On containment relations in directed graphs

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Abstract

Containment relations in graphs such as induced subgraphs, minors, or immersions can be naturally extended to the world of directed graphs. In this thesis, we present new results on several containment relations in digraphs.

The first result is on *butterfly minors*. It is a containment relation in digraphs which can be considered as an extension of the minor relation in graphs. To obtain a butterfly minor, we may contract edges that do not yield "new" directed cycles. This relation was first introduced in [12]. We prove that the class of all semi-complete digraphs is well-quasi-ordered under this containment relation.

The second result is on *strong minors*. It is another containment relation in digraphs which also can be considered as an extension of the minor relation in graphs. To obtain a strong minor, we may contract a strongly-connected subdigraph to a vertex. We prove that the class of all semi-complete digraphs is well-quasi-ordered under this containment relation. We also prove that the same wqo statements fail to hold for some slightly larger classes of digraphs containing all semi-complete digraphs.

The third result is on *immersions*. A digraph is *immersed* in another if the vertices of the first digraph are mapped to vertices of the second, and edges of the first to directed paths of the second, joining the corresponding pairs of vertices and pairwise edge-disjoint. For digraphs, determining if a fixed digraph can be immersed in an input digraph is a hard problem in general. However, for some small fixed digraphs, we can test this in polynominal time. We prove the existence of such algorithms.

The fourth result is on subtournaments. A set of tournaments $\mathcal{T}$ is said to be *heroic* if every tournament, not containing any member of $\mathcal{T}$ as a subtournament, has bounded chromatic number. (The chromatic number of a tournament is the smallest number of colors needed to color the vertices of the tournament so that there is no monochromatic cyclic triangle.) In particular, if a heroic set has size one, then the element of the set is called a *hero*. In [1], the authors were able to construct all heroes. Here, we present some results on general heroic sets.

Acknowledgements

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To my parents.

Contents

List of Figures

Definitions and Notations

Here, we give definitions and notations that will be used throughout the thesis. Readers familiar with graph terminologies may skip this section. Non-trivial definitions will be repeated.

We only deal with finite graphs and digraphs in this thesis, but sometimes we allow loops or parallel edges. Let G be a graph. We denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. We write $e = uv \in E(G)$ if u and v are two end vertices of e. We say an edge e is a *loop* if two ends of e are the same. We say two non-loop edges e and f are *parallel* if two ends of e are the ends of f. We say G is *simple* if it does not have any loops or parallel edges. For two vertices u and v, we say v is a *neighbor* of u or v is *adjacent* to u if there exists an edge uv in G. We denote by $N_G(u)$ the set of neighbors of u in G. The *degree* of u in G is the number of neighbors of u in G, and denoted by $d_G(u)$. We denote by $\delta_G(u)$ the set of edges incident with u in G. For a set $X \subseteq V(G)$ of vertices, $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$. For two disjoint sets $X, Y \subseteq V(G)$ of vertices, we denote by $\delta_G(X, Y)$ the set of all edges such that one end is in X and the other end is in Y. We say X is *complete to* Y if every vertex in X is adjacent to every vertex in Y.

Let G be a digraph. We write $e = uv \in E(G)$ if u is the tail, and v is the head of e. For an edge uv, we say v is *out-adjacent from* u, and u is *in-adjacent to* v. In particular, uv and vu are distinct edges in a digraph. Sometimes we say uv is *leaving* u and *entering* v. We say an edge e is a *loop* if two ends of e are the same. We say two non-loop edges e and f are *parallel* if the tail of e is the tail of f, and the head of e is the head of f. We say two non-loop edges e and f are *anti-parallel* if the tail of e is the head of f, and the head of e is the tail of f. We say G is *simple* if G has no loops or parallel edges.[1] For two vertices u and v, we say v is an *out-neighbor* of u if there exists an edge uv in G, and we say v is an *in-neighbor* of u if there exists an edge vu in G. We denote by $N_G^+(u)$ the set of out-neighbors of u in G, and by $N_G^-(u)$ the set of in-neighbors of u in G. The *out-degree* of u in G is the number of out-neighbors of u in G, and denoted by $d_G^+(u)$. Similarly, the *in-degree* of u in G is the number of in-neighbors of u in G, and denoted by $d_G^-(u)$. We denote by $\delta_G^+(u)$ and $\delta_G^-(u)$ the set of edges leaving u and the set of edges entering u, respectively. For a set $X \subseteq V(G)$ of vertices, $N_G^+(X) = \bigcup_{v \in X} N_G^+(x) \setminus X$ and $N_G^-(X) = \bigcup_{v \in X} N_G^-(x) \setminus X$. For two disjoint sets $X, Y \subseteq V(G)$ of vertices, $\delta_G^+(X, Y)$ is the set of edges from X to Y, and $\delta_G^-(X, Y)$ denotes $\delta_G^+(Y, X)$. We say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y.

For a (di)graph G, we say G is a *null-(di)graph* if $|V(G)| = 0$, and G is *trivial* if $|V(G)| = 1$. (A trivial (di)graph may have loops). For a (di)graph G and a vertex $v \in V(G)$, $G \setminus v$ denotes the

[1] A simple digraph may have anti-parallel edges.

(di)graph obtained from G by deleting the vertex v and all edges incident with v. Similarly, for a (di)graph G and an edge $e \in V(G)$, $G \setminus e$ denotes the (di)graph obtained from G by deleting the edge e (not deleting two ends of e). For a set of vertices $X \subseteq V(G)$, $G \setminus X$ denotes the (di)graph obtained from G by deleting all vertices in X, and $G|X$ denotes $G \setminus (V(G) \setminus X)$. Sometimes for an ordering σ of X, we write $G|\sigma$ for $G|X$. For two (di)graphs G and H, we say G is an *induced sub(di)graph* of H if G can be obtained from H by deleting some vertices. For two (di)graphs G and H, we say G is a *sub(di)graph* of H if G can be obtained from H by deleting some vertices and edges.

For two graphs G and H, we say G is *isomorphic* to H if there exists a bijection from $V(G)$ to $V(H)$ such that for every pair u, v (not necessarily distinct) of vertices of G, the number of edges between u and v are the same as the number of edges between $\phi(u)$ and $\phi(v)$. Similarly for two digraphs G and H, we say G is *isomorphic* to H if there exists a bijection from $V(G)$ to $V(H)$ such that for every pair u, v of vertices of G, the number of edges from u to v are the same as the number of edges from $\phi(u)$ to $\phi(v)$.

Let G be a non-null simple graph with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$ where $n \geq 1$. We call such a graph G a *path*. The vertices v_1 and v_n are said to be *ends* of the path, and say G is a path joining v_1 and v_n or G is a path from v_1 to v_n. The *length* of the path is $|E(G)|$. A non-null graph G is said to be *connected* if for every pair u, v of vertices of G, there exists a path from u to v in G. A *component* of G is a maximal connected subgraph of G. Let G be a non-null graph with $V(G) = \{v_1, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ where $n \geq 1$. We call such a graph G a *cycle*. The *length* of the cycle is $|E(G)|$. Here, we treat a loop as a cycle of length one, and two parallel edges as a cycle of length two. A cycle of length three is called a *triangle*. A non-null simple graph G is called a *forest* if no subgraph of G is a cycle. A forest G is called a *tree* if G is connected. A vertex in a tree is called a *leaf* if it has degree one. A tree G is called a *star* if it has $|V(G)| - 1$ many leaves. The *center* of a star is the non-leaf vertex of the star. A *doubled tree* is a graph obtained from a tree by doubling every edge. In other words, every edge has multiplicity two. A non-null simple graph G is called a *complete graph* if for every pair $u, v \in V(G)$ with $u \neq v$, $uv \in E(G)$. We denote by K_n the complete graph with n vertices. For a graph H, a *clique* is the set of vertices of H pairwise adjacent, and a *stable set* is the set of vertices of H pairwise non-adjacent. The *clique number* $\omega(G)$ is the maximum clique size in G, and the *stability number* $\alpha(G)$ is the maximum stable set size in G. The chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. In other words, $\chi(G)$ is the smallest number of stable sets that $V(G)$ can be partitioned. We say a simple graph G is *bipartite* if $\chi(G) \leq 2$. For a bipartite graph G, a *bipartition* of G is a partition (A, B) of $V(G)$ such that A and B are stable sets in G. A *complete bipartite graph* is a bipartite graph with a bipartition (A, B) such that A is complete to B. We denote by $K_{n,m}$ the complete bipartite graph with $|A| = n$ and $|B| = m$.

We say a non-null simple graph H is *perfect* if $\chi(G) = \omega(G)$ for every induced subgraph G of H. A non-null simple graph G is called a *permutation graph* if for $V(G) = \{v_1, \ldots, v_n\}$, there exists an ordering σ of $V(G)$ so that $v_iv_j \in E(G)$ if and only if $i < j$ and v_i comes before v_j in σ-order. A *partial order* is a reflexive, antisymmetric, and transitive relation. A *quasi-order* is a reflexive and transitive relation. For a partial order or a quasi-order $\preceq$, we say two elements u and v are *comparable* if either $u \preceq v$ or $v \preceq u$, and otherwise we say u and v are *incomparable*. For a given set S with a partial order or a quasi-order, a set $X \subseteq S$ is called a *chain* if elements of X are pairwise comparable, and an *anti-chain* if elements of X are pairwise incomparable. A non-null simple graph G is called a *comparability graph* if there exists a partial order on $V(G)$ such that $uv \in E(G)$ if and only if u and v are comparable. A permutation graph is a comparability graph, and a comparability graph is perfect [10]. For a simple graph G, the *complement* $\overline{G}$ of G is the simple graph with $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Let G be a non-null simple digraph with $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$ where $n \geq 1$. We call such a digraph G a *directed path*. The *length* of the directed path is $|E(G)|$. The vertex v_1 is said to be the *source* of the path and the vertex v_n is said to be the *terminal* of the path, and say G is a directed path from v_1 to v_n. The *length* of the path is $|E(G)|$. A non-null digraph G is said to be *weakly-connected* if the underlying graph is connected. A *weak-component* of G is a maximal weakly-connected subdigraph of G. A non-null digraph G is said to be *strongly-connected* if for every pair u, v of vertices of G, there exists a directed path from u to v. A *strong component* of G is a maximal strongly-connected subdigraph of G. A non-null digraph G is called a *directed cycle* if the vertices of G can be ordered as $v_1, \ldots, v_n$ such that $E(G) = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. The *length* of the directed cycle is $|E(G)|$. Here we treat a loop as a directed cycle of length one, and two anti-parallel edges as a directed cycle of length two. A directed cycle of length three is called a *cyclic triangle*. A digraph G is called *acyclic* if no subdigraph of G is a directed cycle. For every acyclic digraph, there exists an ordering of $V(G)$ such that every edge goes backward with respect to the ordering. For an integer $n \geq 1$, we denote by A_n an acyclic digraph with $2n$ vertices such that the underlying undirected graph is a cycle of length $2n$, so that every vertex has either out-degree zero or in-degree zero. We call this A_n an *alternating* cycle.

A digraph G (not necessarily simple) is called *supertournament* if either $uv \in E(G)$ or $vu \in E(G)$

for every distinct $u, v \in V(G)$. A simple supertournament G is called *semi-complete digraph.* A semi-complete digraph G is called a *tournament* if exactly one of uv and vu is an edge of G for every distinct $u, v \in V(G)$. For a tournament H, an induced subdigraph of H is called a *subtournament.* A semi-complete digraph G is called a *complete digraph* if both uv and vu are edges for every distinct $u, v \in V(G)$. A *transitive tournament* is an acyclic tournament. For a tournament H and $X \subseteq V(H)$, if $H|X$ is a transitive subtournament of H, then X is called a *transitive set.* A tournament is transitive if and only if it does not contain a cyclic triangle as a subtournament. A transitive tournament with three vertices is called a *transitive triangle.*

For a tournament G and an ordering σ of $V(G)$, we denote by G_σ the undirected graph with $V(G_\sigma) = V(G)$ and $uv \in V(G_\sigma)$ if and only if $uv \in E(G)$ and u comes after v in σ-order. For a tournament G, the clique number $\omega(G)$ is $\min_\sigma \omega(G_\sigma)$.[2] For a digraph G, the *stability number* $\alpha(G)$ is the stability number of the underlying undirected graph of G. For instance, a supertournament has stability number one. For a tournament G, the chromatic number $\chi(G)$ is the minimum number of colors needed to color the vertices of G so that there is no monochromatic cyclic triangle. In other words, $\chi(G)$ is the smallest number of transitive sets that $V(G)$ can be partitioned.

Let G be a sub(di)graph of H. We denote by H/G, the (di)graph obtained from H by *contracting* G. More precisely, we obtain H/G from H by identifying G to a single vertex (say w), and for every $u \in V(H) \setminus V(G)$ and $v \in V(G)$, replace all edges uv and vu with uw and wu, respectively. For some $X \subseteq V(H)$, sometimes we write H/X for $H/(H|X)$. Moreover, when G is a one-edge (di)graph $(\{u, v\}, \{uv\})$, we just write H/uv for H/G.

For graphs G, H, we say G is a *minor* of H if there exists a map ϕ on $V(G)$ such that:

- $\phi(v)$ is a non-null connected subgraph of H for each $v \in V(G)$,
- for distinct $u, v \in V(G)$, $\phi(u)$ and $\phi(v)$ are vertex-disjoint,
- for every $u, v \in V(G)$ (not necessarily distinct), if there are k edges in G with two ends u and v, then there are at least k edges in H with one end in $V(\phi(u))$ and the other end in $V(\phi(v))$, and not contained in $E(\phi(x))$ for every $x \in V(G)$.

For graphs G, H, we say G can be *infused* in H or G is an *infusion* of H if there exists a map ϕ such that:

- $\phi(v) \in V(H)$ for each $v \in V(G)$

[2] The definition of tournament clique number is not common. Stéphan Thomassé pointed out that for some tournament, "fractional chromatic number" can be less than its clique number, which is not the case for graphs.[26]

- for each edge $e = uv$ of G, $\phi(e)$ is a (directed) path of H from $\phi(u)$ to $\phi(v)$
- if $e, f \in E(G)$ are distinct, then $\phi(e)$, $\phi(f)$ have no edges in common (they may share vertices).

If we add the condition

- $\phi(u) \neq \phi(v)$ for distinct $u, v \in V(G)$

we call the relation *weak immersion*. Moreover, a weak immersion is called *a strong immersion* if it additionally satisfy the condition

- if $u \in V(G)$ and $e \in E(G)$, and e is not incident with v in G, then $\phi(v)$ is not a vertex of the path $\phi(e)$.

A set $I \subseteq \mathbb{Z}$ of integers is said to be an *integer interval* if $I = [a, b] \cap \mathbb{Z}$ for some real numbers a, b. For two sets A, B, the *symmetric difference* $A \Delta B$ is $(A \setminus B) \cup (B \setminus A)$. For a map $\phi : A \to B$ and a set $X \subseteq A$, $\phi|X$ denotes the map from X to B defined by $\phi|X(x) = \phi(x)$.

Chapter 1

Introduction

1.1 Digraphs and Well-Quasi-Order

The theory of minors is quite developed for graphs. One of the deepest theorems in graph minors theory is that the class of all graphs is "well-quasi-ordered" under minor relation due to the celebrated work of Neil Robertson and Paul Seymour [23]. A *quasi-order* $Q = (E(Q), \leq_Q)$ consists of a class $E(Q)$ and a reflexive transitive relation $\leq_Q$ on $E(Q)$. A quasi-order Q is called a *well-quasi-order* or *wqo* if for every infinite sequence $q_1, q_2, \ldots$ of elements of $E(Q)$, there exist $j > i \geq 1$ such that $q_i \leq_Q q_j$. Robertson and Seymour proved the following wqo statement for graphs through more than 20 papers.

Theorem 1.1.1 *The class of all graphs is a wqo under minor containment.*

They also proved in [24] that the class of all graphs is a wqo under weak immersion.

Theorem 1.1.2 *The class of all graphs is a wqo under weak immersion.*

Theorem 1.1.1 implies that for every minor closed property, there are only finitely many minor-minimal graphs without the property. (It is not hard to see that the same statement fails to hold for other containment relation in graphs such as induced subgraphs, subgraphs, or topological minors.) For instance, the property of being planar is a minor-closed property, and indeed a theorem of Wager [27] says that there are two minor-minimal non-planar graphs, namely K_5 and $K_{3,3}$. Also, since there is a polynomial time minor-testing algorithm ([22], [14]), any minor-closed property can be tested in a polynomial time.

What about digraphs? Is the class of all digraphs a wqo under some containment relation? Of course, the answer is no if the relation is induced subdigraphs or subdigraphs, since the answer is negative even for graphs. The answer remains to be negative even when we restrict our world to the class of all tournaments. Therefore, if we hope to prove a wqo statement, we need to restrict the class even more. For instance, Latka proved in [18] that the class of all tournaments omitting a certain tournament is a wqo under subtournaments.

One can ask the same question for other containment relations in digraphs. For instance, immersion relation for graphs can be naturally extended to digraphs by simply replacing paths with directed paths. However, the analogue of 1.1.2 fails to hold for general digraphs. But this time, if we restrict our world to the class of all tournaments, then it is a wqo under immersion. In [5], Chudnovsky and Seymour proved the following.

Theorem 1.1.3 *The class of all tournaments is a wqo under strong immersion.*

Then, what about minors? For minors, it is not even clear how we should extend it to digraphs. Because in digraphs, contracting an edge may yield a directed cycle, even starting from an acyclic digraph. This seems undesirable for a theory of excluded minors. One way to avoid this is to only permit the contraction of certain edges. For example, if an edge uv is the only edge with tail u or the only edge with head v, then contracting uv does not yield new directed cycles. Another way to avoid the issue is to define a minor relation via contracting strongly-connected subdigraphs. Contracting a strongly-connected subdigraph does not yield new directed cycles, either.

We will call the relation defined by using the former concept butterfly minors[1], and we will call the relation defined by using the latter concept strong minors. Both relations can be considered as extensions of undirected minors.

Not surprisingly, the analogue of 1.1.1 for digraphs with butterfly minors is false, and the same with strong minors is also false. However, again if we restrict our world to the class of all semi-complete digraphs, then they become true. We will prove the following theorems in Chapter 2 and Chapter 3.

Theorem 1.1.4 *The class of all semi-complete digraphs is a wqo under butterfly minors.*

[1]The notion of butterfly minors was first introduced in [12].

Theorem 1.1.5 *The class of all semi-complete digraphs is a wqo under strong minors.*

In fact, there is a sense that the class of all semi-complete digraphs is a "maximal" class of digraphs that the wqo statement holds with respect to strong minors. We consider two natural generalizations of the class of all semi-complete digraphs, and prove that the wqo statements fail to hold for both of them.

1.2 Testing for immersion

In [2], Chudnovsky, Fradkin, and Seymour proved that for every fixed digraph H, there is a polynomial time algorithm to test whether H can be strongly (or weakly) immersed in a given semi-complete digraph.

However, if the input is a general digraph and H is large enough, then the problem is NP-complete. For example, let H be a digraph with two vertices v_1, v_2 and four edges, namely a loop at v_1, a loop at v_2, and edges v_1v_2, v_2v_1. Then testing H-immersion is a NP-complete problem [2].

Although it is hopeless to find an efficient immersion testing algorithm for digraphs, if the target digraph is small enough, then sometimes there is a polynomial time immersion testing algorithm. For integers, $i, j \geq 0$, we denote by $I_{i,j}$ the digraph with two vertices v_1, v_2 and i parallel edges from v_1 to v_2, and j parallel edges from v_2 to v_1. For an integer $k \geq 1$, we denote by A_k the acyclic digraph with $2k$ vertices such that the underlying undirected graph is a cycle of length $2k$, and every vertex has either out-degree zero or in-degree zero. We call this A_k an *alternating* cycle. In chapter 4, we prove that testing $I_{1,2}$-immersion and testing A_k-immersion (for fixed k) can be done in polynomial time. The following theorem is the main result in Chapter 4.

Theorem 1.2.1 *Let k be an integer. Then there exists an integer t such that for every digraph G, either A_k can be infused in G or G has cutwidth at most t.*

1.3 Tournaments and coloring

Let $\mathcal{G}$ be a set of tournaments, we say a set $\mathcal{T}$ of tournaments is *heroic in* $\mathcal{G}$ if every tournament in $\mathcal{G}$, not containing any member of $\mathcal{T}$ as a subtournament, can be partitioned into a bounded number of transitive tournaments. If $\mathcal{G}$ is the class of all tournaments, then we just say $\mathcal{T}$ is heroic. In particular, if $\mathcal{T}$ has size one, then the element of $\mathcal{T}$ is called a *hero*. In [1], the authors explicitly construct all heroes. In chapter 5, we study finite heroic sets.

A heroic set can be defined for graphs as well [6]. A set of graphs $\mathcal{H}$ is called *heroic* if every graph, not containing any member of $\mathcal{H}$ as an induced subgraph, has a bounded cochromatic number. It is an open conjecture independently proposed by Gyárfás and Sumner that:

Conjecture 1.3.1 *Let K be a clique and T be a tree. Then the set $\{K, T\}$ is heroic.*

Assuming 1.3.1, Chudnovsky and Seymour characterized all finite heroic sets in [6].

Theorem 1.3.2 *If 1.3.1 is true, then a finite set of graphs is heroic if and only if it contains a clique partition graph, a complete multipartite graph, a forest, and the complement of a forest.*

In chapter 5, we propose a conjecture, which is an analogue of 1.3.2 for tournaments. Specifically, we introduce four classes of tournaments that every finite heroic set must intersect. We will see that $\{H_1, H_2\}$ is heroic in some large classes of tournaments where H_1 and H_2 are tournaments from the first class and the second class, respectively. We also show some examples of heroic sets.

1.4 Acknowledgement

The result on strong minors in Chapter 3 is a joint work with Paul Seymour, and the material is submitted for publication [15]. The result on immersion in Chapter 4 is a joint work with Maria Chudnovsky and Paul Seymour, and the coloring result in Chapter 5 is a joint work with Maria Chudnovsky, Ringi Kim, and Paul Seymour.

Chapter 2

Butterfly Minors

2.1 Introduction

In this chapter, we introduce a containment relation in digraphs which can be considered as an extension of the minor relation in graphs, and we prove that the class of all semi-complete digraphs is well-quasi-ordered under this containment relation. We call this relation *butterfly minors.* This concept was first introduced in [12]. To obtain a butterfly minor, we may contract edges that do not yield "new" directed cycles.

In digraphs, contracting an edge may yield a directed cycle, even starting from an acyclic digraph, and this seems undesirable for a theory of excluded minors (see Figure 2.1).

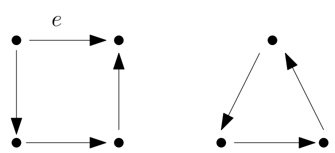

Figure 2.1: Contracting e yields a "new" directed cycle.

One way to avoid this is to only permit the contraction of certain edges. For example, if an edge uv is the only edge with tail u or the only edge with head v, then contracting uv does not yield a new directed cycle. We use this concept to define butterfly minor relation. Another way to avoid the issue is to define a minor relation via contracting strongly-connected subgraphs. We will discuss about this in next chapter.

An edge uv of a digraph is said to be *contractible* if it is the only edge with tail u or the only edge with head v.

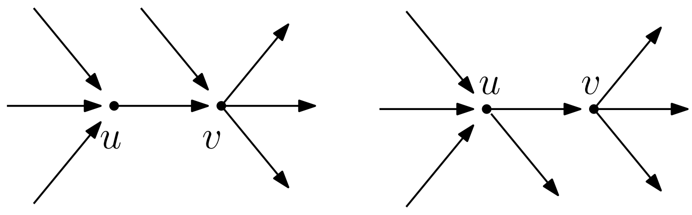

Figure 2.2: Contractible edges

Then, contracting a contractible edge does not yield "new" directed cycles.

Theorem 2.1.1 *Let H be a digraph, and let $e = uv$ be a contractible edge in H. Let ϕ be a function on the set of all directed cycles in H defined by $\phi(C) = C/e$. Then every directed cycle of H/e is an image of ϕ.*

Proof. Let w be the contracted vertex in H/e, and let C be a directed cycle of H/e. We may assume $w \in V(C)$, since otherwise C is a directed cycle in H ($\phi(C) = C/e = C$). If C is a loop (say f) at w, then e and f are anti-parallel in H since e is contractible in H. Therefore $\phi(C') = C/e = C$ where C' is the directed cycle with two edges e and f in H. Therefore we may assume the length of C is at least two. Let aw, wb be the two edges in C incident with w. Then, either

1. $au, ub \in E(H)$ or,
2. $av, vb \in E(H)$ or,
3. $au, vb \in E(H)$ or,
4. $av, ub \in E(H)$.

For the first two cases, the cycle C is still in H, except that either u or v plays the roll of w, and hence C is an image of ϕ. For the third case, take the union of $C \setminus w$ and the directed three-edge-path from a to b (au, uv, vb) in H. Then this union C' is a directed cycle in H, and $\phi(C') = C/e = C$. For the last case, notice that there is an edge leaving u different from uv, namely ub. Similarly, there is an edge entering v different from uv, namely av. This contradicts the fact that uv is contractible. This proves 2.1.1. ∎

Therefore contracting a contractible edge does not yield a new directed cycle, and this suggests a notion of minors in digraphs. Before we give precise definition of the butterfly minor relation, let us look at several ways of defining the minor relation in graphs. For two graphs G and H,

- $G \preceq_1 H$ if G can be obtained from a subgraph of H by repeatedly contracting an edge to a vertex.
- $G \preceq_2 H$ if G can be obtained from a subgraph of H by repeatedly contracting a connected subgraph to a vertex.
- $G \preceq_3 H$ if there exists a function ϕ on $V(G)$ such that for every $v \in V(G)$, $\phi(v)$ is a non-null connected subgraph of H, and if $u, v \in V(G)$ and $u \neq v$, then $\phi(u)$ and $\phi(v)$ are vertex-disjoint, and for every $u, v \in V(G)$ (not necessarily distinct), if there are k edges in G with two ends u and v, then there are at least k edges in H with one end in $V(\phi(u))$ and the other end in $V(\phi(v))$, and not contained in $E(\phi(x))$ for every $x \in V(G)$.

It is obvious that $G \preceq_1 H$ implies $G \preceq_2 H$. Conversely, "contacting a connected subgraph to a vertex" can be done by repeatedly contracting all edges in the connected subgraph. Therefore $G \preceq_1 H$ if and only if $G \preceq_2 H$. And it is easy to see that $G \preceq_3 H$ is another expression of $G \preceq_2 H$ by using the map ϕ, that tells us which connected subgraph of H corresponds to which vertex in G. Therefore $G \preceq_1 H$ if and only if $G \preceq_2 H$ if and only if $G \preceq_3 H$.

Notice that for graphs, we can arbitrarily switch the order of deleting operations and contracting operations. However, for digraphs, we cannot arbitrarily switch the order because some edges are not contractible until we delete some appropriate edges, and also, a contractible edge e in H may not be contractible in H/f for some contractible edge f (see Figure 2.3). But we can always delete first, and contract later because a contractible edge e in H is contractible in every subdigraph of H containing e.

Now, we define the butterfly minor relation. For a digraph H, a weakly-connected subdigraph G is said to be *contractible in* H if $|E(G)| = |V(G)| - 1$ and the edges of G can be numbered as $e_1, \ldots, e_{|E(G)|}$ such that e_1 is contractible in H, and each e_i is contractible edge in $H/e_1 \ldots /e_{i-1}$ ($i = 2, \ldots, |E(G)| - 1$). For two digraphs G and H,

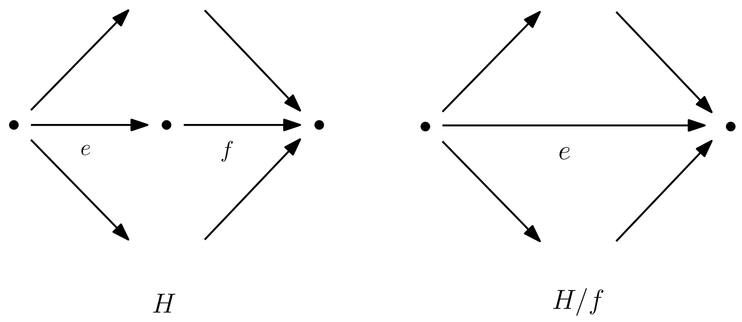

Figure 2.3: e, f are contractible in H, but e is not contractible in H/f.

- $G \preceq_4 H$ if G can be obtained from a subdigraph H' of H by repeatedly contracting a contractible edge to a vertex.
- $G \preceq_5 H$ if G can be obtained from a subdigraph H' of H by repeatedly contracting a contractible subdigraph to a vertex.
- $G \preceq_6 H$ if there exists a function ϕ on $V(G)$ and a subdigraph H' of H such that for every $v \in V(G)$, $\phi(v)$ is a contractible subdigraph in H', and if $u, v \in V(G)$ and $u \neq v$, then $\phi(u)$ and $\phi(v)$ are vertex-disjoint, and for every $u, v \in V(G)$ (not necessarily distinct), if there are k edges in G with tail u and head v, then there are at least k edges in H with head in $V(\phi(u))$ and tail in $V(\phi(v))$, and not contained in $E(\phi(x))$ for every $x \in V(G)$.

A contractible edge in H' is a contractible subdigraph in H', therefore $G \preceq_4 H$ implies $G \preceq_5 H$. Conversely, "contacting a contractible subdigraph to a vertex" can be done by repeatedly contracting every edge in the contractible subdigraph by definition. Therefore $G \preceq_4 H$ if and only if $G \preceq_5 H$. And again, it is easy to see that $G \preceq_6 H$ is another expression of $G \preceq_5 H$ by using the map ϕ, that tells us which contractible subdigraph of H corresponds to which vertex in G. Therefore, $G \preceq_4 H$ if and only if $G \preceq_5 H$ if and only if $G \preceq_6 H$. We say G is a *butterfly minor* of H if one of $G \preceq_4 H$, $G \preceq_5 H$, and $G \preceq_6 H$ holds.

For (undirected) minors, Robertson and Seymour proved that the following wqo statement for graphs in [23].

Theorem 2.1.2 *The class of all graphs is a wqo under minor containment.*

For digraphs, the analogue of 2.1.2 is not true with respect to butterfly minor relation. We will see an example of infinite antichain in the following section. However, if we restrict the class of all digraphs to the class of all semi-complete digraphs, then the butterfly minor relation defines a wqo. Therefore the same is true for the class of all tournaments.

Theorem 2.1.3 *The class of all semi-complete digraphs is a wqo under butterfly minor containment.*

In [5], Chudnovsky and Seymour proved the class of all semi-complete digraphs is a wqo under immersion relation, by using a digraph parameter called "cut-width". Here, we use another parameter called *path-width* in order to prove the analogous statement for butterfly minors. Path-width for undirected graphs was introduced in [21], and it has a natural extension to digraphs, discussed for instance in [9]. We will see that it is useful for the proof of 2.1.3.

2.2 Directed path-width

Before we discuss about path-decompositions and path-width for digraphs, let us see the definitions of them for graphs. Roughly speaking, a path-decomposition of a graph G is a representation of G as a "thickened" path graph, and the path-width of G measures how much the path was thickened to form G. More precisely, for a graph G, we say $P = (W_1, \ldots, W_r)$ is a *path-decomposition* of G if:

- $r \geq 1$ and $\bigcup_{1 \leq i \leq r} W_i = V(G)$,
- *(betweenness condition)* for $1 \leq h < i < j \leq r$, $W_h \cap W_j \subseteq W_i$, and
- *(cut condition)* if $uv \in E(G)$, then some W_i contains both u and v.

The betweenness condition can be replaced by the following equivalent statement.

- for each $v \in V(G)$, $\{i\ : v \in W_i\}$ is an integer interval.

Also, the cut condition can be replaced by the following equivalent statement.

- for each i with $1 < i < r$, there is no edge between $\bigcup_{h<i} W_h \setminus W_i$ and $\bigcup_{j>i} W_j \setminus W_i$ in G.

For a given path-decomposition P of a graph G, we define the *path-width* of P as

$$\max_{1 \leq i \leq r} |W_i| - 1,$$

and denote it by $pw(P)$. The *path-width* of G is the minimum $pw(P)$ over all path-decompositions P of G, and denoted by $pw(G)$. For example, if G is a path of length at least one, then $pw(G) = 1$.

We can naturally extend these concepts to digraphs. For a digraph G, we say $P = (W_1, \ldots, W_r)$ is a *path-decomposition* of G if:

- $r \geq 1$ and $\bigcup_{1 \leq i \leq r} W_i = V(G)$,
- *(betweenness condition)* for $1 \leq h < i < j \leq r$, $W_h \cap W_j \subseteq W_i$, and
- *(cut condition)* if $uv \in E(G)$, then there exist i, j with $1 \leq i \leq j \leq r$ such that $u \in W_j$ and $v \in W_i$.

Again, the betweenness condition can be replaced by the following equivalent statement.

- for each $v \in V(G)$, $\{i \ : v \in W_i\}$ is an integer interval.

Also, the cut condition can be replaced by the following equivalent statement.

- for each i with $1 < i < r$, there is no edge from $\bigcup_{h<i} W_h \setminus W_i$ to $\bigcup_{j>i} W_j \setminus W_i$ in G.

(Notice that there can be "backward" edges from $\bigcup_{j>i} W_j \setminus W_i$ to $\bigcup_{h<i} W_h \setminus W_i$ in G.)

For a given path-decomposition P of a digraph G, we define the *path-width* of P as

$$\max_{1 \leq i \leq r} |W_i| - 1,$$

and denote it by $pw(P)$. The *path-width* of G is the minimum $pw(P)$ over all path-decompositions P of G, and denoted by $pw(G)$. For example, a non-null loopless digraph G is acyclic if and only if $pw(G) = 0$ (the vertices of an acyclic digraph can be ordered such that there is no "backward" edge).

For the purpose of this chapter, we focus on the connection between the path-width parameter and the minor relation. First, we see that having bounded path-width is a minor-closed property, and then we will see that the same is true for directed path-width and butterfly minors.

Theorem 2.2.1 *If a graph has path-width at most k, then so do all its minors.*

Proof. Let $P = (W_1, \dots, W_r)$ be a path-decomposition of a graph G with $pw(P) \le k$. Then P is a path-decomposition of $G \setminus e$ for each edge $e \in E(G)$, and $(W_1 \setminus v, \dots, W_r \setminus v)$ is a path-decomposition of $G \setminus v$ for each vertex $v \in V(G)$. Therefore path-width does not increase by deletions.

Thus, it remains to show that if G has path-width at most k, then so does G/e for each $e = uv \in E(G)$. Let w be the contracted vertex in G/e, and let $I_u = \{i \ : u \in W_i\}$ and $I_v = \{i \ : v \in W_i\}$. Then, $I_u \cap I_v \ne \emptyset$ from the cut condition for P. Therefore $I_u \cup I_v$ is an integer interval. Now define W_i' for each $i = 1, \dots, r$ as follows.

$$W_i' = \begin{cases} \big(W_i \setminus \{u, v\}\big) \cup \{w\} & \text{if } i \in I_u \cup I_v \\ W_i & \text{otherwise.} \end{cases}$$

We claim that $P' = (W_1', \dots, W_r')$ is a path-decomposition of G/e with $pw(P') \le k$. The betweenness condition holds since $I_w = \{i \ : w \in W_i'\} = I_u \cup I_v$ is an integer interval. For the cut condition, we only need to consider the edges incident with w in G/e. For every edge $wa \in E(G/e)$, the corresponding edge in G is either va or ua. By the cut condition for P, there exist some W_i containing both v and a or both u and a. Therefore W_i' contains both w and a. Finally, $pw(P') \le pw(P) \le k$ since $|W_i'| \le |W_i|$ for every $i = 1, \dots, r$. This proves 2.2.1. ∎

Theorem 2.2.2 *If a digraph has path-width at most k, then so do all its butterfly minors.*

Proof. By the same argument in the proof of 2.2.1, path-width does not increase by deletions. Therefore it is enough to show that if a digraph G has path-width at most k, then so does G/e where $e = uv$ is a contractible edge of G. Let w be the contracted vertex in G/e, and let $P = (W_1, \dots, W_r)$ be a path-decomposition of a digraph G with $pw(P) \le k$. We prove that there is a path-decomposition P' of G/e such that $pw(P') \le k$.

Let $I_u = \{i \ : u \in W_i\}$ and $I_v = \{i \ : v \in W_i\}$. First, suppose $I_u \cap I_v \ne \emptyset$. Then, for each $i = 1, \dots, r$, define W_i' by

$$W_i' = \begin{cases} \big(W_i \setminus \{u, v\}\big) \cup \{w\} & \text{if } i \in I_u \cup I_v \\ W_i & \text{otherwise.} \end{cases}$$

We claim that $P' = (W'_1, \ldots, W'_r)$ is a path-decomposition of G/e with $pw(P') \leq k$. Note that the interval $I_w = \{i\ : w \in W'_i\}$ of the new vertex w is an integer interval since $I_w = I_u \cup I_v$ and $I_u \cap I_v \neq \emptyset$. Therefore P' satisfies the betweenness condition. For the cut condition, we only need to consider the edges incident with w in G/e. For every edge $wa \in E(G/e)$, the corresponding edge in G is either va or ua. By the cut condition for P, there exist $i \leq j$ such that $W_j \ni v$ and $W_i \ni a$, or $W_j \ni u$ and $W_i \ni a$. In either case, $W'_j \ni a$ and $W'_i \ni w$, and hence wa satisfies the cut condition. Similarly, every edge $bw \in E(G/e)$ satisfies the cut condition by similar argument. Therefore P' is a path-decomposition of G/e and $pw(P') \leq pw(P)$ since $|W'_i| \leq |W_i|$ for every $i = 1, \ldots, r$.

Therefore, we may assume I_u and I_v are disjoint. Then, from the cut condition for P, $i < j$ for every $j \in I_u$ and $i \in I_v$. Now we consider two cases. Recall that either uv is the only edge leaving u or the only edge entering v. In the former case, for each $i = 1, \ldots, r$, define W'_i by

$$W'_i = \begin{cases} \big(W_i \setminus \{v\}\big) \cup \{w\} & \text{if } i \in I_v \\ W_i \setminus \{u\} & \text{otherwise.} \end{cases}$$

And in the latter case, define W'_i by

$$W'_i = \begin{cases} \big(W_i \setminus \{u\}\big) \cup \{w\} & \text{if } i \in I_u \\ W_i \setminus \{v\} & \text{otherwise.} \end{cases}$$

We claim that $P' = (W'_1, \ldots, W'_r)$ is a path-decomposition of G/e with $pw(P') \leq k$. The betweenness condition holds for P' by definition. Therefore we only need to consider the cut conditions of the edges incident with w.

First, suppose uv is the only edge leaving u in G. Then for every edge $wa \in E(G/e)$, the corresponding edge in G must be va since ua cannot be an edge of G. By the cut condition for P, there exist $i \leq j$ such that $W_j \ni a$ and $W_i \ni v$. Therefore $W'_j \ni a$ and $W'_i \ni w$. Hence, wa satisfies the cut condition. For every edge $bw \in E(G/e)$, the corresponding edge in G is either bu or bv. By the cut condition for P, there exist $i \leq j$ such that

- $W_j \ni b$ and $W_i \ni v$, or
- $W_j \ni b$ and $W_i \ni u$.

In the former case, $W'_j \ni b$ and $W'_i \ni w$ for $i \leq j$ in G/e and hence, bw satisfies the cut condition. In the latter case, take an element i' of I_v. Then, $i' < i$ because I_u and I_v are disjoint. Therefore

$W'_j \ni b$ and $W'_{i'} \ni w$ for $i' \leq j$ in G/e, and hence bw satisfies the cut condition. Therefore every edge incident with w satisfies the cut condition, and hence P' is a path-decomposition of G/e. Also, $pw(P') \leq pw(P)$ since $|W'_i| \leq |W_i|$ for every $i = 1, \ldots, r$. Therefore, we are done if uv is the only edge leaving u in G.

Second, suppose uv is the only edge entering v in G. Let $\overline{G}$ be the digraph obtained from G by reversing all edges. Then, in $\overline{G}$, vu is the only edge leaving v, and $\overline{P} := (W_r, \ldots, W_1)$ is a path-decomposition of $\overline{G}$. Therefore this is exactly the previous case with $\overline{G}$ and $\overline{P}$ except u and v are switched. Since $pw(G) = pw(\overline{G})$, we are done for the second case as well. This proves 2.2.2. ▮

We introduce an application of 2.2.2 before we discuss further about directed path-width.

2.2.1 The falsity of the wqo statement for general digraphs

Here, we introduce a counterexample of the well-quasi-order statement for general digraphs with respect to butterfly minor relation.

A simple digraph G with $|V(G)| = n \geq 5$ is called a *cycle squared* of order n if $V(G)$ can be numbered as $v_1, \ldots, v_n$ such that

- $v_i v_j \in E(G)$ if $j - i \equiv 1 \ (mod\ n)$,
- $v_i v_j \in E(G)$ if $i - j \equiv 2 \ (mod\ n)$, and
- there is no other edge in G.

The following figure is an example of a cycle squared. The edges of the outer cycle (short edges) form a clockwise directed cycle, and the other edges (long edges) all go counter-clockwise.

This particular digraph has path-width three if n is sufficiently large, but every proper butterfly minor of it has path-width less than three. Therefore a long cycle squared does not contain a small cycle squared as a butterfly minor, and hence they form an infinite antichain with respect to butterfly minor relation.

Theorem 2.2.3 *For each integer $n \geq 9$, let D_n be a cycle squared of order n. Then, D_j does not contain D_i as a butterfly minor for every i, j with $9 \leq i < j$.*

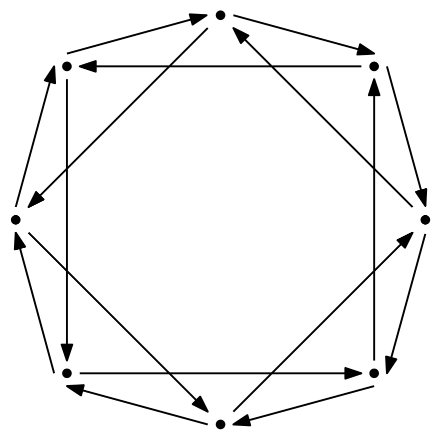

Figure 2.4: A cycle squared of order eight

We first prove a lemma.

Lemma 2.2.4 *Let G be a digraph with $pw(G) < 3$ and $|V(G)| \geq 9$. Then, there exists $X \subseteq V(G)$ with $|X| \leq 2$ such that $V(G) \setminus X$ can be partitioned into A and B with $|A|, |B| \geq 3$ so that there is no edge from A to B in $G \setminus X$.*

Proof. Let $P = (W_1, \ldots, W_r)$ be a path-decomposition of G with $pw(P) < 3$. We may assume $W_i \neq W_{i+1}$ for every $i = 1, \ldots, r-1$. Let

$$a := \min\{\, i : \ \big|\big(\bigcup_{1\leq j\leq i} W_j\big) \setminus W_i\big| \geq 3\big\},$$

$$X = W_{a-1} \ \cap \ W_a,$$

$$A = \big(\bigcup_{1\leq j\leq a} W_j\big) \setminus W_a = \big(\bigcup_{1\leq j\leq a-1} W_j\big) \setminus X, \quad \textit{and}$$

$$B = \big(\bigcup_{a\leq j\leq r} W_j\big) \setminus X = V(G) \setminus X \setminus A.$$

We claim that these X, A and B are the sets we are looking for. First, $|X| \leq 2$ since $|W_{a-1}|, |W_a| \leq 3$ and $W_{a-1} \neq W_a$. Second, $|A| \geq 3$ by definition of a. Third, by the minimality of a,

$$|A| = \big|\big(\bigcup_{1\leq j\leq a} W_j\big) \setminus W_a\big| = \big|\big(\bigcup_{1\leq j\leq a-1} W_j\big) \setminus W_{a-1}\big| + |W_{a-1} \setminus W_a| < 3 + 2.$$

Therefore $|A|$ is either three or four. Third,

$$|B| = |V(G) \setminus X \setminus A| \geq 9 - 2 - 4 = 3.$$

Finally, there is no edge from A to B by the cut condition for P. This proves 2.2.4. ∎

Now, we prove 2.2.3.

Proof of 2.2.3. We first prove that if a cycle squared is large enough, then its path-width is at least three.

(1) *$pw(D_n) \geq 3$ for every $n \geq 9$.*

Suppose $pw(D_n) < 3$ for some $n \geq 9$. Then, there exist $X, A, B \subseteq V(D_n)$ satisfying the statement of 2.2.4. In particular, $D_n \setminus X$ is not strongly-connected.

Let $v_1, \ldots, v_n$ be the numbering of the vertices of D_n such that $v_i v_j \in E(D_n)$, if and only if either $j - i \equiv 1 \ (mod\ n)$ or $i - j \equiv 2 \ (mod\ n)$. First, note that $D_n \setminus v_i$ is strongly-connected for every v_i. Therefore $|X| \neq 1$, and hence $|X| = 2$.

Without loss of generality, we may assume $v_1 \in X$. Then it is easy to check that $G \setminus X$ is not strongly-connected if and only if

$$X = \{v_1, v_4\} \ \ or \ \ \{v_1, v_{n-2}\}$$

Again, without loss of generality, we may assume $X = \{v_1, v_4\}$. Then, there are three strong components in $G \setminus X$, namely $\{v_2\}, \{v_3\}$, and $G \setminus \{v_1, v_2, v_3, v_4\}$. Then, both A and B intersect the biggest strong component $G \setminus \{v_1, v_2, v_3, v_4\}$ since both $|A|$ and $|B|$ are at least three. Therefore there exists an edge from A to B, which is a contradiction. This proves (1).

(2) *Every proper butterfly minor of D_n is a butterfly minor of $D_n \setminus e$ for some $e \in E(D_n)$.*

No edge of D_n is contractible since every vertex has out-degree two, and in-degree two. Therefore deleting operation is necessary to obtain a proper butterfly minor of D_n. This proves (2).

(3) *For each edge e of D_n, $pw(D_n \setminus e) < 3$ where $n \geq 9$.*

Without loss of generality, we may assume $e = v_1v_2$ or v_3v_1. For convenience, consider the indices $(1, \ldots, n)$ of the vertices as the elements of the additive cyclic group of order n.

First, suppose $e = v_1v_2$. For each $i = 1, \ldots, n-2$, let

$$W_i = \{v_i, v_{i-1}, v_{i-2}\}.$$

Then, $P = (W_1, \ldots, W_{n-2})$ is a path-decomposition of $D_n \setminus e$ with $pw(P) = 2$ (see Figure 2.5).

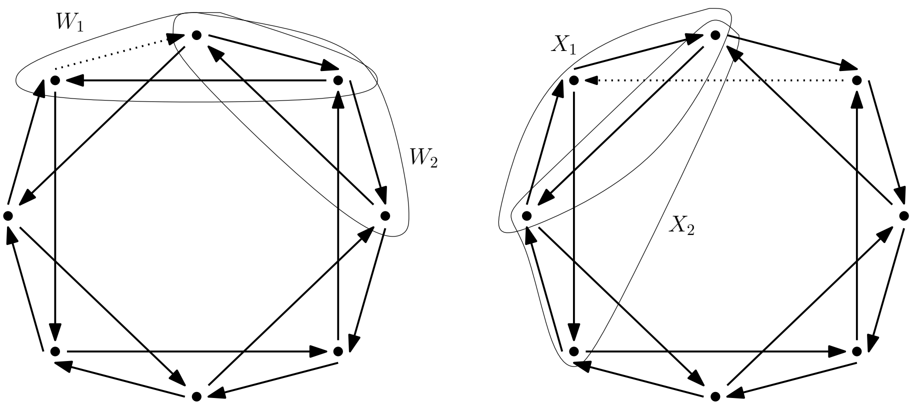

Figure 2.5: Path-decompositions P and P'.

Second, suppose $e = v_3v_1$. For each $i = 1, \ldots, n-2$, let

$$X_i = \{v_2, v_{i+2}, v_{i+3}\}.$$

Then again, $P' = (X_1, \ldots, X_{n-2})$ is a path-decomposition of $D_n \setminus e$ with $pw(P) = 2$. (Notice that W_i's "move" clockwise while X_i's "move" counter-clockwise.) This proves (3).

Now, suppose D_i is a butterfly minor of D_j for some i, j with $9 \leq i < j$. From (2), D_i is a butterfly minor of $D_j \setminus e$ for some $e \in E(D_j)$. However, $pw(D_i) \geq 3$ from (1), and $pw(D_j \setminus e) \leq 2$ from (3). This contradicts 2.2.2. This proves 2.2.3. ∎

Therefore the wqo statement for general digraphs with respect to butterfly minors is false. However, the same statement for the class of all semi-complete digraphs is true. Next, we discuss about directed path-width more in order to prove the wqo statement for semi-complete digraphs.

2.2.2 The path-width theorem for semi-complete digraphs

For graphs, a "large" forest has "large" path-width. Conversely, if a graph does not contain a fixed forest as a minor, then its path-width is bounded above by some constant that only depends on the forest. More precisely, the following is proved by Robertson and Seymour in [21].

Theorem 2.2.5 *For every set $\mathcal{G}$ of graphs, the following are equivalent:*

1. *There exists k such that every member of $\mathcal{G}$ has path-width at most k.*
2. *There exists a forest F such that no member of $\mathcal{G}$ has F as a minor.*

This is particularly useful for the proof of the wqo statement, when G_1 is a forest where the infinite sequence of graphs is $G_1, G_2, \ldots$ [21]. Because here, we only need to prove the wqo statement for graphs with bounded path-width, since we may assume no member of $\{G_2, G_3, \ldots\}$ has G_1 as a minor. For semi-complete digraphs, the corresponding G_1 does not have to be a special digraph. Essentially, this is why the proof for semi-complete digraphs is much easier than the proof for graphs.

To see this, we introduce an analogue of 2.2.5 for directed path-width and the class of all semi-complete digraphs from [9]. For a digraph G, let A, B and C be mutually disjoint subsets of $V(G)$. We say (A, B, C) is a *k-triple* if

- $|A| = |B| = |C| = k$,
- $ab \in E(G)$ for every $a \in A$ and $b \in B$,
- $bc \in E(G)$ for every $b \in B$ and $c \in C$, and
- A, C can be numbered as $\{a_1, \ldots, a_k\}$ and $\{c_1, \ldots, c_k\}$ respectively such that $c_i a_i \in E(G)$ for $i = 1, \ldots, k$.

Theorem 2.2.6 *Suppose there exists a k-triple (A, B, C) in a digraph G. Then G contains every semi-complete digraph with k vertices as a butterfly minor.*

Proof. Let $\{a_1, \ldots, a_k\}$ and $\{c_1, \ldots, c_k\}$ be the numberings of A and C such that $c_i a_i \in E(G)$ for $i = 1, \ldots, k$. Take an ordering $\{b_1, \ldots, b_k\}$ of B. We may assume $V(G) = A \cup B \cup C$ by deleting the other vertices. We also delete every edge of G except the following three types of edges.

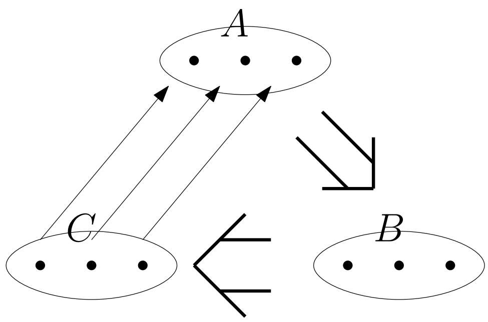

Figure 2.6: k-triple

- $a_i b_j$ for $1 \leq i, j \leq k$ and $i \neq j$,
- $b_i c_i$ for $i = 1, \ldots, k$, and
- $c_i a_i$ for $i = 1, \ldots, k$.

Then $b_i c_i$ is contractible for every $i = 1, \ldots, k$ since it is the only edge leaving b_i. We contract each of $b_i c_i$ to a vertex (say d_i). And then, each $d_i a_i$ is again contractible for every $i = 1, \ldots, k$ since it is the only edge leaving d_i. We contract each of $d_i a_i$ to a vertex (say v_i).

Now, we end up with a digraph G' with k vertices $(v_1, \ldots, v_k)$. Notice that each $a_i b_j \in E(G)$ corresponds to $v_i v_j \in E(G')$ for every pair i, j. Therefore every semi-complete digraph with k vertices is a subdigraph of G' and hence, a butterfly minor of G. This proves 2.2.6. ▮

The following theorem from [9] says a semi-complete digraph has "large" path-width if and only if it has a "large" k-triple.

Theorem 2.2.7 *For every set $\mathcal{S}$ of semi-complete digraphs, the following are equivalent:*

1. *There exists k such that every member of $\mathcal{S}$ has path-width at most k.*
2. *There is a digraph H such that no subdivision of H is a subdigraph of any member of $\mathcal{S}$.*
3. *There exists k such that for each $G \in \mathcal{S}$, there is no k-triple in G.*
4. *There exists k such that for each $G \in \mathcal{S}$, there do not exist k vertices of G that are pairwise k-connected.*
5. *There is a digraph H such that for each $G \in \mathcal{S}$, G does not contain H as a butterfly minor.*

Proof. The equivalence of the first four statements was proved by Fradkin and Seymour in [9]. Here, we prove $1 \Rightarrow 5 \Rightarrow 3$ to extend the theorem.

Suppose 1 holds for k and $\mathcal{S}$ and let H be a digraph with $pw(H) > k$. Then 5 holds by 2.2.2. Now, suppose 5 holds for H and $\mathcal{S}$. Let H' be a simple digraph containing H as a butterfly minor (We can obtain such H' by subdividing each multiple edge of H). Then 5 holds for H' and $\mathcal{S}$ as well. By 2.2.6, G has no $|V(H')|$-triple for each $G \in \mathcal{S}$. Therefore 3 holds. This proves 2.2.7. ∎

Notice that 2.2.7 says for any fixed digraph H, the class of all semi-complete digraphs not containing H as a butterfly minor has bounded path-width. Thus, to prove 2.1.3, it is sufficient to prove it for semi-complete digraphs with bounded path-width.

Theorem 2.2.8 *For all $k \geq 0$, the class of all semi-complete digraphs with path-width $\leq k$ is a wqo under butterfly minor containment.*

Proof of 2.1.3, assuming 2.2.8. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs. We may assume G_i does not contain G_1 as a butterfly minor for each $i \geq 2$. From 2.2.7, there exists k such that every member of $\mathcal{S} = \{G_2, G_3, \ldots\}$ has path-width at most k. From 2.2.8, there exist $j > i \geq 2$ such that G_i is a minor of G_j. This proves 2.1.3. ∎

The following sections are devoted to proving 2.2.8. We first show the existence of a "linked" path-decomposition; and use it to prove a slightly more general version of 2.2.8. For the purpose of the following sections, we give some notations for a given path-decomposition $P = (W_1, \ldots, W_r)$. We denote $\min\limits_{1 \leq i \leq r} |W_i|$, $\max\limits_{1 \leq i \leq r} |W_i|$, W_1, and W_r by $m(P)$, $M(P)$, $F(P)$, and $L(P)$, respectively.[1]

2.3 A WQO for semi-complete digraphs

In this section, we make a particular choice of path-decomposition. Roughly speaking, we will break this path-decomposition into a sequence of small path-decompositions in the natural way, so that we can apply Higman's sequence theorem to this sequence. This particular choice of path-decomposition will be helpful to apply Higman's sequence theorem properly.

[1]For convenience, we use $M(P)(= pw(P) + 1)$ instead of $pw(P)$ in later sections.

2.3.1 Linked path-decompositions

A directed path P in a semi-complete digraph G is said to be *induced* if $v_iv_j \notin E(G)$ for $j - i \geq 2$ where $v_1, \dots, v_n$ are the vertices of P in order. Note that $G|V(P)$ is strongly-connected unless it is a one-edge directed path. (This fact is particularly important in the next chapter.) For two sets A and B, let $A \Delta B$ be the symmetric difference $(A \setminus B) \cup (B \setminus A)$. For a digraph G, we say (C, D) is a *separation of G of order s* if:

- $C \cup D = V(G)$,
- $|C \cap D| = s$, and
- there is no edge $uv \in E(G)$ such that $u \in C \setminus D$ and $v \in D \setminus C$.

If $A, B \subseteq V(G)$, a separation (C, D) of G *separates* A, B if $A \subseteq C$ and $B \subseteq D$. The following is a theorem of Menger [19].

Theorem 2.3.1 *Let G be a digraph, let $A, B \subseteq V(G)$, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

1. *there are k directed paths from A to B, pairwise vertex-disjoint.*
2. *there is a separation (C, D) of G of order $< k$ with $A \subseteq C$ and $B \subseteq D$.*

Now we define linked path-decomposition. A path-decomposition $P = (W_1, \dots, W_r)$ of a digraph G is *linked* if:

- *(increment condition)* $|W_i \; \Delta \; W_{i+1}| = 1$ for $i = 1, \dots, r-1$,
- *(cardinality condition)* $|W_1| = |W_r| = m(P)$, and
- *(linked condition)* if $|W_i| \geq t$ for every i with $h \leq i \leq j$, then there exist t vertex-disjoint directed paths from W_h to W_j.

Observe that for every $v \notin W_1$, there exists a unique i with $1 \leq i \leq r-1$ such that $\{v\} = W_{i+1} \backslash W_i$, and for every $v \notin W_r$, there exists a unique j with $1 \leq j \leq r-1$ such that $\{v\} = W_j \backslash W_{j+1}$.

Therefore the increment condition implies $r-1 = 2\big(|V(G)| - m(P)\big)$. In particular, r is bounded above by $2|V(G)|+1$. We now prove the existence of a linked path-decomposition in a semi-complete digraph.

Theorem 2.3.2 *Let G be a semi-complete digraph and $A, B \subseteq V(G)$ with $|A| = |B| = m \geq 0$. Suppose there exist m vertex-disjoint directed paths from A to B in G, and there exists a path-decomposition (not necessarily linked) P of G such that $F(P) = A$, $L(P) = B$, and $M(P) \leq k$ for some k. Then there exists a linked path-decomposition P' such that $F(P') = A$, $L(P') = B$, and $M(P') \leq k$.*

In particular, every semi-complete digraph G with $pw(G) \leq k$ has a linked path-decomposition P with $pw(P) \leq k$ and $m(P) = 0$.

Proof. Observe that we can obtain a path-decomposition P' with $F(P') = A$, $L(P') = B$ and $M(P') \leq k$ satisfying the increment condition, by modifying P as follows: we remove (one of) any two consecutive sets that are equal, and insert appropriate sets between sets that differ by more than one vertex. The cardinality condition for P' holds because every W_i meets each of the m vertex-disjoint paths, and hence, $|W_i| \geq m$ for every i.

Now, among all the path-decompositions $P = (W_1, \ldots, W_r)$ of G with $F(P) = A$, $L(P) = B$, and $M(P) \leq k$ satisfying the increment condition and the cardinality condition, we take one with $(n_0, n_1, \ldots, n_k)$ "lexicographically maximal", where $n_j = |\{i : |W_i| = j\}|$ for $j = 0, \ldots, k$. (More precisely, take one with n_0 as large as possible; subject to that, take n_1 as large as possible; and so on.) We can take such a path-decomposition since $\sum_{i=0}^{k} n_i \ (= r)$ is bounded above by $2|V(G)|+1$. Let $P' = (W_1, \ldots, W_r)$ be the path-decomposition we choose. We show P' satisfies the linked condition.

Suppose that $|W_i| \geq t$ for every i with $h \leq i \leq j$ and there do not exist t vertex-disjoint directed paths from W_h to W_j. Then from 2.3.1, there is a separation (C, D) of order less than t that separates $\bigcup_{i \leq h} W_i$, $\bigcup_{i \geq j} W_i$. Take such a separation (C, D) with minimum order s. We claim that there exist two path-decompositions

$$P^C = (W_1^C, \ldots, W_j^C)$$

of $G|C$ with $W_i^C = W_i$ for $1 \leq i \leq h$, $W_j^C = C \cap D$, and $M(P^C) \leq k$, and

$$P^D = (W_h^D, \ldots, W_r^D)$$

of $G|D$ with $W_h^D = C \cap D$, $W_i^D = W_i$ for $j \le i \le r$, and $M(P^D) \le k$. We will show that the "concatenation" of the two path-decompositions yields a path-decomposition lexicographically better than P', which contradicts our choice of P'.

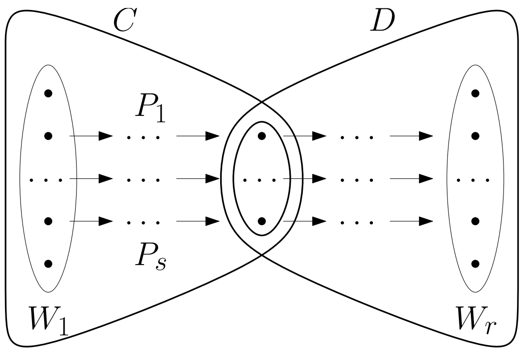

Figure 2.7: C and D

We construct P^C as follows. Note that there exist s vertex-disjoint paths from W_h to W_j by the minimality of s. Take s vertex-disjoint directed paths $P_1, \dots, P_s$ from $W_1 \cup \dots \cup W_h$ to $W_j \cup \dots \cup W_r$ with minimal union. For $1 \le l \le s$, the minimality of the union of $P_1, \dots, P_s$ implies that P_l is induced and no vertex of P_l belongs to $W_1 \cup \dots \cup W_h$ except its first vertex. Since there is no edge from $W_1 \cup \dots \cup W_{h-1}$ to $W_{h+1} \cup \dots \cup W_r$ in $G \setminus W_h$, it follows that the first vertex of P_l belongs to W_h. Similarly, the last vertex of P_l belongs to W_j, and no other vertex of P_l belongs to $W_j \cup \dots \cup W_r$. Recall that $|V(P_l) \cap (C \cap D)| = 1$. Let $p_l \in V(P_l) \cap (C \cap D)$.

(1) *For* $1 \le l \le s$, $\{i : W_i \cap (D \cap V(P_l)) \ne \emptyset,\ 1 \le i \le j\}$ *is an integer interval containing* j.

Fix l and let $G_l = G|(D \cap V(P_l))$, and let I_l be the set in question. I_l obviously contains j. First, suppose G_l is not strongly-connected. Then, G_l has exactly two vertices u, v and one edge uv, since P_l is induced. By the cut condition for the edge uv, there exist a, b with $1 \le a \le b \le r$ such that $u \in W_b$ and $v \in W_a$. Since no vertex of P_l except its last is in $W_j \cup \dots \cup W_r$, it follows that $b < j$. Since $v \in W_a \cap W_j$ and $a \le b \le j$, it follows that $v \in W_b$. In summary, there exists b with $1 \le b < j$ such that $u, v \in W_b$.

On the other hand, $\{i : W_i \cap \{u\} \ne \emptyset,\ 1 \le i \le j\}$ and $\{i : W_i \cap \{v\} \ne \emptyset,\ 1 \le i \le j\}$ are both integer intervals by the betweenness condition. Since they intersect, I_l is also an integer interval since it is the union of the two intersecting intervals. Therefore (1) holds when G_l is not strongly-connected.

Second, suppose G_l is strongly-connected, and suppose I_l is not an integer interval. Take indices $h < i < j$ such that $h, j \in I_l$ and $i \notin I_l$. Let $u \in W_h \cap V(G_l)$ and $v \in W_j \cap V(G_l)$. Since $\{t : u \in W_t\} \subseteq \{1, \dots, i-1\}$ and $\{t : v \in W_t\} \subseteq \{i+1, \dots, r\}$, two sets $\{t : u \in W_t\}$ and $\{t : v \in W_t\}$ do not intersect. Since G_l is strongly-connected and $V(G_l) \cap W_i = \emptyset$, there is a directed path from u to v in $G \setminus W_i$. However, this contradicts the cut condition since there is no edge from $\bigcup_{a<i} W_a$ to $\bigcup_{b>i} W_b$ in $G \setminus W_i$. This proves (1).

For each i with $1 \leq i \leq j$, define

$$W_i^C = (W_i \cap C) \ \cup \ \{p_l : W_i \cap (D \cap V(P_l)) \neq \emptyset, \ 1 \leq l \leq s\}.$$

Let $P^C = (W_1^C, \dots, W_j^C)$.

(2) *P^C is a path-decomposition of $G|C$ with $W_i^C = W_i$ for $1 \leq i \leq h$, $W_j^C = C \cap D$, and $M(P^C) \leq k$.*

First, it is easy to check that $\bigcup_{i=1}^{j} W_i^C = C$. Second, the betweenness condition follows from (1). Third, for the cut condition, we only need to consider the edges incident with p_l in $G|C$ and this is trivial since

$$\{i : p_l \in W_i, 1 \leq i \leq j\} \subseteq \{i : p_l \in W_i^C, 1 \leq i \leq j\}.$$

Therefore P^C is a path-decomposition of $G|C$.

For $1 \leq i \leq h$, $W_i^C = W_i$ since $W_i \subseteq C$. And $W_j^C = C \cap D$ since $W_j \subseteq D$. Finally, $M(P_C) \leq M(P) \leq k$ since $|W_i^C| \leq |W_i|$ for every i with $1 \leq i \leq j$. This proves (2).

Similarly, let $P^D = (W_h^D, \dots, W_r^D)$, where

$$W_i^D = (W_i \cap D) \ \cup \ \{p_l : W_i \cap C \cap V(P_l) \neq \emptyset, \ 1 \leq l \leq s\}$$

for each i with $h \leq i \leq r$. Then it is a path-decomposition of $G|D$ with $W_h^D = C \cap D$, $W_i^D = W_i$ for $j \leq i \leq r$, and $M(P_D) \leq k$.

Let P^* be the path-decomposition of G obtained by concatenating P^C and P^D and refining it (removing (one of) any two consecutive sets that are equal, and insert appropriate sets between sets that differ by more than one vertex) to satisfy the increment condition. Then P^* is "lexicographically better" than P'. To see this, notice that every W_a with $|W_a| \leq s$ is a term in the sequence P^* because

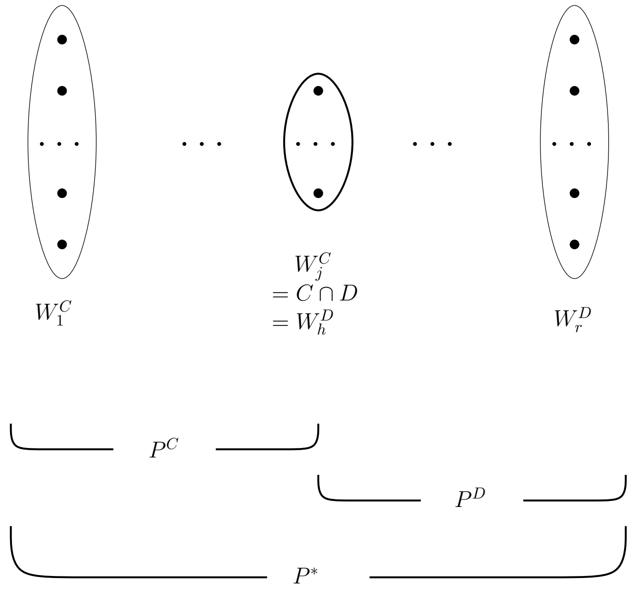

Figure 2.8: Concatenation of P^C and P^D

$W_i^C = W_i$ for $1 \leq i \leq h$ and $W_i^D = W_i$ for $j \leq i \leq r$, and $|W_i| > s$ for $h \leq i \leq j$, and there exists at least one more set of size s, namely $C \cap D$. This proves 2.3.2. ∎

2.3.2 Rooted butterfly minors

Now, we define digraphs with roots. For a digraph G, we fix a linked path-decomposition P of G together with $m(P)$ vertex-disjoint directed paths from $F(P)$ to $L(P)$, and consider the vertices of $F(P)$ and $L(P)$ as roots. More precisely, for integers m, k with $k \geq m \geq 0$, we say $D = (G, P, R)$ is a *(m, k)-digraph* if:

- G is a semi-complete digraph,
- P is a linked path-decomposition of G with $m(P) = m$ and $M(P) \leq k$, and
- $R = (R_1, \ldots, R_m)$ is a sequence of m vertex-disjoint induced directed paths from $F(P)$ to $L(P)$ in G.

Note that $|F(P) \cap V(R_i)| = |L(P) \cap V(R_i)| = 1$ for each $i = 1, \ldots, m$. We say the vertex in $F(P) \cap V(R_i)$ is the *i-th source root* of D and the vertex in $L(P) \cap V(R_i)$ is the *i-th terminal root*

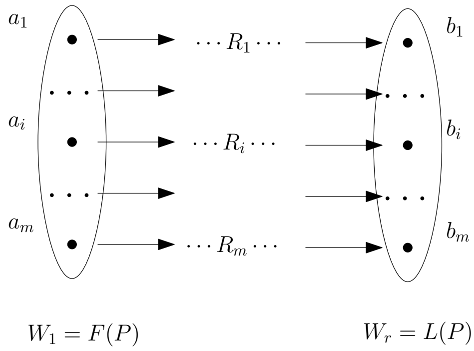

Figure 2.9: A (m,k)-digraph

of D. We denote by $\mathcal{G}_m^k$ the collection of all (m,k)-digraphs. We say D is *trivial* if $r = 1$ where $P = (W_1, \ldots, W_r)$. In particular, $|V(G)| = m$ if D is trivial.

Now, we define the butterfly minor relation on $\mathcal{G}_m^k$. Let $D = (G,P,R)$, $D' = (G',P',R') \in \mathcal{G}_m^k$. Let a_i and a'_i be the i-th source roots of D and D', respectively, and similarly let b_i and b'_i be the i-th terminal roots of D and D', respectively. We say D is a *butterfly minor* of D' if there exists a butterfly minor mapping ϕ from G to G' such that:

- $a'_i \in V(\phi(a_i))$ and $b'_i \in V(\phi(b_i))$ for $i = 1, \ldots, m$.

Again, we call ϕ a *butterfly minor mapping* from D to D'.

2.3.3 Decomposition of (m,k)-digraphs

Next, we define a "decomposition" of a (m,k)-digraph. We will see that the class of all non-decomposable (m,k)-digraphs is a wqo if the class of all $(m+1,k)$-digraphs is a wqo. Then, we will apply Higman's sequence theorem to prove that the class of all (m,k)-digraphs is a wqo if the class of all non-decomposable (m,k)-digraphs is a wqo.

Let $D = (G,P,R)$ be a (m,k)-digraph with $P = (W_1, \ldots, W_r)$, $R = (R_1, \ldots, R_m)$ and suppose $|W_s| = m$ for some s with $1 < s < r$. Let $A = \bigcup_{i \le s} W_i$ and $B = \bigcup_{s \le j} W_j$. Define $D|A = (G_A, P_A, R_A)$ by:

- $G_A = G|A$,

- $P_A = (W_1, \dots, W_s)$, and
- $R_A = (R_1|A, \dots, R_m|A)$.

Similarly, define $D|B = (G_B, P_B, R_B)$ by:

- $G_B = G|B$,
- $P_B = (W_s, \dots, W_r)$, and
- $R_B = (R_1|B, \dots, R_m|B)$.

Then $D|A$ and $D|B$ are both non-trivial (m, k)-digraphs. We write $D = D_A \oplus D_B$ and say D is *decomposable*. We denote by $\mathcal{ND}_m^k$ the class of all non-trivial non-decomposable (m, k)-digraphs. More precisely, we say $D = (G, P, R) \in \mathcal{G}_m^k$ is in $\mathcal{ND}_m^k$ if:

- $r \geq 3$, and $|W_i| > m$ for every i with $2 \leq i \leq r - 1$ where $P = (W_1, \dots, W_r)$.

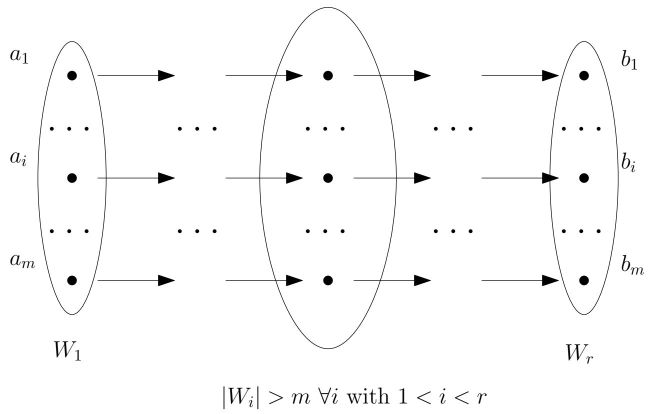

Figure 2.10: A non-decomposable (m, k)-digraph

Note that $P' = (W_2, \dots, W_{r-1})$ is a linked path-decomposition of G with $m(P') = m+1$, $F(P') = W_2 \supseteq W_1$, and $L(P') = W_{r-1} \supseteq W_r$. Let R' be a sequence of $m+1$ vertex-disjoint induced directed paths from W_2 to W_{r-1}. (The existence of R' is guaranteed since P is a linked path-decomposition.) Therefore each $D = (G, P, R) \in \mathcal{ND}_m^k$ yields at least one member $D' = (G, P', R') \in \mathcal{G}_{m+1}^k$. (Notice that it could be the case that some path in R' joins the i-th source root of D to the j-th terminal root of D for some $j \neq i$.)

Theorem 2.3.3 *Let m, k be integers with $k > m \geq 0$. Suppose $\mathcal{G}_{m+1}^k$ is a wqo under butterfly minor containment. Then $\mathcal{ND}_m^k$ is a wqo under butterfly minor containment as well.*

Proof. Let $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{ND}_m^k$. For each $D_i = (G_i, P_i, R_i)$, let $D'_i = (G_i, P'_i, R'_i) \in \mathcal{G}_{m+1}^k$ as described earlier. Notice that every source root of D_i is also a source root of D'_i, and every terminal root of D_i is also a terminal root of D'_i because $W_2 \supseteq W_1$ and $W_{r-1} \supseteq W_r$.

For each $i \geq 1$, let $\sigma_i, \tau_i : \{1, \ldots, m\} \to \{1, \ldots, m+1\}$ be injections defined by

- the t-th source root of D_i equals the $\sigma_i(t)$-th source root of D'_i.
- the t-th terminal root of D_i equals the $\tau_i(t)$-th terminal root of D'_i.

Since there are only finitely many possible pairs (σ_i, τ_i), there exists some (σ, τ) such that $(\sigma_i, \tau_i) = (\sigma, \tau)$ for infinitely many i. Let $I = \{i : (\sigma_i, \tau_i) = (\sigma, \tau)\}$ be such an infinite set. Then, it is sufficient to prove that there exists $i < j$ with $i, j \in I$ such that D_i is a butterfly minor of D_j. Therefore we may assume $\sigma_i = \sigma$ and $\tau_i = \tau$ for every $i \geq 1$.

Since $\mathcal{G}_{m+1}^k$ is a wqo under butterfly minor containment, D'_i is a butterfly minor of D'_j for some $i < j$ with a butterfly minor mapping ϕ. We claim that this ϕ is also a butterfly minor mapping from D_i to D_j. First, ϕ is a butterfly minor mapping from G_i to G_j since $G_i = G'_i$ and $G_j = G'_j$. Second, for each $t = 1, \ldots, m$, let a_t^i and a_t^j be the t-th source roots of D_i and D_j, respectively. Then, a_t^i and a_t^j are the $\sigma(t)$-th source root of D'_i and D'_j, respectively. Since ϕ is a butterfly minor mapping from D'_i to D'_j, $a_t^j \in V(\phi(a_t^i))$. Similarly, $b_t^j \in V(\phi(b_t^i))$ where b_t^i and b_t^j are the t-th terminal root of D_i and D_j. Therefore ϕ is a butterfly minor mapping from D_i to D_j as well. This proves 2.3.3. ∎

Now, we decompose a non-trivial $D \in \mathcal{G}_m^k$ into non-decomposable (m, k)-digraphs to apply Higman's sequence theorem.

Lemma 2.3.4 *Let $D = (G, P, R) \in \mathcal{G}_m^k$ be a non-trivial (m, k)-digraph. Then $D = D_1 \oplus \cdots \oplus D_t$ (perhaps with $t = 1$) such that $D_i \in \mathcal{ND}_m^k$ for $i \in \{1, \ldots, t\}$.*

Proof. Let $P = (W_1, \ldots, W_r)$, and let $i_1(=1) < \cdots < i_t(=r)$ be the indices such that $|W_{i_j}| = m$ for every $j = 1, \ldots, t$. We proceed by induction on t. If $t = 1$, then D itself is non-decomposable by definition. Assume $t > 2$, and let $D_t = D| \bigcup_{i \geq i_{t-1}} W_i$ and $D' = D| \bigcup_{i \leq i_{t-1}} W_i$ Then, $D_t \in \mathcal{ND}_m^k$ by definition, and by the induction hypothesis, $D' = D_1 \oplus \ldots \oplus D_{t-1}$ for some $D_i \in \mathcal{ND}_m^k$ for $i \in \{1, \ldots, t-1\}$. Therefore $D = D' \oplus D_t = D_1 \oplus \ldots \oplus D_t$ as desired. This proves 2.3.4. ∎

2.3.4 The main proof

Lemma 2.3.5 *Let $D, D', D_A, D_B, D'_A, D'_B \in \mathcal{G}_m^k$ be non-trivial (m,k)-digraphs such that:*

- $D = D_A \oplus D_B$,
- $D' = D'_A \oplus D'_B$, *and*
- D_A *and* D_B *are butterfly minors of* D'_A *and* D'_B*, respectively.*

Then, D is a butterfly minor of D'.

Proof. Let ϕ_A, ϕ_B be butterfly minor mappings from D_A to D'_A and from D_B to D'_B, respectively. Let $D_A = (G_A, P_A, R_A), D_B = (G_B, P_B, R_B), D'_A = (G'_A, P'_A, R'_A), D'_B = (G'_B, P'_B, R'_B)$, and let $X = \{x_1, \ldots, x_m\} = L(P_A) = F(P_B)$, and let $Y = \{y_1, \ldots, y_m\} = L(P'_A) = F(P'_B)$ such that $y_i \in V(\phi_A(x_i)) \cap V(\phi_B(x_i))$ for each $i = 1, \ldots, m$.

Let G'' be the union of G'_A and G'_B together with all edges from $V(G'_B) \setminus \big(\bigcup_{1 \le i \le m} V(\phi_B(x_i))\big)$ to $V(G'_A) \setminus \big(\bigcup_{1 \le i \le m} V(\phi_A(x_i))\big)$ (See Figure 2.11).

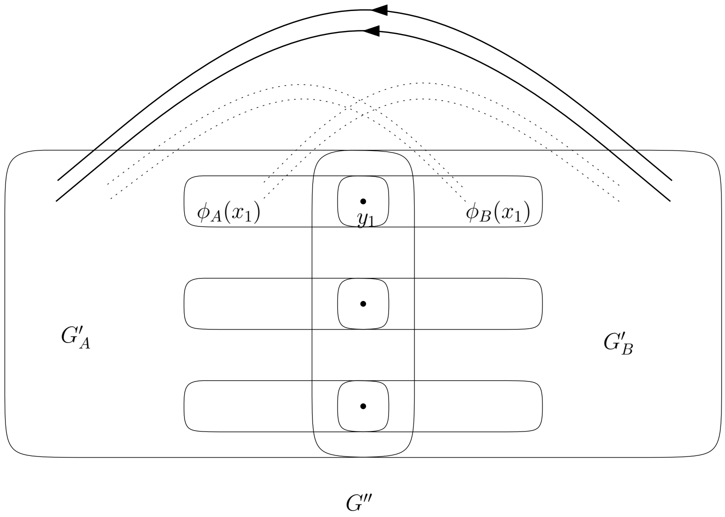

Figure 2.11: $\phi_A(x_i) \cup \phi_B(x_i)$ is contractible in G''.

Then, each $\phi_A(x_i) \cup \phi_B(x_i)$ is contractible in G''. Define ϕ from G to G'' by

$$\phi(v) = \begin{cases} \phi_A(v) & \text{if } v \in V(G_A) \setminus X \\ \phi_B(v) & \text{if } v \in V(G_B) \setminus X \\ \phi_A(v) \cup \phi_B(v) & \text{if } v \in X. \end{cases}$$

Then, ϕ is a butterfly minor mapping from G to G''. Since G'' is a subdigraph of G', G is a butterfly minor of G' as well. Therefore D is a butterfly minor of D', and this proves 2.3.5. ∎

Lemma 2.3.6 *Let $D, D', D_A, D_C, D'_A, D'_B, D'_C \in \mathcal{G}_m^k$ be non-trivial (m,k)-digraphs such that:*

- $D = D_A \oplus D_C$,
- $D' = D'_A \oplus D'_B \oplus D'_C$, *and*
- D_A *and* D_C *are butterfly minors of* D'_A *and* D'_C, *respectively.*

Then, D is a butterfly minor of D'.

Proof. Let $D'_A = (G'_A, P'_A, R'_A), D'_B = (G'_B, P'_B, R'_B), D'_C = (G'_C, P'_C, R'_C)$. Let $X = \{x_1, \ldots, x_m\} = L(P'_A) = F(P'_B)$, and $Y = \{y_1, \ldots, y_m\} = L(P'_B) = F(P'_C)$, and let $R'_B = \{R_1, \ldots, R_m\}$ such that R_i is a directed path from x_i to y_i $(i = 1, \ldots, m)$.

Let G''_A be the digraph obtained from G'_A by deleting all edges in $G'_A|X$. Let G''_B be the disjoint union of $R_1, \ldots, R_m$ (and hence, a subdigraph of G'_B). Take the union of G''_A, G''_B, and G'_C together with all edges from $V(G''_C)$ to $V(G''_A)$.

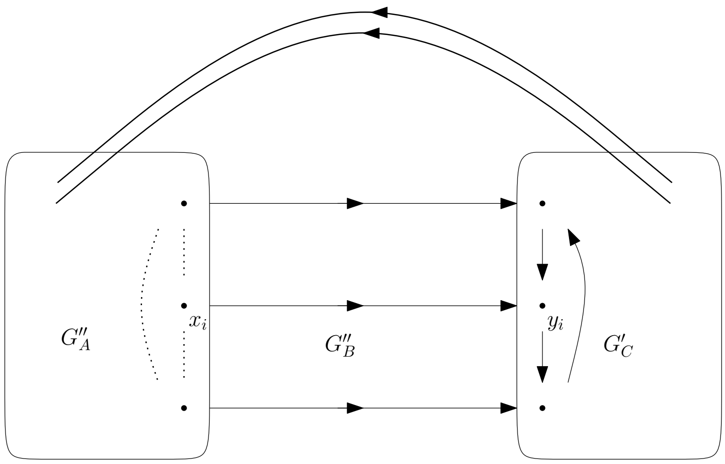

Figure 2.12: Each x_i has out-degree one.

Then, each of $R_1, \ldots, R_m$ is contractible by the definition of G''_A and G''_B (we can sequentially contract edges of R_i. see Figure 2.12). Contract each path and delete multiple edges or loops to obtain a simple digraph G''.

(1) *G'' is a semi-complete digraph.*

Let $v_1, \ldots, v_m$ be the contracted vertices. We only need to check the existence of an edge between each of v_i and each vertex of G''. Fix v_i and consider a vertex $v \in V(D'_A) \setminus X$. Then, there is an edge between v_i and v since the edge between x_i and v in G'_A is not deleted. Next, consider a vertex $v \in \{v_1, \ldots, v_{i-1}, v_i, \ldots, v_m\}$. Then, there is an edge between v_i and v, since the edge between y_i and y in G'_C is not deleted. Finally, consider a vertex $v \in V(D'_C) \setminus Y$. Notice that v is out-adjacent to every vertex of $X(\subseteq V(G'_A))$, and the corresponding edge is not deleted. Therefore there is an edge between v_i and v. This proves (1).

We can naturally define a linked path-decomposition P'' of G'' by omitting the sets in P'_B from P'. Also, we can define R'' from R' by contracting each path of R'_B from the corresponding path in R'. Then, $D'' = (G'', P'', R'')$ is a (m, k)-digraph with $D'' = D'_A \oplus D'_C$. Then, D_A and D_C are butterfly minors of D'_A and D'_C, respectively. Therefore D is a butterfly minor of D'' by 2.3.5. Therefore D is a butterfly minor of D' as well. This prove 2.3.6. ∎

Next, we introduce Higman's sequence theorem. For a quasi-order $Q = (E(Q), \leq_Q)$, define a quasi-order $Q^{<\omega}$ on the set of all finite sequences of elements of $E(Q)$. For two elements $p = (p_1, \ldots, p_a)$ and $q = (q_1, \ldots, q_b)$ of $Q^{<\omega}$, $p \leq_{Q^{<\omega}} q$ if and only if:

- $a \leq b$, and
- there exist $1 \leq \alpha_1 < \ldots < \alpha_a \leq b$ such that $p_i \leq_Q q_{\alpha_i}$ for every $i = 1, \ldots, a$.

It is proved in [11] that

Theorem 2.3.7 *If Q is a wqo, then so is $Q^{<\omega}$.*

Let Q_1 and Q_2 be quasi-orders. The *cartesian product* $Q = Q_1 \times Q_2$ has the element set $E(Q_1) \times E(Q_2)$ ordered by $p = (p_1, p_2) \leq_Q q = (q_1, q_2)$ if:

- $p_1 \leq_{Q_1} q_1$, and
- $p_2 \leq_{Q_2} q_2$.

Then the cartesian product of two wqo is also a wqo. This is an immediate corollary of 2.3.7.

Corollary 2.3.8 *Let Q_1 and Q_2 be wqo. Then so is $Q = Q_1 \times Q_2$.*

The following is also an easy corollary of 2.3.7 and 2.3.8.

Corollary 2.3.9 *Suppose Q_1, Q_2 and Q_3 are wqo. Let $Q = Q_1 \times Q_2^{<\omega} \times Q_3$. Then Q is a wqo.*

Next, we prove a key lemma for 2.2.8.

Lemma 2.3.10 *Let m, k be integers with $k > m \geq 1$. Suppose $\mathcal{G}^k_{m+1}$ is a wqo under butterfly minor containment. Then $\mathcal{G}^k_m$ is a wqo under butterfly minor containment.*

Proof. Let $D_1, D_2, \ldots$ be an infinite sequence of $\mathcal{G}^k_m$. If there are infinitely many trivial D_i in the sequence, then some trivial D_i is a butterfly minor of some trivial D_j for some $i < j$ because every trivial D_i has m vertices, and hence there are finitely many semi-complete digraphs up to isomorphism. Therefore we may assume D_i is non-trivial for every $i \geq 1$. Decompose each D_i as

$$D_i = D_i^1 \oplus \ldots \oplus D_i^{t_i}$$

as in 2.3.4. Since $\mathcal{G}^k_{m+1}$ is a wqo, $\mathcal{ND}^k_m$ is a wqo by 2.3.3.

We apply 2.3.9 for $Q_1 = Q_3 = \mathcal{ND}^k_m$, $Q_2 = {\mathcal{ND}^k_m}^{<\omega}$, and the sequence

$$\big(D_1^1, (D_1^2, \ldots, D_1^{t_1-1}), D_1^{t_1}\big),\ \big(D_2^1, (D_2^2, \ldots, D_2^{t_1-1}), D_2^{t_1}\big), \ldots.$$

Then there exist $i < j$ with $t_i \leq t_j$ and $1 = \alpha_1 < \ldots < \alpha_{t_i} = t_j$ such that:

- D_i^p is a butterfly minor of $D_j^{\alpha_p}$ with a butterfly minor mapping ϕ_p for every $1 \leq p \leq t_i$.

Then, by applying 2.3.6, D_i is a butterfly minor of D_j. This proves 2.3.10. ∎

Theorem 2.3.11 *Let m, k be integers with $k \geq m \geq 0$. Then $\mathcal{G}^k_m$ is a wqo under butterfly minor containment.*

Proof. For fixed k, we use induction on $k - m$. For the base case $k = m$, every member $D \in \mathcal{G}^k_m$ is trivial, and hence the statement holds. Therefore we may assume $k < m$. By induction hypothesis, we may assume $\mathcal{G}^k_{m+1}$ is a wqo. Then, by applying 2.3.10, $\mathcal{G}^k_m$ is a wqo under butterfly minors. This proves 2.3.11. ∎

Proof of 2.2.8. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs with path-width at most k. Let P_i be a linked path-decomposition of G_i with $m(P_i) = 0$ and $pw(P_i) \leq k$. Let $R_i = ()$ be the empty sequence, and let $D_i = (G_i, P_i, R_i)$. Since $D_i \in \mathcal{G}_0^{k+1}$ for each $i \geq 1$, there exist $j > i \geq 1$ such that D_i is a butterfly minor of D_j by 2.3.11. Therefore G_i is a butterfly minor of G_j. This proves 2.2.8. ∎

2.3.5 Classes of digraphs containing all semi-complete digraphs

We have seen that the class of all semi-complete digraphs is a wqo under butterfly minors while the class of all digraphs is not. We can ask the same question for some larger classes of digraphs containing all semi-complete digraphs. For instance, the class of all supertournaments and the class of all simple digraphs with stability number at most two contain every semi-complete digraphs. Unfortunately, we do not know the answer for those two slightly more general classes. (Although, it is not very hard to see that the class of all acyclic supertournaments and the class of all acyclic simple digraphs with stability number at most two are well-quasi-ordered under butterfly minors.) We suspect that the answers for them are negative so that the class of all semi-complete digraphs is a "maximal" class of digraphs that the wqo statement holds in some sense. In the next chapter, we will prove that this is indeed the case for "strong minors".

Conjecture 2.3.12 *The class of all supertournaments is not a wqo under butterfly minors.*

Conjecture 2.3.13 *The class of all simple digraphs with stability number at most two is not a wqo under butterfly minors.*

Chapter 3

Strong Minors

3.1 Introduction

In this chapter, we introduce another containment relation in digraphs which also can be considered as an extension of the minor relation in graphs. We will call this relation *strong minors.* To obtain a strong minor, we may contract strongly-connected subdigraphs rather than edges. (A digraph G is *strongly-connected* if G is non-null and there exists a directed path from u to v for every $u, v \in V(G)$.) The main result of this chapter is the class of all semi-complete digraphs is well-quasi-ordered under strong minors. The strong minor relation is similar with butterfly minor relation but slightly different. We will see that many arguments we have seen in the previous chapter work for strong minors as well with small modifications. Hence in this chapter, we will focus on the difference between two relations and try to avoid the same details.

First, recall the following definition of (undirected) minors.

- $G \preceq_2 H$ if G can be obtained from a subgraph of H by repeatedly contracting a connected subgraph to a vertex.

For digraphs, there are two different concepts of "connectedness". A digraph G is said to be *weakly-connected* if the underlying undirected graph is connected, and a digraph G is said to be *strongly-connected* if for every $u, v \in V(G)$, there exists a directed path from u to v and a directed path from v to u in G.

As we have seen in the last chapter, contracting an edge may yield a directed cycle in digraphs. Therefore contracting a weakly-connected subdigraph may yield new directed cycles. However, contracting a strongly-connected subdigraph does not yield new directed cycles.

Theorem 3.1.1 *Let H be a digraph, and let G be a strongly-connected subdigraph of H. Let ϕ be a function on the set of all directed cycles in H defined by $\phi(C) = C/G$. Then every directed cycle of H/G is an image of ϕ.*

Proof. Let v be the contracted vertex in H/G, and let C be a directed cycle in H/G. We may assume $v \in V(C)$ since otherwise, C is a cycle in H, and hence $\phi(C) = C/G = C$. Suppose C has length at least two. Let uv, vw be the edges of C adjacent with v. Then, there exist $v', v'' \in V(G)$ such that $uv', v''w$ are edges in H. Since G is strongly-connected, there exists a directed path P from v' to v'' in G. Now take the union of $C \setminus v$, uv', P, and $v''w$ in H. Then, this union C' is a directed cycle in H and $\phi(C') = C/G = C$. The same argument works when C is a loop at v. This proves 3.1.1. ∎

Therefore contracting a strongly-connected subdigraph does not yield new directed cycles, and this suggests a notion of minors in digraphs. For two digraphs G and H,

- $G \preceq_7 H$ if G can be obtained from a subdigraph of H by repeatedly contracting a strongly-connected subdigraph to a vertex.
- $G \preceq_8 H$ if there exists a function ϕ on $V(G)$ such that for every $v \in V(G)$, $\phi(v)$ is a non-null strongly-connected subdigraph of H, if $u, v \in V(G)$ and $u \neq v$, then $\phi(u)$ and $\phi(v)$ are vertex-disjoint, and for every $u, v \in V(G)$ (not necessarily distinct), if there are k edges in G with tail u and head v, then there are at least k edges in H with head in $V(\phi(u))$ and tail in $V(\phi(v))$, and not contained in $E(\phi(x))$ for any $x \in V(G)$.

It is easy to see that $G \preceq_8 H$ is another expression of $G \preceq_7 H$ by using the map ϕ, that tells us which strongly-connected subgraph of H corresponds to which vertex in G. We say G is a *strong minor* of H if $G \preceq_7 H$ or $G \preceq_8 H$. In the following sections, we will use both definitions $G \preceq_7 H$ and $G \preceq_8 H$ for convenience. We call the map ϕ in the definition of $G \preceq_8 H$ a *strong minor mapping* from G to H.

Strong minor relation is different from butterfly minor relation. For instance, there exists a pair of digraphs G_1 and H_1 such that G_1 is a butterfly minor of H_1, but not a strong minor of H_1. Conversely, there exists a pair of digraphs G_2 and H_2 such that G_2 is a strong minor of H_2, but not a butterfly minor of H_2 (see Figure 3.1).

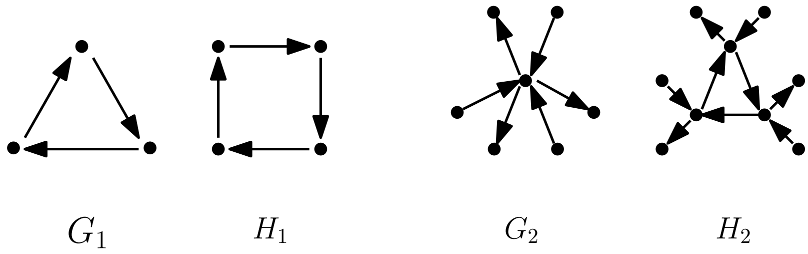

Figure 3.1: Butterfly minors vs. Strong minors

The class of all digraphs is not a wqo under strong minors. For example, a small directed cycle is not a strong minor of a big directed cycle. To see this, notice that the only non-trivial strongly-connected subdigraph of a directed cycle is the directed cycle itself. Therefore we cannot contract any strongly-connected subdigraph to obtain a smaller directed cycle from a big directed cycle. Also, a smaller directed cycle is not a subdigraph of a big directed cycle. This implies that an infinite sequence of directed cycles of distinct lengths is an example of an infinite antichain with respect to strong minors.

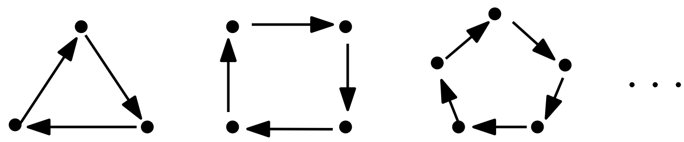

Figure 3.2: An infinite antichain with respect to strong minors

Therefore the wqo statement for general digraphs is false with respect to strong minors. However, the same statement for the class of all semi-complete digraphs is again true.

Theorem 3.1.2 *The class of all semi-complete digraphs is a wqo under strong minor containment.*

Again, the path-width parameter for digraphs plays an important role for the proof. First of all, we prove an analogue of 2.2.2 with respect to strong minors.

Theorem 3.1.3 *If a digraph has path-width at most k, then so do all its strong minors.*

Proof. We have seen in the proof of 2.2.2 that the path-width does not increase by deletions. Thus, it is enough to show that if a digraph G has path-width at most k, then so does G/H where H is a strongly-connected subdigraph of G. Let $P = (W_1, ..., W_r)$ be a path-decomposition of G, and let $I_H = \{i\ : W_i \cap V(H) \neq \emptyset\}$.

(1) *I_H is an integer interval.*

Suppose I_H is not an integer interval. Take indices $h < i < j$ such that $h, j \in I_H$ and $i \notin I_H$. Let $u \in W_h \cap V(H)$ and $v \in W_j \cap V(H)$. Since $\{t\ : u \in W_t\} \subseteq \{1, \ldots, i-1\}$ and $\{t\ : v \in W_t\} \subseteq \{i+1, \ldots, r\}$, two sets $\{t\ : u \in W_t\}$ and $\{t\ : v \in W_t\}$ do not intersect. Since H is strongly-connected and $V(H) \cap W_i = \emptyset$, there is a directed path from u to v in $G \setminus W_i$. However, this contradicts the cut condition since there is no edge from $\bigcup_{a<i} W_a$ to $\bigcup_{b>i} W_b$ in $G \setminus W_i$. This proves (1).

Let $P = (W_1, \ldots, W_r)$ be a path-decomposition of G with $pw(P) \leq k$, and let w be the contracted vertex in G/H. Define $W_i' \subseteq V(G/H)$ by

$$W_i' = \begin{cases} (W_i \setminus V(H)) \cup \{w\} & \text{if } i \in I_H \\ W_i & \text{otherwise.} \end{cases}$$

We claim that $P' = (W_1', \ldots, W_r')$ is a path-decomposition of G/H with $pw(P') \leq k$. The betweenness condition follows from (1). For the cut condition, we only need to consider edges incident with w in G/H. For an edge $uw \in E(G/H)$, consider the corresponding edge $uv \in E(G)$. By the cut condition for P, there exist $i \leq j$ such that $W_j \ni u$ and $W_i \ni v$. Therefore $W_j' \ni u$ and $W_i' \ni w$. The same argument applies for edges with tail w. Finally, $pw(P') \leq pw(P) \leq k$ since $|W_i'| \leq |W_i|$ for every $i = 1, \ldots, r$. This proves 3.1.3. ∎

Next, we prove an analogue of 2.2.6 with respect to strong minors.

Theorem 3.1.4 *Let (A, B, C) be a k-triple of a digraph G. Then G contains every semi-complete digraph with k vertices as a strong minor.*

Proof. Let $\{a_1, \ldots, a_k\}$ and $\{c_1, \ldots, c_k\}$ be numberings of A and C such that $c_i a_i \in E(G)$ for $i = 1, \ldots, k$. Take an ordering $\{b_1, \ldots, b_k\}$ of B. Then $G|\{a_i, b_i, c_i\}$ is strongly-connected for each i. Let G' be the digraph obtained from $G|(A \cup B \cup C)$ by contracting $G|\{a_i, b_i, c_i\}$ to a single vertex for each i. Then $|V(G')| = k$ and $uv \in E(G')$ for every distinct $u, v \in V(G')$. Therefore every semi-complete digraph with k vertices is a subgraph of G' and hence, a strong minor of G. This proves 3.1.4. ∎

Next, we extend 2.2.7 even more.

Theorem 3.1.5 *For every set $\mathcal{S}$ of semi-complete digraphs, the following are equivalent:*

1. *There exists k such that every member of $\mathcal{S}$ has path-width at most k.*
2. *There is a digraph H such that no subdivision of H is a subdigraph of any member of $\mathcal{S}$.*
3. *There exists k such that for each $G \in \mathcal{S}$, there is no k-triple in G.*
4. *There exists k such that for each $G \in \mathcal{S}$, there do not exist k vertices of G that are pairwise k-connected.*
5. *There is a digraph H such that for each $G \in \mathcal{S}$, G does not contain H as a butterfly minor.*
6. *There is a digraph H such that for each $G \in \mathcal{S}$, G does not contain H as a strong minor.*

Proof. We have seen the equivalence of the first five statements in 2.2.7. Here, we prove $1 \Rightarrow 6 \Rightarrow 3$ to extend the theorem.

Suppose 1 holds for k and $\mathcal{S}$ and let H be a digraph with $pw(H) > k$. Then 6 holds by 3.1.3. Now, suppose 6 holds for H and $\mathcal{S}$. Let H' be a simple digraph containing H as a strong minor. Then 6 holds for H' and $\mathcal{S}$ as well. By 3.1.4, G has no $|V(H')|$-triple for each $G \in \mathcal{S}$. Therefore 3 holds. This proves 3.1.5. ∎

Therefore, to prove 3.1.2, again it is sufficient to prove it for digraphs with bounded path-width.

Theorem 3.1.6 *For all $k \geq 0$, the class of all semi-complete digraphs with path-width $\leq k$ is a wqo under strong minor containment.*

Proof of 3.1.2, assuming 3.1.6. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs. We may assume G_i does not contain G_1 as a strong minor for each $i \geq 2$. From 3.1.5, there exists k such that every member of $\mathcal{S} = \{G_2, G_3, \ldots\}$ has path-width at most k. From 3.1.6, there exist $j > i \geq 2$ such that G_i is a minor of G_j. This proves 3.1.2. ∎

In the following sections, we will prove 3.1.6 focused on the modifications we have to make from the arguments in the previous chapter. Roughly speaking, other than roots, we also need labels for all vertices to handle the case when an induced path is a one-edge path, and hence, not contractible. We also give some counterexamples to the analogue of 3.1.2 for some super-classes of the class of all semi-complete digraphs.

3.2 A WQO for semi-complete digraphs

Again, we need linked path-decompositions. We will break the linked path-decomposition into a sequence of small linked path-decompositions, so that we can apply Higman's sequence theorem to this sequence. The main problem is that an induced directed path may not be contractible if it is a one-edge path. To handle this, we define even more general objects than (m, k)-digraphs.

3.2.1 Labeled minors with roots

In this section, for a wqo Q and a semi-complete digraph G, we assign an element of $E(Q)$ to each vertex of G, and we fix a linked path-decomposition P of G together with $m(P)$ vertex-disjoint directed paths from $F(P)$ to $L(P)$. We define the strong minor relation for these slightly more general objects and prove a well-quasi-order theorem for them. Then 3.1.6 will follow as an corollary.

For integers m, k with $k \geq m \geq 0$ and a well-quasi-order Q, we say $D = (G, P, R, l)$ is a (Q, m, k)-*digraph* if:

- G is a semi-complete digraph,
- P is a linked path-decomposition of G with $m(P) = m$ and $M(P) \leq k$,
- $R = (R_1, \ldots, R_m)$ is a sequence of m vertex-disjoint induced directed paths from $F(P)$ to $L(P)$ in G, and
- l is a mapping from $V(G)$ to $E(Q)$.

Note that $|F(P) \cap V(R_i)| = |L(P) \cap V(R_i)| = 1$ for each $i = 1, \ldots, m$. Again, we say the vertex in $F(P) \cap V(R_i)$ is the *i-th source root* of D and the vertex in $L(P) \cap V(R_i)$ is the *i-th terminal root* of D. We denote by $\mathcal{G}_m^k(Q)$ the collection of all (Q, m, k)-digraphs. We say D is *trivial* if $r = 1$ where $P = (W_1, \ldots, W_r)$. Note that $|V(G)| = m$ if D is trivial.

Now we define the strong minor relation on $\mathcal{G}_m^k(Q)$. Let $D = (G, P, R, l)$, $D' = (G', P', R', l')$ $\in \mathcal{G}_m^k(Q)$. Let a_i and a'_i be the i-th source roots of D and D', respectively, and similarly let b_i and b'_i be the i-th terminal roots of D and D', respectively ($i = 1, \ldots, m$). We say D is a *strong minor* of D' if there exists a strong minor mapping ϕ from G to G' such that:

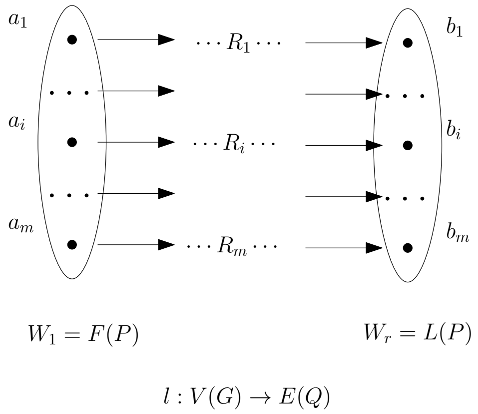

Figure 3.3: A (Q, m, k)-digraph

- $a'_i \in V(\phi(a_i))$ and $b'_i \in V(\phi(b_i))$ for $i = 1, \ldots, m$, and
- for every $v \in V(G)$, $l(v) \leq_Q l'(u)$ for some $u \in V(\phi(v))$.

Again, we call ϕ a *strong minor mapping* from D to D'.

3.2.2 Decomposition of (Q, m, k)-digraphs

Next, we define a "decomposition" of a (Q, m, k)-digraph. For a given wqo Q, we will see that the class of all non-decomposable (Q, m, k)-digraphs is a wqo if the class of all $(Q', m + 1, k)$-digraphs is a wqo for every wqo Q'.

Let $D = (G, P, R, l)$ be a (Q, m, k)-digraph with $P = (W_1, \ldots, W_r)$, $R = (R_1, \ldots, R_m)$ and suppose $|W_s| = m$ for some s with $1 < s < r$. Let $A = \cup_{i \leq s} W_i$ and $B = \cup_{s \leq j} W_j$. Define $D|A = (G_A, P_A, R_A, l_A)$ by:

- $G_A = G|A$,
- $P_A = (W_1, \ldots, W_s)$, and
- $R_A = (R_1|A, \ldots, R_m|A)$.
- $l_A = l|A$.

Similarly, define $D|B = (G_B, P_B, R_B, l_B)$ by:

- $G_B = G|B$,
- $P_B = (W_s, \ldots, W_r)$, and
- $R_B = (R_1|B, \ldots, R_m|B)$.
- $l_B = l|B$.

Then $D|A$, and $D|B$ are both (Q, m, k)-digraphs. We write $D = D_A \oplus D_B$ and say D is *decomposable*. We denote by $\mathcal{ND}_m^k(Q)$ the class of all non-trivial non-decomposable (Q, m, k)-digraphs. More precisely, we say $D = (G, P, R, l) \in \mathcal{G}_m^k(Q)$ is in $\mathcal{ND}_m^k(Q)$ if:

- $r \geq 3$, and $|W_i| > m$ for every i with $2 \leq i \leq r-1$ where $P = (W_1, \ldots, W_r)$.

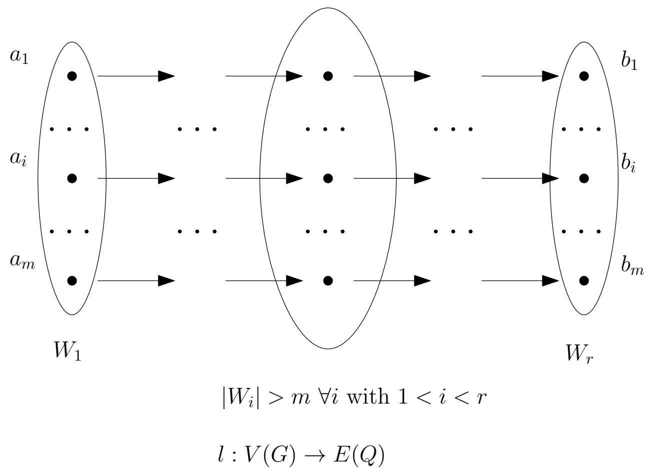

Figure 3.4: A non-decomposable (Q, m, k)-digraph

Note that $P' = (W_2, \ldots, W_{r-1})$ is a linked path-decomposition of G with $m(P') = m + 1$, $F(P') = W_2 \supseteq W_1$, and $L(P') = W_{r-1} \supseteq W_r$. Let R' be a sequence of $m+1$ vertex-disjoint induced directed paths from W_2 to W_{r-1}. (The existence of R' is guaranteed since P is a linked path-decomposition.) Therefore each $D = (G, P, R, l) \in \mathcal{ND}_m^k(Q)$ yields at least one member $D' = (G, P', R', l) \in \mathcal{G}_{m+1}^k$. (Notice that it could be the case that some path in R' joins the i-th source root of D to the j-th terminal root of D for some $j \neq i$.)

The analogue of 2.3.3 for (Q, m, k)-digraphs is also true.

Theorem 3.2.1 *Let m, k be integers with $k > m \geq 0$. Suppose $\mathcal{G}^k_{m+1}(Q)$ is a wqo under strong minor containment for every wqo Q. Then $\mathcal{ND}^k_m(Q)$ is a wqo under strong minor containment for every wqo Q.*

Proof. Let Q be a wqo and $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{ND}^k_m(Q)$. For each $D_i = (G_i, P_i, R_i, l_i)$, let $D'_i = (G_i, P'_i, R'_i, l_i) \in \mathcal{G}^k_{m+1}(Q)$ as described earlier. Notice that every source root of D_i is also a source root of D'_i, and every terminal root of D_i is also a terminal root of D'_i since $W_2 \supseteq W_1$ and $W_{r-1} \supseteq W_r$.

For each $i \geq 1$, let $\sigma_i, \tau_i : \{1, \ldots, m\} \to \{1, \ldots, m+1\}$ be injections defined by

- the t-th source root of D_i equals the $\sigma_i(t)$-th source root of D'_i.
- the t-th terminal root of D_i equals the $\tau_i(t)$-th terminal root of D'_i.

Since there are only finitely many possible pairs (σ_i, τ_i), there exists some (σ, τ) such that $(\sigma_i, \tau_i) = (\sigma, \tau)$ for infinitely many i. Let $I = \{i : (\sigma_i, \tau_i) = (\sigma, \tau)\}$ be such an infinite set. Then, it is sufficient to prove that there exists $i < j$ with $i, j \in I$ such that D_i is a strong minor of D_j. Therefore we may assume $\sigma_i = \sigma$ and $\tau_i = \tau$ for every $i \geq 1$.

Since $\mathcal{G}^k_{m+1}(Q)$ is a wqo under strong minor containment, D'_i is a strong minor of D'_j for some $i < j$ with some strong minor mapping ϕ. We claim that this ϕ is also a strong minor mapping from D_i to D_j. First, ϕ is a strong minor mapping from G_i to G_j since $G_i = G'_i$ and $G_j = G'_j$. Second, for each $t = 1, \cdots, m$, let a^i_t and a^j_t be the t-th source root of D_i and D_j, respectively. Then, a^i_t and a^j_t are the $\sigma(t)$-th source root of D'_i and D'_j, respectively. Since ϕ is a strong minor mapping from D'_i to D'_j, $a^j_t \in V(\phi(a^i_t))$. Similarly, $b^j_t \in V(\phi(b^i_t))$ where b^i_t and b^j_t are the t-th terminal root of D_i and D_j. Third, for every $v \in V(G_i)$, $l_i(v) \leq_Q l_j(u)$ for some $u \in V(\phi(v))$ since the label of a vertex v in D_i is the same as the label of v in D'_i. Therefore ϕ is a strong minor mapping from D_i to D_j. This proves 3.2.1. ∎

3.2.3 Non-contractible (Q, m, k)-digraphs

We say $D = (G, P, R, l) \in \mathcal{G}^k_m(Q)$ is *contractible* if

- $G|V(R_j)$ is strongly-connected for every $j \in \{1, \ldots, m\}$ where $R = (R_1, \ldots, R_m)$.

In other words, D is not contractible if there exists some j such that R_j is a one-edge directed path uv, and vu is not an edge of G. We denote by $\mathcal{NC}_m^k(Q)$ the set of all non-contractible (Q, m, k)-digraphs.

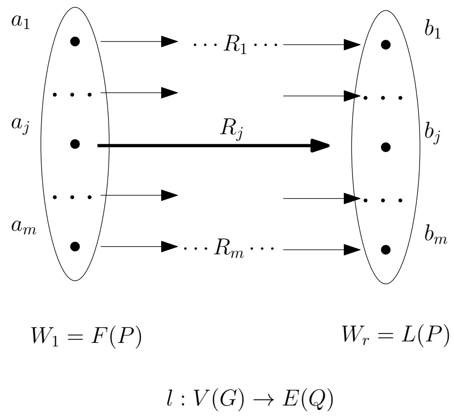

Figure 3.5: A non-contractible (Q, m, k)-digraph; R_j is a one-edge directed path.

Let $D = (G, P, R, l) \in \mathcal{NC}_m^k(Q)$. We will see that D yields at least one member $D' = (G', P', R', l') \in \mathcal{G}_{m-1}^{k-1}(Q')$ (Q' depends on Q, we will define it carefully). Let j be the index such that $G|V(R_j)$ is a one-edge path uv (u is the j-th source root, and v is the j-th terminal root). Note that every W_i contains either u or v where $P = (W_1, \ldots, W_r)$. Let $G' = G \setminus \{u, v\}$, $\hat{W}_i = W_i \setminus \{u, v\}$ for $i = 1, \ldots, r$, and $\hat{P} = (\hat{W}_1, \ldots, \hat{W}_r)$. Then $\hat{P}$ is a path-decomposition (not necessarily linked) of G' with $M(\hat{P}) \leq k-1$. Note that we have $m-1$ vertex-disjoint paths from $F(P)\backslash u$ to $L(P)\backslash v$. From 2.3.2, there exists a linked path-decomposition P' of G' with $F(P') = F(P) \setminus u$, $L(P') = L(P) \setminus v$, $m(P') = m - 1$, and $M(P') \leq k - 1$.

Also, the sequence R' obtained from R by omitting R_j is a sequence of $m - 1$ vertex-disjoint induced directed paths from $F(P')$ to $L(P')$. We defined G', P', and R', so far. Now we define the label l'. This label l' is designed to handle the edges between $\{u, v\}$ and $V(G) \setminus \{u, v\}$. Let Q' be a well-quasi-order defined by

- $E(Q') = E(Q) \times \{0, 1, 2\} \times \{0, 1, 2\}$, and
- $(q, x, y) \leq_{Q'} (q', x', y')$ if and only if $q \leq_Q q'$ and $x = x'$ and $y = y'$.

For each $w \in V(G) \setminus \{u, v\}$, let

$$x(w) = \begin{cases} 0 & \text{if } wu \in E(G), uw \notin E(G) \\ 1 & \text{if } wu \notin E(G), uw \in E(G) \\ 2 & \text{if } wu \in E(G), uw \in E(G) \end{cases}$$

$$y(w) = \begin{cases} 0 & \text{if } wv \in E(G), vw \notin E(G) \\ 1 & \text{if } wv \notin E(G), vw \in E(G) \\ 2 & \text{if } wv \in E(G), vw \in E(G). \end{cases}$$

Let l' be a mapping from $V(G')$ to $E(Q')$ defined by

- $l'(w) = (l(w), x(w), y(w))$ for each $w \in V(G) \setminus \{u, v\}$.

Then $D' = (G', P', R', l') \in \mathcal{G}_{m-1}^{k-1}(Q')$ and we see that each $D \in \mathcal{ND}_m^k(Q)$ yields at least one member D' in $\mathcal{G}_{m-1}^{k-1}(Q')$.

Theorem 3.2.2 *Let m, k be integers with $k \geq m \geq 1$. Suppose $\mathcal{G}_{m-1}^{k-1}(Q)$ is a wqo under strong minor containment for every wqo Q. Then $\mathcal{NC}_m^k(Q)$ is a wqo under strong minor containment for every wqo Q.*

Proof. Let Q be a wqo and $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{NC}_m^k(Q)$. For each $D_i = (G_i, P_i, R_i, l_i)$, let $D'_i = (G'_i, P'_i, R'_i, l'_i) \in \mathcal{G}_{m-1}^{k-1}(Q')$ as described earlier and let u_i and v_i be the source root and the terminal root of (G_i, P_i, R_i, l_i) such that $G'_i = G_i \setminus \{u_i, v_i\}$.

Since Q is a wqo, there exists an infinite increasing sequence of integers $1 \leq s_1 < s_2 < \cdots$ such that if $i < j$, then $l_{s_i}(u_{s_i}) \leq_Q l_{s_j}(u_{s_j})$. Therefore we may assume $l_i(u_i) \leq_Q l_j(u_j)$ for every $i < j$ by narrowing down the infinite set. Even further, we may assume $l_i(v_i) \leq_Q l_j(v_j)$ for every $i < j$ by the same argument.

Since $\mathcal{G}_{m-1}^{k-1}(Q')$ is a wqo under strong minor containment, D'_i is a strong minor of D'_j with a strong minor mapping ϕ' for some i, j with $1 \leq i < j$. Define ϕ from D_i to D_j as

$$\phi(w) = \begin{cases} (\{u_j\}, \emptyset) & w = u_i \\ (\{v_j\}, \emptyset) & w = v_i \\ \phi'(w) & w \in V(G'_i). \end{cases}$$

We claim that this ϕ is a strong minor mapping from D_i to D_j.

(1) *$\phi(v)$ is strongly-connected for every $v \in V(G_i)$, and $\phi(v)$ and $\phi(u)$ are vertex-disjoint for every $u \neq v$.*

If v is either u_i or v_i, then $\phi(v)$ consists of a single vertex, hence it is strongly-connected. Otherwise, $\phi(v) = \phi'(v)$ and $\phi'(v)$ is strongly-connected by the definition of ϕ'. Also, $\phi(u)$ and $\phi(v)$ are vertex-disjoint since ϕ' is a strong minor mapping. This proves (1).

(2) *for every $uv \in E(G_i)$, there exists an edge from $V(\phi(u))$ to $V(\phi(v))$ in G_j not contained in $E(\phi(w))$ for any $w \in V(G_i)$.*

If both u and v are vertices of G'_i, then there exists such an edge in G'_j (hence in G_j), so we may assume either u is u_i or v is v_i.

If $u = u_i$ and $v = v_i$, then there exists such an edge in G_j, namely $u_j v_j$ ($V(\phi(u_i)) = \{u_j\}, V(\phi(v_i)) = \{v_j\}$). And vu is not an edge of G_i. Therefore we only need to consider the edges incident with either u_i or v_i, but not both.

Now, we use the label l'. Suppose both $u_i v$ and vu_i are edges of G_i for some v. Then, the second coordinate of the label $l'_i(v)$ must be 2 by definition. Since ϕ' is a strong minor mapping from D'_i to D'_j, there must be a vertex $w \in V(\phi'(v))$ such that the second coordinate of the label $l'_j(w)$ is 2. This means we have two edges $u_j w$ and wu_j in G_j, so we are done. The same argument works for the other pairs since l' is designed for this. This proves (2).

Therefore ϕ is a strong minor mapping from G_i to G_j, and it is also a strong minor mapping from D_i to D_j. This proves 3.2.2. ∎

3.2.4 Union, and concatenation

For two subclasses $\mathcal{A}, \mathcal{B}$ of $\mathcal{G}^k_m(Q)$, denote by $\mathcal{A} \oplus \mathcal{B}$ the class of all (Q, m, k)-digraphs D, which can be decomposed as $D_A \oplus D_B$ where $D_A \in \mathcal{A}$ and $D_B \in \mathcal{B}$.

Theorem 3.2.3 *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}^k_m(Q)$ are both wqo under strong minor containment, then $\mathcal{A} \cup \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ are both wqo under strong minor containment.*

Proof. $\mathcal{A} \cup \mathcal{B}$ is a wqo under strong minor containment because every infinite sequence in $\mathcal{A} \cup \mathcal{B}$ contains either an infinite subsequence in $\mathcal{A}$ or an infinite subsequence in $\mathcal{B}$.

For $\mathcal{A} \oplus \mathcal{B}$, let $D_1, D_2, \ldots$ be an infinite sequence in $\mathcal{A} \oplus \mathcal{B}$. Let $D_i = D_i^a \oplus D_i^b$ where $D_i^a = (G_i^a, P_i^a, R_i^a, l_i^a) \in \mathcal{A}$ and $D_i^b = (G_i^b, P_i^b, R_i^b, l_i^b) \in \mathcal{B}$ for each $i \geq 1$. Then there exist $i < j$ such that:

- D_i^a is a strong minor of D_j^a with the strong minor mapping ϕ_a, and
- D_i^b is a strong minor of D_j^b with the strong minor mapping ϕ_b.

Define a mapping ϕ from D_i to D_j by

$$\phi(w) = \begin{cases} \phi_a(w) & w \in V(G_i^a) \setminus V(G_i^b) \\ \phi_b(w) & w \in V(G_i^b) \setminus V(G_i^a) \\ \phi_a(w) \cup \phi_b(w) & w \in V(G_i^a) \cap V(G_i^b). \end{cases}$$

Since the union of two strongly-connected subdigraphs with non-empty intersection is also strongly-connected, $\phi_a(w) \cup \phi_b(w)$ is strongly-connected in G_j for each $w \in V(G_i^a) \cap V(G_i^b)$. Then it is easy to check that ϕ is a strong minor mapping from D_i to D_j. This proves 3.2.3. ∎

3.2.5 Links, and the main proof

We say a contractible (Q, m, k)-digraph D ($\notin \mathcal{NC}_m^k(Q)$) is a *link* if:

- $D \in \mathcal{ND}_m^k(Q) \ \cup \ (\mathcal{NC}_m^k(Q) \oplus \mathcal{ND}_m^k(Q))$.

We denote by $\mathcal{L}_m^k(Q)$ the collection of all links in $\mathcal{G}_m^k(Q)$. The following is an easy corollary of 3.2.3.

Theorem 3.2.4 *Suppose $\mathcal{NC}_m^k(Q)$ and $\mathcal{ND}_m^k(Q)$ are wqo under strong minor containment for some wqo Q. Then $\mathcal{L}_m^k(Q)$ is a wqo under strong minor containment as well.*

Now, we decompose $D \in \mathcal{G}_m^k(Q)$ into links (possibly except the last term) to apply Higman's sequence theorem.

Theorem 3.2.5 *Let $D = (G, P, R, l)$ be a non-trivial (Q, m, k)-digraph. Then $D = D_1 \oplus \ldots \oplus D_t$ (perhaps with $t = 1$) such that:*

- *$D_i \in \mathcal{L}_m^k(Q)$ for $i \in \{1, \ldots, t-1\}$, and*
- *$D_t \in \mathcal{L}_m^k(Q) \ \cup \ \mathcal{NC}_m^k(Q)$.*

Proof. We may assume D is contractible since otherwise $D \in \mathcal{NC}_m^k(Q)$ and the result holds with $t = 1$. Let $P = (W_1, \ldots, W_r)$. We proceed by induction on r. For the base case $r = 3$, D belongs to $\mathcal{ND}_m^k(Q)$ and hence, D itself is a link.

Let $1 = n_1 < \cdots < n_s = r$ be the indices such that

$$|W_{n_1}| = \cdots = |W_{n_s}| = m$$

Let $j > 1$ be the smallest index such that the initial segment $D|(\cup_{i=1}^{n_j} W_i)$ is contractible (such j exists since D is contractible).

If $j = 2$, then the initial segment $D|(\cup_{i=1}^{n_2} W_i)$ belongs to $\mathcal{ND}_m^k(Q)$, and hence it is a link. If $j > 2$, then

$$D|(\cup_{i=1}^{n_j} W_i) = D|(\cup_{i=1}^{n_{j-1}} W_i) \ \oplus \ D|(\cup_{i=n_{j-1}}^{n_j} W_i)$$

$$\in \mathcal{NC}_m^k(Q) \ \oplus \ \mathcal{ND}_m^k(Q).$$

Therefore, in either case, $D|(\cup_{i=1}^{n_j} W_i)$ is a link. If $j = s$, then D is a link, and we are done. Otherwise, $D|(\cup_{i=n_j}^{r} W_i)$ is non-trivial and satisfies the statement by the induction hypothesis. Therefore

$$D = D|(\cup_{i=1}^{n_j} W_i) \ \oplus \ D|(\cup_{i=n_j}^{r} W_i)$$

satisfies the statement as well. This proves 3.2.5. ∎

Lemma 3.2.6 *Let $D, D', D_A, D_B, D'_A, D'_B \in \mathcal{G}_m^k(Q)$ for some wqo Q and integers m, k. Suppose:*

- *$D = D_A \oplus D_B$,*
- *$D' = D'_A \oplus D'_B$, and*
- *D_A and D_B are strong minors of D'_A and D'_B, respectively.*

Then, D is a strong minor of D'.

Proof. Let ϕ_A, ϕ_B be strong minor mappings from D_A to D'_A and from D_B to D'_B, respectively. Let $X = \{x_1, \ldots, x_m\} = V(G_A) \cap V(G_B)$.

Since the union of two strongly-connected graphs with non-empty intersection is also strongly-connected, $\phi_A(x_i) \cup \phi_B(x_i)$ is strongly-connected subdigraph of G'. Define ϕ from G to G'' by

$$\phi(v) = \begin{cases} \phi_A(v) & \text{if } v \in V(G_A) \setminus X \\ \phi_B(v) & \text{if } v \in V(G_B) \setminus X \\ \phi_A(v) \cup \phi_B(v) & \text{if } v \in X. \end{cases}$$

Then, ϕ is a strong minor mapping from G to G'. Therefore D is a strong minor of D', and this proves 3.2.6. ∎

Lemma 3.2.7 *Let $D_A, D_C, D'_A, D'_C \in \mathcal{G}^k_m(Q)$ and $D'_B \in \mathcal{L}^k_m(Q)$ for some wqo Q and integers m, k. Suppose:*

- *$D = D_A \oplus D_C$,*
- *$D' = D'_A \oplus D'_B \oplus D'_C$, and*
- *D_A and D_C are strong minors of D'_A and D'_C, respectively.*

Then, D is a strong minor of D'.

Proof. Note that D'_B is contractible since D'_B is a link. Let $D'_B = (G'_B, P'_B, R'_B, l'_B)$ and let $R'_B = (R_1, \ldots, R_m)$. Let D'' be the (Q, m, k)-digraph obtained from D' by contracting each path of $R_1, \ldots, R_m$. Then, $D'' \cong D'_A \oplus D'_C$ is a strong minor of D'. From 3.2.6, D is a strong minor of D'', and hence a strong minor of D' as well. This proves 3.2.7. ∎

We restate 2.3.9.

Lemma 3.2.8 *Suppose Q_1, Q_2 and Q_3 are wqo. Let $Q = Q_1 \times Q_2^{<\omega} \times Q_3$. Then Q is a wqo.*

Next, we prove a key lemma for 3.1.6.

Lemma 3.2.9 *Let m, k be integers with $k > m \geq 1$. Suppose $\mathcal{G}^{k-1}_{m-1}(Q)$ and $\mathcal{G}^k_{m+1}(Q)$ are both wqo under strong minor containment for every wqo Q. Then $\mathcal{G}^k_m(Q)$ is a wqo under strong minor containment for every wqo Q.*

Proof. Let $D_1, D_2, \ldots$ be an infinite sequence of $\mathcal{G}_m^k(Q)$. We may assume D_i is non-trivial for every $i \geq 1$ because every trivial D_i has m vertices. Decompose each D_i as

$$D_i = D_i^1 \oplus \ldots \oplus D_i^{t_i}$$

as in 3.2.5. By 3.2.1, 3.2.2, and 3.2.4, $\mathcal{L}_m^k(Q)$ and $\mathcal{NC}_m^k(Q)$ are both well-quasi-ordered under strong minor containment. Therefore we may assume $t_i \geq 3$ for every $i \geq 1$. We apply 3.2.8 for $Q_1 = Q_2 = \mathcal{L}_m^k(Q)$, $Q_3 = \mathcal{L}_m^k(Q) \cup \mathcal{NC}_m^k(Q)$. Then there exist $i < j$ with $t_i \leq t_j$ and $1 = \alpha_1 < \ldots < \alpha_{t_i} = t_j$ such that:

- D_i^p is a strong minor of $D_j^{\alpha_p}$ with a strong minor mapping ϕ_p for every $1 \leq p \leq t_i$.

Then, by applying 3.2.7, it follows that D_i is a strong minor of D_j. This proves 3.2.9. ∎

Theorem 3.2.10 *Let m, k be integers with $k \geq m \geq 0$. Then $\mathcal{G}_m^k(Q)$ is a wqo under strong minor containment for every wqo Q.*

Proof. We proceed by induction on k. For fixed k, we use induction on $k - m$. For the base case $k = m$, every member $D \in \mathcal{G}_m^k(Q)$ is trivial and hence the statement holds. For the inductive step, since $\mathcal{G}_{m-1}^{k-1}(Q)$ and $\mathcal{G}_{m+1}^k(Q)$ are wqo by the inductive hypotheses, the statement follows from 3.2.9. This proves 3.2.10. ∎

Proof of 3.1.6. Let $G_1, G_2, \ldots$ be an infinite sequence of semi-complete digraphs with path-width at most k. Let Q be a wqo with $E(Q) = \{0\}$. For each $i \geq 1$, let P_i be a linked path-decomposition of G_i with $m(P_i) = 0$ and $pw(P_i) \leq k$. Let $R_i = ()$ be the empty sequence, let l_i be the constant mapping from $V(G_i)$ to $\{0\}$, and let $D_i = (G_i, P_i, R_i, l_i)$. Since $D_i \in \mathcal{G}_0^{k+1}(Q)$ for each $i \geq 1$, there exist $j > i \geq 1$ such that D_i is a strong minor of D_j by 3.2.10. Therefore G_i is a strong minor of G_j. This proves 3.1.6. ∎

3.3 Counterexamples

In this section, we give counter-examples to the analogue of 3.1.2 for some super-classes of the class of all semi-complete digraphs. We introduce two generalizations of semi-complete digraphs. Note that a semi-complete digraph is a "simple" super-tournament, and also, it is a simple digraphs with "stability number at most one". One generalization is the class of all super-tournaments, and another is the class of simple digraphs with stability number at most two.

3.3.1 Supertournaments

We give a counter-example to show that the class of all super-tournaments is not a wqo under strong minor containment; and indeed, the subclass of all super-tournaments with no three edges mutually parallel is not a wqo.

For $i \geq 3$, let T_i be a transitive tournament with i vertices $v_1, \dots, v_i$ such that $v_a v_b \in E(T_i)$ if and only if $a < b$. Let G_i be a super-tournament obtained from T_i by doubling the following i edges:

$$v_1 v_2,\ v_2 v_3,\ \dots,\ v_{i-1}, v_i,\ v_1 v_i.$$

("Doubling" means adding a new edge with the same head and tail as the given edge.) We claim that G_i is not a strong minor of G_j for $j > i \geq 3$. First, we cannot contract anything from G_j because G_j has no directed cycles. Therefore G_j must have G_i as a subdigraph in order to contain it as a strong minor. However, note that the underlying undirected graph of G_i has a cycle of length i with all edges doubled, while G_j does not. Therefore G_i is not a subdigraph of G_j and hence, not a strong minor of G_j.

3.3.2 Simple digraphs with $\alpha = 2$

We give a counter-example for the class of simple digraphs with stability number at most two. For $i \geq 2$, let $A_i = \{a_1, a_2, a_3\}$, $B_i = \{b_1, b_2, b_3\}$, $C_i = \{c_1, \dots, c_i\}$, and $D_i = \{d_1, \dots, d_i\}$. Let G_i be a simple digraph with stability number two defined as follows (see Figure 3.6.)

- $V(G_i)$ is the disjoint union of A_i, B_i, C_i and D_i,
- $G_i|A_i$, $G_i|B_i$ are directed triangles,
- $G_i|C_i$, $G_i|D_i$ are transitive tournaments,
- A_i is complete to C_i,
- D_i is complete to B_i,

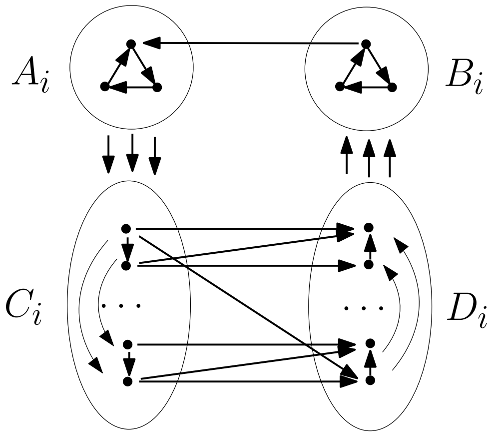

Figure 3.6: G_i

- b_1a_1 is the only edge between A_i and B_i.
- Each edge between C_i and D_i goes from C_i to D_i, and the bipartite graph underlying $(C_i \cup D_i, \delta^+(C_i, D_i))$ is a Hamiltonian cycle.
- There are no other edges between $A \cup C_i$ and $B \cup D_i$.

We claim that there do not exist $j > i \geq 2$ such that G_i is a strong minor of G_j. Suppose G_i is a strong minor of G_j with strong minor mapping ϕ. First, observe the following fact.

- If H is a strongly-connected subdigraph of G_j with $|V(H)| \geq 2$, then either $A_j \subseteq V(H)$ or $B_j \subseteq V(H)$ or $b_1a_1 \in E(H)$.

Therefore once we contract a non-trivial strongly-connected subdigraph H of G_j, there do not exist two disjoint directed cycles. That means we cannot contract anything in G_j if we hope to obtain G_i as a strong minor. Therefore ϕ must be a subdigraph mapping and $\phi(A_i) = G_j|A_j$ and $\phi(B_i) = G_j|B_j$ in order to preserve the existence of two disjoint directed cycles with an edge between them. Then $\phi(C_i)$ and $\phi(D_i)$ are subdigraphs of $G_j|C_j$ and $G_j|D_j$, respectively. However, the underlying bipartite graph of $(C_j \cup D_j, \delta^+(C_j, D_j))$ is a cycle of length $2j$, and hence does not contain a cycle of length $2i$ as a subgraph, a contradiction. This proves our claim.

Chapter 4

Immersions

4.1 Introduction

In this chapter, we deal with immersion relations in digraphs. A (di)graph H is *immersed* in another (di)graph G if the vertices of H are mapped to vertices of G and the edges of H to (directed) paths of G, joining the corresponding pairs of vertices and pairwise edge-disjoint. Unlike minors, immersions can be easily extended to digraphs by simply replacing paths with directed paths. There are several versions of immersion relations [8], [9]. Let us see the precise definitions of them. Let G, H be (di)graphs. An *infusion* of H in G is a map ϕ such that:

- $\phi(v) \in V(G)$ for each $v \in V(H)$
- for each edge $e = uv$ of H, $\phi(e)$ is a (directed) path of G from $\phi(u)$ to $\phi(v)$
- if $e, f \in E(H)$ are distinct, then $\phi(e)$, $\phi(f)$ have no edges in common (they may share vertices).

If we add the condition

- $\phi(u) \neq \phi(v)$ for distinct $u, v \in V(H)$,

we call the relation *weak immersion.* Moreover, a weak immersion is called *a strong immersion* if it additionally satisfies the condition

- if $u \in V(H)$ and $e \in E(H)$, and e is not incident with v in H, then $\phi(v)$ is not a vertex of the path $\phi(e)$.

In [24], Robertson and Seymour proved that the class of all graphs is a wqo under weak immersion.

Theorem 4.1.1 *The class of all graphs is a wqo under weak immersion.*

It was a conjecture of Nash-Williams (the analogue of 4.1.1 for strong immersion is still open). Not surprisingly, the analogue of 4.1.1 for digraphs is false even with infusion. To see this, for an integer $k \geq 1$, let A_k be the acyclic digraph with $2k$ vertices such that the underlying undirected graph is a cycle of length $2k$ and every vertex has either out-degree zero or in-degree zero. Then, A_n cannot be infused in A_m for every $m > n$. However, the wqo statement holds for tournaments. In [5], Chudnovsky and Seymour proved the following.

Theorem 4.1.2 *The class of all tournaments is a wqo under strong immersion.*

In this chapter, we focus on algorithmic problems on immersions. For graphs, the problem of testing H-immersion in an input graph G can be solved in polynomial-time for all H. However, the same problem for digraphs is NP-complete for most digraphs H. (It can be solved in polynomial-time if the input has bounded independence number, see [8] for instance.) Therefore, we focus on small digraphs H, such that there exists a polynomial time algorithm to test H-immersion in general input digraphs.

4.2 Testing for immersion

In [2], Chudnovsky, Fradkin, and Seymour proved that for every fixed digraph H, there is a polynomial time algorithm to test whether H can be strongly (or weakly) immersed in a given semi-complete digraph.

However, if the input is a general digraph, and H is large enough, then the problem is NP-complete. For example, let H be a digraph with two vertices v_1, v_2 and four edges, namely a loop at v_1, a loop at v_2, and edges v_1v_2, v_2v_1. Using the NP-completeness result in [7], it is proved in [2] that H-immersion testing problem is NP-complete. For integers, $i, j \geq 0$, we denote by $I_{i,j}$ the digraph with two vertices v_1, v_2 and i parallel edges from v_1 to v_2 and j parallel edges from v_2 to v_1. Then, one can show that testing $I_{2,2}$ is also an NP-complete problem by similar argument in [2].

Although the problem is hard, if the fixed digraph H is small enough, then there exists a polynomial time algorithm. Here, we present such polynomial time algorithms. First, testing $I_{0,k}$ immersion is easy for every integer $k \geq 1$ because $I_{0,k}$ can be immersed in an input digraph G if and only if there exists k-edge disjoint paths from u to v for some pair of vertices u, v in G. Therefore all we

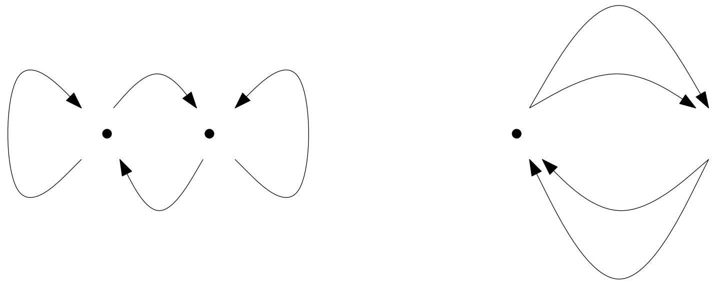

Figure 4.1: Digraphs G such that G-immersion testing problems are NP-complete.

need to do is to check whether there exists an edge-cutset of size less than k from u to v in G, for every pair u, v.

For an integer $k \geq 1$, let L_k be a digraph with a single vertex v, and k loops at v. Then, testing L_k-immersion is also easy. For a given input digraph G, and a vertex u of G, we denote by G_u the digraph obtained from G by splitting the vertex u. More precisely,

- $V(G_u) = (V(G) \setminus \{u\}) \cup \{u_1, u_2\}$, and
- $E(G_u) = E(G \setminus u) \cup \{u_1 w \ : \ w \in N_G^+(u)\} \cup \{w u_2 \ : \ w \in N_G^-(u)\}$.

Then, L_k can be immersed in G if and only if there exists a vertex u of G, such that there are k-edge disjoint paths from u_1 to u_2 in G_u. Therefore what we need to check is, for each vertex u of G, whether there is an edge-cutset of size less than k from u_1 to u_2 in G_u.

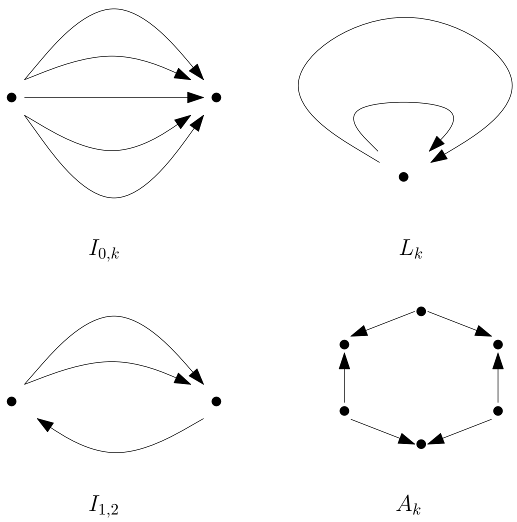

Figure 4.2: Digraphs G such that G-immersion testing problems are polynomial time solvable.

4.2.1 Testing $I_{1,2}$

Here, we present a structure theorem for digraphs not containing $I_{1,2}$-immersion, and prove that there is a polynomial time algorithm to test $I_{1,2}$-immersion.

First, we state a classical theorem for strongly-connected digraphs [16]. Let G be a digraph. An *ear* of G is a directed path (or a directed cycle) P of length at least one in G, such that all internal vertices of P have in-degree and out-degree one in G (if P is a directed cycle, then the internal vertices are all vertices of P except one). In particular, every edge of G is an ear. An *ear-decomposition* of G is a sequence of digraphs $G_0, G_1, \ldots, G_k$, where G_0 is a digraph with one vertex and no edge, $G_k = G$, and G_{i+1} has an ear P such that deleting P except the end vertices of P from G_{i+1} yields G_i for every $i = 0, \ldots, k-1$. Then the following statement holds [16].

Theorem 4.2.1 *A digraph G is strongly-connected if and only if G has an ear-decomposition.*

We say an ear P is a *path-ear* in case P is a directed path. Otherwise P is a directed cycle and we call P a *cycle-ear*. Let G be a strongly-connected digraph with its ear-decomposition $G_0, G_1, \ldots, G_k$, where G_{i+1} can be obtained by adding a cycle-ear from G_i for every $i = 0, \ldots, k-1$. We call such ear-decomposition *all-cycle ear-decomposition.* We use 4.2.1 to prove the following structure theorem for digraphs not containing $I_{1,2}$-immersion.

Theorem 4.2.2 *Let G be a digraph. Then $I_{1,2}$ cannot be strongly (or weakly) immersed in G if and only if for every strong component C of G, C has an all-cycle ear-decomposition.*

Proof. Let G be a strongly-connected digraph with all-cycle ear-decomposition. Observe that G is an Eulerian digraph since for every vertex, its out-degree is the same as its in-degree. In particular, for every distinct $u, v \in V(G)$, there exist two edge-disjoint directed paths one from u to v and the other from v to u. On the other hand, for every distinct $u, v \in V(G)$, there are no two edge-disjoint directed paths both from u to v, by the definition of all-cycle ear-decomposition.

Let G be a digraph. Suppose there exists a strong component C of G such that C does not have an all-cycle ear-decomposition. Let $G_0, G_1, \ldots, G_k$ be an ear-decomposition of C where G_{i+1} can be obtained by adding a path-ear from G_i for some i.

Choose such i as small as possible, so that G_i has an all-cycle ear-decomposition. Let P be the path-ear in G_{i+1}, and let u and v be the source and the terminal of P, respectively ($u \neq v$). From the above observation, there exist two edge-disjoint paths in G_i, one from u to v and the other from v to u. Then, together with P, we have $I_{1,2}$-immersion in G_{i+1} (we have three edge-disjoint directed paths, two of them are from u to v and the other one from v to u).

Conversely, suppose every strong component C of G has all-cycle ear decomposition. Notice that if G has a $I_{1,2}$-immersion, then the image must be in a single strong component of G. Therefore we may assume G is strongly-connected. Then, by the above observation, for every distinct $u, v \in V(G)$, there are no two edge-disjoint paths both from u to v. Therefore $I_{1,2}$ cannot be immersed in G. This proves 4.2.2. ▮

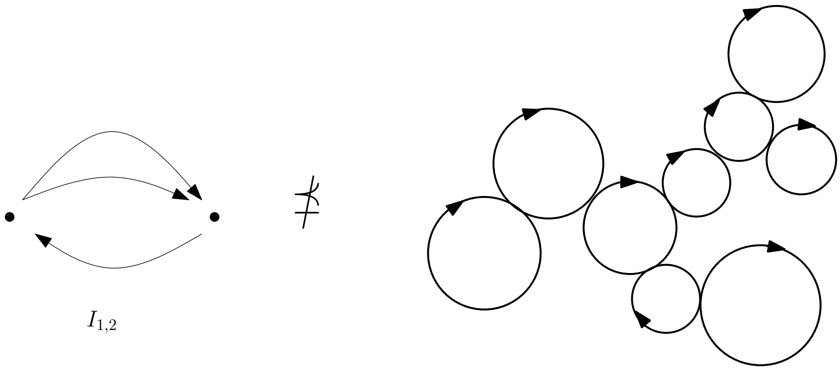

Figure 4.3: A strongly-connected digraph does not have $I_{1,2}$-immersion if and only if it has all-cycle ear-decomposition.

Lemma 4.2.3 *Suppose G is a loopless strongly-connected digraph, and G has an all-cycle ear-decomposition. Then, either $|V(G)| = 1$ or G is a directed cycle, or there exists a vertex v such that $G \setminus v$ is not weakly-connected.*

Proof. Let $G_0, G_1, \ldots, G_k$ be the all-cycle ear-decomposition of G. If $G = G_0$, then $|V(G)| = 1$. If $G = G_1$, then G is a directed cycle. Otherwise, let v be the unique vertex in $|V(G_{k-1}) \cap V(G)|$. Since G is loopless, $G \setminus v$ has at least two non-null weak component. This proves 4.2.3. ▮

Now, we give an $I_{1,2}$-immersion testing algorithm.

Theorem 4.2.4 *There is an algorithm as follows:*

- **Input :** *A digraph G.*
- **Output :** *Decides whether $I_{1,2}$ can be strongly (or weakly) immersed in G or not.*

- **Running time :** $O(|V(G)|^2(|V(G)| + |E(G)|))$.

Proof. For a given digraph G, we can compute its strong components in running time $O(|V(G)| + |E(G)|)$ (see for instance, [25]). Notice that $I_{1,2}$ can be immersed in G if and only if there exists a strong component C of G such that $I_{1,2}$ can be immersed in G. Therefore we run the algorithm for each strong component of G. We may assume G is loopless since deleting loops does not change the answer. For each strong component C, check if C is a directed-cycle. If it is, then since $I_{1,2}$ cannot be immersed in a directed cycle, the answer for C is no. If not, we search for a cut-vertex in C. That is, for each vertex $v \in V(G)$, check whether $G \setminus v$ is weakly-connected or not. This can be done in running time $O(|V(G)|(|V(G)| + |E(G)|))$. If there is no such cut-vertex, then by 4.2.2 and 4.2.3, $I_{1,2}$ can be immersed in C, and the answer is yes. Suppose there exists a cut-vertex v. For each cut-vertex v and a weak component C' of $C \setminus v$, check if $C|C' \cup \{v\}$ is a directed cycle. If there are no such pair (v, C'), then again, $I_{1,2}$ can be immersed in C, and the answer is yes. If there is such a pair (v, C'), then deleting this cycle-ear (we delete at least one internal vertex since C' is loopless) does not change the answer. So far, we either have the answer or the number of vertices of G is decreased by at least one. Therefore by iterating this procedure, we can get the answer in running time $O(|V(G)|^2(|V(G)| + |E(G)|)$. This proves 4.2.4. ∎

4.2.2 Testing an alternating cycle

For an integer $k \geq 1$, we denote by A_k the acyclic digraph with $2k$ vertices such that the underlying undirected graph is a cycle of length $2k$, and every vertex has either out-degree zero or in-degree zero. We call this A_k an *alternating* cycle.

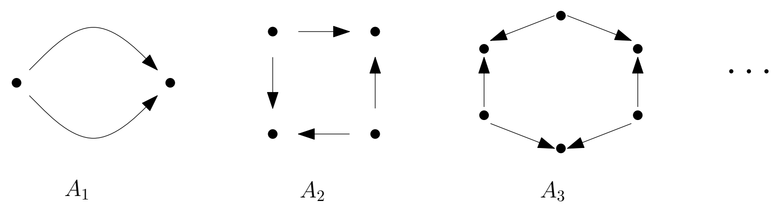

Figure 4.4: Alternating cycles

A digraph has *cutwidth* at most k if its vertices can be ordered $\{v_1, \ldots, v_n\}$ in such a way that for each j, there are at most k edges from $\{v_j, \ldots, v_n\}$ to $\{v_1, \ldots, v_{j-1}\}$. For instance, an acyclic digraph has cutwidth zero.

Here, we prove that there exists a constant t (only depends on k) such that for every digraph G, either A_k can be infused in G or cutwidth of G is at most t. Using this fact, we show that there is a polynomial time algorithm to test an A_k-infusion. First, we state a theorem from [20] proved by Reed, Robertson, Seymour, and Thomas.

Theorem 4.2.5 *For every integer $k \geq 0$ there exists an integer $t_k \geq 0$ such that for every digraph G, either G has k vertex-disjoint directed cycles, or G can be made acyclic by deleting at most t_k vertices.*

The following is an easy corollary of 4.2.5.

Theorem 4.2.6 *For every integer $k \geq 0$ there exists an integer $t_k \geq 0$ such that for every digraph G, either G has k edge-disjoint directed cycles, or G can be made acyclic by deleting at most t_k edges.*

Proof. Let H be the line digraph of G, that is:

- $V(H) = E(G)$,
- for $e, f \in E(G)$, $ef \in E(H)$ if and only if the head of e is the tail of f.

Suppose G does not have k edge-disjoint directed cycles. Then, H does not have k vertex-disjoint directed cycles. Therefore by 4.2.5, H can be made acyclic by deleting t_k vertices (of H). Then, G can be made acyclic by deleting the corresponding t_k edges in G. This proves 4.2.6. ▮

Let G be a digraph, and let $A, B \subseteq V(G)$ be two disjoint non-empty sets of vertices of G. Let P_1 be a directed path from A to B in G, and let P_2 be a directed path from B to A in G, and every internal vertex of P_1 and P_2 does not belong to A or B (P_1 and P_2 are not necessarily edge-disjoint). Suppose $R_1, R_2, \ldots, R_s$ (this might be empty collection) are the vertex-disjoint directed paths such that $P_1 \cap P_2 = R_1 \cup R_2 \cdots \cup R_s$, and $R_1, R_2, \ldots, R_s$ are in P_1-order (in other words, if we walk along P_1, we pass through $R_1, R_2, \ldots, R_s$ in order). We call $P_1 \cup P_2$ a *ladder connecting A and B in G* if: $R_1, R_2, \ldots, R_s$ appears in P_2 in the reverse order (in other words, if we walk along P_2, we pass through $R_t, \ldots, R_1$ in order).

Lemma 4.2.7 *Let G be a strongly-connected digraph and let A, B be two disjoint non-empty sets of vertices of G. Then, there exists a ladder connecting A and B in G.*

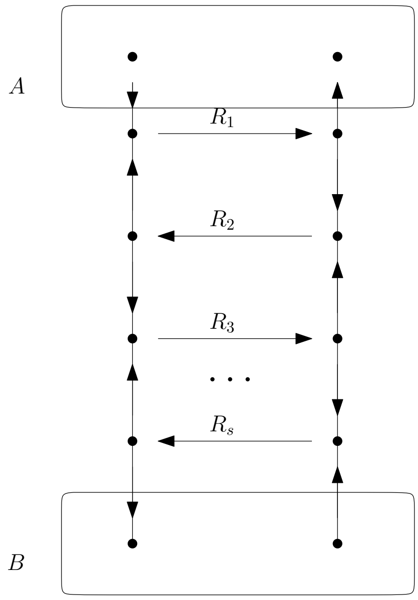

Figure 4.5: A ladder connecting A and B

Proof. Since G is strongly connected, there exists a directed path P_1 from A to B and a directed path P_2 from B to A. Take such P_1 and P_2 with $P_1 \cup P_2$ minimal. By the minimality, every internal vertex of P_1 and P_2 does not belong to A or B. Let $R_1, R_2, \ldots, R_s$ be the vertex-disjoint directed paths both in P_1 and P_2. Label them so that they are in P_1-order. Let $R_{i_1}, R_{i_2}, \ldots, R_{i_s}$ be their P_2-order. Suppose $i_1, i_2, \ldots, i_s$ is not monotone decreasing. Choose j such that $i_j < i_{j+1}$. Then, we can delete the edges of the sub-directed path of P_1 from the terminal of R_{i_j} to the source of $R_{i_{j+1}}$, and reroute P_1 by using the sub-directed path of P_2, which contradicts the minimality of $P_1 \cup P_2$. This proves 4.2.7. ∎

We say a digraph G has *linkage level at least one* if it is strongly-connected and $|E(G)| \geq 1$. Inductively, for an integer $k \geq 2$, we say a digraph G has its *linkage level at least* k if $V(G)$ can be partitioned into two nonempty sets A and B such that:

- both $G|A$ and $G|B$ have their linkage level at least $k-1$,
- G is the union of $G|A$ and $G|B$ together with a ladder connecting A and B.

For a digraph G with $u, v \in V(G)$ (not necessarily distinct), we denote by $G + uv$ the digraph obtained from G by adding an edge uv (if $uv \in E(G)$, then we add a parallel edge).

Lemma 4.2.8 *Let G be a digraph with linkage level at least one (strongly-connected and $|E(G)| \geq 1$). For every pair of vertices $u, v \in V(G)$ (not necessarily distinct), $G + uv$ contains A_1-infusion using the edge uv.*

Proof. There exists a directed path from u to v since G is strongly-connected (in case $u = v$, take a directed cycle containing u). Therefore in $G + uv$, there are two disjoint directed paths from u to v, and this implies $G + uv$ contains an A_1-infusion using uv. This proves 4.2.8. ∎

We generalize 4.2.8 inductively.

Lemma 4.2.9 *Let $k \geq 1$ be an integer. Let G be a digraph with linkage level at least k. For every pair of vertices $u, v \in V(G)$ (not necessarily distinct), each of $A_1, A_2, \ldots, A_k$ can be infused in $G+uv$ using the edge uv. In particular, if G has a subdigraph with linkage level at least $k + 1$, then G has A_k-infusion.*

Proof. The statement holds for $k = 1$ by 4.2.8. Let (A, B) be a partition of $V(G)$ such that both $G|A$ and $G|B$ have their linkage level at least $k - 1$, and G is the union of $G|A$ and $G|B$ together with a ladder connecting A and B. Let P_1 be the directed path from $a_1 \in A$ to $b_1 \in B$, and let P_2 be the directed path from $b_2 \in B$ to $a_2 \in A$ as in the definition of ladder. In each of the following four cases, we first prove the statement when P_1 and P_2 are edge-disjoint, and then, prove it when they are not. (Proving when P_1 and P_2 are edge-disjoint is more difficult.)

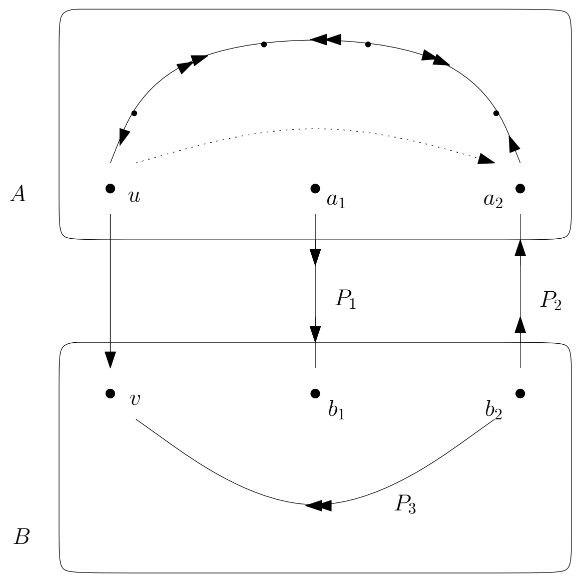

Figure 4.6: Case 1

Case 1: $u \in A$**,** $v \in B$**.**

In this case, we do not need to assume that P_1 and P_2 are edge-disjoint or not, since we do not use P_1 in the proof. By induction hypothesis, if we add a fake edge ua_2 to $G|A$, then there exists an A_i infusion in $G|A + ua_2$ using the edge ua_2 for each $i = 1, \ldots, k-1$. Take a directed path P_3 from b_2 to v in $G|B$ (if $b_2 = v$, then take a directed cycle in $G|B$ containing v). Now, for each A_i-infusion using the edge ua_2, delete the edge ua_2 and glue three edge-disjoint directed paths uv, P_3, and P_2. Then, this form an A_{i+1}-infusion in G using uv $(i = 1, \ldots, k-1)$. From 4.2.8, there exists an A_1-infusion in G using uv as well.

Case 2: $u \in A$**,** $v \in V(P_1)$**.**

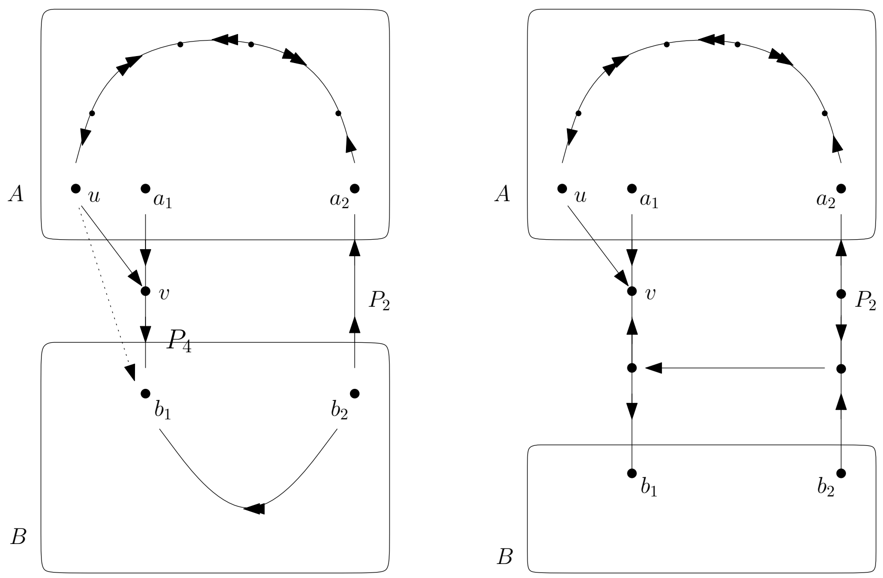

Figure 4.7: Case 2

First, suppose P_1 and P_2 are edge-disjoint. Let P_4 be the directed path from v to b_1 in P_1. From the previous case, we know there exists an A_i-infusion using the fake edge ub_1 for each $i = 1, \ldots, k$. For each of these A_i-infusions, delete the fake edge ub_1 and replace it with the directed path $uv + P_4$ from u to b_1. Then, it is an A_i-infusion using uv.

Second, suppose P_1 and P_2 are not edge-disjoint. then it is even easier to obtain A_i-infusion by using the ladder to gain two more change-over points (see Figure 4.7).

Case 3: $u \in A$, $v \in V(P_2)$.

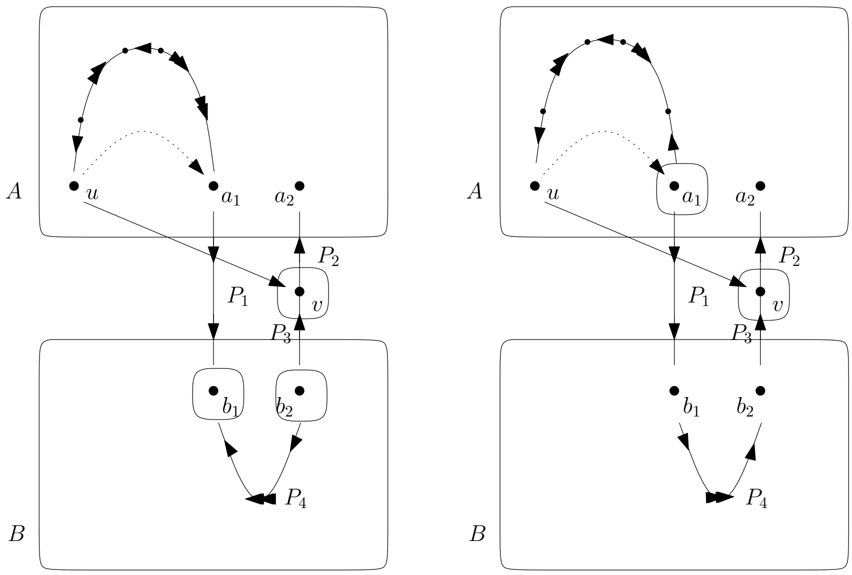

Figure 4.8: Case 3: The direction of P_4 depends on whether a_1 is a change-over vertex or not.

First, suppose P_1 and P_2 are edge-disjoint. By induction hypothesis, if we add a fake edge ua_1 to $G|A$, then there exists an A_i infusion in $G|A + ua_1$ using the edge ua_1 for each $i = 1, \ldots, k-1$. Take a directed path P_3 from b_2 to v in P_2. Now, for each A_i-infusion using the edge ua_1, if a_1 is the change-over vertex, we take as a directed path from b_2 to b_1, and if not, then take a directed path from b_1 to b_2 (say P_4). Now delete the edge ua_1 from the A_i infusion using ua_1 and glue edge-disjoint directed paths uv, P_3, P_4, and P_1. In the former case (a_1 is the change-over vertex in A_i infusion), we lost one change-over point (a_1), and we gain three more change-over points, namely v, b_2, and b_1. In the latter case (a_1 is not the change-over vertex in A_i infusion), we lost no change-over point and we gain two more change-over points, namely v and a_1. Therefore in either case, we have A_{i+1} infusion using uv.

Second, suppose P_1 and P_2 are not edge-disjoint. Then, again it is easier to obtain A_i-infusion by using the ladder to gain two more change-over points.

Case 4: $u, v \in A$.

In this case, for $i = 1, \ldots, k-1$, we already have A_i infusion using extra uv in $G|A + uv$ by induction hypothesis. Therefore it is enough to prove the existence of an A_k infusion using uv in

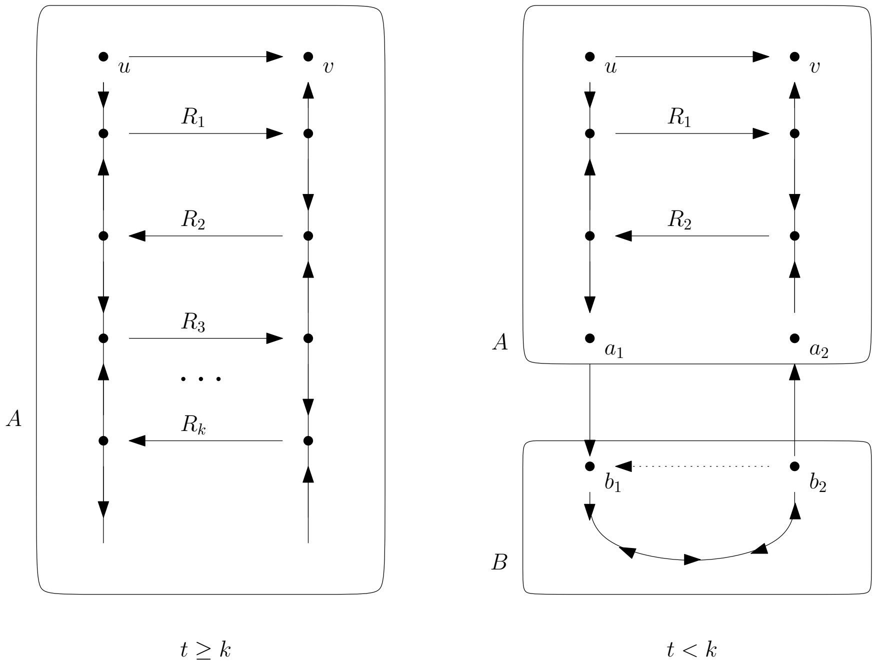

Figure 4.9: Case 4

$G + uv$. Take a directed path P_3 from u to a_1 and a directed path P_4 from a_2 to v in $G|A$ (P_3 and P_4 may not be edge-disjoint.) Choose such P_3 and P_4 with minimal union. Then, $P_3 \cup P_4$ is a ladder connecting $\{u, v\}$ and $\{a_1, a_2\}$. Let $R_1, R_2, \ldots, R_t$ be the directed sub-paths in both P_3 and P_4 in P_3-order. If $t \geq k$, then it is easy to see that there exists an A_k-infusion in $G|A + uv$ using uv (see Figure 4.9). Therefore we may assume $t < k$. In this case, in $G|B$, add a fake edge b_2b_1 and find an A_{k-t} infusion using b_2b_1 in $G|B + b_2b_1$, and delete the fake edge b_2b_1 from it. Then, by gluing this alternating path, we can obtain A_k infusion in $G + uv$ (see Figure 4.9).

By symmetry, the same argument works for the other cases. This proves 4.2.9. ∎

Theorem 4.2.10 *Let k be an integer. Then there exists an integer t such that for every digraph G, either A_k can be infused in G or G has cutwidth at most t.*

Proof. Set $t = kt_{k+1}$ where t_k is as in 4.2.6. We show that the statement holds with this t. We assume that G has cutwidth larger than t, and prove the existence of a subdigraph of G with linkage level at least $k + 1$. Then, we are done since the subdigraph has A_k-infusion by 4.2.9.

Now suppose G has cutwidth larger than t, and let m_1 be the maximum number of vertex-disjoint subdigraphs of linkage level at least one (strongly-connected and $|E(G)| \geq 1$) in G. Take such vertex-disjoint subdigraphs $E_1, \ldots, E_{m_1}$ with maximal union.

(1) *Every E_i can be made acyclic by deleting all its loops and at most t_{k+1} non-loop edges. In particular, its cutwidth is at most t_{k+1}.*

Fix i. We may assume E_i is loopless. By the maximality of m_1, there are no two vertex-disjoint directed cycles in E_i. Then, there are no $k+1$ edge-disjoint directed cycles in E_i because otherwise, they all meet each other and we get A_k-infusion by simply walking along a directed cycle and making $2k$-changes of directions via visiting each of k edge-disjoint cycles. Therefore we can make E_i acyclic by deleting t_{k+1} edges. This proves (1).

Now, let G_1 be the digraph obtained from G by contracting each of $E_1, \ldots, E_{m_1}$ to a vertex (say $v_1, \ldots, v_{m_1}$ are the corresponding contracted vertices).

(2) *Every subdigraph of G_1 with linkage level at least one, contains at least two vertices from $\{v_1, \ldots, v_{m_1}\}$.*

Every subdigraph of G_1 with linkage level at least one, contains at least one member of $\{v_1, \ldots, v_{m_1}\}$ by the maximality of m_1. Suppose some subdigraph E of G_1 with linkage level at least one, contains only one member from $\{v_1, \ldots, v_m\}$. Say v_1 is the vertex in E. Then, $E_1 \cup E$ is a subdigraph of linkage level at least one in G ($E_1 \cup E$ is strongly-connected), and vertex-disjoint from each of $E_2, \ldots, E_{m_1}$. This contradicts the maximality of the union. This proves (2).

(3) *Deleting $t - t_{k+1}$ edges cannot make G_1 acyclic.*

From (2), G_1 is loopless. Suppose there exists a set X of edges of G_1 with $|X| \leq t - t_{k+1}$ such that $G_1 \setminus X$ is acyclic. Let σ be the acyclic ordering of $V(G_1 \setminus X)$ such that every edge goes forward in σ-order. For each i, take an ordering σ_i of E_i so that the cutwidth of E_i is at most t_k (this is possible by (1)). Now, for each v_i in σ, we substitute σ_i for v_i. Then, we get an ordering of $V(G)$, and it is clear that in this ordering, the cutwidth of G is at most $t_{k+1} + |X| \leq t$. This contradicts our assumption. This proves (3).

Then, we iterate this procedure to get $G_2, G_3, \ldots$, etc. More precisely, let m_2 be the maximum number of vertex-disjoint subdigraphs of linkage level at least one in G_1. Take such vertex-disjoint subdigraphs $E'_1, \ldots, E'_{m_2}$ with maximal union. Let G_2 be the digraph obtained from G_1 by contracting each of $E'_1, \ldots, E'_{m_2}$. Then, by the same logic, each E'_i has cutwidth at most t_{k+1}, and deleting $t - 2t_{k+1}$ edges cannot make G_2 acyclic. We iterate this procedure until we obtain G_k. Then, deleting $t - kt_{k+1} = 0$ edges cannot make G_k acyclic, and this means there exists a subdigraph of G_k which linkage level at least one.

Let C be the subdigraph of G_k with linkage level at least one. Then C meets at least two contracted vertices u, v from G_{k-1}. Let E_u, E_v be the two subdigraphs of G_{k-1} correspond to u, v, respectively. We un-contract those two vertices u, v from C, and call it C' (C' is a subdigraph of G_{k-1} containing E_u and E_v). Then, since C' is strongly-connected, and E_u, E_v are vertex-disjoint, there exists a ladder connecting E_u and E_v in C'. Therefore C' is a linkage level at least two in G_{k-1}. We keep doing this un-contracting procedure until we reach $G_0 = G$ level (the concept of linkage level is designed for this). This proves the existence of a subdigraph of linkage level at least $k+1$ in G as desired. This proves 4.2.10. ∎

Now we explain the A_k-infusion testing algorithm. Let G be an input digraph. First, for every set X of edges of G with $|X| \leq (k-1)t_{k+1}$, delete X from G and test if every strong component of $G \setminus X$ can be made acyclic by deleting all its loops and at most t_{k+1} non-loop edges (this can be done in polynomial-time). If it is impossible, then by (1) and (3) in the proof of 4.2.10, G contains A_k-infusion. Otherwise, we get an ordering of $V(G)$ with its cutwidth at most kt_{k+1}. Now we run k-edge disjoint paths problem for digraphs with bounded cutwidth, in order to check if G has A_k-infusion (for every possible embedding of vertices, check if there are edge disjoint paths joining the corresponding pairs). This can be done in polynomial time by using dynamic programming. We roughly describe how it works. For each partition ($\{v_1, \ldots, v_i\}$,$\{v_{i+1}, \ldots, v_n\}$) of the cutwidth ordering, there are at most t many edges going backward. Notice that if there is a solution of k-edge-disjoint paths problem, each of those at most t edges either used for linking i-th pair ($i \in \{1, \ldots, k\}$) or not used. Therefore there are at most $(k+1)^t$ possible combinations (telling which edge is used for which pair) for backward edges passing the cut. Also, notice that if there is a solution, it uses at most $t+k$ many forward edges passing the cut. Therefore there are at most $\binom{|E(G)|}{t+k}(k+1)^{t+k}$ possible combinations for forward edges. This means for each cut, there are at most $\binom{|E(G)|}{t+k}(k+1)^{t+k}(k+1)^t$ possible candidates correspond to feasible solutions. Now we use dynamic programming. From the knowledge of the feasible candidates of the previous cut, we can enumerate every feasible candidates of the next cut by comparing all possible pairs. This can be done in polynomial time.

Chapter 5

Tournaments and Coloring

5.1 Introduction

Let $\mathcal{G}$ be a class of tournaments. We say a set $\mathcal{T}$ of tournaments is *heroic in* $\mathcal{G}$ if every tournament in $\mathcal{G}$, not containing any member of $\mathcal{T}$ as a subtournament, can be partitioned into a bounded number of transitive tournaments. If $\mathcal{G}$ is the class of all tournaments, then we just say $\mathcal{T}$ is heroic. In particular, if $\mathcal{T}$ has size one, then the element of $\mathcal{T}$ is called a *hero*. In [1], the authors explicitly construct all heroes. Here, we study finite heroic sets.

A heroic set can be defined for graphs as well [6]. A set of graphs $\mathcal{H}$ is called *heroic* if every graph, not containing any member of $\mathcal{H}$ as an induced subgraph, has a bounded cochromatic number. It is an open conjecture independently proposed by Gyárfás and Sumner that:

Conjecture 5.1.1 *Let K be a clique and T be a tree. Then the set $\{K, T\}$ is heroic.*

Assuming 5.1.1, Chudnovsky and Seymour characterized all finite heroic sets in [6].

Theorem 5.1.2 *If 5.1.1 is true, then a finite set of graphs is heroic if and only if it contains a clique partition graph, a complete multipartite graph, a forest, and the complement of a forest.*

Here, we propose a conjecture, which is an analogue of 5.1.2 for tournaments, and we prove it for some special cases. We start with some definitions. For a tournament G, the *chromatic number* $\chi(G)$ is the smallest number of disjoint transitive subtournaments that G can be partitioned. It is the same as the smallest number of colors needed to color the vertices of G so that no directed triangle is monochromatic. We say a class of tournament $\mathcal{G}$ has bounded chromatic number if there exists an integer k such that $\chi(G) \leq k$ for every $G \in \mathcal{G}$. For tournaments G, H, we say G is H-free if G has no subtournament isomorphic to H. For classes of tournaments $\mathcal{G}$ and $\mathcal{H}$, we say $\mathcal{G}$ is $\mathcal{H}$-free if G is H-free for every $G \in \mathcal{G}$ and every $H \in \mathcal{H}$. If $\mathcal{H}$ has one element H, then we write $\mathcal{G}$ is H-free instead of $\mathcal{H}$-free. A set $\mathcal{H}$ of tournaments is heroic if the class of all $\mathcal{H}$-free tournaments has bounded chromatic number. For a tournament G and an ordering σ of $V(G)$, we denote by G_σ the "back-edge" (undirected) graph such that:

- $V(G_\sigma) = V(G)$, and
- $uv \in E(G_\sigma)$ if $uv \in E(G)$ and v comes before u in σ-order.

For a tournament G, the *clique number* $\omega(G)$ is

$$\omega(G) := \min_{\sigma} \omega(G_\sigma).$$

Theorem 5.1.3 *Let G be a tournament. Then $\omega(G) \leq \chi(G)$.*

Proof. Let $k = \chi(G)$, and let $C_1, \ldots, C_k$ be a partition of $V(G)$ such that every $G|C_i$ is a transitive tournament. Let σ_i be the ordering of C_i such that $(G|C_i)_{\sigma_i}$ has no edge ($i = 1, \ldots, k$). Let σ be the ordering of $V(G)$ obtained by concatenating $\sigma_1, \ldots, \sigma_k$ in order. Then $\omega(G_\sigma) \leq k$ since every clique in G_σ meets each C_i with at most one vertex. Therefore $\omega(G) \leq \omega(G_\sigma) \leq k$. This proves 5.1.3. ∎

The definition of clique number for tournaments is not common. It makes sense as it provides a lower bound of the chromatic number, but as Stéphan Thomassé pointed out, it fails to be a lower bound of the fractional chromatic number, which is not the case for graphs [26]. For example, for a cyclic triangle C_3, let $T = \Delta(C_3, C_3, C_3)$. Then it is not hard to see that $\omega(T) = 3$. However, there are nine transitive subtournaments of size four covering each vertex four times, which means that the fractional chromatic number is at most $\frac{9}{4}$. However, the clique number for tournaments will be useful here, especially when we deal with the fourth class in later section.

We introduce four classes of tournaments that every finite heroic set must meet. We will first see that $\{H_1, H_2\}$ is heroic in some large classes of tournaments where H_1 and H_2 are from the first class and the second class, respectively.

5.2 Tournaments with large chromatic number

5.2.1 The first class : $\mathcal{S}$

For a tournament G and $X, Y \subseteq V(G)$, we write $X \Rightarrow Y$ if X and Y are disjoint and every vertex in X is adjacent to every vertex in Y. We write $v \Rightarrow Y$ for $\{v\} \Rightarrow Y$, and $X \Rightarrow v$ for $X \Rightarrow \{v\}$. For a tournament G, a non-empty partition (X, Y, Z) of $V(G)$ is called a *trisection* if $X \Rightarrow Y$, $Y \Rightarrow Z$, and $Z \Rightarrow X$, and we write $G = \Delta(A, B, C)$ where A, B, C are tournaments isomorphic to $G|X, G|Y, G|Z$, respectively.

We denote by S_1 the tournament with a single vertex, and for each integer $k \geq 2$, we inductively define S_k as $S_k = \Delta(S_{k-1}, S_{k-1}, S_1)$.

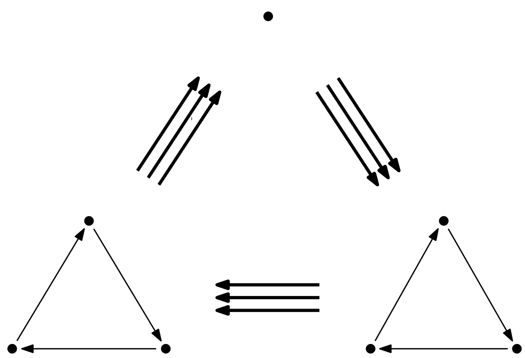

Figure 5.1: S_3

It is not hard to see that $\chi(S_k) \geq k$ (see theorem 2.1 in [1]).

Theorem 5.2.1 *For all integers $k \geq 0$, $\chi(S_k) \geq k$.*

A tournament G is called *type-one* if it is a subtournament of S_k for some k. We denote by $\mathcal{S}$ the class of all type-one tournaments. The following is an immediate corollary of 5.2.1.

Theorem 5.2.2 *Every heroic set of tournaments contains a type-one tournament.*

In fact, $\mathcal{S}$ has a member with arbitrary large clique number as well.

Theorem 5.2.3 *For all integers $k \geq 0$, there exists some k' such that $\omega(S_{k'}) \geq k$.*

We postpone the proof of 5.2.3 to later section.

5.2.2 The second class : $\mathcal{W}$

We say a graph G is a *doubled tree* if it can be obtained from a tree T by doubling each edge of T (i.e, every edge of G has its multiplicity two). A doubled tree has an Eulerian tour since every vertex has an even degree. Take an Eulerian tour of a doubled tree and let $v_1, \ldots, v_n (= v_1)$ be the sequence of the vertices of G in the Eulerian tour order. We define a tournament W_S as follows.

- $V(W_S) = \{u_1, \ldots, u_n\}$,
- for $i < j$, $u_j u_i \in E(W_S)$ if $v_i = v_j$, and
- otherwise $u_i u_j \in E(W_S)$.

We call such a tournament W_S a *tree-walk* tournament. We first give an example of a tree-walk tournament, namely C_5'. We denote by C_5 the tournament with five vertices and every vertex has in-degree two and out-degree two (there is a unique such tournament up to isomorphism). We denote by C_5' the tournament that can be obtained from C_5 by reversing the edge $u_1 u_2$ as in the Figure 5.2. (reversing $u_1 u_3$ from C_5 yields a tournament not isomorphic to C_5'.)

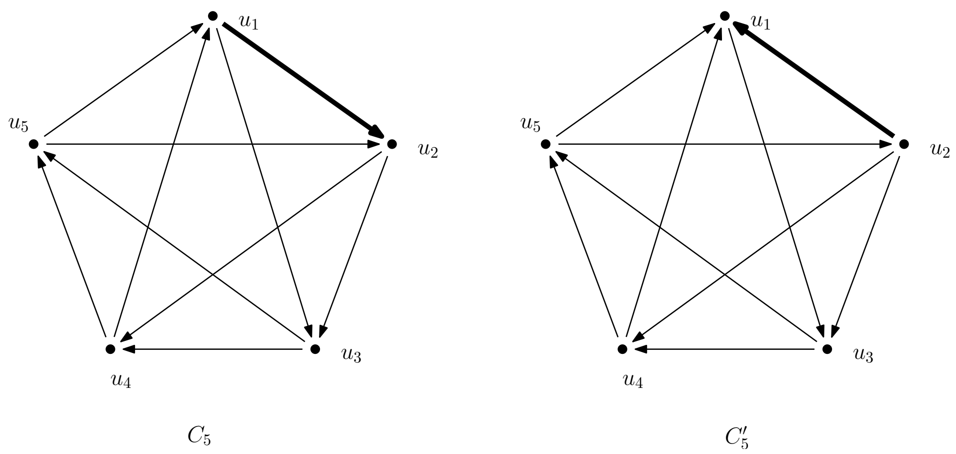

Figure 5.2: C_5 and C_5'

This C_5' is a tree-walk tournament since it can be obtained from a sequence v_1, v_2, v_1, v_3, v_1 of an Eulerian tour of a doubled tree T as in the Figure 5.3. It is not hard to see that C_5' is not type-one, and hence not a hero.

A tournament is said to be *type-two* if it is a subtournament of some tree-walk tournament. We denote by $\mathcal{W}$ the class of all type-two tournaments. Then $\mathcal{W}$ has unbounded chromatic number, and unbounded chromatic number.

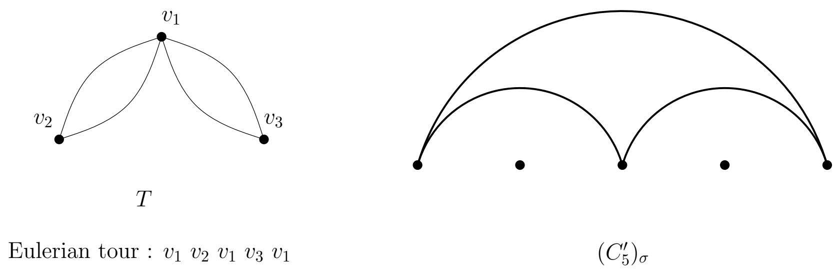

Figure 5.3: C_5' is a tree-walk tournament.

Theorem 5.2.4 *For all integers $k \geq 0$, there exists a tournament $W \in \mathcal{W}$ with $\chi(W) > k$.*

Theorem 5.2.5 *For all integers $k \geq 0$, there exists a tournament $W \in \mathcal{W}$ with $\omega(W) > k$.*

We postpone the proof of 5.2.4 and 5.2.5 to later section. The following is an immediate corollary of 5.2.4.

Theorem 5.2.6 *Every heroic set of tournaments contains a type-two tournament.*

Notice that not every type-two tournament is type-one. For instance, C_5' is type-two but not type-one. Conversely, not every type-one tournament is type-two. For instance, S_3 is type one but not type-two. In particular, neither C_5' nor S_3 is a hero. (But $\{C_5', S_3\}$ is heroic. we will see this in the last section.) In fact, a tournament G is a hero if and only if it is both type-one and type-two.

Theorem 5.2.7 *A tournament G is a hero if and only if it is both type-one and type-two.*

Proof. Suppose G is a hero. Then it is a subtournament of some type-one tournament and a subtournament of some type-two tournament because two classes have unbounded chromatic number. Since two classes are both closed under subtournaments, G is type-one and type-two. Conversely, suppose G is both type-one and type-two. We prove that G is a hero by applying induction on $|V(G)|$. We may assume $|V(G)| \geq 3$ and G is strongly-connected because it is proved in [1] that G is a hero if and only if every strong component of G is a hero. Since G is type-two, it is isomorphic to a subtournament G' of $S_k = \Delta(S_{k-1}, S_{k-1}, 1)$ for some $k \geq 2$. Choose such k as small as possible. Let $A, B, \{v\}$ be the partition of $V(S_k)$ such that $V(S_k|A) \cong V(S_k|B) \cong S_{k-1}$, $A \Rightarrow B$, $B \Rightarrow v$, and $v \Rightarrow A$.

(1) $v \in V(G')$.

Suppose $v \notin V(G')$. Then, either $V(G') \subseteq A$ or $V(G') \subseteq B$ since G' is strongly-connected. Then G' is a subtournament of S_{k-1} and this contradicts to the minimality of k. This proves (1).

Note that $V(G'|A)$ and $V(G'|B)$ are both non-empty since G' is strongly-connected. Therefore $G' = \Delta(V(G'|A), V(G'|B), v)$.

(2) *Either $V(G'|A)$ or $V(G'|B)$ is a transitive tournament.*

Suppose not. Then there exist two cyclic triangles C_1 and C_2 in $V(G'|A)$ and $V(G'|B)$, respectively. Then, together with v, C_1 and C_2 form a subtournament isomorphic to S_3 in G'. This contradicts the fact that G is type-two (S_3 is not type-two). This proves (2).

We may assume $V(G'|A)$ is a transitive tournament. By induction hypothesis, $V(G'|B)$ is a hero. In [1], the following is proved.

(3) Suppose G_A is a transitive tournament, and G_B is a hero. Then $G = \Delta(G_A, G_B, 1)$ is a hero.

Therefore $G' = \Delta(V(G'|A), V(G'|B), v)$ is a hero, and hence so is G. This proves 5.2.7. ∎

5.3 Clouds

In this section, we introduce a large class of tournaments, and prove that every type-two tournament is a hero in the class. We say a tournament G has *cloud-depth zero* if G is a transitive tournament. Inductively, we say a tournament G has *cloud-depth at most k* if there exists an ordering σ of $V(G)$ and a partition $(C_1, \ldots, C_r)$ of $V(G)$ such that each $G|C_i$ has cloud-depth at most $k-1$, and there is no edge between C_i and C_j $(i \neq j)$ in G_σ. We call each $G|C_i$ a *cloud* of G. We denote by $\mathcal{C}_k$ the class of all tournaments with cloud-depth at most k. It is not hard to see that every tournament G has its cloud-depth at most $|V(G)|$.

Theorem 5.3.1 *If $\mathcal{G} \subseteq \mathcal{C}_1$ (cloud-depth one) is W-free for some type-two tournament W, then there exists a constant $f(W)$ such that $\chi(G) \leq f(W)$ for every $G \in \mathcal{G}$.*

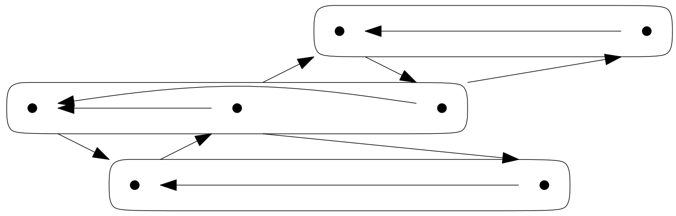

Figure 5.4: Every “backward” edge belongs to a cloud.

Proof. We may assume W is a tree-walk tournament. We apply induction on $|V(W)|$. (The base case $|V(W)| = 1$ is trivial.) Let T be a doubled tree, and let $x_1, \dots, x_m (= x_1)$ be the sequence of an Eulerian tour of T defining W. Let $x \in V(T)$ be the vertex that is equal to $x_1 = x_m$. (In the sequence, x appears $r := |N_T(x)| + 1$ times.) Then in the sequence $x_1, \dots, x_m$, the vertices coming after a particular x and before the next x, are all in the same component of $T \setminus x$. Let $T_1, T_2, \cdots, T_r$ be the components of $T \setminus x$ such that every vertex of T_i comes before every vertex of T_j $(j > i)$ in the sequence $x_1, \dots, x_m$. Then, each T_i is again a doubled tree, and each of the sub-intervals not containing x corresponds to an Eulerian tour of T_i. Let W_i be the subtournament of W that corresponds to T_i.

By induction hypothesis, the statement is true for W_i-free tournaments. Let

$$c := \max\{f(W_1), \cdots, f(W_r)\}.$$

Then, it is enough to prove the following.

Claim : Every W-free tournament is $2r(cr + 1)$-colorable. (In other words, we can set $f(W) = 2r(cr + 1)$.)

Let $G \in \mathcal{C}_1$ be W-free. Take the cloud ordering $\sigma = (v_1, \dots, v_n)$ of $V(G)$, and let $C_1, \dots, C_s$ be the clouds of G. In other words, $(C_1, \dots, C_s)$ is a partition of $V(G)$ such that each $G|C_i$ is a transitive tournament (cloud depth zero), and there is no edge between C_i and C_j with $i \neq j$ in G_σ.

Now, we break σ into many sub-intervals with $\chi \geq cr + 1$ greedily. More precisely, let σ_1 be the shortest prefix of σ such that $\chi(G|\sigma_1) \geq cr + 1$, and σ_2 is the shortest prefix of $\sigma \setminus \sigma_1$ such that $\chi(G|\sigma_2) \geq cr + 1$, and so on. Let $B_i = V(G)|\sigma_i$ for $i = 1, \dots, t + 1$. (Then, $\chi(G|B_i) = cr + 1$ for each $i = 1, \dots, t$ and $\chi(G|B_{t+1}) \leq cr + 1$.)

Take a proper $cr+1$-coloring of each of $G|B_{2i-1}$ using colors from $\{1, \cdots cr+1\}$ and take a proper $cr+1$-coloring of each of $G|B_{2i}$ using colors from $\{cr+2, \cdots, 2(cr+1)\}$. Let ϕ be the $2(cr+1)$-coloring of $V(G)$ by taking the union of those colorings. (It is not necessarily a proper coloring of G.)

Now, fix a cloud C and relabel the indices of the clouds except C as $C_1, \cdots, C_{s-1}$. Say an edge uv in $G_\sigma|C$ is *long* if there exists some sub-interval B_j coming after v and before u in the σ-order. (In other words, $q \geq p+2$ where $B_q \ni u$ and $B_p \ni v$.) Let $LB(C)$ be the subgraph of $G_\sigma|V(C)$ that consists of all its long edges (the vertex set of $LB(C)$ is $V(C)$).

(1) *$LB(C)$ has its clique number at most r.*

For the sake of contradiction, suppose $\{u_1, \cdots, u_{r+1}\}$ is a clique of size $r+1$ in $LB(C)$. We may assume $u_1, \ldots, u_{r+1}$ are in the σ-order. Since $u_{i+1}u_i$ is long, there exists a sub-interval B_j between u_i and u_{i+1} in σ. By relabeling, let B_i be the block between u_i and u_{i+1} for each i ($i = 1, \ldots, r$). Then, since $\chi(G|B_i) = cr+1$ and $G|C$ is a transitive tournament,

$$\chi(G|(B_i \setminus C)) \geq (cr+1) - 1 = cr$$

Now, we can find $F_1, \ldots, F_r$ (*fingers*) satisfying

- $F_i \subseteq B_i \setminus V(C)$,
- $\chi(G|F_i) \geq c$, and
- every cloud C_j meets at most one of $F_1, \ldots, F_r$.

This can be done by the following procedure. Choose the smallest x such that for some B_i,

$$\chi\big(G \mid \big((C_1 \cup \cdots \cup C_x) \cap B_i\big)\big) \geq c.$$

Set $F_i = (C_1 \cup \cdots \cup C_x) \cap B_i$ and remove $(C_1 \cup \cdots \cup C_x)$ from G. Then, $\chi(G|(B_j \setminus C))$ decreases by at most c for each $j \neq i$. We repeat the same procedure by choosing the smallest x' such that for some $B_{i'}$ with $i' \neq i$, $\chi\big(G \mid \big((C_{x+1} \cup \cdots \cup C_{x'}) \cap B_{i'}\big)\big) \geq c$ and set $F_{i'} = (C_{x+1} \cup \cdots \cup C_{x'}) \cap B_{i'}$. We can repeat this procedure until we find all of $F_1, \cdots, F_r$ since $\chi(B_j)$ is initially set to be large enough to complete the procedure.

Now, there is no edge between F_i and F_j in G_σ ($i \neq j$) since no cloud meets two of them, and each $G|F_i$ contains every W_j ($j = 1, \ldots, r$) since $\chi(G|F_i) \geq c$. In particular, $G|F_i$ has a subtournament isomorphic to W_i. Take $F_i' \subseteq F_i$ such that $G|F_i'$ is isomorphic to W_i. Let

$$X = \{u_1, \ldots, u_{r+1}\} \cup F_1' \cup \ldots \cup F_r'.$$

Then, $G|X$ is isomorphic to W (every u_i corresponds to v and each F_i corresponds to W_i). This contradicts the assumption that G is W-free. Therefore $LB(C)$ has its clique number at most r, and this proves (1).

(2) *$LB(C)$ has its (undirected) chromatic number at most r.*

Suppose u, v, w are in σ-order and wv, vu are two (undirected) edges of $LB(C)$. Then wv,vu are directed (backward w.r.t σ) edges in G as well. Then, wu is an edge in G as well, since $G|C$ is a transitive tournament. Also, it is a long (undirected) edge since both wv and vu are long. Therefore wu is an edge of $LB(C)$ as well. Therefore $LB(C)$ is a comparability graph (the corresponding partial order is $u \preceq v$ if either $u = v$ or u comes after v in σ and uv is long.) Therefore $LB(C)$ is perfect, and hence $\chi(LB(C)) \leq r$ from (1). This proves (2).

Take a proper r-coloring of $LB(C)$ (this is a proper coloring of a graph). We do the same for the other clouds (use the same r colors). Let ψ be the r-coloring of $V(G)$ by taking the union of them.

(3) *$\phi \times \psi$ is a proper coloring of G.*

Suppose x, y, z are in σ-order such that $G|\{x, y, z\}$ is a monochromatic cyclic triangle. They are not from the same cloud because every cloud is transitive. Also, they are not from three distinct clouds because edges between clouds all go forward with respect to σ. Therefore exactly two of x, y, z are in the same cloud, and those two must be x and z, since otherwise either $x \Rightarrow \{y, x\}$ or $\{x, y\} \Rightarrow z$. Therefore, xy, yz, zx are edges of G. In particular, zx is a backward edge (w.r.t σ) in a cloud C. Since x and z have the same ψ-color, it is not long. Therefore for $B_i \ni x$ and $B_j \ni z$, either $j = i$ or $j = i + 1$. If $j = i$, then y belongs to the same B_i as well. But, ϕ is a proper coloring on each $G|B_i$, so this is impossible. If $j = i + 1$, then the ϕ color of x (which belongs to $\{1, \cdots, c\}$) and the ϕ

color of z (which belongs to $\{c+1, \cdots, 2c\}$) are different, which also contradicts the monochromatic assumption. This proves (3).

Therefore $\phi \times \psi$ is a proper $2r(cr+1)$-coloring (c and r only depends on W). Therefore we can set $f(W) = 2r(cr+1)$. This proves 5.3.1. ∎

Theorem 5.3.2 *Let $k \geq 0$ be an integer. If $\mathcal{G} \subseteq \mathcal{C}_k$ (cloud-depth k) is W-free for some type-two tournament W, then there exists a constant $f(k, W)$ such that $\chi(G) \leq f(k, W)$ for every $G \in \mathcal{G}$.*

Proof. We proceed by induction on k. The statement is trivial for $k = 0$. From 5.3.1, the statement is true for $k = 1$ as well. Let G be a W-free tournament, and let $C_1, \ldots, C_r$ be clouds (depth at most $k-1$) of G. By induction hypothesis, each of $G|C_i$ is $f(k-1, W)$-colorable. Take a proper coloring of each $G|C_i$ by using colors from $\{1, \ldots, f(k-1, W)\}$. Let ϕ be the $f(k-1, W)$-coloring of G by taking the union of them. (It is not necessarily a proper coloring.)
Now, consider a color class $G|\phi^{-1}(j)$. Then, $G|\phi^{-1}(j)$ is a member of $\mathcal{C}_1$ (cloud-depth one) since each $C_i \cap \phi^{-1}(j)$ is a transitive set (cloud-depth zero). Therefore $G|\phi^{-1}(j)$ is $f(1, W)$-colorable. Color each of ϕ-color class by using colors from $\{1, \cdots, f(1, W)\}$. Let ψ be the $f(1, W)$-coloring of $V(G)$ by taking the union of them.

(1) *$\phi \times \psi$ is a proper coloring of G.*

Suppose x, y, z have the same color. In particular, they are in the same ϕ-color class. Therefore $G|\{x, y, z\}$ is not a cyclic triangle because they have the same ψ-color. This proves (1).

Therefore G is $f(k-1, W) \times f(1, W)$-colorable and we can set $f(k, W) = f(1, W)^k$. This proves 5.3.2. ∎

5.4 Substitutions

For a class $\mathcal{G}$ of tournaments, denote by $cl(\mathcal{G})$ the class of all tournaments that can be obtained from members of $\mathcal{G}$ by (repeated) substitutions.

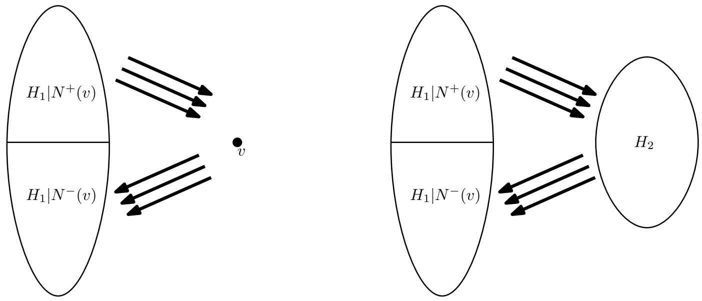

Figure 5.5: Substitution

Theorem 5.4.1 *Let α, k be integers. Let $\mathcal{G}$ be a class of tournaments with $\chi(G) \leq \alpha$ for every $G \in \mathcal{G}$. Let S be a type-one tournament, and let W be a type-two tournament. Suppose $\mathcal{H} \subseteq cl(\mathcal{G})$ is $\{S, W\}$-free. Then there exists a constant $f(S, W, \alpha)$ such that $\chi(G) \leq f(S, W, \alpha)$ for every $G \in \mathcal{H}$.*

Proof. We may assume $S = S_k(= \Delta(S_{k-1}, S_{k-1}, 1))$ for some k and W is a tree-walk tournament. Let T be a doubled tree, and let $x_1, \ldots, x_m(= x_1)$ be the sequence of an Eulerian tour of T defining W. We apply induction on $k + |V(T)|$. If $k \leq 1$ or $|V(T)| \leq 2$, then the statement is trivial. Let $G \in \mathcal{H}$. For fixed k and T, we apply induction on $|V(G)|$. As in the proof of 5.3.1, let $x \in V(T)$ be the vertex that is equal to $x_1 = x_m$, and let $T_1, T_2, \ldots, T_r$ be the components of $T \setminus x$, and let W_i be a subtournament of W that corresponds to T_i. Now set

$$c = \max\{f(S_{k-1}, W, \alpha), f(S_k, W_1, \alpha), \cdots, f(S_k, W_r, \alpha)\}.$$

It is enough to prove the following.

Claim : G is αcr-colorable. (In other words, we can set $f(S, W, \alpha) = \alpha cr$.)

Since $G \in \mathcal{H}$, G can be obtained from $G_0 \in \mathcal{G}$ by substituting $G_1, G_2, \ldots, G_n(\in cl(\mathcal{G}))$ for the vertices $v_1, v_2, \ldots, v_n$ of G_0, respectively. By relabeling if necessary, let $G_1, \ldots, G_s$ be the ones with chromatic number at least c among $G_1, \ldots, G_n$. Then, each of $G_1, \ldots, G_s$ contains all of S_{k-1}, $W_{T_1}, \ldots, W_{T_r}$ by the definition of c and induction hypotheses.

(1) *No two vertices of $\{v_1, \ldots, v_s\}$ are in the same cyclic triangle of G_0. In particular, $\{v_1, \ldots, v_s\}$ is a transitive set in G_0.*

For the sake of contradiction, let v_i, v_j, and w induce a cyclic triangle in G_0. Take a subtournament isomorphic to S_{k-1} from each of G_i and G_j. Now, together with w, those two subtournaments yields a subtournament isomorphic to S_k in G, which is a contradiction. This proves (1).

By relabeling, we may assume $v_1, \ldots, v_s$ are in the transitive order ($v_iv_j \in E(G_0)$ if $i < j$). Let $X = V(G_1) \cup \cdots \cup V(G_s)$, $Y = V(G_{s+1}) \cup \cdots \cup V(G_n)$, and $Z = V(G) \setminus (X \cup Y) = G_0 \setminus \{v_1, \ldots, v_n\}$. ($(X, Y, Z)$ is a partition of $V(G)$.) By (1), for each $w \in \{v_{s+1}, \ldots, v_n\} \cup Z \subseteq V(G_0)$, $\{u, v_1, \ldots, v_s\}$ is a transitive set. In particular, $Y \cup Z \subseteq V(G)$ can be partitioned into $s+1$ sets $A_1, \ldots, A_{s+1}$ with respect to the transitive order of $\{v_1, \ldots, v_s\}$. In other words, $A_1 \Rightarrow \{v_1, \ldots, v_s\}$ and for each $i = 2, \ldots, s$, $\{v_1, \ldots, v_{i-1}\} \Rightarrow A_i$ and $A_i \Rightarrow \{v_i, \ldots, v_s\}$, and $\{v_1, \ldots, v_s\} \Rightarrow A_{s+1}$.

(2) $G|(Y \cup Z)$ *is* αc*-colorable.*

G_0 is α-colorable since $G_0 \in \mathcal{G}$. In particular, $G_0|(Z \cup \{v_{s+1}, \ldots, v_n\})$ is α-colorable. Let ϕ be the proper α-coloring of $G_0|(Z \cup \{v_{s+1}, \ldots, v_n\})$. Let $\overline{\phi}$ be the natural extension (not necessarily a proper coloring) of ϕ to $Y \cup Z$ by assigning $\overline{\phi}(u) = \phi(v_i)$ for every $u \in V(G_i)$ ($i = s+1, \ldots, n$).

On the other hand, each G_i is c-colorable for $i = s+1, \ldots, n$. Take a proper c-coloring for each G_i using colors from $\{1, \ldots, c\}$, and let ψ be the c-coloring of Y by taking the union of them. Let $\overline{\psi}$ be the extension of ψ to $Y \cup Z$ by simply assigning $\overline{\psi}(u) = 1$ for every $u \in Z$.

Then, $\overline{\phi} \times \overline{\psi}$ is a proper αc-coloring of $Y \cup Z$. To see this, suppose u, v, and w induce a monochromatic cyclic triangle. Since ϕ is a proper coloring on G_0, at least two of them are from the same homogeneous set G_i. If exactly two of them, say u and v, are in G_i, then either $w \Rightarrow \{u, v\}$ or $\{u, v\} \Rightarrow w$ since G_i is a homogeneous set, so this is impossible. If all three are in the same G_i, then it cannot be a cyclic triangle since ψ is a proper coloring on each G_i. Therefore this case is also impossible. Therefore there is no monochromatic cyclic triangle in G. This proves (2).

Let $\Phi(= \overline{\phi} \times \overline{\psi})$ be a proper αc-coloring on $G|(Y \cup Z)$ obtained from (2). For each Φ-color class C of $Y \cup Z$, take an ordering σ_C of C as follows.

- Every vertex of $A_i \cap C$ comes before every vertex of $A_j \cap C$ for $i < j$, and
- if $u, v \in A_i \cap C$ for some i and $uv \in E(G)$, then u comes before v (this can be done since $G|C$ is transitive).

Now we look at $(G|C)_{\sigma_C}$.

(4) $(G|C)_{\sigma_C}$ *has its clique number at most* r.

The proof of (4) is almost the same as (1) in 5.3.1. Suppose there exists a clique of size $r+1$. Then by the definition of σ_C, the vertices of the clique are from distinct A_i's. Together with corresponding G_i's ($i = 1, \ldots, s$) containing each "finger", we can find W subtournament in G. A contradiction. This proves (4).

(5) $(G|C)_{\sigma_C}$ *has its chromatic number at most* r.

$(G|C)_{\sigma_C}$ is a permutation graph, since C is a transitive tournament. Therefore $(G|C)_{\sigma_C}$ is perfect. From (4), the chromatic number is at most r. This proves (5).

Take an r-coloring of $(G|C)_{\sigma_C}$ (this is a coloring of a graph), and similarly for the other Φ-color classes (use the same r colors). Let Ψ be the r-coloring of $Y \cup Z$ by taking the union of those r-colorings. Then $\Phi \times \Psi$ is a αcr-coloring on $Y \cup Z$. Of course, it is a proper coloring of $Y \cup Z$ since Φ itself is a proper coloring of $Y \cup Z$.

Now, take a proper αcr-coloring of G_i for each $i = 1, \ldots, s$. This can be done by induction hypothesis since $|V(G_i)| < |V(G)|$, and we use the same αcr colors used in $\Phi \times \Psi$. Let ξ be the αcr-coloring of X by taking the union of them. Note that ξ is a proper coloring on X by (1).

(6) $\xi \cup (\Phi \times \Psi)$ *is a proper* αcr*-coloring of* G.

Suppose x, y, z induce a monochromatic cyclic triangle. Since ξ and $(\Phi \times \Psi)$ are proper colorings on X and $Y \cup Z$, either one or two of the three vertices are in $Y \cup Z$. Suppose one of them, say x, is in $Y \cup Z$. Then, y and z are in the same G_i by (1), but if they are in the same G_i, then either $x \Rightarrow \{y, z\}$ or $\{y, z\} \Rightarrow x$ since G_i is a homogeneous set. So, this is impossible. Suppose two of them, say x and y, are in $Y \cup Z$. We may assume $x \in A_h$ and $y \in A_j$ with $h \leq j$. Then, since $G|\{x, y, z\}$ is a cyclic triangle, $h \neq j$, and hence $h < j$. Moreover, $z \in V(G_i)$ where $h + 1 \leq i \leq j$ and xz, xy, yx are edges of G. In particular, yx is an edge. Recall that x and y are in the same Φ-color class C. From the definition of σ_C, yx is an (undirected) edge in $(G|C)_{\sigma_C}$. But this contradicts that x and y are in the same Ψ-color class. Therefore this is impossible as well. This proves (6).

Therefore $\xi \cup (\Phi \times \Psi)$ is a proper αcr-coloring of G. This proves 5.4.1. ∎

Now, we prove 5.2.3.

Theorem 5.4.2 *Let $k \geq 1$ be an integer. There exists a type-one tournament S with $\omega(S) \geq k$.*

Proof. Suppose the maximum clique number is attained at some number m. Take a tree-walk tournament W that is not type-one. For instance, we can take $W = C'_5$. Take a and b such that $\omega(S_a) = m$, and $\chi(S_b) \geq 2 \times f(S_a, W, 2)$ where f is as in 5.4.1.

Claim : $\omega(S_{b+1}) > m$

Note that $V(S_{b+1})$ can be partitioned into A, B, and $\{v\}$ such that $S_{b+1} = \Delta(A, B, v)$, and $S_{b+1}|A \cong S_{b+1}|B \cong S_b$.

Let τ be an arbitrary ordering of $V(S_{b+1})$. Let A_1 be the set of vertices of A coming before v, and let A_2 be the set of vertices of A coming after v in τ-order.

(1) *Either $\omega((S_{b+1}|A_1)_\tau)$ or $\omega((S_{b+1}|A_2)_\tau)$ is at least m.*

Either $\chi(S_{b+1}|A_1)$ or $\chi(S_{b+1}|A_2)$ is at least $f(S_a, W, 2)$ since $\chi(S_{b+1}|A) = \chi(S_b) \geq 2f(S_a, W, 2)$. From 5.4.1, either $S_{b+1}|A_1$ or $S_{b+1}|A_2$ contains a subtournament isomorphic to either S_a or W. Since W is not type-one, the subtournament must be isomorphic to S_a. Therefore either $\omega(S_{b+1}|A_1)$ or $\omega(S_{b+1}|A_2)$ is m. In particular, either $\omega((S_{b+1}|A_1)_\tau)$ or $\omega((S_{b+1}|A_2)_\tau)$ is at least m. This proves (1).

Similarly, $\omega((S_{b+1}|B_1)_\tau)$ or $\omega((S_{b+1}|B_2)_\tau)$ is at least m where B_1 is the set of vertices of B coming before v, and B_2 is the set of vertices of B coming after v in τ-order.

Since v is complete to A, and complete from B, we may assume $\omega((S_{b+1}|A_1)_\tau) < m$ and $\omega((S_{b+1}|B_2)_\tau) < m$ since otherwise we have a clique of size $m+1$ containing v.

Therefore we may assume $\omega((S_{b+1}|A_2)_\tau) \geq m$ and $\omega((S_{b+1}|B_1)_\tau) \geq m$. But then, we have a clique of size $2m$ in $(S_{b+1})_\tau$ since A is complete to B and every vertex of A_1 comes after every vertex of B_2.

Therefore there exists a clique of size at least $m+1$ in $(S_{b+1})_\tau$ for every τ, and this contradicts to the maximality of m. This proves 5.4.2. ∎

Now, we prove 5.2.4 and 5.2.5.

Theorem 5.4.3 *Let $k \geq 1$ be an integer. There exists a type-two tournament W with $\chi(W) \geq k$.*

Proof. For the sake of contradiction, suppose the maximum chromatic number is attained at some number m. Take a tree-walk tournament W with $\chi(W) = m$ and let T be a doubled-tree, and let $x_1, \ldots, x_m$ be the sequence of an Eulerian tour of T defining W. Let T' be a doubled-tree obtained by substituting each copy of T for each leaf of a (doubled) star with m leaves. Take an Eulerian tour of T' starting from the center vertex of T' (the center of the star before applying substitutions) and obtain a tree-walk tournament W'.

Claim : $\chi(W') > m$**.**

Suppose W' is m-colorable. Let $u_1, \ldots, u_{m+1}$ be the vertices of W' correspond to the center of T'. Then there exist u_i and u_j $(i < j)$ with the same color. By definition of W', there exists some copy of W in W' such that $u_i \Rightarrow V(W)$ and $V(W) \Rightarrow u_j$. Also, there exists a vertex $w \in V(W)$ such that the color of w is the same as the color for u_i and u_j. Then, $W'|\{u_i, u_j, w\}$ is a monochromatic cyclic triangle, which is a contradiction. This proves the claim and the claim contradicts the minimality of m. This proves 5.4.3. ∎

Theorem 5.4.4 *Let $k \geq 1$ be an integer. There exists a type-two tournament W with $\omega(W) \geq k$.*

Proof. For the sake of contradiction, suppose the maximum clique number is attained at some number m. Take W_0 with $\omega(W_0) = m$, and let $M := 2 \times f(S_2, W_0, 2)$ where f is as in 5.4.1. Take W_1 with $\chi(W_1) \geq M$. This can be done by 5.4.3. Let T_1 be a doubled-tree defining W_1. Let T be a doubled-tree obtained by substituting each copy of T_1 for each leaf of a (doubled) star with m leaves. Now, an Eulerian tour starting from the center vertex of T (the center of the star) defines some tree-walk tournament W.

Claim : $\omega(W) > m$**.**

Let $\sigma = r_1, \sigma_1, r_2, \sigma_2, r_3, \cdots, r_m, \sigma_m, r_{m+1}$ be the sequence defining W where $r_1, \ldots, r_{m+1}$ correspond to the center of T. Take an arbitrary ordering τ of $V(W)$. We prove $\omega(W_\tau) > m$. If

$r_1, \cdots, r_{m+1}$ are in order (w.r.t τ), then W_τ has a clique of size $m+1$. Therefore we may assume r_j comes before r_i in τ for some $i < j$. Note that there exists a copy of W_1 complete from r_i and complete to r_j. Take such a copy W'.

In τ-order, let X be the vertices of W' coming before r_i and let Y be the vertices coming after r_i. Then, either $\chi(X)$ or $\chi(Y)$ is at least $f(2, W_0, 2)$ since $\chi(W_1) \geq M = 2f(2, T_0, 2)$. Suppose $\chi(X) \geq f(2, W_0, 2)$. From 5.4.1, X either has S_2 or W_0. But S_2 is not a subtournament of any tree-walk tournament, and hence, X contains W_0. This means there is a clique of size m in X (w.r.t τ). Then together with r_i, which comes after every element of X, we have a clique of size $m+1$. Suppose $\chi(Y) \geq f(2, T_0, 2)$. Similarly from 5.4.1, Y contains W_0, and together with r_j, which comes before every element of Y, we have a clique of size $m+1$. Therefore $\omega(W) > m$, and this contradicts the assumption. This proves 5.4.4. ∎

It is proved in [4] that if a class of graphs $\mathcal{G}$ is χ-bounded, then the class $cl(\mathcal{G})$ obtained from members of $\mathcal{G}$ by (repeated) substitutions is also χ-bounded. We prove an analogue of that for tournaments.

Theorem 5.4.5 *Let $\mathcal{G}$ be a χ-bounded class of tournaments. Then $cl(\mathcal{G})$ is χ-bounded.*

Proof. Let $\mathcal{G}$ be a χ-bounded class of tournament by a function h ($\chi(G) \leq h(w(G))$ for every $G \in \mathcal{G}$. Let $\mathcal{G}_k$ be the class of all tournaments in $cl(\mathcal{G})$ with its clique number at most k. Then, from 5.2.5, and 5.2.3, there exists a type-one tournament S and a type-two tournament W such that $\omega(S), \omega(W) > k$. Then, since $G \in \mathcal{G}_k$ is $\{S, W\}$-free, $\chi(\mathcal{G}_k) \leq f(S, W, h(k))$ by 5.4.1, where f is as in 5.4.1. Note that S and W only depend on k. Therefore $cl(\mathcal{G})$ is χ-bounded by g where $g(k) = f(S, W, h(k))$. This proves 5.4.5. ∎

5.5 More classes meeting every finite heroic sets

Unfortunately, it is not enough to get bounded chromatic number by excluding one type-one tournament and one type-two tournament. We introduce two more classes that every finite heroic sets must meet.

5.5.1 The third class : $\mathcal{F}$

Unlike the other classes, every member of the third class $\mathcal{F}$ has its chromatic number at most two. Moreover, for every member $F \in \mathcal{F}$, there exists some ordering σ of $V(F)$ such that F_σ is a forest.

Despite that the class has bounded chromatic number, we will see that every finite heroic set must contain at least one member from $\mathcal{F}$. To define $\mathcal{F}$, we first introduce a notion and a theorem from [1]. Let G be a tournament, and let $\sigma = (v_1, \ldots, v_n)$ be an ordering of $V(G)$. Let $\phi : V(G) \to \mathbb{Z}$ be an injective map such that $\phi(v_i) < \phi(v_j)$ for every $i < j$. If $e = \{u, v\}$ is an edge of G_σ, then we write $\phi(e) = |\phi(u) - \phi(v)|$. Let $r, s \geq 1$ be integers. Two distinct edges e, f of G_σ are said to be (r, s)-*comparable* (under ϕ) if

- there is a path P of B with at most s edges, with $e, f \in E(P)$, and
- $\phi(e) \leq r\phi(f)$ and $\phi(f) \leq r\phi(e)$.

Additionally, two distinct edges e, f of the same component of G_σ are said to be r-*comparable* if the last condition holds. The following is proved in [1].

Theorem 5.5.1 *For all integers $r, s, k \geq 1$, there exists a tournament G and an ordering σ of $V(G)$ and $\phi : V(G) \to \mathbb{Z}$ as described earlier, such that:*

- *no two edges of G_σ are (r, s)-comparable, and*
- *$\chi(G) > k$.*

Let $\mathcal{H}$ be a finite heroic set of tournaments. Let k be an integer such that $\chi(G) \leq k$ for every $\mathcal{H}$-free G. By 5.5.1, for all integers $r, s \geq 1$, there exists a member H of $\mathcal{H}$ such that H is a subtournament of some G satisfying the statement of 5.5.1 with r, s, k. In particular, there exists an ordering $\sigma'(= \sigma|V(H))$ of $V(H)$ such that no two edges of $H_{\sigma'}$ are (r, s)-comparable. Therefore, if $\mathcal{H}$ is finite, then there exists $H \in \mathcal{H}$ and an ordering σ of $V(H)$ such that for every integer $r \geq 1$,

- no two edges in the same component of H_σ are r-comparable.

We call such a tournament H *type-three* and denote by $\mathcal{F}$ the class of all type-three tournaments. Then, $\mathcal{F}$ is closed under taking subtournaments by definition.

Say a tournament G is *forest orderable* if there is an ordering σ of $V(G)$ such that G_σ is a forest.

Theorem 5.5.2 *Let H be a type-three tournament. Then H is forest orderable.*

Proof. Let σ be the ordering of $V(H)$ such that no two edges in the same component of H_σ are 2-comparable with respect to some ϕ. Suppose H_σ contains a cycle C. For all distinct edges e, f of C, either $\phi(e) > 2\phi(f)$ or $\phi(f) > 2\phi(e)$. Let $E(C) = \{e_1, \ldots, e_k\}$ where $\phi(e_i) > 2\phi(e_{i+1})$ for $1 \leq i \leq k$. Then,

$$\phi(e_1) > \phi(e_2) + \phi(e_3) + \cdots + \phi(e_k),$$

which is impossible because $e_2, e_3, \ldots, e_k$ are the edges of a path of H_σ between the ends of e_1. This proves 5.5.2. ▮

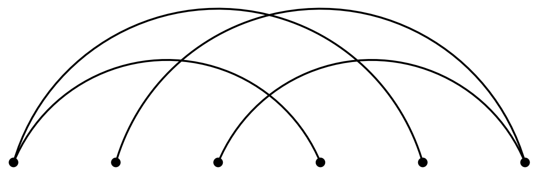

Figure 5.6: $\Delta(2, 2, 2)$ is forest orderable, but not type-three.

Note that not every forest orderable tournament is type-three. For instance, for a two-vertex tournament T_2, $\Delta(T_2, T_2, T_2)$ is forest orderable, but not type-three. A forest orderable tournament has its chromatic number at most two since a forest is two-colorable, and the coloring of the forest is a proper coloring for the tournament as well. Therefore the chromatic number of $\mathcal{F}$ is bounded, but every finite heroic set must contain at least one member from $\mathcal{F}$.

Theorem 5.5.3 *Every finite heroic set of tournaments contains a type-three tournament.*

In the previous section, we have seen that G is a hero if and only if it is both type-one and type-two. The same statement for type-one and type-three is also true.

Theorem 5.5.4 *A tournament G is a hero if and only if it is both type-one and type-three.*

Proof. Suppose G is a hero. Then it must be type-one and type-three by 5.2.2 and 5.5.3. Conversely, suppose G is type-one and type-three. Note that $\chi(S_3) = 3$ but every type-three tournament has its chromatic number at most two. Therefore S_3 is not type-three, and hence G does not contain S_3. Therefore by the same argument of the proof of 5.2.7, G must be a hero. This proves 5.5.4. ▮

5.5.2 The fourth class : $\mathcal{R}$

Unfortunately, for some $S \in \mathcal{S}$, $W \in \mathcal{W}$, and $F \in \mathcal{F}$, there exists some $\{S, W, F\}$-free tournament with arbitrary large chromatic number. The last class $\mathcal{R}$ contains some of them.

Let $n > k \geq 1$ be integers, and let $X = \{x_1, \ldots, x_n\}$ be an n-set. We consider all k-tuples $(x_{i_1}, \ldots, x_{i_k})$ of X such that $1 \leq i_1 < \cdots, i_k \leq n$. Let V be the set of all such k-tuples ($|V| = \binom{n}{k}$). Let σ be the "lexicographic" ordering of V. More precisely, $(x_{i_1}, \ldots, x_{i_k})$ comes before $(x_{j_1}, \ldots, x_{j_k})$ in σ-order if for some $\beta \geq 1$, $i_\beta < j_\beta$ and $i_\alpha = j_\alpha$ for every $\alpha < \beta$. Now, let H be the graph with

$V(H) = V$ and $(x_{i_1}, \ldots, x_{i_k})(x_{j_1}, \ldots, x_{j_k}) \in E(H)$ if and only if $j_1 = i_2$, $j_2 = i_3$, …, $j_{k-1} = i_k$. We call such a graph *(n,k)-graph.* Let G be the tournament with $V(G) = V$ such that $G_\sigma = H$. We call such a tournament *(n,k)-tournament.* A tournament G is *type-four* if it is a subtournament of some (n,k)-tournament.

Theorem 5.5.5 *Let k, $c \geq 1$ be integers. For sufficiently large n, (n,k)-graph G has chromatic number larger than c.*

Proof. It is an easy corollary of the hypergraph-version of Ramsey's theorem. The hypergraph-version of Ramsey's theorem says for all integers k,c,m, if we color the hyper-edges of a sufficiently large enough complete k-regular hypergraph H with c colors, then there exists a set $M \subseteq V(H)$ of size m such that every k-subset of M has the same color.

We apply the statement for $m = k+1$. For sufficiently large n, if we color the vertices of (n,k)-graph G with c colors, then there exists a set $M \subseteq X$ (say $M = \{x_{i_1}, \ldots, x_{i_{k+1}}\}$ where $i_1 < \cdots < i_{k+1}$) of size $k+1$ such that all k-tuples of M are monochromatic in G. In particular, the two vertices $(x_{i_1}, \ldots, x_{i_k})$ and $(x_{i_2}, \ldots, x_{i_{k+1}})$ have the same color, but they are adjacent in G. Therefore if n is large enough, then G is not c-colorable. This proves 5.5.5. ∎

Theorem 5.5.6 *Let k, $c \geq 1$ be integers. For sufficiently large n, (n,k)-tournament G has chromatic number larger than c.*

Proof. From 5.5.5, for sufficiently large n, (n,k)-graph $H = G_\sigma$ has chromatic number larger than $2c$. Now suppose $\chi(G) \leq c$. Take a proper c-coloring of G, and let C be a color class. Since $G|C$ is a transitive tournament, the corresponding $H|C$ is a permutation graph, and hence it is perfect. Note that $H|C$ is triangle-free by the definition of (n,k)-graph. Therefore $\chi(H|C) \leq 2$. Since there are c many color classes (of G), by taking the product color, we get $\chi(H) \leq 2c$. This contradicts our assumption. This proves 5.5.6. ∎

5.5.7 is an immediate corollary of 5.5.6.

Theorem 5.5.7 *Every heroic set of tournaments contains a type-four tournament.*

Now we explain why we need type-four.

Theorem 5.5.8 *Let $k \geq 13$ be an integer. For sufficiently large n, an (n,k)-tournament is $\{S, W, F\}$-free for some type-one tournament S, type-two tournament W, and type-three tournament F.*

Proof. From 5.2.3 and 5.2.5, there exist a type-one tournament S with $\omega(S) \geq 3$ and a type-two tournament W with $\omega(W) \geq 3$. We define a type-three tournament F as follows. First, let F' be an undirected path with

- $V(F') = \{v_1, \ldots, v_{13}\}$, and
- $E(F') = \{v_1v_{13}, v_{13}v_2, v_2v_{12}, v_{12}v_3, v_3v_{11}, v_{11}v_4, v_4v_{10}, v_{10}v_5, v_5v_9, v_9v_6, v_6v_8, v_8v_7\}$.

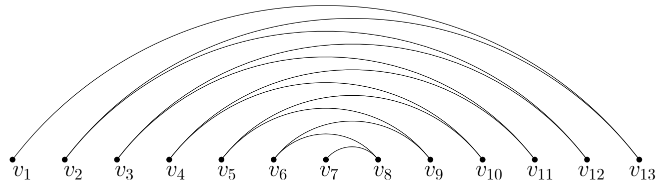

Figure 5.7: $F' = F_\sigma$

Let σ be the ordering $v_1, v_2, \ldots, v_{12}, v_{13}$ of $V(F')$. Let F be the tournament satisfying $F_\sigma = F'$.

(1) *F is type-three.*

We show that for every fixed integer $r \geq 1$, there exists ϕ such that no two edges of F_σ are r-comparable. This can be easily done by first assigning the values $\phi(v_7) = 0$ and $\phi(v_8) = 1$ so that $\phi(v_8v_7) = 1$, and sequentially assigning $\phi(v_6)$, $\phi(v_9), \ldots, \phi(v_1)$ so that $\phi(v_6v_8) = r+1$, $\phi(v_9v_6) = (r+1)^2$, $\phi(v_5v_9) = (r+1)^3$, etc. This proves (1).

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set with $n > 13$. Define a (n,k)-tournament G where $n > k \geq 13$ as described earlier. Let $N = \binom{n}{k}$. Let $\tau = t_1(= (x_1, x_2, \ldots, x_k)), t_2, \ldots, t_N(= (x_{n-k+1}, \ldots, x_n))$ be the lexicographic ordering of $V(G)$ such that G_τ is the corresponding (n,k)-graph.

(2) *Suppose $1 \leq a < b < c < d \leq N$. If three of $t_at_d, t_dt_b, t_bt_c, t_at_c$ are edges of G_τ then the other one is also an edge of G_τ.*

Let $t_a = (x_{a_1}, \ldots, x_{a_k})$, $t_b = (x_{b_1}, \ldots, x_{b_k})$, $t_c = (x_{c_1}, \ldots, x_{c_k})$, and $t_d = (x_{d_1}, \ldots, x_{d_k})$. Since $a < b < c < d$ and at least three of $t_at_d, t_dt_b, t_bt_c, t_at_c$ are edges of G_τ, at least three of the following hold.

$$(a_2, a_3, \ldots, a_k) = (d_1, d_2, \ldots, d_{k-1}),$$

$$(b_2, b_3, \ldots, b_k) = (d_1, d_2, \ldots, d_{k-1}),$$

$$(b_2, b_3, \ldots, b_k) = (c_1, c_2, \ldots, c_{k-1}),$$

$$(a_2, a_3, \ldots, a_k) = (c_1, c_2, \ldots, c_{k-1}).$$

Then, it is easy to see that the other equality holds as well, which means that the corresponding edge is in G_τ. This proves (2).

(3) *If C is an odd cycle of G_τ with length l, then $l \geq 15$.*

Let $t = (x_{a_1}, \ldots, x_{a_k})$ be a vertex in C. Say an entry x_{a_i} is in *even-position* in t if i is even and say it is in *odd-position* in t if i is odd. Let $C_1 \subseteq V(C)$ be the set of vertices t' such that x_{a_7} is in even-position in t', and let $C_2 \subseteq V(C)$ be the set of vertices t'' such that x_{a_7} is in odd-position in t''. Then, C_1 and C_2 are both stable sets by the definition of (n, k)-graph. Since $V(C)$ is an odd-cycle and not 2-colorable, there exists a vertex $s \in V(C)$ such that s does not have x_{a_7} as its entry. Then, both of the two paths from s to t in C have length at least seven (recall that $k \geq 13$). Therefore $l \geq 14$. Since l is odd, $l \geq 15$. This proves (3).

(4) G is F-free.

Suppose G contains F. Let $1 \leq a_1 < a_2 < \cdots < a_{13} \leq N$ be the indices such that $G|\{t_{a_1}, \ldots, t_{a_{13}}\}$ is isomorphic to F. Let $T = \{t_{a_1}, \ldots, t_{a_{13}}\}$. Let ϕ be the isomorphism from $V(F) = \{v_1, \ldots, v_{13}\}$ to T.

From (3), $G_\tau|T$ contains no odd cycle. Therefore $G_\tau|T$ is bipartite. Let (B, C) be a partition of T such that $G_\tau|B$ and $G_\tau|C$ are both stable sets. We may assume $\phi(v_1) \in B$ by switching B and C if necessary.

Suppose $\phi(v_{13}) \in B$. Then, $\phi(v_3), \phi(v_4), \ldots, \phi(v_{12}) \in C$ since each of them form a cyclic triangle together with $\phi(v_1)$ and $\phi(v_{13})$ ($G|B$ is a transitive tournament). However, this is impossible because now $\phi(v_3), \phi(v_{11}), \phi(v_{12})$ form a cyclic triangle in $G|C$. Therefore $\phi(v_{13})$ must be in C.

By the similar argument, we can deduce that:

$$\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4) \in B,$$

$$\phi(v_{10}), \phi(v_{11}), \phi(v_{12}), \phi(v_{13}) \in C.$$

(The number thirteen was chosen for this.)

Since $G_\tau|B$ is a stable set, $\phi(v_1), \phi(v_2), \phi(v_3), \phi(v_4)$ are in τ-order. Similarly, $\phi(v_{10}), \phi(v_{11})$,$\phi(v_{12})$, $\phi(v_{13})$ are in τ-order as well.

Now, suppose $\phi(v_2)$ comes before $\phi(v_{12})$ in τ-order. Then, $\phi(v_1), \phi(v_2), \phi(v_{12}), \phi(v_{13})$ are in τ-order. However, in $G_\tau|\{\phi(v_1), \phi(v_2), \phi(v_{12}), \phi(v_{13})\}$, there are only three edges, namely $\phi(v_1)\phi(v_{13})$, $\phi(v_{13})\phi(v_2)$, $\phi(v_2)\phi(v_{12})$. This contradicts (2). Therefore $\phi(v_2)$ comes after $\phi(v_{12})$ in τ-order.

Now, $\phi(v_{10}), \phi(v_{11}), \phi(v_{12}), \phi(v_2), \phi(v_3), \phi(v_4)$ are in τ-order. However, in $G_\tau|\{\phi(v_{10}),\ \phi(v_{11}),$ $\phi(v_2), \phi(v_3)\}$, there are only three edges, namely $\phi(v_{10})\phi(v_2)$, $\phi(v_{10})\phi(v_3)$, $\phi(v_{11})\phi(v_2)$. Again, this contradicts (2). Therefore G is F-free, and this proves (4).

G is S-free and W-free as well because $\omega(G) \leq \omega(G_\tau) = 2$ while $\omega(S), \omega(W) \geq 3$. This proves 5.5.8. ∎

Again, G is a hero if and only if it is both type-one and type four.

Theorem 5.5.9 *A tournament G is a hero if and only if it is both type-one and type-four.*

Proof. As we have seen in the proof of 5.5.8, for every type-four tournament G, there exists an ordering $\sigma = (t_1, \ldots, t_n)$ of $V(G)$ such that G_σ is triangle-free, and for every $1 \leq a < b < c < d \leq n$, if at least three of $t_at_c, t_at_d, t_bt_c, t_bt_d$ are edges of G_σ, then the other one is also an edge of G_σ. Say such a σ, *type-four ordering.*

(1) S_3 is not type-four.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the vertices of S_3 such that:

- $v_1 \Rightarrow \{v_2, v_3, v_4\} \Rightarrow \{v_5, v_6, v_7\} \Rightarrow v_1$,
- $v_2v_3, v_3v_4, v_4v_2, v_5v_6, v_6v_7, v_7v_5 \in E(S_3)$.

Suppose S_3 is type-four. Then, there exists an type-four ordering σ of $V(S_3)$. Note that at least one of v_2, v_3, v_4 comes after v_1 in σ-order since $(S_3)_\sigma$ is triangle-free. Similarly, at least one of v_5, v_6, v_7 comes before v_1 in σ-order. Without loss of generality, we may assume v_5, v_1, v_2 are in σ-order. Suppose v_3 comes between v_5 and v_2. Then, $(S_3)_\sigma|\{v_5, v_3, v_2\}$ is a triangle, which is impossible.

Suppose v_3 comes before v_5, then $(S_3)_\sigma|\{v_3, v_5, v_1, v_2\}$ has only three edges v_3v_1, v_3v_2, v_3v_1 and the extra edge v_5v_1 is missing, which is also impossible. Therefore v_3 comes after v_2.

Suppose v_4 comes between v_5 and v_3, Then, $(S_3)_\sigma|\{v_5, v_4, v_3\}$ is a triangle, which is impossible. Suppose v_4 comes before v_5, then $(S_3)_\sigma|\{v_4, v_5, v_1, v_3\}$ has only three edges v_4v_1, v_4v_3, v_4v_1 and the extra edge v_5v_1 is missing, which is also impossible. Therefore v_4 comes after v_3.

But, then $(S_3)_\sigma|\{v_1, v_2, v_4\}$ is a triangle, which is a contradiction. This proves (1).

Now, suppose G is a hero. Then it must be type-one and type-four by 5.2.2 and 5.5.7. Conversely, suppose G is both type-one and type-four. From (1), G is S_3-free. Therefore by the same argument of the proof of 5.2.7, G must be a hero. This proves 5.5.9. ∎

Finally, we state the main conjecture.

Conjecture 5.5.10 *A finite set of tournaments is heroic if and only if it contains a type-one, a type-two, a type-three, and a type-four tournament.*

"Only if" part of 5.5.10 is true by 5.2.2, 5.2.6, 5.5.3, and 5.5.7.

5.6 Heroic sets of size two

Here, we give an example of a heroic set of size two, and we give a necessary condition for being a heroic set of size two. First, we state a conjecture which can be considered as an special case of 5.5.10.

Conjecture 5.6.1 *Suppose both G_1 and G_2 are non-heroes. Then, $\{G_1, G_2\}$ is heroic if and only if exactly one of G_1 and G_2 is type-one and the other one is type-two, type-three, type-four at the same time.*

We explain why the "only if" part of 5.6.1 is true. Suppose both G_1 and G_2 are non-heroes, but $\{G_1, G_2\}$ is heroic. We may assume G_1 is type-one. Then, since G_1 is non-hero, from 5.2.7, 5.5.4, 5.5.9, G_1 cannot be type-two or type-three or type four. Therefore G_2 is type-two, type-three, and type-four at the same time by 5.2.6, 5.5.3, 5.5.7. This proves the "only if" part of 5.6.1.

Next, we prove that if $G_2 \cong C_5'$, then $\{G_1, G_2\}$ is heroic.

5.6.1 A heroic set : $\{S, C_5'\}$

Let $k \geq 1$ be an integer. We call a tournament G with $V(G) = \{v_1, \ldots, v_{2k+1}\}$ a *uniform circular-interval* tournament (consider the indices $\{1, \ldots, 2k+1\}$ as elements of the additive cyclic group of order $2k+1$) if $v_iv_j \in E(G)$ if and only if $j - i \in \{1, \ldots, k\}$. In [17], Liu proved the following structure theorem for C_5'-free tournaments.

Theorem 5.6.2 *Suppose a tournament G is C_5'-free. Then either G has a non-trivial homogeneous set, or G is a uniform circular-interval tournament, or $V(G)$ can be partitioned into three sets X, Y, Z such that $G|(X \cup Y)$ and $G|(Y \cup Z)$ and $G|(X \cup Z)$ are transitive tournaments.*

Using 5.6.2, we prove the following.

Theorem 5.6.3 *Suppose a tournament G is $\{S_k, C_5'\}$-free for some $k \geq 3$. Then, $\chi(G) \leq 3^{k-2}$.*

Proof. We prove by induction on $|V(G)| + k$. From 5.6.2, either G has a non-trivial homogeneous set, or G is a uniform circular-interval tournament, or $V(G)$ can be partitioned into three sets X, Y, Z such that $G|(X \cup Y)$ and $G|(Y \cup Z)$ and $G|(X \cup Z)$ are transitive tournaments.

Then, we may assume G has a non-trivial homogeneous set, since otherwise G is 2-colorable. Then, G can be obtained from some G_0, with no non-trivial homogeneous set, by substituting $G_1, G_2, \ldots, G_n$ for the vertices $v_1, v_2, \ldots, v_n$ of G_0, respectively. Then, G_0 is either uniform circular-interval tournament or $V(G_0)$ can be partitioned into three sets X, Y, Z such that $G_0|(X \cup Y)$, $G_0|(Y \cup Z)$, and $G_0|(X \cup Z)$ are transitive.

Case 1: G_0 is a uniform circular-interval tournament.

It is easy to see that every edge of G_0 is in a cyclic triangle of G_0. Therefore, if there exist $i \neq j$ such that both G_i and G_j contain a copy of S_{k-1}, then by taking two copies of S_{k-1} from G_i and G_j, we can obtain S_k in G. Since G is S_k-free, this is impossible. Therefore at most one of $G_1, \ldots, G_n$ contains S_{k-1}. We may assume G_1 does. From induction hypothesis, G_1 is 3^{k-2}-colorable. Let ϕ be a proper 3^{k-2}-coloring of G_1 using colors from $\{1, 2, 3\}^{k-2}$. Note that both $N^+_{G_0}(v_1)$ and $N^-_{G_0}(v_1)$ are transitive sets, and each of $G_2, \ldots, G_n$ is 3^{k-3}-colorable by induction hypothesis. Take a proper 3^{k-3}-coloring of G_i for each $i = 2, \ldots, n$ using colors from $\{1, 2, 3\}^{k-3}$. Say ψ be the union coloring.

Then, we can extend ϕ to G by simply assigning the color $\{1\} \times \psi$ for the out-neighbors of $V(G_0)$ in $V(G)$, and the color $\{2\} \times \psi$ for the in-neighbors of $V(G_0)$ in $V(G)$. Therefore G is 3^{k-2}-colorable.

Case 2 : $V(G_0)$ can be partitioned into three sets X, Y, Z such that $G_0|(X \cup Y)$, $G_0|(Y \cup Z)$, and $G_0|(X \cup Z)$ are transitive.

By relabeling if necessary, let $v_1, \ldots, v_i \in X$, $v_{i+1}, \ldots, v_j \in Y$, $v_{j+1}, \ldots, v_n \in Z$. By induction hypothesis, each of $G_1, \ldots, G_n$ is 3^{k-2}-colorable. If G_i contains S_{k-1}, then properly color all vertices of $V(G_i)$ by using colors from $\{1, 2, 3\}^{k-2}$. If $i \in \{1, \ldots, i\}$ and G_i is S_{k-1}-free, then properly color all vertices of $V(G_i)$ by using colors from $\{1\} \times \{1, 2, 3\}^{k-3}$. Similarly, if $i \in \{i+1, \ldots, j\}$ and G_i is S_{k-1}-free, then properly color all vertices of $V(G_i)$ by using colors from $\{2\} \times \{1, 2, 3\}^{k-3}$, and if $i \in \{j + 1, \ldots, n\}$ and G_i is S_{k-1}-free, then properly color all vertices of $V(G_i)$ by using colors from $\{3\} \times \{1, 2, 3\}^{k-3}$.

Take the union coloring ϕ on $V(G_1) \cup \cdots \cup V(G_n)$. For the vertices x in X not yet colored, we assign $\phi(x) = (1, 1, \ldots, 1) \in \{1, 2, 3\}^{k-2}$. Similarly we assign $\phi(y) = (2, 2, \ldots, 2)$ for the vertices $y \in Y$ not yet colored, and $\phi(z) = (3, 3, \ldots, 3)$ for the vertices $z \in Z$ not yet colored.

Then ϕ is a proper 3^{k-2}-coloring of $V(G)$. To see this, suppose there exists a monochromatic cyclic triangle. The three vertices in the triangle are not from the same $V(G_i)$ since ϕ is a proper coloring on each of $V(G_i)$. Also, if two of them are from the same $V(G_i)$, then the third vertex is either complete to the first two or complete from them, which is impossible. Therefore none of them are in the same homogeneous set. Then, since $G_0|(X \cup Y)$, $G_0|(Y \cup Z)$, and $G_0|(X \cup Z)$ are transitive, the cyclic triangle meets each of X, Y, Z at a vertex. Say $x \in X$, $y \in Y$, $z \in Z$ are the vertices, and we may assume the first coordinate of $\phi(x), \phi(y), \phi(z)$ is all one. Then, the homogeneous set G_y containing y contains S_{k-1} since the first coordinate of $\phi(y)$ is not two, and the homogeneous set G_z containing z contains S_{k-1} since the first coordinate of $\phi(z)$ is not three. Then by taking two copies of S_{k-1} from G_y and G_z, we obtain a copy of S_k in G by adding x. Since G is S_k-free, this is impossible. Therefore there is no monochromatic cyclic triangle, and hence ϕ is a proper 3^{k-2}-coloring. This proves 5.6.3. ∎

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