SELECTED PROBLEMS IN QUANTUM FIELD THEORY IN
DIFFERENT DIMENSIONS

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Abstract

This thesis examines quantum field theories in different dimensions. We investigate three models defined in different dimensions and spacetime backgrounds: the generalized large $q$ Sachdev-Ye-Kitaev (SYK) model in one dimension, the holographic dual theory to the giant Wilson loops in anti-de Sitter space (AdS) of two dimensions, and the field theory in de Sitter space (dS) of general dimensions.

First, we study the generalized large $q$ SYK model. The effective action of the model in the large $q$ limit is derived and a universal expression for the thermodynamic quantities of the model is presented. We also consider the chaos exponent using the retarded kernel method and find an efficient way to compute the Lyapunov exponent numerically for the generalized large $q$ SYK model.

Next, we consider the 1/2-BPS Wilson loops in the large-rank symmetric and antisymmetric representations, which have holographic dual descriptions in terms of D3-branes and D5-branes in $AdS_5 \times S^5$. We study the spectrums of the fluctuations on the D3-branes and the D5-branes. Using the AdS$_2$/dCFT$_1$ correspondence, we compute the correlation functions of operator insertions on the Wilson loop from perturbation theory on the D-branes. The results in special kinematic configurations agree with the prediction of localization.

Finally, we study the scalar field and the Dirac spinor field in the global $dS_d$ space. The effective actions of the scalar field and the spinor field are computed using the in-/out-state formalism. We show that there is particle production in even dimensions for both scalar field and spinor field. The in-out vacuum amplitude $Z_{\text{in}/\text{out}}$ is divergent at late times. By using dimensional regularization, we extract the finite part of $\log Z_{\text{in}/\text{out}}$ in the even $d$ cases and the logarithmically divergent part of $\log Z_{\text{in}/\text{out}}$ in the odd $d$ cases. We show that the regularized in-out amplitude equals the ratio of determinants associated with different quantizations in $AdS_d$ upon the identification of certain parameters in the two theories.
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Chapter 1

Introduction

Quantum field theory (QFT) is one of the major frameworks for the modern theoretical physics. One of the most successful applications of QFT is the development of the Standard Model. The Standard Model is a QFT in four dimensions with the gauge group $SU(3) \times SU(2) \times U(1)$, which describes the physics of electromagnetic, weak and strong interactions. Although the spacetime of the observed universe is four-dimensional, QFT in other dimensions has been studied in various occasions: The Sachdev-Ye-Kitaev (SYK) model [1] in one dimension has attracted attentions in the past few years as a holographic model for NAdS$_2$/NCFT$_1$ correspondence [2,3]. The ’t Hooft model [4] in two dimensions has been examined to understand the features of the strong interaction. The ADD model [5] in $4+n$ dimensions was proposed to explain the weakness of the gravitational interaction. Therefore, the study of QFT in different dimensions is crucial to a better understanding of the physics of our universe. This thesis examines three models defined in different dimensions and spacetime backgrounds: the generalized large q SYK model in one dimension, the holographic dual theory to the giant Wilson loops in anti-de Sitter space (AdS) of two dimensions, and the field theory in de Sitter space (dS) of general dimensions. In the rest of this chapter, we review the basic facts and properties of these three models.

1.1 Sachdev-Ye-Kitaev model

The standard SYK model [1–3,6] is a one-dimensional quantum mechanical model which can be used to describe the physics of the near extremal black holes [7,8]. It is also the first example demonstrating NAdS$_2$/NCFT$_1$ holographic duality [9–12]. More detailed reviews on the model can be found in [13–15].
The standard SYK model consists of \( N \) Majorana fermions \( \chi_i \) where \( i = 1, \ldots, N \). There are random interactions between \( q \) fermions at a time and the Hamiltonian is

\[
H = \sum_{1 \leq i_1 < \ldots < i_q \leq N} (i)^{\frac{q}{2}} J_{i_1 \ldots i_q} \chi_{i_1} \ldots \chi_{i_q}, \tag{1.1}
\]

where \( q \) is an even integer. The random interaction strength \( J_{i_1 \ldots i_q} \) satisfies the Gaussian distribution:

\[
\langle J_{i_1 \ldots i_q} J_{k_1 \ldots k_q} \rangle = \frac{J^2(q-1)!}{N^{q-1}} \delta_{i_1 \ldots i_q}^{k_1 \ldots k_q}. \tag{1.2}
\]

The dependence on \( N \) in (1.2) is chosen so that in the large \( N \) limit the Feynman diagrams for computing \( n \)-point functions are dominated by melonic diagrams. An example of the melonic diagram is shown in Figure 1.1.

To solve the model, we change to Euclidean space and define the averaged two-point function

\[
G(\tau) = \frac{1}{N} \sum_{i=1}^{N} \langle T \chi_i(\tau) \chi_i(0) \rangle, \tag{1.3}
\]

where the time-translation symmetry has been used. The time ordering symbol \( T \) in (1.3) is defined as

\[
\langle T \chi_i(\tau) \chi_i(0) \rangle = \theta(\tau) \langle \chi_i(\tau) \chi_i(0) \rangle - \theta(-\tau) \langle \chi_i(0) \chi_i(\tau) \rangle. \tag{1.4}
\]

In the large \( N \) limit where the melonic diagrams dominate, the Schwinger-Dyson equation becomes

\[
\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \quad \Sigma(\tau) = J^2 G(\tau)^{q-1}, \tag{1.5}
\]

where \( G(\omega) \) and \( \Sigma(\omega) \) are the Fourier transformations of \( G(\tau) \) and \( \Sigma(\tau) \) respectively. The Schwinger-Dyson equation (1.5) can also be derived from the following effective action of bilocal fields \( G(\tau_1, \tau_2) \)
and \( \Sigma(\tau_1, \tau_2) \) as classical equations of motion

\[
S_{\text{eff}} = -N \log \text{Pf}(\partial_\tau - \Sigma) + \frac{N}{2} \int d\tau_1 d\tau_2 \left[ \Sigma(\tau_1, \tau_2)G(\tau_1, \tau_2) - \frac{J^2}{q}G(\tau_1, \tau_2)^q \right].
\]  

(1.6)

In general, it is difficult to solve (1.5) analytically. One approach is to solve (1.5) perturbatively using the \( 1/q \) expansion by taking the large \( q \) limit [3]. We will discuss this approach in more details in Chapter 2. Another approach is to solve (1.5) in the infrared (IR) regime and take the strong coupling limit, where (1.5) simplifies to

\[
\int d\tau G(\tau_1, \tau) \Sigma(\tau, \tau_2) = -\delta(\tau_1 - \tau_2), \quad \Sigma(\tau_1, \tau_2) = J^2G(\tau_1, \tau_2)^q - 1.
\]  

(1.7)

Equation (1.7) is invariant under the following transformation

\[
G(\tau_1, \tau_2) \rightarrow [f'(\tau_1)f'(\tau_2)]^\Delta G(f(\tau_1), f(\tau_2)),
\]

(1.8)

\[
\Sigma(\tau_1, \tau_2) \rightarrow [f'(\tau_1)f'(\tau_2)]^{\Delta(q-1)} \Sigma(f(\tau_1), f(\tau_2)),
\]

(1.9)

with \( \Delta = \frac{1}{q} \). In particular, this means that the theory has an emergent conformal symmetry in this limit and \( \Delta \) is the conformal dimension of the fermion in IR. This symmetry is only approximate since the full effective action (1.6) is not invariant under the transformation (1.8). The equation (1.7) can be solved by the following conformal two-point function:

\[
G(\tau) = \frac{b_\Delta}{|\tau|^{2\Delta}} \text{sgn}(\tau),
\]  

(1.10)

where \( b_\Delta \) is determined from the following equation:

\[
(b_\Delta)^q = \left( \frac{1}{2} - \Delta \right)^{\frac{\tan(\pi\Delta)}{J^2\pi}}.
\]

(1.11)

Although (1.7) is conformally invariant, the conformal solution (1.10) breaks the full conformal symmetry down to SL(2,R) symmetry:

\[
\tau \rightarrow f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad ad - bc = 1.
\]

(1.12)

So this emergent conformal symmetry is both spontaneously broken by the conformal solution (1.10) and explicitly broken by the \( \partial_\tau \) term in the full action (1.6). If we separate the full action (1.6)
into conformally-invariant part and non-invariant part, the leading contribution to the non-invariant part of the full action is the Schwarzian action

\[ S = -\frac{CN}{f} \int d\tau \text{Sch}(f, \tau), \tag{1.13} \]

where \( C \) is some constant and \( \text{Sch}(f, \tau) \) is the Schwarzian derivative defined by

\[ \text{Sch}(f, \tau) \equiv \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2. \tag{1.14} \]

The Schwarzian action (1.13) is also invariant under the SL(2,R) symmetry. Therefore, the emergent conformal symmetry in IR is both spontaneously and explicitly broken down to SL(2,R) and we can regard the reparametrization modes \( f(\tau) \) as pseudo-Goldstone bosons associated to this broken symmetry. The unbroken SL(2,R) symmetry gives rise to the maximally chaotic behavior of SYK in IR [16,17].

### 1.2 Holographic Wilson loops and AdS\(_2/d\text{CFT}_1\)

Wilson loops are among the most fundamental observables in any gauge theory. Furthermore, in supersymmetric gauge theories, one can study the supersymmetric generalization of the ordinary Wilson loops, i.e., supersymmetric Wilson loops. The expectation value of these supersymmetric Wilson loops can be computed exactly using the method of supersymmetric localization. Since the proposal of the AdS/CFT correspondence [18], which states the duality between type IIB string theory in \( AdS_5 \times S^5 \) background and \( \mathcal{N} = 4 \) supersymmetric Yang-Mills (SYM) theory with gauge group \( SU(N) \) on the four-dimensional boundary, the supersymmetric Wilson loops in \( \mathcal{N} = 4 \) SYM theory have been studied intensively as tests for the holographic correspondence [19,20].

The supersymmetric Wilson loop operator in \( \mathcal{N} = 4 \) SYM theory is defined as

\[ W_R[C] = \frac{1}{\dim R} \text{tr}_R \text{P} \exp \left[ \oint dt (iA_\mu \dot{x}^\mu + |\dot{x}| \theta^I \Phi^I) \right], \tag{1.15} \]

where \( x^\mu(t) \) parametrizes the closed loop \( C \) and \( \theta^I(t) \) is a unit vector in \( \mathbb{R}^6 \) which couples to the six scalars \( \Phi^I \) in the gauge theory. In (1.15), \( P \) is the path-ordering operator, \( R \) is the representation of the \( U(N) \) gauge group, and \( \dim R \) is the dimension of the representation. In this section, we restrict our attention to the case in which \( R \) is the fundamental representation. The case where \( R \) is the totally symmetric or antisymmetric representation will be studied in Chapter 3.
Figure 1.2: The holographic dual description of the Wilson loop. The red contour $C$ represents the Wilson loop at the boundary. The surface $\Sigma$ is the minimal surface of the open string worldsheet which ends on the contour $C$ at the boundary.

The Wilson loop operator in (1.15) is only locally supersymmetric for general contour $C$ and couplings to the scalars $\Phi^I$. However, in special cases where the contour $C$ is a straight line or a circle and $\theta^I$ is a constant vector, the Wilson loop is globally supersymmetric and 1/2-BPS, i.e., it preserves only half of the full superconformal supersymmetry of $\mathcal{N} = 4$ SYM theory.

The holographic dual description of the 1/2-BPS Wilson loop in the fundamental representation is given by an open string in $\text{AdS}_5 \times S^5$ background whose worldsheet ends on the contour defined by the Wilson loop at the boundary. In the strong coupling limit, the expectation value of the Wilson loop operator can be computed from the open string worldsheet described by the minimal surface, which is obtained by extremizing the Nambu-Goto action for the open string

$$S_{NG} = T \int d\sigma d\tau \sqrt{\det \left( G_{MN} \partial_\alpha X^M \partial_\beta X^N \right)}.$$  \hspace{1cm} (1.16)

In (1.16), $T$ is the string tension, $G_{MN}$ is the metric of the Euclidean $\text{AdS}_5 \times S^5$ space, $X^M$ represents the coordinates of the open string in the ten-dimensional spacetime, and the string worldsheet is parametrized by $\{\sigma, \tau\}$. The expectation value of the 1/2-BPS Wilson loop is then given by

$$\langle W[C] \rangle = e^{-S_{NG}(\Sigma)},$$  \hspace{1cm} (1.17)
where $\Sigma$ is the minimal surface that ends on contour $C$ at the boundary.

Besides the expectation value of the 1/2-BPS Wilson loop, one can also analyze correlation functions of local operators inserted on the Wilson loop, which have been studied at both weak and strong coupling [21–24]. In the case where the 1/2-BPS Wilson loop is an infinite straight line, such correlation functions can be defined as

$$\langle \langle O_1(\tau_1) \cdots O_m(\tau_m) \rangle \rangle \equiv \frac{1}{N} \langle \text{tr} P \left[ O_1(\tau_1) \cdots O_m(\tau_m) e^{i \oint (iA_\mu \dot{x}^\mu + \Phi^6 | \dot{x})} d\tau \right] \rangle,$$

where we have set $\theta^I \Phi^I = \Phi^6$ and the Wilson line is along the Euclidean time $\tau$ direction. The operators $O(\tau)$ in (1.18) transform in the adjoint representation of the gauge group. To study such correlation functions, one needs to first understand the symmetries preserved by the 1/2-BPS Wilson line. Since the Wilson line in (1.18) is only coupled to the scalar $\Phi^6$, it preserves the $SO(5)$ symmetry which rotates the five scalars $\Phi^I$ with $I = 1, \ldots, 5$. Furthermore, we have the $SO(3)$ symmetry which corresponds to the rotations around the Wilson line and the $SL(2, R)$ symmetry which corresponds to the conformal transformation in one dimension. Combined with the supersymmetries preserved by the Wilson line, the symmetry group for the 1/2-BPS Wilson line is the superconformal group $OSP(4^*|4)$. One can then classify the operator insertions on the Wilson line by the representations that they transform in under the group $OSP(4^*|4)$. The correlation functions (1.18) are also constrained to be invariant under the one-dimensional conformal group $SL(2, R)$. As a result, we can regard them as the correlation functions of a defect conformal field theory (dCFT) on the Wilson line [21,24,25]. In this regard, we can view $O(\tau)$ as operators in this dCFT1.

In the strong coupling limit, the correlation functions (1.18) can be also computed holographically using the perturbation theory on the dual string worldsheet. For the 1/2-BPS Wilson line, the minimal surface of the dual string worldsheet has the geometry of $AdS_2$. By expanding the Nambu-Goto action (1.16) in terms of the fluctuations of the string worldsheet around this minimal surface solution, one then find that the operators $O(\tau)$ in dCFT1 are dual to these fluctuation modes of the string worldsheet. In fact, these fluctuation modes can be viewed as fields on AdS2 background. From this perspective, the duality between $O(\tau)$ and the fluctuation modes can be regarded as an example of AdS2/dCFT1 correspondence. One caveat is that in this correspondence the AdS2 background is fixed so that no gravitational interactions are included on the AdS side. This is refered as a rigid holography [26], which has been studied recently in [27–33]. Using this AdS2/dCFT1 correspondence, one can extract various defect CFT data from the operator product expansions of
the four-point functions computed in the AdS$_2$ field theory [24]. Furthermore, the results of these four-point functions in special configurations reproduce the correlation functions computed using the localization method in the strong coupling limit, which serves as a consistency check of the correspondence [34].

1.3 Quantum field theory in de Sitter space

The de Sitter (dS) space is the vacuum solution of the Einstein’s equations with a positive cosmological constant $\Lambda$, i.e., the solution to the equation

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$  (1.19)

Cosmological observations from the last century have indicated that the expansion of the universe is accelerating [35–37], which suggests the existence of a positive cosmological constant. Furthermore, the dS space plays an important role in the study of inflationary cosmology [38–40], where the period of the exponential expansion of the universe can be approximated by the dS spacetime. From this point of view, the study of QFT in the dS background plays an important role in understanding the physics of our universe. From the theoretical perspective, it is also interesting to study QFT in dS space. Just like anti-de Sitter (AdS) space which has been studied intensively in the context of the AdS/CFT correspondence [18], dS space also has a maximal isometry group. As a result, the canonical quantization of free fields is possible in dS space and the effective action can be computed analytically [41–48].

The $d$-dimensional dS space $dS_d$ can be viewed as a submanifold embedded in the $d + 1$-dimensional Minkowski spacetime $\mathbb{R}^{1,d}$, which has the metric:

$$ds_{d+1}^2 = -dX_0^2 + dX_1^2 + \cdots + dX_d^2.$$  (1.20)

Then, the dS space with radius $l$ is the hypersurface parametrized by the equation

$$-X_0^2 + X_1^2 + \cdots + X_d^2 = l^2,$$  (1.21)

which is a hyperboloid in $\mathbb{R}^{1,d}$. It is the solution to (1.19) with the cosmological constant

$$\Lambda = \frac{(d-1)(d-2)}{2l^2} = \frac{(d-1)(d-2)}{2} H^2.$$  (1.22)
Figure 1.3: An illustration of $dS_2$ space viewed as a hyperboloid in $\mathbb{R}^{1,2}$. Time $t$ is along the vertical direction. At a fixed time, the cross section is a circle $S^1$.

where $H = 1/l$ is the Hubble constant. The global coordinates which cover the entire $dS_d$ space can be defined by

$$X_0 = \frac{\sinh(Ht)}{H}, \quad X_i = u_i \frac{\cosh(Ht)}{H}, \quad (i = 1, \ldots, d)$$

(1.23)

where $u_i$ specify points on the unit $S^{d-1}$. The induced metric of $dS_d$ from (1.20) in terms of the global coordinates is

$$ds^2 = -dt^2 + H^{-2} \cosh^2(Ht) d\Omega_{d-1}^2,$$

(1.24)

where $d\Omega_{d-1}^2$ denotes the metric of a unit $d - 1$ sphere. The Ricci curvature $R$ of $dS_d$ space is

$$R = H^2(d - 1)d.$$

(1.25)

One of the most studied objects for quantized fields on dS background is the one-loop effective action $W$ [41,42,48–51] defined as

$$e^{iW} = \langle \text{vac, out} | \text{vac, in} \rangle.$$  

(1.26)

It has been known for a long time that dS space has a one-parameter family of vacua called $\alpha$-vacua which are invariant under the dS isometry group [43–45]. These vacuum states can be related to each other via MA transformations [43,44]. Therefore, when computing the effective action (1.26),
one needs to specify the two vacuum states $|\text{vac, in}\rangle$ and $|\text{vac, out}\rangle$. Accordingly, one can define a family of Feynman propagators

$$G_F(x,y) = \langle \text{vac, out}|T\phi(x)\phi(y)|\text{vac, in}\rangle,$$

(1.27)

which can be related to the effective action (1.26) [41,48–50]. One of the guiding principles for selection of the vacuum states is the composition principle proposed by Polyakov [52]. The composition principle implies that the Feynman propagator (1.27) for time-like separated points $x$ and $y$ should behave like

$$G_F(x,y) \sim e^{-imL(x,y)} \quad \text{as} \quad L(x,y) \to \infty,$$

(1.28)

where $L(x,y)$ is the length of the time-like geodesic connecting $x$ and $y$. The effective actions studied most in the literatures [41,42,49] were computed using the in-/in-state (Schwinger-Keldysh) formalism with the Bunch-Davies vacuum, which does not obey the composition principle. It is the in-/out-state formalism, which we will use in Chapter 4 to study the effective action, that complies with the composition principle [52].

### 1.4 Overview

In Chapter 2, we study a type of models which are generalizations of the SYK model in the $q \to \infty$ limit. We derive the effective action of such models and use it to study the thermodynamic properties of these models. Furthermore, we reduce the problem of finding the Lyapunov exponent for such type of models to the quantum mechanical problem of solving the spectrum of the bounded states. We present an efficient way to compute the Lyapunov exponent numerically. This chapter is based on the work [53] collaborated with Z. Yang.

In Chapter 3, we examine the 1/2-BPS Wilson loops in the large-rank symmetric and anti-symmetric representations, which have holographic dual descriptions in terms of D3-branes and D5-branes in $AdS_5 \times S^5$. We study the spectrums of the fluctuations on the D3-branes and the D5-branes. A selection of three- and four-point functions of fluctuations are computed using the pertubation theory on the D-branes. The results in special kinematic configurations agree with the prediction of localization using the $AdS_2$/dCFT$_1$ correspondence. This chapter is based on the work [54] collaborated with S. Giombi and S. Komatsu.

In Chapter 4, we compute the effective actions of a scalar field and a Dirac spinor field in the global de Sitter space of any dimension $d$ using the in-/out-state formalism. We show that there is
particle production in even dimensions for both scalar field and spinor field. We also find that the in-out vacuum amplitude \( Z_{\text{in/out}} \) is divergent at late times. By using dimensional regularization, we extract the finite part of \( \log Z_{\text{in/out}} \) in the even \( d \) cases and the logarithmically divergent part of \( \log Z_{\text{in/out}} \) in the odd \( d \) cases. Furthermore, we show that the regularized in-out amplitude equals the ratio of determinants associated with different quantizations in \( AdS_d \) upon the identification of certain parameters in the two theories. This chapter is based on the work [55].
Chapter 2

Generalized large $q$ SYK model

As reviewed in Section 1.1, the standard SYK model contains $N$ Majorana fermions with random $q$-local interactions. In this chapter, we consider a variant of this model by including various types of all-to-all interactions and solve the theory in the double-scaling limit [56–60] where we take $q$ and $N$ to infinity and keep $q^2/N$ fixed and small. We show that the large $N$ effective action (2.14) describes a two dimensional scalar field with a general potential. Using the effective action, we derive its thermodynamic relation and the Lyapunov exponent of the out-of-time-ordered correlation function (OTOC). We find that the OTOC is given by the Lorentzian propagator of the scalar field and therefore is controlled by its potential. In the generalized large $q$ SYK model we are considering, the chaos exponents correspond to the energies of the bound states and we will show that in such models there exists a unique Lyapunov exponent.

The work in this chapter can be used to understand the qualitative behavior of Lyapunov exponent under relevant deformations. Our generic expectation about the Lyapunov exponent is that it is governed by IR modes of the theory. This means irrelevant deformations will not induce any change of the Lyapunov exponent as well relevant deformations will become important as we lower the temperature. And when the relevant deformation grows, the dynamics will be dominated by the lowest dimensional operator, e.g if we add a mass term the theory will be gapped and there is no chaos behavior [61,62]; if the lowest dimensional operator is four fermion interaction or higher, the theory flows to the standard SYK model with maximal Lyapunov exponent. It is the behavior of the transition along the RG flow that we want to address.

This chapter is organized as follows: in Section 2.1, we set up the analytic investigation of the generalized large $q$ SYK model and derive the double-scaling effective action; in Section 2.2 and
Section 2.3 we derive its thermodynamic relation and study the Lyapunov exponent; in Section 2.4, we explore some concrete examples including large \( q \) and \( 2q \) model, and a scaling model where the interactions contain all \( q \)-fermion interactions with a particular distribution of the coupling strength; in Section 2.5, we show that the Lyapunov exponent is unique.

### 2.1 Generalized large \( q \) SYK model

In Section 1.1, we have shown that there are several important features of the standard SYK model in the limit of \( N \) going to infinity: First, Feynman diagrams are dominated by melonic diagrams; Second, the theory is exactly solvable if we further assume \( q \) is large; Third, when \( q > 2 \), the model develops conformal symmetry in IR; Fourth, the conformal symmetry is slightly broken leaving only a \( \text{SL}(2,\mathbb{R}) \) subgroup unbroken, and the pseudo-Goldstone boson is controlled by the Schwarzian action; Last, the unbroken \( \text{SL}(2,\mathbb{R}) \) symmetry gives rise to the maximally chaotic behavior of SYK in IR \([16,17]\).

If one is only interested in the low energy behavior of this theory then only the lowest dimensional operator matters so it can be classified to be either gapped (when the lowest dimension operator is two-local) or maximally chaotic (when the lowest dimension operator is four-local or higher). However, since here we want to talk about the Lyapunov exponent at all temperature scales, different types of interactions will indeed change its behavior. On that account we extend our Hamiltonian to include different types of interactions denoted by \( q_i \) (1 \( \leq \) \( i \) \( \leq \) \( K \)):

\[
H = \sum_{q=1}^{qK} \sum_{1 \leq i_1 < \ldots < i_q \leq N} (i_j^{\frac{q}{2}} J_{i_1 \ldots i_q} \chi_{i_1} \cdots \chi_{i_q}).
\]  

We are taking \( J_{i_1 \ldots i_q} \) to be independent gaussian disorder variables with zero mean and variance \( J_q^2 \frac{(q-1)!}{N^{q-1}} \). In the limit of large \( N \), we can use melon diagrams (Figure 2.1) to write down the Schwinger-Dyson (SD) equation in Euclidean time \( \tau \):

\[
\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \quad \Sigma(\tau) = \sum_{q=1}^{qK} J_q^2 [G(\tau)]^{q-1}.
\]  

We have defined the disordered average two-point function in a thermal state

\[
G(\tau_1 - \tau_2) \equiv \sum_i \frac{1}{N} \langle T\chi_i(\tau_1)\chi_i(\tau_2) \rangle_\beta,
\]  

\[12\]
where \( \langle T\chi_i(\tau_1)\chi_i(\tau_2) \rangle_\beta \) is defined as

\[
\langle T\chi_i(\tau_1)\chi_i(\tau_2) \rangle_\beta = \theta(\tau_1 - \tau_2) \text{Tr} [\rho(\beta)\chi_i(\tau_1)\chi_i(\tau_2)] - \theta(\tau_2 - \tau_1) \text{Tr} [\rho(\beta)\chi_i(\tau_2)\chi_i(\tau_1)],
\]

(2.4)

with \( \rho(\beta) = e^{-\beta H} \) being the density matrix. In (2.2) and (2.3), we have used the time translation invariance to simplify the time dependence of the two-point function. In (2.2), \( \Sigma(\tau) \) is the self energy and is represented by the black bulb in Figure 2.1. The self energy is a sum of different products of two-point function determined by our interaction. Allowing some of the \( J_{q_i} \)'s to be zero, we can assume \( \{q_i\} \) is a sequence with increment 2 from \( q_1 \) to \( q_K \). Clearly, from the SD equation, the low energy limit is only determined by \( q_1 \) in the summation of self energy. To solve the SD equation at all temperature scales analytically, we can take the large \( q \) limit. To be more precise, we consider the case that all the \( q_i \) scale to infinity at the same rate:

\[
q_i = \alpha_i q; \quad \frac{1}{q} \ll \alpha_1 \ll 1 \ll \alpha_K \ll q; \quad \delta\alpha = \alpha_{i+1} - \alpha_i = \frac{2}{q}.
\]

(2.5)

Considering \( J^2(\alpha) = \frac{\pi}{2} J^2_{\alpha q} \), one can write down an integral expression of the SD equation:

\[
\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \quad \Sigma(\tau) = \lim_{q \to \infty} \sum_i \delta\alpha J^2(\alpha_i)G(\tau)^{\alpha_i q - 1} = \int_0^\infty d\alpha J^2(\alpha)G(\tau)^{\alpha q - 1}.
\]

(2.6)

where the lower bound \( \alpha_1 \) and upper bound \( \alpha_K \) are taken to be 0 and \( \infty \) respectively. This SD equation can be formally obtained by the following symbolic Hamiltonian:

\[
H = \int_0^\infty d\alpha (i) \frac{\alpha q}{q} \sum_M J_M(\alpha)\chi_M^{aq},
\]

(2.7)
where \((\sum_{M} J_M(\alpha) \chi_M^{\alpha q})^2\) stands for a SYK type \(\alpha q\)-local interactions. \(M = \{i_1 ... i_{\alpha q}\}\) enumerates all possible \(\alpha q\) fermions. And the interactions strength satisfies:

\[
\langle J_M(\alpha) J_{M'}(\beta) \rangle = \frac{J^2(\alpha) \Gamma(\alpha q)}{N_{\alpha q} - 1} \delta(\alpha - \beta) \delta_{M,M'}.
\]  

(2.8)

We call this model (2.7) the generalized large \(q\) SYK model.\(^1\) One can also try to keep some of the \(q_i\) finite, which corresponds to deforming the Hamiltonian with a finite \(q\) deformation. One such example is to deform the large \(q\) SYK with mass term, which has a low energy interpretation of a double trace deformation of JT gravity [63–66]. We can use the following ansatz to solve the SD equations\(^2\)

\[
G(\tau) = \frac{1}{2} \text{sgn}(\tau) \left[ 1 + \frac{1}{q} g(\tau) + \ldots \right],
\]

(2.9)

\[
\Sigma(\tau) = \int_0^\infty d\alpha J^2(\alpha) 2^{1-\alpha q} \text{sgn}(\tau) e^{\alpha g(\tau)} (1 + \ldots).
\]

(2.10)

The SD equations imply that

\[
g''(\tau) = U(g) \equiv 2 \int_0^\infty d\alpha J^2(\alpha) e^{\alpha g(\tau)},
\]

(2.11)

where \(J^2(\alpha) = q 2^{1-\alpha q} J^2(\alpha)\) and we define \(U(g)\) such that the SD equations become Newton’s equation for a particle under a classical force \(U(g)\). The KMS condition demands that the particle bounces back to its original location after time \(\beta\), i.e. \(g(0) = g(\beta) = 0\) (See Figure 2.2). We also see that \(U(g)\) can be very general since it is a Laplace transformation of an arbitrary positive function \(J^2(\alpha)\). If we further define the potential \(W(g) = -\int_{-\infty}^g dg U(\hat{g})\), then it follows that

\[
g'(\tau) = -\sqrt{2} \left[ W(g_m) - W(g) \right], \quad \tau(g) = \begin{cases} 
\int_0^g \frac{dg}{\sqrt{2W(g_m) - W(g)}} & \tau < \frac{\beta}{2} \\
\beta - \int_0^{g_m} \frac{dg}{\sqrt{2W(g_m) - W(g)}} & \tau > \frac{\beta}{2}
\end{cases}
\]

(2.12)

where \(g_m\) is the location at which the particle bounces \((g_m \leq g(\tau) \leq 0)\) back and it is related to \(\beta\) by:

\[
\beta = \sqrt{2} \int_{g_m}^0 \frac{dg}{\sqrt{W(g_m) - W(g)}}
\]

(2.13)

\(^1\)Strictly speaking, this Hamiltonian is only defined if the \(J(\alpha)\) is centered at even integer values of \(\alpha q\). In large \(q\) limit, as we have shown above, one can just treat it as a positive continuous function of \(\alpha\).

\(^2\)Higher order in \(\frac{1}{q}\) was considered in [67].
From the definition of $U(g)$, the potential $W(g)$ has the property that all its derivatives are negative and in particular it is monotonic and always less than zero. Therefore, when $g_m$ becomes more and more negative, the temperature of the system approaches to zero. So the system flows from UV to IR as we move leftwards along the $g$-axis as shown in Figure 2.2.

Figure 2.2: The figure depicts the scattering of a particle from the potential $W(g)$. The total time it takes for the particle to return to the origin defines $\beta$. Increasing $\beta$ corresponds to scattering the particle with a higher energy and the particle penetrates deeper into the $g < 0$ region. The thermodynamic quantities are determined by the potential in the region $g_m \leq g \leq 0$ while the Lyapunov exponent is determined by the potential in the region $g \leq g_m$.

The large $q$ Schwinger-Dyson equation can also be derived from the large $N$ effective action obtained from the original fermion path integral using the replica method. In the standard large $q$ SYK model, one arrives at a Liouville field theory [63]. Using the same derivation with some small modifications, our effective action becomes:

$$S_E = -S_0 + \frac{N}{8q^2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left[ \frac{1}{2} \partial_{\tau_1} g(\tau_1, \tau_2) \partial_{\tau_2} g(\tau_1, \tau_2) + W(g(\tau_1, \tau_2)) \right],$$

(2.14)

where $S_0 = \frac{N}{2} \log 2$ is the entropy of $N$ free Majorana fermions. The derivation of the effective action is included in appendix A.1. The $g$ field is defined in equation (2.9). It is symmetric with respect to $\tau_1$ and $\tau_2$ as dictated by fermion statistics. Also from the KMS condition, it is periodic with inverse temperature $\beta$. Finally, since the theory is free at UV, it vanishes at coincident points: $g(\tau, \tau) = 0$. 

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We can also define the following coordinates

$$\tilde{\tau} = \tau_1 + \tau_2, \quad \tilde{\sigma} = \tau_1 - \tau_2,$$

(2.15)

which can be considered as describing the kinematic space. In terms of the kinematic space coordinates, the action (2.14) describes the most general type of scalar field theory, though we need to keep in mind that the possible forms of the potential $W(g)$ are restricted. From this effective action, we can calculate the two-point function $G(\tau_1, \tau_2)$ to leading order $\frac{1}{q}$ by simply taking its expectation value:

$$G(\tau_1, \tau_2) = \frac{1}{2} \text{sgn}(\tau_{12}) + \text{sgn}(\tau_{12}) \frac{1}{2q} \langle g(\tau_1, \tau_2) \rangle,$$

(2.16)

where $\langle g(\tau_1, \tau_2) \rangle$ represents doing functional integral over $g$ with the effective action (2.14). In the large $N$ limit, the expectation value is determined from its classical solution and is the same as (2.12) from the SD equation after using the translation symmetry.

Below, we will use this action and its classical solution to study the thermodynamics and chaos behavior of the generalized large $q$ SYK model.

### 2.2 Thermodynamic quantities

In the leading large $N$ approximation, the free energy can be obtained by simply evaluating the on-shell action (2.14) of its classical solution. In particular, we will study the classical solution which has the translation symmetry so that $g(\tau_1, \tau_2) = g(\tau_{12})$. The on-shell action for such classical solution (2.12) then becomes (subtracted by $S_0$):

$$\beta F = \frac{N \beta}{8q^2} \int_0^\beta d\tau \left[ -\frac{1}{2} \partial_\tau g(\tau) \partial_\tau g(\tau) + W(g(\tau)) \right].$$

(2.17)

It will be convenient to define the constant $N = \frac{N}{8q^2}$. We can use (2.12) to get rid of the kinetic term and arrive at the following expression for free energy:

$$F = N \int_0^\beta d\tau \left[ -\frac{1}{2} \partial_\tau g(\tau) \partial_\tau g(\tau) + W(g(\tau)) \right] = N \beta W(g_m) - 2\sqrt{2N} \int_{g_m}^0 dg \sqrt{W(g_m) - W(g)},$$

(2.18)

where $g_m$ as we have discussed in Section 2.1 is the locus where the particle bounces back. Since $g_m$ is uniquely determined by the temperature $\beta$, we shall treat $g_m$ as the thermodynamic variable. We can calculate the energy using the following argument: if we consider $J^2(\alpha) = J^2 f(\alpha)$ where $J$ is
a constant that defines the scale, then $\beta$ and $J$ should appear together in the expression. The only $J$ dependence in (2.14) is from the potential $W(g)$ and this gives us

$$J \partial J (-\beta F) = -2N \int d\tau_1 d\tau_2 W(g(\tau_1, \tau_2)) = -\beta E.$$  \hspace{1cm} (2.19)

Therefore, we can also express the energy $E$ in terms of the potential $W(g)$ and the thermodynamic variable $g_m$ as:

$$E = 2N \beta W(g_m) - 2\sqrt{2}N \int_{g_m}^{0} dg \sqrt{W(g_m) - W(g)}.$$  \hspace{1cm} (2.20)

We find that the expression of the free energy $F$ and the expression of the energy $E$ only differs by $N \beta W(g_m) + TS_0$ which is equal to the product of temperature and entropy. Therefore, all the thermodynamic quantities can be determined from the potential $W(g)$ and the variable $g_m$ in the generalized large $q$ SYK model:\footnote{It is interesting that the final result has an explicit dependence on the potential. A similar behavior happens for 2d dilaton gravity where the thermodynamic relation only depends on a similar quantity called prepotential \cite{68}.}

$$F = -N \beta W(g_m) + E - \frac{S_0}{\beta}, \quad S = S_0 + N \beta^2 W(g_m).$$  \hspace{1cm} (2.21)

It is not obvious from the expressions (2.21) that the energy $E$ and the entropy $S$ would satisfy the thermodynamic relation. However, we can check explicitly that these expressions indeed satisfy the thermodynamic relation $\partial_{\beta} E = \frac{1}{\beta} \partial_{g_m} S$. Because the thermodynamic variable $g_m$ is uniquely determined by $\beta$, we can equivalently check the relation $\partial_{g_m} E = \frac{1}{\beta} \partial_{g_m} S$. The explicit calculations give

$$\frac{1}{\beta} \partial_{g_m} S = N \left( 2W(g_m) \frac{d\beta(g_m)}{dg_m} + \beta \partial_{g} W(g_m) \right) = 2N \frac{d}{dg_m} \left( \beta W(g_m) \right) - N \beta \partial_{g} W(g_m),$$  \hspace{1cm} (2.22)

and

$$\partial_{g_m} E = 2N \frac{d}{dg_m} \left( \beta W(g_m) \right) - 2\sqrt{2}N \frac{d}{dg_m} \left( \int_{g_m}^{0} dg \sqrt{W(g_m) - W(g)} \right)$$

$$= 2N \frac{d}{dg_m} \left( \beta W(g_m) \right) - N \beta \partial_{g} W(g_m).$$  \hspace{1cm} (2.23)

We see that our expressions for the thermodynamic quantities of the generalized large $q$ SYK model do satisfy the thermodynamic relation.
2.3 Chaos exponent

In this section we study the chaos behavior of the generalized large $q$ SYK model. We need to consider the following out-of-time-ordered correlation function (OTOC) in Lorentzian time [2]

\[
F(t_1, t_2) = \frac{1}{N^2} \sum_{i,j=1}^{N} \text{Tr} \left[ y \chi_i(t_1) y \chi_j(0) y \chi_i(t_2) y \chi_j(0) \right], \quad y \equiv \rho(\beta)^{1/4}. \tag{2.24}
\]

The fermions in (2.24) are separated by a quarter of the thermal circle. Instead of calculating $F(t_1, t_2)$ directly, we could obtain it by the analytic continuation of the Euclidean correlation function $\langle G(t_1, t_2) G(0,0) \rangle_{\beta}$. Since $G(t_1, t_2)$ is related to $g(t_1, t_2)$ by (2.16), we need to compute the Euclidean correlation function $\langle g(t_1, t_2) g(0,0) \rangle_{\beta}$ using the large $N$ effective action (2.14). It is more convenient if we change to the kinematic space coordinates (2.15) and analytically continue back to Lorentzian signature. The resulted Lorentzian action is

\[
iS_M = iN \int dt d\sigma \left[ \frac{1}{4} \left( \partial_t g(t, \sigma) \partial_t g(t, \sigma) - \partial_{\sigma} g(t, \sigma) \partial_{\sigma} g(t, \sigma) \right) - \frac{1}{2} W(g) \right], \tag{2.25}
\]

where $t$ and $\sigma$ are the analytic continuations of the kinematic space coordinates $\tilde{t}$ and $\tilde{\sigma}$ defined in (2.15). In the large $N$ limit, the leading connected piece in $\langle g(t, \sigma) g(0, 0) \rangle_{\beta}$ ($t = t_1 + t_2$ and $\sigma = t_1 - t_2$) will be given by the two-point function of the fluctuation $g(t, \sigma)$ around the classical solution $g_c(\sigma)$: $\langle g(t, \sigma) g(0,0) \rangle_{\beta}$. The quadratic effective action for $g(t, \sigma)$ is

\[
iS_{gg} = iN \int dt d\sigma \frac{1}{4} \left[ \partial_t g(t, \sigma) \partial_t g(t, \sigma) - \partial_{\sigma} g(t, \sigma) \partial_{\sigma} g(t, \sigma) - \partial_{g_c}^2 W(g_c(\sigma)) g^2(t, \sigma) \right]. \tag{2.26}
\]

The two-point function $K(t, \sigma) = \langle g(t, \sigma) g(0,0) \rangle_{\beta}$ is just the propagator of $g$, which satisfies the differential equation

\[
\left[ -\partial_t^2 + \partial_{\sigma}^2 - \partial_{g_c}^2 W(g_c(\sigma)) \right] K(t, \sigma) = \frac{2i}{N} \delta(t) \delta(\sigma). \tag{2.27}
\]

Since we want to study the exponential growth of $K(t, \sigma)$ at late time, we will use the ansatz $K(t, \sigma) = e^{\frac{\lambda t^2}{4}} f_{\lambda}(\sigma)$ as guaranteed by the translation symmetry of $t$ in (2.27) and look in the regime $t \gg 1$. After plugging the ansatz into (2.27), we obtain the following differential equation for $f_{\lambda}(\sigma)$:

\[
\left( \frac{\lambda^2}{4} - \partial_{\sigma}^2 \right) f_{\lambda}(\sigma) = -\partial_{g_c}^2 W(g_c(\sigma)) f_{\lambda}(\sigma), \tag{2.28}
\]

which is same as the Schrödinger equation for a particle moving in the potential $\partial_{g_c}^2 W(g_c(\sigma))$. Thus, the problem of finding the Lyapunov exponent for the generalized large $q$ SYK model becomes
a quantum mechanical problem of finding the spectrum of the bound states for a particle in the
$\partial g^2 W(g_c(\sigma))$ potential. Although the form of potential $W$ as a function of $g$ is simple, the form
of the classical solution $g_c(\sigma)$ is usually complicated. Therefore, it is more useful if we can express
(2.28) purely in terms of variable $g$. We can obtain a relation between the two variables $g$ and $\sigma$ by
analytically continuing (2.12):

$$\tau = \frac{\beta}{2} + i\sigma = -\int_0^g d\tilde{g} \frac{1}{\sqrt{2W(g_m) - W(\tilde{g})}} = \frac{\beta}{2} - i\int_{g_m}^g d\tilde{g} \frac{1}{\sqrt{2W(\tilde{g}) - W(g_m)}}. \tag{2.29}$$

Here the variable $g$ is in the range $(-\infty, g_m)$, so the particular branch we have chosen corresponds
to $\sigma \in (0, \infty)$. For $\sigma \in (-\infty, 0)$, we need to choose the other branch to do the analytic continuation.
Here, we will just work with the branch corresponding to $\sigma > 0$ so we can express $\sigma$ in terms of $g$
by the following equation:

$$\sigma = \int_{g}^{g_m} d\tilde{g} \frac{1}{\sqrt{2W(\tilde{g}) - W(g_m))}}, \tag{2.30}$$

with $g \in (-\infty, g_m)$. Recall that when we calculate the various thermodynamic quantities in Section
2.2, we only need the information about the potential $W(g)$ in the region $g \in (g_m, 0)$ while here we
see that the chaos behavior is completely determined by the part of the potential $W(g)$ in the region
$g \in (-\infty, g_m)$ in Figure 2.2. This manifests the fact that the chaos behavior is controlled by the IR
degrees of freedom of the system. Now we can use (2.30) to write the differential equation (2.28)
purely in terms of $g$ variable with $g$ in the range of $(-\infty, g_m)$ as

$$2\sqrt{W(g) - W(g_m)} \partial_{g} \left( \sqrt{W(g) - W(g_m)} \partial_{\sigma} f_{\lambda}(g) \right) = \left( \frac{\lambda^2}{4} + \partial_{g}^2 W(g) \right) f_{\lambda}(g), \tag{2.31}$$

which we can use to calculate numerically the Lyapunov exponent without solving (2.12) directly.
The boundary condition for the ground state wavefunction is $f_{\lambda}(g_m) = 0$.

### 2.4 Results of specific models

#### 2.4.1 Large $q$ and $2q$ model

We consider the model which corresponds to the SYK model with a large $q$ and $2q$ interactions.
Specifically, the Hamiltonian for the model is

$$H = (i)^q \sum_{1 \leq i_1 < \cdots < i_q \leq N} j_{i_1 \cdots i_q} \chi_{i_1} \cdots \chi_{i_q} + (i)^{2q} \sum_{1 \leq i_1 < \cdots < i_{2q} \leq N} k_{i_1 \cdots i_{2q}} \chi_{i_1} \cdots \chi_{i_{2q}}, \tag{2.32}$$
where the coefficients satisfy
\[
\langle j_{i_1 \ldots i_q}^2 \rangle = \frac{2^{q-1} \mathcal{J}^2(q-1)!}{N^{q-1}}, \quad \langle k_{i_1 \ldots i_{2q}}^2 \rangle = \frac{2^{2q-1} \mathcal{K}^2(2q-1)!}{2^q N^{2q-1}}.
\] (2.33)

Using our formalism, this corresponds to a generalized large \( q \) SYK model with the following \( U(g) \):
\[
g''(\tau) = U(g) = 2 \mathcal{J}^2 e^g + \mathcal{K}^2 e^{2g}.
\] (2.34)

The corresponding potential \( W(g) \) is then \( W(g) = -2 \mathcal{J}^2 e^g - \frac{1}{2} \mathcal{K}^2 e^{2g} \). We can express the inverse temperature \( \beta \) in terms of the thermodynamic variable \( g_m \) defined in Section 2.2 by using (2.13). We then obtain the following relation:
\[
\beta = \frac{2e^{-g_m} \Theta}{\sqrt{4\mathcal{J}^2 + e^{g_m} \mathcal{K}^2}}.
\] (2.35)

where the \( \Theta \) is defined as
\[
\cos \Theta = \frac{(-2 + 4e^{g_m}) \mathcal{J}^2 + e^{2g_m} \mathcal{K}^2}{2 \mathcal{J}^2 + e^{g_m} \mathcal{K}^2}.
\] (2.36)

For fixed \( \mathcal{J} \) and \( \mathcal{K} \), we see that as \( g_m \) goes from 0 to \( -\infty \), the angle \( \Theta \) ranges from 0 to \( \pi \) and \( \beta \) increases from 0 to \( \infty \). Since \( \mathcal{K} \) appears in both (2.35) and (2.36) with \( e^{g_m} \) prefactor, at low temperature \( (g_m \ll 0) \) \( \mathcal{K} \) is strongly suppressed, which is expected because in IR the \( q \)-fermion interaction dominates. The entropy for this model can be calculated using (2.21) and we get
\[
S = S_0 - 2N\Theta^2.
\] (2.37)

At zero temperature, the correction becomes \( -\frac{\pi^2}{6q^2} \) which is identical to the standard large \( q \) SYK model as expected.

We can numerically calculate the Lyapunov exponent for this model using (2.31). On the other hand, we can actually solve (2.34) analytically and then use perturbation theory to calculate the Lyapunov exponent at the low temperature regime where the \( q \)-fermion interaction dominates and at the high temperature regime where the \( 2q \)-fermion interaction dominates. The details of the perturbative calculations are included in the Appendix A.3. We show in Figure 2.3 the numerical result for the Lyapunov exponent at different temperature scales and compare it with the analytic results from the perturbation theory. The red (dotted) curve shows the behavior of the Lyapunov exponent at the temperature where the \( 2q \)-fermion interaction dominates, and the perturbation calculation shows that adding the relevant interaction decreases the chaos exponent until the relevant
perturbation dominates which is described by the green (dashed) curve. At lower temperature the theory is described by standard SYK with single $q$-fermion interactions.

Figure 2.3: The figure shows the log plot of the Lyapunov exponent against $\beta$ for the large $q$ and $2q$ model with $\mathcal{J} = 1$ and $K = 100$. The red (dotted) curve shows the result from the perturbative calculation when $\beta K \gg 1$ and $(\beta \mathcal{J})^2 \ll \beta K$, while the green (dashed) curve shows the result from the perturbative calculation when $\beta \mathcal{J} \gg 1$ and $(\beta \mathcal{J})^2 \gg \beta K$.

2.4.2 Scaling model

In this section, we consider the model with the couplings $\mathcal{J}^2(\alpha) = \mathcal{J}^2 \alpha^n$ which we will refer as the scaling model for the reason shown afterwards. This model corresponds to the following $U(g)$ and $W(g)$:

\[ U(g) = 2 \int_0^\infty \frac{d\alpha \mathcal{J}^2 \alpha^n e^{\alpha g}}{(-g)^{n+1}}, \quad W(g) = -\frac{2 \mathcal{J}^2 \Gamma(n)}{(-g)^n}. \]  

(2.38)

The thermodynamic quantities can be calculated using (2.20) and (2.21):

\[ S = S_0 - \frac{2N \beta^2 \mathcal{J}^2 \Gamma(n)}{\left(-g_m\right)^n}, \quad E = \frac{8N \beta^2 \mathcal{J}^2 \Gamma(n)}{\left(n - 2\right)\left(-g_m\right)^n}. \]  

(2.39)

The relation between $g_m$ and $\beta$ is given by (2.13) as

\[ \beta \mathcal{J} = \sqrt{\frac{\pi}{\Gamma(n) \Gamma\left(\frac{1}{n}\right)}} \left(-g_m\right)^{\frac{n+2}{2}}. \]  

(2.40)

\[^4\text{This is an approximation only applicable in the large } q \text{ limit where the lower cutoff should be order of } \frac{1}{q}. \text{ There could also be an upper bound for the number of fermion interactions, in that situation our result applies to the intermediate temperature region where those high dimension operators become negligible.}\]
So we can express $S$ and $\beta E$ in terms of $\beta J$ as

$$
S = S_0 - 2N(\beta J)^{\frac{1}{2n}} C_n, \quad \beta E = \frac{8N}{n-2} (\beta J)^{\frac{1}{2n}} C_n,
$$

(2.41) with $C_n$ a constant coefficient that only depends on $n$. We see that this model has the interesting feature that the entropy $S - S_0$ and the energy $E$ scale with $\beta$. It is for this reason that we refer the model as the scaling model.

The Lyapunov exponent can be calculated numerically using (2.31). If we introduce the new variable $x \in [1, \infty)$ defined by $g = xg_m$ and use the relation (2.40), then the equation (2.31) becomes

$$
\left( 2 - \frac{2}{x^n} \right) f''(x) + \frac{n}{x^{n+1}} f'(x) + \frac{n(n+1)}{x^{n+2}} f(x) = \left( \frac{\lambda \beta}{2\pi} \right)^2 A_n f(x),
$$

(2.42) with $A_n = \frac{\pi^{\frac{2}{n}} (\frac{1}{n})^{\frac{1}{2n}}}{2^{\frac{1}{2n}} (\frac{n+1}{n} + 1)}$. We see that the rescaled chaos exponent $\frac{\lambda \beta}{2\pi}$ does not change with temperature in this model, a feature absent in other SYK models. We plot in figure 2.4 the rescaled chaos exponent against the power $n$ and we observe that the rescaled chaos exponent approaches to the chaos bound as $n$ increases.

![Chaos Exponent with different n](image)

Figure 2.4: The figure shows the Lyapunov exponent for the scaling model with different values of $n$. The rescaled Lyapunov exponent approaches the chaos bound as $n$ increases.

### 2.5 Eigenvalue structure of the chaos exponent equation

As we have shown in Section 2.3, the problem of calculating the chaos exponent in the generalized large $q$ SYK model with the potential $W(g)$ is equivalent to the quantum mechanics problem of
finding the energy spectrum of the bound states for a particle moving in the potential $\partial^2 g W(g)$. From the definition (2.11) of $U(g)$ and $U(g) = -\partial_g W(g)$, we see that the possible form of $W(g)$ is quite restricted in the generalized large $q$ SYK model. Specifically, $W(g)$ has to be the Laplace transformation of a negative distribution and thus has the following properties:

1. $W(g)$ goes to a constant as $g \to -\infty$. This constant is arbitrary so we can always set it to be zero.

2. Any number of derivatives of $W$ is always negative. In particular, $W(g)$ is monotonically decreasing.

3. $W(g)$ is well defined for $g \in (-\infty, 0)$.

As a result, the potential $\partial^2 g W(g)$ will always have a bound state. This implies that there is always an exponential growth at the late time for OTOC in the generalized large $q$ SYK model.

It is natural to ask if there is any subleading exponential growth in the late time OTOC for the model. In terms of the equivalent quantum mechanics problem, this translates to the question if there exists any other bound states besides the ground state. Here we shall argue that there is no such bound states, so no subleading chaos growth in the generalized large $q$ SYK model. Although it is difficult to solve the bound state spectrum directly for general potential $\partial^2 g W(g)$, we notice that there always exits a scattering state with $\lambda_L = 0$. The wavefunction of this state is given by

$$f(\sigma) = g'(\sigma), \text{ or } f(g) = \sqrt{W(g) - W(g_m)}. \quad (2.43)$$

We can verify this directly by plugging into the chaos exponent equation (2.27) and recall that $g''(t) = -U(g) = \partial_g W(g)$. Such a mode with zero chaos exponent has to exist because of energy conservation and we see explicitly that the eigenfunction is generated by taking a time derivative. Furthermore, this scattering state has only one zero point at $g = g_m$ since $W(g)$ monotonically decreases. By the node theorem from quantum mechanics, this should be the first excited state and therefore the spectrum of the system consists of only one single bound state, which means a unique Lyapunov exponent.

---

5We thank Y.Gu and D.Stanford for helpful discussion on this.
2.6 Conclusion

In this chapter, we studied the generalized large $q$ SYK models. We derived the expressions for the thermodynamic quantities such as energy (2.20) and entropy (2.21) and wrote down the general equation to calculate the chaos exponent (2.31) in such models. We pointed out that the equation (2.31) is convenient to do numerical calculations and analyzed its eigenvalue structures. In particular our analysis showed that there exists only one Lyapunov exponent in the generalized large $q$ SYK models and we expect this is a general feature for ladder diagram dominated models. We studied two particular models: the first is the large $q$ and $2q$ model where the chaos exponent displays initial decrease under relevant deformation; the second is the scaling model where the chaos exponent is a constant ratio of the maximum value at all temperatures.
Chapter 3

Giant Wilson loops and AdS$_2$/dCFT$_1$

As reviewed in Section 1.2, the 1/2-BPS Wilson loop is defined on a circle or a straight line and is known to preserve the $OSp(4^*|4)$ subgroup of the full superconformal symmetry of $\mathcal{N} = 4$ SYM [69,70]. In particular, it is invariant under the one-dimensional conformal group $SL(2,R)$ [25,71], and can be regarded as providing an example of defect conformal field theory (dCFT) [21,24,25]. From this point of view, important observables to analyze are the correlation functions of insertions on the Wilson loop with or without local operators in the bulk, and much work has been done to compute them at weak and strong coupling [21–24]. The AdS/CFT correspondence relates the correlation functions on the 1/2-BPS Wilson loop to the correlators of the fluctuations on the dual string worldsheet with AdS$_2$ induced geometry. In the large $N$ limit, the fluctuations on the string worldsheet are decoupled from the closed string modes in the bulk of AdS$_5$, and the setup provides a simple example of AdS$_2$/dCFT$_1$ correspondence. In [24], a set of four-point functions were computed from perturbation theory on the string worldsheet and various defect CFT data were extracted from the operator product expansions. In special kinematical configurations, the results also reproduced the strong-coupling limit of the correlation functions computed from the localization, thereby providing important consistency checks of both approaches [34].

In this chapter, we generalize the work in [24] by computing correlation functions on the 1/2-BPS Wilson loops in higher-rank representations using the AdS$_2$/dCFT$_1$ correspondence. In particular we consider totally symmetric or antisymmetric representations of size of order $N$. These Wilson loops are known to be dual to the D-branes—the D3-branes for the symmetric representations and the D5-branes for the antisymmetric representations—and are analogues of the Giant Gravitons [72,73], which are D-branes dual to local operators with large $R$-charge of order $N$. For this reason, they are
sometimes referred to as *Giant Wilson loops*, a terminology we adopt in this chapter. Much like the Wilson loop in the fundamental representation, they are examples of (super)conformal defects with the $OSp(4^*|4)$ symmetry, and can be studied by supersymmetric localization as was demonstrated in [74–76] for the expectation values (correlation functions of single trace operators in the presence of the Giant Wilson loops were studied in [77]).

The focus of this chapter is to study the correlation functions of the fluctuations on the D-branes in AdS. In particular we focus on the elementary excitations in the $AdS_5$ and $S^5$ directions. The former corresponds to the so-called displacement operator while the latter corresponds to a single scalar insertion on the Wilson loop. For the D5-brane, dual to the antisymmetric Wilson loop, we also analyze the correlation functions of higher Kaluza-Klein modes coming from the $S^4$ worldvolume of the D5-brane. These operators carry higher angular momenta on $S^5$ and correspond to protected scalar insertions with higher $R$-charges. In special kinematics where the correlator preserves a fraction of supersymmetry, the results from the D-brane analysis agree, both for D3 and D5 cases, with the strong-coupling limit of the results of supersymmetric localization.

The rest of this chapter is organized as follows: In Section 3.1, we briefly review the basic facts on the giant Wilson loops and their holographic dual description. In Section 3.2, we compute the correlation functions of fluctuations on the D5-brane, dual to the Wilson loop in the antisymmetric representation. We compute two-, three- and four-point functions of elementary fluctuations on the D5-brane and also a subset of correlation functions that involve the Kaluza-Klein modes on $S^4$. In Section 3.3, we perform a similar analysis for the D3-brane.

### 3.1 Setup and Generalities

In this section, we quickly review and summarize the basic facts about the BPS Wilson loops, their holographic dual descriptions, and their relation to the defect CFT.

#### 3.1.1 Giant Wilson loops and holographic dual

**Higher-rank Wilson loops and D-branes** The 1/2-BPS Wilson loop in $\mathcal{N} = 4$ SYM is the maximally supersymmetric generalization of the ordinary Wilson loop. It can be defined on a straight line or a circle and couples to a single scalar field:

$$W_R = \frac{1}{\dim R} \text{tr}_R P e^{\int (iA_\mu \dot{x}^\mu + \Phi^6 |\dot{x}|)d\tau}$$  \hspace{1cm} (3.1)
Here \( R \) is the representation of the \( U(N) \) gauge group and \( \dim R \) is its dimension. In this chapter, we consider totally symmetric or antisymmetric representations and take the size of the representation, which is the number of boxes in the Young diagram, to be of order \( N \).

In the large \( N \) limit, such Wilson loops are known to be dual to D-branes \([69,74,76,78,79]\). More precisely the Wilson loop in the large-rank symmetric representation is dual to the D3-brane on the \( AdS_2 \times S^2 \) subspace inside \( AdS_5 \) \([74]\) while the one in the antisymmetric representation is dual to the D5-brane on \( AdS_2 \times S^4 \), where \( S^4 \) is a subspace inside \( S^5 \) \([79]\). In both cases, the size of the representation \( k \) is related to the fundamental string charge on the D-brane and determines the size of the “internal space” of the brane (which is \( S^2 \) for the symmetric representation and \( S^4 \) for the antisymmetric representation). The fact that the antisymmetric representation has a cutoff in size translates to the geometric fact that the volume of \( S^5 \) is finite and the D5-brane has a cutoff in size.

**Defect conformal field theory and classification of operators**  
Being defined on a circle or a straight line, the 1/2-BPS Wilson loop preserves a \( SL(2,R) \) subgroup of the four-dimensional conformal group \([25,71]\). Once fermionic symmetries are included, this is extended to the \( OSp(4^*|4) \) 1d (defect) superconformal group \([24,69,70]\). Because of this property, the 1/2-BPS Wilson loop has been analyzed extensively also from the point of view of the defect CFT \([21–24,70]\). So far, most of the studies have focused on the Wilson loop in the fundamental representation, but the loops in higher representations also provide equally well-defined examples of conformal defects.

From the defect CFT point of view, natural observables are the correlation functions of operators on the defect. As is the case with the fundamental Wilson loop, such operators can be defined by inserting the fields of \( \mathcal{N} = 4 \) SYM inside the Wilson loop trace:

\[
\langle \langle O_1(\tau_1) \cdots O_m(\tau_m) \rangle \rangle = \frac{1}{\langle W_R \rangle} \left( \frac{1}{\dim R} \langle \text{tr}_R \left[ O_1 \cdots O_m e^{f(z^\mu \partial \mu + \Phi \xi | \xi |)} \right] \rangle \right). \tag{3.2}
\]

There is however one important difference between the fundamental Wilson loop and the Wilson loops in higher-rank representations. In the case of the fundamental Wilson loop, there is essentially an unique way to build the insertions \( O_j \) from the fundamental fields of \( \mathcal{N} = 4 \) SYM. Namely we take the fields in \( \mathcal{N} = 4 \) SYM and simply multiply them as \( N \times N \) matrices,

\[
(\Phi^2)_{ac} = \sum_b (\Phi)_{ab} (\Phi)_{bc}. \tag{3.3}
\]

To express (3.3) in more group-theoretic terms, it is useful to decompose \( \Phi \) into the generators of
the fundamental representation $T^f_A$ as

$$\Phi_{ab} = \sum_A \Phi_A \left( T^f_A \right)_{ab} \quad (A = 1, \ldots, N^2),$$  \hspace{1cm} (3.4)

Then the product (3.3) can be expressed as

$$\left( \Phi^2 \right)_A \equiv d^f_{ABC} \Phi_B \Phi_C,$$  \hspace{1cm} (3.5)

where the tensor $d^f_{ABC}$ is defined by

$$T^f_A T^f_B = d^f_{ABC} T^f_C.$$  \hspace{1cm} (3.6)

On the other hand, for the higher-rank representations, there are two natural approaches to define the insertions. The first approach is to replace (3.5) and (3.6) with their higher-rank counterparts. Namely we consider

$$\left( \Phi^{[2]} \right)_A \equiv d^R_{ABC} \Phi_B \Phi_C,$$  \hspace{1cm} (3.7)

where the tensor $d^R_{ABC}$ is defined by

$$T^R_A T^R_B = d^R_{ABC} T^R_C,$$  \hspace{1cm} (3.8)

and $T^R_A$’s are the generators in the representation $R$. The operator (3.7) can be inserted inside the Wilson loop trace as

$$\text{tr}_R P \left[ \sum_A \Phi^{[2]}_A T^R_A \exp \left( \oint i A_\mu \dot{x}^\mu + \cdots \right) \right].$$  \hspace{1cm} (3.9)

Since the Wilson loop trace is computed in the representation $R$, such operators arise naturally by bringing together two single insertion of $\Phi$’s on the Wilson loop.

The second approach is to use the multiplication rule for the fundamental Wilson loop and then insert the product inside the Wilson loop trace. Namely we take (3.5) and insert it as

$$\text{tr}_R P \left[ \sum_A \Phi^2_A T^R_A \exp \left( \oint i A_\mu \dot{x}^\mu + \cdots \right) \right].$$  \hspace{1cm} (3.10)

Obviously, the two insertions $\Phi^{[2]}$ and $\Phi^2$ are different (except in the case of the fundamental representation). To understand their physical meaning, it is useful to represent the higher-rank Wilson loop as a collection of fundamental Wilson loops joined together by a projector to the
The Wilson loop in the higher-rank representation can be thought of as a collection of fundamental Wilson loops (black straight lines in the figure) joined together by a projector (denoted by $P_R$ in the figure). In this representation, the insertion of a single scalar $\Phi$ is a sum over insertions of $\Phi$ (denoted by a dot) on each fundamental loop. To insert two $\Phi$'s, there are two possibilities: The first possibility is to simply consider a product of two $\Phi$'s and is given by a double sum $\Phi^{[2]}$. The other possibility is to insert $\Phi^2$ to each fundamental loop. The former operator ($\Phi^{[2]}$) is a two-particle operator while the latter ($\Phi^2$) is a single-particle operator.

representation $R$ (see Figure 3.1). In this representation, the insertion of a single field $\Phi$ corresponds to a sum over insertions of $\Phi$ onto each constituent fundamental loop. Now, if we bring together two of such insertions, we obtain $\Phi^{[2]}$, which is given by a double sum as depicted in Figure 3.1. In this case, the two insertions of $\Phi$ generally live on different fundamental loops as depicted in the figure. On the other hand, the insertion of $\Phi^2$ corresponds to directly inserting two $\Phi$'s onto each constituent fundamental loop.

This representation also provides a holographic interpretation of these operators. As mentioned above, the Giant Wilson loop is dual to a D-brane and the excitations on the brane are described by open strings attached to it. Combined with the fact that each fundamental Wilson loop represents a single string worldsheet, this suggests that the operator $\Phi^{[2]}$ corresponds to excitations of two separate strings, while the operator $\Phi^2$ corresponds to a single string excitation with higher mass.

We find that $\Phi^2$ and its higher charge analogs are related to “single-particle” excitations on the D-branes, while insertions like $\Phi^{[2]}$ to multi-particle ones. We call the operator ($\Phi^{[2]}$) a two-particle operator/insertion while we call ($\Phi^2$) a single-particle operator/insertion.

Protected scalar operators and displacement operator The main subject of this chapter is the calculation of correlation functions of certain protected operator insertions on the Wilson loop. In particular, we focus on two important class of operators.

The first set of operators are made out of five scalar fields $\Phi_a$ ($a = 1, \ldots, 5$)

$$O_L(\tau, u) \equiv (u \cdot \Phi)^L(\tau),$$

(3.11)

where $u$ is a five-dimensional null vector satisfying $(u \cdot u) = 0$. These operators belong to a short
multiplet of the defect superconformal group and have protected scaling dimension $\Delta = L$ \cite{24,70}. The correlation functions of such operators are constrained by the conformal symmetry and the $R$-symmetry. In particular, the two- and the three-point functions are fixed up to overall constants $n_{L_i}$ and $c_{L_1, L_2, L_3}$,

$$\langle \langle O_{L_1}(\tau_1, u_1)O_{L_2}(\tau_2, u_2) \rangle \rangle = n_{L_1} \frac{\delta_{L_1, L_2}(u_1 \cdot u_2)^{L_1}}{(2 \sin \frac{\tau_{12}}{2})^{2L_1}},$$

$$\langle \langle O_{L_1}(\tau_1, u_1)O_{L_2}(\tau_2, u_2)O_{L_3}(\tau_3, u_3) \rangle \rangle = c_{L_1, L_2, L_3} \frac{(u_1 \cdot u_2)^{L_{123}}(u_2 \cdot u_3)^{L_{231}}(u_3 \cdot u_1)^{L_{312}}}{(2 \sin \frac{\tau_{12}}{2})^{2L_{123}}(2 \sin \frac{\tau_{23}}{2})^{2L_{231}}(2 \sin \frac{\tau_{31}}{2})^{2L_{312}},}$$

with $\tau_{ij} \equiv \tau_i - \tau_j$ and $L_{ijk} \equiv (L_i + L_j - L_k)/2$. Here we wrote the results for the correlators on the circular loop. The results for the straight line Wilson loop can be obtained by a simple replacement

$$2 \sin \frac{\tau_{ij}}{2} \rightarrow |\tau_i - \tau_j|.$$ \hspace{1cm} (3.13)

On the other hand, the four-point functions are expressed in terms of the conformal and the $R$-symmetry cross ratios as

$$\langle \langle O_{L_1}(\tau_1, u_1)O_{L_2}(\tau_2, u_2)O_{L_3}(\tau_3, u_3)O_{L_4}(\tau_4, u_4) \rangle \rangle = \frac{1}{(2 \sin \frac{\tau_{12}}{2})^{L_{12}}(2 \sin \frac{\tau_{34}}{2})^{L_{34}}(2 \sin \frac{\tau_{13}}{2})^{L_{13}}(2 \sin \frac{\tau_{24}}{2})^{L_{24}}} \frac{L_{1} - L_{2}}{L_{3} - L_{4}} \frac{(\sin \frac{\tau_{12}}{2})^{L_{1}}}{(\sin \frac{\tau_{34}}{2})^{L_{3}}})^{L_{2}} G(\chi, u) \quad (3.14)$$

The function $G(\chi, u)$ can be further expressed as

$$G(\chi, u) = (u_1 \cdot u_4)^{L_{4} - E}(u_1 \cdot u_3)^{L_{3} - E}(u_2 \cdot u_3)^{L_{2}}(u_3 \cdot u_4)^{E} G(\chi, \xi, \zeta) \quad (3.15)$$

with $2E \equiv L_2 + L_3 + L_4 - L_1$. The $\chi$, $\xi$ and $\zeta$ are the cross ratios defined as

$$\chi \equiv \frac{\sin \frac{\tau_{12}}{2} \sin \frac{\tau_{34}}{2}}{\sin \frac{\tau_{13}}{2} \sin \frac{\tau_{24}}{2}}, \quad \xi \equiv \frac{(u_1 \cdot u_3)(u_2 \cdot u_4)}{(u_1 \cdot u_2)(u_3 \cdot u_4)}, \quad \zeta \equiv \frac{(u_1 \cdot u_4)(u_2 \cdot u_3)}{(u_1 \cdot u_2)(u_3 \cdot u_4)}. \quad (3.16)$$

Note that on the straightline, the cross ratio is given by

$$\chi \equiv \frac{\tau_{12} \tau_{34}}{\tau_{13} \tau_{24}}. \quad (3.17)$$

Although the functional form of $G$ cannot be fixed purely from the symmetry, the superconformal
symmetry imposes the Ward identity \[70\]

\[
\left( \partial_{z_1} + \frac{1}{2} \partial_{\chi} \right) G \left( \chi, \frac{1}{z_1 z_2}, \frac{(1 - z_1)(1 - z_2)}{z_1 z_2} \right) \bigg|_{z_1 = \chi} = 0,
\]

\[
\left( \partial_{z_2} + \frac{1}{2} \partial_{\chi} \right) G \left( \chi, \frac{1}{z_1 z_2}, \frac{(1 - z_1)(1 - z_2)}{z_1 z_2} \right) \bigg|_{z_2 = \chi} = 0.
\]

(3.18)

We will later check that the correlators computed on the D-brane side indeed satisfy these identities.

The other set of operators that we discuss in this chapter are the displacement operators \( F_{tj} \equiv iF_{tj} + D_j \Phi^6 \) along the directions \( j = 1, 2, 3 \) transverse to the Wilson loop \[24, 70\]. They have the protected dimension \( \Delta = 2 \) and the transverse spin \( S = 1 \). These operators correspond to infinitesimal deformations of the Wilson loop orthogonal to the contour. They are in the same ultrashort multiplet as \( O_1 \) and together give eight bosonic operators (which on the D-brane side correspond to certain combinations of the fluctuations in the eight directions transverse to \( AdS_2 \) and of the worldvolume gauge field excitations).

**Comparison of the protected spectrum at weak and strong coupling**  
In addition to \( O_1 \) and \( F_{tj} \), there is an infinite set of protected single-particle operators with higher \( R \)-charge \( O_L \) \( (L \geq 2) \). For the D5-brane, which is dual to the antisymmetric loop, there are natural candidates of their holographic dual: Since the D5-brane is extended in \( S^4 \) inside \( S^5 \), it has infinitely many Kaluza-Klein modes upon reducing to \( AdS_2 \) \[80, 81\]. They have integer angular momenta (dual to \( R \)-charges) and are natural candidates for \( O_L \).

The situation is quite different for the D3-brane. Since the D3-brane is point-like on \( S^5 \), it does not have the Kaluza-Klein modes with higher angular momenta on \( S^5 \) \[82\]. The only excitations that have higher angular momenta are then multi-particle states. However, we expect that \( O_L \) is dual to a single-particle state. This poses a sharp puzzle: On the gauge theory side, we have an infinite set of protected operator \( O_L \)'s but they seem to be absent on the D-brane side. The analysis from the supersymmetric localization shows that the operators \( O_L \) with \( L \geq 2 \) do exist in the spectrum of the Wilson loop defect CFT dual to the D3-brane, but their couplings to \( O_1 \) are exponentially suppressed at strong coupling \[54\]. Therefore, one can not see these higher charge operators could on the D-brane side.

Note that a similar puzzle exists also for the higher transverse spin operators that arise from products of the displacement operator \( F_{tj} \). The D3-brane dual to the symmetric representation is extended in the \( S^2 \) subspace inside \( AdS_5 \). Therefore, it has infinitely many single-particle excitations on \( AdS_2 \) that have higher \( AdS \) angular momenta \[82\]. Natural candidates for such operators on the
gauge theory side are products of the displacement operators \((F_{ij})^S\) inserted on the Wilson loop, which indeed exist at weak coupling. On the other hand, such excitations are absent in the D5-brane since it is not extended in the directions transverse to \(AdS_2\) inside \(AdS_5\). Therefore we again have an apparent paradox, now with the roles of the D3-brane and the D5-brane exchanged. However, this puzzle is not as sharp as the one mentioned earlier since the operators \((F_{ij})^S\) are not protected and they can disappear from the spectrum at strong coupling simply by acquiring infinite anomalous dimensions.

### 3.1.2 1/8 BPS Wilson loops and topological sector

The defect CFT defined by the 1/2-BPS Wilson loop contains a supersymmetric subsector whose correlation functions are position-independent [34,83–87]. For the Wilson loops in the fundamental representation, such correlators were computed exactly using the supersymmetric localization\(^1\) in [34,85]. The results provide non-perturbative defect CFT data, which are important inputs for the conformal bootstrap analysis [70,92].

**1/8 BPS Wilson loops and 2d Yang-Mills** We first consider a broader class of supersymmetric Wilson loops which are 1/8 BPS. They can be defined on an arbitrary contour \(C\) on a \(S^2\) subspace inside \(R^4\) (or \(S^4\)) in the following way:

\[
W_{1/8} = \frac{1}{N} \text{tr}_R \mathbb{P} \exp \left[ \oint_C \left( i A_j + \epsilon_{kjl} x^k \Phi^l \right) dx^j \right] \quad (i, j, k = 1, 2, 3). \tag{3.19}
\]

Here \(x_i\)'s are the embedding coordinates of \(S^2\) of unit radius, \(x_1^2 + x_2^2 + x_3^2 = 1\). Thanks to the specific choice of the coupling to the scalars \(\Phi_i\)'s, they preserve four supercharges in general. If the contour is placed along the great circle of \(S^2\), it preserves sixteen supercharges and becomes half-BPS.

An advantage of studying this specific class of supersymmetric Wilson loops is their equivalence to the two-dimensional Yang-Mills theory (2d YM) in the zero-instanton sector: It was first conjectured based on perturbation theory and AdS/CFT [93,94] and later derived from the supersymmetric localization [95] that the expectation value of the 1/8 BPS Wilson loops coincides with that of the

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\(^1\)Recently the localization computation [88,89] was extended to a large class of observables that include various kinds of defects and the correlation functions on \(\mathbb{R}P^4\) in [87,90]. The formalism was then applied to the D5-brane defect one-point functions in [91].
Figure 3.2: 1/8 BPS Wilson loop on $S^2$. The Wilson loop (denoted by a thick red curve) divides the $S^2$ into two regions, one with area $A$ and the other with area $4\pi - A$.

standard Wilson loops in 2d YM$^2$ defined on the same contour,

$$W_{1/8} \leftrightarrow W_{2dYM} \equiv \frac{1}{N} \text{tr}_R \text{Pexp} \left( \oint_C iA_j dx^j \right), \quad (3.20)$$

under the identification of the coupling constants,

$$g_{2d}^2 = -\frac{g_{YM}^2}{2\pi}. \quad (3.21)$$

**Topological correlators on the Wilson loop** In addition to the expectation values of the Wilson loops, there are other observables that preserve a fraction of supersymmetry and therefore can be computed by 2d YM. The ones relevant in this chapter are the following correlation functions of scalar fields inside a Wilson loop trace,

$$W[\tilde{\Phi}^L_1, \tilde{\Phi}^L_2, \ldots, \tilde{\Phi}^L_n] \equiv \frac{1}{N} \text{tr}_R \text{P}[\tilde{\Phi}^{L_1}(\tau_1) \cdots \tilde{\Phi}^{L_n}(\tau_n) e^{\oint_C (iA_j + \epsilon_{kjl} x^k \Phi^l) dx^j}]. \quad (3.22)$$

Here $\tilde{\Phi}$ is a position-dependent linear combination of the scalars

$$\tilde{\Phi}(x) = x_1 \Phi^1 + x_2 \Phi^2 + x_3 \Phi^3 + i \Phi_4, \quad (3.23)$$

and $\tilde{\Phi}^L$ is a single-particle insertion made out of $L$ such fields. We used a normal ordering symbol $:\tilde{\Phi}^L: \text{P}$ to emphasize the absence of the self-contractions within each operator. One important feature of these correlation functions is their position-independence, which follows from the fact that the spatial translation of $\tilde{\Phi}(x)$ is $Q$-exact [83, 87]. In the rest of this chapter, we often denote these

---

$^2$The equivalence to the 2d YM was later tested extensively against various perturbative computations [83, 84, 96–101].
operators by

$$\tilde{O}_L \equiv :\tilde{\Phi}^L:.$$ \hfill (3.24)

When the Wilson loop is circular and preserves the 1/2-BPS supersymmetry, they can be obtained from the scalar insertions $O_L$ in (3.11) by setting the polarization $u = (\cos \tau, \sin \tau, 0, 0)$, where $\tau \in [0, 2\pi]$ is the position of the operator on the circle. This connection allows us to extract the defect CFT data from the topological correlators, see e.g. Section 2.3 of [85] for more details.

The simplest class of such correlators are the correlation functions of the insertions of a single scalar. They are known to correspond to the insertions of a dual field strength of the two-dimensional Yang-Mills theory [95],

$$\tilde{\Phi} \leftrightarrow i \ast F_{2d},$$ \hfill (3.25)

which in turn is related to an infinitesimal deformation of the contour of the Wilson loop. Thanks to this correspondence, we can compute the correlators of multiple $\tilde{\Phi}$’s by taking the area derivatives of the Wilson loop expectation value,

$$\langle W[\tilde{\Phi} \cdots \tilde{\Phi}] \rangle = \frac{\partial^n \langle W \rangle}{(\partial A)^n}. \hfill (3.26)$$

### 3.2 Correlation functions in dCFT$_1$ from the D5-brane

#### 3.2.1 D5-brane solution in $AdS_5 \times S^5$

In this section, we review the D5-brane solution in the $AdS_5 \times S^5$ background [79, 102]. The bosonic part of the Euclidean D5-brane action takes the form

$$S_{D5} = T_{D5} \int d^6\xi \sqrt{\det(G + F)} + iT_{D5} \int F \wedge C_4.$$ \hfill (3.27)

where $G$ is the induced metric, and we have absorbed a factor of $2\pi \alpha'$ into the worldvolume gauge field. The D5-brane tension $T_{D5}$ is given by

$$T_{D5} = \frac{N\sqrt{\lambda}}{8\pi^4}. \hfill (3.28)$$

To write down the D5-brane solution, we use the following parametrization of the $AdS_5 \times S^5$ space:

$$ds^2 = du^2 + \cosh^2 u ds^2_{AdS_2} + \sinh^2 u d\Omega_2^2 + d\theta^2 + \sin^2 \theta d\Omega_4^2.$$ \hfill (3.29)
The four-form $C_4$ which produces the five-form flux can be written as [79]:

$$C_4 = \left( -\frac{u}{2} + \frac{1}{8} \sinh 4u \right) dH_2 \wedge d\Omega_2 + \left( \frac{3}{2} \theta - \sin 2\theta + \frac{1}{8} \sin 4\theta \right) d\Omega_4, \quad (3.30)$$

where $dH_2$ is the volume element of the $AdS_2$ space. The embedding of the D5-brane in the $AdS_5 \times S^5$ background is parametrized by

$$u = 0, \quad \theta = \theta_k, \quad (3.31)$$

where the angle $\theta_k$ is related to the fundamental string charge $k$ via:

$$k = \frac{N}{\pi} \left( \theta_k - \frac{1}{2} \sin 2\theta_k \right). \quad (3.32)$$

The induced worldvolume geometry of the D5-brane is then $AdS_2 \times S^4$, and the induced metric is

$$ds^2_{D5} = ds^2_{AdS_2} + \sin^2 \theta_k d\Omega^2_4. \quad (3.33)$$

For the case of the Wilson loop on an infinite straight line at the boundary, we can take the metric of $AdS_2$ to be that of the Poincare half-plane

$$ds^2_{AdS_2} = \frac{1}{r^2}(d\tau^2 + dr^2). \quad (3.34)$$

In these coordinates, the Euclidean worldvolume gauge field strength of the classical solution is given by

$$F = i \frac{\cos \theta_k}{r^2} d\tau \wedge dr. \quad (3.35)$$

In addition to the bulk action, we also need to add the following boundary term to implement the correct boundary conditions [74, 79, 103]

$$S^A_{bdy} = - \int d\tau \int d\Omega_4 A_\tau \pi_A, \quad (3.36)$$

where $\pi_A$ is the conjugate momentum to $A_\tau$

$$\pi_A = \frac{\partial L_{D5}}{\partial F_{\tau\tau}}. \quad (3.37)$$

Adding this boundary term corresponds to choosing boundary conditions such that the momentum $\pi_A$ is fixed at the boundary (while $A_\tau$ is dynamical). Indeed, the integral of $\pi_A$ over $S^4$ is related
to the fundamental string charge $k$ by $^3$

$$k = -2\pi i \alpha' \int_{S^4} \partial L_{D5} \frac{\partial F_{rr}}{\partial F_{rr}} = \frac{N}{\pi} (\theta_k - \sin \theta_k \cos \theta_k). \quad (3.38)$$

Let us review how the expectation value of the circular Wilson loop in the large antisymmetric representation is obtained from the classical D5-brane action. The solution described above applies equally well to the circular loop, provided we use the Poincare disk metric of $AdS_2$ instead of (3.34). The expectation value of the Wilson loop is obtained as

$$\langle W_{A_k} \rangle = \exp \left( -S_{D5} + S_{bdy}^A \right). \quad (3.39)$$

Plugging in the solution above, we find

$$S_{D5} + S_{bdy}^A = T_{D5} \text{vol}(AdS_2) \text{vol}(S^4) \sin^3 \theta_k \quad (3.40)$$

Using the well-known regularized value of the hyperbolic disk volume $\text{vol}(AdS_2) = -2\pi i$ as well as $\text{vol}(S^4) = 8\pi^2/3$ and the value of the D5-brane tension, one finds $^7$

$$\langle W_{A_k} \rangle = \exp \left( \frac{2N\sqrt{\lambda}}{3\pi} \sin^3 \theta_k \right), \quad (3.41)$$

which agrees with the localization prediction $^7$.

If we expand the action (3.27) in powers of the fluctuations around the D5-brane solution and perform $KK$-reduction:

$$S_B = \int \frac{d\tau dr}{r^2} L_B, \quad L_B = L^{(2)} + L^{(3)} + L^{(4)} + \ldots, \quad (3.42)$$

the resulting action can be viewed as a 2d field theory on $AdS_2$ background with a manifest symmetry of $SL(2,R) \times SO(3) \times SO(5)$. The dual of this bulk $AdS_2$ theory is the defect $CFT_1$ defined by operator insertions on the straight (or circular) 1/2-BPS Wilson loop. In most of the calculations below, we will focus on the straight line geometry, but all results can be easily translated to the circle.

$^3$The factor of $2\pi \alpha'$ is because in (3.27) we have absorbed this factor into the gauge field, and the factor of $i$ is due to the Euclidean signature.

$^4$Equivalently, one can add to the action a boundary term corresponding to a Legendre transform in the AdS radial direction $^7$.$^{103}$. This gives the same result as using directly the regularized volume of $AdS_2$. 

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3.2.2 Spectrum of excitations around the D5-brane

In this section, we expand the D-brane action around the D5-brane solution and find the spectrum of fluctuations, focusing on bosonic fields only. Since the spectrum has been computed in \([80,81]\), we briefly review the calculation here. For the study of the spectrum, instead of parameterizing \(AdS_5 \times S^5\) as (3.29), we change to \(x^i\) coordinates so that the metric reads

\[
ds^2 = \frac{(1 + \frac{1}{4}x^2)^2 d\bar{s}^2_{AdS_2} + dx^i dx^i}{(1 - \frac{1}{4}x^2)^2} + d\theta^2 + \sin^2 \theta d\Omega_4^2,
\]

(3.43)

where \(i = 1, \ldots, 3\) refers to the transverse directions. The previous \(u\) coordinate is related to \(x^2 = x^i x^i\) by

\[
\frac{x^2}{(1 - \frac{1}{4}x^2)^2} = \sinh^2 u.
\]

(3.44)

We use Greek letters \((\mu, \nu)\) for \(AdS_2\) coordinates and Greek letters \((\alpha, \beta)\) for \(S^4\) coordinates.

Now we consider the effective action for fluctuations \(\delta x^i, \delta \theta\) and \(f\) around the D5-brane solution, where \(f\) is a 2-form in 6d spacetime representing the fluctuations of the background field strength. We expand everything to quartic order in fluctuations as we need to compute various four-point functions later. The variation of the metric in powers of fluctuations is

\[
\delta(ds^2) = \left( \delta x^i + \frac{1}{2} \delta x^4 \right) d\bar{s}^2_{AdS_2} + \left( 1 + \frac{1}{2} \delta x^2 \right) (d\delta x^i)(d\delta x^i) + A(\delta \theta) d\Omega_4^2 + (d\delta \theta)^2,
\]

(3.45)

where

\[
A(\delta \theta) = \sin 2\theta_k \delta \theta + \cos 2\theta_k \delta \theta^2 - \frac{2}{3} \sin 2\theta_k \delta \theta^3 - \frac{1}{3} \cos 2\theta_k \delta \theta^4.
\]

(3.46)

The variation of \(C_4\) in powers of fluctuations is

\[
\delta C_4 = -\frac{1}{8} (12\theta_k - 8 \sin 2\theta_k + 4 \sin 4\theta_k) - 4 \sin^4 \theta_k \delta \theta - 8 \cos \theta_k \sin^3 \theta_k \delta \theta^2
\]

\[
- \frac{8}{3} (1 + 2 \cos 2\theta_k) \sin^2 \theta_k \delta \theta^3 + \frac{2}{3} (\sin 2\theta_k - 2 \sin 4\theta_k) \delta \theta^4.
\]

(3.47)

The mass spectrum can be then obtained by expanding the action (3.27) to quadratic order in fluctuations around the D5-brane solution.

\(\delta x^i\) sector The quadratic Euclidean action for the \(\delta x^i\) sector is

\[
S_{\delta x}^{(2)} = T_{D5} \sin^3 \theta_k \int d^6 \xi \sqrt{|g|} \frac{1}{r^2} \left[ \partial_{\mu} \delta x^i \partial^\mu \delta x^i + \nabla_\alpha \delta x^i \nabla^\alpha \delta x^i + 2 \delta x^i \delta x^i \right],
\]

(3.48)
where $g_4$ is the metric for $S^4$. To keep the $SO(5)$ symmetry manifest, we expand the fields using the spherical harmonics defined by symmetric traceless tensor. Specifically, if we let $Y^a$ to be the five-dimensional vector specifying $S^4$:

$$
\sum_{a=1}^{5} Y^a(\Omega_4)Y^a(\Omega_4) = 1,
$$

(3.49)

then the $\delta x^i$ field is expanded as

$$
\delta x^i(\tau, r, \Omega_4) = \sum_{l=0}^{\infty} (\delta x^i)_{a_1 \ldots a_l} (\tau, r) Y^{a_1} \ldots Y^{a_l},
$$

(3.50)

where $(\delta x^i)_{a_1 \ldots a_l}$ is a symmetric traceless tensor field and the repeated indices are summed. In particular, we have

$$
\nabla^2_{S^4} (\delta x^i)_{a_1 \ldots a_l} Y^{a_1} \ldots Y^{a_l} = -l(l+3)(\delta x^i)_{a_1 \ldots a_l} Y^{a_1} \ldots Y^{a_l}.
$$

(3.51)

The quadratic action for these expanded fields is

$$
S^{(2)}_{\delta x} = \sum_{l=0}^{\infty} V_l T_{D5} \sin^3 \theta_k 
\int \frac{dr d\tau}{r^2} \frac{1}{2} \left[ \partial_\mu (\delta x^i)_{a_1 \ldots a_l} \partial^\mu (\delta x^i)_{a_1 \ldots a_l} + (l+2)(l+1)(\delta x^i)_{a_1 \ldots a_l} (\delta x^i)_{a_1 \ldots a_l} \right].
$$

(3.52)

The factor $V_l$ comes from the integral of spherical harmonics over $S^4$ and is defined by

$$
\int d\Omega_4 (u_1 \cdot Y)^l (u_2 \cdot Y)^l \equiv V_l (u_1 \cdot u_2)^l,
$$

(3.53)

where $u^a$ denotes a five-dimensional null vector. This integral can be done analytically and we find $V_l$ to be

$$
V_l = \frac{16\pi^2 2^l (l+1)!!}{(2l+3)!}.
$$

(3.54)

**$\delta \theta$ and $a_\mu$ sector** In order to decouple $a_\mu$ from the gauge fields along the $S^4$ directions, we need to impose the gauge condition

$$
\nabla^\alpha a_\alpha = 0.
$$

(3.55)
The quadratic Euclidean action for $\delta \theta$ and $a_{\mu}$ is

$$S_{\delta \theta, f}^{(2)} = T D^{5} \int d^{5} \xi \sqrt{g_{4}} \left[ \frac{\sin^{3} \theta_{k}}{2} \left( \partial_{\mu} \delta \theta \partial^{\mu} \delta \theta + \nabla_{\alpha} \delta \theta \nabla^{\alpha} \delta \theta - 4 \delta \theta^{2} \right) + \frac{\sin \theta_{k}}{2} \left( \frac{1}{2} f_{\mu \nu} f^{\mu \nu} + \nabla_{\alpha} a_{\mu} \nabla^{\alpha} a^{\mu} \right) + 2i \sin^{2} \theta_{k} \delta \theta \varepsilon^{\mu \nu} f_{\mu \nu} \right],$$

(3.56)

where $\varepsilon_{\mu \nu} = \sqrt{g} \varepsilon_{\mu \nu}$ is the Levi-Civita tensor\textsuperscript{5}.

We expand the fields $a_{\mu}$ and $\delta \theta$ in terms of the symmetric traceless tensor fields:

$$a_{\mu}(\tau, r, \Omega_{4}) = \sum_{l=0}^{\infty} (a_{\mu})_{a_{1} \ldots a_{l}}(\tau, r) Y^{a_{1}} \ldots Y^{a_{l}}, \quad \delta \theta(\tau, r, \Omega_{4}) = \sum_{l=0}^{\infty} \delta \theta_{a_{1} \ldots a_{l}}(\tau, r) Y^{a_{1}} \ldots Y^{a_{l}}.$$  

(3.57)

Then the equations of motion for $(a_{\mu})_{a_{1} \ldots a_{l}}$ and $\delta \theta_{a_{1} \ldots a_{l}}$ derived from the action (3.56) are

$$-\partial_{r} f_{a_{1} \ldots a_{l}} + l(l + 3)(a_{r})_{a_{1} \ldots a_{l}} - 4i \sin \theta_{k} \partial_{r} \delta \theta_{a_{1} \ldots a_{l}} = 0,$$

$$\partial_{r} f_{a_{1} \ldots a_{l}} + l(l + 3)(a_{r})_{a_{1} \ldots a_{l}} + 4i \sin \theta_{k} \partial_{r} \delta \theta_{a_{1} \ldots a_{l}} = 0,$$

$$-\nabla_{\mu} \nabla^{\nu} \delta \theta_{a_{1} \ldots a_{l}} + (l + 4)(l - 1) \delta \theta_{a_{1} \ldots a_{l}} + \frac{4i}{\sin \theta_{k}} f_{a_{1} \ldots a_{l}} = 0,$$

(3.58)

where we have defined $f_{a_{1} \ldots a_{l}} \equiv \varepsilon^{\mu \nu} \partial_{\mu} (a_{\nu})_{a_{1} \ldots a_{l}}$ to simplify the notation.

Taking derivatives on both sides of the first two equations in (3.58), we obtain the following set of equations:

$$\nabla_{\mu} \nabla^{\nu} f_{a_{1} \ldots a_{l}} - (l^{2} + 3l + 16)f_{a_{1} \ldots a_{l}} + 4i \sin \theta_{k}(l + 4)(l - 1) \delta \theta_{a_{1} \ldots a_{l}} = 0,$$

$$\nabla_{\mu} \nabla^{\nu} \delta \theta_{a_{1} \ldots a_{l}} - (l + 4)(l - 1) \delta \theta_{a_{1} \ldots a_{l}} - \frac{4i}{\sin \theta_{k}} f_{a_{1} \ldots a_{l}} = 0.$$  

(3.59)

By diagonalizing (3.59), we find two types of modes with the mass spectrum

$$\begin{cases} 
O_{a_{1} \ldots a_{l}} = \delta \theta_{a_{1} \ldots a_{l}} - \frac{if_{a_{1} \ldots a_{l}}}{(4 + l) \sin \theta_{k}}, \quad \text{with} \ m_{l}^{2} = l(l - 1), \quad (l = 1, 2, \ldots) \\
X_{a_{1} \ldots a_{l}} = (l - 1) \sin \theta_{k} \delta \theta_{a_{1} \ldots a_{l}} + if_{a_{1} \ldots a_{l}}, \quad \text{with} \ m_{l}^{2} = (l + 3)(l + 4), \quad (l = 0, 1, \ldots).
\end{cases}$$

(3.60)

The $O_{a_{1} \ldots a_{l}}$ modes start with $l = 1$ because the $l = 0$ mode $O_{0}$ is not dynamical as the equations of motion for this mode are

$$\partial_{r} O_{0} = \partial_{r} O_{0} = 0.$$  

(3.61)

\textsuperscript{5}$\varepsilon_{\mu \nu}$ is antisymmetric with $\varepsilon_{\tau r} = 1$
From (3.60), we can express $\delta \theta_{a_1 \cdots a_l}$ and $f_{a_1 \cdots a_l}$ in terms of $O_{a_1 \cdots a_l}$ and $X_{a_1 \cdots a_l}$

$$\delta \theta_{a_1 \cdots a_l} = \frac{X_{a_1 \cdots a_l} + (4 + l) \sin \theta_k O_{a_1 \cdots a_l}}{(2l + 3) \sin \theta_k},$$

$$f_{a_1 \cdots a_l} = \frac{i(l + 4)}{(2l + 3)} \left[ -X_{a_1 \cdots a_l} + (l - 1) \sin \theta_k O_{a_1 \cdots a_l} \right].$$ (3.62)

We will denote the $l = 0$ mode of $f_{a_1 \cdots a_l}$ simply as $f_0$ in the later sections.

$\alpha$ sector The quadratic Euclidean action for gauge fields along $S^4$ directions is

$$S_a^{(2)} = T_{D5} \sin \theta_k \int d^6 \xi \frac{\sqrt{g_4}}{r^2} \frac{1}{2} \left[ \partial_\mu a_\alpha \partial^\mu a_\alpha - a_\alpha (g_4^{\alpha \beta} \nabla^2_{S^4} - R_{4}^{\alpha \beta}) a_\beta \right],$$ (3.63)

where $g_4^{\alpha \beta}$ is the metric for $S^4$ and $R_{4}^{\alpha \beta} = 3g^{\alpha \beta}$ is the Ricci tensor for $S^4$. Since the gauge condition (3.55) is imposed to decouple $a_\alpha$ from $a_\mu$, we need to expand $a_\alpha$ in terms of the transverse vector spherical harmonics on $S^4$ as

$$a_\alpha(\tau, r, \Omega_4) = \sum_{l=1}^\infty a_l(\tau, r)(\hat{Y}_\alpha)_{lm}(\Omega_4).$$ (3.64)

The transverse vector spherical harmonics $(\hat{Y}_\alpha)_{lm}$ satisfies following properties [104,105]:

$$\nabla^2_{S^4}(\hat{Y}_\alpha)_{lm} = -(l^2 + 3l - 1)(\hat{Y}_\alpha)_{lm}, \quad \nabla^a_{S^4}(\hat{Y}_\alpha)_{lm} = 0, \quad (l = 1, 2, \ldots).$$ (3.65)

The quadratic action for the $a_l$ modes is

$$S_{a_l}^{(2)} = \sum_{l=1}^\infty T_{D5} \sin \theta_k \int \frac{dr d\tau}{r^2} \frac{1}{2} \left[ \partial_\mu a_l \partial^\mu a_l + (l + 2)(l + 1)a_l a_l \right].$$ (3.66)

3.2.3 Dual operators and two-point functions

The holographic dictionary for the bulk fluctuation modes has been established in [80]. In this section, we briefly review the dual operators for each fluctuation mode. We summarize the results in table 3.1.

$\delta x^i$ sector From the mass spectrum of the $(\delta x^i)_{a_1 \cdots a_l}$ modes, we see that the mode $(\delta x^i)_{a_1 \cdots a_l}$ should be dual to an operator of dimension $\Delta_l = l + 2$ which transforms under $SO(3)$ as a vector. In particular, the three $l = 0$ modes which we shall denote as $\delta x_0^i$ are dual to the displacement operator $F_{it}$ in the ultrashort supermultiplet of $OSp(4^*|4)$. The higher $l$ modes ($l \geq 1$) are dual to
Fluctuation modes & Dual operator & $\Delta$ & $SO(3)$ & $SO(5)$ \\
\hline
$O_a$ & $O_1$ & 1 & 0 & (0, 1) \\
\hline
$\delta x^i_0$ & $F_i^l = Q^2O_1$ & 2 & 1 & (0, 0) \\
\hline
$O_{a_1\ldots a_l} (l \geq 2)$ & $O_l$ & $l$ & 0 & (0, l) \\
(\delta x^i)_{a_1\ldots a_l} (l \geq 1) & $Q^2O_{l+1}$ & $l + 2$ & 1 & (0, l) \\
$X_{a_1\ldots a_l} (l \geq 0)$ & $Q^4O_{l+2}$ & $l + 4$ & 0 & (0, l) \\
$\alpha_l (l \geq 1)$ & $Q^2O_{l+1}$ & $l + 2$ & 0 & (2, l - 1) \\
\hline

Table 3.1: In this table we summarize the quantum numbers of the operator dual to each fluctuation mode. $\Delta$ gives the conformal dimension of the dual operator. The quantum numbers of the dual operator under $SO(3)$ and $SO(5)$ symmetry are given in terms of the Dynkin labels of the corresponding representations.

The operators in a short multiplet of $OSp(4^*|4)$ (see [80] and table 3.1).

$\delta \theta$ and $a_\mu$ sector  In this sector, there are two families of modes $O_{a_1\ldots a_l}$ and $X_{a_1\ldots a_l}$. From the mass spectrum, we see that the mode $O_{a_1\ldots a_l}$ should be dual to an operator of dimension $\Delta_l = l$ while the mode $X_{a_1\ldots a_l}$ should be dual to an operator of dimension $\Delta_l = l + 4$. In both cases, the dual operator transforms in the symmetric representations of $SO(5)$. The modes $O_{a_1\ldots a_l}$ are dual to the protected operator $O_l$ in the defect CFT which played the central role in the localization analysis (in particular, the $l = 1$ mode $O_a$ is dual to $O_1$ in the ultrashort multiplet of $OSp(4^*|4)$). On the other hand, the modes $X_{a_1\ldots a_l}$ are dual to supersymmetry descendants of the operator $O_l$, i.e. they belong to the same short multiplet of $OSp(4^*|4)$.

$a_\alpha$ sector  From the mass spectrum, we see that the $a_l$ mode should be dual to an operator of dimension $\Delta_l = l + 2$, which is again in the short multiplet of $OSp(4^*|4)$ headed by $O_l$.

The two-point functions  From (3.60) we see that both $O_{a_1\ldots a_l}$ and $X_{a_1\ldots a_l}$ are linear combinations of $\delta \theta$ and the 2d field strength $f_{\mu\nu}$. Therefore, the boundary value for $\delta \theta$ and $f_{\mu\nu}$ should be fixed when varying the action. To ensure that the solutions to the equations of motion (3.58) are stationary under the variations satisfying these boundary conditions, we need to add the following boundary term

$$S_{bdy}^{(2)} = -T_{D5} \int_{r = r_0} d\tau d\Omega_4 \left[ 4i \sin^2 \theta_k \delta \theta a_\tau + \sin \theta_k r^2 a_\tau (\partial_\tau a_\tau - \partial_\tau a_\tau) \right],$$  (3.67)
where \( r_0 \) is the location of the boundary. In fact, this boundary term can be also derived from expanding the boundary term (3.36) to quadratic order in fluctuations.

To compute the tree level two-point function \( \langle \langle O_L, \omega_1, 0 \rangle \rangle \), we use the following normalization of the bulk-to-boundary propagator [106]

\[
S_{\text{on-shell}}^{(2)} = - V_i T_{D5} \int \frac{dr}{r - r_0} \left[ \frac{\sin^3 \theta_k}{2} \delta \theta_{a_1 \ldots a_l} \partial_r \delta \theta_{a_1 \ldots a_l} + \frac{8 \sin^3 \theta_k}{l(l + 3)} \delta \theta_{a_1 \ldots a_l} \partial_r \delta \theta_{a_1 \ldots a_l} \right. \\
\left. - \frac{2i \sin^2 \theta_k}{l(l + 3)} \delta \theta_{a_1 \ldots a_l} \partial_r f_{a_1 \ldots a_l} - \frac{2i \sin^2 \theta_k}{l(l + 3)} \partial_r \delta \theta_{a_1 \ldots a_l} - \frac{\sin \theta_k}{2l(l + 3)} \partial_r f_{a_1 \ldots a_l} \right] \\
= - \frac{V_i T_{D5}}{2} \int d\tau \left[ \frac{(4 + l)^2 \sin^3 \theta_k}{(3 + 2l)^2} O_{a_1 \ldots a_l} \partial_r O_{a_1 \ldots a_l} + \frac{\sin \theta_k}{(2l + 3)(l + 3)} X_{a_1 \ldots a_l} \partial_r X_{a_1 \ldots a_l} \right]. \\
(3.68)
\]

We use the following normalization of the bulk-to-boundary propagator [106]

\[
K_\Delta(r, \tau; \tau') = C_\Delta \left[ \frac{r}{r^2 + (\tau - \tau')^2} \right]^\Delta, \quad C_\Delta = \frac{\Gamma(\Delta)}{\sqrt{\pi} \Gamma(\Delta - 1/2)}. \\
(3.69)
\]

With this normalization, we find that the tree level two-point function of the dual boundary operator \( O_L \) is

\[
\langle \langle O_L, \omega_1, 0 \rangle \rangle = \delta_{L_1, L_2} \frac{T_{D5} \sin^3 \theta_k \pi^2 (L_1 + 4)^2 (2L_1 - 1) \Gamma^2(L_1) (\omega_1 \cdot \omega_2)^{L_1}}{2^{2L_1 + 2(2L_1 + 3)^2 \Gamma(L_1 + 1/2) \Gamma(L_1 + 3/2)}} (\tau_{12})^{2L_1} \\
= \delta_{L_1, L_2} C_{L_1} \frac{(\omega_1 \cdot \omega_2)^{L_1}}{(\tau_{12})^{2L_1}}. \\
(3.70)
\]

As we will also need the tree level two-point function of \( F_{4i} \) later, we provide the result here:

\[
\langle \langle F_{4i}, \omega_1, 0 \rangle \rangle = \langle \delta x_{0i}(\tau_1) \delta x_{0j}(\tau_2) \rangle_{\text{AdS}_2} = \delta^{ij} \frac{16 \pi T_{D5} \sin^3 \theta_k}{r_{12}^4}. \\
(3.71)
\]

### 3.2.4 Three-point functions of \( S^5 \) fluctuations

In this section, we compute the three-point function

\[
\langle \langle O_{L_1}(\tau_1, \omega_1) O_{L_2}(\tau_2, \omega_2) O_{L_3}(\tau_3, \omega_3) \rangle \rangle, \\
(3.72)
\]
from the expanded D-brane action. This requires the knowledge of the cubic interaction vertices of \( \delta \theta \) and \( a_\mu \), which we find to be

\[
L_{f}^{(3)} = T_{D5} \int d\Omega_4 \left[ \cos \theta_k \left( \nabla_\alpha a_\mu \nabla^\alpha a^\mu + f_{\mu \nu} f^{\mu \nu} \right) \delta \theta - \frac{i \cot \theta_k}{4} \varepsilon^{\mu \nu} f_{\mu \nu} \left( \nabla_\alpha a_\mu \nabla^\alpha a^\mu + \frac{1}{2} f_{\mu \nu} f^{\mu \nu} \right) \right. \\
+ \sin \theta_k \left( \frac{1}{2} \nabla_\alpha \delta \theta \nabla^\alpha \delta \theta - 2 \delta \theta^2 \right) \delta \theta + \frac{i \sin 2 \theta_k}{2} \varepsilon^{\mu \nu} \nabla_\alpha a_\mu \nabla^\alpha \delta \theta \delta \theta \\
- \frac{i \sin 2 \theta_k}{8} \varepsilon^{\mu \nu} f_{\mu \nu} \left( \nabla_\alpha \delta \theta \nabla^\alpha \delta \theta - 12 \delta \theta^2 \right) \right]. \tag{3.73}
\]

The relevant cubic coupling for \( \langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_L \rangle \) can be then extracted from (3.73) after we substitute the expressions (3.62) into (3.73).

Using the \( SO(5) \) symmetry, the general three-point function of three \( \mathcal{O}_L \) operators can be written as

\[
\langle \mathcal{O}_{L_1} (\tau_1, \mathbf{u}_1) \mathcal{O}_{L_2} (\tau_2, \mathbf{u}_2) \mathcal{O}_{L_3} (\tau_3, \mathbf{u}_3) \rangle = f_{L_1 L_2 L_3} (\tau_1, \tau_2, \tau_3) \times (\mathbf{u}_1 \cdot \mathbf{u}_2)^{L_{12} | 3} (\mathbf{u}_2 \cdot \mathbf{u}_3)^{L_{23} | 3} (\mathbf{u}_1 \cdot \mathbf{u}_3)^{L_{13} | 2}, \tag{3.74}
\]

where \( L_{ij|k} \equiv (L_i + L_j - L_k)/2 \). The \( f_{L_1 L_2 L_3} \) can be computed from the bulk cubic coupling (3.73) and we find it to be

\[
f_{L_1 L_2 L_3} = \frac{2 (4 + L_1) (4 + L_2) (4 + L_3) L_{12} | 3 L_{23} | 1 L_{13} | 2 (\Sigma^2 - 1)(3 + \Sigma)}{L_1 L_2 L_3 (3 + 2 L_1) (3 + 2 L_2) (3 + 2 L_3)} \\
\times T_{D5} \sin^2 \theta_k \cos \theta_k V_{L_1 L_2 L_3} \times \int \frac{dr d\tau}{r^2} \mathcal{K}(r, \tau; \tau_1) \mathcal{K}(r, \tau; \tau_2) \mathcal{K}(r, \tau; \tau_3), \tag{3.75}
\]

where \( \Sigma \equiv L_1 + L_2 + L_3 \). We have defined \( V_{L_1 L_2 L_3} \) to be

\[
\int d\Omega_4 (\mathbf{u}_1 \cdot \mathbf{Y})^{L_1} (\mathbf{u}_2 \cdot \mathbf{Y})^{L_2} (\mathbf{u}_3 \cdot \mathbf{Y})^{L_3} = V_{L_1, L_2, L_3} \left[ (\mathbf{u}_1 \cdot \mathbf{u}_2)^{L_{12} | 3} (\mathbf{u}_2 \cdot \mathbf{u}_3)^{L_{23} | 3} (\mathbf{u}_1 \cdot \mathbf{u}_3)^{L_{13} | 2} \right], \tag{3.76}
\]

which can be computed as shown in the Appendix, and we find that

\[
V_{L_1, L_2, L_3} = \frac{1 + (-1)^{L_1 + L_2 + L_3}}{2} \frac{8 \pi^2 (\sqrt{2})^{\Sigma} (\Sigma + 2) L_1! L_2! L_3! \left( \frac{\Sigma}{2} \right)!}{(\Sigma + 3)! L_{12} | 3! L_{23} | 1! L_{13} | 2!}. \tag{3.77}
\]
The result for the bulk integral in (3.75) is [106]

$$\int \frac{dr\,d\tau}{r^2} K_{L_1}(r, \tau; \tau_1) K_{L_2}(r, \tau; \tau_2) K_{L_3}(r, \tau; \tau_3)$$

$$= \frac{\Gamma(\frac{\tau_1}{2} - \frac{1}{2}) \Gamma(\tau_2) \Gamma(L_{23}/2)}{2\pi \Gamma(L_1 - \frac{1}{2}) \Gamma(L_2 - \frac{1}{2}) \Gamma(L_3 - \frac{1}{2}) (\tau_1 \tau_2)^{L_{23}/2} (\tau_3 L_{12})^{L_{12}} (\tau L_{13})^{L_{13}}}. \tag{3.78}$$

Putting everything together, the 3-point function is given by

$$\llangle \mathcal{O}_{L_1}(\tau_1, u_1) \mathcal{O}_{L_2}(\tau_2, u_2) \mathcal{O}_{L_3}(\tau_3, u_3) \rrangle = C_{L_1, L_2, L_3} \frac{\tau_1 \cdot u_1 \cdot u_2 \cdot u_3}{(\tau_1 \tau_2)^{L_{23}/2} (\tau_3 L_{12})^{L_{12}} (\tau L_{13})^{L_{13}}} \tag{3.79}$$

with the 3-point structure constants taking the simple factorized form

$$C_{L_1, L_2, L_3} = 8(1 + (-1)^{L_1 + L_2 + L_3})\pi^2 T_{D5} \sin^2 \theta_k \cos \theta_k \prod_{i=1}^3 \frac{\Gamma(L_i)(4 + L_i)}{2\pi \Gamma(L_i - \frac{1}{2})(2L_i + 3)}. \tag{3.80}$$

Note that, although the bulk 3-point integral (3.78) has a pole when one of the $L_{ij,k}$ is zero, the pole is canceled by the $L_{ij,k}$ factor in (3.75). As a result, the three-point function is always finite, when computed by analytic continuation in the charges.

To compare with the prediction of localization, we can do a conformal transformation to the circular Wilson loop and set the polarizations to

$$u_i = (\cos \tau_i, \sin \tau_i, 0, 0). \tag{3.81}$$

The normalized three-point function with the topological configuration is then

$$\frac{\llangle \tilde{\mathcal{O}}_{L_1} \tilde{\mathcal{O}}_{L_2} \tilde{\mathcal{O}}_{L_3} \rrangle}{\sqrt{\llangle \tilde{\mathcal{O}}_{L_1} \tilde{\mathcal{O}}_{L_1} \rrangle \llangle \tilde{\mathcal{O}}_{L_2} \tilde{\mathcal{O}}_{L_2} \rrangle \llangle \tilde{\mathcal{O}}_{L_3} \tilde{\mathcal{O}}_{L_3} \rrangle}}$$

$$= \frac{(1 + (-1)^{L_1 + L_2 + L_3})}{2} \sqrt{\frac{(L_1 + \frac{1}{2})(L_2 + \frac{1}{2})(L_3 + \frac{1}{2})}{2\pi^3 T_{D5}}} \frac{\cos \theta_k}{(\sin \theta_k)^{3/2}}. \tag{3.82}$$

Using the relation $2\pi^3 T_{D5} = N\sqrt{\lambda}/(4\pi) = Ng$, we find the result agrees with the prediction of localization [54].

As we have pointed out previously, for $L_{ij,k} = 0$, the bulk integral (3.78) is divergent while the prefactor in the first line of (3.75) has a zero. The zero actually results from the vanishing of the bulk cubic coupling. In fact, the three-point function is called extremal in this case and one expects the corresponding bulk cubic coupling to vanish [107]. For the case that $L_1 = L_2 = 1$ and $L_3 = 2$ which is relevant to the calculation in Section 3.2.8, by expanding the bulk action explicitly we find
that the corresponding cubic coupling from (3.73) is

\[
\frac{T_{DS} 8\pi^2 \sin \theta_k \sin 2\theta_k}{245} \int \frac{dr d\tau}{r^2} \left( 18 \partial_\mu O_a \partial^\mu O_b O_{ab} + \partial_\mu O_a O_b \partial^\mu O_{ab} - 17 O_a O_b O_{ab} \right),
\]  

(3.83)

which indeed vanishes on-shell. To avoid this subtlety and reproduce the result (3.82) from the bulk calculation, we shall use the following approach. At the boundary the single particle operator \(O_2\) can be mixed with the two-particle operator \(O_1^* O_1\): Therefore, from the bulk point of view, it is reasonable to consider the bulk dual for the boundary operator \(O_2\) to be the linear combination

\[
O_{ab}' \equiv O_{ab} + \frac{c}{T_{DS}} O_a O_b.
\]  

(3.84)

The coefficient \(c\) is fixed by demanding that the direct bulk computation of the three-point function \(\langle \langle O_{ab}' O_c O_d \rangle \rangle\) reproduces the result (3.79). As we have shown in (3.83), the bulk coupling between \(O_{ab}\) and \(O_a\) vanishes on-shell. Therefore, the bulk calculation of the three-point function \(\langle \langle O_{ab}' O_c O_d \rangle \rangle\) only receives contribution from the part \(\langle \langle O_2 O_b: O_c O_d \rangle \rangle\). Evaluating this by simple Wick contractions, we obtain

\[
\frac{c}{T_{DS}} \langle \langle :O_1(\tau_1, u_1)^2: O_1(\tau_2, u_2) O_1(\tau_3, u_3) \rangle \rangle = \frac{c}{T_{DS}} \cdot 2c_1^2 \frac{(u_1 \cdot u_2)(u_1 \cdot u_3)}{\tau_{12}^2 \tau_{13}^2}
\]  

(3.85)

where \(c_1\) is the 2-point function coefficient defined in (3.70), for \(L = 1\). Requiring that this matches (3.79) for \(L_1 = 2, L_2 = L_3 = 1\), we find

\[
c = \frac{27}{56\pi^2 \sin^4 \theta_k}.
\]  

(3.86)

A similar analysis can be carried out for extremal three-point functions involving higher charge operators. We will see in Section 3.2.8 that the contribution of the two-particle state in \(O_{ab}'\) is necessary in order to obtain the correct result for the 4-point function \(\langle O_2 O_2 O_1 O_1 \rangle\).

3.2.5 Four-point function of AdS\(_5\) fluctuations

In this section, we compute the connected part of the four-point function

\[
\langle \langle F_t^{i_1} (\tau_1) F_t^{i_2} (\tau_2) F_t^{i_3} (\tau_3) F_t^{i_4} (\tau_4) \rangle \rangle = \langle \delta x_0^{i_1} (\tau_1) \delta x_0^{i_2} (\tau_2) \delta x_0^{i_3} (\tau_3) \delta x_0^{i_4} (\tau_4) \rangle_{AdS_5}. 
\]  

(3.87)
The relevant quartic vertices from expanding the D5-brane action are

\[
L^{(4)}_{xxx} = \frac{\pi^2 T_{D5} \sin^3 \theta_k}{3} \left[ (\partial_\mu \delta x^i_0 \partial^\mu \delta x^j_0)^2 - 2 (\partial_\mu \delta x^i_0 \partial_\nu \delta x^j_0) (\partial^\mu \delta x^i_0 \partial^\nu \delta x^j_0) \\
+ 2 (\partial_\mu \delta x^i_0 \partial^\mu \delta x^j_0) \delta x^2_0 - 4 \cot^2 \theta_k (\partial_\mu \delta x^i_0 \partial^\mu \delta x^j_0 + 2 \delta x^2_0) \right],
\]

which leads to the contact diagram in Figure 3.3. The contribution from the exchange diagram in Figure 3.3 results from the following cubic vertices:

\[
L_{xxf0} = -\frac{4i \pi^2 T_{D5} \sin \theta_k \cos \theta_k}{3} (\partial_\nu \delta x^i_0 \partial^\mu \delta x^j_0) f_0, \\
L_{xx\delta\theta_0} = \frac{16 \pi^2 T_{D5} \sin^2 \theta_k \cos \theta_k}{3} (\partial_\nu \delta x^i_0 \partial^\mu \delta x^j_0 + 2 \delta x^2_0) \delta \theta_0.
\]

(3.89)

When computing the contribution from the exchange diagram, the field \(\delta x^i_0\) is put on-shell. Therefore, we can use the equations of motion for the external fields \(\delta x^i_0\) and simplify the cubic vertices to

\[
-\frac{4i \pi^2 T_{D5} \sin \theta_k \cos \theta_k}{3} \delta x^i_0 \partial^\mu \delta x^j_0 \left[ -\partial_\mu f_0 - 4i \sin \theta_k \partial_\mu \delta \theta_0 \right].
\]

(3.90)

We emphasize here that this cubic coupling is only correct when \(\delta x^i_0\) is on-shell. To compute the exchange diagram, we need to use the bulk propagator \(G_{pq}(\tau, r; \tau', r')\) with \(p, q \in \{\tau, r, \theta\}\) defined by the bulk two-point functions:

\[
G_{\mu\nu}(\tau, r; \tau', r') = \left< (a_\mu)_0(\tau, r)(a_\nu)_0(\tau', r') \right>, \\
G_{\mu\theta}(\tau, r; \tau', r') = \left< (a_\mu)_0(\tau, r) \delta \theta_0(\tau', r') \right>, \\
G_{\theta\theta}(\tau, r; \tau', r') = \left< \delta \theta_0(\tau, r) \delta \theta_0(\tau', r') \right>.
\]

(3.91)
Since the quadratic action is not diagonal in $a_\mu$ and $\delta \theta$, the bulk propagator $G_{pq}$ satisfies the following equations derived from (3.58):

\begin{align}
-\nabla^\mu (\varepsilon^{\alpha\beta} \partial_\alpha G_{\beta\gamma'}) - 4i \sin \theta_k \nabla^\mu G_{\theta\gamma'} &= \frac{3r^2}{8\pi^2 T_{D5} \sin \theta_k} \varepsilon^{\mu\gamma'} \delta^2(\tau, r; \tau', r'), \\
-\nabla^\mu (\varepsilon^{\alpha\beta} \partial_\alpha G_{\beta\theta}) - 4i \sin \theta_k \nabla^\mu G_{\theta\theta} &= 0, \\
-\nabla_\mu \nabla^\mu G_{\theta p} - 4G_{\theta p} + \frac{4i}{\sin \theta_k} \varepsilon^{\mu\nu} \partial_\mu G_{\nu p} &= \frac{3r^2}{8\pi^2 T_{D5} \sin^3 \theta_k} \delta_{\theta p} \delta^2(\tau, r; \tau', r'),
\end{align}

where we have suppressed the dependence of $G_{pq}$ on the coordinates to simplify the notation. Due to the structure of (3.90), we find that the exchange diagram can be reduced to a contact diagram with the following effective quartic coupling:

\begin{equation}
L_{\text{exchange}} = \frac{\pi^2 T_{D5} \sin \theta_k \cos^2 \theta_k}{3} (\partial_\mu \delta x^i_0 \partial^\mu \delta x^i_0 + 2 \delta x^2_0)^2.
\end{equation}

It follows that the connected part of the four-point function can be computed effectively from a single contact diagram with the quartic coupling:

\begin{equation}
L_{\text{xxxx}}^{\text{eff}} = \frac{8\pi^2 T_{D5} \sin^3 \theta_k}{3} \left[ \frac{1}{8} (\partial_\mu \delta x^i_0 \partial^{\mu} \delta x^i_0)^2 - \frac{1}{4} (\partial_\mu \delta x^j_0 \partial_\nu \delta x^j_0) (\partial^{\mu} \delta x^i_0 \partial^{\nu} \delta x^i_0) \\
+ \frac{1}{4} (\partial_\mu \delta x^j_0 \partial^{\mu} \delta x^j_0) \delta x^2_0 + \frac{1}{2} \delta x^2_0 \delta x^2_0 \right].
\end{equation}

This effective quartic coupling in fact takes the identical form as the one appeared in the fundamental string case [24], but with a different prefactor. Using the result in [24], we find that the connected part of the normalized four-point function is (to get the normalized correlation function, we divide by the two-point function normalization factor in (3.71)):

\begin{equation}
\langle \delta x^i_0 (\tau_1) \delta x^j_0 (\tau_2) \delta x^k_0 (\tau_3) \delta x^l_0 (\tau_4) \rangle = \frac{3}{8\pi^2 T_{D5} \sin^3 \theta_k} \frac{G^{i_1 i_2 i_3 i_4}_4 (\chi)}{\tau_{12}^{1/4} \tau_{34}^{1/4}},
\end{equation}

where the expression of $G^{i_1 i_2 i_3 i_4}_4 (\chi)$ is given in Appendix B.1.

Note that if we take the string limit defined by

\begin{equation}
\frac{k}{N} \to 0, \quad (\theta_k)^3 \to \frac{3\pi k}{2N},
\end{equation}

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the four-point function (3.95) then becomes

$$\langle \delta x_0^{i_1}(\tau_1)\delta x_0^{i_2}(\tau_2)\delta x_0^{i_3}(\tau_3)\delta x_0^{i_4}(\tau_4) \rangle \rightarrow \frac{2\pi}{k\sqrt{\lambda}} \frac{O_{4x}^{i_1,i_2,i_3,i_4}(\chi)}{\tau_{12}^{5} \tau_{24}^{5}}. \quad (3.97)$$

Comparing with the result in [24], we see that the D-brane result reduces to the result calculated from $k$ weakly coupled coincident strings.

### 3.2.6 Two AdS$_5$ and two $S^5$ fluctuations of D5-brane

In this section, we compute the connected part of the four-point function

$$\langle \mathcal{F}^{i_1}_{\tau} \mathcal{F}^{i_2}_{\tau} \Phi_{a_1}(\tau_3) \Phi_{a_2}(\tau_4) \rangle = \langle \delta x_0^{i_1}(\tau_1)\delta x_0^{i_2}(\tau_2)O_{a_1}(\tau_3)O_{a_2}(\tau_4) \rangle_{AdS_2}. \quad (3.98)$$

In previous section, we have shown that the four-point function $\langle \delta x_0^{i_1}\delta x_0^{i_2}\delta x_0^{i_3}\delta x_0^{i_4} \rangle$ has the same form as in the fundamental string case. Then the supersymmetry uniquely fixes the four-point function $\langle \delta x_0^{i_1}\delta x_0^{i_2}O_{a_1}O_{a_2} \rangle$. In fact, we expect it to have the same form as the correlator $\langle \delta x_0^{i_1}\delta x_0^{i_2}\delta y_{a_1}\delta y_{a_2} \rangle$ computed in the fundamental string case [24] but with the same prefactor as in (3.94). We verify this by explicitly calculating the four-point function using the effective action for the fluctuations.

There is a quartic coupling from the expanded D-brane action

$$L_{x_0o}^{(4)} = \frac{2\pi^2 T_{D5} \sin^3 \theta_k}{15} \left[ \frac{(4 - \cos 2\theta_k)}{\sin^2 \theta_k} \partial_{\mu}O_{a}\partial^{\mu}O_{a}\partial_{\nu}\delta x_{0}^{i}\partial^{\nu}\delta x_{0}^{i} \right. $$

$$\left. - 10 \partial_{\mu}O_{a}\partial_{\nu}\delta x_{0}^{i}\partial^{\mu}\delta x_{0}^{i} + 6 \cot^2 \theta_k \partial_{\mu}O_{a}\partial^{\mu}O_{a}\delta x_{0}^{i}\delta x_{0}^{i} \right] + \frac{16 \cot^2 \theta_k O_{a}\partial_{\mu}\delta x_{0}^{i}\partial^{\mu}\delta x_{0}^{i} + 32 \cot^2 \theta_k O_{a}\delta x_{0}^{i}\delta x_{0}^{i}}{15} \right], \quad (3.99)$$

which leads to the contact diagram in Figure 3.4. The exchange diagram in Figure 3.4 results from the cubic couplings (3.90) and$^5$

$$L_{oof_{0}} = \frac{2i\pi^2 T_{D5} \sin 2\theta_k}{15} (3\partial_{\mu}O_{a}\partial^{\mu}O_{a} + 16O_{a}O_{a}) f_{0}, \quad (3.100)$$

$$L_{oos\theta_{0}} = -\frac{8\pi^2 T_{D5} \sin \theta_k}{15} (\partial_{\mu}O_{a}\partial^{\mu}O_{a}\delta \theta_{0} - 4O_{a}\partial_{\mu}O_{a}\partial^{\mu}\delta \theta_{0} + 4O_{a}O_{a}\delta \theta_{0}). \quad (3.101)$$

Due to the special form of the coupling (3.90), the exchange diagram in Figure 3.4 can be again

$^5$There are also cubic couplings between $\langle \delta x_{0}^{i} \rangle$, $\delta x_{0}^{i}$ and $O_{a}$, which leads to the Witten diagram with bulk $\langle \delta x_{0}^{i} \rangle$ fields being exchanged. However, if we put $\delta x_{0}^{i}$ and $O_{a}$ on-shell, then this cubic coupling vanishes.
Figure 3.4: Witten diagrams to compute the connected part of $\langle \delta x_{0}^{i_{1}} \delta x_{0}^{i_{2}} O_{a_{1}} O_{a_{2}} \rangle$. The $l = 0$ modes of $\delta \theta$ and $a_{\mu}$ fields are exchanged in the exchange diagram.

reduced to a contact diagram with the effective quartic coupling:

$$L_{\text{exchange}} = -\frac{2\pi^2 T_{D5} \cos^2 \theta_k \sin \theta_k}{15} \left(3 \partial_{\mu} O_{a} \partial^{\mu} O_{a} + 16 O_{a} a_{a} \right) \left( \partial_{\nu} \delta x_{0}^{i} \partial^{\nu} \delta x_{0}^{i} + 2 \delta x_{0}^{i} \delta x_{0}^{i} \right). \quad (3.102)$$

Combining (3.99) and (3.102), we see that the four-point function can be computed from a contact diagram with the effective quartic coupling

$$L_{\text{eff}}^{x\mu o} = \frac{8 \pi^2 T_{D5} \sin^3 \theta_k}{3} \left( \frac{1}{4} \partial_{\mu} O_{a} \partial^{\mu} O_{a} \partial_{\nu} \delta x_{0}^{i} \partial^{\nu} \delta x_{0}^{i} - \frac{1}{2} \partial_{\mu} O_{a} \partial_{\nu} O_{a} \partial^{\mu} \delta x_{0}^{i} \partial^{\nu} \delta x_{0}^{i} \right). \quad (3.103)$$

The form is exactly what we expect from the fundamental string case and the prefactor agrees with (3.94). Using the result in [24], we find that the connected part of the normalized four-point function takes the form:

$$\langle \delta x_{0}^{i_{1}} (\tau_{1}) \delta x_{0}^{i_{2}} (\tau_{2}) O_{a_{1}} (\tau_{3}) O_{a_{2}} (\tau_{4}) \rangle = \delta^{i_{1} i_{2}} \delta_{a_{1} a_{2}} \frac{3}{8 \pi^2 T_{D5} \sin^3 \theta_k} \frac{G_{2x2y}(\chi)}{\tau_{12}^{\frac{1}{2}} \tau_{34}^{\frac{1}{2}}}, \quad (3.104)$$

where the expression for $G_{2x2y}(\chi)$ is given in Appendix B.1.

### 3.2.7 Four $S^5$ fluctuations of D5-brane

In this section, we compute the four-point function

$$\langle \langle \Phi_{a_{1}} (\tau_{1}) \Phi_{a_{2}} (\tau_{2}) \Phi_{a_{3}} (\tau_{3}) \Phi_{a_{4}} (\tau_{4}) \rangle \rangle = \langle O_{a_{1}} (\tau_{1}) O_{a_{2}} (\tau_{2}) O_{a_{3}} (\tau_{3}) O_{a_{4}} (\tau_{4}) \rangle_{\lambda} ds_{2}. \quad (3.105)$$

The supersymmetry fixes $\langle O_{a_{1}} O_{a_{2}} O_{a_{3}} O_{a_{4}} \rangle$ to take the same form as $\langle y_{a_{1}} y_{a_{2}} y_{a_{3}} y_{a_{4}} \rangle$ in [24]. The Witten diagrams for the D-brane calculation are shown in Figure 3.5. The contact diagram in
Figure 3.5: Witten diagrams to compute the connected part of $\langle O_a O_a O_a O_a \rangle$. Both $l = 0$ and $l = 2$ modes of $\delta \theta$ and $a_\mu$ fields are exchanged in the exchange diagrams.

Figure 3.8 results from the quartic coupling from the expanded action:

$$L_{oooo}^{(4)} = \frac{\pi^2 T^2 D_5 \sin \theta_k}{105} \left[ -80 \cos^2 \theta_k O_a O_a O_b O_b - 14(13 - 8 \sin^2 \theta_k) \partial_\mu O_a \partial^\mu O_a O_b O_b \\
- (11 - 46 \sin^2 \theta_k) \partial_\mu O_a \partial^\mu O_a \partial_\nu O_b \partial^\nu O_b - 2(4 + 31 \sin^2 \theta_k) \partial_\mu O_a \partial_\nu O_a \partial^\mu O_b \partial^\nu O_b \right]. \quad (3.106)$$

The other two diagrams in Figure 3.5 involve the exchange of $l = 0$ and $l = 2$ modes of $\delta \theta$ and $a_\mu$ fields.

**Exchange of $l = 0$ modes** In this case, the cubic couplings involved are (3.100) and (3.101). Using the fact that $O_a$ is put on-shell in the calculation of the Witten diagram, we can write the cubic couplings as

$$\frac{2i \pi^2 T^2 D_5 \sin 2\theta_k}{15} \left[ 3 O_a \partial_\mu O_a (-\partial^\mu f_0 - 4i \sin \theta_k \partial^\mu \delta \theta_0) \right. \\
- 4i \sin \theta_k O_a O_a \left( -\nabla^2 \delta \theta_0 - 4 \delta \theta_0 + \frac{4i}{\sin \theta_k} f_0 \right). \quad (3.107)$$
In this form, we see that the exchange diagram can be reduced to a contact diagram with the effective quartic coupling:

\[ L_{\text{exc}, l=0} = \frac{\pi^2 T_{D5} \cos^2 \theta_k \sin \theta_k}{75} \left( 64 \partial_\mu O_a O_b O_b + 128 \partial_\mu O_a \partial^\mu O_a O_b O_b + 9 \partial_\mu O_a \partial^\mu O_a \partial_\nu O_b \partial^\nu O_b \right). \]  

(3.108)

**Exchange of \( l = 2 \) modes**  In this case, the cubic couplings involved are

\[ L_{oo f_2} = \frac{i8\pi^2 T_{D5} \sin 2\theta_k}{105} \left\{ \partial_\mu O_a \partial^\mu O_b f_{ab} + 5 \delta_\mu \nu O_a \partial_\mu O_b (a_\nu)_{ab} + 11 \frac{1}{2} O_a O_b f_{ab} \right\}, \]  

(3.109)

\[ L_{oo \theta_2} = \frac{8\pi^2 T_{D5} \sin 2\theta_k}{105} \left( 8 \partial_\mu O_a \partial^\mu O_b \delta_{ab} + 3 O_a \partial_\mu O_b \partial^\mu \delta_{ab} - 3 O_a O_b \delta_{ab} \right). \]  

(3.110)

Using integration by parts and the on-shellness of the external \( O_a \), the cubic couplings can be brought to the form

\[ \frac{8\pi^2 T_{D5} \sin 2\theta_k}{105} \left\{ -i O_a \partial_\mu O_b [-\partial^\mu f_{ab} + 10 \delta_\mu \nu (a_\nu)_{ab} - 4i \sin \theta_k \partial^\mu \delta_{ab}] \right. \]

\[ \left. - \frac{\sin \theta_k}{2} O_a O_b \left( -\nabla^2 \delta_{ab} + 6 \delta_{ab} + \frac{4i}{\sin \theta_k} f_{ab} \right) \right\}. \]  

(3.111)

To compute the exchange diagram, we need to use the bulk propagator \( G^{ab a'b'}_{pq}(\tau, r; \tau', r') \) with \( p, q \in \{ \tau, r, \theta \} \) defined similarly as in (3.91). The bulk propagator \( G^{ab a'b'}_{pq} \) satisfies the following equations derived from (3.58):

\[-\nabla^\mu (\varepsilon^{\alpha \beta} \partial_\alpha G^{ab a'b'}_{pq}) + 10 \delta^{\mu \nu} G^{ab a'b'}_{pq} \]  

\[ \left. + 4i \sin \theta_k \nabla^\mu G^{ab a'b'}_{pq} = \frac{105 M^{ab a'b'}_{pq} \pi^2 T_{D5} \sin \theta_k}{16 \pi^2 T_{D5} \sin \theta_k} \right\} \]  

(3.112)

\[-\nabla^\mu (\varepsilon^{\alpha \beta} \partial_\alpha G^{ab a'b'}_{\tau' \theta}) + 10 \delta^{\mu \nu} G^{ab a'b'}_{\tau' \theta} \]  

\[ \left. - 4i \sin \theta_k \nabla^\mu G^{ab a'b'}_{\tau' \theta} = 0 \right\} \]  

(3.112)

\[-\nabla^\mu G^{ab a'b'}_{\tau \theta} + 6 G^{ab a'b'}_{\tau \theta} \]  

\[ \left. + 4i \frac{\sin \theta_k}{\sin \theta_k} \right\} \]  

(3.112)

where \( M^{ab a'b'}_{pq} \) is defined as

\[ M^{ab a'b'}_{pq} \equiv \frac{1}{2} \left( \delta^{ab} \delta^{bb'} + \delta^{ab'} \delta^{ba'} - \frac{2}{5} \delta^{ab} \delta^{a'b'} \right). \]  

(3.113)
By examining the structure of the cubic coupling (3.111), we see that the exchange diagram can be again reduced to a contact diagram with the effective quartic coupling:

\[
L_{\text{exc},l=2} = \frac{2\pi^2 T_{D5}\cos^2 \theta_k \sin \theta_k}{525} \left( 20 \partial_\mu O_a \partial_\nu O_a \partial^{\nu'} O_b \partial^{\nu} O_b - 4 \partial_\mu O_a \partial^{\mu} O_a \partial_\nu O_b \partial^{\nu} O_b + 7 \partial_\mu O_a \partial^{\mu} O_a \partial_\nu O_b \partial^{\nu} O_b - 24 O_a O_a O_b O_b \right).
\]  

(3.114)

The four-point function Combining (3.108) and (3.114) with (3.106), we find that the four-point function can be computed from a contact diagram with the effective quartic coupling:

\[
L^{\text{eff}}_{\text{oool}} = \frac{8\pi^2 T_{D5}}{3} \left( -\frac{1}{4} \partial_\mu O_a \partial^{\mu} O_a O_b O_b + \frac{1}{8} \partial_\mu O_a \partial^{\mu} O_a \partial_\nu O_b \partial^{\nu} O_b 
- \frac{1}{4} \partial_\mu O_a \partial_\nu O_b \partial^{\mu} O_b \partial^{\nu} O_b \right).
\]  

(3.115)

The structure of the vertices is the same as what we expect from the fundamental string case [24]. Computing the Witten diagram with quartic coupling (3.115), we find that the connected part of the normalized four-point function is

\[
\langle O^{a_1}(\tau_1)O^{a_2}(\tau_2)O^{a_3}(\tau_3)O^{a_4}(\tau_4) \rangle = \frac{3}{8\pi^2 T_{D5}} \frac{G^{a_1 a_2 a_3 a_4}(\chi)}{\tau_1^2 \tau_3^2 \tau_{34}^2},
\]  

(3.116)

where the expression of \(G^{a_1 a_2 a_3 a_4}(\chi)\) is given in Appendix B.1.

To compare with the prediction of localization, we again transform to the circular Wilson loop and setting the polarizations to (3.81). The normalized four-point function then becomes

\[
\frac{\langle \hat{O}_1 \hat{O}_1 \hat{O}_1 \hat{O}_1 \rangle}{\langle \hat{O}_1 \hat{O}_1 \rangle^2} = -\frac{9}{16\pi^2 T_{D5} \sin^3 \theta_k} = -\frac{9}{8N g \sin^3 \theta_k},
\]  

(3.117)

which agrees with the prediction of localization [54].

3.2.8 Four \(S^5\) fluctuations including higher KK modes

In this section, we compute the four-point function which includes the \(l = 2\) KK modes:

\[
\langle \Phi^{a_1 b_1}(\tau_1)\Phi^{a_2 b_2}(\tau_2)\Phi^{a_3}(\tau_3)\Phi^{a_4}(\tau_4) \rangle = \langle O^{a_1 b_1}(\tau_1)O^{a_2 b_2}(\tau_2)O^{a_3}(\tau_3)O^{a_4}(\tau_4) \rangle_{AdS_2},
\]  

(3.118)
where $O^{ab}$ is defined in (3.84). The four-point function can be written as the sum of two pieces

$$
\langle O^{a_1 b_1} O^{a_2 b_2} O^{a_3} O^{a_4} \rangle = \langle O^{a_1 b_1} O^{a_2 b_2} O^{a_3} O^{a_4} \rangle + \frac{c^2}{T_{D5}} \langle O^{a_1 b_1} O^{a_2 b_2} O^{a_3} O^{a_4} \rangle.
$$

(3.119)

We shall first compute the piece $\langle O^{a_1 b_1} O^{a_2 b_2} O^{a_3} O^{a_4} \rangle$. The Witten diagrams involved are given in Figure 3.6. The contact diagram comes from the quartic couplings in the expansion of the D-brane action:

$$
L^{(4)}_{O_{a_1 O_{a_2} O_{a_3} O_{a_4}}} = \frac{16\pi^2 T_{D5} \sin \theta_k}{5145} \left[ \left( -71 + 55 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_a \partial^\mu O_{bc} \partial^\nu O_{bc} \\
+ \left( 23 - 40 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_a \partial_\xi O_{bc} \partial^\nu O_{bc} \\
+ \left( 28 - 35 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_a \partial^\mu O_{ac} \partial^\nu O_{bc} \\
+ \left( -79 + 47 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_a \partial_\xi O_{bc} \partial^\nu O_{bc} \\
+ \left( -61 + 65 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_b \partial_\nu O_{ac} \partial^\nu O_{bc} \\
- \left( 277 + 88 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_a \partial_\xi O_{bc} \partial^\nu O_{bc} \\
+ \left( 80 - 46 \cos 2\theta_k \right) \partial_\mu O_a \partial_\nu O_b \partial_\xi O_{ac} \partial^\nu O_{bc} \\
- 36(13 + 6 \cos 2\theta_k) \partial_\mu O_a \partial_\nu O_a \partial^\mu O_{ac} \partial^\nu O_{bc} \\
- 840 \cos^2 \theta_k O_a \partial_\mu O_a \partial_\nu O_{ac} \partial^\nu O_{bc} \right].
$$

(3.120)

We note here that in deriving (3.120) we have used the fact that both $O_a$ and $O_{ab}$ are on-shell in the computation of the Witten diagram so that the equations of motion can be applied. The other diagrams in Figure 3.6 involve the exchange of higher KK modes of $\delta \theta$ and $a_\mu$ fields.

**Exchange of $l = 0$ modes** The cubic vertices appear in the exchange diagram are (3.107) and

$$
L_{O_2 O_2 f_0} = \frac{72i\pi^2 T_{D5} \sin 2\theta_k}{1715} \left( \partial_\mu O_{ab} \partial^\mu O_{ab} + 22O_{ab} O_{ab} \right) f_0,
$$

(3.121)

$$
L_{O_2 O_2 \delta \theta_0} = -\frac{48i\pi^2 T_{D5} \sin 2\theta_k \sin \theta_k}{1715} \left( \partial_\mu O_{ab} \partial^\mu O_{ab} \delta \theta_0 - 20O_{ab} \partial_\mu O_{ab} \partial^\mu \delta \theta_0 + 32O_{ab} O_{ab} \delta \theta_0 \right).
$$

(3.122)
Figure 3.6: Witten diagrams to compute the connected part of \( \langle O_{a_1} O_{a_2} O_{a_3} b_3 O_{a_4} b_4 \rangle \). The \( l = 0, 1, 2, 3 \) modes of \( \delta \theta \) and \( a_\mu \) fields are exchanged in the exchange diagrams.

As in the previous cases, the exchange diagram can be reduced to a contact diagram with the effective quartic coupling:

\[
L_{\text{exc}, l=0} = \frac{216 \pi^2 T_5 \cos^2 \theta_k}{8575} \left( \partial_{\mu} O_a \partial_\nu O_a \partial_{\mu} O_{bc} \partial_\nu O_{bc} + \frac{122}{3} \partial_{\mu} O_a \partial_\mu O_a O_{bc} O_{bc} + \frac{80}{3} O_a O_a O_{bc} O_{bc} \right). \tag{3.123}
\]

**Exchange of \( l = 1 \) modes**  The cubic vertices appear in the exchange diagram are

\[
L_{\alpha_{12} f_1} = \frac{8i \pi^2 T_5 \sin^2 \theta_k}{245} \left[ 3 \partial_{\mu} O_a \partial_\mu O_{ab} f_b - 16 \varepsilon^{\mu\nu} \partial_{\mu} O_a O_{ab} (a_\nu)_b \\
+ 12 \varepsilon^{\mu\nu} O_a \partial_\mu O_{ab} (a_\nu)_b + 42 O_a O_{ab} f_b \right], \tag{3.124}
\]

\[
L_{\alpha_{12} \delta \theta_1} = \frac{8 \pi^2 T_5 \sin \theta_k \sin \theta_k}{245} \left( \partial_{\mu} O_a \partial_\mu O_{ab} \delta \theta_b + 20 \partial_{\mu} O_a O_{ab} \partial_\mu \delta \theta_b \\
+ 13 O_a \partial_{\mu} O_{ab} \partial_\mu \delta \theta_b - 34 O_a O_{ab} \delta \theta_b \right). \tag{3.125}
\]

By using the on-shellness of the external \( O_a \) and \( O_{ab} \) when computing the diagram, we can express the coupling in the form:

\[
\frac{8i \pi^2 T_5 \sin \theta_k}{245} \left\{ (-2 \partial_{\mu} O_a O_{ab} + 5 O_a \partial_{\mu} O_{ab}) \left[ - \partial_{\mu} f_b + 4 \varepsilon^{\mu\nu} (a_\nu)_b - 4i \sin \theta_k \partial_{\mu} \delta \theta_b \right] \\
- 10i \sin \theta_k O_a O_{ab} \left( - \nabla^2 \delta \theta_b + \frac{4i}{\sin \theta_k} f_b \right) \right\}. \tag{3.126}
\]
We also need the bulk propagator $G_{\rho \sigma}^{a a'} (\tau, r; \tau', r')$, which satisfies the equations:

\[
- \nabla^\mu (\varepsilon^{\alpha \beta} \partial_\alpha G_{\beta \gamma}^{a a'}) + 4 \varepsilon^{\mu \nu} G_{\nu \gamma}^{a a'} - 4i \sin \theta_k \nabla^\mu G_{\gamma}^{a a'} = \frac{15 \delta^a a' r^2}{8 \pi^2 T D_5 \sin \theta_k} \varepsilon^{\mu \gamma} \delta^2(\tau, r; \tau', r'), \tag{3.127}
\]

\[
- \nabla^\mu (\varepsilon^{\alpha \beta} \partial_\alpha G_{\beta \theta}^{a a'}) + 4 \varepsilon^{\mu \nu} G_{\nu \theta}^{a a'} - 4i \sin \theta_k \nabla^\mu G_{\theta}^{a a'} = 0, \tag{3.128}
\]

\[
- \nabla_\mu \nabla^\mu G_{\gamma}^{a a'} + \frac{4i}{\sin \theta_k} \varepsilon^{\mu \nu} \partial_\mu G_{\nu \gamma}^{a a'} = \frac{15 \delta^a a' r^2}{8 \pi^2 T D_5 \sin \theta_k} \delta_{\theta \delta} \delta^2(\tau, r; \tau', r'). \tag{3.129}
\]

From the form of the coupling (3.126), it follows that the exchange diagram can be reduced to a contact diagram with the effective quartic coupling:

\[
L_{\text{exc}, l=1} = \frac{432 \pi^2 T D_5 \cos^2 \theta_k \sin \theta_k}{12005} \left( \partial_\mu O_a \partial_\nu O_b \partial^\mu O_{ac} \partial^\nu O_{bc} + \frac{196}{9} \partial_\mu O_a \partial^\mu O_b O_{ac} O_{bc} 
\right. \\
\left. + \frac{580}{9} \partial_\mu O_a O_b \partial^\mu O_{ac} O_{bc} + 100 O_a O_b O_{ac} O_{bc} \right). \tag{3.130}
\]

**Exchange of $l = 2$ modes** The cubic couplings in the exchange diagram are (3.111) and

\[
L_{2, \text{odd} \theta_2} = \frac{16 \pi^2 T D_5 \sin 2 \theta_k}{5145} \left[ \partial_\mu O_a \partial^\mu O_{ac} f_{bc} + 20 \varepsilon^{\mu \nu} O_{ab} \partial_\mu O_{ac} (a_\nu)_bc + 112 O_{ab} O_{ac} f_{bc} \right], \tag{3.131}
\]

\[
L_{2, \text{odd} \theta_2} = \frac{32 \pi^2 T D_5 \sin 2 \theta_k}{5145} \sin \theta_k \left( 13 \partial_\mu O_{ab} \partial^\mu O_{ac} \delta_{bc} + 30 \partial_\nu O_{ab} \partial^\nu O_{ac} \delta_{bc} - 64 O_{ab} O_{ac} \delta_{bc} \right). \tag{3.132}
\]

As before, the exchange diagram can be reduced to a contact diagram with the effective quartic coupling:

\[
L_{\text{exc}, l=2} = - \frac{32 \pi^2 T D_5 \cos^2 \theta_k \sin \theta_k}{5145} \left( \partial_\mu O_a \partial^\mu O_{ab} \partial_\nu O_{ab} \partial^\nu O_{bc} - \frac{1}{5} \partial_\mu O_a \partial^\mu O_a \partial_\nu O_{ab} \partial^\nu O_{bc} 
\right. \\
\left. + 119 \partial_\nu O_a \partial^\nu O_{ab} O_{ac} O_{bc} - \frac{119}{5} \partial_\mu O_a \partial^\mu O_a O_{ab} O_{bc} + 90 O_a O_{ab} O_{ac} O_{bc} 
\right. \\
\left. - 18 O_a O_{ab} O_{bc} O_{bc} \right). \tag{3.133}
\]

**Exchange of $l = 3$ modes** The cubic couplings appear in the exchange diagram are

\[
L_{3, \text{odd} \theta_3} = - \frac{32 \pi^2 T D_5 \sin 2 \theta_k}{735} \left[ \partial_\mu O_a \partial^\mu O_{ab} f_{abc} - 3 \varepsilon^{\mu \nu} \partial_\mu (O_a O_{bc}) (a_\nu)_bc - 7 O_a O_{bc} f_{abc} \right], \tag{3.134}
\]

\[
L_{3, \text{odd} \theta_3} = \frac{32 \pi^2 T D_5 \sin 2 \theta_k}{735} \left[ 9 \partial_\mu O_a \partial^\mu O_{bc} \delta_{abc} + \frac{3}{2} \partial_\mu (O_a O_{bc}) \partial^\mu \delta_{abc} - 5 O_a O_{bc} \delta_{abc} \right]. \tag{3.135}
\]
Using the on-shellness of the external $O_a$ and $O_{ab}$ when computing the diagram, we can expressed the cubic vertices as

\[
-\frac{16i\pi^2 T_{D5}}{35} \sin 2\theta_k \left\{ \partial_\mu (O_a O_{bc}) \left[ -\partial_\mu f_{abc} + 18\varepsilon^{\mu\nu} (a_\nu)_a b c - i 4 \sin \theta_k \partial_\nu \delta_{abc} \right] 
- 4i \sin \theta_k O_{a} O_{bc} \left( -\nabla^2 \delta_{abc} + 14 \delta_{abc} + \frac{4i}{\sin \theta_k} f_{abc} \right) \right\}.
\] (3.136)

The bulk propagator $G_{pq}^{abc, a'b'c'}(\tau, \tau')$ needed in the computation satisfies the following equations:

\[
-\nabla^\mu (\varepsilon^{\alpha\beta} \partial_\alpha G_{\beta\gamma}^{abc, a'b'c'}) + 18\varepsilon^{\mu\nu} G_{\nu\gamma}^{abc, a'b'c'} = \frac{315 M^{abc, a'b'c'}}{16\pi^2 T_{D5} \sin \theta_k} \varepsilon^{\mu\nu} \delta^2 (\tau, \tau') \] (3.137)

\[
-\nabla^\mu (\varepsilon^{\alpha\beta} \partial_\alpha G_{\beta\gamma}^{abc, a'b'}) + 18\varepsilon^{\mu\nu} G_{\nu\gamma}^{abc, a'b'} = 0 \] (3.138)

\[
-\nabla_\mu \nabla^\nu G_{\mu\nu}^{abc, a'b'c'} + 14 \delta_{abc} = \frac{315 M^{abc, a'b'c'}}{16\pi^2 T_{D5} \sin \theta_k} \delta_{\mu\nu} \delta^2 (\tau, \tau') \] (3.139)

where $M^{abc, a'b'c'}$ is defined as

\[
M^{abc, a'b'c'} = \frac{1}{6} \left[ \delta^{a'b'} \delta^{ca'} + \delta^{a'a'} \delta^{cb'} \delta^{b'c'} + \delta^{b'b'} \delta^{ca'} + \delta^{b'a'} \delta^{cb'} \delta^{c'a'} + \delta^{c'c} \delta^{ab'} \delta^{a'b'} + \delta^{a'a} \delta^{b'b} \delta^{c'c} + \delta^{ab'} \delta^{b'a'} \delta^{c'c} \right. \\
\left. + \delta^{a'a} \delta^{b'b} \delta^{c'c} - \frac{2}{3} \left( \delta^{ab} \delta^{ca} \delta^{b'c'} + \delta^{bc} \delta^{a'b'} \delta^{c'a'} + \delta^{ca} \delta^{b'b'} \delta^{a'c'} + \delta^{ab} \delta^{ca} \delta^{b'c'} \delta^{a'c'} + \delta^{bc} \delta^{ca} \delta^{b'b'} \delta^{a'c'} + \delta^{ab} \delta^{ca} \delta^{b'b'} \delta^{c'a'} + \delta^{bc} \delta^{ca} \delta^{b'b'} \delta^{c'a'} + \delta^{ab} \delta^{ca} \delta^{b'b'} \delta^{c'a'} \right) \right].
\] (3.140)

From the form of (3.136), we see that the exchange diagram can be reduced to a contact diagram with the effective quartic coupling:

\[
L_{exc, l=3} = \frac{128 \pi^2 T_{D5} \cos^2 \theta_k \sin \theta_k}{5145} \left( \partial_\mu O_a \partial_\nu O_a \partial_\rho O_{bc} \partial_\sigma O_{bc} \right. \\
\left. - \frac{4}{1} \partial_\mu O_a \partial_\nu O_b \partial_\rho O_{ac} \partial_\sigma O_{bc} - \frac{2}{1} \partial_\mu O_a \partial_\rho O_{ac} \partial_\sigma O_{bc} - \frac{3}{2} \partial_\mu O_a \partial_\sigma O_{ac} O_{bc} - \frac{3}{4} \partial_\nu O_a \partial_\sigma O_{ac} O_{bc} - \frac{54}{1} \partial_\nu O_a \partial_\rho O_{ac} O_{bc} - \frac{10}{1} O_a O_{bc} O_{bc} \right)
\] (3.141)

The four-point function Summing up all the diagrams, we find that the connected part of the four-point function can be computed from a single contact diagram with the effective quartic
coupling:

\[
L_{010122}^{f} = \frac{24\pi^2 T D_5 \sin^3 \theta_k}{245} \left( \partial_{\mu} O_a \partial^\mu O_a \partial_{\nu} O_{bc} \partial^\nu O_{bc} - 2 \partial_{\mu} O_a \partial_{\nu} O_a \partial^\mu O_{bc} \partial^\nu O_{bc} \\
- 2 \partial_{\mu} O_a \partial^\mu O_{bc} \partial_{\nu} O_{ac} \partial^\nu O_{bc} + 2 \partial_{\mu} O_a \partial_{\nu} O_a \partial^\mu O_{ac} \partial^\nu O_{bc} - 2 \partial_{\mu} O_a \partial_{\nu} O_{bc} \partial_{\nu} O_{ac} \partial^\mu O_{bc} \\
- 6 \partial_{\mu} O_a \partial^\mu O_{bc} O_{bc} \partial_{\nu} O_{ac} O_{bc} + 4 \partial_{\mu} O_a \partial^\mu O_a O_{ac} O_{bc} - 8 \partial_{\mu} O_a \partial_{\nu} O_a \partial^\mu O_{ac} O_{bc} \right). \tag{3.142}
\]

This leads to the following result for the connected part of the unnormalized four-point function:

\[
\langle O_{\tau_1}^{a_1} O_{\tau_2}^{a_2} O_{\tau_3}^{a_3} O_{\tau_4}^{a_4} \rangle = - \frac{24\pi^2 T D_5 \sin^3 \theta_k}{245} (C_{\Delta=2} C_{\Delta=1})^2 Q_{a_0 b_0 c_0 d_0}^{a_1 b_1 a_2 b_2 a_3 a_4}, \tag{3.143}
\]

where

\[
Q_{a_0 b_0 c_0 d_0}^{a_1 b_1 a_2 b_2 a_3 a_4} = 8 \left[ -5 D_{2211} - 4 \tau_{12}^2 D_{3311} + 4 \tau_{13}^2 D_{3221} + 4 \tau_{14}^2 D_{3212} + 4 \tau_{23}^2 D_{2321} + 4 \tau_{24}^2 D_{2312} + 2 \tau_{34}^2 D_{2222} + 8 \left( \tau_{12}^2 \tau_{34}^2 - \tau_{13}^2 \tau_{24}^2 - \tau_{14}^2 \tau_{23}^2 \right) D_{3322} \right] \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{a_3 a_4} \\
+ 8 \left[ -5 D_{2211} + 4 \tau_{12}^2 D_{3311} + 4 \tau_{13}^2 D_{3221} + 4 \tau_{14}^2 D_{3212} + 4 \tau_{23}^2 D_{2321} + 2 \tau_{34}^2 D_{2222} - 8 \left( \tau_{12}^2 \tau_{34}^2 - \tau_{13}^2 \tau_{24}^2 + \tau_{14}^2 \tau_{23}^2 \right) D_{3322} \right] \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{a_3 a_4} \\
+ 8 \left[ -5 D_{2211} + 4 \tau_{12}^2 D_{3311} + 4 \tau_{13}^2 D_{3221} + 4 \tau_{14}^2 D_{3212} + 4 \tau_{23}^2 D_{2321} + 2 \tau_{34}^2 D_{2222} - 8 \left( \tau_{12}^2 \tau_{34}^2 + \tau_{13}^2 \tau_{24}^2 - \tau_{14}^2 \tau_{23}^2 \right) D_{3322} \right] \delta_{a_1 a_4} \delta_{b_1 b_2} \delta_{a_2 a_3}, \tag{3.144}
\]

where the function $D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$ is defined in Appendix B.1. In terms of the conformal cross-ratios, the four-point function can be expressed as

\[
\langle O_{\tau_1}^{a_1} O_{\tau_2}^{a_2} O_{\tau_3}^{a_3} O_{\tau_4}^{a_4} \rangle = - \frac{24\pi^3 T D_5 \sin^3 \theta_k}{245} \frac{(C_{\Delta=2} C_{\Delta=1})^2}{\tau_{12}^2 \tau_{34}^2} G_{a_1 b_1 a_2 b_2 a_3 a_4}(\chi). \tag{3.145}
\]

The function $G_{a_1 b_1 a_2 b_2 a_3 a_4}(\chi)$ is defined as

\[
G_{a_1 b_1 a_2 b_2 a_3 a_4}(\chi) = \left[ G_1(\chi) \delta_{a_1 a_2} \delta_{b_1 b_2} \delta_{a_3 a_4} + G_2(\chi) \delta_{a_1 a_3} \delta_{b_1 b_2} \delta_{a_2 a_4} + G_3(\chi) \delta_{a_1 a_4} \delta_{b_1 b_2} \delta_{a_2 a_3} \right]. \tag{3.146}
\]
where

\[
G_1(\chi) = \frac{3}{(\chi - 1)^3} \left[ -4 + 12\chi - 9\chi^2 - 2\chi^3 + 5\chi^4 - 2\chi^5 \\
+ (-4 + 14\chi - 18\chi^2 + 10\chi^3 - 6\chi^5 + 6\chi^6 - 2\chi^7) \log |1 - \chi| \\
+ (6 - 6\chi + 2\chi^2) \chi^4 \log |\chi| \right],
\]

(3.147)

\[
G_2(\chi) = \frac{3}{2(\chi - 1)^3} \left[ 4\chi - 15\chi^2 + 11\chi^3 + 9\chi^4 - 15\chi^5 + 6\chi^6 \\
+ (4 - 12\chi + 14\chi^2 - 10\chi^3 + 16\chi^5 - 18\chi^6 + 6\chi^7) \log |1 - \chi| \\
+ (-16 + 18\chi - 6\chi^2) \chi^5 \log |\chi| \right],
\]

(3.148)

\[
G_3(\chi) = \frac{3}{2(\chi - 1)^3} \left[ -4\chi + 5\chi^2 + 9\chi^3 - 8\chi^4 + 4\chi^5 \\
+ (-4 + 12\chi - 14\chi^2 + 6\chi^3 + 6\chi^4 - 10\chi^5 + 4\chi^6) \log |1 - \chi| \\
+ (-16 + 14\chi - 4\chi^2) \frac{\chi^5}{\chi - 1} \log |\chi| \right].
\]

(3.149)

The second piece in (3.119) can be computed easily and we find

\[
\frac{c^2}{T_{D5}^2} \langle O^{a_1} O^{b_1}; (\tau_1) O^{a_2} O^{b_2}; (\tau_2) O^{a_3} (\tau_3) O^{a_4} (\tau_4) \rangle \\
= \frac{c^2}{T_{D5}^2 \tau_{12} \tau_{34}} \left[ 2\delta^{a_1 a_2} \delta^{b_1 b_2} \delta^{a_3 a_4} + 4\chi^2 \delta^{a_1 a_3} \delta^{b_1 b_2} \delta^{a_2 a_4} + 4 \frac{\chi^2}{(1 - \chi)^2} \delta^{a_1 a_4} \delta^{b_1 b_2} \delta^{a_2 a_3} \right].
\]

(3.150)

The first term in the bracket of (3.150) does not contribute to the connected part of the four-point function as it is proportional to \(\langle\langle O_1 O_1; O_1 O_1 \rangle\rangle\) \langle\langle O_1; O_1 \rangle\rangle\).

Summing up the contribution from the two pieces in (3.119), we find the connected part of the normalized four-point function is

\[
\langle\langle O_2(\tau_1, u_1) O_2(\tau_2, u_2) O_1(\tau_3, u_3) O_1(\tau_4, u_4) \rangle\rangle = \frac{(u_1 \cdot u_2)^2 (u_3 \cdot u_4)}{\tau_{12} \tau_{34}^2} G(\chi, \xi, \zeta),
\]

(3.151)

where

\[
G(\chi, \xi, \zeta) = -\frac{1}{16\pi^3 T_{D5} \sin^4 \theta_k} \left[ G_1(\chi) + \xi G_2(\chi) + \zeta G_3(\chi) - 45 \cot^2 \theta_k \chi^2 \left( \xi + \frac{\zeta}{(1 - \chi)^2} \right) \right].
\]

(3.152)

One can check explicitly that the function \(G(\chi, \xi, \zeta)\) indeed satisfies the superconformal identities (3.18). By transforming to the circular Wilson loop and setting the polarizations to (3.81), we find
the normalized four-point function becomes

$$\frac{\langle \tilde{O}_2 \tilde{O}_2 \tilde{O}_1 \tilde{O}_1 \rangle}{\langle \tilde{O}_2 \tilde{O}_2 \rangle \langle \tilde{O}_1 \tilde{O}_1 \rangle} = -\frac{15(1 - 6 \cot^2 \theta_k)}{16 \pi^3 T_{D5} \sin^3 \theta_k} = -\frac{15(1 - 6 \cot^2 \theta_k)}{8Ng \sin^3 \theta_k}. \quad (3.153)$$

Remarkably, this is in precise agreement with the prediction of localization [54].

### 3.3 Correlation functions in dCFT$_1$ from the D3-brane

#### 3.3.1 D3-brane solution in $AdS_5 \times S^5$

In this section we review the D3-brane solution in $AdS_5 \times S^5$ background [74]. The bosonic part of the Euclidean action for the D3-brane is given by

$$S_{D3} = T_{D3} \int d^4 \sigma \sqrt{\det(G + F)} - T_{D3} \int P[C_4], \quad (3.154)$$

where $P$ stands for the pullback. The D3-brane tension is

$$T_{D3} = \frac{N}{2\pi^2}. \quad (3.155)$$

To write down the D3-brane solution, it is convenient to parametrize the $AdS_5 \times S^5$ space as

$$ds^2_{AdS_5 \times S^5} = \cosh^2 u \, ds^2_{AdS_2} + \sinh^2 u \, d\Omega_2^2 + du^2 + \frac{dy^a dy^a}{(1 + \frac{1}{4}y^2)^2}. \quad (3.156)$$

The four-form potential $C_4$ is

$$C_4 = \left(-\frac{u}{2} + \frac{\sinh 4u}{8}\right) \frac{\sin \theta}{r^2} d\tau \wedge dr \wedge d\theta \wedge d\phi, \quad (3.157)$$

where $(\tau, r)$ are the Poincare coordinates for the Euclidean $AdS_2$ (suitable in the case of straight Wilson line at the boundary), and $(\theta, \phi)$ are the coordinates for $S^2$. The embedding of the D3-brane solution in $AdS_5 \times S^5$ is given by the $AdS_2 \times S^2$ hyper-surface parametrized by $u = u_k$ in $AdS_5$ and an arbitrary point on $S^5$. For simplicity, we can choose $y^a_0 = 0$. The value of $u_k$ is related to the fundamental string charge $k$ dissolved on the brane via [74]

$$\sinh u_k = \frac{k\sqrt{\lambda}}{4N}. \quad (3.158)$$
The background gauge field strength is

\[ F = i \frac{\cosh u_k}{r^2} d\tau \wedge dr. \]  

(3.159)

As in the D5-brane case, we need to add the following boundary term to the action to implement the correct boundary conditions [74,82]

\[ S_{\text{bdy}}^A = - \int d\tau \int d\Omega_2 A_\tau \pi_A, \]  

(3.160)

with \( \pi_A \) being the conjugate momentum to \( A_\tau \):

\[ \pi_A = \frac{\partial L_{D3}}{\partial F_\tau}. \]  

(3.161)

As explained in the D5 brane case above, the boundary term ensures that the momentum conjugate to \( A \) is held fixed at the boundary. This is related to fundamental string charge as

\[ k = -2\pi i \alpha' \int_{S^2} \frac{\partial L_{D3}}{\partial F_\tau} = \frac{4N}{\sqrt{\lambda}} \sinh u_k, \]  

(3.162)

and fixing \( k \) means fixing the rank of the symmetric representation of the Wilson loop operator.

The expectation value of the circular Wilson loop at strong coupling can be obtained by using the hyperbolic disk coordinates on \( AdS_2 \) and evaluating the D3 brane classical action supplemented by the boundary term (3.160)

\[ \langle W_{S_k} \rangle = \exp \left( -S_{D3} - S_{\text{bdy}}^A \right). \]  

(3.163)

Using the solution above, we find

\[ S_{D3} + S_{\text{bdy}}^A = \frac{1}{2} T_{D3} \text{vol}(AdS_2) \text{vol}(S^2) \left( u_k + \sinh u_k \cosh u_k \right). \]  

(3.164)

This yields (as for the D5 brane, we use \( \text{vol}(AdS_2) = -2\pi \) instead of adding a boundary term for the AdS radial coordinate):

\[ \langle W_{S_k} \rangle = \exp \left( 2N \left( u_k + \sinh u_k \cosh u_k \right) \right). \]  

(3.165)

This agrees with the localization prediction [74,76].
3.3.2 Spectrum of excitations around the D3-brane

To obtain the mass spectrum, we need to consider the quadratic action for the fluctuations $\delta y^a$, $\delta u$ and $f$ around the D3-brane solution, where $f$ is a 2-form representing the fluctuations of the background field strength. Since the spectrum has been computed in [82], we briefly review the calculation here. The variation of the metric up to quartic order in fluctuations is

$$\delta (d^2 s^2) = (\sinh 2u_k \delta u + \cosh 2u_k \delta u^2 + \frac{2}{3} \sinh 2u_k \delta u^3 + \frac{1}{3} \cosh 2u_k \delta u^4) (d\sigma^2_{AdS_2} + d\Omega_2^2) + (d\delta u)^2 + (1 - \frac{1}{2} \delta y^2)(d\delta y)^2.$$ (3.166)

The variation of the four-form $C_4$ up to quartic order in fluctuations is

$$\delta C_4 = \frac{\sin \theta}{r^2} (\sinh^2 2u_k \delta u + \sinh 4u_k \delta u^2 + \frac{4}{3} \cosh 4u_k \delta u^3 + \frac{4}{3} \sinh 4u_k \delta u^4) d\tau \wedge dr \wedge d\theta \wedge d\phi.$$ (3.167)

We use Greek letters $(\mu, \nu)$ for the coordinates of $AdS_2$ and Greek letters $(\alpha, \beta)$ for the coordinates of $S^2$. The mass spectrum can be obtained by expanding the action (3.154) to quadratic order in fluctuations around the D3-brane solution.

**$\delta y^a$ sector** The quadratic Euclidean action for $\delta y^a$ sector is

$$S^{(2)}_{\delta y} = \frac{T_{D3} \sinh 2u_k}{2} \int d^4\xi \frac{\sqrt{g_2}}{r^2} \left( \partial_{\mu} \delta y^a \partial^{\mu} \delta y^a + \nabla_{\alpha} \delta y^a \nabla^{\alpha} \delta y^a \right),$$ (3.168)

where $g_2$ is the metric for $S^2$. Similar to the D5-brane case, we expand the field $\delta y^a$ in terms of symmetric traceless tensor fields:

$$\delta y^a(\tau, r, \Omega_2) = \sum_{l=0}^{\infty} (\delta y^a)_{i_1\ldots i_l} (\tau, r) Y^{i_1} \ldots Y^{i_l},$$ (3.169)

where $Y^i$ is the three-dimensional vector specifying $S^2$:

$$\sum_{i=1}^{3} Y^i(\Omega_2) Y^i(\Omega_2) = 1.$$ (3.170)

We have

$$\nabla^2_{S^2} (\delta y^a)_{i_1\ldots i_l} Y^{i_1} \ldots Y^{i_l} = -l(l+1)(\delta y^a)_{i_1\ldots i_l} Y^{i_1} \ldots Y^{i_l}.$$ (3.171)
The quadratic action for the \((\delta y^a)_{i_1 \ldots i_l}\) modes is

\[
S^{(2)}_{\delta y} = \sum_{l=0}^{\infty} \frac{V_l T_{D3} \sinh 2u_k}{2} \int \frac{dr dr' \sinh 2u_k}{r^2} \left[ \partial_\mu (\delta y^a)_{i_1 \ldots i_l} \partial^\mu (\delta y^a)_{i_1 \ldots i_l} + l(l + 1)(\delta y^a)_{i_1 \ldots i_l} (\delta y^a)_{i_1 \ldots i_l} \right].
\] (3.172)

The factor \(V_l\) comes from the integral of spherical harmonics over \(S^2\) and is defined by

\[
\int d\Omega_2 (u_1 \cdot Y)^l (u_2 \cdot Y)^l \equiv V_l (u_1 \cdot u_2)^l,
\] (3.173)

where \(u\) is a three-dimensional null vector. Using the same method as in the D5-brane case, we find that

\[
V_l = \frac{4\pi (l!)^2 2^l}{(2l + 1)!}.
\] (3.174)

\(\delta u\) sector The quadratic Euclidean action in \(\delta u\) sector is

\[
S^{(2)}_{\delta u} = \frac{T_{D3} \sinh 2u_k}{2} \int d^4 \xi \sqrt{g} \frac{1}{r^2} \left( \partial_\mu \delta u \partial^\mu \delta u + \nabla_\alpha \delta u \nabla^\alpha \delta u \right).
\] (3.175)

Expanding the \(\delta u\) field in terms of the symmetric traceless tensor fields

\[
\delta u(\tau, r, \Omega_2) = \sum_{l=0}^\infty \delta u_{i_1 \ldots i_l}(\tau, r) Y^{i_1} \ldots Y^{i_l},
\] (3.176)

we find that the quadratic action for these modes is

\[
S^{(2)}_{\delta y} = \sum_{l=0}^{\infty} \frac{V_l T_{D3} \sinh 2u_k}{2} \int \frac{dr dr' \sinh 2u_k}{r^2} \left[ \partial_\mu \delta u_{i_1 \ldots i_l} \partial^\mu \delta u_{i_1 \ldots i_l} + l(l + 1)\delta u_{i_1 \ldots i_l} \delta u_{i_1 \ldots i_l} \right].
\] (3.177)

Gauge field sector In order to decouple \(a_\mu\) from the gauge fields along \(S^2\) direction, we impose the gauge condition:

\[
\nabla^\alpha a_\alpha = 0.
\] (3.178)

The field \(a_\mu\) can be expanded in terms of symmetric traceless tensor fields expand while the field \(a_\alpha\) needs to be expanded using transverse vector spherical harmonics \((\hat{Y}_\alpha)_{lm}\):

\[
a_\mu(\tau, r, \Omega_2) = \sum_{l=0}^\infty (a_\mu)_{i_1 \ldots i_l}(\tau, r) Y^{i_1} \ldots Y^{i_l}, \quad a_\alpha(\tau, r, \Omega_2) = \sum_{l=1}^\infty a_l(\tau, r)(\hat{Y}_\alpha)_{lm}(\Omega_2). \quad (3.179)
\]
The transverse vector spherical harmonics satisfy the following properties \([104,105]\)

\[
\nabla^2 S_2 (\hat{Y}_\alpha)_{lm} = -(l^2 + l - 1)(\hat{Y}_\alpha)_{lm}, \quad \nabla^a S_2 (\hat{Y}_\alpha)_{lm} = 0, \quad (l = 1, 2, \ldots).
\]

The quadratic action for the \((a_\mu)_{i_1 \ldots i_l}\) modes is

\[
S^{(2)}_{a_\mu} = \sum_{l=1}^{\infty} \frac{V_l T_{D3} \coth u_k}{2} \int \frac{d\tau dr}{r^2} \left[ \frac{1}{2} (f_{\mu\nu})_{i_1 \ldots i_l} (f^{\mu\nu})_{i_1 \ldots i_l} + l(l+1) (a_\mu)_{i_1 \ldots i_l} (a^\nu)_{i_1 \ldots i_l} \right],
\]

where we have omitted the \(l = 0\) mode of \(a_\mu\) because it is not dynamical. The quadratic action for the \(a_l\) modes is

\[
S^{(2)}_a = \sum_{l=1}^{\infty} \pi T_{D3} \coth u_k \int \frac{d\tau dr}{r^2} \left[ \frac{1}{2} \partial_\mu a_l \partial^\mu a_l + l(l+1)a_l^2 \right].
\]

### 3.3.3 Dual operators and two-point functions

In this section, we discuss the dual operators for the bulk fluctuation modes. Unlike in the D5-brane case, although there have been discussions on the holographic dictionary in \([82]\), we think there remain some questions on the identification of the dual operators. In table 3.2, we summarize the quantum numbers of the dual operators.

**\(\delta y^a\) sector** From the mass spectrum, we see that the mode \((\delta y^a)_{i_1 \ldots i_l}\) should be dual to an operator of dimension \(\Delta_l = l + 1\) which transforms as a \(SO(5)\) vector. In particular, the five \(l = 0\) modes which we shall denote as \(\delta y^a_0\) are dual to the five scalars \(\Phi^a\) in the ultrashort supermultiplet of \(OSp(4^*\mid 4)\). Note that the protected operators \(O_L\) with \(L > 1\) do not appear as single-particle states in the D3 brane spectrum, unlike the D5 brane case discussed above, which agrees with the localization analysis \([54]\). Note that, as in the fundamental string case \([24,34]\), one still has protected "multi-particle" operators with \(\Delta = l\) in the totally symmetric representation of \(SO(5)\) built from symmetrized products of \(\delta y^a\).

**\(\delta u\) and \(a_\mu\) sector** From the mass spectrum, we see that both \(\delta u_{i_1 \ldots i_l}\) and \((a_\mu)_{i_1 \ldots i_l}\) should be dual to the operators of dimension \(\Delta_l = l + 1\) which transform in the spin-\(l\) representation of \(SO(3)\). In particular, \(\delta u_0\) should be dual to an operator of dimension \(\Delta = 1\) which is a singlet under both \(SO(3)\) and \(SO(5)\). There is no natural candidate for a protected operator with these quantum numbers on the gauge theory side. A possible resolution to this puzzle is that \(\delta u_0\) belongs to a semi-short multiplet of \(OSp(4^*\mid 4)\) (this can be thought of as a long multiplet at the unitarity bound, see \([70]\)), and as soon as we move away from the strict strong coupling limit, this operator may
Table 3.2: In this table we summarize the quantum numbers of the operator dual to each fluctuation mode. \( \Delta \) gives the conformal dimension of the dual operator. The quantum numbers of the dual operator under \( SO(3) \) and \( SO(5) \) symmetry are given in terms of the Dynkin labels of the corresponding representations.

<table>
<thead>
<tr>
<th>Fluctuation modes</th>
<th>( \Delta )</th>
<th>( SO(3) )</th>
<th>( SO(5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\delta y^a)_i) ((l \geq 0))</td>
<td>( l + 1 )</td>
<td>( l )</td>
<td>((0,1))</td>
</tr>
<tr>
<td>(\delta u_{i_1 \cdots i_l} ) ((l \geq 0))</td>
<td>( l + 1 )</td>
<td>( l )</td>
<td>((0,0))</td>
</tr>
<tr>
<td>((a_\mu)_{i_1 \cdots i_l} ) ((l \geq 1))</td>
<td>( l + 1 )</td>
<td>( l )</td>
<td>((0,0))</td>
</tr>
<tr>
<td>(a_l ) ((l \geq 1))</td>
<td>( l + 1 )</td>
<td>( l )</td>
<td>((0,0))</td>
</tr>
</tbody>
</table>

As a test of our identification of \( \chi_i \) as dual to the displacement operator, we will verify that it has the correct four-point functions by computing them in Section 3.3.5 and 3.3.6.

The two-point functions As in the D5-brane case, we need to include the following boundary term to ensure the correct boundary conditions for the gauge fields

\[
- \int d\tau \int d\Omega_4 \coth u_k r^2 a_\tau (\partial_\tau a_\tau - \partial_\tau a_\tau). \tag{3.184}
\]
Including this boundary term, we find that the tree level two-point function of the operator $F_{ti}$ is

$$
\langle\langle F_{ti}(\tau_1) F_{tj}(\tau_2) \rangle\rangle = \langle \chi_i(\tau_1) \chi_j(\tau_2) \rangle_{AdS_2} = \delta_{ij} \frac{2T_{D3} \sinh 2u_k}{\tau_{12}^2}.
$$

(3.185)

The tree level two-point function for the operator $\Phi^a$ is

$$
\langle\langle \Phi^a(\tau_1) \Phi^b(\tau_2) \rangle\rangle = \langle \delta y^a_0(\tau_1) \delta y^b_0(\tau_2) \rangle_{AdS_2} = \delta_{ab} \frac{2T_{D3} \sinh 2u_k}{\tau_{12}^2}.
$$

(3.186)

### 3.3.4 Four-point function of $S^5$ fluctuations of D3-brane

In this section, we compute the connected part of the tree level four-point function

$$
\langle\langle \Phi^{a_1}(\tau_1) \Phi^{a_2}(\tau_2) \Phi^{a_3}(\tau_3) \Phi^{a_4}(\tau_4) \rangle\rangle = \langle \delta y^{a_1}_0(\tau_1) \delta y^{a_2}_0(\tau_2) \delta y^{a_3}_0(\tau_3) \delta y^{a_4}_0(\tau_4) \rangle_{AdS_2}.
$$

(3.187)

The Witten diagrams involved are shown in the Figure 3.7. The quartic coupling in the contact diagram is obtained from expanding the D3-brane action and we find

$$
L^{(4)}_{\text{yyyy}} = \pi T_{D3} \left[ \left( \coth u_k - \frac{4}{\sinh 2u_k} \right) \partial_{\mu} \delta y^{a}_0 \partial_{\nu} \delta y^{a}_0 \partial_{\mu} \delta y^{b}_0 \partial_{\nu} \delta y^{b}_0 - 2 \tanh u_k \partial_{\mu} \delta y^{a}_0 \partial_{\mu} \delta y^{a}_0 \delta y^{b}_0 \delta y^{b}_0 - \sinh 2u_k \partial_{\mu} \delta y^{a}_0 \partial_{\mu} \delta y^{a}_0 \delta y^{b}_0 \delta y^{b}_0 \right].
$$

(3.188)

The cubic couplings in the exchange diagrams are

$$
L_{\text{yuy0}} = 4\pi T_{D3} \sinh^2 u_k \partial_{\mu} \delta y^{a}_0 \partial_{\mu} \delta y^{a}_0 \delta u_0,
$$

$$
L_{\text{yyf0}} = -2\pi T_{D3} \coth u_k \partial_{\mu} \delta y^{a}_0 \partial_{\mu} \delta y^{a}_0 \delta y^{b}_0 f_0,
$$

(3.189, 3.190)

where $f_0 = \varepsilon_{\mu\nu} \partial_{\mu}(a_{\nu})_0$.

To compute the exchange-diagrams, we need the bulk propagator $G_{uu}(\tau, r; \tau', r')$ for $\delta u_0$ and $G_{\mu\nu}(\tau, r; \tau', r')$ for $(a_{\mu})_0$. These propagators satisfy the following equations:

$$
-\nabla^\mu (\varepsilon^{\alpha\beta} \partial_\alpha G_{\beta\gamma}) = \frac{\tanh u_k}{4\pi T_{D3}} \frac{r^2}{e^{\mu, \gamma} \delta(\tau - \tau') \delta(r - r')},
$$

(3.191)

$$
-\nabla^\mu \nabla_\mu G_{uu} = \frac{r^2}{2\pi T_{D3} \sinh 2u_k} \delta(\tau - \tau') \delta(r - r'),
$$

(3.192)

where we have suppressed the dependence of the propagators on the coordinates. As in D5-brane case, due to the special form of the cubic coupling, we find that the exchange diagrams can be
Figure 3.7: Witten diagrams to compute the connected part of \( \langle \delta y^a_1 \delta y^a_2 \delta y^a_3 \delta y^a_4 \rangle \). The \( l = 0 \) modes of \( \delta u \) and \( u_\mu \) fields are exchanged in the exchange diagram.

Reduced to a single contact diagram with the effective coupling

\[
L_{\text{exc}}^{\text{eff}} = \pi T_{D3} \left( \frac{1}{\sinh 2u_k} \partial_\mu \delta y^a_0 \partial^\mu \delta y^a_0 \partial_\nu \delta y^b_0 \partial^\nu \delta y^b_0 + \frac{\sinh^3 u_k}{\cosh u_k} \frac{\partial_\mu \delta y^a_0 \partial^\mu \delta y^a_0 \partial_\nu \delta y^b_0 \partial^\nu \delta y^b_0}{\nu_1 \nu_2 \nu_3 \nu_4} \right),
\]

by using integration by parts and the on-shellness of the external \( \delta y^a_0 \). Combining (3.193) and (3.188), we see that the connected part of the four-point function can be computed from a single contact diagram with the effective quartic coupling

\[
L_{\text{yyyy}}^{\text{eff}} = 4\pi T_{D3} \tanh u_k \left( \frac{1}{8} \partial_\mu \delta y^a_0 \partial^\mu \delta y^a_0 \partial_\nu \delta y^b_0 \partial^\nu \delta y^b_0 - \frac{1}{4} \partial_\mu \delta y^a_0 \partial_\nu \delta y^a_0 \partial^\mu \delta y^b_0 \partial^\nu \delta y^b_0 \right) - \frac{1}{4} \partial_\mu \delta y^a_0 \partial^\mu \delta y^a_0 \delta y^b_0 \delta y^b_0.
\]

(3.194)

This effective coupling has the identical form as (3.115) in the D5-brane case except the prefactor.

Using the same normalization for the bulk-to-boundary propagator as in the D5-brane case, we find that the connected part of the normalized four-point function is

\[
\langle \delta y^a_1 (\tau_1) \delta y^a_2 (\tau_2) \delta y^a_3 (\tau_3) \delta y^a_4 (\tau_4) \rangle = \frac{1}{4\pi T_{D3} \sinh u_k} \left( G_{a_1 a_2 a_3 a_4}(\chi) \right) \frac{G_{a_1 a_2 a_3 a_4}(\chi)}{r_{12}^2 r_{13}^2 r_{14}^2 r_{23}^2 r_{24}^2 r_{34}^2}.
\]

(3.195)

We can again compare this result to the localization analysis by transforming to the circle and choosing the “topological” configuration of the polarization vectors. The result is

\[
\frac{\langle \hat{O}_1 \hat{O}_1 \hat{O}_1 \hat{O}_1 \rangle}{\langle \hat{O}_1 \hat{O}_1 \rangle^2} = -\frac{3}{8\pi^2 T_{D3} \sinh u_k} \left( \frac{3}{4N \sinh u_k} \right) = -\frac{3}{4N \sinh u_k}.
\]

(3.196)

This again precisely agrees with the localization prediction for the 4-point function, which in this case just reduces to taking simple area derivatives of the Wilson loop expectation value (given by
(3.165) with the replacement $\lambda \rightarrow \lambda A(4\pi - A)/(4\pi^2)$.

Note that if we take the string limit defined by

$$
\frac{k}{N} \to 0, \quad u_k \to \frac{k\sqrt{\lambda}}{4N},
$$

(3.197)

then the normalized four-point function (3.195) becomes

$$
\langle \delta y_0^a_1(\tau_1)\delta y_0^a_2(\tau_2)\delta y_0^a_3(\tau_3)\delta y_0^a_4(\tau_4) \rangle \to \frac{2\pi}{k\sqrt{\lambda}} \frac{G^{a_1a_2a_3a_4}_{4g}}{r_{12}r_{34}},
$$

(3.198)

As in the D5-brane case, the D3-brane result reduces to the result calculated from $k$ weakly coupled string.

### 3.3.5 Two AdS$_5$ and two S$^5$ fluctuations of D3-brane

In this section, we compute the connected part of the tree level four-point function

$$
\langle \langle F_{i_1i}(\tau_1)F_{i_2i}(\tau_2)\Phi^{a_1}(\tau_3)\Phi^{a_2}(\tau_4) \rangle \rangle = \langle \chi_{i_1}(\tau_1)\chi_{i_2}(\tau_2)\delta y_0^{a_1}(\tau_3)\delta y_0^{a_2}(\tau_4) \rangle_{AdS_2}.
$$

(3.199)

Since the four-point function $\langle \delta y_0^{a_1}\delta y_0^{a_2}\delta y_0^{a_3}\delta y_0^{a_4} \rangle$ has the same form as in the fundamental string case, the supersymmetry then uniquely fixes the result for $\langle \chi_{i_1}\chi_{i_2}\delta y_0^{a_1}\delta y_0^{a_2} \rangle$ if $\chi_i$ is dual to the displacement operator $F_{ii}$. We shall show that it is indeed the case below. The diagrams involved in the calculation are shown in Figure 3.8. The contact diagram in Figure 3.8 results from the quartic coupling in the expanded D3-brane action

$$
L_{\chi y y}^{(4)} = \frac{2\pi T_{D3}}{36} \left[ \frac{(3\cosh 2u_k - 1)}{\sinh 2u_k} \partial_\mu \chi_i \partial^\mu \chi_i \partial_\nu \delta y_0^a \partial^\nu \delta y_0^a 
- 6 \tanh u_k \partial_\mu \chi_i \partial_\nu \chi_i \partial^\mu \delta y_0^a \partial^\nu \delta y_0^a + \frac{(3 + \cosh 4u_k)}{\sinh 2u_k} \chi_i \chi_i \partial_\mu \delta y_0^a \partial^\mu \delta y_0^a \right].
$$

(3.200)

The other two exchange diagrams in Figure 3.8 involve the exchange of $l = 0$ and $l = 1$ modes of the bulk fields.
Exchange of $l = 0$ modes

In this case, the cubic couplings involved are (3.189), (3.190) and

$$L_{\chi \chi u_0} = \frac{\pi T_{D3}}{9} (\cosh 2u_k + 3)(2\chi_i \bar{\partial}_\mu \chi_i \partial^\mu \delta u_0 + \partial^\mu \chi_i \partial_\mu \chi_i \delta u_0 + 2\chi_i \delta u_0),$$

(3.201)

$$L_{\chi \chi f_0} = \frac{i\pi T_{D3}}{9 \sinh u_k} (\bar{\partial}_\mu \chi_i \partial^\mu \chi_i + 2\chi_i \chi_i) f_0.$$ 

(3.202)

As before, these exchange diagrams can be reduced to contact diagrams after using the on-shellness of the external $\chi_i$ and $\delta y_0^a$ and performing integration by parts. The end result can be summarized as a single contact diagram with the effective quartic coupling

$$L_{\text{exc},l=0} = -\frac{\pi T_{D3}}{18} \left[ (\cosh 2u_k + 3) \tanh u_k \bar{\partial}_\mu \delta y_0^a \partial^\mu \delta y_0^a \chi_i \chi_i 
- \frac{2}{\sinh 2u_k} \left( \bar{\partial}_\mu \delta y_0^a \partial^\mu y_0^a \partial_\nu \chi_i \partial^\nu \chi_i + 2\bar{\partial}_\mu \delta y_0^a \partial^\mu \delta y_0^a \chi_i \chi_i \right) \right].$$

(3.203)

Exchange of $l = 1$ modes

In this case, the cubic coupling involved is

$$L_{yy_1 \chi} = \frac{4\pi T_{D3}}{3} \sqrt{\frac{2}{3}} (\cosh^2 u_k \chi_i \bar{\partial}_\mu \delta y_0^a \partial^\mu y_1^a + \delta y_0^a \bar{\partial}_\mu \delta y_0^a \partial^\mu \chi_i).$$

(3.204)
Using the fact that $\delta y_a^i$ and $\chi_i$ are put on-shell in the calculation of the Witten diagram, we can rewrite the cubic coupling as

$$\frac{2\pi T_{D3}}{3} \sqrt{\frac{2}{3}} \chi_i \delta y_0^a (-\nabla^2 + 2)(\delta y^a)^i_i.$$ (3.205)

To compute the diagram, we need the bulk propagator $G_{ij'}^{ab'}(\tau, r; \tau', r')$ for $\delta y_a^i$, which satisfies the equation

$$(-\nabla^\mu \nabla_\mu + 2) G_{ij'}^{ab'} = \delta_{ij'} \delta^{ab'} \frac{3r^2}{2\pi T_{D3} \sinh 2u_k} \delta(\tau - \tau') \delta(r - r').$$ (3.206)

Due to the form of the cubic coupling (3.205), we see that the exchange diagram can be reduced to a contact diagram with the effective coupling

$$L_{exc, l}^{\text{eff}} = -\frac{\pi T_{D3}}{9} \sinh^2 u_k \tanh u_k, \quad L_{exc, l}^{\text{eff}} = -\frac{\pi T_{D3}}{9} \sinh^2 u_k \tanh u_k. \quad (3.207)$$

The four-point function Combining (3.203) and (3.207) with (3.200), we find that the four-point function can be computed from a single contact diagram with the effective quartic coupling

$$L_{xxyy}^{\text{eff}} = \frac{2\pi T_{D3}}{3} \tanh u_k \left( \frac{1}{4} \partial_\mu \chi_i \partial_\mu \chi_i \partial_\nu \delta y_0^a \partial_\nu \delta y_0^a - \frac{1}{2} \partial_\mu \chi_i \partial_\nu \chi_i \partial_\mu \delta y_0^a \partial_\nu \delta y_0^a \right).$$ (3.208)

The effective coupling has the same form as (3.103), which we have expected from the supersymmetry. It follows that the connected part of the normalized tree level four-point function is

$$\langle \chi_i(\tau_1) \chi_i(\tau_2) \delta y_0^a(\tau_3) \delta y_0^a(\tau_4) \rangle = \delta_{i_1 i_2} \delta^{a_1 a_2} \frac{1}{4\pi T_{D3} \sinh u_k \cosh^3 u_k} \frac{G_{2\chi y}(\chi)}{r_{12}^{1} r_{34}^{2}}.$$ (3.209)

We note that the prefactor in (3.209) also agrees with (3.195).

### 3.3.6 Four AdS$_5$ fluctuations of D3-brane

In this section, we compute the connected part of the tree level four-point function

$$\langle F_l^{i_1}(\tau_1) F_l^{i_2}(\tau_2) F_l^{i_3}(\tau_3) F_l^{i_4}(\tau_4) \rangle = \langle \chi^{i_1}(\tau_1) \chi^{i_2}(\tau_2) \chi^{i_3}(\tau_3) \chi^{i_4}(\tau_4) \rangle_{AdS_5}.$$ (3.210)
Figure 3.9: Witten diagrams to compute the connected part of $\langle \chi_i \chi_j \chi_k \chi_l \rangle$. Both $l = 0$ and $l = 2$ modes of $\delta u$ and $a_\mu$ fields are exchanged in the exchange diagrams.

The relevant quartic vertices from expanding the D3-brane action are:

$$L^{(4)}_{\chi \chi \chi \chi} = \frac{\pi T_{D3}}{1080} \tanh u_k \left[ (14 + \coth^2 u_k) \partial_\mu \chi_i \partial^\mu \chi_i \partial_\nu \chi_j \partial^\nu \chi_j ight. \\
- (22 + 8 \coth^2 u_k) \partial_\mu \chi_i \partial_\nu \chi_i \partial^\mu \chi_j \partial^\nu \chi_j + \frac{(-15 + 5 \cosh 2u_k)}{\sinh^2 u_k} \chi_i \chi_i \partial_\mu \chi_j \partial^\mu \chi_j \\
+ \frac{(-18 + 10 \cosh 2u_k)}{\sinh^2 u_k} \chi_i \chi_i \chi_j \chi_j \right], \tag{3.211}$$

which leads to the contact diagram in Figure 3.9. The exchange diagrams in Figure 3.9 involves the exchange of $l = 0$ and $l = 2$ particles.

**Exchange of $l = 0$ modes** The cubic couplings involved are (3.201) and (3.202). As before, since the external $\chi_i$ in the calculation of the exchange diagram is put on-shell, we can use the equations of motion for $\chi_i$ and integration by parts to rewrite the cubic couplings as

$$\frac{\pi T_{D3}}{18} \left[ (\cosh 2u_k + 3) \chi_i^2 (-\nabla^2 \delta u_0) + \frac{2i}{\sinh u_k} (\partial_\mu \chi_i \partial^\mu \chi_i + 2 \chi_i^2) f_0 \right]. \tag{3.212}$$
It follows that the exchange diagrams can be reduced to a contact diagram with the effective coupling

\[
L_{\text{exc}, l=0} = \frac{\pi T_{D3}}{45 \sinh 2u_k} \left[ (3 + \cosh 2u_k)^2 (\chi_i \chi_j \partial_\mu \chi_j \partial_\mu \chi_j + 2 \chi_i \chi_j \chi_j \chi_j) + 2 \left( \partial_\mu \chi_i \partial_\mu \chi_i \partial_\mu \chi_j \chi_j + 4 \chi_i \chi_i \partial_\mu \chi_j \chi_j + 4 \chi_i \chi_i \chi_j \chi_j \right) \right].
\]  

(3.213)

Exchange of \( l = 2 \) modes  In this case, the cubic couplings involved are

\[
L_{\chi \chi u_2} = \frac{2 \pi T_{D3}}{45} \left[ 2 \cosh 2u_k (\chi_i \partial_\mu \chi_j + \partial_\mu \chi_i \partial_\mu \chi_j) \partial_\mu \delta u_{ij} + (5 \cosh 2u_k - 3) \chi_i \chi_j \delta u_{ij} \right] + (\cosh 2u_k + 3) \partial_\mu \chi_i \partial_\mu \chi_j \delta u_{ij},
\]

\[
L_{\chi \chi f_2} = -\frac{2i \pi T_{D3}}{45 \sinh u_k} \left[ 6 \varepsilon^{\mu \nu} \chi_i \partial_\mu \chi_j (\alpha_\nu)_{ij} + (\chi_i \chi_j + 2 \partial_\mu \chi_i \partial_\mu \chi_j) f_{ij} \right].
\]  

(3.214)

Using integration by parts and the on-shellness of the external \( \chi_i \) in the calculation of the diagram, the cubic couplings can be expressed as

\[
\frac{\pi T_{D3}}{45} (\cosh 2u_k - 3) \left[ \chi_i \chi_j (-\nabla^2 + 6) \delta u_{ij} - \frac{4i}{\sinh u_k} \chi_i \partial_\mu \chi_j (-\nabla^\mu f_{ij} + 6 \varepsilon^{\mu \nu} (\alpha_\nu)_{ij}) \right].
\]  

(3.215)

To compute the exchange diagram, we also need the bulk propagator \( G^{ij;ij'}_{\mu \nu}(\tau, r; \tau', r') \) for \( \delta u_{ij} \) and \( G^{ij;ij'}_{\mu \nu}(\tau, r; \tau', r') \) for \( (\alpha_\mu)_{ij} \), which satisfy the equations

\[
(\nabla^\mu \nabla_\mu + 6) G^{ij;ij'}_{\mu \nu} = \frac{15 r^2 M^{ij;ij'}_{\mu \nu}}{4 \pi T_{D3} \sinh 2u_k} \delta^2 (\tau, r; \tau', r'),
\]

\[
-\nabla^\mu (\varepsilon^{\alpha \beta} \partial_\alpha G^{ij;ij'}_{\beta \gamma}) + 6 \varepsilon^{\mu \beta} G^{ij;ij'}_{\mu \gamma} = \frac{15 \tanh u_k r^2}{8 \pi T_{D3}} \varepsilon^{\mu \gamma} M^{ij;ij'} \delta^2 (\tau, r; \tau', r'),
\]  

(3.216)

(3.217)

where \( M^{ij;ij'} \) is defined as

\[
M^{ij;ij'} = \frac{1}{2} \left( \delta^{ii'} \delta^{jj'} + \delta^{ij'} \delta^{ji} - \frac{2}{3} \delta^{ij} \delta^{jj'} \right).
\]  

(3.218)

From the form of the cubic coupling (3.215), we see that the exchange diagram can be again reduced to a contact diagram with the effective quartic coupling

\[
L_{\text{exc}, l=2} = \frac{\pi T_{D3}}{648 \sinh 2u_k} \left[ -(\cosh 2u_k - 3)^2 (\chi_i \chi_i \partial_\mu \chi_j \partial_\mu \chi_j + 2 \chi_i \chi_j \chi_j \chi_j) + \frac{48}{5} \partial_\mu \chi_i \partial_\mu \chi_j \partial_\mu \chi_j \partial_\mu \chi_j - \frac{16}{5} \partial_\mu \chi_i \partial_\mu \chi_i \partial_\mu \chi_j \partial_\mu \chi_j \right.

\[
- 8 \chi_i \chi_i \partial_\mu \chi_j \partial_\mu \chi_j - \frac{112}{5} \chi_i \chi_j \chi_j \chi_j \right].
\]  

(3.219)
The four-point function Combining (3.213) and (3.219) with (3.211), we find that the four-point function can be computed from a single contact diagram with the effective quartic coupling

\[ L_{\chi\chi\chi\chi}^{\text{eff}} = \frac{\pi T_{D3}}{9} \left( \frac{1}{8} \partial^\mu \chi_i \partial^\nu \chi_i \partial^\rho \chi_j \partial^\sigma \chi_j - \frac{1}{4} \partial^\mu \chi_i \partial^\nu \chi_i \partial^\rho \chi_j \partial^\sigma \chi_j 
\right. 
\left. + \frac{1}{4} \chi_i \chi_i \partial^\mu \chi_j + \frac{1}{2} \chi_i \chi_i \chi_j \chi_j \right), \tag{3.220} \]

which has the same form as (3.94). This agrees again with our expectation from the supersymmetry. It follows that the connected part of the normalized four-point function is

\[ \langle \chi^{i_1}(\tau_1)\chi^{i_2}(\tau_2)\chi^{i_3}(\tau_3)\chi^{i_4}(\tau_4) \rangle = \frac{1}{4\pi T_{D3} \sinh u_k \cosh^4 u_k} \frac{G_4^{i_1i_2i_3i_4}(\chi)}{\tau_1^2 \tau_2^2 \tau_3^2 \tau_4^2}, \tag{3.221} \]

with the same prefactor as in (3.195) and (3.209).

3.4 Conclusion

In this chapter, we have studied the correlation functions of insertions on the 1/2-BPS Wilson loop in \( \mathcal{N} = 4 \) SYM. In particular we have focused on the Giant Wilson loops—the Wilson loops in large-rank symmetric or antisymmetric representations whose sizes are of order \( N \). We have computed the correlation functions using the dual description in terms of D-branes. Both for the antisymmetric and the symmetric representations, we computed the four-point functions of elementary fluctuations on the D-brane, which are dual to either the displacement operators or the single scalar insertions on the Wilson loop. For the Wilson loops in the antisymmetric representations that are dual to the D5-branes on \( \text{AdS}_2 \times S^4 \), we also computed a set of correlation functions involving the Kaluza-Klein modes coming from the reduction of the \( S^4 \) worldvolume. In a special supersymmetric configuration, these correlators reproduce the results of supersymmetric localization [54], providing nontrivial evidence for the holographic duality.

It would be interesting to consider the Wilson loops in even larger representations; namely the representations whose sizes are of order \( N^2 \). Such Wilson loops are known to be dual to so-called bubbling geometries [108–110]. In this case, the insertions on the Wilson loop are expected to be described by supergravity states propagating in such geometries. It would be interesting to make this statement precise by computing the defect CFT correlators both in the gauge theory and in supergravity.
In this chapter, we study the effective actions of both a massive scalar field and a massive Dirac spinor field in the global coordinates of $dS_d$ space using the in-/out-state formalism. As reviewed in Section 1.3, the metric of $dS_d$ space in terms of the global coordinates is

$$ds^2 = -dt^2 + \cosh^2(t)d\Omega_{d-1}^2,$$  \hspace{0.5cm} (4.1)
cut-off $T$, we expect the structure of the effective action $W$ to be as

$$W \sim c_1 \int d^{d-1}x \sqrt{\tilde{g}} + c_2 \int d^{d-1}x \sqrt{\tilde{g}} \tilde{R} + \cdots + W_{finite}$$  \hspace{1cm} (4.2)

for $d$ even, and as

$$W \sim c_1 \int d^{d-1}x \sqrt{\tilde{g}} + c_2 \int d^{d-1}x \sqrt{\tilde{g}} \tilde{R} + \cdots + c \log \tilde{R} + W_{finite}$$  \hspace{1cm} (4.3)

for $d$ odd. In (4.2) and (4.3), $\tilde{g}$, $\tilde{R}$, and $\tilde{R}$ are the determinant of the metric, the scalar curvature, and the radius of the spatial slice at the cut-off time $T$ respectively. For odd $d$, due to the presence of the logarithmically divergent term, the finite piece $W_{finite}$ is ambiguous. Using dimensional regularization, we will compute the finite term for even $d$ and the coefficient of the logarithmically divergent term for odd $d$. It turns out that the regularized in-out vacuum amplitude has the same expression as the ratio of determinants associated with different quantizations in AdS space. The computation of such ratio of determinants is related to the double-trace deformation in AdS/CFT correspondence [112–116].

The contents of this chapter are organized as follows. In Section 4.1, we briefly review the calculation of the effective action using the in-/out-state formalism. Section 4.2 contains the calculation for a massive real scalar field and Section 4.3 contains the calculation for a massive Dirac spinor field. In Section 4.4 we focus on the imaginary part of the effective action and obtain the particle production rate in even dimensions. In Section 4.5 we use dimensional regularization to obtain a closed-form expression for the vacuum amplitude. For even $d$ we get a finite answer, while for odd $d$ we compute the coefficient of the logarithmically divergent term. We further show that this expression equals to the ratio of determinants associated with different quantizations in AdS space in both scalar and spinor cases. In the concluding section we briefly discuss the possible connections between these two quantities.

4.1 In-/out- state formalism

In this section, we briefly review the in-/out-state formalism for calculating the effective action [117–119]. The effective action $W$ is defined by the scattering amplitude

$$Z_{in/out} = e^{iW} = \langle 0, \text{out} | 0, \text{in} \rangle, \quad W = \int d^d x \sqrt{-g} \mathcal{L}_{eff},$$  \hspace{1cm} (4.4)
where $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are the in-going and the out-going vacuum states respectively. Specifically, for a general massive field $\Phi$, we can expand it in terms of either in-going modes or out-going modes over the set of quantum numbers $\lambda$ as

$$\Phi = \sum_\lambda a_{\lambda,\text{in}} \Phi_{\lambda^+} + b_{\lambda,\text{in}}^\dagger \Phi_{\lambda^-} = \sum_\lambda a_{\lambda,\text{out}} \Phi_{\lambda^+} + b_{\lambda,\text{out}}^\dagger \Phi_{\lambda^-}. \quad (4.5)$$

Here $\Phi_{\lambda^+}$ and $\Phi_{\lambda^-}$ are the positive-frequency and negative-frequency in-going modes while $\Phi_{\lambda^+}$ and $\Phi_{\lambda^-}$ are the positive-frequency and negative-frequency out-going modes. In terms of the asymptotic behavior, we have

$$\Phi_{\lambda^\pm} \sim e^{\pm i\mu t} \quad \text{as} \quad t \to -\infty, \quad (4.6)$$
$$\Phi_{\lambda^\pm} \sim e^{\pm i\mu t} \quad \text{as} \quad t \to +\infty, \quad (4.7)$$

where $\mu$ is some effective mass. Then $|0, \text{in}\rangle$ is the state annihilated by $a_{\lambda,\text{in}}$ and $b_{\lambda,\text{in}}$ for each $\lambda$ while $|0, \text{out}\rangle$ is the state annihilated by $a_{\lambda,\text{out}}$ and $b_{\lambda,\text{out}}$.

For a scalar field $\Phi$, the in-going modes $\Phi_{\lambda^\pm}$ and the out-going modes $\Phi_{\lambda^\pm}$ are related by the Bogoliubov transformation [43,118]:

$$\Phi_{\lambda^+} = \mu_\lambda \Phi_{\lambda^+} + \nu_\lambda \Phi_{\lambda^-},$$
$$\Phi_{\lambda^-} = \nu_\lambda^* \Phi_{\lambda^+} + \mu_\lambda^* \Phi_{\lambda^-}. \quad (4.8)$$

with the coefficients $\mu_\lambda$ and $\nu_\lambda$ satisfying the Bogoliubov relation

$$|\mu_\lambda|^2 - |\nu_\lambda|^2 = 1, \quad (4.9)$$

as required by the commutation relations for bosons. We emphasize here that we have assumed the transformation (4.8) is diagonal in $\lambda$, which is true in the case studied. Similarly for a spinor field $\Psi$, the in-going modes $\Psi_{\lambda^\pm}$ and the out-going modes $\Psi_{\lambda^\pm}$ are related by the transformation [118]

$$\Psi_{\lambda^+} = \mu_\lambda \Psi_{\lambda^+} + \nu_\lambda \Psi_{\lambda^-},$$
$$\Psi_{\lambda^-} = -\nu_\lambda^* \Psi_{\lambda^+} + \mu_\lambda^* \Psi_{\lambda^-}. \quad (4.10)$$
The coefficients $\mu_\lambda$ and $\nu_\lambda$ now satisfy the relation

$$|\mu_\lambda|^2 + |\nu_\lambda|^2 = 1,$$

(4.11)
as required by the commutation relations for fermions. Again we have assumed the transformation (4.10) is diagonal in $\lambda$, as this is the case of interest.

In the in-/out-state formalism, the exact one-loop effective action $W$ can be then expressed in terms of the Bogoliubov coefficients $\mu_k$ as

$$W = \int d^d x \sqrt{-g} \mathcal{L}_{\text{eff}} = \sum_\lambda d(\lambda) W_\lambda,$$

(4.12)
where $d(\lambda)$ is the degeneracy and $W_\lambda$ is given by

$$W_\lambda = \begin{cases} 
  i \ln \mu_\lambda^* & \text{(complex scalar),} \\
  -i \ln \mu_\lambda & \text{(Dirac spinor).}
\end{cases}$$

(4.13)

### 4.2 Massive scalar field in global de Sitter space

In this section, we consider a real scalar field $\Phi$ with mass $M$ in the global patch of $dS_d$ space. We briefly review the calculation here as it has been done in various papers [43,45–48,51]. The action of the scalar field is

$$-\frac{1}{2} \int d^d x \sqrt{-g} \left( \partial_\mu \Phi \partial^\mu \Phi + M^2 \Phi^2 \right).$$

(4.14)
From the action, we have that the Klein-Gordon equation for $\Phi$ is

$$\left[ -\frac{1}{\cosh^{d-1}(t)} \partial_t \left( \cosh^{d-1}(t) \partial_t \right) + \frac{1}{\cosh^2(t)} \nabla^2_{\Omega_{d-1}} - M^2 \right] \Phi = 0$$

(4.15)
where $\nabla^2_{\Omega_{d-1}}$ is the Laplacian on the unit $d-1$ sphere. To solve the equation, we expand $\Phi$ using the real spherical harmonics $Y_l(\Omega_{d-1})$ which satisfies

$$\nabla^2_{\Omega_{d-1}} Y_l(\Omega_{d-1}) = -l(l + d - 2) Y_l(\Omega_{d-1}),$$

(4.16)
with degeneracies:

$$D_{(d-1)}(l) = \frac{(l + d - 3)!}{l!(d-2)!} (2l + d - 2), \quad (l = 0, 1, \ldots).$$

(4.17)

1 For a real massive scalar, there is a factor of $\frac{1}{2}$ as a complex scalar field can be viewed as two real scalar fields.
Using the expansion \( \Phi(t, \Omega_{d-1}) = \sum_l \phi_l(t)Y_l(\Omega_{d-1}) \), we find that for each mode \( l \) the function \( \phi_l(t) \) satisfies the equation:

\[
\left( \partial_t^2 + (d - 1) \tanh(t) \partial_t + \frac{l(l + d - 2)}{\cosh^2(t)} + M^2 \right) \phi_l(t) = 0.
\] (4.18)

Following [43,45,51], we could write the two independent solutions as either

\[
\phi_l(\pm)(t) = \cosh(t) \exp \left[ \left( -l - \frac{d - 1}{2} \mp i\mu \right) t \right]
\]

\[
F\left( l + \frac{d - 1}{2}, l + \frac{d - 1}{2} \pm i\mu, 1 \pm i\mu, -e^{-2t} \right),
\]

or

\[
\phi_l(\pm)(t) = \cosh(t) \exp \left[ \left( l + \frac{d - 1}{2} \mp i\mu \right) t \right]
\]

\[
F\left( l + \frac{d - 1}{2}, l + \frac{d - 1}{2} \mp i\mu, 1 \mp i\mu, -e^{-2t} \right),
\]

where \( F \) is the hypergeometric function \( {}_2F_1 \) and \( \mu = \sqrt{M^2 - \frac{(d-1)^2}{4}} \). Here we have restricted our attention to the case \( M^2 > (d - 1)^2/4 \), where the solution oscillates at the past and the future infinity. The two solutions (4.19) have the asymptotic behaviors as

\[
\phi_l(\pm)(t) \to \exp \left( -\frac{d - 1}{2} t \mp i\mu t \right) \text{ as } t \to +\infty,
\]

while the two solutions (4.20) have the asymptotic behaviors as

\[
\phi_l(\pm)(t) \to \exp \left( \frac{d - 1}{2} t \mp i\mu t \right) \text{ as } t \to -\infty.
\]

Therefore, we could identify the two solutions (4.19) as the positive/negative-frequency out-modes while the two solutions (4.20) as the positive/negative-frequency in-modes. The in-modes and the out-modes are related by the Bogoliubov transformation (4.8) as

\[
\phi_l^+ = \mu_l \phi_l^+ + \nu_l \phi_l^-,
\]

\[
\phi_l^- = \nu_l^* \phi_l^+ + \mu_l^* \phi_l^-,
\]

\[
(4.23)
\]

\[\text{The solutions for } M^2 < (d - 1)^2/4 \text{ can be obtained by analytic continuation in } \gamma. \text{ In this case, the modes do not oscillate and can be interpreted as the source of the operator in the dual boundary CFT [120].}\]
where we have suppressed the dependence on \( t \). Using the transformation formula for the hypergeometric function [121], we find the Bogoliubov coefficients to be

\[
\mu_l = \frac{\Gamma(1 - i\mu)\Gamma(-i\mu)}{\Gamma(l + \frac{d-1}{2} - i\mu)\Gamma(-l - \frac{d-3}{2} - i\mu)}, \quad (l = 0, 1, \ldots),
\]

\[
\nu_l = \frac{i\cos(l\pi + \frac{d}{2}\pi)}{\sinh(\pi\mu)},
\]

with degeneracies \( D_{(d-1)}(l) \). One can check that these coefficients indeed obey the relation

\[
|\mu_l|^2 - |\nu_l|^2 = 1,
\]

as required by the commutation rules. In particular, \( \nu_l = 0 \) when \( d \) is odd, which implies that \( |0, \text{in}\rangle \) and \( |0, \text{out}\rangle \) define the same state.

### 4.3 Massive Dirac spinor field in global de Sitter space

In this section, we consider a massive Dirac spinor field \( \Psi \) with mass \( M \) in the global patch of \( dS_d \) space. The action for the Dirac field \( \Psi \) is

\[
- \int d^d x \bar{\Psi} (\nabla + M) \Psi,
\]

where \( \nabla \equiv \gamma^a (e_a)^\mu \nabla_\mu \). This leads to the Dirac equation

\[
\gamma^a (e_a)^\mu \left( \partial_\mu - \frac{1}{8} \omega_{\mu bc} [\gamma^b, \gamma^c] \right) \Psi + M \Psi = 0.
\]

Here \( \gamma^a \) for \( (a = 0, \ldots, d - 1) \) are the gamma matrices which satisfy the Dirac algebra

\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \mathbf{1}
\]

with \( \eta^{ab} \) of the signature \((-+, \ldots, +)\). The \( \{e_a\} \) is a vielbein on \( dS_d \) and the spin connection \( \omega_{abc} \) is defined as

\[
\omega_{abc} = (e_a)^\mu \left[ \partial_\mu (e_b)^\nu + \Gamma^\nu_{\mu \gamma} (e_b)^\gamma \right] (e_c)^\nu,
\]

where \( \{\Gamma^\nu_{\mu \gamma}\} \) is the Christoffel symbols of the Levi-Civita connection for the metric (1.24). We follow the method used in [122] to solve the Dirac equation. If we let \( \{\bar{e}_i\} \) be a vielbein on the \( S^{d-1} \),
then we could define \{e_a\} as
\[
e_0 = \partial_t, \quad e_j = \frac{1}{\cosh(t)} \tilde{e}_j, \quad (j = 1, \ldots, d - 1) \quad (4.30)
\]
The only non-zero components of the spin connection \(w_{abc}\) are
\[
\omega_{ijk} = \frac{1}{\cosh(t)} \tilde{\omega}_{ijk},
\omega_{i0k} = -\omega_{ik0} = \tanh(t) \delta_{ik}, \quad (i, j, k = 1, \ldots, d - 1), \quad (4.31)
\]
where \(\tilde{\omega}_{ijk}\) is the spin connection on \(S^{d-1}\) corresponding to the frame \{\tilde{e}_i\}.

Since the construction of the representations for Clifford algebra in even and odd dimensions is slightly different, we shall discuss the two cases separately below. Our construction of the representations of the Clifford algebra (4.28) follows [122].

### 4.3.1 Even dimension

We construct the gamma matrices satisfying (4.28) in the following way: We let \{\gamma^a\} be the set of \(d\) matrices of dimension \(2^{d/2}\) defined by
\[
\gamma^0 = \begin{pmatrix} 0 & i \mathbf{1} \\ i \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & i \tilde{\Gamma}^j \\ -i \tilde{\Gamma}^j & 0 \end{pmatrix}, \quad (j = 1, \ldots, d - 1) \quad (4.32)
\]
where \(\mathbf{1}\) is the identity matrix of dimension \(2^{d/2-1}\) and the \(d - 1\) matrices \(\tilde{\Gamma}^j\) also of dimension \(2^{d/2-1}\) satisfy the following Clifford algebra:
\[
\tilde{\Gamma}^j \tilde{\Gamma}^k + \tilde{\Gamma}^k \tilde{\Gamma}^j = 2 \delta^{jk} \mathbf{1}, \quad (j, k = 1, \ldots, d - 1). \quad (4.33)
\]

With the representations of the gamma matrices defined in (4.32) and (4.31), the Dirac equation (4.27) becomes
\[
\gamma^0 \left( \partial_t + \frac{d - 1}{2} \tanh(t) \right) \Psi + \frac{1}{\cosh(t)} \begin{pmatrix} 0 & i \tilde{\nabla} \\ -i \tilde{\nabla} & 0 \end{pmatrix} \Psi + M \Psi = 0 \quad (4.34)
\]
where $\hat{\nabla}$ is the Dirac operator on $S^{d-1}$. If we represent $\Psi$ with two components given by

$$
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
$$

(4.35)

then the Dirac equation (4.34) decomposes to the following set of equations:

$$
i \left( \partial_t + \frac{d-1}{2} \tanh(t) + \frac{1}{\cosh(t)} \hat{\nabla} \right) \psi_2 + M \psi_1 = 0,
$$

$$
i \left( \partial_t + \frac{d-1}{2} \tanh(t) - \frac{1}{\cosh(t)} \hat{\nabla} \right) \psi_1 + M \psi_2 = 0.
$$

(4.36)

By eliminating either $\psi_1$ or $\psi_2$ in (4.36), we obtain

$$\psi''_1 + (d-1) \tanh(t) \psi'_1 + \left[ M^2 + \frac{(d-1)^2}{4} - \frac{(d-1)(d-3)}{4 \cosh^2(t)} \right] \psi_1
- \text{sech}^2(t) \hat{\nabla}^2 \psi_1 + \frac{\tanh(t)}{\cosh(t)} \hat{\nabla} \psi_1 = 0,$n

(4.37)

$$\psi''_2 + (d-1) \tanh(t) \psi'_2 + \left[ M^2 + \frac{(d-1)^2}{4} - \frac{(d-1)(d-3)}{4 \cosh^2(t)} \right] \psi_2
- \text{sech}^2(t) \hat{\nabla}^2 \psi_2 - \frac{\tanh(t)}{\cosh(t)} \hat{\nabla} \psi_2 = 0,$n

(4.38)

where prime denotes derivatives with respect to $t$. We only need to solve the equation for $\psi_1$ and the solution for $\psi_2$ could be obtained from (4.36). Using the eigenfunctions of the Dirac operator $\hat{\nabla}$ on $S^{d-1}$ for $d$ even which are defined by [122]

$$
\hat{\nabla} \chi^{(\pm)}_{lm}(\Omega_{d-1}) = \pm i \left( l + \frac{d-1}{2} \right) \chi^{(\pm)}_{lm}(\Omega_{d-1}), \quad (l = 0, 1, \ldots),
$$

(4.39)

with degeneracies given by

$$
D^{(\pm)}_{d-1}(l) = \frac{2^{(d-2)/2} (d+l-2)!}{l! (d-2)!}, \quad \text{for even } d.
$$

(4.40)

we can separate variables by considering the expansion

$$
\psi_1(t, \Omega_{d-1}) = \sum_{l,m} \phi_i(t) \chi^{(\pm)}_{lm}(\Omega_{d-1}) + \varphi_i(t) \chi^{(-)}_{lm}(\Omega_{d-1}).
$$

(4.41)
Inserting the expansion (4.41) into (4.37) and using (4.39), we obtain the equations for \( \phi_l \) and \( \varphi_l \) as

\[
\phi'_l + (d - 1) \tanh(t) \phi'_l + \left[ M^2 + \frac{(d - 1)^2}{4} + \frac{(d - 1)(2l + 1) + 2l^2}{2 \cosh^2(t)} + \frac{i(d - 1 + 2l) \tanh(t)}{2 \cosh(t)} \right] \phi_l = 0.
\]

(4.42)

\[
\varphi'_l + (d - 1) \tanh(t) \varphi'_l + \left[ M^2 + \frac{(d - 1)^2}{4} + \frac{(d - 1)(2l + 1) + 2l^2}{2 \cosh^2(t)} - \frac{i(d - 1 + 2l) \tanh(t)}{2 \cosh(t)} \right] \varphi_l = 0.
\]

(4.43)

To solve equations (4.42) and (4.43), we perform a change of variable to \( z = \text{i} \sinh(t) \) and consider the following ansatz:

\[
\phi_l(z) = (1 + z)^{l/2}(1 - z)^{(l+1)/2} g_\phi(z),
\]
\[
\varphi_l(z) = (1 + z)^{(l+1)/2}(1 - z)^{l/2} g_\varphi(z).
\]

(4.44)

Plugging the ansatz, we find the two independent solutions for \( g_\phi \) and \( g_\varphi \) as

\[
g_{\phi 1}(z) = \left(1 - \frac{z}{2}\right)^{-d/2-l} \hat{F} \left(-iM, iM, 1 - \frac{d}{2} - l; \frac{1 - z}{2}\right),
\]

(4.45)

\[
g_{\phi 2}(z) = \hat{F} \left(\frac{d}{2} - iM + l, \frac{d}{2} + iM + l, 1 + \frac{d}{2} + l; \frac{1 - z}{2}\right),
\]

(4.46)

\[
g_{\varphi 1}(z) = \left(1 - \frac{z}{2}\right)^{-d/2-l+1} \hat{F} \left(1 - iM, 1 + iM, 2 - \frac{d}{2} - l; \frac{1 - z}{2}\right),
\]

(4.47)

\[
g_{\varphi 2}(z) = \hat{F} \left(\frac{d}{2} - iM + l, \frac{d}{2} + iM + l, \frac{d}{2} + l; \frac{1 - z}{2}\right),
\]

(4.48)

where for convenience we have defined the rescaled hypergeometric function \( \hat{F} \) as

\[
\hat{F} \left(a, b, c; x\right) = \frac{\Gamma(c - b)}{\Gamma(1 - b)\Gamma(c)} \, _2F_1 \left(a, b, c, x\right).
\]

(4.49)
We can use the equation (4.36) to find the corresponding solutions for \( \psi_2 \) component. The general solution for \( \Psi \) can be written as:

\[
\Psi = \sum_{l,m} C'_{lm} \begin{pmatrix}
\phi_1(t) \chi^+_{lm} (\Omega_{d-1}) \\
-i\phi_1(t) \chi^+_{lm} (\Omega_{d-1})
\end{pmatrix} + D_{lm} \begin{pmatrix}
\phi_2(t) \chi^+_{lm} (\Omega_{d-1}) \\
+i\phi_2(t) \chi^+_{lm} (\Omega_{d-1})
\end{pmatrix} \\
+ \sum_{l,m} C''_{lm} \begin{pmatrix}
\varphi_1(t) \chi^-_{lm} (\Omega_{d-1}) \\
i\varphi_1(t) \chi^-_{lm} (\Omega_{d-1})
\end{pmatrix} + D''_{lm} \begin{pmatrix}
\varphi_2(t) \chi^-_{lm} (\Omega_{d-1}) \\
i\varphi_2(t) \chi^-_{lm} (\Omega_{d-1})
\end{pmatrix},
\]  

(4.50)

where \( C(C') \) and \( D(D') \) are arbitrary constants.

By examining the asymptotic behaviors of the general solution (4.50), we could identify the positive-/negative-frequency in-modes as the modes with the following asymptotic behaviors

\[
\Psi^+, t \sim e^{\left(\frac{d-1}{2} - iM\right)t} \begin{pmatrix}
\chi^+_{lm} (\Omega_{d-1}) \\
-\chi^+_{lm} (\Omega_{d-1})
\end{pmatrix} \quad \text{or} \quad e^{\left(\frac{d-1}{2} + iM\right)t} \begin{pmatrix}
\chi^-_{lm} (\Omega_{d-1}) \\
-\chi^-_{lm} (\Omega_{d-1})
\end{pmatrix},
\]

(4.51)

at the past infinity while the positive/negative-frequency out-modes as the modes with the asymptotic behaviors

\[
\Psi^-, t \sim e^{\left(\frac{d-1}{2} - iM\right)t} \begin{pmatrix}
\chi^-_{lm} (\Omega_{d-1}) \\
\chi^+_{lm} (\Omega_{d-1})
\end{pmatrix} \quad \text{or} \quad e^{\left(\frac{d-1}{2} + iM\right)t} \begin{pmatrix}
\chi^-_{lm} (\Omega_{d-1}) \\
\chi^+_{lm} (\Omega_{d-1})
\end{pmatrix} \quad \text{as} \quad t \to -\infty,
\]

(4.52)

at the future infinity. In terms of these in-/out-modes, the Bogoliubov transformation (4.10) is
expressed as

\[ \Psi_l^+ = \mu_l \Psi_l^+ + \nu_l \Psi_l^- , \]
\[ \Psi_l^- = -\nu_l^* \Psi_l^+ + \mu_l^* \Psi_l^- , \] (4.53)

and we find the corresponding Bogoliubov coefficients to be

\[ \mu_l = \frac{\Gamma(\frac{1}{2} - iM)^2}{\Gamma(1 - l - \frac{d}{2} - iM)\Gamma(\frac{d}{2} + l - iM)} , \quad (l = 0, 1, \ldots) \]
\[ \nu_l = \mp i \cos(l\pi + \frac{d}{2}\pi) \frac{\cosh(\pi M)}{\cosh(\pi M)} , \] (4.54)

where (−) sign is taken for the modes with \( \chi_{lm}^+ \) components while the (+) sign is taken for the modes with \( \chi_{lm}^- \) components. One can verify that these coefficients satisfy the relation

\[ |\mu_l|^2 + |\nu_l|^2 = 1 . \] (4.55)

as required by the commutation rules. The degeneracy for each mode \( l \) is

\[ D_{d-1}(l) = \frac{2\frac{d}{2}(d + l - 2)!}{l!(d - 2)!} , \quad \text{for even } d . \] (4.56)

### 4.3.2 Odd dimension \( (d \geq 3) \)

In this case the dimension of the gamma matrices is \( 2^{(d-1)/2} \), same as the \( d - 1 \) dimension representation. If we let \( \{\tilde{\Gamma}^i\} \) be the set of \( d - 1 \) matrices of dimension \( 2^{(d-1)/2} \) which satisfies the Clifford algebra (4.33), then the set of matrices

\[ \gamma^0 = \begin{pmatrix} i1 & 0 \\ 0 & -i1 \end{pmatrix} , \quad \gamma^j = \tilde{\Gamma}^j , \quad (j = 1, \ldots, d - 1) \] (4.57)

satisfies the Dirac algebra (4.28). Using the representations of the gamma matrices (4.57) and (4.31), the Dirac equation (4.27) becomes

\[ \gamma^0 \left( \partial_t + \frac{d - 1}{2} \tanh(t) \right) \Psi + \frac{1}{\cosh(t)} \tilde{\nabla} \Psi + M \Psi = 0 , \] (4.58)
where \( \tilde{\nabla} \) is the Dirac operator on \( S^{d-1} \). Instead of solving (4.58) directly, it is easier for us to act with the operator \( \tilde{\nabla} - M \) on both sides and solve the following equation instead

\[
- \left( \partial_t + \frac{d-1}{2} \tanh(t) \right)^2 \Psi - \frac{\tanh(t)}{\cosh(t)} \gamma^0 \tilde{\nabla} \Psi + \frac{1}{\cosh^2(t)} \tilde{\nabla}^2 \Psi - M^2 \Psi = 0. \tag{4.59}
\]

The reason is that the operator \( \gamma^0 \tilde{\nabla} \) commutes with \( \tilde{\nabla}^2 \) while it is not true for \( \gamma^0 \) and \( \tilde{\nabla} \) in (4.58), which follows from

\[
\gamma^0 \tilde{\nabla} + \tilde{\nabla} \gamma^0 = 0. \tag{4.60}
\]

If we consider the eigenfunctions of the Dirac operator \( \tilde{\nabla} \) on \( S^{d-1} \) for \( d \) odd which satisfy \[122\]

\[
\tilde{\nabla} \chi_{lm}^- (\Omega_{d-1}) = -i \left( l + \frac{d-1}{2} \right) \chi_{lm}^- (\Omega_{d-1}), \quad (l = 0, 1, \ldots) \tag{4.61}
\]

with degeneracies

\[
D_d^{(-)}(l) = \frac{2^{(d-1)/2}(d + l - 2)!}{l!(d-2)!} \quad \text{for odd } d, \tag{4.62}
\]

then one can verify that the functions \( \chi_{lm}^+ = \gamma^0 \chi_{lm}^- \) are also the eigenfunctions of \( \tilde{\nabla} \) with eigenvalues

\[
\tilde{\nabla} \chi_{lm}^+ (\Omega_{d-1}) = i \left( l + \frac{d-1}{2} \right) \chi_{lm}^+ (\Omega_{d-1}), \quad (l = 0, 1, \ldots). \tag{4.63}
\]

Using \( \chi_{lm}^{(\pm)} \), we can construct the following functions

\[
\hat{\chi}_{lm}^- (\Omega_{d-1}) = \frac{1}{\sqrt{2}} \left[ \chi_{lm}^- (\Omega_{d-1}) + \chi_{lm}^+ (\Omega_{d-1}) \right], \tag{4.64}
\]

\[
\hat{\chi}_{lm}^+ (\Omega_{d-1}) = \gamma^0 \hat{\chi}_{lm}^- (\Omega_{d-1}), \tag{4.65}
\]

which are the common eigenfunctions of \( \gamma^0 \tilde{\nabla} \) and \( \tilde{\nabla}^2 \) with eigenvalues

\[
\tilde{\nabla}^2 \hat{\chi}_{lm}^{(\pm)} (\Omega_{d-1}) = - \left( l + \frac{d-1}{2} \right)^2 \hat{\chi}_{lm}^{(\pm)} (\Omega_{d-1}), \tag{4.66}
\]

\[
\gamma^0 \tilde{\nabla} \hat{\chi}_{lm}^{(\pm)} (\Omega_{d-1}) = \pm i \left( l + \frac{d-1}{2} \right) \hat{\chi}_{lm}^{(\pm)} (\Omega_{d-1}). \tag{4.67}
\]

Now we can expand \( \Psi \) in terms of those functions as

\[
\Psi(t, \Omega_{d-1}) = \sum_{l,m} \hat{\phi}_l(t) \hat{\chi}_{lm}^+ (\Omega_{d-1}) + \hat{\psi}_l(t) \hat{\chi}_{lm}^- (\Omega_{d-1}), \tag{4.68}
\]

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and substitute it into (4.59). As a result, we find that \( \hat{\phi}_l \) and \( \hat{\varphi}_l \) satisfy the same equations (4.42) and (4.43) for \( \phi_l \) and \( \varphi_l \) respectively. By further checking the Dirac equation (4.58), one find that the general solution for \( \Psi \) can be written as the ones satisfy

\[
\Psi = \sum_{l,m} C_{lm} [\phi_{l1}(t) \tilde{\chi}^+_l(\Omega_{d-1}) + \varphi_{l1}(t) \tilde{\chi}^-_l(\Omega_{d-1})] \\
+ D_{lm} [\phi_{l2}(t) \tilde{\chi}^+_l(\Omega_{d-1}) - \varphi_{l2}(t) \tilde{\chi}^-_l(\Omega_{d-1})],
\]

(4.69)

where \( C \) and \( D \) are arbitrary constants and \( \phi_l \) and \( \varphi_l \) are the same functions defined in the even \( d \) case. By examing the asymptotic behaviors of the general solution (4.69), we could identify the positive-/negative-frequency in-modes as those behave like

\[
\Psi^+_l \sim e^{\left(\frac{d-1}{2} - iM\right)t} \left[ \tilde{\chi}^+_l(\Omega_{d-1}) + i\tilde{\chi}^-_l(\Omega_{d-1}) \right], \\
\Psi^-_l \sim e^{\left(\frac{d-1}{2} + iM\right)t} \left[ \tilde{\chi}^+_l(\Omega_{d-1}) - i\tilde{\chi}^-_l(\Omega_{d-1}) \right] \quad \text{as} \quad t \to -\infty,
\]

(4.70)

at the past infinity while the positive-/negative-frequency out-modes as those behave like

\[
\Psi^{+}_l \sim e^{\left(-\frac{d-1}{2} - iM\right)t} \left[ \tilde{\chi}^+_l(\Omega_{d-1}) - i\tilde{\chi}^-_l(\Omega_{d-1}) \right], \\
\Psi^{-}_l \sim e^{\left(-\frac{d-1}{2} + iM\right)t} \left[ \tilde{\chi}^+_l(\Omega_{d-1}) + i\tilde{\chi}^-_l(\Omega_{d-1}) \right] \quad \text{as} \quad t \to +\infty,
\]

(4.71)

at the future infinity. In terms of these in-/out-modes and the relation (4.53), we find the corresponding Bogoliubov coefficients to be

\[
\mu_l = \frac{\Gamma\left(\frac{1}{2} - iM\right)^2}{\Gamma(1 - l - \frac{d}{2} - iM)\Gamma(\frac{d}{2} + l - iM)}, \quad (l = 0, 1, \ldots) \\
\nu_l = \frac{i \cos(l\pi + \frac{d}{2}\pi)}{\cosh(\pi M)} = 0 \quad \text{for odd } d,
\]

(4.72)

which has the same expression as in the even \( d \) case. The degeneracy for each mode \( l \) is

\[
D_{d-1}(l) = \frac{2^{(d-1)/2}(d + l - 2)!}{l!(d - 2)!} \quad \text{for odd } d.
\]

(4.73)

### 4.4 Particle production in even dimensions

Before proceeding to calculate the effective action, we could see there is a difference between even dimensions and odd dimensions. In [45] it has been shown that the Bogoliubov coefficient \( \nu_l \) vanishes.
when \( d \) is odd in the massive scalar case. From our calculation, we see that it is also true in the massive spinor case. This implies that the in-vacuum and the out-vacuum are the same state in odd dimensions, so there is no production of either scalar or spinor particles. In contrast, there is always particle production in even dimensions for both scalar and spinor field. The particle production rate per spacetime volume \( \mathcal{P} \) is related to the imaginary part of the effective action \( W \) by

\[
\mathcal{P} = \lim_{V_d \to \infty} \frac{2}{V_d} \text{Im} W, \tag{4.74}
\]

where \( V_d \) is the spacetime volume of \( dS_d \).

Using the formula (4.12) and the Bogoliubov coefficients calculated in Sections 4.2 and 4.3, we obtain the effective action for a massive real scalar

\[
W_b = \frac{i}{2} \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \ln \Gamma(1 + i\mu) + \ln \Gamma(i\mu) - \ln \pi + \ln \sin(-l\pi - \frac{d-3}{2}\pi + i\mu\pi) - \ln \Gamma \left( l + \frac{d-1}{2} + i\mu \right) + \ln \Gamma \left( l + \frac{d-1}{2} - i\mu \right) \right], \tag{4.75}
\]

and the effective action for a massive Dirac spinor

\[
W_f = -i \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ 2 \ln \Gamma \left( \frac{d}{2} + iM \right) - \ln \pi + \ln \sin(-l\pi - \frac{d-2}{2}\pi + iM\pi) - \ln \Gamma \left( l + \frac{d}{2} + iM \right) + \ln \Gamma \left( l + \frac{d}{2} - iM \right) \right]. \tag{4.76}
\]

In arriving at (4.75) and (4.76), we have used the following identity for the gamma function

\[
\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad (z \notin \mathbb{Z}). \tag{4.77}
\]

By using the integral representation for \( \ln \Gamma(z) \) [123]:

\[
\ln \Gamma(z) = \int_0^\infty \frac{ds}{s} \left[ \frac{e^{-zs} - e^{-s}}{1 - e^{-s}} + (z - 1)e^{-s} \right], \quad (\text{Re}(z) > 0), \tag{4.78}
\]

we obtain that for odd \( d \), the effective actions are

\[
W_b = \frac{1}{2} \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \int_0^{\infty} \frac{ds}{s} \frac{\sin(\mu s)}{\sinh(\frac{s}{2})} \left( e^{-\frac{s}{2}} - e^{(-\frac{d}{2}-l+1)s} \right) + \frac{(-1)^{\frac{d+1}{2}+l} - 1}{2} \pi \right], \tag{4.79}
\]

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\[ W_f = \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \int_0^{\infty} \frac{ds}{s} \sin(Ms) \sinh\left(\frac{s}{2}\right) \left( e^{-\frac{s}{2} - l + \frac{1}{2}}s - 1 \right) + \left( -1 \right)^{\frac{d-2}{2} + l} \frac{\pi}{2} \right], \quad (4.80) \]

while for even \( d \) the effective actions are

\[ W_b = \frac{1}{2} \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \int_0^{\infty} \frac{ds}{s} \sin(\mu s) \sinh\left(\frac{s}{2}\right) \left( e^{-\frac{s}{2}} - e^{(-\frac{s}{2} + l + 1)s} \right) + \left( -1 \right)^{\frac{d-2}{2} + l} \frac{\pi}{2} \right] + i \ln \coth(\pi \mu), \quad (4.81) \]

\[ W_f = \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \int_0^{\infty} \frac{ds}{s} \sin(Ms) \sinh\left(\frac{s}{2}\right) \left( e^{-\frac{s}{2} - l + \frac{1}{2}}s - 1 \right) + \left( -1 \right)^{\frac{d-2}{2} + l} \frac{\pi}{2} \right] + i \ln \coth(\pi M). \quad (4.82) \]

Due to the infinite summation over the angular quantum number \( l \), the scalar and spinor effective actions are divergent in both even and odd dimensions. However, in odd dimensions the effective action is pure real while in even dimensions it has an divergent imaginary part besides the divergent real part:

\[ \text{Im } W_b = \frac{1}{2} \sum_{l=0}^{\infty} D_{(d-1)}(l) \ln \coth(\pi \mu), \quad (4.83) \]

\[ \text{Im } W_f = \sum_{l=0}^{\infty} D_{(d-1)}(l) \ln \coth(\pi M). \quad (4.84) \]

Since the divergence of the imaginary part is proportional to the summation of the degeneracies for both scalar and spinor field, we could regularize it by introducing a cut-off \( N \gg 1 \) on the angular quantum number \( l \). To relate this cut-off \( N \) to the cut-off \( T \) on the time discussed at the beginning of this chapter, we use the method developed in [43,111]. This method is based on the analysis of the real time particle creation process, which requires that we evolve system from a finite initial time \(-T\) to a finite final time \( T\) and the \( T \to \infty \) limit is taken at the end. By examining the wave equation (4.18), (4.42) and (4.43), we see that the cut-off \( N \) corresponds to a cut-off on the physical momentum

\[ k_{\text{phys}} \sim \frac{N}{\cosh(T)} \quad (4.85) \]

at the time when the initial state is prepared. Although it seems that the cut-off \( N \) directly corre-

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\(^3\)Since the summation for the real part of the effective action is more complicated, we will use a more covariant approach to regularize the effective action in Section 4.5.
sponds to a UV cut-off on the physical momentum, this is not true since we also need to take the $T \to \infty$ limit in the end. In fact if we demand that the cut-off for $k_{\text{phys}}$ is fixed at the finite initial time when the state is prepared, we need to change the cut-off $N$ accordingly when taking the limit $T \to \infty$. Specifically, the change of $T$ by $\delta T$ requires a change of $N$ by

$$\delta N \approx N \delta T, \text{ or } N \approx e^T. \quad (4.86)$$

Therefore, the divergences in the summations (4.83) and (4.84) result from the IR divergence of the spacetime volume.

If we change the time cut-off $T$ by $\delta T$, then the spacetime volume changes by

$$\delta V \approx \frac{2\pi^{\frac{d}{2}}}{\Gamma(1 + \frac{d}{2})} \cosh^{d-1}(T) \delta T. \quad (4.87)$$

In the meantime, the cut-off $N$ needs to be changed by $\delta N$ as we have argued before. This results a change in $\text{Im} W$ by

$$\delta \text{Im} W_b \approx \frac{2 \ln \coth(\pi \mu)}{\Gamma(d - 1)} N^{d-2} \delta N, \quad (4.88)$$

$$\delta \text{Im} W_f \approx \frac{2^{\frac{d}{2}} \ln \coth(\pi M)}{\Gamma(d - 1)} N^{d-2} \delta N. \quad (4.89)$$

Using (4.86) and the definition of $\mathcal{P}$ in (4.74), we find that the particle production rate for a massive real scalar is

$$\mathcal{P}_b \approx \frac{2^{d-1} \Gamma(\frac{d}{2} + 1)}{d \pi^\frac{d}{2} \Gamma(d - 1)} \ln \coth(\pi \mu), \quad (d \text{ even}), \quad (4.90)$$

and the particle production rate for a massive Dirac spinor is

$$\mathcal{P}_f \approx \frac{2^{\frac{d}{2}-1} \Gamma(\frac{d}{2} + 1)}{d \pi^\frac{d}{2} \Gamma(d - 1)} \ln \coth(\pi M), \quad (d \text{ even}). \quad (4.91)$$

In the large-mass/weak-curvature limit ($M \gg H$), the particle production rate for both scalar and spinor fields goes like

$$\mathcal{P} \sim e^{-M/T_H}. \quad (4.92)$$

where $T_H = H/(2\pi)$ is the Hawking-de Sitter temperature. This agrees with the results calculated in [48,50] using the Green’s function method.
4.5 The vacuum amplitude and the ratio of determinants

In Section 4.4, we find that the expressions for the effective action (4.75) and (4.76) are divergent due to the infinite summation over the angular quantum number $l$. From the structures of divergence, i.e., (4.2) for even $d$ and (4.3) for odd $d$, we expect that the effective action contains a logarithmically divergent term in odd dimensions but no such term in even dimensions. Therefore, the finite piece of the effective action is unambiguous in even dimensions. In the language of dS/CFT, we expect that the coefficient of the logarithmically divergent term in odd dimensions might be connected to the conformal anomaly present in the boundary CFT [124–127].

In this section, we compute the finite term of the effective action in even dimensions and the coefficient of the logarithmically divergent term of the effective action in odd dimensions. We follow the method developed in [114] and use dimensional regularization\(^4\) to regularize the summation over $l$ in (4.75) and (4.76). In dimensional regularization, the logarithmically divergent term corresponds to the pole in $\epsilon$ [124], where we set the dimension $d = \text{integer} - \epsilon$. Along the way, we show that the regularized vacuum amplitude $Z_{\text{in/out}}^b$ in $dS_d$ has the same expression as the ratio of the functional determinants associated with different quantizations in $AdS_d$. The calculation of such ratio of the determinants have appeared in the study of double-trace deformation in AdS/CFT correspondence [112–116].

4.5.1 Real massive scalar

The vacuum amplitude $Z_{\text{in/out}}^b$ can be obtained from $W_b$ using the relation \( \log Z_{\text{in/out}}^b = iW_b \). Therefore, we have

\[
\log Z_{\text{in/out}}^b = -\frac{1}{2} \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \ln \Gamma(1 + i\mu) + \ln \Gamma(i\mu) - \ln \pi + \ln \sin(-l\pi - \frac{d-3}{2}\pi + i\mu\pi) \right.
\]

\[
\left. - \ln \Gamma \left( l + \frac{d-1}{2} + i\mu \right) + \ln \Gamma \left( l + \frac{d-1}{2} - i\mu \right) \right].
\]

For reasons that will become clear later, we define $\nu = i\mu = i\sqrt{M^2 - \frac{(d-1)^2}{4}}$ and consider the derivative of $\log Z_{\text{in/out}}^b$ with respect to $\nu$:

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b = \frac{1}{4\nu} \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \psi(l + \frac{d-1}{2} + \nu) + \psi(l + \frac{d-1}{2} - \nu) \right],
\]

\(^4\)In the appendix, we show that the similar result can be obtained using another regularization method.
where $\psi(z)$ is the digamma function. In (4.94) we have neglected all the terms that are proportional to $\sum_{l=0}^{\infty} D_{(d-1)}(l)$. The reason is that $\sum_{l=0}^{\infty} D_{(d-1)}(l) = 0$ under dimensional regularization \[114\]. We briefly review the argument here. The degeneracy $D_{(d-1)}(l)$ can be rewritten as

$$D_{(d-1)}(l) = \frac{2l + d - 2 (d-2)l}{d-2} \frac{\Gamma(a)}{l!},$$

(4.95)

where $(a)_l = \Gamma(a + l)/\Gamma(a)$ is the Pochhammer symbol. Now using the following expansion for $(1 - x)^a$:

$$(1 - x)^a = \sum_{l=0}^{\infty} \frac{(-a)_l}{l!} x^l,$$

(4.96)

we have

$$\sum_{l=0}^{\infty} D_{(d-1)}(l) = 2(1 - 1)^{-(d-1)} + (1 - 1)^{-(d-2)},$$

(4.97)

which is 0 for $d < 1$. As the summation (4.94) is also convergent when $d < 1$, we can analytically continue the result from this region. To proceed, we use the integral representation for $\psi(z)$:

$$\psi(z) = \int_0^{\infty} ds \left( \frac{e^{-s} - e^{-sz}}{s - e^{-s}} \right),$$

(4.98)

and perform the summation over $l$. The remaining integral over $s$ can be done analytically and the final expression is

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b = -\frac{1}{2} \Gamma(1 - d) \left[ \frac{\Gamma(\nu + \frac{d-1}{2})}{\Gamma(1 + \nu - \frac{d-1}{2})} - \frac{\Gamma(-\nu + \frac{d-1}{2})}{\Gamma(1 - \nu - \frac{d-1}{2})} \right].$$

(4.99)

On the other hand, the Euclidean one-loop effective action for a massive real scalar field in $AdS_d$ is

$$Z^b_{\text{class}} = Z^b_{\text{class}} \cdot \left[ \det_{\pm} (-\nabla^2 + m^2) \right]^{-\frac{1}{2}}.$$ (4.100)

where $Z^b_{\text{class}}$ is the classical partition function and $\pm$ refers to the bulk quantization corresponding to the boundary operator with dimension $\Delta_{\pm}$ defined by

$$\Delta_{\pm} = \frac{d-1}{2} \pm \nu', \quad \nu' = \sqrt{(d-1)^2 + 4m^2}.$$ (4.101)

Instead of $\left[ \det_{\pm} (-\nabla^2 + m^2) \right]^{-\frac{1}{2}}$, it’s much easier to calculate:

$$\frac{\partial}{\partial m^2} \log \left[ \det_{\pm} (-\nabla^2 + m^2) \right]^{-\frac{1}{2}} = -\frac{1}{2} \int dr dx^{d-1} \sqrt{g} G_{\Delta_{\pm}}^b (r; r, x),$$ (4.102)
where $G_{\Delta \pm}^b$ is the propagator for the scalar field. Using dimensional regularization, we have [115]

$$G_{\Delta \pm}^b(r, x; r, x) = (4\pi)^{-\frac{d}{2}} \Gamma(1 - \frac{d}{2}) \frac{\Gamma(\pm \nu' + \frac{d-1}{2})}{\Gamma(1 \pm \nu' - \frac{d-1}{2})},$$  \hspace{1cm} (4.103)$$

and the spacetime volume to be

$$V_d = \pi^{\frac{d-1}{2}} \Gamma(- \frac{d-1}{2}).$$  \hspace{1cm} (4.104)$$

Thus, under dimensional regularization [114] we obtain

$$\frac{\partial}{\partial m^2} \log \frac{[\det_+ (-\nabla^2 + m^2)]^{-\frac{1}{2}}}{[\det_- (-\nabla^2 + m^2)]^{-\frac{1}{2}}} = -\frac{1}{2} \Gamma(1 - d) \left[ \frac{\Gamma(\nu' + \frac{d-1}{2})}{\Gamma(1 + \nu' - \frac{d-1}{2})} - \frac{\Gamma(-\nu' + \frac{d-1}{2})}{\Gamma(1 - \nu' - \frac{d-1}{2})} \right].$$  \hspace{1cm} (4.105)$$

If we identify $\nu'$ with $\nu$, then we can establish the equality

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_b^{\text{in/out}} = \frac{\partial}{\partial m^2} \log \frac{[\det_+ (-\nabla^2 + m^2)]^{-\frac{1}{2}}}{[\det_- (-\nabla^2 + m^2)]^{-\frac{1}{2}}} = \frac{1}{2\nu'} \frac{\partial}{\partial \nu'} \log Z_b^{\nu'},$$  \hspace{1cm} (4.106)$$

under dimensional regularization.

Although there appears to be a pole for all physical dimension $d$ in (4.99), the expression actually only has a pole in $d$ when $d$ is odd:

$$-\frac{1}{2} \Gamma(1 - d) \left[ \frac{\Gamma(\nu + \frac{d-1}{2})}{\Gamma(1 + \nu - \frac{d-1}{2})} - \frac{\Gamma(-\nu + \frac{d-1}{2})}{\Gamma(1 - \nu - \frac{d-1}{2})} \right] = \frac{\sin(\pi \nu)}{2 \cos(\frac{\pi d}{2})} \Gamma(\nu + \frac{d-1}{2}) \Gamma(-\nu + \frac{d-1}{2}) \Gamma(d).$$  \hspace{1cm} (4.107)$$

Now we go to the physical dimension by letting $d \to d - \epsilon$.

**d even** In this case, (4.107) is finite:

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_b^{\text{in/out}} = \frac{\pi}{2\nu} \frac{(-1)^{\frac{d}{2}+1} \Gamma(\nu + \frac{d-1}{2}) \Gamma(-\nu + \frac{d-1}{2})}{\Gamma(d) \Gamma(\nu) \Gamma(-\nu)}.$$  \hspace{1cm} (4.108)$$

In the language of dS/CFT, the boundary CFT also has no conformal anomaly and the finite term in the partition function of the CFT is well defined. The result (4.108) gives the difference of this finite term in the UV and IR CFT [112–114].
In this case, (4.107) has a pole in $\epsilon$

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b = \frac{1}{\epsilon} \frac{(-1)^{\frac{d+1}{2}} \Gamma(\nu + \frac{d-1}{2}) \Gamma(-\nu + \frac{d+1}{2})}{\nu \Gamma(\nu) \Gamma(-\nu)} + \mathcal{O}(1),
\]

which signals the logarithmic divergence. In the language of dS/CFT, the boundary CFT has conformal anomaly in this case. The change of the conformal anomaly due to the double-trace deformation can be computed from the change of the central charge between the UV and IR CFT. The residue at the pole in (4.109) reproduces this change of the central charge on the boundary CFT \cite{113} up to a constant prefactor.

### 4.5.2 Massive Dirac spinor

For a massive Dirac spinor, we have

\[
\log Z_{\text{in/out}}^f = \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ 2 \ln \Gamma\left(\frac{l}{2} + iM\right) - \ln \pi + \ln \sin(-l\pi - \frac{d-2}{2} \pi + iM\pi) \right.
\]

\[
- \ln \Gamma\left(l + \frac{d}{2} + iM\right) + \ln \Gamma\left(l + \frac{d}{2} - iM\right) \right].
\]

(4.110)

Similar to the scalar case, we denote $\nu = iM$ and consider the derivative of $\log Z_{\text{in/out}}^f$ with respect to $\nu$:

\[
\frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^f = - \sum_{l=0}^{\infty} D_{(d-1)}(l) \left[ \psi\left(l + \frac{d}{2} + \nu\right) + \psi\left(l + \frac{d}{2} - \nu\right) \right].
\]

(4.111)

We recall that the degeneracies can be written as

\[
D_{(d-1)}(l) = \dim \gamma^d \frac{(d-1)_l}{l!}.
\]

(4.112)

where $\dim \gamma^d$ is the dimension of the gamma matrices in $d$-dimensional spacetime. Again the $l$–independent terms in (4.111) is neglected because $\sum_{l=0}^{\infty} D_{(d-1)}(l) = 0$ using dimensional regularization. Following the same method used in the scalar case, we find that the final expression is

\[
\frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^f = \dim \gamma^d \Gamma(1-d) \left[ \frac{\Gamma(\nu + \frac{d}{2})}{\Gamma(1 + \nu - \frac{d}{2})} + \frac{\Gamma(-\nu + \frac{d}{2})}{\Gamma(1 - \nu - \frac{d}{2})} \right].
\]

(4.113)

On the other hand, the Euclidean one-loop effective action for a massive Dirac spinor field in $\text{AdS}_d$ is:

\[
Z_f^\pm = Z_{\text{class}}^f \cdot \left[ \det_{\pm}(\nabla + m) \right],
\]

(4.114)
where $Z_{\text{class}}^f$ is the classical partition function and $\pm$ refers to the bulk quantization corresponding to the boundary operator with dimension $\Delta_\pm$ defined by

$$\Delta_\pm = \frac{d-1}{2} \pm \nu', \quad \nu' = m. \quad (4.115)$$

As in the scalar case, it is much easier to compute

$$\frac{\partial}{\partial m} \log \left[ \det_+ (\bar{\nabla} + m) \right] = - \int dr dx^{d-1} \sqrt{\bar{g}} \text{Tr} \left[ G^f_{\Delta_\pm}(r, x; r, x) \right]. \quad (4.116)$$

where $G^f_{\Delta_\pm}$ is the propagator for the spinor field. Using dimensional regularization, we have [115,128]

$$\text{Tr} \left[ G^f_{\Delta_\pm}(r, x; r, x) \right] = \mp \dim \gamma^d (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\frac{d}{2} + \nu')}{\Gamma(1 - \frac{d}{2} + \nu')}, \quad (4.117)$$

Multiplying the regularized volume for spacetime (4.104), we obtain

$$\frac{\partial}{\partial m} \log \left[ \det_+ (\bar{\nabla} + m) \right] \left[ \det_- (\bar{\nabla} + m) \right] = \dim \gamma^d \Gamma(1 - d) \left[ \frac{\Gamma(\nu' + \frac{d}{2})}{\Gamma(1 + \nu' - \frac{d}{2})} + \frac{\Gamma(-\nu' + \frac{d}{2})}{\Gamma(1 - \nu' - \frac{d}{2})} \right]. \quad (4.118)$$

If we identify $\nu'$ with $\nu$, we can establish the equality

$$\frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^f = \frac{\partial}{\partial m} \log \left[ \frac{\det_+ (\bar{\nabla} + M)}{\det_- (\bar{\nabla} + M)} \right] = \frac{\partial}{\partial \nu} \log \frac{Z^f_{\text{in}}}{Z^f_{\text{out}}}, \quad (4.119)$$

under dimensional regularization. Similar to the scalar case, the expression (4.113) only has pole in $d$ when $d$ is odd

$$\dim \gamma^d \Gamma(1 - d) \left[ \frac{\Gamma(\nu + \frac{d}{2})}{\Gamma(1 + \nu - \frac{d}{2})} - \frac{\Gamma(-\nu + \frac{d}{2})}{\Gamma(1 - \nu - \frac{d}{2})} \right]$$

$$= \dim \gamma^d \cos(\pi \nu) \frac{\Gamma(\nu + \frac{d}{2})\Gamma(-\nu + \frac{d}{2})}{\cos\left(\frac{\pi d}{2}\right) \Gamma(d)}. \quad (4.120)$$

Now we go to physical dimension by letting $d \to d - \epsilon$.

**$d$ even** In this case, (4.120) is finite:

$$\frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^f = \dim \gamma^d \frac{\pi(-1)^{\frac{d}{2}} \Gamma(\nu + \frac{d}{2})\Gamma(-\nu + \frac{d}{2})}{\Gamma(d) \Gamma(\nu + \frac{d}{2})\Gamma(-\nu + \frac{d}{2})}. \quad (4.121)$$
As in the scalar case, the boundary CFT has no anomaly and the result (4.121) computes the difference of the finite term in UV and IR CFT [116].

\[ d \text{ odd} \quad \text{In this case, (4.120) has a pole in } \epsilon: \]

\[ \frac{\partial}{\partial \nu} \log Z_{in/out} = \frac{1}{\epsilon} \dim \gamma^d \frac{(\nu + \frac{d}{2})! (\nu - \frac{d}{2})!}{\Gamma(\nu + \frac{d}{2}) \Gamma(\nu - \frac{d}{2})} + \mathcal{O}(1). \]

As in the scalar case, the boundary CFT has conformal anomaly which can be computed from its central charge. Up to a constant prefactor, the residue at the pole reproduces the change of the central charge in the UV and IR CFT connected by the RG flow due to the double-trace deformation on the boundary CFT [115,116].

### 4.6 Conclusion

In this chapter we have used in-/out-state formalism to calculate the effective action for both a real scalar field and a Dirac spinor field in the global patch of dS space in any dimension. It has been known for a long time [45] that there is no imaginary contribution to the effective action of a scalar field in the odd-dimensional dS space. In this chapter we have shown that it is also true for the effective action of a spinor field in the odd-dimensional dS space. In [48] the authors have given a heuristic argument for why there is no imaginary contribution for the scalar field in odd dimensions. We think this argument can be adapted to the spinor case as well. In even dimensions, there is an imaginary part in the effective action in both scalar and spinor field cases. Such imaginary part signals the event of particle production. In both cases, we have calculated the corresponding particle production rate and we have found that in the large-mass/weak-curvature limit both rates approach \( e^{-M/T_H} \) where \( T_H \) is the Hawking-de Sitter temperature.

Using dimensional regularization, we have extracted the finite term of the effective action in even dimensions and the coefficient of the logarithmically divergent term in odd dimensions. We also have shown that the regularized in-out vacuum amplitude \( Z_{in/out} \) in global \( dS_d \) has the same expression as the ratio of the functional determinants associated with different quantizations in \( AdS_d \) upon identification of certain parameters in the two theories. It is intriguing that there is a relation between the vacuum amplitude in \( dS_d \) and the ratio of determinants in \( AdS_d \). We don’t know if it is just a coincidence or there is a deeper connection underlying. Nevertheless, we want to point out that the summations in (4.94) and (4.111) have appeared exactly in the dual \( CFT_{d-1} \).
calculation [113–116] in the study of double trace deformation. In the $CFT_{d-1}$ computation for odd $d$, the coefficient of the logarithmic divergence is related to the change of the central charge from the UV fixed point to the IR fixed point [113,115]. The CFT at the two fixed points correspond to the two different quantizations in the bulk AdS space. In the language of dS/CFT [120], the in-/out-modes in our calculation also correspond to different quantizations in the dual CFT. It could be possible that the in-out vacuum amplitude in dS space is related to the double-trace deformation on the boundary CFT as the time evolution in the bulk corresponds to the RG flows in the dual CFT in the context of dS/CFT [129]. It would be interesting if we could find the exact connection between the two.
Appendix A

Appendix for Chapter 2

A.1 Derivation of the effective action

In this section, we compute the free energy which is equivalent to the effective action by using the replica trick [130]

\[
\beta F = -\ln Z = -\lim_{M \to 0} \ln \frac{Z^M}{M} \tag{A.1}
\]

where the bar indicates averaging over the disorder and \(Z^M\) is the partition function of \(M\) copies of the system. Specifically, we have

\[
\overline{Z^M} = \int \mathcal{D}J_I(\alpha) \int \mathcal{D}\chi_i^a P[J_I] \exp \left[ -\sum_a \int d\tau \left( \frac{1}{2} \sum_i \chi_i^a \partial_\tau \chi_i^a + \int_0^\infty \frac{d\alpha}{\Gamma(\alpha q + 1)} \sum_I J_I(\alpha) \chi_i^{a_{\alpha q}} \right) \right] \tag{A.2}
\]

where \(a\) is the replica index, \(a \in \{1, \ldots, M\}\), \(i \in \{1, \ldots, N\}\), and \(I\) in \(J_I(\alpha)\) is the collective index representing \(i_1 \ldots i_{\alpha q}\), and \(P[J_I]\) is the probability distribution for \(J_I(\alpha)\) which gives (2.8). We can regard \(J_I\) as a dynamical field with propagator (2.8) and integrate it out. The result is

\[
\overline{Z^M} = \int \mathcal{D}\chi_i^a \exp \left[ -\sum_{a,i} \frac{1}{2} \int d\tau \chi_i^a \partial_\tau \chi_i^a 
+ \frac{1}{4} \int_0^\infty d\alpha \frac{J^2(\alpha)N}{\alpha q^2} \sum_{a,b} \int d\tau_1 d\tau_2 \left( \frac{2}{N} \sum_i \chi_i^a(\tau_1) \chi_i^b(\tau_2) \right)^{\alpha q} \right] \tag{A.3}
\]
Next we introduce the collective fields

\[ G^{ab}(\tau_1, \tau_2) = \frac{1}{N} \sum_i \chi_i^a(\tau_1)\chi_i^b(\tau_2) \quad (A.4) \]

and insert into (A.3) the following delta function

\[ \int d\Sigma^{ab}(\tau_1, \tau_2) \exp \left[ -\frac{N}{2} \Sigma^{ab}(\tau_1, \tau_2) \left( G^{ab}(\tau_1, \tau_2) - \frac{1}{N} \sum_i \chi_i^a(\tau_1)\chi_i^b(\tau_2) \right) \right] \quad (A.5) \]

This leads to

\[ \mathcal{Z}^M = \int D\Sigma DG \int D\chi_i^a \exp \left[ -\frac{1}{2} \sum_{a,i} \int d\tau \chi_i^a \partial_\tau \chi_i^a + \frac{1}{2} \sum_{a,b,i} \int d\tau_1 d\tau_2 \Sigma^{ab}(\tau_1, \tau_2)\chi_i^a(\tau_1)\chi_i^b(\tau_2) 
\right. 
\left. - \frac{N}{2} \sum_{a,b} \int d\tau_1 d\tau_2 \left( \Sigma^{ab}(\tau_1, \tau_2)G^{ab}(\tau_1, \tau_2) - \int_0^\infty d\alpha \frac{J^2(\alpha)}{2\alpha q^2} (2G^{ab}(\tau_1, \tau_2))^{\alpha q} \right) \right] \quad (A.6) \]

Now we can integrate out the fermions and arrive at

\[ \mathcal{Z}^M = \int D\Sigma DG \exp \left\{ N \sum_{a,b} \left[ \text{Tr} \log(\delta^{ab}\partial_\tau - \Sigma^{ab}) 
\right. 
\left. - \frac{1}{2} \int d\tau_1 d\tau_2 \left( \Sigma^{ab}(\tau_1, \tau_2)G^{ab}(\tau_1, \tau_2) - \frac{1}{2q^2} \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} (2G^{ab}(\tau_1, \tau_2))^{\alpha q} \right) \right] \right\} \quad (A.7) \]

Finally, we shall assume a replica symmetric saddle point so that \( G^{ab}(\tau_1, \tau_2) = \delta^{ab}G(\tau_1, \tau_2) \) and \( \Sigma^{ab}(\tau_1, \tau_2) = \delta^{ab}\Sigma(\tau_1, \tau_2) \), then we have

\[ \mathcal{Z}^M = \int D\Sigma DG \exp \left( -MS_E \right) \quad (A.8) \]

where the large \( N \) effective action \( S_E \) is

\[ -\frac{S_E}{N} = \frac{1}{2} \text{Tr} \log(\partial_\tau - \Sigma) - \frac{1}{2} \int d\tau_1 d\tau_2 \left[ \Sigma(\tau_1, \tau_2)G(\tau_1, \tau_2) - \frac{1}{2q^2} \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} [2G(\tau_1, \tau_2)]^{\alpha q} \right] \quad (A.9) \]

Using (2.9) and (2.10), the determinant term can be expanded in large \( q \) limit as

\[ \text{Tr} \log(\partial_\tau - \Sigma) = \text{Tr} \log(G_0^{-1}) - \text{Tr}(G_0 * \Sigma) - \frac{1}{2} \text{Tr}(G_0 * \Sigma * G_0 * \Sigma) + \ldots \quad (A.10) \]
with $G_0(\tau_1, \tau_2) = \frac{1}{2} \text{sgn}(\tau_1 - \tau_2)$. The $\text{Tr} \log(G_0^{-1})$ term gives the entropy of free fermions while the $\text{Tr}(G_0 \ast \Sigma)$ term vanishes. Ignoring the constant piece, we have to order $q^{-2}$:

$$
\frac{S_E}{N} \simeq \frac{1}{4} \text{Tr}(G_0 \ast \Sigma \ast G_0 \ast \Sigma) + \frac{1}{2q} \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) G_0(\tau_1, \tau_2) g(\tau_1, \tau_2) \\
- \frac{1}{4q^2} \int d\tau_1 d\tau_2 \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} e^{\alpha g(\tau_1, \tau_2)}
$$

(A.11)

If we define

$$
\Phi(\tau_1, \tau_2) = [G_0 \ast \Sigma](\tau_1, \tau_2)
$$

(A.12)

Then it follows that

$$
\Sigma(\tau_1, \tau_2) = \partial_{\tau_1} \Phi(\tau_1, \tau_2)
$$

(A.13)

and (A.11) becomes

$$
\frac{S_E}{N} \simeq \frac{1}{4} \text{Tr}(\Phi \ast \Phi) + \frac{1}{2q} \int d\tau_1 d\tau_2 \partial_{\tau_1} \Phi(\tau_1, \tau_2) G_0(\tau_1, \tau_2) g(\tau_1, \tau_2) \\
- \frac{1}{4q^2} \int d\tau_1 d\tau_2 \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} e^{\alpha g(\tau_1, \tau_2)}
$$

(A.14)

After integrating out $\Phi$, we obtain (after subtracting the constant piece)

$$
\frac{S_E}{N} = \frac{1}{4q^2} \int d\tau_1 d\tau_2 \partial_{\tau_1} (G_0(\tau_1, \tau_2) g(\tau_1, \tau_2)) \partial_{\tau_2} (G_0(\tau_1, \tau_2) g(\tau_1, \tau_2)) \\
- \frac{1}{4q^2} \int d\tau_1 d\tau_2 \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} e^{\alpha g(\tau_1, \tau_2)}
$$

$$
= \frac{1}{16q^2} \int d\tau_1 d\tau_2 \partial_{\tau_1} g(\tau_1, \tau_2) \partial_{\tau_2} g(\tau_1, \tau_2) - \frac{1}{4q^2} \int d\tau_1 d\tau_2 \int_0^\infty d\alpha \frac{J^2(\alpha)}{\alpha} e^{\alpha g(\tau_1, \tau_2)}
$$

$$
= \frac{1}{16q^2} \int d\tau_1 d\tau_2 \partial_{\tau_1} g(\tau_1, \tau_2) \partial_{\tau_2} g(\tau_1, \tau_2) + \frac{1}{8q^2} \int d\tau_1 d\tau_2 W(g(\tau_1, \tau_2))
$$

(A.15)

To arrive at the first line, we use the property that $g(\tau_1, \tau_2) = g(\tau_2, \tau_1)$ from the definition of $G(\tau_1, \tau_2)$ and the last line follows from the definition of $U(g)$ and $W(g)$.

### A.2 Thermodynamics of the standard large $q$ SYK model

In the large $q$ limit of the standard SYK model [3], we have $U(g) = 2J^2 e^g$ and $W(g) = -2J^2 e^g$. Then it follows from (2.13) that

$$
\beta J = 2 \exp\left(-\frac{g_m}{2}\right) \tan^{-1}\left(\sqrt{e^{-g_m} - 1}\right)
$$

(A.16)
If we define $\nu$ such that $\tan\left(\frac{\pi \nu}{2}\right) = \sqrt{e^{-g_m} - 1}$, then we have the relations

$$\beta J = \frac{\pi \nu}{\cos\left(\frac{\pi \nu}{2}\right)}, \quad 2J^2 \cos^2\left(\frac{\pi \nu}{2}\right) = -W(g_m) \tag{A.17}$$

Using (A.17) and (2.21), we have

$$S = S_0 - \frac{N}{4q^2} \pi^2 \nu^2 \tag{A.18}$$

We can use (A.17) to obtain the low-temperature expansion of $\nu$ as

$$\nu = 1 - \frac{2}{\beta J} + o\left(\frac{1}{(\beta J)^2}\right) \tag{A.19}$$

which reproduces the $-\frac{\pi^2}{3q^2}$ term of the zero-temperature entropy of the standard large-$q$ model result.

### A.3 Perturbative calculations in large q and 2q model

The equation (2.34) can be solved analytically and we have the solution

$$e^{g(t)} = \frac{2\nu^2}{\sqrt{J^4 + \nu^2 K^2} \cos(2\nu t - \nu \beta) + J^2}, \quad \text{with} \quad \frac{2\nu^2 - J^2}{\sqrt{J^4 + \nu^2 K^2}^2} = \cos(\nu \beta) \tag{A.20}$$

Since $\nu$ is dimensionful, if we introduce $\nu = \frac{\omega}{\beta}$ with $\omega \sim o(1)$, then the equation for $\nu$ becomes

$$\frac{2\omega^2 - (\beta J)^2}{\sqrt{(\beta J)^4 + \omega^2 (\beta K)^2}^2} = \cos(\omega) \tag{A.21}$$

from which we see that the transition from two conformal points occurs at the temperature $(\beta J)^2 \sim \beta K$.

If we plug the solution (A.20) into (2.28) and define $x = 2\nu \sigma$, together with $J^2 = A \cos \theta$ and $\nu K = A \sin \theta$ where $\theta \in (0, \frac{\pi}{2})$, we then have

$$-\partial_x^2 f_\lambda(x) - \frac{\cos \theta}{\cosh x + \cos \theta} f_\lambda(x) - \frac{2 \sin^2 \theta}{(\cosh x + \cos \theta)^2} f_\lambda(x) = -\left(\frac{\lambda}{4\nu}\right)^2 f_\lambda(x) \tag{A.22}$$

Although we are not able to solve (A.22), we can consider doing perturbation around the two conformal points and evaluate the correction to the Lyapunov exponent.
We first consider the case that \( \epsilon = (\frac{\beta J}{\beta K})^2 \ll 1 \) and \( \beta K \gg 1 \), then we have

\[
\omega = \frac{\pi}{2} + \frac{2}{\pi} \epsilon - \frac{\pi}{\beta K} + o(\frac{1}{(\beta K)^2}), \quad \theta = \frac{\pi}{2} - \frac{2}{\pi} \epsilon + o(\frac{1}{(\beta K)^2}) \tag{A.23}
\]

The left-hand-side of perturbed equation (A.22) becomes

\[
-\partial_x^2 f_\lambda(x) - \frac{2}{\cosh^2 x} f_\lambda(x) - \epsilon \left( \frac{1}{\cosh x} - \frac{4}{\cosh^3 x} \right) f_\lambda(x) \tag{A.24}
\]

Using the unperturbed ground state solution \( f_\lambda(x) = \frac{1}{\sqrt{2 \cosh x}} \), we get

\[
\lambda = 4\nu(1 - \frac{1}{2} \epsilon) = \frac{2\pi}{\beta} \left[ 1 - \frac{2}{\beta K} + \left( \frac{1}{2} - \frac{4}{\pi^2} \right) \frac{(\beta J)^2}{\beta K} \right] + o(\frac{1}{(\beta K)^2}) \tag{A.25}
\]

We can also consider the case that \( \frac{1}{\beta J} \ll \gamma = (\frac{\beta J}{\beta K})^2 \ll 1 \), then we have

\[
\theta = \omega \gamma + o(\gamma^2), \quad \omega = \pi - \pi \gamma + o(\frac{1}{\gamma J}) \tag{A.26}
\]

The l.h.s of the perturbed equation becomes

\[
-\partial_x^2 f_\lambda(x) - \frac{1}{2 \cosh^2 x} f_\lambda(x) - \frac{\gamma^2 \pi^2}{2} - \frac{4 - \cosh x}{2 \cosh^3 x} f_\lambda(x) \tag{A.27}
\]

However, in this case we see that the perturbation starts at \( o(\gamma^2) \), which implies that the eigenvalue of the ground state is \( -\frac{1}{4} + o(\gamma^2) \). So we have at \( o(\gamma) \)

\[
\lambda = 2\nu = \frac{2\pi}{\beta} (1 - \gamma + o(\gamma^2)) = \frac{2\pi}{\beta} \left[ 1 - \frac{\beta K}{(\beta J)^2} + o((\frac{\beta J}{\beta K})^2) \right] \tag{A.28}
\]

**A.4 Derivation of the chaos exponent from retarded kernel**

The \( F(t_1, t_2) \) function satisfies the following equation which comes from a set of ladder diagrams

\[
F(t_1, t_2) = \int dt_3 dt_4 K_R(t_1, t_2; t_3, t_4) F(t_3, t_4) \tag{A.29}
\]
The retarded kernel $K_R(t_1, t_2; t_3, t_4)$ is defined by

$$K_R(t_1, t_2; t_3, t_4) = G_R(t_13)G_R(t_24) \int_0^\infty d\alpha J^2(\alpha)(\alpha q - 1)2^{2-\alpha q}[G_{lr}(t_{34})]^{\alpha q-2}$$  \hspace{1cm} (A.30)

$$= \theta(t_{13})\theta(t_{24})\partial_y U(g(\frac{\beta}{2} + it))$$  \hspace{1cm} (A.31)

where $G_R(t)$ is the retarded propagator, which in the large $q$ limit is just $\theta(t)$, and $G_{lr}$ is the Wightman correlator with points separated by half of the thermal circle ($G_{lr}(t) = G(\frac{\beta}{2} + it))$.

To solve (A.29), we use the growth ansatz

$$F(t_1, t_2) = e^{\lambda_L(t_1+t_2)/2}f_\lambda(t_{12})$$  \hspace{1cm} (A.32)

then the Lyapunov exponent is just the values of $\lambda_L$ such that $f(t)$ is an eigenfunction of $K_R$ with eigenvalue one by solving (A.29).

By substituting (A.31) and (A.32) into (A.29) and taking derivatives with respect to $t_1$ and $t_2$, we obtain the following equation

$$\left[\frac{\lambda_L^2}{4} - \partial_\sigma^2\right]f_\lambda(\sigma) = \partial_y U(g(\frac{\beta}{2} + i\sigma))f_\lambda(\sigma)$$  \hspace{1cm} (A.33)

with $\sigma = t_{12}$. The calculation of the Lyapunov exponent becomes the quantum mechanics problem of solving the bound state energy with the potential $\partial_y U(g(\frac{\beta}{2} + i\sigma))$.  

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Appendix B

Appendix for Chapter 3

B.1 Relevant functions in the holographic calculation

In this appendix, we give the definitions and the expressions for the various functions that appear in the D-brane computation of the correlation functions. The $D$-function appears in the computation of tree level four-point functions that only involve contact diagrams [131–133]. In the general case of $AdS_{d+1}$, the $D$-function is defined as

$$D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \int \frac{dr d^d\vec{x}}{r^{d+1}} \prod_{i=1}^4 \left( \frac{r}{r^2 + (\vec{x} - \vec{x}_i)^2} \right)^{\Delta_i}. \quad (B.1)$$

The various functions appear in the result of the four-point functions have been first computed in [24] and we simply quote the results below.

The function $G_{2x2y}(\chi)$ is given by

$$G_{2x2y}(\chi) = -\frac{2}{\pi} \left[ 1 - \left( \frac{1}{2} - \frac{1}{\chi} \right) \log |1 - \chi| \right]. \quad (B.2)$$

Both $G_{44}^{i_1i_2i_3i_4}(\chi)$ and $G_{44}^{a_1a_2a_3a_4}(\chi)$ can be decomposed into singlet ($S$), symmetric traceless ($T$) and antisymmetric ($A$) parts as

$$G_{44}^{i_1i_2i_3i_4}(\chi) = G_{44}^{(S)}(\chi) \delta^{i_1i_2} \delta^{i_3i_4} + G_{44}^{(T)}(\chi) \left( \delta^{i_1i_3} \delta^{i_2i_4} + \delta^{i_1i_4} \delta^{i_2i_3} - \frac{2}{3} \delta^{i_1i_2} \delta^{i_3i_4} \right)$$

$$+ G_{44}^{(A)}(\chi) \left( \delta^{i_1i_3} \delta^{i_2i_4} - \delta^{i_2i_3} \delta^{i_1i_4} \right), \quad (B.3)$$
with

\[
G^{(S)}_{4x}(\chi) = \frac{1}{6\pi} \left[ -\frac{(24\chi^8 - 90\chi^7 + 125\chi^6 - 76\chi^5 + 125\chi^4 - 306\chi^3 + 438\chi^2 - 288\chi + 72)}{3(\chi - 1)^4} - \frac{2(4\chi^6 - \chi^5 - 6\chi + 12)}{\chi} \log |1 - \chi| \right.
\]
\[+ \left. \frac{2\chi^4(4\chi^6 - 21\chi^5 + 45\chi^4 - 50\chi^3 + 30\chi^2 - 6\chi + 2)}{(\chi - 1)^5} \log |\chi| \right] , \tag{B.4}
\]

\[
G^{(T)}_{4x}(\chi) = \frac{1}{4\pi} \left[ -\frac{(48\chi^4 - 198\chi^3 + 313\chi^2 - 230\chi + 115)\chi^4}{6(\chi - 1)^4} - \frac{(8\chi - 5)\chi^4}{(\chi - 1)^5} \log |\chi| \right. 
\]
\[+ \left. \frac{(8\chi^6 - 45\chi^5 + 105\chi^4 - 130\chi^3 + 90\chi^2 - 30\chi + 10)\chi^4}{(\chi - 1)^5} \log |\chi| \right] , \tag{B.5}
\]

\[
G^{(A)}_{4x}(\chi) = \frac{1}{4\pi} \left[ -\frac{(\chi - 2)(48\chi^6 - 90\chi^5 + 91\chi^4 + 4\chi^3 - 17\chi^2 + 18\chi - 6)\chi}{6(\chi - 1)^4} 
\]
\[+ \frac{(8\chi^5 - 3\chi^4 + 2)\log |1 - \chi| + (\chi - 2)(8\chi^4 - 27\chi^3 + 41\chi^2 - 28\chi + 14)\chi^5}{(\chi - 1)^5} \log |\chi| \right]. \tag{B.6}
\]

and

\[
G^{a_1a_2a_3a_4}_{4y}(\chi) = G^{(S)}_{4y}(\chi)\delta^{a_1a_2}\delta^{a_3a_4} + G^{(T)}_{4y}(\chi)\left(\delta^{a_1a_2a_3}\delta^{a_4} + \delta^{a_2a_3a_4}\delta^{a_1} - \frac{2}{5}\delta^{a_1a_2}\delta^{a_3a_4} \right) 
\]
\[+ G^{(A)}_{4y}(\chi)\left(\delta^{a_1a_2}\delta^{a_3a_4} - \delta^{a_2a_3}\delta^{a_1a_4} \right) , \tag{B.7}
\]

with

\[
G^{(S)}_{4y}(\chi) = \frac{1}{10\pi} \left[ -\frac{2(\chi^4 - 4\chi^3 + 9\chi^2 - 10\chi + 5)}{(\chi - 1)^2} \frac{\chi^2(2\chi^4 - 11\chi^3 + 21\chi^2 - 20\chi + 10)}{(\chi - 1)^3} \log |\chi| \right. 
\]
\[+ \left. \frac{(2\chi^4 - 5\chi^3 - 5\chi + 10)}{\chi} \log |1 - \chi| \right] , \tag{B.8}
\]

\[
G^{(T)}_{4y}(\chi) = \frac{1}{2\pi} \left[ -\frac{\chi^2(2\chi^2 - 3\chi + 3)}{2(\chi - 1)^2} \frac{\chi^4(\chi^2 - 3\chi + 3)}{(\chi - 1)^3} \log |\chi| - \chi^3 \log |1 - \chi| \right] , \tag{B.9}
\]

\[
G^{(A)}_{4y}(\chi) = \frac{1}{2\pi} \left[ \frac{\chi(-2\chi^3 + 5\chi^2 - 3\chi + 2)}{2(\chi - 1)^2} + \frac{\chi^3(\chi^3 - 4\chi^2 + 6\chi - 4)}{(\chi - 1)^3} \log |\chi| 
\]
\[+ \left. (\chi^3 - \chi^2 - 1) \log |1 - \chi| \right] . \tag{B.10}
\]
B.2 Calculation of $V_L$ and $V_{L_1,L_2,L_3}$

In this appendix, we derive the expressions for $V_L$ and $V_{L_1,L_2,L_3}$ appear in the D5-brane calculation.

We consider the following generating function

$$I[J] = \int d\Omega_4 e^{J \cdot Y} = V_{S^3} \int_0^{\pi} d\theta \sin^3 \theta e^{\frac{8\pi^2 |J| \cos \theta}{|J|^2}} \left( \cosh |J| - \frac{\sin |J|}{|J|} \right)$$

(B.11)

where $J$ is a five-dimensional vector and $Y$ is the unit five-dimensional vector specifying $S^4$. We can express (B.11) as a series in power of $J^2$:

$$I[J] = 16\pi^2 \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+3)!} (J^2)^n$$

(B.12)

To compute $V_L$, we set $J = u_1 + u_2$ so that $J^2 = 2 u_1 \cdot u_2$. One can then compute $V_L$ by extracting the coefficient of the $(u_1 \cdot u_2)^L$ term in (B.12) multiplied by $L!$ from expanding the exponential in (B.11):

$$V_L = \frac{16\pi^2 2^L (L!)^2 (L+1)}{(2L+3)!}.$$  

(B.13)

The $V_L$ defined in (3.173) in the D3-brane calculation can be computed analogously.

To compute $V_{L_1,L_2,L_3}$, we set $J = u_1 + u_2 + u_3$ so that $J^2 = 2(u_1 \cdot u_2 + u_2 \cdot u_3 + u_1 \cdot u_3)$. Now we need to extract the coefficient of the term $(u_1 \cdot u_2)^{L_{123}}(u_2 \cdot u_3)^{L_{231}}(u_1 \cdot u_3)^{L_{132}}$ in (B.12) multiplied by $L_1! L_2! L_3!$ from the expansion of the exponential:

$$V_{L_1,L_2,L_3} = \frac{1 + (-1)^{L_1+L_2+L_3}}{2} \frac{8\pi^2 (\sqrt{2})^\Sigma (\Sigma + 2) L_1! L_2! L_3!}{(\Sigma + 3)!} \left( \frac{\Sigma}{2} \right)^{L_3} \binom{L_3}{L_{123}} \binom{L_3}{L_{231}}$$

$$= \frac{1 + (-1)^{L_1+L_2+L_3}}{2} \frac{8\pi^2 (\sqrt{2})^\Sigma (\Sigma + 2) L_1! L_2! L_3!}{(\Sigma + 3)!} \frac{(\frac{\Sigma}{2})!}{L_{123}! L_{231}! L_{132}!},$$

(B.14)

where the last two terms of the first line in (B.14) stand for the binomial coefficients.
Appendix C

Appendix for Chapter 4

C.1 Regularization with Laplacian

In this appendix, we show that the results in section 4.5 could be obtained by other regularization method. Specifically, we regularize the summation over $l$ using the exponential suppression factor $\exp(-\lambda_l \varepsilon)$ in the scalar field case. The $\lambda_l = l(l + d - 2)$ is the eigenvalue of the sphere Laplacian for the angular quantum number $l$. We perform the calculation for the case of $d = 2, 3, 4, 5$. We obtain the same finite terms as those calculated in section 4.5 for $d = 2, 4$. However, for $d = 3, 5$, the results differ by a finite term which could be obtained from the summation of the degeneracies $D_{(d-1)}(l)$. We think this mismatch might have resulted from a different choice of counterterms under the two regularization schemes.

We closely follow the method developed in [134]. The following asymptotic expansion [134] is crucial for our calculation:

$$
\sum_{l=1}^{\infty} l^{-s} e^{-l(l+q)t} \frac{t^{\frac{1-s}{2}}}{l \rightarrow 0} \left[ \Gamma\left(\frac{1}{2} - s\right) - q \Gamma\left(1 - \frac{s}{2}\right) t^{\frac{s}{2}} + \frac{q^3}{2!} \frac{2}{2} \Gamma\left(\frac{3}{2} - \frac{s}{2}\right) t^{1} \right. \\
\left. - \frac{q^3}{3!} \Gamma\left(2 - \frac{s}{2}\right) t^{2} + \frac{q^4}{4!} \Gamma\left(\frac{5}{2} - \frac{s}{2}\right) t^{2} + O(t^2) \right] + \zeta(s).
$$

(C.1)

$d=2$ We first look at the summation of the degeneracies using (C.1), which is

$$
1 + \sum_{l=1}^{\infty} 2 e^{-l^2 \varepsilon} = \sqrt{\pi \varepsilon}^{-\frac{1}{2}}.
$$

(C.2)
So the summation of the \( l \)-independent terms do not contain either \( \log \varepsilon \) term or a finite term. The summation of the remaining terms can be evaluated as

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b = \frac{1}{4\nu} \left[ \psi\left(\frac{1}{2} + \nu\right) + \psi\left(\frac{1}{2} - \nu\right) \right] + \frac{1}{2\nu} \sum_{l=1}^{\infty} \left[ \psi(l + \frac{1}{2} + \nu) + \psi(l + \frac{1}{2} - \nu) \right] e^{-l^2 \varepsilon} \\
= \frac{1}{\nu} \sum_{l=1}^{\infty} \log l e^{-l^2 \varepsilon} + \frac{1}{2\nu} \sum_{l=1}^{\infty} \left[ \psi(l + \frac{1}{2} + \nu) + \psi(l + \frac{1}{2} - \nu) - 2 \log l \right] \\
+ \frac{1}{4\nu} \left[ \psi\left(\frac{1}{2} + \nu\right) + \psi\left(\frac{1}{2} - \nu\right) \right] \\
= \frac{1}{2\nu} \log(2\pi) - \frac{1}{2\nu} \left[ \log(2\pi) + \pi \nu \tan(\pi \nu) \right] \\
= -\frac{\pi}{2} \tan(\pi \nu),
\]

(C.3)

where we have only kept the relevant finite term and the \( \log \varepsilon \) term from the summation. To arrive at the result, we have used (C.1) to compute the first summation on the second line, while the second summation is convergent (goes like \( l^{-2} \) asymptotically) and can be calculated analytically. The result agrees with the finite term obtained using the dimensional regularization.

\( d=4 \) The summation of the degeneracies is

\[
1 + \sum_{l=1}^{\infty} (l + 1)^2 e^{-l(l+1)\varepsilon} \varepsilon \to 0 \quad \frac{\sqrt{\pi}}{4} (e^{-\frac{3}{2}} + e^{-\frac{1}{2}}).
\]  
(C.4)
As in $d = 2$ case, we can neglect the summation of $l$-independent terms. The remaining summation is

$$\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log \mathcal{Z}^{b}_{\text{in/out}} = \frac{1}{4\nu} \left[ \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) \right]$$

$$+ \frac{1}{4\nu} \sum_{l=1}^{\infty} \left( l + 1 \right)^2 \left[ \psi\left( l + \frac{3}{2} + \nu \right) + \psi\left( l + \frac{3}{2} - \nu \right) \right] e^{-l(l+2)\epsilon}$$

$$= \frac{1}{4\nu} \sum_{l=1}^{\infty} f(l) e^{-l(l+2)\epsilon} + \frac{1}{4\nu} \sum_{l=1}^{\infty} \left\{ (l+1)^2 \left[ \psi\left( l + \frac{3}{2} + \nu \right) + \psi\left( l + \frac{3}{2} - \nu \right) \right] - f(l) \right\}$$

$$+ \frac{1}{4\nu} \left[ \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) \right]$$

$$= \frac{1}{96\nu} \left[ -75 + 16\gamma + 36\nu^2 + 96 \log \mathcal{G} + 24 \log(2\pi) - 48\zeta'(-2) \right]$$

$$+ \frac{1}{96\nu} \left[ 75 - 16\gamma - 36\nu^2 - 96 \log \mathcal{G} - 24 \log(2\pi) - \frac{12}{\pi^2} \zeta(3) + 2\pi\nu(1 - 4\nu^2) \tan(\pi

where $\mathcal{G}$ is Glaisher’s constant and $\zeta(s)$ is the Riemann zeta function. We have only kept $\log \epsilon$ and the finite term at the end. The function $f(l)$ equals

$$f(l) = 2l^2 \log l + 2l(1 + 2 \log l) + \frac{1}{12} (37 - 12\nu^2 + 24 \log l) + \frac{2}{3l}.$$  \hspace{1cm} (C.6)

Again, the result agrees with the one obtained using the dimensional regularization.

$d=3$ In this case, we expect there to be $\log \epsilon$ term and we want to compute its coefficient. The summation of the degeneracies is

$$1 + \sum_{l=1}^{\infty} (2l + 1)e^{-l(l+1)\epsilon} \stackrel{\epsilon \to 0}{=} \frac{\epsilon^{-1}}{3}. \hspace{1cm} (C.7)$$
Thus, the summation of the \( l \)-independent terms will not contribute to the \( \log \epsilon \) term. The remaining summation is

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b = \frac{1}{4\nu} \left[ \psi(1 + \nu) + \psi(1 - \nu) \right] + \frac{1}{4\nu} \sum_{l=1}^{\infty} \left( 2l + 1 \right) \left[ \psi(l + 1 + \nu) + \psi(l + 1 - \nu) \right] e^{-l(l+1)\epsilon}
\]

\[
= \frac{1}{4\nu} \sum_{l=1}^{\infty} f(l) e^{-l(l+1)\epsilon} + \frac{1}{4\nu} \sum_{l=1}^{\infty} \left\{ (2l + 1) \left[ \psi(l + 1 + \nu) + \psi(l + 1 - \nu) \right] - f(l) \right\} + \frac{1}{4\nu} \left[ \psi(1 + \nu) + \psi(1 - \nu) \right],
\]

where \( f(l) \) is given by

\[
f(l) = 4l \log l + 2(1 + \log l) - \frac{2(3\nu^2 - 1)}{3l}.
\]

As the second summation in (C.8) is convergent, the \( \log \epsilon \) term can only come from the first summation. We have

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b \bigg|_{\log \epsilon} = \frac{1}{4\nu} \left( \nu^2 - \frac{1}{3} \right) \log \epsilon.
\]

The dimensional regularization \((d = 3 - \epsilon)\) result is

\[
\frac{1}{2\nu} \frac{\partial}{\partial \nu} \log Z_{\text{in/out}}^b \bigg|_{\epsilon} = -\frac{1}{2\nu} \nu^2 \epsilon^{-1}.
\]

The constant term in the parentheses of (C.10) equals

\[
\sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\epsilon} \bigg|_{\text{finite}} = \frac{1}{3}
\]

\(d=5\)  Again we want to calculate the coefficient of the \( \log \epsilon \) term. The summation of the degeneracies is

\[
1 + \sum_{l=1}^{\infty} \frac{(l + 1)(l + 2)(2l + 3)}{6} e^{-l(l+3)\epsilon} \bigg|_{\epsilon \to 0} = \frac{1}{6} \epsilon^{-2} + \frac{1}{3} \epsilon^{-1} + \frac{29}{90}
\]
So the summation of the \( l \)-independent terms will not contribute to \( \log \varepsilon \) term. The remaining summation is

\[
\frac{1}{2 \nu} \frac{\partial}{\partial \nu} \log Z_{b \text{in/out}}^b = \frac{1}{4 \nu} \left[ \psi(2 + \nu) + \psi(2 - \nu) \right] \\
+ \frac{1}{4 \nu} \sum_{l=1}^{\infty} \frac{(l + 1)(l + 2)(2l + 3)}{6} \left[ \psi(l + 2 + \nu) + \psi(l + 2 - \nu) \right] e^{-l(l+3)\varepsilon} \\
= \frac{1}{4 \nu} \sum_{l=1}^{\infty} f(l) e^{-l(l+3)\varepsilon} + \frac{1}{4 \nu} \sum_{l=1}^{\infty} \left\{ \frac{(l + 1)(l + 2)(2l + 3)}{6} \left[ \psi(l + 2 + \nu) \\
+ \psi(l + 2 - \nu) \right] - f(l) \right\} + \frac{1}{4 \nu} \left[ \psi(2 + \nu) + \psi(2 - \nu) \right].
\]

Again, only the first summation in (C.14) can contribute to \( \log \varepsilon \) as the other summation is convergent. Thus, we get

\[
\frac{1}{2 \nu} \frac{\partial}{\partial \nu} \log Z_{b \text{in/out}}^b \bigg|_{\log \varepsilon} = \frac{1}{4 \nu} \left( \frac{\nu^2(\nu - 1)(\nu + 1)}{12} - \frac{29}{90} \right) \log \varepsilon,
\]

while the dimensional regularization \((d = 5 - \epsilon)\) result is

\[
\frac{1}{2 \nu} \frac{\partial}{\partial \nu} \log Z_{b \text{in/out}}^b \bigg|_{\frac{1}{\epsilon}} = -\frac{1}{2 \nu} \frac{\nu^2(\nu - 1)(\nu + 1)}{12} \epsilon^{-1}.
\]

Similar to the \( d = 3 \) case, the constant term in the parentheses of (C.15) equals

\[
\sum_{l=0}^{\infty} \frac{(l + 1)(l + 2)(2l + 3)}{6} e^{-l(l+3)\varepsilon} \bigg|_{\text{finite}} = \frac{29}{90}.
\]
Bibliography


