner product and norm

$$\langle f, g \rangle := \int_I f(x)g(x) \, dx, \quad \|f\|_2 := \sqrt{\langle f, f \rangle}.$$  

Let $AC = AC(I)$ denote the set of functions that are locally absolutely continuous on $I$'s closure, and let

$$H^1_V = H^1_V(I) := \{f \in AC : \|f\|_2, \|f'\|_2, \|V^{1/2}f\|_2 < \infty\}.$$  

Our purpose in this definition is to introduce the following objects:

1. $(f, g) \mapsto \mathcal{E}(f, g)$, the sesquilinear form associated with the deterministic operator $H = \frac{1}{2}\Delta + V$.

2. $(f, g) \mapsto \xi(fg)$, the sesquilinear form associated with the noise.

3. $D(\mathcal{E}) \subset L^2$, the form domain on which $\mathcal{E}$ and $\xi$ are defined.

Then, we define $\hat{\mathcal{E}}(f, g) := \mathcal{E}(f, g) + \xi(fg)$, which is the sesquilinear form associated with $\hat{H}$. We now define these objects for every case in Assumption DB.

**Case 1:**

$$\begin{cases} 
D(\mathcal{E}) := H^1_V, \\
\mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle + \langle fg, V \rangle \\
\xi(fg) := -\langle f'g + fg', \Xi \rangle 
\end{cases}$$
Case 2-D:
\[
\begin{align*}
D(E) := & \{ f \in H^1_V : f(0) = 0 \} \\
\mathcal{E}(f, g) := & \frac{1}{2} \langle f', g' \rangle + \langle fg, V \rangle \\
\xi(fg) := & -\langle f'g + fg', \Xi \rangle
\end{align*}
\]
\[
D(E) := H^1_V.
\]
Case 2-R:
\[
\begin{align*}
D(E) := & \{ f \in H^1_V : f(0) = f(b) = 0 \} \\
\mathcal{E}(f, g) := & \frac{1}{2} \langle f', g' \rangle - \alpha f(0)g(0) + \langle fg, V \rangle \\
\xi(fg) := & -\langle f'g + fg', \Xi \rangle
\end{align*}
\]
Case 3-D:
\[
\begin{align*}
\mathcal{E}(f, g) := & \frac{1}{2} \langle f', g' \rangle + \langle fg, V \rangle \\
\xi(fg) := & -\langle f'g + fg', \Xi \rangle
\end{align*}
\]
\[
D(E) := H^1_V.
\]
Case 3-R:
\[
\begin{align*}
\mathcal{E}(f, g) := & \frac{1}{2} \langle f', g' \rangle - \frac{\alpha}{2} f(0)g(0) - \frac{\beta}{2} f(b)g(b) + \langle fg, V \rangle \\
\xi(fg) := & -\langle f'g + fg', \Xi \rangle + f(b)g(b)\Xi(b)
\end{align*}
\]
\[
D(E) := \{ f \in H^1_V : f(b) = 0 \}
\]
Case 3-M:
\[
\begin{align*}
\mathcal{E}(f, g) := & \frac{1}{2} \langle f', g' \rangle - \frac{\alpha}{2} f(0)g(0) + \langle fg, V \rangle \\
\xi(fg) := & -\langle f'g + fg', \Xi \rangle
\end{align*}
\]

Remark 2.1.4. While it is clear that the form \( \mathcal{E} \) is well defined on smooth and compactly supported functions, the same is not immediately obvious for our choices of \( D(\mathcal{E}) \). We prove in Proposition 2.2.2 that the form \( \mathcal{E} \) can be continuously extended to functions in \( D(\mathcal{E}) \), and thus is well defined on this domain.

Remark 2.1.5. As noted by Bloemendal and Virág in [BV13, Remark 2.5 and (2.11)], the Dirichlet boundary conditions can be specified in the form domain \( D(\mathcal{E}) \), but the Robin conditions must be enforced by the form itself, since the derivative of an
absolutely continuous function is only defined almost everywhere. Taking Case 3-R as an example, by a formal integration by parts we have

$$\int_{0}^{b} f(x)g''(x) \, dx = f(b)(-g'(b)) + f(0)g'(0) + \langle f', g' \rangle.$$

Substituting $g'(0) = -\alpha g(0)$ and $-g'(b) = -\beta g(0)$ then yields $E(f, g)$.

Our main result regarding $\hat{H}$’s definition with a form is as follows.

**Proposition 2.1.6.** Suppose that Assumptions DB, PG, and FN hold. Almost surely, there exists a unique self-adjoint operator $\hat{H}$ with dense domain $D(\hat{H}) \subset L^2$ such that the following conditions hold.

1. $D(\hat{H}) \subset D(E)$.

2. For every $f, g \in D(\hat{H})$, one has $\langle f, \hat{H}g \rangle = \hat{E}(f, g)$.

3. $\hat{H}$ has compact resolvent.

**Remark 2.1.7.** In Case 1, the statement of Proposition 2.1.6 is to the best of our knowledge completely new. In Case 2, the closest result is [Min15, Theorem 2], which assumes that $I = (0, \infty)$ with Dirichlet boundary condition, that $V$ is continuous, and that $\Xi$ is a fractional Brownian motion. In Case 3, the closest result seems to be [FN77, §2], which only considers the case $V = 0$ with Dirichlet boundary conditions and $\Xi$ a Brownian motion.

An immediate corollary of Proposition 2.1.6 is the ability to study the spectrum of $\hat{H}$ using the variational characterization coming from the form $\hat{E}$:

**Definition 2.1.8.** Let $A$ be a self-adjoint operator with discrete spectrum. We use $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ to denote the eigenvalues of $A$ in increasing order, and we use $\psi_1(A), \psi_2(A), \ldots$ to denote the associated eigenfunctions.
Corollary 2.1.9. Under the assumptions of Proposition 2.1.6,

1. $-\infty < \lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \cdots \nearrow +\infty$;
2. the $\psi_k(\hat{H})$ form an orthonormal basis of $L^2$; and
3. for every $k \in \mathbb{N}$,

$$
\lambda_k(\hat{H}) = \inf_{\psi \in D(\mathcal{E}), \psi \perp \psi_1(\hat{H}), \cdots, \psi_{k-1}(\hat{H})} \frac{\hat{\mathcal{E}}(\psi, \psi)}{\|\psi\|_2^2},
$$

with $\psi_k(\hat{H})$ being the minimizer of the above infimum with unit $L^2$ norm.

2.1.2 Semigroup

We now state our main result regarding the Feynman-Kac formula for the semigroup generated by $\hat{H}$. Thanks to Proposition 2.1.6 and Corollary 2.1.9, we know that under Assumptions DB, PG, and FN, the semigroup of $\hat{H}$ is the family of bounded self-adjoint operators with spectral expansions

$$
e^{-t\hat{H}} f = \sum_{k=1}^{\infty} e^{-t\lambda_k(\hat{H})} \langle \psi_k(\hat{H}), f \rangle \psi_k(\hat{H}), \quad t > 0, \; f \in L^2. \quad (2.1.4)$$

In order to state our Feynman-Kac formula for $e^{-t\hat{H}}$, we introduce some notations and further assumptions.

Preliminary Definitions

We begin with some preliminary definitions regarding the covariance of the noise $\xi$ and the stochastic processes required to define our Feynman-Kac kernels.

Definition 2.1.10 (Covariance). Let us denote by $\text{PC}_c = \text{PC}_c(I)$ the set of functions $f : I \mapsto \mathbb{R}$ that are càdlàg and compactly supported on $I$’s closure. We say that
$f \in \text{PC}_c$ is a step function if it can be written as

$$f = \sum_{i=1}^{k} c_i 1_{[x_i, x_{i+1})}, \quad c_i \in \mathbb{R}, \quad -\infty < x_1 < x_2 < \cdots < x_{k+1} < \infty. \quad (2.1.5)$$

To simplify forthcoming definitions and statements, we often extend the domain of $f \in \text{PC}_c$ to all of $\mathbb{R}$, with the convention that $f(x) = 0$ for all $x \notin I$.

Let $\gamma : \text{PC}_c \to \mathbb{R}$ be an even almost-everywhere-defined function or Schwartz distribution (even in the sense that $\langle f, \gamma \rangle = \langle rf, \gamma \rangle$ for every $f$, where $rf(x) = f(-x)$ denotes the reflection map), such that the bilinear map

$$\langle f, g \rangle_{\gamma} := \int_{\mathbb{R}^2} f(x)\gamma(x-y)g(y) \, dx \, dy, \quad f,g \in \text{PC}_c \quad (2.1.6)$$

is a semi-inner-product. We denote the seminorm induced by (2.1.6) as

$$\|f\|_{\gamma} := \sqrt{\langle f, f \rangle_{\gamma}}, \quad f \in \text{PC}_c.$$

**Remark 2.1.11.** If $\gamma$ is not an almost-everywhere-defined function, then the integral over $\gamma(x-y)$ in (2.1.6) may not be well defined. In such cases, we rigorously interpret (2.1.6) as $\langle f \ast vg, \gamma \rangle = \langle rf \ast g, \gamma \rangle$.

**Definition 2.1.12** (Stochastic Processes, etc.). We use $B$ to denote a standard Brownian motion on $\mathbb{R}$, $X$ to denote a reflected standard Brownian motion on $(0, \infty)$, and $Y$ to denote a reflected standard Brownian motion on $(0, b)$.

Let $Z = B$, $X$, or $Y$. For every $t > 0$ and $x, y \in I$, we define the conditioned processes

$$Z^x := (Z|Z(0) = x) \quad \text{and} \quad Z^x_{t} := (Z|Z(0) = x \ \text{and} \ Z(t) = y), \quad \text{for } \ t > 0.$$
and we use $E^x$ and $E^{x,y}_t$ to denote the expected value with respect to the law of $Z^x$ and $Z^{x,y}_t$, respectively.

We denote the Gaussian kernel by

$$G_t(x) := \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}, \quad t > 0, \ x \in \mathbb{R}.$$ 

We denote the transition kernels of $B$, $X$, and $Y$ as $\Pi_B$, $\Pi_X$, and $\Pi_Y$, respectively. That is, for every $t > 0$,

$$\Pi_B(t; x, y) := G_t(x - y) \quad x, y \in \mathbb{R},$$

$$\Pi_X(t; x, y) := G_t(x - y) + G_t(x + y) \quad x, y \in (0, \infty),$$

$$\Pi_Y(t; x, y) := \sum_{z \in 2b \pm y} G_t(x - z) \quad x, y \in (0, b).$$

Let $Z = B$, $X$, or $Y$. For any time interval $[u, v] \subset [0, \infty)$, we let $a \mapsto L^a_{[u,v]}(Z)$ ($a \in I$) denote the continuous version of the local time of $Z$ (or its conditioned versions) on $[u, v]$, that is,

$$\int_u^v f(Z(s)) \, ds = \int_I L^a_{[u,v]}(Z) f(a) \, da = \langle L_{[u,v]}(Z), f \rangle \quad (2.1.7)$$

for any measurable function $f : I \to \mathbb{R}$. In the special case where $u = 0$ and $v = t$, we use the shorthand $L_t(Z) := L_{[0,t]}(Z)$. When there may be ambiguity regarding which conditioning of $Z$ is under consideration, we use $L_{[u,v]}(Z^x)$ and $L_{[u,v]}(Z^{x,y}_t)$.

As a matter of convention, if $Z = X$ or $Y$, then we distinguish the boundary local time from the above, which we define as

$$L^c_{[u,v]}(Z) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_u^v 1_{\{c - \varepsilon < Z(s) < c + \varepsilon\}} \, ds$$

for $c \in \partial I$ (i.e., $c = 0$ if $Z = X$ or $c \in \{0, b\}$ if $Z = Y$), with $L^c_t(Z) := L^c_{[0,t]}(Z)$. 

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Remark 2.1.13. Since we use the continuous version of local time, \( a \mapsto L^a_{[u,v]}(Z) \) is continuous and compactly supported on \( I \)'s closure; in particular, \( L^a_{[u,v]}(Z) \in \text{PC}_c \).

Noise

We now articulate the assumptions that the noise \( \xi \) must satisfy for our Feynman-Kac formula to hold. We recall from the introduction that we think of \( \xi \) as a Gaussian process with covariance \( E[\xi(x)\xi(y)] = \gamma(x - y) \), with \( \gamma \) as in Definition 2.1.10. Interpreting \( \xi(f)" = \int_{\mathbb{R}} f(x)\xi(x) \, dx \) for a function \( f \), this suggests that, as a random Schwartz distribution, \( \xi \) is a Gaussian process with covariance \( E[\xi(f)\xi(g)] = \langle f, g \rangle_\gamma \).

In similar fashion to Assumption FN, we want to interpret \( \xi \) as the distributional derivative of some continuous process \( \Xi \), that is, corresponding to (1.3.2). If \( \xi \)'s covariance is given by the semi-inner-product \( \langle \cdot, \cdot \rangle_\gamma \), then this suggests that \( \Xi \)'s covariance is equal to

\[
E[\Xi(x)\Xi(y)] = \begin{cases} 
\langle 1_{[0,x)}, 1_{[0,y)} \rangle_\gamma & \text{if } x, y \geq 0 \\
\langle 1_{[0,x)}, -1_{[y,0)} \rangle_\gamma & \text{if } x \geq 0 \geq y \\
\langle -1_{[x,0)}, 1_{[0,y)} \rangle_\gamma & \text{if } y \geq 0 \geq x \\
\langle 1_{[x,0)}, 1_{[y,0)} \rangle_\gamma & \text{if } 0 \geq x, y.
\end{cases}
\]

(2.1.8)

This leads us to the following Assumption:

Assumption SN. (Semigroup Noise) The Gaussian process \( \Xi : \mathbb{R} \to \mathbb{R} \) satisfies Assumption FN. Moreover, there exists a \( \gamma : \text{PC}_c \to \mathbb{R} \) as in Definition 2.1.10 that satisfies the following conditions.

1. \( \Xi \)'s covariance is given by (2.1.8).
2. There exists $c_\gamma > 0$ and $1 \leq q_1, \ldots, q_\ell \leq 2$ such that

$$\|f\|_2^2 \leq c_\gamma \left( \|f\|_{q_1}^2 + \cdots + \|f\|_{q_\ell}^2 \right), \quad f \in \text{PC}_c,$$

(2.1.9)

where $\|f\|_q := \left( \int_{\mathbb{R}} \left| f(x) \right|^q \, dx \right)^{1/q}$ denotes the usual $L^q$ norm.

Then, for every $f \in \text{PC}_c$, we define

$$\xi(f) := \int_{\mathbb{R}} f(x) \, d\Xi(x),$$

(2.1.10)

where $d\Xi$ denotes stochastic integration with respect to $\Xi$ interpreted in the pathwise sense of Karandikar [Kar95] (see Section 2.2.2 for the details of this construction).

**Remark 2.1.14.** Though this is not immediately obvious from the above definition, the pathwise stochastic integral (2.1.10) actually coincides with $\xi(f)$ as defined in Definition 2.1.3 for every $f \in \mathbb{H}_V^1$. We note, however, that the extension of $\xi$ to $\text{PC}_c$ need not be linear on all of $\text{PC}_c$, and thus may not be a Schwartz distribution in the proper sense on that larger domain. Our interest in defining the stochastic integral in a pathwise sense is that it allows to construct $\xi$ as a random map from $\text{PC}_c$ to $\mathbb{R}$ that satisfies the following properties.

1. We can consider the conditional distribution of $\xi(L_t(Z))$ given a fixed realization of $\Xi$, assuming independence between $Z$ and $\Xi$.

2. $f \mapsto \xi(f)$ is a centered Gaussian process on $\text{PC}_c$ with covariance $\langle \cdot, \cdot \rangle_\gamma$.

We point to Remark 2.1.20, Section 2.2.2, and Appendix A.1 for more details.

**Remark 2.1.15.** The requirement that $\Xi$ be a continuous process with stationary increments in Assumption SN is redundant: Firstly, the covariance (2.1.8) implies that $\Xi(x) - \Xi(y)$ corresponds to $\xi(1_{[x,y]})$, which is stationary since the semi-inner-product $\langle \cdot, \cdot \rangle_\gamma$ is translation invariant. Secondly, if we construct $\Xi$ using abstract
existence theorems for Gaussian processes (which is possible since \( \langle \cdot, \cdot \rangle_\gamma \) is a semi-inner-product), then the assumption (2.1.9) implies that \( \Xi \) has a continuous version by Kolmogorov’s theorem for path continuity (see Section 2.2.3 for details). We nevertheless state these properties as assumptions for clarity.

Feynman-Kac Kernels

We now introduce the Feynman-Kac kernels that describe \( \hat{H} \)'s semigroup.

**Definition 2.1.16.** In Cases 2 and 3, let us define the quantities

\[
\bar{\alpha} := \begin{cases} 
-\infty & \text{(Case 2-D)} \\
\alpha & \text{(Case 2-R)} 
\end{cases} \\
(\bar{\alpha}, \bar{\beta}) := \begin{cases} 
(-\infty, -\infty) & \text{(Case 3-D)} \\
(\alpha, \beta) & \text{(Case 3-R)} \\
(\alpha, -\infty) & \text{(Case 3-M)}
\end{cases}
\]

where \( \alpha, \beta \in \mathbb{R} \) are as in (2.1.1) and (2.1.2). For every \( t > 0 \), we define the (random) kernel \( \hat{K}(t) : I^2 \to \mathbb{R} \) as

\[
\hat{K}(t; x, y) := \begin{cases} 
\Pi_B(t; x, y) \mathbb{E}_t^{x,y} \left[ e^{-\langle L_t(B), V \rangle - \xi(L_t(B))} \right] & \text{(Case 1)} \\
\Pi_X(t; x, y) \mathbb{E}_t^{x,y} \left[ e^{-\langle L_t(X), V \rangle - \xi(L_t(X)) + \alpha \xi(X)} \right] & \text{(Case 2)} \\
\Pi_Y(t; x, y) \mathbb{E}_t^{x,y} \left[ e^{-\langle L_t(Y), V \rangle - \xi(L_t(Y)) + \alpha \xi(Y) + \beta \xi(Y)} \right] & \text{(Case 3)}
\end{cases}
\]

where we assume that \( \Xi \) is independent of \( B, X, \) or \( Y \), and \( \mathbb{E}_t^{x,y} \) denotes the expected value conditional on \( \Xi \).

**Remark 2.1.17.** Let \( Z = X \) or \( Y \). In the above definition, we use the convention

\[
-\infty \cdot \mathcal{L}_i^c(Z) = \begin{cases} 
0 & \text{if } \mathcal{L}_i^c(Z) = 0 \\
-\infty & \text{if } \mathcal{L}_i^c(Z) > 0
\end{cases}
\]
for any $c \in \partial I$ as well as $e^{-\infty} = 0$. Thus, if we let $\tau_c(Z) := \inf\{t \geq 0 : Z(t) = c\}$ denote the first hitting time of $c$, then we can interpret $e^{-\infty} \mathcal{L}_t(Z) = 1_{\{\tau_c(Z) > t\}}$. In particular, if we remove the term $\xi(L_t(Z))$ from the kernel (2.1.11), then we recover the classical Feynman-Kac formula for the semigroup of $H$. See Section 2.4.1 for more details.

**Notation 2.1.18.** Given a Kernel $J : I^2 \to \mathbb{R}$ (such as $\hat{K}(t)$), we also use $J$ to denote the integral operator induced by the kernel, that is,

$$Jf(x) := \int_I J(x, y)f(y) \, dy.$$ 

We say that $J$ is Hilbert-Schmidt if $\|J\|_2 < \infty$, and trace class if $\text{Tr}[|J|] < \infty$.

**Main Result**

Our main result is as follows.

**Theorem 2.1.19** (Feynman-Kac Formula). Suppose that Assumptions DB, PG, and SN hold. Almost surely, $e^{-tH}$ is a Hilbert-Schmidt/trace class integral operator for every $t > 0$. Moreover, for every $t > 0$, the following holds with probability one.

1. $e^{-tH} = \hat{K}(t)$.
2. $\text{Tr}[e^{-tH}] = \int_I \hat{K}(t; x, x) \, dx < \infty$.

**Remark 2.1.20.** We point to Section 2.2.2 and Appendix A.1 for a justification of the well-posedness of the conditional expectation in (2.1.11) and that the kernel $\hat{K}(t)$ is Borel measurable, thus making quantities such as

$$\int_I \hat{K}(t; x, y) f(y) \, dy, \quad \int_{I^2} \hat{K}(t; x, y)^2 \, dx dy, \quad \text{and} \quad \int_I \hat{K}(t; x, x) \, dx$$

(where $f \in L^2$) well defined.
Remark 2.1.21. The closest analogs of Theorem 2.1.19 in the literature are [GLS19, Proposition 1.8 (a)] and [GS18b, Corollary 2.2], which concern Case 2 in the special case where $V(x) = x$ and $\Xi$ is a Brownian motion. All other cases are new.

2.1.3 Optimality and Examples

We finish Section 2.1 by discussing the optimality of the growth condition (2.1.3) in our results and by providing examples of covariance functions/distributions $\gamma$ that satisfy Assumption SN.

Optimality of Potential Growth

On the one hand, one of the key aspects of our proof of Proposition 2.1.6 for unbounded domains $I$ is to show that the growth rate of the squared increment process $x \mapsto (\Xi(x + 1) - \Xi(x))^2$ is dominated by $V$ as $|x| \to \infty$ (see (2.3.3) and the passage that follows). Given that the growth rate of stationary centered Gaussian processes (such as $\Xi(x + 1) - \Xi(x)$) is at most of order $\sqrt{\log|x|}$ (e.g., Corollary A.2.2), and that in many cases there is also a matching lower bound (e.g., Remark A.2.4), the growth condition (2.1.3) appears to be the best one can hope for with the method we use to prove Proposition 2.1.6. It would be interesting to see if this condition is necessary for $\hat{H}$ to have compact resolvent (perhaps by using the Sturm-Liouville interpretation (1.3.4)). That being said, for the deterministic operator $H = -\frac{1}{2}\Delta + V$ on $I = (0, \infty)$, it is well known that having a spectrum of discrete eigenvalues that are bounded below is equivalent to $\int_x^{x+\delta} V(y) \, dy \to \infty$ as $x \to \infty$ for all $\delta > 0$; hence it is natural to expect that $V$ must have some kind of logarithmic growth to balance the Gaussian potential.

On the other hand, condition (2.1.3) is necessary to have that $\mathbf{E}[\|\hat{K}(t)\|_2^2] < \infty$ for $t > 0$ close to zero, which is crucial in our proof of Theorem 2.1.19. Given that the deterministic semigroup $e^{-tH}$ is not trace class for small $t > 0$ when (2.1.3) does
not hold, we do not expect it is possible to improve Theorem 2.1.19 in that regard. We refer to Remark 2.4.24 for more details.

**Examples**

Given the simplicity of Assumption FN, it is straightforward to come up with examples of Gaussian noises to which Proposition 2.1.6 can be applied. In contrast, Assumption SN is a bit more involved. In what follows, we provide examples of covariance functions/distributions $\gamma$ that satisfy Assumption SN.

**Example 2.1.22.** Let $\gamma : \text{PC}_c \to \mathbb{R}$ be an even almost-everywhere-defined function or Schwartz distribution.

1. **(Bounded)** If $\gamma \in L^\infty(\mathbb{R})$, then we call $\xi$ a bounded noise. Depending on $\gamma$’s regularity, in many such cases $\xi$ can actually be realized as a Gaussian process on $\mathbb{R}$ with measurable sample paths and covariance $E[\xi(x)\xi(y)] = \gamma(x - y)$.

2. **(White)** If $\gamma = \sigma^2 \delta_0$ for some $\sigma > 0$, where $\delta_0$ denotes the delta Dirac distribution, then $\xi$ is a Gaussian white noise with variance $\sigma^2$. This corresponds to stochastic integration with respect to a two-sided Brownian motion $W$ with variance $\sigma^2$:

$$\xi(f) = \int_{\mathbb{R}} f(x) \, dW(x).$$

3. **(Fractional)** If $\gamma(x) := \sigma^2 \mathfrak{f}(2\mathfrak{f} - 1)|x|^{(2\mathfrak{f})-2}$ for $\sigma > 0$ and $\mathfrak{f} \in (1/2, 1)$, then $\xi$ is a fractional noise with variance $\sigma^2$ and Hurst parameter $\mathfrak{f}$. This noise corresponds to stochastic integration with respect to a two-sided fractional Brownian motion $W^{\mathfrak{f}}$ with variance $\sigma^2$ and Hurst parameter $\mathfrak{f}$:

$$\xi(f) = \int_{\mathbb{R}} f(x) \, dW^{\mathfrak{f}}(x).$$
4. (\(L^p\)-Singular) Let \(\ell \in \mathbb{N}\) and \(1 \leq p_1, \ldots, p_\ell < \infty\). As a generalization of bounded and fractional noise, we say that \(\xi\) is an \(L^p\)-singular noise if

\[
\gamma = \gamma_1 + \cdots + \gamma_\ell + \gamma_{\infty},
\]

where \(\gamma_i \in L^{p_i}(\mathbb{R})\) for \(1 \leq i \leq \ell\) and \(\gamma_{\infty} \in L^{\infty}(\mathbb{R})\). Indeed, the \(\gamma_i\) may have one or several \(p_i\)-integrable point singularities, such as \(\gamma_i(x) \sim |x|^{-\epsilon}\) as \(x \to 0\) for some \(\epsilon \in (0, 1/p_i)\), or \(\gamma_i(x) \sim (-\log |x|)^\epsilon\) as \(x \to 0\) for \(\epsilon > 0\).

Our last result in this Section is the following.

**Proposition 2.1.23.** For every covariance \(\gamma\) in Example 2.1.22, there exists a Gaussian process \(\Xi\) that satisfies Assumption SN.

### 2.2 Proof Outline

In this section, we provide an outline of the proofs of our main results. Most of the more technical results, which we state here as a string of propositions, are accounted for in Sections 2.3 and 2.4. Throughout Section 2.2, we assume that Assumptions DB and PG are met.

#### 2.2.1 Outline for Proposition 2.1.6

In this outline, we assume that Assumption FN holds. Let \(C_0^\infty = C_0^\infty(I)\) be the set of real-valued smooth functions \(\phi : I \to \mathbb{R}\) such that

1. \(\text{supp}(\phi)\) is a compact subset of \(I\) in Cases 1, 2-D, and 3-D;
2. \(\text{supp}(\phi)\) is a compact subset of \(I\)'s closure in Cases 2-R and 3-R; and
3. \(\text{supp}(\phi)\) is a compact subset of \([0, b)\) in Case 3-M.
We begin with a classical result in the theory of Schrödinger operators. (For definitions of the functional analysis terminology used in this section, we refer to [RS80, Section VIII.6], [Sim15, Section 7.5], or [Tes09, Section 2.3].)

**Proposition 2.2.1.** The form $E$ is closed and semibounded on $D(E)$, and $C_0^\infty$ is a form core for $E$. $H$ is the unique self-adjoint operator on $L^2$ whose sesquilinear form is $E$. Moreover, $H$ has compact resolvent.

The following is a generalization of a result that first appeared in [RRV11].

**Proposition 2.2.2.** $\xi(f^2) = -2\langle f', f, \Xi \rangle$ is finite and well defined for all $f \in D(E)$. Moreover, for every $\theta > 0$, there exists a finite random $c = c(\theta) > 0$ such that $|\xi(f^2)| \leq \theta E(f, f) + c\|f\|_2^2$ for all $f \in D(E)$.

Proposition 2.2.2 states that $\xi$ is an infinitesimally form-bounded perturbation of $E$. Therefore, according to the KLMN theorem (e.g., [RS75, Theorem X.17] or [Sim15, Theorem 7.5.7]), $\hat{E} = E + \xi$ is closed and semibounded on $D(E)$, and $C_0^\infty$ is a form core for $\hat{E}$. Thus, by [RS80, Theorem VIII.15], there exists a unique self-adjoint operator $\hat{H}$ satisfying conditions (1) and (2) in the statement of Proposition 2.1.6. Since $H$ has compact resolvent and $\hat{H}$ is infinitesimally form-bounded by $H$, the fact that $\hat{H}$ has compact resolvent follows from standard variational estimates (e.g., [RS78, Theorem XIII.68]).

### 2.2.2 Outline for Theorem 2.1.19

We now go over the outline of the proof of our main result. Throughout, we assume that Assumption SN holds. The outline presented here is separated into five steps. In the first step we provide details on the construction of the pathwise stochastic integral (2.1.10). In the second step, we introduce smooth-noise approximations of $\hat{H}$ and $\hat{K}(t)$ that serve as the basis of our proof of Theorem 2.1.19. Then, in the last three steps we prove Theorem 2.1.19 using these smooth approximations.
Step 1. Stochastic Integral

If \( f \in PC_c \) is a step function of the form (2.1.5), then we can define a pathwise stochastic integral in the usual way:

\[
\xi(f) = \int_{\mathbb{R}} f(x) \, d\Xi(x) := \sum_{i=1}^{k} c_i \left( \Xi(x_{i+1}) - \Xi(x_i) \right).
\]

Thanks to (2.1.8), straightforward computations reveal that for such \( f \) we have the isometry \( \mathbb{E} \left[ \xi(f)^2 \right] = \|f\|_\gamma^2 \). According to (2.1.9), step functions are dense in \( PC_c \) with respect to \( \|f\|_\gamma^2 \), and thus we may then uniquely define a stochastic integral \( \xi^*(f) \) for arbitrary \( f \in PC_c \) as the \( L^2(\Omega) \) limit of \( \xi(f_n) \), where \( f_n \) is a sequence of step functions that converges to \( f \) in \( \| \cdot \|_\gamma \) and \( L^2(\Omega) \) denotes the space of square integrable random variables on the same probability space on which \( \Xi \) is defined.

We now discuss how \( \xi(f) \) for general \( f \in PC_c \) can be defined in a pathwise sense as per Karandikar [Kar95]. Given \( f \in PC_c \), for every \( n \in \mathbb{N} \), define \( k(n) \) and 

\[
-\infty < \tau_{1}^{(n)} \leq \tau_{2}^{(n)} \leq \cdots \leq \tau_{k(n)+1}^{(n)} < \infty
\]

as the quantities

\[
\tau_{1}^{(n)} := \inf \{ x \in \mathbb{R} : f(x) \neq 0 \}, \quad \tau_{k(n)+1}^{(n)} := \sup \{ x \in \mathbb{R} : f(x) \neq 0 \}
\]

and

\[
\tau_{k}^{(n)} := \inf \{ x \geq \tau_{k-1}^{(n)} : |f(x) - f(\tau_{k-1}^{(n)})| \geq 2^{-n} \}, \quad 1 < k \leq k(n).
\]

Then, we define the approximate step function

\[
f^{(n)} := \sum_{k=1}^{k(n)} f\left( \tau_{k}^{(n)} \right) 1_{[\tau_{k}^{(n)}, \tau_{k+1}^{(n)}]}
\]
as well as the pathwise stochastic integral

$$\xi(f) = \int_{\mathbb{R}} f(x) \, d\Xi(x) := \begin{cases} 
\lim_{n \to \infty} \xi(f^{(n)}) & \text{if the limit exists} \\
0 & \text{otherwise.} 
\end{cases}$$  \ (2.2.1)

On the one hand, as argued in Appendix A.1 (see also [Kar95, Section 1]), the pathwise definition of $f \mapsto \xi(f)$ in (2.2.1) enables $\hat{K}(t)$’s definition as conditional expectation of $\xi(L_t(Z))$ given $\Xi$. On the other hand, $\xi(f)$ retains its meaning as a stochastic integral, since for every $f \in \text{PC}_c$, it holds that $\xi(f) = \xi^*(f)$ almost surely. Indeed, by combining the $L^2(\Omega)-\|\cdot\|_\gamma$ isometry of $\xi^*$, the definition of $\tau_k^{(n)}$, and (2.1.9), we get that

$$E[(\xi(f^{(n)}) - \xi^*(f))^2] = \|f^{(n)} - f\|_\gamma^2 \leq c_\gamma \left( \sum_{i=1}^{\ell} \|f^{(n)} - f\|_{q_i}^2 \right) \leq c_\gamma 2^{-2n} \left( \sum_{i=1}^{\ell} |\text{supp}(f)|^{2/q_i} \right);$$

since this is summable in $n$ we conclude that $\xi(f^{(n)}) \to \xi^*(f)$ almost surely, as desired.

**Remark 2.2.3.** Let $f \in \text{PC}_c$ be smooth and compactly supported. By a summation by parts, we note that

$$\xi(f^{(n)}) = \sum_{k=1}^{k(n)} f(\tau_k^{(n)}) \left( \Xi(\tau_{k+1}^{(n)}) - \Xi(\tau_k^{(n)}) \right) = -\sum_{k=2}^{k(n)} \Xi(\tau_k^{(n)}) \left( f(\tau_k^{(n)}) - f(\tau_{k-1}^{(n)}) \right)$$

for all $n \in \mathbb{N}$. Since $\Xi$ is continuous and $f$ is of bounded variation, the above converges to the usual Riemann-Stieltjes integral:

$$\lim_{n \to \infty} \xi(f^{(n)}) = -\int_{\mathbb{R}} \Xi(x) \, df(x) = -\int_{\mathbb{R}} f'(x) \Xi(x) \, dx.$$
In particular, the pathwise stochastic integral defined in (2.2.1) can be seen as an extension of the Schwartz distribution $\Xi'$ to all of $\text{PC}_c$, in the sense that

$$\xi(f) = -\langle f', \Xi \rangle \quad \text{for all } f \text{ smooth and compactly supported.} \quad (2.2.2)$$

However, as noted in an earlier remark, it is not clear that $\xi$ preserves its linearity in $f$ on all of $\text{PC}_c$.

**Step 2. Smooth Approximations**

A key ingredient in the proof of Theorem 2.1.19 consists of using smooth approximations of $\Xi'$ for which the classical Feynman-Kac formula can be applied, thus creating a connection between $\hat{H}$ as defined via a quadratic form and the kernels $\hat{K}(t)$.

**Definition 2.2.4.** Let $\varrho : \mathbb{R} \to \mathbb{R}$ be a mollifier, that is,

1. $\varrho$ is smooth, compactly supported, nonnegative, even (i.e., $\varrho(x) = \varrho(-x)$), and such that $\int \varrho(x) \, dx = 1$; and

2. if we define $\varrho_\varepsilon(x) := \varepsilon^{-1} \varrho(x/\varepsilon)$ for every $\varepsilon > 0$, then $\varrho_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ in the space of Schwartz distributions, where $\delta_0$ denotes the delta Dirac distribution.

For every $\varepsilon > 0$, we define the stochastic process $\Xi_\varepsilon := \Xi * \varrho_\varepsilon(x)$, where $*$ denotes the convolution.

**Remark 2.2.5.** Since $\varrho_\varepsilon$ is smooth, the process $\Xi'_\varepsilon = \Xi * (\varrho_\varepsilon)'$ has continuous sample paths. Thanks to (2.1.8), straightforward computations reveal that $\Xi'_\varepsilon$ is a stationary centered Gaussian process with covariance

$$E[\Xi'_\varepsilon(x)\Xi'_\varepsilon(y)] = E[\Xi * (\varrho_\varepsilon)'(x) \Xi * (\varrho_\varepsilon)'(y)]$$

$$= \int_{\mathbb{R}^2} E[\Xi(a)\Xi(b)] \varrho'_\varepsilon(a-x)\varrho'_\varepsilon(b-y) \, da \, db = (\gamma * \varrho_\varepsilon^2)(x-y) \quad (2.2.3)$$
for every $x, y \in \mathbb{R}$, where the last equality follows from integration by parts.

Moreover, following up on Remark 2.2.3, we note that the pathwise stochastic integral $\xi$ is coupled to the random Schwartz distribution

$$f \mapsto \int_{\mathbb{R}} f(x) \Xi'_\varepsilon(x) \, dx, \quad f \in \text{PC}_c$$

in the following way: For every $f \in \text{PC}_c$, the function $f * \varrho_\varepsilon$ is smooth and compactly supported, and thus by (2.2.2) we have that

$$\langle f, \Xi'_\varepsilon \rangle = \langle f, (\Xi * \varrho_\varepsilon)' \rangle = -\langle (f * \varrho_\varepsilon)', \Xi \rangle = \xi(f * \varrho_\varepsilon). \tag{2.2.4}$$

**Definition 2.2.6.** For every $\varepsilon > 0$, let us define the operator $\hat{H}_\varepsilon := H + \Xi'_\varepsilon$ along with its associated sesquilinear form

$$\hat{E}_\varepsilon(f, g) := E(f, g) + \langle fg, \Xi'_\varepsilon \rangle,$$

and the random kernel

$$\hat{K}_\varepsilon(t; x, y) := \begin{cases} 
\Pi_B(t; x, y) \mathbf{E}_t^{x,y} \exp[-(L_t(B) + \Xi'_\varepsilon)] & \text{(Case 1)} \\
\Pi_X(t; x, y) \mathbf{E}_t^{x,y} \exp[-(L_t(X) + \Xi'_\varepsilon) + \bar{\alpha}_0 B_t(x)] & \text{(Case 2)} \\
\Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \exp[-(L_t(Y) + \Xi'_\varepsilon) + \bar{\alpha}_1 \beta B_t(Y)] & \text{(Case 3)}
\end{cases}$$

Since $\Xi'_\varepsilon$ has regular sample paths, applying classical operator theory to $\hat{H}_\varepsilon$ yields the following result.

**Proposition 2.2.7.** For every $\varepsilon > 0$, the following holds almost surely.

1. $\hat{H}_\varepsilon$ is self-adjoint with compact resolvent.
2. For every $t > 0$, $e^{-t\hat{H}_\epsilon}$ is a self-adjoint Hilbert-Schmidt/trace class operator, and we have the Feynman-Kac formula $e^{-t\hat{H}_\epsilon} = \hat{K}_\epsilon(t)$. In particular,

\begin{align*}
\hat{K}_\epsilon(t; x, y) &= \hat{K}_\epsilon(t; y, x), & t > 0, \ x, y \in I; \\
\int_I \hat{K}_\epsilon(t; x, z)\hat{K}_\epsilon(t; z, y) \, dz &= \hat{K}_\epsilon(t + \bar{t}; x, y), & t, \bar{t} > 0, \ x, y \in I; \\
\hat{K}_\epsilon(t)f &= \sum_{i=1}^k e^{-t\lambda_k(\hat{H}_\epsilon)}\langle \psi_k(\hat{H}_\epsilon), f \rangle \psi_k(\hat{H}_\epsilon), & f \in L^2. \tag{2.2.7}
\end{align*}

Moreover, we can show that the objects introduced in Definition 2.2.6 serve as good approximations of $\hat{H}$ and $\hat{K}(t)$ in the following sense.

**Proposition 2.2.8.** Almost surely, every vanishing sequence in $(0, 1]$ has a further subsequence $(\epsilon_n)_{n \in \mathbb{N}}$ along which

\[
\lim_{n \to \infty} \lambda_k(\hat{H}_{\epsilon_n}) = \lambda_k(\hat{H}) \quad \text{and} \quad \lim_{n \to \infty} \|\psi_k(\hat{H}_{\epsilon_n}) - \psi_k(\hat{H})\|_2 = 0 \tag{2.2.8}
\]

for all $k \in \mathbb{N}$, up to possibly relabeling the eigenfunctions of $\hat{H}$ if it has repeated eigenvalues.

**Proposition 2.2.9.** For every $t > 0$, it holds that

\[
\lim_{\epsilon \to 0} \mathbb{E}\left[\|\hat{K}_\epsilon(t) - \hat{K}(t)\|_2^2\right] = 0 \tag{2.2.9}
\]

and

\[
\lim_{\epsilon \to 0} \mathbb{E}\left[\left(\int_I \hat{K}_\epsilon(t; x, x) - \hat{K}(t; x, x) \, dx\right)^2\right] = 0. \tag{2.2.10}
\]

**Step 3. Feynman-Kac Formula**

We are now in a position to prove Theorem 2.1.19. We begin by proving that for every $t > 0$, $e^{-t\hat{H}} = \hat{K}(t)$ almost surely. Let us fix some $t > 0$. By Proposition 2.2.9,
there is a probability one event and a vanishing sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) such that

\[
\lim_{n \to \infty} \|\hat{K}_{\varepsilon_n}(t) - \hat{K}(t)\|_2 = 0 \tag{2.2.11}
\]

and Proposition 2.2.8 hold. For the remainder of this step, we assume that we are working with an outcome in this probability one event. In particular, for such an outcome we can take a subsequence of \((\varepsilon_n)_{n \in \mathbb{N}}\) (which we also denote by \((\varepsilon_n)_{n \in \mathbb{N}}\) for simplicity) such that (2.2.8) holds.

Since the space \(L^2(I \times I)\) of Hilbert-Schmidt integral operators on \(L^2\) is complete, (2.2.11) means that \(\|\hat{K}(t)\|_2 < \infty\). In particular, \(\hat{K}(t)\) is compact. Furthermore, given that convergence in Hilbert-Schmidt norm implies weak operator convergence and every \(\hat{K}_{\varepsilon_n}(t) = e^{-t\hat{H}_{\varepsilon_n}}\) is nonnegative and symmetric, this implies that \(\hat{K}(t)\) is nonnegative and symmetric, hence self-adjoint (e.g., [Wei80, Theorems 4.28 and 6.11]). By the spectral theorem for compact self-adjoint operators (e.g., [Tes12, Theorems 5.4 and 5.6]), we then know that there exists a random orthonormal basis \((\Psi_k)_{k \in \mathbb{N}} \subset L^2\) and a point process \(\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \cdots \geq 0\) such that \(\hat{K}(t)\) satisfies

\[
\hat{K}(t)f = \sum_{k=1}^{\infty} \Lambda_k \langle \Psi_k, f \rangle \Psi_k, \quad f \in L^2.
\]

Consequently, to prove that \(e^{-t\hat{H}} = \hat{K}(t)\), we need only show that \(\hat{K}(t)\)’s spectral expansion is equivalent to (2.1.4).

On the one hand, since the Hilbert-Schmidt norm dominates the operator norm, it follows from (2.2.11) that \(\|\hat{K}_{\varepsilon_n}(t) - \hat{K}(t)\|_{op} \to 0\); hence \(e^{-t\lambda_k(\hat{H}_{\varepsilon_n})} \to \Lambda_k\) for all \(k \in \mathbb{N}\) by (2.2.7). Given that \(\lambda_k(\hat{H}_{\varepsilon_n}) \to \lambda_k(\hat{H})\) by (2.2.8), we conclude that \(\Lambda_k = e^{-t\lambda_k(\hat{H})}\)
for all $k \in \mathbb{N}$. On the other hand, we note that

$$
\| \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}_{\varepsilon_n}) - \hat{K}(t) \psi_k(\hat{H}) \|_2 \\
\leq \| \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}_{\varepsilon_n}) - \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}) \|_2 + \| \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}) - \hat{K}(t) \psi_k(\hat{H}) \|_2 \\
\leq \| \hat{K}_{\varepsilon_n}(t) \|_2 \| \psi_k(\hat{H}_{\varepsilon_n}) - \psi_k(\hat{H}) \|_2 + \| \hat{K}_{\varepsilon_n}(t) - \hat{K}(t) \|_2.
$$

This vanishes as $n \to \infty$ with probability one for all $k \in \mathbb{N}$. Moreover, the spectral expansion (2.2.7) and (2.2.8) imply that

$$
\lim_{n \to \infty} \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}_{\varepsilon_n}) = \lim_{n \to \infty} e^{-t\lambda_k(\hat{H}_{\varepsilon_n})} \psi_k(\hat{H}_{\varepsilon_n}) = e^{-t\lambda_k(\hat{H})} \psi_k(\hat{H})
$$
in $L^2$; hence $\hat{K}(t) \psi_k(\hat{H}) = e^{-t\lambda_k(\hat{H})} \psi_k(\hat{H})$. Thus $(e^{-t\lambda_k(\hat{H})}, \psi_k(\hat{H}))_{k \in \mathbb{N}}$ can be taken as the eigenvalue-eigenfunction pairs for $\hat{K}(t)$, concluding the proof that $\hat{K}(t) = e^{-t\hat{H}}$ almost surely.

**Step 4. Trace Formula**

Next, we prove Theorem 2.1.19 (2), that is, for every $t > 0$, $\text{Tr}[e^{-t\hat{H}}] = \int_I \hat{K}(t; x, x) \, dx < \infty$ almost surely. Let $t > 0$ be fixed. By Proposition 2.2.9, we can find a sparse enough vanishing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$
\lim_{n \to \infty} \| \hat{K}_{\varepsilon_n}(t/2) - \hat{K}(t/2) \|_2, \quad \lim_{n \to \infty} \left| \int_I \hat{K}_{\varepsilon_n}(t; x, x) - \hat{K}(t; x, x) \, dx \right| = 0 \quad (2.2.12)
$$

almost surely. Since $e^{-t\hat{H}}$ is by definition a semigroup, we have that

$$
\text{Tr}[e^{-t\hat{H}}] = \sum_{k=1}^{\infty} \left( e^{-t/2}\lambda_k(\hat{H}) \right)^2 = \| e^{-t/2}\hat{H} \|_2^2. \quad (2.2.13)
$$

Then, by combining the symmetry and semigroup properties (2.2.5) and (2.2.6), the almost sure convergences (2.2.12), and the almost sure equality $\hat{K}(t/2) = e^{-t/2}\hat{H}$
established in the previous step of this proof, we obtain that

\[
\|e^{-t/2}\hat{H}\|_2^2 = \|\hat{K}(t/2)\|_2^2 = \lim_{n \to \infty} \|\hat{K}_{\varepsilon_n}(t/2)\|_2^2
\]

\[
= \lim_{n \to \infty} \int I^2 \hat{K}_{\varepsilon_n}(t/2; x, y)^2 \, dy \, dx = \lim_{n \to \infty} \int_I \left( \int I \hat{K}_{\varepsilon_n}(t/2; x, y) \hat{K}_{\varepsilon_n}(t/2; y, x) \, dy \right) \, dx
\]

\[
= \lim_{n \to \infty} \int_I \hat{K}_{\varepsilon_n}(t; x, x) \, dx = \int_I \hat{K}(t; x, x) \, dx
\]

almost surely. Since we know that \(\|\hat{K}(t/2)\|_2 < \infty\) almost surely from the previous step, this concludes the proof of Theorem 2.1.19 (2).

**Step 5. Last Properties**

We now conclude the proof of Theorem 2.1.19 by showing that, almost surely, \(e^{-t\hat{H}}\) is a Hilbert-Schmidt/trace class integral operator for every \(t > 0\). By combining (2.2.13) with the fact that every Hilbert-Schmidt operator on \(L^2\) has an integral kernel in \(L^2(I \times I)\) (e.g., [Wei80, Theorem 6.11]), we need only prove that, almost surely, \(e^{-t\hat{H}}\) is trace class for all \(t > 0\).

In the previous step of this proof, we have already shown the weaker statement that, for every \(t > 0\), \(\text{Tr}[e^{-t\hat{H}}] < \infty\) almost surely. By a countable intersection we can extend this to the statement that there exists a probability-one event on which \(\text{Tr}[e^{-t\hat{H}}] < \infty\) for every \(t \in \mathbb{Q} \cap (0, \infty)\). Since \(\lambda_k(\hat{H}) \to \infty\) as \(k \to \infty\), there exists some \(k_0 \in \mathbb{N}\) such that \(\lambda_k(\hat{H}) > 0\) for every \(k > k_0\). Since \(\sum_{k=1}^{k_0} e^{-t\lambda_k(\hat{H})}\) is finite for every \(t\) and \(\sum_{k=k_0+1}^{\infty} e^{-t\lambda_k(\hat{H})}\) is monotone decreasing in \(t\), the fact that \(\text{Tr}[e^{-t\hat{H}}] < \infty\) holds for \(t \in \mathbb{Q} \cap (0, \infty)\) implies that it holds for all \(t > 0\), concluding the proof of Theorem 2.1.19.

**Remark 2.2.10.** In contrast to the proofs of [GLS19, Proposition 1.8 (a)] and [GS18b, Corollary 2.2] (which we recall apply to Case 2 with \(V(x) = x\) and white noise), the argument presented here uses smooth approximations of \(\hat{K}(t)\) rather than random matrix approximations. Since the present chapter does not deal with con-
vergence of random matrices, this choice is natural, and it allows to sidestep several technical difficulties involved with discrete models. With this said, the proof of (2.2.8) is inspired by the convergence result for the spectrum of random matrices in [BV13, Section 2] and [RRV11, Section 5]. We refer to Section 2.4 for the details.

2.2.3 Outline for Proposition 2.1.23

The main technical result in the proof of Proposition 2.1.23 is the following estimate, which first appeared in [GLGY19].

**Proposition 2.2.11.** Using the notations of Example 2.1.22, there exists a constant $c_\gamma > 0$ such that for every $f \in \text{PC}_c$, it holds that

$$
\|f\|^2_\gamma \leq \begin{cases} 
    c_\gamma \|f\|^2_1 & \text{(bounded noise)} \\
    c_\gamma \|f\|^2_2 & \text{(white noise)} \\
    c_\gamma (\|f\|^2_2 + \|f\|^2_1) & \text{(fractional noise with } \mathbb{H} \in (\frac{1}{2}, 1)) \\
    c_\gamma (\sum_{i=1}^\ell \|f\|^2_1 / (1 - 1/2p_i) + \|f\|^2_2) & \text{(L^p-singular noise with } p_i \geq 1). 
\end{cases}
$$

(2.2.14)

Whenever $\gamma$ is such that $\langle \cdot, \cdot \rangle_\gamma$ is a semi-inner-product, we know from standard existence theorems that there exists a Gaussian process $\Xi$ on $\mathbb{R}$ with covariance (2.1.8). As argued in Remark 2.1.15, such a process must have stationary increments. To see that such $\Xi$ have continuous versions, we note that for any $1 \leq q \leq 2$ and and $x < y$ such that $y - x \leq 1$, one has $\|1_{[x,y]}\|^4_q = (y - x)^{4/q}$ with $4/q > 1$. Thus, given that $1/(1 - 1/2p) \in (1, 2]$ for every $p \geq 1$, it follows from Proposition 2.2.11 that there exists some constants $c, r > 0$ such that

$$
\mathbb{E}[(\Xi(x) - \Xi(y))^4] = 3! \|1_{[x,y]}\|^4_\gamma \leq c |x - y|^{1+r}
$$

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for every $x < y \in \mathbb{R}$. The existence of a continuous version then follows from the classical Kolmogorov criterion (e.g., [MR06, Section 14.1]).

## 2.3 Proof of Proposition 2.1.6

In this section, we complete the proof of Proposition 2.1.6 outlined in Section 2.2.1 by proving Propositions 2.2.1 and 2.2.2.

### 2.3.1 Proof of Proposition 2.2.1

Much of this proof is standard functional analysis; hence several details are omitted.

To see that $E$ is closed on $D(E)$, simply note that Sobolev spaces and the $L^2$ space with measure $V(x)dx$ are complete. Since $V \geq 0$, $\|f\|_2^2 + \|V^{1/2}f\|_2^2 \geq 0$. This automatically implies that $E$ is semibounded (in fact, positive) in Cases 1, 2-D, and 3-D. In the other cases, semiboundedness follows from the next lemma.

**Lemma 2.3.1.** Suppose that $f \in AC \cap L^2$. For every $\kappa > 0$, there exists $c = c(\kappa) > 0$ such that $f(0)^2 \leq \kappa \|f\|_2^2 + c \|f\|_2^2$ and in Case 3 also $f(b)^2 \leq \kappa \|f\|_2^2 + c \|f\|_2^2$.

**Proof.** Consider first Cases 1 and 2. Since $f$ is continuous and square-integrable, $f(x) \to 0$ as $x \to \infty$, and thus

$$f(0)^2 \leq \int_{0}^{\infty} \left| \left( f(x)^2 \right)' \right| \, dx = 2 \int_{0}^{\infty} |f(x)||f'(x)| \, dx.$$  

The result then follows from the fact that for every $\kappa > 0$, we have the inequality $|z\bar{z}| \leq \frac{\kappa}{2} z^2 + \frac{1}{2\kappa} \bar{z}^2$. Suppose then that we are in Case 3. Let us define the function $h(x) = 1 - x/b$. Then,

$$f(0)^2 = f(0)^2 h(0) \leq \int_{0}^{b} \left| \left( f(x)^2 h(x) \right)' \right| \, dx \leq 2 \int_{0}^{b} |f(x)f'(x)| h(x) \, dx + \frac{1}{b} \|f\|_2^2.$$  

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Since $h \leq 1$, the same inequality used in Cases 1 and 2 yields the result. To prove the claim involving $f(b)$, we apply the same method with $h(x) = x/b$.

To see that $C_0^\infty$ is a form core, it suffices to note that the latter is dense in Sobolev spaces and $L^2(I, V(x)dx)$. The standard proof of this uses convolution against mollifiers and then smooth cutoff functions.

It only remains to prove that $H$ has compact resolvent. In Case 3, this property follows from the fact that $H$ is in this case a regular Sturm-Liouville operator. In Cases 1 and 2, this property follows from the fact that $V(x) \to \infty$ as $x \to \pm \infty$: Indeed, $H$ is in those cases limit point (e.g., [Zet05, Chapter 7.3 and Theorem 7.4.3]), and compactness of the resolvent is given by [RS78, Theorem XIII.67] or the Molchanov criterion as stated in [Zet05, Page 213].

### 2.3.2 Proof of Proposition 2.2.2

Since $C_0^\infty$ is a form core for $\mathcal{E}$, it suffices to prove the bound for $f \in C_0^\infty$; the bound can then be extended to $H_1^1$ by continuity. We claim that we need only prove that for every $\theta > 0$, there exists a random $c \in (0, \infty)$ such that

$$|\xi(f^2)| \leq \theta \left( \frac{1}{2} \|f'\|_2^2 + \|V^{1/2}f\|_2^2 \right) + c\|f\|_2^2.$$  

(2.3.1)

In Cases 1, 2-D, and 3-D, this is in fact equivalent to Proposition 2.2.2. To see how (2.3.1) implies the desired estimate in other cases, let us consider, for example, Case 2-R: According to (2.3.1),

$$|\xi(f^2)| \leq \tilde{\theta} \left( \frac{1}{2} \|f'\|_2^2 + \|V^{1/2}f\|_2^2 \right) + \bar{c} \|f\|_2^2 = \tilde{\theta} \mathcal{E}(f,f) + \frac{\tilde{a}^2}{2} f(0)^2 + \bar{c} \|f\|_2^2,$$
and then controlling $f(0)^2$ with Lemma 2.3.1 yields the desired estimate (with the straightforward substitution $\theta := \bar{\theta}(1 + \frac{\alpha}{2})$). Cases 3-R and 3-M can be dealt with in the same way.

Let us then prove (2.3.1). We begin with Cases 1 and 2. Following [Min15, RRV11], we define the integrated process

$$\tilde{\Xi}(x) := \int_x^{x+1} \Xi(y) \, dy, \quad x \in \mathbb{R}$$

so that we can write $\Xi(x) = \tilde{\Xi}(x) + (\Xi(x) - \tilde{\Xi}(x))$ and obtain

$$\xi(f^2) = -\langle 2f' f, \Xi \rangle = \langle f^2, \tilde{\Xi}' \rangle + 2\langle f' f, \tilde{\Xi} - \Xi \rangle$$

by an integration by parts. By Assumption FN, the processes $x \mapsto \tilde{\Xi}(x)$ and $x \mapsto \tilde{\Xi}(x) - \Xi(x)$ are continuous stationary centered Gaussian processes on $\mathbb{R}$, and thus it follows from Corollary A.2.2 that there exists a large enough finite random variable $C > 0$ such that, almost surely,

$$|\tilde{\Xi}'(x)|, (\Xi(x) - \tilde{\Xi}(x))^2 \leq C \log(2 + |x|) \quad (2.3.2)$$

for all $x \in I$. Since $V(x) \gg \log |x|$ as $|x| \to \infty$, for every $\theta > 0$, there exists $\bar{c}_1, \bar{c}_2 > 0$ depending on $\theta$ such that

$$C \log(2 + |x|) \leq \frac{\theta}{2}(\bar{c}_1 + V(x)), \quad \sqrt{C \log(2 + |x|)} \leq \frac{\theta}{2} \sqrt{\bar{c}_2 + V(x)} \quad (2.3.3)$$

for all $x \in I$. On the one hand, (2.3.2) and the above inequality imply that

$$\int_I f(x)^2|\tilde{\Xi}'(x)| \, dx \leq \frac{\theta}{2} \|V^{1/2} f\|_2^2 + \frac{\theta \bar{c}_1}{2} \|f\|_2^2.$$
On the other hand, the same inequalities and $|z\bar{z}| \leq \frac{1}{2}(z^2 + \bar{z}^2)$ imply

$$\int_I |f'(x)f(x)||\Xi(x) - \Xi(x)| \, dx \leq \frac{\theta}{2} \int_I |f'(x)f(x)|\sqrt{c_2 + V(x)} \, dx$$

$$\leq \frac{\theta}{2} \left( \int_I f'(x)^2 \, dx + \int_I f(x)^2(c_2 + V(x)) \, dx \right) \leq \frac{\theta}{2} \left( \|f'\|_2^2 + \|V^{1/2}f\|_2^2 \right) + \frac{\theta c_2}{2} \|f\|_2^2,$$

concluding the proof.

Suppose then that we are in Case 3. Since $\Xi$ is almost surely continuous by Assumption FN, the random variable $C := \sup_{0 \leq x \leq b} |\Xi(x)|$ is finite, and thus

$$|\xi(f^2)| \leq 2C \int_0^b |f'(x)||f(x)| \, dx + |\Xi(b)|f(b)^2.$$

The arguments of Lemma 2.3.1 then yield an upper bound of the form $\frac{\theta}{2}\|f'\|_2^2 + c\|f\|_2^2$, which is better than (2.3.1).

### 2.4 Proof of Theorem 2.1.19 & Proposition 2.1.23

In this section, we complete the outline of proof for Theorem 2.1.19 and Proposition 2.1.23 in Sections 2.2.2 and 2.2.3 by proving Propositions 2.2.7–2.2.9 and 2.2.11. This is done in Sections 2.4.6–2.4.10 below. Before we do this, however, we need several technical results regarding the deterministic semigroup $e^{-tH}$ and the behaviour of the local times $L_t(Z)$ and $\mathcal{L}_t(Z)$. This is done in Sections 2.4.1–2.4.5.

#### 2.4.1 Feynman-Kac Formula for Deterministic Operators

According to the classical Feynman-Kac formula, we expect that $e^{-tH} = K(t)$ for the kernels $K(t)$ defined as follows:
**Definition 2.4.1.** With the same notations as in Definitions 2.1.12 and 2.1.16, for every $t > 0$, we define the kernel $K(t) : I^2 \to \mathbb{R}$ as

$$K(t; x, y) := \begin{cases} 
\Pi_B(t; x, y) E^x_y \left[ e^{-\langle L_t(B), V \rangle} \right] & \text{(Case 1)} \\
\Pi_X(t; x, y) E^x_y \left[ e^{-\langle L_t(X), V \rangle + \bar{\alpha} L_0(t)} \right] & \text{(Case 2)} \\
\Pi_Y(t; x, y) E^x_y \left[ e^{-\langle L_t(Y), V \rangle + \bar{\alpha} L_0(t) + \bar{\beta} L_b(t)} \right] & \text{(Case 3)}
\end{cases} \quad (2.4.1)$$

To prove this, we begin with a reminder regarding the Kato class of potentials.

**Definition 2.4.2.** We define the Kato class, which we denote by $\mathcal{K} = \mathcal{K}(I)$, as the collection of nonnegative functions $f : I \to \mathbb{R}$ such that

$$\sup_{x \in I} \int_{\{y \in I : |x - y| \leq 1\}} f(y) \, dy < \infty. \quad (2.4.2)$$

We use $\mathcal{K}_{\text{loc}} = \mathcal{K}_{\text{loc}}(I)$ to denote the class of $f$’s such that $f1_K \in \mathcal{K}$ for every compact subset $K$ of $I$’s closure.

**Remark 2.4.3.** There is a large diversity of equivalent definitions of the Kato class, some of which are probabilistic. See, for instance, [Sim82, Section A.2].

**Theorem 2.4.4.** If $V \in \mathcal{K}_{\text{loc}}$, then $e^{-tH} = K(t)$ for all $t > 0$. Moreover,

$$K(t; x, y) = K(t; y, x), \quad t > 0, \ x, y \in I; \quad (2.4.3)$$

$$\int_I K(t; x, z) K(t; z, y) \, dz = K(t + \bar{t}; x, y), \quad t, \bar{t} > 0, \ x, y \in I. \quad (2.4.4)$$

**Proof.** The proof of $e^{-tH} = K(t)$ in Case 1 can be found in [Szn98, Theorem 4.9]. For Cases 2-D and 3-D, we refer to [CZ95, (34) and Theorem 3.27]. For Case 3-R, we have [Pap90, (3.3’) and (3.4), Theorem 3.4 (b), and Lemmas 4.6 and 4.7]. Though we expect $e^{-tH} = K(t)$ for Cases 2-R and 3-M to be folklore, the precise statement we were looking for was not found in the literature and is proved in Appendix A.3.
The proof of (2.4.3) and (2.4.4) could similarly only be found for Cases 1, 2-D, 3-D and 3-R in the literature; given that the same simple classical argument works for all cases, we prove the result in its entirety in Appendix A.3.

It is easy to see from (2.4.2) that locally integrable functions are in $\mathcal{K}_{loc}$ so that, by Assumption PG, $V \in \mathcal{K}_{loc}$. Therefore, we have the following immediate consequence of Theorem 2.4.4:

**Corollary 2.4.5.** Theorem 2.4.4 holds under Assumption PG.

### 2.4.2 Reflected Brownian Motion Couplings

The local time process of the Brownian motion $B$ is much more well studied than that of its reflected versions $X$ or $Y$. Thus, it is convenient to reduce statements regarding the local times of the latter into statements concerning the local time of $B$. In order to achieve this, we use the following couplings of $B$ with $X$ and $Y$.

**Half-Line**

For any $x > 0$, we can couple $B$ and $X$ in such a way that $X^x(t) = |B^x(t)|$ for every $t \geq 0$. In particular, for any functional $F$ of Brownian paths, one has

$$E^x[F(X)] = E^x[F(|B|)].$$

(2.4.5)

Under the same coupling, we observe that for every positive $x$, $y$, and $t$, one has

$$X^x_{t,y} \overset{d}{=} (|B^x| \mid B^x(t) \in \{-y, y\}).$$

Note that

$$P[B^x(t) = y \mid B^x(t) \in \{-y, y\}] = \frac{\mathcal{G}_t(x - y)}{\mathcal{G}_t(x - y) + \mathcal{G}_t(x + y)} = \frac{\Pi_B(t; x, y)}{\Pi_X(t; x, y)},$$

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and similarly,

\[ P[B^x(t) = -y \mid B^x(t) \in \{-y, y\}] = \frac{\Pi_B(t; x, -y)}{\Pi_X(t; x, y)}. \]

Therefore, for any path functional \( F \), it holds that

\[
\Pi_X(t; x, y) \mathbb{E}_t^{x,y}[F(X)] = \Pi_X(t; x, y) \mathbb{E}[F(|B^x|) \mid B^x(t) \in \{-y, y\}]
= \Pi_B(t; x, y) \mathbb{E}_t^{x,y}[F(|B|)] + \Pi_B(t; x, -y) \mathbb{E}_t^{x,-y}[F(|B|)].
\]

(2.4.6)

According to the strong Markov property and the symmetry about 0 of Brownian motion, we note the equivalence of conditionings

\[
(\mid B^x \mid \mid B^x(t) = -y) \overset{\text{d}}{=} (\mid B^x \mid \mid \tau_0(B^x) < t \text{ and } B^x(t) = y),
\]

(2.4.7)

where we define the hitting time \( \tau_0 \) as in Remark 2.1.17. Indeed, we can obtain the left-hand side of (2.4.7) from the right-hand side by reflecting \( (B^x \mid B^x(t) = -y) \) after it first hits zero and then taking an absolute value (see Figure 2.1 below for an illustration). Since

\[
P[\tau_0(B^x) < t \mid B^x(t) = y]^{-1} \Pi_B(t; x, -y) = e^{2xy/t} \Pi_B(t; x, -y) = \Pi_B(t; x, y)
\]

(this is easily computed from the joint density of the running maximum and current value of a Brownian motion [RY99, Chapter III, Exercise 3.14]), we see that

\[
\Pi_B(t; x, -y) \mathbb{E}_t^{x,-y}[F(|B|)] = \Pi_B(t, x, y) \mathbb{E}_t^{x,y}[1_{\{\tau_0(B) < t\}}F(|B|)].
\]
Thus (2.4.6) becomes

\[ \Pi_X(t; x, y) E_t^{x,y}[F(X)] = \Pi_B(t; x, y) E_t^{x,y}[(1 + 1_{\tau_0(B)<t}) F(|B|)]. \] (2.4.8)

Finally, given that \( \Pi_B(t; x, y)/\Pi_X(t; x, y) \leq 1 \), if \( F \geq 0 \), then (2.4.8) yields the inequality

\[ E_t^{x,y}[F(X)] \leq 2E_t^{x,y}[F(|B|)]. \] (2.4.9)

Figure 2.1: Reflection Principle: The path of \( B_t^{x,-y} \) (black) and its reflection after the first passage to zero (red).

**Bounded Interval**

For any \( x \in (0, b) \), we can couple \( Y^x \) and \( B^x \) by reflecting the path of the latter on the boundary of \( (0, b) \), that is,

\[ Y^x(t) := \begin{cases} B^x(t) - 2kb & \text{if } B^x(t) \in [2kb, (2k + 1)b], \quad k \in \mathbb{Z}, \\ |B^x(t) - 2kb| & \text{if } B^x(t) \in [(2k - 1)b, 2kb], \quad k \in \mathbb{Z}. \end{cases} \] (2.4.10)
(See Figure 2.2 below for an illustration of this coupling.) Under this coupling, it is clear that for any \( z \in (0, b) \), we have

\[
L_t^z(Y^x) = \sum_{a \in 2b\mathbb{Z} \pm z} L_t^a(B^x).
\]  (2.4.11)

Figure 2.2: Path of \( B^x \) (black) and its reflection on the boundary of \((0, b)\) (red).

### 2.4.3 Boundary Local Time

In this section, we control the exponential moments of the boundary local time of the reflected paths \( X \) and \( Y \).

**Lemma 2.4.6.** For every \( \theta, t > 0 \) and \( c \in \{0, b\} \), it holds that

\[
\sup_{x \in (0, \infty)} \mathbb{E}_x^x [e^{\theta \xi_t^0(X)}], \sup_{x \in (0, b)} \mathbb{E}_x^x [e^{\theta \xi_t^0(Y)}] < \infty.
\]  (2.4.12)

**Proof.** We begin by proving (2.4.12) in Case 2 (i.e., the process \( X \)). By (2.4.5) it suffices to prove that

\[
\sup_{x \in (0, \infty)} \mathbb{E}_x^x [e^{\theta \xi_t^0(B)}] < \infty
\]
for every $\theta, t > 0$, where

$$\mathcal{L}_t^0(B) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{-\varepsilon < B(s) < \varepsilon\}} \, ds.$$ 

On the one hand, by Brownian scaling, we have the equality in law

$$\mathcal{L}^0_t(B^x) \overset{d}{=} t^{1/2} \mathcal{L}^0_1(B^{x^{1/2}}_t). \tag{2.4.13}$$

On the other hand, according to [Pit99, (1)], for every $x, y \in \mathbb{R}$ and $\ell > 0$, one has

$$\mathbb{P}[\mathcal{L}_1^0(B^x_\cdot) \in d\ell, B^x(1) \in dy] = \frac{(|x| + |y| + \ell)e^{-((|x|+|y|+\ell)^2/2)}}{\sqrt{2\pi}} \, d\ell dy;$$

integrating out the $y$ variable then yields

$$\mathbb{P}[\mathcal{L}_1^0(B^x) \in d\ell] = \frac{2e^{-(|x|+\ell)^2/2}}{\sqrt{2\pi}}. \tag{2.4.14}$$

Thanks to (2.4.13) and (2.4.14), we see that

$$\sup_{x \in (0, \infty)} \mathbb{E}_t^{x}[e^{\theta \mathcal{L}_t^0(B)}] \leq \mathbb{E}_1^{0}[e^{\theta t^{1/2} \mathcal{L}_t^0(B)}] < \infty \tag{2.4.15}$$

for every $\theta, t > 0$; hence (2.4.12) holds in Case 2.

The proof of (2.4.12) for Case 3 (i.e., the process $Y$) follows directly from [Pap88, (2.18) and (3.11')], which states that there exists constants $K, K' > 0$ (depending on $\theta$) such that $\mathbb{E}_t^{x}[e^{\theta \mathcal{L}_t^0(Y)}] \leq K't^{Kt}$ for all $t > 0$ and $x \in (0, b)$. \qed

Next, we aim to extend the result of Lemma 2.4.6 to the local time of the bridge processes $Z^{x,x}_t$. Before we can do this, we need the following estimate on $\Pi_Z$. 61
Lemma 2.4.7. For every $t > 0$, it holds that

$$s_t(Z) := \sup_{x,y \in I} \frac{\Pi_Z(t/2; x, y)}{\Pi_Z(t; x, x)} < \infty. \quad (2.4.16)$$

Proof. In all three cases, $\Pi_Z(t; x, x) \geq 1/\sqrt{2\pi t}$, and thus it suffices to prove that $\sup_{x,y \in I} \Pi_Z(t; x, y) < \infty$. In Cases 1 & 2, this is trivial. In Case 3, we recall that, by definition,

$$\Pi_Y(t; x, y) := \sum_{z \in 2aZ \pm y} \mathbb{G}_t(x - z) = \frac{1}{\sqrt{2\pi t}} \left( \sum_{k \in \mathbb{Z}} e^{-(x+y-2bk)^2/2t} + e^{-(x-y-2bk)^2/2t} \right).$$

According to the integral test for series convergence, we note that for every $b, t > 0$ and $z \in \mathbb{R}$, it holds that

$$\sum_{k = \lceil -z/2b \rceil}^{\infty} \frac{e^{-(z+2bk)^2/2t}}{\sqrt{2\pi t}} \leq \frac{e^{-(z+2b[-z/2b])^2/2t}}{\sqrt{2\pi t}} + \int_{[-z/2b]}^{\infty} \frac{e^{-(z+2bu)^2/2t}}{\sqrt{2\pi t}} \, du \leq \frac{1}{\sqrt{2\pi t}} + \frac{1}{b},$$

and similarly for the sum from $k = -\infty$ to $[-z/2b]$. From this we easily obtain that $\sup_{(x,y) \in (0,b)^2} \Pi_Y(t; x, y) < \infty$. \hfill \Box

We finish this section with the following.

Lemma 2.4.8. For every $\theta, t > 0$ and $c \in \{0, b\}$, it holds that

$$\sup_{x \in (0, \infty)} \mathbb{E}^x_t \left[ e^{\theta \mathbb{G}_t^0(X)} \right], \sup_{x \in (0,b)} \mathbb{E}^x_t \left[ e^{\theta \mathbb{G}_t^c(Y)} \right] < \infty. \quad (2.4.17)$$

Proof. As it turns out, (2.4.17) follows from Lemma 2.4.6. The trick that we use to prove this makes several other appearances in this thesis: Since the exponential function is nonnegative, for every $\theta > 0$, an application of the tower property and the
Doob $h$-transform yields

$$
\mathbb{E}_t^{x,x} \left[ e^{\theta L_t^x(Z)} \right] = \mathbb{E} \left[ \mathbb{E}_t^{x,x} \left[ e^{\theta L_t^x(Z)} \big| Z_t^{x,x}(t/2) \right] \right]
$$

$$
= \int_I \mathbb{E}_t^{x,x} \left[ e^{\theta L_t^x(Z)} \big| Z_t^{x,x}(t/2) = y \right] \frac{\Pi_Z(t/2; x, y) \Pi_Z(t/2; y, x)}{\Pi_Z(t; x, x)} \, dy. \tag{2.4.18}
$$

If we condition on $Z_t^{x,x}(t/2) = y$, then the path segments

$$
(Z_t^{x,x}(s) : 0 \leq s \leq t/2) \quad \text{and} \quad (Z_t^{x,x}(s) : t/2 \leq s \leq t)
$$

are independent of each other and have respective distributions $Z_{t/2}^{x,y}$ and $Z_{t/2}^{y,x}$. Since $\Pi_Z(t/2; \cdot, \cdot)$ is symmetric for every $t > 0$, the time-reversed process $s \mapsto Z_{t/2}^{y,x}(t/2 - s)$ (with $0 \leq s \leq t/2$) is equal in distribution to $Z_{t/2}^{x,y}$. Thus,

$$
\mathbb{E}_t^{x,x} \left[ e^{\theta L_t^x(Z)} \big| Z_t^{x,x}(t/2) = y \right] = \mathbb{E}_t^{x,x} \left[ e^{\theta L_{[0,t/2]}(Z) + L_{[t/2,t]}(Z)} \big| Z_t^{x,x}(t/2) = y \right]
$$

$$
= \mathbb{E}_{t/2}^{x,y} \left[ e^{\theta L_{t/2}^x(Z)} \right]^2 \leq \mathbb{E}_{t/2}^{x,y} \left[ e^{2\theta L_{t/2}^x(Z)} \right], \tag{2.4.19}
$$

where the equality in (2.4.19) follows from independence and the fact that local time is invariant with respect to time reversal, and the last term in (2.4.19) follows from Jensen’s inequality.

Let us define the constant $s_t(Z) < \infty$ as in (2.4.16). According to (2.4.18) and (2.4.19), we then have that for $t > 0$,

$$
\mathbb{E}_t^{x,x} \left[ e^{\theta L_t^x(Z)} \right] \leq s_t(Z) \int_I \mathbb{E}_{t/2}^{x,y} \left[ e^{2\theta L_{t/2}^x(Z)} \right] \Pi_Z(t/2; x, y) \, dy = s_t(Z) \mathbb{E}_{t/2}^{x} \left[ e^{2\theta L_{t/2}^x(Z)} \right]. \tag{2.4.20}
$$

Hence the present result is a direct consequence of Lemma 2.4.6. \qed

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2.4.4 Self-Intersection Local Time

In this section, we obtain bounds on the exponential moments of the self-intersection local time of $B$, $X$, and $Y$. Such results for $B^x$ are well known (see, for instance, [Che10, Section 4.2]). For $X$ and $Y$ and the bridge processes, we rely on the couplings introduced in Section 2.4.2 and the midpoint sampling trick used in the proof Lemma 2.4.8, respectively. Before we state our result, we need the following.

**Lemma 2.4.9.** For every $\theta, u, v > 0$ and $q \geq 1$, it holds that

$$\sup_{x \in I} \mathbb{E}^x \left[ e^{\theta \| L_u + v (Z) \|_q^2} \right] \leq \left( \sup_{x \in I} \mathbb{E}^x \left[ e^{2\theta \| L_u (Z) \|_q^2} \right] \right) \left( \sup_{x \in I} \mathbb{E}^x \left[ e^{2\theta \| L_v (Z) \|_q^2} \right] \right)$$

**Proof.** Let $x, y \in I$ be fixed. Conditional on $Z^x(u) = y$, the path segments

$$\left( Z^x(s) : 0 \leq s \leq u \right) \quad \text{and} \quad \left( Z^x(u + t) : 0 \leq t \leq \infty \right)$$

are independent of each other and have respective distributions $Z^x_{u,y}$ and $Z^y$. Therefore, by the tower property, we have that

$$\mathbb{E}^x \left[ e^{\theta \| L_{u+v} (Z) \|_q^2} \right]$$

$$= \int_I \mathbb{E}^x \left[ e^{\theta \| L_{u+v} (Z) \|_q^2} \bigg| Z^x(u) = y \right] \Pi_Z(u; x, y) \ dy$$

$$\leq \int_I \mathbb{E}^x \left[ e^{2\theta \| L_u (Z) \|_q^2 + 2\theta \| L_{v, u+v} (Z) \|_q^2} \bigg| Z^x(u) = y \right] \Pi_Z(u; x, y) \ dy$$

$$= \int_I \mathbb{E}^{x,y} \left[ e^{2\theta \| L_u (Z) \|_q^2} \right] \mathbb{E}^y \left[ e^{2\theta \| L_v (Z) \|_q^2} \right] \Pi_Z(u; x, y) \ dy$$

$$\leq \left( \sup_{y \in I} \mathbb{E}^y \left[ e^{2\theta \| L_v (Z) \|_q^2} \right] \right) \int_I \mathbb{E}^{x,y} \left[ e^{2\theta \| L_u (Z) \|_q^2} \right] \Pi_Z(u; x, y) \ dy$$

$$= \left( \sup_{y \in I} \mathbb{E}^y \left[ e^{2\theta \| L_v (Z) \|_q^2} \right] \right) \mathbb{E}^x \left[ e^{2\theta \| L_u (Z) \|_q^2} \right],$$

where the inequality on the third line follows from the triangle inequality in $L^q$ with $(z + \bar{z})^2 \leq 2(z^2 + \bar{z}^2)$, and the equality on the fourth line follows from conditional
independence of the path segments. The result then follows by taking a supremum over $x$.

Lemma 2.4.10. Let $1 \leq q \leq 2$. For every $\theta, t > 0$, one has

$$\sup_{x \in I} E^x \left[ e^{\theta \|L_t(Z)\|_q^2} \right] < \infty. \quad (2.4.21)$$

Proof. We begin by noting that $\|L_t(Z)\|_1 = t$ by (2.1.7), and thus the result is trivial if $q = 1$. To prove the result for $1 < q \leq 2$, we claim that it suffices to show that there exists nonnegative random variables $R_1, R_2 \geq 0$ with finite exponential moments in some neighbourhood of zero, as well as constants $\kappa_1, \kappa_2 > 1$ such that

$$\sup_{x \in I} E^x \left[ e^{\theta \|L_t(Z)\|_q^2} \right] \leq E \left[ e^{\theta \kappa_1 R_1} \right] \quad (2.4.22)$$

or

$$\sup_{x \in I} E^x \left[ e^{\theta \|L_t(Z)\|_q^2} \right] \leq E \left[ e^{\theta \kappa_1 R_1} \right]^{1/2} E \left[ e^{\theta \kappa_2 R_2} \right]^{1/2} \quad (2.4.23)$$

for all $t > 0$. To see this, suppose (2.4.22) holds, and let $\theta_0 > 0$ be such that $E[e^{\theta R_1}] < \infty$ for all $\theta < \theta_0$. Then, for any fixed $\theta > 0$,

$$\sup_{x \in I} E^x \left[ e^{\theta \|L_t(Z)\|_q^2} \right] \leq E \left[ e^{\theta \kappa_1 R_1} \right] < \infty$$

for every $t < (\theta_0/\theta)^{1/\kappa_1}$. In particular, if $u, v \leq (\theta_0/2\theta)^{1/\kappa_1}$, we get from Lemma 2.4.9 that

$$\sup_{x \in I} E^x \left[ e^{\theta \|L_{u+v}(Z)\|_q^2} \right] \leq \left( \sup_{x \in I} E^x \left[ e^{2\theta \|L_u(Z)\|_q^2} \right] \right) \left( \sup_{x \in I} E^x \left[ e^{2\theta \|L_v(Z)\|_q^2} \right] \right) < \infty.$$ 

Thus, (2.4.21) now holds for $t < 2(\theta_0/2\theta)^{1/\kappa_1} = 2^{1-1/\kappa_1}(\theta_0/\theta)^{1/\kappa_1}$. Since $\kappa_1 > 1$, $2^{1-1/\kappa_1} > 1$, and thus by repeating this procedure infinitely often, we obtain by
induction that (2.4.21) holds for all \( t > 0 \), as desired. Essentially the same argument gives the result if we instead have (2.4.23).

We then prove (2.4.22)/(2.4.23). We argue on a case-by-case basis. Let us begin with Case 1. If we couple \( B^x = x + B^0 \) for all \( x \), then changes of variables with a Brownian scaling imply that

\[
\|L_t(B^x)\|_q^2 = \|L_t(B^0)\|_q^2 = t^2 \left( \int_\mathbb{R} L_t^{-1/2}(B^0)^q \, da \right)^{2/q} = t^{1+1/q} \|L_t(B^0)\|_q^2
\] (2.4.24)

for every \( q > 1 \). Thanks to the large deviation result [Che10, Theorem 4.2.1], we know that for every \( q > 1 \), there exists some \( c_q > 0 \) such that

\[
P[\|L_1(B^0)\|_q^2 > u] = e^{-c_q u^{q/(q-1)(1+o(1))}}, \quad u \to \infty. \tag{2.4.25}
\]

Thus, in Case 1 (2.4.22) holds with \( R_1 = \|L_1(B^0)\|_q^2 \) and \( \kappa_1 = 1 + 1/q \).

Consider now Case 2. By coupling \( X^x(t) = |B^x(t)| \) for all \( t > 0 \), we note that for every \( a > 0 \), one has \( L_t^a(X^x) = L_t^a(|B^x|) = L_t^a(B^x) + L_t^{-a}(B^x) \). Therefore,

\[
\|L_t(X^x)\|_q^2 = \left( \int_0^\infty L_t^a(X^x)^q \, da \right)^{2/q} \leq 2^{2(q-1)/q} \left( \int_0^\infty L_t^a(B^x)^q + L_t^{-a}(B^x)^q \, da \right)^{2/q} = 2^{2(q-1)/2} \|L_t(B^x)\|_q^2 \tag{2.4.26}
\]

Thus, the proof in Case 2 follows from Case 1.

Finally, consider Case 3. Recall the coupling of \( Y^x \) and \( B^x \) in (2.4.10), which yields the local time identity (2.4.11). The argument that follows is inspired from the
proof of [CL04, Lemma 2.1]: Under the coupling (2.4.11),
\[
\left( \int_0^b L_t^z(Y^z)^q \, dz \right)^{1/q} = \left( \int_0^b \left( \sum_{k \in 2\mathbb{N}} L_t^{k+z}(B^x) + L_t^{k-z}(B^x) \right)^q \, dz \right)^{1/q} \\
\leq 2^{(q-1)/q} \sum_{k \in 2\mathbb{N}} \left( \int_{-b}^b L_t^{k+z}(B^x)^q \, dz \right)^{1/q}.
\]

Let us denote the maximum and minimum of $B^x$ as
\[
M^x(t) := \sup_{s \in [0,t]} B^x(s) \quad \text{and} \quad m^x(t) := \inf_{s \in [0,t]} B^x(s).
\]

In order for the integral $\int_{-b}^b L_t^{k+z}(B^x)^2 \, dz$ to be different from zero, it is necessary that $M^x(t) \geq k - b$ and $m^x(t) \leq k + b$, that is, $M^x(t) + b \geq k \geq m^x(t) - b$. Consequently, for every $q > 1$, one has
\[
\sum_{k \in 2\mathbb{N}} \left( \int_{-b}^b L_t^{k+z}(B^x)^q \, dz \right)^{1/q} \\
= \sum_{k \in 2\mathbb{N}} \left( \int_{-b}^b L_t^{k+z}(B^x)^q \, dz \right)^{1/q} 1_{\{M^x(t) + b \geq k \geq m^x(t) - b\}} \\
\leq \left( \sum_{k \in 2\mathbb{N}} \int_{-b}^b L_t^{k+z}(B^x)^q \, dz \right)^{1/q} \left( \sum_{k \in 2\mathbb{N}} 1_{\{M^x(t) + b \geq k \geq m^x(t) - b\}} \right)^{q-1/q} \\
= \left( \int_{\mathbb{R}} L_t^a(B^x)^q \, da \right)^{1/q} \left( \sum_{k \in 2\mathbb{N}} 1_{\{M^x(t) + b \geq k \geq m^x(t) - b\}} \right)^{q-1/q} \\
\leq c_1 t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{q-1/q} (M^x(t) - m^x(t) + c_2)^{q-1/q} \\
\leq c_1 t^{1/q} \left( c_2 \left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{q-1/q} + \left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \cdot (M^x(t) - m^x(t)) \right)^{q-1/q} \right)
\]

where $c_1, c_2 > 0$ only depend on $b$ and $q$: The inequality on the third line follows from Hölder’s inequality; the equality on the fourth line follows from the fact that $\sum_{k \in 2\mathbb{N}} \int_{-b}^b L_t^a(B^x)^q \, da$ is equal to $\int_{\mathbb{R}} L_t^a(B^x)^q \, da$; the inequality on the fifth line follows from the fact that $\int_{\mathbb{R}} L_t^a(B^x)^q \, da$ is bounded by $\left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{q-1} \| L_t(B^x) \|_1$, 

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where $\|L_t(B^x)\|_1 = t$; and the inequality on the last line follows from the fact that 

$$
(M^x(t) - m^x(t) + c_2)^{\frac{q-1}{q}} \leq (M^x(t) - m^x(t) + c_2)^{\frac{q-1}{q}} + c_2^{\frac{q-1}{q}}.
$$

By Brownian scaling and translation invariance, we have that 

$$
t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{\frac{q-1}{q}} \overset{d}{=} t^{1/2 + 1/2q} \left( \sup_{a \in \mathbb{R}} L_1^a(B^0) \right)^{\frac{q-1}{q}}
$$

and 

$$
t^{1/q} \left( \sup_{a \in \mathbb{R}} L_t^a(B^x) \cdot (M^x(t) - m^x(t)) \right)^{\frac{q-1}{q}} \overset{d}{=} t \left( \sup_{a \in \mathbb{R}} L_1^a(B^0) \cdot (M^0(1) - m^0(1)) \right)^{\frac{q-1}{q}}.
$$

Given that $4(\frac{q-1}{q}) \leq 2$ for all $q \in (1, 2]$ and that there exists $\theta_0 > 0$ small enough so that 

$$
E \left[ \exp \left( \theta_0 \sup_{a \in \mathbb{R}} L_1^a(B^0)^2 \right) \right], E \left[ \exp \left( \theta_0 (M^0(1) - m^0(1))^2 \right) \right] < \infty,
$$

(e.g., the proof of [CL04, Lemma 2.1] and references therein) we finally conclude by Hölder’s inequality that (2.4.23) holds in Case 3 with 

$$
R_1 = 4c_1^2 \sup_{a \in \mathbb{R}} L_1^a(B^0)^{2\frac{q-1}{q}}, \quad R_2 = 4 \left( \sup_{a \in \mathbb{R}} L_1^a(B^0) \cdot (M^0(1) - m^0(1)) \right)^{2\frac{q-1}{q}},
$$

and $\kappa_1 = 1 + 1/q$ and $\kappa_2 = 2$. 

\begin{proof}

\end{proof}

\begin{lemma}

Let $1 \leq q \leq 2$. For every $\theta, t > 0$, one has 

$$
\sup_{x \in I} \mathbb{E}^{x,x}_t \left[ e^{\theta \|L_t(Z)\|_q^2} \right] < \infty.
$$

\end{lemma}

\begin{proof}

Once again, the present result follows from Lemma 2.4.10. To see this, we use the same trick employed in the proof of Lemma 2.4.8: For every $\theta > 0$, the tower property and the Doob $h$-transform yields 

$$
\mathbb{E}^{x,x}_t \left[ e^{\theta \|L_t(Z)\|_q^2} \right] = \int_I \mathbb{E}^{x,x}_t \left[ e^{\theta \|L_t(Z)\|_q^2} | Z_t^{x,x}(t/2) = y \right] \frac{\Pi_Z(t/2; x,y) \Pi_Z(t/2; y,x)}{\Pi_Z(t; x,x)} dy.
$$

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Arguing as in the passage following (2.4.18),

\[
\mathbb{E}_t^{x,x}[e^{\theta \|L_t(Z)\|_2^2} | Z^{x,x}_t(t/2) = y] = \mathbb{E}_t^{x,x}[e^{\theta \|L_{t/2}(Z)+L_{t/2}(Z)\|_2^2} | Z^{x,x}_t(t/2) = y] \\
\leq \mathbb{E}_t^{x,x}[e^{2\theta \|L_{t/2}(Z)\|_2^2+\|L_{t/2}(Z)\|_2^2} | Z^{x,x}_t(t/2) = y] \\
= \mathbb{E}_{t/2}^{x,y}[e^{2\theta \|L_{t/2}(Z)\|_2^2}]^2 \leq \mathbb{E}_{t/2}^{x,y}[e^{4\theta \|L_{t/2}(Z)\|_2^2}],
\]

where the inequality on the second line follows from a combination the triangle inequality and \((z + \bar{z})^2 \leq 2(z^2 + \bar{z}^2)\), the equality on the third line follows from independence and invariance of local time under time reversal, and the inequality on the third line follows from Jensen’s inequality.

With \(s_t(Z)\) as in (2.4.16), similarly to (2.4.20) we then have the upper bound

\[
\mathbb{E}_t^{x,x}[e^{\theta \|L_t(Z)\|_2^2}] \leq s_t(Z) \mathbb{E}^x[e^{4\theta \|L_{t/2}(Z)\|_2^2}]
\]

for every \(t > 0\); whence the present result readily follows from Lemma 2.4.10. \(\square\)

### 2.4.5 Compactness Properties of Deterministic Kernels

We now conclude the proofs of our technical results with some estimates regarding the integrability/compactness of the deterministic kernels (2.4.1). In this section and several others, to alleviate notation, we introduce the following shorthand.

**Notation 2.4.12.** For every \(t > 0\), we define the path functional

\[
\mathcal{A}_t(Z) := \begin{cases} 
   -\langle L_t(B), V \rangle & \text{(Case 1)} \\
   -\langle L_t(X), V \rangle + \bar{\alpha} \mathcal{L}_t^0(X) & \text{(Case 2)} \\
   -\langle L_t(Y), V \rangle + \bar{\alpha} \mathcal{L}_t^0(Y) + \bar{\beta} \mathcal{L}_t^1(Y) & \text{(Case 3)}
\end{cases}
\]
Lemma 2.4.13. For every $p \geq 1$ and $t > 0$,

$$\int I \Pi_Z(t; x, x) E_t^{x,x} [e^{y\mathbb{A}_t(Z)}]^{1/p} \, dx < \infty.$$ 

Proof. Let us begin with Case 1. By Assumption PG, for every $c_1 > 0$, there exists $c_2 > 0$ large enough so that $V(x) \geq c_1 \log(1 + |x|) - c_2$ for every $x \in \mathbb{R}$. Therefore, we have

$$\Pi_Z(t; x, x) E_t^{x,x} [e^{y\mathbb{A}_t(B)}]^{1/p} \leq \frac{e^{c_2 t}}{\sqrt{2\pi t}} E_t^{0,0} \left[ \exp \left( -pc_1 \int_0^t \log (1 + |B(s)|) \, ds \right) \right]^{1/p}$$

By using the inequalities

$$\log(1 + |x + z|) \geq \log(1 + |x|) - \log(1 + |z|) \geq \log(1 + |x|) - |z|, \quad (2.4.29)$$

which are valid for all $z \in \mathbb{R}$, we get the further upper bound

$$\frac{e^{c_2 t - c_1 t \log(1 + |x|)}}{\sqrt{2\pi t}} E_t^{0,0} \left[ \exp \left( pc_1 \int_0^t |B(s)| \, ds \right) \right]^{1/p}.$$

On the one hand, a Brownian scaling implies that

$$E_t^{0,0} \left[ \exp \left( pc_1 \int_0^t |B(s)| \, ds \right) \right] = E_t^{0,0} \left[ \exp \left( t^{3/2}p_1 \int_0^1 |B(s)| \, ds \right) \right] \leq E \left[ \exp \left( t^{3/2}p_1 S \right) \right], \quad (2.4.30)$$

where $S = \sup_{s \in [0,1]} |B_1^{0,0}(s)|$. Note that $s \mapsto |B_1^{0,0}(s)|$ is a Bessel bridge of dimension one (see, for instance, [RY99, Chapter XI]). Consequently, we know that (2.4.30) is
finite for any \( t, p, c_1 > 0 \) thanks to the tail asymptotic for \( S \) in [GS96, Remark 3.1] (the Bessel bridge is denoted by \( \varrho \) in that paper). On the other hand, for any \( t > 0 \), we can choose \( c_1 > 0 \) large enough so that

\[
\int_{\mathbb{R}} e^{-c_1 t \log (1 + |x|)} \, dx = \int_{\mathbb{R}} (1 + |x|)^{-c_1 t} \, dx < \infty,
\]

concluding the proof in Case 1.

For Case 2, by Hölder’s inequality, we have that

\[
\Pi_X(t; x, x) \mathbb{E}_t^{x,x} \left[ e^{\rho \mathbb{H}_t(X)} \right]^{1/p} \leq \Pi_X(t; x, x) \mathbb{E}_t^{x,x} \left[ e^{-\langle L_t(X), 2pV \rangle} \right]^{1/2p} \sup_{x \in (0, \infty)} \mathbb{E}_t^{x,x} \left[ e^{2\rho \mathbb{H}_t(X)} \right]^{1/2p}.
\]

The supremum of exponential moments of local time can be bounded by a direct application of Lemma 2.4.8. Then, by (2.4.9), we have that

\[
\int_0^b \Pi_Y(t; x, x) \mathbb{E}_t^{x,x} \left[ e^{-\langle L_t(Y), pV \rangle + p\rho \mathbb{H}_t(Y) + p\beta \mathbb{H}_t(Y) \rangle} \right]^{1/p} \, dx \leq b \left( \sup_{x \in (0, b)} \Pi_Y(t; x, x) \right) \left( \sup_{x \in (0, b)} \mathbb{E}_t^{x,x} \left[ e^{p\rho \mathbb{H}_t(Y) + p\beta \mathbb{H}_t(Y)} \right]^{1/p} \right).
\]

This term can be controlled in the same way as Case 1.

For Case 3, since \( I = (0, b) \) is finite and \( V \geq 0 \) (hence \( e^{-\langle L_t(Y), pV \rangle} \leq 1 \)),

\[
\int_0^b \Pi_Y(t; x, x) \mathbb{E}_t^{x,x} \left[ e^{-\langle L_t(Y), pV \rangle + p\rho \mathbb{H}_t(Y) + p\beta \mathbb{H}_t(Y) \rangle} \right]^{1/p} \, dx \leq b \left( \sup_{x \in (0, b)} \Pi_Y(t; x, x) \right) \left( \sup_{x \in (0, b)} \mathbb{E}_t^{x,x} \left[ e^{p\rho \mathbb{H}_t(Y) + p\beta \mathbb{H}_t(Y)} \right]^{1/p} \right).
\]

This is finite by Lemmas 2.4.7 and 2.4.8. \( \square \)
2.4.6 Proof of Proposition 2.2.7

Suppose we can prove that for every $\varepsilon > 0$, the potential $V + \Xi'_\varepsilon$ satisfies Assumption PG with probability one (up to a random additive constant, making it nonnegative). Then, by Proposition 2.2.1, the $\tilde{H}_\varepsilon$ are self-adjoint with compact resolvent. Moreover, $\tilde{K}_\varepsilon(t) = e^{-t\tilde{H}_\varepsilon}$ and the properties (2.2.5)–(2.2.7) then follow from Corollary 2.4.5, and the fact that $e^{-t\tilde{H}}$ is trace class follows from Lemma 2.4.13 in the case $p = 1$. Thus, it only remains to prove the following:

**Lemma 2.4.14.** For every $\varepsilon > 0$, there exists a random $c = c(\varepsilon) \geq 0$ such that the potential $V + \Xi'_\varepsilon + c$ satisfies Assumption PG with probability one.

**Proof.** Since $\Xi'_\varepsilon$ is continuous, $V + \Xi'_\varepsilon$ is locally integrable on $I$’s closure. Moreover, if we prove that $|\Xi'_\varepsilon(x)| \ll \log |x|$ as $x \to \pm \infty$, then the continuity of $\Xi'_\varepsilon$ also implies that $V + \Xi'_\varepsilon$ is bounded below and is such that

$$\lim_{x \to \pm \infty} \frac{V(x) + \Xi'_\varepsilon(x)}{\log |x|} = \infty;$$

hence we can take

$$c(\varepsilon) := \max \left\{ 0, -\inf_{x \in I} \left( V(x) + \Xi'_\varepsilon(x) \right) \right\} < \infty.$$

The fact that $|\Xi'_\varepsilon(x)| \ll \log |x|$ follows from Corollary A.2.2, since $\Xi'_\varepsilon$ is stationary. □

2.4.7 Proof of Proposition 2.2.8

Our proof of this result is similar to [BV13, Section 2] and [RRV11, Section 5], save for the fact that we use smooth approximations instead of discrete ones. We provide the argument in full. Let us endow $H^1_V$ (Definition 2.1.3) with the inner product $\langle f, g \rangle_* := \langle f', g' \rangle + \langle fg, V + 1 \rangle$ as well as the associated norm $\|f\|_*^2 := \|f'\|_2^2 + \|(V + 1)^{1/2}f\|_2^2$. 

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Remark 2.4.15. It is easy to see that $\| \cdot \|_*$ is equivalent to the “+1-norm” induced by $\mathcal{E}$ (c.f., [Sim15, 7.5.11]). Thus, $C_0^\infty$ (which we recall was defined in Section 2.2.1) is dense in $(D(\mathcal{E}), \| \cdot \|_*)$.

Arguing as in [BV13, Fact 2.2] and [RRV11, Fact 2.2], we have the following compactness property of $\| \cdot \|_*$.

Lemma 2.4.16. If $(f_n)_{n \in \mathbb{N}} \subset D(\mathcal{E})$ is such that sup$_n \| f_n \|_* < \infty$, then there exists $f \in D(\mathcal{E})$ and a subsequence $(n_i)_{i \in \mathbb{N}}$ along which

1. $\lim_{i \to \infty} \| f_{n_i} - f \|_2 = 0$;
2. $\lim_{i \to \infty} \langle g, f'_{n_i} \rangle = \langle g, f' \rangle$ for every $g \in L^2$;
3. $\lim_{i \to \infty} f_{n_i} = f$ uniformly on compact sets; and
4. $\lim_{i \to \infty} \langle g, f_{n_i} \rangle_* = \langle g, f \rangle_*$ for every $g \in D(\mathcal{E})$.

Remark 2.4.17. In [BV13, RRV11], this result is proved only for Case 2. However, exactly the same argument carries over to Cases 1 and 3 without additional difficulty.

Remark 2.4.18. It is easy to see by definition of $\langle \cdot, \cdot \rangle_*$ that if $f_n \to f$ in the sense of Lemma 2.4.16 (1)–(4), then for every $g \in C_0^\infty$, one has

$$\lim_{n \to \infty} \mathcal{E}(g, f_n) = \mathcal{E}(g, f).$$

We can reformulate Proposition 2.2.2 in terms of $\| \cdot \|_*$ thusly:

Lemma 2.4.19. There exist finite random variables $c_1, c_2, c_3 > 0$ such that

$$c_1\|f\|_*^2 - c_2\|f\|_2^2 \leq \hat{\mathcal{E}}(f, f) \leq c_3\|f\|_*^2, \quad f \in D(\mathcal{E}).$$

We also have the following finite $\varepsilon$ variant:
**Lemma 2.4.20.** There exist finite random variables $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ such that for every $\varepsilon \in (0, 1]$,

$$\tilde{c}_1 \|f\|_2^2 - \tilde{c}_2 \|f\|_2^2 \leq \tilde{E}_\varepsilon(f, f) \leq \tilde{c}_3 \|f\|_2^2,$$

for every $f \in D(\mathcal{E})$.

**Proof.** By repeating the proof of Proposition 2.2.2, we only need to prove that for every $\theta > 0$, there exists $c > 0$ large enough so that

$$|\langle f^2, \Xi'_\varepsilon \rangle| \leq \theta \left( \frac{1}{2} \|f'\|_2^2 + \|V^{1/2} f\|_2^2 \right) + \|f\|_2^2,$$

for every $\varepsilon \in (0, 1]$ and $f \in C_0^\infty$. Let us define

$$\tilde{\Xi}_\varepsilon(x) := \int_x^{x+1} \Xi_\varepsilon(y) \, dy.$$

Arguing as in the proof of Proposition 2.2.2, it suffices to show that

$$\sup_{\varepsilon \in (0, 1]} \sup_{0 \leq x \leq b} |\Xi_\varepsilon(x)| < \infty$$

almost surely and that there exist finite random variables $C > 0$ and $u > 1$ independent of $\varepsilon \in (0, 1]$ such that for every $x \in \mathbb{R}$,

$$\sup_{y \in [0, 1]} |\Xi_\varepsilon(x + y) - \Xi_\varepsilon(x)| \leq C \sqrt{\log(u + |x|)}.$$

Let $K > 0$ be such that $\text{supp}(\varrho) \subset [-K, K]$ so that $\text{supp}(\varrho_\varepsilon) \subset [-K, K]$ for all $\varepsilon \in (0, 1]$. On the one hand, since the $\varrho_\varepsilon$ integrate to one,

$$\sup_{\varepsilon \in (0, 1]} \sup_{0 \leq x \leq b} \left| \int_{\mathbb{R}} \Xi(x - y) \varrho_\varepsilon(y) \, dy \right| \leq \sup_{-K \leq x \leq b + K} |\Xi(x)| < \infty.$$
On the other hand, by Corollary A.2.2 and Remark A.2.3, for every \( x \in I \) and \( \varepsilon \in (0, 1] \), one has

\[
\sup_{y \in [0,1]} \left| \int_{\mathbb{R}} \left( \Xi(x + y - z) - \Xi(x - z) \right) \varrho_\varepsilon(z) \, dz \right|
\leq \sup_{w \in [x-K,x+K]} \sup_{y \in [0,1]} |\Xi(w + y) - \Xi(w)| \leq \sup_{w \in [x-K,x+K]} C \sqrt{\log(2 + |w|)},
\]

which yields the desired estimate.

**Remark 2.4.21.** We see from Lemma 2.4.20 that the forms \( (f, g) \mapsto \langle fg, \Xi'_\varepsilon \rangle \) are uniformly form-bounded in \( \varepsilon \in (0, 1] \) by \( \mathcal{E} \), in the sense that there exists a \( 0 < \theta < 1 \) and a random \( c > 0 \) independent of \( \varepsilon \) such that

\[
|\langle f^2, \Xi'_\varepsilon \rangle| \leq \theta \mathcal{E}(f, f) + c\|f\|_2^2, \quad f \in D(\mathcal{E}), \; \varepsilon \in (0, 1].
\]

Among other things, this implies by the variational principle (see, for example, the estimate in [RS78, Theorem XIII.68]) that for every \( k \in \mathbb{N} \) and \( \varepsilon \in (0, 1] \), one has

\[
(1 - \theta)\lambda_k(H) - c \leq \lambda_k(\hat{H}_\varepsilon) \leq (1 + \theta)\lambda_k(H) + c. \tag{2.4.31}
\]

Finally, we need the following convergence result.

**Lemma 2.4.22.** Almost surely, for every \( f, g \in C_0^\infty \), it holds that

\[
\lim_{\varepsilon \to 0} \langle fg, \Xi'_\varepsilon \rangle = \xi(fg). \tag{2.4.32}
\]

Moreover, if \( (\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1] \) converges to zero, \( \sup_n\|f_n\|_* < \infty \), and \( f_n \to f \) in the sense of Lemma 2.4.16 (1)-(4), then almost surely,

\[
\lim_{n \to \infty} \langle f_n g, \Xi'_{\varepsilon_n} \rangle = \xi(fg) \tag{2.4.33}
\]
for every \( g \in C_0^\infty \).

**Proof.** For (2.4.32), it suffices to prove that

\[
\lim_{\varepsilon \to 0} \langle (f'g + fg') * \varrho_\varepsilon, \Xi \rangle = \langle f'g + fg', \Xi \rangle.
\]

Since \( f'g + fg' \) is compactly supported and \( \Xi \) is continuous (hence bounded on compacts), the result follows by dominated convergence.

Let us now prove (2.4.33). Using again the fact that \( g \) and \( g' \) are compactly supported, we know that there exists a compact \( K \subset \mathbb{R} \) (in Case 3 we may simply take \( K = [0, b] \)) such that

\[
\langle f'_n g + f_n g', \Xi * \varrho_{\varepsilon_n} \rangle = \langle f'_n g + f_n g', \Xi * \varrho_{\varepsilon_n} 1_K \rangle
\]

and similarly with \( f_n \) replaced by \( f \) and \( \Xi * \varrho_{\varepsilon_n} \) replaced by \( \Xi \). Given that, as \( n \to \infty \),

\( \Xi * \varrho_{\varepsilon_n} 1_K \to \Xi 1_K \) in \( L^2 \), \( f'_n g + f_n g' \to f'g + fg' \) weakly in \( L^2 \), and \( \sup_n \|f'_n g + f_n g'\|_2 < \infty \), we conclude that

\[
\lim_{n \to \infty} \langle f'_n g + f_n g', \Xi * \varrho_{\varepsilon_n} \rangle = \langle f'g + fg', \Xi \rangle.
\]

Hence (2.4.33) holds. \( \square \)

We finally have all the necessary ingredients to prove the spectral convergence. We first prove that there exists a subsequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) such that

\[
\liminf_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \geq \lambda_k(\hat{H}) \tag{2.4.34}
\]

for every \( k \in \mathbb{N} \).

**Remark 2.4.23.** For the sake of readability, we henceforth denote any subsequence and further subsequences of \( (\varepsilon_n)_{n \in \mathbb{N}} \) as \( (\varepsilon_n)_{n \in \mathbb{N}} \) itself.
According to (2.4.31), the $\lambda_k(\hat{H}_\varepsilon)$ are uniformly bounded, and thus it follows from the Bolzano-Weierstrass theorem that, along a subsequence $\varepsilon_n$, the limits

$$\lim_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) =: l_k$$

exist and are finite for every $k \in \mathbb{N}$, where $-\infty < l_1 \leq l_2 \leq \cdots$. Since the eigenvalues are bounded, it follows from Lemma 2.4.20 that the eigenfunctions $\psi_k(\hat{H}_\varepsilon)$ are bounded in $\| \cdot \|_\ast$-norm uniformly in $\varepsilon \in [0, 1)$, and thus there exist functions $f_1, f_2, \ldots$ and a further subsequence along which $\psi_k(\hat{H}_{\varepsilon_n}) \to f_k$ for every $k$ in the sense of Lemma 2.4.16 (1)–(4). By combining Remark 2.4.18 and (2.4.33), this means that

$$l_k \langle g, f_k \rangle = \lim_{n \to \infty} \hat{E}_{\varepsilon_n}(g, \psi_k(\hat{H}_{\varepsilon_n})) = \hat{E}(g, f_k)$$

for all $k \in \mathbb{N}$ and $g \in C_0^\infty$. That is, $(l_k, f_k)_{k \in \mathbb{N}}$ consists of eigenvalue-eigenfunction pairs of $\hat{H}$, though these pairs may not exhaust the full spectrum. Since the $l_k$ are arranged in increasing order, this implies that $l_k \geq \lambda_k(\hat{H})$ for every $k \in \mathbb{N}$, which proves (2.4.34).

We now prove that we can take $\hat{H}'$’s eigenfunctions in such a way that, along a further subsequence,

$$\limsup_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \lambda_k(\hat{H}) \quad \text{and} \quad \lim_{n \to \infty} \| \psi_k(\hat{H}_{\varepsilon_n}) - \psi_k(\hat{H}) \|_2 = 0 \quad (2.4.35)$$

for every $k \in \mathbb{N}$. We proceed by induction. Suppose that (2.4.35) holds up to $k - 1$ (if $k = 1$ then we consider the base case). Let $\psi$ be an eigenfunction of $\lambda_k(\hat{H})$ orthogonal to $\psi_1(\hat{H}), \ldots, \psi_{k-1}(\hat{H})$, and for every $\theta > 0$, let $\varphi_\theta \in C_0^\infty$ be such that $\| \varphi_\theta - \psi \|_\ast < \theta$. Let us define the projections

$$\pi_{\varepsilon_n}(\varphi_\theta) := \varphi_\theta - \sum_{\ell=1}^{k-1} \langle \psi_\ell(\hat{H}_{\varepsilon_n}), \varphi_\theta \rangle \psi_\ell(\hat{H}_{\varepsilon_n})$$
of \( \varphi_\theta \) onto the orthogonal of \( \psi_1(\hat{H}_{\varepsilon_n}), \ldots, \psi_{k-1}(\hat{H}_{\varepsilon_n}) \) (if \( k = 1 \), then we simply have \( \pi_{\varepsilon_n}(\varphi_\theta) = \varphi_\theta \)). Then, by the variational principle, for any \( \theta > 0 \),

\[
\limsup_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \limsup_{n \to \infty} \frac{\hat{\mathcal{E}}_{\varepsilon_n}(\pi_{\varepsilon_n}(\varphi_\theta), \pi_{\varepsilon_n}(\varphi_\theta))}{\| \pi_{\varepsilon_n}(\varphi_\theta) \|_2^2}. \tag{2.4.36}
\]

Given that \( \| \psi_\ell(\hat{H}_{\varepsilon_n}) - \psi_\ell(\hat{H}) \|_2 \to 0 \) for every \( \ell \leq k - 1 \), one has

\[
\lim_{\theta \to 0} \lim_{n \to \infty} \pi_{\varepsilon_n}(\varphi_\theta) = \psi
\]

in \( L^2 \). Moreover, the convergence of the \( \lambda_\ell(\hat{H}_{\varepsilon_n}) \) and Lemma 2.4.20 imply that the \( (\psi_\ell(\hat{H}_{\varepsilon_n}))_{\ell=1,\ldots,k-1} \) are uniformly bounded in \( \| \cdot \|_* \)-norm, and thus

\[
\lim_{\theta \to 0} \limsup_{n \to \infty} \left\| \sum_{\ell=1}^{k-1} \langle \psi_\ell(\hat{H}_{\varepsilon_n}), \varphi_\theta \rangle \psi_\ell(\hat{H}_{\varepsilon_n}) \right\|_* = 0.
\]

We recall that, by Lemma 2.4.20, the maps \( f \mapsto \hat{\mathcal{E}}(f, f) \) are continuous with respect to \( \| \cdot \|_* \) uniformly in \( \varepsilon \in (0, 1] \). Consequently,

\[
\limsup_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \limsup_{\theta \to 0} \limsup_{n \to \infty} \frac{\hat{\mathcal{E}}_{\varepsilon_n}(\varphi_\theta, \varphi_\theta)}{\| \varphi_\theta \|_2^2},
\]

since (2.4.36) holds for any \( \theta > 0 \). Then, if we use (2.4.32) to compute the supremum limit in \( n \), followed by Lemma 2.4.19 for the limit in \( \theta \) (recall that \( \| \varphi_\theta - \psi \|_* \to 0 \) as \( \theta \to 0 \)), we conclude that

\[
\limsup_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \hat{\mathcal{E}}(\psi, \psi) = \lambda_k(\hat{H}).
\]

Since \( \liminf_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \geq \lambda_k(\hat{H}) \) by the previous step, we now know that \( \lambda_k(\hat{H}_{\varepsilon_n}) \to \lambda_k(\hat{H}) \) as \( n \to \infty \). Thus, according to Lemma 2.4.20, the eigenfunctions \( (\psi_k(\hat{H}_{\varepsilon_n}))_{n \in \mathbb{N}} \) are uniformly bounded in \( \| \cdot \|_* \)-norm. Thus, there exists \( \bar{\psi} \in D(\mathcal{E}) \) such that \( \psi_k(\hat{H}_{\varepsilon_n}) \to \bar{\psi} \) in the sense of Lemma 2.4.16 along a further subsequence.
Combining this with Remark 2.4.18 and (2.4.33), and the fact that \( \lambda_k(\hat{H}_{\varepsilon_n}) \to \lambda_k(\hat{H}) \), we then also have

\[
\hat{\mathcal{E}}(g, \tilde{\psi}) = \lim_{n \to \infty} \hat{\mathcal{E}}_{\varepsilon_n}(g, \psi_k(\hat{H}_{\varepsilon_n})) = \lim_{n \to \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \langle g, \psi_k(\hat{H}_{\varepsilon_n}) \rangle = \lambda_k(\hat{H}) \langle g, \tilde{\psi} \rangle
\]

for all \( g \in C_0^\infty \). In particular, \( \tilde{\psi} \) must be an eigenfunction for \( \lambda_k(\hat{H}) \), which is orthogonal to \( \psi_1(\hat{H}), \ldots, \psi_{k-1}(\hat{H}) \). Thus we may take \( \psi_k(\hat{H}) := \tilde{\psi} \), concluding the proof of the proposition since Lemma 2.4.16 includes \( L^2 \) convergence.

### 2.4.8 Proof of Proposition 2.2.9, Part 1

We begin by proving (2.2.9).

#### Step 1. Computation of Expected \( L^2 \) Norm

Our first step in the proof of (2.2.9) is to obtain a formula for \( \mathbb{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|^2] \) that is amenable to analysis. Using once again the notation \( \mathfrak{A}_t \) from (2.4.28), we have by Fubini’s theorem that

\[
\mathbb{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|^2] = \int_{I^2} \mathbb{E}[\hat{K}_\varepsilon(t; x, y)^2] - 2\mathbb{E}[\hat{K}_\varepsilon(t; x, y)\hat{K}(t; x, y)] + \mathbb{E}[\hat{K}(t; x, y)^2] \, dy \, dx
\]

\[
= \int_{I^2} \Pi_{Z}(t; x, y)^2 \mathbb{E}\left[ e^{\mathfrak{A}_t(Z_{1,x,y}) + \mathfrak{A}_t(Z_{2,x,y})} \left( \mathbb{E}_\Xi\left[ e^{\langle L_t(Z_{1,x,y}) + L_t(Z_{2,x,y}), \Xi \rangle} \right] + \mathbb{E}_\Xi\left[ e^{-\xi(L_t(Z_{1,x,y}) - \xi(L_t(Z_{2,x,y}))} \right] \right) \right] \, dy \, dx,
\]

where \( Z_{i,x,y} \) \((i = 1, 2)\) are i.i.d. copies of \( Z_{t,x,y} \) that are independent of \( \Xi \), and \( \mathbb{E}_\Xi \) denotes the expected value with respect to \( \Xi \) conditional on \( Z_{i,x,y} \). For every functions \( f_1, f_2 \in PC_c \), the variable \( \xi(f_1) + \xi(f_2) \) is Gaussian with mean zero and variance \( \sum_{i,j=1}^2 \langle f_i, f_j \rangle_{\gamma} = \|f_1 + f_2\|_\gamma^2 \). Thanks to (2.2.4), a straightforward Gaussian moment
generating function computation yields that the above is equal to

\[ \int_{I^2} \Pi_Z(t; x, y)^2 \mathbb{E} \left[ e^{\frac{3}{4}L_s(Z_{t,x}^{1,x,y}) + \frac{1}{2}L_s(Z_{t,x}^{2,x,y})} \left( e^{\frac{1}{2}L_s(Z_{t,x}^{1,x,y})} \phi_s + L_s(Z_{t,x}^{1,x,y}) \phi_s \right)^2 \right] dy dx. \]

Arguing in exactly the same way as the proof of (2.4.4) in Appendix A.3.2, we finally obtain the formula

\[ \mathbb{E} \left[ \| \hat{K}_\varepsilon(t) - \hat{K}(t) \|^2 \right] = \int_{I} \Pi_Z(2t; x, x) \mathbb{E}^{x,x}_{2t} \left[ e^{\frac{3}{4}L_s(Z)} \left( e^{\frac{1}{2}L_s(Z)} \phi_s \right)^2 \right] dx. \quad (2.4.37) \]

**Step 2. Convergence Inside Expectation**

With (2.4.37) in hand, our second step to prove (2.2.9) is to show that, for every \( x \in I \), we have the almost sure limit

\[ \lim_{\varepsilon \to 0} e^{\frac{1}{2}L_s(Z_{2t,x}^{t,x})} \phi_s = 2e^{\frac{1}{2}L_s(Z_{2t,x}^{t,x})} \phi_s + L_{[t, 2t]}(Z_{2t,x}^{t,x}) \phi_s = 0. \quad (2.4.38) \]

This is a simple consequence of (2.1.9) coupled with the fact that if \( f \in L^q \) for some \( q \geq 1 \), then \( \| f * \phi_\varepsilon - f \|_q \to 0 \) as \( \varepsilon \to 0 \).

**Step 3. Convergence Inside Integral**

Our next step is to prove that for every \( x \in I \), we have the limit in expectation

\[ \lim_{\varepsilon \to 0} \mathbb{E}^{x,x}_{2t} \left[ e^{\frac{3}{4}L_s(Z)} \left( e^{\frac{1}{2}L_s(Z)} \phi_s \right)^2 \right] = 0. \quad (2.4.39) \]
Thanks to (2.4.38), for this it suffices to prove that the prelimit variables in (2.4.39)
are uniformly integrable in $\varepsilon > 0$, which itself can be reduced to the claim that
\[
\sup_{\varepsilon > 0} E_{2t}^{x,x} \left[ e^{2\mathfrak{A}_{2t}(Z)} \left( e^{\frac{1}{4} \| L_{2t}(Z) * \varrho_\varepsilon \|^2_{\gamma} - 2e^{\frac{1}{4} \| L_{t}(Z) * \varrho_\varepsilon + L_{t,2t}(Z) \|^2_{\gamma}} + e^{\frac{1}{4} \| L_{2t}(Z) \|^2_{\gamma}} \right)^2 \right] < \infty.
\]
By combining Hölder’s inequality with $(z - 2\bar{z} + \bar{z})^2 \leq 16(z^2 + \bar{z}^2 + \bar{z}^2)$, it is enough
to prove that
\[
E_{2t}^{x,x} \left[ e^{4\mathfrak{A}_{2t}(Z)} \right] < \infty \tag{2.4.40}
\]
and
\[
\sup_{\varepsilon > 0} E_{2t}^{x,x} \left[ e^{\| L_{2t}(Z) * \varrho_\varepsilon \|^2_{\gamma}} \right], \quad \sup_{\varepsilon > 0} E_{2t}^{x,x} \left[ e^{\| L_{t}(Z) * \varrho_\varepsilon + L_{t,2t}(Z) \|^2_{\gamma}} \right], \quad E_{2t}^{x,x} \left[ e^{\| L_{2t}(Z) \|^2_{\gamma}} \right] < \infty. \tag{2.4.41}
\]
By combining the assumption $V \geq 0$ (hence $e^{-4(L_{2t}(Z),V)} \leq 1$) and Lemma 2.4.8, we immediately obtain (2.4.40). Next, it follows from (2.1.9) that
\[
E_{2t}^{x,x} \left[ e^{\| L_{2t}(Z) \|^2_{\gamma}} \right] \leq E_{2t}^{x,x} \left[ e^{c_\gamma \sum_{i=1}^\ell \| L_{2t}(Z) \|^2_{q_i}} \right].
\]
This is finite by Lemma 2.4.11 since $1 \leq q_i \leq 2$ for all $1 \leq i \leq \ell$. According to
Young’s convolution inequality, the fact that the $\varrho_\varepsilon$ integrate to one implies that
$\| f * \varrho_\varepsilon \|_{q} \leq \| f \|_{q} \| \varrho_\varepsilon \|_{1} \leq \| f \|_{q}$. Thus, it follows from (2.1.9) that
\[
\sup_{\varepsilon > 0} E_{2t}^{x,x} \left[ e^{\| L_{2t}(Z) * \varrho_\varepsilon \|^2_{\gamma}} \right] \leq E_{2t}^{x,x} \left[ e^{c_\gamma \sum_{i=1}^\ell \| L_{2t}(Z) \|^2_{q_i}} \right] < \infty.
\]
Since \( \| \cdot \|_\gamma \) is a seminorm, it satisfies the triangle inequality, and thus

\[
\| L_t(Z^x_{2t})^* e_\varepsilon + L_{[t,2t]}(Z^x_{2t}) \|_\gamma^2 \leq 2 \| L_t(Z^x_{2t})^* e_\varepsilon \|_\gamma^2 + 2 \| L_{[t,2t]}(Z^x_{2t}) \|_\gamma^2,
\]

Given that \( L_t(Z^x_{2t}) \) and \( L_{[t,2t]}(Z^x_{2t}) \) are both smaller than \( L_{2t}(Z^x_{2t}) \), applying once again (2.1.9) and Young’s inequality yields

\[
\sup_{\varepsilon > 0} E^{x,x}_{2t} \left[ e^{\| L_t(Z)^* e_\varepsilon + L_{[t,2t]}(Z) \|_\gamma^2} \right] \leq E^{x,x}_{2t} \left[ e^{4c_\gamma \sum \| L_{2t}(Z) \|_{q_i}^2} \right],
\]

which is finite by Lemma 2.4.11. We therefore conclude that (2.4.41) holds, and thus (2.4.39) as well.

**Step 4. Convergence of Integral**

Our final step in the proof of (2.2.9) is to show that (2.4.37) converges to zero. Given (2.4.39), by applying the dominated convergence theorem, it suffices to find an integrable function that dominates

\[
\Pi_{Z}(2t, x, x) E^{x,x}_{2t} \left[ e^{A_{2t}(Z)} \left( \varepsilon^\frac{1}{2} \| L_{2t}(Z)^* e_\varepsilon \|_\gamma^2 + 2 \varepsilon^\frac{1}{2} \| L_t(Z)^* e_\varepsilon + L_{[t,2t]}(Z) \|_\gamma^2 + e^\frac{1}{2} \| L_{2t}(Z) \|_\gamma^2 \right) \right]
\]

for every \( \varepsilon > 0 \). By Hölder’s inequality, this is bounded by

\[
\Pi_{Z}(2t; x, x) E^{x,x}_{2t} \left[ e^{2A_{2t}(Z)} \right]^{1/2} \sup_{\varepsilon > 0, x \in I} E^{x,x}_{2t} \left[ \left( \varepsilon^\frac{1}{2} \| L_{2t}(Z)^* e_\varepsilon \|_\gamma^2 + 2 \varepsilon^\frac{1}{2} \| L_t(Z)^* e_\varepsilon + L_{[t,2t]}(Z) \|_\gamma^2 + e^\frac{1}{2} \| L_{2t}(Z) \|_\gamma^2 \right)^2 \right]^{1/2}.
\]

uniformly in \( \varepsilon > 0 \). Thanks to Lemma 2.4.13 with \( p = 2 \), the first line of (2.4.43) is integrable. Then, by arguing in exactly the same way as in Section 2.4.8 (i.e., Young’s convolution inequality, (2.1.9), etc.), the term on the second line of (2.4.43)
is bounded by
\[
\sup_{x \in I} C \left[ e^{\theta c_2 \sum_{i=1}^{t} \|L_{2t}(Z)\|_{q_i}^2} \right]^{1/2}
\]
for some constants \( C, \theta > 0 \). This is finite by Lemma 2.4.11. We therefore conclude that (2.4.42) is dominated by an integrable function for all \( \varepsilon > 0 \); hence (2.2.9) holds.

**Remark 2.4.24.** Arguing as in (2.4.37), we have the formula
\[
\mathbb{E}[\|\hat{K}(t)\|_{2}^{2}] = \int_{I} \Pi_{Z}(2t; x, x) \mathbb{E}_{2t}^{x,x} \left[ e^{\frac{1}{2} \|L_{2t}(Z)\|_{2}^{2}} \right] dx. \tag{2.4.44}
\]

Considering Case 1 for simplicity, it follows from (the reverse) Hölder’s inequality that for every \( p > 1 \), the above is bounded below by
\[
\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \mathbb{E}_{2t}^{x,x} \left[ e^{-\langle L_{2t}(B), V/p \rangle} \right]^{p} \mathbb{E}_{2t}^{0,0} \left[ e^{-\frac{1}{2p} \|L_{2t}(B)\|_{2}^{2}} \right]^{-(p-1)} dx
\]
for every \( x \in \mathbb{R} \). If \( V(x) \leq c_1 \log(1 + |x|) + c_2 \) for some \( c_1 > 0 \) and large enough \( c_2 > 0 \), then an argument similar to the proof of Lemma 2.4.13 (using the bound \( \log(1 + |z + \bar{z}|) \leq \log(1 + |z|) + |\bar{z}| \) instead of (2.4.29)) yields the further lower bound
\[
\zeta_t \int_{\mathbb{R}} (1 + |x|)^{-c_1 t} dx
\]
for some finite \( \zeta_t > 0 \) that only depends on \( t \); this blows up whenever \( t \leq 1/c_1 \). Thus, if we do not assume (2.1.3), then there is always some \( t_0 > 0 \) such that \( \mathbb{E}[\|\hat{K}(t)\|_{2}^{2}] = \infty \) for all \( t \leq t_0 \). Essentially the same argument implies that \( \|e^{-tH}\|_{2} = \infty \) for all \( t \leq t_0 \) for the deterministic operator \( H \) as well.
2.4.9 Proof of Proposition 2.2.9, Part 2

We now prove (2.2.10). By Fubini,

\[
E \left[ \left( \int_{I} \hat{K}_\varepsilon(t; x, x) - \hat{K}(t; x, x) \, dx \right)^2 \right] = \int_{I^2} E[\hat{K}_\varepsilon(t; x, x)\hat{K}_\varepsilon(t; y, y)]

- 2E[\hat{K}_\varepsilon(t; x, x)\hat{K}(t; y, y)] + E[\hat{K}(t; x, x)\hat{K}(t; y, y)] \, dx \, dy.
\]

Arguing as in the previous section, the above is seen to be equal to

\[
\int_{I^2} \Pi_{Z(t; x, x)\Pi(t; y, y)} E \left[ e^{\mathcal{K}(Z_t^{1;x,x}) + \mathcal{K}(Z_t^{2;y,y})} \left( e^{\frac{1}{2} \| L_t(Z_t^{1;x,x})\theta_z + L_t(Z_t^{2;y,y})\theta_z \|^2} - 2e^{\frac{1}{2} \| L_t(Z_t^{1;x,x})\theta_z + L_t(Z_t^{2;y,y})\|^2} \right) \right] \, dx \, dy,
\]

where \( Z_t^{1;x,x} \) and \( Z_t^{2;y,y} \) are independent processes with respective distributions \( Z_t^{x,x} \) and \( Z_t^{y,y} \). At this point, essentially the same argument that we used to prove (2.2.9) in the previous section yields (2.2.10).

2.4.10 Proof of Proposition 2.2.11

The argument that follows first appeared in [GLGY19]; we provide it in full for convenience since it is rather short. We argue on a case-by-case basis. Suppose first that \( \xi \) is a bounded noise. Then,

\[
\int_{\mathbb{R}^2} |f(a)\gamma(a - b)f(b)| \, dadb \leq \|\gamma\|_\infty \|f\|_1^2.
\]

Next, if \( \xi \) is a white noise, then \( \| \cdot \|_\gamma = \sigma^2 \| \cdot \|_2 \).

Consider then fractional noise with Hurst parameter \( H \). There exists some constant \( \bar{c} > 0 \) such that

\[
\|f\|_\gamma^2 \leq \bar{c} \int_{\mathbb{R}^2} \frac{|f(a)f(b)|}{|a - b|^{2-2H}} \, dadb.
\]
Note that we can decompose

$$\int_{\mathbb{R}^2} \frac{|f(a)f(b)|}{|a-b|^{2-2\gamma}} \, dadb = \int_{\{|b-a|<1\}} \frac{|f(a)f(b)|}{|a-b|^{2-2\gamma}} \, dadb + \int_{\{|b-a|\geq 1\}} \frac{|f(a)f(b)|}{|a-b|^{2-2\gamma}} \, dadb. \quad (2.4.45)$$

On the one hand, by Young’s convolution inequality, the first integral on the right-hand side of (2.4.45) is bounded above by

$$\left(\int_{-1}^{1} \frac{1}{|z|^{2-2\gamma}} \, dz\right) \left(\int_{\mathbb{R}} f(a)^2 \, da\right) = \left(\int_{-1}^{1} \frac{1}{|a|^{2-2\gamma}} \, da\right) \|f\|_2^2.$$

On the other hand, the second integral on the right-hand side of (2.4.45) is bounded by

$$\left(\int_{\mathbb{R}} |f(a)| \, da\right)^2 = \|f\|_1^2.$$

Thus, we obtain (2.2.14) with the constant $c_\gamma = \max\left\{ \tilde{c} \int_{-1}^{1} |a|^{2\gamma-2} \, da, 1 \right\} < \infty$.

Lastly, let $\xi$ be an $L^p$-singular noise with decomposition $\gamma = \gamma_1 + \cdots + \gamma_\ell + \gamma_\infty$. Then, the bound on $\|f\|_\gamma^2$ is a consequence of Young’s convolution inequality:

$$\|f\|_\gamma^2 \leq \sum_{i=1}^{\ell} \int_{\mathbb{R}^2} |f(a)\gamma_i(a-b)f(b)| \, dadb + \int_{\mathbb{R}^2} |f(a)\gamma_\infty(a-b)f(b)| \, dadb$$

$$\leq \sum_{i=1}^{\ell} \|\gamma_i\|_{p_i} \|f\|_{q_i}^2 + \|\gamma_\infty\|_{\infty} \|f\|_1^2,$$

where $\frac{1}{q_i} + \frac{1}{q_i} + \frac{1}{p_i} = 2$, or equivalently, $q_i = 1/(1 - \frac{1}{2p_i})$, so that we can take the constant $c_\gamma = \max\{\|\gamma_1\|_{p_1}, \ldots, \|\gamma_\ell\|_{p_\ell}, \|\gamma_\infty\|_{\infty}\}$. 85
Chapter 3

Number Rigidity

3.1 Main Results

3.1.1 Statement of Results

Definition 3.1.1. Let $\gamma$ be as in Definition 2.1.10. We say that $\gamma$ is compactly supported if there exists a compact set $A \subset \mathbb{R}$ such that $\langle f, \gamma \rangle = 0$ whenever $f(x) = 0$ for every $x \in A$.

Our main result is as follows.

Theorem 3.1.2. Suppose that Assumptions DB, PG, and SN hold. In Case 3, $\hat{H}$’s eigenvalues are always number rigid. In Cases 1 & 2, if $b > 1$ is such that

$$\limsup_{t \to 0} t^{-b} \left( \sup_{x \in I} E^x \left[ \left\| L_t(Z) \right\|_{\gamma}^{2b} \right]^{1/\theta} \right) < \infty \quad \tag{3.1.1}$$

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for every $\theta > 0$, then $\hat{H}$’s eigenvalues are number rigid if the following growth condition on the potential $V$ holds:

$$
\begin{align*}
\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/(2b-1)}} &= \infty \quad \text{(if $\gamma$ is compactly supported)} \\
\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/(b-1)}} &= \infty \quad \text{(otherwise).}
\end{align*}
$$

Remark 3.1.3. If Assumptions DB, PG, and SN hold, then there always exists some $b > 1$ such that (3.1.1) holds (namely, combine (2.1.9) and (3.2.1)). Thus, under these assumptions, Theorem 3.1.2 always provides a nontrivial class of potentials for which rigidity holds.

Applying the growth condition (3.1.2) to the four noises discussed earlier in Example 2.1.22 yields the following result.

Theorem 3.1.4. Let $\xi$ be one of the four types of noises considered in Example 2.1.22. Then, (3.1.1) holds with

$$
b := \begin{cases} 
2 & \text{(bounded)} \\
3/2 & \text{(white)} \\
1 + \mathcal{H} & \text{(fractional with index $\mathcal{H} \in (\frac{1}{2}, 1)$)} \\
2 - 1/2p & \text{($L^p$-singular with $p = \min p_i$)}.
\end{cases}
$$

Consequently, in Cases 1 & 2, $\hat{H}$’s eigenvalues are number rigid if the following growth conditions on $V$ are satisfied:

1. (Bounded) If $\xi$ is a bounded noise, then

$$
\begin{align*}
\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/3}} &= \infty \quad \text{(if $\gamma$ is compactly supported)} \\
\lim_{|x| \to \infty} \frac{V(x)}{|x|^2} &= \infty \quad \text{(otherwise).}
\end{align*}
$$
2. (White) If $\xi$ is a white noise, then

$$\lim_{|x| \to \infty} \frac{V(x)}{|x|} = \infty.$$  \hspace{1cm} (3.1.5)

3. (Fractional) If $\xi$ is a fractional noise with Hurst index $H \in (\frac{1}{2}, 1)$, then

$$\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2/3}} = \infty.$$  \hspace{1cm} (3.1.6)

4. (Lp-Singular) If $\xi$ is an $L^p$-singular noise with $p := \min_i p_i$, then

$$\begin{cases} 
\lim_{|x| \to \infty} \frac{V(x)}{|x|^{2p/(3p-1)}} = \infty & (if \ \gamma \ \text{is compactly supported}) \\
\lim_{|x| \to \infty} \frac{V(x)}{|x|^{4p/(2p-1)}} = \infty & (otherwise). 
\end{cases}$$  \hspace{1cm} (3.1.7)

An outline of the proofs of Theorems 3.1.2 and 3.1.4 is proved in Section 3.2. As explained in that section, the main technical ingredient in this proof is Theorem 3.2.2, which provides quantitative upper bounds on the variance of the linear statistic $\sum_k e^{-t\lambda_k(H)}$ as $t \to 0$. The result then follows from an application of the rigidity criterion Corollary 1.4.7 by proving that

$$\lim_{t \to 0} \text{Var} [\text{Tr}[e^{-tH}]] = \lim_{t \to 0} \text{Var} \left[ \int_I \hat{K}(t; x, x) \, dx \right] = 0$$

under the conditions stated in Theorem 3.1.2.

### 3.1.2 Questions of Optimality

The growth assumptions (3.1.2) raise natural questions concerning the optimality of Theorem 3.1.2 in Cases 1 & 2. More precisely, it would be interesting to have an
answer to the following open problem, so as to fully characterize the potentials for which number rigidity can be proved using our Feynman-Kac method.

**Problem 3.1.5.** Suppose that Assumptions DB, PG, and SN hold. Given a fixed noise $\xi$, characterize the potentials $V$ such that

$$\lim_{t \to 0} \text{Var}[\text{Tr}[e^{-t\hat{H}}]] = 0$$

in Cases 1 & 2.

In this section, we provide counterexamples that prove that, in two situations (namely, white noise and bounded noise with non-compactly-supported $\gamma$), Theorem 3.1.2 is optimal in the sense that a better general sufficient condition than (3.1.2) cannot be obtained using the vanishing of $\text{Var}[\text{Tr}[e^{-t\hat{H}}]]$.

**Stochastic Airy Operator**

By using the fact that SAO$_2$’s eigenvalues generate the Airy-2 point process, we prove in Section 3.4 the following result.

**Proposition 3.1.6.** It holds that

$$\lim_{t \to 0} \text{Var}[\text{Tr}[e^{-t\text{SAO}_2/2}]] = (4\pi)^{-1}.$$  

Since SAO$_2$/2 has deterministic potential $V(x) = x$ on $(0, \infty)$ and a white noise, this proves that a general statement like Theorem 3.1.2 cannot be improved in the case of white noise (c.f., (3.1.5)). Regarding a complete characterization of the type asked in Problem 3.1.5, the following conjecture then seems natural.

**Conjecture 3.1.7.** Suppose that Assumptions DB and PG hold, and let $\xi$ be a white noise. In Cases 1 & 2, if there exists $\kappa, \nu > 0$ such that $V(x) \leq \kappa|x| + \nu$ for all $x \in I$, 

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then we have that
\[
\liminf_{t \to 0} \text{Var}[\text{Tr}[e^{-t\hat{H}}]] > 0.
\]

**Randomly Shifted Harmonic Oscillator**

Consider the operator

\[
\text{HO}f := \frac{1}{2}\Delta f(x) + cx^2 f(x)
\]

acting on \(\mathbb{R}\), where \(c > 0\) is some constant. The Hamiltonian of the form (3.1.9) is well-known in the literature as the quantum Harmonic Oscillator (e.g., [Tak08, Section 2.6]). Let \(g \sim N(0, 1)\) be a standard Gaussian random variable. Then, the constant noise defined as \(\xi(x) := g\) for all \(x \in \mathbb{R}\) is a bounded noise with non-compactly-supported covariance function \(\gamma(x) = 1\) (\(x \in \mathbb{R}\)). Thus, the random Schrödinger operator \(\text{HO} + g\) satisfies Assumptions DB, PG, and SN, but fails the growth assumption (3.1.4) since its potential is not superquadratic. We have the following counterexample, which provides another situation where our general sufficient condition is optimal.

**Proposition 3.1.8.** The eigenvalues of \(\text{HO} + g\) are not number rigid.

In light of this result, it is natural to wonder if (3.1.4) not only characterizes the vanishing of the variance (3.1.8) for bounded noise, but in fact provides a characterization of rigidity. However, given that the noise in this particular example is in some sense very degenerate (i.e., \(\xi\) is constant, and thus only applies a random Gaussian shift to the spectrum of \(\text{HO}\)), it is not clear if Proposition 3.1.8 should be considered meaningful evidence for either of these claims.
3.2 Proof Outline

3.2.1 Outline for Theorem 3.1.4

We begin with the outline for Theorem 3.1.4, since it contains the proof that (3.1.1) always holds for some $b > 1$. We want to prove that (3.1.1) holds with the exponent $b$ in (3.1.3). In order to prove this result, we require the following slight improvement of Proposition 2.2.11.

**Proposition 3.2.1.** In the case of a fractional noise with parameter $\mathfrak{f} \in (1/2, 1)$, there exists a constant $c_\gamma > 0$ such that for every $f \in \mathcal{PC}_c$ and $t > 0$, it holds that

$$\|f\|_\gamma^2 \leq c_\gamma t^\mathfrak{f} \left( t^{-1/2} \|f\|_2^2 + t^{-1} \|f\|_1^2 \right)$$

Combining the above with Proposition 2.2.11, it suffices to prove that for every $1 \leq q \leq 2$, one has

$$\limsup_{t \to 0} t^{-(1+1/q)} \left( \sup_{x \in I} E^x \left[ \|L_t(Z)\|_q^{2\theta/\theta} \right]^{1/\theta} \right) < \infty. \quad (3.2.1)$$

We prove this result case by case.

In Case 1, (3.2.1) follows directly from the scaling property (2.4.24) and the fact that $\|L_1(B^0)\|_q^2$ has finite exponential moments of all orders for $1 \leq q \leq 2$ (e.g., (2.4.25)). In Case 2, we combine (2.4.26) with the proof of (3.2.1) in Case 1. Finally, in Case 3, we know from the proof of Lemma 2.4.10 in Case 3 that there exists nonnegative random variables $R_1$ and $R_2$ with finite exponential moments in a neighbourhood of zero such that

$$\sup_{x \in (0,b)} E^x \left[ \|L_t(Y)\|_q^{2\theta} \right] \leq C E \left[ (t^{1+1/q} R_1 + t^2 R_2)^\theta \right]$$
for some constant $C > 0$ for every $t > 0$. Since $q \geq 1$, we have that $1 + 1/q \leq 2$, which concludes the proof of (3.2.1) in Case 3.

3.2.2 Outline for Theorem 3.1.2

The main technical ingredient in the proof of Theorem 3.1.2 is the following quantitative bound on the variance of the trace of $e^{-tH}$ for vanishing $t > 0$.

**Theorem 3.2.2.** Let $b > 1$ be the exponent such that (3.1.1) holds. In Cases 1 & 2, assume that there exists $\kappa, \nu, a > 0$ such that

$$V(x) \geq |\kappa x|^a - \nu \quad \text{for every } x \in I. \quad (3.2.2)$$

In Cases 1 & 2, there exists a constant $C_a > 0$ that only depends on $a$ such that

$$\text{Var}[\text{Tr}[e^{-tH}]] \leq \begin{cases} C_a \frac{e^{2\nu t}}{\kappa} t^{b-1/2-1/a} & \text{if } \gamma \text{ is compactly supported} \\ C_a \frac{e^{2\nu t}}{\kappa^2} t^{b-1-2/a} & \text{otherwise} \end{cases} \quad (3.2.3)$$

for every small enough $t > 0$. In Case 3, there exists $C > 0$ such that

$$\text{Var}[\text{Tr}[e^{-tH}]] \leq Ct^{b-1} \quad (3.2.4)$$

for small enough $t > 0$.

With this in hand, Theorem 3.1.2 now follows from a relatively simple argument: By Corollary (1.4.7), it suffices to prove that

$$\lim_{t \to 0} \text{Var}[\text{Tr}[e^{-tH}]] = 0. \quad (3.2.5)$$

We show that the limit (3.2.5) holds under the conditions stated in Theorem 3.1.2 case by case.
In Case 3, (3.2.5) follows from (3.2.4) and \( b > 1 \). Consider now Cases 1 & 2. If \( V(x)/|x|^a \to \infty \) for some \( a > 0 \), then for every \( \kappa > 0 \), we can take \( \nu_\kappa > 0 \) large enough so that \( V(x) \geq |\kappa x|^a - \nu_\kappa \) for every \( x \in I \). As per (3.1.2), we choose \( a \) such that

\[
\begin{cases}
    b - 1/2 - 1/a = 0 & \text{(if } \gamma \text{ is compactly supported)} \\
    b - 1 - 2/a = 0 & \text{(otherwise)},
\end{cases}
\]

and thus (3.2.3) yields

\[
\limsup_{t \to 0} \text{Var}[\text{Tr}[e^{-tH}]] \leq \begin{cases}
    C_a/\kappa & \text{(if } \gamma \text{ is compactly supported)} \\
    C_a/\kappa^2 & \text{(otherwise)}.
\end{cases}
\]

Since \( \kappa > 0 \) was arbitrary, we then obtain (3.2.5) by taking \( \kappa \to \infty \).

### 3.3 Proof of Theorems 3.1.2 & 3.1.4

We now conclude the proof of Theorems 3.1.2 and 3.1.4 by proving Theorem 3.2.2 and Proposition 3.2.1. We begin with some notations.

**Notation 3.3.1.** For the remainder of Section 3.3, we use \( C, c > 0 \) to denote constants independent of \( \kappa, \nu, \) and \( a \) whose exact values may change from line to line, and we use \( C_a > 0 \) to denote such constants that depend only on \( a \).
Notation 3.3.2. Let $Z$ be as in Definition 2.1.12, and let $\bar{Z}$ be an independent copy of $Z$. For every $t > 0$, we define the following random functions: for $x, y \in I$,

\[ A_t(x, y) := -\langle L_t(Z_t^{x,x}) + L_t(\bar{Z}_t^{y,y}), V \rangle, \]

\[ B_t(x, y) := \begin{cases} 
0 & \text{(Case 1)} \\
\bar{\alpha} \Sigma^0_t(x_t^{x,x}) + \bar{\alpha} \Sigma^0_t(\bar{X}_t^{y,y}) & \text{(Case 2)} \\
\bar{\alpha} \Sigma^0_t(Y_t^{x,x}) + \bar{\beta} \Sigma^b_t(Y_t^{y,y}) + \bar{\alpha} \Sigma^0_t(\bar{Y}_t^{y,y}) + \bar{\beta} \Sigma^b_t(\bar{Y}_t^{y,y}) & \text{(Case 3)}.
\end{cases} \]

\[ C_t(x, y) := \frac{\|L_t(Z_t^{x,x})\|_2^2 + \|L_t(\bar{Z}_t^{y,y})\|_2^2}{2}, \]

\[ D_t(x, y) := \langle L_t(Z_t^{x,x}), L_t(\bar{Z}_t^{y,y}) \rangle, \gamma, \]

\[ P_t(x, y) := \Pi_2(t; x, x) \Pi_2(t, y, y). \]

### 3.3.1 Theorem 3.2.2, Step 1. Variance Formula

The first step in the proof of Theorem 3.2.2 is the following variance formula, which follows from a combination of the definition of $\hat{K}(t)$ in Definition 2.1.16 and Theorem 2.1.19 (2).

**Lemma 3.3.3.** Following Notation 3.3.2, it holds that

\[ \text{Var} \left[ \text{Tr} [e^{-t\hat{H}}] \right] = \int_{I^2} P_t(x, y) E \left[ e^{(A_t + B_t + C_t)(x, y)} \left( e^{D_t(x, y)} - 1 \right) \right] \, dx \, dy. \quad (3.3.1) \]

**Proof.** Recall the shorthand $A_t(Z)$ introduced in (2.4.28). By Theorem 2.1.19 (2), and a similar computation to (2.4.37) and (2.4.44), it follows from Fubini’s theorem that

\[ E \left[ \text{Tr} [e^{-t\hat{H}}] \right] = \int_I \Pi_2(t; x, x) E_t^{x,x} \left[ e^{A_t(Z)} \left( e^{-\xi(L_t(Z))} \right) \right] \, dx \]

\[ = \int_I \Pi_2(t; x, x) E_t^{x,x} \left[ e^{A_t(Z)} \frac{1}{2} \|L_t(Z)\|_2^2 \right] \, dx. \]
Using Fubini’s theorem once again, we get

\[
E[\text{Tr}[e^{-t\hat{H}}]^2] = \int_{I^2} \mathcal{P}_t(x,y) E \left[ e^{\left( \mathfrak{A}_t(Z_{t}^{x,x}) + \mathfrak{A}_t(Z_{t}^{y,y}) + \frac{1}{2} \| L_t(Z_{t}^{x,x}) \|^2 + \frac{1}{2} \| L_t(Z_{t}^{y,y}) \|^2 \right)} \right] \, dx \, dy
\]

\[
= \int_{I^2} \mathcal{P}_t(x,y) E \left[ e^{(A_t+B_t+C_t)(x,y)} \right] \, dx \, dy. \tag{3.3.2}
\]

A similar computation yields

\[
E[\text{Tr}[e^{-t\hat{H}}]^2] = \int_{I^2} \mathcal{P}_t(x,y) E \left[ e^{-\left( L_t(Z_{t}^{x,x}) + L_t(Z_{t}^{y,y}) \right)} E \left[ e^{-\xi(L_t(Z_{t}^{x,x})) + \xi(L_t(Z_{t}^{y,y}))} \right] \right] \, dx \, dy.
\]

Given that \( \xi(L_t(Z_{t}^{x,x})) + \xi(L_t(Z_{t}^{y,y})) \) is Gaussian with mean zero and variance

\[
\| L_t(Z_{t}^{x,x}) \|^2 \gamma + \| L_t(Z_{t}^{y,y}) \|^2 \gamma + 2(L_t(Z_{t}^{x,x}), L_t(Z_{t}^{y,y})) \gamma,
\]

we may now write

\[
E[\text{Tr}[e^{-t\hat{H}}]^2] = \int_{I^2} \mathcal{P}_t(x,y) E \left[ e^{(A_t+B_t+C_t+D_t)(x,y)} \right] \, dx \, dy.
\]

Finally, the result follows by subtracting (3.3.2) from the above. \( \square \)

3.3.2 Theorem 3.2.2, Step 2. Bounded Terms

Thanks to (3.3.1), it follows from Hölder’s inequality that

\[
\text{Var}[\text{Tr}[e^{-t\hat{H}}]] \leq \int_{I^2} \mathcal{P}_t(x,y) E \left[ e^{4A_t(x,y)} \right]^{1/4} E \left[ e^{4B_t(x,y)} \right]^{1/4}
\]

\[
\times E \left[ e^{4C_t(x,y)} \right]^{1/4} E \left[ (e^{D_t(x,y)} - 1)^4 \right]^{1/4} \, dx \, dy. \tag{3.3.3}
\]
The second step in the proof of Theorem 3.2.2 is to show that the terms involving \( B_t(x, y) \) and \( C_t(x, y) \) in (3.3.3) are uniformly bounded for small \( t \). Before we state our result to this effect, we need the following refinement of Lemma 2.4.7 for small \( t > 0 \).

**Lemma 3.3.4.** There exist constants \( 0 < c < C \) such that for every \( t \in (0, 1] \),

\[
ct^{-1/2} \leq \inf_{x \in I} \Pi_Z(t; x, x) \quad \text{and} \quad \sup_{x, y \in I} \Pi_Z(t; x, y) \leq C t^{-1/2}.
\]

In particular, if we define \( s_t(Z) \) as in (2.4.16), then

\[
\sup_{t \in (0, 1]} s_t(Z) < \infty. \tag{3.3.4}
\]

**Proof.** In Case 1, the result follows directly from the fact that \( \Pi_B(t; x, y) \leq 1 / \sqrt{2\pi t} \) and \( \Pi_B(t; x, x) = 1 / \sqrt{2\pi t} \) for all \( x, y \) and \( t \). A similar argument holds for Case 2. Consider now Case 3. We recall that

\[
\Pi_Y(t; x, y) = \frac{1}{\sqrt{2\pi t}} \left( \sum_{k \in \mathbb{Z}} e^{-(x-2bk+y)^2/2t} + e^{-(x-2bk-y)^2/2t} \right).
\]

On the one hand, note that \( t \mapsto e^{-z/t} \) is increasing in \( t > 0 \) for every \( z \geq 0 \); hence for every \( t \in (0, 1] \), one has

\[
\sup_{(x, y) \in (0,b)^2} \left( \sum_{k \in \mathbb{Z}} e^{-(x-2bk+y)^2/2t} + e^{-(x-2bk-y)^2/2t} \right) \leq \sup_{(x, y) \in (0,b)^2} \left( \sum_{k \in \mathbb{Z}} e^{-(x-2bk+y)^2/2t} + e^{-(x-2bk-y)^2/2t} \right) < \infty.
\]

On the other hand, by isolating the \( k = 0 \) term in \( \sum_{k \in \mathbb{Z}} e^{-(2bk)^2/2t} \),

\[
\inf_{x \in (0,b)} \left( \sum_{k \in \mathbb{Z}} e^{-(2x-2bk)^2/2t} + e^{-(2bk)^2/2t} \right) \geq \left( \inf_{x \in (0,b)} \sum_{k \in \mathbb{Z}} e^{-(2x-2bk)^2/2t} \right) + 1 \geq 1,
\]

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concluding the proof.

Lemma 3.3.5. For any $\theta > 0$,

\[
\limsup_{t \to 0} \sup_{x,y \in I} E \left[ e^{\theta B_t(x,y)} \right] < \infty, \tag{3.3.5}
\]

\[
\limsup_{t \to 0} \sup_{x,y \in I} E \left[ e^{\theta C_t(x,y)} \right] < \infty. \tag{3.3.6}
\]

Proof. We start by proving (3.3.5). In Case 1 the result is trivial. In Case 2, by independence, we have that

\[
E \left[ e^{\theta B_t(x,y)} \right] = E_t^{x,x} \left[ e^{\theta \alpha L^0_0(X)} \right] E_t^{y,y} \left[ e^{\theta \alpha L^0_0(X)} \right],
\]

and thus it suffices to prove that

\[
\limsup_{t \to 0} \sup_{x \in I} E_t^{x,x} \left[ e^{\theta L^0_0(X)} \right]
\]

for every $\theta > 0$. By using essentially the same argument leading up to (2.4.20) together with the bound (3.3.4), it suffices to prove that for every $\theta > 0$,

\[
\lim_{t \to 0} E^x \left[ e^{\theta L^0_0(X)} \right] < \infty.
\]

By coupling $X^x(s) = |B^x(s)|$ for all $s \geq 0$, we have that $L^0_0(X^x) = L^0_0(B^x)$. Thus, the desired claim follows from the scaling identity (2.4.15) and dominated convergence. In Case 3, we similarly argue that it suffices to prove

\[
\limsup_{t \to 0} \sup_{x \in I} E^x \left[ e^{\theta L^0_0(Y)} \right],
\]

which follows from the bound $\sup_{x \in I} E^x \left[ e^{\theta L^0_0(Y)} \right] \leq K e^{K't}$ from [Pap88] used in the proof of Lemma 2.4.6.
We now prove (3.3.6). By independence,

\[
E[e^{\theta C_t(x,y)}] = E^x_t \left[ e^{\frac{\theta}{2} \|L_t(Z)\|_\gamma^2} \right] E^y_t \left[ e^{\frac{\theta}{2} \|L_t(Z)\|_\gamma^2} \right]
\]

Therefore, by using the midpoint-conditioning argument leading up to (2.4.27), it suffices to prove that

\[
\limsup_{t \to 0} \sup_{x \in I} E^x_t \left[ e^{\frac{\theta}{2} \|L_t(Z)\|_\gamma^2} \right] < \infty
\]

for all \( \theta > 0 \), which, by (2.1.9), can be further reduced to

\[
\limsup_{t \to 0} \sup_{x \in I} E^x_t \left[ e^{\frac{\theta}{2} \|L_t(Z)\|_\gamma^2} \right] < \infty
\]

for every \( \theta > 0 \) and \( 1 \leq q \leq 2 \). This follows directly from (2.4.22)/(2.4.23) (which is proved in Lemma 2.4.10) and dominated convergence. \( \square \)

3.3.3 Theorem 3.2.2, Step 3. Vanishing Term

By applying Lemma 3.3.5 and the upper bound on \( \Pi_Z(t; x, y) \) in Lemma 3.3.4, we can refine (3.3.3) to the following upper bound for small enough \( t > 0 \):

\[
\text{Var}[\text{Tr}[e^{-t\hat{H}}]] \leq Ct^{-1} \int_{I^2} E \left[ e^{4A_t(x,y)} \right]^{1/4} E \left[ \left( e^{D_t(x,y)} - 1 \right)^4 \right]^{1/4} \, dx \, dy. \tag{3.3.7}
\]

The third step in the proof of Theorem 3.2.2 is to understand the vanishing rate of \( e^{D_t(x,y)} - 1 \) as \( t \to 0 \).

Lemma 3.3.6. Let \( b \) be as in (3.1.1). For any \( \theta > 0 \),

\[
\sup_{x,y \in I} E \left[ \left| e^{D_t(x,y)} - 1 \right|^\theta \right]^{1/\theta} \leq Ct^b \quad \text{for small enough } t > 0.
\]
Proof. By Cauchy-Schwarz and $|z \bar{z}| \leq \frac{1}{2}(z^2 + \bar{z}^2)$, we have that

$$|D_t(x, y)| \leq \frac{1}{2}(\|L_t(Z^{x,x}_t)\|_\gamma^2 + \|L_t(\bar{Z}^{y,y}_t)\|_\gamma^2).$$

By combining this with $|e^z - 1| \leq e|z| - 1 \leq |z|e|z|$ and the triangle inequality,

$$\mathbb{E}\left[|e^{D_t(x, y)} - 1|^{1/\theta}\right]^{\frac{1}{2}} \leq C \left(\mathbb{E}\left[\|L_t(Z^{x,x}_t)\|^{2\theta}_\gamma e^{(\theta/2)(\|L_t(Z^{x,x}_t)\|_\gamma^2 + \|L_t(\bar{Z}^{y,y}_t)\|_\gamma^2)}\right]^{1/\theta} + \mathbb{E}\left[\|L_t(\bar{Z}^{y,y}_t)\|^{2\theta}_\gamma e^{(\theta/2)(\|L_t(Z^{x,x}_t)\|_\gamma^2 + \|L_t(\bar{Z}^{y,y}_t)\|_\gamma^2)}\right]^{1/\theta}\right).$$

By using independence of $Z$ and $\bar{Z}$ and applying Hölder’s inequality, we get the further upper bound

$$C \left(\mathbb{E}_t^{x,x}\left[\|L_t(Z)\|^{4\theta}_\gamma\right]^{1/2\theta} \mathbb{E}_t^{x,x}\left[e^{\theta\|L_t(Z)\|_\gamma^2}\right]^{1/2\theta} \mathbb{E}_t^{y,y}\left[e^{(\theta/2)(\|L_t(Z)\|_\gamma^2)}\right]^{1/\theta} + \mathbb{E}_t^{y,y}\left[\|L_t(Z)\|^{4\theta}_\gamma\right]^{1/2\theta} \mathbb{E}_t^{y,y}\left[e^{\theta\|L_t(Z)\|_\gamma^2}\right]^{1/2\theta} \mathbb{E}_t^{x,x}\left[e^{(\theta/2)(\|L_t(Z)\|_\gamma^2)}\right]^{1/\theta}\right).$$

Arguing as in the proof of (3.3.6), for every $\theta > 0$,

$$\sup_{x \in I} \mathbb{E}_t^{x,x}\left[e^{\theta\|L_t(Z)\|_\gamma^2}\right] \leq C$$

for small enough $t > 0$. At this point, to complete the proof of Lemma 3.3.6, it suffices to show that for every $\theta > 0$,

$$\limsup_{t \to 0} t^{-b}\left(\sup_{x \in I} \mathbb{E}_t^{x,x}\left[\|L_t(Z)\|^{2\theta}_\gamma\right]^{1/\theta}\right) < \infty. \tag{3.3.8}$$

We claim that (3.3.8) is a consequence of (3.1.1). To see this, we once again condition on the midpoint of the process $Z^{x,x}_t$: with $\mathfrak{s}_t(Z) < \infty$ as in (2.4.16), we
obtain that for any $t \in (0, 1]$,

$$
\mathbb{E}_t^{x,x}[\|L_t(Z)\|_\gamma^{2\theta}]
\leq \int I_t \mathbb{E}_t^{x,x}[\|L_t(Z)\|_\gamma^{2\theta} \mid Z_t^{x,x}(t/2) = z] \frac{\Pi_Z(t/2; x, z) \Pi_Z(t/2; z, x)}{\Pi_Z(t; x, x)} \, dz
\leq s_t(Z) \int I_t \mathbb{E}_t^{x,x}[\|L_{t/2}(Z) + L_{[t/2,t]}(Z)\|_\gamma^{2\theta} \mid Z_t^{x,x}(t/2) = z] \Pi_Z(t/2; x, z) \, dz
\leq 2^{\max\{2\theta, 1\}} s_t(Z) \int I_t \mathbb{E}_t^{x,x}[\|L_{t/2}(Z)\|_\gamma^{2\theta}] \Pi_Z(t/2; x, z) \, dz
= C \mathbb{E}_t^{x}[\|L_{t/2}(Z)\|_\gamma^{2\theta}].
$$

where the equality on the second line follows from the Doob h-transform (see (2.4.18)); the inequality on the fourth line follows from first applying the triangle inequality to bound $\|L_{t/2}(Z) + L_{[t/2,t]}(Z)\|_\gamma^{2\theta}$ by $2^{\max\{2\theta, 1\}}(\|L_{t/2}(Z)\|_\gamma^{2\theta} + \|L_{[t/2,t]}(Z)\|_\gamma^{2\theta})$, and then using the fact that, under the conditioning $Z_t^{x,x}(t/2) = z$, the local time processes $L_{t/2}(Z_t^{x,x})$ and $L_{[t/2,t]}(Z_t^{x,x})$ are i.i.d. copies of $L_{t/2}(Z_t^{x,z})$. \qed

### 3.3.4 Theorem 3.2.2, Step 4. Final Estimates

We now have all necessary ingredients to conclude the proof of Theorem 3.2.2. We argue the result case by case.

We start with Case 3, as it is the easiest. By Assumption PG, $V$ is nonnegative; hence $e^{A_t(x,y)} \leq 1$. By (3.3.7), we then have

$$
\text{Var} \left[ \text{Tr} [e^{-tH}] \right] \leq C t^{-1} \int_{(0,b)^2} \mathbb{E} \left[ (e^{D_t(x,y)} - 1)^4 \right]^{1/4} \, dx \, dy,
$$

which proves (3.2.4) by Lemma 3.3.6.
Next, we prove the result in Case 1. By coupling $B_t^{x,x} := x + B_t^{0,0}$ and similarly for $B_t^{y,y}$, it follows from (3.2.2) that

$$A_t(x, y) \leq 2\nu t - \kappa a \int_0^t \left( |x + B_t^{0,0}(s)|^a + |y + B_t^{0,0}(s)|^a \right) ds.$$  (3.3.9)

Then, by the change of variables $s \mapsto st$ and a Brownian scaling, we obtain that the right-hand side of (3.3.9) is equal to

$$2\nu t - \kappa a \int_0^1 \left( |t^{\frac{1}{a}} x + t^{\frac{1}{a}} B_t^{0,0}(st)|^a + |t^{\frac{1}{a}} y + t^{\frac{1}{a}} B_t^{0,0}(st)|^a \right) ds$$

$$\overset{d}{=} 2\nu t - \kappa a \int_0^1 \left( |t^{\frac{1}{a}} x + t^{\frac{1}{a}} + \frac{1}{a} B_1^{0,0}(s)|^a + |t^{\frac{1}{a}} y + t^{\frac{1}{a}} + \frac{1}{a} B_1^{0,0}(s)|^a \right) ds.$$

To alleviate notation, let us introduce the shorthands

$$\mathcal{B}_t(x) := |t^{\frac{1}{a}} x + t^{\frac{1}{a}} + \frac{1}{a} B_1^{0,0}(s)|^a, \quad \mathcal{B}_t(y) := |t^{\frac{1}{a}} y + t^{\frac{1}{a}} + \frac{1}{a} B_1^{0,0}(s)|^a.$$

Consider first the case of general $\gamma$ (i.e., not necessarily compactly supported). If we combine Lemma 3.3.6 and (3.3.7) with (3.3.9), then we obtain that, for small $t > 0$,

$$\text{Var} \left[ \text{Tr}[e^{-t\hat{H}}] \right] \leq C e^{2\nu t} t^{b-1} \int_{\mathbb{R}^2} E \left[ e^{-\kappa a \int_0^t \mathcal{B}_t(x) + \mathcal{B}_t(y)} ds \right] dxdy$$

$$= C e^{2\nu t} t^{b-2/a} \int_{\mathbb{R}^2} E \left[ e^{-\kappa a \int_0^t \mathcal{B}_{t-1/a_x}(s) + \mathcal{B}_{t-1/a_y}(s)} ds \right] dxdy$$

where in the second line we used the change of variables $(x, y) \mapsto t^{-1/a}(x, y)$. If we write

$$\mathcal{F}_t(x, y) := e^{-\kappa a \int_0^t \left( \mathcal{B}_{t-1/a_x}(s) + \mathcal{B}_{t-1/a_y}(s) \right)} ds,$$  (3.3.11)
then for every fixed $x, y \in \mathbb{R}$,

$$\lim_{t \to \infty} \mathcal{F}_t(x, y) = e^{-|\kappa x|^a - |\kappa y|^a} \quad (3.3.12)$$

almost surely. Moreover, for every $z, \bar{z} \in \mathbb{R}$,

$$|z + \bar{z}|^a \geq |z + \bar{z}|_{\min\{a,1\}} - 1 \geq |z|_{\min\{a,1\}} - |\bar{z}|_{\min\{a,1\}} - 1,$$

and thus

$$\sup_{t \in (0,1]} \mathcal{F}_t(x, y) \leq \exp\left( - |\kappa x|_{\min\{a,1\}} - |\kappa y|_{\min\{a,1\}} \right) \times \exp\left( \kappa_{\min\{a,1\}} \left( 2 + \sup_{s \in [0,1]} |B^{0,0}_1(s)|_{\min\{a,1\}} + \sup_{s \in [0,1]} |\bar{B}^{0,0}_1(s)|_{\min\{a,1\}} \right) \right). \quad (3.3.13)$$

Recall that the process $s \mapsto |B^{0,0}_1(s)|$ is a Bessel bridge of dimension one. As argued in the proof of Lemma 2.4.13, we know by [GS96, Remark 3.1] that Bessel bridge maxima have finite exponential moments of all orders. Consequently, given that $(x, y) \mapsto \exp(-|\kappa x|_{\min\{a,1\}} - |\kappa y|_{\min\{a,1\}})$ is integrable on $\mathbb{R}^2$, it follows from the dominated convergence theorem that

$$\lim_{t \to 0} \int_{\mathbb{R}^2} \mathbb{E}[\mathcal{F}_t(x, y)] \, dx \, dy = \int_{\mathbb{R}^2} e^{-|\kappa x|^a - |\kappa y|^a} \, dx \, dy = \left( \frac{2 \Gamma(1 + 1/a)}{\kappa} \right)^2.$$

Combining this limit with (3.3.10) yields the upper bound

$$\text{Var}[\text{Tr}[e^{-t\hat{H}}]] \leq C_a e^{2\nu t} \kappa^a t^{a-1-2/a} \quad \text{for small } t > 0 \quad (3.3.14)$$

for general $\gamma$.

Consider now Case 1 when $\gamma$ is compactly supported, that is, there exists some $K > 0$ such that $\langle f, \gamma \rangle = 0$ whenever $f(z) = 0$ for every $z \in [-K, K]$. In this case, in
order for $\mathcal{D}_t(x,y) = \langle L_t(B_{t,x}^x, L_t(\bar{B}_{t,y}^y)) \rangle$ to be nonzero, it is necessary that

$$\begin{cases}
\max_{0 \leq s \leq t} B_{t,x}^x(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_{t,y}^y(s) \quad \text{(if } x \leq y) \\
\max_{0 \leq s \leq t} \bar{B}_{t,y}^y(s) + K \geq \min_{0 \leq s \leq t} B_{t,x}^x(s) \quad \text{(if } x \geq y). 
\end{cases}$$

In the case where $x \leq y$, this means that

$$\mathbb{E}\left[ \left( e^{\mathcal{D}_t(x,y)} - 1 \right)^4 \right] = \mathbb{E}\left[ 1_{\{\max_{0 \leq s \leq t} B_{t,x}^x(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_{t,y}^y(s)\}} \left( e^{\mathcal{D}_t(x,y)} - 1 \right)^4 \right]^{1/4} \leq \mathbb{P}\left[ \max_{0 \leq s \leq t} B_{t,x}^x(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_{t,y}^y(s) \right]^{1/8} \mathbb{E}\left[ \left( e^{\mathcal{D}_t(x,y)} - 1 \right)^8 \right]^{1/8} \leq C t^b \mathbb{P}\left[ \max_{0 \leq s \leq t} B_{t,x}^x(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_{t,y}^y(s) \right]^{1/8},$$

where the third line follows from Hölder’s inequality and the last line from Lemma 3.3.6. Then, by combining a Brownian scaling with the fact that the maxima of Brownian bridges have sub-Gaussian tails, we obtain

$$\mathbb{P}\left[ \max_{0 \leq s \leq t} B_{t,x}^x(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_{t,y}^y(s) \right]^{1/8} = \mathbb{P}\left[ \max_{0 \leq s \leq 1} B_{1,x}^{0,0}(s) + \max_{0 \leq s \leq 1} \bar{B}_{1,y}^{0,0}(s) \geq (y - x - K) / t^{1/2} \right]^{1/8} \leq C e^{-(y-x-K)^2/2d t}.
$$

A similar bound is obtained when $x \geq y$. Consequently, a minor modification of the argument leading up to (3.3.10) yields, for small $t > 0$,

$$\text{Var}[\text{Tr}[e^{-t \hat{B}}]] \leq C e^{2d t} t^{b-1} \int_{\mathbb{R}^2} \mathbb{E}\left[ e^{-\kappa \oint_0^t (\partial_t x(s) + \partial_t y(s)) \, ds} \right] e^{-\frac{(y-x-K)^2}{2d t}} \, dx \, dy.$$
By applying the change of variables \((x, y) \mapsto t^{1/a}(x, y)\), we see that the integral on the right-hand side of the above equation is equal to

\[
t^{-2/a} \int_{\mathbb{R}^2} E[\mathcal{F}_t(x, y)] e^{-\frac{|x-y| - t^{1/a}K}{2ct^{1+2/a}}} \, dx \, dy
\]

\[
= \sqrt{2\pi c} \cdot t^{1/2-1/a} \int_{\mathbb{R}^2} E[\mathcal{F}_t(x, y)] \frac{e^{-\frac{|x-y| - t^{1/a}K}{2ct^{1+2/a}}}}{\sqrt{2\pi ct^{1+2/a}}} \, dx \, dy, \quad (3.3.15)
\]

where we reuse the notation \(\mathcal{F}_t\) introduced in (3.3.11). Given that

\[
(|x - y| - t^{1/a}K)^2 \geq \min \{(x - y - t^{1/a}K)^2, (x - y + t^{1/a}K)^2\},
\]

we observe that

\[
e^{-\frac{|x-y| - t^{1/a}K}{2ct^{1+2/a}}} \leq e^{-(x-y-t^{1/a}K)^2/2ct^{1+2/a}} + e^{-(x-y+t^{1/a}K)^2/2ct^{1+2/a}}
\]

which yields

\[
\frac{e^{-\frac{|x-y| - t^{1/a}K}{2ct^{1+2/a}}}}{\sqrt{2\pi ct^{1+2/a}}} \leq \mathcal{G}_{ct^{1+2/a}}(x - y - t^{1/a}K) + \mathcal{G}_{ct^{1+2/a}}(x - y + t^{1/a}K),
\]

where we recall that \(\mathcal{G}_t\) denotes the Gaussian kernel (Definition 2.1.12). By combining this observation with (3.3.13) and substituting into (3.3.15), we conclude that \(\text{Var}[\text{Tr}[e^{-t\hat{H}}]]\) is bounded above by

\[
C_a e^{2ct} t^{b-1/2-1/a} \left( \int_{\mathbb{R}^2} e^{-|x|^{\min(a, 1)} - |y|^{\min(a, 1)}} \mathcal{G}_{ct^{1+2/a}}(x - y - t^{1/a}K) \, dx \, dy + \int_{\mathbb{R}^2} e^{-|x|^{\min(a, 1)} - |y|^{\min(a, 1)}} \mathcal{G}_{ct^{1+2/a}}(x - y + t^{1/a}K) \, dx \, dy \right).
\]

By a change of variables and the fact that the Gaussian kernel is an approximate identity (i.e., \(\mathcal{G}_t \to \delta_0\) as \(t \to 0\)), the integrals above have the following limits by
dominated convergence

\[
\lim_{t \to 0} \int_{\mathbb{R}} e^{-|\kappa(x \pm t^{1 \slash a} K)|^{\min{\{a,1\}}}} \left( \int_{\mathbb{R}} e^{-|\kappa y|^{\min{\{a,1\}}}} g_{ct+2 \slash a}(x-y) \ dy \right) \ dx \\
= \int_{\mathbb{R}} e^{-2|\kappa x|^{\min{\{a,1\}}}} \ dx = \frac{2^{1-1 \slash \min{\{a,1\}}} \Gamma\left(1+1 \slash \min{\{a,1\}}\right)}{\kappa}.
\]

Combining this result with (3.3.15) yields

\[
\text{Var}[\text{Tr}[e^{-tH}]] \leq \frac{C_a e^{2\nu t}}{\kappa} \ t^{b-1/2-1/a} \quad \text{for small } t > 0 \quad (3.3.16)
\]

whenever \( \gamma \) is compactly supported, completing the proof of (3.2.3) for Case 1.

We now finish the proof of Theorem 3.2.2 with Case 2. Once again we begin with general \( \gamma \). Since \( V(x) \geq |\kappa x|^a - \nu \),

\[
\mathbb{E}\left[e^{4A_t(x,y)}\right]^{1/4} \leq e^{2\nu t} \mathbb{E}\left[e^{-4\kappa a \int_0^t (|X_t^{x,x}(s)|^a + |X_t^{y,y}(s)|^a) \ ds}\right]^{1/4}.
\]

Thanks to (2.4.9), we then have that

\[
\mathbb{E}\left[e^{4A_t(x,y)}\right]^{1/4} \leq C e^{2\nu t} \mathbb{E}\left[e^{-4\kappa a \int_0^t (|x+B_0^{0,0}(s)|^a + |x+B_0^{0,0}(s)|^a) \ ds}\right]^{1/4}.
\]

Combining this with Lemma 3.3.6, we obtain (3.3.14) in Case 2 for general \( \gamma \) by using the same argument as in Case 1.

Finally, consider Case 2 when \( \gamma \) is compactly supported. Similarly to Case 1, by Hölder’s inequality we have the upper bound

\[
\mathbb{E}\left[(e^{D_t(x,y)} - 1)^4\right]^{1/4} \leq P[\langle L_t(X_t^{x,x}), L_t(X_t^{y,y})\rangle_{\gamma} \neq 0]^{1/8} \mathbb{E}\left[(e^{D_t(x,y)} - 1)^8\right]^{1/8}.
\]
By (2.4.9), we get the further upper bound
\[
\mathbb{P}[\langle L_t([B_t^{x,x}]), L_t([\bar{B}_t^{y,y}]) \rangle_{\gamma} \neq 0]^{1/8} \leq 2^{1/8} \mathbb{P}[\langle L_t([B_t^{x,x}]), L_t([\bar{B}_t^{y,y}]) \rangle_{\gamma} \neq 0]^{1/8}.
\]

As \( L_t^a([B_t^{x,x}]) = L_t^a(B_t^{x,x}) + L_t^{-a}(B_t^{x,x}) \) for all \( a > 0 \) and similarly for \( \bar{B}_t^{y,y} \),
\( \langle L_t([B_t^{x,x}]), L_t([\bar{B}_t^{y,y}]) \rangle_{\gamma} \) can be expanded as the sum
\[
\int_{(0,\infty)^2} L_t^a(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y}) \, db \, da + \int_{(0,\infty)^2} L_t^{-a}(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y}) \, db \, da
\]
\[
+ \int_{(0,\infty)^2} L_t^{-a}(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y}) \, db \, da + \int_{(0,\infty)^2} L_t^a(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y}) \, db \, da.
\]

Define the set \( S := (-\infty,0)^2 \cup (0,\infty)^2 \). Since \( \gamma \) is even, by a simple change of variables, the first two terms in the above sum add up to
\[
\int_{S} L_t^a(B_t^{x,x})\gamma(a-b)L_t^b(\bar{B}_t^{y,y}) \, db \, da,
\]
and the last two terms add up to
\[
\int_{S} L_t^{-a}(B_t^{x,x})\gamma(a-b)L_t^{-b}(\bar{B}_t^{y,y}) \, db \, da.
\]

Assume that \( 0 < x \leq y \). In order for (3.3.17) to be nonzero, it is necessary that
\[
\max_{0 \leq s \leq t} B_t^{x,x}(s) + K \geq \min_{0 \leq s \leq t} \bar{B}_t^{y,y}(s),
\]
and for (3.3.18) to be nonzero, it must be the case that
\[
- \min_{0 \leq s \leq t} \bar{B}_t^{y,y}(s) + K \geq \min_{0 \leq s \leq t} B_t^{x,x}(s).
\]
Thus, by a union bound, followed by Brownian scaling and the fact that Brownian bridge maxima have sub-Gaussian tails, we see that

\[
P[|\langle L_t(|B_t^{x,x}|), L_t(|\bar{B}_t^{y,y}|)\rangle_{\gamma} > 0]^{1/8} \\
\leq P\left[\max_{0 \leq s \leq 1} B_1^{0,0}(s) + \max_{0 \leq s \leq 1} \bar{B}_1^{0,0}(s) \geq \frac{y - x - K}{t^{1/2}}\right]^{1/8} \\
+ P\left[\max_{0 \leq s \leq 1} B_1^{0,0}(s) + \max_{0 \leq s \leq 1} \bar{B}_1^{0,0}(s) \geq \frac{x + y - K}{t^{1/2}}\right]^{1/8} \\
\leq C \left(e^{-((x-y)-K)^2/2ct} + e^{-((x+y)-K)^2/2ct}\right).
\]

The same bound holds for \(y \leq x\). At this point, given that for any function \(F\),

\[
\int_0^\infty F(x, y)\left(e^{-((x-y)-K)^2/2ct} + e^{-((x+y)-K)^2/2ct}\right) dy = \int_\mathbb{R} F(x, |y|)e^{-((x-y)-K)^2/2ct} dy,
\]

we obtain (3.3.16) for compactly supported \(\gamma\) by using the same argument as Case 1.

### 3.3.5 Proof of Proposition 3.2.1

Arguing as in the proof of Proposition 2.2.11,

\[
\|f\|_\gamma^2 \leq \bar{c} \int_{\mathbb{R}^2} \frac{|f(a)f(b)|}{|a-b|^{2-2H}} dadb
\]

for some constant \(\bar{c} > 0\). By applying the change of variables \((a, b) \mapsto t^{1/2}(a, b)\) to the right-hand side of this equation and splitting the integral as in (2.4.45), we are led to

\[
\int_{\mathbb{R}^2} \frac{|f(a)f(b)|}{|a-b|^{2-2\delta}} dadb = t^{\delta} \int_{\mathbb{R}^2} \frac{|f(t^\delta a)f(t^\delta b)|}{|a-b|^{2-2\delta}} dadb \\
= t^{\delta} \left(\int_{\{|b-a|<1\}} \frac{|f(t^\delta a)f(t^\delta b)|}{|a-b|^{2-2\delta}} dadb + \int_{\{|b-a|\geq 1\}} \frac{|f(t^\delta a)f(t^\delta b)|}{|a-b|^{2-2\delta}} dadb\right).
\]
By Young’s convolution inequality the first integral (over \(|b - a| < 1\)) is bounded above by
\[
\left( \int_{-1}^{1} \frac{1}{|z|^{2-2H}} \, dz \right) \left( \int_{\mathbb{R}} f(tz) \, da \right) = \left( \int_{-1}^{1} \frac{1}{|z|^{2-2H}} \, dz \right) t^{-\frac{1}{2}} \|f\|_{2}^{2},
\]
where the right-hand side comes from the change of variables \(a \mapsto t^{-\frac{1}{2}}a\). By the same change of variables, the second integral (over \(|b - a| \geq 1\)) is bounded by
\[
\left( \int_{\mathbb{R}} |f(tz)| \, da \right)^{2} = t^{-1} \|f\|_{1}^{2},
\]
concluding the proof of the proposition.

3.4 Optimality Counterexamples

3.4.1 Proof of Proposition 3.1.6

We recall that for every \(\beta > 0\), the Airy-\(\beta\) point process, which we denote by \(\mathfrak{Ai}_\beta\), is defined as the eigenvalue point process of \(-\text{SAO}_\beta\), where we use \(\text{SAO}_\beta\) to denote the stochastic Airy operator of parameter \(\beta\) with Dirichlet boundary condition at the origin (see Definition 1.4.5). In the special case \(\beta = 2\), the Airy-\(\beta\) process has an alternative definition. That is, \(\mathfrak{Ai}_2\) is the determinantal point process with kernel

\[
\mathfrak{K}(x, y) := \begin{cases} 
\frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y} & \text{if } x \neq y \\
\text{Ai}'(x)^2 - x\text{Ai}(x)^2 & \text{if } x = y,
\end{cases}
\]

where \(\text{Ai}\) denotes the Airy function
\[
\text{Ai}(x) := \frac{1}{\pi} \int_{0}^{\infty} \cos \left( \frac{u^{3}}{3} + xu \right) \, du, \quad x \in \mathbb{R},
\]
which we recall solves the ODE

\[ \text{Ai}''(x) = -x \text{Ai}(x), \quad x \in \mathbb{R}. \]  

(3.4.2)

Denote \( f_t(x) := e^{tx} \) for all \( t > 0 \). By classical formulas for the variance of linear statistics of Determinantal point processes (e.g., [Gho15, Equation (8)]), it holds that\(^1\)

\[ \text{Var}[ \text{Tr}[e^{-2t\hat{H}}]] = \text{Var}[\text{Ai}_2(f_t)] = \frac{1}{2} \int_{\mathbb{R}^2} (e^{tx} - e^{ty})^2 \mathcal{K}(x, y)^2 \, dx \, dy. \]  

(3.4.3)

Expanding the square and using the identity \( \mathcal{K}(x, x) = \int_{\mathbb{R}^2} \mathcal{K}(x, y)^2 \, dy \) (as \( \mathcal{K} \) is a symmetric projection kernel [TW94, Lemma 2]), we obtain

\[ \text{Var}[\text{Ai}_2(f_t)] = \int_{\mathbb{R}} e^{2tx} \mathcal{K}(x, x) \, dx - \int_{\mathbb{R}^2} e^{(x+y)} \mathcal{K}(x, y)^2 \, dx \, dy. \]

The computation that follows is largely inspired from [Oko02]. We provide it in full for convenience. By a combination of the fundamental theorem of calculus and the identity (3.4.2), we have that

\[ \mathcal{K}(x, y) = \int_0^\infty \text{Ai}(u + x) \text{Ai}(u + y) \, du. \]

Consequently, it follows from Fubini’s theorem that we can write (3.4.3) as the difference \( E_1(t) - E_2(t) \), where

\[ E_1(t) := \int_{\mathbb{R}} e^{2tx} \left( \int_0^\infty \text{Ai}(u + x)^2 \, du \right) \, dx = \int_0^\infty \left( \int_{\mathbb{R}} e^{2tx} \text{Ai}(u + x)^2 \, dx \right) \, du, \]

\(^1\)The variance formula in question is usually stated for compactly supported test functions. The formula can easily be extended to \( f_t \) by dominated convergence, thanks to standard asymptotics for the Airy function such as [AS64, 10.4.59–10.4.62].
and

\[ E_2(t) := \int_{\mathbb{R}^2} e^{t(x+y)} \left( \int_0^\infty \int_0^\infty \text{Ai}(u+x)\text{Ai}(u+y)\text{Ai}(v+x)\text{Ai}(v+y) \, du \, dv \right) \, dx \, dy \]

\[ = \int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}} e^{tx}\text{Ai}(u+x)\text{Ai}(v+x) \, dx \right)^2 \, du \, dv. \]

We note that the application of Fubini’s theorem in \( E_1(t) \) is valid because the integrand is nonnegative, and in \( E_2(t) \) it suffices to verify that

\[ \int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}} e^{tx} |\text{Ai}(u+x)\text{Ai}(v+x)| \, dx \right)^2 \, du \, dv < \infty. \]

For this, we note that, according to \cite[Lemma 2.6]{Oko02}, one has

\[ \int_{\mathbb{R}} e^{tx}\text{Ai}(x+u)\text{Ai}(x+v) \, dx = \frac{1}{2\sqrt{\pi}t} \exp \left( \frac{t^3}{12} - \frac{u+v}{2} t - \frac{(u-v)^2}{4t} \right) \quad (3.4.4) \]

and thus it follows from Cauchy-Schwarz that

\[ \int_{\mathbb{R}} e^{tx} |\text{Ai}(u+x)\text{Ai}(v+x)| \, dx \leq \left( \int_{\mathbb{R}} e^{tx} \text{Ai}(u+x)^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} e^{tx} \text{Ai}(v+x)^2 \, dx \right)^{1/2} \]

\[ = \frac{1}{2\sqrt{\pi}t} \exp \left( \frac{t^3}{12} - \frac{u+v}{2} t \right) \]

as desired.

With \( E_1(t) \) and \( E_2(t) \) established, (3.4.4) allows to further reduce

\[ E_1(t) = \int_0^\infty \frac{\exp \left( \frac{3u^3}{2} - 2tu \right)}{2\sqrt{2\pi}t} \, du = \frac{e^{2\pi^3/2}}{4\sqrt{2\pi}t^{3/2}}. \]
and

\[ E_2(t) = \int_0^\infty \int_0^\infty \exp \left( \frac{t^3}{6} - (u + v)t - \frac{(u-v)^2}{2t} \right) \frac{du}{4\pi t} \frac{dv}{4\pi t} = \frac{e^{2t^3/3}}{4\sqrt{2\pi} t^{3/2}} \left( 1 - \text{erf} \left( \frac{t^{3/2}}{\sqrt{2}} \right) \right), \]

where we use \( \text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw \) to denote the error function. Consequently

\[ \lim_{t \to 0} E_1(t) - E_2(t) = \lim_{t \to 0} \frac{e^{2t^3/3}}{4\sqrt{2\pi} t^{3/2}} \text{erf} \left( \frac{t^{3/2}}{\sqrt{2}} \right) = \frac{1}{4\pi}, \]

concluding the proof of Proposition 3.1.6.

### 3.4.2 Proof of Proposition 3.1.8

According to [Tak08, Proposition 2.2 (ii)], the eigenvalues of the quantum harmonic oscillator are of the form

\[ \lambda_k(\text{HO}) = c_1 k + c_2, \quad k \in \mathbb{N} \cup \{0\} \]

for some constants \( c_1, c_2 > 0 \). Thus, the eigenvalue point process of \( \text{HO} + g \) is equal to \( \Lambda = (c_1 k + c_2 + g)_{k \geq 0} \), where we recall that \( g \sim N(0,1) \).

Let \( K > c_1 \) be fixed, and consider the interval \( A = [c_2, K] \subset \mathbb{R} \). Note that we have the equality of events

\[ \{ \Lambda(A) > 0 \text{ and } \Lambda((-\infty,c_2)) = 0 \} = \{ g \in [0,K] \}, \]

and thus there is a nonzero probability that the smallest element of \( \Lambda \) is in the interval \( A \). Since the spacing between consecutive points in \( \Lambda \) is a constant (i.e., \( c_1 \)) that is smaller than the length of \( A \), whenever the event (3.4.5) occurs, it is impossible to
differentiate between $\Lambda(A) = 1$ and $\Lambda(A) > 1$ simply by observing the configuration of points outside of $A$ (see Figure 3.1 for an illustration). Thus, $\Lambda$ cannot be number rigid.

Figure 3.1: The red dots represent two realizations of the spectrum of HO + $g$ such that the event (3.4.5) holds. The two realizations illustrate that it is possible for the configuration of points outside of $A = [c_2, K]$ to be the same while $\Lambda(A)$ is not.
Chapter 4
Random Matrices

4.1 Setup and Main Results

4.1.1 Random Matrix Models

We begin by introducing our random matrix models. Let \((m_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers, and for every \(n \in \mathbb{N}\), let \(\mathbb{L}_n := \{0, 1, 2, \ldots, n\}\) and \(\tilde{\mathbb{L}}_n := m_n^{-1}\mathbb{L}_n\). Associating \(\mathbb{R}^{n+1}\) with the space of real-valued functions on \(\tilde{\mathbb{L}}_n\), we view \((n+1) \times (n+1)\) matrices as operators on \(\tilde{\mathbb{L}}_n\). Our aim is to obtain operators acting on \((0, \infty)\) in the large \(n\) limit; hence it is natural to require that \(m_n \to \infty\) and \(m_n = o(n)\) as \(n \to \infty\). Due to technical considerations, we make the following more restrictive assumption.

**Assumption** \(m_n\). There exists \(0 < C \leq 1\) and \(1/13 < \delta < 1/2\) such that

\[
Cn^\delta \leq m_n \leq C^{-1}n^\delta, \quad n \in \mathbb{N}.
\]  

(4.1.1)

Let \((w_n)_{n \in \mathbb{N}}\) be a sequence of real numbers satisfying the following condition.

**Assumption** \(w_n\). There exists some \(w \in \mathbb{R}\) such that

\[
\lim_{n \to \infty} m_n(1 - w_n) = w.
\]  

(4.1.2)

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For every $n \in \mathbb{N}$, let us define the $(n + 1) \times (n + 1)$ tridiagonal matrices

$$\Delta_n^d := m_n^2 \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix},$$

(4.1.3)

$$\Delta_n^r := m_n^2 \begin{bmatrix} -2 + w_n & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix},$$

(4.1.4)

as well as the $(n + 1) \times (n + 1)$ random tridiagonal matrix

$$Q_n := \begin{bmatrix} D_n(0) & U_n(0) \\ L_n(0) & D_n(1) & U_n(1) \\ & L_n(1) & \ddots & \ddots \\ & & \ddots & \ddots & U_n(n - 1) \\ & & & L_n(n - 1) & D_n(n) \end{bmatrix},$$

(4.1.5)

where $D_n(a)$, $U_n(a)$, and $L_n(a)$ are real-valued random variables for every $n \in \mathbb{N}$ and $0 \leq a \leq n$ (or $0 \leq a \leq n - 1$).

**Notation 4.1.1.** Throughout this chapter, we index the entries of a $(n + 1) \times (n + 1)$ matrix $M$ as $M(a, b)$ for $0 \leq a, b \leq n$. Similarly, a vector $v \in \mathbb{R}^{n+1}$ is indexed as $v(a)$ for $0 \leq a \leq n$.  

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Notation 4.1.2. To simplify notation, we often make statements about properties of the random entries $D_n(a)$, $U_n(a)$, and $L_n(a)$ “for all $0 \leq a \leq n$,” with the understanding that $a$ cannot exceed $n - 1$ for $U_n(a)$ and $L_n(a)$.

We think of $\Delta^d_n$ and $\Delta^r_n$ as approximations of the Laplacian operator $\Delta$ on $(0, \infty)$ with Dirichlet boundary condition or Robin boundary condition $f'(0) = w f(0)$ at zero respectively. $Q_n$ approximates pointwise multiplication by a deterministic potential plus a Gaussian white noise. To this effect, we assume that the entries of $Q_n$ satisfy the following: For $E \in \{D, U, L\}$, we have that

$$E_n(a) = V^E_n(a) + \xi^E_n(a), \quad 0 \leq a \leq n,$$

where the $V^E_n(a)$ are deterministic and the $\xi^E_n(a)$ are random. We call $V^E_n$ the potential terms and $\xi^E_n$ the noise terms. We expect that $V^D_n + V^U_n + V^L_n$ converges to a deterministic function and that $\xi^D_n + \xi^U_n + \xi^L_n$ resembles the discrete derivative of a Brownian motion on $\tilde{L}_n$ (i.e., independent variables with small mean and variance of order $m_n$). We make these requirements (and more) precise in a series of assumptions below (namely, Assumptions PT1–PT3 for $V^E_n$, Assumptions NT1–NT3 for $\xi^E_n$, and Assumption R in the case of the Robin boundary condition).

Thanks to (4.1.6), we expect that $\frac{1}{2}(-\Delta^d_n + Q_n)$ and $\frac{1}{2}(-\Delta^r_n + Q_n)$ converge in some sense to an operator of the form

$$\hat{H} := -\frac{1}{2} \Delta + V + W'$$

on $(0, \infty)$ with the appropriate boundary condition at zero, where $V$ is a deterministic potential and $W$ is a Brownian motion. (We recall that such results are the subject of [BV13, RRV11].) Our aim is to concoct sequences of random matrices that converge to the semigroup generated by $\hat{H}$. For this purpose, we make the following definition.
Definition 4.1.3 (Random Matrix Models). For every $n \in \mathbb{N}$ and $t > 0$, we define

$$
\hat{K}_d^n(t) := \left( I_n - \frac{-\Delta_d^n + Q_n}{3m_n^2} \right)^{\lfloor m_n^2 (3t/2) \rfloor}, \quad \hat{K}_r^n(t) := \left( I_n - \frac{-\Delta_r^n + Q_n}{3m_n^2} \right)^{\lfloor m_n^2 (3t/2) \rfloor}.
$$

(4.1.8)

Indeed, $\hat{K}_d^n(t) \approx e^{-\frac{(3t/2)(\Delta_d^n + Q_n)/3}{2}} = e^{-t(-\Delta_d^n + Q_n)/2} \approx e^{-t\hat{H}}$ as $n \to \infty$, and similarly for $\hat{K}_r^n(t)$.

Remark 4.1.4. It may seem at first glance more natural to define $\hat{K}_d^n(t)$ as

$$
\left( I_n - \frac{-\Delta_d^n + Q_n}{2m_n^2} \right)^{\lfloor m_n^2 t \rfloor},
$$

(4.1.9)

or to simply use the matrix exponential $\exp(-t(-\Delta_d^n + Q_n)/2)$ (and similarly for $\hat{K}_r^n(t)$). More generally, we suspect that for any sufficiently well-behaved sequence of functions $(F_{n,t})_{n \in \mathbb{N}}$ such that $F_{n,t}(x) \to e^{-tx/2}$, one has $F_{n,t}(-\Delta_d^n + Q_n) \to e^{-t\hat{H}}$. The difficulty involved in carrying this out rigorously in the generality aimed in this chapter is to choose $F_{n,t}$'s that are both amenable to combinatorial analysis and applicable to general tridiagonal models. The main insight in this chapter is that the matrix models introduced in Definition 4.1.3 are in this sense better suited than the more “obvious” candidates discussed in this remark. We refer to Section 4.3.3 for more details on this distinction.

4.1.2 Continuum Limit

The continuum limit that we expect to obtain for (4.1.8) is the random semigroup $\hat{K}(t)$ in Definition 2.1.16; the limit of $\hat{K}_d^n(t)$ should correspond to Case 2-D in Assumption DB, and $\hat{K}_r^n(t)$ to Case 2-R with $\alpha = -w$. Since only these two cases are under consideration in the present chapter, we use the following notation in order to maintain consistency:
Definition 4.1.5. Let $V : (0, \infty) \to \mathbb{R}$, and let $\xi$ be a white noise with variance $\sigma^2 > 0$ in the sense of Assumption SN/Example 2.1.22. For every $t > 0$, we define the Kernels $\hat{K}^d(t)$ and $\hat{K}^r(t)$ as

\begin{align}
\hat{K}^d(t; x, y) &:= \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} E_{t}^{x,y}\left[1_{\{\tau_0(B) > t\}} e^{-(L_t(B),V) - \xi(L_t(B))}\right], \\
\hat{K}^r(t; x, y) &:= \left(\frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} + \frac{e^{-(x+y)^2/2t}}{\sqrt{2\pi t}}\right) E_{t}^{x,y}\left[e^{-(L_t(X),V) - \xi(L_t(X)) - w_0^2(B)}\right]
\end{align}

(4.1.10) (4.1.11)

for every $x, y \in (0, \infty)^2$, where we denote the processes $B$ and $X$ and the expectation $E_t^{x,y}$, the local times $L_t$ and $L_t$, and the hitting time $\tau_0$ as in Definition 2.1.12 and Remark 2.1.17 respectively.

Notation 4.1.6. Let $\beta > 0$. If $V(x) = x/2$ and $\sigma^2 = 1/\beta$ in in the above definition, then we use the notation $\text{SAS}^d_{\beta}(t) := \hat{K}^d(t)$ and $\text{SAS}^r_{\beta}(t) := \hat{K}^r(t)$, since in this case the operators in question coincide with the stochastic Airy semigroup defined in [GS18b, GLS19]. The infinitesimal generator of $\text{SAS}^d_{\beta}(t)$ and $\text{SAS}^r_{\beta}(t)$ is $\frac{1}{2}\text{SAO}^d_{\beta}$, with the appropriate boundary condition. Our interest in identifying this special case comes from applications to random matrices and $\beta$-ensembles (see Section 4.2).

In order to make sense of the claim that $\hat{K}^d_n(t)$ and $\hat{K}^r_n(t)$ converge to $\hat{K}^d(t)$ and $\hat{K}^r(t)$, we must ensure that these objects act on the same space. We make this precise in the following way:

Notation 4.1.7. We think of $\hat{K}^d_n(t)$ and $\hat{K}^r_n(t)$ as acting on functions on $\tilde{L}_n$. We can extend this to a class of functions on $(0, \infty)$ by associating $\tilde{L}_n$ with the vector space of step functions

$$\sum_{a=0}^{n} v(a) \cdot 1_{(a/m_n, (a+1)/m_n)}, \quad v \in \mathbb{R}^{n+1}. \quad (4.1.12)$$
Then, we may further extend this action to any locally integrable function $f : (0, \infty) \to \mathbb{R}$ through the projection

$$
\pi_n f := m_n^{1/2} \sum_{a=0}^{n} \int_{a/m_n}^{(a+1)/m_n} f(x) \, dx \cdot 1_{[a/m_n,(a+1)/m_n)}.
$$

Thus, for any $(n + 1) \times (n + 1)$ matrix $M$ and locally integrable functions $f, g$, we define $Mf$ as the vector/step function $M\pi_n f$, and we define

$$
\langle f, Mg \rangle := m_n \sum_{0 \leq a, b \leq n} \left( \int_{a/m_n}^{(a+1)/m_n} f(x) \, dx \right) M(a, b) \left( \int_{b/m_n}^{(b+1)/m_n} g(x) \, dx \right). \tag{4.1.13}
$$

### 4.1.3 Technical Assumptions

We are now finally in a position to state our main results and the assumptions under which they apply. We begin with the assumptions; our main theorems are stated in Section 4.1.4 below.

**Assumptions on the Potential Terms**

**Assumption PT1.** (Potential Convergence.) There exists nonnegative continuous functions $V^D, V^U, V^L : (0, \infty) \to \mathbb{R}$ such that

$$
\lim_{n \to \infty} V^E_n ([m_n x]) = V^E(x), \quad x \geq 0
$$

uniformly on compact sets for every $E \in \{D, U, L\}$. Moreover, the function

$$
V := \frac{1}{2} (V^D + V^U + V^L), \quad x \geq 0 \tag{4.1.14}
$$

satisfies $V(x)/\log x \to \infty$ as $x \to \infty$ (i.e., the potential growth Assumption PG).
Assumption PT2. (Growth Upper Bounds.) For every $E \in \{D, U, L\}$ we have the following: For large enough $n$,

$$0 \leq V_n^E(a) \leq 2m_n^2, \quad 0 \leq a \leq n, \quad (4.1.15)$$

and if $C_n = o(n)$ as $n \to \infty$, then

$$\max_{a \leq C_n} V_n^E(a) = o(m_n^2), \quad n \to \infty. \quad (4.1.16)$$

Assumption PT3. (Growth Lower Bounds.) At least one of $E \in \{D, U, L\}$ satisfies the following: For every $\theta > 0$, there exists $c = c(\theta) > 0$ and $N = N(\theta) \in \mathbb{N}$ such that for every $n \geq N$,

$$\theta \log(1 + a/m_n) - c \leq V_n^E(a) \leq m_n^2, \quad 0 \leq a \leq n. \quad (4.1.17)$$

Moreover, at least one of $E \in \{D, U, L\}$ (not necessarily the same as (4.1.17)) satisfies the following: With $d$ as in (4.1.1), there exists $d/2(1 - d) < \alpha \leq 2d/(1 - d), \varepsilon > 0$, and positive constants $\kappa$ and $C > 0$ such that

$$\kappa(a/m_n)^\alpha \leq V_n^E(a) \leq m_n^2, \quad Cn^{1-\varepsilon} \leq a \leq n \quad (4.1.18)$$

for $n$ large enough.

Assumptions on the Noise Terms

Assumption NT1. (Independence.) For every $n \in \mathbb{N}$, the variables $\xi_n^D(0), \ldots, \xi_n^D(n)$ are independent, and likewise for $\xi_n^U(0), \ldots, \xi_n^U(n-1)$ and $\xi_n^L(0), \ldots, \xi_n^L(n-1)$. We emphasize, however, that $\xi_n^D$, $\xi_n^U$, and $\xi_n^L$ need not be independent of each other (for instance, if $Q_n$ is symmetric, then $\xi_n^U = \xi_n^L$).
**Assumption NT2.** (Moment Asymptotics.) For every $E \in \{D, U, L\}$, we have the following:

$$|E[\xi_n^E(a)]| = o\left(m_n^{-1/2}\right) \quad \text{as} \quad (n-a) \to \infty,$$

(4.1.19)

and there exists constants $C > 0$ and $0 < \gamma < 2/3$ such that

$$E[|\xi_n^E(a)|^q] \leq m_n^{q/2}C^q q^{\gamma q}$$

(4.1.20)

for every $0 \leq a \leq n$, integer $q \in \mathbb{N}$, and $n$ large enough.

**Assumption NT3.** (Noise Convergence.) There exists Brownian motions $W^D$, $W^U$, and $W^L$ such that

$$\lim_{n \to \infty} \left( \frac{1}{m_n} \sum_{a=0}^{m_n x} \xi_n^E(a) \right)_{E=D,U,L} = \left( W^E(x) \right)_{E=D,U,L}, \quad x \geq 0$$

(4.1.21)

in joint distribution with respect to the Skorokhod topology. We assume that

$$W := \frac{1}{2}(W^D + W^U + W^L)$$

(4.1.22)

is also a Brownian motion with some variance $\sigma^2 > 0$. Furthermore, if $\varphi_1, \ldots, \varphi_k$ are continuous and compactly supported functions and $(\varphi_1^{(n)})_{n \in \mathbb{N}}, \ldots, (\varphi_k^{(n)})_{n \in \mathbb{N}}$ are such that $\varphi_i^{(n)} \to \varphi_i$ uniformly for every $1 \leq i \leq k$, then

$$\lim_{n \to \infty} \left( \sum_{a \in \mathbb{N}_0} \varphi_i^{(n)}(a/m_n) \xi_n^E(a) \right)_{E=D,U,L, 1 \leq i \leq k} = \left( \int_{(0, \infty)} \varphi_i(a) \, dW^E(a) \right)_{E=D,U,L, 1 \leq i \leq k}$$

(4.1.23)

in joint distribution, and also jointly with (4.1.21).
Assumption for the Robin Boundary Condition

The following assumption will only be made when considering $\hat{K}_n^r(t)$:

**Assumption R.** (Robin Boundary.) The variables $\left(D_n(0)/m_n^{1/2}\right)_{n \in \mathbb{N}}$ are uniformly sub-Gaussian, in the sense that there exists $C, c > 0$ independent of $n$ such that

$$\sup_{n \in \mathbb{N}} E\left[ e^{y|D_n(0)|/m_n^{1/2}} \right] \leq C e^{cy^2}, \quad y \geq 0.$$ 

**Remark 4.1.8.** If $\gamma < 1/2$ in (4.1.20), then Assumption R follows directly from Assumption NT2.

### 4.1.4 Main Theorems

**Theorem 4.1.9.** Suppose that Assumption $m_n$ holds, as well as the potential terms and noise terms Assumptions PT1–PT3 and NT1–NT3. Let $\hat{K}_n^d(t)$ be defined as in (4.1.10), where $V$ is given by (4.1.14) and the variance $\sigma^2$ of the white noise $\xi$ is the same as that of the Brownian motion $W$ in (4.1.22). Then, $\hat{K}_n^d(t) \to \hat{K}^d(t)$ as $n \to \infty$ in the following two senses:

1. For every $t_1, \ldots, t_k > 0$ and $f_1, g_1, \ldots, f_k, g_k : (0, \infty) \to \mathbb{R}$ uniformly continuous and bounded,

   $$\lim_{n \to \infty} \left( \langle f_i, \hat{K}_n^d(t_i) g_i \rangle \right)_{1 \leq i \leq k} = \left( \langle f_i, \hat{K}^d(t_i) g_i \rangle \right)_{1 \leq i \leq k}$$

   in joint distribution and mixed moments.

2. For every $t_1, \ldots, t_k > 0$,

   $$\lim_{n \to \infty} \left( \text{Tr}[\hat{K}_n^d(t_i)] \right)_{1 \leq i \leq k} = \left( \text{Tr}[\hat{K}^d(t_i)] \right)_{1 \leq i \leq k}$$

   in joint distribution and mixed moments.
Theorem 4.1.10. Suppose that Assumptions $m_n$ and $w_n$ hold, as well as Assumptions PT1–PT3, NT1–NT3, and $R$. Let $\hat{K}^r(t)$ be defined as in (4.1.11), where $V$ is given by (4.1.14) and $\xi$’s variance is the same as (4.1.22). Then, $\hat{K}^r_n(t) \to \hat{K}^r(t)$ as $n \to \infty$ in the following sense: For every $t_1, \ldots, t_k > 0$ and $f_1, g_1, \ldots, f_k, g_k : (0, \infty) \to \mathbb{R}$ uniformly continuous and bounded,

$$\lim_{n \to \infty} \left( \langle f_i, \hat{K}^r_n(t_i) g_i \rangle \right)_{1 \leq i \leq k} = \left( \langle f_i, \hat{K}^r(t_i) g_i \rangle \right)_{1 \leq i \leq k}$$

in joint distribution and mixed moments.

Remark 4.1.11. Unlike Theorem 4.1.9, Theorem 4.1.10 contains no statement on the convergence of traces. Similarly to the lack of trace convergence in [GLS19], this is due to the fact that we were unable to construct a strong coupling of a certain Markov chain and its occupation measures with the reflected Brownian bridge $X^{x,x}_t$ and its local time process. Throughout this chapter, we make several remarks and conjectures concerning this trace convergence, its consequences, and the related strong invariance result (see Conjectures 4.1.12 and 4.5.11, and Remarks 4.2.2 and 4.2.6).

Conjecture 4.1.12. In the setting of Theorem 4.1.10, for every $t_1, \ldots, t_k > 0$,

$$\lim_{n \to \infty} (\text{Tr}[\hat{K}^r_n(t_i)])_{1 \leq i \leq k} = (\text{Tr}[\hat{K}^r(t_i)])_{1 \leq i \leq k}$$

in joint distribution and mixed moments.

Remark 4.1.13. Theorems 4.1.9 and 4.1.10 remain valid if we define

$$\hat{K}^d_n(t) = \left( I_n - \frac{-\Delta^d_n + Q_n}{3m^2_n} \right)^{\vartheta(n,t)}$$

$$\hat{K}^r_n(t) = \left( I_n - \frac{-\Delta^r_n + Q_n}{3m^2_n} \right)^{\vartheta(n,t)}$$

for $\vartheta(n,t) := [m^2_n(3t/2)] \pm 1$, instead of (4.1.8). Thus, up to making this minor change, there is no loss of generality in assuming that $[m^2_n(3t/2)]$ is always even
or odd if that is more convenient (this distinction comes in handy in the proof of Proposition 4.2.1 below). We refer to Remark 4.6.2 for more details.

4.2 Applications to Random Matrices

The results proved in this section are as follows (we state our results in Section 4.2.1 and provide their proofs in Sections 4.2.2–4.2.5): Proposition 4.2.1 states that, if \(-\Delta_n^d + Q_n\) is diagonalizable with real eigenvalues, then Theorem 4.1.9 (2) implies convergence of eigenvalues. Following this, we state in Theorem 4.2.5 sufficient conditions for the convergence of eigenvalues of non-symmetric matrices. In Corollaries 4.2.9, 4.2.11, 4.2.13, and 4.2.15, we show that generalized versions of the edge-rescaled \(\beta\)-Hermite and right-edge-rescaled \(\beta\)-Laguerre ensembles satisfy the technical assumptions for our main theorems. Then, in Corollary 4.2.17, we provide an example of non-symmetric tridiagonal matrices that belong to the stochastic Airy operator/Tracy-Widom universality class.

4.2.1 Statements of Results

Convergence of Eigenvalues

Let \(V\) and \(W\) be as in (4.1.14) and (4.1.22). Define the operator \(\hat{H} = -\frac{1}{2}\Delta + V + W'\) on \((0, \infty)\) with Dirichlet boundary condition (i.e., Case 2-D) as in Proposition 2.1.6, and let \(-\infty < \lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \cdots \searrow \infty\) be its eigenvalues, as per Corollary 2.1.9.

Proposition 4.2.1. Suppose that

1. Assumptions \(m_n, PT1–PT3\) and \(NT1–NT3\) hold;

2. for large enough \(n\), the matrix \(-\Delta_n^d + Q_n\) is diagonalizable with real eigenvalues \(\lambda_{n;1} \leq \lambda_{n;2} \leq \cdots \leq \lambda_{n;n+1}\); and
3. there exists $\delta > 0$ such that

$$P[\lambda_{n,n+1} \geq (6 - \delta)m_n^2 \text{ for infinitely many } n] = 0. \quad (4.2.1)$$

For every $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{1}{2}(\lambda_{n,1}, \ldots, \lambda_{n,k}) = (\lambda_1(\tilde{H}), \ldots, \lambda_k(\tilde{H})) \quad \text{in joint distribution.} \quad (4.2.2)$$

**Remark 4.2.2.** Proposition 4.2.1 is only stated for the Dirichlet boundary condition since it depends on the trace convergence of Theorem 4.1.9 (2). If Conjecture 4.1.12 holds, then the same argument used to prove Proposition 4.2.1 would imply that the eigenvalues of $-\Delta^t_n + Q_n$ converge to that of $\tilde{H}$ with Robin boundary condition.

We have the following convenient sufficient condition for (4.2.1), which is easily seen to be satisfied for every example considered in the remainder of this section.

**Proposition 4.2.3.** Suppose that there exists $\bar{\delta} > 0$ and $N \in \mathbb{N}$ such that

$$\max_{0 \leq a \leq n} \left( 2 + \frac{V_n^D(a)}{m_n} \right) + \left| \frac{V_n^U(a)}{m_n^2} - 1 \right| + \left| \frac{V_n^L(a-1)}{m_n^2} - 1 \right| \leq 6 - \bar{\delta}. \quad (4.2.3)$$

for every $n \geq N$. Then, (4.2.1) holds.

**Non-Symmetric Matrices**

We recall the following elementary result in matrix analysis (e.g. [HJ13, 3.1.P22; see also Page 585]).

**Lemma 4.2.4.** Let $M$ be a $(n+1) \times (n+1)$ real-valued tridiagonal matrix. If $M(a,a+1)M(a+1,a) > 0$ for every $0 \leq a \leq n-1$, then $M$ is similar to a Hermitian matrix. In particular, $M$ is diagonalizable with real eigenvalues.
As a consequence of Theorem 4.1.9 (2), Proposition 4.2.1, and the above lemma, we immediately obtain the following result.

**Theorem 4.2.5.** Suppose that Assumptions $m_n$, PT1–PT3 and NT1–NT3 hold. Suppose that there exists $N \in \mathbb{N}$ large enough so that $Q_n$’s entries satisfy

$$(U_n(a) - m_n^2)(L_n(a) - m_n^2) > 0, \quad 0 \leq a \leq n - 1, \ n \geq N. \quad (4.2.4)$$

Then, for $n \geq N$, the eigenvalues $\lambda_{n:1} \leq \lambda_{n:2} \leq \cdots \leq \lambda_{n:n+1}$ of $-\Delta_n^d + Q_n$ are real. In particular, if (4.2.1) holds, then we have the convergence of eigenvalues (4.2.2).

**Remark 4.2.6.** Following up on Remark 4.2.2, we see that if Conjecture 4.1.12 holds, then we have an analog of Theorem 4.2.5 for $-\Delta_n^t + Q_n$.

**Question 4.2.7.** It would be interesting to see if an analog of Theorem 4.2.5 can be proved in the case where $-\Delta_n^d + Q_n$ is diagonalizable with complex eigenvalues. We leave this as an open question.

**Generalized $\beta$-Hermite Ensembles**

**Definition 4.2.8** (Generalized $\beta$-Hermite Ensemble). For every $n \in \mathbb{N}$, let

$$\bar{g}_n(0), \ldots, \bar{g}_n(n), g_n(0), \ldots, g_n(n - 1)$$

be a collection of independent random variables satisfying the following conditions.

1. There exists constants $C > 0$ and $0 < \gamma < 2/3$ such that

$$\mathbb{E}[|\bar{g}_n(a)|^q], \mathbb{E}[|g_n(a)|^q] < C^q q^{\gamma q} \quad (4.2.5)$$

for every $n, q \in \mathbb{N}$ and $0 \leq a \leq n$. 

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2. As \((n - a) \to \infty\),

\[ |\mathbb{E}[ar{g}_n(a)]|, |\mathbb{E}[g_n(a)]| = o((n - a)^{-1/3}). \]

3. There exists constants \(\sigma, \bar{\sigma} > 0\) such that, as \((n - a) \to \infty\),

\[ \mathbb{E}[\bar{g}_n(a)^2] = \bar{\sigma}^2 + o(1) \quad \text{and} \quad \mathbb{E}[g_n(a)^2] = \sigma^2 + o(1). \]

Then, for any \(n \in \mathbb{N}\), define the random matrix

\[
H_n^d := \begin{bmatrix}
-\bar{g}_n(0) & \chi_n(0) \\
\chi_n(0) & -\bar{g}_n(1) & \chi_n(1) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \chi_n(n-1) \\
& & & \chi_n(n-1) & -\bar{g}_n(n)
\end{bmatrix}, \tag{4.2.6}
\]

where \(\chi_n(a) := \sqrt{n-a} - g_n(a)\).

\(H_n^d\) is the random matrix model studied in [GS18b]. As noted in [GS18b, Lemma 2.1], the \(\beta\)-Hermite ensemble studied in [DE02, ES07, RRV11] is a special case of (4.2.6). In order to capture the edge fluctuations of \(H_n^d\), we consider the rescaling

\[
R_n := n^{1/6}(2\sqrt{n}I_n - H_n^d). \tag{4.2.7}
\]

We have the following consequence of Theorem 4.1.9, which is a restatement of the main result of [GS18b] in our setting.

**Corollary 4.2.9.** For every \(t > 0\), let

\[
\hat{K}_n^d(t) := \left( I_n - \frac{R_n}{3n^{2/3}} \right)^{\lfloor n^{2/3}(3t/2) \rfloor} = \left( \frac{I_n}{3} + \frac{H_n^d}{3\sqrt{n}} \right)^{\lfloor n^{2/3}(3t/2) \rfloor}.
\]

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Then, \( \hat{K}_d^d(t) \to \hat{K}_d(t) \) in the sense of Theorem 4.1.9, where \( V(x) = x/2 \) and \( W \) (resp. \( \xi \)) has variance \( \frac{1}{4} \sigma^2 + \sigma^2 \). In particular, if we define \( \beta := \left( \frac{1}{4} \sigma^2 + \sigma^2 \right)^{-1} > 0 \), then \( \hat{K}_d^d(t) \to \text{SAS}_d^d(t) \).

**Definition 4.2.10** (Generalized Spiked \( \beta \)-Hermite Ensemble). Let \( \bar{g}_n \) and \( g_n \) be as in Definition 4.2.8, with the additional requirement that there exists \( C, c > 0 \) such that

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{y |\bar{g}_n(0)|} \right] \leq C e^{cy^2}, \quad y \geq 0. \tag{4.2.8}
\]

Let \( (\mu_n)_{n \in \mathbb{N}} \) be a sequence of real numbers such that

\[
\lim_{n \to \infty} n^{-1/6} (\sqrt{n} - \mu_n) = w \in \mathbb{R}. \tag{4.2.9}
\]

For every \( n \in \mathbb{N} \), define the random matrix

\[
H_n^r := \begin{bmatrix}
-\bar{g}_n(0) + \mu_n & \chi_n(0) \\
\chi_n(0) & -\bar{g}_n(1) & \chi_n(1) \\
\end{bmatrix}
\]

\[
\vdots \\
\chi_n(1) & \ddots & \ddots & \ddots \\
\vdots \\
\chi_n(n-1) & \chi_n(n-1) & -\bar{g}_n(n)
\]

where \( \chi_n(a) = \sqrt{n - a} - g_n(a) \).

\( H_n^r \) is the matrix model studied in [GLS19], and it generalizes the spiked \( \beta \)-Hermite ensemble with a critical (i.e., of size \( \sqrt{n} \)) rank-one additive perturbation studied in [BV13, (1.5)] (see also [Péc06]). Its edge fluctuations are captured by

\[
R_n^r := n^{1/6} (2\sqrt{n} I_n - H_n^r). \tag{4.2.11}
\]

We have the following restatement of [GLS19] in our setting:
Corollary 4.2.11. For every $t > 0$, let

$$
\tilde{K}_n^r(t) := \left( I_n - \frac{R_n^r}{3n^{2/3}} \right)^{\lfloor n^{2/3}(3t/2) \rfloor} = \left( \frac{I_n}{3} + \frac{H_n^r}{3\sqrt{n}} \right)^{\lfloor n^{2/3}(3t/2) \rfloor}.
$$

Then, $\tilde{K}_n^r(t) \to \tilde{K}^r(t)$ in the sense of Theorem 4.1.10, where $V(x) = x/2$ and $W$ (resp. $\xi$) has variance $\frac{1}{4}\bar{\sigma}^2 + \sigma^2$. In particular, if we define $\beta := \left( \frac{1}{4}\bar{\sigma}^2 + \sigma^2 \right)^{-1} > 0$, then $\tilde{K}_n^r(t) \to \text{SAS}^r_{\beta}(t)$.

Generalized $\beta$-Laguerre Ensembles

Definition 4.2.12 (Generalized $\beta$-Laguerre Ensemble). Let $\bar{g}_n$ and $g_n$ be as in Definition 4.2.8. Let $p = p(n) > n$ be an increasing sequence such that

$$
\nu := \lim_{n \to \infty} n/p \in [0, 1].
$$

Define for each $n \in \mathbb{N}$ the random matrix

$$
\tilde{L}_n^d := \begin{bmatrix}
\bar{\chi}_n(0) \\
\chi_n(0) & \bar{\chi}_n(1) \\
& \ddots \\
& & \ddots \\
& & & \ddots \\
& & & & \chi_n(n - 1) & \bar{\chi}_n(n)
\end{bmatrix},
$$

where $\bar{\chi}_n(a) := \sqrt{p - a} - \bar{g}_n(a)$ and $\chi_n(a) := \sqrt{n - a} - g_n(a)$. Then, we define

$$
L_n^d := (\tilde{L}_n^d)^\top \tilde{L}_n^d.
$$

$L_n^d$ is a generalization of the $\beta$-Laguerre ensemble studied in [DE02, ES07, RRV11]. The right-edge (i.e., largest eigenvalues) fluctuations of $L_n^d$ are captured by

$$
\Sigma_n := \frac{m^2_n}{m_p^2} \left( (\sqrt{n} + \sqrt{p})^2 I_n - L_n^d \right),
$$

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where
\[ m_n = \left( \frac{\sqrt{n \beta}}{\sqrt{n} + \sqrt{\beta}} \right)^{2/3}. \]  

(4.2.13)

**Corollary 4.2.13.** For every \( t > 0 \), let
\[ \hat{K}^d_n(t) := \left( I_n - \frac{\sum_n}{3m_n^2} \right)^{\lfloor m_n^2(3t/2) \rfloor}. \]

Then, \( \hat{K}^d_n(t) \to \hat{K}^d(t) \) in the sense of Theorem 4.1.9, where \( V(x) = x/2 \) and \( W \) (resp. \( \xi \)) has variance \( \bar{\sigma}^2 + \sigma^2 \). If we write \( \beta := (\bar{\sigma}^2 + \sigma^2)^{-1} > 0 \), then \( \hat{K}^d_n(t) \to \text{SAS}^d_\beta(t) \).

**Definition 4.2.14** (Generalized Spiked \( \beta \)-Laguerre Ensemble). Let \( \bar{g}_n, g_n, \) and \( p = p(n) \) be as in Definition 4.2.12, with the additional requirement that there exists \( C, c > 0 \) such that
\[ \sup_{n \in \mathbb{N}} E \left[ e^{y |\bar{g}_n(0)|} \right], \sup_{n \in \mathbb{N}} E \left[ e^{y |g_n(0)|} \right] \leq C e^{cy^2}, \quad y \geq 0, \]  

(4.2.14)

and let \( m_n \) be as in (4.2.13). Let \( (\ell_n)_{n \in \mathbb{N}} \) be a sequence of real numbers such that
\[ \lim_{n \to \infty} m_n \left( 1 - \sqrt{p/n} (\ell_n - 1) \right) = w \in \mathbb{R}. \]  

(4.2.15)

For every \( n \in \mathbb{N} \), let
\[ \tilde{L}^r_n := \begin{bmatrix} \sqrt{\ell_n} \bar{\chi}_n(0) \\ \chi_n(0) & \bar{\chi}_n(1) \\ & \chi_n(1) & \cdots \\ & & \ddots & \ddots \\ & & & \chi_n(n - 1) & \bar{\chi}_n(n) \end{bmatrix} \]
and \( L_r^n := (\tilde{L}_r^n)^\top \tilde{L}_r^n \), where \( \tilde{\chi}_n(a) := \sqrt{p-a} - \bar{g}_n(a) \) and \( \chi_n(a) := \sqrt{n-a} - g_n(a) \).

The entries of \( L_r^n \) are the same as that of \( L_d^n \) in Definition 4.2.12, except for

\[
L_r^n(0, 0) = \ell_n \bar{\chi}_n(0)^2 + \chi_n(0)^2.
\]

\( L_r^n \) is a generalization of the critical (i.e., of size \( 1 + \sqrt{\nu} \)) rank-one spiked model of the \( \beta \)-Laguerre ensemble (c.f., [BBAP05] and [BV13, (1.2)]), and its right-edge fluctuations are captured by

\[
\Sigma_r^n := \frac{m_n^2}{\sqrt{np}} \left( (\sqrt{n} + \sqrt{p})^2 I_n - L_r^n \right),
\]

with \( m_n \) as in (4.2.13).

**Corollary 4.2.15.** For every \( t > 0 \), let

\[
\hat{K}_n^r(t) := \left( I_n - \frac{\Sigma_r^n}{3m_n^2} \right)^{\lfloor m_n^2(3t/2) \rfloor}.
\]

Then, \( \hat{K}_n^r(t) \to \hat{K}_r(t) \) in the sense of Theorem 4.1.10, where \( V(x) = x/2 \) and \( W \) (resp. \( \xi \)) has variance \( \bar{\sigma}^2 + \sigma^2 \). If we write \( \beta := (\bar{\sigma}^2 + \sigma^2)^{-1} > 0 \), then \( \hat{K}_n^r(t) \to \text{SAS}_\beta(t) \).

**A Non-Symmetric Example**

**Definition 4.2.16** ("Non-Symmetric \( \beta \)-Hermite Ensemble"). For every \( n \in \mathbb{N} \), let

\[
\bar{g}_n(0), \ldots, \bar{g}_n(n), g_n(0), \ldots, g_n(n-1), \bar{g}_n(0), \ldots, \bar{g}_n(n-1)
\]

be a collection of independent random variables that satisfy the moment conditions of Definition 4.2.8 (1) and (2), and such that

\[
\mathbf{E}[\bar{g}_n(a)^2] = \bar{\sigma}^2 + o(1), \quad \mathbf{E}[g_n(a)^2] = \sigma^2 + o(1), \quad \text{and} \quad \mathbf{E}[\bar{g}_n(a)^2] = \bar{\sigma}^2 + o(1)
\]
as \((n - a) \to \infty\) for some \(\bar{\sigma}, \sigma, \tilde{\sigma} > 0\). Suppose further that, for large enough \(n\),
\(g_n(a), \tilde{g}_n(a) < \sqrt{n - a}\) for every \(0 \leq a \leq n - 1\).

For every \(n \in \mathbb{N}\), we define the \((n + 1) \times (n + 1)\) random tridiagonal matrix

\[
\tilde{H}_n^d := \begin{bmatrix}
-\bar{g}_n(0) & \chi_n(0) & & \\
\tilde{\chi}_n(0) & -\bar{g}_n(1) & \chi_n(1) & \\
\tilde{\chi}_n(1) & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \chi_n(n-1) \\
\tilde{\chi}_n(n-1) & -\bar{g}_n(n) \\
\end{bmatrix}, \tag{4.2.16}
\]

where \(\chi_n(a) := \sqrt{n - a - g_n(a)}\) and \(\tilde{\chi}_n(a) := \sqrt{n - a - \tilde{g}_n(a)}\) for every \(0 \leq a \leq n - 1\).

In order to capture the edge fluctuations of \(\tilde{H}_n^d\), we consider the rescaled version

\[
\tilde{R}_n := n^{1/6}(2\sqrt{n} I_n - \tilde{H}_n^d).
\]

**Corollary 4.2.17.** For every fixed \(k \in \mathbb{N}\), the \(k\) smallest eigenvalues of \(\tilde{R}_n\) converge in joint distribution to the \(k\) smallest eigenvalues of \(\text{SAO}_\beta^d\), where \(\beta := 4(\bar{\sigma}^2 + \sigma^2 + \tilde{\sigma}^2)^{-1}\).

### 4.2.2 Proof of Propositions 4.2.1 and 4.2.3

We begin with Proposition 4.2.1. As argued in [GS18b, Section 6] and [Sod15, Section 5], it suffices to prove the convergence of Laplace transforms

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{n+1} e^{-t_i \lambda_{n,j}/2} \right)_{0 \leq i \leq k} = \left( \sum_{j=1}^{\infty} e^{-t_i \lambda_j(\tilde{H})} \right)_{0 \leq i \leq k}, \quad t_1, \ldots, t_k > 0
\]

in joint distribution. On the one hand, if \(-\Delta_n^d + Q_n\) is diagonalizable, then

\[
\text{Tr}[\tilde{K}_n^d(t)] = \sum_{j=1}^{n+1} \left( 1 - \frac{\lambda_{n,j}}{3m_n^2} \right)^{m_n^2(3t/2)}
\]

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for every $t > 0$. On the other hand, by Theorem 2.1.19, for every $t > 0$, 

$$\text{Tr} [\hat{K}^d (t)] = \sum_{j=1}^{\infty} e^{-t \lambda_j (\hat{H})} < \infty.$$ 

Consequently, by Theorem 4.1.9 (2), we need only prove that 

$$\lim_{n \to \infty} \left( \sum_{j=1}^{n+1} e^{-t \lambda_{nj}/2} - \left( 1 - \frac{\lambda_{nj}}{3m_n^2} \right)^{|m_n^2(3t_i/2)|} \right)_{0 \leq i \leq k} = (0, \ldots, 0) \quad (4.2.17)$$

in joint distribution.

By the Skorokhod representation theorem, if $\hat{K}_n^d (t) \to \hat{K}^d (t)$ in the sense of Theorem 4.1.9 (2), then there exists a coupling of the sequence $(\lambda_{nj})_{1 \leq j \leq n+1, n \in \mathbb{N}}$ and $(\lambda_j (\hat{H}))_{j \in \mathbb{N}}$ such that 

$$\lim_{n \to \infty} \sum_{j=1}^{n+1} \left( 1 - \frac{\lambda_{nj}}{3m_n^2} \right)^{|m_n^2(3t_i/2)|} = \sum_{j=1}^{\infty} e^{-t \lambda_j (\hat{H})} < \infty \quad (4.2.18)$$

almost surely for $1 \leq i \leq k$. By Remark 4.1.13, there is no loss of generality in assuming that $|m_n^2(3t_i/2)|$ is even for all $n$; hence 

$$\sum_{j=1}^{n+1} \left( 1 - \frac{\lambda_{nj}}{3m_n^2} \right)^{|m_n^2(3t_i/2)|} = \sum_{j=1}^{n+1} \left| 1 - \frac{\lambda_{nj}}{3m_n^2} \right|^{|m_n^2(3t_i/2)|}. $$

Let us fix $0 < \delta < 1$ and $0 < \varepsilon < \varnothing$, where $\varnothing$ is as in (4.1.1). We consider four different regimes of eigenvalues of $-\Delta_n^d + Q_n$:

1. $J_{n;1} := \{ j : \lambda_{nj} < -n^\varepsilon \}$;
2. $J_{n;2} := \{ j : -n^\varepsilon \leq \lambda_{nj} < n^\varepsilon \}$;
3. $J_{n;3} := \{ j : n^\varepsilon \leq \lambda_{nj} < (6 - \delta) |m_n^2(3t_i/2)| \}$; and
4. $J_{n;4} := \{ j : (6 - \delta) |m_n^2(3t_i/2)| \leq \lambda_{nj} \}$. 

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Firstly, note that
\[ \sum_{j \in J_{n;1}} \left| 1 - \frac{\lambda_{n;j}}{3m_n^2} \right|^{m_n^2(3t/2)} \geq |J_{n;1}| \left( 1 + \frac{n^\varepsilon}{3m_n^2} \right)^{m_n^2(3t/2)}, \]
where $|J_{n;1}|$ denotes the cardinality of $J_{n;1}$. If $|J_{n;1}| > 0$ for infinitely many $n$, then this quantity diverges, contradicting the convergence of (4.2.18). Hence $J_{n;1}$ does not contribute to (4.2.17).

Secondly, recall the elementary inequalities
\[
0 < e^z - \left( 1 + \frac{z}{m} \right)^m < \left( 1 + \frac{z}{m} \right)^m \left( \left( 1 + \frac{z}{m} \right)^z - 1 \right), \quad \forall z, m > 0
\]
and
\[
0 < e^{-z} - \left( 1 - \frac{z}{m} \right)^m < \left( 1 - \frac{z}{m} \right)^m \left( \left( 1 - \frac{z}{m} \right)^{-z} - 1 \right), \quad \forall m > z > 0,
\]
which imply that
\[
\left| \sum_{j \in J_{n;2}} e^{-t\lambda_{n;j}/2} - \left( 1 - \frac{\lambda_{n;j}}{3m_n^2} \right)^{m_n^2(3t/2)} \right| 
\leq \left( 1 + \frac{n^\varepsilon}{3m_n^2} \right)^{n^\varepsilon} - 1 \right) \sum_{j \in J_{n;2}} \left| 1 - \frac{\lambda_{n;j}}{3m_n^2} \right|^{m_n^2(3t/2)}.
\]
Since $n^{2\varepsilon} = o(m_n^2)$, we have $\left( 1 + \frac{n^\varepsilon}{3m_n^2} \right)^{n^\varepsilon} = 1 + o(1)$, and thus (4.2.18) implies that the contribution of $J_{n;2}$ to (4.2.17) vanishes.

Thirdly, one the one hand, we have that
\[
\sum_{j \in J_{n;3}} e^{-t\lambda_{n;j}/2} \leq |J_{n;3}| e^{-t n^\varepsilon/2},
\]
and on the other hand, since $|1 - z| \leq \max\{e^{-z}, e^{z-2}\}$ ($z \in \mathbb{R}$), we see that

$$
\sum_{j \in J_{n;3}} \left| 1 - \frac{\lambda_{n;j}}{3m_n^2} \right|^{m_n^2(3t_i/2)} \leq |J_{n;3}| \max_{j \in J_{n;3}} \left\{ \exp\left(-\frac{m_n^2(3t_i/2)}{3m_n^2} \right) \right\},
$$

$$
\exp\left(\frac{m_n^2(3t_i/2)}{3m_n^2} \lambda_{m;j} \right) - 2\left[m_n^2(3t_i/2)\right] \leq \exp\left(-\frac{\delta}{3} \left[m_n^2(3t_i/2)\right]\right).
$$

Note that $|J_{n;3}| \leq n + 1$ and that there exists a constant $C > 0$ independent of $n$ such that for every $j \in J_{n;3},$

$$
\exp\left(-\frac{m_n^2(3t_i/2)}{3m_n^2} \lambda_{m;j} \right) \leq e^{-Cn\epsilon},
$$

$$
\exp\left(\frac{m_n^2(3t_i/2)}{3m_n^2} \lambda_{m;j} \right) - 2\left[m_n^2(3t_i/2)\right] \leq \exp\left(-\frac{\delta}{3} \left[m_n^2(3t_i/2)\right]\right).
$$

Consequently, the contribution of $J_{n;3}$ to (4.2.17) vanishes.

Finally, we know from (4.2.1) that there is eventually no eigenvalue in $J_{n;4}$, and thus it has no contribution to (4.2.17), completing the proof Proposition 4.2.1.

We now prove Proposition 4.2.3. According to the Gershgorin disc theorem (e.g., [Zha13, Corollary 9.11]),

$$
\frac{\lambda_{n;n+1}}{m_n^2} \leq \max_{0 \leq a \leq n} \left( 2 + \frac{D_n(a)}{m_n^2} + \left| -1 + \frac{U_n(a)}{m_n^2} \right| + \left| -1 + \frac{L_n(a - 1)}{m_n^2} \right| \right).
$$

By combining this with (4.2.3) and the triangle inequality, we get

$$
\frac{\lambda_{n;n+1}}{m_n^2} \leq 6 - \bar{\delta} + \sum_{E=D,U,L} \max_{0 \leq a \leq n} \frac{|\xi_n^E(a)|}{m_n^2}
$$

for large enough $n$. By a union bound, (4.1.20), and Markov’s inequality, we see that

$$
P \left[ \max_{0 \leq a \leq n} \frac{|\xi_n^E(a)|}{m_n^2} \geq \tilde{\delta} \right] = O \left( \frac{n}{m_n^{3q/2}} \right).
$$
for any $\tilde{\delta} \in (0, \bar{\delta})$ and $q \in \mathbb{N}$. By (4.1.1), we can take $q$ large enough so that $
ord{n/m_n^3q/2} < \infty$; the result then follows by the Borel-Cantelli lemma.

### 4.2.3 Proof of Corollaries 4.2.9 and 4.2.11

We begin with the proof of Corollary 4.2.9. Straightforward computations reveal that

$$R_n(a, a) = 2n^{2/3} + n^{1/6} \bar{g}_n(a) \quad \text{and} \quad R_n(a, a + 1) = -n^{2/3} + n^{1/6}(\sqrt{n} - x_n(a)).$$

Therefore, if we let $m_n = n^{1/3}$ (which satisfies (4.1.1)), then $R_n$ is of the form $-\Delta_n^D + Q_n$, where the diagonal $D_n$ only has noise terms $\xi_n^D(a) := n^{1/6} \bar{g}_n(a)$, and the off-diagonal $E_n = U_n, L_n$ have potential and noise terms

$$V_n^E(a) := n^{1/6}(\sqrt{n} - \sqrt{n - a}) \quad \text{and} \quad \xi_n^E(a) := n^{1/6} \bar{g}_n(a).$$

We first check the assumptions regarding the potential terms. Note that

$$V_n^E(a) = n^{2/3} \left(1 - \sqrt{1 - \frac{a}{n}} \right).$$

Thus Assumption PT2 is met. Elementary calculus shows that for any $0 < \kappa < 1/2$ and $c > 0$, the function

$$x \mapsto c^2 \left(1 - \sqrt{1 - \frac{x}{c^3}} \right) - \kappa x$$

is nonnegative on $x \in [0, c^3]$. Taking $c = m_n$, this implies that Assumption PT3 is met with $E = L, U$ in both (4.1.17) and (4.1.18). Finally, for $E = L, U$ and $x \geq 0$,

$$V_n^E([n^{1/3}x]) = n^{2/3} \left(1 - \sqrt{1 - \frac{[n^{1/3}x]}{n}} \right) \to \frac{x}{2}.$$
as \( n \to \infty \), and thus we have pointwise convergence to the function \( V^E(x) = x/2 \).

Since \( V^E_n(\lfloor n^{1/3} x \rfloor) \) is nondecreasing in \( x \) for every \( n \), the convergence is uniform on compacts. Then, we are led to \( V(x) = \frac{1}{2} (V^U(x) + V^L(x)) = x/2 \), as desired.

We now check the noise terms assumptions. Given that \( m_{n-1/2} \xi_n^D = \bar{g}_n \) and \( m_{n-1/2} \xi_n^E = g_n \), Assumptions NT1 and NT2 follow directly from Definition 4.2.8. Noting that \( \xi_n^D/m_n = \bar{g}_n/n^{1/6} \) and \( \xi_n^E/m_n = g_n/n^{1/6} \), the joint convergences (4.1.21) and (4.1.23) follow from Proposition A.4.1 and the fact that \( \bar{g}_n \) and \( g_n \) are independent. We therefore obtain two independent Brownian motions \( W^D \) and \( W^U = W^L \) of respective variances \( \bar{\sigma}^2 \) and \( \sigma^2 \), which further implies that \( W = \frac{1}{2} (W^D + W^U + W^L) \) is a Brownian motion with variance \( \frac{1}{4} \bar{\sigma}^2 + \sigma^2 \).

We now prove Corollary 4.2.11. If we let \( w_n = \mu_n/\sqrt{n} \), then \( R_n \) is of the form \(-\Delta_n + Q_n\). (4.2.8) implies that \( w_n \) satisfies Assumption \( w_n \), and (4.2.9) guarantees that Assumption \( R \) is met.

### 4.2.4 Proof of Corollaries 4.2.13 and 4.2.15

We begin with Corollary 4.2.13. According to (4.2.12), we have the limit

\[
\lim_{n \to \infty} n^{-1/3} m_n = \frac{1}{(1 + \sqrt{\nu})^{2/3}},
\]

hence (4.1.1) holds with \( \vartheta = 1/3 \). Assuming that \( g_n(n) = \chi_n(n) = 0 \) for notational simplicity, it is straightforward to verify that

\[
\Sigma_n(a, a) = 2m_n^2 + \frac{m_n^2}{\sqrt{np}} ((n + p) - \bar{\chi}_n(a)^2 - \chi_n(a)^2)
\]

\[
\Sigma_n(a, a + 1) = -m_n^2 + m_n^2 \left( 1 - \frac{\chi_n(a) \bar{\chi}_n(a + 1)}{\sqrt{np}} \right).
\]
Thus, we can write $\Sigma_n$ in the form $-\Delta_n^d + Q_n$, where $D_n$ and $E_n = U_n, L_n$ have potential and noise terms given by

$$V_n^D(a) = 2 \frac{m_n^2}{\sqrt{np}} a,$$
$$\xi_n^D(a) = \frac{m_n^2}{\sqrt{np}} \left( 2(\sqrt{p} - a \bar{\theta}_n(a)) + \sqrt{n - a g_n(a)} \right) - \bar{\theta}_n(a)^2 - g_n(a)^2,$$
$$V_n^E(a) = m_n^2 \left( 1 - \sqrt{\left( 1 - \frac{a}{n} \right) \left( 1 - \frac{a - 1}{p} \right)} \right),$$
$$\xi_n^E(a) = \frac{m_n^2}{\sqrt{np}} \left( \sqrt{n - a \bar{\theta}_n(a + 1)} + \sqrt{p - a - 1 g_n(a)} \right) - \bar{\theta}_n(a + 1) g_n(a).$$

We first check the potential terms assumptions. (4.1.15) and (4.1.16) are immediate from the above expansions. Since

$$\left( 1 - \sqrt{\left( 1 - \frac{a}{n} \right) \left( 1 - \frac{a - 1}{p} \right)} \right) \geq \left( 1 - \sqrt{1 - \frac{a}{n}} \right),$$

the same argument used in the proof of Corollary 4.2.9 implies that (4.1.17) and (4.1.18) both hold with $E = U, L$. Next, by writing $n = \nu p (1 + o(1))$, we observe that we have the following pointwise limits in $x \geq 0$:

$$\lim_{n \to \infty} V_n^D(\lfloor m_n x \rfloor) = V^D(x) := \frac{2\sqrt{\nu} x}{(1 + \sqrt{\nu})^2},$$
$$\lim_{n \to \infty} V_n^E(\lfloor m_n x \rfloor) = V^E(x) := \frac{(1 + \nu)x}{2(1 + \sqrt{\nu})^2}, \quad E = U, L.$$

Once again the monotonicity in $x$ of the functions involved implies uniform convergence on compacts, and we have $\frac{1}{2} (V^D(x) + V^U(x) + V^L(x)) = x/2$.

We now verify the noise terms assumptions. The independence of the $\bar{\theta}_n(a)$ and $g_n(a)$ implies that Assumption NT1 holds. Next, note that

$$m_n = O(n^{1/3}) \quad \text{and} \quad \frac{m_n^2}{\sqrt{n}}, \frac{m_n^2}{\sqrt{p}} = O(n^{1/6}) = O(m_n^{1/2}) \quad (4.2.19)$$
as $n \to \infty$. Thus, by combining the inequality

$$|z_1 + z_2 + z_3 + z_4|^p \leq 4^{p-1}(|z_1|^p + |z_2|^p + |z_3|^p + |z_4|^p), \quad p \geq 1, \ z_i \in \mathbb{R} \ (4.2.20)$$

with (4.2.5), we conclude that (4.1.19) and (4.1.20) hold. Let $\bar{\Xi}$ and $\Xi$ be defined as the joint limits in distribution

$$\bar{\Xi}(x) := \lim_{n \to \infty} \frac{1}{m_n^{1/2}} \sum_{a=0}^{\lfloor m_n x \rfloor} \bar{g}_n(a) \quad \text{and} \quad \Xi(x) := \lim_{n \to \infty} \frac{1}{m_n^{1/2}} \sum_{a=0}^{\lfloor m_n x \rfloor} g_n(a),$$

as per Proposition A.4.1. (Indeed, $m_n/n^{1/3}$ converges to a constant.) $\bar{\Xi}$ and $\Xi$ are independent Brownian motions with respective variances $\bar{\sigma}^2$ and $\sigma^2$. For $a = o(n)$, we note that

$$\lim_{n \to \infty} \frac{m_n^{3/2}}{\sqrt{np}} \sqrt[p]{p-a} = \frac{1}{1+\sqrt{\nu}} \quad \text{and} \quad \lim_{n \to \infty} \frac{m_n^{3/2}}{\sqrt{np}} \sqrt{n-a} = \frac{\sqrt{\nu}}{1+\sqrt{\nu}}.$$

Thus by independence of $\Xi$ and $\bar{\Xi}$ we have the joint convergences in distribution (4.1.21) and (4.1.23), where

$$W_D = \left( \frac{2}{1+\sqrt{\nu}} \right) \Xi + \left( \frac{2\sqrt{\nu}}{1+\sqrt{\nu}} \right) \bar{\Xi},$$

$$W_E = \left( \frac{1}{1+\sqrt{\nu}} \right) \Xi + \left( \frac{\sqrt{\nu}}{1+\sqrt{\nu}} \right) \bar{\Xi}, \quad E = U, L,$$

and $W = \frac{1}{2}(W^D + W^U + W^L) = \bar{\Xi} + \Xi$, the latter having a variance of $\bar{\sigma}^2 + \sigma^2$ by independence of $\Xi$ and $\bar{\Xi}$.

We now prove Corollary 4.2.15. The entries of $\Sigma_r^E$ are the same as that of $\Sigma_n$ except for the $(0, 0)$ entry, which is

$$\Sigma_r^E(0, 0) = 2m_n^2 + \frac{m_n^2}{\sqrt{np}}((n+p) - \ell_n \bar{\chi}_n(0)^2 - \chi_n(0)^2).$$
Thus, it is readily checked that $\Sigma_n^r$ is of the form $-\Delta_n^r + Q_n$, where

$$w_n = \sqrt{p/n} (\ell_n - 1), \quad V_n^D(0) = 0,$$

$$D_n(0) = \frac{m_n^2}{\sqrt{np}} \left( 2(\sqrt{p}\ell_n \tilde{g}_n(0) + \sqrt{n} g_n(0)) - \ell_n \tilde{g}_n(0)^2 - g_n(0)^2 \right),$$

and all of the other entries are the same as for $\Sigma_n$. By (4.2.15), Assumption $w_n$ is met. Note that (4.2.15) also implies that $\ell_n = 1 + \sqrt{\gamma} + O(m_n^{-1})$. Thus, by combining the estimate (4.2.19) with (4.2.14), we conclude that Assumption R holds by a straightforward application of Hölder’s inequality.

### 4.2.5 Proof of Corollary 4.2.17

It is easy to see that $\tilde{R}_n$ is of the form $-\Delta_n^d + Q_n$ (with $m_n = n^{1/3}$), where

$$U_n(a) = n^{1/6}(\sqrt{n} - \sqrt{n-a} + g_n(a)),$$

$$L_n(a) = n^{1/6}(\sqrt{n} - \sqrt{n-a} + \tilde{g}_n(a)),$$

Given that $-\sqrt{n-a} + g_n(a), -\sqrt{n-a} + \tilde{g}_n(a) < 0$ (by assumption on $g_n$ and $\tilde{g}_n$), $\tilde{R}_n$ satisfies (4.2.4). We can prove that $\tilde{R}_n$ satisfies Assumptions $m_n$, $w_n$, and PT1–NT3 using the same arguments as in the proof of Corollary 4.2.9; hence the result follows from Theorem 4.2.5 ((4.2.3) is easily seen to hold here).

### 4.3 From Matrices to Feynman-Kac Functionals

In this section, we derive probabilistic representations for $\langle f, \hat{K}_n^d(t)g \rangle$, $\text{Tr}[\hat{K}_n^d(t)]$, and $\langle f, \hat{K}_n^r(t)g \rangle$ that serve as finite-dimensional analogs of the same quantities involving the continuous kernels (4.1.10) and (4.1.11).
4.3.1 Dirichlet Boundary: Lazy Random Walk

**Definition 4.3.1** (Lazy Random Walk). Let $S = (S(u))_{u \in \mathbb{N}_0}$ ($\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) be a lazy random walk, i.e., the increments $S(u) - S(m - 1)$ are i.i.d. uniform random variables on $\{-1, 0, 1\}$. For every $a, b, u \in \mathbb{N}_0$, we denote $S_a := (S | S(0) = a)$ and $S_{a,b}^u := (S | S(0) = a$ and $S(u) = b)$.

**Inner Product**

Let $M$ be a $(n + 1) \times (n + 1)$ random tridiagonal matrix, let $v \in \mathbb{R}^{n + 1}$ be a vector, and let $\vartheta \in \mathbb{N}$ be a fixed integer. By definition of matrix product, for every $0 \leq a \leq n$,

$$((\frac{1}{3}M)^\vartheta v)(a) = \frac{1}{3^\vartheta} \sum_{a_1, \ldots, a_{\vartheta - 1}} M(a, a_1)M(a_1, a_2) \cdots M(a_{\vartheta - 1}, a_\vartheta)v(a_\vartheta), \quad (4.3.1)$$

where the sum is taken over all $a_1, \ldots, a_\vartheta \in \mathbb{N}_0$ such that $(a, a_1, \ldots, a_\vartheta)$ forms a path on the lattice $\mathbb{L}_n = \{0, 1, 2, \ldots, n\}$ with self-edges (i.e., $|a_i - a_{i-1}| \in \{0, 1\})$. The probability that $S^a$ is equal to any such path is $3^{-\vartheta}$, and thus we see that

$$((\frac{1}{3}M)^\vartheta v)(a) = \mathbb{E}^a \left[ 1_{\{\tau^{(n)}(S) > \vartheta\}} \left( \prod_{u=0}^{\vartheta - 1} M(S(u), S(u + 1)) \right) v(S(\vartheta)) \right], \quad (4.3.2)$$

where the random walk $S$ is independent of the randomness in $M$, $\mathbb{E}^a$ denotes the expected value with respect to the law of $S^a$ conditional on $M$, and

$$\tau^{(n)}(S) := \min\{u \geq 0 : S(u) = -1 \text{ or } n + 1\}.$$

We can think of the contribution of $M$ to (4.3.2) as a type of random walk in random scenery process on the edges of $\mathbb{L}_n$, that is, each passage of $S$ on an edge contributes to the multiplication of the corresponding entry in $M$. In particular, if we define the
edge-occupation measures

$$\Lambda^{(a,b)}(S) := \sum_{u=0}^{\vartheta-1} 1_{\{S(u) = a \text{ and } S(u+1) = b\}}, \quad 0 \leq a, b \leq n,$$

(4.3.3)

then we have that

$$\prod_{u=0}^{\vartheta-1} M(S(u), S(u+1)) = \prod_{a,b \in \mathbb{Z}} M(a,b) \Lambda^{(a,b)}(S).$$

(4.3.4)

We now apply the above discussion to the study of $\hat{K}^d_n(t)$. We observe that

$$\left( I_n - \frac{-\Delta_n^d + Q_n}{3m_n^2} \right) (a, a) = \frac{1}{3} \left( 1 - \frac{D_n(a)}{m_n^2} \right) \quad 0 \leq a \leq n,$$

(4.3.5)

$$\left( I_n - \frac{-\Delta_n^d + Q_n}{3m_n^2} \right) (a, a+1) = \frac{1}{3} \left( 1 - \frac{U_n(a)}{m_n^2} \right) \quad 0 \leq a \leq n-1,$$

(4.3.6)

$$\left( I_n - \frac{-\Delta_n^d + Q_n}{3m_n^2} \right) (a+1, a) = \frac{1}{3} \left( 1 - \frac{L_n(a)}{m_n^2} \right) \quad 0 \leq a \leq n-1.$$

(4.3.7)

Let $t > 0$ and $n \in \mathbb{N}$ be fixed, and let us denote $\vartheta = \vartheta(n,t) := \left\lfloor m_n^2 (3t/2) \right\rfloor$. By combining (4.3.5)–(4.3.7), the combinatorial analysis in (4.3.1)–(4.3.4), and the embedding $\pi_n$ defined in Notation 4.1.7, we see that

$$\langle f, \hat{K}^d_n(t)g \rangle = \int_0^{(n+1)/m_n} f(x) E\left[ m_n x \right] \int_{S(\vartheta)/m_n}^{(S(\vartheta)+1)/m_n} g(y) \, dy \, dx,$$

(4.3.8)

where $S$ is independent of $Q_n$, we define the random functional

$$F_{n,t}^d(S) := 1_{\{x^{(a)}(S) > \vartheta\}} \prod_{a \in \mathbb{N}_0} \left( 1 - \frac{D_n(a)}{m_n^2} \right) \Lambda^{(a,a)}(S) \cdot \left( 1 - \frac{U_n(a)}{m_n^2} \right) \Lambda^{(a+1,a+1)}(S) \left( 1 - \frac{L_n(a)}{m_n^2} \right) \Lambda^{(a+1,a)}(S),$$

(4.3.9)

and for any $x \geq 0$, $E\left[ m_n x \right]$ denotes the expected value with respect to $S\left[ m_n x \right]$, conditional on $Q_n$.  

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Trace

Letting $M$ be as in the previous section, it is easy to see that

$$
\text{Tr}\left[(\frac{1}{3}M)\vartheta\right] = \sum_{a=0}^{n} \mathbb{P}[S^a(\vartheta) = a] \mathbb{E}_\vartheta^{a,a} \left[ \prod_{u=0}^{\vartheta-1} M(S(u), S(u+1)) \right]
$$

where $S$ is independent of $M$, and $\mathbb{E}_\vartheta^{a,a}$ denotes the expected value with respect to the law of $S_{\vartheta}^{a,a}$, conditional on $M$. Given that $\mathbb{P}[S^a(\vartheta) = a] = \mathbb{P}[S^0(\vartheta) = 0]$ is independent of $a$, if we apply a Riemann sum on the grid $\frac{m-1}{n}\mathbb{Z}$ to the previous expression for $\text{Tr}[\left[(\frac{1}{3}M)\vartheta\right]]$, we note that

$$
\text{Tr}\left[(\frac{1}{3}M)\vartheta\right] = m_n \mathbb{P}[S^0(\vartheta) = 0] \int_0^{(n+1)/m_n} \mathbb{E}_\vartheta^{m_n x, m_n x} \left[ \prod_{u=0}^{\vartheta-1} M(S(u), S(u+1)) \right] \, dx.
$$

Applying this to the model of interest $\hat{K}^d_n(t)$, we then see that

$$
\text{Tr}[\hat{K}^d_n(t)] = m_n \mathbb{P}[S^0(\vartheta) = 0] \int_0^{(n+1)/m_n} \mathbb{E}_\vartheta^{m_n x, m_n x} \left[ F_{\vartheta,n,t}^d(S) \right] \, dx, \quad (4.3.10)
$$

where $\vartheta = \vartheta(n, t) = [m_n^2(3t/2)]$, $S$ is independent of $Q_n$, $\mathbb{E}_\vartheta^{m_n x, m_n x}$ denotes the expected value of $S_{\vartheta}^{m_n x, m_n x}$ conditional on $Q_n$, and $F_{\vartheta,n,t}^d$ is as in (4.3.9).

4.3.2 Robin Boundary: “Reflected” Random Walk

**Definition 4.3.2.** Let $T = (T(u))_{u \in \mathbb{N}_0}$ be the Markov chain on the state space $\mathbb{N}_0$ with the following transition probabilities:

$$
\mathbb{P}[T(u+1) = a + b | T(u) = a] = \frac{1}{3}, \quad \text{if} \ a \in \mathbb{N}_0 \setminus \{0\} \text{ and } b \in \{-1, 0, 1\},
$$

$$
\mathbb{P}[T(u+1) = 0 | T(u) = 0] = \frac{2}{3}, \quad \text{and} \quad \mathbb{P}[T(u+1) = 1 | T(u) = 0] = \frac{1}{3}.
$$
We denote $T^a := (T|T(0) = a)$ and $T^a_{u,b} := (T|T(0) = a$ and $T(u) = b)$.

Let $M$ be a $(n+1) \times (n+1)$ tridiagonal matrix, and let $\tilde{M}$ be defined as

$$\tilde{M}(a,b) = \begin{cases} \frac{2}{3}M(a,b) & \text{if } a = b = 0, \\ \frac{1}{3}M(a,b) & \text{otherwise.} \end{cases}$$

For any $\vartheta \in \mathbb{N}$, $0 \leq a \leq n$, and vector $v \in \mathbb{R}^{n+1}$,

$$\langle \tilde{M}^\vartheta v(a) = E^\vartheta \left[ 1_{\left\{ \tau^\vartheta(T) > \vartheta \right\}} \left( \prod_{a,b \in \mathbb{N}_0} M(a,b)^{\Lambda_{\vartheta}^{(a,b)}(T)} \right) v(T(\vartheta)) \right] \right)$$

(4.3.11)

with $T$ independent of $M$, $E^\vartheta$ denoting the expected value of $T^a$ conditioned on $M$, and we define $\Lambda_{\vartheta}^{(a,b)}(T)$ in the same way as (4.3.3).

We now apply this to the study of the matrix model $\hat{K}_t^n(t)$. We note that the entries of $I_n - (\Delta_n^t + Q_n)/3m^2_n$ are the same as (4.3.5)–(4.3.7) except for the $(0,0)$ entry, which is equal to

$$\left( I_n - \frac{\Delta_n^t + Q_n}{3m^2_n} \right)(0,0) = \frac{1}{3} \left( 1 + w_n - \frac{D_n(0)}{m^2_n} \right)$$

$$= \frac{1}{3} \left( 2 - (1 - w_n) - \frac{D_n(0)}{m^2_n} \right) = \frac{2}{3} \left( 1 - \frac{(1 - w_n) - D_n(0)}{2m^2_n} \right).$$

Therefore, if we let $\vartheta = \vartheta(n,t) := \lfloor m^2_n(3t/2) \rfloor$, then

$$\langle f, \hat{K}_t^n(t)g \rangle$$

$$= \int_0^{(n+1)/m_n} f(x) E^{[m_n x]} \left[ F_{n,t}^{\vartheta}(T) m_n \int_{T(\vartheta)/m_n}^{(T(\vartheta+1))/m_n} g(y) dy \right] dx,$$  

(4.3.12)
where \( T \) is independent of \( Q_n \), we define the random functional

\[
F_{n,t}^x(T) := 1_{\{\tau^x(T) > \theta\}} \left( 1 - \frac{1 - w_n}{2} - \frac{D_n(0)}{2m_n^2} \right) \Lambda_{\theta}^{(0,0)}(T)
\]

\[
\cdot \left( \prod_{a \in \mathbb{N}} \left( 1 - \frac{D_n(a)}{m_n^2} \right) \Lambda_{\theta}^{(a,a)}(T) \right)
\]

\[
\cdot \left( \prod_{a \in \mathbb{N}_0} \left( 1 - \frac{U_n(a)}{m_n^2} \right) \Lambda_{\theta}^{(a,a+1)}(T) \left( 1 - \frac{L_n(a)}{m_n^2} \right) \Lambda_{\theta}^{(a+1,a)}(T) \right),
\]

and \( E^{m_n x} \) is the expected value of \( T^{m_n x} \) conditional on \( Q_n \).

### 4.3.3 A Brief Comparison with Other Matrix Models

Now that we have laid out the basis of the combinatorial analysis associated with our matrix model, we take this opportunity to compare our model with the ones discussed in Remark 4.1.4.

We begin with the matrix exponential \( \exp(-t(-\Delta_n^d + Q_n)/2) \). If \( Q_n \) is diagonal, then we can express the matrix exponential in terms of a Feynman-Kac formula involving the continuous-time simple random walk on \( \mathbb{Z} \) with exponential jump times. This formula is very similar to (4.1.10) and (4.1.11) (e.g., (1.1.10) with \( \mathbb{Z} \) being the random walk on \( \mathbb{Z} \) with exponential jump times) and is arguably easier to work with than (4.3.9) or (4.3.13). However, for general tridiagonal \( Q_n \), the Feynman-Kac formula becomes much more unwieldy: namely, the generator of the associated random walk now depends on the entries of \( Q_n \), making a general unified treatment seemingly more difficult.
As for the matrix model (4.1.9), we note that

\[
\left( I_n - \frac{-\Delta_n^d + Q_n}{2m_n^2} \right) (a, a) = \frac{-D_n(a)}{m_n^2} \quad 0 \leq a \leq n,
\]

\[
\left( I_n - \frac{-\Delta_n^d + Q_n}{2m_n^2} \right) (a, a + 1) = \frac{1}{2} \left( 1 - \frac{U_n(a)}{m_n^2} \right) \quad 0 \leq a \leq n - 1,
\]

\[
\left( I_n - \frac{-\Delta_n^d + Q_n}{2m_n^2} \right) (a + 1, a) = \frac{1}{2} \left( 1 - \frac{L_n(a)}{m_n^2} \right) \quad 0 \leq a \leq n - 1.
\]

If \( D_n(a) = 0 \) for all \( n \) and \( a \), then a combinatorial analysis similar to the one performed earlier in this section can relate (4.1.9) to a functional of simple symmetric random walks on \( \mathbb{Z} \) (i.e., i.i.d. uniform ±1 increments). More generally, if \( D_n(a) \) is of smaller order than \( m_n^2 - U_n(a) \) and \( m_n^2 - L_n(a) \) for large \( n \) (e.g., in the \( \beta \)-Hermite ensemble), then a similar analysis holds, but with additional technical difficulties (see [GLS19, Section 3.1] and [GS18b, Section 3] for the details). However, if \( D_n(a) \) is allowed to be of the same order as \( m_n^2 - U_n(a) \) and \( m_n^2 - L_n(a) \) (e.g., in the \( \beta \)-Laguerre ensembles), then the analysis of [GS18b] and [GLS19] no longer directly applies.

### 4.4 Strong Couplings for Lazy Random Walk

Equations (4.3.8) and (4.3.10) suggest Theorem 4.1.9 relies on understanding how Brownian motion and its local time arises as the limit of the lazy random walk and its edge-occupation measures. This is the subject of this section.

**Definition 4.4.1.** For every \( x \geq 0 \), let \( \tilde{B}^x \) be a Brownian motion started at \( x \) with variance 2/3, and for every \( t > 0 \), let \( \tilde{B}^x \left| \tilde{B}^x(t) = x \right. \). We define the local time process for \( \tilde{B} \) in the same way as \( B \).

The main result of this section is the following.

**Theorem 4.4.2.** Let \( t > 0 \) and \( x \geq 0 \) be fixed. For every \( 0 \leq s \leq t \) and \( n \in \mathbb{N} \), let \( \vartheta_s = \vartheta_s(n) := \lfloor m_n^2 s \rfloor \) and \( x^n := \lfloor m_n x \rfloor \). We use the shorthand \( \vartheta := \vartheta_t \). For every
\( y \in \mathbb{R} \), let \( (y_n, \bar{y}_n)_{n \in \mathbb{N}} \) be equal to one of the three sequences

\[
\left( \lfloor m_n y \rfloor, \lfloor m_n y \rfloor \right)_{n \in \mathbb{N}}, \quad \left( \lfloor m_n y \rfloor, \lfloor m_n y \rfloor + 1 \right)_{n \in \mathbb{N}}, \quad \text{or} \quad \left( \lfloor m_n y \rfloor + 1, \lfloor m_n y \rfloor \right)_{n \in \mathbb{N}}. \tag{4.4.1}
\]

Finally, suppose that \( (Z_n, Z) = (S_x^n, B^x) \), or \( (S_{x^n} x^n, B_i^{x,x}) \) for each \( n \in \mathbb{N} \). For every \( 0 < \varepsilon < 1/5 \), there exists a coupling of \( Z_n \) and \( Z \) such that the following holds almost surely as \( n \to \infty \)

\[
\sup_{0 \leq s \leq t} \frac{|Z_n(y_s) - Z(s)|}{m_n} = O \left( m_n^{-1} \log m_n \right), \tag{4.4.2}
\]

\[
\sup_{0 \leq s \leq t, y \in \mathbb{R}} \frac{|\Lambda^{y_n, \bar{y}_n}(Z_n) - L^y_s(Z)|}{m_n} = O \left( m_n^{-1/5 + \varepsilon} \log m_n \right). \tag{4.4.3}
\]

Classical results on strong couplings of local time (such as [BK93]) concern the vertex-occupation measures of a random walk:

\[
\Lambda^u_a(S) := \sum_{j=0}^u 1_{\{S(j) = a\}}, \quad a \in \mathbb{Z}, \ u \in \mathbb{N}. \tag{4.4.4}
\]

Indeed, for any measurable \( f : \mathbb{R} \to \mathbb{R} \), the vertex-occupation measures satisfy

\[
\sum_{j=0}^u f(S(j)) = \sum_{a \in \mathbb{Z}} \Lambda^u_a(S) f(a), \tag{4.4.5}
\]

making a direct comparison with local time more convenient by (2.1.7). Thus, our strategy of proof for Theorem 4.4.2 has two steps: We first use standard methods to construct a strong coupling of the vertex-occupation measures of \( S_x^n \) and \( S_{x^n} x^n \) with the local time of their corresponding continuous processes. Then, we prove that the occupation measure of a given edge \((a, b)\) is very close to a multiple of the occupation measure of the vertices \(a\) and \(b\). More precisely:
Proposition 4.4.3. For every $0 < \varepsilon < 1/5$, there exists a coupling such that

$$\sup_{0 \leq s \leq t, \ y \in \mathbb{R}} \left| \frac{\Lambda_{\vartheta s}^{[m n y]}(Z_n)}{m_n} - L_{\vartheta s}^{y}(Z) \right| = O \left( m_n^{-1/5 + \varepsilon} \log m_n \right)$$

(4.4.6)

and (4.4.2) hold almost surely as $n \to \infty$.

Proposition 4.4.4. Almost surely, as $n \to \infty$, one has

$$\sup_{0 \leq u \leq \vartheta} \frac{1}{m_n} \left| \Lambda_{u}^{(a,b)}(Z_n) - \frac{\Lambda_{u}^{a}(Z_n)}{3} \right| = O( m_n^{-1/2} \log m_n ).$$

Notation 4.4.5. In Propositions 4.4.3 and 4.4.4, and the remainder of Section 4.4, whenever we state a result for $Z_n$ and $Z$, we mean that the result in question applies to $(Z_n, Z) = (S_{x_n}^{x}, \tilde{B}_{x}^{x})$ and $(S_{y}^{x_n}, \tilde{B}_{y}^{x,x})$.

4.4.1 Condition for Strong Local Time Coupling

We begin with a criterion for local time couplings. The following lemma is essentially the content of the proof of [BK93, Theorem 3.2]; we provide a full proof since we need a modification of the result in Section 4.5.

Lemma 4.4.6. For any $0 < \delta < 1$, the following holds almost surely as $n \to \infty$:

$$\sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{Z}} \left| \frac{\Lambda_{u}^{a}(Z_n)}{m_n} - L_{u/m_n^2}^{a}(Z) \right| = O \left( \sup_{0 \leq s \leq t, |y-z| \leq m_n^{-\delta}} \left| L_{s}^{y}(Z) - L_{s}^{z}(Z) \right| \right)$$

$$+ m_n^{2\delta} \sup_{0 \leq s \leq t} \left| \frac{Z_n(\vartheta s)}{m_n} - Z(s) \right| + \sup_{0 \leq u \leq \vartheta, |a-b| \leq m_n^{1-\delta}} \left| \frac{\Lambda_{u}^{a}(Z_n) - \Lambda_{u}^{b}(Z_n)}{m_n} \right|$$

$$+ \sup_{0 \leq u \leq \vartheta, a \in \mathbb{Z}} \frac{\Lambda_{u}^{a}(Z_n)}{m_n^{2-\delta}} \right).$$

Proof. Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ be fixed, and for each $\varepsilon > 0$, define the function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ as follows.

1. $f_\varepsilon(a/m_n) = 1/\varepsilon$;
2. \( f_\varepsilon(z) = 0 \) whenever \(|z - a/m_n| > \varepsilon\); and

3. define \( f_\varepsilon(z) \) by linear interpolation for \(|z - a/m_n| \leq \varepsilon\).

Since \( f_\varepsilon \) integrates to one, for every \( 0 \leq u \leq \vartheta \), we have that

\[
\left| \int_0^{u/m_n^2} f_\varepsilon(Z(s)) \, ds - \frac{L_a/m_n^2}{m_n} \sum_{j=1}^{u} f_\varepsilon(Z_n(j)/m_n) \right| = \int_{\mathbb{R}} f_\varepsilon(y) \left(L_u/m_n^2(Z) - L_a/m_n^2(Z)\right) \, dy
\]

\[
\leq \sup_{|y-a/m_n| \leq \varepsilon} \left| L_u/m_n^2(Z) - L_a/m_n^2(Z) \right|.
\]

Note that \(|f_\varepsilon(z) - f_\varepsilon(y)|/|z - y| \leq \frac{1}{\varepsilon^2}\) for all \( z, y \in \mathbb{R} \); hence, for every \( 0 \leq u \leq \vartheta \),

\[
\left| \int_0^{u/m_n^2} f_\varepsilon(Z(s)) \, ds - \frac{1}{m_n^2} \sum_{j=1}^{u} f_\varepsilon(Z_n(j)/m_n) \right| = \left| \int_0^{u/m_n^2} f_\varepsilon(Z(s)) - f_\varepsilon(Z_n(\vartheta)/m_n) \, ds \right|
\]

\[
\leq \frac{t}{\varepsilon^2} \sup_{0 \leq s \leq u/m_n^2} \left| Z_n(\vartheta)/m_n - Z(s) \right|.
\]

Finally,

\[
\frac{1}{m_n^2} \sum_{j=1}^{u} f_\varepsilon(Z_n(j)/m_n) - \frac{\Lambda_a(Z_n)}{m_n} = \frac{1}{m_n^2} \sum_{b \in \mathbb{Z}} f_\varepsilon(b/m_n) \Lambda_a^b(Z_n) - \frac{\Lambda_a(Z_n)}{m_n}
\]

\[
= \frac{1}{m_n} \sum_{b \in \mathbb{Z}} f_\varepsilon(b/m_n) \left( \frac{\Lambda_a(Z_n) - \Lambda_a^b(Z_n)}{m_n} \right)
\]

\[
+ \frac{\Lambda_a(Z_n)}{m_n} \left( \frac{1}{m_n} \sum_{b \in \mathbb{Z}} f_\varepsilon(b/m_n) - 1 \right).
\]

By a Riemann sum approximation,

\[
\left| \frac{1}{m_n} \sum_{b \in \mathbb{Z}} f_\varepsilon(b/m_n) - 1 \right| = O \left( \frac{1}{\varepsilon m_n} \right),
\]

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and thus we conclude that
\[
\frac{1}{m_n^2} \sum_{j=1}^{u} f_{\varepsilon}(Z_{n}(j)/m_n) - \frac{\Lambda^a_{u}(Z_{n})}{m_n} = O \left( \sup_{|a-b| \leq \varepsilon m_n} \frac{\Lambda^b_{u}(Z_{n}) - \Lambda^a_{u}(Z_{n})}{m_n} + \frac{\Lambda^a_{u}(Z_{n})}{\varepsilon m_n^2} \right).
\]

The result then follows by taking a supremum over \(0 \leq u \leq \vartheta\) and \(a \in \mathbb{Z}\), and taking \(\varepsilon = \varepsilon(n) = m_n^{-\delta}\).

4.4.2 Proof of Proposition 4.4.3

We begin with the proof of (4.4.2):

\[\text{Lemma 4.4.7. There exists a coupling such that (4.4.2) holds. In particular, for any} \quad 0 < \delta < 1/2, \quad \text{it holds almost surely as} \quad n \to \infty \quad \text{that}\]

\[
m_n^{2\delta} \sup_{0 \leq s \leq t} \left| \frac{Z_n(s)}{m_n} - Z(s) \right| = O(m_n^{-1+2\delta} \log m_n).
\]

\[\text{Proof. Suppose first that} \quad x = 0 \quad \text{so that} \quad (Z_n, Z) = (S^0, \tilde{B}^0) \quad \text{or} \quad (S^0, \tilde{B}^0, \tilde{B}^0, 0, t). \quad \text{According to the classical KMT coupling (e.g., [LL10, Section 7]) for Brownian motion and its extension to the Brownian bridge (e.g., [Bor78, Theorem 2], or the more recent [DW19]), it holds that}\]

\[
\sup_{0 \leq u \leq \vartheta} \left| \frac{Z_n(u)}{m_n} - Z(u/m_n^2) \right| = O(m_n^{-1} \log m_n)
\]

almost surely. Thus it only remains to prove that

\[
\sup_{0 \leq s \leq t} \left| Z(\vartheta s/m_n^2) - Z(s) \right| = O(m_n^{-1} \log m_n).
\]

For \(Z = \tilde{B}^0\), this is Lévy’s modulus of continuity theorem. For \(Z = \tilde{B}^{0,0}_t\), we note that the laws of \((\tilde{B}^{0,0}_t(s))_{s \in [0,t/2]}\) and \((\tilde{B}^{0,0}_t(t - s))_{s \in [0,t/2]}\) are absolutely continuous with respect the the law of \((\tilde{B}^0(s))_{s \in [0,t/2]}\).
Suppose now that \( x > 0 \). We can define \( S^x_n := x^n + S^0 \) and \( S^x_n := x^n + S^0 \), and similarly for \( \tilde{B} \). Since \( x^n/m_n = x + O(m_n^{-1}) \), our proof in the case \( x = 0 \) yields

\[
\sup_{0 \leq s \leq t} \left| \frac{S^x_n(\vartheta_s)}{m_n} - \tilde{B}^x(s) \right| = O(m_n^{-1} \log m_n + m_n^{-1})
\]

and similarly for the bridge, as desired. \( \square \)

With (4.4.2) established, the proof (4.4.6) is a straightforward application of the criterion provided by Lemma 4.4.6:

**Lemma 4.4.8.** For every \( \delta > 0 \) and \( 0 < \varepsilon < \delta/2 \),

\[
\sup_{0 \leq s \leq t, |y-z| \leq m_n^{\delta}} \left| L^y_s(Z) - L^z_s(Z) \right| = O(m_n^{-\delta/2+\varepsilon} \log m_n)
\]

almost surely as \( n \to \infty \).

**Proof.** The result for \( \tilde{B}^x \) is a direct application of [BK93, Equation (3.7)] (see also [Tro58, (2.1)]). We obtain the same result for \( \tilde{B}^{x,x}_i \) by the absolute continuity of \( (\tilde{B}^{x,x}_i(s))_{s \in [0,t/2]} \) and \( (\tilde{B}^{x,x}_i(t-s))_{s \in [0,t/2]} \) with respect to \( (\tilde{B}^x(s))_{s \in [0,t/2]} \), and the fact that local time is additive and invariant under time reversal. \( \square \)

**Lemma 4.4.9.** For every \( \delta > 0 \) and \( 0 < \varepsilon < \delta/2 \),

\[
\sup_{0 \leq u \leq \vartheta, |a-b| \leq m_n^{-1-\delta}} \left| \Lambda^a_u(Z_n) - \Lambda^b_u(Z_n) \right| = O(m_n^{-\delta/2+\varepsilon} \log m_n)
\]

almost surely as \( n \to \infty \).

**Proof.** According to [BK93, Proposition 3.1], for every \( \delta > 0 \) and \( 0 < \varepsilon < \delta/2 \), it holds that

\[
P \left[ \sup_{0 \leq u \leq \vartheta, a,b \in \mathbb{Z}} \frac{m_n^{-1} \left| \Lambda^a_u(S^x_n) - \Lambda^b_u(S^x_n) \right|}{|a/b - 1|^{2-\varepsilon} \wedge 1} \geq \lambda \right] = O(e^{-c\lambda} + m_n^{-14}) \quad (4.4.7)
\]
for every \( \lambda > 0 \), where \( c > 0 \) is independent of \( n \) and \( \lambda \). We recall that \( m_n \succ n^\vartheta \) with \( 1/13 < \vartheta \), which implies in particular that \( m_n^{-14} \) is summable in \( n \). Thus, if we take \( \lambda = \lambda(n) = C \log m_n \) for a large enough \( C > 0 \), then Borel-Cantelli yields

\[
\sup_{0 \leq u \leq \vartheta, \, |a-b| \leq m_n^{1-\delta}} \left| \frac{\Lambda^a_u(S^{x_n}) - \Lambda^b_u(S^{x_n})}{m_n} \right| = O \left( m_n^{\delta(n^{-1/2}) \log m_n} \right)
\]

almost surely, proving the result for \( Z_n = S^{x_n} \). In order to extend the result to \( Z_n = S^{x_n,x_n} \) we apply the local CLT (i.e., \( \mathbb{P}[S^{x_n}(\vartheta) = x^n]^{-1} = O(m_n) \); e.g., [GK68, §49]) with the elementary inequality \( \mathbb{P}[E_1|E_2] \leq \mathbb{P}[E_1]/\mathbb{P}[E_2] \) to (4.4.7):

\[
\mathbb{P} \left[ \sup_{0 \leq u \leq \vartheta, \, a,b \in \mathbb{Z}} \frac{m_n^{-1} \left| \Lambda^a_u(S^{x_n,x_n}) - \Lambda^b_u(S^{x_n,x_n}) \right|}{(a/m_n - b/m_n)^{1/2-\eta} \wedge 1} \geq \lambda \right] = O(m_ne^{-C\lambda} + m_n^{-13})
\]

for all \( \lambda > 0 \). Since \( \sum_n m_n^{-13} < \infty \) the result follows by Borel-Cantelli.

**Lemma 4.4.10.** For every \( 0 < \delta < 1 \), it holds almost surely as \( n \to \infty \) that

\[
\sup_{0 \leq u \leq \vartheta, \, a \in \mathbb{Z}} \frac{\Lambda^a_u(Z_n)}{m_n^{2-\delta}} = O \left( m_n^{-1+\delta} \log m_n \right).
\]

**Proof.** Note that, for any \( n, u \in \mathbb{N} \), \( |S^{x_n}(u)| \leq |x^n| + O(m_n^2) \). Therefore, by taking a large \( b \) in (4.4.7) (i.e., large enough so that \( \Lambda^b_u(S^{x_n}) = 0 \) surely), we see that

\[
\mathbb{P} \left[ \sup_{0 \leq u \leq \vartheta, \, a \in \mathbb{Z}} \frac{\Lambda^a_u(S^{x_n})}{m_n} \geq \lambda \right] = O(e^{-\lambda} + m_n^{-14}) \quad (4.4.8)
\]

for all \( \lambda > 0 \). The proof then follows from the same arguments as in Lemma 4.4.9.

By combining Lemmas 4.4.6–4.4.10, we obtain that

\[
\sup_{0 \leq u \leq \vartheta, \, a \in \mathbb{Z}} \left| \frac{\Lambda^a_u(Z_n)}{m_n} - L_{a/m_n^2}(Z) \right| = O(m_n^4 \log m_n),
\]
where, for every $0 < \delta < 1/2$ and $0 < \varepsilon < \delta/2$, we have

$$t = t(\delta, \varepsilon) := \max \{ -1 + 2\delta, -\delta/2 + \varepsilon \}.$$  

For any fixed $\varepsilon > 0$, the smallest possible $t(\delta, \varepsilon)$ occurs at the intersection of the lines $\delta \mapsto -1 + 2\delta$ and $\delta \mapsto -\delta/2 + \varepsilon$. This is attained at $\delta = 2(1 + \varepsilon)/5$, in which case $t = -1/5 + 4\varepsilon/5$. At this point, in order to get the statement of Proposition 4.4.3, we must show that

$$\sup_{0 \leq s \leq t, y \in \mathbb{R}} \left| L_{\delta, m}^{\lfloor m_n y \rfloor/m_n} (Z) - L_{s}^{y} (Z) \right| = O(m_n^{-1/5+\varepsilon} \log m_n)$$

as $n \to \infty$ for any $\varepsilon > 0$. This follows by a combination of Lemma 4.4.8 with $\delta = 1$ and the estimate [Tro58, (2.3)], which yields

$$\sup_{0 \leq s, \delta \leq t, |s-\delta| \leq m_n^{-2}, y \in \mathbb{R}} \left| L_{s}^{y} (Z) - L_{\delta}^{y} (Z) \right| = O(m_n^{-2/3} (\log m_n)^{2/3}).$$

(The results of [Tro58] are only stated for the Brownian motion, but this can be extended to the bridge by the absolute continuity argument used in Lemma 4.4.8.)

### 4.4.3 Proof of Proposition 4.4.4

We may assume without loss of generality that $x = 0$. We begin with the case of the unconditioned random walk $Z_n = S^0$.

Let $(\zeta_n^a)_{n \in \mathbb{N}, a \in \mathbb{Z}}$ be a collection of i.i.d. random variables with uniform distribution on $\{-1, 0, 1\}$. We can define the random walk $S^0$ as follows: For every $u, v \in \mathbb{N}$ and $a \in \mathbb{Z}$, if $S^0(u) = a$ and $\Lambda^a_n(S^0) = v$, then $S^0(u + 1) = S^0(u) + \zeta^a_n$. In doing so, up to an error of at most 1, it holds that

$$\Lambda_n^{(a,b)}(S^0) = \sum_{j=1}^{\Lambda^a_n(S^0)} 1_{\{\zeta_j^a = b-a\}}, \quad a, b \in \mathbb{Z}.$$
Hence, by the Borel-Cantelli lemma, it is enough to show that for any \( z \in \{-1, 0, 1\} \),

\[
\sum_{n \in \mathbb{N}} \mathbb{P} \left[ \sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{Z}} \left| \sum_{j=1}^{\vartheta} 1_{\{a_j = z\}} - \frac{\Lambda_a^u(S^0)}{3} \right| \geq Cm_n^{1/2} \log m_n \right] < \infty \tag{4.4.9}
\]

for some suitable finite constant \( C > 0 \). In order to prove this, we need two auxiliary estimates. Let us denote the range of a random walk by

\[
\mathcal{R}_u(S) := \max_{0 \leq j \leq u} S(j) - \min_{0 \leq j \leq u} S(j), \quad u \in \mathbb{N}_0. \tag{4.4.10}
\]

**Lemma 4.4.11.** For every \( \varepsilon > 0 \),

\[
\sum_{n \in \mathbb{N}} \mathbb{P} \left[ \mathcal{R}_\vartheta(S^0) \geq m_n^{1+\varepsilon} \right] < \infty.
\]

**Proof.** According to [Che10, (6.2.3)], there exists \( C > 0 \) independent of \( n \) such that

\[
\mathbb{E}[\mathcal{R}_\vartheta(S^0)^q] \leq (Cm_n)^q \sqrt{q!}, \quad q \in \mathbb{N}_0. \tag{4.4.11}
\]

Consequently, for every \( r < 2 \) and \( C > 0 \),

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{C(\mathcal{R}_\vartheta(S^0)/m_n)^r} \right] < \infty, \tag{4.4.12}
\]

The result then follows from Markov’s inequality. \( \square \)

**Lemma 4.4.12.** If \( C > 0 \) is large enough,

\[
\sum_{n \in \mathbb{N}} \mathbb{P} \left[ \sup_{a \in \mathbb{Z}} \Lambda_a^0(S^0) \geq Cm_n \log m_n \right] < \infty.
\]

**Proof.** This follows directly from (4.4.8) since \( \sum_n m_n^{-14} < \infty \). \( \square \)
According to Lemmas 4.4.11 and 4.4.12, to prove (4.4.9), it is enough to consider the sum of probabilities in question intersected with the events

\[ D_n := \left\{ R_{\vartheta}(S^0) \leq m_n^{1+\varepsilon}, \sup_{0 \leq u \leq \vartheta, a \in \mathbb{Z}} \Lambda_n^a(S^0) \leq C m_n \log m_n \right\} \]

for some large enough \( C > 0 \). By a union bound,

\[
P\left[ \left\{ \sup_{0 \leq u \leq \vartheta, a \in \mathbb{Z}} \left| \sum_{u=1}^{\Lambda_n^a(S^0)} 1_{\{\zeta^a_u = z\}} - \frac{\Lambda_n^a(S^0)}{3} \right| \geq c m_n^{1/2} \log m_n \right\} \cap D_n \right] \leq \sum_{-m_n^{1+\varepsilon} \leq a \leq m_n^{1+\varepsilon}, 0 \leq h \leq C m_n \log m_n} P\left[ \left\{ \sum_{j=1}^{h} 1_{\{\zeta^a_j = z\}} - \frac{h}{3} \geq c m_n^{1/2} \log m_n \right\} \right]. \tag{4.4.13}\]

By Hoeffding’s inequality,

\[
P\left[ \left| \sum_{j=1}^{h} 1_{\{\zeta^a_j = z\}} - \frac{h}{3} \right| \geq \tilde{C} m_n^{1/2} \log m_n \right] \leq 2e^{-2 \tilde{C}^2 \log m_n / C}
\]

uniformly in \( 0 \leq h \leq C m_n \log m_n \). Since the sum in (4.4.13) involves a polynomially bounded number of summands in \( m_n \) and the latter grows like a power of \( n \),

for any \( q > 0 \), we can choose \( \tilde{C} > 0 \) so that (4.4.9) is of order \( O(n^{-q}) \). \( \tag{4.4.14} \)

This concludes the proof of Proposition 4.4.4 in the case \( Z_n = S^0 \) by Borel-Cantelli.

In order to extend the result to the case \( Z_n = S_{\vartheta}^{x^n, x^n} \), it suffices to prove that (4.4.9) holds with the additional conditioning \( \{S^0(\vartheta) = 0\} \). The same local limit
theorem argument used at the end of the proof of Lemma 4.4.9 implies that

\[
\begin{align*}
&\mathbb{P} \left[ \sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{Z}} \left| \sum_{j=1}^{\Lambda_{0}^{a}(S_{0}^{x^{n}, x^{n}})} 1_{\{\zeta_{j}^{a}=\vartheta\}} - \frac{\Lambda_{u}^{a}(S_{0}^{x^{n}, x^{n}})}{3} \right| \geq Cm_{n}^{1/2} \log m_{n} \right] \\
&= O \left( m_{n} \mathbb{P} \left[ \sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{Z}} \left| \sum_{j=1}^{\Lambda_{0}^{a}(S_{0})} 1_{\{\zeta_{j}^{a}=\vartheta\}} - \frac{\Lambda_{u}^{a}(S_{0})}{3} \right| \geq Cm_{n}^{1/2} \log m_{n} \right] \right).
\end{align*}
\]

The result then follows from (4.4.14) by taking a large enough \( q \).

## 4.5 Strong Coupling for Reflected Walk

We now provide the counterpart of Theorem 4.4.2 for the Markov chain \( T \) in Definition 4.3.2 that is needed for Theorem 4.1.10.

**Definition 4.5.1.** Let \( \tilde{X} \) be a reflected Brownian motion on \((0, \infty)\) with variance \( \frac{2}{3} \). For every \( x \geq 0 \), we denote \( \tilde{X}^{x} := (\tilde{X}|\tilde{X}(0) = x) \), and we define the local time \( L_{t} \) and the boundary local time \( \mathcal{L}_{t} \) of \( \tilde{X} \) in the same way as for \( X \).

Our main result in this section is the following.

**Theorem 4.5.2.** Let \( t > 0 \) and \( x \geq 0 \) be fixed. Let \( \vartheta, \vartheta_{s}, (0 \leq s \leq t), x^{n}, \text{ and } (y^{n}, \bar{y}^{n}) \) \((y > 0)\) be as in Theorem 4.4.2. For every \( 0 < \varepsilon < 1/5 \), there exists a coupling of \( T^{x^{n}} \) and \( \tilde{X}^{x} \) such that

\[
\begin{align*}
&\sup_{0 \leq s \leq t} \left| \frac{T_{x^{n}}^{x}(\vartheta_{s})}{m_{n}} - \tilde{X}^{x}(s) \right| = O \left( m_{n}^{-1} \log m_{n} \right), \quad (4.5.1) \\
&\left| \frac{\Lambda_{0}^{(0,0)}(T^{x^{n}})}{m_{n}} - \frac{4\mathcal{L}_{0}^{x}(\tilde{X}^{x})}{3} \right| = O \left( m_{n}^{-1/2} (\log m_{n})^{3/4} \right), \quad (4.5.2) \\
&\sup_{0 \leq s \leq t, \ y > 0} \left| \frac{\Lambda_{0}^{(y, \bar{y})}(T^{x^{n}})}{m_{n}} (1 - \frac{1}{3} 1_{(y^{n}, \bar{y}^{n})=(0,0)}) - \frac{L_{s}^{y}(\tilde{X}^{x})}{3} \right| = O \left( m_{n}^{-1/5+\varepsilon} \log m_{n} \right).
\end{align*}
\]
almost surely as \( n \to \infty \).

**Remark 4.5.3.** In contrast with Theorem 4.4.2, Theorem 4.5.2 does not include a strong invariance result for the \( T \)'s bridge process \( T^x_\theta : x^n \). We discuss this omission (and state a related conjecture) in Section 4.5.5 below.

The first step in the proof for Theorem 4.5.2 is to use a modification of the Skorokhod reflection trick developed in [GLS19, Section 2] to reduce (4.5.1) to the KMT coupling stated in (4.4.2). As it turns out, this step also provides a proof of (4.5.2). The second step is to introduce a suitable modification of Lemma 4.4.6 that provides a criterion for the strong convergence of the vertex-occupation measures of \( T \) with the local time of \( \tilde{X} \). The third step is to prove an analog of Proposition 4.4.4. We summarize the last two steps in the following propositions:

**Proposition 4.5.4.** Almost surely, as \( n \to \infty \), one has

\[
\begin{align*}
sup_{0 \leq u \leq \theta} \left| \frac{1}{m_n} \Lambda^{(a,b)}_u(T^x_n) - \frac{\Lambda^a_u(T^x_n)}{3} \right| &= O(m_n^{-1/2} \log m_n), \\
and \\
\sup_{0 \leq u \leq \theta} \left| \frac{1}{m_n} \Lambda^{(0,0)}_u(T^x_n) - \frac{2\Lambda^0_u(T^x_n)}{3} \right| &= O(m_n^{-1/2} \log m_n).
\end{align*}
\]

**Proposition 4.5.5.** For every \( 0 < \varepsilon < 1/5 \), under the same coupling as (4.5.1), it holds almost surely as \( n \to \infty \) that

\[
\begin{align*}
sup_{0 \leq s \leq t, \: y > 0} \left| \frac{\Lambda_{\theta_s}^{[m_n y]}(T^x_n)}{m_n} - L^y_s(\tilde{X}^x) \right| &= O(m_n^{-1/5+\varepsilon} \log m_n).
\end{align*}
\]
4.5.1 Proof of (4.5.1)

Definition 4.5.6 (Skorokhod Map). Let $Z = (Z(t))_{t \geq 0}$ be a continuous-time stochastic process. We define the Skorokhod map of $Z$, denoted $\Gamma_Z$, as the process

$$\Gamma_Z(t) := Z(t) + \sup_{s \in [0,t]} (-Z(s))_+, \quad t \geq 0,$$

where $(-)_+ := \max\{0, -\}$ denotes the positive part of a real number.

Notation 4.5.7. In the sequel, whenever we discuss the Skorokhod map of the random walk $S$, $\Gamma_S$, we mean the Skorokhod map applied to the continuous-time process $s \mapsto S(\vartheta_s)$ for $0 \leq s \leq t$.

Note that $Z \mapsto \Gamma_Z$ is 2-Lipschitz with respect to the supremum norm on compact time intervals. Therefore, (4.5.1) is a direct consequence of (4.4.2) if we provide couplings $(T, S)$ and $(\tilde{X}, \tilde{B})$ such that $T^x_n(\vartheta_s) = \Gamma_{S^x_n}(s)$ and $\tilde{X}^x(s) = \Gamma_{\tilde{B}^x}(s)$.

Let us begin with the coupling of $\tilde{X}^x$ and $\tilde{B}^x$. Note that we can define $\tilde{X}^x := |\tilde{B}^x|$, where $\tilde{B}$ is a Brownian motion with variance $2/3$. Since the quadratic variation of $\tilde{B}^x$ is $t \mapsto (2/3)t$, it follows from Tanaka’s formula that

$$\tilde{X}^x(t) = x + \int_0^t \text{sgn}(\tilde{B}^x(s)) \, d\tilde{B}^x(s) + \frac{2\mathcal{L}_1^0(\tilde{B}^x)}{3}, \quad t \geq 0$$

(e.g., [RY99, Chapter VI, Theorem 1.2 and Corollary 1.9]), where

$$\mathcal{L}_1^0(\tilde{B}^x) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{-\varepsilon < \tilde{B}^x(s) < \varepsilon\}} \, ds = \mathcal{L}_1^0(\tilde{X}^x).$$

If we define

$$\tilde{B}^x_t := x + \int_0^t \text{sgn}(\tilde{B}^0(s)) \, d\tilde{B}^0(s), \quad t \geq 0,$$

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which is a Brownian motion with variance $2/3$ started at $x$, then we get from [RY99, Chapter VI, Lemma 2.1 and Corollary 2.2] that $\tilde{X}_t^x = \Gamma_{\tilde{B}^x(t)}$ and

$$\frac{2\mathcal{G}^0(\tilde{X}_t^x)}{3} = \sup_{s \in [0, t]} (-\tilde{B}^x(s))_+ \quad (4.5.4)$$

for every $t \geq 0$, as desired.

We now provide the coupling of $T^{x_n}$ and $S^{x_n}$. (See Figure 4.1 below for an illustration of the procedure we are about to describe.) Let $\mathcal{C}$ be the set of step functions of the form

$$A(s) = \sum_{u=0}^{\vartheta} A_u 1_{[u, u+1)}(s), \quad (4.5.5)$$

where $A_0, A_1, \ldots, A_{\vartheta} \in \mathbb{Z}$ are such that $A_0 = x_n$ and $A_{u+1} - A_u \in \{-1, 0, 1\}$ for all $u$. Let $\mathcal{C}_+ \subset \mathcal{C}$ be the subset of such functions that are nonnegative. For every $A \in \mathcal{C}$, let us define

$$\mathcal{H}_0(A) := \sum_{u=0}^{\vartheta-1} 1_{\{A_u = A_{u+1} = 0\}}. \quad (4.5.6)$$

By definition of $S$ and $T$, we see that for any $A \in \mathcal{C}$,

$$\begin{align*}
P\left[ (S^{x_n}(\vartheta_s))_{0 \leq s \leq t} = (A(\vartheta_s))_{0 \leq s \leq t} \right] &= \frac{1}{3^\vartheta}, \\
P\left[ (T^{x_n}(\vartheta_s))_{0 \leq s \leq t} = (A(\vartheta_s))_{0 \leq s \leq t} \right] &= \frac{2^{\mathcal{H}_0(A)}}{3^\vartheta} 1_{\{A \in \mathcal{C}_+\}}.
\end{align*}$$

It is clear that $A \mapsto \Gamma_A$ maps $\mathcal{C}$ to $\mathcal{C}_+$ and that this map is surjective since $\Gamma_A = A$ for any $A \in \mathcal{C}_+$. Thus, in order to construct a coupling such that $T^{x_n}(\vartheta_s) = \Gamma_{S^{x_n}(s)}$, it suffices to show that for every $A \in \mathcal{C}_+$, there are exactly $2^{\mathcal{H}_0(A)}$ distinct functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$. 

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Let $A \in \mathcal{C}_+$. If $\mathcal{H}_0(A) = 0$, then there is no $\tilde{A} \neq A$ such that $\Gamma_{\tilde{A}} = A$, as desired. Suppose then that $\mathcal{H}_0(A) = h > 0$. Let $0 \leq u_1, \ldots, u_h \leq \vartheta - 1$ be the integer coordinates such that $A_{u_j} = A_{u_{j+1}} = 0, 1 \leq j \leq h$. Then, $\Gamma_{\tilde{A}} = A$ if and only if the following conditions hold:

1. $\tilde{A}_{u_{j+1}} - \tilde{A}_{u_j} = 0$ or $\tilde{A}_{u_{j+1}} - \tilde{A}_{u_j} = -1$ for all $1 \leq j \leq h$, and

2. $\tilde{A}_{u+1} - \tilde{A}_u = A_{u+1} - A_u$ for all integers $u$ such that $u \notin \{u_1, \ldots, u_h\}$.

Note that, up to choosing whether the increments $\tilde{A}_{u_{j+1}} - \tilde{A}_{u_j}$ ($1 \leq j \leq h$) are equal to 0 or $-1$, the above conditions completely determine $\tilde{A}$. Moreover, there are $2^h$ ways of choosing these increments, each of which yields a different $\tilde{A}$. Therefore, there are in general $2^{\mathcal{H}_0(A)}$ distinct functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$, as desired.

Figure 4.1: On the left is a step function $A \in \mathcal{C}_+$ (where $x^n = 2$). The segments contributing to $\mathcal{H}_0(A)$ are blue. On the right are two (out of $2^{\mathcal{H}_0(S)} = 8$) step functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$.

### 4.5.2 Proof of (4.5.2)

Since the map $Z \mapsto \sup_{s \in [0,t]} (-Z(s))^+$ is Lipschitz with respect to the supremum norm on $[0,t]$, if we prove that the coupling of $T$ and $S$ introduced in Section 4.5.1 is such that

$$\Lambda^{(0,0)}(T^{x^n})_+ - 2 \max_{0 \leq s \leq t} \left( -S^{x^n}(\vartheta_s) \right)_+ = O \left( m_n^{-1/2} \log m_n \right)^{3/4}$$

(4.5.7)
almost surely as $n \to \infty$, then (4.5.2) is proved by a combination of (4.4.2) and (4.5.4).

Note that if $T^x_n(\vartheta_s) = A(\vartheta_s)$ for $s \leq t$, where $A \in \mathcal{C}_+$ is a step function of the form (4.5.5), then $\Lambda_{\vartheta}^{(0,0)}(T^{x_n}) = \mathcal{H}_0(A)$, as defined in (4.5.6). By analyzing the construction of the coupling of $T$ and $S$ in Section 4.5.1, we see that, conditional on the event $\{\Lambda_{\vartheta}^{(0,0)}(T^{x_n}) = h\}$ ($h \in \mathbb{N}_0$), the quantity $\max_{0 \leq s \leq t} \left( -S^{x_n}(\vartheta_s) \right)_+$ is a binomial random variable with $h$ trials and probability $1/2$. With this in mind, our strategy is to prove (4.5.7) using a binomial concentration bound similar to (4.4.13). For this, we need a good control on the tails of $\Lambda_{\vartheta}^{(0,0)}(T^{x_n})$:

**Proposition 4.5.8.** There exists constants $C, c > 0$ independent of $n$ such that for every $y \geq 0$,

$$\sup_{n \in \mathbb{N}, x \geq 0} \mathbb{P} \left[ \Lambda_{\vartheta}^{(0,0)}(T^{x_n}) \geq m_n y \right] \leq C e^{-cy^2}.$$ 

In particular, there exists $C > 0$ large enough so that

$$\sum_{n \in \mathbb{N}} \mathbb{P} \left[ \Lambda_{\vartheta}^{(0,0)}(T^{x_n}) \geq C m_n \sqrt{\log m_n} \right] < \infty. \quad (4.5.8)$$

Indeed, with this result in hand, we obtain by Hoeffding’s inequality that

$$\mathbb{P} \left[ (h/2) - \max_{0 \leq s \leq t} \left( -S^{x_n}(\vartheta_s) \right)_+ \geq \tilde{C} m_n^{1/2} (\log m_n)^{3/4}/2 \left| \Lambda_{\vartheta}^{(0,0)}(T^{x_n}) = h \right| \right]$$

$$\leq 2 e^{-\tilde{C}^2 \log m_n / 2C}$$

uniformly in $0 \leq h \leq C m_n \sqrt{\log m_n}$. By taking $\tilde{C} > 0$ large enough, we conclude that (4.5.2) holds by an application of the Borel-Cantelli lemma combined with (4.5.8).

**Proof of Proposition 4.5.8.** Let $T$ and $S$ be coupled as in Section 4.5.1, and let

$$\mu_{\vartheta}(S) := \sum_{u=0}^{\vartheta-1} 1\left\{ S(u+1) \leq \min\{S(0), S(1), \ldots, S(u)\} \right\},$$

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that is, the number of times that $S$ is smaller or equal to its running minimum over the first $\vartheta$ steps. Then, we see that

$$
\Lambda^{(0,0)}_\vartheta(T) = \sum_{u=0}^{\vartheta-1} \mathbf{1}_{\left\{ S(u) \leq 0, \ S(u+1) \leq \min\{S(0),S(1),\ldots,S(u)\} \right\}} \leq \mu_\vartheta(S).
$$

Given that $\mu_\vartheta(S)$ is independent of $S$’s starting point, it suffices to prove that

$$
\sup_{n \in \mathbb{N}} \mathbb{P}\left[ \mu_\vartheta(S^0) \geq mn y \right] \leq Ce^{-cy^2}, \quad y \geq 0 \quad (4.5.9)
$$

for some constants $C, c > 0$.

If $y > mn t$, then $mn y \geq \vartheta$, hence $\mathbb{P}\left[ \mu_\vartheta(S^0) \geq mn y \right] = 0$. Thus, it suffices to prove (4.5.9) for $y \leq mn t$. Our proof of this is inspired by [MM03, Lemma 7]: Let $0 = t_0 < t_1 < t_2 < \cdots$ be the weak descending ladder epochs of $S^0$, that is,

$$
t_{u+1} := \min\{v > t_u : S^0(v) \leq S^0(t_u)\}, \quad u \in \mathbb{N}_0.
$$

Then, for any $\nu > 0$,

$$
\mathbb{P}\left[ \mu_\vartheta(S^0) \geq mn y \right] = \mathbb{P}\left[ t_{[mn y]} \leq \vartheta \right]
\leq \mathbb{P}\left[ S^0(t_{[mn y]}) \geq \min_{0 \leq u \leq \vartheta} S^0(u) \right]
\leq \mathbb{P}\left[ S^0(t_{[mn y]}) \geq -\nu mn y \right] + \mathbb{P}\left[ \min_{0 \leq u \leq \vartheta} S^0(u) < -\nu mn y \right].
$$

On the one hand, we note that $S^0(t_{[mn y]})$ is equal in distribution to the sum of $[mn y]$ i.i.d. copies of $S^0(t_1)$, which we call the the ladder height of $S^0$. Moreover, it is easily seen that the ladder height has distribution $\mathbb{P}[S^0(t_1) = 0] = 2/3$ and $\mathbb{P}[S^0(t_1) = -1] = 1/3$. In particular, $\mathbb{E}[S^0(t_{[mn y]})] = -[mn y]/3$. Thus, if we choose $\nu$ small enough (namely $\nu < 1/3$), then by combining Hoeffding’s inequality with
\( m_n \geq y/t \), we obtain

\[
P \left[ S^0(\text{t}_{m_n,y}) \geq -\nu m_n y \right] = P \left[ S^0(\text{t}_{m_n,y}) + \left\lceil \frac{m_n y}{3} \right\rceil \geq -\nu m_n y + \left\lceil \frac{m_n y}{3} \right\rceil \right] \leq C_1 e^{-c_1 m_n y} \leq C_1 e^{-c_1 y^2/t}
\]

for some \( C_1, c_1 > 0 \) independent of \( n \).

On the other hand, by Etemadi’s and Hoeffding’s inequalities,

\[
P \left[ \min_{0 \leq u \leq \varrho} S^0(u) < -\nu m_n y \right] \leq P \left[ \max_{0 \leq u \leq \varrho} |S^0(u)| > \nu m_n y \right] \leq 3 \max_{0 \leq u \leq \varrho} P \left[ |S^0(u)| > \nu m_n y/3 \right] \leq C_2 e^{-c_2 y^2}
\]

for some \( C_2, c_2 > 0 \) independent of \( n \), concluding the proof of (4.5.9) for \( y \leq m_n t \).

**4.5.3 Proof of Proposition 4.5.4**

By replicating the binomial concentration argument in the proof of Proposition 4.4.4, it suffices to prove that

\[
\sum_{n \in \mathbb{N}} P \left[ \mathcal{R}_\varrho(T^x_n) \geq m_n^{1+\varepsilon} \right] < \infty \quad (4.5.10)
\]

for every \( \varepsilon > 0 \), where we define \( \mathcal{R}_\varrho(T^x_n) \) as in (4.4.10), and

\[
\sum_{n \in \mathbb{N}} P \left[ \sup_{a \in \mathbb{Z}} \Lambda^a_{\varrho}(T^x_n) \geq C m_n \log m_n \right] < \infty \quad (4.5.11)
\]

provided \( C > 0 \) is large enough. In order to prove this, we introduce another coupling of \( S \) and \( T \), which will also be useful later in the chapter:

**Definition 4.5.9.** Let \( a \in \mathbb{N}_0 \) be fixed. Given a realization of \( T^a \), let us define the time change \( \left( \tilde{T}^a(u) \right)_{u \in \mathbb{N}_0} \) as follows:
1. \( \tilde{\varrho}^a(0) = 0 \).

2. If \( T^a(\tilde{\varrho}^a(u)) \neq 0 \) or \( T^a(\tilde{\varrho}^a(u) + 1) \neq 0 \), then \( \tilde{\varrho}^a(u + 1) = \tilde{\varrho}^a(u) + 1 \).

3. If \( T^a(\tilde{\varrho}^a(u)) = 0 \) and \( T^a(\tilde{\varrho}^a(u) + 1) = 0 \) then we sample

\[
P[\tilde{\varrho}^a(u + 1) = \tilde{\varrho}^a(u) + 1] = \frac{1}{4} \quad \text{and} \quad P[\tilde{\varrho}^a(u + 1) = \tilde{\varrho}^a(u) + 2] = \frac{3}{4},
\]

independently of the increments in \( T^a \).

In words, we go through the path of \( T^a \) and skip every visit to the self-edge \((0, 0)\) with probability \(3/4\). Then, we define \( \varrho^a \) as the inverse of \( \tilde{\varrho}^a \), which is well defined since the latter is strictly increasing.

By a straightforward geometric sum calculation, it is easy to see that we can couple \( S \) and \( T \) in such a way that

\[
T^{x^n}(u) = |S^{x^n}(\tilde{\varrho}^{x^n}(u))|, \quad u \in \mathbb{N}_0. \tag{4.5.12}
\]

For the remainder of the proof of Proposition 4.5.4 we adopt this coupling.

On the one hand, \( \mathcal{R}_\varrho(T^{x^n}) = \mathcal{R}_{\varrho^{x^n}}(|S^{x^n}|) \leq \mathcal{R}_\varrho(S^{x^n}) \). Thus (4.5.10) follows directly from Lemma 4.4.11. On the other hand, for every \( a \neq 0 \),

\[
\Lambda^a_u(T^{x^n}) = \Lambda^a_{\tilde{\varrho}^{x^n}}(S^{x^n}) + \Lambda^{-a}_{\tilde{\varrho}^{x^n}}(S^{x^n}) \leq \Lambda^a_u(S^{x^n}) + \Lambda^{-a}_u(S^{x^n}), \tag{4.5.13}
\]

and

\[
\Lambda^0_u(T^{x^n}) = \Lambda^{(0, 0)}_u(T^{x^n}) + \Lambda^{(0, -1)}_{\tilde{\varrho}^{x^n}}(S^{x^n}) + \Lambda^{(0, 1)}_{\tilde{\varrho}^{x^n}}(S^{x^n}) \\
\leq \Lambda^{(0, 0)}_u(T^{x^n}) + \Lambda^0_u(S^{x^n}). \tag{4.5.14}
\]

Thus (4.5.11) follows from (4.5.8) and Lemma 4.4.12.
4.5.4 Proof of Proposition 4.5.5

The following extends Lemma 4.4.6 to $T$.

Lemma 4.5.10. For any $0 < \delta < 1$, the following holds almost surely as $n \to \infty$:

\[
\sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{N}_0} \left| \frac{\Lambda_a(T^{x_n})}{m_n} - \frac{L^{a/m_n}}{m_n} (\bar{X}^x) \right| = O \left( \sup_{y,z \geq 0, \ |y-z| \leq m_2^{-\delta}} \left| L^y_s(\bar{X}^x) - L^z_s(\bar{X}^x) \right| \right)
\]
\[
+ m_2^{2\delta} \sup_{0 \leq s \leq \vartheta} \left| \frac{T^{x_n}(\vartheta_s)}{m_n} - \bar{X}^x(s) \right| + \frac{m_2^{2\delta}}{m_n} \sup_{a,b \in \mathbb{N}_0, \ |a-b| \leq m_2^{-\delta}} \left| \frac{\Lambda_a^a(T^{x_n}) - \Lambda_b^b(T^{x_n})}{m_n} \right|
\]
\[
+ \frac{m_2^{2\delta}}{m_n} \sup_{0 \leq u \leq \vartheta, \ a \in \mathbb{N}_0} \left| \frac{\Lambda_a(T^{x_n})}{m_n} \right| \left( m_1^{1-2\delta} \log m_n \right).
\]

Proof. Let $a \in \mathbb{N}$ be fixed. For every $\varepsilon > 0$, let $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined as in the proof of Lemma 4.4.6, and let us define $g_\varepsilon : (0, \infty) \to \mathbb{R}$ as

\[
g_\varepsilon(z) := f_\varepsilon(z) \left( \int_0^\infty f_\varepsilon(z) \, dz \right)^{-1}, \quad z \geq 0.
\]

$g_\varepsilon$ integrates to one on $(0, \infty)$, and $|g_\varepsilon(z) - g_\varepsilon(y)|z - y| \leq \frac{2}{\varepsilon^2}$ for $y, z \geq 0$. By repeating the proof of Lemma 4.4.6 verbatim with $g_\varepsilon$ instead of $f_\varepsilon$, we obtain the result. \[\square\]

We now apply Lemma 4.5.10. (4.5.1) yields

\[
m_2^{2\delta} \sup_{0 \leq s \leq \vartheta} \left| \frac{T^{x_n}(\vartheta_s)}{m_n} - \bar{X}^x(s) \right| = O \left( m_1^{1-2\delta} \log m_n \right).
\]

As for the regularity of the vertex-occupation measures and local time of $T^{x_n}$ and $\bar{X}^x$, they follow directly from the proof of Proposition 4.4.3 using Lemma 4.4.6 by applying some carefully chosen couplings of $T^{x_n}$ with $S^{x_n}$, and $\bar{X}^x$ with $\tilde{B}^x$.
We begin with the latter. If we define $\tilde{X}^x(s) = |\tilde{B}^x(s)|$, then for every $y \geq 0$ and $s \geq 0$, we have that $L^y_s(\tilde{X}^x) = L^y_s(\tilde{B}^x) + L^{-y}_s(\tilde{B}^x)$. Consequently,

$$ |L^y_s(\tilde{X}^x) - L^z_s(\tilde{X}^x)| \leq |L^y_s(\tilde{B}^x) - L^z_s(\tilde{B}^x)| + |L^{-y}_s(\tilde{B}^x) - L^{-z}_s(\tilde{B}^x)|. \quad (4.5.15) $$

The regularity estimates for $L^y_s(\tilde{X}^x)$ then follow from the same results for $L^y_s(\tilde{B}^x)$.

To prove the desired estimates on the occupation measures, we use the coupling introduced in Definition 4.5.9. This immediately yields an adequate control of the supremum of $\Lambda^a(T^{x^n})$ by (4.5.11). As for regularity, one the one hand, we note that

$$ |\Lambda^a_u(T^{x^n}) - \Lambda^b_u(T^{x^n})| \leq |\Lambda^a_{\phi'_n}(S^{x^n}) - \Lambda^b_{\phi'_n}(S^{x^n})| + |\Lambda^{a'}_{\phi'_n}(S^{x^n}) - \Lambda^{b'}_{\phi'_n}(S^{x^n})| $$

for any $a, b \neq 0$. On the other hand, for any $a \neq 0$,

$$ |\Lambda^0_u(T^{x^n}) - \Lambda^a_u(T^{x^n})| \leq \frac{1}{2} |\Lambda^0_u(T^{x^n}) - \Lambda^a_{\phi'_n}(S^{x^n})| + \frac{1}{2} |\Lambda^0_u(T^{x^n}) - \Lambda^{-a}_{\phi'_n}(S^{x^n})|.$$

Hence we get the desired estimate by Lemma 4.4.9 if we prove that

$$ \sup_{0 \leq u \leq \theta} \left| \frac{1}{2} \Lambda^0_u(T^{x^n}) - \Lambda^0_{\phi'_n}(S^{x^n}) \right| = O\left(m^{-1/2} \log m_n \right) \quad (4.5.16) $$

almost surely as $n \to \infty$. By Propositions 4.4.4 and 4.5.4, (4.5.16) can be reduced to

$$ \sup_{0 \leq u \leq \theta} 3 \left| \frac{1}{4} \Lambda^{(0,0)}_u(T^{x^n}) - \Lambda^{(0,0)}_{\phi'_n}(S^{x^n}) \right| = O\left(m^{-1/2} \log m_n \right). \quad (4.5.17) $$

By Definition 4.5.9, conditional on $\Lambda^{(0,0)}_u(T^{x^n})$, we note that $\Lambda^{(0,0)}_{\phi'_n(u)}(S^{x^n})$ is a binomial random variable with $\Lambda^{(0,0)}_u(T^{x^n})$ trials and probability $1/4$. Hence we obtain (4.5.17) by combining (4.5.8) with Hoeffding's inequality similarly to (4.4.13).

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4.5.5 Coupling of $T_{\vartheta}^{a,b}$

In light of Theorems 4.4.2 and 4.5.2, the following conjecture is natural.

**Conjecture 4.5.11.** The statement of Theorem 4.5.2 holds with every instance of $T^{x^n}$ replaced by $T_{\vartheta_{t_1}}^{x^n,x^n}$, and every instance of $\tilde{X}^{x^n}$ replaced by $\tilde{X}_{t_1}^{x^n,x^n}$.

However, if we couple $T$ in $S$ as in Section 4.5.1, then conditioning on the endpoint of $T$ corresponds to an unwieldy conditioning of the path of $S$:

$$P[T^a(\vartheta) = a] = P\left[ S^a(\vartheta) = \max_{0 \leq u \leq \vartheta} (-S^a(u))_+ + a \right].$$

There seems to be no existing strong invariance result (such as KMT) applicable to this conditioning. Consequently, it appears that a proof of Conjecture 4.5.11 relies on a strong invariance result for conditioned random walks that is outside the scope of the current literature, or that it requires an altogether different reduction to a classical coupling (which we were not able to find).

### 4.6 Weak Convergence to Dirichlet Semigroup

In this section, we prove Theorem 4.1.9 (1). For the remainder of this section, we fix some times $t_1, \ldots, t_k > 0$ and uniformly continuous and bounded functions $f_1, g_1, \ldots, f_k, g_k$.

**4.6.1 Step 1: Convergence of Mixed Moments**

Consider a mixed moment

$$\mathbb{E}\left[ \prod_{i=1}^k (f_i, \hat{K}_{t_i}^d g_i)^{n_i} \right], \quad n_1, \ldots, n_k \in \mathbb{N}_0.$$
Up to making some $f_i$'s, $g_i$'s, and $t_i$'s equal to each other and reindexing, there is no loss of generality in writing the above in the form

$$E \left[ \prod_{i=1}^{k} \langle f_i, \hat{K}^d_n(t_i)g_i \rangle \right]. \quad (4.6.1)$$

By applying Fubini's theorem to (4.3.8), we can write (4.6.1) as

$$\int_{[0,(n+1)/m_n)^k} \left( \prod_{i=1}^{k} f_i(x_i) \right) \cdot E \left[ \prod_{i=1}^{k} F^d_{n,t_i}(S^i_{x^i_n}) m_n \int_{S^i_{x^i_n}(\vartheta_i)/m_n}^{(S^i_{x^i_n}(\vartheta_i)+1)/m_n} g_i(y) \, dy \right] \, dx_1 \cdots dx_k \quad (4.6.2)$$

and the corresponding limiting expression is

$$E \left[ \prod_{i=1}^{k} \langle f_i, \hat{K}^d(t_i)g_i \rangle \right] = \int_{(0,\infty)^k} \left( \prod_{i=1}^{k} f_i(x_i) \right) \cdot E \left[ \prod_{i=1}^{k} 1_{\{\tau_0(B^i;x^i_i)>t\}} e^{-(L_i(B^i;x^i_i),V)-\int L_i^a(B^i;x^i_i) \, dW(a)} g_i(B^i;x^i_i(t)) \right] \, dx_1 \cdots dx_k, \quad (4.6.3)$$

where

1. $\vartheta_i = \vartheta_i(n,t_i) := \lfloor m_n^2(3t_i/2) \rfloor$ for every $n \in \mathbb{N}$ and $1 \leq i \leq k$;

2. $x^i_n := \lfloor m_n x_i \rfloor$ for every $n \in \mathbb{N}$ and $1 \leq i \leq k$;

3. $S^1_{x^1_n}, \ldots, S^k_{x^k_n}$ are independent copies of $S$ with respective starting points $x^1_n, \ldots, x^k_n$, and

4. $B^1_{x^1}, \ldots, B^k_{x^k}$ are independent copies of $B$ with respective starting points $x_1, \ldots, x_k$.

We further assume that the $S^i_{x^i_n}$ are independent of $Q_n$, and that the $B^i_{x^i}$ are independent of $W$. The proof of moment convergence is based on the following:
Proposition 4.6.1. Let $x_1, \ldots, x_n \geq 0$ be fixed. There is a coupling of the $S^{i:x_i^n}$ and $B^{i:x_i}$ such that the following limits hold jointly in distribution over $1 \leq i \leq k$:

1. $\lim_{n \to \infty} \sup_{0 \leq s \leq t_i} \left| \frac{S^{i:x_i^n}([m_n^2(3s/2)])}{m_n} - B^{i:x_i}(s) \right| = 0$.

2. $\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \left| \frac{\Lambda_{\hat{y}_1}^{(y_n, \hat{y}_n)}(S^{i:x_i^n})}{m_n} - \frac{1}{2} L^y_{t_i}(B^{i:x_i}) \right| = 0$, jointly in $(y_n, \bar{y}_n)_{n \in \mathbb{N}}$ equal to the three sequences in (4.4.1).

3. $\lim_{n \to \infty} m_n \int_{S^{i:x_i^n}(\hat{y}_1)/m_n}^{S^{i:x_i^n}(\hat{y}_1)+1/m_n} g_i(y) \, dy = g_i(B^{i:x_i}(t))$.

4. The convergences in (4.1.21).

5. $\lim_{n \to \infty} \sum_{a \in \mathbb{N}_0} \frac{\Lambda_{\hat{y}_1}^{(a_E, \hat{a}_E)}(S^{i:x_i^n}) \xi^n_a(a)}{m_n} = \frac{1}{2} \int_{(0,\infty)} L^y_{t_i}(B^{i:x_i}) \, dW^E(y)$, jointly in $E \in \{D, U, L\}$, where for every $a \in \mathbb{N}_0$,

$$
(a_E, \hat{a}_E) := \begin{cases} 
(a, a) & \text{if } E = D, \\
(a, a + 1) & \text{if } E = U, \\
(a + 1, a) & \text{if } E = L.
\end{cases}
$$

Proof. According to Theorem 4.4.2 in the case of the lazy random walk, we can couple $S^{i:x_i^n}$ with a Brownian motion with variance $2/3$ started at $x_i$, $\tilde{B}^{i:x_i}$, in such a way that

$$
\frac{S^{i:x_i^n}([m_n^2(3s/2)])}{m_n} \to \tilde{B}^{i:x_i}(3s/2) \quad \text{and} \quad \frac{\Lambda_{\hat{y}_1}^{(y_n, \hat{y}_n)}(S^{i:x_i^n})}{m_n} \to \frac{1}{3} L^y_{3t_i/2}(\tilde{B}^{i:x_i})
$$

uniformly almost surely. Let $B^{i:x_i}(s) := \tilde{B}^{i:x_i}(3s/2)$. By the Brownian scaling property, $B^{i:x_i}$ is standard, and $L^y_{3t_i/2}(\tilde{B}^{i:x_i}) = \frac{3}{2} L^y_{t_i}(B^{i:x_i})$. Hence (1) and (2) hold almost surely. Since $g_i$ is uniformly continuous, (3) holds almost surely by (1) and the Lebesgue differentiation theorem. With this given, (4) and (5) follow from Assumption NT3.
Remark 4.6.2. Since the strong invariance principles in Theorem 4.4.2 are uniform
in the time parameter, it is clear that Proposition 4.6.1 remains valid of we take
\( \vartheta_i := [m_n^2(3t_i/2)] \pm 1 \) instead of \( [m_n^2(3t_i/2)] \). Referring back to Remark 4.1.13, there
is no loss of generality in assuming that the \( \vartheta_i \) have a particular parity. The same
comment applies to our proof of Theorem 4.1.9 (2) and Theorem 4.1.10.

Convergence Inside the Expected Value

We first prove that for every fixed \( x_1, \ldots, x_k \geq 0 \), there exists a coupling such that

\[
\lim_{n \to \infty} \prod_{i=1}^{k} F_{n,t_i}^{d_i}(S^{i,x_i}_n) m_n \int_{S^{i,x_i}_n(\vartheta_i)/m_n}^{(S^{i,x_i}_n(\vartheta_i)+1)/m_n} g_i(y) \, dy
= \prod_{i=1}^{k} 1_{\{\tau_0(B^{i,x_i}) > t_i\}} e^{-\langle L_{t_i}(B^{i,x_i}), V \rangle - \int L_{t_i}^a(B^{i,x_i}) dW(a)} g_i(B^{i,x_i}(t)) \quad (4.6.5)
\]
in probability. According to the Skorokhod representation theorem (e.g., [Bil99, The-
orem 6.7]), there is a coupling such that Proposition 4.6.1 holds almost surely. For
the remainder of Section 4.6.1, we adopt such a coupling.

Since \( m_n^{-1} S^{i,x_i}_n([m_n^2(3s/2)]) \to B^{i,x_i}(s) \) uniformly on \( s \in [0, t_i] \), and \( m_n^2 = o(n) \),

\[
\lim_{n \to \infty} 1_{\{\tau_0(B^{i,x_i}) > t_i\}} 1_{\{\tau^{(n)}(S^{i,x_i}_n) > \vartheta_i\}} = 1_{\{\tau_0(B^{i,x_i}) > t_i\}}
\]

almost surely. By combining this with Proposition 4.6.1 (3), it only remains to
prove that the terms involving the matrix entries \( D_n, U_n \), and \( L_n \) in the functional
\( F_{n,t_i}^{d_i} \) converge to \( e^{-\langle L_{t_i}(B^{i,x_i}), V \rangle - \int L_{t_i}^a(B^{i,x_i}) dW(a)} \). To this effect, we note that for all
\( E \in \{D, U, L\} \), one has

\[
\prod_{a \in \mathbb{N}_0} \left( 1 - \frac{E_n(a)}{m_n^2} \right) A_{\vartheta_i}^{(a_E, a_E), (S^{i,x_i}_n)} = \exp \left( \sum_{a \in \mathbb{N}_0} A_{\vartheta_i}^{(a_E, a_E), (S^{i,x_i}_n)} \log \left( 1 - \frac{E_n(a)}{m_n^2} \right) \right),
\]

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where we recall that \((a_E, \tilde{a}_E)\) are defined as in (4.6.4). By using the Taylor expansion \(\log(1 + z) = z + O(z^2)\), this is equal to

\[
\exp\left(-\sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{E_n(a)}{m_n^2} + O\left(\sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{E_n(a)^2}{m_n^4}\right)\right) .
\] (4.6.6)

We begin by analyzing the leading order term in (4.6.6). On the one hand, the uniform convergence of Proposition 4.6.1 (2) (which implies in particular that \(y \mapsto \Lambda_{\partial_i}^{(y_n, \tilde{y}_n)}(S_i^a) / m_n\) and \(y \mapsto L(y(B_i))\) are supported on a common compact interval almost surely) together with the fact that \(V_n^E([m_n y]) \to V^E(y)\) uniformly on compacts (by Assumption PT1) implies that

\[
\lim_{n \to \infty} \sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{V_n^E(a)}{m_n^2} = \lim_{n \to \infty} \int_0^\infty \frac{\Lambda_{\partial_i}^{(y_n, \tilde{y}_n)}(S_i^a) V_n^E([m_n y])}{m_n} \, dy = \frac{1}{2} \langle L_{t_i}(B_i), V^E \rangle
\] (4.6.7)

almost surely (where we choose the appropriate sequence \((y_n, \tilde{y}_n)\) as defined in (4.4.1) depending on \((a_E, \tilde{a}_E))\). By combining this with Proposition 4.6.1 (5), we get

\[
\lim_{n \to \infty} \sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{E_n(a)}{m_n^2} = \frac{1}{2} \langle L_{t_i}(B_i), V^E \rangle + \frac{1}{2} \int_0^\infty L_{t_i}^a(B_i) \, dW^E(a) \quad (4.6.8)
\]

almost surely.

Next, we control the error term in (4.6.6). By using \((z + \tilde{z})^2 \leq 2(z^2 + \tilde{z}^2)\), for this it suffices control

\[
\sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{V_n^E(a)^2}{m_n^4} \quad \text{and} \quad \sum_{a \in \mathbb{N}_0} \Lambda_{\partial_i}^{(a_E, \tilde{a}_E)}(S_i^a) \frac{\xi_n^E(a)^2}{m_n^4}.
\]
separately. On the one hand, the argument used in (4.6.7) yields
\[
\sum_{a \in \mathbb{N}_0} \Lambda_{\delta_i}^{(a,\tilde{a}_E)}(S^{i,x_n^i}) \frac{V_n^E(a)^2}{m_n^4} = m_n^{-2} \cdot \frac{1}{2}(1 + o(1)) \langle L_t(B^{x_i}), (V^E)^2 \rangle.
\]
Since $V^E$ is continuous and $L_t(B^{x_i})$ is compactly supported with probability one, this converges to zero almost surely. On the other hand, by definition of (4.3.3),
\[
\sum_{a \in \mathbb{N}_0} \Lambda_{\delta_i}^{(a,b)}(S^{i,x_n^i}) \leq \vartheta_i = O(m_n^2)
\]
uniformly in $b \in \mathbb{Z}$. Therefore, it follows from the tower property and (4.1.20) that
\[
E \left[ \sum_{a \in \mathbb{N}_0} \Lambda_{\delta_i}^{(a,b)}(S^{i,x_n^i}) \right] \leq O(m_n^{-1});
\]
hence we have convergence to zero in probability.

By combining the convergence of the leading terms (4.6.8), our analysis of the error terms, and (4.1.14) and (4.1.22), we conclude that (4.6.5) holds.

Convergence of the Expected Value

Next, we prove that
\[
\lim_{n \to \infty} E \left[ \prod_{i=1}^k F_{n,t_i}^{d}(S^{i,x_n^i}) m_n \int_{S^{i,x_n^i}(\vartheta_i)/m_n} g_i(y) \, dy \right] = E \left[ \prod_{i=1}^k 1_{\{\tau_0(B^{i,x_i}) > t\}} e^{-L_{t_i}(B^{i,x_i},V)} - \int L_{t_i}^{a}(B^{i,x_i}) \, dW(a) g_i(B^{i,x_i}(t)) \right] (4.6.9)
\]
pointwise in $x_1, \ldots, x_k \geq 0$. Given (4.6.5), we must prove that the sequence of variables inside the expected value on the left-hand side of (4.6.9) are uniformly
integrable. For this, we prove that

\[
\sup_{n \geq N} \mathbb{E} \left[ \prod_{i=1}^{k} \left( F^{d}_{n,t_i}(S^{i,x^n_i}) m_n \int_{S^{i,x^n_i}(\vartheta_i)/m_n} g_i(y) \, dy \right)^2 \right] \\
\leq \sup_{n \geq N} \prod_{i=1}^{k} \mathbb{E} \left[ \left( F^{d}_{n,t_i}(S^{i,x^n_i}) m_n \int_{S^{i,x^n_i}(\vartheta_i)/m_n} g_i(y) \, dy \right)^{2k} \right]^{1/k} < \infty
\]

for large enough \( N \), where the first upper bound is due to Hölder’s inequality.

Since the \( g_i \)'s are bounded,

\[
m_n \int_{S^{i,x^n_i}(\vartheta_i)/m_n} g_i(y) \, dy \leq \|g_i\|_{\infty} < \infty,
\]

uniformly in \( n \), and thus we need only prove that

\[
\sup_{n \geq N} \mathbb{E} \left[ \left| F^{d}_{n,t_i}(S^{i,x^n_i}) \right|^{2k} \right] < \infty, \quad 1 \leq i \leq k. \tag{4.6.10}
\]

Since indicator functions are bounded by 1, their contribution to (4.6.10) may be ignored. For the other terms, we note that for \( E \in \{D, U, L\} \) we can write

\[
1 - \frac{E_n(a)}{m_n^2} = \frac{m_n^2 - V_n^E(a) - \xi_n^E(a)}{m_n^2} = \left( 1 - \frac{V_n^E(a)}{m_n^2} \right) \left( 1 - \frac{\xi_n^E(a)}{m_n^2 - V_n^E(a)} \right). \tag{4.6.11}
\]

By (4.1.15), for large \( n \) we have \( |1 - V_n^E(a)/m_n^2| \leq 1 \), hence by applying Hölder’s inequality in (4.6.10), we need only prove that

\[
\sup_{n \geq N} \mathbb{E} \left[ \prod_{a \in \mathbb{N}_0} \left| 1 - \frac{\xi_n^E(a)}{m_n^2 - V_n^E(a)} \right|^{6k\Lambda^{(a_{E^a,E})}(S^{i,x^n_i})} \right] < \infty, \quad E \in \{D, U, L\}. \tag{4.6.12}
\]

Let us fix \( E \in \{D, U, L\} \) and define

\[
\zeta_n(a) := \frac{\xi_n^E(a)}{m_n^{1/2}} \quad \text{and} \quad r_n(a) := \frac{m_n^{1/2}}{m_n^2 - V_n^E(a)}.
\]

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By (4.1.20), we know that there exist $C > 0$ and $0 < \gamma < 2/3$ such that $E[|\zeta_n(a)|^q] \leq C^q q^{\gamma q}$ for every $q \in \mathbb{N}$ and $n$ large enough. Thus, since the variables $\xi^{E}_n(0), \ldots, \xi^{E}_n(n)$ are independent, it follows from the upper bound [GS18b, (4.25)] that there exist $C' > 0$ and $2 < \gamma' < 3$ both independent of $n$ such that (4.6.12) is bounded above by

$$E \left[ \exp \left( C' \left( \sum_{a \in \mathbb{N}_0} |r_n(a)| \Lambda_{\vartheta_i}(S^{i:x_n^i}) \right) \right) \right]$$

for every $q \in \mathbb{N}$ and $n$ large enough. Thus, since the variables $\xi^{E}_n(0), \ldots, \xi^{E}_n(n)$ are independent, it follows from the upper bound [GS18b, (4.25)] that there exist $C' > 0$ and $2 < \gamma' < 3$ both independent of $n$ such that (4.6.12) is bounded above by

$$E \left[ \exp \left( C' \left( \sum_{a \in \mathbb{N}_0} |r_n(a)| \Lambda_{\vartheta_i}(S^{i:x_n^i}) \right) \right) \right] < \infty,$$

where we use the trivial bound $\Lambda^a_{\vartheta}(a,b)\Lambda^b_{\vartheta}(b,a) \leq \Lambda^a_{\vartheta}$ for all $a,b$.

For any fixed $x$, we know that $S^{ix_n^i}(u) = O(m_n^2)$ uniformly in $0 \leq u \leq \vartheta_i$ because $\vartheta_i = O(m_n^2)$. Thus, the only values of $a$ for which $\Lambda^a_{\vartheta_i}$ is possibly nonzero are at most of order $O(m_n^2) = o(n)$. For any such values of $a$, the assumption (4.1.16) implies that $V_n^E(a) = o(m_n^2)$, hence $r_n(a) = O(m_n^{-3/2})$. By combining all of these estimates with (4.1.19), (4.6.12) is then a consequence of the following proposition.

**Proposition 4.6.3.** Let $\vartheta = \vartheta(n,t) := [m_n^2 t]$ and $x := [m_n x]$ for some $t > 0$ and $x \geq 0$. For every $C > 0$ and $1 \leq q < 3$,

$$\sup_{n \in \mathbb{N}, \ x \geq 0} E \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda^a_{\vartheta}(S^{x_n^i})^q}{m_n^3} \right) \right] < \infty.$$

Since the proof of Proposition 4.6.3 is rather long and technical, we provide it later in Section 4.6.3 so as to not interrupt the flow of the present argument.

**Convergence of the Integral**

We now complete the proof that (4.6.2) converges to (4.6.3). With (4.6.9) established, it only remains to justify passing the limit inside the integral in $dx_1 \cdots dx_k$. In order to prove this, we aim to use the Vitali convergence theorem (e.g., [FL07, Theorem

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For this, we need a more refined version of the uniform integrability estimate used in the proof of (4.6.9). By Hölder’s inequality,

\[
\left( \prod_{i=1}^{k} f_i(x_i) \right) \mathbb{E} \left[ \prod_{i=1}^{k} F_{n,t_i}^d(S_i;x_i^n) m_n \int_{S_i;x_i^n(\bar{\nu}_i)/m_n}^{(S_i;x_i^n(\bar{\nu}_i)+1)/m_n} g_i(y) \, dy \right] \\
\leq \prod_{i=1}^{k} \| f_i \|_{\infty} \| g_i \|_{\infty} \mathbb{E} \left[ |F_{n,t_i}^d(S_i;x_i^n)^k|^{1/k} \right].
\]

(4.6.14)

Our aim is to find a suitable upper bounds for the functions

\[ x \mapsto \mathbb{E} \left[ |F_{n,t_i}^d(S_i;x_i^n)^k|^{1/k} \right], \quad 1 \leq i \leq k. \]

In order to achieve this, we fix a small \( \varepsilon > 0 \) (precisely how small will be determined in the following paragraphs), and we consider separately the two cases \( x \in [0, n^{1-\varepsilon}/m_n) \) and \( x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n) \).

Let us first consider the case \( x \in [0, n^{1-\varepsilon}/m_n) \). Note that for any \( E \in \{ D, U, L \} \),

\[
1_{\{\tau(n)(S) > \varepsilon\}} \prod_{a \in \mathbb{N}_0} \left| 1 - \frac{E_n(a)}{m_n^2} \right|^{\Lambda_{a,E}^{(a,E,\bar{a}E)}(S)} \\
= 1_{\{\tau(n)(S) > \varepsilon\}} \prod_{a \in \mathbb{Z}} \left| 1 - \frac{E_n(|a|)}{m_n^2} \right|^{\Lambda_{a,E}^{(a,E,\bar{a}E)}(S)} \leq \prod_{a \in \mathbb{Z}} \left| 1 - \frac{E_n(|a|)}{m_n^2} \right|^{\Lambda_{a,E}^{(a,E,\bar{a}E)}(S)}. \]

Then, by combining Hölder’s inequality with a rearrangement similar to (4.6.11), we see that \( \mathbb{E} \left[ |F_{n,t_i}^d(S_i;x_i^n)^k|^{1/k} \right] \) is bounded above by the product of the terms

\[
\prod_{E \in \{D, U, L\}} \mathbb{E} \left[ \prod_{a \in \mathbb{Z}} \left| 1 - \frac{\xi_{E_n}(a)}{m_n^2} \right|^{6k \Lambda_{a,E}^{(a,E,\bar{a}E)}(S_i;x_i^n)} \right]^{1/6k}, \quad (4.6.15)
\]

\[
\prod_{E \in \{D, U, L\}} \mathbb{E} \left[ \prod_{a \in \mathbb{Z}} \left| 1 - \frac{V_{E_n}(a)}{m_n^2} \right|^{6k \Lambda_{a,E}^{(a,E,\bar{a}E)}(S_i;x_i^n)} \right]^{1/6k}. \quad (4.6.16)
\]
Since \( m_n x = O(n^{1-\varepsilon}) = o(n) \), the random walk \( S^{i:x_n} \) can only attain values of order \( o(n) \) in \( \vartheta_i = O(m_n^2) = o(n) \) steps. Thus, for \( E \in \{D,U,L\} \), it follows from (4.1.16) that \( V_n^E(a) = o(m_n^2) \) for any value attained by the walk when \( x \in [0,n^{1-\varepsilon}/m_n) \).

By using the same argument as for (4.6.12) (namely, the inequality [GS18b, (4.25)] followed by Proposition 4.6.3), we conclude that (4.6.15) is bounded by a constant for large \( n \). For (4.6.16), let us assume without loss of generality that \( V_n^D \) is the sequence (or at least one of the sequences) that satisfies (4.1.17). According to (4.1.15), we have

\[
\prod_{E \in \{U,L\}} E \left[ \prod_{a \in \mathbb{Z}} \left| 1 - \frac{V_n^E(|a|)}{m_n^2} \right| \right]^{1/6k} \leq 1
\]

for large enough \( n \). For the terms involving \( V_n^D \), since \( |1 - y| \leq e^{-y} \) for any \( y \in [0,1] \), it follows from (4.1.17) that, up to a constant \( C \) independent of \( n \) (depending on \( \theta \) through \( c = c(\theta) \) in (4.1.17)), we have the upper bound

\[
E \left[ \prod_{a \in \mathbb{Z}} \left| 1 - \frac{V_n^D(|a|)}{m_n^2} \right| ^{6k \Lambda_{\vartheta_i}^{(a,a)}(S^{i:x_n})} \right]^{1/6k} \leq C \left[ \exp \left( -\frac{6k \theta}{m_n^2} \sum_{a \in \mathbb{Z}} \log(1 + |a|/m_n) \Lambda_{\vartheta_i}^{(a,a)}(S^{i:x_n}) \right) \right]^{1/6k}
\]

for large enough \( n \). If we define \( S^{i:x^n} := x^n + S^0 \) for all \( x \geq 0 \), then \( \Lambda_{\vartheta_i}^{(a,a)}(S^{i:x_n}) = \Lambda_{\vartheta_i}^{(a-x^n,a-x^n)}(S^0) \). By combining this change of variables with the inequality

\[
\log(1 + |z + \bar{z}|) \geq \log(1 + |z|) - \log(1 + |\bar{z}|) \geq \log(1 + |z|) - |\bar{z}|,
\]

which is valid for all \( z, \bar{z} \in \mathbb{R} \), we obtain that, up to a multiplicative constant independent of \( n \), (4.6.17) is bounded by

\[
E \left[ \exp \left( -\frac{6k \theta}{m_n^2} \sum_{a \in \mathbb{Z}} \left( \log(1 + x) - |a/m_n| \right) \Lambda_{\vartheta_i}^{(a,a)}(S^0) \right) \right]^{1/6k}.
\]

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Noting that $\Lambda_{a,a}^{(a,a)} \leq \Lambda_{a,a}^a$ for every $a \in \mathbb{Z}$ and that the vertex-occupation measures satisfy (4.4.5), an application of Hölder’s inequality then implies that (4.6.17) is bounded above by the product of the two terms

$$E \left[ \exp \left( -\frac{12k\theta \log(1+x)}{m_n^2} \sum_{a \in \mathbb{Z}} \Lambda_{a,a}^{(a,a)}(S^0) \right) \right]^{1/12k}$$

(4.6.18)

$$E \left[ \exp \left( \frac{12k\theta}{m_n^2} \sum_{0 \leq u \leq \vartheta_i} |S^0(u)| \right) \right]^{1/12k}.$$  (4.6.19)

Recall the definition of the range $\mathcal{R}_{\vartheta_i}(S^0)$ in (4.4.10). Since

$$\mathcal{R}_{\vartheta_i}(S^0) \geq \max_{0 \leq u \leq \vartheta_i} |S^0(u)|,$$

we conclude that there exists $C > 0$ independent of $n$ such that (4.6.19) is bounded by the exponential moment $E \left[ e^{C \mathcal{R}_{\vartheta_i}(S^0)/m_n} \right]^{1/12k}$. Thus, by (4.4.12), we see that (4.6.19) is bounded by a constant independent of $n$. It now remains to control (4.6.18). To this end, we note that $\sum_{a \in \mathbb{Z}} \Lambda_{a,a}^{(a,a)}(S^0)$, which represents the total number of visits on the self-edges of $\mathbb{Z}$ by $S^0$ before the $\vartheta_i$th step, is a Binomial random variable with $\vartheta_i$ trials and probability $1/3$. Thus, for small enough $\nu > 0$, it follows from Hoeffding’s inequality that

$$P \left[ \sum_{a \in \mathbb{Z}} \Lambda_{a,a}^{(a,a)}(S^0) < \nu m_n^2 \right] \leq e^{-cm_n^2}$$

(4.6.20)

for some $c > 0$ independent of $n$. By separating the expectation in (4.6.18) with respect to whether or not the walk has taken less than $\nu m_n^2$ steps on self-edges, we may bound it above by

$$\left( e^{-12k\nu \theta \log(1+x)} + e^{-cm_n^2} \right)^{1/12k} \leq (1 + x)^{-\nu \theta} + e^{-(c/12k)m_n^2}.$$
Combining all of these bounds together with the fact that $m_n$ is of order $n^\alpha$ by (4.1.1), we finally conclude that for every $1 \leq i \leq k$, there exist constants $c_1, c_2, c_3 > 0$ independent of $n$ such that, for large enough $n$,

$$
E \left[ \left| F_{n,t_i}^d(S^{i:x^n}) \right|^k \right]^{1/k} \leq c_1 \left( (1 + x)^{-c_2 \theta} + e^{-c_3 n^{2\alpha}} \right), \quad x \in [0, n^{1-\varepsilon}/m_n]. \tag{4.6.21}
$$

**Remark 4.6.4.** We emphasize that $c_2$ does not depend on $\theta$, and thus the assumption (4.1.17) implies that we can make $c_2\theta$ arbitrarily large by taking a large enough $\theta$. In particular, if we take $\theta > 1/c_2$, then $(1 + x)^{-c_2 \theta}$ is integrable on $[0, \infty]$.

We now turn to the estimate in the case where $x \in [n^{1-\varepsilon}/m_n, (n + 1)/m_n]$. By taking $\varepsilon > 0$ small enough (more specifically, such that $1 - \varepsilon > 2d$, with $d$ as in (4.1.1)), we can ensure that $m_n x \geq n^{1-\varepsilon}$ implies that, for any constant $0 < C < 1$, we have $S^{i:x^n}(u) \geq Cn^{1-\varepsilon}$ for all $0 \leq u \leq \theta_i$ and $n$ large enough. Let us assume without loss of generality that $V_{n}^{D}$ satisfies (4.1.18). Provided $\varepsilon > 0$ is small enough (namely, at least as small as the $\varepsilon$ in (4.1.18)), for any $a \in \mathbb{N}_0$ that can be visited by the random walk, we have that $V_{n}^{D}(a) \geq \kappa(Cn^{1-\varepsilon}/m_n)^\alpha$; hence

$$
\left| 1 - \frac{D_n(a)}{m_n^2} \right| \leq \frac{m_n^2 - V_{n}^{D}(a)}{m_n^2} + \frac{|\xi_{n}^{D}(a)|}{m_n^2} \leq \frac{m_n^2 - \kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2} + \frac{|\xi_{n}^{D}(a)|}{m_n^2} = \left( 1 - \frac{\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2} \right) \left( 1 + \frac{|\xi_{n}(a)|}{m_n^2 - \kappa(Cn^{1-\varepsilon}/m_n)^\alpha} \right). \tag{4.6.22}
$$

According to (4.1.1), we know that $(n^{1-\varepsilon}/m_n)^\alpha \asymp n^{\alpha(1-d)-\alpha\varepsilon}$. Since $\alpha$ is chosen such that $d/2 < \alpha(1 - d) \leq 2d$ in Assumption PT3, we can always choose $\varepsilon > 0$ small enough so as to guarantee that

$$
n^{\alpha/2} = o(n^{\alpha(1-d)-\alpha\varepsilon}) \quad \text{and} \quad (n^{1-\varepsilon}/m_n)^\alpha = o(n^{2d}) = o(m_n^2). \tag{4.6.23}
$$
As a consequence of the second equation in (4.6.23), for $n$ large enough (4.6.22) yields

$$\left| 1 - \frac{D_n(a)}{m_n^2} \right| \leq \left( 1 - \frac{\kappa(Cn^{1-\varepsilon}/m_n)^{\alpha}}{m_n^2} \right) \left( 1 + \frac{2|\xi_n(a)|}{m_n^2} \right).$$

As for $E \in \{U, L\}$, we have from (4.1.15) that

$$\left| 1 - \frac{E_n(a)}{m_n^2} \right| \leq \left| \frac{m_n^2 - V_n^E(a)}{m_n^2} \right| + \left| \frac{\xi_n^E(a)}{m_n^2} \right| \leq 1 + \left| \frac{\xi_n^E(a)}{m_n^2} \right|.$$

Thus, for any $x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n)$ and large enough $n$, it follows from Hölder’s inequality that the expectation $E \left[ |F_{n,t_i}^d(S_{i,x}^{n})|^k \right]^{1/k}$ is bounded above by the product of the following terms:

\[
\begin{align*}
E \left[ \prod_{a \in \mathbb{Z}} \left( 1 - \frac{\kappa(Cn^{1-\varepsilon}/m_n)^{\alpha}}{m_n^2} \right) \frac{4k\Lambda^a_{\varphi}(S_{i,x}^{n})}{m_n^2} \right]^{1/4k} \tag{4.6.24} \\
E \left[ \prod_{a \in \mathbb{Z}} \left( 1 + \frac{2|\xi_n^D(|a|)|}{m_n^2} \right) \frac{4k\Lambda^{(a,\delta)}_{\varphi}(S_{i,x}^{n})}{m_n^2} \right]^{1/4k} \tag{4.6.25} \\
\prod_{E \in \{U, L\}} E \left[ \prod_{a \in \mathbb{Z}} \left( 1 + \frac{|\xi_n^E(|a|)|}{m_n^2} \right) \frac{4k\Lambda^{(a,E)}_{\varphi}(S_{i,x}^{n})}{m_n^2} \right]^{1/4k}. \tag{4.6.26}
\end{align*}
\]

By repeating the bound (4.6.20) and the argument thereafter, we conclude that there exist $c_4, c_5 > 0$ independent of $n$ such that (4.6.24) is bounded by $e^{-c_4n^{(1-\varepsilon)-\alpha\varepsilon}} + e^{-c_5n^{2\gamma}}$. For (4.6.25), let us define $\zeta_n(a) := |\xi_n^D(a)|/m_n^{1/2}$. By applying [GS18b, (4.25)] in similar fashion to (4.6.13), we see that (4.6.25) is bounded above by

\[
\begin{align*}
E \left[ \exp \left( C' \left( \frac{1}{m_n^{1/2}} \sum_{a \in \mathbb{Z}} \Lambda^a_{\varphi}(S_{i,x}^{n}) \right) - \frac{1}{m_n^{1/2}} \sum_{a \in \mathbb{Z}} \Lambda^a_{\varphi}(S_{i,x}^{n}) \right) \right] \\
+ \frac{1}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^a_{\varphi}(S_{i,x}^{n})^2 + \frac{1}{m_n^{\gamma/2}} \sum_{a \in \mathbb{Z}} \Lambda^a_{\varphi}(S_{i,x}^{n})^{\gamma/2} \right) \tag{4.6.27}
\end{align*}
\]

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for some $C' > 0$ and $2 < \gamma' < 3$ independent of $n$. By (4.1.20), the moments $E[|\zeta_n(a)|]$ are uniformly bounded in $n$, and thus

$$
\frac{1}{m_n^{1/2}} \sum_{a \in \mathbb{Z}} \Lambda_{g_i}^{n}(S_{i,n}^{x_n}) E[|\zeta_n(|a|)|] = O(m_n^{1/2}) = O(n^{\gamma'/2}).
$$

By applying the uniform exponential moment bounds of Proposition 4.6.3 to the remaining terms in (4.6.27), we conclude that there exists a constant $c_6 > 0$ independent of $n$ such that (4.6.25) is bounded by $e^{c_6 n^{\gamma'/2}}$. A similar bound applies to (4.6.26). Then, by using the first equality in (4.6.23) and combining the inequalities for (4.6.24)–(4.6.26), we see that there exist $\bar{c}_4, \bar{c}_5 > 0$ independent of $n$ such that

$$
E \left[ |F_{n,t_i}(S_{i,n}^{x_n})|^k \right]^{1/k} \leq e^{-\bar{c}_4 n^{(1-\delta)-\alpha \epsilon}} + e^{-\bar{c}_5 n^{2 \delta}}, \quad x \in [n^{1-\epsilon}/m_n, (n+1)/m_n] \quad (4.6.28)
$$

By combining (4.6.21) and (4.6.28), we conclude that, for large $n$, the integral of the absolute value of (4.6.14) on the set $[0, (n+1)/m_n)^k$ is bounded above by

$$
\left( \prod_{i=1}^k \|f_i\|_{\infty} \|g_i\|_{\infty} \right) \left( c_1 \int_0^{n^{1-\epsilon}/m_n} (1+x)^{-c_2 \theta} + e^{-c_3 n^{2 \delta}} \, dx + \int_{n^{1-\epsilon}/m_n}^{(n+1)/m_n} e^{-\bar{c}_4 n^{(1-\delta)-\alpha \epsilon}} + e^{-\bar{c}_5 n^{2 \delta}} \, dx \right)^k
$$

for some $c_1, c_2, c_3, \bar{c}_4, \bar{c}_5 > 0$ independent of $n$. If we take $\theta > 0$ large enough so that $(1+x)^{-c_2 \theta}$ is integrable, then the sequence of functions

$$
\left( \prod_{i=1}^k 1_{[0, (n+1)/m_n]}(x_i) f_i(x_i) \right) E \left[ \prod_{i=1}^k F_{n,t_i}(S_{i,n}^{x_n}) m_n \int_{S_{i,n}^{x_n}(\vartheta_i) / m_n}^{(S_{i,n}^{x_n}(\vartheta_i) + 1) / m_n} g_i(y) \, dy \right]
$$

is uniformly integrable in the sense of [FL07, Theorem 2.24-(ii),(iii)], concluding the proof of the convergence of moments in Theorem 4.1.9 (1).
4.6.2 Step 2: Convergence in Distribution

Up to writing each $f_i$ and $g_i$ as the difference of their positive and negative parts, there is no loss of generality in assuming that $f_i, g_i \geq 0$. The convergence in joint distribution follows from the convergence in moments proved in Section 4.6.1. The argument we use to prove this is essentially the same as [GS18b, Lemma 4.4]:

For any $\bar{R} \in [-\infty, 0]$ and $\bar{R} \in [0, \infty]$, let us define

$$\hat{K}_n^{R,\bar{R}}(t) g(x) := E^{[m_n x]} \left[ (R \lor F_{n,t}^d(S)) \land \bar{R} \right] m_n \int_{S(t)/m_n}^{S(\theta) + 1/m_n} g(y) \, dy,$$

and

$$\hat{K}_n^{R,\bar{R}}(t) g(x) := E^x \left[ (R \lor 1_{\{\tau_0(B) > t\}} e^{-\langle L_t(B), V \rangle - \xi(L_t(B))} \lor \bar{R} ) g(B(t)) \right],$$

where we use the convention $R \lor y \land \bar{R} := \max\{R, \min\{y, \bar{R}\}\}$ for any $y \in \mathbb{R}$. We note a few elementary properties of these truncated operators:

1. $\hat{K}_n^{-\infty,\infty}(t) = \hat{K}_n^d(t)$, and $\hat{K}_n^{-R,\infty}(t) = \hat{K}_n^d(t)$ for all $R \leq 0$.

2. Arguing as in Section 4.6.1, for every $R \in [-\infty, 0]$ and $\bar{R} \in [0, \infty]$,

$$\lim_{n \to \infty} \langle f_i, \hat{K}_n^{R,\bar{R}}(t_i) g_i \rangle = \langle f_i, \hat{K}_n^{R,\bar{R}}(t_i) g_i \rangle, \quad 1 \leq i \leq k \quad (4.6.29)$$

in joint moments.

3. If $|R|, \bar{R} < \infty$, then the $\langle f_i, \hat{K}_n^{R,\bar{R}}(t_i) g_i \rangle$ are bounded uniformly in $n$; hence the moment convergence of (4.6.29) implies convergence in joint distribution.

Let $R > -\infty$ be fixed. Since $\langle f_i, \hat{K}_n^{R,\infty}(t_i) g_i \rangle \to \langle f_i, \hat{K}_n^{R,\infty}(t_i) g_i \rangle$ in joint moments, the sequences in question are tight (e.g., [Bil95, Problem 25.17]). Therefore, it suffices to prove that every subsequence that converges in joint distribution has $\langle f_i, \hat{K}_n^{R,\infty}(t_i) g_i \rangle$ as a limit (e.g., [Bil95, Theorem–Corollary 25.10]). Let $A_1^R, \ldots, A_k^R$
be limit points of \((f_1, \hat{K}_1^R(t_1)g_1), \ldots, (f_k, \hat{K}_k^R(t_1)g_k)\). Since \(f_i, g_i \geq 0\), the variables \((f_i, \hat{K}_n^R(t_i)g_i)\) and \((f_i, \hat{K}_R(t_i)g_i)\) are increasing in \(R\). Therefore, for every \(\bar{R} < \infty\), we have

\[
(A_1^R, \ldots, A_k^R) \geq (\langle f_1, \hat{K}_1^R(t_1)g_1 \rangle, \ldots, \langle f_k, \hat{K}_k^R(t_k)g_k \rangle)
\] (4.6.30)

in the sense of stochastic dominance in the space \(\mathbb{R}^k\) with the componentwise order (e.g. [KKO77, Theorem 1 and Proposition 3]). By the monotone convergence theorem,

\[
\lim_{R \to \infty} \langle f_i, \hat{K}_R(t_i)g_i \rangle = \langle f_i, \hat{K}_\infty(t_i)g_i \rangle, \quad 1 \leq i \leq k
\]

almost surely; hence the stochastic dominance (4.6.30) also holds for \(\bar{R} = \infty\). Since \(A_i^R\) and \((f_i, \hat{K}_R(t_i)g_i)\) have the same mixed moments, we thus infer that their joint distributions coincide. In conclusion, for any finite \(\bar{R}\), we have that

\[
\lim_{n \to \infty} \langle f_i, \hat{K}_n^R(t_i)g_i \rangle = \langle f_i, \hat{K}_n^\infty(t_i)g_i \rangle
\]

in joint distribution. In order to get the result for \(\bar{R} = -\infty\), we use the same stochastic domination argument by sending \(R \to -\infty\).

### 4.6.3 Proof of Proposition 4.6.3

If we prove that

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \exp \left( \frac{C}{m_n} \sum_{a \in d} \frac{\Lambda_a(S^n)^q}{m_n^q} \right) \right] < \infty,
\]

then we get the desired result by a simple change of variables. Similarly to [GS18b, Proposition 4.3], a crucial tool for proving this consists of combinatorial identities involving the quantile transform for random walks derived in [AFP15]. However, such results only apply to the simple symmetric random walk.
In order to get around this requirement, we decompose the vertex-occupation measures in terms of the edge-occupations measures as follows: By combining

$$\Lambda^a_\vartheta(S^0) = \Lambda^{(a,a-1)}_\vartheta(S^0) + \Lambda^{(a,a)}_\vartheta(S^0) + \Lambda^{(a,a+1)}_\vartheta(S^0) + 1_{\{S(0) = a\}}, \quad a \in \mathbb{Z}$$

with the inequality $(z + \bar{z})^q \leq 2^{q-1}(z^q + \bar{z}^q)$ (for $z, \bar{z} \geq 0$ and $q \geq 1$), it suffices by an application of Hölder’s inequality to prove that the exponential moments of

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \left( \Lambda^{(a,a-1)}_\vartheta(S^0) + \Lambda^{(a,a+1)}_\vartheta(S^0) \right)^q$$

(4.6.31)

and

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^{(a,a)}_\vartheta(S^0)^q$$

(4.6.32)

are uniformly bounded in $n$.

**Non-Self-Edges**

Let us begin with (4.6.31).

**Definition 4.6.5.** Let $\mathcal{G}$ be a simple symmetric random walk on $\mathbb{Z}$, that is, the increments $\mathcal{G}(u) - \mathcal{G}(m - 1)$ are i.i.d. uniform on $\{-1, 1\}$. For any $a, b \in \mathbb{Z}$ and $u \in \mathbb{N}_0$, we denote $\mathcal{G}^a := (\mathcal{G}|\mathcal{G}(0) = a)$ and $\mathcal{G}^{a,b}_u := (\mathcal{G}|\mathcal{G}(0) = a \text{ and } \mathcal{G}(u) = b)$ (note that the latter only makes sense if $|b - a|$ and $u$ have the same parity).

For every $u \in \mathbb{N}_0$, let

$$\mathcal{H}_u(S^0) := \sum_{a \in \mathbb{Z}} \Lambda^{(a,a)}_u(S^0),$$

(4.6.33)
i.e., the number of times $S^0$ visits self-edges by the $u^{th}$ step. Then, it is easy to see that we can couple $S^0$ and $S^0$ in such a way that

$$S^0(u) = S^0(u - \mathcal{H}_u(S^0)), \quad u \in \mathbb{N},$$

i.e., $S^0$ is the same path as $S^0$ with the visits to self-edges removed. If we define the edge-occupation measures for $S^0$ in the same way as (4.4.4), then it is clear that the coupling of $S$ and $S$ satisfies

$$\Lambda^{(a,a-1)}_q(S) + \Lambda^{(a,a+1)}_q(S) \leq \Lambda^a_q(S).$$

Thus, for (4.6.31) we need only prove that the exponential moments of

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^a_q(S^0)^q m_n^q$$

are uniformly bounded in $n$.

By the total probability rule, we note that

$$E \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^a_q(S^0)^q \right) \right] = \sum_{b \in \mathbb{Z}} E \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^a_q(S^0)^q \right) \right] P[S^0(\vartheta) = b].$$

According to the proof of [GS18b, Proposition 4.3] (more specifically, [GS18b, (4.19)]) and the following paragraph, explaining the distribution of the quantity denoted $M(N, \tilde{T})$ in [GS18b, (4.19)]], there exists a constant $\bar{C} > 0$ that only depends on $C$, $q$, and the number $t$ in $\vartheta = [m_n^2 t]$ such that

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^a_q(S^0)^q m_n^q \leq \bar{C} \left( (\Lambda^a_q / m_n)^q + ((|b| + 2)/m_n)^q \right),$$

(4.6.35)
where $\mathcal{R}_{\theta}^{0,b}$ is equal in distribution to the range of $\mathcal{S}_{\theta}^{0,b}$, that is,

$$\mathcal{R}_{\theta}^{0,b} \overset{d}{=} \mathcal{R}_{\theta}(\mathcal{S}_{\theta}^{0,b}) := \max_{0 \leq u \leq \theta} \mathcal{S}_{\theta}^{0,b}(u) - \min_{0 \leq u \leq \theta} \mathcal{S}_{\theta}^{0,b}(u).$$

Hence, if $\mathcal{R}_{\theta}(\mathcal{S}^0)$ denotes the range of the unconditioned random walk $\mathcal{S}^0$, then

$$\mathbb{E} \left[ \exp \left( \frac{\bar{C}}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda_{\theta}^q(\mathcal{S}^0)^q}{m_n^q} \right) \right] \leq \mathbb{E} \left[ \exp \left( C \left( \frac{(\mathcal{R}_{\theta}(\mathcal{S}^0))^{q-1}}{m_n^{q-1}} + \frac{(\mathcal{S}^0(\theta) + 2)^{q-1}}{m_n^{q-1}} \right) \right) \right].$$

Since $q - 1 < 2$, the result then follows from the same moment estimate leading up to (4.4.12), but by applying [Che10, (6.2.3)] to the random walk $\mathcal{S}^0$ instead of $S^0$.

**Self-Edges**

We now control the exponential moments of (4.6.32). By referring to the uniform boundedness of the exponential moments of (4.6.31) that we have just proved, we know that for any $b \in \{-1, 1\}$, the exponential moments of

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda_{\theta}^{(a,a+b)}(S^0)^q}{m_n^q} \quad \text{and} \quad \frac{1}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda_{\theta}^{(a+b,a)}(S^0)^q}{m_n^q}$$

are uniformly bounded in $n$. Thus, by (4.2.20), the exponential moments of

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \left( \Lambda_{\theta}^{(a+1,a)}(S^0) + \Lambda_{\theta}^{(a-1,a)}(S^0) + \Lambda_{\theta}^{(a,a+1)}(S^0) + \Lambda_{\theta}^{(a,a-1)}(S^0) \right)^q$$

are uniformly bounded in $n$. Thus, by (4.2.20), the exponential moments of

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are uniformly bounded in \( n \). Consequently, it suffices to prove that there exist \( c, \bar{c} > 0 \) such that for every \( n \in \mathbb{N} \) and \( y \) large enough (independently of \( n \)),

\[
P \left[ \sum_{a \in \mathbb{Z}} \Lambda_{\vartheta}^{(a,a)}(S^0)^q > y \right] 
\leq P \left[ \sum_{a \in \mathbb{Z}} \left( \Lambda_{\vartheta}^{(a+1,a)}(S^0) + \Lambda_{\vartheta}^{(a-1,a)}(S^0) + \Lambda_{\vartheta}^{(a,a+1)}(S^0) + \Lambda_{\vartheta}^{(a,a-1)}(S^0) \right)^q > cy - \bar{c} \right].
\]

(4.6.36)

We now prove (4.6.36).

**Definition 4.6.6.** If \( \vartheta \) is even, let \( S_0, S_1, \ldots, S_{\vartheta/2-1} \) be defined as the path segments

\[
S_u = (S^0(2u), S^0(2u + 1), S^0(2u + 2)), \quad 0 \leq u \leq \vartheta/2 - 1.
\]

If \( \vartheta \) is odd, then we similarly define \( S_0, S_1, \ldots, S_{(\vartheta-1)/2-1}, S_{(\vartheta-1)/2} \) as

\[
S_u = \begin{cases} 
(S^0(2u), S^0(2u + 1), S^0(2u + 2)) & \text{if } 0 \leq u \leq (\vartheta - 1)/2 - 1, \\
(S^0(2u), S^0(2u + 1)) & \text{if } u = (\vartheta - 1)/2.
\end{cases}
\]

In words, we partition the path formed by the first \( \vartheta \) steps of \( S^0 \) into successive segments of two steps, with the exception that the very last segment may contain only one step if \( \vartheta \) is odd (see Figure 4.2 below for an illustration of this partition).

**Definition 4.6.7.** Let \( S_u \) be a path segment as in the previous definition. We say that \( S_u \) is a **type 1** segment if there exist some \( a \in \mathbb{Z} \) and \( b \in \{-1, 1\} \) such that

\[
S_u = \begin{cases} 
(a, a, a + b), \\
(a + b, a, a), \text{ or} \\
(a, a),
\end{cases}
\]
we say that $S_u$ is a **type 2** segment if there exists some $a \in \mathbb{Z}$ such that

$$ S_u = (a, a, a), $$

and we say that $S_u$ is a **type 3** segment if there exists some $a \in \mathbb{Z}$ such that

$$ S_u = (a, a + 1, a). $$

Given a realization of the first $\vartheta$ steps of the lazy random walk $S^0$, we define the transformed path $(\hat{S}^0(u))_{0 \leq u \leq \vartheta}$ by replacing every type 2 segment $(a, a, a)$ in $(S^0(u))_{0 \leq u \leq \vartheta}$ by the corresponding type 3 segment $(a, a + 1, a)$, and vice versa. (See Figure 4.2 below for an illustration of this transformation). Given that this path transformation is a bijection on the set of all possible realizations of $(S^0(u))_{0 \leq u \leq \vartheta}$, $(\hat{S}^0(u))_{0 \leq u \leq \vartheta}$ is also a lazy random walk.

![Figure 4.2: The partition into two-step segments is represented by dashed gray lines. Type 2 segments are red, and type 3 segments are blue. The two paths represent $S^0$ and $\hat{S}^0$, as related to each other by the permutation of type 2 and 3 segments.](image)

Every contribution of $S^0$ to $\sum_a \Lambda_{\vartheta}^{(a,a)}(S^0)$ comes from type 1 and 2 segments. Moreover, if a type 1 segment $S_u$ is not at the end of the path and adds a contribution of one to $\Lambda_{\vartheta}^{(a,a)}(S^0)$ for some $a \in \mathbb{Z}$, then it must also add one to

$$ \Lambda_{\vartheta}^{(a+1,a)}(S^0) + \Lambda_{\vartheta}^{(a-1,a)}(S^0) + \Lambda_{\vartheta}^{(a,a+1)}(S^0) + \Lambda_{\vartheta}^{(a,a-1)}(S^0). \quad (4.6.37) $$
Lastly, for every type 2 segment, a contribution of two to \( \Lambda^{(a,a)}(S^0) \) for some \( a \in \mathbb{Z} \) is turned into a contribution of two to (4.6.37) in \( \hat{S}^0 \). In short, we observe that there is at most one \( a_0 \in \mathbb{Z} \) (i.e., the one level, if any, where a type 1 segment occurs at the very end of the path of \( S^0(u), u \leq \vartheta \)) such that

\[
\Lambda^{(a,a)}(S^0) \leq \Lambda^{(a+1,a)}(\hat{S}^0) + \Lambda^{(a-1,a)}(\hat{S}^0) + \Lambda^{(a,a+1)}(\hat{S}^0) + \Lambda^{(a,a-1)}(\hat{S}^0)
\]

for every \( a \in \mathbb{Z} \setminus \{a_0\} \), and

\[
\Lambda^{(a_0,a_0)}(S^0) \leq \Lambda^{(a_0+1,a_0)}(\hat{S}^0) + \Lambda^{(a_0-1,a_0)}(\hat{S}^0) + \Lambda^{(a_0,a_0+1)}(\hat{S}^0) + \Lambda^{(a_0,a_0-1)}(\hat{S}^0) + 1
\]

Given that \((z + 1)^q \leq 2^{q-1}z^q + 2^{q-1}\) for every \( z, q \geq 1 \), we obtain (4.6.36).

### 4.7 Trace Convergence to Dirichlet Semigroup

In this section, we prove Theorem 4.1.9 (2). This proof is very similar to that of Theorem 4.1.9 (1), except that we deal with random walks and Brownian motions conditioned on their endpoint.

#### 4.7.1 Step 1: Convergence of Moments

We begin with a generic mixed moment of traces, which we can always write in the form

\[
\mathbb{E}\left[ \prod_{i=1}^{k} \text{Tr}\left[K_{x_i}^{d}(t_i)\right] \right].
\]

By Fubini’s theorem, this is equal to

\[
\int_{[0,(n+1)/m_n]^k} \mathbb{E}\left[ \prod_{i=1}^{k} m_n \mathbf{P}[S^0(\vartheta_i) = 0] F_{n,t_i}^{d}\left(S^i_{x_i}^{n},x_i^{n}\right) \right] \, dx_1 \cdots dx_k,
\]

(4.7.1)
and by the trace formula in Theorem 2.1.19 the corresponding continuum limit is

\[
\mathbb{E} \left[ \prod_{i=1}^{k} \text{Tr} \left[ \tilde{K}^d(t_i) \right] \right] = \int_{(0, \infty)} \mathbb{E} \left[ \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi t_i}} \mathbf{1}_{\{\tau_0(B_i^{t_i}(x_i, x_i)) > t_i\}} e^{-\langle L_{t_i}(B_i^{t_i}(x_i, x_i)), V \rangle - \int L_{t_i}^1(B_i^{t_i}(x_i, x_i)) dW(a)} \right] \, dx_1 \cdots dx_k,
\]

where \( \vartheta_i \) and \( x^n_i \) are as in Section 4.6.1, and

1. \( S_{\vartheta_i}^{1:x_i^n, x_i^n}, \ldots, S_{\vartheta_k}^{k:x_k^n, x_k^n} \) are independent copies of random walk bridges \( S_{\vartheta}^{x,x} \) with \( x = x^n_i \) and \( \vartheta = \vartheta_i \);

2. \( B_{t_1}^{1:x_1, x_1}, \ldots, B_{t_k}^{k:x_k, x_k} \) are independent copies of standard Brownian bridges \( B_t^{x,x} \) with \( x = x_i \) and \( t = t_i \).

Also, \( S_{\vartheta_i}^{i:x_i^n, x_i^n} \) are independent of \( Q_n \), and \( B_{t_i}^{i:x_i, x_i} \) are independent of \( W \).

According to the local central limit theorem,

\[
\lim_{n \to \infty} m_n \mathbb{P}[S^0(\vartheta_i) = 0] = \frac{1}{\sqrt{2\pi t_i}}, \quad 1 \leq i \leq k.
\]

Moreover, we have the following analog of Proposition 4.6.1:

**Proposition 4.7.1.** The conclusion of Proposition 4.6.1 holds with every instance of \( S^{i:x_i^n, x_i^n} \) replaced by \( S_{\vartheta_i}^{i:x_i^n, x_i^n} \), and every instance of \( B^{i:x_i, x_i} \) replaced by \( B_{t_i}^{i:x_i, x_i} \).

**Proof.** Arguing as in the proof of Proposition 4.6.1, this follows from coupling \( S^{i:x_i^n, x_i^n} \) with a Brownian bridge \( B_{3t_i/2}^{i:x_i, x_i} \) with variance 2/3 using Theorem 4.4.2, and then defining \( B_{t_i}^{i:x_i, x_i}(s) := B_{3t_i/2}^{i:x_i, x_i}(3s/2) \). \( \square \)
With these results in hand, by repeating the arguments in Section 4.6.1, for any $x_1, \ldots, x_k \geq 0$, we can find a coupling such that

$$\lim_{n \to \infty} m_n \mathbb{P}[S^0(\vartheta_i) = 0] F_{n,t_i}^d (S^{i;x_n^n,i^n}) = \frac{1}{\sqrt{2\pi t_i}} \mathbf{1}_{\{\tau_0(B^{1;i,x_i}_{t_i},x_i) > t\}} \exp \left(-\frac{1}{2} \langle L_{t_i}(B^{1;i,x_i}_{t_i},x_i), V \rangle - \int L_{a_{t_i}}(B^{1;i,x_i}_{t_i},x_i) dW(a) \right)$$

in probability for $1 \leq i \leq k$. Then, by arguing as in Section 4.6.1 (more specifically, the estimate for (4.6.10)), we get the convergence

$$\lim_{n \to \infty} \mathbb{E} \left[ \prod_{i=1}^k m_n \mathbb{P}[S^0(\vartheta_i) = 0] F_{n,t_i}^d (S^{i;x_n^n,i^n}) \right] = \mathbb{E} \left[ \prod_{i=1}^k \frac{1}{\sqrt{2\pi t_i}} \mathbf{1}_{\{\tau_0(B^{1;i,x_i}_{t_i},x_i) > t\}} \exp \left(-\frac{1}{2} \langle L_{t_i}(B^{1;i,x_i}_{t_i},x_i), V \rangle - \int L_{a_{t_i}}(B^{1;i,x_i}_{t_i},x_i) dW(a) \right) \right]$$

pointwise in $x_1, \ldots, x_k$ thanks to the following proposition, which we prove at the end of this section.

**Proposition 4.7.2.** Let $\vartheta = \vartheta(n, t) := \lfloor m_n^2 t \rfloor$ and $x^n := \lfloor m_n x \rfloor$ for some $t > 0$ and $x \geq 0$. For every $C > 0$ and $1 \leq q < 3$,

$$\sup_{n \in \mathbb{N}, x \geq 0} \mathbb{E} \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda^q_{\vartheta}(S^{x^n,i^n})}{m_n^q} \right) \right] < \infty.$$

It only remains to prove that we can pass the limit outside the integral (4.7.1). We once again use [FL07, Theorem 2.24]. For this, it is enough to prove that, for $n$
large enough, there exist constants \( c_1, c_2, c_3, \bar{c}_4, \bar{c}_5 > 0 \) such that

\[
\int_{[0, (n+1)/m_n]^k} \left| \mathbb{E} \left[ \prod_{i=1}^{k} F_{n,t_i}^{d} \left( S_{\varnothing_i}^{x^n_i,x^n_i} \right) \right] \right| \, dx_1 \cdots dx_k \\
\leq \prod_{i=1}^{k} \int_0^{(n+1)/m_n} \mathbb{E} \left[ |F_{n,t_i}^{d} (S_{\varnothing_i}^{x^n_i,x^n_i})|^k \right]^{1/k} \, dx \\
\leq \left( c_1 \int_0^{n^{1-\varepsilon}/m_n} \left( (1+x)^{-c_2 \theta} + e^{-c_3 n^{2\varepsilon}} \right) \, dx \\
+ \int_{n^{1-\varepsilon}/m_n}^{(n+1)/m_n} \left( e^{-c_4 n^{(1-\delta) - \alpha \varepsilon}} + e^{-2\bar{c}_5 n^{2\varepsilon}} \right) \, dx \right)^k
\] (4.7.2)

where \( \theta \) is taken large enough so that \( (1+x)^{-c_2 \theta} \) is integrable. To this end, for every \( \varnothing \in \mathbb{N} \), let us define \( R_{\varnothing}(S_{\varnothing,0}) \) as the range of \( S_{\varnothing,0} \). By replicating the estimates in Section 4.6.1, we see that (4.7.2) is the consequence of the following two propositions, concluding the proof of the convergence of moments.

**Proposition 4.7.3.** Let \( \varnothing = \varnothing(n, t) := \lfloor m_n^2 t \rfloor \) for some \( t > 0 \). For every \( C > 0 \),

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{C R_{\varnothing}(S_{\varnothing,0}) / m_n} \right] < \infty.
\]

**Proposition 4.7.4.** Let \( \varnothing = \varnothing(n, t) := \lfloor m_n^2 t \rfloor \) for some \( t > 0 \). For small enough \( \nu > 0 \), there exists some \( c > 0 \) independent of \( n \) such that

\[
P \left[ \sum_{a \in \mathbb{Z}} \Lambda_{\varnothing, a+b}^{(n,a+b)} (S_{\varnothing,0}) < \nu m_n^2 \right] \leq e^{-cm_n^2}, \quad b \in \{-1, 0, 1\}.
\]

**Proof of Proposition 4.7.3.** Let us define

\[
\mathcal{M}(S_{\varnothing,0}) := \max_{0 \leq u \leq \varnothing} |S_{\varnothing,0}(u)|.
\]

It is easy to see that \( R_{\varnothing}(S_{\varnothing,0}) \leq 2 \mathcal{M}(S_{\varnothing,0}) \), and thus it suffices to prove that the exponential moments of \( \mathcal{M}(S_{\varnothing,0}) / m_n \) are uniformly bounded in \( n \).
Let $\mathcal{S}$ be as in Definition 4.6.5, and define

$$
\mathcal{M}(\mathcal{S}^{0,0}_v) := \max_{0 \leq u \leq v} |\mathcal{S}^{0,0}_u|, \quad v \in 2\mathbb{N}_0.
$$

According to [GS18b, (4.7)] (up to normalization, the quantity denoted $\tilde{M}(N, \tilde{T})$ in [GS18b, (4.7)] is essentially the same as what we denote by $\mathcal{M}(\mathcal{S}^{0,0}_v)$; see the definition of the former on [GS18b, Page 2302]) we know that for every $0 < q < 2$ and $C > 0$,

$$
\sup_{u \in \mathbb{N}} \mathbb{E} \left[ e^{C(\mathcal{M}(\mathcal{S}^{0,0}_u)/\sqrt{u})^q} \right] < \infty. \tag{4.7.3}
$$

Let us define

$$
\mathcal{H}(\mathcal{S}^{0,0}_u) := \sum_{a \in \mathbb{Z}} \Lambda^{(a,a)}(\mathcal{S}^{0,0}_u), \quad u \in 2\mathbb{N}_0. \tag{4.7.4}
$$

For any $h \in \mathbb{N}_0$, we can couple the bridges of $S$ and $\mathcal{S}$ in such a way that

$$
(\mathcal{S}^{0,0}_\theta(u)|\mathcal{H}(\mathcal{S}^{0,0}_u) = h) = \mathcal{S}^{0,0}_{\theta - h}(u - \mathcal{H}(\mathcal{S}^{0,0}_u)).
$$

In words, we obtain $\mathcal{S}^{0,0}_{\theta - h}$ from $\mathcal{S}^{0,0}_\theta(u)$ by removing all segments that visit self-edges. Since visits to self-edges do not contribute to the magnitude of $\mathcal{S}^{0,0}_\theta$,

$$
(\mathcal{M}(\mathcal{S}^{0,0}_\theta)|\mathcal{H}(\mathcal{S}^{0,0}_\theta) = h) = \mathcal{M}(\mathcal{S}^{0,0}_{\theta - h}).
$$

Thus, (4.7.3) for $q = 1$ implies that

$$
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{C\mathcal{M}(\mathcal{S}^{0,0}_\theta)/m_n} \right] \\
= \sup_{n \in \mathbb{N}} \sum_{h \in \mathbb{N}_0} \mathbb{E} \left[ e^{C\mathcal{M}(\mathcal{S}^{0,0}_\theta)/m_n} \left| \mathcal{H}(\mathcal{S}^{0,0}_\theta) = h \right| \right] \mathbb{P}[\mathcal{H}(\mathcal{S}^{0,0}_\theta) = h] \tag{4.7.5} \leq \sup_{n \in \mathbb{N}} \sup_{1 \leq u \leq \theta} \mathbb{E} \left[ e^{(\sqrt{n}/m_n)C\mathcal{M}(\mathcal{S}^{0,0}_u)/\sqrt{n}} \right] < \infty
$$

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for every $C > 0$, as desired.

**Proof of Proposition 4.7.4.** Note that

$$P \left[ \sum_{a \in \mathbb{N}_0} \Lambda^{(a,a+b)}_\vartheta (S^0_0) < \nu m_n^2 \right] \leq P \left[ \sum_{a \in \mathbb{N}_0} \Lambda^{(a,a+b)}_\vartheta (S^0_0) < \nu m_n^2 \right] P \left[ S^0_0 = 0 \right]^{-1}.$$ 

By the local central limit theorem, $P[S^0_0 = 0]^{-1} = O(m_n)$, and thus the result follows from the same binomial concentration argument used for (4.6.20).

**Proof of Proposition 4.7.2.** In similar fashion to the proof of Proposition 4.6.3, it suffices to prove that the exponential moments of

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \left( \Lambda^{(a,a-1)}_\vartheta (S^0_{\vartheta}) + \Lambda^{(a,a+1)}_\vartheta (S^0_{\vartheta}) \right)$$

and

$$\frac{1}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^{(a,a)}_\vartheta (S^0_{\vartheta})$$

are uniformly bounded in $n$.

We start with (4.7.6). Under the coupling in the proof of Proposition 4.7.3,

$$\left( \sum_{a \in \mathbb{Z}} \left( \Lambda^{(a,a-1)}_\vartheta (S^0_{\vartheta}) + \Lambda^{(a,a+1)}_\vartheta (S^0_{\vartheta}) \right) \right) \mathcal{H}(S^0_{\vartheta}) = h \right) \leq \sum_{a \in \mathbb{Z}} \Lambda^{a}_{\vartheta-h} (S^0_{\vartheta-h})$$

for every $h \in \mathbb{N}_0$. By conditioning on $\mathcal{H}(S^0_{\vartheta})$ as in (4.7.5), we need only prove that

$$\sup_{n \in \mathbb{N}} E \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{Z}} \Lambda^{a}_{\vartheta} (S^0_{\vartheta}) \right) \right] < \infty.$$ 

By using (4.6.35) in the case $b = 0$ (i.e., [GS18b, (4.19)]), this follows from (4.7.3).

With a bound for (4.7.6) established, the exponential moments of (4.7.7) can be controlled by using the same argument in Section 4.6.3 (the path transformation used
therein does not change the endpoint of the path that is being modified; hence the transformed version of $S^0_\theta$ is a random walk bridge).

\[ \blacksquare \]

### 4.7.2 Step 2: Convergence in Distribution

The convergence in distribution follows from the convergence of mixed moments by using the same truncation/stochastic domination argument as in Section 4.6.2.

### 4.8 Weak Convergence to Robin Semigroup

In this section, we prove Theorem 4.1.10. This follows roughly the same steps as the proof of Theorem 4.1.9 (1).

#### 4.8.1 Step 1: Convergence of Moments

Expression for Mixed Moments and Convergence Result

By Fubini’s theorem, any mixed moment $E \left[ \prod_{i=1}^{k} \langle f_i, \hat{K}^d_n(t_i) g_i \rangle \right]$ can be written as

\[
\int_{(0,(n+1)/m_n)}^k \left( \prod_{i=1}^{k} f_i(x_i) \right) \cdot E \left[ \prod_{i=1}^{k} F_{n,t_i}^d(T^{i,x_i}_n) m_n \int_{T^{i,x_i}_n(\vartheta_i) + 1/m_n}^\infty g_i(y) \, dy \right] \, dx_1 \cdots dx_k, \quad (4.8.1)
\]

and the corresponding continuum limit is

\[
E \left[ \prod_{i=1}^{k} \langle f_i, \hat{K}^d(t_i) g_i \rangle \right] = \int_{(0,\infty)}^k \left( \prod_{i=1}^{k} f_i(x_i) \right) \cdot E \left[ \prod_{i=1}^{k} e^{-\langle L_{t_i}(X^{i,x_i}),V \rangle - \int L_{t_i}(X^{i,x_i}) dW(a) - \frac{w_0 T^{i,x_i}}{L_{t_i}(X^{i,x_i})} g_i(X^{i,x_i}(t))} \right] \, dx_1 \cdots dx_k, \quad (4.8.2)
\]

where $\vartheta_i$ and $x_i^n$ are as in Section 4.6.1,
1. $T^1:x^n_1, \ldots, T^k:x^n_k$ are independent copies of the Markov chain $T$ with respective starting points $x^n_1, \ldots, x^n_k$, and

2. $X^1:x_1, \ldots, X^k:x_k$ are independent copies of $X$ with respective starting points $x_1, \ldots, x_k$.

$T^i:x^n_i$ are independent of $Q_n$ and $X^i:x_i$ are independent of $Q$.

**Proposition 4.8.1.** Let $x_1, \ldots, x_n \geq 0$ be fixed. The following limits hold jointly in distribution over $1 \leq i \leq k$:

1. \[ \lim_{n \to \infty} \sup_{0 \leq s \leq t_i} \left| \frac{T^i:x^n_i(|m_n^2(3s/2)|)}{m_n} - X^i:x_i(s) \right| = 0. \]

2. \[ \lim_{n \to \infty} \sup_{y > 0} \left| \frac{\Lambda(y \cdot \bar{y}_n)(T^i:x^n_i)}{m_n} (1 - \frac{1}{2}1\{(y_n, \bar{y}_n) = (0,0)\}) - \frac{1}{2}L^y_{t_i}(X^i:x_i) \right| = 0, \]
   jointly in $(y_n, \bar{y}_n)_{n \in \mathbb{N}}$ as in (4.4.1).

3. \[ \lim_{n \to \infty} \left| \frac{\Lambda^{(0,0)}(T^i:x^n_i)}{m_n} - 2\sigma^0_{t_i}(X^i:x_i) \right| = 0. \]

4. \[ \lim_{n \to \infty} m_n \int_{(T^i:x^n_i(\vartheta_i)+1)/m_n} g_i(y) \, dy = g_i(X^i:x_i(t)). \]

5. The convergences in (4.1.21).

6. \[ \lim_{n \to \infty} \sum_{a \in \mathbb{N}_0} \frac{\Lambda_{\vartheta_i}(a_E, \bar{a}_E)(X^i:x^n_i)}{m_n} \xi^E_n(a) \frac{1}{m_n} = \frac{1}{2} \int_{(0,\infty)} L^y_{t_i}(T^i:x_i) \, dW^E(y) \]
   for $E \in \{D, U, L\}$, where, for every $a \in \mathbb{N}_0$, $(a_E, \bar{a}_E)$ are as in (4.6.4).

**Proof.** Arguing as in Proposition 4.6.1, the result follows by using Theorem 4.5.2 to couple the $T^i:x^n_i$ with reflected Brownian motions with variance $2/3$, $\tilde{X}^i:x^n_i$, and then defining $X^i:x^n_i(s) := \tilde{X}^i:x^n_i(3s/2)$, which yields a standard reflected Brownian motion such that $L^y_{3t_i/2}(X^i:x_i) = \frac{3}{2}L^y_{t_i}(X^i:x_i)$ and $\xi^0_{3t_i/2}(\tilde{X}^i:x_i) = \frac{3}{2}\xi^0_{t_i}(X^i:x_i).$ \qed
Convergence Inside the Expected Value

We begin with the proof that for every $x_1, \ldots, x_k \geq 0$, there is a coupling such that

$$
\lim_{n \to \infty} \prod_{i=1}^{k} F_{n,t_i}(T_i^{x_i n}) m_n \int_{T_i^{x_i n}(\theta_i) / m_n}^{(T_i^{x_i n}(\theta_i) + 1) / m_n} g_i(y) \, dy
= \prod_{i=1}^{k} e^{-(L_{t_i}(X_i^{x_i t_i}), V) - \int L_{t_i}(X_i^{x_i t_i}) \, dW(a) - w L_{t_i}^{0}(X_i^{x_i t_i})} g_i(X_i^{x_i t_i}(t))
$$

in probability. Proposition 4.8.1 provides a coupling such that

$$
\prod_{i=1}^{k} \mathbb{1}_{\{\tau_n(T_i^{x_i n}) > \theta_i\}} \left( \prod_{a \in \mathbb{N}} \left( 1 - \frac{D_n(a)}{m_n^2} \right) \Lambda_{\theta_i}^{(0,a)}(T_i^{x_i n}) \right)
\cdot \left( \prod_{a \in \mathbb{N}_0} \left( 1 - \frac{U_n(a)}{m_n^2} \right) \Lambda_{\theta_i}^{(a+1,a)}(T_i^{x_i n}) \right)
$$

converges in probability to $\prod_{i=1}^{k} e^{-(L_{t_i}(X_i^{x_i t_i}), V) - \int L_{t_i}(X_i^{x_i t_i}) \, dW(a) - w L_{t_i}^{0}(X_i^{x_i t_i})}$ Combining this with Proposition 4.8.1 (4), it only remains to show that

$$
\lim_{n \to \infty} \prod_{i=1}^{k} \left( 1 - \frac{(1 - w_n)}{2} - \frac{D_n(0)}{2m_n^2} \right) \Lambda_{\theta_i}^{(0,0)}(T_i^{x_i n}) = \prod_{i=1}^{k} e^{-w L_{t_i}^{0}(X_i^{x_i t_i})}.
$$

To this effect, the Taylor expansion $\log(1 + z) = z + O(z^2)$ yields

$$
\left( 1 - \frac{(1 - w_n)}{2} - \frac{D_n(0)}{2m_n^2} \right) \Lambda_{\theta_i}^{(0,0)}(T_i^{x_i n}) = \exp \left( -\Lambda_{\theta_i}^{(0,0)}(T_i^{x_i n}) \left( \frac{(1 - w_n)}{2} + \frac{D_n(0)}{2m_n^2} + O \left( \frac{(1 - w_n)^2}{4} + \frac{D_n(0)^2}{2m_n^4} \right) \right) \right).
$$

By Proposition 4.8.1 (3) and Assumption $w_n$,

$$
\lim_{n \to \infty} \Lambda_{\theta_i}^{(0,0)}(T_i^{x_i n}) \left( \frac{(1 - w_n)}{2} + \frac{D_n(0)}{2m_n^2} \right) = w L_{t_i}^{0}(X_i^{x_i t_i})
$$
and
\[ \lim_{n \to \infty} \Lambda^{(0,0)}_{\vartheta_i} (T^n_{i|x_n}) \left( \frac{(1 - w_n)^2}{4} + \frac{D_n(0)^2}{2m_n^4} \right) = 0 \]
almost surely, as desired.

**Convergence of the Expected Value**

Next we prove
\[ \lim_{n \to \infty} \mathbb{E} \left[ \prod_{i=1}^{k} F_{n,t_i} (T^n_{i|x^n_i}) \right] \mathbb{E} \left[ \int_{T^n_{i|x_i} (\vartheta_i)/m_n} g_i(y) \, dy \right] \]
for large enough \( N \). To achieve this we combine Proposition 4.5.8 and the following:

**Proposition 4.8.2.** Let \( \vartheta = \vartheta(n, t) = [m^2 t] \) for some \( t > 0 \). For every \( C > 0 \) and \( 1 < q < 3 \),
\[ \sup_{n \in \mathbb{N}, \, x \geq 0} \mathbb{E} \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{N}} \frac{\Lambda^a(T^n)^q}{m_n^q} \right) \right] < \infty. \]

**Proof.** If we couple \( X \) and \( S \) as in Definition 4.5.9, then we see that
\[ \mathbb{E} \left[ \exp \left( \frac{C}{m_n} \sum_{a \in \mathbb{N}} \frac{\Lambda^a(T^n)^q}{m_n^q} \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{2^{q-1} C}{m_n} \sum_{a \in \mathbb{N} \setminus \{0\}} \frac{\Lambda^a(S^0)^q}{m_n^q} \right) \right] \]
\[ \leq \mathbb{E} \left[ \exp \left( \frac{2^{q-1} C}{m_n} \sum_{a \in \mathbb{Z}} \frac{\Lambda^a(S^0)^q}{m_n^q} \right) \right]. \]
Thus Proposition 4.8.2 follows directly from Proposition 4.6.3.
Indeed, the argument used to prove (4.6.10) show that the contribution of the terms of the form (4.3.14) and (4.3.15) to (4.8.5) can be controlled by Proposition 4.8.2. Thus, it suffices to prove that for every $C > 0$, there is some $N \in \mathbb{N}$ large enough so that

$$
\sup_{n \geq N, \, x \geq 0} \mathbb{E} \left[ \left| 1 - \left( 1 - \frac{w_n}{2} \right) - \frac{D_n(0)}{2m_n^2} \right|^{C \Lambda_{\vartheta}(0,0) \varphi(T^n x) \vartheta_{\Lambda}(T^n x; x_n)} \right] < \infty. \tag{4.8.6}
$$

By using the bound $|1 - z| \leq e^{\|z\|}$, it suffices to control the exponential moments of

$$\Lambda_{\vartheta}(0,0)(T^n x)|1 - w_n| \tag{4.8.7}$$

and

$$\frac{\Lambda_{\vartheta}(0,0)(T^n x)|D_n(0)|}{m_n^2}. \tag{4.8.8}$$

We begin with (4.8.7). According to Proposition 4.5.8, for every $C > 0$,

$$\sup_{n \in \mathbb{N}, \, x \geq 0} \mathbb{E} \left[ e^{C \Lambda_{\vartheta}(0,0)(T^n x)/m_n} \right] < \infty.$$

Thus, given that $|1 - w_n| = O(m_n^{-1})$ by Assumption $w_n$, we conclude that

$$\sup_{n \in \mathbb{N}, \, x \geq 0} \mathbb{E} \left[ e^{C \Lambda_{\vartheta}(0,0)(T^n x)|1 - w_n|} \right] < \infty.$$

Let us now consider (4.8.8). By the tower property and Assumption R, there exist $\bar{C}, \bar{c} > 0$ independent of $n$ such that

$$\mathbb{E} \left[ e^{C \left( \frac{\Lambda_{\vartheta}(0,0)(T^n x)}{m_n^{3/2}} \right) \left( \frac{|D_n(0)|}{m_n^{1/2}} \right)^2} \right] \leq \bar{C} \mathbb{E} \left[ e^{\bar{c} (C^{2/m_n}) \left( \frac{\Lambda_{\vartheta}(0,0)(T^n x)}{m_n} \right)^2} \right].$$

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Since $\bar{c}(C^2/m_n) \to 0$, it follows from Proposition 4.5.8 that

$$\sup_{n \geq N, \ x \geq 0} \mathbb{E} \left[ e^{\bar{c}(C^2/m_n) \left( \Lambda_{\tilde{a}}^{(n)\circ}(T^{i,x^n})/m_n \right)^2} \right] < \infty$$

for large enough $N$, concluding the proof of (4.8.6).

**Convergence of the Integral**

With (4.8.4) established, once more we aim to prove that (4.8.1) converges to (4.8.2) by using [FL07, Theorem 2.24]. Similarly to Section 4.6.1, for this we need upper bounds of the form

$$\mathbb{E} \left[ \left| F_{n,t_i}(T^{i,x^n}) \right|^k \right]^{1/k} \leq c_1 \left( (1+x)^{-c_2 \theta} + e^{-c_3 n^2 \alpha} \right), \quad x \in [0, n^{1-\varepsilon}/m_n) \quad (4.8.9)$$

and

$$\mathbb{E} \left[ \left| F_{n,t_i}(T^{i,x^n}) \right|^k \right]^{1/k} \leq e^{-c_4 n^{\alpha(1-\delta)} - \alpha \epsilon} + e^{-c_5 n^2 \alpha}, \quad x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n), \quad (4.8.10)$$

where $\varepsilon, c_1, c_2, c_3, \bar{c}_4, \bar{c}_5 > 0$ are independent of $n$ and $\theta > 0$ is taken large enough so that $(1+x)^{-c_2 \theta}$ is integrable.

We begin with $x \in [0, n^{1-\varepsilon}/m_n)$. By replicating the analysis leading up to (4.6.15) and (4.6.16), we are led to bounding $\mathbb{E} \left[ \left| F_{n,t_i}(T^{i,x^n}) \right|^k \right]^{1/k}$ by the product of the following five terms:

$$\mathbb{E} \left[ 1 - \frac{(1 - w_n)}{2} - \frac{D_n(0)}{2m^2_n} \right]^{1/7k}, \quad (4.8.11)$$

$$\prod_{E \in \{U,L\}} \mathbb{E} \left[ \prod_{a \in \mathbb{N}_0} \left| 1 - \frac{\xi^E_n(a)}{m^2_n - V^E_n(a)} \right|^{7k \Lambda_{\tilde{a}}^{(n)\circ}(T^{i,x^n})} \right]^{1/7k}, \quad (4.8.12)$$
\[
E \left[ \prod_{a \in \mathbb{N}} \left( 1 - \frac{\xi_n^D(a)}{m_n^2 - V_n^D(a)} \right)^{7k \Lambda_{\bar{\vartheta}_i}^{(a,a)}(T^i;x^n)} \right]^{1/7k}, \quad (4.8.13)
\]

\[
\prod_{E \in \{U, L\}} E \left[ \prod_{a \in \mathbb{N}_0} \left( 1 - \frac{V_E^n(a)}{m_n^2} \right)^{7k \Lambda_{\bar{\vartheta}_i}^{(a,a)}(T^i;x^n)} \right]^{1/7k}, \quad (4.8.14)
\]

\[
E \left[ \prod_{a \in \mathbb{N}} \left( 1 - \frac{V_n^D(a)}{m_n^2} \right)^{7k \Lambda_{\bar{\vartheta}_i}^{(a,a)}(T^i;x^n)} \right]^{1/7k}. \quad (4.8.15)
\]

Suppose without loss of generality that \( V_n^D \) satisfies (4.1.17). (4.8.11) can be controlled with (4.8.6); (4.8.12) and (4.8.13) can be controlled with Proposition 4.8.2; and (4.8.14) can be controlled with (4.1.15). For (4.8.15), up to a constant independent of \( n \), we get from (4.1.17) the upper bound

\[
E \left[ \exp \left( -\frac{7k\theta}{m_n^2} \sum_{a \in \mathbb{N}} \log(1 + |a|/m_n) \Lambda_{\bar{\vartheta}_i}^{(a,a)}(T^i;x^n) \right) \right]^{1/7k}. \quad (4.8.16)
\]

Let us couple \( T^i;x^n \) and \( S^{x^n} = x^n + S^0 \) as in Definition 4.5.9. The same argument used to control (4.6.17) implies that (4.8.16) is bounded above by the product of

\[
E \left[ \exp \left( -\frac{14k\theta \log(1 + x)}{m_n^2} \sum_{a \in \mathbb{Z} \setminus \{0\}} \Lambda_{\bar{\vartheta}_i}^{(a,a)}(S^0) \right) \right]^{1/14k}, \quad (4.8.17)
\]

\[
E \left[ \exp \left( \frac{14k\theta}{m_n^2} \sum_{0 \leq u \leq \vartheta_i^n} \frac{|S^0(u)|}{m_n} \right) \right]^{1/14k}. \quad (4.8.18)
\]

Since \( \vartheta_i^n \leq \vartheta_i \), we can prove that (4.8.18) is bounded by a constant independent of \( n \) by using (4.4.12) directly. As for (4.8.17), we have the following proposition:

**Proposition 4.8.3.** Let \( \vartheta = \vartheta(n,t) := \lfloor m_n^2 t \rfloor \) for some \( t > 0 \). For every \( x \geq 0 \), let us couple \( T^{x^n} \) and \( S^{x^n} := x^n + S^0 \) as in Definition 4.5.9. For small enough \( \nu > 0 \),
there exist $C, c > 0$ independent of $x$ and $n$ such that

$$\sup_{x \geq 0} \mathbf{P} \left[ \sum_{a \in \mathbb{Z} \setminus \{0\}} \Lambda_{\frac{a}{x^n}}^{(a, a+b)} (S^0) < \nu m^2_n \right] \leq Ce^{-cm^2_n}, \quad b \in \{-1, 0, 1\}.$$  

**Proof.** By Proposition 4.5.8, for any $0 < \delta < 1$, we can find $\bar{C}, \bar{c} > 0$ such that

$$\sup_{x \geq 0} \mathbf{P} \left[ \Lambda^{(0,0)} (T^{x^n}) \geq \delta \vartheta \right] \leq \bar{C} e^{-\bar{c}m^2_n}.$$  

Given that $\vartheta - \varrho x^n \leq \Lambda^{(0,0)} (T^{x^n})$, it suffices to prove that

$$\sup_{x \geq 0} \mathbf{P} \left[ \sum_{a \in \mathbb{Z} \setminus \{0\}} \Lambda_{\frac{a}{(1-\delta)\vartheta}}^{(a, a+b)} (S^0) < \nu m^2_n \right] \leq Ce^{-cm^2_n}, \quad b \in \{-1, 0, 1\}$$

for large enough $N$. This follows by Hoeffding’s inequality. \hfill \Box

Indeed, by arguing as in the passage following (4.6.20), Proposition 4.8.3 implies that (4.8.17) is bounded above by $c_1 \left( (1 + x)^{-c_2 \theta} + e^{-c_3 n^{2\theta}} \right)$ for some $c_1, c_2, c_3 > 0$ independent of $n$ (and $c_2$ independent of $\theta$), hence (4.8.9) holds.

We now prove (4.8.10). Let $x \in [n^{1-\varepsilon}/m_n, (n + 1)/m_n)$. Assuming without loss of generality that $V_n^D$ satisfies (4.1.18), by arguing as in Section 4.6.1, we get that $\mathbf{E} \left[ |F_{n, t_i}^{\tau} (T^{i:x^n})|^k \right]^{1/k}$ is bounded by the product of the four terms

$$\mathbf{E} \left[ \left| 1 - \frac{1 - w_n}{2} \right|^{5k\Lambda^{(0,0)}_{\vartheta_i} (T^{i:x^n})} \right]^{1/5k}$$

$$\mathbf{E} \left[ \prod_{a \in \mathbb{N}} \left( 1 - \frac{\kappa (Cn^{1-\varepsilon} / m_n)^{\alpha} \Lambda_{\vartheta_i}^{(a, a)} (T^{i:x^n})}{m_n^2} \right)^{5k\Lambda^{(a,a)}_{\vartheta_i} (T^{i:x^n})} \right]^{1/5k}$$

$$\mathbf{E} \left[ \prod_{a \in \mathbb{N}} \left( 1 + \frac{2|\varepsilon_n D(a)|}{m_n^2} \right)^{5k\Lambda^{(a,a)}_{\vartheta_i} (T^{i:x^n})} \right]^{1/5k}$$

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\[ \prod_{E \in \{U, L\}} \mathbb{E} \left[ \prod_{a \in \mathbb{N}_0} \left( 1 + \frac{\xi^{E}(a)}{s^{E}(a)} \right)^{\frac{5kA_{\Lambda}(aE, aE)(T^{\varepsilon_n})}{m_{\varepsilon_n}^2}} \right]^{1/5k} \]

By combining Propositions 4.8.2 and 4.8.3 with (4.8.6), the same arguments used in Section 4.6.1 yields (4.8.10), concluding the proof of the convergence of moments.

4.8.2 Step 2: Convergence in Distribution

The convergence in joint distribution follows from the convergence of moments by using the same truncation/stochastic dominance argument as in Section 4.6.2, thus concluding the proof of Theorem 4.1.10.
Appendix A

A.1 Measurability of Kernel

We begin by proving that, in Case 1, for every realization of $\Xi$ as a continuous function, $(x, y) \mapsto \hat{K}(t; x, y)$ can be made a Borel measurable function on $\mathbb{R}^2$.

**Notation A.1.1.** Let $C_L = C_L([0, t])$ be the set of continuous functions $f : [0, t] \to \mathbb{R}$ with a continuous local time (in the sense of (2.1.7)), equipped with the uniform topology. Let $C = C(\mathbb{R})$ be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the uniform-on-compacts topology; and let $C_0 = C_0(\mathbb{R})$ be the space of continuous and compactly supported functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the uniform topology. We use $P_t^{0,0}$ to denote the probability measure of the Brownian bridge $B_t^{0,0}$ on $C_L$, and assume that $C$ is equipped with the probability measure of $\Xi$.

By Fubini’s theorem, it suffices to prove that there exists a measurable map

$$F : \mathbb{R}^2 \otimes C_L \otimes C \to \mathbb{R}$$
such that for every \((x, y) \in \mathbb{R}^2, \omega \in C_L, \) and \(\bar{\omega} \in C\), we can interpret

\[
e^{-\langle L_t(B^x_t,B^y_t), V \rangle - \xi(L_t(B^x_t,B^y_t))} := F((x, y), \omega, \bar{\omega}) \quad \text{(A.1.1)}
\]

(here, \(\bar{\omega} \in C\) corresponds to a realization of \(\Xi\), and \(((x, y), \omega) \in \mathbb{R}^2 \otimes C_L\) corresponds to a realization of the Brownian bridge \(B^x_t, B^y_t\) with deterministic endpoints \(x, y\) and random dynamics given by the Brownian path \(B^0_0, B^0_t\)). Indeed, if this holds, then for every realization of the noise \(\bar{\omega} \in C\), we can define the Borel measurable function

\[
\hat{K}(t; x, y) := \int_{C_L} \Pi_B(t; x, y) F((x, y), \omega, \bar{\omega}) \, d\mathbf{P}^{0,0}_t(\omega), \quad x, y \in \mathbb{R},
\]

which corresponds to the expected value of \(\Phi_t^X(B^0_t, B^0_t) e^{-\langle L_t(B^x_t,B^y_t), V \rangle - \xi(L_t(B^x_t,B^y_t))}\) given \(\Xi\).

Given a realization \(\omega \in C_L\) of \(B^0_0, B^0_t\) and \(x, y \in \mathbb{R}\), we can construct a realization of \(B^x_t, B^y_t\) by using the measurable map \(F_1 := \mathbb{R}^2 \otimes C_L \rightarrow C_L\) defined as

\[
F_1((x, y), \omega) := \left( \omega(s) + \frac{(t-s)x}{t} + \frac{s y}{t} : 0 \leq s \leq t \right).
\]

Next, we define \(F_2 : C_L \rightarrow C_0\) as the measurable function that maps \(\omega \in C_L\) to its local time process \(x \mapsto L^x_t(\omega)\). Finally, let

\[
F_3(f, \bar{\omega}) = \int_{\mathbb{R}} f(x) \, d\bar{\omega}(x) := \begin{cases} 
\lim_{n \to \infty} F^{(n)}_3(f, \bar{\omega}) & \text{if the limit exists} \\
0 & \text{otherwise}
\end{cases}
\]

be the limit of the measurable maps \(F^{(n)}_3 : C_0 \otimes C \rightarrow \mathbb{R}\) defined as

\[
F^{(n)}_3(f, \bar{\omega}) := \sum_{k=1}^{k(n)} f\left(\tau_k^{(n)}\right) \left(\bar{\omega}\left(\tau_k^{(n)}\right) - \bar{\omega}\left(\tau_k^{(n)}\right)\right),
\]

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as per Section 2.2.2/Karandikar [Kar95]. We may then define \((A.1.1)\) using the compositions of measurable maps

\[
F((x, y), \omega, \bar{\omega}) := e^{-(F_2 \circ F_1((x, y), \omega), V) - F_3 \circ F_2 \circ F_1((x, y), \omega, \bar{\omega})}.
\]

In order to prove that the diagonal \(x \mapsto K(t; x, x)\) is Borel measurable, we apply the same argument, except that \(x = y\). Then, in order to prove the measurability in Cases 2 and 3, we can use the same argument, except that we add a few additional steps to construct the conditioned processes

\[
(B^x \mid B^x(t) \in \{y, -y\}) \quad \text{or} \quad (B^x \mid B^x(t) \in 2b\mathbb{Z} \pm y),
\]

and then use the couplings discussed in Section 2.4.2 to construct \(X^{x,y}_t\) and \(Y^{x,y}_t\) and their local times from the latter in the space of continuous and compactly supported functions on \([0, \infty)\) and \([0, b]\), respectively.

### A.2 Tails of Gaussian Suprema

Throughout this section, we assume that \((X(x))_{x \in T}\) is a continuous centered Gaussian process on some index space \(T\). We have the following result regarding the behaviour of the tails of \(X\)’s supremum.

**Theorem A.2.1** ([MR06, Theorem 5.4.3 and Corollary 5.4.5]). Let us define

\[
v^2 := \sup_{x \in T} \mathbb{E}[X(x)^2] \quad \text{and} \quad m := \text{Med} \left[ \sup_{x \in T} X(x) \right],
\]

where \(\text{Med}\) denotes the median. It holds that

\[
P \left[ \sup_{x \in T} X(x) \leq t \right] \leq 1 - \Phi \left( \frac{(t - m)}{v} \right) \leq e^{-\frac{(t - m)^2}{2v^2}}, \quad t \geq 0,
\]

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where $\Phi$ denotes the standard Gaussian CDF.

Using this Gaussian tails result, we can control the asymptotic growth of functions involving Gaussian suprema.

**Corollary A.2.2.** Let $\mathcal{X} = \mathbb{R}$, and suppose that $\mathcal{X}$ is stationary. There exists a finite random variable $C > 0$ such that, almost surely,

$$|\mathcal{X}(x)| \leq C \sqrt{\log(2 + |x|)}, \quad x \in \mathbb{R}.$$ 

**Proof.** For every $n \in \mathbb{Z} \setminus \{0\}$ and $c > 0$, define the events

$$E_n^{(c)} := \left\{ \sup_{x \in [n, n+1]} |\mathcal{X}(x)| \geq c \sqrt{\log |n|} \right\}.$$ 

By the Borel-Cantelli lemma, it suffices to prove that $\sum_n \mathbb{P}[E_n^{(c)}] < \infty$ for a large enough $c > 0$. Since $\mathcal{X}$ is stationary, for every $n$, it holds that

$$\sup_{x \in [n, n+1]} \mathbb{E}[\mathcal{X}(x)^2] = \mathbb{E}[\mathcal{X}(0)^2] =: \sigma^2$$

and

$$\text{Med}\left[ \sup_{x \in [n, n+1]} \mathcal{X}(x) \right] = \text{Med}\left[ \sup_{x \in [0,1]} \mathcal{X}(x) \right] =: \mu.$$ 

Thus, by Theorem A.2.1, $\mathbb{P}[E_n^{(c)}] \leq \exp\left( (c \sqrt{\log |n|} - \mu)^2 / 2\sigma^2 \right)$. Since this is summable in $n$ for large enough $c > 0$, the result is proved.

**Remark A.2.3.** By examining the proof of Corollary A.2.2, we note that we can easily also prove the stronger statement that, almost surely,

$$\sup_{y \in [x, x+1]} |\mathcal{X}(y)| \leq C \sqrt{\log(2 + |x|)},$$

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since
\[
\sup_{y \in [x,x+1]} |\mathcal{X}(y)| \leq \sup_{y \in [(x],[x]+1]} |\mathcal{X}(y)| + \sup_{y \in ([x]+1,[x]+2]} |\mathcal{X}(y)|.
\]

**Remark A.2.4.** In the setting of Corollary A.2.2, if we also assume that
\[
\lim_{|x| \to \infty} \mathbb{E}[\mathcal{X}(0)\mathcal{X}(x)] = 0,
\]
then we can prove that the upper bound in Corollary A.2.2 is optimal, in the sense that we also have a matching lower bound of the form
\[
\sup_{|y| \leq x} |\mathcal{X}(y)| \geq \tilde{C} \sqrt{\log(2 + |x|)}, \quad x \in \mathbb{R}
\]
for some \(0 < \tilde{C} < C\) (see, e.g., [CM95, Section 2.1] and references therein).

### A.3 Deterministic Feynman-Kac Computations

In this appendix, we prove (2.4.3) and (2.4.4), as well as the the two missing cases of \(e^{-tH} = K(t)\) in Theorem 2.4.4, namely, Cases 2-R and 3-M.

#### A.3.1 Symmetry

On the one hand, since the Gaussian kernel \(\mathcal{G}_t\) is even, the transition kernels satisfy
\[
\Pi_Z(t;x,y) = \Pi_Z(t;y,x) \text{ for every } t > 0 \text{ and } x,y \in I.
\]
On the other hand, given that
\[
(Z^x_y(t-s) : 0 \leq s \leq t) \overset{d}{=} (Z^y_x(s) : 0 \leq s \leq t)
\]
\[
\mathbb{E}^x_y[F(Z)] = \mathbb{E}^y_x[F(Z)] \text{ for any path functional } F \text{ that is invariant under time reversal.}
\]
In particular, since local time is invariant under time reversal, we have (2.4.3).
A.3.2 Semigroup Property

As argued in the proof of Lemma 2.4.8, for every \( x, y \in I \) and \( t, \bar{t} > 0 \), if we condition the path \( Z_{t+\bar{t}}^{x,y} \) on \( Z_{t+\bar{t}}^{x,x} = z \), then the path segments

\[
(Z_{t+\bar{t}}^{x,y}(s) : 0 \leq s \leq t) \quad \text{and} \quad (Z_{t+\bar{t}}^{x,y}(s) : t \leq s \leq t + \bar{t}) \quad (A.3.1)
\]

are independent and have respective distributions \( Z_{t+\bar{t}}^{x,z} \) and \( Z_{t+\bar{t}}^{z,y} \). We moreover note that \( Z_{t+\bar{t}}^{x,x} \) has density

\[
z \mapsto \frac{\Pi_Z(t; x, z)\Pi_Z(t; z, y)}{\Pi_Z(t + \bar{t}; x, y)}
\]

by the Doob \( h \)-transform. Given that the functions \( f \mapsto \langle f, V \rangle \), \( f \mapsto \bar{\alpha}f \) and \( f \mapsto \bar{\beta}f \) are all linear in \( f \), and that local time is additive, in the sense that

\[
L_{[u,v]}(Z) + L_{[v,w]}(Z) = L_{[u,w]}(Z)
\]

for all \( 0 < u < v < w \), (2.4.4) is then a consequence of Fubini’s theorem and the following rearrangement: Letting \( Z_{t}^{1;x,z} \) and \( Z_{t}^{2;z,y} \) denote independent processes with respective distributions \( Z_{t}^{x,z} \) and \( Z_{t}^{z,y} \) for all \( z \), we have that \( \int I K(t; x, z)K(t; z, y) \, dz \) is equal to (recall the notation \( \mathfrak{A}_t \) from (2.4.28))

\[
\Pi_Z(t + \bar{t}; x, y) \int_I E \left[ e^{\mathfrak{A}_t(Z_{t+\bar{t}}^{1;x,z}) + \mathfrak{A}_t(Z_{t+\bar{t}}^{2;z,y})} \frac{\Pi_Z(t; x, z)\Pi_Z(t; z, y)}{\Pi_Z(t + \bar{t}; x, y)} \right] \, dz
\]

\[
= \Pi_Z(t + \bar{t}; x, y) \int_I E \left[ e^{\mathfrak{A}_t(Z_{t+\bar{t}}^{x,y})} \right] \left. Z_{t+\bar{t}}^{x,y}(t) = z \right] \frac{\Pi_Z(t; x, z)\Pi_Z(t; z, y)}{\Pi_Z(t + \bar{t}; x, y)} \, dz
\]

\[
= K(t + \bar{t}; x, y),
\]

as desired.
A.3.3 Case 3-M

Let us assume that we are considering Case 3-M, that is, the operator \( H = -\frac{1}{2}\Delta + V \) is acting on \((0, b)\) with mixed boundary conditions (as in Assumption DB) and

\[
K(t; x, y) = \Pi_Y(t; x, y) \mathbf{E}_{t}^{x,y} \left[ e^{-\langle L_t(Y), V \rangle + \alpha \mathcal{L}_t(Y) - \frac{1}{2} \mathcal{L}_t(Y)} \right].
\]

As argued in [Pap90, Pages 62 and 63], it can be shown that

1. \( K(t) \) is a strongly continuous semigroup on \( L^2 \); and

2. if, for every \( n \in \mathbb{N} \), we define

\[
K_n(t; x, y) = \Pi_Y(t; x, y) \mathbf{E}_{t}^{x,y} \left[ e^{-\langle L_t(Y), V \rangle + \alpha \mathcal{L}_t(Y) - n \mathcal{L}_t(Y)} \right],
\]

then for every \( t > 0 \), \( \| K_n(t) - K(t) \|_{op} \to 0 \) as \( n \to \infty \).

Item (1) above implies that \( K(t) \) has a generator, so it only remains to prove that this generator is in fact \( H \). By Lemma 2.4.13 in the case \( p = 1 \), we know that the \( K_n(t) \) and \( K(t) \) are compact. Therefore, if we let \( H_n \) be the operator \(-\frac{1}{2}\Delta + V\) on \((0, b)\) with Robin boundary

\[
f'(0) + \alpha f(0) = -f'(b) - nf(0) = 0,
\]

then by repeating the argument in Section 2.2.2, we need only prove that \( H_n \to H \) as \( n \to \infty \) in the sense of convergence of eigenvalues and \( L^2 \)-convergence of eigenfunctions.

If we define the matrices

\[
A := \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

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and the vector function \( F(x) := \begin{bmatrix} f'(x) & f(x) \end{bmatrix}^\top \), then we can represent \( H \)'s boundary conditions in matrix form as \( AF(0) + BF(b) = 0 \). Similarly, if we let

\[
C_n := \begin{bmatrix} 0 & 0 \\ 1/n & 1 \end{bmatrix},
\]

then \( H_n \)'s boundary conditions are represented as \( AF(0) + C_n F(b) = 0 \). Clearly \( \|B - C_n\| \to 0 \) as \( n \to \infty \), and thus it follows from [Zet05, Theorems 3.5.1 and 3.5.2] that for every \( k \in \mathbb{N} \), \( \lambda_k(H_n) \to \lambda_k(H) \) and \( \psi_k(H_n) \to \psi_k(H) \) uniformly on compacts. Since the domain \((0, b)\) is bounded, this implies \( L^2 \)-convergence of the eigenfunctions, concluding the proof.

### A.3.4 Case 2-R

Let us now assume that \( H \) acts on \((0, \infty)\) with Robin boundary at the origin and that

\[
K(t; x, y) = \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[ e^{-(L_t(X),V) + \alpha L_0 t(X)} \right].
\]

The same arguments used in [Pap90, Theorem 3.4 (b)] imply that this semigroup is strongly continuous on \( L^2 \), and we know it is compact by Lemma 2.4.13.

For every \( n \in \mathbb{N} \), let \( H_n = -\frac{1}{2} \Delta + V \), acting on \((0, n)\) with mixed boundary conditions

\[
f(0) + \alpha f'(0) = f(n) = 0.
\]

By the previous section, the semigroup generated by this operator is given by

\[
K_n(t; x, y) = \Pi_{Y_n}(t; x, y) \mathbf{E}_t^{x,y} \left[ e^{-(L_t(Y_n),V) + \alpha L_0 t(Y_n) - \infty \cdot L_n t(Y_n)} \right],
\]
where \( Y_n \) is a reflected Brownian motion on \((0, n)\). Arguing as in the previous section, it suffices to prove that \( K_n(t) \rightarrow K(t) \) in operator norm and \( H_n \rightarrow H \) in the sense of eigenvalues and eigenfunctions.

We begin with the semigroup convergence. We first note that the quantity \( \|K_n(t) - K(t)\|_{op} \) is ambiguous, since \( K_n(t) \) and \( K(t) \) do not act on the same space. However, by using an argument similar to (2.4.6), we can extend the kernel \( K_n(t) \) to \((0, \infty)^2\) by defining

\[
\tilde{K}_n(t; x, y) = \Pi_X(t; x, y) E^{x,y}_t \left[ 1_{\{\tau_{[n,\infty)}(X) > t\}} e^{-\langle L_n(X), V \rangle + \alpha L_{n,0}(X)} \right],
\]

where \( \tau_{[n,\infty)} \) is the first hitting time of \([n, \infty)\). This transformation does not affect the eigenvalues, and the eigenfunctions are similarly extended from functions on \((0, n)\) vanishing on the boundary to functions on \((0, \infty)\) that are supported on \((0, n)\). We have that

\[
\|\tilde{K}_n(t) - K(t)\|_2^2 = \int_0^\infty \tilde{K}_n(2t; x, x) - 2\tilde{K}_n,0(2t; x, x) + K(2t; x, x) \, dx,
\]

where

\[
\tilde{K}_n,0(2t; x, x) = \Pi_X(2t; x, x) E^{x,x}_{2t} \left[ 1_{\{\tau_{[n,\infty)}(X) > t\}} e^{-\langle L_{2t}(X), V \rangle + \alpha L_{0,2t}(X)} \right].
\]

Thus it suffices to prove that

\[
\lim_{n \to \infty} \int_0^\infty \tilde{K}_n,0(2t; x, x) \, dx, \lim_{n \to \infty} \int_0^\infty \tilde{K}_n(2t; x, x) \, dx = \int_0^\infty K(2t; x, x) \, dx.
\]

Since \( X_{2t}^{x,x} \) is almost surely continuous, hence bounded, the result is a straightforward application of monotone convergence (both with \( E^{x,x} \) and the \( dx \) integral).
We now prove convergence of eigenvalues and eigenvectors. Let $\mathcal{E}$ denote the form of $H$ and $D(\mathcal{E})$ its domain, as defined in Definition 2.1.3 for Case 2-R. We note that we can think of $H_n$ as the operator with the same form $\mathcal{E}$ but acting on the smaller domain

$$D_n := \{ f \in H^1_V((0, \infty)) : f(x) = 0 \text{ for every } x \geq n \} \subset D(\mathcal{E}).$$

These domains are increasing, in that $D_1 \subset D_2 \subset \cdots \subset D(\mathcal{E})$. A straightforward modification of the convergence argument presented in Section 2.4.6 gives the desired result (at least through a subsequence).

**A.4 Noise Convergence**

**Proposition A.4.1.** For every $n \in \mathbb{N}$, let $\varsigma_n(1), \ldots, \varsigma_n(n)$ be a sequence of independent random variables satisfying the following conditions:

1. $|\mathbb{E}[\varsigma_n(a)]| = o((n - a)^{-1/3})$ as $n - a \to \infty$.

2. There exists $\sigma > 0$ such that $\mathbb{E}[\varsigma_n(a)^2] = \sigma^2 + o(1)$ as $n - a \to \infty$.

3. There exists $C > 0$ and $0 < \gamma < 2/3$ such that $\mathbb{E}[|\varsigma_n(a)|^p] < C^{p \gamma^p}$ for every $n \in \mathbb{N}$, $p \in \mathbb{N}$, and $0 \leq a \leq n$.

For every $n \in \mathbb{N}$, let

$$W_n(x) := \frac{1}{n^{1/6}} \sum_{a=0}^{[n^{1/3}x]} \varsigma_n(a), \quad x \geq 0.$$
$W_n$ converges to a Brownian motion $W$ with variance $\sigma^2$ as $n \to \infty$ in the Skorokhod topology. Moreover, if $\varphi_i$ and $\varphi_i^{(n)}$ ($1 \leq i \leq k$) are as in Assumption NT3, then

$$\lim_{n \to \infty} \left( \sum_{a \in \mathbb{N}_0} \varphi_i^{(n)}(a/n^{1/3}) \frac{\varsigma_n(a)}{n^{1/6}} \right)_{1 \leq i \leq k} = \left( \int_{\mathbb{R}_+} \varphi_i(a) \, dW(a) \right)_{1 \leq i \leq k}$$

(A.4.1)

in distribution jointly with $W_n \to W$.

**Proof.** This result follows from the same arguments used in the proof of [GS18b, Proposition 4.4] (more specifically, see [GS18b, (4.20)–(4.23) and (4.33)–(4.35)]). We note that the random variables denoted as $\zeta = \zeta_n$ in [GS18b] correspond to $\zeta_n(a) := \varsigma_n(n - a)$ in our setting. Moreover, we note that in [GS18b, Proposition 4.4], the joint convergence (A.4.1) is only explicitly stated for the case where $\varphi_i$ is the local time of independent Brownian motions and $\varphi_i^{(n)}$ its random walk approximation. However, the proof of [GS18b, Proposition 4.4] only uses the fact that the $\varphi_i$ are continuous and compactly supported and that the $\varphi_i^{(n)}$ converge uniformly, hence the result. \qed

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