ESSAYS ON THE MACROECONOMICS OF LABOR

MARKETS

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Abstract

This dissertation studies the macroeconomic implications of the reallocation of workers within and between labor markets. Chapter 1 starts from the fact that unemployment rates differ widely across local labor markets. I document that high local unemployment follows from elevated local job losing rates, even for similar workers. I then propose a theory in which spatial differences in job loss arise endogenously through the location decision of employers. Labor market frictions distort the latter, providing a rationale for commonly used place-based policies. The estimated model accounts for the cross-sectional dispersion in unemployment rates and the key role of job loss. Finally, I show that both real-world and optimal place-based policies yield sizable welfare gains at the local and aggregate level.

Chapter 2, joint with Esteban Rossi-Hansberg, then investigates the individuals’ location decisions in more depth. We argue the location choice of individuals resembles investing in a location asset. This asset costs the locations rent, and a pays off better job and schooling opportunities. Savers go to expensive locations with good future opportunities. Borrowers go to cheap locations that offer few other advantages. The location asset is used by credit constrained individuals. We provide an analytical model to make this idea precise and to derive a number of related implications, that we confirm with French individual panel data from tax returns.

Chapter 3, joint with Niklas Engbom, Simon Mongey and Gianluca Violante, finally studies how the allocation of workers across firms in the aggregate labor market affects aggregate productivity. We develop a frictional labor market model with firm and worker dynamics. Multi-worker firms choose whether to shrink or expand in response to shocks to their decreasing returns to scale technology. Growing entails posting vacancies, filled either by the unemployed or by employees poached from other firms. Tractability is obtained by proving that all workers and firms’ decisions are characterized by productivity and size. The estimated model yields empirically consistent cross-sectional patterns of net poaching, and rationalizes U.S. labor market dynamics around the Great Recession.
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Chapter 1

The Geography of Unemployment

1.1 Introduction

Unemployment rates vary enormously across local labor markets. In 2017 in Versailles, an affluent French city close to Paris, only five out of a hundred workers were unemployed. In southern Marseille, that ratio exceeded twelve out of a hundred. Such differences can be found in most developed countries, including the United States.\(^1\) Despite their magnitude, these spatial gaps persist over decades. Being exposed to unemployment in a strong labor market is undoubtedly harmful, but spending a lifetime in a distressed labor market can have dire consequences for workers. While local governments devote billions of dollars every year to attract jobs, there is but scant guidance as to the determinants of spatial unemployment differentials. Why is the unemployment rate persistently high in some places, while it remains low in others? What are the welfare implications of this spatial dispersion for workers? Can place-based policies improve the prospects of local residents as well as labor market conditions in the aggregate economy?

This paper proposes answers to these questions with four contributions. First, I offer new empirical evidence showing that spatial unemployment differentials result from spatial

\(^1\)In 2017, the unemployment rate was 5.4% in Boston, Massachusetts. It was 13% in Flint, Michigan. See the OECD (2005) report for more OECD countries.
gaps in the rate of job loss, tied to employers rather than to workers. Second, I propose a theory of the location choice of employers with labor market frictions that accounts for spatial differences in job stability. Third, I structurally estimate the framework on French administrative data. Fourth, I quantify the local and aggregate welfare gains from place-based policies in general equilibrium. This paper thus consists of four parts, one for each contribution, which I now describe in more detail.

In the first part of the paper, I examine how local labor market flows differ between locations. I use French administrative matched employer-employee data to assess whether differences in unemployment rates across commuting zones reflect differences between inflow (job losing) versus outflow (job finding) rates. Strikingly, differences in job losing rates emerge as the primary source of spatial unemployment differentials, accounting for 86% of the variation. In contrast, job finding rates are close to constant across locations. Controlling for industry and worker composition does not affect these results, which also hold in publicly available data in the United States. The dominant role of the job losing rate indicates that locations have high unemployment because workers repeatedly lose their job there, not because finding a job is particularly hard. This result contrasts with aggregate unemployment fluctuations, as well as with existing models of spatial unemployment, that have focused on the job finding rate.\(^2\) The limited role of worker composition suggests that spatial gaps in the rate of job loss arise because of systematic differences in job stability between employers.

In the second part of the paper, I propose an analytical theory to account for spatial gaps in job losing rates. Workers choose freely where to live and work, and employers choose where to open jobs.\(^3\) They meet in frictional local labor markets, with housing in limited supply. Jobs differ in their initial productivity. Job productivity subsequently fluctuates due to idiosyncratic shocks, leading to endogenous job loss. As a result, initially more

\(^2\)Changes in the job finding rate have been found to be the dominant force in aggregate unemployment fluctuations along the business cycle. See Shimer (2005), Hall (2005), Fujita and Ramey (2009) and Krusell et al. (2017).

\(^3\)As is common in the search literature, there is no difference between employers, firms and jobs in my framework.
productive jobs are more stable. The distinct interaction of the location choice of employers and labor market frictions gives rise to labor market pooling complementarities that lie at the heart of the model’s implications. Employers trade off higher wages against high vacancy contact rates across locations. More productive employers make higher profits when operating. Thus, they forego relatively more than unproductive employers while waiting for a worker: productive employers have a higher opportunity cost of time. Hence, they prefer locating in slack labor markets where they fill vacancies rapidly but wages are high. In contrast, unproductive employers are priced out where wages are high but forego lower profits where the vacancy contact rate is low. Thus, they self-select into low wage areas where the labor market is tight and vacancies are filled slowly. As a result, sorting emerges in spatial equilibrium.

Labor market flows reflect the spatial sorting of employers. The job losing rate is high where employers are unproductive. Crucially, job finding rates depend on two components. The first is the rate at which workers contact employers, which rises when there are more vacancies relative to unemployed workers. But not all contacts result in a viable job. The second component of job finding rates is the conditional probability that a contact is successful. I argue that both components offset each other. In locations with many unemployed workers, there are also more employers since labor is cheaper. On net, the labor market is tighter, workers contact many employers, and employers contact few workers. Only unproductive employers locate in high-unemployment locations. There, expected wages are close to reservation wages. Thus, the probability that a contact results in a viable job is low, offsetting the higher worker contact rate. When both forces closely balance, the job finding rate is flat across locations.

I then show that the spatial equilibrium features misallocation. Because of labor market pooling complementarities, productive employers over-value the benefits from locating close to each other. Labor market frictions prevent productive employers from attracting as many workers as would be socially optimal, should they enter in a location with low productivity
employers. Hence, productive employers find it privately optimal to concentrate too much in top locations with a larger pool of workers relative to vacancies. A utilitarian planner thus chooses an optimal policy that incentivizes productive employers to relocate towards high unemployment areas. A profit subsidy that rises with the local job losing rate implements the optimal allocation, providing a rationale for commonly used place-based policies that target high unemployment locations.

The third part of the paper develops and structurally estimates a quantitative version of the framework. The main additions are local productivity and amenity differences; migration frictions that hinder workers’ ability to move; and heterogeneity in human capital, which grows while workers accumulate experience at work, and depreciates while they are unemployed. To isolate the importance of labor market pooling externalities, I abstract from other sources of agglomeration. Despite its richness, the quantitative model produces estimating equations that allow for transparent identification leveraging the many dimensions of the French administrative data. In particular, the estimation directly targets neither the cross-sectional variance of local unemployment rates nor its breakdown into job losing and job finding rates.

The estimated model accounts for the primary margins of spatial unemployment differentials. It generates over 90% of the cross-sectional variance of local unemployment rates in the data. It also closely replicates the respective contributions of job losing and job finding rates. 86% stems from the job losing rate in the data, against 85% in the model. Pooling externalities are crucial to rationalize the location choice of employers, and hence job losing rate differences. Shutting down pooling externalities reduces the spatial variation in unemployment rates by over 80%. Over-identification checks support these results by highlighting that the model can match a number of non-targeted moments.

I then propose a set of validation exercises to support the core structure of the theory. First, I verify the key link between labor productivity differences and job losing rates. Leveraging firm-level balance sheet data for the near universe of French businesses, I show that
labor productivity is 37% lower in locations with a one percentage point higher job losing rate. This gap rises by an additional 43% for establishments less than two years old, that have a higher proportion of new jobs. In contrast, labor productivity growth is flat across locations. Second, I use survey data to unpack how job finding rates reflect worker contact rates and contact-to-job probabilities. Consistent with the view that more productive employers locate where there are few vacancies per job seeker, I find that worker contact rates rise with the unemployment rate. In contrast, the proportion of contacts resulting in a viable job falls with the unemployment rate.

The fourth and last part of the paper conducts two policy counterfactuals. The first exercise explores the local and aggregate effects of economy-wide place-based policies. I contrast the optimal policy with the French Enterprise Zones (EZ) program – which consisted in heavy subsidies for businesses opening jobs in high unemployment areas. In both cases, the policy is funded at the federal level. Qualitatively, the optimal policy resembles the French EZ policy. Quantitatively, however, it is much larger. By massively relocating productive jobs towards high job losing rate areas, the optimal policy cuts the local unemployment rate by over 10 percentage points in the most afflicted locations. It also achieves over 20% welfare gains for their residents. While half of those gains follow directly from the unemployment rate reduction, the other half are indirect amplification effects due to human capital improvements. Although the optimal policy primarily redistributes jobs across locations, it ameliorates aggregate outcomes: the aggregate unemployment rate falls by 0.5 percentage points and utilitarian welfare rises by 5.1%. In contrast, the model indicates that the French EZ program reduced unemployment only in targeted areas by 1 percentage point, leading to 3% local welfare gains but little aggregate effects.

The second policy counterfactual concludes the paper by assessing whether locally funded, discretionary place-based policies can achieve local outcomes similar to federal programs. Specifically, it examines how attracting a “Million Dollar Plant” (MDP) affects a location. In

\[4\text{The “Zone Franches Urbaines”.}\]
the model, a MDP calibrated to the estimates in Greenstone et al. (2010) lowers commuting-zone level unemployment by 0.2 percentage points, largely driven by improved job stability. Gross welfare of residents rises by 1.15%. In practice, attracting a MDP comes at a cost. To discipline that trade-off, I also compute the optimal MDP subsidy in the model financed by non-distortionary local taxes. The results indicate that locally funded policies can achieve as much as 20% net welfare gains in high unemployment locations.

**Literature.** This paper adds to four strands of literature. First and most closely related is the body of work that examines persistent spatial unemployment differentials. Kline and Moretti (2013), Şahin et al. (2014) and Marinescu and Rathelot (2018) study spatial variants of the Diamond (1982), Mortensen (1982), and Pissarides (1985) model. These papers focus on the role of the job finding rate and abstract from job losing rate differentials. Kline and Moretti (2013) find that subsidies to high unemployment areas reduce welfare. In contrast, I stress that a different theory is needed after documenting that job losing rate gaps are the key empirical determinant of spatial unemployment differentials. As a result, I find that subsidies to high unemployment areas raise welfare, reconciling theory with real-world place-based policies.

Second, this paper adds to the large literature that studies the location decisions of agents. A first subset thereof has focused on workers’ location decisions based on income prospects (Roback, 1982, Kennan and Walker, 2011a, Desmet and Rossi-Hansberg, 2013, Bilal and Rossi-Hansberg, 2018). A second set of papers studies firms’ location choices (Combes et al., 2012, Gaubert, 2018). Both literatures abstract from unemployment, while I show that inclusion thereof has distinct policy implications. A final strand of the literature

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7For structural change over time, see Diamond (2016), Giannone (2017) Caliendo et al. (2017), Glaeser et al. (2018), and Couture et al. (2019).
proposes theoretical assignment models to study the sorting between workers and employers (Sattinger, 1993, Shimer and Smith, 2000, Davis and Dingel, 2014, Eeckhout and Kircher, 2018), which the present paper builds on.

Third, this paper adds to the body of work that studies the efficiency properties of search models (Hosios, 1990, Mortensen and Pissarides, 1994).\textsuperscript{8} The labor market pooling externality can be seen as a spatial analogue of Acemoglu (2001). He shows that when high and low productivity jobs coexist in the labor market, too many low productivity jobs open in the aggregate labor market because they fail to internalize that they divert workers away from productive jobs. In my model, a similar force pushes less productive jobs to inefficiently locate in places that are too productive for them.\textsuperscript{9}

Finally, this paper is closely tied to the large literature on agglomeration and congestion externalities. Going back to at least Marshall (1920), externalities operating at the local level have formed the basis for place-based policies. Recent empirical analyses of the latter have found mixed employment effects across several countries (Glaeser and Gottlieb, 2008, Hanson, 2009, Neumark and Simpson, 2014, Mayer et al., 2015, Slattery and Zidar, 2019). Several recent papers propose spatial models with either or both worker and firm mobility to analyze place-based policies but all abstract from unemployment (Ossa, 2017, Fajgelbaum et al., 2018, Fajgelbaum and Gaubert, 2018, Slattery, 2019). In many cases, agglomeration economies call for subsidies to high income locations, which contrasts with many real-world place-based policies. While the overall net policy should account for that largest possible set of agglomeration and congestion externalities, I highlight and quantify a particular mechanism whereby labor market pooling externalities favor subsidies to low income locations. The idea that redistributing a given set of jobs across heterogeneous local labor markets can improve aggregate outcomes even in the absence of technological spillovers goes back at least

\textsuperscript{8}See Jarosch (2015) and Mangin and Julien (2018) for recent contributions.

\textsuperscript{9}In contemporaneous work, Brancaccio et al. (2019) emphasize a similar mechanism in the context of transport markets.
to Bartik (1991), and has been recently revived by Austin et al. (2018). This paper proposes a theory of frictional local labor markets that makes this idea precise.

The remainder of the paper is structured as follows. Section 1.2 presents the data and empirical analysis. Section 1.3 builds a simple model of spatial unemployment differentials with endogenous job loss and characterizes the spatial equilibrium. Section 1.4 lays out the quantitative extensions and the estimation strategy. Section 1.5 describes the estimation results and proposes additional validation exercises. Section 1.6 presents the policy counterfactuals. The last section concludes. Proofs and additional details can be found in the Appendix.

1.2 Descriptive evidence

This section first describes the data. Next, I highlight that spatial unemployment gaps are large and persistent. Then, I show that spatial unemployment gaps are primarily driven by spatial differences in job losing rates, independently from local worker and industry composition. My main analysis focuses on France where I can exploit the richness of administrative data, but I also confirm the main findings in the United States.

Data

Worker flows in and out of unemployment are central components of labor market studies. Aggregate time series exercises typically break down the contribution of job losing and job finding rates in accounting for the unemployment rate. While they are jointly determined equilibrium variables, separating their contributions is a useful diagnostic device that informs the underlying economic mechanisms.

Adapting this approach to a geographic setting is challenging. On the one hand, large repeated cross-sections like the Census or the American Community Survey (ACS) are ill-suited for the measurement of worker flows. On the other hand, surveys with a short panel
dimension such as the Current Population Survey (CPS) typically have a much smaller cross-section. This limitation leads to measurement error concerns, particularly for the outflow from unemployment, and prevents any compositional split. In addition, panel surveys often stop tracking movers who change location.

To circumvent these difficulties, I turn to administrative matched employer-employee data from France. I use a combination of the DADS and of the French Labor Force Survey (LFS) between 1997 and 2007. The DADS have two advantages. First, they are a representative dataset covering almost one million individuals in any cross-section. Second, it is a panel that consists of the entire work history of individuals, with rich demographic, geographic and firm-level information. Thus, the DADS are well-suited to study the employment versus non-employment status of individuals across cities. The sample size allows to break down the analysis by city, industry, and finely disaggregated worker groups to control for composition.

One drawback of the DADS is that it only allows to discriminate between employment and non-employment. To address this limitation, I first restrict my sample to males between 30 and 52 years old. This group has a high and stable labor force participation rate, which allows to abstract from life-cycle changes therein. Second, I complement the DADS with the LFS. I compute conditional transition probabilities between employment, non-employment and unemployment in the LFS, by broad city and worker group. I then use those conditional transition probabilities from the LFS to probabilistically discriminate between non-employment and unemployment in the DADS. In practice, this imputation has a limited impact on the results because non-employment and unemployment are closely related in the cross-section of locations. Therefore, the results are very similar if I use only

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10 Once broken down by city or commuting zone, the CPS data has about one hundred individuals by city, and thus only about five unemployed individuals in any cross-section.

11 DADS: “Déclaration Administrative de Données Sociales.” The LFS is the “Enquête Emploi.”

12 Consistent with the International Labour Office’s definition, I define an employed individual as one who has a job. A non-employed individual is one who is not working for a wage. An unemployed individual is one who is not working but is actively looking for a job and available to start work within two weeks.

13 This imputation exercise resembles Blundell et al. (2008) who use the Panel Study of Income Dynamics to complement consumption categories in the Consumption Expenditure Survey. For instance, if an individual goes through an employment to non-employment transition in the DADS, I define her employment status after the transition (unemployment or no-employment) based on the LFS transition probabilities.
the unemployment information in the LFS. I aggregate the resulting sample at the quarterly frequency. Table A.1 in Appendix A.1.1 shows that aggregate statistics in this sample and in the LFS alone are similar.

I complement these datasets with several other data sources. To compute city-level and establishment-level variables, I use a repeated cross-section version of the DADS that covers the universe of French workers. For some over-identifying exercises in Section 1.5.4, I use firm-level balance sheet data covering the near universe of French business for the same period. I also use a single cross-section of housing prices from an online realtor, MeilleursAgents.com.

I define a location as a commuting zone as defined by the French statistical institute INSEE. A commuting zone is an area where most of the residents work at jobs located in that same area. There are 328 commuting zones, and they partition the French territory. This definition is most natural as a spatial notion of a local labor market. In what follows, location, commuting zone and city are used interchangeably. I construct a measure of skill from occupation and age data because the main DADS panel dataset does not have education data. Skill is defined as the average age and occupation wage premium for a worker, derived from a Mincer regression. Appendix A.1.1 provides more details.

For the United States, I use the CPS. I define a location as a metropolitain statistical area, and use a similar definition of skill as in France. I focus on white males between 30 and 52 years old that are household heads, and use the CPS’s definition of unemployment.

1.2.1 Dispersion and persistence of spatial unemployment differentials

I start by showing that local unemployment rates are widely dispersed and highly persistent across locations in France. Figure 1.1 (a) maps commuting-zone level unemployment rates in mainland France. Darker shades of blue encode higher unemployment rates. Figure 1.1

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14 “Institut National de la Statistique et des Études Économiques”.
15 I also check that using education to define skill in the CPS leaves the results unchanged.
(a) highlights that commuting zones with unemployment rates above 10% or below 5% can be found throughout the country. The cross-sectional standard deviation is 2.3 percentage points, almost twice as much as the time-series standard deviation of the aggregate unemployment rate (1.3 percentage points).

Figure 1.1: Unemployment rates in France, by commuting zone and over time

(a) Local unemployment rates, 1997-2007 averages

(b) Persistence of local unemployment rates

To assess the persistence of spatial unemployment differentials, I then split the sample in two subperiods, 1997-2001 and 2002-2007. Figure 1.1 (b) plots the local unemployment rate in the second subperiod against the unemployment rate in the first subperiod for every city. The blue circles represent a city, with the size proportional to population. Figure 1.1 (b) reveals that local unemployment rates are highly persistent, as they line up closely around the orange 45 degree line. The 5-year autocorrelation is 0.86.16

Figure 1.1 confirms earlier findings from Kline and Moretti (2013) and Amior and Manning (2018) for the United States. I now turn to the main empirical contribution of this

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16In Appendix A.1, I show that controlling for economy-wide industry business cycles increases local persistence, with a conditional autocorrelation of 0.99.
paper: unpacking how worker flows in and out of unemployment differ between commuting zones.

1.2.2 Worker flows in and out of unemployment

Inflows from local employment, from non-participation and in-migration from other locations all contribute to local unemployment. Similarly, outflows into local employment, into non-participation and out-migration reduce the number of unemployed workers. In what follows, I borrow standard terminology from the literature and call the rate at which employed workers flow into unemployment the job losing rate. Similarly, I call the rate at which unemployed workers flow into employment the job finding rate.

To guide the analysis, start with a simple two-state accounting model. Suppose that employed workers in city $c$ face a constant job losing rate $s_c$ per unit of time (i.e. separation rate to unemployment), and that unemployed workers face a constant job finding rate $f_c$ per unit of time. Abstract from movements in and out of the labor force and migration for now. In steady state, the local equilibrium unemployment rate $u_c$ satisfies

$$\log \frac{u_c}{1-u_c} = \log s_c - \log f_c. \quad (1.1)$$

Both $s_c$ and $f_c$ can be directly measured in the data using transition probabilities between employment and unemployment.$^{17}$

To examine the respective contributions of the job losing and job finding rates to local unemployment, panel (a) in Figure 1.2 plots the logarithm of the job losing rate $s_c$ against the logarithm of the unemployment-to-employment ratio $\frac{u_c}{1-u_c}$ for France. The data align closely along the 45 degree line in orange, indicating that local job losing rates are the primary determinants of spatial unemployment differentials. Panel (b) in Figure 1.2 plots the

$^{17}$In principle, these quarterly transition probabilities must be corrected for time aggregation in order to obtain instantaneous transition rates. In practice, Figure A.4 in Appendix A.1.3 shows that the time-aggregation correction leaves the variance share of the job losing share virtually unchanged in France and the United States.
Figure 1.2: Local job losing and finding rates against unemployment-to-employment ratios in France.

(a) Job losing rate  
(b) Job finding rate  

logarithm of the job finding rate against the logarithm of the unemployment-to-employment ratio. In contrast to the job losing rate, the job finding rate appears almost flat across locations.\textsuperscript{18}

In practice, movements in and out of the labor force, migration and local transitional dynamics could introduce a wedge between the left-hand-side and right-hand-side of equation (1.1). They account for part of the dispersion around the 45 degree line in Figure 1.2 (a). To assess exactly how much variation stems from the job losing rate, Appendix A.1.3 extends equation (1.1) to a three-state model with labor force participation, which can then be used to for an exact variance decomposition. In France, the job losing rate accounts for 86% of the cross-sectional variation in spatial unemployment rate. The job finding rate accounts for almost all of the remaining 14%. Figure A.3 in Appendix A.1.3 replicates the same exercise in the United States. There, the job losing rate contributes 73%.

\textsuperscript{18}Similarly, Figure A.2 in Appendix A.1.3 shows that the job-to-job mobility rate is not systematically associated with the local unemployment rate.
1.2.3 Worker and industry composition

In principle, differences in the local industry mix and worker skill mix may account for some or all of the differences in unemployment and job losing rates in Figure 1.2. To unpack the contribution of worker and industry composition, I estimate econometric models of the following form:

\[ Y_{i,t} = \alpha_{C(i,t)} + \beta_{J(i,t)} + \gamma_{S(i)} + e_{i,t}, \]

(1.2)

where \( C \) denotes a city, \( J \) denotes a 3-digit industry, \( S \) denotes a skill group, \( i \) denotes a worker identifier, and \( t \) is a quarter. \( Y \) is an outcome of interest, for instance the local unemployment, job losing or finding rates, or some functional transformation thereof. \( \alpha_C \) is a city effect, \( \beta_J \) an industry effect, and \( \gamma_S \) a skill effect. \( e_{i,t} \) is a conditionally mean zero residual. I first estimate linear probability models with 232 industry fixed effects and 300 skill fixed effects. Then, I replicate the exercise from Figure 1.2 with the estimated city fixed effects \( \hat{\alpha}_c \).\(^{19}\)

Figure 1.3 reveals that industry and worker composition do not contribute significantly to spatial unemployment differentials. In addition, Figure 1.3 shows that, even after controlling for local composition, the job losing rate remains the dominant source of spatial unemployment gaps. In fact, its variance share increases to 91% when using the estimated city fixed effects.

To make sure that my results are not driven by small sample biases or functional form assumptions, I also estimate probit models as well as correlated random effect models in Figures A.6 and A.7 in Appendix A.1.4. In all cases, industry and worker composition do not account for more than 20% of spatial differences in job losing or job finding rates.

One remaining concern is that unobserved worker heterogeneity within the 300 skill groups drives part of the spatial differences in job losing rates. I test for that possibility

\(^{19}\)For variable \( Y \), I impose that the city fixed effects have a mean equal to the unconditional mean of \( Y \).
Figure 1.3: Local job losing and finding rates against unemployment-to-employment ratios in France. City fixed effects net of local industry and worker composition.

(a) Job losing rate

(b) Job finding rate

by also estimating equation (1.2) with worker effects in the French data.\textsuperscript{20} Figure A.8 in Appendix A.1.4 indicates that over 70% of the spatial variation still stems from city effects in that case.

1.2.4 Conditional correlations

I briefly conclude this section by discussing two additional dimensions of the spatial distribution of worker flows that will inform the quantitative features of the model. First, I describe conditional correlations between local labor market flows and two local observables: commuting zone wages and population density. Table A.2 in Appendix A.1.5 indicates that spatial gaps in job losing rates are negatively correlated with local wages. Job losing rates appear uncorrelated with local population density conditional on wages and compositional controls. Second, Figure A.9 in Appendix A.1.6 shows that spatial gaps in job losing rates

\textsuperscript{20}This procedure imposes stronger data requirements. Mobility conditional on city, industry and worker effects must be random. See Card et al. (2013) for details.
are starkest during the first two years of a job, and stabilize thereafter.

Overall, the results in this section indicate that spatial differences in job losing rates are by far the largest contributor to spatial unemployment rate differentials in France and in the United States. In addition, these spatial differences are not explained by the local industry mix or composition of the workforce, suggesting systematic differences on the employer side. These findings are, to the best of my knowledge, new to the literature. They elude existing models of local unemployment that have focused on the job finding rate. In contrast, the job losing rate lies at the heart of the theory I propose below.

1.3 A model of spatial unemployment differentials

This section develops an analytical theory of spatial unemployment differentials. I build on Kline and Moretti (2013)’s model of frictional unemployment in spatial equilibrium. I add two key ingredients. First, heterogeneous employers decide where to locate. More productive employers prefer filling vacancies faster in locations with slack labor markets where hiring is easy. Hence, spatial sorting emerges in equilibrium. Second, job loss is endogenous and tied to employers. Thus, spatial gaps in job losing rates arise. Distinct normative implications follow, that I present after describing the structure of the model in more detail.

1.3.1 Setup

Time is continuous. There is a single final good used as the numeraire and freely traded across locations.

Geography. There is a continuum of ex-ante heterogeneous locations endowed with one unit of housing. Locations differ in productivity $\ell$ with cumulative distribution function $F_{\ell}$ on a connected support $[\ell, \tilde{\ell}]$, with density $F'_{\ell}$. Thus, a location is characterized by its productivity $\ell$ rather than its particular name. Local productivity differences $\ell$ are useful
to fix ideas and provide a natural ordering of locations, but are not necessary for the main mechanism.

**Workers.** There is a unit mass of infinitely-lived homogeneous workers. Their preferences over streams of consumption of the final good $c_t$ and housing services $h_t$ are

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \left( \frac{c_t}{1-\omega} \right)^{1-\omega} \left( \frac{h_t}{\omega} \right)^\omega \, dt \right],$$

with $\omega > 0$. Workers consume their income each period. They only search when unemployed.\(^{21}\) Workers are freely mobile across locations.

**Employers and jobs.** As is common in the search literature, the productive unit is an employer-worker match. Thus, the notions of firms, employers and jobs are interchangeable in the model.\(^{22}\) An employer pays a fixed cost $c_e$ to open a new job. After paying $c_e$, the employer draws a job quality (or expected productivity) $z$ that informs its initial productivity draw. The population distribution of quality $z$ is $F_z$, with connected support $[\underline{z}, \bar{z}]$ and density $F'_z$. After observing job quality $z$, employers choose a location $\ell$ to open their job and search for workers by posting a single vacancy in the local labor market. After they match with a worker, they draw their initial match productivity $y_0$ from a conditional distribution $G_0(y_0|z)$ that depends on the employer’s quality $z$. Drawing a higher $z$ implies that the job will be more productive on average, in a sense made precise in Assumption 1 below. After observing this initial draw, the matched pair decides to start producing together or not. If not, the worker returns to unemployment, and the job disappears.

\(^{21}\)I do not incoroporate job-to-job mobility, for two reasons. First, job-to-job moves do not directly affect the unemployment rate as they relocate workers from one job to another. Second, I show in Figure A.2 in Appendix A.1.3 that, just like the job finding rate, the job-to-job mobility rate is not systematically correlated with the unemployment rate. Together, these remarks suggest that including job-to-job moves would have small quantitative effects on counterfactuals affecting spatial unemployment differentials.

\(^{22}\)The model can also be seen as one in which there are large, constant-returns-to-scale firms that open many jobs at cost $c_e$ per job. For models with a well-defined notion of firm size through decreasing returns to scale and search frictions, see Bilal et al. (2019b), Schaal (2017) and Elsby and Michaels (2013). I explore the role of the boundary of the firm in the data in Section 1.5.4.
An active job with productivity $y_t$ in a location $\ell$ produces $y_t\ell$: there is a technological complementarity between the local productivity $\ell$ and the job’s idiosyncratic productivity $y_t$. Over time, the productivity of the job fluctuates according to a geometric Brownian motion

$$d\log y_t = -\delta dt + \sigma dW_t,$$

(1.3)

where $\delta > 0$: productivity depreciates on average. This assumption is required for endogenous separations to take place, as well as for a well-defined steady-state distribution to arise.\(^{23}\) $\sigma$ is the volatility of shocks. A geometric Brownian motion is the continuous-time analogue of an otherwise standard random walk with drift. Importantly, the productivity process is identical in all locations, so that any spatial differences in job loss must originate from differences between employers. For values to remain finite, I impose that $\rho + \delta > \frac{\sigma^2}{2}$. If the match breaks up, the job disappears.

**Local labor markets.** Unemployed workers search for jobs only in the location where they live, and employers search for workers only in the location where their job is open. Workers randomly meet vacancies in a single labor market in each location. Meetings occur according to a Cobb-Douglas matching function $\mathcal{M}(U(\ell), V(\ell)) = mU(\ell)^\alpha V(\ell)^{1-\alpha}$, where $U(\ell)$ is the local number of unemployed workers, and $V(\ell)$ is the local number of vacancies in that market.\(^{24}\)

Local market tightness is $\theta(\ell) = V(\ell)/U(\ell)$. Workers’ local contact rate is then $f(\theta(\ell)) = m\theta(\ell)^{1-\alpha}$ while the vacancy contact rate for employers is $q(\theta(\ell)) = m\theta(\ell)^{-\alpha}$. The contact rate may differ from the realized finding rate if some contacts do not result in a new job. Denote by $f_R(\ell)$ and $q_R(\ell)$ the realized rates.

---

\(^{23}\) $\delta < 0$ reflects the difference between parameters governing productivity growth at new jobs relative to incumbent jobs in endogenous growth models such as Engbom (2018).

\(^{24}\) The main results extend to the case of a non-Cobb-Douglas constant returns to scale matching function if its elasticities are bounded away from 0 and 1.
**Flow value of unemployment.** Unemployed workers in location $\ell$ consume $b\ell$. This specification captures the idea that unemployment benefits are a constant replacement rate of past wages, because wages will scale with local productivity $\ell$. It also helps with analytical tractability.\textsuperscript{25}

**Wage determination.** Workers and employers bargain à la Rubinstein (1982), with worker bargaining power $\beta$. For simplicity, I assume that renegotiation occurs each instant.\textsuperscript{26}

**Ownership.** A representative mutual fund owns housing and claims to employers’ profits. The mutual fund rents land to workers at equilibrium rents $r(\ell)$. It also collects profits from employers. For simplicity, I assume in this section that absentee owners receive the profits from housing rents and firms.\textsuperscript{27}

### 1.3.2 Value functions and wage determination

In what follows, the economy is in steady-state.

**Unemployment and employment.** Let $U$ be the value of unemployment. Because unemployed workers are freely mobile, their value must be equalized across all locations that they populate. The Inada property of the matching function ensures that any populated location must have some unemployed workers. Thus, the value of unemployment is equal across all populated locations.\textsuperscript{28}

\textsuperscript{25}The specification can also be seen as home production or self-employment with the same production function as firms, but with an efficiency $b$. Because the model features aggregate constant returns to scale in production, defining unemployment benefits to be directly a constant replacement rate of past wages leads to multiplicity.

\textsuperscript{26}As in Bilal et al. (2019b), wage setting protocols such as renegotiation by mutual consent (Cahuc et al., 2006, Postel-Vinay and Turon, 2010b) leave the surplus of matched pairs unchanged and the resulting allocations coincide with those under renegotiation each instant. Wage dynamics differ.

\textsuperscript{27}Alternatively, the proceeds from land rents and profits can be rebated to workers as a flat earnings subsidy. In that case the cross-sectional implications are unchanged. To keep the focus on the efficiency properties of the location choice of employers and abstract from distributional considerations between owners and workers, I use the flat earnings subsidy rebate in the quantitative exercises.

\textsuperscript{28}Specifically, the Inada property reads $\lim_{U \downarrow 0} \frac{\partial M}{\partial U}(U,V) = +\infty$. 
To keep the exposition simple in the main text, I consider wage functions $w^*(y, \ell)$ that only depend on productivity $y$ and the location $\ell$. As shown in Appendix A.2.1, this restriction is without loss of generality. Let $V^E(y, \ell)$ be the value of employment at wage $w^*(y, \ell)$ in location $\ell$. $U$ and $V^E$ satisfy the recursions

$$\rho U = b\ell r(\ell)^{-\omega} + f(\ell)E_\ell \left[ \max \{V^E(y_0, \ell) - U, 0\} \right]$$  \hspace{1cm} (1.4)$$

$$\rho V^E(y, \ell) = w^*(y, \ell)r(\ell)^{-\omega} + (L_y V^E)(y, \ell).$$  \hspace{1cm} (1.5)$$

The first term on the right-hand-side reflects workers’ flow value when unemployed or employed. Because of Cobb-Douglas preferences their housing choice, workers spend a constant share $\omega$ of their income on housing. Hence, workers’ flow value is income adjusted by local housing prices $r(\ell)$. The second term on the right-hand-side of equation (1.4) reflects unemployed workers’ future employment opportunities. At rate $f(\ell)$, they meet potential employers. The latter then draw the initial productivity $y_0$. Provided it is sufficiently high, the worker is hired and the matched pair starts producing together. Because initial productivity $y_0$ is unknown prior to meetings, the value of employment opportunities reflects the expected value from employment conditional on the pool of employers in location $\ell$. The second term on the right-hand-side of equation (1.5) reflects the expected continuation value of employment due to productivity shocks. Given the geometric Brownian motion assumption (1.3), the functional operator $L_y$ is defined by

$$L_y V^E = \left( \frac{\sigma^2}{2} - \delta \right) y \frac{\partial V^E}{\partial y} + \frac{\sigma^2}{2} y^2 \frac{\partial^2 V^E}{\partial y^2}.$$

**Employers.** The value of a matched employer with productivity $y$ in location $\ell$ solves

$$\rho J(y, \ell) = y\ell - w^*(y, \ell) + (L_y J)(y, \ell).$$
Employers value flow profits \( y\ell - w^*(y, \ell) \) as well as the contribution of future productivity changes.

**Joint surplus and wage determination.** A common solution method in search models is to focus on the joint surplus from an employed worker and her employer. The wage then drops out, which allows to use the standard solution to Rubinstein (1982)’s split-the-pie game. In the present geographic setting, workers consume local housing with heterogeneous prices across locations. Therefore, workers’ marginal valuation of a dollar (or euro) depends on local housing prices. In contrast, employer’s marginal valuation of a dollar does not. Therefore, adding up the worker’s and the firm’s surplus does not have a well-defined interpretation.

Despite this apparent complication, I show in Lemma 9 in Appendix A.2.3 that the bargaining game still delivers a simple solution for wages, because both parties have differing but constant marginal valuations of a dollar. The only requirement is to put each side’s value on a common numeraire scale. This property obtains because wage determination in the alternating offers game depends on each side’s valuation a dollar relative to their own outside option, not relative to the other side’s valuation.

Lemma 9 greatly simplifies the analysis and extends otherwise standard bargaining results. It allows to restrict attention to a single object, that I dub the adjusted surplus. It is defined as

\[
S(y, \ell) = J(y, \ell) + r(\ell)^\omega \cdot (V^E(y, \ell) - U)
\]  

(1.6)

It is independent from wages. Appendix A.2.1 shows that it follows a recursion similar to that of employers. Lemma 9 then implies that wages adjust so that workers and employers each receive a constant adjusted share of the adjusted surplus, Namely,

\[
r(\ell)^\omega \cdot (V^E(y, \ell) - U) = \beta S(y, \ell) , \quad J(y, \ell) = (1 - \beta)S(y, \ell) .
\]  

(1.7)
In particular, both sides agree when to break up the match when the adjusted surplus drops to zero. In that case, a separation occurs. Existing matches therefore solve a forward-looking optimal stopping problem, which is detailed in Appendix A.2.3. There, I characterize its solution which is described in the following lemma, useful for future reference.

**Lemma 1.** *(Adjusted surplus)*

There exists a unique solution adjusted surplus, given by

\[
\rho S(y, \ell) = \ell(b + v(\ell))S \left( \frac{y}{y(\ell)} \right)
\]

for \( y \geq y(\ell) \), and \( S(y, \ell) = 0 \) for \( y \leq y(\ell) \), where

\[
\frac{y(\ell)}{y_0} = b + v(\ell), \quad v(\ell) = \frac{f(\ell)r(\ell)\E_e[\max\{V_E(y_0, \ell) - U, 0\}]}{\ell}, \quad S(Y) = \frac{\tau Y + Y^{-\tau}}{1 + \tau} - 1,
\]

and \( \tau, y_0 \) are transformation of \( \rho, \delta, \sigma \) given in Appendix A.2.3.

The local endogenous separation cutoff \( y(\ell) \) increases as the worker’s local value of unemployment relative to housing prices, \( b + v(\ell) \), rises. The latter is the equilibrium outcome of local market tightness \( \theta(\ell) \) and the local mix of employers. The adjusted surplus \( S \) is an increasing function of current productivity \( y \) relative to the local endogenous cutoff \( y(\ell) \). The nonlinearity in the function \( S \) arises because of the option value of separation, which rises as productivity \( y \) approaches the cutoff \( y(\ell) \). Hence, the adjusted surplus \( S \) satisfies both the value matching and smooth-pasting conditions at the cutoff: \( S(y(\ell), \ell) = \frac{\partial S}{\partial y}(y(\ell), \ell) = 0 \).\(^{29}\)

It is also useful to define workers’ reservation wage \( w(\ell) \) in each location, in efficiency units of local productivity \( \ell \). It satisfies

\[
w(\ell) = w_0 y(\ell), \quad (1.8)
\]

\(^{29}\)The term \( Y^{-\tau} \) rises as \( Y \) approaches 1 from above. When an adverse productivity shock pushes the match below the cutoff, both parties are better off separating rather than producing at below cutoff productivity, thereby insuring the pair against negative shocks. As productivity approaches the cutoff from above, the probability of productivity dropping below the cutoff rises, and so must the option value of separation.
where $w_0 = (1 - \beta)\rho / y_0 + \beta$. When the local separation threshold is higher, matches break up at higher productivity levels because, relative to local housing prices, workers value more the option to search for a different job in the same local labor market. Thus, the local reservation wage is higher.

Given reservation wages $w(\ell)$, the free mobility condition takes a simple form,

$$U = \frac{\ell w(\ell)}{w_0 y_0 r(\ell)^\omega}.$$  \hspace{1cm} (1.9)

Across locations, higher housing prices compensate either higher productivity or a higher local reservation wage. Employed workers do not move because their value exceeds the common value of unemployment. With those results at hand, it is now possible to characterize the location choice of employers.

### 1.3.3 The location choice of employers

An employer with a quality $z$ contemplates the expected value from entering in each location, and chooses the location that delivers the highest payoff. When it matches, the employer receives a share $1 - \beta$ of the adjusted surplus. The employer’s expected payoff in each location $\bar{J}(z, \ell)$ then follows from integrating over the job’s initial productivity distribution $G_0(y_0|z)$, adjusted for the vacancy contact rate $q(\ell)$:

$$\rho \bar{J}(z, \ell) = q(\ell)(1 - \beta) \int S(y_0, \ell) dG_0(y_0|z)$$  \hspace{1cm} (1.10)

To facilitate the exposition, I assume that the starting distribution $G_0$ is Pareto in the main text. I show that the Pareto assumption is empirically plausible in Section 1.5.4. Nonetheless, I also provide more general distributional conditions under which my results hold in Appendix A.2.4.
Assumption 1. *(Initial productivity distribution)*

Assume that the conditional starting distribution is Pareto with support \([Y, +\infty)\),

\[
G_0(y_0 | z) = 1 - \left( \frac{Y}{y_0} \right)^{\frac{1}{1-z}}, \quad z \in (0, 1).
\]

Under Assumption 1, Lemma 1 implies that the expected payoff of job \(z\) in location \(\ell\) satisfies

\[
\log \left( (\bar{\rho} \bar{J}(z, \ell))^{\frac{1}{1-z}} \right) = \frac{z}{1-z} \log \bar{S}(z) + \frac{z}{1-z} \log \ell + \frac{z}{1-z} \log q(\ell) - \log w(\ell) \quad (1.11)
\]

where \(\bar{\rho} = \rho + \frac{\beta}{1-\beta} y_0\) and \(\bar{S}(z) = (Y/y_0)^{1/z} \frac{z}{1-z} \frac{\tau z}{1-\tau z + 1} \).

The four terms on the right-hand-side of equation (1.11) reveal four forces that shape how employers value different locations. The first term encodes the absolute advantage of employers according to their job quality \(z\). High quality jobs draw from a better starting distribution, have higher productivity on average and earn higher profits regardless of their location. This term does not affect the location choice of employers.

The second term reflects standard technological complementarities in production. From the production function, more productive employers benefit relatively more from high local productivity \(\ell\). As a result, they value locating in more productive locations relatively more than unproductive employers. While local productivity differences \(\ell\) are useful to fix ideas because they define a natural ordering of locations, they are not the central ingredient of the model. In fact, I show in Corollary 1 below that one can think of productivity \(\ell\) being identical to 1 in every location, and \(\ell\) simply indexing geographically distinct but otherwise ex-ante identical locations.

In contrast, the third term in equation (1.11) lies at the core of the mechanism this paper proposes. It reveals that more productive employers value relatively more locations where hiring is easy – where the vacancy contact rate \(q(\ell)\) is high. Because more productive employers generate higher profits, waiting longer until they meet a worker and start producing is relatively more costly for them. Higher foregone profits translate into a higher opportu-
nity cost of time for more productive employers. Importantly, some contacts do not result in a viable match, so that the vacancy filling rate and the vacancy contact rate differ. The probability that a contact results in a match, \( \left( \frac{Y}{y(\ell)} \right)^{1/z} \), depends on both the employer type \( z \) (first term) and on local reservation wages \( w(\ell) \) through the separation threshold \( y(\ell) \) (last term).

The vacancy contact rate \( q(\ell) = m\theta(\ell)^{-\alpha} \) is an equilibrium object that depends on local market tightness \( \theta(\ell) \). Ultimately, it depends on the pool of employers and workers who choose to locate in \( \ell \). Therefore, I follow Marshall (1920)'s terminology and call the complementarity between the employer’s productivity \( z \) and the location’s vacancy contact rate \( q(\ell) \) a pooling complementarity. In contrast to technological complementarities which can be found in the assignment literature without frictions, the pooling complementarity arises at the confluence of the location choice of heterogeneous employers and frictional local labor markets.

Finally, the fourth term in equation (1.11) reflects the expected cost of labor in a particular location \( \ell \), which can be summarized by the reservation wage \( w(\ell) \). All employers prefer locations with low labor costs where the reservation wage is low.

In equilibrium, local reservation wages are related to local vacancy contact rates though labor market tightness \( \theta(\ell) \). Therefore, employers face a trade-off between local vacancy contact rates and local wages. From the pooling complementarity, more productive employers value high vacancy contact rates relatively more. As a result, productive employers are willing to pay more for locating in places with a slack labor market and a high vacancy contact rate. In contrast, unproductive employers are priced out in high wage locations, while they forego lower profits by waiting for workers in locations with tight labor markets.

The differential valuation of locations by different employers plays the role of a single-crossing condition. The value of employers reflects their forward-looking optimal separation decision, which is fully determined by current productivity \( y \) relative to the endogenous threshold \( \underline{y}(\ell) \) as per Lemma 1. From the Pareto distribution, the probability of drawing a
below-threshold starting productivity decays more slowly for high quality employers.\footnote{Indeed, \( \frac{\partial^2 \log(1-G_0(y_0|z))}{\partial z \partial \log y_0} > 0. \)} Thus, both the Pareto assumption and the structure of the forward-looking optimal separation problem give rise to the single-crossing property. Appendix A.2.4 proposes more general distributional conditions under which employer payoffs exhibit the single-crossing property. For instance, it also arises if the starting distribution is a mass point at \( y_0 = z \) and the starting productivities \( z \) are sufficiently far from the largest cutoff \( \max_\ell y(\ell) \).

An employer with quality \( z \) thus solves

\[
\ell^*(z) = \arg\max_\ell \frac{z}{1-z} \log \ell + \frac{z}{1-z} \log q(\ell) - \log w(\ell) \tag{1.12}
\]

Although every active job faces a dynamic optimal stopping problem in each location, the explicit solutions in Lemma 1 allow to simplify the location choice problem to one that shares many features with standard static assignment problems. Examples thereof can be found in Sattinger (1993) and Davis and Dingel (2014).\footnote{For an in-depth exposition of the underlying theory, see Topkis (1998), Villani (2003) and Galichon (2016).} Apart from the underlying dynamic production decision, a distinction with those studies arises. Traditional assignment problems resolve the sorting between two-sided markets with exogenously given quantities. In contrast, in the present model, local labor markets clear through the adjustment of labor market tightness \( \theta(\ell) \). The latter in turn feeds back into the vacancy contact rate, thereby adding an additional layer of general equilibrium effects to the payoffs that determine the assignment. This feedback acts as an agglomeration force, with two implications. First, cities with different ex-post characteristics emerge in equilibrium even in the absence of ex-ante heterogeneity.\footnote{See Gaubert (2018) for a similar idea when employers’ technology directly depends on city population.} Second, well-known multiplicity issues may arise.\footnote{See Grossman and Rossi-Hansberg (2012) for an example of multiple equilibria in a spatial context with agglomeration economies and exogenous differences across locations.}

I define an assignment pair as a pair of functions \( \ell \mapsto (z(\ell), w(\ell)) \), where \( z(\ell) \) is the assignment function of employers to locations. It is the inverse of \( \ell^*(z) \). In this paper, I call
the assignment function, while $M$ is the matching function that determines contacts in the labor market. $w(\ell)$ is the equilibrium reservation wage that supports this location choice. To facilitate the exposition in the main text, I restrict attention to assignments that exhibit weak positive assortative matching, i.e. for which $z$ is increasing. In Appendix A.2.4, I show that this is only a mild restriction, for two reasons. First, when the matching function elasticity $\alpha$ is not too large, only assignments with positive assortative matching can exist. Second, any other potential steady-state assignment is dynamically unstable for any value of $\alpha$, in a sense made precise in Appendix A.2.4.

**Proposition 1. (Sorting)**

Suppose that Assumption 1 holds. Fix the equilibrium value of unemployment $U$ and the mass of new jobs $M_e$. There exists a unique solution $\ell \mapsto (z(\ell), w(\ell))$ to (1.12) among all possible assignments with increasing $z$. There exists a threshold $\alpha > 0$ such that for all $\alpha \in [0, \alpha]$, this solution is unique among all possible assignments. $z$ and $w$ are strictly increasing functions.

Proposition 1 establishes uniqueness of the assignment with positive assortative matching between local productivity $\ell$ and firm quality $z$: more productive jobs go to more productive locations. It also shows that this assignment is the only possible one when the matching function elasticity $\alpha$ is not too large. Proposition 9 in Appendix A.2.4 extends this result to all dynamically stable assignments and under more general distributional conditions for $G_0$.

The equilibrium response of local reservation wages $w(\ell)$ to the location choice of employers sustains the positive assignment. Reservation wages adjust up to the point where the marginal employer is indifferent between locations $\ell$ and $\ell + d\ell$. This adjustment reflects two forces. First, reservation wages reflect expected future wages conditional on starting work, which depend on equilibrium employer quality $z(\ell)$. Therefore, reservation wages rise with $\ell$, but less than one-for-one relative to wages of employed workers due to discounting. Second, reservation wages also reflect the job finding rate $f_R(\ell)$ and labor market tightness $\theta(\ell)$. As employers sort across locations, more workers locate in places with high expected wages and high employer quality $z(\ell)$. In response, labor market tightness $\theta(\ell)$ falls there, reducing
the job contact rate $f(\theta(\ell))$. Since the value of search $v(\ell)$ reflects both the rising expected wages conditional on work and the falling job contact rate $f(\theta(\ell))$, reservation wages $w(\ell)$ rise with $\ell$, but again less than one-for-one relative to $z(\ell)$. By characterizing the allocation of heterogeneous jobs to locations, these results deliver predictions for spatial unemployment differentials. I turn thereto in the next section.

1.3.4 Endogenous job loss and unemployment

In every location, the job losing rate depends on three forces: the average starting productivity at new jobs, the productivity separation threshold, and how fast productivity depreciates from the starting productivity down to the threshold. The productivity depreciation rate is governed by the productivity process (1.3) and is constant across locations by assumption. Therefore, any differences in local job losing rates must arise because of differences in the ratio between the starting productivity and the separation threshold. Both are related to the equilibrium assignment function $z(\ell)$ and reservation wage $w(\ell)$.

To make this argument precise and determine how many workers lose their job per unit of time, it is necessary to solve for the invariant distribution of employment across productivities in each location $\ell$. Denote $g(y, \ell)$ its density function. In steady-state, $g(y, \ell)$ solves the Kolmogorov Forward Equation (KFE),

$$0 = (L_y^* g)(y, \ell) + n(\ell)g_0(y, \ell), \quad y > y(\ell),$$

(1.13)

where $g_0(\cdot, \ell)$ is the density associated with the entry distribution $G_0(y_0|z(\ell))$, which in turn depends on the equilibrium quality of jobs $z(\ell)$ that open in location $\ell$. $n(\ell)$ is the endogenous inflow of unemployed workers into employment. The operator $L_y^*$ encodes how productivity shocks shape the distribution.\textsuperscript{34} Under the geometric Brownian motion assumption (1.3), it

\textsuperscript{34}It is the formal adjoint of the operator $L_y$. See Appendix A.2.5 for a heuristic derivation of the KFE (1.13), and Oksendal (1992) for a formal derivation.
is given by

\[(L^*_y g)(y) = -\left(\frac{\sigma^2}{2} - \delta\right) \frac{\partial}{\partial y} \left( y g(y, \ell) \right) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} \left( y^2 g(y, \ell) \right).\]

By construction, the density \(g\) must integrate to unity in each location: 

\[1 = \int_{y(\ell)}^{\infty} g(y, \ell) dy.\]

Because of Brownian shocks, the distribution must satisfy the additional boundary condition 

\[g(y(\ell), \ell) = 0.\]

There always exists a closed-form solution to the KFE (1.13) with the boundary condition. To facilitate exposition in the main text, Lemma 2 describes that solution under Assumption 1. The generalized solution is given in Appendix A.2.5.

**Lemma 2.** (Employment distribution)

Let \(\kappa = \frac{2\delta}{\sigma^2}\). Under Assumption 1, the solution to the KFE (1.13) with \(g(y(\ell), \ell) = 0\) satisfies

\[g(y, \ell) = \frac{\kappa}{\kappa z(\ell) - 1} \left[ \left( \frac{y}{y(\ell)} \right)^{-\frac{1}{\kappa z(\ell)}} - \left( \frac{y}{y(\ell)} \right)^{-\kappa} \right], \quad \forall y \geq y(\ell).\]

The steady-state distribution has two components. The first component reflects the productivity distribution of new jobs. The invariant productivity distribution inherits the right tail from the starting distribution \(1/z(\ell)\). The right tail is thicker in locations with high quality \(z(\ell)\). The second component reflects the productivity process. When the negative drift \(\delta\) is higher, \(\kappa\) is higher, implying that the distribution is more left-skewed as productivity depreciates faster. When volatility \(\sigma\) is higher, \(\kappa\) is lower and the distribution is more right-skewed: more jobs receive large positive shocks, while large negative shocks are truncated due to endogenous job loss. Finally, the entry rate \(n\) does not appear because it simply scales the overall mass of employed workers, as in Hopenhayn and Rogerson (1993a).

Having solved for the invariant distribution in each location \(\ell\), it is possible to determine the endogenous job losing rate \(s(\ell)\) (or separation rate into unemployment). Given the

\[35\text{In a small time period, the Brownian motion shocks dominate the negative drift. Because these shocks are symmetric, half of the workers close to the cutoff are pushed into unemployment in any small time period. Compounded over a non-zero time interval, this process leaves no workers at the cutoff. A formal proof is provided in Appendix A.2.5}\]
steady-state distribution, the local endogenous job losing rate depends on how many workers
are close to the cutoff. In Appendix A.2.5, I show that it is

\[ s(\ell) = \frac{\sigma^2}{2} \frac{\partial g}{\partial y}(y(\ell), \ell) . \] (1.14)

Recall that close to the cutoff, only workers who receive a negative shock become unemployed.
Because Brownian shocks dominate in any small time period and are symmetric, only half
of shocks result in job loss, hence the division by two. Because \( g(y(\ell), \ell) = 0 \), the number
of job losers follows from the second order contribution of the mass of workers close to the
cutoff, which is \( \frac{\partial g}{\partial y}(y(\ell), \ell) \).

Expression (1.14) for the local job losing rate is useful when combined with the explicit
solution to the distribution in Lemma 2. Together, they produce a simple solution to the
local endogenous job losing rate as well as for labor market flows at the local level. Again
to facilitate exposition in the main text, only the expression under the Pareto assumption is
presented in the main text. Appendix A.2.5 describes the general solution.

**Proposition 2.** (Spatial unemployment differentials)

Under Assumption 1, the local job losing, finding and unemployment rates in location \( \ell \) are

\[ s(\ell) = \frac{\delta}{z(\ell)} , \quad f_R(\ell) = f(\theta(\ell)) \times \left( \frac{Y}{y(\ell)} \right)^{1/z(\ell)} , \quad u(\ell) = \frac{s(\ell)}{s(\ell) + f_R(\ell)} . \]

In addition, the job losing rate is decreasing in \( \ell \).

The Pareto case is particularly transparent. When the negative drift \( \delta \) is higher, pro-
ductivity depreciates faster everywhere and the endogenous job loss rate increases in all
locations.\(^{36}\) In low \( \ell \) locations, local jobs are of low quality \( z(\ell) \). Hence, they draw from a

\(^{36}\)Perhaps surprisingly, the volatility of the productivity process \( \sigma \) does not affect the job losing rate with Pareto entry. This reflects two opposing forces. When volatility \( \sigma \) increases, matches receive larger negative shocks, which may push them into breaking up more frequently – the direct volatility channel. But matches are also subject to larger positive shocks, which raises the option value of producing and lowers the cutoff – the option value channel. In general, this second channel operates through the cutoff \( y \) that appears in the general expression for the job losing rate in Appendix A.2.5. When firms enter with a Pareto distribution,
left-skewed distribution and enter close to the endogenous threshold. Thus, they fall below the threshold early on and the local job losing rate is high. In high $\ell$ locations, local jobs are of high quality $z(\ell)$ and hence enter highly productive. Because of both discounting and the general equilibrium adjustment of labor market tightness, reservation wages $w(\ell)$ rise slower than the assignment function $z(\ell)$ across locations. Hence, the endogenous separation threshold $y(\ell)$ increases less than one-for-one across locations relative to $z(\ell)$. As a result, the ratio between the average starting productivity and the threshold $y(\ell)$ is larger and productivity takes more time to fall below the local threshold $y(\ell)$. Therefore, the job losing rate is low in high $\ell$ locations. Overall, positive assortative matching between firm quality $z$ and local productivity $\ell$ implies that the job losing rate is decreasing in local productivity.

By contrast, the job finding rate is the outcome of two opposing forces. It is the product of the worker contact rate and the probability that a given contact results in a job, the contact-to-job probability. First, the worker contact rate depends positively on labor market tightness $\theta(\ell)$. As more productive employers $z(\ell)$ benefit more from higher vacancy contact rates $q(\theta(\ell))$, the worker contact rate $f(\theta(\ell))$ is negatively correlated with $z(\ell)$. However, the contact-to-job probability $(Y/y(\ell))^{1/z(\ell)}$ pushes in the other direction. In locations with more productive employers $z(\ell)$, contacts are more likely to result in a job because new matches draw from a better productivity distribution, and because the endogenous separation threshold $y(\ell)$ rises less than one-for-one with local employer productivity. Both forces need not offset each other exactly, but when they almost do, the job finding rate is close to flat across locations.

Recall that, if anything, the job finding rate is moderately negatively correlated with the job losing rate in the data (Figure 1.2). In the model, pooling complementarities incentivize employers with stable jobs to locate where there are few vacancies per job seeker. As a result, the worker contact rate is positively correlated with the job losing rate. Thus, the model can rationalize the moderate negative correlation between job losing and job finding.

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the direct volatility channel and the option value channel exactly offset each other and changes in volatility do not affect the endogenous job losing rate in steady-state.
rates only if the contact-to-job probability more than offsets the direct correlation with the worker contact rate. In Section 1.5.4, I validate this implication of the model. Using survey data, I show that the worker contact rate is indeed positively correlated with the job losing rate while the probability that a contact results in a job pushes the other way more than one-for-one.

1.3.5 Equilibrium and comparative statics

Having characterized how the location choice of employers shapes spatial unemployment differentials, I close the economy in the decentralized equilibrium. Local housing and labor markets must clear in each location \( \ell \),

\[
\begin{align*}
    r(\ell) &= \omega L(\ell) \left( u(\ell)b\ell + (1 - u(\ell))\bar{w}(\ell) \right), \\
    \theta(\ell) &= \frac{M_e F'_z(z(\ell))z'(\ell)}{u(\ell)L(\ell)F'_\ell(\ell)},
\end{align*}
\]

(1.15)

where \( L(\ell) \) is population in location \( \ell \), and \( \bar{w}(\ell) = \int w^*(y, \ell)g(y, \ell)dy \) is the average wage in location \( \ell \).

Local housing prices reflect local expenditures on housing. The labor market clearing condition simply states that labor market tightness is the ratio between the number of vacancies and the number of unemployed workers in locations with productivity \( \ell \). The number of unemployed workers is the unemployment rate times total population across the \( F'_\ell(\ell)d\ell \) locations with productivity in \([\ell, \ell + d\ell]\). The number of vacancies in a location reflects the total number of new jobs, \( M_e \), but also the spatial sorting of employers. There are fewer employers in locations where the assignment function \( z \) is steep. In that case, a given mass of employers is stretched across a wider set of locations.

Finally, employers enter freely each period, so that the cost of entry is equal to the expected value from entering, and total population in the economy must add up to unity,

\[
\begin{align*}
    c_e &= \int \bar{J}(z, \ell^*(z))dF_z(z), \\
    1 &= \int L(\ell)dF_\ell(\ell).
\end{align*}
\]

(1.16)
A decentralized equilibrium is comprised of a mass of entering employers $M_e$, a value of unemployment $U$, an assignment function $z(\ell)$, a reservation wage function $w(\ell)$, wages of employed workers $w^*(y, \ell)$, an employment distribution $g(y, \ell)$, a distribution of unemployment $u(\ell)$ and market tightness $\theta(\ell)$, housing prices $r(\ell)$, and a population distribution $L(\ell)$, such that (1.4), (1.5), (1.7), the definitions in Lemma 1, (1.8), (1.9), (1.12), (1.13), (1.14), (1.15), and (1.16) hold. The following proposition characterizes existence and uniqueness of the decentralized equilibrium.

**Proposition 3.** (Existence and uniqueness)

Under Assumption 1, there exists a decentralized steady-state equilibrium with weak positive assortative matching. There exist $d_z, d_{\ell} > 0$ such that, for $|\bar{z} - \bar{z}| < d_z$ and $|\bar{\ell} - \bar{\ell}| < d_{\ell}$, the equilibrium is unique.

Proposition 3 guarantees that there exists a unique steady-state equilibrium with weak positive assortative matching, when there is not too much dispersion in spatial and productivity primitives. Although this result only proves uniqueness in a particular region of the parameter space, I check in simulations of the estimated model that my algorithm selects an equilibrium that continuously converges to one in the local uniqueness region as I shrink the dispersion in $F_\ell, F_z$ to zero. To facilitate exposition, Proposition 3 focuses on the Pareto case in Assumption 1. In the Appendix, I extend these results to more general entry distributions $G_0$ as well as to all dynamically stable steady-states.

With sufficient conditions for existence and uniqueness at hand, it is possible to study comparative statics across equilibria. To shed further light on how spatial unemployment differentials depend on the labor market pooling complementarity, it is useful to consider a particular limiting economy in which ex-ante spatial differences in $\ell$ become arbitrarily small. In that case, only the pooling complementarity may determine sorting as well as any ex-post differences across locations. Corollary 1 below shows that spatial differentials in job losing and unemployment rates arise even in the absence of any substantial ex-ante heterogeneity between locations.
Corollary 1. (Non-vanishing spatial differences with ex-ante identical locations)

Suppose that the conditions in Proposition 3 hold and that the matching function elasticity \( \alpha \) is strictly positive. Then the variance of local job losing and unemployment rates remain strictly positive and bounded above zero as the variance in exogenous differences \( \ell \) goes to zero.

This results highlights that the pooling complementarities suffice to sustain sorting in equilibrium, irrespectively of technological complementarities.\(^{37}\) When technological differences \( \ell \) vanish, locations are ex-ante identical and ex-post differences emerge endogenously. In particular, job losing and unemployment rates differ across locations. This is possible because congestion in local housing markets allows for differences in reservation wages across locations. Figure 1.4 depicts the structure of the equilibrium in that case. In the limit, locations can be re-indexed by labor market slackness \( 1/\theta \), which is on the x-axis. The y-axis shows the endogenous separation threshold \( y(\theta) \) as a function of labor market tightness, as well as expected starting productivity. From the solution to the KFE in Proposition 2, the ratio between the average starting productivity and the separation threshold is \( H(z(\theta)) = \frac{\kappa}{(\kappa-1)(1-z(\theta))} \). Consistently with Proposition 2, it rises with the assignment function \( z(\theta) \). Thus, it also rises with market slackness \( 1/\theta \). In contrast, if housing played no role \( \omega = 0 \), all locations would become ex-post identical.

Corollary 1 indicates that labor market pooling complementarities combined with congestion in local housing markets have the potential to lead to large differences in job losing and unemployment rates by incentivizing heterogeneous employers to self-select in space. I now assess whether the resulting location choices are efficient.

\(^{37}\)At a more formal level, taking the limit of arbitrarily small differences selects one particular equilibrium in the limit without any exogenous spatial heterogeneity. When exogenous spatial differences are exactly zero, there is an infinity of equilibria because locations can be arbitrarily reshuffled. However, there are only two possible spatial distribution of equilibrium outcomes: the mixing distribution in which all locations are identical, and the separating distribution in which locations differ due to sorting. The separating distribution survives because of labor market pooling complementarities. Taking the limit under vanishing spatial heterogeneity always selects the separating distribution. In addition, the mixing distribution is trembling-hand unstable.
1.3.6 Efficiency and planning allocation

To fix ideas, recall that in a single-location model of the labor market such as Mortensen and Pissarides (1994), the only sources of inefficiency are the overall entry and separation margins. These arise because of a single missing price: the price of labor market tightness. Both margins are efficient only when employers are compensated for opening and shutting down jobs by exactly as much as they congest the matching function. This is the case when the Hosios (1990) condition $\alpha = \beta$ holds. The same logic carries through to the model with many locations for the overall entry and separation decisions.

With geography, employers must make an additional decision: the location choice. It introduces an additional margin of inefficiency. There are many labor markets to choose from, but there is still no price for market tightness in any local labor market. Thus, there is not one, but many missing markets. Efficiency requires that employers are compensated by exactly as much as they congest the matching function in each location. However, due to the spatial heterogeneity in profitability and the spatial sorting, the congestion effect on the matching function varies across space.

To understand the nature of this spatial externality, consider two locations $\ell_1 < \ell_2$. Each location is populated with jobs $z_1 = z(\ell_1) < z(\ell_2) = z_2$. Consider a marginal job $z \in (z_1, z_2)$
contemplating opening in locations $\ell_1$ or $\ell_2$. If job $z$ enters in location $\ell_2$, it is worse than the average local job. Due to labor market frictions however, it does meet as many workers as its more productive competitors. By opening in location $\ell_2$, job $z$ exerts a negative externality on all other open jobs there because it diverts workers away from them. This externality is also socially harmful, as workers are redirected towards a less productive job, $z < z_2$. Symmetrically, the marginal job $z$ exerts a negative externality on other jobs in location $\ell_1$ if it enters there. However, the externality is socially beneficial in this case, as workers are redirected towards a more productive job, $z > z_1$. In both cases, the magnitude of the externality depends on the quality of local jobs $z_1$ or $z_2$, and on the quality of the newcomer $z$.

On net, the marginal job has an incentive to free-ride the favorable hiring conditions in location $\ell_2$, because its vacancy contact rate does not reflect that it is worse than average there. Wages are bargained ex-post and thus do not fully price contact rates. As a result, employers will concentrate too much in the best labor markets relative to the social optimum. This inefficiency trickles down across locations and generates misallocation throughout the economy.

The externality thus emerges at the confluence of three features of the model. First, geography creates many labor markets. Second, employers are heterogenous and choose where to locate. Third, labor markets are frictional and matches are formed with some degree of randomness. The externality arises because heterogeneous employers would be pooled in the same matching function, should they deviate off equilibrium play. Thus, I call it a labor market pooling externality. To make these arguments precise, I now define the planner’s problem.

**Planning problem.** A utilitarian planner maximizes a possibly weighted sum of values of all individuals in the economy, taking the search frictions as given. The decentralized equilibrium is inefficient when there exists no set of utilitarian weights such that the allocations
under the decentralized equilibrium and under the planning solution coincide. Otherwise, the decentralized equilibrium is efficient. Because the planner can freely reallocate the final good across locations while workers can only consume their income in the decentralized equilibrium, only one set of utility weights delivers planning allocations that may coincide with the decentralized equilibrium. These weights are defined in equation (A.26), Appendix A.2.8.

The planner controls where to send unemployed and possibly employed workers to search for jobs. The planner also decides when to break up matches, and is subject to the same search frictions as in the decentralized equilibrium. Because idiosyncratic productivity shocks are persistent, the planner must take the entire distribution of employment across productivities and locations as a state variable. If the planner does not know this distribution, she may not break up matches optimally. This distribution is an infinite-dimensional object. Nevertheless, a well-defined planner problem can be established with carefully chosen functional spaces for the distribution, described in Appendix A.2.8. To do so, I build on the work of Moll and Nuño (2018) who propose a method to solve planning problems with infinite-dimensional heterogeneity.\(^{38}\) Because it involves additional notation, I relegate the formal definition of the planning problem to Appendix A.2.8 and simply characterize it in the main text. Denote with \(SP\) superscripts variables in the planning solution, and with \(DE\) superscripts variables in the decentralized equilibrium.

**Proposition 4. (Planning solution)**

- With utility weights from (A.26), sorting (Proposition 1), local labor market flows (Proposition 2), and existence and uniqueness (Proposition 3) results extend to the planning solution under the same conditions.

- The decentralized equilibrium is inefficient for all values of \(\alpha, \beta \in (0,1]\).

\(^{38}\)My approach departs from theirs in two aspects. First, because of endogenous separations, their results do not directly apply and I must start from first principles. Second, minor modifications are needed for the general methodology to be fully consistent. For instance, the distribution must actually lie in a Sobolev-Strichartz space rather than a Lebesgue space. I provide details in Appendix A.2.8. I thank Ben Moll and Galo Nuño for useful related discussions.
• Suppose $\beta = \alpha$ and that the supports of $F_\ell, F_z$ are not too large as in Proposition 3. Then for all $\ell$:
  - $z^{SP}(\ell) \geq z^{DE}(\ell)$ with equality if and only if $\ell \in \{\ell, \bar{\ell}\}$.
  - $\frac{\partial \log w^{DE}_\ell}{\partial \ell} > \frac{\partial \log w^{SP}_\ell}{\partial \ell}$

• The planning allocation coincides with the allocation in a decentralized equilibrium in which search is directed.

Proposition 4 first establishes that the basic sorting, labor market flows, existence and uniqueness properties of the decentralized equilibrium also hold in the planning solution. Second, it formalizes the discussion above by stressing that the decentralized equilibrium is always inefficient, even when the Hosios (1990) condition $\alpha = \beta$ holds. To illustrate the externality, I compare the private value of jobs entering a particular location in the decentralized equilibrium, to the planner’s value of sending the same job to the same location. Conditional on the same separation threshold $y(\ell)$, these values satisfy

$$
\left( \frac{J^{DE}(z, \ell)}{J^{SP}(z, \ell)} \right)^{1-\alpha} = \frac{S(z^{DE}(\ell))}{S(z)} \cdot \left( \frac{Y}{y(\ell)} \right)^{\frac{1}{z^{DE}(\ell)} - \frac{1}{2}}. 
$$

(1.17)

The planner’s valuation of opening job $z$ in location $\ell$ only depends on the quality of that particular job, $z$. In contrast, the private value from entering in the same location $\ell$ for job $z$ also depends on the quality other local jobs $z^{DE}(\ell)$. This difference exactly encodes the labor market pooling externality, acting though the vacancy contact rate. Because $z^{DE}(\ell)$ is increasing, employers over-value opening jobs in locations where other employers are productive. As the planner considers all possible assignment functions $z^{SP}_\ell(\ell)$, it internalizes that mixing different jobs in the same location is not optimal. In contrast, deviating away from sorting is a viable alternative for employers in the decentralized equilibrium.

The comparison between the assignments $z^{DE}$ and $z^{SP}$ follows. For any location $\ell$, the local employer is not productive enough in the decentralized equilibrium relative to
the planner’s choice. Indeed, the more productive employers are too concentrated in high productivity locations $\ell' > \ell$ in the decentralized equilibrium. As a result, shadow reservation wages rise too fast in the decentralized equilibrium.

Finally, recall that the spatial externality arises because there is no price for labor market tightness in any location. I outline an alternative setup with directed search in Appendix A.2.8. The key assumptions are that firms are able to commit to fully state-contingent contracts and that workers can perfectly allocate between submarkets within each location should they offer different contracts. Employers then internalize that by entering in a local labor market with higher quality than their own, they depress their contact rate as workers direct their search away towards the more productive jobs. As a result, they post wage contracts that exactly price congestion effects and the decentralized equilibrium is efficient. Whether search is directed or random is ultimately an empirical question with data requirements that go beyond the scope of this paper. In principle, reality is likely to lie between both models.

Nevertheless, I propose two checks to lend credibility to this paper’s welfare implications. First, I allow employers to post many vacancies in the extended model of Section 1.4. More productive employers post more vacancies than less productive ones. Thus, they contact relatively more workers, mitigating the strength of the externality, akin to directed search. The vacancy cost elasticity then determines where the model lies between random and directed search. At the estimated cost, I find large welfare effects from place-based policies. Second, Table A.8 in Appendix A.5 shows that re-estimating the model under the directed search assumption delivers too little dispersion in local unemployment rates relative to the data and misses the variance decomposition into job losing and finding rates described in section 1.2. Conditional on the rest of the model and in this spatial context, the data thus supports the random search assumption among those two extreme cases.
1.3.7 Optimal policy

Given that the the decentralized equilibrium does not attain the first best, a natural question is whether it can be restored using standard policy instruments. An optimal policy should achieve the following. First, it should correct the pooling externality by incentivizing employers to open jobs in low profitability locations. Second, it should enforce the Hosios (1990) condition. I introduce place-based policies into the model in Appendix A.2.9 and show in Proposition 5 that they can be used to bring the economy back to its first-best.

Proposition 5. (Optimal policy)

Constrained efficiency is restored with a combination of place-based policies:

- A labor subsidy increasing in local productivity $\ell$ if and only if $\beta < \alpha$.
- A profit subsidy decreasing in local productivity $\ell$.
- A lump-sum transfers to owners.\(^{39}\)

The labor subsidy implements the Hosios (1990) condition. As in Kline and Moretti (2013), spatial variation in workers’ value of search makes that policy place-specific. Similarly to their results, labor needs to be taxed more heavily in low productivity locations on the empirically relevant side of the Hosios (1990) condition $\beta < \alpha$.\(^{40}\) Because this particular trade-off has been extensively studied, I focus primarily on the externality in the location choice of jobs.

The spatial misallocation that results from the labor market pooling externality results in an optimal profit subsidy that resembles real-world place-based policies. The Empowerment Zone program in the United States and its French equivalent both grant large effective profit subsidies for firms opening jobs in distressed areas – in practice, they guarantee tax exemptions relative to a baseline tax rate. In the model, the profit subsidy corrects the labor market pooling externality that equation (1.17) obviates. Subsidies must rise as local

\(^{39}\)Alternatively, if there are no absentee owners and profits are rebated to workers with a flat earnings subsidy, then a flat earnings tax replaces the lump-sum tax on owners.

\(^{40}\)Kline and Moretti (2013) deem this conclusion to be “rather counter-intuitive”.

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productivity $\ell$ diminishes, and thus rise with the local job losing rate as per Proposition 2. From Section 1.2, those locations have high unemployment in the data. Provided the model can tie together high job losing rates and high unemployment rates, it propose a structural justification for subsidizing high unemployment areas: high productivity employers fail to internalize their positive labor market spillovers there. To the best of my knowledge, this is the first paper to propose a structural justification for such policies based on frictional labor markets and two-sided mobility of workers and employers. Finally, lump-sum transfers to owners balance the government’s budget.

So far the spatial and individual heterogeneity in the model has remained minimal. To quantitatively account for local labor market flows and the welfare effects of place-based policies, I enrich this baseline framework in Section 1.4 below.

### 1.4 Extended model and estimation

In this section, I first describe the extensions of the model. I then establish how the results from Section 1.3 extend to the richer environment. Finally, I detail the estimation strategy.

#### 1.4.1 Quantitative setup

**Geography.** There is ample empirical evidence that locations differ in residential amenities. Better amenities attract more workers which may congest the labor market. Incorporating amenities thus allows to capture the joint spatial variation in population, wages, and unemployment. Hence, I now assume that locations differ both in productivity $p$ and amenities $a$. Locations are indexed by productivity-amenity pairs $\ell = (p, a)$, and are exogenously distributed with cumulative function $F_\ell$ on a connected support.

**Housing supply.** The magnitude of welfare gains from place-based policies that attract jobs and workers depends on how much local congestion offsets the direct gains from the
policy. To better capture this force, I introduce perfectly competitive land developers using the final good to produce housing on a unit endowment of land with an isoelastic production function. It results in a local housing supply given by \( H(r(\ell)) = H_0 r(\ell)^n \).

**Migration frictions.** The externality that arises due to labor market pooling complementarities relies on the general equilibrium response of worker mobility to changes in local economic conditions. Thus, the migration elasticity of workers crucially affects the welfare gains from place-based policies. Instead of being freely mobile, workers now receive the opportunity to move at Poisson rate \( \mu \geq 0 \). When hit by this “moving opportunity”, they receive a set of preference shocks for locations \( \{\varsigma_t\}_t \) that are Fréchet-distributed with shape parameter \( 1/\varepsilon \), and choose where to locate.\(^{41}\) Those shocks stay constant until the next moving opportunity arrives.

**Preferences.** The flow utility function becomes \( u(c, h, a, \varsigma) = \left( \frac{c}{1-\omega} \right)^{1-\omega} \left( \frac{h}{\omega} \right)^\omega a \varsigma \).

**Non-participation.** Workers stochastically exit the labor force at Poisson rate \( \Delta > 0 \). When they do, they are replaced by a single new worker. Entry and exit from the labor force stabilizes the human capital distribution described below.

**Learning and human capital.** An important channel through which unemployment harms workers above and beyond direct earnings losses is by hindering their ability to accumulate labor market experience. When out of work, not only do individuals fail to accumulate valuable knowledge, but their human capital tends to depreciate over time. In a spatial context with limited worker mobility, these scarring effects in high unemployment areas produce pockets of low human capital labor. There, high quality jobs may be less likely to open, further worsening local labor market conditions and magnifying spatial disparities.

\(^{41}\)The shifter is normalized to \( 1/\Gamma(1-\varepsilon) \), where \( \Gamma \) is Euler’s Gamma function, because it is not separately identified from amenities \( a \).
Thus, learning effects and localized unemployment interact through the location choice of employers and may amplify welfare gains from place-based policies.\textsuperscript{42}

To parsimoniously capture this idea, I assume that workers now differ in their human capital $k$. When employed, workers’ human capital grows at rate $\lambda \geq 0$. When unemployed, their human capital grows at rate $\lambda - \varphi$. $\varphi \geq 0$ encodes the relative depreciation rate of human capital for unemployed workers. Consistently with the idea that young workers enter the labor force with human capital that reflects the average human capital in the economy, I assume that the distribution of human capital of new workers $k_t$ also shifts at rate $\lambda$: the rescaled distribution $k_t e^{-\lambda t}$ for new workers does not depend on calendar time $t$, and is denoted $F_k$.\textsuperscript{43} I also assume that workers with different human capital in the same location search in the same labor market: potential employers cannot discriminate between workers with different human capital prior to meeting with them.

**Production.** I allow employers to use housing in production, to capture the idea that local congestion due to higher population affects production costs. Filled jobs with idiosyncratic productivity $y$ in a location with local productivity $p$ thus use housing $h$ and human capital $k$ of their employee to produce, with production function $(ypk)^{1+\psi} h^{\psi/\psi + 1}$.

**Recruiting intensity.** Finally, I let employers adjust their recruiting efforts. This channel potentially mitigates the strength of labor market pooling externalities. Thus, employers with open jobs are now allowed to post many vacancies $v$ at cost $\frac{cv}{1+1/\gamma} v^{1+1/\gamma}$.

\textsuperscript{42}Human capital differences also allow the model to jointly account for the role of sorting in wage and unemployment differentials. Wages reflect human capital, the sorting of which thus contributes to spatial wage differentials directly. In contrast, because human capital is transferable between jobs, the separation decision is independent of human capital. As a result, the local mix of human capital does not directly affect spatial job loss differentials.

\textsuperscript{43}This assumption can be understood as young workers learning from older workers prior to entry in the labor force. The economy is therefore on a balanced growth path determined by $\lambda$. In levels, the distribution of knowledge of new workers is a “travelling wave with constant shape”. I also assume that $F_k$ has a density with full support equal to $R_+$. These assumptions also help with tractability.
1.4.2 Characterization

The extensions preserve the basic structure of the location choice of employers. I show in Appendix A.3 that when migration opportunities are rare enough $\mu \ll 1$ and the relative depreciation rate of human capital is small enough $\phi \ll 1$, the location choice of job $z$ in equation (1.12) becomes

$$\arg\max_{(p,a)=\ell} \frac{z}{1-z} \left\{ \log\left( p^\Omega a^{-\psi P} \right) \cdot C(w(\ell), z(\ell))^{\psi P} \right\} + \log q(\ell) + \log \left( \bar{h}(u(\ell)) \right)^Q \right\} - \log w(\ell)$$

(1.18)

where $P = \frac{1}{\omega + \psi + \epsilon(1 + \eta + \psi)}$ and $Q = \frac{\omega + \epsilon(1 + \eta)}{\omega + \psi + \epsilon(1 + \eta + \psi)}$, and the average human capital in location $\ell$ is a decreasing function of the local unemployment rate $u(\ell)$. It is equal to $\bar{h}(u(\ell)) = \frac{\Delta}{\bar{\Delta} + \varphi u(\ell)}$ up to a general equilibrium constant. Recall that $\ell = (p,a)$ now indexes productivity-amenity pairs. The function $C$ is defined in Appendix A.3 and is increasing in each argument.

Equation (1.18) first highlights that in the extended model, technological complementarities depend on a combination of productivity $p$ and amenities $a$. Higher productivity $p$ makes locations more lucrative for jobs, but higher local amenities reduce profitability. Higher amenities bring in more workers, raising housing prices and driving up production costs. This housing price channel explains why the amenity contribution enters with an elasticity $\psi$. Anticipating a result showing that this single index is a local sufficient statistic for the model’s outcomes, I identify a pair $\ell = (p,a)$ with the combined index of local advantage

$$\ell(p,a) \equiv p^\Omega a^{-\psi P}.$$  \hspace{1cm} (1.19)

In addition, local expenditures on housing also depend on local wages, as captured by $C(w(\ell), z(\ell))$.

Second, equation (1.18) shows that labor market pooling complementarities remain unchanged and still depend only on the local vacancy contact rate. Similarly, the expected cost of labor continues to be summarized by local reservation wages $w(\ell)$. 

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Third, equation (1.18) reveals the contribution of learning at the workplace for the location choice of employers. Average local human capital \( \tilde{h}(u(\ell)) \) falls as the local unemployment rate rises as workers are more frequently scarred by unemployment. When scarring effects \( \varphi \) are stronger relative to how frequently the workforce turns over (\( \Delta \)), a given unemployment rate is associated with worse average local human capital.

Because the structure of the location choice of employers in equation (1.18) closely resembles its more stylized version in equation (1.12), virtually all the analytical results from Section 1.3 carry through.

**Proposition 6. (Characterization of the extended model)**

When the migration rate \( \mu \) and the scarring effects of unemployment \( \varphi \) are not too large, Propositions 1, 2, 3, 4, 5 and Corollaries 1, 2 obtain in the extended framework under the same conditions, with three modifications. First, replace the local unemployment rate by \( u(\ell) = \frac{s(\ell)+\mu+\Delta + f_R(\ell)}{s(\ell)+\mu+\Delta + f_R(\ell)} \). Second, replace \( \ell \) with the combined index of local advantage \( \ell(p,a) \). Third, population depends on both \( \ell(p,a) \) and on a conditional on \( \ell(p,a) \): \( L(p,a) \equiv L(\ell(p,a),a) \).

Population cannot be summarized solely by the local advantage index \( \ell(p,a) \) because workers value amenities directly, while employers value amenities only through local housing prices. As a result, amenities generate variation in population even conditional on the local advantage index \( \ell(p,a) \). Thus, in the model as in the results from Section 1.2.4, local unemployment correlates negatively with local wages, but not with population conditional on wages. I provide more details in Appendix A.3.1. Having laid out the structure of the extended framework, I turn to the structural estimation.\(^{44}\)

\(^{44}\)Appendix A.3.2 discusses additional possible extensions and how they may affect the results: industry heterogeneity, labor market segmented by skill, technological and amenity spillovers, and trade costs.
1.4.3 Estimation strategy and identification

Despite its rich structure, the quantitative model is transparent enough to produce estimating equations for almost all key parameters. In particular, no simulation will be required until the last step, which estimates the entry cost. To make this argument precise, I discuss how each parameter can be recovered recursively given the data I choose. A proposition at the end of this subsection summarizes the formal identification of the model. Different specific estimators are used for different parameters, but all can be nested into an overarching Generalized Methods of Moments (GMM) estimator. In total, there are 19 parameters to be estimated: $\rho, \Delta, \omega, \psi, \delta, \sigma, \beta, b, Y, \eta, \mu, \varepsilon, \alpha, \gamma, c_v, m, \lambda, \varphi$; together with two distributions $F_z, F_{p,a}$. I do not specify functional forms in the main estimation as they can be recovered non-parametrically.

The 19 parameters can be divided into three groups. Parameters in the first group – $\rho, \Delta, \omega, \psi, \mu, b, c_v, m$ – directly map into empirical counterparts or can be normalized, thus only requiring simple Minimum Distance Estimators (MDE). Parameters in the second group – $\delta, \sigma, \beta, Y, \eta, \varepsilon, \alpha, \gamma, \lambda, \varphi$ – require more involved estimating equations, together with different estimators. The third group of parameters only contains the entry cost $c_e$, which is estimated by numerical search (Method of Simulated Moments). Finally, I parametrize the distributions $F_z, F_{p,a}$ after the estimation for simulation purposes, leading to a fourth group of parameters. Before describing how to estimate each group of parameters, I briefly discuss the data used to construct empirical targets.

Data. I use data from France for all years between 1997 to 2007. I choose a quarter as the baseline time period $[t, t + 1)$. For most of the estimation, I use averages over the entire period. For some parameters I break down the sample into two subperiods, and use averages for 1997-2001 and for 2002-2007. I index locations (cities) in the data by $c$. I use aggregate data for expenditure shares on housing for households. I measure expenditures on real estate for firms in the firm-level balance sheet data. Using the DADS-LFS combination, one obtains
measures of local unemployment rates $u_c$, local job losing rates for stayers $s_c$, local job finding rates for stayers $f_{Re}$, local average wages $W_c$, and population shares $L_c$. The DADS-LFS combination also delivers measures of aggregates such as the geographic mobility rate of workers and the average job offer acceptance probability. Finally, the DADS-LFS allow finer disaggregation of job losing rates and wages by tenure and location which will be useful to estimate several parameters in the second group. The last data source is the online realtor MeilleurAgents.com, from which I construct commuting zone housing prices $r_c$.

**First group (8 parameters).** The moving opportunity rate $\mu$ is directly identified from the geographic mobility rate for individuals transitioning into unemployment at the same time.\(^{45}\) $\Delta$ then follows from the flow equation for unemployment:

$$\Delta = \sum_c L_c \left( \bar{u} f_{Re} - s_c - \mu \right),$$

where $\bar{u}$ is the aggregate unemployment rate. The interest rate identifies $\rho$ through the effective discount rate of individuals $\rho + \Delta$. Next, household’s expenditure share on housing $\omega$ can be directly equated to the value reported by INSEE (23%).\(^{46}\) Similarly, the expenditure share on real estate out of value added by employers $\psi$ is equated to my estimate of 11%.\(^{47}\) The remaining parameters can be normalized $b = c_v = m = 1$.\(^{48}\)

**Second group (10 parameters).**

**Productivity process $\delta$ and $\sigma$.** To estimate $(\delta, \sigma)$, I use data on job losing rates and wage growth by tenure. To that end, I leverage a closed-form solution to the time-dependent KFE equation derived in Appendix A.4.2. This solution delivers an explicit expression for

\(^{45}\)In the model, unemployed and employed workers always change location and enter unemployment when they receive the moving opportunity at rate $\mu$. That rate must be time-aggregated quarterly.

\(^{46}\)INSEE’s calculations reflect both renters and homeowners.

\(^{47}\)Balance sheet data lists all rental expenditures, as well as the book value of land, building and structures owned by the firm. I annuitize the value of those properties using a 5% annual interest rate, and add the annuitized value to the rental expenditures. This defines expenditures on real estate.

\(^{48}\)The unemployment income parameter $b$ is not separately identified from productivity $\ell$. The shifter of the vacancy cost function $c_v$ and the matching function efficiency are not separately identified from the entry cost $c_e$. 

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the time-aggregated job losing rate in the first year in each location in the model. Given the measured average job losing rate $s_c$ in city $c$, the job losing rate in the first year of a job in city $c$ is $s_1(s, \hat{\delta})$, where $s_1$ is an explicit decreasing function of $\hat{\delta}$ given $\mu, \Delta$, and is specified in Appendix A.4.2. Intuitively, if the volatility $\sigma$ is much larger than the drift $\delta$, many separations occur at early tenure. Denoting $s_{1c}$ the measured job losing rate in the first year in city $c$, I recover $\hat{\delta}$ directly by estimating

$$s_{1c} = s_1(s, \hat{\delta})$$

(1.20)

with Non-Linear Least Squares (NLLS), treating residuals as measurement error.

Given the estimated ratio $\hat{\delta} = \frac{\delta}{\sigma}$, the same solution to the time-dependent KFE allows to explicitly compute wage growth by tenure when $\beta$ is not too large. Appendix A.4.3 shows that it identifies the common scale of $\delta, \sigma$. Intuitively, when productivity depreciates faster, wages at continuing jobs fall behind wages at new jobs at a faster pace. Thus, a NLLS regression similar to (1.20) estimates $\delta$. When $\beta$ is large, the tenure profile of wages must be computed numerically, and $(\beta, \sigma)$ must be estimated jointly. At the estimated bargaining power $\beta$ the difference is negligible.

**Bargaining power $\beta$.** The labor share in location $c$ is $\beta + \frac{1-\beta}{H(s_c)}$, where $H$ only depends on $\delta$ and $\sigma$. I target the aggregate labor share net of debt servicing expenditures to identify $\beta$ by simple MDE.\(^{49}\)

**Learning rates $\lambda$ and $\varphi$.** Wage changes for workers coming out of unemployment reflect human capital losses, that grow with unemployment duration. Appendix A.4.5 shows

\(^{49}\)I net out debt payments from value added before computing the labor share. This adjustment is correct under the assumption of a national frictionless non-housing capital market rented from households, with a Cobb-Douglas production function in efficient units of labor, non-housing capital, and housing. In that case, the labor share is 0.85 in my sample.
that, for worker $i$ who loses her job at time $t_0$ and finds a new job at time $t_1$, wages satisfy

$$\log W_i(t_1, \ell) = (\lambda - \varphi)(t_1 - t_0) + \Phi(\ell) + \log W_i(t_0, \ell) + \tilde{W}_i(\ell),$$

(1.21)

where $\tilde{W}_i(\ell)$ is a mean-zero random variable that reflects draws from the local new job distribution, and $\Phi(\ell)$ is a location fixed effect. Because productivity draws are independent from unemployment duration, $\tilde{W}_i(\ell)$ does not depend on $t_1 - t_0$. Hence, OLS consistently estimate $\lambda - \varphi$ using equation (1.21). $\lambda$ can then be directly obtained from aggregate real wage growth (see Appendix A.4.5). Thus, I recover $\varphi$.

In practice, mechanisms left out from the model may generate endogeneity issues. To address those concerns, Appendix A.4.5 proposes several other specifications with more flexible controls (for instance, industry fixed effects, worker fixed effects, past wage controls, employed workers as control group). The point estimate of $\varphi$ remains stable around 1% per quarter and statistically significant across specifications.

**Local quality and cutoff.** For the remainder of the estimation, I recover estimates of the local job quality $z(\ell)$ and the local productivity cutoff $y(\ell)$ in each city. They are endogenous outcomes, not fixed primitives of the economy. Given the estimate for $\delta$, local job losing and finding rates directly identify job quality and the threshold in each city as per Proposition 2,

$$z_c = \frac{\delta}{s_c}, \quad y_c = \frac{by_0}{\hat{\rho}} \frac{\hat{\beta} f_{Re} \bar{S}(z_c)}{\hat{\rho} - \beta f_{Re} \bar{S}(z_c)},$$

(1.22)

where $\hat{\rho} = \rho + \Delta + \mu + \varphi - \lambda$, and $y_0$ and the function $\bar{S}$ can be calculated from known parameters.

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50Permanent differences across workers may correlate with their human capital and productivity draws, or with their probability of finding a new job. Workers coming out of unemployment may draw starting productivities that depend on unemployment duration. The new productivity draws may be correlated with current human capital.
Lower bound of initial productivity draws $Y$. With an estimate of $y_c$ at hand, I use data on job search behavior from the LFS to identify $Y$. In Appendix A.4.6, I show how to use this information to recover the contact-to-job probability conditional on meeting. I estimate it to be 20.6%). From the model, the contact-to-job probability in city $c$ is $\left(\frac{BY}{y_c}\right)^{1/zc}$, where $B$ is a known constant. $Y$ is estimated by MDE between the average acceptance probability across locations in the model, and the empirical target of 20.6%.

Housing elasticity $\eta$. At this stage, it is possible to construct demand for housing in each city in the model. Appendix A.4.8 derives a known function $r_0$ such that $\log r_c = r_1 + \frac{1}{1+\eta} \log r_0(W_c, L_c, u_c, z_c, y_c)$. I then obtain $\eta$ with OLS, assuming that measurement error is the only residual.52

Migration elasticity $1/\varepsilon$. Migration shares by destination $\pi(\ell)$ satisfy

$$\log \pi(\ell) = \pi_0 + \frac{1}{\varepsilon} \log \bar{U}(\ell) + \log a$$

where $\bar{U}(\ell) = \frac{\bar{w}(\ell)}{(1-\beta+\beta H(a(\ell))r(\ell))}$ can now be computed in the model, and $\pi_0$ is a general equilibrium constant. Because unobserved amenities $a$ are correlated with $\bar{U}(\ell)$, I split the sample into two subperiods 0 and 1 and first-difference equation (1.23). Then, I use local productivity shocks based on shift-share projections of economy-wide industry shocks as instruments for the change $\log \frac{\bar{U}(\ell_1)}{\bar{U}(\ell_0)}$. I thus estimate $1/\varepsilon$ with Two Stage Least Squares (2SLS) using (1.23) in first differences. The identification assumption is that economy-wide industry-level shocks are orthogonal to local changes in amenities. I further discuss how to map industry-level shocks into the model and the identification assumption in Appendix A.4.9.

\footnote{Data reported in Faberman et al. (2017) suggest an acceptance probability of 29.6% in the United States.}

\footnote{Omitted factors like heterogeneous housing supply elasticities may be a source of endogeneity. With repeated cross-sections of housing prices, difference-in-difference specifications using shift-share shocks as instruments could be used to correct for endogeneity. With only one cross-section, these approaches are not possible.}
Non-parametric distributions of local productivity, amenities and job quality.

At this stage I need to recover non-parametric estimates of local productivity and amenities \((p_c, a_c)\) in each city, as well as the density function of job qualities \(f_z\). Equation (A.44) in Appendix A.4.7 shows that local productivity \(p_c\) follows from inverting the model’s predictions for local wages. Given the migration elasticity estimate, inverting the population equation (A.32) in Appendix A.3.1 then delivers an estimate of local amenities \(a_c\) in each city. Together, the estimates \((p_c, a_c)\) provide a non-parametric estimate of the distribution \(F_{p,a}\). Finally, Appendix A.4.10 shows that the density function of job losing rates across locations identifies \(f_z\) using (1.22).

Matching function and vacancy cost elasticities \(\alpha\) and \(\gamma\). To estimate \(\alpha\) and \(\gamma\), I express local job finding rates as a function of estimated market tightness and employers’ values in equation (A.45) in Appendix A.4.11, together with more details. I use the same shift-share approach in first differences to estimate \(\alpha, \gamma\) jointly with 2SLS.

Together with the details in Appendix A.4, the previous arguments prove identification of the 15 parameters that need not be normalized, together with the distributions of fundamentals in the economy. All the previous estimators can be formally collected into an overarching GMM estimator.

Proposition 7. (Identification)

When \(\mu, \Delta\) and \(\beta\) are not too large, the parameters \(\mu, \Delta, \rho, \omega, \psi, \delta, \sigma, \lambda, \varphi, \eta, \varepsilon, Y, \alpha, \gamma\), as well as the distribution of firms qualities \(F_z\), the joint distribution of local productivities and amenities \(F_{p,a}\), are exactly identified by the GMM estimator. The other parameters can be normalized except the entry cost.

\(^{53}\)Alternatively, amenities could be obtained as residuals from the migration share equation (1.23). Because the estimation relies on observed population shares, I choose to match population rather than migration shares. In practice, they are highly correlated.
Third group (1 parameter). After estimating those 15 parameters, a numerical search estimates the entry cost $c_e$ by targeting the aggregate unemployment rate.\footnote{Given the estimator for $\Delta$, this procedure searches for the entry cost $c_e$ that delivers the right average job finding rate.} For simulations purposes, I impose parametric functional forms for the distributions, to which I turn now.

Fourth group (7 parameters). I estimate a joint lognormal distribution for local amenities and productivities, with respective standard deviations $\sigma_a, \sigma_\ell$ and correlation $\ell,a$. I estimate a Beta distribution for the distribution of employer quality. Its shape parameters are $g_1, g_2$ and its support is $[\underline{z}, \overline{z}]$.

The next section discusses the results from the estimation and reports additional over-identification and validation exercises.

1.5 Results

This section presents the estimation results. First, I discuss the parameter estimates. Second, I show that the model can quantitatively account for spatial unemployment differentials. Third, I conduct a set of over-identification checks to support the model’s predictions. Finally, I propose direct evidence to validate the model’s core structure.

1.5.1 Parameter estimates

Table 1.1 reports the parameter estimates. Overall, they are close to values found in the literature. The housing shares for workers $\omega = 0.23$ is close to the commonly used value of 0.3 for the United States. Similarly, the housing share for firms $\psi = 0.11$ is in the range of estimates reported in Desmet et al. (2018). While the negative drift $\delta$ of the worker-level productivity process is close to the quarterly value of 0.5% implied by the estimates in Engbom (2018), the volatility $\sigma$ is somewhat smaller. The bargaining power $\beta = 0.10$ is close to the estimate in Hagedorn and Manovskii (2008) and references therein. The housing
Table 1.1: Parameter estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Interpretation</th>
<th>Target</th>
<th>Estimator</th>
<th>Estimate</th>
</tr>
</thead>
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<td>$\rho$</td>
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<td>Annual interest rate</td>
<td>MDE</td>
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<td>Aggregate unemployment rate</td>
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<td>Expenditures on housing</td>
<td>MDE</td>
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<tr>
<td>$\psi$</td>
<td>Housing share (firms)</td>
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</tr>
<tr>
<td>$\delta$</td>
<td>Drift of productivity</td>
<td>Job losing rate by tenure</td>
<td>NLLS</td>
<td>0.001</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Volatility of productivity</td>
<td>Wage growth by tenure</td>
<td>NLLS</td>
<td>0.004</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Bargaining power</td>
<td>Labor share</td>
<td>MDE</td>
<td>0.10</td>
</tr>
<tr>
<td>$Y$</td>
<td>Lower bound of init. prod.</td>
<td>Job acceptance probability</td>
<td>MDE</td>
<td>0.88</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Housing elasticity</td>
<td>Housing prices</td>
<td>OLS</td>
<td>3.49</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Migration rate</td>
<td>Migration rate</td>
<td>MDE</td>
<td>0.001</td>
</tr>
<tr>
<td>$1/\varepsilon$</td>
<td>Migration elasticity</td>
<td>Migration shares</td>
<td>2SLS</td>
<td>1.65</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Matching function elasticity</td>
<td>Local job finding rates</td>
<td>2SLS</td>
<td>0.47</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Vacancy cost elasticity</td>
<td>Local job finding rates</td>
<td>2SLS</td>
<td>1.31</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Learning rate</td>
<td>Aggregate growth</td>
<td>MDE</td>
<td>0.002</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Depreciation rate</td>
<td>Unemployment scar</td>
<td>OLS</td>
<td>0.01</td>
</tr>
</tbody>
</table>

$z_\zeta$ | Lowest job quality | Local job losing rates | MDE | 0.07 |

$\tau$ | Highest job quality | Local job losing rates | MDE | 0.41 |

$g_1$ | Shape of job quality distrib. | Local job losing rates | MDE | 1.27 |

$g_2$ | Shape of job quality distrib. | Local job losing rates | MDE | 1.80 |

$\sigma_p$ | St.d. of local productivity | Local wages | MDE | 0.18 |

$\sigma_a$ | St.d. of local amenities | Local population | MDE | 0.77 |

$c_{p,a}$ | Correlation prod.–amenities | Local wages and population | MDE | 0.17 |

Supply elasticity $\eta$ implies a price-to-population elasticity of 0.22, which is within the range of estimates reported in Saiz (2010) for the United States. The key driver of steady-state population adjustments is the shape parameter of the idiosyncratic preference shock distribution $1/\varepsilon$, which also coincides with the migration elasticity. Its value is 1.65, well within the values reported in the literature between 0.5 and 3.\(^{55}\) The matching function elasticity $\alpha$ is 0.47, in the middle of the range reported in Petrongolo and Pissarides (2001), and the vacancy cost elasticity parameter $\gamma$ implies that the cost function is close to quadratic, also

\(^{55}\)The estimate for $\mu$ implies an annual migration rate of 0.4%. This is lower than the overall migration rate in my sample which is about 3%. This discrepancy is due to the fact that many migrants are employed workers moving with a job at hand. However, the quantitative results are similar if I target the overall migration rate rather than the migration rate into unemployment. In steady-state, the migration elasticity is the key driver of population movements, not the migration rate.
in line with existing estimates. Finally, the estimate of the unemployment scar $\varphi$ implies a 4% wage loss for workers who spent a year unemployed – roughly the average duration of unemployment – relative to workers who remained employed throughout the year. This value is somewhat conservative relative to the value implied by the estimate of 10% in Jarosch (2015).\textsuperscript{56}

### 1.5.2 Spatial job loss differentials and unemployment

With the estimated model at hand, I start by describing the decentralized equilibrium in Figure 1.5. Locations are indexed by their local advantage index $\ell(p, a)$ relative to the lowest value thereof.\textsuperscript{57}

\textsuperscript{56}Jarosch (2015) estimates the long-run effect of an initial job loss, whereas I estimate the elasticity of wage losses to unemployment duration. In the data and in Jarosch (2015)'s model, current job loss begets future job losses, thereby increasing the long-run effect of job loss on human capital relative to my estimate.

\textsuperscript{57}The lognormal distribution of productivity and amenities used in the estimation has an unbounded support. For simulation purposes, numerical lower and upper bounds must be chosen. I pick the 0.1% and 99.9% quantiles of the $\ell(p, a)$ distribution. Because the supply of locations left out is then small enough, the equilibrium outcomes are virtually unchanged when widening out those bounds.
Consistent with Proposition 6, the job losing rate declines as the local advantage index increases due to rising job quality $z(\ell)$. Because of the opposing forces highlighted in Proposition 2, the finding rate is non-monotonic in the local advantage index in the decentralized equilibrium. Although it declines in the low $\ell(p,a)$ locations, in proportional terms it falls by about half while the job losing rate soars more than four-fold. As a result, the unemployment rate largely follows the declining pattern of the job losing rate.

Tracking the rise in job quality, wages grow as locations become better suited for production. Thus, average population density $L(\ell)$ also rises. Recall that conditional on the advantage index $\ell$, there is residual variation in amenities $a$ across locations. Conditional on $\ell$, there are thus locations with higher or lower population density than the average $L(\ell)$. Finally, mirroring the falling unemployment rate as per (1.18), average human capital $\bar{h}(u(\ell))$ steeply rises across locations. The somewhat conservative estimate of $\varphi$ still implies human capital gaps over 30% between residents of the best and worst locations, due to the interaction of spatial unemployment differentials and scarring effects of job loss.

I now connect the model’s outcomes back to the motivating evidence. The model generates predictions for the spatial variation in local unemployment rates. By non-parametrically recovering the distribution of job quality $f_z$ from local job losing rates, the estimation imposes some restriction on the spatial variation in job losing rates. This moment results from (a) the distribution $f_z$, but also (b) the equilibrium assignment of heterogeneous employers to locations. In equilibrium, the assignment of employers to locations is an endogenous object. Importantly, it is not constrained by the estimation and may freely adjust in counterfactuals. In addition, the estimation does not restrict the spatial variation in job finding rates apart from the two coefficients in equation (A.45) that identify $\alpha$ and $\gamma$. Therefore, both the spatial variation in the unemployment rate and its split into the job losing and job finding contributions are useful moments to assess the model’s ability to speak to spatial unemployment differentials.
Table 1.2: Aggregate and local unemployment rates in the decentralized equilibrium.

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Baseline</td>
<td>No pooling</td>
</tr>
<tr>
<td>Aggregate unemployment rate</td>
<td>0.076</td>
<td>0.071</td>
</tr>
<tr>
<td>St. dev. unemployment rate</td>
<td>0.022</td>
<td>0.004</td>
</tr>
<tr>
<td>St. dev. log unemp. / emp.</td>
<td>0.281</td>
<td>0.045</td>
</tr>
<tr>
<td>Job losing rate</td>
<td>85 %</td>
<td>-180 %</td>
</tr>
<tr>
<td>Job finding rate</td>
<td>15 %</td>
<td>280 %</td>
</tr>
</tbody>
</table>

The first and fourth columns of Table 1.2 reveal that the model accounts for over 90% of the cross-sectional variance of the unemployment rates. The standard deviation is 0.022 in the model against 0.023 in the data. Table 1.2 also highlights that the model closely replicates the contribution of job losing rates to spatial unemployment differentials, which is 85% in the model and 86% in the data. Table 1.2 finally reports the (targeted) aggregate unemployment rate in the model and the data.

I now assess the relative importance of labor market pooling complementarities versus technological complementarities in shaping spatial unemployment differentials. To that end, I conduct two exercises. First, I shut down the labor market pooling externality in the decentralized equilibrium. This exercise only removes the off-equilibrium-play incentive for employers to choose locations with too high an \( \ell(p, a) \), and hence isolates the contribution of technological differences. In the second exercise, I remove technological differences in productivity \( p \) and amenities \( a \) across locations. The economy then behaves as the limiting one with ex-ante identical locations in Section 1.3.5, and isolates the role of pooling complementarities.

The second column of Table 1.2 reveals that, without pooling complementarities, the standard deviation of local unemployment rates falls from 0.022 to 0.004, which represents only 18% of the standard deviation in the baseline model. This drop is accompanied by a
complete reversal of the role of the job finding and job losing rates. In contrast, the third column of Table 1.2 indicates that even without technological differences across locations, the model can generate 45% of the spatial dispersion in unemployment rates. In that case, the model also qualitatively matches the contribution of job losing and finding rates.

Overall, Table 1.2 shows that pooling complementarities are the primary mechanism generating empirically relevant dispersion in job losing rates across space. Quantitatively, pooling complementarities account for about half the spatial dispersion in unemployment rates. Technological differences contribute one sixth, and the interaction between pooling and technological complementarities contributes a third.

In addition, Table A.8 in Appendix A.5 shows that re-estimating the model under alternative assumptions such as the Hosios (1990) condition or directed search yield predictions that are at odds with the data. In particular, none of the alternative assumptions can account jointly for the large cross-sectional variation in unemployment rates and the respective contributions of the job losing and job finding rates in the data.

Having characterized the model’s ability to account quantitatively for the margins of spatial unemployment differences, I now propose additional over-identification exercises.

1.5.3 Over-identification exercises

This subsection proposes a set of over-identifying exercises. The goal is to support the identification of key parameters using non-targeted moments.

Productivity process. I start by discussing three exercises that lend credibility to the estimates of the productivity process $\delta$ and $\sigma$. First, Figure 1.6 (a) shows that, despite relying on a single degree of freedom ($\delta/\sigma$) to predict job losing rates in the first year as per (1.20), the model closely fits the full cross-sectional variation between cities. Second, I construct job losing rates at all tenures across cities in the model.58 Figure 1.6 (b) shows that

58The estimation targets the average job losing rate in each city, and the average job losing rate in the first year economy-wide. The structure of the model then fills in the gaps.
the model closely accounts for the estimated tenure profiles from Figure A.9, both across different tenures and across different cities.

The third exercise offers direct evidence supporting the estimate of the common scale $\delta$. I use balance sheet data to compute firm-level labor productivity growth relative to aggregate labor productivity growth, which should be close to $\delta$. Focusing on large and high labor productivity firms to minimize survival selection bias, I obtain a relative decline of 0.5% annually, close to the annualized estimate of $\delta$, 0.4%.$^{59}$

**Bargaining power.** To estimate $\beta$, I target the aggregate labor share. The model’s labor share equation also predicts a negative correlation between wages and local labor shares. This negative correlation mirrors the fact that reservation wages rise less than one-for-one relative to employer productivity. Table A.4 in Appendix A.4.12 shows that the regression coefficient of labor share on local wages is -0.19. In the estimated model, it is -0.11. Despite

$^{59}$These firms are the least likely to exit in the data. They are also least likely to exit according to theories of firm dynamics with frictional labor markets as Bilal et al. (2019b).
the simplicity of the bargaining protocol, the model replicates prominent characteristics of rent-sharing across locations.

**Amenities.** A natural check of the non-parametric amenity estimates $a_c$ is to correlate them with local characteristics that should affect the value of living in a particular location. I regress the estimated log amenities on the log of sun hours per month, as well as a the log density of residential service establishments of various kinds. Table A.5 in Appendix A.4.12 shows that more sun hours and a higher density of health, education or commercial services are all positively associated with higher amenities. While these results cannot be interpreted as causal, they support the view that the estimated amenities capture salient features of a location’s residential attractiveness.

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60The number of establishments is adjusted by the area of the commuting zone.  
61For instance, a 10% increase in the number of sun hours per month raises the amenity value of a location by 3.3%. A 10% rise in the density of health establishments increases amenities by 2.2%.
**Housing elasticity.** To assess how well the estimated housing supply elasticity accounts for cross-sectional dispersion in housing prices, Figure 1.7 (a) plots housing prices in the model against housing prices in the data. The estimation targets a single moment, the correlation between local house prices and local income. While there is some residual dispersion, the model’s predictions are centered around the 45 degree line in orange.

**Positive assortative matching.** The final over-identification exercise tests the positive assortative matching prediction from Proposition 1. Using the non-parametric estimates of local job quality $z(\ell)$ and the local cutoff $y(\ell)$, I check whether locations with higher cutoffs also have higher job quality. While this increasing relationship is not enforced in the estimation, Figure 1.7 (b) shows that the data supports it.

### 1.5.4 Model validation

Having proposed over-identification checks for parameter estimates, I now turn to a last set of validation exercises. The goal of this subsection is to lend credibility to four crucial mechanisms of the model before turning to counterfactuals. To that end, I propose direct evidence using firm-level balance sheet data that is not used in the estimation.

**Labor productivity across and within locations.** The first validation exercise emphasizes the link between labor productivity differences and job losing rate differentials. Using the solution to the productivity distribution from Lemma 2, the model produces testable implications that tie labor productivity with job losing rates.

1. *Average labor productivity is higher in locations with lower job losing rates.*
2. *The labor productivity distribution in low job losing rate locations first-order stochastically dominates the distribution in high job losing rate locations.*
3. *The labor productivity distribution has a Pareto tail with index $1/z(\ell)$ in each location.*
4. The ratio of Pareto tails indices between locations is equal to the ratio of job losing rates between the same locations.

Implications 1 and 2 are not special to the Pareto case, while implications 3 and 4 are closely tied to that particular functional form. To test implications 1 to 4, I compute labor productivity in single-establishment firms in the balance sheet data. I then group commuting zones into four job losing rate quartiles, and compute the employment-weighted labor productivity distribution in each group.

Figure 1.8 (a) displays the labor productivity distribution in the bottom and top quartiles of commuting zones, ranked by their job losing rate. The vertical lines are the local averages. Consistent with the first implication of the model, average labor productivity is higher in locations with low job losing rates. Furthermore, the cumulative distribution function of labor productivity in low job losing rate locations is always below the cumulative function in high job losing rate locations. Therefore, the data also supports the second, finer implication that the labor productivity distribution first-order stochastically decreases with the job losing rate.
Panel (b) zooms into the right tail of the productivity distribution by showing the log tail probability as a function of log labor productivity. In both groups of locations, the log tail probability is approximately linear, consistent with the third implication of a Pareto tail. The fourth implication of the model imposes a strong link between the local job losing rate and the shape of the right tail of the labor productivity distribution. I estimate the ratio between the tail indices in each group of locations to be 1.35. It is close to the ratio of group averages of job losing rates, which is 1.58. Together, these results support the structure of the model that ties the heterogeneous productivity of employers to job losing rates, as well as the Pareto assumption.

**Labor productivity by employer age.** The transparent link between employer productivity and job losing rates obtains as the result of two assumptions in the model. First, the productivity process does not differ across locations. Second, differences in local job losing rates all stem from differences in new job productivity. To support these assumptions, I compute labor productivity in level and growth rate, both for entrant firms (less than two years old), and incumbent firms (at least two years old). All jobs are new at a firm that just entered. While it is unclear what fraction of jobs are new at an incumbent firm, it is arguably less than at an entrant firm. The model then predicts that (a) the negative correlation between labor productivity and job losing rate should be more negative for entrants. The model also predicts that (b) labor productivity growth should not correlate with the job losing rate. Table A.6 in the Appendix shows that the data supports both implications (a) and (b). A one percentage point increase in the local job losing rate is associated with 37% lower labor productivity on average across locations. For entrant firms, labor productivity is lower by an additional 43%. In contrast, labor productivity growth rises by an economically and statistically insignificant 0.01.62

62 Any spatial gaps in $\delta/\sigma$ would also introduce a location-specific residual in equation (1.20). The close fit in Figure 1.6 indicates that any spatial variation in $\delta/\sigma$ is likely to be small.
Given this association between employer’s life-cycle and job losing rates, it is natural to ask how much spatial differences in establishment-level exit rates contribute to spatial gaps in job losing rates. Recall that the model does not take a stand on the boundaries of the firm, and hence may be interpreted at the firm, establishment or job level. Figure A.12 in Appendix A.4.13 shows that job loss at surviving establishments is the dominant source of geographical variation in the job losing rate. Surviving establishments account for 89% of the cross-location variance in the job losing rate, while exiting establishment contribute only 11% of the cross-location variation. This finding is not particularly surprising given that establishment-level exit rates rarely exceed 10-15%. Thus, locations have different job losing rates primarily because local establishments lay off workers more frequently without exiting.

**The role of firm-specific job instability.** I now ask how much of spatial gaps in job stability are tied to the firms that open establishments in particular locations, rather than within-firm differences across establishments. To do so, I use firms that open multiple establishments in different cities in the same year. The idea is to compare the job losing rate for new establishments within a given firm to the average job losing rate of their location, both unconditionally and conditionally on the firm’s identity. The structure of the model allows to interpret those moments and delivers three predictions. First, a positive correlation between establishment-level job losing rates and local job losing rates should arise – implication (a). To the extent that firms present large differences in their average job quality, conditioning on firms’ identity should substantially weaken that correlation, implication (b). The correlation should not disappear if there is residual within-firm job quality dispersion, implication (c).

The main test is thus whether the within-firm correlation between establishment job losing rates is lower than the between-firm correlation. If this is not the case, then it is likely that local factors unrelated to the location choice of heterogeneous employers affect spatial

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63 I use the DADS Postes that have firm and establishment identifiers for the universe of French workers.
64 Assume that a firm \( f \) is a discrete set of heterogeneous jobs \( j \), \( \{ z_{fj} \} j \), with mass \( m_{fj} \) for each \( j \). The model then predicts that firm \( f \) opens at least one establishment per distinct value \( z_{fj} \). Firm \( f \) sorts its establishments and jobs across locations: establishments with high \( z_{fj} \) jobs locate in low job losing rate locations, while establishments with low \( z_{fj} \) jobs locate in high job losing rate locations.
job losing rate differences. Table A.7 in Appendix A.4.13 indicates that the data supports all three implications. It reports conditional correlations of establishment-level job losing rates with the average local job losing rate. First, the R-squared of the regression increases from 0.14 to 0.64 with firm fixed effects, suggesting that there are substantial differences in job losing rates across firms common to their many establishments. Second, consistent with implication (a), the regression coefficient without firm fixed effects is 1.34 and statistically significant: a one percentage point increase in the average local job losing rate is associated with a 1.34 increase in new establishments’ job losing rates in the first year. When including firm fixed effects, the regression coefficient drops from 1.34 to 0.30. This 78% reduction indicates that there are substantial differences in job stability inherent to firms which contribute significantly to spatial job instability gaps, supporting implication (b). The remaining 22% are due to either within-firm dispersion in job quality – implication (c) – or to local factors.

**Job finding rate vs. worker contact rate.** Having investigated the determinants of job losing rates across space, I conclude this validation section by unpacking the countervailing effects of worker contact rates and contact-to-job probabilities in shaping job finding rates. In the model, sorting arises largely because more productive employers prefer to locate where filling vacancies is easy. Hence, unemployment rates should correlate positively with worker contact rates. I test this implication using the French Labor Force Survey. Replicating the approach outlined in Section 1.4.3, I estimate both worker contact rates and contact-to-job probabilities across locations. I bin commuting zones into 5 unemployment quintiles to reduce small sample measurement error. Figure 1.9 shows that the worker contact rate is indeed positively correlated with the unemployment rate in the data, the opposite of the job finding rate. Hence, the data indicate that the contact-to-job probability more than offsets the positive correlation between contact rates and unemployment rates.

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65 Consistent with Figure A.9, the coefficient is larger than 1, reflecting larger job losing rates at new jobs.  
66 If anything, increasing the number of bins results in a steeper positive slope for the contact rate.
1.6 Policy counterfactuals

With the estimated model at hand, this final section presents two policy counterfactuals. The first exercise investigates the general equilibrium welfare gains from both optimal and real-world economy-wide place-based policies that are financed at the federal level. The second exercise studies the effects of a widespread type of discretionary place-based policy financed at the local level. Namely, it studies the effects of subsidies to attract a large, productive plant to a particular location – a “Million Dollar Plant” as in Greenstone et al. (2010)

1.6.1 The aggregate effects of place-based policies

The first policy exercise evaluates the local and aggregate effects of economy-wide place-based policies. As per Proposition 6, the optimal policy subsidizes high job losing rate locations,
which also tend to have high unemployment. I focus on the location choice of employers, and start by examining the quasi-optimal policy that corrects the labor market pooling externality. Under this policy, the economy is not fully efficient since the Hosios (1990) condition needs not hold. The quasi-optimal policy takes the form of a profit subsidy, financed with a non-distortionary tax. To compute welfare gains without taking a stand on distributional issues between owners and workers, I use the alternative, equivalent formulation in which profits and rents are redistributed to workers with a non-distortionary flat earnings subsidy, while the policy is subsidized with a flat earnings tax.

I contrast the effects of the quasi-optimal policy with a real-world example of an economy-wide set of place-based policies. Federal programs such as the Empowerment Zones program in the United States proposed considerable tax breaks for firms opening jobs in high unemployment areas. In France, a similar Enterprise Zones (EZ) program was rolled out in 1996 and subsequently expanded. The labor market pooling externality provides a theoretical basis for such policies. By changing employers’ incentives to open jobs in various locations, the policies effectively relocate jobs across space and affect the general equilibrium of the economy. Ideally, the structural estimation would account for the policy during the sample period. However, reliable estimates of local policy expenditures are hard to obtain, making it difficult to net out the effects of the EZ policy in the estimation. In practice, the policy is small and has modest local and general equilibrium effects. Therefore, it is unlikely to substantially affect the parameter estimates and the counterfactuals.

Figure 1.10 displays the cross-sectional patterns of the equilibrium under the quasi-optimal policy and under a budget-equivalent version of the French EZ program. Figure 1.10 reveals that the EZ program subsidy is much smaller than the quasi-optimal one. However, it shares the same qualitative pattern: to incentivize high productivity employers to open jobs in high unemployment locations. Because of its smaller magnitude, it induces only few jobs to change location and has minor effects on the job losing rate. In contrast, the quasi-optimal policy massively relocates productive jobs towards initially high unem-
Figure 1.10: Model’s solution in the decentralized equilibrium, the quasi-optimal policy and the French EZ program.

Figure 1.11 depicts the welfare gains for residents in all locations. Locations are ordered by their unemployment rate in the laissez-faire equilibrium, and grouped into population-weighted quantiles to reflect how many workers experience a given welfare increase. Figure 1.11 reveals that the quasi-optimal policy achieves large welfare gains in initially high unemployment locations. Appendix A.3.1 derives an exact welfare decomposition in the model, which corresponds to the different colored areas in Figure 1.11. The blue area shows that direct gains to the average resident unemployed worker steadily rise with pre-policy local unemployment, and exceed 10% in the most distressed areas. Importantly, in steady-state, unemployed workers all have the same expected utility due to compensating differentials in the form of preference shocks. Therefore, the blue area that describes the welfare gains to unemployed workers conditional on a given preference shock $\zeta = 1$ and conditional on
human capital $k = 1$.\textsuperscript{67} The blue area also corresponds to each location’s contribution to the aggregate welfare gains for unemployed workers in the economy. The green area represents the additional welfare gains to the average employed worker. Thus, the welfare gains to an employed worker with $k = 1$ is the sum of the blue and the green area. Under the quasi-optimal policy, employed workers gain slightly more than unemployed workers. Finally, human capital accumulation in orange benefits both unemployed and employed workers. It amplifies welfare gains more than two-fold in originally high unemployment locations. Because the quasi-optimal policy relocates jobs away from the best locations, residents there experience welfare losses. In contrast, the EZ program has more modest effects, with welfare gains peaking around 3% and concentrated in the targeted high unemployment areas.

To highlight the spatial distribution of these local welfare gains, Figure 1.12 (a) maps the gains from the quasi-optimal policy across all French commuting zones. Because welfare gains are strongly correlated with the local unemployment rate, the southern Mediterranean coast benefits most. In suburban areas close to Paris, several high unemployment commuting zones also benefit substantially. Figure 1.12 (b) shows that local welfare gains are accompanied

\textsuperscript{67}Alternatively, the blue area is equal to the steady-state welfare gains of an unemployed worker who never received the moving opportunity and so stayed in the same location, with $k = 1$. 
Figure 1.12: Local gains from the quasi-optimal policy

(a) Welfare (%)

(b) TFP (%)

by substantial TFP improvements. Finally, note that residential amenities are fixed in counterfactuals. To the extent that other factors such as local crime rates or locally provided public services respond to lower unemployment rates, the welfare gains in the model are likely to be a lower bound relative to welfare gains inclusive of such adjustments.

Leveraging the structure of the model, I aggregate the local welfare gains and compute the aggregate welfare gains from the quasi-optimal policy and the EZ program. Table 1.3 first highlights that the quasi-optimal policy removes the pooling externality and reduces spatial unemployment differentials five-fold as in Table 1.2. This change follows a large relocation of high productivity jobs towards poorer locations. As the aggregate efficiency of the economy rises, the aggregate unemployment rate falls by 0.5 percentage points, a 7% drop. The quasi-optimal policy achieves over 5% aggregate welfare gains. Overall, the quasi-optimal policy largely ameliorates opportunities for unemployed workers. They become shielded against the risk of remaining trapped in distressed labor markets.

Second, Table 1.3 reveals that despite its relatively small size, the EZ program reduced spatial unemployment differentials by 10%. While it had virtually no impact on the aggre-
Table 1.3: Aggregate gains from place-based-policies

<table>
<thead>
<tr>
<th></th>
<th>Laissez-faire</th>
<th>EZ program</th>
<th>Quasi-optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate unemployment rate</td>
<td>0.076</td>
<td>0.076</td>
<td>0.071</td>
</tr>
<tr>
<td>St. dev. unemployment rate</td>
<td>0.022</td>
<td>0.020</td>
<td>0.004</td>
</tr>
<tr>
<td>Aggregate welfare gains (%)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unemployed</td>
<td>0.23</td>
<td>5.08</td>
<td></td>
</tr>
<tr>
<td>Employed</td>
<td>0.09</td>
<td>2.93</td>
<td></td>
</tr>
<tr>
<td>Human capital</td>
<td>0.01</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>Redistribution (% of GDP)</td>
<td>0.13</td>
<td>1.10</td>
<td></td>
</tr>
</tbody>
</table>

Aggregate unemployment rate, it raised aggregate welfare by 0.23%. Most of these gains stem from better human capital accumulation in high unemployment areas. Interestingly, the relative contribution of human capital accumulation is lower under the quasi-optimal policy than under the EZ program. Because it is smaller, the EZ program harmlessly punctures high productivity jobs from the entire economy and helps residents of high unemployment locations. The quasi-optimal policy involves more drastic relocations. Therefore, human capital gains in poorer locations are partly offset by losses in richer locations.

It is not surprising that the EZ policy delivers smaller gains than the quasi-optimal policy. The EZ policy consists in a much smaller subsidy scheme as shown in Figure 1.10. Aggregate expenditures on the EZ policy represent redistributing 0.04% of Gross Domestic Product (GDP). Expenditures under the quasi-optimal policy are over 100 times larger. If scaling up the redistribution-efficiency ratio of the EZ policy was possible, welfare would rise by 5.75% for every percent of GDP redistributed. The redistribution-efficiency ratio of the quasi-optimal policy is close to 1, indicating that decreasing returns rapidly kick in. Indeed, one should expect the planner’s problem to be concave in the profit subsidy around the quasi-optimal policy. Thus, the largest gains for a marginal increase in the profit subsidy should arise close to the laissez-faire.
1.6.2 The local effects of a “Million Dollar Plant”

The previous analysis focused on federally funded place-based policies. This last section explores the effects of locally funded place-based policies. Specifically, I examine the local employment and welfare effects of attracting a large, productive plant – a “Million Dollar Plant” (MDP). As documented by Greenstone et al. (2010) and Slattery (2019), local governments allocate considerable resources to discretionary subsidies to firms that seek a location for a new plant. Local governments are largely motivated by potential employment expansions, both due to the direct jobs created by the MDP, but also due to potential spillover effects. These two channels are sometimes called direct and indirect job creation in this context.

From the perspective of the theory, attracting a large productive plant with subsidies resembles the optimal policy from Proposition 5, although in the MDP case the policy is locally funded. To quantify the potential employment and welfare gains from such a policy, I identify a MDP as a mass point \( \bar{m} \) of high productivity jobs \( \bar{z} \) in the model. I start with a simple positive exercise that abstracts from the cost of attracting a MDP. To that end, I select a location with a 10\% unemployment rate, I target the MDP’s employment and output share in the location.\(^{68}\) Then, I compare the steady-state outcomes in the location before and after the MDP.\(^{69}\)

Table 1.4 displays the results. Interestingly, the model produces Total Factor Productivity (TFP) gains at employers other than the MDP that are close to those estimated by Greenstone et al. (2010). Although the model abstracts from technological spillovers, TFP gains arise because of labor market pooling complementarities. The MDP improves the local job quality mix. Workers throughout the economy in-migrate to benefit from improved earnings prospects, leading to a half percent increase in population. The labor market pooling

\(^{68}\) Their estimates are at the county level, while the model is at the commuting zone level. I map their estimates into commuting zone-level targets by deflating their estimates by the average number of counties per commuting zone in the United States.

\(^{69}\) While local transitional dynamics can be computed, a full discussion thereof is beyond the scope of this paper.
Table 1.4: Employment and welfare gains from the calibrated MDP.

<table>
<thead>
<tr>
<th>Moment</th>
<th>Greenstone et al.</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>MDP employment share (%)</td>
<td>1.70</td>
<td>1.78</td>
</tr>
<tr>
<td>MDP output share (%)</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>TFP gains at other employers (%)</td>
<td>1.80</td>
<td>1.91</td>
</tr>
<tr>
<td>Job multiplier</td>
<td></td>
<td>-0.55</td>
</tr>
<tr>
<td>Unemp. rate change (p.p.)</td>
<td></td>
<td>-0.21</td>
</tr>
<tr>
<td>Job losing</td>
<td></td>
<td>-119%</td>
</tr>
<tr>
<td>Job finding</td>
<td></td>
<td>+19%</td>
</tr>
<tr>
<td>Population change (%)</td>
<td></td>
<td>0.58</td>
</tr>
<tr>
<td>Welfare gains (%)</td>
<td></td>
<td>1.15</td>
</tr>
</tbody>
</table>

Complementarity then kicks in as hiring conditions improve, thereby attracting better jobs even beyond the MDP. While technological spillovers are a common interpretation for the effects of a MDP, the model indicates that labor market pooling complementarities alone can also rationalize the ensuing TFP gains.

Together, the MDP and the better other employers contribute to a 0.2 percentage point reduction in the local unemployment rate. The drop in the job losing rate is the primary driver of this reduction, as jobs at the MDP and at other employers are more stable. As a result, gross welfare of local residents rises by 1.15%. Despite these welfare gains, the calibrated MDP experiment brings in few jobs above and beyond what the MDP directly contributes. Every new MDP job displaces on average half a pre-existing job, in line with existing evidence.\(^\text{70}\)

The exercise so far abstracted from the cost of attracting the MDP. In practice, these plants enjoy considerable subsidies often financed with local taxes rather than an economy-wide tax as in Section 1.6.1. As pointed out by Slattery and Zidar (2019), measuring the exact collection of subsidies that MDPs receive is challenging. To circumvent this difficulty,\(^\text{70}\) Slattery and Zidar (2019) find little support for systematic indirect job creation effects of MDPs. One caveat is that the model does feature neither input-output linkages with trade costs nor technological spillovers. Both may in principle amplify the indirect job creation effects, although the calibrated MDP already matches estimated TFP gains.

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I discipline the cost of a MDP inside the model. For each location, I compute the optimal MDP subsidy that maximizes the welfare of its residents. To do so, I assume that every local government believes it acts in isolation – a complete evaluation of between-location tax competition is beyond the scope of this paper. The MDP benefits from a profit subsidy, which is financed through a local earnings tax.

Figure 1.13 displays the local effects from the optimal MDP in the cross-section of locations ordered by pre-policy unemployment rate. In line with evidence from Slattery (2019), the model indicates that high unemployment locations spend more on subsidies to MDPs. These locations offer higher profit subsidies, sometimes over 10%. They optimally attract more productive MDPs which can take over as much as 50% of local employment, in turn bringing in more productive jobs through labor market pooling complementarities. Local unemployment drops by 10 percentage points in the most distressed areas, largely driven by reductions in the job losing rate. On net, welfare rises by over 10% there. However, optimal discretionary spending comes at a substantial fiscal cost that can exceed a 10% earnings tax.
Given political economy or credit constraints, local government may not be able to take advantage of the large gains depicted in Figure 1.13.

**Conclusion**

This paper proposes an alternative view of spatial unemployment differentials. I have shown that high localized unemployment arises because workers repeatedly lose their job, not because finding a job is particularly hard. Differences in job losing rates emerge as employers with unstable jobs self-select into similar locations, while employers with stable jobs locate in others. I have developed a theory in which labor market pooling complementarities are the central driver of the location choice of heterogeneous employers. As a result, employers with stable jobs over-value locating close to each other due to labor market pooling externalities. This view implies that redistributing from low unemployment locations towards high unemployment locations is welfare improving.

Of course, the idea that pooling complementarities result in too much concentration in the best options available to workers and employers is more general than the particular spatial context put forward in this paper. For instance, investigating the implications of pooling externalities for the allocation of workers and employers across occupations and industries could lead to interesting policy insights. Indeed, in the spatial context alone, pooling externalities quantitatively account for the lion’s share of differences in unemployment across locations.

Consequently, the view of this paper emphasizes that spatial unemployment differentials are not an immutable characteristic of the economic landscape. Instead, place-based policies have the potential to drastically reshape the spatial distribution of unemployment, and ameliorate employment prospects at the aggregate level. While a long tradition of research has found that agglomeration economies call for taxes on poor locations, the implications thereof have remained at odds with a wide range of real-world spatial policies. The view
that labor market pooling externalities lie at the heart of the location decisions of employers helps reconcile theory with policymakers’ intuition that incentivizing businesses to open in distressed areas may help rather than harm individuals. Yet, the inherently local nature of many economic interactions gives rise to many other externalities. Therefore, any policy recommendation should account for as many sources of agglomeration and congestion as possible. As individuals who grow up and live and different places seem to face increasingly divergent economic opportunities, place-based policies appear more relevant than ever.
Chapter 2

Location as an Asset

2.1 Introduction

Few decisions shape an individual’s life more than the location decision. It determines job opportunities, social interactions, schooling and entertainment options, as well as a number of other less central characteristics of someone’s life. The location decision is particularly relevant because of the large heterogeneity in location characteristics, even within a country, a state, a region, or a city. Living in Soho in Manhattan is quite different than living in Queens, and a world apart from living in parts of Newark or Camden, New Jersey. These spatial differences are enormous. Life prospects for a kid growing in Palo Alto are staggeringly different than those for someone growing in central Detroit, even if they come from similar backgrounds and both go to local public schools. The obvious question that arises is then, why do people remain in some of these locations? Why do we fail to see people go to the locations that seem to offer the best prospects for them and for their families?

Three main answers have been offered to these questions in the economics literature. The first one relies on the presence of large migration costs that make moving to better locations not worth the cost.\footnote{Kennan and Walker (2011a) estimate that moving costs as large as $380 thousand 2010 dollars (for young movers, 312 thousand for average ones) are needed to account for observed migration flows using a} The second one argues that local living costs, as reflected
in housing and other local prices, compensate for other local benefits over the residency period. The third one simply argues that agents ‘cannot afford’ to live in some places perhaps due to indivisibilities in housing. The problem with the first explanation is that it is hard to imagine that moving costs are sufficient to bridge the gap between the best and worse neighborhoods in virtually all regions of the world. These largely unobserved costs seem to be just a stand-in for another mechanism. As for the other two explanations, although housing and other local costs can differ substantially across regions, adjusting the size of one’s apartment, commuting from cheaper locations, and buying in big-boxed stores and other national retailers are effective strategies to deal with local prices. Something is missing from this basic notion of static spatial equilibrium where similar marginal movers equalize utility across locations adjusted for moving costs.

In this paper we propose a different way of conceptualizing the location decision of agents. We argue that the location decision can be understood as an asset investment decision. Buying more of the asset involves moving to better locations that cost more today but give better returns tomorrow, while selling the asset implies moving to cheaper locations with little opportunities. The ‘location as an asset’ view can explain why agents prefer locations that seem undesirable from a static spatial equilibrium perspective even in the absence of moving costs. It can also explain why local living costs compensate the benefits from desirable locations for some agents but not for others, even in the absence of non-homotheticities or differences in preferences. The ‘location asset’ should not be confused with ‘an asset at a

state-of-the-art model of location decisions. Diamond et al. (2019) using a policy that implements rent-controls in the San Francisco area find a smaller but still large fixed cost of around $40 thousand.

Another potential reason for these location choices are non-homotheticities in preferences: the less wealthy simply like certain amenities better and the locations that have them are the ones with worse opportunities.

We can think of at least two channels through which returns to location would accrue over time. First, different places may provide different labor market prospects during one’s work-life, as documented by De La Roca and Puga (2017). Second, different locations may offer different rates of intergenerational human capital accumulation. For example, by offering different schooling options, as shown in Chetty and Hendren (2018).
location’, like a house. The location asset is used by all agents, including renters and owners, when they make location choices.\textsuperscript{4}

The ‘location asset’ has some specific features that make it different from other assets and determine its use. As any other asset, unconstrained agents use it only to the extent that the return from doing so dominates that of other assets, in particular, risk free bonds. The unique characteristic of the location asset is that it is not subject to borrowing constraints. Agents can always borrow, namely, transfer resources from the future to the present, by going to a cheaper neighborhood or city with worse opportunities. If an agent is not in the worst possible neighborhood already, she can keep transferring resources from the future to the present by ‘selling’ the location asset.\textsuperscript{5} The other unique characteristic of this asset is that the amount of the asset that an agent can hold is limited by the housing needs, labor supply, fertility decisions and other choices that determine the current cost and the future benefits of living in a particular location. As such, the asset has heterogeneous returns depending on the holder of the asset.

Conceptualizing location decisions as buying and selling a ‘location asset’ is useful to understand mobility decisions. Consider an agent with little or no wealth that receives a front-loaded income shock. For example, a blue-collar worker in the automobile industry in Detroit that gets fired. Where will she go? A good neighborhood with excellent schools for her children and plenty of job opportunities or a run-down neighborhood in Saint Louis? Think first about the consumption-savings decision of this agent. The front-loaded shock makes her want to transfer consumption from the future to the present. In the absence of accumulated wealth, smoothing consumption requires borrowing. The absence of collateral, however, implies that she will be constrained to borrow using standard financial assets. What

\textsuperscript{4}We view housing wealth as financial wealth that is perhaps less liquid. Hence, borrowers will run it down completely before they become financially constrained and start using the location asset to transfer resources intertemporally.

\textsuperscript{5}This feature distinguishes the location asset from human capital. Because mandatory schooling lasts until 16 years old in most U.S. states, individuals cannot ‘short’ human capital as much as the location asset. In addition, although a host of aid and financing programs exists, individuals might face borrowing constraints to finance college and other types of higher education.
is left is to borrow using the location asset and downgrade to a cheaper location with worse opportunities. Hence, constrained agents that receive bad shocks will have a higher demand for locations that offer few opportunities at minimal cost. Similarly, front-loaded positive shocks will make constrained individuals upgrade location so as to save using the location asset.

Some additional aggregate implications follow. For example, changes in the rewards for particular occupations will result in front-loaded shocks for dynastic families, since heads-of-households have already invested in an occupation while their descendants have yet to choose. Hence, these changes in rewards will lead to spatial segregation as auto workers who borrow locate in Detroit and computer programmers and Yoga teachers who save locate in Palo Alto. Furthermore, our view underscores that place-based policies may hurt the currently poor as they reduce the supply of cheap locations where those individuals may prefer to locate.

To make precise our conceptualization of the location decision as an asset that does not face borrowing constraints, we start by proposing a simple two period economy where agents have heterogeneous asset holdings, incomes, and levels of skills. Agents have access to a risk free bond but face a standard borrowing constraint that prevents them from borrowing beyond an exogenous amount. Individuals choose a consumption profile and a location, which in turn determines their current rent and income next period as a function of their skill. There is a continuum of locations that differ in the marginal return of a unit of skill. In equilibrium, wealthy agents locate in their ideal city conditional on their skill, while constrained individuals, either because they have low assets levels or back-loaded incomes, locate in cities that pay less but where rents are lower. Namely, they borrow using the location asset. Back and front-loaded shocks have the effects described above.

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6Our view also implies that low rates of return in financial market (e.g. low interest rates) result in low rates of return of the ‘location asset’ and, therefore, larger price differentials across locations, reminiscent of the low interest rate period after the 2008 financial crisis where some of the differentials in house prices increased.
We then present a fully-fledged infinite horizon dynamic model with similar characteristics in order to generalize our findings and provide a framework that is potentially closer to quantitative analysis. Agents now face an idiosyncratic income process. The main advantage of this framework relative to our simple two-period framework is that wealth is now endogenous and we can compute an invariant wealth distribution, and, perhaps more importantly, that we can use it to understand the reaction of constrained and unconstrained dynasties to transitory and permanent income shocks over multiple periods. The drawback of this more complex framework is that our analysis is based on numerical simulations only. As a result of an idiosyncratic temporary income shock, unconstrained individuals first run down their financial assets until they are at the borrowing constraint. Once there, they start borrowing using the location asset and so downgrade their location in order to minimize fluctuations in their level of consumption. This downgrading of location continues until individuals reach the worse location they are willing to go to, or the income shock reverts to the high value. Once the temporary shock has reverted, individuals go back to the initial location progressively.\(^7\)

The implications of the ‘location as an asset’ view are sharp. Negative (positive) front-loaded shocks should make constrained individuals downgrade (upgrade) location, while unconstrained agents should not change their location. To contrast these predictions with empirical evidence we use detailed individual panel data from France. We use a longitudinal 8\% panel of workers which allows us to track the same individual over several years. By merging tax return data from both households and employers, our dataset provides us with three key variables, along a number of other characteristics. Crucially, we observe the wage

\(^7\)Our infinite horizon model shares many features with dynamic portfolio problems with investors who face a credit constraint on risk-free bonds. Thus, we build a Huggett (1993) economy with a second asset: the ‘location asset’. In particular, our model could be viewed as one in which possibly constrained entrepreneurs choose in which project to invest (the location), subject to a collateral constraint; or one in which successive overlapping generations choose how much education to buy subject to a constrained borrowing-saving trade-off. Related work includes but is not limited to, Angeletos (2007) and Moll (2014). Our framework is distinct from those in two dimensions. First, we model both risk-aversion and idiosyncratic additive income shocks on the investor side, leading individuals to use the location asset to smooth consumption when they are close to the constraint. Second, individuals in our model always wish to hold a convex combination of both assets, due to the endogenously nonlinear returns of the ‘location asset’.
earned by individuals, their detailed location information, as well as their income from financial assets. We infer an individual’s stock of financial assets from her annual financial income. Consistent with our model, we order locations according to the average income of local residents to obtain a ranking of locations. Overall, our dataset constitutes one of the first large-scale administrative datasets with detailed information on financial assets, high-resolution location, and matched employer-employee labor market characteristics for a large economy like France.

We first document that moving to locations with a higher rank pays off gradually over time. Building on De La Roca and Puga (2017), we focus on the dynamic gains in the labor market across finely disaggregated neighborhoods. We find that the returns to moving to the best location relative to moving to the worst double after 10 years for observationally equivalent individuals. Comparing similar individuals, movers to the best location relative to movers to the worst location receive wages that are 10% higher upon moving, but that gap widens to 20% after 10 years. These findings support the view that moving to better locations pays off gradually over time, thus underscoring the importance of the investment aspect of location.

We then empirically investigate the location decisions of individuals that receive negative income shocks. We use an event-study design to track how the rank of an individual’s location changes over time. The results are stark. Conditional on municipality, income, occupation, age, and home ownership, after a negative income shock of at least 25%, individuals that move and start at the bottom quintile of the financial wealth distribution (and so are presumably financially constrained) downgrade their location by about 2 percentile points relative to movers at the top quintile.

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8 Lack of data on school quality in France prevents us from estimating intergenerational rates of human capital accumulation across locations due to heterogeneous schooling options. Therefore, our results should be interpreted as a lower bound on the dynamic gains of location.

9 Glaeser and Mare (2001) also find some evidence in the U.S. for dynamic gains from migrating to larger cities. Baum-Snow and Pavan (2012) show, in a structural model, that differences in initial wages are important drivers of pay differentials between small cities, while differences in wage growth explain a large part of pay differentials between larger cities in the U.S.
Our results are robust to a number of potential concerns. First, low-wealth individuals might simply exhibit systematically different location trajectories than wealthy individuals for reasons unrelated to the use of the ‘location asset’. Yet, prior to the income shock, we find no significant difference in the relative location behaviour of agents in different wealth quintiles. Second, individuals may partially anticipate their income process, potentially muting the estimated response. To isolate an income shock that is less likely to be anticipated, we restrict attention to individuals in the context of a “mass layoff” in their firm (an event in which their employer shrinks by 25%). In this case, the estimated magnitude of the relative downgrading of the location of financially constrained individuals is larger. Third, one might speculate that the relative downgrading of an individual’s location depending on their wealth level might potentially be related to changes in their relative consumption of local amenities or adjustments in their commuting patterns. We show theoretically that, conditional on initial location, consumption of amenities leads to the opposite implications.\(^\text{10}\) In addition, when we control for post-shock local amenities and commuting distance, our results are essentially unchanged. Hence, the downgrading of location is not simply reflecting the static choice to consume less amenities or commute more.

The predictions of the theory can be verified in relative differences across wealth groups, as discussed, as well as in levels. We show that following the negative income shock, low-wealth individuals on average move to lower ranked locations, while high-wealth individuals do not adjust their location much. As predicted by our mechanism, the location decision of individuals should be mirrored by their holdings of financial assets, since unconstrained individuals smooth consumption with the asset that has the lowest return at the margin. We find that low-wealth individuals who receive the shock downgrade their location but do not adjust their holdings of financial assets, consistent with them being close to the credit

\(^{10}\)Downgrading location due to a simple static choice to consume less urban amenities as a result of the income shock is not consistent with our results. The reason is that, absent financial constraints, if two individuals live in the same location but have different incomes, the high income individual is the one that loses more at impact from the shock and for whom location is more elastic. Hence, if anything, the high income individual necessarily downgrades more. We prove this result formally in Section 2.6.
constraint. In contrast, wealthy individuals who receive the shock do not downgrade their location but reduce their holdings of financial assets. Together, our empirical findings provide clear evidence of the use of the ‘location asset’ to intertemporally smooth the consumption of income shocks.

There is a large literature documenting the large variation in income levels and other outcomes across locations.\footnote{See Wilson (1987), Denton and Massey (1998), Cutler and Glaeser (1997), Desmet Rossi Hansberg (2013) Altonji and Mansfield (2011), and Hsieh and Moretti (2019) among many others.} Kennan and Walker (2011b) argue forcefully that inter-state migration decisions are made based on income prospects, but are also influenced importantly by geographic differences. In fact, Diamond\textsuperscript{2016} and Giannone\textsuperscript{2017} show that the U.S. has experienced increasing skill segregation, indicating that spatial gaps are not diminishing. Bilal (2020) emphasizes that spatial unemployment differentials are large and persistent, and lead to substantial human capital gaps as workers in high-unemployment areas are repeatedly scarred by unemployment.\footnote{Qualitatively, this channel provides one explanation for the differential returns to mobility that we document. Quantitatively, scarring effects from unemployment can account for about half of the differential returns to mobility.} Kaplan and Schulhofer-Wohl (2017) show that mobility in the U.S. is declining.\footnote{Kaplan and Schulhofer-Wohl (2017) link the decline in U.S. mobility to falling wage differentials within occupations.} Going one step further, Fogli and Guerrieri (2018) argue that spatial segregation is related to income inequality because it affects the returns to human capital and therefore offsprings’ education.

Most equilibrium analysis of individual location choices is either cast in partial equilibrium and so does not consider the valuation side of the ‘location as an asset’ view (like Kennan and Walker 2011b, or Diamond\textsuperscript{2016}) or static and based on a simple spatial equilibrium condition that does not include the investment aspect of location decisions (like Desmet Rossi Hansberg\textsuperscript{2013} Allen and Arkolakis 2014, or Redding 2016). Giannone\textsuperscript{2017} and Desmet Et Al\textsuperscript{2018} do provide dynamic general equilibrium setups with costly migration, but migration decisions only provide static gains or losses. In Caliendo and Parro (2019) and Bilal (2020), agents solve forward looking problems in deciding their
location but they simply consume their income and so do not solve a consumption-savings decision or accumulate wealth.

The view of investment as an asset was hinted at initially by Sjaastad (1962). Lucas (2004), Morten (2019) and Cavalcanti-Ferreira et al. (2018) also present evidence and arguments to view migration as a stepping-stone or a form of self-insurance. Some of the most detailed studies of mobility for low income, and likely constrained individuals, are consistent with the ‘location as an asset’ view. For example, in the “Moving to Opportunity” randomized experiment, conditioning aid on upgrading location reduced the use of housing vouchers by about a third (21 percentage points). Furthermore, while the literature using this experiment initially found that economic outcomes were not affected by an upgrade in location (Duncan et al. 2013), the most recent studies have found strong evidence that the outcomes for children that moved when young are positive (Chetty and Hendren 2018, and Davis et al. 2015), consistent with our emphasis on the investment dimension of location decisions rather than on the current benefits. Using tax records, Chetty and Hendren 2018 found a trade-off between child future earnings and rents. They estimate that a 1% increase in a child’s future earnings can be achieved by moving to a location with a median rent that is $176 higher. The ‘location as an asset’ view argues that constrained agents might not want to take what seems like a good bargain, since they are constrained and want to borrow not invest further.

The rest of the paper is organized as follows. The next section, Section 2.2, introduces the simplest model necessary to make precise our notion of location decisions as investment decisions. This simple two period model is then extended to an infinite horizon model in Section 2.3. In that section we present examples of the implied dynamic consumption, asset, and location paths of individuals. Section 2.4 presents our empirical analysis using the French individual level panel to show that agent’s location decision respond to income

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14Fernandez and Rogerson (1998) and Fogli and Guerrieri (2018) discuss the trade-off between location and children education.
shocks as our theory predicts. Section 2.5 concludes. An Appendix includes the technical proofs, additional robustness tests, and detailed data descriptions.

2.2 A Simple Model

We aim to provide the simplest setup in which our ‘location as an asset’ view can be made precise. Because we need location to be an investment, we need a model with at least two periods. Hence, we model an economy over periods 0 and 1. The economy consists of a unit mass of individuals that differ in their skill, \( s \in [s, \bar{s}] \), and their income in period 0 and 1, \( \{y_t\}_{t=0}^1 \in [\underline{y}, \bar{y}] \). The income of the individual in period 0 includes her labor income plus any wealth she is initially endowed with. In sum, an individual is characterized by a triplet \((y_0, y_1, s)\). We denote the joint probability density function over these outcomes by \( f \) and the cumulative distribution by \( F \).

There is a continuum of locations or ‘cities’. We classified cities according to the complementarity of the returns from living in them with the skills of individuals. We denote locations by an index \( z \in [\underline{z}, \bar{z}] \) with \( \underline{z} \geq 0 \). The density of cities with characteristic \( z \) is given by \( h \) with cumulative density \( H \). The skill of an individual determines the benefits from locating in cities. We assume that the returns for an individual of skill \( s \) to living in city \( z \) are given by \( zs \). Agents can move freely across locations. Hence, the supermodularity of this function will lead to positive assortative matching conditional on other individual characteristics, as we describe below.

The population density, \( L(z) \), of individuals living in cities of type \( z \), as well as land rents, \( q(z) \), in those cities are determined endogenously. We assume that the cost of supplying housing increases with population size due to some form of decreasing returns. Hence,

\[
q(z) = Q(L(z)) \text{ for } z \in [\underline{z}, \bar{z}]
\]
where \( Q(0) = 0 \) and \( Q \) strictly increasing. That is, housing is free in locations without population and rents are strictly increasing in city size.

Individuals have access to a risk free bond with gross interest \( R > 1 \). We assume that this world interest rate is exogenous and determined in world markets.\(^{15}\) Agents are subject to a standard borrowing constraint that limits their asset holdings between period 0 and 1, \( a \), to be above some level \( a \). Hence, if, for example \( a = 0 \), agents can only save but not borrow with the financial asset.

### 2.2.1 Asset and Location Choices

Households maximize lifetime utility with a discount factor given by \( \beta \leq 1 \). For simplicity we specify the period utility function as \( u(c) = \log c \) but virtually all our results go through for any concave utility function that satisfies Inada conditions. The problem of a household is then to choose consumption in each period, purchases of the risk free bond, and location in period 1, to solve

\[
V(y_0, y_1, s) = \max_{c_0, c_1, a, z} \log c_0 + \beta \log c_1
\]

Subject to

\[
\begin{align*}
    c_0 + a + q(z) &= y_0, \\
    c_1 &= zs + y_1 + Ra, \\
    a &\geq a.
\end{align*}
\]

That is, individuals maximize utility subject to budget constraints each period, as well as the borrowing constraint. In period zero, an agent’s income includes anything he earns today and all of his wealth. Note that we have abstracted from any returns from the complementarity between an agent’s skill and the city where she starts (say, \( z_0s \)). We think of this term as also being embedded in \( y_0 \). Not explicitly recognizing this term explicitly avoids carrying

\(^{15}\)Technically, we only need \( R > 0 \), which we can allow without loss of generality. In addition, it would be simple to endogenize the interest rate \( R \) through an asset market clearing condition without changing any of our core results.
$z_0$ as a state variable in the consumer problem. This is without loss of generality given that free mobility implies that current location only affects an agent’s decisions through current income.

Note also that we make the agent pay rent one period in advance. So land rent for their chosen $z$ location, $q(z)$, enters the left-hand-side of the period 0 budget constraint only. Rent paid for living in location $z_0$ in period 0 is not modeled and would simply be included in the resulting period 0 income. Making household pay rent one period in advance underscores the investment nature of the location choice. Namely, it recognizes that the good jobs, amenities, or education associated with living in a good location are enjoyed over time and not necessarily immediately after arriving there. Furthermore, although we do not incorporate other current urban costs as commuting, congestion, or crime, or urban benefits as amenities, all of them can be thought of as included in the net current price of location $q(z)$.

The problem above abstracts from a flexible housing demand choice since it makes anyone living in location $z$ pay the same cost $q(z)$. We decided not to incorporate this margin explicitly because, absent adjustment costs, the housing demand choice is a static choice that does not eliminate or prevent the use of the location asset (although it can affect its return).\textsuperscript{16} The problem in (2.7) also abstracts from income risk. In the next section we write a multi-period extension with uncertainty about the realization of the income process. However, in this simple model without uncertainty, the location asset is used to transfer consumption across time, but not across states of nature or for precautionary purposes. Of course, in a richer environment the location asset could also be used for these alternative purposes.

The first-order conditions of the problem in (2.7) imply the standard ‘Financial Euler equation’

$$
\frac{c_1^*(y_0, y_1, s)}{\beta c_0^*(y_0, y_1, s)} \geq R \text{ for all } (y_0, y_1, s),
$$

\textsuperscript{16}In Appendix B.2.2 we develop an extension where agents can choose the size of their house and pay a prize $q(z) + p\ell$ for renting $\ell$ units of housing in location $z$. 

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with equality if and only if the borrowing constraint is not binding, namely \( a^*(y_0, y_1, s) > a \). We denote all individual optimal choices with an asterisk (*)..

Absent borrowing constraints, the desired asset holdings of an individual \((y_0, y_1, s)\), denoted by \( \tilde{a}(y_0, y_1, s) \), are given by their income net of rents in period zero \((y_0 - q(z))\) minus permanent consumption, which is given by \( \left( y_0 + \frac{y_1 + \tilde{z}^s}{R} - q(\tilde{z}^s) \right) / (1 + \beta) \).\(^{17}\) Namely,

\[
\tilde{a}(y_0, y_1, s) = y_0 - q(\tilde{z}^s(y_0, y_1, s)) - \frac{y_0 + \frac{y_1 + \tilde{z}^s(y_0, y_1, s)}{R} - q(\tilde{z}^s(y_0, y_1, s))}{1 + \beta}.
\]

Thus, actual savings in the financial asset are given by

\[
a^*(y_0, y_1, s) = \max\{\tilde{a}(y_0, y_1, s), a\}.
\]

Free mobility implies that individuals are never constrained in the ‘location asset’. Hence, for all agents, the location decision yields a ‘Mobility Euler equation’ given by

\[
\frac{c_1^*(y_0, y_1, s)}{\beta c_0^*(y_0, y_1, s)} = \frac{s}{q'(\tilde{z}^*(y_0, y_1, s))}, \tag{2.3}
\]

for all \((y_0, y_1, s)\).

Hence agents can optimize their intertemporal consumption path by choosing their holdings of financial assets and what we have dubbed the ‘location asset’. To make the analogy with a standard asset more precise, we can propose two interpretations. First, one in which each location \(z\) constitutes an asset, and agents moving to location \(z\) buy the asset, and the ones moving out sell it. How much of it they buy is limited by their housing demand and labor supply. Here, for simplicity, we have limited labor supply and housing demand to be equal to one. The return of the asset depends on the skill of the individual, \(s\), and is given by the right-hand-side of equation (2.3), namely, \( s/q'(\tilde{z}^*) \).

\(^{17}\)Whenever it is clear by the context we abbreviate optimal choices and do not write the dependence on the agent’s type. Namely, we might write \( \tilde{z}^* \) instead of \( \tilde{z}^*(y_0, y_1, s) \).
An alternative interpretation is to consider only a single asset with unit cost. The quantity purchased of the asset is equal to the housing costs, \( q \), and returns of the asset depend both on the quantity purchased and the skill of the individual. Again, those returns are given by \( s/q'(z^*) \). Under both these interpretations, the individual’s problem (2.7) can be seen as a standard portfolio choice problem in which the risk-free bond is subject to a borrowing constraint, and the return to the ‘location asset’ is endogenously nonlinear and specific the individual’s skill.

We are ready to define a competitive equilibrium in our economy.

**Definition 1.** Given a distribution \( F \) of triplets \((y_0, y_1, s) \in [y_0, y_0] \times [y_1, y_1] \times [s, s]\) and an interest rate \( R \), an equilibrium is a set of individual decision functions \( c^*_0, c^*_1, a^* : [y_0, y_0] \times [y_1, y_1] \times [s, s] \rightarrow \mathbb{R}_+ \) and \( z^* : [y_0, y_0] \times [y_1, y_1] \times [s, s] \rightarrow [z, \bar{z}] \), and rent and population functions \( q, L : [z, \bar{z}] \rightarrow \mathbb{R}_+ \) such that

- individuals solve the problem in (2.7) and

- land rents are such that \( q(z) = Q(L(z)) \) for \( z \in [z, \bar{z}] \) where city population \( L(z) \) satisfies

\[
\int_{z}^{\bar{z}} L(z) H(dz) = \int_{y_0}^{y_0} \int_{y_1}^{y_1} \int_{s}^{s} \mathbf{1}[z^*(y_0, y_1, s) \leq z] F(dy_0, dy_1, ds) \quad \text{for all } z \in [z, \bar{z}]
\]

(2.4)

and \( \mathbf{1} \) denotes the indicator function.

Condition (2.4) guarantees that the number of people in locations worse that \( z \) (the left-hand-side of the condition) is equal to the number of people that choose to live in those locations (the right-hand-side of the condition). Note that Condition (2.4) has to hold for all \( z \in [z, \bar{z}] \) and so it implicitly determines the population density function \( L(z) \).
2.2.2 Equilibrium Allocation and House Rents

In order to understand agent’s location choices, consider a city $z$ in which an unconstrained individual $(y_0, y_1, s)$ lives. Because $a^*(y_0, y_1, s) > a$, equation (2.2) holds with equality and so the returns she faces on the financial and the location asset need to be equal. That is,

$$R = \frac{s}{q'(z^*(y_0, y_1, s))}.$$  

This implies that unconstrained individuals sort into cities on the basis of their skill component $s$ only. Then, if $q(\cdot)$ is a strictly increasing function (something we show below), there exists a matching function $Z^U(s) = z^*(y_0, y_1, s)$ for unconstrained individuals, such that

$$R = \frac{s}{q'(Z^U(s))}.$$  

Furthermore, when $q(\cdot)$ is convex (which we also show below), $Z^U(s)$ is strictly increasing. Of course, whether individuals are constrained on the financial asset depends on their income path and skill, and the resulting location choice. For example, a flat income path with $y_0$ high relative to the values of future income, $y_1$, and skill, $s$, implies that the individual is not constrained.

Now consider an individual with the same $y_1$ and $s$ but low enough $y'_0 < y_0$ such that she is constrained. This individual has a larger marginal rate of substitution than the interest rate, so the Financial Euler equation (2.2) holds with strict inequality. Since the agent can still use the location asset, and so (2.3) holds, this implies that $s/q'(Z^U(s)) = R < s/q'(Z^C(y'_0, y_1, s))$ where $Z^C(y'_0, y_1, s)$ is the constrained agent’s location choice. Note that the constrained agent’s location choice depends on all the individual characteristics, not just $s$. Hence, for $q(\cdot)$ strictly increasing, $Z^U(s) > Z^C(y'_0, y_1, s)$. Constrained individuals locate in cities with lower land rents and lower returns to skill than unconstrained individuals with the same skills. The reason is that they use the location asset rather than the financial asset.
to adjust their intertemporal consumption path. More specifically, they borrow using the location asset to transfer resources to the present, something financial markets do not allow them to do.

\( Z^C(y_0, y_1, s) \) is increasing in \( y_0 \) and in fact will converge to \( Z^U(s) \) as we increase \( y_0 \). In contrast, it is decreasing in \( y_1 \), since larger future income results in larger need to borrow from the future and therefore more use of the location asset to do so. Finally, more skilled individuals locate in better cities, whether constrained or unconstrained, due to the skill complementary we introduce in individual earnings. Note that the reason the individual location choice is always uniquely determined is our setup is the supermodular income in \( z \) and \( s \). In contrast, if agents had identical skills, they would be indifferent about where to locate when unconstrained, but their use of the location asset to transfer consumption to the present would still determine their location choice when constrained. We formalize this discussion in the following lemma that characterizes the location decision of agents.

**Lemma 3.** There exists a pair of matching functions \( Z^U(s) \) and \( Z^C(y_0, y_1, s) \) such that individual \((y_0, y_1, s)\) chooses city

- \( z^*(y_0, y_1, s) = Z^U(s) \) if \( y_0 \geq Y_0(y_1, s) \), so she is unconstrained, and
- \( z^*(y_0, y_1, s) = Z^C(y_0, y_1, s) < Z^U(s) \) if \( y_0 < Y_0(y_1, s) \), so she is constrained,

where

\[
Y_0(y_1, s) = \{ y_0 | a^*(y_0, y_1, s) > \bar{a} \}
\]

\[
= (1 + \beta^{-1})\bar{a} + q(Z^U(s)) + \frac{y_1 + sZ^U(s)}{\beta R}
\]

and \( Z^U \) and \( Z^C \) are determined by a system of ordinary differential equations described in Appendix B.1.1.

**Proof.** See Appendix B.1.1. □
Lemma 3 characterizes the threshold for current income $y_0$ that determines whether an individual is constrained using the function $Y_0(y_1, s)$. Because the rent function is increasing in $z$ as we show below, and since $Z^U(s)$ is increasing in $s$, this threshold is increasing in both arguments. More future income makes unconstrained individuals want to consume more in the present and therefore makes the constraint on borrowing more binding. Similarly, more skilled individuals will earn more in the future and will live in more expensive cities, making the constraint more binding.

Of course, given the monotonicity of $Z^U(s)$ and $Z^C(y_0, y_1, s)$ in $s$, we can define the inverse as $S^U(z) = Z^U^{-1}(z)$ and $S^C(y_0, y_1, z) = Z^C^{-1}(y_0, y_1, z)$. These functions then tell us the skill of the set of constrained and unconstrained individuals that live in a given city $z$. In equilibrium, unconstrained individuals always locate in better cities than constrained ones, hence there exists a threshold $\hat{z}$ such that for $z < \hat{z}$ all individuals in the city are constrained and above that we have a mixed of constrained and unconstrained individuals. The best city, $\bar{z}$, is an exception and has no constrained agents. The following corollary states these results formally.

**Corollary 1.** There exists a threshold $\hat{z}$ such that individuals in city $z \geq \hat{z}$ are either

1. unconstrained with skill $s = S^U(z)$ and $y_0 \geq Y_0(y_1, S^U(z))$, or
2. constrained with $s = S^C(y_0, y_1, z) > S^U(z)$, and

$$
Y_0(y_1, S^U(z)) \geq y_0 \\
S^C(y_0, y_1, z) = \frac{S^U(z)(y_1 + Ra)}{\beta R (y_0 - a - q(z)) - zS^U(z)}
$$

In cities $z < \hat{z}$, all individuals are constrained, and $S^C(y_0, y_1, z) = \frac{q'(z)(y_1 + Ra)}{\beta (y_0 - a - q(z)) - zq'(z)}$.

**Proof.** Direct corollary of Lemma 3.

Figure 2.1 represents these results graphically. We have discussed all the elements in the figure except for $\tilde{z}$ that represents the lowest city that has non-negative housing rents.
Namely, $\bar{z}$ is implicitly defined by $q(\bar{z}) = 0$. If $q(z)$ is strictly increasing in $z$, any city with $z < \bar{z}$ is not feasible. Note that the upper bound of the correspondence of skills that live in the city is given by $S^C(y_0, y_1, z)$ evaluated at the lowest current income (denoted by $y_0$) and highest future income ($\hat{y}_1$). Namely, the most constrained individual in the city, which is the highest skilled individual using the location asset the most. Note that below $\hat{z}$ the city has only constrained individuals, and only the lowest skilled individuals locate in the worst city $\bar{z}$ (as long as $\bar{z}$ is low enough).

Figure 2.1: Allocation of skills to cities

We can also represent graphically the set of current income levels, $y_0$, of individuals that locate in a given city. Of course, current income and initial wealth are indistinguishable in our two-period setup. We do so in Figure 2.2. In city $z$, all individuals with incomes $y_0 \geq Y_0(y_1, S^U(z))$ are unconstrained and locate according to their skill level only. Other individuals that locate in those cities are constrained and have low income, and either high
skills, high future income or both. Because lower current income leads individuals to choose worse cities, it must be that the lowest income present in a given city $z$ is the income of the individual with the highest incentives to save in the location asset. Namely, the highest skill agent with the lowest future income present in the city. This lower bound, denoted by $y_0(z)$ in the figure, can be found by evaluating the expression for $S^C$ in equation 2.6 in Corollary 1 at $\bar{s}$ and the lowest future income $\underline{y}_1$.

Figure 2.2: Allocation of income groups to cities

We finish the discussion of an equilibrium in our simple two-period economy with a characterization of the house rent schedule. As we alluded already above, land rents are increasing in $z$ since higher $z$ cities yield higher income for individuals of all skills. Furthermore, the complementarity between $z$ and $s$ implies that the highest skilled unconstrained individuals locate there, which implies that rents grow more than proportionally with city type, as does the income of its unconstrained residents. Hence, rents are convex. Figure 2.3 illustrates such a rent function for two values of the interest rate, $R$.

\[\text{Figure 2.3: Rent schedule for different interest rates, } R.\]

\[\text{18The thresholds } \hat{z} \text{ and } \hat{z} \text{ in the figure refer to the high } R \text{ case.}\]
In cities with unconstrained individuals the slope of the rent function is given by the \( S^U(z)/R \). Namely, the slope of the rent function is determined by the skill of unconstrained individuals in the city and is inversely proportional to the interest rate. Thus, a low interest rate implies that the house rent schedule is steeper. Since the return of the location asset for an unconstrained individual with skill \( s \) is \( s/q'(Z^U(s)) \), this naturally also implies a lower return of the competing location asset by no-arbitrage. That is, lower returns in the financial market result in steeper rents that reduce the return of the location asset (see Figure 2.3). Furthermore, a lower interest rate \( R \) implies that more agents wish to borrow and hence are constrained. This implies more downgrading and segregation. So the model predicts that periods of low interest rates should be periods of increasing rent differentials across cities and more segregation, reminiscent of the pre and post-2008 crisis housing markets around the world. We formalize these results in the following lemma.

**Lemma 4.** The equilibrium house rent function has the following properties:

- \( q(z) \) is increasing and convex,
• for $z \geq \hat{z}$, $q'(z) = \frac{S'(z)}{R}$, and

• $\frac{\partial q'(z)}{\partial R} < 0$ for $z \geq \hat{z}$ if $\bar{s} - \tilde{s}$ is sufficiently small.

Proof. See Appendix B.1.2.

Note that our results on the use of the location asset do not require either skill heterogeneity or the assumed supermodularity between $s$ and $z$. Consider the case in which there is only one skill, say $s_0$, and so everyone obtains the same benefit from living in a given city. In this case, there is complete spatial segregation between constrained and unconstrained individuals. Namely, there is a threshold location $\hat{z}$, so that all unconstrained individuals are indifferent between locations $z \geq \hat{z}$, and constrained individuals locate in places $z < \hat{z}$, where the return to the location asset is higher than the interest rate.

### 2.2.3 Optimal Allocation

The equilibrium allocation of the model described above is inefficient due to the presence of borrowing constraints. The inefficiency is reflected in the use of the location asset by constrained individuals. Their use of the location asset ameliorates the effect of the financial constraint. However, because it reduces total output in the second period by driving agents to locations where they earn less, the resulting allocation is still inefficient relative to an economy without financial constraints.

Finding an efficient allocation can be broken down in two parts. First, the problem of allocating individuals across locations to maximize discounted second period output net of housing costs, and second the allocation of consumption in both periods across individuals of different types. We focus on the solution of the first part of the planner’s problem. The second part is a redistribution problem that depends on the chosen social welfare function and for any standard welfare function the solution is increasing in the total output generated by the allocation of agents across locations.
Given the assumed supermodularity between $s$ and $z$, the planner allocation necessarily involves a one-to-one increasing matching function. Namely, the solution exhibits positive assortative matching. Hence, in contrast to the equilibrium allocation, only one type of agent locates in a given city. We show this rigorously in the following lemma.

**Lemma 5.** Consider the problem of a planner in a small open economy that does not face credit constraints and has access to an asset with exogenous return $R$. Then:

- the planner allocates individuals according to an increasing matching function $Z^{SP}(s)$, and
- the decentralized allocation yield strictly less (i) present value of output, and (ii) present value of output net of housing costs

**Proof.** See Appendix B.1.3.

**2.2.4 Placed-Based Policies**

The equilibrium described above determines the distribution of population across cities, $L(z)$, for all $z \in [\bar{z}, \bar{z}]$ with $L(z) > 0$ for $z \in [\tilde{z}, \bar{z}]$. In the equilibrium allocation, agents with low values of $s$ that are constrained decide to locate in the lower range of cities because they use the location asset to borrow. We now want to consider the effect that place based-policies might have on welfare for the different types of agents. Place-based policies aim to improve the characteristics of some of the worse locations in the economy. This is naturally costly, and implies taxing other locations. Therefore, as a stylized representation of these policies, consider policies that shrink the range of characteristics of equilibrium cities $[\tilde{z}, \bar{z}]$ to a singleton $\{z_0\}$, keeping the mass of cities constant. We choose $z_0$ to guarantee that the average income that individuals derive from cities stays unchanged, namely, $E[sz] = z_0E[s]$.

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19The expectation on the left-hand-side is taken with respect to the equilibrium allocation in space in the competitive equilibrium with different cities, before the policy change. The expectation on the right-hand-side involves only the exogenous marginal distribution of skill.
this policy captures the essential elements of place-based policies if they are implemented without generating any aggregate loss of resources. Note also that positive sorting between skills and city types implies that

\[ z_0 = E \left[ z \times \frac{s}{E[s]} \right] > E[z]. \]

That is, the targeted city type is better than the average.

To explain our general result below it is useful to start with an example where \( s = 0 \). Namely, the lowest skilled individuals in the economy have zero skill and, therefore, derive no benefits from living in better cities. These individuals in equilibrium locate in the worst cities in the economy, \( \tilde{z} \), and pay zero rent \( q(\tilde{z}) = 0 \). Naturally, such individuals will be worse off if we implement our place-based policy. In the equilibrium with place-based policies rents are positive and identical in all cities, but for the lowest skilled individuals the benefits of locating in the improved cities are still zero. Hence, anyone with \( s = 0 \) necessary losses from the policy. By continuity, there is a range of individuals with \( s > 0 \) that are also worse off after the policy. If they have some skills, they benefit in terms of future income, but the increase in rents still dominates. Or, in other words, the policy prevents them from borrowing with the location asset. Something they would like to do.

As long as \( s = 0 \), the logic above applies for any policy that reduces the range of cities at the bottom of the distribution. Namely, any policy that improves the worst city that agents have access to (and therefore increases its equilibrium rents). Of course, this logic also relies on keeping the mass of cities constant. This is intuitive, place-based policies that improve the worse cities in the equilibrium allocation but that allow for the creation of new low-\( z \) cities would achieve little.

The logic described above for the case of \( s = 0 \) can be extended to a more general setting with \( s > 0 \), when \( Q(L(z)) = L^\eta \) with \( \eta < 1 \).\(^\text{20}\) In this case we can characterize the set of

---

\(^{20}\)This is a natural assumption that holds, for example, in the standard circular monocentric city model with a central business district and commuting (as in Desmet and Rossi-Hansberg, 2013). In that case, \( \eta = 1/2 \).
individuals that lose using the matching functions. The individuals that are guaranteed to lose are the ones between the lowest skill, and the skill of the individuals that locate, in the original equilibrium, in the average city. The reason is, again, that up to that point the convexity of housing prices implies that the increase in rents associated with the policy does not compensate the future gain in income for these agents. That is, these agents get low returns for the location asset, so they like to use it to borrow, not to save. This is particularly true for constrained individuals, so the set of skills of constrained individuals that lose is larger than the set of unconstrained individuals that lose from the policy. The next lemma presents the formal result.

**Lemma 6.** Suppose house rents are concave in population, i.e. \( Q(L(z)) = L^n \) with \( n < 1 \). Then a place-based policy that makes all cities have characteristic \( z_0 \) makes

- all unconstrained agents with \( s \in [\underline{s}, S^U(E[z])] \) worse off and
- all constrained agents \((y_0, y_1, s)\) with \( s \in [\underline{s}, S^C(y_0, y_1, E[z])] \) worse off.

Since \( S^C(y_0, y_1, E[z]) > S^U(E[z]) \), the set of skills of constrained individuals that are worse off is larger.

**Proof.** See Appendix B.1.4.

### 2.2.5 The Location Effect of Front and Back-Loaded Shocks

The results above can also be used to describe how agents react to shocks of different types. We are particularly interested in income shocks that affect the relative slope of an individual’s income path. Namely, shocks that affect income today, \( y_0 \), relative to income tomorrow, \( y_1 + s z \). These shocks will induce agents to adjust their savings using the financial and location assets. In Section 2.4 we study how workers in France reallocate across regions as a result of such an income shock. These shocks are front-loaded since, by design, they reduce income today but not necessarily income tomorrow. Hence, we can contrast the model’s predictions
with our observations for France. Other front-loaded shocks include unemployment shocks and declines in the compensation of particular occupations or particular industries. In the latter case, the shock is front-loaded because individuals and their descendants can adjust their future occupation and industry, but are stuck in the short run.

Consider an individual \((y_0, y_1, s)\) that experiences an idiosyncratic negative front-loaded shock that decreases \(y_0\) to \(y'_0 < y_0\) but increases \(y_1\) to \(y'_1 \geq y_1\). Because the shock is idiosyncratic, it does not affect the equilibrium matching function or rent schedule. The results in Lemma 3 imply that agents that are constrained will use the location asset more and will downgrade their location, since \(Z^C(y'_0, y'_1, s) < Z^C(y_0, y_1, s)\). Unconstrained individuals that become constrained due to the shock also downgrade their location, since \(Z^C(y'_0, y'_1, s) < Z^U(s)\). In contrast, unconstrained individuals that remain unconstrained (individuals such that \(y_0 > y'_0 > Y_0(y'_1, s) \geq Y_0(y_1, s)\)) stay where they are, since \(Z^U(s)\) is independent of the income path. Hence, constrained individuals, or those that become constrained, borrow more using the location asset, while unconstrained individuals use the financial asset to transfer consumption to the present. Of course, since what matters for the argument is the slope of the income path, a positive back-loaded shock has a similar effect on location choices and the use of the location asset.

A positive front-loaded shock or a negative back-loaded shock have exactly the reverse effect. Constrained individuals, or individuals that become unconstrained, save with the location asset and upgrade location. Individuals that were, and remain, unconstrained use the financial market to save and do not change their use of the location asset.

Note that permanent adverse (or positive) shocks can also imply a change in the slope of the income profile. For example a permanent shock that changes both \(y_0\) and \(y_1\) induces borrowing if \(y'_0 - y'_1/\beta R < y_0 - y_1/\beta R\). Such a shock then generates the same qualitative effects on the use of the location asset as a front-loaded negative shock. In contrast, if \(y'_0 - y'_1/\beta R > y_0 - y_1/\beta R\), the shock induces extra savings and so has a similar qualitative...
effect than a back-loaded negative shock. Of course, if \( y'_0 - y'_1 / \beta R = y_0 - y_1 / \beta R \), location decisions remain unchanged.

As a last possibility consider an individual that acquires more skill, namely, an increase in \( s \). Because \( s \) increases income in the future, some of the implications of the increase in \( s \) are similar to those of a back-loaded positive shock. On top of this, an increase in \( s \) increases the return of the location asset relative to the financial asset which implies that agents want to save more using the location asset. Hence, they want to upgrade their city. Lemma 3 tells us that the the second effect always dominates, given that both \( Z^C (y_0, y_1, s) \) and \( Z^U (s) \) are increasing in \( s \).

In the context of our simple model and the results described above, we can think of an income shock as changing current income from \( y_0 \) to \( y'_0 \), either by becoming unemployed or transitioning to a lower-paying job. In the long run, the worker’s income prospects remain unchanged, and so next period she again earns \( y_1 \). If the worker receives unemployment benefits that are, say, a fraction \( \kappa < 1 \) of her last salary, then \( y'_0 = y_0 \kappa \) and the shock is a front-loaded negative shock that makes individuals downgrade if constrained and not relocate if unconstrained. We contrast this exact implications with French data in Section 2.4.

### 2.2.6 Amenities

The model we have proposed so far emphasizes the investment dimension of location choices. Agents choose a better location to obtain the future benefits \( zs \). Of course, part of the location choice might also be related to the quality of living, or amenities, particular locations offer. We now investigate how our predictions change when we introduce amenities in the model. Suppose locations offer residential amenities that rise with \( z \), and that are valued by individuals. Then, to the extent that amenities are a normal good, their consumption increases in income; namely, amenities incentivize individuals with higher income to live in
locations with higher $z$. As a result, when amenities are relevant, a negative income shock induces all types of individuals to downgrade their location.

We claim, however, that the presence of amenities implies that unconstrained, wealthy individuals should downgrade more than constrained, low-wealth individuals after a comparable income shock, conditional on both individuals being in the same location to begin with. The reason is straightforward: a low-wealth constrained individual locates in the same place as a high-wealth unconstrained individual only if the constrained individual has a higher return to location $s$. As a result, the low-wealth, constrained, individual is less location-elastic to income shocks than the wealthy individual. Thus, the presence of amenities yields the opposite relative location implications from an income shock than the ‘Location as an Asset’ view (explained in Section 2.2.5).

To make this intuition precise, we extend our basic model to incorporate amenities. Suppose now that individuals solve

$$V(y_0, y_1, s) = \max_{c_0, c_1, a, z} \log c_0 + \beta \log c_1 + Az$$

s.t. $c_0 + a + q(z) = y_0,$

$c_1 = zs + y_1 + Ra,$

$a \geq a.$

This specification enriches our baseline model with an extra utility term $Az$, which implies that locations with better income prospects $z$ also provide more residential amenities. The parameter $A$ governs the relative value of amenities to non-durable consumption.

The only, but critical, change to the individual’s optimality conditions is that now their ‘Mobility Euler equation’ is given by

$$\frac{c_1^*(y_0, y_1, s)}{\beta c_0^*(y_0, y_1, s)} \left[ 1 - \frac{A/q'(z^*(y_0, y_1, s))}{1/c_0^*(y_0, y_1, s)} \right] = \frac{s}{q'(z^*(y_0, y_1, s))}. \quad (2.7)$$

21Our specification assumes a perfect correlation between income prospects and amenities to make the amenity channel as stark as possible.
If agents enjoy the amenities conveyed by the city \( (A > 0) \), going to a city with a larger \( z \), gives an extra utility benefit of \( A/q'(z^*(y_0, y_1, s)) \) per unit of extra rent. Dividing by the marginal utility of current consumption \( (1/c_0^*(y_0, y_1, s)) \) determines the price discount in terms of current rent implied by amenities. For every marginal dollar paid in rent, agents effectively face only a fraction of the cost, since they obtain the additional benefits from amenities. Hence, the term in brackets, which is equal to 1 when \( A = 0 \) and is smaller than 1 when \( A > 0 \), multiplied by the marginal rate of substitution between non-durable consumption today and tomorrow must now equal the future monetary return to the location asset. Conditional on a rent function \( q(\cdot) \), amenities result in a lower effective price for the location asset, and since future benefits are not affected, a larger rate of return.\(^{22}\)

To formalize our theory’s predictions when locations also provide residential amenities, we consider two individuals \( P \) and \( R \) with initial income \( y_0^P < y_0^R \), and \( y_1^P = \tau y_0^P \), \( y_1^R = \tau y_0^R \). Individual \( P \) has lower initial income and/or wealth than individual \( R \), but both individuals expect the same income growth over time. We are interested in the change in the location decision of individual \( j \in \{P, R\} \) after a proportional income shock \( y_{0j}^{\nu j} = \nu y_0^j \), where \( \nu \leq 1 \), conditional on both individuals choosing to locate in \( z \) in the absence of the shock. Formally, we consider the derivative with respect to \( \nu \), evaluated at \( \nu = 1 \),

\[
D^{\nu j}(z, A) = y_0 \frac{\partial z^*(y_{0j}^{\nu j}, y_{1j}^{\nu j}, S(y_{0j}^{\nu j}, y_{1j}^{\nu j}, z))}{\partial y_0},
\]

where \( S \) is equal to either \( S^C \) or \( S^U \) depending on the constrained status of individuals.\(^{23}\) The following lemma states our results.

**Lemma 7.** If \( A > 0 \), and given a rent function \( q(z) \),

\(^{22}\)Of course, in equilibrium, the rent function \( q(\cdot) \) does adjust since rents are increasing in population density.

\(^{23}\)We omit the dependence of individuals’ optimal choices on \( A \) for notational brevity.
• in the absence of credit constraints \( \underline{a} = -\infty \),

\[
D^R(z, A) - D^P(z, A) > D^R(z, 0) - D^P(z, 0) = 0;
\]

• in the presence of credit constraints that bind for \( P \) but not for \( R \), when \( A \) is not too large, \( D^R(z, A) - D^P(z, A) < 0 \) is increasing in \( A \).

**Proof.** See Appendix B.2.1.

Lemma 7 characterizes the location response of individuals ranked by their initial income and/or wealth. In order to isolate the contribution of amenities to location choices *ceteris paribus*, we fix the housing rent function \( q(z) \). The first result states that, when individuals are not constrained, the presence of amenities makes wealthy individuals more location-elastic to income shocks than low-wealth individuals. Wealthy individuals are found to be relatively more elastic than their poorer counterparts when \( A > 0 \), because their \( s \) must be smaller in order for them to live in the same location. Therefore, the basic amenities channel works against the ‘Location as an Asset’ channel. The second result reveals that this conclusion carries through to a mixed model with credit constraints and amenities. As an individuals’ amenity valuation increases from 0, the difference between a constrained individuals’ location response and unconstrained individuals’ location response diminishes.\(^{24}\)

Lemma 7 is particularly useful to interpret the empirical results in Section 2.4. There, we use controls for local amenities when we estimate the differential effect of income shocks on individuals’ location. Of course, one can always argue that individuals value amenities in ways that we are not able to fully control for. Reassuringly, however, Lemma 7 implies that any estimated differential effect of income shocks on individuals’ location will have to be a *lower bound* on the true effect of the ‘Location as an Asset’ mechanism. In order to illustrate the exact predictions of the ‘Location as an Asset’ view that we will bring to the

\(^{24}\)In Appendix B.2, we show that these conclusions continue to hold for non-proportional income shocks.
data, we now present a fully dynamic quantitative model with asset accumulation, location decisions, and credit constraints.

2.3 The Infinite Horizon Model

In this section we extend our model to an infinite horizon economy. The key differences with the model presented in the previous section is that now agents live forever and receive an idiosyncratic income stream $y_t$. Depending on their skill, location, asset holdings, and income, they make consumption and savings decisions. To do so they use the financial market subject to a borrowing constraint and the location asset by choosing where to live. As before, cities differ in their return to skill and their rent. Also as before, one can view individuals as solving a two-asset portfolio choice problem subject to a borrowing constraint on the risk-free bond. In contrast to the two period model, the infinite horizon version determines the invariant distribution of wealth in the population and therefore the wealth composition of cities as well.

2.3.1 Model Setup

In any period $t$, infinitely-lived individuals receive an idiosyncratic income shock $y_t$, which follows a first-order Markov chain with states $y_1, ..., y_N$ and a given transition matrix, $\Lambda$. Throughout we assume that individuals have a permanent skill $s$.\footnote{\textsuperscript{25}It is feasible to relax this restriction and introduce idiosyncratic skill shocks, although at some computational cost.} In period $t$, an individual in location $z_t$ with an asset level $a_t$, chooses how much to consume, $c_t$, how much to save, $a_{t+1}$, in a one period risk-free bond with interest rate $R$, and where to live next period, $z_{t+1}$. Agents can move freely across locations. Their income in period $t$ is $y_t + sz_t$. To go to location $z_{t+1}$, they need to pay the rent $q(z_{t+1})$ one period in advance, i.e. in period $t$. Finally, we assume that the risk-free bond is subject to a standard credit constraint $a_{t+1} \geq a$.\footnote{\textsuperscript{25}It is feasible to relax this restriction and introduce idiosyncratic skill shocks, although at some computational cost.}
Given an increasing and concave flow utility function $u$ satisfying Inada conditions, and a discount factor $\beta < 1$, individuals maximize

$$V(a_t, z_t, y_t, s) = \max_{\{a_{t+1}, z_{t+1}\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

(2.8)

s.t.  \[ c_t + a_{t+1} + q(z_{t+1}) = y_t + sz_t + Ra_t, \]

\[ a_{t+1} \geq \underline{a}. \]

If we denote optimal choices with an asterisk, as in the two period model, the solution to this dynamic optimization problem yields a financial Euler equation

$$\frac{u'(c^*_t(a_t, z_t, y_t, s))}{\beta E_t[u'(c^*_{t+1}(a_{t+1}, z_{t+1}, y_{t+1}, s))]} \geq R$$

that holds with equality if and only if $a^*_{t+1}(a_t, z_t, y_t, s) > \underline{a}$. Also similarly, free mobility implies a mobility Euler equation given by

$$\frac{u'(c^*_t(a_t, z_t, y_t, s))}{\beta E_t[u'(c^*_{t+1}(a_{t+1}, z_{t+1}, y_{t+1}, s))]} = \frac{s}{q'(z^*_t+1(a_t, z_t, y_t, s))}$$

which implies that

$$\frac{s}{q'(z^*_t+1(a_t, z_t, y_t, s))} \geq R,$$

(2.9)

with equality if and only if $a^*_{t+1}(a_t, z_t, y_t, s) > \underline{a}$. Note that, as before, for non-constrained individuals city choice $z^*_t+1(a_t, z_t, y_t, s)$ only depends on skill $s$.

Denote by $F_t$ the joint distribution of the four-tuple $(a_t, z_t, y_t, s)$ in period $t$. Then the distribution of people across cities, $L_t(z)$ is given by

$$\int_{\bar{z}}^{\tilde{z}} L_t(z) H(dz) = \sum_{i=1}^{N} \int_{\bar{z}}^{\bar{z}} \int_{\bar{z}}^{\tilde{z}} 1_{[z^*(a, z, y, s) \leq z]} F_t(da,dz,y_t,ds) \text{ for all } z \in [\underline{z}, \bar{z}]$$
and rents are given by \( q(z) = Q(L(z)) \). This economy converges to a steady state where the distribution \( F_t \) is constant over time.

An equilibrium of the model above can be computed numerically. We do so for a CRRA utility function, for a uniform distribution of cities, and for a particular house rent schedule.\(^{26}\) We choose reasonable parameters values that allow us to illustrate the main forces at work. The exact values, specifications, and solution method are described in Appendix B.3.

### 2.3.2 A Quantitative Illustration of the Use of the Location Asset

Figure 2.4 presents the results of a simulation of this model. We focus on the reaction of a particular individual to a transitory income shock. The figure presents five panels, each of them displaying a different variable. For comparison purposes we present the behavior of an individual that can move (solid dark lines), and therefore use the location asset, and the behavior of an individual that cannot move from her preferred location when unconstrained, \( Z_U(s_0) \) (dashed light lines). The difference between these two cases represents the way in which the location asset helps the individual deal with the transitory income shock. We plot the effects for a particular individual with a fixed skill level.

The first panel in Figure 2.4 simply plots the income shock over time. The agent can be in two income states: high, \( y_H = 0.2 \), and low, \( y_L = 0.05 \).\(^{27}\) In period one, the agent transitions from the high to the low income level. It stays there until the ninth period when he transitions back to the high income. This income process is identical for both scenarios, with and without mobility.

The second panel plots the level of financial assets. We start the individual at assets that are equal to the transitory income level in the high state. The individual also receives an

\(^{26}\)In principle, specifying a given house rent schedule is without loss of generality, because we can find a skill distribution that would lead to this particular house rent schedule as an equilibrium outcome. Of course, endogenizing the house rent schedule is essential to perform aggregate counterfactual simulations. In the exercises below, we only consider counterfactuals that change the state of a particular individual and therefore do not affect the aggregate equilibrium allocation and prices.

\(^{27}\)We could think of the low state as unemployment, and the high state as non-employment. Our calibration of the transition matrix \( \Lambda \) then implies a steady-state non-employment rate of about 14\%, in line with employment rates for prime-age males in France.
income proportional to her skill and the city where she lived, $z_t s_0$. This additional income represents most of the individual’s income. The transitory path represents between 5 and
15% of the agent’s total income. As a result of the shock, the agent consumes part of her financial assets and therefore the asset balance declines until it hits zero, which is the level of the financial constraint. That is, individuals cannot borrow at all in financial markets. This decline in financial assets happens a bit faster when individuals can use the location asset, since in that case they know that when they hit the financial constraint they will be able to smooth consumption by moving. In period 3, the agent that cannot move hits the borrowing constraint and stays there for several periods. The agent that can use the location asset hits the borrowing constraint one period earlier. When the income shock reverses in period 10, without the location asset, the agent immediately starts saving and building a financial asset stock. In contrast, because at that point the location asset pays a higher return than the financial asset, the individual that can use the location asset, uses it to save. Such an individual stays stuck at the constraint for an extra two periods while it moves to better locations. Eventually, she reaches her desired location, the return she perceives on the location asset goes down, and she starts saving with the financial asset. Note that the presence of the location asset makes the individual stay longer at the financial constraint!

The third panel plots the location of the agent over time. The ideal unconstrained location of the agent is at city $Z^U(s_0) = 0.88$. The agent that cannot move simply stays there throughout. The one that can move stays there until financial assets hit the financial constraint. Once she runs down financial assets to zero, she starts borrowing using the location asset. That is, she starts downgrading her location progressively. In this case the total downgrade is about 40%. This downgrading continues until the agent either reaches the minimum location she is willing to live in, or the shock reverses. In the plot, it continues until the 9th period, the last period the agent obtains the low income. After the shock reverses to the high income state, the agent starts upgrading her location progressively. The last period where she is financially constrained, she reaches her unconstrained preferred city and starts saving with the financial asset only.$^{28}$ Note that location downgrading happens throughout the transition, the change in the city component of income due to the use of the location asset reaches 0.18, which is of the same order of magnitude than the high idiosyncratic income state.
smoothly over time since there is no fixed cost of migration. In the presence of fixed costs, if income shocks are large enough, agents would still use the location asset, although only sporadically.

The fourth panel in Figure 2.4 shows the share of housing expenditure in total income. The average share lies between 20% and 40%, similar to the data (Davis and Ortalo-Magne 2011). At impact, the share of housing expenditure jumps up due to the fall in income. It starts falling once agents hit the credit constraint and start using the location asset. It keeps falling as the agents borrow more with the location asset and downgrade their location. It starts increasing when the agent begins to save with the location asset and eventually stabilizes at the same level as for the immobile agent.

Finally, the bottom panel in the figure shows the agent consumption path with and without mobility. As we have underscored, the use of the location asset allows the agent to smooth consumption better since it can borrow even when she is at the financial constraint. The result is a consumption path that declines more slowly and smoothly than without the location asset. Because borrowing with the location asset involves sacrificing future income, given that this is an unexpectedly long shock, the total fall in consumption is also eventually somewhat larger. Once the shock reverses, the path out of the consumption slump is also a bit smoother for agents that can use the location asset. Overall, these dynamic behavior patterns vary substantially with and without the location asset.

The ability to use the location asset results in expected welfare gains for the agent.\textsuperscript{29} The presence of some gains is obvious given that the location asset provides a way of relaxing the friction imposed by the financial constraint and the agent can always decide not to move. In Figure 2.5 we present the percentage gain in consumption, and in consumption equivalent welfare from using the location asset. The values are calculated starting from

\textsuperscript{29}The gains from using the location asset for one particular path of realizations can be either negative or positive. For example, in Figure 2.4, the negative shock lasts for nine periods. This increases the set of periods where agents that use the location asset obtain less consumption. However, this particular path is relatively unlikely. Other paths with shorter duration of the negative transitory shock yield larger benefits from the use of the location asset, and are more likely. In expectation, there are gains since the agent has a larger, more flexible choice set.
the ideal city for unconstrained individuals, $Z^U(s)$, and we keep the skill of the individual fixed, as in Figure 2.4. Figure 2.5 then plots the relative consumption and welfare from using the location asset as a function of the starting asset level, as well as the invariant distribution of assets in the right panel. It presents the gains for agents with a current high or low income realization. Clearly, because we are not estimating the parameters of the model for a particular circumstance, the level of the gains provides only a rough indication of what is at stake from using the location asset. In contrast, the qualitative patterns are more interesting. Most consumption gains happen close to the constraint for low-income individuals who are dis-saving. These consumption gains quickly fade away as we consider individuals with higher levels of assets. However, because those consumption gains occur precisely in the high marginal utility states, they translate into welfare gains of 0.5% close to the constraint. The figure also shows that agents in the low income state benefit more than agents in the high income state, as they are the most likely to use the ‘location asset’.

The empirical exercises in Section 2.4 presents evidence on how individuals use the ‘location asset’ using an event study design that follows the location and asset holdings of agents that experience income shocks and start with different levels of wealth at the same location. Figure 2.4 shows that, in our theory when hit by a negative income shock, individuals first dis-save in their financial assets. Once they hit the credit constraint, they start dis-saving in the ‘location asset’ by downgrading their location. Of course, this pattern also manifests itself when comparing individuals who reside in a given location, but have different wealth levels. Our next exercise presents this event study in our modelled economy. The implied qualitative patterns are the implications that will be looking for in the parallel exercises in

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30 Because financially constrained individuals borrow with the location asset, our two-asset model predicts more individuals at the financial constraint than the standard one-asset formulation. This has interesting implications for macroeconomic policy. For instance, tax rebate shocks would be partly saved by financially constrained individuals by upgrading location.

31 Gains in flow consumption can be as high as 10% for low-income individuals close to the constraint that live in their unconstrained preferred location. These gains are larger than the ones depicted in Figure 2.4. This is the case because individuals usually start downgrading location one period before they hit the constraint. Note also that the small kinks in the consumption gains are due to kinks in the consumption policy functions, when individuals hit the constraint next period.
the data. We select individuals who satisfy the following criteria, as we will also do in the data. First, they must be in either the bottom quintile, or the top quintile of the invariant asset distribution in the economy. Second, they must reside in the same given location $z_0$, and currently be in the high income state. We then hit all these individuals with a front-loaded income shock: they remain in the low state for two periods, before reverting to the high state.

Figure 2.6 shows these individuals’ asset and location over time, for different subgroups. In column (a), we select only individuals of the same skill $s = s_0 = 1$, where $s_0 = SU(z_0)$ is the skill of the unconstrained individuals who reside in location $z_0$. Selecting individuals of the same skill $s_0$ in both groups implies that everyone must be unconstrained, and so the low-wealth individuals must hold just enough financial assets to be unconstrained but sufficiently little to be in the bottom quintile. The second row of column (a) shows that, as a response to the negative income shock, wealthy individuals dis-save their financial assets to smooth consumption. By contrast, low-wealth individuals do not dis-save much because

\[ \text{We use } z_0 = Z_U(s_0) \text{ with } s_0 = 1.\]
they hit the credit constraint rapidly. As shown in the last row, these low-wealth, credit-constrained, individuals smooth consumption using the ‘location asset’ and downgrading their location. By contrast, wealthy individuals stay in their unconstrained location $z_0$. After the idiosyncratic component of income reverts to the high state, the initially wealthy individuals start saving again in financial assets. The credit-constrained individuals save in the ‘location asset’ by upgrading their location. Because we select individuals of the same skill $s_0$ among both low-wealth and wealthy individuals in column 4, $z_0$ is the unconstrained location for individuals of both groups. Thus, the low-wealth individuals also revert to $z_0$ in period 4 when they all accumulate assets and become unconstrained.

In column (b) of Figure 2.6, we select only wealth-poor individuals who are exactly constrained $a_0 = g$ when entering the high income state in period 0. Constrained and un-
constrained individuals choose to live in the same location only if the unconstrained wealthy individuals have a lower $s$ and are, therefore, less location-elastic. Hence, we cannot condition on $s$. The second row of column (b) shows that the asset downgrading of initially low-wealth individuals is minimal, since individuals now hold only the assets they managed to accumulate in period zero, when they had high income. Since low-wealth constrained individual have a higher $s$ than wealthy ones, they are already ‘borrowing’ with the location asset. As in column (a), however, they downgrade location even more to weather the low income shock. Once the idiosyncratic component of income reverts to the high state, however, they upgrade their location towards an even better location than where they started (this is evident in the last row of column (b) in period 4). In fact, we know that they will start accumulating assets and will stop upgrading their location only when they reach a better location than their wealthy counterparts, since we know they have a higher $s$.

In column (c) we consider all individuals who satisfy our initial criteria. As a result, the impulse response of assets and location are a weighted average of those in columns (a) and (b). Overall, all three columns reveal similar patterns. In sum, Figure 2.6 illustrates the main implications we will look for in the empirical exercise. First, conditional on residing in the the same location, and following a front-loaded income shock, we should expect wealth-poor individuals to downgrade their location but keep close to constant and negligible amounts of financial assets. We should also expect wealth-rich individuals to dis-save financial assets while remaining in the same location. Conditioning on the type $s$, and whether poor individuals are at the constraint, or simply close to it, seems less important. We now explore some of the implications of our view of location choices using French data on individual income, asset, and location paths.

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33 Of course, because we have assumed that there is no cost of mobility at all, in our model agents optimize their location every period. Small moving costs would make adjustments to the agents’ location, and therefore borrowing and saving with the location asset, more infrequent (although still beneficial). In addition, because the borrowing constraint generates a concave value function in wealth, small moving costs would reduce the frequency of moves more for low-wealth individuals relative to high-wealth individuals. Together with shocks to skill this would help explain jointly the sorting patterns across space and mobility patterns across income groups.
2.4 Location and Moving Choices in France

We have discussed in detail several implications of our view of location decisions as investing in a location asset. In particular, constrained individuals will downgrade their location as a result of a negative front-loaded income shock while their assets remain minimal and unchanged. In contrast, unconstrained individuals will not react to these shocks by moving, but instead by reducing their wealth. In this section we contrast both these predictions with individual level data.\footnote{In addition to the implications on the location destination and assets of individuals who experience income shocks, our theory has implications on who decides to move at all. Decisions to move, however, are also directly impacted by fixed moving costs and municipality-level shocks that change job opportunities and prices in an agent’s origin municipality. For example, although the ‘Location as an Asset’ view implies that low-wealth individuals should move more as a result of an income shock, fixed moving costs imply that they should move less, since their marginal utility of consumption is higher. These confounding factors, however, do not affect the decision of where to move, and the implications on wealth dynamics for constrained and unconstrained individuals conditional on moving. Hence, we focus on these latter implications in our empirical exercise.}

We use tax return data from a longitudinal panel representative of all households in the French economy from 2008 to 2015. We have both household and individual identifiers. Crucially for our analysis, we observe the households’ annual financial asset income. It is broken down into various categories that include bank accounts, financial vehicles such as mutual funds and stocks, as well as housing (rent payments minus outstanding mortgage payments).\footnote{These measures do not include the flow value of a owner-occupied house with no outstanding mortgage. In principle, it could be backed out from reported property taxes, but such an approach would require either strong assumptions or other data sources, specifically because property taxes vary across locations. Since owner-occupied housing is a highly illiquid asset, we instead simply control for home-ownership in our analysis. We also note that France’s pension system is pay-as-you-go, and therefore pensions are not relevant to our analysis.}

We match this household tax income dataset to employer tax return data using individual identifiers. Using employer tax returns allows us to obtain precise information about individuals wage income at the employment year-spell level. Since the address reported in the household income tax data is often inaccurate, it is critical for us to use the residence and workplace municipality reported by the current employer of individuals. Municipalities in France compare to ZIP codes in the US: there are 36569 municipalities in France, with an
average area of 15 squared kilometers and 435 inhabitants. The employer tax return data allows us to also observe a number of worker characteristics like age, gender, occupation, and birthplace.

Put together, these data sources constitute one of the first large-scale administrative datasets with information on financial assets, high-resolution location, and matched employer-employee labor market characteristics for a large economy like France. Nevertheless, contrasting our data with our theoretical predictions involves several choices. First, since the data does not have direct information on the stock of assets of households, we simply assume that the income flow from financial assets is increasing in the value of assets. We then bin households into five quintiles of our measure of financial income, which under the assumption is equivalent to grouping them by financial assets, and study outcomes across these quintiles. For interpretation purposes, sometimes it is convenient to have a measure of the level of financial assets and not simply the wealth rank of individuals. Hence, we divide the flow income from all financial and housing assets by a common interest rate of 5% to back out an implied stock of assets.\footnote{If we had reliable data on the returns from the various categories of assets, we could use more granular portfolio weights to back out total assets.} Consistent with our theory, our interpretation is that households at the bottom quintiles of the financial income distribution are more likely to be constrained. Figure 2.8 below shows that assets for the bottom quintiles are close to zero and slightly negative for the bottom quintile.

Second, we need to determine which locations are more complementary with skill, or more attractive. To address this challenges we use our theoretical model. In our theory there is positive assortative matching between a worker’s skill and her earnings, which we observe. Furthermore, as implied by the model in Section 2.2, residents of cities with higher $z$ have higher average incomes. Hence, we can determine the $z$–rank of cities using the rank of their average income (see Figure 2.1).
2.4.1 The Impact of Location on Wages

In order for location to resemble an investment decision, it is essential that some of the benefits (or costs) of living in a given location accrue over time. The ideal experiment to test if the returns of moving to a better municipality increase over time would randomly allocate identical workers across different locations, and would compare wages over time of workers who were allocated to good locations relative to those who were allocated to bad locations. In practice, however, finding instruments that achieve such a random spatial allocation is difficult. Therefore, we turn to an event study specification in which we control for as many observable characteristics as possible. We isolate male movers between 25 and 62 years old, and estimate

\[
\log \frac{w_{i,t}}{w_{i,-1}} = \alpha_{G(i,t)} + \beta_t P(z_{i0}) + \varepsilon_{i,t},
\]

pooled over all individuals \(i\) and years \(t\). \(P(z_{i0})\) is the percentile of the municipality where individuals migrated in \(t = 0\). The dependent variable \(\log \frac{w_{i,t}}{w_{i,-1}}\) denotes wage growth between the period just before the move (period \(-1\)) and period \(t\). The difference specification controls for any time-invariant worker characteristic (a worker fixed effect). \(\alpha_{G(i,t)}\) controls non-parametrically for age, year, 2-digit origin occupation, and origin municipality fixed effects interacted with linear time trends and with a post-move dummy. Occupation and municipality are measured before the move at \(t = -1\). Finally, estimating this equation on movers only avoids picking up unobserved heterogeneity between movers and stayers. For this exercise, we use data for an 8% representative panel of French workers between 2002 and 2015. Appendix B.4.1 provides a description of our dataset.

The investment dimension of mobility is captured by the difference \(\beta_t - \beta_{-1}\).\(^{37}\) The identifying assumption that lends a causal interpretation to this estimate is that there are no (a) worker-specific trends that are systematically correlated with the location decision at

\(^{37}\)The initial effect, \(\beta_0 - \beta_{-1}\), could capture, on top of the immediate effect from moving, a short term investment component that is not realized immediately but takes less than 2 years to be reflected in wages.
Figure 2.7: Effect of migration on wages over time.

Note: Plot of the $\beta_t - \beta_{t-1}$ coefficients, for $t = -5...8$, and observed daily real wages. $t = 0$ is the first move of a worker and is the instantaneous effect of location. Standard errors clustered by commuting zone and 2-digit occupation. Vertical bars depict 95% confidence intervals. Depending on the specification, the set of controls includes: fixed effects for the time-0 municipality, interacted with a post-move dummy and with a linear slope; fixed effects for the time-0 2-digit occupation, interacted with a post-move dummy and with a linear slope; and 5-year age bin fixed effects, interacted with a post-move dummy and with a linear slope.

$t = 0$ and subsequent wage growth, and (b) no unobserved shocks that are systematically correlated with mobility decisions and wage growth between 0 and $t$, conditional on our controls. In a robustness exercise, we directly control for (a) by including worker-specific time trends in the estimation. However, if individuals receive an idiosyncratic worker-level shock at $t = -1$ that makes their wage grow systematically faster in better municipalities, in a way that is orthogonal to worker fixed effects, the trend controls, and pre-move wages, then we would not be able to interpret $\beta_t - \beta_0$ in a causal way.
Figure 2.7 shows our baseline estimation results. It displays the point estimates relative to period $-1$. The estimate for $t = 0$ reveals that moving to the best location in France conditional on our controls leads to about 11% higher wages than moving to the worst location. Comparing the estimate at $t = 8$ to the estimate at $t = 0$ shows that the return to migration almost doubles after 8 years: wages are then 21% higher. This increase represents the dynamic gains from location.

Table 2.1: Wage growth before and after move.

<table>
<thead>
<tr>
<th></th>
<th>Basic (1)</th>
<th>Basic (2)</th>
<th>Worker slopes (3)</th>
<th>Worker slopes (4)</th>
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<td>Pre-move level</td>
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<td>-0.011*</td>
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<td>pre slope = post slope</td>
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</table>

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Column (1) and (3) include as controls: fixed effects for the time-0 municipality, interacted with a post-move dummy and with a linear slope. Columns (2) and (4) add fixed effects for the time-0 2-digit occupation, interacted with a post-move dummy and with a linear slope; and 5-year age bin fixed effects, interacted with a post-move dummy and with a linear slope.

In Table 2.1, we show that our results are robust to controlling for individual trends. Specifically, we net out worker-specific fixed effects and linear time trends estimated on the
pre-period $t \leq -1$ only, before running equation (2.10).\footnote{We first project $\log \frac{w_{i,t}}{w_{i,t-1}} = \gamma_i + \delta_i \times t + u_{i,t}$, for $t \leq -1$, and re-run equation (2.10) with $\log \frac{w_{i,t}}{w_{i,t-1}} - \hat{\gamma}_i - \hat{\delta}_i \times t$ as our dependent variable.} Here, we estimate specifications with a level effect and a linear trend after the move in order to increase statistical power.\footnote{This exercise is therefore equivalent to comparing the average post-move slope to the average pre-move slope in Figure 2.7.} Columns (1) and (2) reproduce our baseline estimates, and show that wages grow 1 to 1.4% faster per year in $z = 1$ locations relative to $z = 0$ locations. The point estimate of this slope effect is remarkably stable after netting out worker-specific trends: it remains between 0.8 and 1.1% per year. However, our estimates with worker trends are more noisy because we estimate workers’ pre-move slope on 3 to 5 observations per worker. Therefore, the measurement error in our dependent variable increases substantially. Consistent with mean-zero measurement error in the dependent variable, the point estimate remains similar although the standard errors increase substantially.

Note that we focus on dynamic benefits from location in wages because we can measure this effect with our data. Of course, high-$z$ locations convey other dynamic benefits like better schools and learning from more knowledgeable or able peers and neighbors. Therefore, the dynamic location effect we have documented, probably understates the actual dynamic benefits from living in a high-$z$ location. We conclude that location in fact has a payment structure that resembles an intertemporal asset.

### 2.4.2 Location Decisions after Income Shocks

The previous subsection showed that location can be viewed as an asset. We now turn to exploring if agents actively use this asset. To do so, we study individual changes in residential locations as a result of an income shock.

Figure 2.8 provides some basic statistics for our dataset. The left panel plots the average financial wealth of individuals by wealth quintile. As is common in empirical wealth distributions, it is heavily skewed. Individuals in the bottom quintile have negative wealth,
while individuals in the top quintile own over 200,000 euros on average. The right panel shows the annual migration rate of the different quintiles across municipalities and commuting zones. Perhaps surprisingly, but consistent with our theory, individuals in the bottom wealth quintile move more frequently than their wealthier counterparts.

Figure 2.8: Financial assets and fraction of movers in a year by financial assets quintile.

As described in Section 2.2.5 and illustrated quantitatively in Figure 2.6, the main implication of our model is that, upon receiving a front-loaded income shock, low-wealth constrained individuals should downgrade their location relative to individuals in the same location who are at the top of the wealth distribution and, therefore, financially unconstrained. We construct annual wage income growth for each individual, and call a negative income shock a decline in annual wage income that is no less than 25%. In Figure B.1 in Appendix B.4.2, we show that income initially falls, and then mean-reverts over time following the shock. Thus, these income shocks are indeed front-loaded income shocks.

We use an event-study design. Parallel to the quantitative exercise in Section 2.3.2, we compare the location of individuals who receive the shock in the bottom wealth quintile to individuals who also receive the shock but their assets put them in the top quintile. Our control group then consists of individuals in the same wealth quintile, but who did not receive

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the shock. We start by estimating the following regression:

\[
P(z_{it}) - P(z_{i0}) = \sum_{q=1}^{5} \sum_{n=0}^{1} \sum_{t=-2}^{4} \alpha_{q,n,t} \cdot I_{Q(a_{i0})=q} \cdot I_{N(i)=n} \cdot I_t + \beta \cdot X_{it} + \varepsilon_{it}, \tag{2.11}
\]

where \(i\) indexes individuals in the sample and \(t\) is the current year. \(P(z_{it})\) is the percentile of \(i\)'s location at time \(t\), \(I_{Q(a_{i0})=q}\) is a set of asset quintiles indicators. \(I_{N(i)=n}\) is a set of indicators for the negative income shock at time 0. \(I_t\) is a set of time indicators, \(X_{it}\) is a vector of pre- and post-move worker controls and fixed effects. It includes year, asset quintile, time-0 wage income, destination amenities, commuting distance, origin municipality, occupation, age, and home-ownership fixed effects. Finally, \(\varepsilon_{it}\) is a mean zero error term which we assume has the standard mean independence properties.

We are particularly interested in the difference between the location of low-wealth individuals who receive the shock and the location of high-wealth individuals who also receive the shock: \(\alpha_{1,1,t} - \alpha_{5,1,t}\). Since some of the agents in the bottom quintile are financially constrained, if not all, the theory predicts that agents that are in the lowest quintile of the wealth distribution should downgrade relative to those in the top quintile as a result of the income shock. So our ‘location as an asset’ view implies that \(\alpha_{1,1,t} - \alpha_{5,1,t} < 0\) for \(t \geq 0\).

Figure 2.9 presents the results for \(\alpha_{1,1,t} - \alpha_{5,1,t}\) for all individuals. All specifications include income controls and 36,000 municipality fixed effects to control for the location of individuals. As implied by the ‘Location as an Asset’ theory, the estimated difference is negative and significant. The magnitude of the difference is close to 0.2 percentage point in the first year, and remains similar for the next two years, it drops to zero in the fourth year. The estimated differences in location choices are not very sensitive to adding fixed effects for 2-digit occupation (64), age bin (6), and home-ownership status (2). All standard errors are clustered at the commuting zone by 2-digit occupation level.

One possible concern with our interpretation of the results is that location decisions are the result of a static trade-off rather than a dynamic investment choice. In particular, poor
Figure 2.9: Differential location effect of a negative income shock (Q1 - Q5).

Note: Difference between location of individuals with low financial assets (Q1) and individuals with high financial assets (Q5) $\alpha_{1,t} - \alpha_{5,t}$ following a negative income shock relative to individuals who do not receive the shock. $t = 0$ is the year before the income shock. Standard errors clustered by commuting zone and 2-digit occupation. Vertical bars depict 95% confidence intervals. Depending on the specification, the set of controls includes: fixed effects for the time-0 municipality, log wage income at period 0, fixed effects for the time-0 2-digit occupation, 5-year age bin fixed effects, a home-ownership (HO) fixed effect, our measure of amenities for the current location, and log commuting distance at the current residence and workplace.
individuals could have decided to consume less amenities relative to wealthy residents, and therefore move to relatively lower-zi locations, which likely rank lower in terms of amenity as well. As we argued in Section 2.2.6,40 this reasoning is faulty since we condition on initial location and the shock is at least as large for wealthy individuals (as shown in Figure B.1 in Appendix B.4.2). If only amenities choices determine location decisions, wealthy individuals must necessarily be more elastic than low-wealth individuals, since otherwise we would not observe them in the same location in equilibrium. Hence, wealthy individuals would downgrade more, not less as observed in Figure 2.9. If amenities matter in addition to the use of the ‘location asset’, Figure 2.9 reflects the balance of these two forces, and shows that the net effect is close to the one predicted by our theory.41

Nonetheless, to be excessively cautious, we include a measure of local amenities for the destination municipality as controls in the estimation.42 Finally, to guarantee that our results are also not driven by an increase in worker’s commuting time in response to income shocks, we control for commuting distance after the income shock.43 Thus, we are comparing mobility patterns of individuals holding constant commuting distance. Figure 2.9 reveals that explicitly controlling for amenities and commuting distance barely affects our estimates.

40See also Appendix B.2.3
41One more subtle alternative arises if individuals receive idiosyncratic amenity shocks for locations each period. In a given period, we could observe wealthy individuals with a particularly high realization of the amenity shock in the same location as a wealth-poor individual with an average realization of the amenity shock. Wealthy individuals would then mean-revert to their average preferred location over time. This type of mean-reversion cannot account for our results because it would arise irrespectively of the income shock. Therefore, any baseline mean-reversion across wealth quintile is controlled for by the first-difference of our difference-in-difference design. Consistently, we find no evidence of differential location decisions in the periods prior to the shock.
42To compute amenities at the residence municipality level, we use data from the Base Permanente des Equipements 2007 - the closest year available before our sample with financial income starts - on the number of 136 types of establishments in health services (e.g. hospitals), education services (e.g. pre-schools), public services (e.g. police stations), and commercial services (e.g. perfumeries). We first compute the number of these establishments per capita in each municipality. Then, we extract the first principal components of the corresponding covariance matrix. For each municipality, we obtain the loading on this principal component. We choose the sign of the principal component such that the loadings correlate positively with our measure of zi. Finally, we rank these loadings between 0 and 1.
43To construct a measure of commuting distance, we simply compute the geodesic distance between centroids of residence and workplace municipalities.
The modest magnitude of the effects we detect masks two attenuating forces. First, individuals may anticipate the negative income shock. In that case, they may move preemptively, mechanically reducing the effects we estimate. Second, only a fraction of individuals move. As a result, all the stayers pull our estimates towards zero. To check whether our estimates are sensitive to those two forces, we run equation (2.11) on restricted samples of individuals. First, we restrict attention on individuals who receive the shock as part of a mass layoff event, in which their employer’s employment shrunk by at least 25%. Our assumption is that an income loss that happens concurrently to a mass layoff event is more likely to be unexpected and is therefore less likely to be preceded by preemptive moves. Second, we also restrict attention to movers. Figure 2.10 shows our estimates on these two sub-samples. The magnitudes increases to 1 to 2 percentage points, consistent with the view that unexpected income shocks generate larger moves. Furthermore, we find a more gradual downgrading of relative location, consistent with the lack of anticipation. Overall, the results across all samples paint a consistent picture.

Table 2.2 shows the average post-shock effect across all specifications. In particular, column (5) reveals that restricting attention to movers during mass layoff events increases our coefficient of interest to 3.4 percentage points. Column (6) shows that there is no evidence of differential pre-trends. Consistent with Figures 2.9 and 2.10, Table B.2 in Appendix B.4.3 shows that the results are virtually unchanged after controlling for destination amenities and commuting distance. Table B.3 in Appendix B.4.3 shows results for all quintiles. As expected, the effect is clearly larger for the bottom quintile of the wealth distribution. While Figure 2.10 shows only the difference between the location of individuals in the top quintile and the bottom quintile (corresponding to the \( Q_1 \times \) Shock coefficient in Table 2.2), the coefficient on Shock in Table 2.2 estimates the location response of wealthy individuals who receive the shock relative to wealthy individuals who do not receive the shock. The ‘location as an asset’ view implies that wealthy individuals do not move to systematically lower-ranked locations as a result of the shock, since they prefer to use the financial asset to
2.4.3 Financial Wealth Dynamics after Income Shocks

So far we have shown that low-wealth individuals use location in a way that is consistent with our ‘Location as an Asset’ view, i.e. that they smooth consumption when they receive a negative front-loaded income shock by downgrading their location. We have also shown that wealthy individuals do not change their location as a result of a similar shock. We interpreted those results as evidence that low-wealth individuals cannot borrow in financial markets and therefore use location as an asset. Similarly, we argued that wealthy individuals smooth consumption by withdrawing from their financial assets instead of downgrading their location. We now test directly the second implication of our ‘Location as an Asset’ view;
Table 2.2: Effect of an income shock on location rank (p.p.) by financial assets quintiles.

<table>
<thead>
<tr>
<th></th>
<th>Post shock</th>
<th>Pre shock</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Mass layoffs</td>
<td>-0.22***</td>
<td>-0.18***</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Movers</td>
<td>-0.00</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.03)</td>
</tr>
</tbody>
</table>

Controls and FE

- Year, Q1-Q5, Q2-Q4 × Shock
- Inc., Mun., Occ., Age, HO

<table>
<thead>
<tr>
<th></th>
<th>Obs.</th>
<th>R²</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5139677</td>
<td>0.001</td>
<td>1538559</td>
<td>0.140</td>
<td>3064728</td>
<td>0.125</td>
</tr>
<tr>
<td></td>
<td>675975</td>
<td>0.370</td>
<td>385757</td>
<td>0.368</td>
<td>2957728</td>
<td>0.095</td>
</tr>
</tbody>
</table>

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * p < 0.10, ** p < 0.05, *** p < 0.01. Average difference between location of individuals with low financial assets (Q1) and high financial assets (Q5) α1,1 - α5,1, as well as location of individuals with high financial assets (Q5) α5,1, following a negative income shock, relative to individuals who did not receive the income shock. We pool all years in the sample. The baseline set of controls includes: year fixed effects, dummies for each financial asset quintile, and dummies for financial asset quintiles interacted with the ‘Shock’ dummy. Additional controls include log wage income at time 0, time-0 municipality fixed effects, time-0 2-digit occupation fixed effects, 5-year age bin fixed effects, and a home-ownership fixed effect (HO).

namely, that low-wealth individuals do not adjust their financial wealth as a result of the negative shock, while wealthy individuals do reduce their holdings of financial assets. To do so, we run the same event study as in equation (2.11), but we replace the dependent variable with financial wealth.

Table 2.3 presents the results. Individuals in the top wealth quintile who experience the shock reduce their holdings of financial wealth by about 20 thousand euros relative to individuals in the top wealth quintile who do not experience and income shock (i.e. the coefficient on the ‘Shock’ variable is -20.85). In contrast, individuals in the bottom quintile of the distribution do not change their financial wealth in a statistically significant way.\(^{44}\)

Table B.4 in Appendix B.4.4 shows that our results remain very similar after controlling for amenities and commuting distance. As with location, we find no evidence of pre-trends. We

\(^{44}\)To obtain the change in the wealth of individuals in the bottom quintiles, sum the coefficients on Shock and on Q1 × Shock. For instance, in column (1), it is -2,150 euros.
find somewhat smaller and more noisy effects when we condition on a sample of movers, particularly when we do not control for amenities and commuting. This might be the result of changes in the quality and cost of the house that serves as primary residence after a move, which is not accounted for in our measure of financial income. It might also be simply the result of a significantly smaller sample size in a specification with a large number of fixed effects.

Table 2.3: Effect of an income shock on financial assets (1,000 euros) by financial assets quintiles.

<table>
<thead>
<tr>
<th></th>
<th>Post shock</th>
<th>Pre shock</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Mass layoffs</td>
<td>18.60***</td>
<td>20.55***</td>
</tr>
<tr>
<td>Movers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1 × Shock</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shock</td>
<td>-20.85***</td>
<td>-20.35***</td>
</tr>
<tr>
<td></td>
<td>(4.60)</td>
<td>(4.76)</td>
</tr>
</tbody>
</table>

Controls and FEs

- Year, Q1-Q5, Q2-Q4 × Shock ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓
- Inc., Mun., Occ., Age, HO ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

| Obs.        | 5139270   | 5138156   | 3064573   | 675851    | 378521    | 2957584   |
| R²          | 0.001     | 0.009     | 0.011     | 0.029     | 0.108     | 0.012     |

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * p < 0.10, ** p < 0.05, *** p < 0.01. Average difference between financial assets of individuals with low financial assets (Q1) and high financial assets (Q5) \( \alpha_{1,1} - \alpha_{5,1} \), as well as location of individuals with high financial assets (Q5) \( \alpha_{5,1} \), following a negative income shock, relative to individuals who did not receive the income shock. We pool all years in the sample. The baseline set of controls includes: year fixed effects, dummies for each financial asset quintile, and dummies for financial asset quintiles interacted with the ‘Shock’ dummy. Additional controls include log wage income at time 0, time-0 municipality fixed effects, time-0 2-digit occupation fixed effects, 5-year age bin fixed effects, and a home-ownership fixed effect (HO).

To summarize, Figure 2.11 shows the response of location and financial assets to an income shock in levels and estimated year by year.45 These results can be directly compared to those in the quantitative exercise in Figure 2.6. The similarity is uncanny. Not only is the behaviour of location and assets exactly as predicted, but controlling for an agent’s type does not seem to matter much once we condition on initial location. We conclude that the

45We omit confidence intervals for readability, but the significance of the effect is established in Tables 2.2 and 2.3.
Figure 2.11: Location and wealth effect of a negative income shock by financial assets quintile.

Note: Location (left panel (a)) and financial assets (right panel (b)) of individuals with low financial assets (Q1) and individuals with high financial assets (Q5), relative to individuals who did not receive the income shock. We plot the estimated effects $\alpha_{1,t}$ and $\alpha_{5,t}$ for both location and wealth, for three different sets of controls. $t = 0$ is the year before the income shock. Standard errors omitted for readability. Depending on the specification, the set of controls includes: fixed effects for the time-0 municipality, log wage income at period 0, fixed effects for the time-0 2-digit occupation, 5-year age bin fixed effects, a home-ownership (HO) fixed effect, our measure of amenities for the current location, and log commuting distance at the current residence and workplace.

joint changes in location and wealth after an front-loaded negative income shock support our ‘Location as an Asset’ view.

2.4.4 Changing Location Within and Between Commuting Zones

The geographic unit of analysis we have used so far is a municipality. These municipalities are small, and so they allow us to compare workers that in fact live in the same location (e.g. housing prices vary substantially across municipalities within a commuting zone). Furthermore, since our measure of $z$ is based on the average income of residents in a municipality, and many of these residents work in neighboring municipalities, it already captures the relevant commuting zone-level variation in earnings. In addition, when analyzing the mobility decisions of agents across municipalities, we control for commuting distance and amenities in the new destination (see Table B.2 in Appendix B.4.3). Therefore, our results are not
driven by low-wealth individuals deferentially moving to locations with longer commutes or lower amenities conditional on having similar labor market opportunities in a commuting zone.\textsuperscript{46} Hence, if municipalities are smaller than the relevant local labor markets, we expect our effects to be mostly driven by moves between, rather than within, commuting zones, which are formed by collections of municipalities.

We now investigate whether our results are driven by individuals who move between or within commuting zones. We split movers based on their destination commuting zone and report results for each group. Table 2.4 displays our results on mobility and financial assets for movers within and across commuting zones. Columns (1) and (3) present quantitatively small and statistically insignificant effects for individuals who move within commuting zones. In contrast, the results in Columns (2) and (4) for movers across commuting zone are, as expected, larger and qualitatively similar to the ones presented in Table 2.2. Consistent with the view that commuting zones are the appropriate notion of a local labor market and that individuals’ response to income shocks is based on local labor market opportunities, we find that our effects are driven by individuals who move \textit{between} commuting zones, not \textit{within} them.

\section*{2.5 Conclusions}

This paper provides an alternative view of individual location decisions. We have argued that we can understand location decisions as an investment that allows individuals to transfer resources across periods even when they are constrained in financial markets. Individuals that are constrained to borrow in financial markets use the location asset to borrow and live in locations that offer relatively poor work and educational opportunities but are cheap in terms of housing costs and other local expenses. Hence, our view of location choices

\textsuperscript{46}As we argue in 2.2.6 the differential implication across constrained and unconstrained individuals in a model with amenities would be the opposite. Wealthy individuals would downtrade more as a result of a front-loaded negative income shock.
underscores the importance of the incentives to smooth consumption and the extent to which individuals face financial constraints as essential to understand where they live.

We show that the implications of our model can rationalize the location and moving choices observed in France when individuals experience income shocks. More generally, our view can help explain why some individual locate in areas that seem so undesirable otherwise. The fact that many individuals choose to live in such locations, rather than in areas that offer more opportunities, might seem puzzling from a static perspective, but is a perfectly reasonable choice through the lens of our dynamic theory. In most cases the previous literature has relied on unobserved, and implausibly large, migration costs to explain these choices. In contrast, our view can rationalize this behavior even when migration is perfectly free. The change in perspective is relevant for policy. As we have argued, using
place-based policies to improve some of the worse locations can harm some of the less skilled agents in the economy.

Of course, the ‘location as an asset’ view is more general than the particular model we put forward in this paper and can be contrasted more fully with the data. For example, modelling location choices in an overlapping generations model with multiple locations could help us understand the implications of our view for life-cycle patterns and investment in the skills of descendants. Modelling location choices as changing the properties of an agents income process (by, for example, affecting the likelihood of becoming unemployed) would allow us to study the value of the location asset to manage risk. In addition, in general equilibrium, the agents that decide to locate in a particular region determine, at least partly, the characteristic of the region. Incorporating this form of external effects could lead to interesting insights for policy. Finally, embedding this type of consumption-savings decision with borrowing constraints in a fully-fledged quantitative spatial model with skill complementarities, factor price determination, as well as mobility and trade costs, could help decompose the role of the location asset in determining net mobility patterns relative to other forces. It could also help us understand how the use of location as an asset affects the evaluation of global phenomena that affect factor rewards in particular locations, occupations, and industries.
Chapter 3

Firm and Worker Dynamics in a Frictional Labor Market

3.1 Introduction

Aggregate production in the economy is divided into millions of firms, each facing idiosyncratic fluctuations in its productivity and demand. Understanding the process of labor reallocation across these production units is important for several reasons. In the long run, reallocating labor away from unproductive firms toward more productive firms enhances aggregate productivity and growth. In the short run, the propagation of sectoral and aggregate shocks depends on how quickly labor flows across firms and between unemployment and employment. From a normative perspective, understanding the potential welfare losses or gains due to reallocation is necessary for assessing the efficacy of policies that subsidize jobless workers, protect employment, or advantage particular sectors/firms.

The labor reallocation process has three key properties. First, it has distinct layers: the entry and exit of firms, the creation and destruction of positions (i.e., jobs) at existing firms, and the turnover of workers across jobs at existing firms. Second, it is intermediated by labor markets that are frictional, as revealed by the coexistence of vacancies and job seekers.
Third, around half of worker turnover occurs through direct job-to-job transitions: most new hires come from another firm rather than from unemployment.

Conceptually, therefore, addressing labor reallocation requires a framework with (i) a theory of the firm (i.e., its boundaries) and of firm dynamics (entry, growth, separations, exit); and (ii) a theory of worker flows intermediated by frictional labor markets that allows for on-the-job search and job-to-job mobility (i.e., poaching). Quantitatively, such a framework should account for a new body of time series and cross-sectional evidence—emerging from matched employer-employee data—that describes the relationship between firm characteristics and the direction and composition of worker flows.¹

In this paper, we present a new model with these traits. A firm is a profit maximizing owner of a technology with decreasing returns to scale and stochastic productivity, that chooses optimally whether to enter and when to exit the market.² Firms grow by posting costly vacancies that are randomly matched to either unemployed or employed workers. Worker flows occur when matched workers determine that the value of working at the matched firm exceeds their value of unemployment or employment in their current firm. In general, with decreasing returns to scale in production, these values are a complicated function of a high dimensional state vector that includes distributions of wages or worker values inside the firm. This makes the problem intractable.

The first contribution of our paper is to set out a parsimonious set of assumptions that are sufficient for tractability. Our assumptions place a minimal structure on bargaining and surplus sharing such that, as a result, the state vector becomes manageable. Three assumptions are common to many single-worker firm environments: (i) lack of commitment in firing and quit decisions; (ii) wage contract renegotiation by mutual consent; (iii) Bertrand competition among employers for employed jobseekers. Two further assumptions are required

¹If we consider hires for a particular firm type (e.g., young, small and fast-growing), by composition we mean the split between hires from unemployment and those from employment. Within hires from employment, direction refers to the characteristics of the employers between which workers are reallocated.
²Or, equivalently, a monopolistic producer facing a downward sloping demand curve with a stochastic shifter. These two interpretation are isomorphic in our model.
in our new multi-worker firm environment: (iv) internal wage renegotiations between the firm and its incumbent workers are a zero-sum game, i.e. no surplus gets lost; and (v) privately efficient vacancy posting—for which we offer an explicit microfoundation. Under these assumptions firm and workers’ decisions are privately efficient, as if the firm and incumbent workers maximize their *joint value*. The state variables of the joint value function are only firm size \((n)\) and productivity \((z)\), allowing us to cleanly study firm and worker dynamics in a frictional labor market.

Two other ingredients are vital to achieve tractability. First, we work in continuous time. In a small interval of time only one random event may occur. A firm, for example, only needs to deal with one of its employee meeting another firm, not all combinations of its employees meeting other firms. Second, we take the continuous limit of a discrete workforce. Worker flows are determined by comparing the change in joint surplus that would arise if a worker either joins or leaves a firm. With a continuous measure of workers, this *marginal surplus* can be conveniently expressed as a partial derivative of total surplus.

We show that total and marginal surplus are sufficient for characterizing firm and worker dynamics. Marginal surplus pins down hiring: facing a convex vacancy cost, firms post vacancies until the marginal cost of a vacancy is equal to the expected marginal surplus of hiring. Marginal surplus also pins down separations: facing a decreasing marginal product of labor, firms fire workers until the marginal surplus of a worker equals the value of unemployment. When total surplus is less (more) than the firm’s private outside option the firm exits (enters). Finally, in equilibrium, marginal surpluses determine the direction of worker flows. Workers climb a *job ladder* in marginal surplus, quitting when on-the-job search delivers a match with a higher marginal surplus firm. An intuitive Bellman equation accounts for the evolution of surplus, while a law of motion reflecting frictional labor reallocation accounts for the evolution of the firm size and productivity distribution.\(^3\)

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\(^3\)This representation uniquely pins down firm and worker dynamics, the subject of this paper, but is consistent with multiple wage determination mechanisms that determine how this joint value is split. Wages, therefore, are not allocative in that the distribution of firms and flows of workers across firms is independent.
Our second contribution is to exploit the tractability of this simple representation to analytically characterize equilibrium firm and worker reallocation. First, we characterize firm dynamics and job turnover graphically in \((n, z)\)-space by describing the regions in which a firm exits, fires and hires. Firms that exit and fire always destroy jobs. Hiring firms may either grow on net (creating jobs) or shrink on net (destroying jobs) because some of their workers quit to firms with a higher rank in the marginal surplus ladder. Second, we decompose net growth of hiring firms into the different types of gross flows: hires and separation from/to unemployment and from/to employment via poaching. This decomposition varies systematically with the firm states \((n, z)\) since they determine the firm marginal surplus. Third, we study the limiting behaviors of our economy. As \textit{decreasing returns to scale} vanish, the economy converges to one in which single-worker firms operate in a frictional labor market (à la Postel-Vinay and Robin, 2002). As \textit{frictions} vanish, the economy converges to one in which multi-worker firms operate in a competitive labor market (à la Hopenhayn, 1992). We show that this limit obtains only in the presence of on-the-job search, which provides the key force toward equating marginal products across firms. As opposed to an economy with constant returns to scale, our economy features a non-degenerate size distribution in the limit.

Our third contribution is to exploit the tractability of our framework to implement the model quantitatively. We estimate the model by Simulated Method of Moments, targeting cross-sectional moments of the size distribution of firms, firm dynamics, job flows and worker flows for the U.S. economy. We argue that parameters are well-identified. We then validate the model, conducting a data-model comparison of how firm-level hiring is split into inputs, i.e. between vacancy rates and vacancy yields (as in Davis et al., 2013) and outputs, i.e. between hires from employment and from unemployment (as in Bagger et al., 2019). A firm experiencing a positive productivity shock increases hiring by posting more vacancies, but more so by filling those vacancies faster as a higher marginal surplus makes it more attractive of wage dynamics. In order to study the model’s implication for wage dynamics, one has to make additional assumptions. We return on this point in Section 3.2.
to jobseekers. For the same reason, hires from employment increase disproportionately more than hires from unemployment. These patterns are quantitatively consistent with the data.

Finally, we use the parameterized model to address three quantitative questions that require an environment with decreasing returns to scale and on-the-job search. No other existing structural equilibrium model can address them.

We begin by quantifying the misallocation effects of labor market frictions. A doubling of match efficiency, which approximately corresponds to a doubling of contact rates, raises output by 15 percent relative to our estimated benchmark. Our key finding is that four fifths of the increase in output is caused by lower labor misallocation due to faster job-to-job transitions rather than by higher scale of production due to more employment.

Next, we direct our attention to a new set of facts about job-to-job flows and net poaching. Job-to-job flows and net poaching vary systematically by firm characteristics in the data (Haltiwanger et al., 2018). Young firms poach workers from older firms, but firm size is only weakly correlated with net poaching. Our theory offers an interpretation of these facts. A young firm is far from its optimal size, and since decreasing returns generates a high marginal surplus, this firm will be near the top of the job ladder. However, a firm may be small because unproductive, or because young even if productive. These two types of small firms will be on opposite ends of the job ladder. To guide future measurement we show that labor productivity and firm growth are observables that are strongly positively correlated with marginal surplus, so predictive of net poaching and job ladder rank.

We conclude the quantitative analysis with an application to the U.S. Great Recession. Two distinguishing features were the sharp drop in firm entry and a decline in job-to-job reallocation of workers that led to a ‘failure of the job ladder’, i.e. a slow down of the process through which workers climb toward better firms (Siemer, 2014; Moscarini and Postel-Vinay, 2016). Our model suggests that the former accounts for the latter. A transitory shock to the discount rate (a commonly used shortcut for worsening financial frictions) lowers the value of entry and shrinks the population of young, high marginal surplus firms with high equilibrium
net poaching rates. With fewer firms at the top of the job ladder and less vacancy posting among these firms, labor reallocation up the ladder breaks down. The resulting misallocation causes a persistent slump in output.

Overall, these applications demonstrate that our new theoretical framework offers a useful platform to jointly analyze the microeconomic dynamics of firms and workers in a frictional labor market and how these relate to macroeconomic fluctuations.

**Literature**

Our paper connects two strands of literature. The common core between the two is the idea that diminishing returns in production and heterogeneity in productivity are the dominant forces that deliver a non-degenerate firm-size distribution. This idea goes back at least to Lucas (1978) span of control model.

The first strand is the large literature on equilibrium models of single-product firm dynamics with competitive labor markets. Classic examples are Hopenhayn (1992), Hopenhayn and Rogerson (1993b), and Luttmer (2011). Recent examples, with applications to the Great Recession, are Arellano et al. (2016), Clementi and Palazzo (2010) and Sedláček (2014).

Like these models, our framework features entry, exit, and non degenerate distributions of firm size and age. Unlike these models, the employment adjustment costs that firms face are endogenous. They depend on the firm’s probability of poaching and the expected transfers required to hire a worker away from a competing firm. Both are a function of the firm rank on the marginal surplus ladder, which itself is an equilibrium object. The frictionless limit of our model, where these endogenous costs vanish, is a version of Hopenhayn (1992).

The second literature comprises a number of papers that model multi-worker firms in frictional labor markets. Here, two parallel approaches have been taken: directed search and random search.

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4For a review of the literature see also Luttmer (2010).
The directed search models of Kaas and Kircher (2015) and Schaal (2017) generate firm employment dynamics resembling those in the micro data. Building on Menzio and Shi (2011), the model in Schaal (2017) also allows for on the job search, and thus is the closest counterpart of our framework within the directed search approach. A drawback of directed search is that the probability that a firm hires from another firm or from unemployment is not determined. As a result, this class of models cannot speak to systematic variation in net poaching rates or the composition of hires across firm types that have been documented in micro data. A model consistent with these facts is one of the objectives of our analysis.

In the random search strand, Elsby and Michaels (2013) and Acemoglu and Hawkins (2014) solve models where firms face decreasing returns in production, stochastic productivity, linear vacancy costs, and wages determined by Nash bargaining. Both generate employment relationships with a large average surplus and small marginal surplus. Elsby and Michaels (2013) demonstrate that the latter property yields a volatile job-finding rate over the cycle, while the former avoids an excessively high separation rate, thus resolving the tension identified by Shimer (2005) in the Diamond-Mortensen-Pissarides framework. Gavazza et al. (2018) generalize this model by introducing a hiring effort decision and financial constraints and show that it accounts for the sharp drop in aggregate recruiting intensity around the Great Recession. All of these models abstract from search on-the-job.

Random search models with wage posting feature both on-the-job search and a firm-size distribution despite constant returns to scale. These follow Burdett and Mortensen (1998)...

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5It is worth remarking that these two papers had very different objectives to ours. Kaas and Kircher (2015) illustrate that a key advantage of directed search, the efficiency and block-recursivity properties of equilibrium, extends to models with 'large' firms. Schaal (2017) proves this property is also robust to the addition of on-the-job-search and studies aggregate uncertainty shocks in the context of the Great Recession.

6A key feature of the equilibrium is that net hiring costs are equated across firms through free entry, which implies that firms are indifferent across the markets in which they search for workers. The probability that a separation from a firm is to employment or unemployment, however, is determined.

7Bertola and Caballero (1994) derive closed form results under a linear approximation to both marginal product and convex vacancy costs, and a two state Markov process for productivity.

8Fujita and Nakajima (2016) introduce on-the-job search and study the dynamics of job-job flows over the business cycle. However, in their model all workers are always indifferent between searching/working and staying/moving because, to solve for the equilibrium, they must assume that the worker outside option is always the value unemployment.
and its generalizations to out of steady-state dynamics in Moscarini and Postel-Vinay (2013) and Moscarini and Postel-Vinay (2016), Coles and Mortensen (2016), Engbom (2018), Gouin-Bonenfant (2018) and Audoly (2019). In these models the size distribution is non degenerate only because of the existence of search frictions. As frictions disappear, all workers become employed at the most productive firm. In our framework, instead, we can decompose how much of size dispersion is due to technology and how much is due to frictions.

Within the random search literature, we build on the set-up developed by Postel-Vinay and Robin (2002), which pairs Bertrand competition for workers among employers with wage renegotiation under mutual consent. This environment has become another workhorse of the literature due to its tractability and empirically plausible wage dynamics. Kiyotaki and Lagos (2007) develop a version of this protocol which is a step closer to us. Their firms have fixed capacity of exactly one position and thus feature an extreme version of decreasing returns to scale. In their model, when a matched firm meets another worker, it also engages in a negotiation with its current incumbent. Internal renegotiation is a prominent feature of our model. Our contribution is to generalize this sequential auction protocol to multi-worker firms, show how one can still solve the model’s equilibrium through the notion of joint surplus, and in doing so maintain a great deal of tractability. As opposed to the original Postel-Vinay and Robin (2002) framework the probability of hiring is not a function of the exogenous distribution of firm productivity, but is determined by the endogenous distribution of marginal surpluses, which itself depends on how the equilibrium of the frictional labor market has allocated workers across heterogeneous firms.

The final expression for joint surplus that features among our equilibrium conditions is reminiscent of that in Lentz and Mortensen (2012), a version of Klette and Kortum (2004) with on-the-job search in which a firm’s demand for labor is limited by demand for its portfolio of products. While they assume that all decisions are based on joint firm-workers values,

\footnote{Another implication of such environments is that large firms, which pay higher wages in the model, should poach from small firms, while the data suggest otherwise.}\footnote{Recent examples are Postel-Vinay and Turon (2010a), Jarosch (2015), Lindenlaub and Postel-Vinay (2016), Borovicková (2016), and Lise and Robin (2017).}
we derive this result from primitives, provide a characterization of the equilibrium and illustrate how to use the model for a quantitative analysis of newly documented empirical patterns. Our central finding that a job ladder in marginal surplus arises in equilibrium is closely related to contemporaneous work by Elsby and Gottfries (2019) who elegantly characterize a special case of our environment with linear vacancy costs and no firm entry/exit. In this setting a firm’s values and its decisions are only a function of a single state variable reflecting the marginal product of labor. In our model marginal surplus is related to the marginal product of labor, but includes continuation values that depend on average surplus due to exit. In our calibrated model, we find that the rank-correlation between marginal surplus and marginal product is high. To the extent that marginal and average products of labor are strongly correlated, our result implies that the average product of labor—which is easily measurable—is an informative proxy for the rank of a firm in the job ladder, even in environments richer than Elsby and Gottfries (2019).

Outline. In Section 3.2, we establish the environment and our key assumptions on how firm and workers share value following various stochastic events. In Section 3.3, we state our joint value representation and, to provide intuition on how we achieve tractability, we apply our assumptions in a simplified, static framework. Section 3.4 returns to the fully dynamic model and, after defining an equilibrium, characterizes firm dynamics and worker flows. In Section 3.5 we estimate the model with US data, discuss identification and validate our parameterization. Section 3.6 present our three quantitative exercises. Section 3.7 concludes. The Appendix contains all proofs and details on the computation of the model’s equilibrium.

3.2 Model

In this section we describe the characteristics of the agents in the economy, how meetings take place in the labor market, and the timing of events. We then lay out our key assumptions regarding the contractual environment, i.e. on how the value generated by production is
shared. Finally, we state our main result: the joint value representation of the economy. Under this representation, all allocations are privately efficient, meaning that they maximize the joint value of all the agents involved in the decision.

### 3.2.1 Physical environment

Time is continuous and there is no aggregate uncertainty. There are two types of agents. An exogenous mass $\pi$ of ex-ante identical, infinitely-lived workers that are risk neutral, discount the future at rate $\rho$ and are endowed with one unit of time each period which is inelastically supplied to production. An infinite mass of homogeneous potential firms, of which an endogenous mass become operating firms.

**Production technology.** There is a single homogeneous good. Workers may either be employed or unemployed. Unemployed workers produce $b$ units of the final good. Employed workers are organized into firms which are heterogeneous in their productivity $z \in Z$. A firm employing $n$ workers produces $y(z, n)$ units of the final good, where $y(z, n)$ is strictly increasing in $z$ and $n$ and concave in $n$, i.e. $y_{nn}(z, n) \leq 0$.\(^{11}\), \(^{12}\)

**Firm demographics.** A potential firm becomes an operating firm by paying a fixed cost $c_0$. Paying the fixed cost $c_0$ produces a draw of productivity $z$ from the distribution $\Pi_0(z)$ and $n_0$ workers, taken from unemployment. If a potential firm enters and becomes an operating firm then its productivity $z$ evolves stochastically. At any point in time a firm may exit, at which point all of its workers become unemployed and the firm produces $\vartheta > 0$ units of the good.

\(^{11}\)In addition, we assume that for any $z$ the Inada conditions hold with respect to $n$: (i) $y(z, 0) = 0$, (ii) $\lim_{n \to 0} y_n(z, n) = +\infty$, and (iii) $\lim_{n \to +\infty} y_n(z, n) = 0$.

\(^{12}\)Key for firms’ decisions is that this specification yields a decreasing returns to scale revenue function $r(z, n) = Py(z, n)$, with the normalization $P = 1$. An alternative foundation for a decreasing returns to scale revenue function is constant returns to scale in production of a differentiated final good which would yield a decreasing marginal revenue as under monopolistic competition. If firms’ goods are imperfectly substitutable and household preferences are CES with elasticity $\eta > 1$, the firm faces a demand curve $p(z, n) \propto y(z, n)^{-1/\eta}$, such that the revenue function is $r(z, n) = p(z, n) \times y(z, n) \propto y(z, n)^{(\eta-1)/\eta}$. Our framework therefore accommodates imperfect substitutability in the goods market which is a key ingredient of trade models and macroeconomic models with nominal rigidities.
final good which we refer to as its \textit{scrap value}.\footnote{A positive scrap value plays the same role as a fixed operation cost in generating endogenous exit.} Denote the mass of entrants $m_0$ and the mass of operating firms $m$.

**Matching technology.** Workers join firms from both employment and unemployment through a frictional matching process. The total number of meetings between firms and workers is given by a CRS aggregate matching technology $m(s, v)$. Inputs to the matching technology are total vacancies $v$ and total units of search efficiency $s = u + \xi (\bar{n} - u)$, where the parameter $\xi$ determines the relative search efficiency of employed workers. Search is random in the following sense. A firm pays a cost $c(v; z, n)$ to post $v$ vacancies, where $c$ is increasing and convex in $v$ and $c(0; \cdot, \cdot) = 0$. Each vacancy of the firm is matched with a worker at rate $q(s, v) = m(s, v)/v$. With probability $\phi = (u/s)$ the worker is unemployed, with probability $(1 - \phi)$ the worker is employed. A worker faces no cost of search. An unemployed worker meets a firm at rate $\lambda^U(s, v) = m(s, v)/s$. An employed worker meets a firm at rate $\lambda^E(s, v) = \xi \lambda^U(s, v)$. The rates $q$ and $\lambda^U$ can be expressed in terms of \textit{market tightness} $\theta = (v/s)$ If constituted, the match of a worker to a firm exogenously expires at rate $\delta$, upon which the worker becomes unemployed.

**States.** Let $x$ be the vector of state-variables for the firm. This vector includes all individual state variables of all workers at the firm. For now, we do not specify exactly what is in $x$ and, along the way, define a number of functions that map $x$ at instant $t$ into a new state vector at $t + dt$. The vector $x$ is common knowledge among all workers and the firm. Let $i$ be an indicator function (possibly also a vector) that selects the particular entries of $x$ that identify the worker within a firm (i.e., $i$ is the unique identity of a worker in the firm $x$).\footnote{For example, $x$ is a complete description of IBM and of all its workers. It might contain IBM productivity $z$, its size $n$, and all those features of the contracts of the current employees that are needed to forecast IBM’s value and the value of each of its workers. The state $(x, i)$ should be read: here is IBM, characterized by $x$, and we are assessing the characteristics of the worker named $i$ within IBM.} Let $H(x)$ be the measure of $x$ across firms in the economy, $v(x)$ the number of vacancies created by a firm with state $x$, and $n(x)$ employment at firm $x$. The total mass of vacancies
and employed workers in the economy are
\[ v = \int v(x) dH(x), \quad n = \pi - u = \int n(x) dH(x). \]

Probability densities that will show up in firm and worker problems describe the vacancy-weighted and employment weighted distributions of firms:
\[ h_v(x) = \frac{v(x)h(x)}{v}, \quad h_n(x) = \frac{n(x)h(x)}{n}. \]

**Information.** Information in the economy is complete. Workers and firms know the relevant aggregate variables, i.e. \( u, m \), the measure \( H(x) \) and distributions \( H_v(x), H_n(x) \). The states \( x \) and \( x' \) of firms in competition for a worker are observable to both firms and to all incumbent workers of the two firms. Similarly, unemployed workers know the vector \( x \) of the firm they meet when searching, and incumbents of firm \( x \) know whether the firm has met with an unemployed worker.

**Timing.** We separate the within-\( dt \) timing of events in the model into two parts.

First, events up to the opening of the labor market are described in Figure 3.1. A firm’s productivity \( z \) is first realized. Next, incumbent workers are fired, choose whether to quit the firm, or their employment contracts are renegotiated. Next, the firm decides whether to stay in operation or exit. An operating firm produces \( y(z, n) \), pays wages according to contracts with its workers, and posts vacancies in the labor market.

Second, the mutually exclusive events that may occur to a worker or firm are described in Figure 3.2. The first branch in Figure 3.2 describes events that may occur to an unemployed worker. The second and third branch distinguish between direct and indirect events that may affect the value of incumbent worker \( i \). Direct events involve worker \( i \) meeting with another firm, or the destruction of the worker’s job. Indirect events involve worker \( i \)’s co-worker \( j \) meeting with another firm, or the destruction of a co-worker’s job. The final branch describes

\[ ^{15} \text{The mutual exclusivity property is a consequence of continuous time.} \]
Beginning of period \([t, t+\Delta t]\)

- State \(x_t\)
- Productivity \(z_t\)
- Workers \(n_t\)
- Wages \(\{w_{it}\}_{i=1}^{n_t}\)
- ...

Separation
- Workers quit
- Firms fire
- Renegotiation

Stay/Exit

Operation
- Produce
- Pay wages
- Post vacancies

Productivity shocks

Labor Market Opens

Figure 3.1: Timing of events prior to the opening of the labor market

Labor Market Opens

Direct events to unemployed worker

Meeting

Take-leave offer

Reject

Unemployment

Accept

New firm

Direct events to employed worker \(i\)

Meeting

Sequential auction

Stay

Possible wage gain for \(i\)

Switch

New firm for \(i\)

Direct events to employed workers \(j \neq i\)

Meeting

Sequential auction

Stay

New state for \(i\)

Switch

- New firm for \(j\)
- New state for \(j\)

Events to firm

Meet unemployed

Take-leave offer

Hire

- Possible wage cut for incumbents
- New state for all workers

Meet employed

Sequential auction

Not hire

- Possible wage cut for incumbents
- New state for all workers

Figure 3.2: Labor market: Set of mutually exclusive possible labor market events

events that directly impact the firm. The firm may meet an employed or unemployed worker, emerge either with a new hire or not and new allocation of values to its workers, reflected in updates to the state \(x\). Following any of these events, the state vector \(x\) changes, potentially affecting the value of the match to worker \(i\). Through the following assumptions, we put structure on the states in which these events occur and how values evolve in each case.
3.2.2 Contractual Environment

In this section we state a set of assumptions on the contractual environment sufficient to derive our main theoretical results. It is useful to begin from the definition of a wage contract. A contract between the firm and one of its workers is a binding agreement which specifies a constant wage, i.e. a fixed payment from the firm to the worker, in exchange for labor services. The contract satisfies five assumptions:

(A-LC) **Limited commitment.** All parties are subject to limited commitment. In particular,

(a) **Layoffs** - Firms can fire workers at will.

(b) **Quits** - Workers can always quit into unemployment or to another firm when they meet one.

(c) **Collective agreements** - Workers cannot commit to any other worker inside the firm. *De facto* this assumption rules out transfers among workers.

(A-MC) **Mutual consent.** The wage (contract) can be renegotiated only by mutual consent, i.e. only if one party can credibly threaten to dissolve the match (the firm by firing, the worker by quitting). A threat is credible when one of the two parties has an outside option that provides her with a value that is higher than the value under the current contract.

(A-EN) **External negotiation.** An *external negotiation* is a situation where, through search, the firm comes into contact with an external job seeker or an incumbent worker comes into contact with another firm. In external negotiations, all offers are *take-it-or-leave-it*.

- In a meeting an unemployed worker, the firm makes a take-leave offer to the worker.

- In a meeting with an employed worker, the two firms Bertrand compete through a *sequential auction*. First, the poaching firm makes the take-leave wage offer.
Second, the target firm makes a take-leave counteroffer to the worker. Third, the worker decides.

(A-IN) **Internal negotiation.** An *internal negotiation* is any other situation where contracts between firm and any incumbent workers are modified (following (A-MC), an internal negotiation takes place when any party has a credible threat). The only parties involved in an internal negotiation are those that have a threat and those that are under that threat. We assume that—with respect to worker and firm values—the internal negotiation is a *zero-sum game* and that participation is individually rational for all parties.\(^\text{16}\) Apart from these assumptions we leave internal negotiation unrestricted.

(A-VP) **Vacancy posting.** The firm posts the privately efficient amount of vacancies, which is the one that maximizes the sum of the values of the firm and its workers. Below we propose one possible micro-foundation for (A-VP).

**Discussion.** First, our simple wage contracts are rooted in incomplete contract theory, in which a key tenet is that contracting is only allowed on features that are verifiable to a third party e.g. a court. In our context, the only verifiable and hence contractible features are the wage, whether the firm made the wage payment, and whether the worker provided labor services. As a result, more complex state contingent contracts are ruled out. In the context of such incomplete contracts, renegotiation under mutual consent is a natural assumption consistent with many existing legal frameworks (as argued by Malcomson, 1999), and in the terminology of MacLeod and Malcomson (1989) yields *self-enforcing contracts*.

Second, (A-LC, a,b), (A-EN) and (A-MC) amount to the contractual environment in Postel-Vinay and Robin (2002). The authors show that they lead to a convenient joint value representation in the one-worker-one-firm model. We now discuss how (A-IN) and

\(^{16}\)We adopt the standard definition of a zero-sum game: each individual’s gain or loss is exactly offset by losses and gains of other participants. We also adopt the standard definition of individual rationality: after internal negotiation each player who remains employed at the firm receives at least the outside option that was present before internal negotiation.
(A-VP) are sufficient to extend this convenient representation to an environment with a diminishing marginal product of labor.

Our zero-sum game assumption on internal negotiation (A-IN) allows for a large class of possible micro-foundations for the internal renegotiation game. Each would imply different wage dynamics. The central takeaway is that, no matter the details of such a game, if (A-IN) is satisfied then our following representation of allocations as determined by joint value dynamics holds. Since this paper focuses on the allocations that result from firm and worker dynamics in a frictional labor market, we leave for future research a detailed theoretical and empirical investigation of the implications of different internal renegotiation games.\(^\text{17}\)

Absent (A-VP), the firm would have strong incentives to over-post vacancies relative to the privately efficient amount. The incentives to over post come in two forms: over-hiring and generating what we call *swapping threats*. The firm may post vacancies in order to over-hire, lowering marginal product and so credibly threaten some of its incumbents with wage cuts, as extensively discussed by Stole and Zwiebel (1996) and Brügemann et al. (2018). The firm may also post vacancies with no intention of hiring—which only occurs when marginal products are decreasing—hoping to use a match to threaten to *swap* an incumbent worker with a job seeker, extracting a wage cut from the incumbent. Proceeding under either would require the full distribution of wages as a state variable, ruling out tractability. Assumption (A-VP) resolves these issues.\(^\text{18}\)

The presence of these inefficiencies and the need for an assumption like (A-VP) is unique to an environment with DRS, on-the-job search and endogenous vacancy posting. In a model with constant returns and on-the-job search, over-hiring does not arise due to a constant

\(^{17}\)As a start, a companion note available on our websites, (Bilal et al., 2019a) shows how wages would be pinned down under a particular internal renegotiation mechanism. We assume that workers make take-leave offers to the firm in internal renegotiation. Since this satisfies (A-IN), then allocations are consistent with our general representation (1). In this setting we can compute wages without having to keep track of the entire within-firm wage distribution under the assumption that exit is exogenous. We show that if, instead, the firm made take-leave offers to workers in internal renegotiation—which also satisfies (A-IN), complex transfers would be needed to implement (A-VP) and compute wages.

\(^{18}\)In a different environment that allows full commitment to paying a fixed wage, as in Hawkins (2015), wage cuts are assumed away which eliminates privately inefficient vacancy posting.
marginal product of labor (Postel-Vinay and Robin, 2002). Constant returns also implies that filling a vacancy with a matched job seeker is always profitable, removing the swapping threat. In a model with extreme decreasing returns—a capacity constraint of one worker—and on-the-job search but without endogenous vacancy posting there is no inefficient vacancy posting to resolve (Kiyotaki and Lagos, 2007). In a multi-worker firm model with decreasing returns and endogenous vacancies but without on-the-job search, incumbents are all hired from unemployment and with the same outside option are paid the same wage (Elsby and Michaels, 2013; Acemoglu and Hawkins, 2014). Swapping is not a threat because the job seeker and incumbent are paid the same wage. An over-hiring inefficiency is present, but with a degenerate distribution of wages within the firm, accommodating this inefficiency does not impede tractability. On-the-job-search generates a distribution of wages inside the firm due to the origin of hire and accumulated outside offers. If not addressed, the over-hiring inefficiency would render the model intractable.

We propose one possible micro-foundation that implements assumption (A-VP). The idea is to remove any gains to the firm from expected future wage cuts that would otherwise encourage excess vacancy posting. We assume that workers anticipate that firm’s behavior and offer a preemptive wage cut that leaves the firm indifferent between the efficient vacancy policy and the firm’s privately optimal policy. We formalize this assumption below.

(A-VPI) After the firm announces its proposed vacancies for \( dt \), a randomly selected incumbent worker has the opportunity to make a take-leave counter-offer to the firm. The counter-offer specifies acceptable wages for all (or some) incumbents in exchange for an alternative spot vacancy policy.\(^{19}\) We formalize this assumption below.

\(^{19}\)Alternative implementations could be based on the introduction of ‘social norms’ that prevent firms from cutting the wage of a worker and swapping an incumbent worker with a new worker. Because they would involve deviations from lack of commitment (A-LC), we do not emphasize these alternative implementations in this paper.

\(^{20}\)This assumption does not require commitment because it is a ‘spot contract’ between the parties involved: a transfer in exchange for an immediate action.
Having described the economy’s environment and the contract space, we now state our main result.

### 3.3 Joint value representation

In this section we describe the main theoretical result of the paper. For presentation purposes, the environment is specialized in two ways. First, each firm employs a continuum of workers \( n \). Second, productivity follows a diffusion

\[
dz_t = \mu(z_t)dt + \sigma(z_t)dW_t. \tag{3.1}
\]

**Result.** All allocative decisions in the economy—entry, exit, vacancy posting and mobility of workers between firms—are determined by the joint value. The joint value \( \Omega(z, n) \) is the sum of the present discounted value of an operating firm’s profits plus the present discounted value of lifetime utility of its workers, and satisfies the following, where \( U \) is lifetime utility of an unemployed worker:

\[
\begin{align*}
\rho \Omega (z, n) &= \max_{v \geq 0} y(z, n) - c(v; z, n) \quad (3.1) \\
EU \ \text{destruction:} &- \delta n \left[ \Omega_n(z, n) - U \right] \\
UE \ \text{hire:} &+ \phi q(\theta)v \left[ \Omega_n(z, n) - U \right] \\
EE \ \text{hire:} &+ (1 - \phi)q(\theta)v \int \max \left\{ \Omega_n(z, n) - \Omega_n(z', n'), 0 \right\} dH_n(z', n') \\
\text{Shock:} &+ \mu(z)\Omega_z(z, n) + \frac{\sigma(z)^2}{2} \Omega_{zz}(z, n).
\end{align*}
\]

Firms’ operation requires \((z, n)\) to be interior to an exit boundary. An additional boundary condition determines when separations occur:

\[
\begin{align*}
\text{Exit boundary:} & \quad \Omega(z, n) \geq \vartheta + nU, \quad , \\
\text{Layoff boundary:} & \quad \Omega_n(z, n) \geq U. \quad (3.2)
\end{align*}
\]

---

21 As shown in Appendix C.6, our results also hold with an integer-valued workforce and when the productivity process is a jump-diffusion.

22 More formally, Appendix C.6 states the full Hamilton-Jacobi-Bellman-Variational-Inequality formulation of the joint value problem.
The first term in (3.1) is simply output net of vacancy costs. Next, the firm exogenously loses a worker at rate $\delta n$ with a net loss of $\Omega_n - U$ to the initial coalition of firm and workers. The change in value has two pieces: the change in value of the firm and its non-separating workers which is simply the marginal value of the lost worker ($-\Omega_n$) and the value of unemployment attained by the separated worker ($+U$). The firm hires by posting vacancies which are matched to a worker at rate $q(\theta)$, the probability that this worker is unemployed is $\phi$. The firm always hires unemployed workers, which increases the value of the firm and incumbents by $\Omega_n$ but requires a pledge of $U$ to the new worker.

The firm also hires from and loses workers to other firms by poaching. Workers at other firms are met according to the employment-weighted distribution of productivity and size, $H_n$. Upon meeting, the net coalition value increases by $\Omega_n(z, n) - \Omega_n(z', n')$, so poaching is successful if the firm’s marginal value is largest. Note that $\Omega_n(z', n')$ is the highest value the other firm will offer to its incumbent, and hence it is the take-leave offer the poaching firm will make, as long as it is lower than $\Omega_n(z, n)$.\textsuperscript{23}

The firm’s current workers may also quit to higher marginal value firms. The firm and non-poached workers will lose $\Omega_n(z, n)$ and so are prepared to increase the poached worker’s value by this amount to retain them. Knowing this, the external firm offers the poached worker exactly $\Omega_n(z, n)$ in order to hire them. The joint value—that of firm, non-poached workers and poached worker—is therefore unchanged and reminiscent of Postel-Vinay and Robin (2002), no ‘EE Quit’ term appears in (3.1).

Boundary conditions (3.2) describe firm exit and layoffs. Firms keep operating if the value of doing so exceeds the total value of exit: the private scrap value $\vartheta$ plus unemployment for all its workers. If productivity falls, the marginal value of a worker will fall, but must remain above the opportunity cost of employment. To ensure this, firms layoff workers to sustain $\Omega_n(z, n) \geq U$.

\textsuperscript{23}This term of the Bellman equation reads as if the poaching coalition, which induces a breach of contract between the worker and the losing coalition, compensates the latter exactly for its loss of value associated with the quit. This scheme is reminiscent of the result in Diamond and Maskin (1979) (also present in Kiyotaki and Lagos, 2007) that compensatory damages in breach of contracts restore efficiency.

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This joint value representation has three appealing properties.

### 3.3.1 Properties

1. **Parsimony.** Firm and worker policies are characterized by a low-dimensional state vector: productivity and size. Given decreasing returns to scale in production and on-the-job search, this simplification is a contribution. With decreasing returns spillovers exist as bargaining moves from one worker to the next. This problem has been addressed in the literature by following the approach of Stole and Zwiebel (1996), recently revisited by Brügemann et al. (2018), which delivers tractability. However this approach fails when workers have heterogeneous outside options such as due to on-the-job search. Previous frameworks have therefore restricted their analysis to the case of homogeneous outside options which requires confining attention only to labor market transitions between employment and unemployment, ignoring job-to-job flows and poaching altogether. In models with on-the-job search and heterogeneous outside options these bargaining spillovers are assumed away either (i) by constant returns to scale, which reduces decision making units to one-worker-one-firm pairs and impedes a proper study of firm dynamics; or (ii) by the combination of full commitment to complex state-contingent contracts and directed search. Our contribution is to prove that a plausible set of minimal assumptions on the contractual environment (featuring limited commitment) is sufficient to micro-found a parsimonious representation of allocations.

2. **Private efficiency.** All agents’ decisions (entry, exit, separations, vacancies, and hires) maximize their joint value. Put differently, in external and internal negotiations all privately attainable gains from trade are exploited. For the parties involved, no transfer could yield a Pareto improvement. We have therefore shown how the Coase theorem arises in our context without the need to assume full commitment and complex state contingency in contracting.\(^{24}\)

\(^{24}\)We do not solve for the socially efficient allocations in this paper, but note that the decentralized and planner’s allocations will not coincide. Besides the standard congestion externality à la Hosios, an additional composition externality arises. As in Acemoglu (2001), low-productivity firms do not internalize that when posting vacancies on-the-job search will result in them diverting workers away from high-productivity firms.
(3) **Job ladder.** In one-worker-one-firm models, it is the firm’s *exogenous* productivity that fully determines its position on the job ladder. Here the ladder is in *endogenous marginal values* of labor $\Omega_{n}(z,n)$. These equilibrium objects are determined by the current marginal product of labor together with expectations of future productivity, worker mobility and exit. Hence life-cycle firm dynamics and worker dynamics across firms determine their equilibrium distribution. In particular, the equilibrium allocation is affected by a job ladder in marginal values that endogenously lowers the cost of hiring for firms at the top of the ladder.

**Proof.** The proof of the joint value representation for the full dynamic model, which needs much additional notation, is in Appendix C.6. To convey the economics of how our assumptions lead to this result, we use a static model. The approach and the arguments are the same as in the dynamic model, but the proof is much shorter. This static example covers the construction of each term in (3.1), one by one. We describe the *UE hire* term here, and detail the construction of the other terms in Appendix C.1.

### 3.3.2 Static example

**Set up.** Consider a firm with decreasing returns to scale technology $y(z,n)$ such that $y(z,0) = 0$. Suppose the firm starts with productivity $z$ and $n = 1$ worker. The current contract between the firm and the incumbent specifies a wage $w_1 \in (b, y(z,1))$, where $b = U$ is the value of unemployment. At this point, the incumbent worker does not have a credible threat to quit into unemployment nor the firm has a credible threat to fire the worker. Then, the labor market opens. For now we also assume that the firm has sunk the cost of a vacancy $c$, in the Appendix we explicitly consider the decision to post a vacancy.

**UE Hire.** We describe how to obtain the ‘UE hire’ term in (3.1). Assume the firm’s vacancy meets an unemployed worker. Four different cases can arise from the combination

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These distorted vacancy decisions affect the equilibrium distributions of workers across firms $H_n$ which, in turn, influences the hiring opportunities of all other firms.
of hiring/not hiring and renegotiating/not renegotiating the wage with the incumbent. Our assumption on external negotiation (A-EN) requires that in all cases the take-leave wage offer of the firm to the outside worker is $w_2 = b$. Our internal negotiation assumption (A-IN) requires that the joint value with and without renegotiation is the same and simply equals output $y(z,n)$. Let $w_1^*$ be the incumbent wage after the internal negotiation.

If the firm hires the new worker, its profits are as follows:

$$\begin{align*}
\begin{aligned}
&y(z,2) - w_1 - b, & y(z,2) - w_1^* - b, \\
&\text{Without renegotiation} & \text{With renegotiation}
\end{aligned}
\end{align*}$$

If the firm does not hire the new worker, its profits are

$$\begin{align*}
\begin{aligned}
&y(z,1) - w_1, & y(z,1) - w_1^* \\
&\text{Without renegotiation} & \text{With renegotiation}
\end{aligned}
\end{align*}$$

We now describe which case occurs. This requires understanding when our mutual consent assumption (A-MC) coupled with limited commitment on layoffs (A-LC) bind. In particular, the firm may obtain a credible threat to trigger renegotiation of $w_1$. Since we are interested in allocations only, we focus first on when a hire occurs.

**Hire.** A hire without renegotiation occurs when the following two conditions hold:

$$\begin{align*}
\begin{aligned}
&y(z,2) - w_1 - b \geq y(z,1) - b, & y(z,2) - w_1 - b \geq y(z,1) - w_1 \\
&\text{No credible threat} & \text{Optimal to hire w/o renegotiation}
\end{aligned}
\end{align*} \quad (3.3)$$

The first condition illustrates that the threat to fire the incumbent worker is not credible, which under (A-MC) implies no renegotiation. Keeping the incumbent worker at $w_1$ and employing the outside worker at $b$ delivers a higher value to the firm than the threat of “swapping”: firing worker one and hiring the unemployed worker in his place. Given no renegotiation, the second condition ensures hiring is privately optimal for the firm.
A hire with renegotiation occurs when the following two conditions hold:

\[
\begin{align*}
y(z,2) - w_1 - b &< y(z,1) - b, \\
y(z,2) - w_1^* - b &> y(z,1) - w_1^*.
\end{align*}
\] (3.4)

The firm has now a credible threat to fire the incumbent worker according to (A-LC). This is possible only under decreasing returns to scale: even though \( w_1 < y(z,1) \), the first inequality in (3.4) implies \( w_1 > y(z,2) - y(z,1) \), i.e. the incumbent wage is above its own marginal product. Employing the outside worker at \( b \) and keeping the incumbent worker at \( w_1 \) delivers a lower value than ‘firing and swapping’. The second condition is necessary for hiring to be optimal under the renegotiated wage \( w_1^* \) to the incumbent worker.

Under the zero sum game assumption (A-IN), the renegotiated wage \( w_1^* \) only redistributes value between the incumbent worker and the firm and does not affect total value.\(^{25}\) In addition, it must be individually rational, and so \( w_1^* \in [b, y(z,2) - y(z,1)] \). Without further assumptions we cannot say exactly what this wage is, but we can nonetheless pin down allocations.

Rearranging the optimal hiring conditions, we observe that both are satisfied as long as

\[
y(z,2) - y(z,1) > b.
\] (3.5)

Note that without internal renegotiation (A-IN), the hiring condition would differ in the two cases. If wages could not be cut and the firm had a credible threat, the incumbent worker would be fired and the firm would always hire the unemployed worker \( (y(z,1) > b) \). As a result, to determine when a hire occurs, one would need to know the incumbent’s wage to distinguish between the two cases (thus, in the general model with \( n \) workers, the whole wage distribution). Similarly, if a fraction of output were to be lost because of the internal negotiation, a violation of (A-IN), the hiring conditions in (3.3) and (3.4) would differ and, again one would need to know wages to determine whether a hire occurs.

\(^{25}\)Two relevant cases that would violate this condition are (i) if worker’s effort depends on the wage and enters the production function, and (ii) concave utility.
We can write inequality (3.5) in terms of joint value. Workers’ values are simply equal to their wage $w_i$ for $i \in \{1, 2\}$. The firm’s value is simply equal to its profits. The fact that wages are valued linearly by both worker and firm implies that the joint value $\Omega(z, n)$ is independent of wages:

$$\Omega(z, n) = y(z, n) - \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i$$, for any $(w_i)_{i=1}^{n}$.

Using the definition of joint value, equation (3.5) characterizes when the UE hire occurs:

$$\Omega(z, 2) - \Omega(z, 1) > U.$$  \hspace{1cm} (3.6)

Thus, the decision of hiring from unemployment does not depend on wages, but only on productivity, size, and the value of unemployment $U = b$.

**No hire.** For completeness, consider the cases where no hiring occurs. No hire with renegotiation occurs when the following two conditions hold:

$$y(z, 1) - b > y(z, 1) - w_1$$, \hspace{1cm} (3.7)\hspace{1cm} y(z, 1) - w^*_1 \geq y(z, 2) - w^*_1 - b$$

When incentive compatible for the firm to not expand its workforce, the firm always has a credible threat to swap out its incumbent worker since $w_1 > b$. In this case we can pin down $w^*_1$ from the worker’s individual rationality constraint. If $w^*_1 > b$, then the firm would still have a credible threat to swap the worker, hence $w^*_1 = b$. Since this outcome represents a redistribution of value between firm and worker then, consistent with (A-IN), the joint value remains $\Omega(z, 1)$.

Finally, the no-hiring condition in (3.7) can be re-written as in (3.5) with the opposite inequality, $\Omega(z, 2) - \Omega(z, 1) \leq U$.

---

26The value before renegotiation was $\Omega(z, 1) = z - w_1 + w_1 = z$. The joint value after renegotiation is $\Omega(z, 1) = z - w^*_1 + w^*_1 = z$.  

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The firm hires from unemployment when its vacancy meets an unemployed worker and the marginal value of the job seeker exceeds the value of unemployment:

\[
\frac{\Omega(z, 2) - \Omega(z, 1)}{2 - 1} > U. \tag{3.8}
\]

In addition, the joint value of the firm and its workers rises by \(\frac{\Omega(z, 2) - \Omega(z, 1)}{2 - 1} - U\) when the hire occurs. This is exactly the UE hire term in the HJB equation (3.1). In the case of a hire, incumbent wages may or may not be renegotiated but this has no impact on whether hiring occurs, or how the joint value changes. When this condition fails, the firm does not hire, wages are renegotiated, but the joint value remains constant. All decisions require knowledge of \((z, n)\) only, but not of incumbents’ wages.

Appendix C.1 shows how our assumptions deliver all the other terms of (3.1): EE hires, vacancy posting, layoff, quits, entry and exit. It also contains an extension of the UE hire case to a scenario in which the hiring firm has \(n = 2\) incumbents and either one or both may be under threat of layoff.

### 3.4 Equilibrium and characterization

Returning to the joint value representation (3.1) and (3.2), we define an equilibrium and characterize firm behavior and allocations.

#### 3.4.1 Surplus formulation

A convenient formulation of (3.1) is in terms of joint surplus, defined as \(S(z, n) = \Omega(z, n) - nU\), such that

\[
S_n(z, n) = \Omega_n(z, n) - U, \quad S_z(z, n) = \Omega_z(z, n), \quad S_{zz}(z, n) = \Omega_{zz}(z, n).
\]

The marginal (joint) surplus \(S_n(z', n')\) at a competitor is sufficient to characterize how surplus changes over an EE hire. We therefore directly compute the value of a vacancy using the
employment-weighted distribution of marginal surplus in the economy: $H_n(S'_n)$. Recall that $ho U = b$. With these definitions (3.1) becomes

$$
\rho S(z, n) = \max_{v \geq 0} \ y(z, n) - nb - \delta n S_n(z, n) - c(v; z, n)
$$

$$
+ q(\theta) v \left[ \phi S_n(z, n) + (1 - \phi) \int_0^{S_n(z, n)} S_n(z, n) - S'_n \ dH_n(S'_n) \right]
$$

$$
+ \mu(z) S_z(z, n) + \frac{\sigma^2(z)}{2} S_{zz}(z, n)
$$

subject to the two boundary conditions expressed in terms of surplus:

$$
\text{Exit boundary: } S(z, n) \geq \vartheta, \quad \text{Layoff boundary: } S_n(z, n) \geq 0. 
$$

The entry decision can be written as: $\int S(z, n_0) \ d\Pi_0(z) \geq c_0$.

**One-worker-one-firm models.** The Bellman equation (3.9) is a natural extension of expressions of firm value found in earlier single worker job ladder models. In these settings, constant returns to scale production imply that firms can be treated as arbitrary groups of one-worker-one-firm pairs, each with match output $y(z)$. The surplus from such a firm-worker match in our model follows closely Postel-Vinay and Robin (2002) and Lise and Robin (2017). It can be obtained as a special case of (3.9) when the functions $y$ and $c$ are linear in $n$:

$$
\rho S(z) = \max_{v \geq 0} \ y(z) - b - c(v; z) - \delta S(z)
$$

$$
+ q(\theta) v \left[ \phi S(z) + (1 - \phi) \int_0^{S(z)} \left( S(z) - S' \right) dH_n(S') \right] + \mu(z) S_z(z) + \frac{\sigma^2(z)}{2} S_{zz}(z)
$$

Surplus depends only on exogenous productivity $z$, and with one worker firms the unweighted and employment weighted measures of firms are identical. The expected return to a vacancy is therefore computed by integrating over $H_n(S') = H(z)$. In our framework $H_n(S')$ is an
equilibrium outcome, while here it coincides with the exogenous productivity distribution. Thus, the rank of a firm on the job ladder is determined only by its productivity \( z \).

### 3.4.2 Equilibrium

A stationary equilibrium with positive entry consists of: (i) a joint surplus function \( S(z,n) \); (ii) a vacancy policy \( v(z,n) \); (iii) a law of motion for firm level employment \( \frac{dn}{dt}(z,n) \); (iv) a stationary distribution of firms \( H(z,n) \); (v) vacancy and employment weighted distributions of marginal surplus \( H_v(S_n) \) and \( H_n(S_n) \); (vi) a positive mass of entrants \( m_0 \), (vii) a vacancy meeting rate \( q(\theta) \) and conditional probability of meeting an unemployed worker \( \phi \), such that:

(i) Total surplus \( S(z,n) \) satisfies the HJB equation (3.9) and boundary conditions (3.10).

(ii) The vacancy policy \( v(z,n) \) satisfies the first order condition:

\[
c_v(v(z,n); z,n) = q(\theta) \left[ \phi S_n(z,n) + (1 - \phi) \int_0^{S_n(z,n)} (S_n(z,n) - S_n') \, dH_n(S_n') \right].
\]

(iii) The law of motion for firm level employment is

\[
\frac{dn}{dt}(z,n) = \begin{cases} 
-\frac{n}{\delta} & n < n^*_E(z) \\
q(\theta)v(z,n)\left[ \phi + (1 - \phi)H_n(S_n(z,n)) \right] & n \in \left[n^*_E(z), n^*_L(z)\right] \\
-n\left[ \delta + \lambda E(\theta)(1 - H_v(S_n(z,n))) \right] & n \geq n^*_L(z),
\end{cases}
\]

where the notation \( \frac{dn}{dt} \) denotes a jump of size \( n \), and where the layoff and exit thresholds satisfy

\[
S_n\left(z, n^*_L(z)\right) = 0 , \quad S\left(z, n^*_E(z)\right) = \vartheta ,
\]

From (3.10)

\[
S_z\left(z, n^*_E(z)\right) = 0 , \quad S_n\left(z, n^*_E(z)\right) = 0 \quad \text{if} \quad \frac{dn}{dt}(z,n^*_E(z)) < 0
\]

Smooth pasting

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(iv) The vacancy and employment weighted distributions of marginal surplus $H_v(S_n)$ and $H_n(S_n)$ are consistent with $H(z, n)$:

$$H_v(S_n) = \int_{[S_n(z, n) \leq S_n]} \frac{v(z, n)}{v} dH(z, n) \quad , \quad v = \int v(z, n) dH(z, n)$$

$$H_n(S_n) = \int_{[S_n(z, n) \leq S_n]} \frac{n(z, n)}{n} dH(z, n) \quad , \quad n = \int n(z, n) dH(z, n)$$

(v) The measure of firms $H(z, n)$ is stationary, and admits a density function $h(z, n)$ that satisfies:

$$0 = -\frac{\partial}{\partial n} \left( \frac{dn}{dt} (z, n) h(z, n) \right) - \frac{\partial}{\partial z} \left( \mu(z) h(z, n) \right) + \frac{\partial^2}{\partial z^2} \left( \frac{\sigma(z)^2}{2} h(z, n) \right) + m_0 \pi_0(z) \Delta(n)$$

where $\Delta$ is the Dirac delta “function” which is zero everywhere except $n = n_0$ where it is infinite.\(^{27}\)

(vi) Free-entry implies that the entry condition holds with equality, which determines entry $m_0$:

$$c_0 = \int S(z, n_0) d\Pi_0(z),$$

(vii) The vacancy meeting rate $q(\theta)$ and conditional probability of meeting an unemployed worker $\phi$ are consistent with the aggregate matching function given unemployment ($u = \bar{n} - \int n dH(n, z)$) and aggregate vacancies ($v = \int v(z, n) dH(z, n)$).

The numerical procedure to compute the equilibrium of the model is described in Appendix C.4.

### 3.4.3 Vacancy policy

From (3.9), the first order condition for the firm’s vacancy decision gives

$$q(\theta) R(S_n(z, n)) = c_v(v; z, n) \quad , \quad \text{where} \quad R(S_n) = \phi S_n + (1 - \phi) \int_0^{S_n} \left( S_n - S_n' \right) dH_n(S_n')$$

\(^{27}\)For notational brevity we have slightly abused notation by writing the Kolmogorov-Forward equation in the space of Schwarz distributions.
The return on a vacancy is independent of $v$, and is a strictly increasing and strictly convex function of only marginal surplus:

$$R'(S_n) = [\phi + (1 - \phi)H_n(S_n)] \cdot 1 + (1 - \phi)h_n(S_n) \cdot 0,$$

$$R''(S_n) = (1 - \phi)h_n(S_n).$$

Since the function $c$ is convex in $v$ and $c(0; z, n) = 0$, optimal vacancies are uniquely determined. On the intensive margin, a rise in $S_n$ increases the return to hiring an unemployed or employed worker one-for-one. On the extensive margin, increasing $S_n$ widens the set of firms from which the firm will poach, increasing the probability of a hire by $(1 - \phi)h_n(S_n)$, but hiring from these additional firms yields zero additional value as the target firm’s marginal surplus associated with the worker is close to that of the poaching firm.

### 3.4.4 Endogenous hiring cost

The literature on firm dynamics models employment adjustment costs parametrically. Search frictions and the job ladder induce, instead, an endogenous hiring cost function which depends on both equilibrium market tightness and on the rank of the firm on the job ladder.

The hiring rate per vacancy for a firm with marginal surplus $S_n(z, n)$ is $p = q(\theta)[\phi + (1 - \phi)H_n(S_n)]$. Attaining $\tilde{h}$ hires therefore requires $\tilde{h}/p$ vacancies and costs $C(h, n, z, S_n)$, given by

$$C(h, z, n, S_n) = c\left(v(\tilde{h}, S_n); z, n\right) = c\left(\frac{\tilde{h}}{q(\theta)\left[\phi + (1 - \phi)H_n(S_n)\right]}; z, n\right).$$

(3.13)

The reduced form hiring cost function implied by the model is convex in $\tilde{h}$ and decreasing in marginal surplus. It is also determined by two equilibrium objects: overall market tightness via $q(\theta)$ and the macroeconomic distribution of marginal surplus $H_n(S_n)$. The cost function (3.13) therefore makes clear the role of frictions and on-the-job search as endogenous sources of adjustment cost.\textsuperscript{28}

\textsuperscript{28}Compare this cost function, for example, to the standard convex adjustment cost in firm dynamics models, which depends only on the net growth rate but not equilibrium objects, or to the effective firm-
3.4.5 Hire and separation policies

Properties of $S(z, n)$. It is useful to establish some properties of the joint surplus function under standard assumptions on technologies. Suppose (i) productivity follows a geometric Brownian motion with $\mu(z) = \mu \cdot z$, $\sigma(z) = \sigma \cdot z$, (ii) the vacancy cost function is isoelastic in vacancies only $c(v) = c_0 v^{1+\gamma}$, and (iii) the production function satisfies $y_z > 0$, $y_n < 0$, $y_{nn} < 0$, $y_{zn} > 0$.\footnote{All of which are satisfied by $y(z, n) = zn^\alpha$ with $\alpha \in (0, 1)$, the functional form assumed in our quantitative analysis.} In Appendix C.2 we show that under these assumptions $P1 − P3$ hold:

(P1) $S$ is increasing and concave in employment: $S_n > 0$, $S_{nn} < 0$

(P2) $S$ is increasing in productivity: $S_z > 0$

(P3) $S$ is supermodular in productivity and labor: $S_{zn} > 0$

Optimal policies. Figure 3.3 exploits these properties to characterize the firm’s policies for alternative levels of productivity. The red dashed line describes the value of hiring minus the scrap value: $\Omega(z, n) − \vartheta$. The lower blue dashed line extending from the origin gives the total value of unemployment to the firms’ employees: $U \times n$. The exit threshold $n^*_E(z)$ is determined by their intersection. At this point the per worker value net of $\vartheta$ is equal to the level employment adjustment cost functions in the directed search model of Kaas and Kircher (2015) or the random search model of Gavazza et al. (2018) which do not feature on the job search and so depend on the distribution of firms in the economy only through the ‘price’ $q(\theta)$.

\footnote{level employment adjustment cost functions in the directed search model of Kaas and Kircher (2015) or the random search model of Gavazza et al. (2018) which do not feature on the job search and so depend on the distribution of firms in the economy only through the ‘price’ $q(\theta)$.}
Figure 3.4: Gross worker flows by employment level, for given productivity

Notes: The solid red curve represents total separations \((EU + EE^-)\) and the dashed red horizontal line exogenous quits \(EU\). The green curve represents total hires \((UE + EE^+)\) and the dashed green curve hires from unemployment \((UE)\).

value unemployment: \((\Omega(z, n_E^*(z)) - \vartheta)/n = U\). As opposed to this condition on average values, the layoff threshold \(n_L^*(z)\) equates the marginal value to \(U\).

The solid red line is the upper envelope describing the pre-separation/exit value \(\Omega(z, n) = \mathcal{K}_{\{n<n_L^*(z)\}} \max\{nU, \Omega(z, n) - \vartheta\} + \mathcal{K}_{\{n\geq n_L^*(z)\}} \left[\Omega(z, n_L^*(z)) + (n - n_L^*(z))U\right]\). For example, if \(n > n_L^*(z)\), the firm fires \((n - n_L^*(z))\) incumbents who each receive \(U\). The joint value, given by the solid red line, is therefore given by

\[\Omega(z, n) = \Omega(z, n_L^*(z)) + (n - n_L^*(z))U.\]

Panel (b) shows that under a lower productivity, the exit and layoff regions extend, while the hiring region shrinks. At an even lower \(z_L < z_M\) it is optimal for the firm to exit for all \(n\) (panel (c)).
3.4.6 Worker reallocation

The model enables us to decompose firms’ job flows (i.e. growth) into the four worker flows discussed in the introduction. Firm job growth in the hiring region is given by

\[
\frac{dn}{n} = q(\theta) \frac{v(z, n)}{n} \left[ \phi + (1 - \phi) H_n(S_n(z, n)) \right] - \left[ \delta + \lambda^E(\theta) \Pi_v(S_n(z, n)) \right].
\]

Under assumptions (i)-(iii) stated above, we can also prove (see Appendix C.2):

(P4) Net employment growth \( \frac{dn}{n} \) is increasing with productivity \( z \) and decreasing with size \( n \).

Figure 3.4 illustrates how the four worker flows which determine net firm growth vary with \( n \) for a given level of \( z \). Consider a firm that is at the layoff frontier, \( n = n^*_L(z) \). Marginal surplus is zero so the firm posts zero vacancies and shrinks due to exogenous separations and poaching. Conditional on a meeting, any worker employed in that firm leaves \((\Pi_v(0) = 1)\), and so separations occur at rate \( \delta + \lambda^E(\theta) \). As the firm shrinks, decreasing returns in production cause the firm’s marginal surplus to increase (P1). In terms of outflows, the firm loses fewer workers to competitors. In terms of inflows, the firm posts vacancies which always generate hires from unemployment and, as marginal surplus increases further, hires from employment too. Firms shrink towards \( n^*_ZC(z) \) where gross flows are positive but there is zero growth. For any given productivity \( z \), the firm with the highest marginal surplus has the smallest size compatible with operating, i.e. size \( n^*_E(z) \), and grows quickly away from \( n^*_E(z) \) with high vacancy posting and net poaching.

Moreover, if \( c(v; z, n) = c(v, S_n) \), then:

(1) Rates of \( EE^- \) and \( EE^+ \) are, respectively, decreasing and increasing in the firm’s growth rate. Faster growing firms have higher rates of net-poaching: \( (EE^+ - EE^-) \).

(2) Share of hires from unemployment decreases (from employment increases) in firm growth rate.
Figure 3.5: Exit, layoff and no-growth frontiers in the \((n, z)\)-space

Notes: This figure plots exit, layoff and no-growth frontiers for two cases: without and with positive scrap value. It also includes examples of hypothetical firm paths, in each case keeping productivity fixed. A firm (black dot) that begins in the layoff region jumps to the layoff frontier, firing \(n - n^*_S(z)\) workers. Subsequent declines in productivity smoothly move the firm along the layoff frontier until, possibly, exit. A firm that is located in the hiring region smoothly converges toward the \(dn = 0\) line by growing or shrinking.

(3) Share of separations to unemployment increases (to employment decreases) in firm growth rate.

The intuition is simply that fast growing firms have high marginal surplus. For example, the pattern in (2) can be observed from Figure 3.4. As one moves leftward along the \(x\)-axis, \(S_n\) and firm’s growth rate increases and \(EE^+\) as a the share of total hires increases goes up as well.

We conclude by noting that this type of analysis on the composition of hires by firm size and productivity cannot be performed in directed search models. As explained in the Introduction, in that class of models, the composition of hires at the firm level is indeterminate.

### 3.4.7 Firm dynamics

Combining the characterization above with properties \(P1 \sim P4\), we can fully theoretically represent firm reallocation (exit), job reallocation (net growth), and worker reallocation (hires and separations) in \((n, z)\)-space. Figure 3.5 attains this by describing the functions
that determine the stay/exit frontier \( n^*_E(z) \), hire/layoff frontier \( n^*_L(z) \), and the zero growth locus \( n^*_ZG(z) \).

Panel (a) considers the model without a scrap value such that there is no endogenous exit. We can tightly characterize the layoff frontier. From (3.10) the layoff frontier has slope \( dZ/dn = -S_{nn}/S_z \). Given properties P1 and P3, the frontier is therefore positively sloped. To understand firm dynamics to the left of this frontier, note that fractionally to the left \( S_n \approx 0 \), vacancy posting is low and the firm shrinks due to \( EE^- \) and \( EU \) flows. The zero growth locus along which \( dn = 0 \) must therefore be located strictly to the left of the layoff frontier. To the right of the zero-growth locus, firms hire but lose even more workers, and so experience net job destruction (\( JD \)). To the left of the zero-growth locus, marginal surplus is sufficiently large that firms are successful in hiring and retaining workers to experience net job creation (\( JC \)), but some endogenous separations through quits also occur. Thus, the model generates both hires for shrinking firms and endogenous separations for growing firms. To the right of the frontier, firms layoff workers, destroying jobs en masse, jumping back to the frontier.

Panel (b) introduces a positive scrap value and endogenous exit. First, consider ignoring smooth-pasting conditions. In this case the exit frontier would have gradient \( dZ/dn = -S_n/S_z \). Since \( S_{nn} < 0 \) (P1) and \( S_z > 0 \), the frontier would have a minimum when \( S_n = 0 \). The exit frontier therefore crosses the layoff frontier at its lowest point, increasing on either side.

Now let’s consider how incorporating smooth pasting conditions affects exit. A necessary condition for optimal exit is that \( S_n = 0 \) on the boundary: if marginal surplus was positive, the firm would not want to exit. Since \( S_n = 0 \) on the layoff boundary and by (P1) \( S \) is strictly concave in \( n \) then it cannot be the case that \( S_n \) is zero again in the hiring region. This has two implications. First, firms cannot be exiting along the downward sloping section of the exit boundary in the \( Hire \& JC \) region. This is consistent with employment dynamics as in this region firms drift to the right: \( dn/n > 0 \). Second, firms cannot be located in the
Hire & JD region below the \( z \) at which \( n^*_{ZG}(z) \) crosses the exit frontier. A firm located here would be drifting toward exit with \( S_n > 0 \), and exit with \( S_n > 0 \) is sub-optimal. As a result, to the right of the intersection of the zero-growth locus and the \( S(z,n) = \vartheta \) locus, the exit frontier is flat.\(^{30}\)

The stationary distribution of firms in the economy has support along the layoff frontier, and to its left. The distribution has zero mass along the left exit frontier. Growing firms do not exit, but shrinking firms may experience productivity shocks that force them to crossing the horizontal section of the \( S(z,n) = 0 \) exit frontier. All firms—except those on the layoff frontier—post vacancies, and hire workers both from employment and unemployment, and lose workers both to employment and unemployment.

The results derived in Section 3.4.6 regarding gross flows fully describe employment dynamics of the firm when interior to these boundaries.\(^{31}\)

### 3.4.8 Frictionless limits

The frictionless limit of our economy is identical to that of a competitive, Hopenhayn-style model of firm dynamics with no dispersion in the marginal product of labor. Absent job-to-job mobility, this limit cannot be obtained. This theoretical limiting behavior benchmarks our exercise in Section 3.6.2 where we quantify the output effects of labor market frictions. In the rest of this section we offer an intuition for these results. The formal proof is in Appendix C.3.

**Frictionless limit without on-the-job-search.** Let \( A \) be matching efficiency, the scalar in front of the matching function, and take \( A \to \infty \). Unemployment decreases and, without on-the-job search, firms can only hire from the shrinking pool of unemployed workers. From

\(^{30}\)Note that the sufficient condition for this flat exit boundary to exist is that the \( S_n = 0 \) locus lies strictly below the \( dn = 0 \) locus. This is always the case, since if \( S_n = 0 \) the firm must always be shrinking and \( S_nz > 0 \) (P3). Therefore, a higher \( S_n \) such that the firm is not shrinking must be associated with a strictly higher \( z \).

\(^{31}\)One can think of Figure 3.4 as describing gross firm hiring along a straight horizontal line drawn in the \( (n,z) \) space of Figure 3.5 and running from the exit to the layoff frontier.
the perspective of the firm, the increase in $A$ raises meeting rates, while the decrease in unemployment reduces meeting rates. In equilibrium these two forces exactly offset because the free entry condition uniquely pins down $q(A)$, independently of $A$. With $q$ unaffected by the increase in $A$, the firm problem is unaltered, so firm employment dynamics are unchanged and, conditional on age, the dispersion in the marginal product of labor is unchanged.\textsuperscript{32} Positive dispersion in marginal products is not a property of a frictionless competitive economy, but that of a competitive economy with adjustment costs. We now show that

**Frictionless limit with on-the-job search.** As $A \to \infty$ unemployment decreases but, with on-the-job search, firms can still hire from the non-shrinking pool of employed workers. Worker search efficiency, therefore, becomes constant, while aggregate feasibility ensures finite vacancies, so $q(A) = A(s/v)^{(1-\alpha)}$ increases in $A$. The increase in $q(A)$ accelerates labor reallocation from low to high marginal surplus firms. With decreasing returns to scale, marginal surplus increases at firms that lose workers, and decreases at the firms that poach them. The limit features the hallmark of a competitive model: zero dispersion in marginal surplus. Job-to-job mobility is the key equalizing force.\textsuperscript{33}

In the limit firm behavior is described by the following Bellman equation

$$\rho S(z) = \max_n y(z,n) - bn + \mu(z) \frac{\partial S}{\partial z}(z) + \frac{\sigma^2(z)}{2} \frac{\partial^2 S}{\partial z^2}(z), \quad S(z) \geq \vartheta$$

which makes clear the key properties of the limit. Without dispersion in marginal surpluses the on-the-job search terms drop out. The allocation is as if firms choose their optimal size each instant, where these hires are realized through immediate job-to-job reallocation. The only state variable is therefore $z$, and the productivity-size distribution is degenerate

\textsuperscript{32}As $A$ increases the only change in the distribution of marginal products in the economy comes from the shifting age composition of firms as entry increases.

\textsuperscript{33}The proof requires characterization of the limiting behavior of the entire general equilibrium. This involves (i) the Bellman equation of the coalition, and (ii) the Kolmogorov-Forward equation of the distribution of coalitions. These two partial differential equations are coupled through the equilibrium distribution of marginal surplus and the firm vacancy policy. Given this complexity, we keep the proof manageable by assuming some additional structure. In particular, we assume that the entry productivity distribution has a sufficiently fat tail.
along \((z, n^*(z))\), where \(y_n(z, n^*(z)) = b\), and marginal products are equalized. Firm exit is determined by a cut-off rule on productivity \(z\). For new firms, the value of jumping from \(n_0\) to \(n^*(z)\) upon entry is finite, positive, and still depends on market tightness \(\theta\).

Thus in the limit the model is isomorphic to Hopenhayn (1992) with respect to job reallocation and firm exit. The only conceptual difference between the frictionless limit and Hopenhayn (1992) is that free-entry determines \(\theta\) in the former, while it pins down the wage in the latter.

### 3.5 Estimation

We estimate the model on U.S. data. Because the model is set and solved in continuous time, we can construct correctly time aggregated measures at any desired frequency.

We make the following functional form assumptions. The vacancy cost function is \(c(v, n) = \bar{\tau} \left( \frac{v}{n} \right)^{\gamma+1} v\) as in Kaas and Kircher (2015), such that the per vacancy cost is increasing in the vacancy rate. The production function is \(y(z, n) = zn^\alpha\). The matching function is Cobb-Douglas with vacancy elasticity \(\beta\): a worker meets a vacancy at rate \(p(\theta) = A\theta^\beta\) and a vacancy meets a worker at rate \(q(\theta) = A\theta^{-(1-\beta)}\). The distribution of entrant productivity draws is Pareto with a minimum of one and shape parameter \(\zeta\). We add exogenous firm exit at rate \(d\).

We set two parameters exogenously and normalize three, as summarized by Table 3.1.

The discount rate \(\rho\) implies an annual real interest rate of five percent. The elasticity of the matching function \(\beta = 0.5\) is based on standard values in the literature. Without loss of

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho) Discount rate</td>
<td>0.004</td>
<td>5% annual real interest rate</td>
</tr>
<tr>
<td>(\beta) Elasticity of matches w.r.t. vacancies</td>
<td>0.5</td>
<td>Petrongolo and Pissarides (2001)</td>
</tr>
<tr>
<td>(\vartheta) Scrap value</td>
<td>250</td>
<td>Normalization (= 1/(\rho))</td>
</tr>
<tr>
<td>(\bar{\tau}) Scalar in the cost of vacancies</td>
<td>100</td>
<td>Normalization</td>
</tr>
<tr>
<td>(m) Number of active firms</td>
<td>(1-0.06)/22</td>
<td>Average firm size (BDS)</td>
</tr>
</tbody>
</table>

Table 3.1: Externally chosen parameters
generality we normalize the scrap value, $\vartheta$ and the scalar in the vacancy posting cost, $\tau$.\footnote{Increasing the scrap value shifts up the exit frontier. Since productivity follows a geometric Brownian motion increasing the exit frontier simply increases mean productivity, so normalizing the scrap value is isomorphic to normalizing productivity which we are obviously free to do. The first order condition for vacancies obtained from (3.9) will have $\tau$ multiply the marginal cost of a vacancy and $A$ multiply the marginal benefit of a vacancy. We therefore cannot identify $\tau$ from $A$.}

The entry cost $c_e$ is always pinned down by an average firm size of 22 in 2016 (U.S. Census Business Dynamics Statistics; BDS). We first identify a number of active firms $m$ that delivers an average firm size of 22 when there is a unit measure of workers and an unemployment rate of six percent: $\tilde{m} = (1 - 0.06)/22 = 0.043$. While $m$ is an equilibrium outcome, the fact that a higher $m$ decreases the value of entry through a tighter labor market implies that there is always a unique $c_e$ that satisfies the free-entry condition under $m = \tilde{m}$.

3.5.1 Internally estimated - Minimum Distance

We estimate the 11 remaining parameters to minimize the objective function

$$G(\psi) = \left(\tilde{m} - m(\psi)\right)' W^{-1} \left(\tilde{m} - m(\psi)\right), \quad \psi = \{\mu, d, \zeta, n_0, \alpha, \gamma, A, \xi, \delta, b\},$$

where $\tilde{m}$ is a vector of empirical moments and $m(\psi)$ are their model counterpart. The matrix $W$ contains squares of the data moments on the main diagonal and zeros elsewhere.\footnote{Our moments are taken from various data sources and in most instances we cannot compute variances of the moments, let alone covariances with other moments.}

We target 11 moments that are relatively standard to firm dynamics and frictional labor market literatures. While $\psi$ is jointly estimated, some moments are particularly informative about some parameters. We briefly outline our logic then study identification more formally.

Table 3.2 summarizes the estimated parameter values and the model fit with respect to the targeted moments.

**Firm dynamics.** The negative drift of productivity, $\mu$, is informed by the exit rate of firms. The larger the drift, the faster firms hit the exit threshold. However, most of these firms that exit are small. The exogenous exit rate $d$ induces large firms to exit and is informed by the employment-weighted exit rate. The standard deviation of productivity shocks, $\sigma$, is in-

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formed by cross-sectional dispersion in TFP, while the shape of the productivity distribution for entrants, $\zeta$, affects productivity dispersion among young firms (Decker et al., 2018).\textsuperscript{36} The size of initial firms $n_0$ is informed by the job creation rate of age one firms. Given a distribution of entrant productivity, a smaller $n_0$ implies firms start further to the left of the zero growth locus, implying a faster rate of job creation. A narrower span of control parameter allows for fewer large firms, so $\alpha$ is informed by the employment share of firms with more than 500 employees. Finally, curvature in vacancy costs, $\gamma$, is informed by cross-sectional dispersion in annual employment growth (Elsby and Michaels, 2013). Conditional on $\sigma$, more convexity lowers responsiveness to productivity shocks, reducing dispersion in growth rates. We note that the data ask for both decreasing returns to scale and convexity in vacancy costs.\textsuperscript{37}

**Frictional labor market.** Matching efficiency, $A$, is set to match the monthly $UE$ rate. If matching is more efficient, workers find jobs faster. The relative search efficiency of employed workers, $\xi$, is then set to match the ratio of monthly $EE$ rate to $UE$ rate. The exogenous idiosyncratic separation rate $\delta$ maps into the monthly $EU$ hazard. Finally, we set $b$ to target a standard value for the flow value of leisure relative to average output per worker (Shimer, 2005).

### 3.5.2 Identification

To illustrate the identification of the model’s parameters, we conduct two exercises. First, we demonstrate that, in a sizable hypercube around the estimated value of our 11-dimension parameter vector, the model is globally identified. Our argument proceeds parameter by parameter. We move each parameter $\psi_i$ in steps in a wide range around $\psi_i^*$. Fixing $\psi_i^*$ we re-optimize all other parameters $\psi_{-i}$ to minimize $G(\psi_{-i}, \psi_i^*)$. We argue that the model is

\textsuperscript{36}A natural alternative would have been to target the productivity gap between entrants (younger than 1 year old) and incumbents. The model does well in this respect. At the estimated parameter vector, this gap is 27 (35) percent in the model (data) (Gavazza et al., 2018).

\textsuperscript{37}Similar degrees of convexity in vacancy costs have been estimated by Kaas and Kircher (2015) and Lise and Robin (2017).
### Table 3.2: Estimated parameters and targeted moments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Moment</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ Mean of productivity shocks</td>
<td>-0.004</td>
<td>Exit rate (unweighted)</td>
<td>0.084</td>
<td>0.076</td>
</tr>
<tr>
<td>$d$ Exogenous exit rate</td>
<td>0.001</td>
<td>Exit Rate (employment weighted)</td>
<td>0.019</td>
<td>0.020</td>
</tr>
<tr>
<td>$\sigma$ Std. of productivity shocks</td>
<td>0.054</td>
<td>Std. deviation of log TFP</td>
<td>0.484</td>
<td>0.500</td>
</tr>
<tr>
<td>$\zeta$ Shape of entry distribution</td>
<td>3.735</td>
<td>Std. deviation of log TFP (age 1-5)</td>
<td>0.362</td>
<td>0.400</td>
</tr>
<tr>
<td>$n_0$ Size of entrants</td>
<td>2.058</td>
<td>Job creation rate at age 1</td>
<td>0.237</td>
<td>0.244</td>
</tr>
<tr>
<td>$\alpha$ Curvature in production</td>
<td>0.587</td>
<td>Employment share 500+</td>
<td>0.551</td>
<td>0.518</td>
</tr>
<tr>
<td>$\gamma$ Curvature of vacancy cost</td>
<td>6.023</td>
<td>Std. deviation of employment growth</td>
<td>0.337</td>
<td>0.420</td>
</tr>
<tr>
<td>$A$ Matching efficiency</td>
<td>0.157</td>
<td>UE rate</td>
<td>0.243</td>
<td>0.242</td>
</tr>
<tr>
<td>$\xi$ Search efficiency of employed</td>
<td>0.142</td>
<td>EE rate / UE rate</td>
<td>0.068</td>
<td>0.076</td>
</tr>
<tr>
<td>$\delta$ Exogenous separation rate</td>
<td>0.017</td>
<td>EU rate</td>
<td>0.018</td>
<td>0.016</td>
</tr>
<tr>
<td>$b$ Flow value of leisure</td>
<td>0.295</td>
<td>Value of leisure to average output</td>
<td>0.367</td>
<td>0.400</td>
</tr>
</tbody>
</table>

**Notes**: Exit rates and size distribution of firms are from HP-filtered Census BDS data between 2013–2016, which refer to firms (not establishments) and are annual. Moments specific to productivity dynamics and dispersion in productivity are taken from Decker et al. (2018) (cross-sectional dispersion). The standard deviation of annual growth rates is taken from Elsby and Michaels (2013). The UE, EE and EU monthly mobility rates are averages of HP-filtered CPS data between 2013–2016, estimated on matched micro data for workers 16 years and older. The flow value of leisure to average output of 40 percent based on the average replacement rate reported in Shimer (2005).

### Figure 3.6: Global identification

**Notes**: For each parameter $\psi_i \in \{\mu, \ldots, b\}$, the black line plots the minima of the minimum distance function $G(\bar{\psi}_i) = G(\psi_i, \bar{\psi}_i)$. when $\psi_i$ is fixed at $\bar{\psi}_i$ (plotted on the x-axis), and minimization is with respect to all other parameters $\psi_{-i}$. The red vertical line marks the estimated value $\psi_i^*$ listed in Table 3.2 identified if $G(\psi_{-i}^*(\psi_i^*), \psi_i^*)$ plotted as a function of $\psi_i^*$, traces a steep “U” with a minimum
Figure 3.7: How informative specific moments are for individual parameters

Notes This figure plots the relationship between each parameter $\psi_i \in \{\mu, \ldots, b\}$ and the moment aligned with the parameter in Table 3.2. For each panel, the $x$-axis plots alternative values of the parameter. The $y$-axis plots the change in the corresponding moment in the steady state of the model obtained when all other parameters are as in Table 3.2.

at $\psi_i^*$. Figure 3.6 plots this exercise and gives us confidence that our parameter vector is a global minimum. This is a straightforward exercise for showing global identification.

Second, we show that our argument for identification is valid locally around $\psi^*$. We discussed how each parameter is especially informed by a particular moment, despite the model being jointly identified. To support this argument, Figure 3.7 plots each one of the 11 moments as a function of the corresponding parameter in Table 3.2, keeping all other parameters at their estimated values. All panels show significant variation in the moment of interest as a function of its respective parameter.

3.5.3 Non-targeted moments

The aim of our theory is to describe the mechanics behind the reallocation of workers, in particular poaching flows, across the distribution of firms. Before exploring the importance of poaching flows directly we show that the model is consistent with data on (i) the distribution
A. Distributions of firms and employment

<table>
<thead>
<tr>
<th>Group</th>
<th>A. Firms</th>
<th>B. Employment</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model</td>
<td>Data</td>
</tr>
<tr>
<td>A. By firm size</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0-19</td>
<td>0.909</td>
<td>0.671</td>
</tr>
<tr>
<td>20-49</td>
<td>0.053</td>
<td>0.069</td>
</tr>
<tr>
<td>50-249</td>
<td>0.032</td>
<td>0.063</td>
</tr>
<tr>
<td>250-499</td>
<td>0.004</td>
<td>0.021</td>
</tr>
<tr>
<td>500+</td>
<td>0.003</td>
<td>0.176</td>
</tr>
</tbody>
</table>

A. By firm age

<table>
<thead>
<tr>
<th>Age</th>
<th>Model</th>
<th>Data</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0.158</td>
<td>0.108</td>
<td>0.020</td>
<td>0.161</td>
</tr>
<tr>
<td>2-3</td>
<td>0.134</td>
<td>0.078</td>
<td>0.024</td>
<td>0.049</td>
</tr>
<tr>
<td>4-5</td>
<td>0.110</td>
<td>0.066</td>
<td>0.027</td>
<td>0.040</td>
</tr>
<tr>
<td>6-10</td>
<td>0.198</td>
<td>0.148</td>
<td>0.072</td>
<td>0.035</td>
</tr>
<tr>
<td>11+</td>
<td>0.400</td>
<td>0.600</td>
<td>0.856</td>
<td>0.026</td>
</tr>
</tbody>
</table>

B. Dynamics of employment-productivity

Figure 3.8: Distribution of firms in data and model

of firms, (ii) job and worker flows across the distribution, (iii) vacancy rates and vacancy yields, and (iv) the composition of hires and separations in response to firm-level productivity shocks.

1. Distribution of firms. Figure 3.8A shows that the model reproduces the skewed firm distribution. By size, in both data and model, around 90 percent of firms are small (less than 20 employees), but these account for only around 18 percent of employment. Symmetrically, firms with more than 500 employees represent around 0.4 percent of firms, but more than 50 percent of employment. By age, less than half of firms are older than 10 years, but these account for more than 80 percent of employment.

Figure 3.8B plots the distribution of firms over size and productivity at ages 1, 5, 10 and 21+ firms. Output is higher when the correlation between employment and productivity is higher. In our model search frictions impede a perfect correlation and, thus, reduce output relative to the frictionless benchmark. Firms all start at $n_0$, but with a sizable dispersion in productivity. After a year the correlation between productivity and size is still low. Firms continue to grow rapidly during the first five years, while productivity gradually drifts down.
Despite this rapid growth, there is still a significant dispersion in employment conditional on productivity. As firms get older, the productivity-size correlation increases, while the exit and layoff frontiers compress the dispersion in size conditional on productivity from the left and right, respectively. These frontiers are clearly delineated in \((n,z)\) space, verifying our theoretical characterization of \((n,z)\)-space (Figure 3.5).

### 2. Firm, job and worker reallocation.

Table 3.3 shows that the model matches the key fact that worker reallocation rates are around three times as large as job reallocation rates. This difference is generated entirely by the presence of job-to-job mobility. That the model appears to generate a steady-state level of replacement hiring consistent with the data positions it well to understand the importance of job-to-job mobility in the Great Recession.

In the cross-section, job flows display a degree of the empirical pattern of ‘up-or-out’ dynamics. Job creation and exit rates peak for the young firms, albeit for exit the pattern is not as pronounced as in the data. The model also accounts for job creation by age. In the data (model) 18 percent (16 percent) of all jobs are created by new firm births and 26
A. Vacancy rates and yields by gross hires

B. Decomposing growth

<table>
<thead>
<tr>
<th>Moment</th>
<th>BFGT (2019)</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Net flows</td>
<td></td>
<td></td>
</tr>
<tr>
<td>From/to employment</td>
<td>0.72</td>
<td>0.90</td>
</tr>
<tr>
<td>From/to unemployment</td>
<td>0.28</td>
<td>0.10</td>
</tr>
<tr>
<td>B. Gross flows</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Increasing hires</td>
<td></td>
<td></td>
</tr>
<tr>
<td>From employment</td>
<td>0.44</td>
<td>0.51</td>
</tr>
<tr>
<td>From unemployment</td>
<td>0.14</td>
<td>0.10</td>
</tr>
<tr>
<td>Decreasing separations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>To employment</td>
<td>0.28</td>
<td>0.39</td>
</tr>
<tr>
<td>To unemployment</td>
<td>0.14</td>
<td>—</td>
</tr>
</tbody>
</table>

Figure 3.9: Vacancy filling and decomposing response to shocks

Notes

Panel A. Data: Establishment-month observations in JOLTS microdata 2002-2018 are pooled in bins, where bins are determined by net monthly growth rate, and have a width of 1 percent. Growth rates computed as in DFH. Within bin \( b \), total hires \( h_b \), total vacancies \( v_b \), total employment \( n_b \) are computed. From these, the gross hiring rate \( h_b / n_b \), implied daily filling rate \( f_b \) and implied daily vacancy posting rate \( v_r = v_b / (v_b + n_b) \) are computed using the daily recruiting model of DFH. 

Model: The filling rate in the model is \( f(n, z) = q(\theta)[\phi + (1-\phi)F(n, z)] \). The daily vacancy rate is \( v_r(n, z) = v(n, z)/(v(n, z) + n) \). Points plotted are logs of these variables, differenced about the bin representing a one percent net growth rate.

Panel B. Data: Authors calculations from Table 8 of Bagger et al. (2019). Model: This column describes the response of firm employment to a 20 percent increase in productivity in the model, split into each margin. Aggregating across firms, the table gives the percentage of the increase in net job creation due to changes in different hiring margins. For example, 89.6 percent of the increase in net job creation is due to increased net poaching: \((EE^+ - EE^-)\).

3. Vacancy rates and yields. As pointed out by Davis et al. (2013), the rate at which firms fill their vacancies varies systematically in the cross-section: firms with larger hiring rates also have higher filling rates. Our model shares this implication. We compare the

percent (25 percent) by firms aged 1-10. We conclude that our framework accounts for the distribution of firms across age and size, job flows across these types of firms, as well as entry and exit dynamics.

Turning to worker flows, the data hiring rates display a negative relationship with both firm age and firm size, but the slope with respect to age is much more pronounced. In line with these facts, hiring rates are strongly decreasing in age whereas the rate at which gross flows decline with size is much more moderate.
model to BLS microdata in Figure 3.9A, taking the same approach to data and model as Davis et al. (2013).\footnote{Data is our own computations from BLS JOLTS microdata, in which we extend the sample period of Davis et al. (2013) from 2002-2006 to 2002-2018. The filling rate and vacancy rate are the \textit{daily filling rate} and \textit{daily vacancy flow rate} implied by the model of daily hiring used in their paper.} First we bin firms by their net growth rate, then within bins compute the average hiring rate $\tilde{h}$, vacancy rate $\tilde{v}$ and filling rate $f$ of vacancies:

$$
\tilde{h}(z,n) = \frac{h(z,n)}{n}, \quad \tilde{v}(z,n) = \frac{v(z,n)}{n}, \quad f(z,n) = q(\theta) \left[ \phi + (1 - \phi)H_n(S_n(z,n)) \right].
$$

The model replicates the key empirical observations of DFH: (i) both the vacancy rate and the filling rate are increasing in the hiring rate, (ii) the relationships are essentially log-linear, and (iii) with respect to the hiring rate, the filling rate is much more elastic than the vacancy rate.

In the model firms with high marginal surplus have higher hiring rates, post more vacancies and fill them more quickly as they poach from more firms. The slope of the vacancy rate is mediated by the degree of convexity of the vacancy cost $\gamma$. The fact that the model matches the data along this dimensions gives us more confidence on the estimated value for this parameter. In our model, on-the-job search is entirely responsible for the positive slope in the filling rate. Note, however, that in the data, nearly 80 percent of increases in the hiring rate are driven by changes in the filling rate, whereas in the model this effect accounts for just above 60 percent. We conclude that there is residual scope for other sources of firm’s search effort margins, collectively interpreted as \textit{recruiting intensity} by DFH and Gavazza et al. (2018).

4. Firm-level response to productivity shocks The \textit{cross-sectional} relationships documented above do not address the extent to which the employment growth of a firm is attained through more hires from unemployment, more poaching hires, fewer poaching separations or fewer separations to unemployment. Our theory has sharp qualitative predictions for this decomposition of growth into changes in constituent gross flows (Figure 3.4). A recent paper using Danish registry data by Bagger et al. (2019) decomposes the response
of firm-level net employment following a permanent value-added shock into these margins. Their key finding is that job creation is achieved predominantly through increasing poaching inflows and decreasing poaching outflows.\textsuperscript{39}

We replicate the exercise underlying the empirics of Bagger et al. (2019) and find that the model shares this central result. Table 3.9B shows that in Danish data 72 percent of the net increase in employment following a value added shock is due to increasing net poaching, and only 28 percent is due to increasing net hiring from unemployment. The model shares this feature, with the bulk of the growth owing to increased net poaching. In terms of gross flows, around 60 percent of growth comes from increasing hires and 40 percent from decreasing separations. Of these, poaching hires from employment plays the dominant role in increased hiring.\textsuperscript{40}

3.6 Three applications

We now exploit our model parameterized to the US labor market to address three questions that require a model featuring proper notions of both firm dynamics and worker dynamics.

3.6.1 Net poaching by firm characteristics

The first question we ask is: who poaches from whom? What does the model tell us with respect to firms’ key characteristics that determine their rank on the job ladder? The direct answer is: marginal surplus. Figure 3.10A plots the distribution of $S_n$ together with the net poaching rate as a function of marginal surplus. The CDF reveals that the equilibrium density $h(S_n)$ displays quite a lot of mass in the middle and relatively long tails. As ex-

\textsuperscript{39}The data burden on producing these results is beyond what is available in the U.S. Two ingredients are necessary: high frequency data on (i) on job-to-job flows, (ii) firm value-added and employment. In the U.S., the LEHD contains the former, but revenue data—which can be used to proxy for value added under certain assumptions—is only available annually in the LBD. Data of this quality is, however, available in France and Sweden, for example.

\textsuperscript{40}In the model the constant $EU$ rate away from the layoff frontier means that separations can only fall on the $EE$ margin.
Figure 3.10: Net poaching and marginal surplus distribution

Notes: **Panel A.** Net poaching rate \( p(S_n) \) by log marginal surplus \( S_n \) and the CDF of log marginal surplus. **Panel B.** Decomposition of the change in net poaching rate as \( S_n \) rises into three components: (i) higher vacancies (red line), (ii) more poaching hires due to higher rank on the job ladder (green line), and (iii) lower poaching separations due to higher rank on the job ladder (blue line).

As expected, net poaching is strictly increasing in \( S_n \) with a marked ‘S shape’. What explains this particular shape? Figure 3.10B helps answering this questions.

Under our assumptions on vacancy costs, the vacancy rate of the firm \( (\tilde{v} = v/n) \) depends only on marginal surplus. The net poaching rate is:

\[
\begin{align*}
    p(S_n) &= \tilde{v}(S_n)q(\theta)(1 - \phi)H_n(S_n) - \lambda E(\theta)\overline{H_v}(S_n), \\
    \tilde{v}(S_n) &= q(\theta)^{1/\gamma}[\phi S_n + (1 - \phi) \int_0^{S_n} S_n - u \ dH_n(u)]^{1/\gamma}.
\end{align*}
\]

As \( S_n \) rises net poaching increases through three channels: (i) a higher return to vacancies leads to higher vacancy posting, increasing \( EE \) hires \( (\uparrow \tilde{v}(S_n)) \); (ii) conditional on any vacancy policy a greater fraction of meetings result in a hire \( (\uparrow H_n(S_n)) \); (iii) firm incumbents bump into fewer vacancies that result in an \( EE \) quit \( (\downarrow \overline{H_v}(S_n)) \). Figure 3.10B plots these three forces using the following decomposition

\[
\begin{align*}
p(S_n) - \lambda E &= \int_0^{S_n} \left[ \frac{\partial \tilde{v}(u)}{\partial u} q(\theta)(1 - \phi)H_n(u) + q(\theta)(1 - \phi) \frac{\partial H_n(u)}{\partial u} \tilde{v}(u) - \lambda E(\theta) \frac{\partial \overline{H_v}(u)}{\partial u} \right] du.
\end{align*}
\]

\[41\text{To see this note that with } c(v, n) \propto (v/n)^{\gamma} v, \text{ marginal cost is } c_v(v, n) \propto (v/n)^{\gamma} \text{ and, as characterized in Section 3.4, the marginal benefit of a vacancy depends only on } S_n.\]
Firms with very low marginal surplus basically do not hire and lose all their employees who are successful in their search on the job, so net poaching for them approaches $-\lambda^E$. In the range of $\log S_n$ between zero and two—any rise in marginal surplus increases net poaching almost entirely through changes in marginal surplus rank. For a given percentage change in $S_n$, firms climb the ranks of the job ladder especially fast in the middle of the distribution and this is reflected in the high slope of net poaching in that range (left panel). The vacancy rate initially rises slowly but high marginal surplus firms want to grow fast and in order to achieve high growth post lots of vacancies.

**Size and age.** We now project this relationship between net poaching and marginal surplus onto observables in order to compare the model to empirical patterns documented by Haltiwanger et al. (2018). A key result is a negligible gradient of net poaching by size, but a steep gradient by age as young poach from old. Figure 3.11 shows that the model matches these patterns quite well. Size is not a particularly good predictor of where a firm sits on

---

$\text{H}_n$ and $\text{H}_v$ are different distributions, but have strikingly similar properties given that vacancies are increasing in marginal surplus and, under our cost function, scale with $n$ conditional on $S_n$. 

180
the marginal surplus job ladder. Consider a vertical slice of Figure 3.5. At a given size some firms are highly productive, have a high $S_n$, have positive net poaching and create jobs on net. Meanwhile some firms are less productive, have a low $S_n$, negative net poaching and destroy jobs on net. In contrast, young firms are on average small and productive, sitting to the left of $dn = 0$ and having not yet had time to grow and are high in the marginal surplus job ladder. They therefore display large, positive net poaching rates.

Figure 3.12 plots marginal surplus and net poaching as a function of two other observable firm characteristics, labor productivity and net employment growth rate. The model predicts a much higher gradient between these two variables and net poaching rates compared to size and age. Marginal surplus is highly correlated with the static marginal product of labor, and the latter is proportional to the average product under our functional form for $y(z, n)$.43 Finally, as documented in Table 3.8B, firms grow mostly through hires from employment. Thus, the model implies a tight positive relation between net growth rate and net poaching rate.

43One generalization of the model that would weaken this relation is the addition of heterogeneity in the scale of production parameter $\alpha$, as in (Gavazza et al., 2018). This would create an additional source of cross-sectional variation in marginal surplus that is orthogonal to $z$. 

---

Figure 3.12: Additional determinants of marginal surplus and net poaching
A. Changes in $A$ to half, double $UE$ rate

<table>
<thead>
<tr>
<th></th>
<th>Baseline</th>
<th>Half $UE/U$</th>
<th>Double $UE/U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \log A$</td>
<td>—</td>
<td>-0.47</td>
<td>0.56</td>
</tr>
<tr>
<td>$UE/U$ rate</td>
<td>0.22</td>
<td>0.11</td>
<td>0.44</td>
</tr>
<tr>
<td>$EE/E$ rate</td>
<td>0.016</td>
<td>0.008</td>
<td>0.032</td>
</tr>
<tr>
<td>$u$ rate</td>
<td>0.069</td>
<td>0.153</td>
<td>0.028</td>
</tr>
<tr>
<td>$corr(n, z)$</td>
<td>0.822</td>
<td>0.755</td>
<td>0.888</td>
</tr>
<tr>
<td>$\Delta \log Y$</td>
<td>—</td>
<td>-0.231</td>
<td>0.152</td>
</tr>
<tr>
<td>$\Delta \log TFP$</td>
<td>—</td>
<td>-0.175</td>
<td>0.126</td>
</tr>
<tr>
<td>$\Delta \log TFP/\Delta \log Y$</td>
<td>—</td>
<td>75.8%</td>
<td>82.9%</td>
</tr>
</tbody>
</table>

Figure 3.13: Effect of increasing match efficiency

3.6.2 Misallocation cost of labor market frictions

Next, we ask the model to quantify the misallocation costs of search frictions. Recall that frictionless limits do not make sense in models (i) with OJS but without DRS because they predict that the most productive firm would hire the entire labor force, or (ii) with DRS but without OJS because, as explained in Section 3.4.8, in these environments one does not recover a competitive economy in the limit.

In our counterfactual we shift the value of matching efficiency $A$ holding all other parameters fixed at our baseline calibration. In each case, we resolve the model and recompute key moments of interest. In Figure 3.13A we focus our quantitative analysis on the range for $A$ between -50% and +50% of our point estimate, which roughly corresponds to observed differences in job finding rates across developed countries (Engbom, 2017), but Figure 3.13B plots model outcomes for a wider range as well.

As frictions vanish, unemployment falls, the dispersion of marginal products across firms shrinks and the correlation between size and productivity rises. Doubling match efficiency (and roughly doubling the UE rate) would increase aggregate output by 15 percent (7 percent net of the value of leisure).
Figure 3.14: Entry and job-to-job hiring rates over the Great Recession: Aggregate and Cross-section

Notes Both panels are constructed from the same data at the metro level. Establishment entry and number of establishments are from the Census BDS data, and used to construct establishment entry rate. Job-to-job hires and employment are from the Census J2J data, and used to construct EE hire rate. The data cover the subset of states that participate in these Census data release programs. These cover more than half of the US population.

To isolate the role of misallocation, we decompose the change in output into the component due to the allocation of workers across firms, and the component due to more employment in the economy as a whole (the scale effect). Imposing an aggregate production function $Y = Z\bar{n}^\alpha$, then across steady states

$$\Delta \log Y = \Delta \log Z + \alpha \Delta \log \bar{n}, \quad Z := \int_{\mathbb{N} \times \mathbb{Z}} z \left(\frac{n}{\bar{n}}\right)^\alpha dH(n, z).$$

The TFP term $Z$ captures misallocation and is constant if the distribution of employment across productive units is constant. Lower misallocation accounts for over 80 percent of the increase in output. Interestingly, the relationship between frictions and output is concave because the unemployment rate is convex in match efficiency. Hence doubling frictions leads to a larger output loss, relative to the symmetric case, because of a stronger scale effect. Nonetheless, for both higher and lower labor market frictions the dominant channel determining output is higher or lower TFP due to labor misallocation.
3.6.3 Firm and worker dynamics in the Great Recession

Two of the defining features of the U.S. Great Recession were: (i) a strong decline in firm entry, and (ii) a sharp reduction in job-to-job worker reallocation associated with a failure of the job ladder. Firm entry (measured as the number of firms less than 1 year old in the BDS) dropped by 25-30 percent between 2007 and 2009 and since then it remained below trend (Siemer, 2014). The $EE$ rate also fell around 25-30 percent over the same period (Shigeru et al., 2019; Haltiwanger et al., 2018).

The decline in job to job transitions implied a marked slowdown of worker movements up the ladder. Haltiwanger et al. (2018) document a decline in net poaching of high wage firms, those who are presumably at the top of the job ladder. Similarly, Moscarini and Postel-Vinay (2016) use a structural model to rank firms on the job ladder and estimate that high-rank firms curtailed their demand for new labor in the recession. As a result, the process of upgrading to better jobs, through job-to-job quits from low-rank to high-rank firms slowed down considerably. In short, as they put it: the job ladder failed, starting from the upper rungs.

These two facts have not been connected in the literature. Our environment suggests a natural mechanism to establish a causal link between the two. New entrants and young firms account for a substantial share of vacancies and have higher marginal surplus than other firms in the economy. Thus they account for a large fraction of the poaching of employed workers. Following a drop in the number of entrants, poaching would fall at the top of the ladder which reduces worker reallocation through the middle of the ladder, and so on down to unemployment.

Figure 3.14 shows that drawing this link is also consistent with the cross-region patterns. We combine newly released Census J2J data with Census BDS data at the metro level. The time-series decline in entry and job-to-job mobility is mirrored in the cross-section of metropolitan labor markets: metro areas with larger decline in establishment entry were associated with larger decline in job-to-job mobility.
The aggregate shock that best describes the Great Recession is one that worsens financial frictions. To proxy for a financial shock in our framework, we solve the model under an unexpected temporary increase in the discount rate $\rho$ (as in Hall, 2017). We calibrate the initial jump and the rate of convergence of $\rho$ to match the 5 ppt increase in the unemployment rate and the six years it took to return to pre-recession levels. Appendix C.4 provides more details on the computation.

Figure 3.15 describes the response of aggregates to the shock. The direct effect of the shock is to lower the valuation of future revenues at all firms. As a result, both average and marginal surplus fall, leading to an increase in $EU$ separations from incumbents as well as from a spike in firm exit. Symmetrically, declining marginal surplus reduces the return on vacancies (3.13): vacancies collapse, job creation contracts, and so $UE$ hires decrease. The combination of higher layoff rates and lower job finding rates induces the observed dynamics.
of unemployment. Young firms, which have a disproportionate fraction of their present value of revenues in the future, are especially hard hit, causing entry to collapses by almost 20 percent.

We illustrate the link between firm dynamics and the frictional labor market by plotting the dynamics of the job ladder in Figure 3.16. The top left panel shows that the job-to-job mobility rate drops upon impact and slowly recovers. The size of the drop is in line with the data. As young firms are disproportionately affected and have high-marginal surpluses, the share of vacancies originating from high-marginal surplus firms drops substantially. The decline of vacancies is much less pronounced in other regions of the ladder because of a general equilibrium rise in the rate at which vacancies get filled. With less vacancies and more unemployed workers, the aggregate vacancy yield rises. As in the data, the vacancy yield grows much more for small firms (Moscarini and Postel-Vinay, 2016).
The shifting vacancy distribution causes net poaching to collapse at high-marginal surplus firms and grow at low-marginal surplus firms. This compositional effect reduces the probability that a worker moves to a high-marginal surplus firm relative to a low-marginal surplus firm causing the observed ‘failure’ of the job ladder.

The collapse in the job ladder grinds down aggregate productivity persistently. Consistent with the data, aggregate output per worker increases for a short period. Initially, as low-productive firms shed more workers than high-productive firms, and as average firm size falls, output per worker increases, reflecting a short-lived cleansing effect of the recession. However, throughout the recession and its slow recovery, job-to-job worker flows towards high-marginal surplus firms slow down, exacerbating the misallocation that arises from labor market frictions. The result is a persistent decline in productivity. Note that, even after unemployment is back to trend, a decade after the onset of the recession, TFP is still 2.5 percent below steady state. The recovery of aggregate productivity is sluggish, with the scars of the recession encoded in the slow moving dynamics of the employment-weighted firm distribution.

3.7 Conclusion

We have set out a new and tractable framework to jointly study firm dynamics and worker reallocation. Consistent with the data—and novel with respect to existing multi-worker firm dynamics models with random search—firms hire from both employment and unemployment. In the limit as frictions vanish, our economy converges to a standard competitive firm dynamics model, in contrast to existing models with constant returns to scale in production where in the limit the most productive firm employs the whole workforce. This limiting behavior allowed us to estimate the productivity loss from labor misallocation due to labor market frictions.
The model features a marginal surplus ladder. Firms with higher marginal surplus attract workers more easily, and thus post more vacancies and grow faster. According to the model, value added per worker, firm growth, and age are tightly connected to marginal surplus, whereas firm size is only weakly correlated. The model, estimated on US micro data on firm dynamics, job reallocation and worker flows, produces the observed patterns of net poaching by size and age. Finally, the model offers a natural interpretation for the collapse of the job ladder during the Great Recession: the sharp drop in firm entry reduced the poaching from young and productive units and this weaker pull from the top trickled-down and muted poaching across the entire job ladder.

There are three natural directions to expand the research agenda. First, incorporating wage determination into the model. In Bilal et al. (2019a) we make a first step in this direction. We plan to analyze data on the evolution of the wage distribution at the firm level (frequency and size of wage cuts, correlation of wage changes across workers, etc.) in order to discriminate between alternative protocols.

Second, adding aggregate uncertainty in order to analyze the cyclicality of labor market flows and of net poaching. As shown, we can compute transitional dynamics very efficiently. Thus, exploiting the impulse response as a numerical derivative (as in Boppart et al., 2018) seems the most direct way to study aggregate fluctuations.

Finally, we applied our model to the US economy, but publicly available data from other countries (e.g., France, Germany, Italy, Sweden) would allow to compute poaching statistics by many additional firm characteristics. An interesting question is whether differences in stylized facts arise across countries and whether such differences can be interpreted as the result of heterogeneity in the degree of labor market frictions, institutions, technology, or other factors.

Because of the contemporaneous presence of a well defined notion of firm boundaries (through decreasing returns in technology or downward sloping demand) and a comprehen-
sive model of workers’ frictional reallocation across firms, our framework can be potentially useful to study a number of questions in macroeconomics, labor and trade.
Appendix A

The Geography of Unemployment

This Appendix is organized as follows. Section A.1 provides additional details on the data, sample construction and descriptive evidence. Section A.2 contains proofs for the baseline model. Section A.3 contains proofs for the quantitative framework, as well as a discussion of additional extensions. Section A.4 provides details for the estimation and identification proof, as well as validation exercises.

A.1 Data and descriptive evidence

A.1.1 Data

DADS panel. The central dataset is the 4% sample of the DADS panel, between 1993 and 2007. Once a worker enters the dataset in any year after 1976, all her subsequent employment spells are recorded. The dataset provides start and end days of each employment spell, the job’s wage, the residence and workplace zipcodes of the individual, four-digit occupation and industry, as well as establishment and firm tax identifiers that can be linked to administrative balance-sheet data.

In addition to the sample restrictions described in the main text, I exclude from the sample individuals during the first year that they appear in it. This restriction ensures that
aggregate fluctuations in non-employment are not driven by higher entry in the sample in a particular year, given that individuals are first observed when they have a job. I also drop individuals from the sample two years after their last job. I keep only the years after 1997 because the entry in the panel is noisier in the initial years 1993-1996. I stop in 2007 to avoid both an important classification changes in 2008 and the Great Recession in 2009.

**DADS cross-section.** The DADS *Postes*, are used by the French statistical institute to construct the DADS *Panel*. They cover the universe of French workers, but in the version available to researchers, worker identifiers are reshuffled every two years. The DADS Postes allow to compute employment, wages, occupational mix as well as exit rates and job losing rates for the near universe of French establishments, which can be located at the zipcode level.

**LFS.** I complement the DADS panel with the LFS. I use the LFS starting in 2003 due to a large survey change in 2002. The LFS is quarterly and tracks individuals for six consecutive quarters. The LFS reports whether an individual is working, unemployed or out of the labor force. As in many surveys, the LFS drops individuals if they move between quarters, which is why the DADS panel is particularly useful. I apply the same demographics restrictions as in the DADS panel. I use the LFS to discriminate between unemployment and non-employment in the DADS panel. To that end, I estimate cell-level quarterly transition probabilities between employment, unemployment and non-participation in the LFS. A cell is an occupation and age group - city group bin. Occupation and ages are binned into 4 groups based on their average wage. Similarly, cities are binned into 4 groups based on their unemployment rate. With the estimated transition probabilities at hand, I probabilistically impute the non-participation vs. unemployment status of individuals in the DADS panel. Table A.1 shows that the DADS panel and the LFS have similar aggregate statistics.
Table A.1: Summary statistics in the DADS and the LFS

<table>
<thead>
<tr>
<th></th>
<th>DADS</th>
<th>LFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unemployment rate</td>
<td>0.077</td>
<td>0.071</td>
</tr>
<tr>
<td>Implied unemp. rate</td>
<td>0.057</td>
<td>0.055</td>
</tr>
<tr>
<td>Participation rate</td>
<td>0.911</td>
<td>0.903</td>
</tr>
<tr>
<td>E-to-U probability</td>
<td>0.011</td>
<td>0.015</td>
</tr>
<tr>
<td>U-to-E probability</td>
<td>0.180</td>
<td>0.261</td>
</tr>
</tbody>
</table>

**Skill definition.** Because the DADS panel does not have education data, I construct a measure of skill based on workers’ occupation and age, I run a Mincer regression of worker wages on basic demographics (age and occupation fixed effects), industry and city fixed effects. I retrieve the age and occupation fixed effects, average them over the individual’s work history. Then I rank those averages between workers, and define that rank as skill. I check that several alternative definitions of skill do not alter the results.

More precisely, I run the following Mincer regression:

$$
\log w_{it} = \alpha_{O(i,t)} + \alpha_{Y(t)} + \alpha_{C(i,t)} + \alpha_{A(i,t)} + \varepsilon_{it}
$$

for employed workers $i$ in quarter $t$. Age is binned into 5-year groups, and occupations are at the 3-digit level.\(^1\) Then define skill as average occupation and age premium

$$\hat{S}_i = \frac{1}{N_{i,O}} \sum_{k=1}^{N_{i,O}} (\hat{\alpha}_{O(i,t)} + \hat{\alpha}_{A(i,t)})$$

Results are similar when including worker fixed effects in the Mincer regression and in the skill measure, or when including industry fixed effects.

**Firm-level balance sheet data.** For several over-identification exercises, I use firm-level balance sheet data. I use the FICUS data ("Fichier Complet Unifié de Suse") which covers

\(^{1}\)In some years only 2-digit occupations are available. For workers who have at least one job in years with a 3-digit occupation, I use that occupation. For a few workers I only have 2-digit occupation information, in which case I use the 2-digit occupation wage premium.
the near universe of nonfarm French businesses. The unit of observation is a firm-year. I link the firm identifier to the DADS postes, which allows to identify all workers in the different establishments of the firm. Except to compute the real estate expenditure share and to examine the location choices of multi-establishment firms, I restrict the analysis to single-establishment firms in order to have a well-defined notion of location. In the sample of single-establishment firms, I use firm age and industry. I can also compute value added per worker (labor productivity), average worker skill at a firm along with other variables used in the over-identification exercises.

### A.1.2 Persistence

Figure A.1 shows persistence in local unemployment rates after netting out country-wide industry cycles. The autocorrelation is 0.99. To remove the contribution of industry cycles at the country level, I first compute country-wide change in employment at the 3-digit industry level $\Delta E_j$ between bot subperiods 0 (1997-2001) and 1 (2002-2007). Then, I construct
a predicted employment change at the commuting zone level by projecting the predicted industry employment changes $\Delta E_j$ at the local level using industry employment shares in each location in the 1997-2001 subperiod $w_{c,j,0}$: $\Delta E_c = \sum_j w_{c,j,0} \times \Delta E_j$. Next, I regress changes in local unemployment rates on this predicted change in employment $\Delta u_c = \beta_0 + \beta_1 \Delta E_c + \Delta \tilde{u}_c$. Finally, I extract the residuals from this regression $\Delta \tilde{u}_c$ and construct a measure of local unemployment net of industry cycles in the second subperiod as $\hat{u}_{c,1} = u_{c,0} + \Delta \tilde{u}_c$. Figure A.1 plots $\hat{u}_{c,1}$ against $u_{c,0}$.

### A.1.3 Transition rates

#### Job-to-job rate

Figure A.2: Local job-to-job mobility rate against unemployment-to-employment ratios. France.
United States

See Figure A.3.

Figure A.3: Local job losing and finding rates against unemployment-to-employment ratios. France and United States.

(a) Job losing rate

(b) Job finding rate

Time-aggregation

Two-state continuous time model. Consider first the case in which each city is isolated and workers never leave or enter the labor force which size is normalized to 1. Assume constant job losing and finding rates $s, f$. Then unemployment and employment in each city evolves according to the ODE system

$$
\dot{u} = se - fu; \quad \dot{e} = fu - se; \quad e = 1 - u
$$

This system has a simple solution

$$
u(t) = u_\infty + (u_0 - u_\infty)e^{-(s+f)t}; \quad e(t) = e_\infty + (e_0 - e_\infty)e^{-(s+f)t}
$$
where \( u_\infty = \frac{s}{s+f} \) and \( e_\infty = \frac{f}{s+f} \). Therefore, the transition probabilities in any given time interval \([0, t]\) are

\[
P_t[E \rightarrow U] = u(t) \big|_{u_0=0} = \frac{s(1 - e^{-(s+f)t})}{s+f} \quad ; \quad P_t[U \rightarrow E] = e(t) \big|_{u_0=1} = \frac{f(1 - e^{-(s+f)t})}{s+f}
\]

Hence, the instantaneous quarterly transition rates can be recovered from time-aggregated transition probabilities from

\[
s = \mathcal{T} \times P_1[E \rightarrow U] \quad ; \quad f = \mathcal{T} \times P_1[U \rightarrow E]
\]

where one quarter is the interval \([t, t+1]\), and the time aggregation factor is

\[
\mathcal{T} = \frac{\log \left( 1 - P_1[E \rightarrow U] - P_1[U \rightarrow E] \right)}{P_1[E \rightarrow U] + P_1[U \rightarrow E]}
\]

**Time-aggregation in the data.** To assess the importance of time-aggregation in the data, Figure A.4 shows the variance decomposition of the log unemployment-to-employment ration into using transition probabilities and time-aggregated transition rates, for France and the United States. The job losing shares remains stable across all specifications and countries. In France, the time-aggregation correction also does not change the job finding and the covariance share. In the United States, these are more sensitive to time aggregation, without changing the main results. There are at least two explanations for this difference. First, the CPS data for the United States contains potentially substantial measurement error in the local job finding probabilities due to small sample issues. This inflates the variance share of job finding flows, which is in turn magnified by the time aggregation correction. Second, gross labor market flows are larger in the United States, leading to potentially more time aggregation bias.
Three-state model. I now consider a three-state version of the model, still with isolated locations and the total number of individuals normalized to 1 in each location. Denote now by $n(t)$ the number of individuals out of the labor force, so that $u(t) + e(t) + n(t) = 1$. There are transitions between all states, such that

\[ \dot{u} = se - fu + rn - du \quad ; \quad \dot{n} = s_n e - f_n n - rn + du \]

where $s_n$ is the separation rate into non-participation, $f_n$ the finding rate out of non-participation, $r$ the re-entry rate (NU) and $d$ the drop-out rate (UN). In steady-state,

\[ du - rn = se - fu \quad ; \quad du - rn = f_n n - s_n e \]
Finally, the unemployment rate $u_R$ is $u_R = \frac{u}{e+n} = \frac{u}{1-n}$. Using $e = 1 - u - n$ and combining both equations,

$$u_R = \frac{s(f_n + r) + rs_n}{f_n(d + f + s) + r(f + s_n + s)}$$

and so

$$\frac{u_R}{1-u_R} = \frac{s(f_n + r) + rs_n}{f_n(d + f) + rf}$$

Then define

$$p = \frac{s(f_n + r) + rs_n}{f_n(d + f) + rf} - \frac{s}{s + f}$$

Therefore, with flows in and out of the labor force within isolated cities $c$, equation (1.1) becomes

$$\log \frac{u_c}{1-u_c} = \log s_c - \log f_c + \log p_c + e_c \quad (A.1)$$

where $e_c$ is a residual that captures migration flows, local dynamics and measurement error.

The exact variance decomposition of the log unemployment-to-employment ratio writes

$$\text{Var} \left[ \log \frac{u_c}{1-u_c} \right] = \text{Cov} \left[ \log \frac{u_c}{1-u_c}, \log s_c \right] + \text{Cov} \left[ \log \frac{u_c}{1-u_c}, -\log f_c \right]$$

$$+ \text{Cov} \left[ \log \frac{u_c}{1-u_c}, \log p_c \right] + \text{Cov} \left[ \log \frac{u_c}{1-u_c}, e_c \right]$$

Figure A.5 reports the results from this variance decomposition.
Equation (1.2) is useful to attribute the spatial variation in $Y$ to city, industry and skill or worker characteristics. An exact variance decomposition follows from taking expectations conditional on city $c$ on each side, and breaking up the resulting variance:

$$\text{Var} [\bar{Y}_c] = \text{Cov} [\bar{Y}_c, \alpha_c] + \text{Cov} [\bar{Y}_c, \mathbb{E}_c [\beta_j]] + \text{Cov} [\bar{Y}_c, \mathbb{E}_c [\gamma_i]]$$  \hspace{1cm} (A.2) \hspace{1cm}

where $\bar{Y}_c = \mathbb{E}_c [Y_{c,j,i}]$ denotes the local average of $Y$. The first term on the right-hand-side of equation (A.2) is the contribution of city-specific heterogeneity to the spatial variation in $Y$. The second term is the contribution of industry heterogeneity, and the third of skill or worker heterogeneity. These two last terms are zero if $\mathbb{E}_c [\beta_j] = \mathbb{E}_c [\gamma_i] = 0$, which occurs when there is no systematic sorting of industries or skills across cities.

Fixed effects regressions estimate the decomposition in equations (1.2)-(A.2), but at a fine level of disaggregation with over 300 cities, 220 industries and 300 skill groups, well-known small-sample biases in the covariances may arise. Small sample biases may arise
similarly to those in worker and firm effects models as in Abowd et al. (1999). Therefore, I also estimate a correlated random effects structure following Borovičková and Shimer (2017), which provides an unbiased estimate of the variance-covariance matrix at the cost of distributional restrictions. The key idea is to use a leave-out estimator. I posit that the random effects follow a jointly normal distribution. Finally, I estimate linear and probit models for robustness to functional form assumptions. See also Kline et al. (2019) as well as Bonhomme et al. (2019) for alternative approaches. I now describe my correlated random effect estimator in more detail.

Consider the de-meaned job losing probability $EU_{c,j,i}$. Assume that

$$EU_{c,j,i} = \alpha_c + \beta_j + \gamma_i + \epsilon_{c,j,i} \quad (A.3)$$

Suppose that $(\alpha_c, \beta_j, \gamma_i)$ is jointly normally distributed in the employed population, with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_c^2 & \rho_{cj} \sigma_c \sigma_j & \rho_{ci} \sigma_c \sigma_i \\ \rho_{cj} \sigma_c \sigma_j & \sigma_j^2 & \rho_{ji} \sigma_j \sigma_i \\ \rho_{ci} \sigma_c \sigma_i & \rho_{ji} \sigma_j \sigma_i & \sigma_i^2 \end{pmatrix}$$

Suppose that $\epsilon_{c,j,i}$ has mean zero conditional on $(c,j,i)$, is normally, identically, and independently distributed across triplets. Given an estimate of $\Sigma$, the variance decomposition in (A.2) follows from conditional Gaussian distributions:

$$\text{Cov}[\bar{Y}_c, \alpha_c] = \sigma_c V$$
$$\text{Cov}[\bar{Y}_c, \mathbb{E}_c[\beta_j]] = \rho_{cj} \sigma_j V$$
$$\text{Cov}[\bar{Y}_c, \mathbb{E}_c[\gamma_i]] = \rho_{ci} \sigma_i V$$

$$V \equiv \sigma_c + \rho_{cj} \sigma_j + \rho_{ci} \sigma_i$$
To estimate $\Sigma$, I use conditional second moments. I first outline the strategy in large samples. Then, I describe how I correct for small sample biases in practice.

**Large samples.** The random effects structure implies

\[
\begin{align*}
\mathbb{E}_c[EU_{c,j,i}] &= \left(1 + \rho_{cj} \frac{\sigma_j}{\sigma_c} + \rho_{cs} \frac{\sigma_s}{\sigma_c}\right) \alpha_c \\
\mathbb{E}_j[EU_{c,j,i}] &= \left(1 + \rho_{cj} \frac{\sigma_c}{\sigma_j} + \rho_{sj} \frac{\sigma_s}{\sigma_j}\right) \beta_j \\
\mathbb{E}_i[EU_{c,j,i}] &= \left(1 + \rho_{ci} \frac{\sigma_c}{\sigma_i} + \rho_{ji} \frac{\sigma_j}{\sigma_i}\right) \gamma_i
\end{align*}
\]

Thus, the random effects correlations can be directly estimated from correlations between conditional means:

\[
\begin{align*}
\text{Corr} \left[\mathbb{E}_c[EU_{c,j,i}], \mathbb{E}_s[EU_{c,j,i}]\right] &= \rho_{cs} \\
\text{Corr} \left[\mathbb{E}_c[EU_{c,j,i}], \mathbb{E}_j[EU_{c,j,i}]\right] &= \rho_{cj} \\
\text{Corr} \left[\mathbb{E}_i[EU_{c,j,i}], \mathbb{E}_j[EU_{c,j,i}]\right] &= \rho_{ji}
\end{align*}
\] (A.4)

Given estimates of the pairwise correlations, recover variances from the 3x3 linear system:

\[
\begin{align*}
\sqrt{\text{Var} \left[\mathbb{E}_c[EU_{c,j,i}]\right]} &= \sigma_c' + \rho_{cj} \sigma_j + \rho_{cs} \sigma_i \\
\sqrt{\text{Var} \left[\mathbb{E}_j[EU_{c,j,i}]\right]} &= \sigma_j + \rho_{cj} \sigma_c + \rho_{ji} \sigma_i \\
\sqrt{\text{Var} \left[\mathbb{E}_i[EU_{c,j,i}]\right]} &= \sigma_i + \rho_{ci} \sigma_c + \rho_{ji} \sigma_j
\end{align*}
\] (A.5)

which can be solved as

\[
\begin{pmatrix}
\sigma_j \\
\sigma_c \\
\sigma_s
\end{pmatrix} = \frac{1}{D} \begin{pmatrix}
1 - \rho_{ci}^2 & \rho_{ci} \rho_{ij} - \rho_{cj} & \rho_{cj} \rho_{ci} - \rho_{ij} \\
\rho_{ci} \rho_{ij} - \rho_{cj} & 1 - \rho_{ij}^2 & \rho_{cj} \sigma_{ij} - \rho_{ci} \\
\rho_{cj} \rho_{ci} - \rho_{ij} & \rho_{cj} \rho_{ij} - \rho_{ci} & 1 - \rho_{cj}^2
\end{pmatrix} \cdot \begin{pmatrix}
\sqrt{\text{Var} \left[\mathbb{E}_j[EU_{c,j,i}]\right]} \\
\sqrt{\text{Var} \left[\mathbb{E}_c[EU_{c,j,i}]\right]} \\
\sqrt{\text{Var} \left[\mathbb{E}_i[EU_{c,j,i}]\right]}
\end{pmatrix}
\]

\[
D \equiv 1 - \rho_{cj}^2 - \rho_{ci}^2 - \rho_{ji}^2 + 2 \rho_{cj} \rho_{ci} \sigma_{si}
\]
Replace $EU_{c,j,i}$ with the inverse probit transformation of $EU$ for the probit model. When estimating with skill groups and worker effects, impose that the correlation between worker effects and skill effects is zero.

**Small samples.** In practice, “naive” estimators of the variances in (A.5) and the correlations in (A.4) are subject to small sample biases. First, variance estimates must be adjusted using the Bessel correction factor. Second, consider the “naive” correlation estimator that simply correlates estimated conditional means. In that case, the common observation in conditional means creates a positive bias.¹ To circumvent this difficulty given the i.i.d. assumption of the residual, it suffices to remove all common observations from the estimated conditional means. This strategy is often called a leave-out estimator.

Finally, because the relevant distribution is the employment-weighted distribution, it remains to specify how mobility is correlated with the fixed effects. To make progress, I follow Card et al. (2013) and Borovičková and Shimer (2017) and assume conditional random mobility. More precisely, I assume that the number of quarters of worker $i$ in city $c$ and industry $j$, $n_{c,j,i}$, is uncorrelated with $\alpha_c$ and $\beta_j$ conditional on $\gamma_i$. This is admittedly a strong assumption, although it will be satisfied in the model. Similarly, I assume that $n_{c,j,i}$ is uncorrelated with $\gamma_i$ and $\alpha_c$ conditional on $\beta_j$, and that it is uncorrelated with $\gamma_i$ and $\beta_j$ conditional on $\alpha_c$.

In practice using cities, industries and skill groups, using the leave-out estimator or not does not affect the results because the common observation bias is small. Thus, the results are robust to relaxing the conditional random mobility assumption. However, when using cities, industries, skills and workers effects, the conditional random mobility must be imposed: there are only few different cities and industries for each worker.

With three groups, the leave-out estimation procedure imposes additional data requirements relative to a two-group situation. To facilitate exposition, I outline the estimator with

---

¹This positive bias is distinct from the negative bias that arises in small sample fixed effect estimators. For fixed effects in samples, the negative bias arises because positive measurement error in one fixed effect is mechanically transmitted as negative measurement error into the other fixed effect of a given pair.
two groups. The logic extends directly to three groups. Suppose there are only cities \( c \) and workers \( i \).

**Within variance.** Following Borovičková and Shimer (2017), start by estimating the within-worker variance \( \text{Var}_i[EU_{ict}] \). First, an unbiased estimator of the conditional mean is

\[
\overline{EU}_i = \frac{1}{N_i} \sum_{c,t} n_{i,c,t} EU_{ict}
\]

where \( N_i = \sum_{c,t} n_{ict} \) is the number of quarters for which individual \( i \) is observed. Now,

\[
EU_{ict} \mid i \sim \mathcal{N}(\gamma_i + \mathbb{E}_i[\alpha_c], \sigma^2_{W,i})
\]

where by definition, \( \sigma^2_{W,i} \) is the within-worker variance. An unbiased estimator of \( \sigma^2_{W,i} \) is then

\[
\hat{\sigma}^2_{W,i} = \frac{N_i}{N_i - 1} \cdot \frac{1}{N_i} \sum_{c,t} n_{ict} (EU_{ict} - \overline{EU}_i)^2
\]

The average within-worker variance is \( \sigma^2_W \equiv \mathbb{E}[\text{Var}_i[EU_{ict}]] \), for which an unbiased estimator is then

\[
\hat{\sigma}^2_W = \frac{1}{N} \sum_{i=1}^N N_i \hat{\sigma}^2_{W,i} = \frac{1}{T} \sum_{i=1}^N \frac{N_i}{N_i - 1} \cdot \sum_{c,t} n_{ict} (EU_{ict} - \overline{EU}_i)^2
\]

where \( N = \sum_{i=1}^N N_i \), and \( \tilde{N} \) is the number of workers in the sample.

**Variance of conditional mean.** To estimate the variance of the conditional mean, use the law of total variance

\[
\text{Var}[\mathbb{E}_i[EU_{ict}]] = \text{Var}[EU_{ict}] - \mathbb{E}[\text{Var}_i[EU_{ict}]]
\]
where the unconditional variance can be estimated with the standard variance estimator.

**Covariance of conditional means** To get an unbiased estimator of the covariance between conditional means, compute conditional means leaving out common terms. Denote $\mathcal{N}_i$ the set of quarters $t$ for which individual $i$ is observed. Denote $\mathcal{N}_{i,-c}$ the set of quarters for which individual $i$ is observed, but in a city different than $c$. Denote $N_{i,-c}$ the number of quarters in $\mathcal{N}_{i,-c}$. Symmetrically, denote $\mathcal{N}_{c,-i}$ the set of individual-quarters pairs that are in city $c$ outside of worker $i$, and $N_{c,-i}$ its cardinality. Then compute

$$
\text{Cov} \left( \frac{1}{N_{i,-c}} \sum_{(t,\ell) \in \mathcal{N}_{i,-c}} n_{i\ell t} EU_{ikt}, \frac{1}{N_{c,-i}} \sum_{(t,k) \in \mathcal{N}_{c,-i}} n_{kct} EU_{kct} \right)
$$

$$
= \text{Cov} \left( \gamma_i + \frac{1}{N_{i,-c}} \sum_{(t,\ell) \in \mathcal{N}_{i,-c}} n_{i\ell t} (\alpha_k + e_{ikt}), \alpha_c + \frac{1}{N_{c,-i}} \sum_{(t,k) \in \mathcal{N}_{c,-i}} n_{kct} (\gamma_k + e_{ikt}) \right)
$$

$$
= \text{Cov}(\gamma_i, \alpha_c)
$$

which, adjusted by the estimated variances, delivers an estimate of the correlation. These estimators can be directly extended to three groups, provided all common observations between one group and the two others are removed from the conditional mean estimators.

**Results.** Figure A.6 displays the results of the variance decomposition for the job losing rate from equation (A.2) for France and the United States. In practice, the results are very close across specifications and countries. Industry and skill composition of cities account for no more than 10-15% of the spatial variation in the job losing rate, while city-specific effects account for over 80% of it. Figure A.7 indicates that the results are similar for the job finding rate.

Finally, figure A.8 the decomposition after including worker effects. Estimating that specification requires to restrict the sample to movers between industry and cities, as well as assuming that worker effects have mean zero conditional on the skill effect. For comparison, Figure A.8 reports the results from similar decompositions without the worker effects, but on
the same restricted sample of workers. The variance share of city effects modestly diminishes to about 75%, but mostly as a result of the sample selection. Including worker effects leaves cities’ contribution essentially unchanged. It affects the contributions of industries and skills to spatial job losing rate differentials.

Figure A.6: Variance decompositions of local job losing rate into city, industry and skill contributions. France and United States.
Figure A.7: Variance decompositions of local job finding rate into city, industry and skill contributions. France and United States.

Figure A.8: Variance decompositions of local job losing rate into city, industry, skill and worker contributions. France only.
A.1.5 Conditional correlations

See Table A.2.

### Table A.2: OLS regressions of worker-level unemployment, job loss and job finding probabilities

<table>
<thead>
<tr>
<th></th>
<th>Unemployment</th>
<th>Job loss</th>
<th>Job finding</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>Log city wage</td>
<td>-1.44***</td>
<td>-0.79*</td>
<td>-0.62***</td>
</tr>
<tr>
<td>(0.40)</td>
<td>(0.32)</td>
<td>(0.15)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Log city pop.</td>
<td>1.63+++</td>
<td>1.40+++</td>
<td>0.50*</td>
</tr>
<tr>
<td>(0.40)</td>
<td>(0.33)</td>
<td>(0.21)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Worker skill</td>
<td>-3.07+++</td>
<td>-0.47+++</td>
<td>0.55+++</td>
</tr>
<tr>
<td>(0.18)</td>
<td>(0.02)</td>
<td>(0.12)</td>
<td>(0.12)</td>
</tr>
</tbody>
</table>

**Fixed Effects**

<table>
<thead>
<tr>
<th></th>
<th>Year</th>
<th>Industry-Year</th>
<th>Worker</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Obs.</th>
<th>$R^2$</th>
<th>W.-$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3005929</td>
<td>0.006</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>3005919</td>
<td>0.070</td>
<td>0.010</td>
</tr>
<tr>
<td></td>
<td>3005306</td>
<td>0.361</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>2699433</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>2699426</td>
<td>0.015</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>2697645</td>
<td>0.105</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>306496</td>
<td>0.002</td>
<td>0.000</td>
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<tr>
<td></td>
<td>306436</td>
<td>0.017</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>296336</td>
<td>0.232</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Standard errors in parenthesis, two-way clustered by city and 3-digit industry. $^+ p < 0.10$, $^* p < 0.05$, $^{**} p < 0.01$, $^{***} p < 0.001$.

City population by km2 (density). Quarterly frequency, 1997-2007. Movers only.

Left-hand-side variables in 100 percentage points. Right-hand-side variables standardized.

A.1.6 Tenure profile

It is well known that job losing rates tend to be highest at early job tenures, and subsequently decline. A natural question is then whether spatial gaps in job losing rates also occur at early job tenures.

To estimate the tenure profile of job loss, I run the following linear probability model

$$EU_{i,q} = \alpha_{C(i,q),\tau(i,q)} + \beta_{J(i,q),\tau(i,q)} + \gamma_{S(i),\tau(i,q)} + \delta_{Y(q)} + e_{i,q}$$

where $i$ indexed workers, $c$ cities, $j$ 3-digit industries and $q$ quarters. $\tau(i,q)$ denotes worker $i$’s years of job tenure in quarter $q$. $Y(q)$ is a year fixed effect. I then retrieve all fixed effects.
The solid blue line in Figure A.9 is then the economy-wide average of all the fixed effects:

\[
A_\tau = \frac{1}{N_\tau} \sum_{i,q \mid \tau(i,q) = \tau} \left( \hat{\alpha}_{C(i,q),\tau(i,q)} + \hat{\beta}_{J(i,q),\tau(i,q)} + \hat{\gamma}_{S(i),\tau(i,q)} \right)
\]

It numerically coincides with the outcome of a simple OLS regression \( \hat{A}_\tau \)

\[
EU_{iq} = \delta_{Y(q)} + A_{\tau(i,q)} + e_{iq}
\]

The dashed orange lines are the city premia. They correspond to

\[
P_{g,\tau} = A_\tau + \frac{1}{N_{g,\tau}} \sum_{i,q \mid \tau(i,q) = \tau \land (i,q) \in N_g} \left( \hat{\alpha}_{C(i,q),\tau(i,q)} - A_\tau \right)
\]

where \( g \) denotes city groups. Cities are binned into four population-weighted raw job losing rate quartiles. \( N_{g,\tau} \) is the number of worker-quarter pairs in group \( g \) with annual job tenure \( \tau \). \( N_{g,\tau} \) denotes the cardinality of \( N_{g,\tau} \).
Figure A.9: Job loss rate by job tenure, aggregate and by city. France.

Figure A.10: Job loss rate by job tenure, aggregate and by city. Difference between top and bottom quartile of cities ranked by job losing rate. France.
Figure A.9 first reveals that average job losing probabilities decline with tenure in all cities. Most of the changes occur in the first two years. Second, it shows that the gap between high and low job losing cities is largest at early job tenures – in the first two years of the job – and stabilize thereafter. Figure A.10 shows that the pattern is similar after controlling for worker fixed effects.

A.2 Baseline model

A.2.1 Value functions

In this Appendix I solve the more general model without assuming that wages depend only on productivity and the location.

Values. When the wage needs not depend on \((y, \ell)\) but only follows a Markov process, workers’ values become

\[
\rho U = b \ell r(\ell) - \omega + f(\ell) \mathbb{E}[V^E(w^*(y_0, \ell), \ell) - U] \quad ; \quad \rho V^E(w, \ell) = w r(\ell) - \omega + (L_w V^E)(w, \ell)
\]

where the expectation is taken over the starting productivity \(y_0\) in location \(\ell\). \(L_w\) is the integro-differential infinitesimal generator that encodes the continuation value of employment due to wage changes. It needs not be explicitly specified at this stage.

Worker surplus. Workers’ surplus from being employed \(V^E - U\) solves

\[
\rho(V^E(w, \ell) - U) = r(\ell)^{-\omega} \left( w - (b + v(\ell)) \ell \right) + L_w(V^E - U)(w, \ell)
\]

where I denote \(v(\ell)\ell = f(\ell) \mathbb{E}[V^E(w^*(y_0, \ell)) - U]\) the efficiency value of search in location \(\ell\).
Employers. The value of a filled job paying wage \( w \) with productivity \( y \) in location \( \ell \) solves

\[ \rho J(w, y, \ell) = y\ell - w + (L_y J)(y, w, \ell) + (L_w J)(y, w, \ell) \]

A.2.2 Bargaining

To characterize wages and values, it is useful to define the adjusted surplus

\[ S(y, \ell) = J(y, w, \ell) + r(\ell)\omega \cdot (V^E(y, w, \ell) - U) \]

which is independent from wages, and solves the recursion

\[ \rho S(y, \ell, a) = \ell \cdot (y - b - v(\ell)) - L_y S \]

for continuing matches. Renegotiation every instant means that employers and workers bargain over flow surpluses

\[ r(\ell)^{-\omega} (w - (b + v(\ell))\ell) \]

Without loss of generality, these flow surpluses can be written as values

\[ W(w) = W_0 w - W_1 \]

\[ F(w) = F_1 - w \]

To solve for wages in the bargaining game, the idea is now to make use Proposition 122.1 p.122, Chapter 7, of Osborne and Rubinstein (1994). The setup of the bargaining game is as follows. There is a parallel time for bargaining, in which the worker and the firm have linear flow preferences over a wage \( w \) given by \( W(w), F(w) \), and discount the future. Denote by \( \delta_F \) the discount factor of the worker in the bargaining space-time, and \( \delta_F \) that of the firm.
Disagreement and admissible wages. If bargaining breaks down, each side gets 0. The admissible bargaining set is all \( w \) such that

\[
B^F \equiv \frac{W_1}{W_0} \leq w \leq F_1 \equiv B^W
\]

where \( B^W, B^F \) denote the worker’s and firm’s best agreement, respectively. Finally, define the Pareto frontier as the set of wages \( w \) such that there is no other wage \( w' \) such that both parties prefer \( w' \) to \( w \) in the initial round: \( F(w') > F(w) \) and \( W(w') > W(w) \). Because of the linearity of flow values, the Pareto frontier is exactly equal to the set of admissible wages.


(A1) – For no two distinct wages \( w \neq w' \), it is the case that \( W(w) = W(w') \) and \( F(w) = F(w') \). Each party’s objective is strictly monotonic in the chosen wage \( w \), so (A1) is satisfied.

(A2) – Getting the other party’s best agreement in the second round is the same as getting in the first round, i.e. \( F(B^W) = \delta_F F(B^W) \) and \( W(B^F) = \delta_W F(B^F) \). Since \( F(B^W) = W(B^F) = 0 \), (A2) is satisfied.

(A3) – The Pareto frontier is strictly monotone: for any efficient/admissible wage \( w \), there is no other wage \( w' \neq w \) such that each side weakly prefers \( w' \). This again directly follows from linearity of payoffs.

(A4) – There is a unique pair of wages \((w^W, w^F)\) such that \( \delta^W(w^W) = W(w^F) \) and \( \delta^F(w^F) = F(w^W) \), and both \((w^W, w^F)\) are efficient. I write down the system of equations

\[
\begin{align*}
\delta^W[W_0w^W - W_1a] &= W_0w^F - W_1 \\
\delta^F[F_1 - w^F] &= F_1 - w^W
\end{align*}
\]
Use the second equation to obtain:

\[ w^F = \frac{w^W}{\delta_F} - \frac{1 - \delta^F}{\delta^F} F_1 \]

Substituting into the first equation:

\[ w^W = \beta^W F_1 + (1 - \beta^W) \frac{W_1}{W_0} \]

where

\[ \beta^W = \frac{1 - \delta^F}{1 - \delta^F \delta^W} \in (0, 1) \]

and so

\[ w^F = \beta^F F_1 + (1 - \beta^F) \frac{W_1}{W_0} \]

where

\[ \beta^F = \frac{(1 - \delta^F) \delta^W}{1 - \delta^F \delta^W} \in (0, 1) \]

Finally, \( w^W, w^F \) are automatically on the Pareto frontier because they are admissible and payoffs are linear, which concludes the proof to the bargaining solution.

Without loss of generality, suppose that the worker moves first. Then the worker’s effective bargaining power is \( \beta = \beta_W \). Finally, note that the bargaining solution solves

\[ \frac{W(w^*)}{W_0} = \beta \cdot \left( F(w^*) + \frac{W(w^*)}{W_0} \right) \quad ; \quad F(w^*) = (1 - \beta) \cdot \left( F(w^*) + \frac{W(w^*)}{W_0} \right) \]
Therefore, it is enough to define an adjusted surplus \( F(w) + \frac{W(w)}{W_0} \) which does not depend on wages. Then rescaled values split this adjusted surplus. This argument proves the following Lemma.

**Lemma 8. (Bargaining solution)**

Suppose a worker and an employer play an alternating offer game à la Rubinstein (1982) with static surpluses \( W(w) = W_0w - W_1 \) and \( F(w) = F_1 - w \), and worker effective bargaining power \( \beta \). Define the adjusted surplus \( S(w) = F(w) + \frac{W(w)}{W_0} \). Then

- The adjusted surplus is independent from wages \( S(w) \equiv S \)
- The equilibrium wage \( w^* \) solves

\[
\frac{W(w^*)}{W_0} = \beta S \quad ; \quad F(w^*) = (1 - \beta)S
\]

**A.2.3 Adjusted surplus**

Using Lemma 8, the solution to the dynamic bargaining problem immediately follows.

**Lemma 9. (Bargaining solution)**

Equilibrium wages \( w^*(y, \ell) \) split the adjusted surplus into constant shares:

\[
J(y, w^*(y, \ell), \ell) = (1 - \beta)S(y, \ell) \quad ; \quad V^E(y, w^*(y, \ell), \ell) - U = \beta r(\ell)^{-\omega} \cdot S(y, \ell)
\]

Because of static renegotiation, wages for continuing matches can then be immediately calculated

\[
w^*(y, \ell) = \left[ (1 - \beta)(b + v(\ell)) + \beta y \right] \ell \quad (A.7)
\]
However, all matches eventually break up. Thus, the adjusted surplus $S$ solves an optimal stopping problem, and thus a Hamilton-Jacobi-Bellman-Variational-Inequality (HJB-VI):³

$$0 = \max \left\{ \left( y - (b + v(\ell)) \right) \ell + (L_yS)(y, \ell) - \rho S(y, \ell), \ S(y, \ell) \right\}, \ \forall \ y \geq 0 \ \ \ (A.8)$$

The structure of the HJB-VI (A.8) has two implications: first, there exists a continuation region in which the HJB (A.6) holds. As will become clear, the joint surplus is strictly increasing in this continuation region. Thus, it takes the form of an interval $[y(\ell), +\infty)$ in each location: there is a cutoff productivity $y(\ell)$ below which the match breaks up. Then, at that cutoff, the surplus must be zero: $S(y(\ell), \ell) = 0$. This condition is sometimes called the value-matching condition. Second, because the cutoff is chosen optimally, a first-order-condition with respect to the cutoff must hold, implying $\frac{\partial S}{\partial y}(y(\ell), \ell) = 0$. This condition is sometimes called the smooth-pasting condition.⁴ In addition, the joint surplus must be smaller than the surplus of a match without any outside option, which is $\frac{y \ell}{\rho + \delta - \sigma^2/2}$.⁵

Together, the HJB (A.6), the value-matching, smooth-pasting conditions and the upper bound determine the value $S(y, \ell)$ and the endogenous separation cutoff $y(\ell)$, which I summarize as

$$\rho S(y, \ell) = \left( y - (b + v(\ell)) \right) \ell + (L_yS)(y, \ell), \ \forall y \geq y(\ell) \ \ \ (A.9)$$

s.t. $S(y(\ell), \ell) = 0$, $\frac{\partial S}{\partial y}(y(\ell), \ell) = 0$, $S(y, \ell) \leq \frac{y \ell}{\rho + \delta - \sigma^2/2}$

Lemma 1 then displays the solution to problem (A.9), with the constants

$$\tau = \frac{2\delta}{\sigma^2} \left\{ \sqrt{1 + \frac{2\rho\sigma^2}{\delta^2}} - 1 \right\} \quad ; \quad \frac{y_0}{\tau} = \frac{1 + \tau}{\rho + \delta - \sigma^2/2}$$

³See Pham (2009) for a formal derivation of the HJB-VI from the sequential formulation.
⁴See Pham (2009) for a formal derivation of the interval property and of the smooth-pasting condition.
⁵Formally, from the sequential formulation, the joint surplus can be expressed as $S(y, \ell) = \ell \mathbb{E}_0 \left[ \int_0^\tau e^{-\rho t}(y_t - (b + v)) \ dt \ | y_0 = y \right]$ where $\tau$ is the stopping time. Taking an upper bound, the surplus must be bounded above by the aforementioned expression.
The solution method follows the arguments in Luttmer (2007). To make notation lighter, I drop location indices $\ell$ and solve without loss of generality

$$\rho S(y) = y - c + L_y S \quad \forall y \geq y$$

s.t. $S(y) = 0$, $S'(y) = 0$, $S(y) \leq \frac{y}{\rho + \delta - \sigma^2/2}$

First re-express the problem in logs $x = \log y$:

$$\rho V(x) = e^x - c - \delta V'(x) + \frac{\sigma^2}{2} T''(x) \quad \forall x \geq x$$

s.t. $V(x) = 0$, $V'(x) = 0$, $V(x) \leq \frac{e^x}{\rho + \delta - \sigma^2/2}$

**The homogeneous equation.** Look for a solution $V(x) = e^{-\tau x}$ to $\rho V(x) = -\delta V'(x) + sV''(x)$ where $s = \sigma^2/2$. This delivers a second-order equation

$$\rho = \delta \tau + s \tau^2$$

Denote $\kappa = \mu/s$ and $\eta = \rho/s$, so that the equation re-writes $\tau^2 + \kappa \tau - 1 = 0$. The assumption on parameters implies $\eta > 1 + \kappa$. The discriminant is $D = \kappa^2 + 4\eta > 0$. The equation hence has two solutions in general:

$$\tau_{\pm} = \frac{-\kappa \pm \sqrt{\kappa^2 + 4\eta}}{2}$$

Both roots can be bounded. First, $\tau_- > 0$. Second, $-\tau_+ > 1$. Indeed, since $\eta > 1 + \kappa$,

$$-\tau_+ = \frac{\sqrt{\kappa^2 + 4\eta} - \kappa}{2} > \frac{\sqrt{\kappa^2 + 4\kappa + 1} - \kappa}{2} = \frac{\sqrt{\kappa^2 + 4\kappa + 1} - \kappa}{2} = \frac{|\kappa + 2 - \kappa|}{2} \geq \frac{\kappa + 2 - \kappa}{2} \geq 1.$$

Therefore, the homogeneous solution with $\tau_+$ violates the upper bound on the value function. The solution with $\tau \equiv \tau_-$ is thus the only possible homogeneous solution.
Thus, slightly abusing notation, the homogeneous equation subject to the upper bound has solutions

\[ V_H(x) = Ae^{-\tau x}, \ A \in \mathbb{R} \]

**Inhomogeneous solution.** Now look for solutions:

\[ V(x) = Ae^{-\tau x} + Be^x - C \]

Substituting in the HJB, the homogeneous term drops out and we find:

\[
\begin{align*}
\rho B &= 1 - \delta B + sB \rightarrow B = \frac{1}{\rho + \delta - s} \\
-\rho C &= -c \rightarrow C = \frac{c}{\rho}
\end{align*}
\]

Notice that \( Be^x \) is the value if the match continues forever, \(-C\) is the annuitized option value. The term \( Ae^{-\tau x} \) then captures the endogenous separation decision.

Because \( e^{-\tau x} \) solves the homogeneous equation, \( A \) is not determined form the HJB. I am left with \((A, x)\) to determine, with the two boundary conditions \( V(x) = 0, \ V'(x) = 0:\)

\[
\begin{align*}
A e^{-\tau x} + Be^x &= C \\
-A \tau e^{-\tau x} + Be^x &= 0
\end{align*}
\]

leading to \((\tau + 1)Be^x = \tau C\), and hence

\[
\begin{align*}
e^x &= \frac{\tau}{\tau + 1} \frac{C}{B} = \frac{\tau}{\tau + 1} \cdot \left( 1 - \frac{1 + \kappa}{\eta} \right) \cdot d \\
A &= \frac{B}{\tau} e^{(1+\tau)x}
\end{align*}
\]
The solution finally writes \( V(x) = \frac{B}{\tau} e^{-\tau x} + B e^{x-\xi} e^{\xi} - C = B e^{\xi} \{ e^{x-\xi} + e^{-\tau (x-\xi)} \} - C \), i.e.

\[
V(x) = \frac{e^x}{\rho + \delta - s} \cdot \{ e^{x-\xi} + e^{-\tau (x-\xi)} \} - \frac{c}{\rho}
\]

Going back to \( y = e^x \) and re-arranging delivers the expression in Lemma 1.

**A.2.4 Sorting**

Given the bargaining solution and the adjusted surplus, the value of an employer \( z \) in location \( \ell \) then satisfies

\[
\rho J(z, \ell) = (1 - \beta) q(\ell) \ell (b + v(\ell)) S(z, y(\ell))
\]

(A.10)

where I denote

\[
S(z, y) = \int S \left( \frac{y_0}{y} \right) G_0(dy_0 | z)
\]

Under Assumption 1, the integral can be explicitly computed and equation (A.10) becomes

\[
\rho J(z, \ell) = (1 - \beta) q(\ell) \ell (b + v(\ell))^{1+\frac{1}{z}} S_0(z)
\]

(A.11)

where

\[
S_0(z) = \left( \frac{\rho Y}{y_0} \right)^{\frac{1}{z}} \frac{z}{1 - z} \frac{\tau z}{\tau z + 1}
\]

Expressing \( b + v(\ell) = \frac{w(\ell)}{1 - \beta + \beta y_0 / \rho} \), I obtain

\[
\rho J(z, \ell) = \frac{1 - \beta}{1 - \beta + \beta y_0 / \rho} q(\ell) \ell w(\ell)^{1-\frac{1}{z}} (1 - \beta + \beta y_0 / \rho)^{1/z} S_0(z)
\]
Defining

\[ \tilde{S}(z) = \tilde{S}_0(z)(1 - \beta + \beta y_0/\rho)^{1/z} \]
\[ = (Y/w_0)^{\frac{1}{z}} \frac{z}{1 - z \tau z + 1} \]

and raising to a power \( \frac{z}{1 - z} \) delivers (1.11). I now turn to the proof of Proposition 1.

**Proof of Proposition 1**

To make notation lighter, denote \( \zeta = 1/z \). New jobs \( \zeta \) solve

\[
\max_{\ell} \left[ \zeta - 1 \right] \log \frac{1}{b + v(\ell)} + \log (\ell q(\ell))
\]

This is a non-standard assignment problem, where labor costs \( v(\ell) \) enter both in the return to a location and as part of the endogenous price that adjusts to mediate the matching. It is useful to consider the inverse functions \( \ell(v), q(v) \) rather than \( \ell(v) \), and view the problem as

\[
\max_{\ell} \left[ \zeta - 1 \right] \log \frac{1}{b + v} + \log (\ell(v)q(v))
\]

where now \( \ell(v), q(v) \) act as the endogenous price that adjust to sustain the matching. The first part of the objective is decreasing and convex in \( v \), and so we expect \( \ell(v) \) to be increasing in equilibrium.

**Continuum property.** To use first-order conditions (FOC) to characterize the assignment, I first show that a closed interval of \( v \)'s exists in equilibrium. Suppose for a contradiction that there is a “hole” in the distribution of equilibrium \( v \)'s. Denote \( v_1 < v_2 \) the lim-sup before the jump and the lim-inf after the jump. Then firms who locate right above the jump would have an incentive to deviate down because the distribution of \( \ell \) is
continuous. Thus, there can be no “hole” in the distribution of \( v \)'s in equilibrium. In this case, the continuous sorting case is the only relevant one for the interior. Therefore, I can treat \( v \) as a continuous variable in the sorting problem.

**Interval property.** Suppose that there is only one \( v \) in equilibrium. Then all firms would locate at a corner since locations are heterogeneous. This cannot be an equilibrium as house rents would be infinite there.

**Sorting.** Because of the supermodularity between \( \zeta \) and \( \log \frac{1}{b+v} \), the solution must feature a one-to-one assignment function between \( \zeta \) and \( v \). From the second-order condition, the matching function that maps \( \log \frac{1}{b+v} \) to \( \zeta \) must be increasing, so the matching function \( \zeta(v) \) must be decreasing: \( \zeta'(v) < 0 \).

**First-order condition.** The FOC is

\[
-(\zeta(\ell) - 1) \frac{v'(\ell)}{b + v(\ell)} + \frac{1}{\ell} + \frac{q'(\ell)}{q(\ell)} = 0
\]

In what follows, I denote \( \bar{S}(\zeta) = \bar{S}(1/\zeta) \) where \( \zeta = 1/z \). Now, from the definition of the worker’s value,

\[
v(\ell)\ell = \rho^{-1} \beta f(\ell)\ell (b + v(\ell)) \left( \frac{B}{b + v(\ell)} \right)^{\zeta(\ell)} \bar{S}(\zeta(\ell))
\]

(A.12)

\[
= \rho^{-1} \beta m^{\frac{1}{\alpha}} q(\ell)^{-\frac{1}{\alpha}} \cdot \ell q(\ell) (b + v(\ell)) \left( \frac{B}{b + v(\ell)} \right)^{\zeta(\ell)} \bar{S}(\zeta(\ell))
\]

Differentiating this identity and using the FOC for the envelope theorem,

\[
\frac{1}{\ell} + \frac{v'(\ell)}{v(\ell)} = -\frac{1}{\alpha} \frac{q'(\ell)}{q(\ell)} + \left( \frac{\bar{S}'}{\bar{S}} + \log \frac{B}{b + v} \right) \zeta'(\ell)
\]

(A.13)

\(^6\)See Galichon (2016).
Substitute back into the FOC to obtain

\[-\frac{v'(\ell)}{v(\ell)} \left[ \alpha + \frac{v(\ell)}{b + v(\ell)} (\zeta(\ell) - 1) \right] + \frac{1 - \alpha}{\ell} + \alpha \left( \frac{\bar{S}'(\zeta(v))}{\bar{S}(\zeta(v))} + \log \frac{B}{b + v} \right) \zeta'(\ell) = 0 \tag{A.14}\]

Now re-index in terms of \(v\) to use \(\zeta'(v) < 0\):

\[-\frac{1}{v(\ell)(v)} \left[ \alpha + \frac{v}{b + v} (\zeta(v) - 1) \right] + \frac{1 - \alpha}{\ell(v)} + \alpha \left( \frac{\bar{S}'(\zeta(v))}{\bar{S}(\zeta(v))} + \log \frac{B}{b + v} \right) \zeta'(v) = 0\]

and therefore,

\[
\frac{1}{v(\ell)(v)} \left\{ \alpha + \frac{v}{b + v} (\zeta(v) - 1) + \alpha v \left( \frac{\bar{S}'(\zeta(v))}{\bar{S}(\zeta(v))} + \log \frac{B}{b + v} \right) \right\} > \frac{1 - \alpha}{\ell(v)}
\]

When \(\alpha = 0\), the bracket on the left-hand-side is always positive. In this case, \(\ell'(v) > 0\), which implies \(\zeta'(\ell) > 0\): there is positive assortative matching (PAM). Therefore, there exists a region of the parameter space where \(\alpha\) is small and positive assortative matching obtains.\(^7\)

**Generalization: starting productivity distribution**

I now state the general set of assumptions required for positive sorting to obtain in equilibrium.

**Assumption 2. (Initial productivity distribution)**

Assume that

\[
\bullet \quad \frac{\partial \log \bar{S}(z,y)}{\partial y} < \alpha
\]

\[
\bullet \quad \frac{\partial \log \bar{S}(z,y)}{\partial z} > 0
\]

\[
\bullet \quad \frac{\partial^2 \log \bar{S}(z,y)}{\partial y \partial z} > 0
\]

\(^7\)Formally, this statement anticipates that the general equilibrium conditions involve only continuously differentiable fixed point functionals.
These assumptions allow to generalize the sorting results.

**Proposition 8.** (Sorting 2)

Suppose that Assumption 2 holds. Then all the implications of Proposition 1 hold.

**Proof.** The structure of the proof closely follows Appendix A.2.4. The differences to check are supermodularity and the FOC. First, the location choice becomes

$$\max_{\ell} \log(\ell q(\ell)) + \log \frac{y(\ell)S(z, y(\ell))}{\bar{S}(z, y(\ell))}$$  \hspace{1cm} (A.15)

Because \(\bar{S}(z, y)\) is log-supermodular in \((z, y)\), PAM between \(z\) and \(y\) obtains. The FOC is

$$\frac{1}{\ell} + \frac{q'(\ell)}{q(\ell)} + \frac{y'(\ell)}{y(\ell)} \left( 1 + \frac{y(\ell)\bar{S}_y(z(\ell), y(\ell))}{\bar{S}(z(\ell), y(\ell))} \right) = 0$$  \hspace{1cm} (A.16)

Re-arranging the worker’s value of search,

$$q(\ell) \propto \left( \frac{y(\ell) - y_1}{y(\ell)\bar{S}(z(\ell), y(\ell))} \right)^{-\frac{\alpha}{1-\alpha}}$$

where \(y_1 = by_0/\rho\). Thus,

$$\frac{1 - \alpha}{\alpha} q'(\ell) = \frac{y'(\ell)}{y(\ell)} \left( 1 + \frac{y(\ell)\bar{S}_y(z(\ell), y(\ell))}{\bar{S}(z(\ell), y(\ell))} - \frac{y(\ell)}{y(\ell) - y_1} \right) + \frac{\bar{S}_z(z(\ell), y(\ell))}{\bar{S}(z(\ell), y(\ell))} z'(\ell)$$

Substituting into the FOC,

$$0 = \frac{1 - \alpha}{\ell} + \frac{y'(\ell)}{y(\ell)} \left( 1 + \frac{y(\ell)\bar{S}_y(z(\ell), y(\ell))}{\bar{S}(z(\ell), y(\ell))} - \alpha \frac{y(\ell)}{y(\ell) - y_1} \right) + \alpha \frac{\bar{S}_z(z(\ell), y(\ell))}{\bar{S}(z(\ell), y(\ell))} z'(\ell) \hspace{1cm} (A.17)$$

Thus, when \(\alpha = 0\), \(y'(\ell) > 0\), which concludes the proof. \(\square\)
A special case. Suppose that the starting distribution is degenerate at \( y_0 = z > \max_\ell y(\ell) \).

In that case,

\[(1 + \tau)y\bar{S}(z, y) = \tau z + y^{1+\tau}z^{-\tau} - (1 + \tau)y\]

Then

\[
\frac{\partial \log y\bar{S}(z, y)}{\partial z} = \tau \frac{1 - z^{-\tau-1}y^{1+\tau}}{y\bar{S}(z, y)} > 0
\]

\[
\frac{\partial \log y\bar{S}(z, y)}{\partial y} = \frac{1 + \tau}{y} \frac{(y/z)^\tau - 1}{z/y + (y/z)^\tau - 1 - \tau} < 0 < \alpha
\]

In addition, it can be shown that this last expression is also increasing in \( z \) on some interval \([K(y(\ell)), +\infty)\). So in that region, \( y\bar{S}(z, y) \) is log-supermodular in \((y, z)\).

Generalization: dynamic stability

In this section, I define a notion of dynamic stability of steady-states to rule out steady-states with negative assortative matching (NAM).

**Definition 1. (Dynamically stable assignment)**

A dynamically stable assignment is a pair of functions \( A : \ell \mapsto (z(\ell), y(\ell)) \) such that (a) \( A \) solves the job location problem (1.12) and (b) \( A \) is the steady-state assignment that arises starting from a uniform assignment, and letting of jobs choose their location at Poisson rate \( R \), in the limit where \( R \to 0 \).

Definition 1 proposes a natural restriction on the set of possible equilibria that may arise. Starting from a uniform assignment of jobs to locations, the equilibrium must be attainable as jobs are slowly allowed to relocate over time. This apparently mild restriction suffices to eliminate potential coordination failures, a common source of mutliplicity in assignment.
problems with agglomeration economies, whose role is played here by the general equilibrium feedback of labor market tightness into employers’ payoffs.\(^8\)

**Proposition 9.** *(Sorting)*

*Under Assumption 2, conditional on the mass of entrants \(M_e\) and the value of unemployment \(U\), there exists a unique globally stable assignment function for job quality \(z(\ell)\) and a unique local cutoff function \(y(\ell)\). \(z\) and \(y\) are strictly increasing functions.*

**Proof.** First, the limit \(R \to 0\) ensures that steady-state values are sufficient to characterize employers’ values: all employers exit with probability one before \(R\) changes sufficiently to affect the values. The proof proceeds by “continuous induction” – formally, I show that the set of times \(T\) such that weak PAM obtains is a non-empty closed and open subset of \(\mathbb{R}_+\), which then implies that it can only be \(\mathbb{R}_+\). First, note that \(T\) is characterized by a weak inequality \(y' \geq 0\). Thus, it is a closed set.

**Initialization.** Consider time 0 at which employers are randomly allocated. For the fraction \(\mathcal{R} dt\) of employers who can choose their location, the location FOC is equation (A.16), but where \(q'/q = 0\). Therefore, \(y' > 0\) immediately follows at time 0, and so \(0 \in T\).

**Recursion.** Let \(t\) be the least upper bound of \(T\). The location FOC for employers allowed to relocate at \(t\) is (A.17), where by definition of \(T\), \(z' \geq 0\) at \(t\). It then immediately follows that \(y' \geq 0\) for a small time interval \([t, t + \varepsilon)\).

Thus, \(T\) is both open and closed in \(\mathbb{R}_+\), and is nonempty. Thus, it is \(\mathbb{R}_+\).

---

\(^8\)Exogenous differences across locations \(\ell\) create incentives for jobs to sort, but so do endogenous differences in the vacancy contact rate \(q(\ell)\). When exogenous differences in productivity \(\ell\) are small, starting from an assignment where jobs are perfectly sorted but in reverse order relative to \(\ell\) may still generate large enough differences in the vacancy contact rate \(q(\ell)\) to sustain that assignment. Jobs’ location choices would thus result in a spatial coordination failure, as aggregate output would be depressed relative to the best possible self-sustaining assignment. While examining these outcomes may be interesting per se, they are not the subject of the present paper. An alternative restriction would be to simply pick the output-maximizing self-sustaining assignment.
A.2.5 Endogenous job loss and unemployment

**Intuition for the KFE (1.13).** To provide some intuition for the KFE (1.13), consider a point \( y > x \). In a small time interval \( dt \), a mass of \((\sigma^2/2 - \delta)y \cdot g(y)dy \cdot dt\) flows out from the interval \([y, y + dy]\) because of the drift. However, a mass \((\sigma^2/2 - \delta)(y - dy) \cdot g(y - dy)dy \cdot dt\) flows in from below when the drift is positive \(\sigma^2/2 - \delta > 0\). In net, the change in mass is \((\sigma^2/2 - \delta)\left[ (y - dy)g(y - dy) - yg(y) \right] \cdot dt\). Divide by \(dy\) to recover a density, and by \(dt\) to recover a change per unit of time, and take \(dy \to 0\) to obtain the first term in the KFE.

**KFE bound**

**Derivation of the KFE bound.** First consider an intuitive version of the proof. Consider a second-order time interval \((dt)^2\). The change in log productivity is \(d^2 \log z_t \approx \sigma dt N\) where \(N\) is a standard normal variable. Thus, half of the workers at the cutoff \(y\) are thrown below the cutoff \(y\) and into unemployment in an interval \((dt)^2\). Starting from \(g(y)\) workers at the cutoff, only a fraction \(2^{-\left\lfloor \frac{1}{2} \right\rfloor}\) of those workers remain there after a time \(dt\). Taking \(dt \to 0\), this fraction must be zero. I now make this intuition precise.

**Proof.** Denote \(x = \log y\), and \(\underline{x} = \log \underline{y}\). Omit \(\ell\) indices for clarity. Let \(f\) be the local invariant density function. Consider the interval \([\underline{x}, \underline{x} + dx]\). The gross flows in and out of this interval between times \(t\) and \(t + dt\) are:

\[
\text{Inflow} = \int_{\underline{x} + dx}^{\infty} f(z) \mathbb{P}[x \leq z + dW_t \leq x + dx]dz = \int_{dx}^{\infty} f(x + y) \mathbb{P}[-y \leq dW_t \leq -y + dx]dy
\]

\[
\text{Outflow} = \int_{\underline{x}}^{\underline{x} + dx} f(z) \mathbb{P}[z + dW_t > \underline{x} + dx \text{ or } z + dW_t < \underline{x}]dz
\]
\[
= \int_{0}^{dx} f(x + y) \left\{ \mathbb{P}[y + dW_t < 0] + \mathbb{P}[y + dW_t > dx] \right\} dy
\]
Then, denoting by $\Phi$ the cumulative distribution function of a standard normal variable,

$$
\text{Net flow}(dx, dt) = -\int_0^x f(x + y) dy + \int_0^\infty f(x + y) \left\{ \Phi \left( \frac{-y + dx}{\sigma \sqrt{dt}} \right) - \Phi \left( \frac{-y}{\sigma \sqrt{dt}} \right) \right\} dy
$$

Then:

$$
\frac{\partial f}{\partial t}(x) = \frac{1}{dx dt} \text{Net flow}(dx, dt)
$$

$$
= -\frac{1}{dx dt} \int_0^x f(x + y) dy + \frac{1}{dx dt} \int_0^\infty f(x + y) \left\{ \Phi \left( \frac{-y + dx}{\sigma \sqrt{dt}} \right) - \Phi \left( \frac{-y}{\sigma \sqrt{dt}} \right) \right\} dy
$$

$$
= -\frac{1}{dt} \int_0^1 f(x + zdx) dz + \frac{1}{dt} \int_0^\infty f(x + zdx) \left\{ \Phi \left( (1 - z)\lambda \right) - \Phi \left( -\lambda z \right) \right\} dz
$$

where $\lambda = \frac{dx}{\sigma \sqrt{dt}}$. Now,

$$
\frac{1}{dt} \int_0^1 f(x + zdx) dz \approx_{dx \ll 1} \frac{f(x)}{dt} + \frac{f'(x)dx}{2dt} + \frac{f''(x)dx^2}{6dt} + \mathcal{O}(dx^3/dt)
$$

So is left to calculate:

$$
\int_0^\infty f(x + zdx) \left\{ \Phi \left( (1 - z)\lambda \right) - \Phi \left( -\lambda z \right) \right\} dz
$$

In integral form and changing variables:

$$
\Phi \left( (1 - z)\lambda \right) - \Phi \left( -\lambda z \right) = \Phi(z\lambda) - \Phi(z\lambda - \lambda) = \int_0^\lambda \varphi(z\lambda - y) dy
$$

where $\varphi$ here denotes the standard normal density function. Then, after some algebra

$$
\int_0^\infty f(x + zdx) \left\{ \Phi \left( (1 - z)\lambda \right) - \Phi \left( -\lambda z \right) \right\} dz = \frac{1}{dx} \int_R dz \varphi(z) \int_{z\sigma \sqrt{dt} + dx}^{z\sigma \sqrt{dt} + dx} 1[y \geq 0] f(x + y) dy
$$
Now, \( \int_a^{a+\varepsilon} f(y)dy \approx f(a)\varepsilon + f'(a)\frac{\varepsilon^2}{2} + \frac{1}{2} f''(a)\frac{\varepsilon^3}{6} + \mathcal{O}(\varepsilon^4) \), and \( \int_{\delta}^{\delta+\varepsilon} f(x+y)dy \approx f(x)\varepsilon + \frac{1}{2} f'(x)\varepsilon(2\delta + \varepsilon) + \frac{1}{6} f''(x)\varepsilon[3\delta^2 + 3\delta\varepsilon + \varepsilon^2] + \ldots \). So

\[
\frac{1}{dx} \int_{\mathbb{R}} dz \varphi(z) \int_{z\varepsilon}^{z\varepsilon + dx} f(x+y)dy = \frac{1}{dx} \int_{0}^{\infty} dz \varphi(z) \int_{z\varepsilon}^{z\varepsilon + dx} f(x+y)dy = (\approx A)
\]

\[
\quad + \frac{1}{dx} \int_{-\lambda}^{0} dz \varphi(z) \int_{0}^{z\varepsilon + dx} f(x+y)dy = (\approx B)
\]

Then:

\[
A \approx \int_{0}^{\infty} dz \varphi(z) \left\{ f(x) + f'(x)dx \left(\frac{2z}{x} + 1\right) + \frac{1}{6} f''(x)dx^2 \left[1 + 3 \left(\frac{z}{x}\right)^2 + 3\frac{z^3}{x}\right]\right\}
\]

Similarly,

\[
B \approx_{\lambda \to +\infty} A_{-\infty}^0 + \frac{f(x)}{\lambda} \int_{-\infty}^{0} \varphi(z)z\,dz + \mathcal{O}(\lambda^{-2})
\]

and so

\[
A + B = f(x) + f'(x)dx + \frac{f''(x)dx^2}{6} - \frac{f(x)}{\lambda\sqrt{2\pi}} + \mathcal{O}(\lambda^{-2} + \ldots)
\]

Thus,

\[
\frac{\partial f}{\partial t}(x) = -\frac{f(x)}{dt\lambda\sqrt{2\pi}} + o(1)
\]

Now, \( \lambda \to \infty \) but \( dx \to 0 \). So \( \lambda dt \sim dxdt^{1/2} \to 0 \). This implies:

\[
\frac{\partial f}{\partial t}(x) = -\infty
\]

and thus

\[
f(x, t) = 0
\]
for all times \( t > 0 \).

**KFE and job losing rates in Lemma 2 and Proposition 2.**

**Pareto case: Proof of Lemma 2** Consider a single location \( \ell \) and omit location subscripts \( \ell \). Thus, in logs \( x = \log y \), the entry mass function is \( g_0(x) = g_0 e^{-\zeta(x-x)} \) where \( \zeta = 1/z \).

Denote \( g \) the invariant entry density, and \( h(x) = g'(x) \). Then the KFE becomes

\[
0 = \delta g(x) + s g'(x) + g_0 e^{-\zeta(x-x)}
\]

The homogeneous solution is \( h_H(x) = Ae^{-\kappa(x-x)} \). Varying the constant, I obtain

\[
sA'(x)e^{-\kappa(x-x)} + g_0 e^{-\zeta(x-x)} = 0
\]

and so

\[
A(x) = A_0 - \frac{g_0}{s} \int_0^{x-x} e^{(\kappa-\zeta)t} dt
\]

\[
= A_0 - \frac{ng_0}{s(\kappa-\zeta)} e^{(\kappa-\zeta)(x-x)}
\]

Therefore,

\[
g'(x) = A_0 e^{-\kappa(x-x)} - \frac{g_0}{s(\kappa-\zeta)} e^{-\zeta(x-x)}
\]

Given the integrability condition for \( g \), the integration constants must cancel out, and

\[
g(x) = B e^{-\kappa(x-x)} + \frac{g_0}{s\zeta(\kappa-\zeta)} e^{-\zeta(x-x)}
\]

\( f(x) = 0 \) then pins down \( B \), so that

\[
g(x) = \frac{g_0}{s\zeta(\kappa-\zeta)} \left[ e^{-\zeta(x-x)} - e^{-\kappa(x-x)} \right] \geq 0
\]
The separation flow is $sg'(x)$, which is given by

$$\varepsilon = \frac{g_0}{\zeta}$$

Normalizing $g$ to 1 pins down $g_0$ which otherwise simply scales with the total mass of employed workers. This allows to compute the exit rate. Normalizing $g$ to 1 yields $1 = \frac{g_0}{s\zeta(\kappa - \zeta)} \cdot \frac{\kappa - \zeta}{\zeta \kappa} = \frac{g_0}{s\zeta^2 \kappa}$. Thus, the invariant density function is

$$f(x) = \frac{\zeta \kappa}{\kappa - \zeta} \left[ e^{-\zeta(x - \bar{z})} - e^{-\kappa(x - \bar{z})} \right]$$

and the exit rate is

$$\varepsilon = s\zeta \kappa = \delta \zeta$$

To express job finding, it suffices to use the definition of workers' value of search. Under Assumption 1, they follow equation (A.12). The realized finding rate is thus

$$f_R(\ell) = f(\ell) \left( \frac{B}{b + v(\ell)} \right)^{1/z(\ell)} = \frac{\rho v(\ell)}{\beta (b + v(\ell)) S(z(\ell))} \equiv \Phi_R(v(\ell), 1/z(\ell))$$

Substituting in the definition of reservation wages delivers the expression for $f_R$ in Proposition 2, with $w_1 = b \left( 1 - \beta + \beta y_0 / \rho \right)$. The expression for the unemployment rate follows from the usual accounting equation. Under Assumption 2, re-arranging the worker's value of search yields

$$f_R(\ell) = \frac{\rho}{\beta b + v(\ell)} \cdot \frac{1 - G_0(y(\ell)|z(\ell))}{S(z(\ell), y(\ell))}$$

**General case: Proof of Lemma 10** The KFE now becomes

$$0 = \delta g'(x) + sg''(x) + g_0(x)$$
The homogeneous solution is the same as before. As before, I vary the constant and look for a solution \( f'(x) = A(x)e^{-\kappa(x-x)} \), so that

\[
A'(x) = -g_0(x)e^{-\kappa(x-x)}
\]

Thus,

\[
A(x) = A_0 + \int_x^\infty g_0(y)e^{\kappa(y-x)}dy
\]

and hence

\[
g'(x) = A_0e^{-\kappa(x-x)} + e^{-\kappa(x-x)} \int_x^\infty g_0(y)e^{\kappa(y-x)}dy
\]

Integrating once more:

\[
g(x) = A + Be^{-\kappa(x-x)} - \int_x^\infty dy e^{-\kappa(y-x)} \int_y^\infty g_0(z)e^{\kappa(z-x)}dz
\]

Integrability imposes \( A = 0 \). \( B \) is determined by \( g(x) = 0 \):

\[
B = \frac{1}{\kappa} \int_x^\infty g_0(y)[e^{\kappa(y-x)} - 1]dy
\]

As before, the total mass of new jobs simply scales the invariant mass distribution. The separation flow is \( sg'(x) \), where

\[
g'(x) = -\kappa B + \int_x^\infty g_0(y)e^{\kappa(y-x)}dy = \int_x^\infty g_0(y)dy
\]
To get the separation rate, normalize $g$ to 1. Denote by $H_0 = \int_{-\infty}^{\infty} g_0(y) dy$ the mass of newly created new jobs and $h_0 = g_0 / H_0$ the entry density of new jobs. Using the expression for $g$ above,

$$\frac{\kappa}{H_0} = \int_{-\infty}^{\infty} e^{-\kappa(x-z)} \int_{z}^{x} e^{\kappa(y-z)} h_0(y) dy - \frac{1}{\kappa} \int_{z}^{\infty} \int_{x}^{\infty} h_0(y) dy = \int_{z}^{\infty} x h_0(x) dx$$

Therefore the separate rate is

$$\frac{\delta}{E h_0 [\log(y/y)]}$$

These arguments prove the following Lemma.

**Lemma 10.** *Employment distribution*

Denote by $g_0(y_0|z(\ell))$ the density function of successful new jobs. Then the invariant distribution $g$ in location $\ell$ is

$$g(y, \ell) = B(\ell) \left( \frac{y}{y(\ell)} \right)^{-\kappa} - \frac{1}{\kappa} \int_{y}^{\infty} g_0(y'|z(\ell)) \left( \frac{y'}{y(\ell)} \right)^{\kappa} \frac{dy'}{y'}$$

where $B(\ell) = \frac{1}{\kappa} \int_{y(\ell)}^{\infty} g_0(y'|z(\ell)) \left( \frac{y'}{y(\ell)} \right)^{\kappa} \frac{dy'}{y'}$, and the job losing rate is

$$s(\ell) = \frac{\delta}{\int_{y(\ell)}^{\infty} \left( \log \frac{y'}{y(\ell)} \right) dy'}$$

**A.2.6 Proof of Proposition 3**

**Pareto case**

Impose Assumption 1 and consider dynamically stable steady-states. Then, PAM obtains.

Denote again $\zeta = 1/z$. Because of PAM, labor market clearing in location $\ell$ writes

$$\theta(\ell) = -\frac{M e f \zeta(\ell) \zeta'(\ell)}{u(\ell) L(\ell) f_\ell(\ell)}$$
and so

\[ M_e f_\zeta(\zeta(\ell)) \zeta'(\ell) = -L(\ell)u(\ell)\theta(\ell)f_\ell(\ell) \]

Using the expression of the finding rate in Proposition 2, re-express labor market tightness as a function of \( v, \zeta \):

\[
\theta(\ell) = \left[ \frac{\rho}{b + v(\ell)} \left( \frac{B}{b + v(\ell)} \right)^{\zeta(\ell)} \right] \frac{1}{1 - \alpha} \\
\equiv \Theta(v(\ell), \zeta(\ell))
\]

In what follows, it is useful to define the notation for the local unemployment rate

\[
u(v(\ell), \zeta(\ell)) = \frac{\delta \zeta(\ell)}{\delta \zeta(\ell) + \Phi_R(v(\ell), \zeta(\ell))}
\]

Land market clearing writes in each location

\[
r(\ell) = \omega L(\ell) \left[ b u(v(\ell), \zeta(\ell)) + (1 - u(v(\ell), \zeta(\ell)))(b + v(\ell))(1 - \beta + \beta \mathcal{E}(\zeta(\ell))) \right]
\]

where \( \mathcal{E}(\zeta) = \frac{y_0}{\rho \kappa / (\kappa - 1)} \) is expected productivity under the invariant distribution from Lemma 2. Substituting into workers’ free mobility condition \( \rho U = \frac{\ell (b + v(\ell))}{r(\ell)^2} \), one can express population as

\[
L(\ell) = U^{-\frac{1}{2}} \tilde{L}(\ell, v(\ell), \zeta(\ell)) \tag{A.18}
\]

with

\[
\tilde{L}(\ell, v(\ell), \zeta(\ell)) = \frac{1}{\omega \rho^{\frac{1}{2}} \nu(v(\ell), \zeta(\ell)) + \left(1 - u(v(\ell), \zeta(\ell))\right)(b + v(\ell))(1 - \beta + \beta \mathcal{E}(\zeta(\ell)))} \]
Substitute back into labor market clearing:

\[ K\zeta'(\ell) = -\frac{f_\ell(\ell)\bar{L}(\ell, v(\ell), \zeta(\ell))\Theta(v(\ell), \zeta(\ell))}{\bar{f}_\zeta(\zeta(\ell))} \tag{A.19} \]

where \( K = U^\frac{1}{2} M_e \) is a combined general equilibrium constant, and

\[
\bar{L}(\ell, v(\ell), \zeta(\ell)) = \bar{L}(\ell, v(\ell), \zeta(\ell))u(v(\ell), \zeta(\ell)) = \frac{1}{\omega^{\frac{1}{2}} b} + \frac{\ell^{\frac{1}{2}} - 1}{\beta \delta(\ell) S(\zeta(\ell))} \left((1 - \beta) + \beta E(\zeta(\ell))\right)
\]

Substituting into the FOC for \( v \):

\[
\frac{v'(\ell)}{v(\ell)} \left[ \alpha + \frac{v(\ell)}{b + v(\ell)}(\zeta(\ell) - 1) \right] = 1 - \frac{1}{\ell} - \frac{1}{K} \times \alpha \left( \frac{\bar{S}'(\zeta(\ell))}{\bar{S}(\zeta(\ell))} \right) + \log \frac{B}{b + v(\ell)} \frac{\bar{L}(\ell, v(\ell), \zeta(\ell))\Theta(v(\ell), \zeta(\ell))}{\bar{f}_\zeta(\zeta(\ell))} \tag{A.20}
\]

Given \( K \), equations (A.19)-(A.20) define a coupled system of ODEs, with two boundary conditions:

\[
\zeta(\ell) = \bar{\zeta} \quad ; \quad \zeta(\bar{\ell}) = \zeta
\]

Inspection of (A.19)-(A.20) indicate that the system satisfies standard regularity conditions for a unique solution to obtain if it has two initial conditions. The present system, however, has one initial and one terminal condition.

**Existence of a solution to the ODE system given \( K \).** Denote \( \underline{v} = v(\ell) \). Given \( K \), inspection of (A.19)-(A.20) reveals that the system is Lipschitz continuous. Given \( \underline{v}, \zeta \) and \( K \), there thus exists a unique solution to (A.19)-(A.20). The idea is now to study how changes in \( \underline{v} \) affect \( \zeta(\bar{\ell}) \) in the solution to that system. Lipschitz continuity ensures that \( \zeta(\bar{\ell}) \) is a continuous function of \( \underline{v} \). Further inspection of (A.19)-(A.20) reveals that as \( v \to 0 \), so
do \( Z, V \). Similarly, as \( v \to +\infty \), so do \( Z, V \). Therefore, the same conclusion holds when \( v \to 0 \) or \( v \to +\infty \). Hence, there exists at least one \( v(K) \) such that \( \zeta(\bar{\ell}) = \zeta \).

**Existence of \( K \).** The equilibrium has a block-recursive structure. Free-entry alone is enough to determine \( K \) without using population adding up. Given \( K \) and thus the solution \((b, \zeta)\), population adding-up immediately determines \( U \) as per (A.18). Thus, it suffices to show that free-entry implies existence of \( K \). Free-entry can be re-written

\[
K \cdot c_e = J_0 \int \left[ B^{z(\ell)}(b + v(\ell))^{1-\zeta(\ell)} v(\ell)^{-\alpha} \tilde{S}(\zeta(\ell)) \right]^{\frac{1}{1-\alpha}} \cdot \ell \cdot \tilde{L}(\ell) u(\ell) \theta(\ell) d\ell
\]

As \( K \to 0 \), (A.19) together with the boundary conditions on \( \zeta \) and an application of Rolle’s theorem to \( \zeta'(\ell) \) implies \( v(K)^{\frac{1}{1-\alpha}} \sim K \to 0 \). As \( K \to +\infty \), a similar argument implies \( v(K)^{\frac{1}{1-\alpha}} \sim K \to +\infty \), where \( \zeta_0 \in [\zeta, \bar{\zeta}] \). Thus, the right-hand-side integral of free-entry is of order \( K^{1-\alpha} \) as \( K \to 0 \), and is of order \( K^{\frac{1}{\zeta_0 + 1 - \omega}} \) as \( K \to +\infty \). Since \( \zeta > 1 \) by assumption, \( \frac{1}{\zeta_0 + 1 - \omega} < 1 \). Therefore, there exists at least one solution \( K \) to the free-entry condition.

**Uniqueness.** Now suppose that the supports of \( F_\ell, F_z \) are small enough. This assumption allows to use a first-order approximation to the ODE system (A.19)-(A.20). In that case, to a first order,

\[
K \zeta'(\ell) \approx -\frac{\tilde{L}(\ell, v, \bar{\zeta}) \Theta(v, \bar{\zeta})}{f_\zeta(\bar{\zeta})} = -L_0 \frac{v^{\frac{1}{1-\alpha}}}{1 + L_1 v} (b + v)^{\frac{1}{\omega} + \frac{\zeta - 1}{1-\alpha}}
\]

where \( L_0, L_1 > 0 \) are transformations of parameters. Integrating, it implies

\[
K = L'_0 \frac{v^{\frac{1}{1-\alpha}}}{1 + L_1 v} (b + v)^{\frac{1}{\omega} + \frac{\zeta - 1}{1-\alpha}}
\]
where \( L'_0 = L_0 \frac{\overline{\zeta} - \zeta}{\zeta - \overline{\zeta}} \) only depends on parameters. Similarly, free entry can be approximated to a first order by

\[
K = J''_0 \frac{v}{v + 1/L_1}(b + v)^{\frac{1}{\alpha}}
\]

(A.23)

where \( J''_0, J_1 \) depend only on parameters. Substituting (A.23) into (A.22), one obtains

\[
1 = L''_0 \overline{\zeta}^{\frac{\alpha}{1-\alpha}}(b + v)^{\frac{\overline{\zeta} - 1}{1-\alpha}}
\]

(A.24)

where \( L''_0 \) depends only on parameters. The right-hand-side of (A.24) is strictly increasing in \( v \), and so (A.24) uniquely pins down \( v \). Then (A.23) uniquely pins down \( K \). Then \( \int L(\ell)F_\ell(d\ell) \) uniquely pins down \( U \).

**Non-Pareto case**

When the entry distribution satisfies Assumption 2, all the previous arguments continue to hold. They only involve additional notation. Hence, I omit them for brevity.

### A.2.7 Limiting economies

**Proof of Corollary 1**

This limiting economy preserve a wide support for \( F_z \) but considers the limit of a small support for \( F_\ell \). In that case, it is more useful to index locations by their value of search \( v \) rather than productivity \( \ell \).

Shrinking the support of \( F_\ell \) implies \( \ell'(v) = 0 \). Thus, the FOC (A.15) implies

\[
\alpha + \frac{v}{b + v}(\zeta(v) - 1) + \alpha v \left( \frac{S'(\zeta(v))}{S(\zeta(v))} + \log \frac{B}{b + v} \right) (-\zeta'(v)) = 0
\]

(A.25)
which defines a non-degenerate assignment $\zeta(v)$ in the limit. Given the boundary conditions, it must be that there is an interval of $v$’s in the limit. The assignment $\zeta(v)$ implies non-vanishing dispersion in job losing and unemployment rates.

**Vanishing search frictions**

Suppose that labor market frictions become small. Should cities be expected to have similar labor market flows? Corollary 2 below shows that cities remain different even with small labor market frictions.

**Corollary 2.** *(Vanishing spatial differences with vanishing search friction)*

Suppose that the conditions in Proposition 3 hold and that $\alpha > 0$. The variance of local job losing and unemployment rates remain strictly positive and bounded above zero as the matching function efficiency becomes large ($m \to +\infty$).

This result highlights that taking into account even small labor market frictions leads to substantial departures from a model that would not feature frictional unemployment. Spatial unemployment differentials survive when search frictions become small because reservation wages rise everywhere as meeting with new employers becomes easier. This adjustment offsets the direct unemployment benefits of higher contact rates.\(^9\)

*Proof.* Consider the limit $m \to +\infty$. Denote $K(m) = m^{\frac{1}{1-\alpha}} M(\rho U)^{\frac{1}{2}}$ and $\theta_0(\ell) = \theta(\ell) m^{\frac{1}{1-\alpha}}$. I now distinguish three cases for the candidate limit as $m \to +\infty$. Impose Assumption 1 for now.

- Suppose that $v \sim 1$ remains finite.

  Then $f_R, u, \hat{L} \sim 1$ and so free-entry ensures that $K(m) \sim 1$.

- Suppose that $v \to +\infty$.

\(^9\)See Bilal et al. (2019b) and Martellini and Menzio (2018) for related results.
Then $f_R, u \sim 1$. Then $\hat{L} \sim v^{\frac{1}{2}} \rightarrow +\infty$. Free-entry then implies $K(m) \sim ||v||^{\frac{1}{2}} \rightarrow +\infty$. Labor market clearing requires $||v||^{\frac{1}{2}} |\zeta'| \sim ||v||^{\frac{1}{2} + \frac{1}{2\alpha - 1}}$ and so $|\zeta'| \rightarrow \infty$, violating the boundary conditions for $\zeta$.

- Suppose that $v \rightarrow 0$. Then $\hat{L} \sim 1$, and from free-entry $K(m) \rightarrow 0$. From population adding up, $U \sim 1$. The definition of $K(m)$ then implies $M_e \rightarrow 0$. This cannot be an equilibrium.

Therefore, in the limit $m \rightarrow +\infty$, $v$ remains finite and the assignment is non-degenerate. The same arguments hold under Assumption 2.

\[\square\]

A.2.8 Planning solution

Optimality conditions

The planner solves chooses the number of unemployed workers $U(t, \ell)$ to locate in each city $\ell$ at time $t$, the rate at which to break up existing matches $\Delta(t, y, \ell)$. The planner also chooses the consumption $c_U(t, \ell), c_E(t, y, \ell), h_U(t, \ell), h_E(t, y, \ell)$ of employed and unemployed workers, as well as the consumption of the owners $C(t)$. For simplicity, I assume that unemployed workers produce $b\ell$ at home. I anticipate that the planner chooses PAM, so that it suffices to let the planner choose the matching function $\zeta(t, \ell)$ together with its slope $\xi(t, \ell)$. I assume for simplicity that the planner cannot transfer resources across time periods. Because I will focus on steady-states, this assumption is without loss of generality.

I denote by $\lambda(\ell)$ the planner’s weight on individuals who live in location $\ell$. Due to complementaries between housing and final good consumption in the utility function, the spatial redistribution of the final good is not neutral. Only one particular set of weights implements an allocation that resembles the decentralized equilibrium, which will be the focus of this paper.\(^{10}\)

\(^{10}\)An alternative assumption to choosing one particular set of weights is that the planner has to provide consumption to workers with locally produced final goods.
The planner’s objective is then

\[
W = \int_0^\infty dt e^{-\rho t} \int d\ell f_\ell(\ell) \lambda(\ell) \left\{ U(t, \ell) \left( \frac{c_U(t, \ell)}{1 - \omega} \right)^{1-\omega} \left( \frac{h_U(t, \ell)}{\omega} \right)^\omega \right. \\
+ \left. \int \mathcal{E}(t, y, \ell) \left( \frac{c_E(t, \ell)}{1 - \omega} \right)^{1-\omega} \left( \frac{h_E(t, \ell)}{\omega} \right)^\omega dy \right\} \\
+ \int_0^\infty e^{-\rho t} C(t) dt
\]

where \( \mathcal{E} \) denotes the mass distribution of employment across productivity \( y \) in location \( \ell \) at time \( t \). The last term is the welfare of the owners. The planner is subject to the constraints

\[
\forall t, 1 = \int d\ell f_\ell(\ell) \left\{ U(t, \ell) + \int \mathcal{E}(t, y, \ell) dy \right\}
\]

\[
\forall t, \ell, 1 = U(t, \ell) h_U(t, \ell) + \int \mathcal{E}(t, y, \ell) h_E(t, y, \ell) dy
\]

\[
\forall t, \ell 0 = \int f_\ell(\ell) \left\{ U(t, \ell) \left( b_\ell - c_U(t, \ell) \right) + \int \mathcal{E}(t, y, \ell) \left( y_\ell - c_E(t, y, \ell) \right) dy \right\}
- C(t) - c_e M_e(t)
\]

\[
\forall y, \ell, t, \frac{\partial \mathcal{E}(t, y, \ell)}{\partial t} = L^*_y \mathcal{E}(t, y, \ell) + n(M_e(t), \xi(t, \ell), U(t, \ell)) g_0(y, \zeta(t, \ell)) - \Delta(t, y, \ell) \mathcal{E}(t, y, \ell)
\]

\[
\forall t, \zeta(t, \ell) = \bar{\zeta}
\]

\[
\forall t, \zeta(t, \ell) = \zeta
\]

\[
\forall t, \int_\ell^\ell \xi(t, x) dx = 1 - F_\xi(\zeta(t, \ell))
\]

\[
\forall t n(M(t), \xi(t, \ell), U(t, \ell)) = m \left( M(t) \xi(t, \ell) \right)^{1-\alpha} U(t, \ell)\alpha
\]

The first constraint simply states that total population is one in the economy. The second constraint clears the land market in each location. The third constraints is the planner’s aggregate resource constraint. The fourth constraint is the time-dependent KFE that encodes how the distribution of employment across productivity evolves over time. The fifth and sixth constraints are the boundary conditions for the assignment function, i.e. the location choice of jobs. The seventh constraint is simply the definition of \( \xi \), which is the slope of the
assignment function that enters into labor market tightness. The eighth constraint simply states the matching function.

The structure of the planning problem is standard. The only non-standard element is that the planner controls a full distribution of workers in each location. This distribution $\mathcal{E}$ is an infinite-dimensional object. To use standard convex optimization methods described in Luenberger (1997), some regularity conditions must be imposed on the functional space in which the distribution $\mathcal{E}$ is allowed to lie. I build on ideas developed in Moll and Nuño (2018), who propose functional spaces for such cases. There are several differences between their approach and the one in this paper. First, their results do not directly apply because of the endogenous separation margin and I must start from first principles. Second, their method in fact requires further restrictions on the functional spaces that those they outline. They propose to use square integrable functions of time and other states (section 2.1.2 p. 154). This restriction is in fact not quite sufficient for their Theorem 2 p. 168 to obtain. The reason is that Luenberger (1997)’s Theorem 1 p. 243 that they refer to also requires that the transition operator that encodes the evolution equation of the state, maps into a Banach space. Yet, there is in general no guarantee that a functional operator like a continuous-time transition operator $L_y^*$ maps the space of square-integrable functions into a Banach space. For it to map into a Banach space, the functional space in which the distribution lies must be further restricted.

It suffices to impose that the distribution $\mathcal{E}$ lies in a Sobolev-Strichartz space, which is a variant of Sobolev spaces:

$$H^{1,2} \equiv \left\{ \mathcal{E} : \text{for all } g \text{ among } \mathcal{E}, \text{ its first } t, y, \ell-weak derivatives, \right.$$  

and second $y, \ell$-weak derivatives,

$$\int_0^\infty e^{-pt} \left( \int \int |\mathcal{E}(t, y, \ell)|^2 dy d\ell \right) dt < \infty \right\}$$

---

11I thank Ben Moll and Galo Nuño for related discussions.
Sobolev-Strichartz spaces are useful precisely because infinitesimal generators such as $L^*_y$ map Sobolev-Strichartz spaces into Lebesgue space (see Tao, 2006).

Finally, the approach I use builds on duality methods similar to Moll and Nuño (2018). However, these duality methods apply without loss of generality in my setup because the distribution endogenously satisfies the boundary condition $\mathcal{E} = 0$ at the lower point of the support. In contrast, Moll and Nuño (2018) apply duality methods for general distributions. However, when the distribution does not vanish at the edge of bounded supports, additional terms should appear in their results.

I am now ready to formulate a current-value Hamiltonian (which is equivalent to a Lagrangian):

$$
H = \int d\ell f_\ell(\ell) \chi(\ell) \left\{ \mathcal{U}(t, \ell) \left( \frac{c_U(t, \ell)}{1 - \omega} \right)^{1-\omega} \left( \frac{h_U(t, \ell)}{\omega} \right)^{\omega} + \int \mathcal{E}(t, y, \ell) \left( \frac{c_E(t, \ell)}{1 - \omega} \right)^{1-\omega} \left( \frac{h_E(t, \ell)}{\omega} \right)^{\omega} dy \right\} - c_e M_e(t) + \int f_\ell(\ell) \left\{ \mathcal{U}(t, \ell) \left( b\ell - c_U(t, \ell) \right) + \int \mathcal{E}(t, y, \ell) \mathcal{U}(t, \ell) \left( y\ell - c_E(t, y, \ell) \right) \right\} \\
+ \int d\zeta d\ell f_\ell(\ell) \left[ \mathcal{E}(t, y, \ell) \mathcal{L} S(t, y, \ell) + n(M(t), \xi(t, \ell), \mathcal{U}(t, \ell)) g_0(y, \zeta(t, \ell)) S(t, y, \ell) - \Delta(t, z) \mathcal{E}(t, z, \ell) S(t, y, \ell) \right] \\
+ \rho U(t) \left[ 1 - \int d\ell f_\ell(\ell) \left( \mathcal{U}(t, \ell) + \int \mathcal{E}(t, y, \ell) dy \right) \right] \\
+ \int \left( \partial_t \pi(t, \ell) \right) F_\xi(\zeta(t, \ell)) d\ell - \int \xi(t, \ell) \pi(t, \ell) \\
+ \tau(t) \left[ 1 - \int \xi(t, \ell) d\ell \right] \\
+ \int d\ell f_\ell(\ell) r(t, \ell) \left\{ 1 - \mathcal{U}(t, \ell) h_U(t, \ell) - \int \mathcal{E}(t, y, \ell) h_E(t, y, \ell) dy \right\}
$$

where I have substituted out the consumption of owners using the aggregate budget constraint. I have integrated by parts the $\xi$ constraint with multiplier $A$, and denoted $\pi(t, \ell) =$
\[ - \int t^\ell A(t, x) dx. \]
I have an ad adding up constraint for total employment in each location. \( S \) is the multiplier attached to the KFE constraint, which I also integrated by parts. I have also combined the multipliers on the resource constraints, without loss of generality.

**Consumption and housing.** Optimality of consumption and housing choices in steady-state delivers

\[
1 = \frac{(1 - \omega) \lambda(\ell)}{c_U(\ell)} u(c_U(\ell), h_U(\ell)) = \frac{(1 - \omega) \lambda(\ell)}{c_E(y, \ell)} u(c_E(y, \ell), h_E(y, \ell))
\]

\[
r(\ell) = \frac{\omega \lambda(\ell)}{h_U(\ell)} u(c_U(\ell), h_U(\ell)) = \frac{\omega \lambda(\ell)}{h_E(y, \ell)} u(c_E(y, \ell), h_E(y, \ell))
\]

Re-arranging,

\[
r h_i = \frac{\omega}{1 - \omega} c_i ; \quad u_i = r^\omega c_i (1 - \omega)^{-1}
\]

which then implies \( r^\omega = \lambda \). Land market clearing in every location re-writes

\[
\frac{\omega}{1 - \omega} r(\ell) = U(\ell) c_U(\ell) + \int \mathcal{E}(y, \ell) c_E(y, \ell) dy
\]

where the second equality follows from the the local budget constraint. Varying the weights \( \lambda(\ell) \) thus \( r(\ell) = \lambda(\ell)^\omega \), and traces out the Pareto frontier of this economy. To keep the focus on the inefficiency in the location choice of employers, I choose the specific set of weights \( \lambda(\ell) \) such that the land market clearing coincides with its decentralized equilibrium counterpart when \( \beta = \alpha \). Namely, I choose \( \lambda(\ell) \) such that

\[
\frac{\lambda(\ell)^\omega}{\omega} = U(\ell) b \ell + \int \mathcal{E}(y, \ell) \left[ (1 - \alpha)(b + v(\ell)) \ell + \alpha y \ell \right] dy \quad (A.26)
\]

where \( v(\ell) \) is defined below.
**Allocation of workers.** I now take FOCs w.r.t. \( U, \Delta \) and impose steady-state. Starting with \( U \):

\[
\lambda(\ell)u(c_U(\ell), h_U(\ell)) + \alpha n(\ell) \int g_0 S + (b - c_U(\ell)) - r(\ell)h_U - \rho U = 0
\]

Using the previous FOCs to obtain that \( \lambda u_U = c_U + rh_U \), and denoting \( v(\ell)\ell r(\ell)^{-\omega} = \alpha n(\ell) \int g_0 S \), I obtain

\[
\rho U = \frac{(b + v(\ell))\ell}{r(\ell)^\omega}
\]

I guess that for now, the definition of \( v \) does not depend on \( r \). The co-state equation for \( E \) is then

\[
\rho S = u(c_E(y, \ell), h_E(y, \ell)) + LS - \rho U + (y\ell - c_E(y, \ell)) - r(\ell)I(\ell)h_E(y, \ell) - \Delta S
\]

Re-arranging similarly to the \( U \) FOC,

\[
\rho S = \frac{(y - (b + v(\ell)))\ell}{r(\ell)^\omega} + LS - \Delta S
\]

Finally, the FOC for \( \Delta \) yields

\[
\Delta = \begin{cases} 
0 & \text{if } S \geq 0 \\
+\infty & \text{if } S < 0
\end{cases}
\]

Therefore, \( X = r(\ell)^\omega S \) solves \( \rho X = (y - (b + v(\ell))\ell)\ell + LX \) in the continuation region. Hence, \( v \) is defined as \( v(\ell)\ell = \alpha n(\ell) \int g_0 X \). Together, these define a pair of equations that does not directly depend on \( r \). Thus, the guess that the definition of \( v \) does not depend on \( r \) is verified.
These multipliers correspond exactly to the shadow values of unemployed and employed workers when $\beta = \alpha$. The planner breaks up matches when the surplus $S$ is negative, and thus the recursion for $S$ has the same solution as in the decentralized equilibrium when replacing $\beta$ with $\alpha$.

**Allocation of jobs.** The FOC for $M_e$ is

$$c_e = (1 - \alpha) \frac{1}{M} \int d\ell f_\ell(\ell) n(\ell) \int g_0(y, \ell) S(y, \ell) dy$$

The FOCs for $\xi$ and $\zeta$ are then

$$[\pi + \tau] \xi = (1 - \alpha) n f_\ell \int g S dz$$

$$0 = f_\ell n \cdot \left( \int \frac{\partial \xi g_0}{g_0} g_0 S dy \right) + \frac{\pi'(\ell)}{\pi(\ell) + \tau} \cdot \left[ \pi(\ell) + \tau \right] f_\zeta(\zeta)$$

Denote $J(\ell) \equiv \tau + \pi(\ell)$ and so simplifying out $f_\ell$

$$n \left( \int \frac{\partial \xi g_0}{g_0} g_0 S dy \right) + \frac{J'(\ell)}{J(\ell)} \cdot \frac{f_\zeta(\zeta)}{\xi} \cdot (1 - \alpha) n \int g_0 S dy = 0$$

and hence

$$(1 - \alpha) \frac{J'}{J_\zeta} = \frac{\int \frac{\partial \xi g_0}{g_0} g_0 S dy}{\int g_0 S dy}$$

Using the known solution to $S$, one obtains

$$\frac{\int \frac{\partial \xi g_0}{g_0} g_0 S dy}{\int g_0 S dy} = \frac{S'(\zeta)}{S(\zeta)} + \log \frac{B_0}{b + v(\ell)}$$
Finally, changing variables to $J(\ell) \equiv J(\zeta(\ell))$:

$$(1 - \alpha) \frac{J'(\zeta)}{J(\zeta)} = \frac{\bar{S}'(\zeta)}{S(\zeta)} + \log \frac{B_0}{b + v(\zeta)}$$

This equation corresponds to an envelope condition of the decentralized equilibrium, which coincides with the FOC/envelope condition from the competitive equilibrium.

Re-write the $\xi$ FOC as

$$J(\ell)\xi(\ell) = (1 - \alpha)q(\ell) \cdot M\xi(\ell) \cdot \int g_0 S$$

where I have used that, by definition of $q(\ell), n(\ell) = q(\ell) \cdot M, \xi(\ell)$. Thus,

$$\frac{J(\ell)}{\rho M_e(1 - \alpha)} = q(\ell)\ell \left( \frac{B_0}{b + v(\ell)} \right)^{\zeta(\ell)} (b + v(\ell))S(\zeta(\ell))$$

Finally, use the definition of $v$ to substitute $q$ out:

$$J(\ell)^{1 - \alpha} \propto \ell^{1 - \alpha} v(\ell)^{-\alpha} \cdot \left( \frac{B_0}{b + v(\ell)} \right)^{\zeta(\ell)} (b + v(\ell))S(\zeta(\ell))$$

Then using the envelope condition from above, I obtain the FOC for $v$:

$$-\frac{v'(\ell)}{v(\ell)} \left[ \alpha + \frac{v(\ell)}{b + v(\ell)}(\zeta(\ell) - 1) \right] + \frac{1 - \alpha}{\ell} + 0 = 0 \quad (A.27)$$

This FOC resembles the one in the decentralized equilibrium, except that it does not have the last term $\left( \frac{S'(\zeta)}{S(\zeta)} + \log \frac{B_0}{b + v(\zeta)} \right) \zeta'(\ell)$. This last term is the labor market pooling externality that the planner internalizes. Finally, I can go back to the entry FOC, which re-writes:

$$c_e = (1 - \alpha) \int d\ell q(\ell)f(\zeta(\ell)|\zeta'(\ell)|\ell \left( \frac{B_0}{b + v(\ell)} \right)^{\zeta(\ell)} (b + v(\ell))\bar{S}(\zeta(\ell))$$

which corresponds to the free-entry condition when $\beta = \alpha$. 

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Proof of Proposition 4

Having laid out the planner’s optimality conditions, I can now turn to the proof of Proposition 4.

Extensions of decentralized equilibrium results. Comparing the $v$ FOC in the decentralized equilibrium (A.14) and in the planning solution (A.27), the labor market pooling externality immediately arises. Except for this discrepancy, inspecting the the planner’s optimality conditions reveal that they are identical to the decentralized equilibrium’s when $\beta = \alpha$. Therefore, Propositions 1, 2 and 3 extend under the same conditions.

Efficiency. Due to the labor market pooling externality term in the $v$ FOC in the decentralized equilibrium (A.14) relative to the planning solution (A.27), the decentralized equilibrium is inefficient as soon as $\alpha > 0$.

Comparison of allocations. In the linearized case of small supports for $F_\ell$, $F_z$ and when $\beta = \alpha$, it is possible to compare the assignment functions. From (A.23) and (A.24), $v$ and $K$ are identical in both the decentralized equilibrium and the planner solution to a first order. But then from the FOC (A.20), $\frac{w(\ell)}{v(\ell)}$ is larger in the decentralized equilibrium due to the labor market pooling externality term. Given that $\frac{w(\ell)}{v(\ell)} \approx \frac{v}{\ell + \ell v_1}$ to a first order, the comparison between reservation wages obtains.

For the comparison between assignmen functions $z$, it is useful to start from (A.19). Re-arranging its first-order approximation delivers the first-order approximation to $v(\ell) \approx \bar{v}(\ell/\ell)^{v_1},$ where $v_1$ is a constant that depends only on parameters. $v_1$ is higher in the decentralized equilibrium due to the labor market pooling externality. A common solution method in ODEs is to “bootstrap” successive approximation to derive higher orders. I follow this method in spirit and substitute back this first-order approximation into (A.19) and re-arrange
to obtain
\[
\zeta(\ell) \approx 1 + \left(1 + \frac{b}{v}(\ell/\ell)_{\nu} \right) \left(1 - \alpha + \alpha I_0 \right)
\]
where \(I_0 > 0\) in the decentralized equilibrium and \(I_0\) in the planner’s solution. Using the boundary conditions, one obtains
\[
\frac{z(x) - \tilde{z}}{z - \tilde{z}} = \frac{x^v_1 - 1}{x^\nu_1 - 1}
\]
where \(x = \frac{\ell}{\zeta}\). This functional form implies \(z^{DE} < z^{SP}\) except at the boundaries.

**Limit of identical locations.** Consider the location FOC(A.25) when there is no dispersion in \(\ell\). It holds only if there is dispersion in \(v\). Without the inefficiency, the last term on the left-hand-side is zero. Therefore, it implies \(\frac{v}{\nu_1} (\zeta(v) - 1) = -\alpha < 0\) which is a contradiction. Therefore, there can be no dispersion in \(v\) in the planner’s solution.

**Directed search.** I first briefly describe the economy with directed search. Then I show how the values of workers and employers change. Finally, I show that the location choice of employers coincides with the planner’s choice.

**Setup.** Employers can commit to fully state-contingent contracts that promise a stream of wage payments. For simplicity, I assume without loss of generality that these contracts must be Markovian. Within each location, there can be a continuum of submarkets indexed by their contract. Workers perfectly observe each contract and each submarket and direct their search across submarkets. Once they choose a submarket, they queue and wait until they meet the employers. Meetings in each submarket are created according to the same matching function as in the random search model.
Values. The value of unemployment satisfies

\[ \rho U = \frac{b \ell}{r(\ell) \omega} + \max_{\theta \in \Theta_\ell} f(\theta) \frac{s(\ell, \theta)}{r(\ell) \omega} \]

where without loss of generality each submarket in location \( \ell \) is indexed by its labor market tightness \( \theta \) which lies in the set \( \Theta_\ell \). \( s(\ell, \theta) \) denotes the promised value to the worker. The value of employment at wage \( w \) is

\[ \rho V^E(w, \ell) = \frac{w}{r(\ell) \omega} + L_w V^E \]

Then, \( V^E - U \) solves

\[ \rho \left( V^E(w, \ell) - U(\ell) \right) = \frac{w - b \ell - V(\ell)}{r(\ell) \omega} + L_w (V^E - U) \]

where I denote

\[ V(\ell) = \max_{\theta \in \Theta_\ell} f(\theta) \frac{s(\theta, \ell)}{r(\ell) \omega} \]

the value of search in location \( \ell \). Denote also \( v(\ell) = \frac{r(\ell)rV(\ell)}{\ell} \) the value of search relative to productivity.

It is also useful to define the adjusted surplus, which satisfies

\[ \rho J(\ell, y) = y \ell - [V(\ell) + b \ell] + L_y S \]

with boundary conditions identical to the random search case. Thus, Lemma 1 applies. The value of emloyer \( \zeta = 1/z \) in location \( \ell \) is then

\[ J(\zeta, \ell) = \max_{\theta \in \Theta_\ell} \left\{ q(\theta) E_{\zeta, \ell} \left[ S(\ell, z) - s(\theta, \ell) \right] \right\} \]

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Substituting the definition of $V$ to express tightness\textsuperscript{12} as a function of the surplus $s,$

\[ J(\zeta, \ell) = V(\ell)^{-\frac{1}{1-\alpha}} m^{\frac{1}{1-\alpha}} \max_s \left\{ s^\alpha \mathbb{E}_{\zeta, \ell} \left[ S(\ell, z) - s \right] \right\} \]

\[ = V(\ell)^{-\frac{1}{1-\alpha}} m^{\frac{1}{1-\alpha}} \max_{\hat{s} = s^\alpha} \left\{ \mathbb{E}_{\zeta, \ell} \left[ S(\ell, z) \right] \cdot \hat{s}^\alpha - \hat{s} \right\} \]

This results in

\[ s(\theta(\zeta, \ell), \ell) = \alpha \mathbb{E}_{\zeta, \ell} [S(\ell, z)] \]

\[ \theta(\zeta, \ell)^{1-\alpha} = \frac{V(\ell)}{\alpha m \mathbb{E}_{\zeta, \ell} [S(\ell, z)]} \]

and hence

\[ J(\zeta, \ell) = \left\{ \frac{(1 - \alpha)^{1-\alpha}}{\alpha^\alpha} m \mathbb{E}_{\zeta, \ell} [S(\ell, z)] V(\ell)^{-\alpha} \right\}^{\frac{1}{1-\alpha}} \]

**Location choice.** Using Lemma 1, the value of having entering in location $\ell$ for employer $\zeta$ is

\[ \rho J(\zeta, \ell) = \left\{ \frac{(1 - \alpha)^{1-\alpha}}{\alpha^\alpha} m \left( \frac{B}{b + v(\ell)} \right)^\zeta (b + v(\ell)) v(\ell)^{-\alpha} \cdot \ell^{1-\alpha} \cdot \mathcal{S}(\zeta) \right\}^{\frac{1}{1-\alpha}} \]

which coincides with the planner’s valuation.

**A.2.9 Optimal policy**

I consider five possible taxes and subsidies:

- A wage tax paid by the employer $\tau_w$
- A profit tax $\tau_\pi$

\textsuperscript{12}From the worker’s indifference condition, $q(s) = V(\ell)^{-\frac{1}{1-\alpha}} m^{\frac{1}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}$
- An unemployment benefits tax $\tau_b$
- A value added tax $\tau_{va}$
- An employment tax $\tau_e \ell$ paid by the employer, where it is useful to define $\tau_n = \frac{\tau_e}{\tau_b \tau_w}$

Using Lemma 8, these taxes affect the decentralized equilibrium as follows.

- Effective output is $\tau_{va} y \ell$
- Unemployment benefits are $b \ell \tau_b$
- The negotiated wage is
  
  $$w^* = (1 - \beta)[b \tau_b + v(\ell) + \tau_e] \ell + \beta \frac{\tau_{va} z \ell}{\tau_w}$$

- Employer values scale with $\tau_n$

These taxes results in flow values for employers

$$J_0(y, \ell) \equiv (1 - \beta)\tau_n (\tau_{va} \cdot y - \tau_e - \tau_w \tau_b b - \tau_w \cdot v(\ell)) \ell = \tau_{va} \tau_n (1 - \beta) \left( z - \frac{\tau_e}{\tau_{va}} - \frac{\tau_w \tau_b}{\tau_{va}} \cdot b - \frac{\tau_w}{\tau_{va}} \cdot v(\ell) \right) \ell$$

and workers

$$V_0 \equiv \beta \left( \frac{\tau_{va} z - \tau_e}{\tau_w} \cdot z - b \tau_b - \mathbb{E}[V^E - U] \right) \ell = \tau_{va} \tau_n \beta \left( z - \frac{\tau_e}{\tau_{va}} - b \cdot \frac{\tau_w \tau_b}{\tau_{va}} - \frac{\tau_w}{\tau_{va}} \mathbb{E}[V^E - U] \right) \ell$$

The endogenous separation cutoff is

$$y \propto \frac{\tau_e}{\tau_{va}} + \frac{\tau_w \tau_b}{\tau_{va}} \cdot b + \frac{\tau_w}{\tau_{va}} v = \frac{\tau_e}{\tau_{va}} \cdot \left( \frac{\tau_e}{\tau_{va}} + \tau_b b + v \right) = \frac{\tau_w \tau_b}{\tau_{va}} \cdot \left( \tau_n + b + \bar{v} \right)$$

where $\bar{v} = \tau_b v$. Finally, solving the worker’s problem, one obtains

$$v(\ell) = \beta f(\ell) \frac{\tau_{va}}{\tau_w} \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell)} \cdot \frac{y(\ell) S(\zeta(\ell))}{y(\ell) S(\zeta(\ell))}$$

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and so

\[ \tilde{v}(\ell) = \frac{\tau_{va}}{\tau_w} \beta \cdot f(\ell) \cdot \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell)} \frac{y(\ell)}{y(\ell)} \cdot \bar{S}(\zeta(\ell)) \]

Denoting \( T = \frac{\tau_{va}}{\tau_w} \) one obtains

Then

\[ \tilde{v} = c_2 Ty - b - \tau_n \]

for a constant \( c_2 > 0 \), and one can use the worker’s value of search to re-write

\[ c_2y - \frac{b + \tau_n}{T} = \beta c_1 f(\ell) \cdot \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell)} \frac{y(\ell)}{y(\ell)} \cdot \bar{S}(\zeta(\ell)) \]

which implies

\[ q(\ell) \propto \left( c_2y - \frac{b + \tau_n}{T} \right)^{-\frac{\alpha}{1-\alpha}} \cdot \beta^{\frac{\alpha}{1-\alpha}} \cdot \left[ \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell)} \frac{y(\ell)}{y(\ell)} \cdot \bar{S}(\zeta(\ell)) \right]^{\frac{\alpha}{1-\alpha}} \]

The employers’s expected value is then

\[ J(\ell, \zeta)^{1-\alpha} = \left( \frac{1 - \beta}{1 - \alpha} \right)^{1-\alpha} \left( \frac{\beta}{\alpha} \right)^{\alpha} (\tau_\pi(\ell)\tau_{va}(\ell))^{1-\alpha} \left[ \frac{c_2y(\ell) - b}{c_2y(\ell) - \frac{b + \tau_n(\ell)}{T(\ell)}} \cdot \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell)-\zeta} \cdot \frac{\bar{S}(\zeta(\ell))}{\bar{S}(\zeta)} \right]^{\alpha} \]

\times J_{SP}(\zeta, \ell, y(\ell))^{1-\alpha}

where \( J_{SP} \) is the planner’s shadow value of job \( \zeta \) in location \( \ell \). To ensure that allocations in the planner’s solution and the decentralized equilibrium coincide, there are three margins to correct. First, the decision to start producing together must be efficient, which can be implemented with the employment tax – the standard Hosios (1990) condition in each location. Second, the overall entry margin must be efficient, which can be implemented with the overall level of the profit tax. Third, the location choice of jobs must be efficient, which can be implemented with the spatial progressivity of the profit tax. When those
three margins are corrected, it is straightforward to check that the decentralized equilibrium is efficient from the equilibrium conditions. Any transfers can be funded through non-distortionary lump-sum taxes on owners. Alternatively, a flat earnings tax (on both wages and unemployment benefits) leaves the allocation undistorted and concentrates the burden on workers.

Set \( T \equiv 1 \). Then there exists a \( \tau_n \) that equated the separation cutoff for the planner and the decentralized equilibrium:

\[
\frac{1}{\beta} \cdot \frac{c_2 y - b - \tau_n}{y} = \frac{1}{\alpha} \cdot \frac{c_2 y - b}{y} \quad \rightarrow \quad \tau_n(\ell) = \frac{\alpha - \beta}{\alpha} v^{\text{SP}}(\ell)
\]

Substituting back into the employer’s problem,

\[
J(\ell, \zeta)^{1-\alpha} = \left( \frac{1 - \beta}{1 - \alpha} \right)^{1-\alpha} \tau_\pi^{1-\alpha}(\ell) \left[ \left( \frac{B_0}{y(\ell)} \right)^{\zeta(\ell) - \zeta} \frac{\bar{S}(\zeta(\ell))}{S(\zeta)} \right]^{\alpha} \times J_{\text{SP}}(\zeta, \ell, y(\ell))^{1-\alpha}
\]

The efficient profit tax then satisfies

\[
(1 - \alpha) \frac{\tau'_\pi(\ell)}{\tau_\pi(\ell)} + \alpha \left( \frac{S'(\zeta(\ell))}{S(\zeta(\ell))} + \log \frac{B}{b + v^{\text{SP}}(\ell)} \right) \left( \zeta^{\text{SP}} \right)'(\ell)
\]

and thus \( \tau'_\pi(\ell) < 0 \). Given the convention that \( \tau_\pi \) is the fraction that the employer keeps after tax, this inequality implies higher marginal profit tax rate in high \( \ell \) locations.

### A.3 Quantitative model

#### A.3.1 Characterization

**Values**

The bargaining solution from Lemma 8 readily extends to the extended model, which allows to define values with a minimal amount of extra notation. Denote \( U(p, a, h) \) the value of unemployment in location \((p, a)\) for a worker with human capital \( h \). With Frechet taste
shocks, the continuation value from migration is

$$M(h) = \left( \int U(p,a,k)^\nu F_{p,a}(dp,da) \right)^{1/\nu}$$

where I denote $\nu = 1/\varepsilon$. Migration shares are

$$\pi(\ell, a, k) = \frac{U(\ell, a, k)^\nu}{M(k)^\nu}$$

Guess that the value of unemployment scales with $k$. Then the value of unemployment solves the recursion

$$(\rho + \Delta + \mu)U(p,a,k) = apr(p,a)^{-(\omega+\psi)} \cdot U_1(p,a)k + (\lambda - \varphi) kU_k + \mu M_0 k$$

where $M_0 = \left( \int U_1(p,a)^\nu F_{p,a}(dp,da) \right)^{1/\nu}$. Because there is a continuum of locations, employed workers who receive the moving opportunity always take it: there is always a location where they taste shock is high enough to make them move. The adjusted surplus then solves

$$(\rho + \Delta + \mu)S(y,k,p,a) = pry(p,a)^{-\psi}k \left[ y - U_1(p,a) \right] + LyS + \lambda kS_k$$

Using Lemma 1, the solution scales with $k$, and is

$$(\rho + \Delta + \mu - \lambda)S(y,k,p,a) = k \cdot pry(p,a)^{-\psi}U_1(p,a) \cdot S \left( \frac{y}{y(p,a)} \right)$$

$$\left( \rho + \Delta + \mu - \lambda \right) \frac{y(p,a)}{y_0} = U_1(p,a)$$

where $y_0$ is calculated using $\bar{\rho} = (\rho + \Delta + \mu - \lambda)$ as the effective discount rate. Because workers’ outside option scales with $h$ under the guess, the separation decision is independent
from \( k \). Going back to the value of unemployment,

\[
\tilde{\rho}U(p, a, k) = \frac{(b + v(p, a))a\ell}{r(p, a)^{\omega + \psi}} k + \mu M_0 k - \phi kU_k
\]

where

\[
\tilde{\rho}v(p, a) = \beta f(p, a)U_1(p, a) \left( \frac{Y}{y(p, a)} \right)^{\zeta^*(p, a)} \tilde{S}(\zeta^*(p, a))
\]

and the guess is verified. In addition,

\[
(\tilde{\rho} + \phi)U_0(p, a) = \frac{(b + v(p, a))a\ell}{r(p, a)^{\omega + \psi}} + \mu M_0
\]

which can be rearranged into

\[
\tilde{\rho}U_0(p, a) = \frac{\tilde{\rho}(b + v(\ell, a))}{\tilde{\rho} + \phi} \cdot \frac{a\ell}{r(\ell)^{\omega + \psi}} + \mu M_0 - \frac{\phi}{\tilde{\rho} + \phi} \mu M_0
\]

and therefore,

\[
U_1(p, a) = \frac{\tilde{\rho}}{\tilde{\rho} + \phi} (b + v(\ell, a)) - \frac{\phi}{\tilde{\rho} + \phi} \mu M_0
\]

Under the (empirically relevant) assumption that \( \mu \ll 1 \), the second term is negligible, and so

\[
U_1(p, a) \approx \frac{\tilde{\rho}}{\tilde{\rho} + \phi} (b + v(p, a))
\]

Going back to the joint surplus,

\[
\tilde{\rho}S(y, k, p, a) \approx k \cdot pr(p, a)^{-\psi} \left[ \frac{\tilde{\rho}}{\tilde{\rho} + \phi} (b + v(p, a)) \right] \cdot S \left( \frac{y}{y(p, a)} \right)
\]

\[
\frac{\tilde{\rho}y(p, a)}{y_0} \approx \frac{\tilde{\rho}}{\tilde{\rho} + \phi} (b + v(p, a))
\]
To a first order approximation in $\mu$, the previous results imply

$$\hat{\rho}S(y, k, p, a) = k \cdot pr(p, a)^{-\psi}(b + v(p, a)) \cdot S\left(\frac{y}{y(p, a)}\right) ; \quad \hat{\rho}\frac{y(p, a)}{y_0} = (b + v(p, a)) \quad (A.28)$$

$$\hat{\rho}U(p, a, k) = \frac{(b + v(p, a))ap}{r(p, a)^{\omega+\psi}}k \equiv U_0(p, a)k \quad ; \quad \pi(p, a, k) = \left(\frac{U_0(p, a)}{M_0}\right)^{\nu} \quad (A.29)$$

where $\hat{\rho} = \tilde{\rho} + \varphi$.

### Human capital across locations

I now characterize the human capital distribution in each location. For now, focus on a single location and omit $(p, a)$ subscripts to facilitate exposition. The probability mass functions of rescale log human capital $h = \log k - \lambda t$ for employed and unemployed workers in a location solve:

$$0 = -sg_E(h) + f_Rg_U(h) - \mu g_E(h) - \Delta g_E(h)$$

$$0 = \varphi g_U'(h) - f_Rg_U(h) + sg_E(h) - \mu g_U(h) + K(h) - \Delta g_U(h)$$

where $K(h)$ is the overall entry distribution inclusive of in-migration and newborns. This simple combination of ODEs obtains because there is no relative human capital growth while employed. This delivers the crucial simplification that separations are independent from the human capital level. Re-arranging the first equation,

$$g_E(h) = \frac{f_R}{\mu + \Delta + s}g_U(h)$$

and so, substituting back into the second equation

$$0 = \varphi g_U'(h) - (\mu + \Delta)\frac{\mu + \Delta + s + f_R}{\mu + \Delta + s}g_U(h) + K(h)$$

$$\equiv C$$

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While this ODE can be solved explicitly, computing the mean human capital is sufficient to characterize equilibrium. Multiply the KFE by $e^h$, integrate over $h$ in $\mathbb{R}$ and integrate the first term by parts to obtain

\[0 = \left[e^h g_U(h)\right]_{-\infty}^{\infty} - \varphi \int_{\mathbb{R}} e^h g_U - C \int_{\mathbb{R}} e^h g_U + \int_{\mathbb{R}} e^h K\]

The first term is 0 at both extremes. Denote $k_0 = \int_{\mathbb{R}} e^h K$. To get average human capital in the location one needs to solve for total population masses in each location: $U, E$. They solve similar KFEs, so that

\[E = \frac{f_R}{\mu + \Delta + s} U\]

and so

\[(\mu + \Delta) \frac{\mu + \Delta + s + f_R}{\mu + \Delta + s} U = K\]

where $K$ is the total mass of entrants. Hence, the unemployment rate is

\[u = \frac{U}{U + E} = \frac{\mu + \Delta + s}{\mu + \Delta + s + f_R} = \frac{\mu + \Delta}{\mu + \Delta + s + f_R} \cdot \frac{\mu + \Delta}{C}\]

while the mass of unemployed is $U = u \cdot \frac{K}{\mu + \Delta}$, and so population is $E + U = \frac{K}{\mu + \Delta}$. Recall that by definition, $\int_{\mathbb{R}} g_U = U$. Thus, average human capital in a location is

\[\mathbb{E}(p, a) \equiv \mathbb{E}[e^h | p, a] \equiv \bar{h}(\ell, a) = \frac{k_0}{U \cdot (\varphi + C)} = (\mu + \Delta) \frac{\mathbb{E}^K[e^h]}{u \cdot (\varphi + C)} = \frac{\mu + \Delta}{\mu + \Delta + u(p, a) \varphi} \cdot \mathbb{E}^K[e^h]\]

where $\mathbb{E}^K[e^h]$ is the expected human capital of new entrants. By definition, the mass of entrants at human capital $h$ is

\[K(h) = \mu \pi(p, a) I(h) + \Delta L(p, a) E(h)\]
where $I$ is the economy-wide invariant distribution, and $E$ is the entry distribution, and $\pi$ are migration shares. So one obtains
\[
E^K[e^h] = \frac{\mu\pi}{\mu\pi + \Delta L}E^I[e^h] + \frac{\Delta L}{\mu\pi + \Delta L}E^E[e^h]
\]

In steady-state, population density is equal to migration shares:
\[
L(p, a) = \pi(p, a)
\]

Therefore,
\[
E^K[e^h] = \frac{\mu}{\mu + \Delta}E^I[e^h] + \frac{\Delta}{\mu + \Delta}E^E[e^h] \equiv x_0E^I + (1 - x_0)E^E
\]

Now,
\[
E^I[e^h] = \int L(p, a)F_{p,a}(dp, da) \cdot E(p, a)
\]

so that one obtains a linear system across locations:
\[
E(p, a) = \frac{z_0}{z_0 + \varphi u(p, a)} \left[ x_0E^I + (1 - x_0)E^E \right]
\]

where $z_0 = \mu + \Delta$. Denote $X(p, a) = (1 + \varphi_0 u(p, a))E(p, a)$ where $\varphi_0 = \varphi/z_0$. Re-write the linear system as
\[
X(p, a) = (1 - x_0)E^E + x_0 \int \frac{L(p', a')}{1 + \varphi_0 u(p', a')}F_{p,a}(dp', da')
\]
This system can be explicitly solved. Multiply by \( Z(p, a) \equiv x_0 \frac{L(p, a)}{1 + \varphi_0 u(p, a)} F_{p, a}(dp', da') \) and integrate to obtain

\[
\int Z(p, a) F_{p, a}(dp, da) = \frac{\int Z(p, a) F_{p, a}(dp, da)}{1 - \int Z(p, a) F_{p, a}(dp, da)} \cdot (1 - x_0) E^E
\]

Then

\[
Z(p, a) = \frac{(1 - x_0) E^E}{1 - \int Z(p', a') F_{p, a}(dp', da')}
\]

which finally implies

\[
E(p, a) = \frac{1}{1 + \varphi_0 u(p, a)} \cdot \frac{(1 - x_0) E^E}{1 - x_0 \int \frac{L(p', a')}{1 + \varphi_0 u(p', a')} \cdot F_{p, a}(dp', da')}
\]

**Labor market flows**

Given (A.28), the expression for labor market flows immediately extends given an assignment \( z(p, a) \) and a value of search \( v(p, a) \). The only change follows from the KFE, in logs:

\[
0 = \delta g'(x) + \frac{\sigma^2 \delta}{2} g''(x) - (\Delta + \mu) g(x)
\]

The associated characteristic equation has only one negative (stable) root,

\[
\kappa = -\frac{1}{2} \left[ \frac{2\delta}{\sigma^2} + 2 \sqrt{\frac{2\mu + \Delta}{\sigma^2}} + \frac{\delta^2}{\sigma^4} \right]
\]

which coincides with the simple solution \( \kappa_0 = \frac{2\delta}{\sigma^2} \) when \( \mu + \Delta = 0 \). Thus, the previous expression for the invariant distribution extends with \( \kappa \) instead of \( \kappa_0 \). In addition, the expression for the average productivity also extends. The exit rate from employment is then \( \delta/z(p, a) + \mu + \Delta \). Using the flow equation for unemployment together with the steady-state migration shares, one obtains \( u(p, a) = \frac{\delta/z(p, a) + \mu + \Delta}{\delta/z(p, a) + \mu + \Delta + f_R(p, a)} \).
Population, housing prices and composite index

Having solved for average human capital $\mathbb{E}(p,a)$ in each location, it is possible to characterize housing prices and thus the value of employers. In steady-state, total population in a location is given by migration shares:

$$L(p,a) = \left( \frac{U_0(p,a)}{M_0} \right)^\nu$$

(A.30)

Housing rents follow from equating total housing demand to local supply. From the Cobb-Douglas structure of the production function, employers spend a fraction $\psi$ of output on housing. Hence, total housing demand in location $(p,a)$ is now

$$H_0r(p,a)^\eta = p\mathbb{E}(p,a)r(p,a)^{-\psi} \cdot L(p,a) \cdot \frac{1}{r(p,a)} \left[ \omega u(p,a)b \right. \\
+ \omega(1-u(p,a))(b + v(p,a))(1 - \beta + \beta \mathbb{E}_{p,a}[y|y > y(p,a)]) \\
\left. + \psi(1-u(p,a))(b + v(p,a))\mathbb{E}_{p,a}[y|y > y(p,a)] \right]$$

$$\equiv p\mathbb{E}(p,a)r(p,a)^{-\psi-1}L(p,a)G(\zeta(p,a),v(p,a))$$

where $\eta$ is the housing supply elasticity, and $\mathbb{E}_{p,a}[h]$ is the average local human capital. For the last equality, I anticipate that local unemployment will still be a function of $v, \zeta$ alone, and that there is PAM in equilibrium. In what follows, normalize $H_0$ to one without loss of generality.

After substituting equation (A.31) into the migration share equation, some algebra yields

$$L = M_0^{-\frac{1}{\varepsilon + \frac{1}{\lambda + \gamma + \delta}}} \cdot \left[ \frac{a \cdot (p\mathbb{E}(p,a))^{1+\eta-\omega} \cdot (b + v(p,a))}{G(v(p,a),\zeta(p,a))^\frac{\omega + \psi}{1+\gamma+\delta}} \right]^{\frac{1}{\varepsilon + \frac{1}{\lambda + \gamma + \delta}}}$$

(A.32)

After substituting equation (A.32) back into (A.31) and some algebra, and so

$$r(p,a) = M_0^{-\frac{1}{\varepsilon + \frac{1}{\lambda + \gamma + \delta}}} \cdot \left\{ a \cdot (p\mathbb{E}(p,a))^{1+\gamma} \cdot (b + v(p,a)) \cdot G(v(p,a),\zeta(p,a))^\delta \right\}^{\frac{1}{(\lambda + \gamma + \delta)\varepsilon + \omega + \psi}}$$
Therefore the expected prefactor in the adjusted surplus is

\[ p \mathbb{E}(p, a) r(p, a)^{-\psi} = M_0^{\psi} \cdot \frac{\left( p \mathbb{E}(p, a)^{\omega + \epsilon(1 + \eta)} a^{-\psi} \right)^{\frac{1}{\omega + \psi + \epsilon(1 + \eta + \psi)}}}{\left[ (b + v(p, a)) G(v(p, \zeta(p, a)))^{\epsilon \psi} \right]^{\frac{1}{\omega + \psi + \epsilon(1 + \eta + \psi)}}} \]  

(A.33)

This equation motivates the definition of the composite index

\[ \ell(p, a) = \left( p^{\omega + \epsilon(1 + \eta)} a^{-\psi} \right)^{\frac{1}{\omega + \psi + \epsilon(1 + \eta + \psi)}} \]  

(A.34)

**Location choice of employers**

Using the adjusted surplus and (A.33), the value of opening a job in location \((p, a)\) for employer \(\zeta = 1/z\) is

\[ J(\zeta, p, a)^{\frac{1}{1 + \gamma}} \propto \mathbb{E}(p, a)^{\frac{\omega + \epsilon(1 + \eta)}{\omega + \psi + \epsilon(1 + \eta + \psi)}} \cdot \ell(p, a) \]

\[ (b + v(p, a))^{1 - \frac{\psi}{\omega + \psi + \epsilon(1 + \eta + \psi)}} \cdot G(v(p, a), \zeta^*(p, a))^{\frac{\epsilon \psi}{\omega + \psi + \epsilon(1 + \eta + \psi)}} \]

\[ q(p, a) \left( \frac{B}{b + v(p, a)} \right)^{\zeta} \hat{S}(\zeta) \]  

(A.35)

where the optimal vacancy posting decision has been maximized out. Re-arranging equation (A.35) delivers equation (1.18). Using the worker’s value of search to substitute out \(q(p, a)\) delivers the employer’s location problem

\[ \max_{p, a} \quad \tilde{\mathbb{E}}(u(v(p, a), \zeta^*(p, a))^{1 - \alpha} \cdot \ell(p, a)^{1 - \alpha} \]

\[ (b + v(p, a))^{1 - \frac{(1 - \alpha)\omega + \psi + \epsilon(1 + \eta + \psi)}{\omega + \psi + \epsilon(1 + \eta + \psi)}} v(p, a)^{-\alpha} \cdot G(v(p, a), \zeta^*(p, a))^{\frac{(1 - \alpha)\epsilon \psi}{\omega + \psi + \epsilon(1 + \eta + \psi)}} \]

\[ \left( \frac{B}{b + v(p, a)} \right)^{\frac{1}{(1 - \alpha)\zeta + a\zeta^*(p, a)}} \hat{S}(\zeta)^{1 - \alpha} \hat{S}(\zeta^*(p, a))^{\alpha} \]  

(A.36)

where \(\tilde{\mathbb{E}}(u(v(p, a), \zeta(p, a)) \equiv \mathbb{E}(p, a)\) but where the dependence on the local unemployment rate has been made explicit.
In principle, employers must take two first-order conditions for their optimal location choice: with respect to each dimension \( i \in \{ p, a \} \). After taking these first-order conditions and re-arranging, one obtains:

\[
\partial_i v = A(v, \zeta, \ell)\partial_i \ell + B(v, \zeta, \ell)\partial_i \zeta^*
\]

for some functions \( A, B \). Now guess that \( \zeta^* \) is a function of \( \ell(p, a) \) only. Then one obtains for \( i \in \{ p, a \} \)

\[
\partial_i v = C(v, \zeta, \ell)\partial_i \ell
\]

Combining equations, standard partial differential equation results imply that \( v \) is a function of \( \ell \) alone. Thus, employers need only choose the unidimensional combined index \( \ell(p, a) \).

With this observation at hand, the structure of equation (A.36) then closely mirrors its baseline model equivalent. Therefore, the assignment results extend under either Assumption 1 or Assumption 2 – the latter would only the expression for \( \bar{S} \). The FOC for the optimal location choice is then

\[
\frac{v'(\ell)}{v(\ell)} \left\{ \alpha + \frac{v(\ell)}{b+v(\ell)} \left( \zeta(\ell) - 1 - \frac{(1-\alpha)\psi}{\omega + \psi + (1 + \psi)\varepsilon} \left[ 1 + \varepsilon \frac{(B + v(x)) G_v}{G} \right] \right) \right\} = 1 - \frac{\alpha}{\ell} + \alpha \left( \frac{\bar{S}'(\zeta(\ell))}{S(\zeta(\ell))} + \log \frac{B}{b+v(\ell)} \right) \zeta'(\ell) + \varepsilon \cdot \frac{(1-\alpha)\psi}{\omega + \psi + (1 + \psi)\varepsilon} G \zeta'(\ell)
\]

\[
+ \frac{(1-\alpha)(\omega + \varepsilon)}{\omega + \psi + \varepsilon(1 + \psi)} \frac{d}{d\ell} \left( \log \frac{D}{D + \varphi u(\ell)} \right)
\]

where

\[
\frac{d}{d\ell} \left( \log \frac{D}{D + \varphi u(\ell)} \right) = \frac{\varphi u(\ell)}{D + \varphi u(\ell)} \left\{ \frac{u(\ell) f_R(\ell)}{b+v(\ell)} \frac{v'(\ell)}{v(\ell)} - \left[ \delta(1 - u(\ell)) + f_R(\ell) \frac{\bar{S}'(\zeta(\ell))}{\bar{S}(\ell)} \right] \zeta'(\ell) \right\}
\]
and where

\[ G_v = \omega b u_v + \omega(1 - u)(1 - \beta + \beta E) - \omega(b + v) u_v (1 - \beta + \beta E) + \psi(1 - u) E - \psi u_v (b + v) E \]

\[ G_{\zeta} = \omega b u_{\zeta} + \omega(1 - u)(b + v) \beta E_{\zeta} - \omega u_{\zeta} (b + v) (1 - \beta + \beta E) + \psi(1 - u) (b + v) E_{\zeta} - \psi u_{\zeta} (b + v) E \]

where here \( E(\zeta) = \mathbb{E}_\zeta[y/\gamma|y \geq y] \). It is then possible to express labor market tightness in a location \( \ell \):

\[ \theta(\ell) = -\frac{M e f_\zeta(\zeta(\ell)) \mathcal{V}(\ell, \zeta(\ell)) \zeta'(\ell)}{u(\ell) \mathcal{L}(\ell) f_\ell(\ell)} \] (A.37)

where optimal vacancies are

\[ \mathcal{V}(\ell, \zeta(\ell)) \propto J(\zeta^*(\ell), \ell)^\gamma \] (A.38)

and the maximized value of employers is

\[ J(\zeta^*(\ell), \ell) \frac{1}{1+\gamma} = \mathbb{E}(u(\ell)) \frac{\omega + \varepsilon(1 + \eta)}{\omega + \psi + \varepsilon(1 + \eta + \psi)} \cdot \ell \]

\[ \cdot (b + v(\ell))^{\frac{1}{1+\eta}} \cdot v(\ell)^{-\frac{1}{1+\alpha}} G(v(\ell), \zeta(\ell))^{\frac{\varepsilon\psi}{\omega + \psi + \varepsilon(1 + \eta + \psi)}} \cdot \frac{B}{b + v(\ell)} \] \( \mathcal{S}(\zeta(\ell)) \] \( \frac{1}{1+\alpha} \) (A.39)

**Labor market clearing and population determination**

After substituting equation (A.34) back into (A.32) and some algebra,

\[ L(p, a) \propto \ell(p, a)^{\frac{1+\eta-\omega}{\omega + \varepsilon(1 + \eta)}} \cdot (b + v(p, a))^{\frac{1+\eta+\psi}{\omega + \psi + \varepsilon(1 + \eta + \psi)}} \cdot a^{\frac{1}{\omega + \psi + \varepsilon(1 + \eta + \psi)}} \]

Then average population density in locations with index \( \ell \), \( \mathcal{L}(\ell) \), is given by

\[ \mathcal{L}(\ell) \propto \frac{\ell^{\frac{1+\eta-\omega}{\omega + \varepsilon(1 + \eta)}} \cdot (b + v(\ell))^{\frac{1+\psi}{\omega + \psi + \varepsilon(1 + \eta + \psi)}} \cdot \mathcal{C}(\ell)}{G(v(\ell), \zeta(\ell))^{\frac{1+\psi}{\omega + \psi + \varepsilon(1 + \eta + \psi)}}} \]
\[
C(\ell) = \mathbb{E} \left[ a^{\omega + \varepsilon(1 + \eta + \psi)} \mid \ell(p, a) = \ell \right]
\]

By construction,

\[
(\omega + \psi + \varepsilon(1 + \eta + \psi)) \log \ell = (\omega + \varepsilon(1 + \eta)) \log \ell - \psi \log a
\]

As an example for \(C(\ell)\), consider the lognormal case of the estimation. Then \((\log a, \log \ell)\) is jointly lognormal, with variance matrix

\[
\begin{pmatrix}
\sigma_a^2 \\
\sigma_a (\omega + \varepsilon(1 + \eta + \psi)) \rho_{ap} \sigma_p - \psi \sigma_a^2
\end{pmatrix}
\]

Using the conditional normal distributions,

\[
\log a \mid \log \ell = \frac{1}{\sigma_\ell^2} \cdot \frac{(\omega + \varepsilon(1 + \eta)) \rho_{ap} \sigma_p \sigma_a - \psi \sigma_a^2}{\omega + \psi + \varepsilon(1 + \eta + \psi)} \log \ell + N
\]

where \(N \sim \mathcal{N}(0, \sigma_a^2(1 - \rho_{a,\ell}^2))\) is independent from \(\log \ell\).\(^{13}\) Therefore, the correction factor is

\[
C(\ell) = C_0 \exp \left( \frac{\sigma_a^2}{\sigma_\ell^2} \cdot \frac{\rho_{ap} \sigma_p}{\sqrt{\omega + \varepsilon(1 + \eta)}} \cdot \log \ell \right)
\]

**Intuition for sufficient statistic**

Here I briefly discuss why the combined index \(\ell(p, a)\) is a local sufficient statistic in equilibrium. Given that the direct contributions of local productivity \(p\) and amenities \(a\) are combined into the single index \(\ell(p, a)\) in the location choice of jobs (1.18), it is natural to conjecture that this single index will be a sufficient statistic for the model's outcomes. However,

\(^{13}\)Note that the correlation is \(\rho_{a,\ell} = \frac{(\omega + \varepsilon) \rho_{ap} \sigma_p - \psi \sigma_a}{\omega + \psi + \varepsilon(1 + \psi)} / \sigma_\ell\)
ever, one potential complication arises. Labor market clearing in each location relates the number of vacancies to the number of unemployed workers. While the volume of local vacancies is a function of $\ell(p, a)$ only as per equation (1.18), the number of locally unemployed workers is not. Because workers also directly care about amenities $a$, their location choices reflect $p$ and $a$ in a combination that does not align with employers’. Thus, the number of locally unemployed workers varies with $\ell(p, a)$ and with amenities $a$ conditional on $\ell(p, a)$.

The key insight is that employers only value locations through the combined index $\ell(p, a)$ as long as labor market tightness $\theta(\ell)$ also only depends on the combined index. As illustrated by Figure A.11, employers are then indifferent between all locations that have the same index $\ell(p, a)$ even if these locations have different amenities $a$. Jobs with the same quality $z$ thus allocate along one-dimensional indifference curves – $\ell(p, a)$ isoquants – to ensure that labor market tightness $\theta(\ell)$ remains constant along the indifference curve. Locations with higher amenities $a$ conditional on the local advantage index $\ell(p, a)$ have both more unemployed workers and more open jobs, but in similar proportions.\footnote{\(\mathcal{D}(p, a)\) in Figure A.11 is defined in the Appendix and encodes how small changes $dp, da$ translate into small changes $d\ell$. Formally, is the determinant of the Jacobian matrix of the mapping $\ell(p, a)$.}

**Efficiency**

All the additional choices in the extended model are efficient. Thus, the normative results extend, with one caveat. Workers have heterogeneous human capital within a location but search in the same labor market. Therefore, low human capital workers who separate into unemployment create a negative externality on high human capital workers who are searching for a job. In general, this provides a motive for the planner to retain workers with low human capital longer on the job.

This source of inefficiency is not the focus of the paper, and thus I do not attempt to derive an optimal policy that would correct it. Rather, note that when $\varphi$ and the support of $F_k$ are small enough, there is little dispersion between human capital levels within a location. In that case, it is possible to show that, the inefficiency is small in the sense that
it is quadratic in \( \varphi, \text{Var}_k \). Finally, it is possible to extend the directed search environment to the richer framework. Because human capital is not observed by employer prior to matching, the optimal contract may in principle depend nonlinearly on human capital if employers try to screen different workers. It is nonetheless possible to show that the optimal contract is still a local wage rate per unit of human capital, which makes comparisons with the random search model straightforward.

Making those arguments precise requires a substantial amount of new notation and lengthy derivations. Thus, they are omitted in the present paper, but are available upon request.
Welfare

The average welfare of unemployed workers in locations $\ell$ satisfies

$$\hat{\rho} \mathbb{E} \left[ \int U(p, a, k) L(p, a) F_{p,a}(dp, da) | \ell \right] \approx \bar{U}_0 \cdot J(M_0) U_2(\ell) D(\ell) \mathbb{E}(\ell)$$

where $\bar{U}_0$ is a transformation of parameters, and

$$J(M_0) = M_0^{\omega + \psi + (1 + \eta + \psi)}; \quad U_2(\ell) = \ell^{(1 + \eta - \omega)}(b + v(\ell))^{1+(1 + \eta + \psi)} G(v(\ell), \zeta(\ell))^{-1/(1 + \eta + \psi)}$$

and

$$D(\ell) = \mathbb{E}[a^{1 + \eta + \psi} \cdot (1 + \omega + \psi + (1 + \eta + \psi)) \mathbb{E}(\ell)]$$

can be computed similarly to $C(\ell)$. However, given the presence of idiosyncratic preference shocks, that act as compensating differentials, the welfare of unemployed workers of the same human capital is equalized across locations in expectation, and is simply $M_0$. To compute $M_0$, note that

$$M_0^{\nu} = \int U_0(p, a)^{\nu} F(dp, da)$$

and so

$$M_0^{1 + \eta + \psi} = \int U_2(\ell)^{1 + \eta - \omega} \ell^{1 + \eta + \psi} D(\ell) F(\ell) d\ell$$

where

$$\tilde{D}(\ell) = \mathbb{E}\left[a^{1 + \eta + \psi} \cdot (1 + \omega + \psi + (1 + \eta + \psi)) \mathbb{E}(\ell) \right]$$
The contribution of unemployed workers to total welfare is thus

$$\bar{U} = A \int J(M_0)U_2(\ell)D(\ell)\mathbb{E}(\ell)u(\ell)\mathcal{L}(\ell)F_\ell(d\ell)$$

for some constant $A > 0$. Similarly, total welfare of employed workers is

$$\bar{V}^E = A \int E \left[ \int \mathbb{E}[V^E(\ell, a, y)]L(\ell, a)dF_{\ell,a}(\ell, a)|x] \cdot \bar{h}(x) \cdot (1 - u(x))\mathcal{L}(x) \cdot dF_x(x) \right]$$

$$= A \int J(M_0)U_2(\ell) \left( 1 + \beta \mathcal{S}(\zeta(\ell)) \right) D(\ell)\mathbb{E}(\ell)(1 - u(\ell))\mathcal{L}(\ell)F_\ell(d\ell)$$

Finally, in the decentralized equilibrium, I must take a stand on how profits from land rents and employers are redistributed. I assume that they are rebated to workers with a flat earnings subsidy. This formulation has two advantages. First, is non-distortionary. Second, the transfer is subsumed into $M_0$ and thus using the standard population adding up and free entry condition suffices to compute the equilibrium.

**A.3.2 Alternative extensions**

Here I discuss several possible extensions of the model and argue that the main theoretical predictions are robust to those alternative specifications.

**Micro-foundation for local productivity.** Assume for simplicity that the production function is $F(y_\ell, \ell, k) = y_\ell \cdot k \cdot \ell$. Assume also homogeneous amenities $a \equiv 1$, and a subsistence level of housing for workers stemming from Stone-Geary preferences. Denote by $K(\ell) = \mathbb{E}[k|\ell]$ the average worker productivity in location $\ell$ under the equilibrium population distribution. It is then straightforward to show that the job’s location problem is the same as in equation (1.12), except that the endogenous productivity $\ell \cdot K(\ell)$ replaces the exogenous productivity $\ell$. Thus, jobs sort in space according to a matching function $z(\ell \cdot K(\ell))$ resulting in a cutoff function $y(\ell \cdot K(\ell))$. 

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In the limit when locations becomes ex-ante homogeneous $\text{Var}(\ell) \to 0$, there remain sustained differences in measured productivity $K(\ell)$ because workers systematically sort in space due to Stone-Geary preferences, as in Corollary 1. Small local differences again act as a coordinating device or sunspot. To see why, recall that when locations becomes ex-ante homogeneous $\text{Var}(\ell) \to 0$, there can be sustained differences in measured productivity $K(\ell)$ only if workers systematically sort in space, as in Corollary 1. To obtain positive sorting, I must introduce Stone-Geary preferences for housing: there is a subsistence level of housing $H_0 > 0$ that workers must rent to live in a location. Workers’ location choices then follow their free-mobility condition

$$
\rho U(k) = \max_\ell \frac{(k\ell)(b + v(\ell K(\ell))) - H_0 r(\ell)}{r(\ell) \omega}
$$

Then, the worker’s object $(k\ell)(b + v(\ell K(\ell)))$ is supermodular in $(k, \ell(b + v(\ell K(\ell))))$, which ensures perfect positive assortative matching between worker productivity $k$ and the location’s value of $\ell(b + v(\ell K(\ell)))$. This last term is increasing in $K(\ell)$, and so the equilibrium sustains positive sorting between individual productivity $k$ and average local productivity $K(\ell)$. These arguments prove the following proposition which closely mirrors Corollary 1.

**Proposition 10.** (Micro-foundation for spatial differences in productivity)
As $\text{Var}(\ell) \to 0$, the equilibrium sorting function $K$ converges to a strictly increasing function $K_0$.

Proposition 10 guarantees that even as differences in exogenous local productivity vanish, the endogenous sorting of heterogeneous workers sustains measured differences in endogenous local productivity $K_0(\ell)$. Small local differences act as a coordinating device or sunspot. Having established Proposition 10, Propositions 1 to 5 extend with minor modifications: the basic sorting patterns of jobs, the implications for job losing and finding rates, as well as the inefficient margins, all remain.
Skill markets. Section 1.2 establishes that, after city residuals, sorting of heterogeneous workers along observable skill characteristics accounts for a little over 20% of the spatial variation in local job losing rates. Suppose that workers differ in their skill $x$, which is exogenously distributed in the population according to a cumulative distribution function $F_x$. Suppose for clarity of exposition that the production function is $F(y_t, \ell, x) = y_t \cdot x \cdot \ell$, no differences in amenities $a \equiv 1$ and free mobility. Assume that firms perfectly observe $x$ and can choose which skill level to hire, i.e. there are segmented labor markets per skill in each location. Then Proposition 1 extends with an endogeneous profitability index $\ell(p, x) = \ell \cdot x$ that combines worker’s skill and local characteristics. Conditional on skill, locations with higher $\ell$ attract high productivity jobs and thus have a lower job loss rate. Conditional on location, high skill workers have lower job loss rates because they match with high productivity jobs, as in the data. Propositions 1 to 5 then extend with minor modifications.

Skills also sort across space due to an endogeneous complementarity between skills and locations stemming from the job’s location problem. To see this, note that jobs’ value conditional on their type $z$ and their choice $\ell, x$ is

$$\frac{J(z, x, \ell)}{J_0} = \left\{ \frac{b + v(\ell, x)}{v(\ell, x)^{\alpha}} \right\} \cdot \left\{ \frac{S(z, y(\ell, x))^{1-\alpha}}{S(z(\ell, x), y(\ell, a, h))^{\alpha}} \right\}^{\frac{1}{1-\alpha}} \cdot (\ell x)$$

The same argument as in Proposition 6 ensures that $y, v, z$ are univariate functions of the endogenous profitability index $\ell x$. Worker’s free mobility then writes

$$y_0 U(x) = \max_{\ell} \frac{\ell x}{r(\ell) \cdot y(\ell x)}$$

There is positive sorting between $x$ and $\ell$ if $y(\ell x)$ is log-supermodular in $(x, \ell)$. Now,

$$\partial_x \left( \log y(\ell x) \right) = \frac{\ell x \cdot v'(\ell x)}{v(\ell x)} \cdot \frac{v(\ell x)}{b + v(\ell x)}$$
From the firm’s FOC, the first term is

\[
\frac{\ell x \cdot v'(\ell x)}{v(\ell x)} = \frac{1 - \alpha}{\alpha + \frac{v(\ell x)}{b+v(\ell x)} \left(1/z(\ell x) - 1 \right) - \alpha \left(\log \frac{B_0}{b+v(\ell x)} + \frac{S'(z(\ell x))}{S(z(\ell x))}\right) \frac{z'(\ell x)}{z(\ell x)^2}}
\]

Thus, when \(\alpha\) is small enough, \(\partial_x \left(\log y(\ell x)\right)\) is increasing in \(\ell\). This ensures log-supermodularity of the worker’s objective and hence positive assortative matching between \(x\) and \(\ell\).

**Industries.** To include heterogeneous industries in the model, it suffices to assume that jobs in each industry \(j\) draw their quality from an industry-specific initial distribution \(F_{z,j}\). This extended model generates differences in job loss rates by industry, and the mixture of those distributions directly maps into the single distribution \(F_z\).

**Agglomeration and congestion spillovers.** It is straightforward to include standard agglomeration and congestion externalities into the model, following Diamond (2016) and Fajgelbaum and Gaubert (2019). Agglomeration externalities would call for subsidies to large, high wage cities, thus going in the opposite direction as the new spatial search externality on the firm side that I highlight in this paper. Congestion externalities would call for subsidies to small, low wage cities. The net optimal policy would account for all these margins.

**Trade costs.** Through the model, I maintain the assumption of a freely traded single final good, due to the lack of good data on commuting zone-level trade flows in France and in the US. In practice, some goods are more tradable than others. It is possible to include intra-national trade in this model. Assume that each job produces a differentiated variety subject to location-pair-specific trade costs \(\tau(\ell, a; \ell', a')\). In each location, perfectly competitive intermediate producers aggregate varieties from all locations with a Constant...
Elasticity of Substitution (CES) aggregator. They produce a non-traded good, which is used as an intermediate in production by firms which use labor in production.

In this model, exogenous local differences in productivity $\ell$ are complemented by a general equilibrium aggregator of bilateral trade costs, and market conditions in all other locations, resulting in an effective local productivity $P(\ell; \tau(\cdot), ...)$. Propositions 1 to 5 then extend directly when replacing the exogenous productivity $\ell$ by the endogenous productivity $P$. While trade adds a layer of general equilibrium conditions, the basic sorting and inefficiency properties continue to hold along with their implications for local labor market flows and policy.

**Identical employers and different workers.** Consider the alternative model where $z$ is attached to a worker instead of a job. Denote $\zeta = 1/z$. Suppose that all locations are ex-ante heterogeneous. Then the worker’s location problem is

$$\rho U(\zeta) = \max_{\ell} \frac{b + v(\zeta, \ell)}{r(\ell)^\omega}$$

where

$$v(\zeta, \ell) = \beta f(\ell) \left( \frac{B_0}{\bar{b} + v(\zeta, \ell)} \right)^\zeta \bar{S}(\zeta)$$

Take $b \to 0$ to simplify, so that

$$v(\zeta, \ell) \propto \left[ f(\ell) \bar{S}(\zeta) \right]^{1/\beta_{1+\zeta}}$$

and so

$$\zeta : \max_{\ell} \frac{1}{1 + \zeta} \log f(\ell) - \omega \log r(\ell)$$
The main difference is that workers take part of their productivity cutoff with them across space. Thus, if there is sorting, there is NAM between $\zeta$ and $f$, and so NAM between separation rates and finding rates. Can this assignment be sustained? From jobs’ free mobility,

$$J = q(\ell) \left( \frac{B_0}{b + v(\zeta(\ell), \ell)} \right)^{\zeta(\ell)} \tilde{S}(\zeta(\ell)) = b \rightarrow 0 q(\ell) \left( \frac{B_0}{v(\zeta(\ell), \ell)} \right)^{\zeta(\ell)} \tilde{S}(\zeta(\ell))$$

and so substituting back into the worker’s location choice,

$$1 \propto f^{-\frac{\alpha}{1+\alpha}} \left( \frac{B_0}{v} \right)^{\zeta} \tilde{S} \propto f^{-\frac{\alpha}{1+\alpha}} \left( \frac{B_0}{f^{\frac{1}{1+\alpha}} \tilde{S}^{\frac{1}{1+\alpha}}} \right)^{\zeta} \tilde{S} \propto f^{-\left[ \frac{\alpha}{1+\alpha} + \frac{\zeta}{1+\alpha} \right]} B_0^{\zeta} S(\zeta)^{\frac{1}{1+\alpha}}$$

and so

$$f(\ell) \propto \left[ B_0^{\zeta(\ell)} S(\zeta(\ell))^{\frac{1}{1+\alpha \zeta(\ell)}} \right]^{\frac{1}{1+\alpha}}$$

When $B_0$ is above 1, then $f(\zeta)$ becomes an increasing function. It can also be non-monotonic depending on parameter values. Thus, whether the sorting can be sustained is not robust to changes in parameter values.

To see this clearly, consider a Taylor expansion when $\zeta(\ell) \gg 1$. It is not hard to see that

$$\log f(\ell) \approx \zeta(\ell) - \infty \zeta(\ell) \cdot (1 - \alpha) \log B_0 + o(1)$$

and so the sign only depends on whether $B_0$ is larger or smaller than 1.
A.4 Estimation

A.4.1 Time-dependent KFE

The first step for the estimation is to compute an explicit solution to the time-dependent KFE:

\[ g_t = L^*_y g - (\Delta + \mu)g \]

where \( t \) denotes tenure at a job, and subscripts denote partial derivatives. Define

\[ g(t, y) = e^{-(\Delta + \mu) t} h(t, y) \]

Then \( g_t = e^{-(\Delta + \mu) t} (h_t - kh) \) so that

\[ h_t = L^*_y h \]  \hspace{1cm} (A.40) \]

The solution to this PDE is known. In logs, \( x = \log y \), define

\[ \Gamma(t, x) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(x+\delta t)^2}{2\sigma^2 t}} \quad ; \quad G(t, x, y) = \Gamma(t, x - y) - e^{\frac{2\delta}{\sigma^2(y-x)}} \cdot \Gamma(t, x + y - 2x) \]

Then it is straightforward to check that

\[ h(t, x) = \int_{x}^{\infty} G(t, x, y) h_0(y) dy \]

is the solution to (A.40) in logs, with initial distribution \( h_0 \). See Luttmer (2007) and references therein for a similar result. The details of the derivation are available upon request. Then, in logs,

\[ g(t, x) = e^{-(\Delta + \mu) t} \int_{x}^{\infty} G(t, x, y) g_0(x_0) dx_0 \]  \hspace{1cm} (A.41)
is the time-dependent distribution of log productivity across employed workers in a location with log cutoff \( x \) given a starting distribution \( g_0 \).

### A.4.2 Tenure profile of job loss

Now fix a location \( \ell \) and omit \( \ell \) indices for simplicity. Normalize \( x \equiv 0 \) without loss of generality. The flow of workers into local unemployment is

\[
\text{Endog. Sep.}(t) = \frac{\sigma^2}{2} \frac{\partial g}{\partial x}(t, x)
\]

which can be calculated at all times using the explicit solution (A.41). Straightforward yet lengthy algebra leads to

\[
\text{Endog. Sep.}(t) = e^{-(\mu + \Delta)t} \left[ \frac{(s/\hat{\delta})}{\sqrt{t}} \varphi\left(\hat{\delta} \sqrt{t}\right) + \frac{(s/\hat{\delta})^2}{2} e^{-(s/\hat{\delta})^2 t} \left\{ e^{st} \Phi\left(-\left(\hat{\delta} + s/\hat{\delta}\right) \sqrt{t}\right) - e^{-st} \Phi\left((\hat{\delta} - s/\hat{\delta}) \sqrt{t}\right) \right\} \right]
\]

where \( s = \delta \zeta \) is the local average job losing rate (into local unemployment), and \( \hat{\delta} = \delta / \sigma \).

To get the time-aggregated job losing rate in the first year, denoted \( s_1(s, \hat{\delta}, D) \), integrate between 0 and 1 against \( g \). Yet lengthier algebra leads to

\[
s_1(s, \hat{\delta}, D) = \frac{(s/\hat{\delta})}{4} \left\{ \frac{e^{-(D+s-(s/\hat{\delta})^2/2)}}{D + s - (s/\hat{\delta})^2/2} \left( s/\hat{\delta} \right) + 4 \text{Erf} \left[ \frac{\sqrt{2D+s-(s/\hat{\delta})^2/2}}{2D + \hat{\delta}^2} \right] + \frac{s/\hat{\delta}}{D + s - (s/\hat{\delta})^2/2} \left( \frac{\hat{\delta} - s/\hat{\delta}}{\sqrt{2D + \hat{\delta}^2}} \right) + e^{-(D+s-(s/\hat{\delta})^2/2)} \text{Erf} \left[ \frac{\hat{\delta} - (s/\hat{\delta})}{\sqrt{2}} \right] \right\} + \frac{(s/\hat{\delta})}{D + s - (s/\hat{\delta})^2/2} \left( \frac{s/\hat{\delta}}{\sqrt{2D + \hat{\delta}^2}} \right) + e^{D+s-(s/\hat{\delta})^2/2} \text{Erfc} \left[ \frac{\hat{\delta} + (s/\hat{\delta})}{\sqrt{2}} \right] \right\}
\]
where \( D = \Delta + \mu \), and Erf denotes the error function (a transformation of the Gaussian cumulative function). In the limit of a small \( D \), it is tedious but straightforward to check that \( s_1 \) is a decreasing function of \( \hat{\delta} \).

### A.4.3 Tenure profile of wages

A similar computation as for job loss first delivers average labor productivity net of human capital by tenure

\[
\bar{y}(t) = s/\hat{\delta}/\sigma - \frac{1}{2} \left( e^{A_0(s,\sigma,\hat{\delta})t} \text{Erf}[A_1(s,\hat{\delta})\sqrt{t}] - \text{Erf}[A_2(\sigma,\hat{\delta})\sqrt{t}] + 1 \right)\text{Erfc}[A_1(s,\hat{\delta})\sqrt{t}] - 2 \right)

\[
+ \frac{e^{A_4(s,\hat{\delta},D)t}}{s/\hat{\delta}/\sigma - 1}
\]

\[
\equiv Y(t,s,\sigma,\hat{\delta},D)
\]

where \( A_0(s,\sigma,\hat{\delta}) = \frac{1}{2} \left( (s/\hat{\delta})^2 - \sigma^2 - 2s + 2\sigma\hat{\delta} \right), A_1(s,\hat{\delta}) = \frac{\hat{\delta} - s/\hat{\delta}}{\sqrt{2}}, A_2(\sigma,\hat{\delta}) = \frac{\hat{\delta} - \sigma}{\sqrt{2}}, A_3(\sigma, D, \hat{\delta}) = \sigma\hat{\delta} + D - \sigma^2 / 2.\)

Average wages of continuing jobs at tenure \( t \) relative to new jobs in a given location at calendar time \( t_{\text{cal}} \) are then

\[
\frac{W(t,t_{\text{cal}})}{W_{\text{new}}(t_{\text{cal}})} = \left( s/\hat{\delta}/\sigma - 1 \right) \left[ (1 - \beta) + \beta y_0 / \hat{\rho} Y(t,s,\sigma,\hat{\delta},D) \right]
\]

\[
\equiv w(t,\hat{\delta},\sigma,s)
\]

When \( \hat{\delta}, D \) are small, lengthy algebra ensures that the slope of \( Y \) with tenure falls as \( \sigma \) rises. When \( \beta \) is small enough,

\[
\log w(t,\hat{\delta},\sigma,s) \approx \text{constant} + \log(s/\hat{\delta}/\sigma - 1) + \log Y(t,s,\sigma,\hat{\delta},D)
\]
Assuming that $\beta$ is small, this equation can be taken to the data with NLLS, using

$$\log \frac{W_{t_{\text{cal}}, t_{\text{tenure}}, c}}{W_{t_{\text{cal}}, c}} = \log w(t_{\text{tenure}}, \hat{\delta}, \sigma, s_c),$$ \hspace{1cm} (A.42)

where $W_{t_{\text{cal}}, t_{\text{tenure}}, c}$ denotes average wages in city $c$ at calendar time $t_{\text{cal}}$ and tenure $t_{\text{tenure}}$. $W_{t_{\text{cal}}, c}$ denotes average wages of new jobs. I time-aggregate (A.42) at the quarterly frequency.

In practice, $\beta$ may not be small, so it must be estimated jointly with $\sigma$. To do so, I numerically search for the pair $(\beta, \sigma)$ that minimizes the sum of square deviations from equations (A.42) and the labor share equation in the main text.

### A.4.4 Labor share

From the bargaining solution, the labor share in location $\ell$ is

$$\text{Labor Share}(\ell) = \frac{(1 - \beta)(b + v(\ell) + \beta y_0 / \rho H(\ell))}{E(\ell)}$$ \hspace{1cm} (A.43)

where $H(\ell) = \mathbb{E}_t[y / y | y \geq y]$ is expected labor productivity in location $\ell$ under the invariant distribution. Using the solution to the KFE, one obtains

$$H(\ell) = \frac{\kappa \zeta(\ell)}{(\kappa - 1)(\zeta(\ell) - 1)} = \frac{\kappa}{(\kappa - 1)(1 - z(\ell))} \equiv H(z(\ell))$$

### A.4.5 Learning parameters

Log real wages are proportional to $K_t R_t^{-\omega}$, where $t$ is calendar time, $K_t$ the average knowledge of the economy and $R_t$ average house prices. In the data, economy-wide log real wages grow by 0.0015 each quarter. In the model, $K_t R_t^{-\omega} \propto K_t^{1 - \omega - \psi}$ up to a constant. Thus, $\lambda = \frac{0.0015}{1 - \omega - \psi} = 0.0023$. 

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Then, notice that all workers who become unemployed in a given location have the same wage when they are laid off: the reservation wage. While they are unemployed, their human capital grows at rate $\lambda - \varphi$. When they find a new job, they draw a productivity from the local new job distribution, which is independent from their history. Therefore, equation (1.21) obtains. In empirical specifications, I follow the literature and restrict the sample to workers that held a job for at least two years before becoming unemployed. This restriction ensures that the estimates are not driven by temporary jobs.

The model abstracts from additional mechanisms that could create a correlation between new productivity draws and workers’ past unemployment or employment history, as in Jarosch (2015). Thus, in practice, equation equation (1.21) may deliver a biased estimate of the depreciation rate of human capital. To address such concerns, I run version of equation (1.21) with additional controls that account flexibly for workers’ past employment history. I also control for worker-level unobserved heterogeneity. For completeness, I also propose a specification where I use employed workers as a control group in a difference-in-difference specification – although this control group introduces an additional endogeneity problem. Results for the estimate of $\lambda - \varphi$ are reported in the first row of Table A.3. The point estimate remains stable across specifications, although controlling for worker-level heterogeneity reduces the coefficient by half.

**A.4.6 Lower bound of productivity draws**

I propose a microfoundation of job search that allows to use data on duration since last job offer to inform $Y$. In the LFS, there is data on duration since last contact with the national unemployment agency (at the time called ANPE, “Agence Nationale Pour l’Emploi”, now called “Pôle Emploi”) and duration since last job offer. The latter is not necessarily an offer that came from the ANPE.

To leverage this data, I assume that individuals contact either the national unemployment agency, or the private sector with intensity $S$. Conditional on a contact, it is a contact with
Table A.3: Unemployment scar estimation. Dependent variable = post-unemployment log wage

<table>
<thead>
<tr>
<th></th>
<th>Unemployed only</th>
<th>DiD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Job loss × Duration</td>
<td>-0.019***</td>
<td>-0.012***</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Job loss</td>
<td></td>
<td>-0.086*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.041)</td>
</tr>
<tr>
<td>Duration</td>
<td></td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.001)</td>
</tr>
<tr>
<td>Pre log wage</td>
<td>0.535***</td>
<td>0.421***</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.021)</td>
</tr>
<tr>
<td>Skill</td>
<td>0.003***</td>
<td>0.003***</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>Fixed Effects</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Year</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2-digit Industry</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>City</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Worker</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Obs.</td>
<td>30952</td>
<td>30952</td>
</tr>
<tr>
<td>R²</td>
<td>0.029</td>
<td>0.270</td>
</tr>
<tr>
<td>W.-R²</td>
<td>0.012</td>
<td>0.257</td>
</tr>
</tbody>
</table>

Standard errors in parenthesis, two-way clustered by city and 2-digit industry.

+ $p < 0.10$,  $^* p < 0.05$,  $^{**} p < 0.01$,  $^{***} p < 0.001$.

the ANPE with probability $s$, and a contact with the private sector with probability $1 - s$. Conditional on contacting the ANPE, workers they receive an offer with probability $\omega$. Conditional on receiving an offer, they accept it with probability $a$. Conditional on a private sector contact, they receive offers with conditional probability $\tau$. They accept them with conditional probability $a$. The key is that the conditional acceptance probability $a$ is the same. Allowing for private sector contacts is also important because only 6.58% of jobs are found through the ANPE.
Unemployment duration in sample of unemployed individuals. The rate at which an individual leaves unemployment in $dt$ is $S(s \omega + (1 - s) \tau) a \cdot dt$. Therefore, the probability that a currently unemployed individual has been unemployed for exactly $n$ small $dt$ periods is

$$p_n \propto [1 - S(s \omega + (1 - s) \tau) a dt]^n$$

Note that a given amount of time is $T = n \cdot dt$. The expected unemployment duration in a sample of unemployed individuals is thus

$$D^U = \left\{ \frac{1}{S(s \omega + (1 - s) \tau) a} \cdot \frac{1}{dt} - 1 \right\} \cdot dt \rightarrow_{dt \to 0} \frac{1}{S(s \omega + (1 - s) \tau) a}$$

So $D^U = e^{S(s \omega + (1 - s) \tau) a}$.

Composition of job findings. The probability that an individual finds a job in a quarter through the ANPE is $s \omega a$, and through other channels $(1 - s) \tau a$. Therefore, the probability that an employed individuals has found a job through ANPE is

$$P^{ANPE} = \frac{s \omega a}{s \omega a + (1 - s) \tau a}$$

At this point, one can thus identify $x \equiv Ss \omega a$ and $y \equiv S(1 - s) \tau a$: $x + y = \frac{1}{P^\nu}$ and $x = P^{ANPE} \times (x + y)$

ANPE contacts. The probability that a currently unemployed individual has last contacted the ANPE $n$ periods ago and did not find a job is thus

$$p_n \propto \left[ 1 - S dt + S(1 - s)(1 - \tau a) dt \right]^n$$
So the expected duration since the last ANPE contact is, similarly to before,

\[ D^C = \frac{1}{\frac{1}{S(1 - (1-s)(1-\tau a))} = \frac{1}{Ss + y}} \]

which identifies \( Ss \) given \( x \) and \( y \): \( Ss = \frac{1}{DC} - y \). Hence, \( X = \omega a = \frac{x}{Ss} \) is known and \( z = \frac{x}{y} = \frac{\omega \cdot s}{1 - s} \), and so \( \omega/\tau \).

**ANPE offers.** Similarly, the probability that a current unemployed worker has last received an offer from ANPE \( n \) periods ago is

\[ q_n \propto \left[ 1 - Sdt + S((1-s)(1-\tau a) + s(1-\omega))dt \right]^n = \left[ 1 - S(1 - (1-s)(1-\tau a) - s(1-\omega))dt \right]^n \]

So the expected duration since the last ANPE offer is

\[ D^O = \frac{1}{S(1 - (1-s)(1-\tau a) - s(1-\omega))} \]

Re-write this as \( \frac{1}{D^O} = Ss\omega + y \), which identifies \( Ss\omega = \frac{1}{D^O} - y \), and therefore \( a = \frac{x}{Ss\omega} \).

**A.4.7 Local wages**

Wages in location \( \ell \) are given by

\[ \bar{w}(\ell) = W_0p \cdot \left( \frac{\Delta}{\Delta + \varphi u(\ell)} \right) r(\ell)^{-\psi_y(\ell)} \left[ (1 - \beta) \frac{\rho}{y_0} \right] + \beta H(s(\ell)) \] \hspace{1cm} (A.44)

where \( W_0 \) is a general equilibrium constant.
A.4.8 Housing elasticity

Using (A.31) together with the solution for average wages in a location $W(\ell)$, housing prices can be expressed as

$$r(\ell)^{1+\eta} = \frac{\bar{w}(\ell)}{1 - \beta + \beta y_n/\hat{\rho}E(s(\ell))} \cdot L(\ell, a)G(s(\ell)/\delta, v(\ell))$$

The right-hand-side defines $r_0$, and involves parameters that have been estimated or data.

A.4.9 Migration elasticity

To circumvent endogeneity in the OLS regression version of (1.23), I use changes in predicted local employment as an instrument. I break down the sample in two subperiods, and, in this section only, I use the notation $\Delta$ to refer to changes between these two periods. Specifically, I use predicted changes in local employment $\Delta E_c$ from Appendix A.1.2. To understand this instrument within the model, assume that there is a set $\mathcal{J} = \{1, ..., J\}$ of industries. Employers in each industry draw from the same productivity distribution $F_z$. Locations $c$ differ in a set of industry-specific productivities $\{p_{jc}\}_j$. Consistent with larger cross-industry flows than cross-location worker flows, suppose that there is a single labor market for all industries within a location. Suppose further that the cross-industry variance in industry productivity $\text{Var}_c(p_{jc})$ is much smaller than the cross-location variance in city productivity $\text{Var}_j(p_{jc})$. This assumption implies that the industrial mix is not strongly predictive of the local unemployment rate, consistent with the data. Under this assumption, the single-industry model is also a close approximation to the multi-industry model.

Now consider a set of industry-wide shocks that change $p_{jc}$ to $p'_{jc} = \hat{p}_{jc}$. Vacancy creation reacts to changes $\hat{p}_j$, so that national employment in industry $j$ is positively correlated with $\hat{p}_j$. Similarly, employment shares $E_{jc,0}$ in the first subperiod are correlated with $p_{jc}$. Suppose that (1) $\hat{p}_j$ are uncorrelated with $p_{jc}$, (2) $\hat{p}_j$ are i.i.d. across industries. Then $\hat{p}_j$ are uncorrelated with changes in amenities $\Delta a_c$ in the population-weighted distribution of
cities and industries, even if $E_c[p_{jc}]$ are correlated with amenities. With a large number of industries and locations, the shift share $\Delta E_c$ is thus correlated with the average change in local productivity $E_c[\Delta p'_{jc}]$. In general equilibrium, employers relocate in each industry, and so $\Delta E_c$ is also correlated with $E_c[\Delta \zeta_{jc}]$. If anything, this correlation makes the instrument stronger. The crucial exclusion restriction is that $\Delta E_c$ is uncorrelated with changes in amenities $\Delta a_c$. Therefore, it constitutes a valid instrument in this augmented model with small industry heterogeneity.

**A.4.10 Productivity distribution**

To estimate $F_z$, I first recover firm quality in each location using (1.22). It is easier to work with the reciprocal of firm quality $z$, denoted $\zeta = 1/z$. Consider locations with profitability in $(\ell - d\ell, \ell]$. Because the job losing rate is strictly decreasing in $\ell$, they are exactly those with a job loss rate in $[s(\ell), s(\ell) + ds(\ell)]$. Due to the model’s sorting implications, the mass of open jobs in those locations is proportional to $f_\zeta(\zeta(\ell)) d\zeta(\ell) = \delta^{-1} f_\zeta(\zeta(\ell)) ds(\ell)$.

For simulations, I estimate a Beta distribution for $\zeta$: $f_\zeta(\zeta) \propto \left(\frac{\zeta - \zeta}{\zeta - \zeta}\right)^{g_2} \left(\frac{\zeta - \zeta}{\zeta - \zeta}\right)^{g_1}$ which is equivalent to a Beta distribution for $z$. I estimate the Beta distribution by minimizing the mean square error between the empirical density function (a histogram) and the Beta density.

**A.4.11 Matching function elasticity**

Start from

$$\theta(\ell) = \left(\frac{f_\zeta(\zeta(\ell)) |\zeta'(\ell)|}{f_\zeta(\ell) \mathcal{U}(\ell)}\right) J(\ell)^\gamma$$

But

$$J(\ell) = q(\ell) \bar{J}(\ell) \propto \theta(\ell)^{-\alpha} \bar{J}(\ell)$$
\[
\hat{J}(\ell) \propto \mathbb{E}(u(\ell))^{\frac{\omega + \epsilon(1 + \eta)}{\psi + \omega + (1 + \psi + \omega)}} \cdot \ell(b + v(\ell))^{1 - \frac{\psi}{\omega + \psi + (1 + \psi + \omega)}} \cdot G(v(\ell), \zeta(\ell))^{\frac{\epsilon \psi}{\omega + \psi + (1 + \psi + \omega)}} \left(\frac{B}{b + v(\ell)}\right)^{\zeta(\ell)} \cdot S(\zeta(\ell))
\]

Therefore,

\[
\theta(\ell)^{1 + \alpha \gamma} \propto \left(\frac{f(\xi(\ell))}{f(\ell)} \cdot U(\ell)\right)^{1 - \alpha} \hat{J}(\ell)^{\gamma}
\]

and so

\[
f_R(\ell) \left(\frac{B}{b + v(\ell)}\right)^{\zeta(\ell)} \propto \left(\frac{f(\xi(\ell))}{f(\ell)} \cdot U(\ell)\right)^{\frac{1 - \alpha}{1 + \alpha \gamma}} \hat{J}(\ell)^{\gamma}\frac{(1 - \alpha)}{1 + \alpha \gamma}
\]

Taking logs delivers

\[
\log \left(\frac{f_R(\ell)}{f(z(\ell))} \cdot \mathbb{E}(u(\ell))^{\frac{\omega + \epsilon(1 + \eta)}{\psi + \omega + (1 + \psi + \omega)}} \cdot \ell(b + v(\ell))^{1 - \frac{\psi}{\omega + \psi + (1 + \psi + \omega)}} \cdot G(v(\ell), \zeta(\ell))^{\frac{\epsilon \psi}{\omega + \psi + (1 + \psi + \omega)}} \left(\frac{B}{b + v(\ell)}\right)^{\zeta(\ell)} \cdot S(\zeta(\ell))\right)
\]

where recall that \( U(\ell) \) denotes the number of unemployed workers in location \( \ell \), and \( \hat{J}(\ell, y(\ell), z(\ell)) \) is now known. At this stage, both right-hand-side variables can be calculated. In the model, equation (A.45) can be estimated with OLS. It is not hard to add location-specific heterogeneity in the matching function efficiency or vacancy costs to the model. In that case a structural residual correlated with the right-hand-side variables arises. In contrast to the previous estimating equations, this structural residual leads to omitted variable bias in equation (A.45).

With OLS, \( \alpha, \gamma \) are separately identified only through functional form differences between the right-hand-side variables because both are functions of the same latent variable \( \ell \). 2SLS also relies on functional form identification. Thus, I use the local shift-share shock and a non-linear transformation thereof as two instruments. Notice also that in the generalized model with omitted variable bias, the latter only affects the estimation of equation (A.45), and not
the previous estimating equations. Indeed, the previous estimating equations condition on the observed job losing rate, which is enough to control for the omitted variables through local job quality $z(\ell)$.

I first-difference (A.45) between the two subperiods. I use as the first instruments the same shift-share shocks $\Delta E_c$. Under the same assumptions as in section A.4.9, it is a valid instrument. To obtain a second instrument, I de-mean $\Delta E_c$ and use $1_{\Delta E_c > 0}$. This is a nonlinear transformation of $\Delta E_c$. Strengthening the identification assumption to conditional independence makes it a valid instrument.

### A.4.12 Over-identification

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(Wage)</td>
<td>-0.192*</td>
<td>-0.114***</td>
</tr>
<tr>
<td></td>
<td>(0.084)</td>
<td>(0.027)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.728***</td>
<td>0.724***</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.003)</td>
</tr>
<tr>
<td>Obs.</td>
<td>348</td>
<td>348</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.066</td>
<td>0.046</td>
</tr>
</tbody>
</table>

Table A.4: Cross-sectional regression of labor share onto log wages.

Standard errors in parentheses

$\dagger p < 0.10, \,* p < 0.05, \,** p < 0.01, \,*** p < 0.001

Standard errors in the model regressions reported for completeness.

They are population objects and do not reflect statistical uncertainty.
Table A.5: Correlation of estimated amenities with observables.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Weather</strong></td>
<td></td>
</tr>
<tr>
<td>Sun hours</td>
<td>0.333*</td>
</tr>
<tr>
<td></td>
<td>(0.151)</td>
</tr>
<tr>
<td><strong>Services</strong></td>
<td></td>
</tr>
<tr>
<td>Basic public</td>
<td>0.051</td>
</tr>
<tr>
<td></td>
<td>(0.132)</td>
</tr>
<tr>
<td>Education</td>
<td>0.113</td>
</tr>
<tr>
<td></td>
<td>(0.107)</td>
</tr>
<tr>
<td>Health</td>
<td>0.223**</td>
</tr>
<tr>
<td></td>
<td>(0.041)</td>
</tr>
<tr>
<td>Commercial</td>
<td>0.098</td>
</tr>
<tr>
<td></td>
<td>(0.085)</td>
</tr>
<tr>
<td><strong>Obs.</strong></td>
<td>288</td>
</tr>
<tr>
<td><strong>$R^2$</strong></td>
<td>0.517</td>
</tr>
</tbody>
</table>

Robust S.E. in parenthesis.

$^+ p < 0.10$, $^* p < 0.05$, $^{**} p < 0.01$.  

Log amenities on log sun hours per month and log service establishments.
A.4.13 Validation

Table A.6: Plant-level labor productivity across space.

<table>
<thead>
<tr>
<th></th>
<th>Log</th>
<th>Growth rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>VA/N</td>
<td>VA/N</td>
<td>VA</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Geography</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Job losing rate</td>
<td>-0.373***</td>
<td>-0.241</td>
<td>-0.140</td>
<td>0.009</td>
<td>-0.002</td>
<td>0.027**</td>
</tr>
<tr>
<td></td>
<td>(0.079)</td>
<td>(0.150)</td>
<td>(0.143)</td>
<td>(0.012)</td>
<td>(0.016)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>Entrant × Job losing rate</td>
<td>-0.433**</td>
<td>-0.317+</td>
<td>-0.516***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.142)</td>
<td>(0.174)</td>
<td>(0.131)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Skill mix controls</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Skill mix</td>
<td>0.287**</td>
<td>0.001</td>
<td></td>
<td>0.005</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.107)</td>
<td>(0.029)</td>
<td></td>
<td>(0.005)</td>
<td>(0.003)</td>
<td></td>
</tr>
<tr>
<td>Entrant × Skill mix</td>
<td>0.085**</td>
<td>0.025</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
<td>(0.039)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fixed Effects</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Year × Entry status</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2-digit industry × Entry status</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

| Obs.   | 785252 | 383099 | 383099 | 692951 | 333873 | 333873 |
| R²     | 0.173  | 0.213  | 0.084  | 0.005  | 0.006  | 0.003  |

Standard errors in parenthesis, two-way clustered by city and 2-digit industry.

+ p < 0.10, * p < 0.05, ** p < 0.01, *** p < 0.001. Annual frequency, 2003-2005.

Davis-Haltiwanger growth rate. Continuing plants only. Entrant defined as less than two year old.

All value added per worker regressions are employment-weighted.

Job loss at survivors and exiters. Local job losing rates can be broken down into the contribution of job loss at surviving establishments and the contribution of exiting establishments. I plot that simple accounting decomposition in Figure A.12. The y-axis values of the blue circles and the green diamonds add up to the 45 degree line in orange. It shows that job loss at surviving establishments is the dominant source of geographical variation in the job losing rate, accounting for 89% of the cross-location variance.
Figure A.12: Job loss from surviving and exiting establishment, by French commuting zones.

Table A.7: Job losing rate at new plants within multi-plant openings.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local job losing rate</td>
<td>1.34**</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>(0.42)</td>
<td>(0.38)</td>
</tr>
</tbody>
</table>

Controls and FEs

- Skill-Year ✓ ✓
- 3-digit industry-Year ✓ ✓
- Firm-Year ✓

Obs. 21676 20046

$R^2$ 0.143 0.635

S.E.s in parenthesis, two-way clustered by commuting zone and 2-digit industry.

$^+ p < 0.10, ^* p < 0.05, ^** p < 0.01, ^*** p < 0.001.$
### A.5 Results: robustness

Table A.8: Robustness to bargaining power and search assumptions.

<table>
<thead>
<tr>
<th></th>
<th>Model</th>
<th></th>
<th>Data</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta \neq \alpha$</td>
<td>$\beta = \alpha$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entry cost estimation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Estimation</td>
<td>$\beta \neq \alpha$</td>
<td>$\beta \neq \alpha$</td>
<td>$\beta = \alpha$</td>
<td>$\beta = \alpha$</td>
</tr>
<tr>
<td>Bargaining power</td>
<td>$\beta \neq \alpha$</td>
<td>$\beta \neq \alpha$</td>
<td>$\beta = \alpha$</td>
<td>$\beta = \alpha$</td>
</tr>
<tr>
<td>Search</td>
<td>Random</td>
<td>Directed</td>
<td>Random</td>
<td>Random</td>
</tr>
<tr>
<td>Aggregate unemployment rate</td>
<td>0.076</td>
<td>0.071</td>
<td>0.127</td>
<td>0.076</td>
</tr>
<tr>
<td>St. dev. unemployment rate</td>
<td>0.022</td>
<td>0.004</td>
<td>0.002</td>
<td>0.004</td>
</tr>
<tr>
<td>Var. log unemp. / emp.</td>
<td>0.079</td>
<td>0.002</td>
<td>0.002</td>
<td>0.012</td>
</tr>
<tr>
<td>Job losing rate</td>
<td>85 %</td>
<td>-180 %</td>
<td>380 %</td>
<td>183 %</td>
</tr>
<tr>
<td>Job finding rate</td>
<td>15 %</td>
<td>280 %</td>
<td>-280 %</td>
<td>-83 %</td>
</tr>
</tbody>
</table>
Appendix B

Location as an Asset

B.1 Appendix: Proofs for the model in Section 2.2

B.1.1 Proof of Lemma 3

We split the proof in three parts:

1. Location decisions of constrained and unconstrained individuals

2. Equilibrium in cities in which at least one unconstrained individual lives

3. Equilibrium in cities with only constrained individuals

Location decisions

Recall that for unconstrained individuals,

\[ R = \frac{s}{q'(z)} \]

Therefore, unconstrained individuals of skill \( s \) locate in cities \( Z^U(s) \) such that

\[ R = \frac{s}{q'(Z^U(s))} \]

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In addition, some constrained individuals may choose cities in which only constrained individuals locate. For those individuals, we cannot use the expression above, and we directly use the mobility Euler equation:

\[
\frac{(y_1 + Ra) + zSC(y_0, y_1, z)}{\beta[y_0 - a - q(z)]} = \frac{SC(y_0, y_1, z)}{q'(z)}
\]

which implies

\[
SC(y_0, y_1, z) = \frac{q'(z)(y_1 + Ra)}{\beta[y_0 - a - q(z)] - zq'(z)}
\]

(B.1)

Notice that for constrained individuals \((y_0, y_1, SC(y_0, y_1, z))\) who locate in a city \(z\) where at least one unconstrained individual with skill \(SU(z)\) lives, we can substitute out \(q'(z) = SU(z)/R\), leading to

\[
SC(y_0, y_1, z) = \frac{SU(z)(y_1 + Ra)}{\beta R (y_0 - a - q(z)) - zSU(z)}
\]

(B.2)

In the sequel, it will be useful to have notation for this relationship in terms of all the endogenous objects. Therefore, we define

\[
X(y_0, y_1, s, ZU(s), q(ZU(s))) = \frac{s(y_1 + Ra)}{\beta R (y_0 - a - q(ZU(s))) - ZU(s)s}
\]

(B.3)

Equation (B.3) describes which constrained individuals \((y_0, y_1, X(y_0, y_1, s, ZU(s), q(ZU(s))))\) choose to locate in city \(ZU(s)\).

To obtain the lowest possible income in a given city, we can re-write equation (B.2) as

\[
y_0 = a + q(z) + \frac{1}{\beta R} [zSU(z) + \frac{(y_1 + Ra)SU(z)}{SC(y_0, y_1, z)}]
\]

(B.4)
This delivers the lower bound on initial income for constrained individuals who locate in city \( z \) with at least an unconstrained individual:

\[
y_0 \geq Y_0(z) = a + q(z) + \frac{1}{\beta R} [z s^U(z) + (y_1 + R a) \times \frac{S^U(z)}{s}]
\]

A similar bound involving \( q'(z) \) holds for cities in which only unconstrained individuals live.

**Equilibrium in cities with at least one unconstrained individual**

We first consider equilibrium in cities with at least one constrained individual. Because at any skill, constrained individuals locate in worse cities that unconstrained individuals, cities with unconstrained individuals have higher \( z \) than those with only constrained individuals. Thus, there exists a cutoff \( \hat{z} \) such that a city has at least one unconstrained individual iff \( z \geq \hat{z} \).

We start by assuming that the matching function \( Z^U(s) \) is increasing at all \( s \). Total population that locates in cities \([Z^U(s), Z^U(s) + Z^U_s(s)ds] \) is the sum of the unconstrained individuals of the same skill and constrained individuals of higher skill. Before expressing total population, we denote by

\[
\tilde{A}(y_0, y_1, s, Z^U(s), q(Z^U(s))) = y_0 - q(Z^U(s)) - \frac{y_0 - q(Z^U(s)) + \frac{y_1 + sZ^U(s)}{R}}{1 + \beta}
\]

desired savings as a function of individual characteristics and the matching function. Using the notation we defined, we can express total population as:

\[
G(s, Z^U(s), q(Z^U(s)), Z^U_s(s))
\equiv \int \int f(y_0, y_1, s) \mathbf{1} \left[ \tilde{A}(y_0, y_1, s, Z^U(s), q(Z^U(s))) > a \right] dy_0 dy_1
\]

\[
+ \int \int \mathbf{1} \left[ \tilde{A}(y_0, y_1, s, Z^U(s), q(Z^U(s))) \leq a \right] \times f(y_0, y_1, X(y_0, y_1, s, Z^U(s), q(Z^U(s)))) \times \frac{d[X(y_0, y_1, s, Z^U(s), q(Z^U(s)))]}{ds} dy_0 dy_1
\]
where it is understood that $\frac{d[X(y_0, y_1, s, Z^U(s), q(Z^U(s)))]}{ds}$ is the total derivative of $s \mapsto X(y_0, y_1, s, Z^U(s), q(Z^U(s)))$ with respect to $s$. We can calculate this last term explicitly:

$$
\frac{d[X(y_0, y_1, s, Z^U(s), q(Z^U(s)))]}{ds} \equiv X_0(y_0, y_1, s, Z^U(s), q(Z^U(s)))
$$

where an $s$ subscript denotes a derivative w.r.t. $s$, and where we define

$$
X_0(y_0, y_1, s, Z^U(s), q(Z^U(s))) = \frac{y_1 + Ra}{\beta R (y_0 - a - q(Z^U(s))) - \overline{Z^U(s)}s} s\overline{Z^U(s)}(y_1 + Ra)
$$

$$
= \frac{\beta R (y_0 - a - q(Z^U(s))) - \overline{Z^U(s)}s} {\beta R (y_0 - a - q(Z^U(s))) (y_1 + Ra)}
$$

and

$$
X_1(y_0, y_1, s, Z^U(s), q(Z^U(s))) = \frac{s^2 Z^U(s)(y_1 + Ra)} {(\beta R (y_0 - a - q(Z^U(s))) - \overline{Z^U(s)}s)^2}
$$

We now make use once again of the mobility Euler equation $q'(Z^U(s)) = R/s$ to re-write

$$
\frac{d[X(y_0, y_1, s, Z^U(s), q(Z^U(s)))]}{ds} = X_0(y_0, y_1, s, Z^U(s), q(Z^U(s)))
$$

$$
+ X_1(y_0, y_1, s, Z^U(s), q(Z^U(s))) \times Z^U(s)
$$

Substituting these expressions into our expression for the supply of individuals in cities $[Z^U(s), Z^U(s) + Z^U(s)ds]$, we obtain

$$
G(s, Z^U(s), q(Z^U(s)), Z^U(s)) = A(s, Z^U(s), q(Z^U(s))) + B(s, Z^U(s), q(Z^U(s))) \times Z^U(s)
$$

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where we defined

\[
A(s, Z^u(s), q(Z^u(s))) = \int \int f(y_0, y_1, s) \mathbb{1} \left[ \tilde{A}(y_0, y_1, s, Z^u(s), q(Z^u(s))) > a \right] dy_0 dy_1 \\
+ \int \int \mathbb{1} \left[ \tilde{A}(y_0, y_1, s, Z^u(s), q(Z^u(s))) \leq a \right] \\
\times f(y_0, y_1, X(y_0, y_1, s, Z^u(s), q(Z^u(s)))) \\
\times X_0(y_0, y_1, s, Z^u(s), q(Z^u(s))) dy_0 dy_1
\]

\[
B(s, Z^u(s), q(Z^u(s))) = \int \int \mathbb{1} \left[ \tilde{A}(y_0, y_1, s, Z^u(s), q(Z^u(s))) \leq a \right] \\
\times f(y_0, y_1, X(y_0, y_1, s, Z^u(s), q(Z^u(s)))) \\
\times X_1(y_0, y_1, s, Z^u(s), q(Z^u(s))) dy_0 dy_1
\]

Now, equating total population supply to total housing supply:

\[
h(Z^u(s))L(Z^u(s))Z^u_s(s) = G(s, Z^u(s), q(Z^u(s)), Z^u_s(s))
\]

where recall that \( h(z) \) is the density of cities with income \( z \). Re-arranging,

\[
Z^u_s(s) = \frac{A(s, q(Z^u(s)), Z^u(s))}{h(Z^u(s))L(Z^u(s)) - B(s, q(Z^u(s)), Z^u(s))}
\]

It is easier at this stage to write the system in terms of the inverse matching function for unconstrained individuals \( S^U(z) \) for the range of cities in which unconstrained individuals live. Using the mobility Euler equation again, we finally obtain a nonlinear system of coupled Ordinary Differential Equations (ODEs):

\[
S^U_z(z) = \frac{h(z)L(z) - B(S^U(z), Q(L(z)), z)}{A(S^U(z), Q(L(z)), z)}
\]

\[
L_z(z) = \frac{R}{S^U(z)Q'(L(z))}
\]
where recall that house prices are given by \( q(z) = Q(L(z)) \). The boundary conditions of this system are \( S^U(\bar{z}) = \bar{s} \), and \( S^U(\hat{z}) \) given by total population supply, as defined below. When \( s > 0 \) and \( f \) is bounded, inspection of this system reveals that it is uniformly Lipschitz continuous. In addition, the solution, if it exists, must be bounded. Indeed, diverging \( S^U \) or \( L(z) \) are ruled out by our compact support assumptions and by the fact that house prices cannot exceed income which is bounded above. Thus, conditional on boundary conditions, standard results ensure existence and uniqueness of a global solution to this system.

Recall that we assumed that the matching function \( Z^U(s) \) was locally increasing. We now show that the matching function \( Z^U(s) \) cannot be decreasing. The ODE without assuming that the matching function is increasing would be \( |S^U(z)| = \frac{h(z)L(z) - B(S(z), Q(L(z)), z)}{A(S(z), Q(L(z)), z)} \). Then, if the matching function has negative slope negative at some \( z_0 \), since the right-hand-side is of constant sign and the matching function \( Z^U(s) \) cannot be flat (otherwise we would have a mass point, ruled out through the price function), the matching function \( S(z) \) cannot have a zero and hence is decreasing everywhere. Thus, house prices are concave throughout the support (from the no-arbitrage condition). Then we have \( q'(z) = S(z)/R < \bar{s}/R \), and hence \( q(z) < q(S^U(\hat{z}))+\bar{s}z/R \). Substituting back into the budget constraint of the individuals with skill in \((\bar{s} - ds, \bar{s}]\), they would have an incentive to increase their city choice, since this would yield a higher return on housing. This violates the Second Order Condition for optimality, and hence cannot hold in equilibrium.

**Equilibrium in cities with only constrained individuals**

We now turn to cities in which only constrained individuals live. We will apply the exact same logic as in the case for cities with at least one unconstrained individuals. We first define notation that is the counterpart of \( S^C(y_0, y_1, z) \), but makes explicit the dependence
on all endogenous objects:

\[ C(y_0, y_1, q(z), q'(z)) = \frac{q'(z)(y_1 + Ra)}{\beta[y_0 - a - q(z)] - zq'(z)} \]

and notice that \( S^C(y_0, y_1, z) = C(y_0, y_1, q(z), q'(z)) \) at the equilibrium house rent schedule.

Total population in location \( z \) must satisfy

\[
\begin{align*}
  h(z)L(z) &= \int \int 1 \left[ \tilde{A}(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \leq a \right] \\
  &\quad \times f(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \times d[C(y_0, y_1, q(z), q'(z))] \times dy_0 dy_1
\end{align*}
\]

We can compute

\[
\frac{d[C(y_0, y_1, q(z), q'(z))]}{dz} = C_0(y_0, y_1, z, q(z), q'(z)) + C_1(y_0, y_1, z, q(z), q'(z)) \times q''(z)
\]

where we define

\[
\begin{align*}
  C_0(y_0, y_1, z, q(z), q'(z)) &= \frac{(1 + \beta)[q^2(y_1 + Ra)]}{\beta[y_0 - a - q(z)] - zq'^2} \\
  C_1(y_0, y_1, z, q(z), q'(z)) &= \frac{y_1 + Ra}{\beta[y_0 - a - q(z)] - zq'(z)}
\end{align*}
\]

and hence

\[
\begin{align*}
  h(z)L(z) &= D(z, q(z), q'(z)) + E(z, q(z), q'(z)) \times q'''(z)
\end{align*}
\]

where

\[
\begin{align*}
  D(z, q(z), q'(z)) &= \int \int 1 \left[ \tilde{A}(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \leq a \right] \\
  &\quad \times f(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \times C_0(y_0, y_1, z, q(z), q'(z)) \times dy_0 dy_1
\end{align*}
\]

\[
\begin{align*}
  E(z, q(z), q'(z)) &= \int \int 1 \left[ \tilde{A}(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \leq a \right] \\
  &\quad \times f(y_0, y_1, C(y_0, y_1, q(z), q'(z))) \times C_1(y_0, y_1, z, q(z), q'(z)) \times dy_0 dy_1
\end{align*}
\]
which defines the nonlinear second-order ODE:

\[ q''(z) = \frac{h(z)Q^{-1}(q(z)) - D(z, q(z), q'(z))}{E(z, q(z), q'(z))} \]

with boundary conditions \( q(\hat{z}^-) = q(\hat{z}^+), \) \( q(z_{min}) = 0 \) that pins down \( z_{min} \). In addition, \( q' \) must be continuous at the limiting point, otherwise there would be scope for arbitrage: \( q'(\hat{z}^-) = q'(\hat{z}^+) = R/SU(\hat{z}) \). The same argument as before ensures existence and uniqueness of the global solution conditional on boundary conditions. Finally, \( SU(\hat{z}) \) is determined by the requirement that \( \int h(z)L(z)dz = 1 \), total population. Thus, an equilibrium exists.

**B.1.2 Proof of Lemma 4**

Suppose that \( \bar{s} = \underline{s} = s \). Unconstrained individuals are indifferent between any of the locations in which there is at least one unconstrained individual. Because constrained individuals always locate in worst cities than any unconstrained individual of the same skill and we have only one skill type, it must be that constrained individuals all locate below \( \hat{z} \). In other words, there is perfect segregation.

In this case, for unconstrained individuals,

\[ R = \frac{s}{q'(z)} \]

This implies that for all cities \( z \geq \hat{z} \),

\[ \frac{d[q'(z)]}{dR} = -\frac{s}{R^2} < 0 \]  \( \text{(B.5)} \)

By continuity, this result extends to the case in which \( \bar{s} - \underline{s} \) is strictly positive but small enough.
B.1.3 Proof of Lemma 5

First, we need to specify the production technology of housing. Suppose housing a produced using the final good $k$ as sole input, according to $H = xk^\theta$, where $\eta = \frac{1-\theta}{\theta}$ and $q_0 = \frac{1}{\theta x^{1/\theta}}$.

Under perfect competition in the housing sector, this production function results in the house rent prices used in the competitive equilibrium. The planner’s problem can then be split into two stages: (1) allocate individuals over space to maximize second period output net of discounted housing creation, and (2) redistribute output for consumption. So if the planner can produce more output net of housing production than the competitive equilibrium, he can achieve any utilitarian or Pareto improvements over the competitive equilibrium. The planner chooses the joint distribution of $(s, z), g(s, z)$, to solve

$$\max_{g,k} \int szg(s,z)dsdz - R\int k(z)dz$$

s.t.  
$$\int g(s,z)dz = f(s)$$
$$\int g(s,z)ds = xk(z)^\theta$$
$$\int g(s,z)dsdz = 1$$

where $f$ is the given marginal skill distribution. Note that the planner discounts house production at the market interest rate, since if it did not use second period output to pay for housing today, it could save those resources which would deliver a gross return of $R$ tomorrow.

First, we re-write this in terms of the shadow price of land that would prevail in the planner’s allocation. We have after some algebra

$$k(z) = \left(\frac{L(z)}{x}\right)^{\frac{1}{\theta}} = (\theta x)^{\frac{1}{1-\theta}} q(z)^{\frac{1}{1-\theta}}$$

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and hence

\[
\max_{g,q} \int szg(s,z)dz - R(\theta x)^{\frac{1}{1-n}} \int q(z)^{1+\frac{1}{n}}dz
\]

s.t. \[
\int g(s,z)dz = f(s),
\]
\[
\int g(s,z)ds = q_0^{-1/\eta}q(z)^{1/\eta}
\]

By construction, the planner’s solution must yield weakly higher output than the competitive equilibrium.

Now, *conditional on* a shadow housing price schedule \(q(z)\), this is a standard optimal transport problem, and given the supermodularity of the surplus \(sz\), the solution is perfect Positive Assortative Matching (PAM): there exists an increasing matching function \(S(z)\) such that

\[
\int_{S(z)}^\bar{s} f(x)dx = q_0^{-1/\eta} \int_z^\bar{z} q(z')^{1/\eta}dz'
\]

i.e.

\[
S(z) = \bar{F}^{-1} \left( \int_z^\bar{z} q(z')^{1/\eta}dz' \right)
\]
\[
f(S(z))S'(z) = Q_0^{-1/\eta}q(z)^{1/\eta}
\]

where \(\bar{F}(s) = 1 - F(s)\) is the skill tail cdf. In addition, from Theorem 4.7 p. 39 in *Galichon2016* we know that the solution is unique.

Now, the planner also chooses \(q\). Clearly the house rent schedule from the competitive equilibrium is in the planner’s choice set. Yet, we know that conditional on the competitive equilibrium’s house rent schedule, the unique maximizer of the planner’s problem features perfect PAM. Since the competitive equilibrium delivers imperfect PAM (the positive mass of constrained individuals do not satisfy strict PAM), the planner’s solution must yield strictly higher gross output than the competitive equilibrium given the same house rent schedule.
In addition, since the planner can always choose the same house rent schedule as the competitive equilibrium, and the sorting of individuals differ strictly between both cases, it must be that output net of housing costs is strictly higher in the planner’s solution. In sum, the planner’s solution yields strictly higher gross and net output compared to the competitive equilibrium.

B.1.4 Proof of Lemma 6

The proof is structured in three steps.

1. Show that city income net of rents is a sufficient statistic to capture welfare losses from the policy

2. Show that city income net of rents declines for all unconstrained individuals below the announced skill threshold

3. Show that this implies that it declines also for constrained individuals below the same skill threshold.

Indirect utility

We first go back to the problem of the individual and define indirect utility. For the unconstrained, consumption is

\[ c_0 = \frac{1}{1 + \beta} \left[ y_0 - q^* + \frac{y_1 + z^* s}{R} \right] \]

\[ c_1 = \beta R c_0 \]
where we denote optimal choices with asterisks (*), and omit dependence on individual characteristics for notational simplicity. Indirect utility of unconstrained individuals is

\[
V^U := \beta \log \beta R + (1 + \beta) \log c_0 \\
= \log \left( \frac{(\beta R)\beta}{(1 + \beta)^{1+\beta}} \right) + (1 + \beta) \log \left( y_0 + \frac{y_1}{R} + \frac{sz^*}{R} - q^* \right)
\]

For the constrained, consumption is

\[
c_0 = y_0 - q^* - a \\
c_1 = y_1 + z^* s + Ra
\]

and their indirect utility is

\[
V^C = \log (y_0 - q^* - a) + \beta \log (y_1 + z^* s + Ra)
\]

Consider a small change in \(q\) (\(dq\)) and \(zs\) (\(d(zs)\)). Then indirect utility changes according to

\[
dV^C = -\frac{1}{c_0} dq + \frac{c_1}{\beta} d(zs)
\]

Therefore, using the financial Euler equation,

\[
c_0 \times dV^C < d \left[ \frac{z^* s}{R} - q^* \right]
\]

Therefore, if the right-hand-side is negative for the policy change (even though the change may be large, we can integrate the inequality across a sequence of infinitesimal changes), constrained individuals lose. In sum, for both constrained and unconstrained individuals, a decline in \(\frac{z^* s}{R} - q^*\) entails a decline in indirect utility.
Income net of rent for unconstrained individuals

Define net income before the policy change as

\[ I(y_0, y_1, s) = sz^*(y_0, y_1, s) - q(z^*(y_0, y_1, s)) \]

and net income after the policy change as

\[ \bar{I}(s) = z_0s - \bar{q} \]

where \( \bar{q} \) is unique the rent after the policy change. For unconstrained individuals, we simplify notation to

\[ I(y_0, y_1, s) \equiv I^U(s) = \frac{sz^U(s)}{R} - q(z^U(s)) \]

because location choice does not depend on \( (y_0, y_1) \) conditional upon being unconstrained. For them, net income is an increasing and convex function of skill \( s \). Indeed, differentiating it w.r.t. \( s \):

\[
\frac{d}{ds} \left( \frac{sz^U(s)}{R} - q(z^U(s)) \right) = \frac{Z^U(s)}{R} + \left( \frac{s}{R} - q'(z^U(s)) \right) \cdot \frac{Z^U_s(s)}{R} > 0
\]

where the last equality comes from the mobility Euler equation.

After the policy change, matching still holds (even though it is degenerate) and hence the same formula applies. In this case the slope calculated in the previous equation is constant in \( s \), and takes the unique value \( z_0/R \).
We now turn to the rent after the policy change, \( \bar{q}_0 \). Using the assumption \( \eta < 1 \), we can easily make comparisons:

\[
\bar{q}_0 = q_0 L_0^\eta \\
= q_0 E[L]^\eta \quad \text{(where } L \text{ is the equilibrium population before policy change)} \\
> q_0 E[L^\eta] \quad \text{(Jensen’s inequality on the concave function } L \mapsto L^\eta) \\
= E[q] \\
> q(E[z]) \quad \text{(Jensen’s inequality on the convex function } z \mapsto q(z))
\]

Now, define \( s_1 < s_0 \) such that \( Z^U(s_1) = E[z] < z_0 = Z^U(s_0) \). For unconstrained individuals with \( s_1 \leq s \leq s_0 \), since \( I_s^U(s) = Z^U(s) \in [E[z], z_0] \), we can integrate to obtain

\[
\frac{E[z](s_0 - s_1)}{R} < I^U(s_0) - I^U(s_1) < \frac{z_0(s_0 - s_1)}{R}
\]

Therefore,

\[
I^U(s_1) > I^U(s_0) - \frac{z_0(s_0 - s_1)}{R} \\
= \frac{s_1z_0}{R} - q(E[z]) \\
> \frac{s_1z_0}{R} - \bar{q}_0 = \bar{I}(s_1)
\]

Hence, we know that at skill \( s_1 \), net income for unconstrained individuals pre-reform is above net income post-reform. In addition, the slope of net income is lower pre-reform for \( s \leq s_1 \): it is \( Z^U(s) \leq E[z] \) pre-reform, compared to \( z_0 > E[z] \) post-reform.

The convexity of \( I^U(s) \) then implies that

\[
I^U(s) > \bar{I}(s) \quad \forall s \leq s_1
\]
i.e. that all unconstrained individuals with lower skill than $s_1$ lose net income form the reform. Since net income is a sufficient statistic for indirect utility, unconstrained individuals with $s \leq S^U(E[z])$ lose from the policy.

**Constrained individuals.**

We can repeat exactly the same argument as for unconstrained individuals. We simply need to allow for dependence on $(y_0, y_1)$ and leverage the monotonicity property of $Z^C$ in skill. Define $s_0(y_0, y_1) < s_1(y_0, y_1)$ such that $Z^C(y_0, y_1, s_1(y_0, y_1)) = E[z] < z_0 = Z^C(y_0, y_1, s_0(y_0, y_1))$. Then the argument carries through, holding $(y_0, y_1)$ fixed: the range is now for all constrained individuals with skill in $[s, \mathcal{S}^C(y_0, y_1, E[z])]$. Since $\mathcal{S}^C(y_0, y_1, z) > S^U(z)$, the range of skills for which constrained individuals lose is larger.

**B.2 Appendix: Extensions with Amenities and Variable Housing Choice**

This section develops extensions to our baseline model. We add amenities and a variable housing choice. We start by deriving the optimality conditions for location choice in that extended model. With an eye towards our empirical exercise, we then proceed to showing comparative statics with respect to income shocks in the absence of credit constraints. In particular, we show that without credit constraints two individuals who start in the same location and both receive the same negative income shock would downgrade location, but the initially high-income individual would downgrade more. This is inconsistent with the empirical evidence we present. Therefore, our results are unlikely to be driven by a simple static amenities choice. In fact, it suggests that if we do not control for wage growth, or amenities in the new location, our estimates are a lower bound on the causal effect of the consumption-smoothing motive alone, net of the contribution of amenity consumption.
to mobility. This lends additional support to the intertemporal consumption-smoothing mechanism we highlight in the main text.

The key observation is that for two individuals to be in the same location, the initially high-income individual must have lower returns to mobility. Thus, following the income shocks, the initially high-income individual would downgrade more. To formally prove this result, we rely on the central insight that in the absence of credit constraint, a dynamic model is essentially isomorphic to a static model because it can be cast in present value terms. We rely on it to show our comparative statics in the two-period model. We then show that the logic is robust to adding more time periods, and extend our results to an infinite-horizon model.

B.2.1 Proof of Lemma 7

We start with the extended model of Section 2.2.6.

\[
\max_{c_0,c_1,a,z} \log(c_0) + \beta \log(c_1) + Az \\
\text{s.t.} \quad c_0 + q(z) + a = y_0 \\
\quad c_1 = y_1 + Ra + sz \\
\quad a \geq a
\]

The optimality conditions are now

\[
\frac{c_1}{\beta c_0} \geq R \text{ with equality iff } a^* > a \\
\frac{c_1}{\beta c_0} = \frac{s}{q'(z)} + \frac{Ac_1}{\beta q'(z)}
\]

We impose \( \beta = R = 1 \) to simplify the exposition, but the results generalize in a straightforward way when \( \beta < 1 < R \)
Unconstrained individuals As before, we combine both budget constraints to obtain
\[ c_1 = y_1 + a + sz = y_1 + sz + y_0 - c_0 - q(z). \]
Using the Euler equation \( c_1 = c_0 \), we then solve for \( c_0, c_1 \) as a function of permanent income \( c_0 = c_1 = \frac{1}{2}I(s, y_0, y_1, z) \), where \( I(s, y_0, y_1, z) \equiv y_0 - q(z) + y_1 + sz \equiv I(y_0, y_1) + sz - q(z) \). The location decision then writes
\[ q'(z) = s + Ac_1, \]
The higher individuals' permanent income, the more amenities they want to consume, and so they locate in better places. We can re-arrange this location FOC as
\[ s = \frac{2q'(z) + Aq(z) - AI(y_0, y_1)}{2 + Az}. \]
In particular, \( q'(z) - s = \frac{A}{2 + Az} \left[ zq'(z) - q(z) + I(y_0, y_1) \right]. \)
We can then compute the response of the location decision of an unconstrained individual to a \( y_0 \) shock, \( D_U = \partial_{y_0} z^* \), by differentiating the location FOC above:
\[ 2q''(z)D_U + Aq'(z)D_U - A = AsD_U. \]
Re-arranging,
\[ D_U = \frac{A}{2q''(z) + A^2 \left[ zq'(z) - q(z) + I(y_0, y_1) \right]} \]

Proportional income shock. We now compare the location response of two unconstrained individuals \( P, R \) who would locate in the same location \( z \) absent the shock. For \( j \in \{P, R\} \),
\[ y_0D_U = \frac{Ay_0}{2q''(z) + A^2 \left[ zq'(z) - q(z) + (1 + \tau)y_0 \right]}. \]
In particular,
\[ \frac{(1 + \tau)A}{2 + Az} d_U y_0 \equiv \frac{\tilde{y}_0}{X(A, z) + \tilde{y}_0}, \]
where \( \tilde{y}_0 = \frac{(1 + \tau)A^2 y_0}{2 + Az} \) and \( X(A, z) = 2q''(z) + A^2 \left[ zq'(z) - q(z) \right] \). \( \frac{\tilde{y}_0}{X(A, z) + \tilde{y}_0} \) is an increasing function of \( \tilde{y}_0 \) as long as \( X(A, z) > 0 \). We show below that the equilibrium convexity of \( q \)
implies this result. Therefore,

\[ D^R = y_0^R P^R > y_0^P P^P = D^P. \]

Proof that \( X(A, z) \geq 0 \). Suppose first that \( q \) is convex. When \( a = -\infty \), every populated location has an unconstrained individual, and so the last populated location has \( q(z) = 0 \). Convexity of \( q(z) \) then implies that \( q(z)/z \equiv p(z) \) is an increasing function. Then \( q'(z) = p(z) + zp'(z) \) and so \( zq'(z)/q(z) = 1 + z^2 p'(z)/q(z) \geq 1 \), and \( X(A, z) > 0 \).

Proof that \( q \) is convex in equilibrium. We know that \( q \) must be convex when \( A = 0 \). Increasing \( A \) continuously keeps \( q \) convex because market clearing conditions are continuous in \( A \). As we increase \( A \), \( q \) can become locally concave only if \( q' \) becomes constant some \( z^* \). As in the baseline case, this implies only unconstrained individuals that satisfy \( s + Ac_1 = q'(z^*) \) optimally locate in a neighbourhood of \( z^* \) of infinite Radon-Nikodym derivative relative to the Lebesgue measure \( dz \). In contrast, individuals in a comparable slice of the distribution of \((y_0, y_1, s)\) locates in a neighborhood of any other \( z \) with a finite Radon-Nikodym derivative relative to the Lebesgue measure. As long as the population distribution of \((y_0, y_1, s)\) is absolutely continuous, the infinite Radon-Nikodym derivative violates land market clearing as it implies zero housing prices, a contradiction. Thus, \( q \) must be convex for any \( A > 0 \).

Constrained individuals  Constrained individuals have \( a^* = a \). Impose \( a = 0 \) for notational simplicity. Their location choice is given by

\[ s + Ac_1 = q'(z) R(s, y_0, y_1, z) \]

\[ R(s, y_0, y_1, z) \equiv \frac{y_1 + sz}{y_0 - q(z)} \]

Our first step is to express \( s \) as a function of \( z, y_0, y_1 \):

\[ s = \frac{q(z)}{y_0 - q(z)} - A \frac{y_1}{1 + Az - \frac{zq'(z)}{y_0 - q(z)}} \]
As a result,

\[ c_1 = y_1 + sz = \frac{y_1}{1 + Az - \frac{zq'(z)}{y_0 - q(z)}}, \quad c_0 = y_0 - q(z), \quad \mathcal{R} = \frac{y_1}{(y_0 - q(z))(1 + Az - zq'(z))} \]

For constrained individuals, we also differentiate the location FOC:

\[ \left( \mathcal{R}q''(z) + q'(z)\mathcal{R}_z \right) D_C + q'(z)\mathcal{R}_{y_0} = A D_C \]

After re-arranging and some algebra:

\[ D_C = \frac{q'(z)}{q''(z)(y_0 - q(z)) + 2q'(z)^2 - 2q'(z)[A(y_0 - q(z))] + [A(y_0 - q(z))]^2} \]

Now, \( D_C \) is increasing in \( A \) iff \( \partial_A \left( -2q'(z)[A(y_0 - q(z))] + [A(y_0 - q(z))]^2 \right) < 0 \), i.e. \( 2(y_0 - q(z))^2A < 2q'(z)(y_0 - q(z)) \), i.e. \( A < \frac{q'(z)}{y_0 - q(z)} \). But from the location FOC, we know that \( 0 \leq \frac{s}{c_1} = \frac{q'(z)}{y_0 - q(z)} - A \), and so the inequality above is always satisfied. Therefore, \( D_C \) is always strictly increasing in \( A \): a constrained individual also downgraded more when receiving a negative income shock when amenities are valued.

**Downgrading of constrained relative to unconstrained individuals**  When \( A \to 0 \),

\[ y_0^R D_U^R \approx y_0^R \frac{A}{2q''(z)} \]

Similarly,

\[ y_0^P D_C^P \approx d(y_0^P, z) + \frac{q'(z)}{q''(z)(y_0^P - q(z)) + 2q'(z)^2} \cdot \frac{2q'(z)(y_0^P - q(z))}{q''(z)(y_0^P - q(z)) + 2q'(z)^2} A \]

Therefore, for a level income shock,

\[ D_U^R - D_C^R \approx d(y_0^P, z) + \frac{1}{2q''(z)} \cdot \left( \frac{XY}{(X + Y)^2} - 1 \right) A \]

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where \( X = q''(z)(y_0^P - q(z)) \) and \( Y = 2q'(z)^2 \). Now, \((X + Y)^2 = X^2 + Y^2 + 2XY = (X - Y)^2 + 4XY \geq 4XY > XY\). Therefore,

\[
D_U^R - D_C^R \approx d(y_0^P, z) - f(y_0^P, z)A,
\]

which is a decreasing function of \( A \) since \( f(y_0^P, z) > 0 \).

For a proportional income shock,

\[
y_0^R D_U^R - y_0^P D_C^R \approx y_0^P \left[ d(y_0^P, z) + \frac{1}{2q''(z)} \left( \frac{XY}{(X + Y)^2} y_0^P - y_0^R \right) \right] A
\]

and the term that multiplies \( A \) is still negative. Therefore, \( y_0^R D_U^R - y_0^P D_C^R \) is decreasing in \( A \) to a first order.

### B.2.2 Generalized two-period model

We now consider a more general version of our two-period model with variable housing choice, amenities, and city income in both periods. Namely, suppose that individuals indexed by \((y, s)\) solve the following problem:

\[
V(y_0, y_1, s) = \max_{c_0,c_1,b_0,h_0,a} \log(A(z) \cdot \ell_0^a c_0^{1-\alpha}) + \beta \log(A(z) \cdot \ell_1^a c_1^{1-\alpha}) \tag{B.6}
\]

\[
s.t. \quad c_0 + a + q(z) + p(z)\ell_0 = y_0 + \tau \Phi(s, z)
\]

\[
c_1 + \theta[q(z) + p(z)\ell_1] = y_1 + Ra + \Phi(s, z)
\]

\[
a \geq a
\]

where, relative to the model in the main text, \( \tau \) governs how much of the mobility returns individual receive immediately, and \( \theta \) governs how much housing costs must be paid in the
second period. When $\Phi(s, z) = sz$, $\tau = \theta = p(z) = 0$, $A(z) = 1$, and $\alpha = 0$, we obtain the model in the main text.

Cobb-Douglas utility in consumption and variable housing implies constant expenditure shares on consumption and housing, so that the individual’s problem can equivalently be written in terms of consumption only. Namely,

$$ V(y_0, y_1, s) = \max_{c_0, c_1, a, z} \log(B(z) \cdot c_0) + \beta \log(B(z) \cdot c_1) $$

s.t.

$$ \frac{c_0}{1 - \alpha} + a + q(z) = y_0 + \tau \Phi(s, z) $$
$$ \frac{c_1}{1 - \alpha} + \theta q(z) = y_1 + Ra + \Phi(s, z) $$

$$ a \geq a $$

where $B(z) = \frac{A(z) p(z)^\alpha}{p(z)^\alpha}$ are perceived amenities after variable housing consumption has been internalized.

The financial Euler equation is then

$$ \frac{c_1}{\beta c_0} \geq R $$

which holds with equality if and only if the individual is financially unconstrained, i.e. $a^* > a$.

The mobility Euler equation becomes

$$ \frac{c_1}{\beta c_0} = \frac{\Phi_z(s, z) - \theta q'(z)}{q'(z) - \tau \Phi_z(s, z)} $$

where the period-1 return has to be netted out from housing costs at that period, and the period-0 cost has to be netted our from the current return of mobility.
B.2.3 Comparative statics in the generalized two-period model without credit constraints

When we remove credit constraints, we can compare the location decisions of individuals who initially locate in the same place, but have different income. The goal is to verify that incentives to change amenities consumption in response to an income shock cannot explain our empirical findings. We show that when two individuals who start in the same location receive a negative income shock, the initially high-income individual should downgrade more. This supports the view that our empirical findings are not driven by amenity or variable housing choice, but rather by consumption-smoothing incentives in the presence of credit constraints. More formally, we show the following result.

**Lemma 11.** Suppose the following assumptions hold:

1. $A(z)$ is continuously differentiable and nondecreasing in $z$

2. $p(z) \equiv p_0$ is constant across locations (materials for construction)

3. The housing production technology results in land prices $q(z) = Q(L(z))$ where $L(z)$ is total population in $z$, and $Q$ is an increasing function such that $Q(0) = 0$ and $\lim_{L \to +\infty} Q(L) = +\infty$

4. The supply of land exceeds population: $\int h(z)dz \geq 1$, where $h(z)$ is the density of land of quality $z \geq 0$

5. Individual income is of the form $I(y, s, z) = y + \Phi(s, z)$, where $\Phi$ is continuously differentiable and $\Phi(s, z) > 0$, $\Phi_z(s, z) > 0$, $\Phi_{sz}(s, z) > 0$

6. There are no credit constraints: $a = -\infty$

Consider two individuals $A$ and $B$ who solve Problem (B.6), with the same future income and location choice. Namely, they have:
• The same period-1 income: $y_1^A = y_1^B$

• Different period-0 incomes: $y_0^A < y_0^B$. A is initially lower-income than B

• The same location choice: $z^A = z^B = z^*$

Suppose that they both receive a negative income shock in period 0, such that both individuals loose income down to $y_0^A' = y_0^B' < y_0^A$. Then the initially high-income individual (B) downgrades location more than the initially low-income (A):

$$z^B < z^A < z^*$$

The logic for this result is simple. Without credit constraints, the dynamic problem above can be reduced to a static present-value problem. As we explain in the proof, we can boil it down to the following maximization:

$$\max_z B(z)[y + \Phi(s, z) - q(z)]$$

where $y$ is permanent income, and $\Phi, q$ are proportional to $\Phi, q$. In that static problem, for two individuals to locate in the same place, it must be that the high-$y$ individuals is low-$s$. Therefore, the initially high-income individual is more location-elastic because she cares less about downgrading location. Thus, after the income shock, she downgrade more.

We structure the proof of this result in several steps. First, we study the static problem and show the comparative statics there. Second, we reduce the dynamic two-period problem to the static problem and deduce the comparative statics.

To assess the robustness of this result, we prove it under several variants. First, as mentioned already, we show it in a static case. Second, we extend it in a similar infinite-horizon model without credit constraints. Third, we show that the result does not depend on the particular assumptions 1 to 6 in Lemma 11, as long as there is strict positive sorting in the economy. We state this last point as follows:
Corollary 2. Suppose only:

7. Primitives are such that individuals choose location according to a matching function $Z(y, s)$, where $y$ is permanent income, and such that $Z_y(y, s) > 0$ and $Z_s(y, s) > 0$ ,

instead of Assumptions 1-6. Then the implications of Lemma 11 continue to hold: consider two individuals A and B who solve Problem (B.6) , with the same future income and location choice. Namely, they have:

- The same period-1 income $y_1^A = y_1^B$
- Different period-0 income $y_0^A < y_0^B$: A is initially lower-income than B
- The same location choice $z^A = z^B = z^*$

Suppose that they both receive a negative income shock in period 0, such that both individuals lose $y_0^A' = y_0^B' < y_0^A$. Then the initially high-income individual (B) downgrades location more than the initially low-income (A):

$$z^{B'} < z^{A'} < z^*$$

Proof of Lemma 11 and Corollary 2: Part 1, Static Problem

We study the following static problem:

$$
\max_{c, h, z} A(z) c^{1-\alpha} \ell^\alpha \\
\text{s.t. } c + q(z) + p(z)\ell = I(y, s, z)
$$

The consumption-housing choice implies that expenditure shares on consumption and housing are constant fraction of income:

$$c = (1 - \alpha)I(y, s, z)$$
$$p(z)h = \alpha I(y, s, z)$$
Substituting housing and consumption decisions into the maximization problem, we obtain a problem in terms of choosing location alone:

$$\max_z B(z)I(y, s, z)$$

where we defined

$$B(z) = \frac{A(z)}{p(z)^\alpha}$$

the perceived amenities of location $z$, after variable housing choice. The FOC is

$$\nu(z) + \frac{sz - zq'(z)}{y + sz - q(z)} = 0$$

where $\nu(z) = \frac{zB'(z)}{B(z)}$ is the elasticity of perceived amenities. Re-arrange it as

$$y = \frac{(\nu(z) + \eta(z))q(z) - (\phi(s, z) + \nu(z))\Phi(s, z)}{\nu(z)}$$

where $\phi(s, z) = \frac{z\Phi_z(s, z)}{\Phi(s, z)}$ is the elasticity of city-income elasticity. Define

$$G(z, s) = \frac{(\nu(z) + \eta(z))q(z) - (\phi(s, z) + \nu(z))\Phi(s, z)}{\nu(z)}$$

**First, suppose that there is positive sorting in equilibrium on both $y$ and $s$.** Namely, we assume that there exists a unique solution $Z(y, s)$ to the FOC

$$y = G(Z(y, s), s),$$

and in addition we assume $Z_y(y, s), Z_s(y, s) > 0$. Use of the implicit function theorem implies that $G_z > 0$ and $G_s < 0$. In particular, $\nu(z) > 0$. Now consider individuals A and B, before and after the shock. Then

$$G(z_t^A, s^A) = G(z_t^B, s^B)$$
Because $G_s < 0 < G_z$, it must be that $z'^A > z'^B$. Thus, the initially high-income individual downgrades more. This proves Corollary 2.

**Second, we show that positive sorting obtains in equilibrium under Assumptions 1-6.** First, the assumptions that $A(z)$ is increasing and that $p(z) = p_0$ ensure that $\nu(z) = \frac{zB'(z)}{B(z)} \geq 0$. Denote

$$u(z; y, s) = B(z)[y + \Phi(s, z) - q(z)]$$

the utility function. Then

$$\frac{z u_z(z; y, s)}{u} \cdot [y + \Phi(s, z) - q(z)] = \nu(z)y + \nu(z)\Phi(s, z) + z\Phi_z(s, z) - \nu(z)q(z) - zq'(z)$$

Second, the assumption of excess land together with $Q(0) = 0$ ensures that there is always a worst city which is empty with zero land price. Thus, in equilibrium,

$$y + \Phi(s, z) - q(z) \geq 0$$

Finally, denote again the optimal location choice $Z(y, s)$ (the matching function). Notice that if $u_z(Z(y, s); y, s) = 0$, then $u_z(Z(y, s); y, s') > 0$ for $s' > s$. Therefore, it must be that the optimal choice $Z(y, s)$ is weakly increasing in $s$: $Z_s(y, s) \geq 0$. If the matching function is locally flat (i.e. the derivative is zero), then the assumption that $\lim_{L \to +\infty} Q(L) = +\infty$ implies that prices are locally infinite. This cannot be an equilibrium, and hence the matching function is strictly increasing. The same logic applies for $Z_y(y, s) > 0$. Therefore, strict positive assortative matching must hold in equilibrium. Thus, the comparative statics proven above follow.
Proof of Lemma 11 and Corollary 2: Part 2. Two-Period Problem

First, maximizing out the variable housing choice that attributes constant expenditure shares in Problem (B.6), we obtain the equivalent problem

\[
V(y_0, y_1, s) = \max_{C_0, C_1, a, z} \log(B(z) \cdot C_0) + \beta \log(B(z) \cdot C_1)
\]

s.t. \(C_0 + a + q(z) = y_0 + \tau \Phi(s, z)\)

\(C_1 + \theta q(z) = y_1 + Ra + \Phi(s, z)\)

where \(B(z) = A(z)/p(z)^a\). Second, using the Euler equation \(C_1 = \beta RC_0\) to express consumption in period 1 as a function of consumption in period 0, and combining both budget constraints into the intertemporal budget constraint, we obtain the equivalent problem

\[
V(y_0, y_1, s) = \max_{C_0, z} (1 + \beta) \log[B(z)C_0]
\]

s.t. \((1 + \beta)C_0 + (1 + \theta R^{-1})q(z) = y_0 + R^{-1}y_1 + [\tau + R^{-1}]\Phi(s, z)\)

Using the budget constraint to expression period-0 consumption as a function of location, we obtain the equivalent problem

\[
V(y_0, y_1, s) = \max_z B(z)[y + \Phi(s, z) - q(z)]
\]

where we defined

\[
\Phi(s, z) = (\tau + R^{-1}) \Phi(s, z)
\]

\(y = y_0 + R^{-1}y_1\)

\(q(z) = (1 + \theta R^{-1}) q(z)\)
The result follows from Part 1 of the proof.

**B.2.4 Comparative statics in the generalized infinite-horizon extension without credit constraints**

We now extend the previous results in our two-period model to an infinite-horizon model without credit constraints. We assume no risk for simplicity. Individuals solve

$$V(a_0, y_0, z_{-1}, s) = \max_{c_t, a_t, h_t, z_t} \sum_{t=0}^{\infty} \beta^t \log \left( A(z_t) c_t^{1-\alpha} \ell_t^\alpha \right)$$

s.t. $c_t + q(z_t) + p(z_t) \ell_t + a_{t+1} = Ra_t + s(z_{t-1} + \tau z_t) + y_t$

where $a_t$ are assets, $z_t$ is location, $c_t$ is consumption of a perishable good, $h_t$ is housing consumption. $s$ is a permanent skill that governs returns to location. $\tau$ governs the fraction of location-specific income that accrues upon arrival in a location. $y_t$ is an exogenous income stream. $R \geq 1$ is an exogenous interest rate on financial assets. $A(z)$ are amenities, $p(z)$ is the price of variable housing, and $q(z)$ is the price of the fixed component of housing.

Relative to the two-period model in which we spread our rents for a unique location choice across the two periods, we effectively set $\theta = 0$ because the natural assumption with many periods is to have rents paid in the current period. Lemma 11 extends in the following way.

**Corollary 3.** Impose either (i) assumptions 1-6 of Lemma 11, or (ii) assumption 7 of Corollary 2 for period-0 location choice. Consider two individuals A and B solving Problem (B.7) in a stationary equilibrium, with the same initial assets, past location and future income and location choice. Namely, they have:

- The same income after period 1: $y_t^A = y_t^B$ for all $t \geq 1$; the same asset holdings $a_0^A < a_0^B$; the same past location $z_{-1}^A = z_{-1}^B$
- Different period-0 income $y_0^A < y_0^B$: A is initially lower-income than B
• The same location choice in period 0: \( z_A^0 = z_B^0 = z_0^* \)

Suppose that they both receive a negative income shock in period 0, such that both individuals lose \( y_A'^0 = y_B'^0 < y_A^0 \). Then the initially high-income individual (B) downgrades location more than the initially low-income (A):

\[
z_B'^0 < z_A'^0 < z_0^*
\]

**Proof.** Using the same logic as in the two-period model, we can (i) maximize out variable housing choice, (ii) use the Euler equation to link consumption across time periods, (iii) iterate forward on the budget constraints and use the transversality condition to re-write Problem (B.7) as an equivalent present-value problem:

\[
V(a_0, \{y_t\}_t, z_{-1}, s) = \max_{\{z_t\}_{t \geq 0}} B(\{z_t\}_t) \left[ Y(a_0, \{y_t\}_t) + \Psi(s, \{z_t\}_t) - Q(\{z_t\}_t) \right]
\]

where we defined

\[
B(\{z_t\}_t) = \exp \left[ \sum_{t=0}^{\infty} \beta^t \log \frac{A(z_t)}{p(z_t)^\alpha} \right]
\]

\[
Y(a_0, \{y_t\}_t) = R \left[ a_0 + \sum_{t=0}^{\infty} R^{-t} y_t + \Phi(s, z_{t-1}) \right]
\]

\[
\Psi(s, \{z_t\}_t) = \sum_{t=0}^{\infty} R^{-t} (\Phi(s, z_t) + \tau \Phi(s, z_{t+1}))
\]

\[
Q(\{z_t\}_t) = \sum_{t=0}^{\infty} R^{-t} q(z_t)
\]

\( \beta R = 1 \) must hold in a stationary equilibrium. Then the utility value of amenities decays at the same rate as income. Taking the FOC with respect to \( z_t \), we obtain:

\[
\nu(z_t) + \frac{(\tau + R^{-1}) \Phi_z(s, z_t) - q'(z_t)}{Y_0 + \Psi(s, \{z_t\}_t) - Q(\{z_t\}_t)} = 0
\]
where $\nu$ is the elasticity of $\frac{A(z)}{p(z)}$. Then it is straightforward to show that:

$$z_0^* = Z(Y_0, \{y_t\}_{t \geq 1}, s)$$

and depends on $y_0$ only through the denominator (first express it as a function of $Y_0 + \Psi(s, \{z_t\}_t) - Q(\{z_t\}_t)$). Denote

$$B(z) = \frac{A(z)}{p(z)^\alpha}$$

$$\Phi(s, z) = (\tau + R^{-1}) \Phi(s, z)$$

$$Y(a_0, \{y_t\}_{t \geq 1}, z, -1) = R \left[ a_0 + \sum_{t=1}^\infty R^{-t} y_t + \Phi(s, z, -1) \right]$$

$$+ \sum_{t=1}^\infty R^{-t} \left( \Phi(s, Z(Y_t, \{y_r\}_{r \geq t}, s)) + \tau \Phi(s, Z(Y_t, \{y_r\}_{r \geq t+1}, s)) \right)$$

$$- \sum_{t=1}^\infty R^{-t} q(Z(Y_t, \{y_r\}_{r \geq t}, s))$$

The problem for solving for $z_0$ as a function of $(y_0, s)$ given $(a_0, \{y_t\}_{t \geq 1}, z, -1)$ is now equivalent to solving

$$V(y_0, s; a_0, \{y_t\}_{t \geq 1}, z, -1) = \max_{z} B(z) [y_0 + Y(a_0, \{y_t\}_{t \geq 1}, z, -1) + \Phi(s, z) - q(z)]$$

The result follows from the proof in the static case (Part 1 of the proof in the two-period case).

**B.3 Appendix: Calibration**

We calibrate our infinite horizon economy to an annual level with two income states $N = 2$ for CRRA utility $u(c) = \frac{c^{1+\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}$. We choose the parameter values in Table B.1.

Most of those values are standard. For instance, if we interpret the low income state $y_1$ as unemployment and the high income state $y_2$ as employment, we can compute the stationary
unemployment rate in this economy through the invariant distribution of the Markov chain transition matrix \( \Lambda' \). At our current values, we obtain a stationary non-employment rate of 14%, consistent with the prime-age male non-employment rate in France.

Our value of the Intertemporal Elasticity of Substitution \( \sigma \) (IES) is within the accepted range. The median skill we use is \( s_0 = 1 \). Given our house rent schedule and the equilibrium city choice, this implies that the idiosyncratic component of income \( y_t \) represents between 5% and 15% of total labor income \( y_t + s_0 z_t \) depending on where individuals are in the state space. Persistent income \( s_0 z_t \) thus represents between 85% and 95%. This reflects the large
observed differences in wages across cities. The differences in location between the best city 1 and the lowest city 0.5 individuals locate in, imply an income change of 0.5, which is of the order of magnitude of the high idiosyncratic income state.

Finally, our house rents schedule is constructed in such a way that unconstrained individuals of skill $s = 1.2$ locate at the best available city, and are free to downgrade as much as they like. It also implies housing expenses of about one third of total labor income, consistent with its empirical counterpart reported in Davis and Ortalo-Magne (2011).

To solve the model numerically, we adapt the method of endogenous grid points of Carroll (2006).

B.4 Appendix: Data Description and Robustness Exercises

B.4.1 Data Description and Sample Selection

Our main data sources are the ‘Déclaration de Données Sociales’ (DADS) Panel, as well as the ‘Données Sociales et Fiscales’ from the ‘Echantillon Demographique Permanent’ (EDP). Both are administrative tax data from the French statistical institute (INSEE).

DADS. The DADS is a matched employer-employee dataset based on tax returns filed by employers. It has rich information on a representative sample of workers who receive taxable labor income in France. It is a panel of all workers in France born in October of even years (approximately 8%). In this dataset we can track the same individual throughout her employment spells for the period 2002-2015. We start in 2002 to observe workers for a long enough period and estimate long-term returns to mobility.

We extract the following variables from the dataset.

- Anonymized individual identifier, common to DADS and EDP,
• total net wage earnings,
• age and gender,
• municipality of residence and workplace,
• 2-digit occupation.

We extract the highest paying employment spell for each individual and each quarter. We then aggregate wage at the annual level, and select municipality and occupation based on the highest paying spell in the year.

**EDP.** The fiscal data in the EDP dataset starts in 2008 and contains income tax return information for French households that are sampled in the DADS or in the baseline EDP sample. The EDP sample contains individuals born in January 2-5, April 1-4, July 1-4, and October 1-4. We link it to the DADS Panel through a common individual identifier.

We extract the following variables from the dataset.

• Anonymized individual identifier, common to DADS and EDP,

• Income from financial assets:
  
  – Annuities,
  
  – Housing rents, net of expenses (mortgage payments, repairs, etc.),
  
  – Stocks, mutual funds, bonds, taxable bank accounts, excluding capital gains,
  
  – Imputed non-taxable income (life insurance, certain types of bank accounts, etc.).

We excluded private equity from the analysis because in many cases it corresponds to ownership of a practice (lawyers, medical doctors, etc.) that it highly illiquid and hard to separate from the worker and sell. We use the residence information from the DADS rather
than the fiscal residence information from the EDP due to well-known concern that the fiscal residence is oftentimes different from the actual residence.\footnote{We indeed find that using the fiscal residence implies a annual migration rate that is an order of magnitude lower than what we find in the DADS, and implausibly low.}

**Additional data.** We complement our main sample with additional data from two sources.

**Amenities.** We use the ‘Base Permamente des Equipements’ in 2007 to construct a measure of amenities. 2007 is the year prior to which the closest year available before our sample with financial income starts. It reports data the number of 136 types of establishments in health services (e.g. hospitals), education services (e.g. pre-schools), public services (e.g. police stations), and commercial services (e.g. perfumeries). We first compute the number of these establishments per capita in each municipality. Then, we extract the first principal components of the corresponding covariance matrix. For each municipality, we obtain the loading on this principal component. We choose the sign of the principal component such that the loadings correlate positively with our measure of amenities. Finally, we rank these loadings between 0 and 1. This rank is our measure of amenities.

**Commuting distance.** We obtain data on the centroids of each municipality in France from a database publicly available from the French government at \url{https://www.data.gouv.fr/en/datasets/listes-des-communes-geolocalisees-par-regions-departements-circonscriptions-nd/}. We then compute the geodesic distance between each residence-workplace municipality pair, and use this distance as our measure of commuting distance.

**Background on the French geography.** The French mainland territory is partitioned in about 96 districts (‘Départements’) and 36,552 municipalities (‘Communes’). Départements are fairly large areas (median area is 8,763 km$^2$ and median population is 531,380 inhabitants), while municipalities are much smaller (median area is slightly above 10 km$^2$, and median population is 432 inhabitants).
Construction of the $z$ variable. To determine how desirable a municipality is, we compute average annual wage earnings in each municipality in the DADS. We then rank municipalities and compute the corresponding percentile for each municipality.

B.4.2 Appendix: Income Shock

Figure B.1: Wage income effect of a negative income shock by financial assets quintile.

(a) In Euros

(b) In Percent

Note: Difference between wage income of individuals with low financial assets (Q1) and individuals with high financial assets (Q5) $\alpha_{1,t} - \alpha_{5,t}$ following a negative income shock relative to individuals who do not receive the shock. $t = 0$ is the year before the income shock. Confidence intervals omitted for readability. The set of controls includes: fixed effects for the time-0 municipality, log wage income at period 0, fixed effects for the time-0 2-digit occupation, 5-year age bin fixed effects, and a home-ownership (HO) fixed effect.
## B.4.3 Appendix: Location Decisions

Table B.2: Effect of an income shock on location rank (p.p.) by financial assets quintiles.

<table>
<thead>
<tr>
<th></th>
<th>Post shock</th>
<th></th>
<th></th>
<th>Pre shock</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>Mass layoffs</td>
<td>-0.19***</td>
<td>-0.57***</td>
<td>-1.05***</td>
<td>-3.22***</td>
</tr>
<tr>
<td>Movers</td>
<td>(0.06)</td>
<td>(0.19)</td>
<td>(0.33)</td>
<td>(1.13)</td>
</tr>
<tr>
<td>Mass &amp; movers</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Q1 × Shock

-0.19***

### Shock

-0.03 0.15 -0.18 0.52 -0.05*

### Controls and FEs

**Year, Q1-Q5, Q2-Q4 × Shock**

- ✓ ✓ ✓ ✓ ✓

**Inc., Mun., Occ., Age, HO**

- ✓ ✓ ✓ ✓ ✓

**Distance, Amenities**

- ✓ ✓ ✓ ✓ ✓

<table>
<thead>
<tr>
<th>Obs.</th>
<th>4782989</th>
<th>2848345</th>
<th>600097</th>
<th>336392</th>
<th>2743274</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>0.163</td>
<td>0.150</td>
<td>0.381</td>
<td>0.382</td>
<td>0.105</td>
</tr>
</tbody>
</table>

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Average difference between location of individuals with low financial assets (Q1) and high financial assets (Q5) $\alpha_{1,1} - \alpha_{5,1}$, as well as location of individuals with high financial assets (Q5) $\alpha_{5,1}$, following a negative income shock, relative to individuals who did not receive the income shock. We pool all years in the sample. The baseline set of controls includes: year fixed effects, dummies for each financial asset quintile, and dummies for financial asset quintiles interacted with the ‘Shock’ dummy. Additional controls include log wage income at time 0, time-0 municipality fixed effects, time-0 2-digit occupation fixed effects, 5-year age bin fixed effects, a home-ownership fixed effect (HO), log current commuting distance, and amenities of the current location.
Table B.3: Effect of an income shock on location rank (p.p.) by financial assets quintiles.

<table>
<thead>
<tr>
<th></th>
<th>Post shock</th>
<th>Pre shock</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Q1 × Shock</td>
<td>-0.22***</td>
<td>-0.18***</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Q2 × Shock</td>
<td>-0.06</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Q3 × Shock</td>
<td>-0.01</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Q4 × Shock</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Shock</td>
<td>-0.00</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.03)</td>
</tr>
</tbody>
</table>

Controls and FEs

Year, Q1-Q5, Q2-Q4 × Shock ✓ ✓ ✓ ✓ ✓ ✓ ✓
Inc., Mun., Occ, Age, HO ✓ ✓ ✓ ✓ ✓ ✓ ✓

| Obs.                  | 5139677   | 5138559   | 3064728   | 675975   | 378575   | 2957728   |
| R²                    | 0.001     | 0.140     | 0.125     | 0.370    | 0.368    | 0.095     |

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * p < 0.10, ** p < 0.05, *** p < 0.01. Average difference between location of individuals with financial assets in first four quintiles (Q1 to Q4) and high financial assets (Q5) \( \alpha_{q,1} - \alpha_{5,1,1} \), as well as location of individuals with high financial assets (Q5) \( \alpha_{5,1} \), following a negative income shock, relative to individuals who did not receive the income shock. We pool all years in the sample. The baseline set of controls includes: year fixed effects, dummies for each financial asset quintile. Additional controls include log wage income at time 0, time-0 municipality fixed effects, time-0 2-digit occupation fixed effects, 5-year age bin fixed effects, and a home-ownership fixed effect (HO).
### Appendix: Financial Assets

Table B.4: Effect of an income shock on financial assets (1,000 euros) by financial assets quintiles.

<table>
<thead>
<tr>
<th></th>
<th>Post shock</th>
<th>Pre shock</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Mass layoffs &amp; movers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1 × Shock</td>
<td>21.79***</td>
<td>25.56***</td>
</tr>
<tr>
<td></td>
<td>(5.30)</td>
<td>(8.64)</td>
</tr>
<tr>
<td>Shock</td>
<td>-21.26***</td>
<td>-22.70***</td>
</tr>
<tr>
<td></td>
<td>(5.17)</td>
<td>(7.09)</td>
</tr>
</tbody>
</table>

**Controls and FEs**

- Year, Q1-Q5, Q2-Q4 × Shock: ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓
- Inc., Mun., Occ., Age, HO: ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓
- Distance, Amenities: ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓

| Obs.      | 4782623 | 2848200 | 599997 | 336348 | 2743136 |
| R²        | 0.008   | 0.010   | 0.031  | 0.016  | 0.009   |

Note: S.E.s in parenthesis, clustered at commuting zone by occupation level. * p < 0.10, ** p < 0.05, *** p < 0.01. Average difference between financial assets of individuals with low financial assets (Q1) and high financial assets (Q5) α1,1 - α5,1, as well as location of individuals with high financial assets (Q5) α5,1, following a negative income shock, relative to individuals who did not receive the income shock. We pool all years in the sample. The baseline set of controls includes: year fixed effects, dummies for each financial asset quintile, and dummies for financial asset quintiles interacted with the ‘Shock’ dummy. Additional controls include log wage income at time 0, time-0 municipality fixed effects, time-0 2-digit occupation fixed effects, 5-year age bin fixed effects, a home-ownership fixed effect (HO), log current commuting distance, and amenities of the current location.
Appendix C

Firm and Worker Dynamics in a Frictional Labor Market

This Appendix is organized as follows. Section C.1 provides intuition for how our assumptions (A) yield a tractable Bellman equation for joint value by completing the analysis of static example started in the main text. Section C.2 provides a characterization of the surplus function. Section C.3 derives the limiting behavior of our economy when frictions vanish. Section C.4 details the algorithms used in the paper to compute and estimate the model.

C.1 Static Example

In this section, we continue the derivation of each term in the joint value equation (3.1). We start by generalizing the UE hire case analyzed in the main text to a firm with multiple incumbents.
**C.1.1 UE hire when the internal renegotiation involves with multiple workers**

It is sufficient to consider the case of two incumbent workers, \( n = 2 \). Without loss of generality, assume that the second worker is paid more than the first, \( w_2 > w_1 \). As in the approach taken earlier, suppose the firm has posted a vacancy that has met an unemployed worker. We have three cases to consider which illustrate how the firm may use a worker outside the firm to sequentially reduce wages of workers inside the firm.

First, the firm hires *without* renegotiation if:

\[
\begin{align*}
y(z, 3) - w_1 - w_2 - b &> y(z, 2) - w_1 - b, \\
\text{No credible threat to } w_2
\end{align*}
\]

Hiring with current wages is preferred to replacing the most expensive incumbent—there is no credible threat—, and given no renegotiation, hiring is optimal. Since \( w_2 > w_1 \), no credible threat to worker 2 implies no credible threat to worker 1.

Second, the firm hires *with* renegotiation with worker 2 if:

\[
\begin{align*}
y(z, 2) - w_1 - b &> y(z, 3) - w_1 - w_2 - b > y(z, 2) - w_2 - b, \\
\text{Credible threat for worker 2 only}
\end{align*}
\]

\[
\begin{align*}
y(z, 3) - w_1 - w_2^* - b &> y(z, 2) - w_1 - w_2^*, \\
\text{Optimal to hire under } (w_1, w_2^*)
\end{align*}
\]

The threat is credible for worker 2, but is not for worker 1, and, conditional on renegotiating to \((w_1, w_2^*)\), hiring is optimal.

Third, the firm hires *with* renegotiation with *both* workers if:

\[
\begin{align*}
y(z, 2) - w_1 - b &> y(z, 2) - w_2 - b > y(z, 3) - w_1 - w_2 - b, \\
\text{Credible threat for both workers}
\end{align*}
\]

\[
\begin{align*}
y(z, 3) - w_1^* - w_2^* - b &> y(z, 2) - w_1^* - w_2^*, \\
\text{Optimal to hire under } (w_1^*, w_2^*)
\end{align*}
\]
In all three cases, the optimal hiring condition can be written as:

$$\Omega(z, 3) - \Omega(z, 2) > U.$$  \hfill (C.1)

This last inequality does not depend on the order of the internal negotiation between firm and workers. In conclusion, the distribution of wages among incumbents again determines the patterns of wage renegotiation, but is immaterial for the sufficient condition for hiring.

Assumption (A-LC-c) that was not present in the one worker example plays a role here. Suppose that the renegotiated wage for worker 2 is pushed all the way down to $b$, making her indifferent between staying and quitting. Worker 1 could transfer a negligible amount to worker 2 in exchange of her quitting, which would raise the firm’s marginal product and, possibly, remove its own threat. This is problematic for the representation because in this latter case the hiring condition becomes $y(z, 2) - y(z, 1) - w_1 - b > y(z, 1) - w_1$, distinct from (C.1). Thus, to know whether a firm hires or not, one would need to know the wage distribution inside the firm. (A-LC-c) is sufficient to rule out transfers among workers and to prevent this scenario from happening.

Note that, this transfer scheme between workers occurring during the internal negotiation changes the joint value, and hence one can think of (A-LC-c) as being subsumed into (A-IN) already.

### C.1.2 EE hire

Now suppose that the worker matched with the firm’s vacancy is currently employed at another firm with productivity $z'$ and a single worker $n' = 1$. The situation is not that different from UE hire, except that the potential hire may have a better outside option in the form of the retention offer made to her by her current employer under (A-EN). To see the similarity for now we fix this wage offer at $\overline{w}$. The same four cases can arise, except
with \( \bar{w} \) playing the role of \( b \).\(^1\) We can therefore reason as before and jump to the result that hiring will occur if and only if the following counterpart to (3.6) holds:

\[
\Omega(z, 2) - \Omega(z, 1) > \bar{w}.
\]

We now determine the poached worker’s outside option \( \bar{w} \). The poached firm’s willingness to pay is a wage \( \tilde{w} \) that makes it indifferent between retaining and releasing the worker: \( y(z', 1) - \tilde{w} = 0 \). Hence, the contacted worker switches to the new employer as long as the poaching firm offers \( \bar{w} = y(z', 1) \). Bertrand competition between the two firms implies that the poaching firm offers \( \bar{w} = y(z', 1) \), which is exactly the marginal value of the worker at the poached firm. As in the case of \( UE \) hire, whether \( EE \) hire occurs can be summarized by joint values:

\[
\frac{\Omega(z, 2) - \Omega(z, 1)}{2 - 1} > \frac{\Omega(z', 1) - \Omega(z', 0)}{1 - 0}.
\]

The \( EE \) hire decision is entirely characterized by knowledge of the pair \((z, n)\) for the two firms.\(^2\) The value gain to the firm and its workers is the difference between the left-hand side and right-hand side of equation (C.2). This comparison of marginal values is precisely the \( EE \) hire term in the HJB equation (1).

Finally, this exercise explains the absence of a \( EE \) quit term in (3.1). The payment received by its poached worker is equal to the poached coalition’s willingness to pay, which is in turn exactly equal to the worker’s marginal value to the coalition. The joint value of the poached coalition therefore does not change as it loses its worker. \( EE \) quit events play an important role in the dynamics of employment at the firm—which we describe below—but no role in the dynamics of \( \Omega(z, n) \).

\(^1\) Renegotiation will happen for different values of \( w_1 \) in the no hire case. Indeed, to establish the presence of a credible threat \( w_1 \) must be compared to \( \bar{w} \) instead of \( b \), but this has no allocative implications for the hiring decisions.

\(^2\) The case when the firm meets a worker at a firm with \((z', 2, w_1, w_2)\) is similar. Suppose the firm meets worker 1. The poached firm has the additional option of cutting \( w_2 \), but this is inconsequential for the argument because it only redistributes value within the poached-from firm.
C.1.3 Vacancy posting

We now explain the private inefficiency in vacancy posting and why (A-VP) is crucial for tractability.

Recall that in the hiring scenarios just analyzed, two cases arise when the firm can credibly force a wage cut: (i) when it hires and the incumbent wage is above the post-hire new marginal product; (ii) when hiring is not profitable, but the firm can credibly ‘fire and swap’, i.e. as long as the reservation wage of the external worker met through search is below the incumbent wage. The firm has therefore incentives to spend resources on vacancy posting only to transfer value between agents, a privately inefficient outcome. The amount spent would depend on the incumbent’s wage, breaking the tractability of our representation. Private efficiency reinstates tractability.

We start with the firm’s preferred vacancy policy. Without loss of generality, suppose firms only meet unemployed workers (hence, upon a meeting, the ‘fire and swap’ threat is always credible). Let \( v \) be the number of vacancies posted, \( c(v) \) the associated cost, and \( qv \) the probability a single vacancy meets a single worker. If no meeting occurs, then as per (A-MC), \( w_1 \) does not change so the value of the firm does not change. The firm maximizing the expected return from vacancy posting net of costs is:

\[
\max_v \quad -c(v) + qv \left[ \max \left\{ y(z,2) - w_1' - b , \ y(z,1) - b \right\} - \left( y(z,1) - w_1 \right) \right],
\]

Following a meeting, three cases may occur. In Case 1, the firm hires and there is no renegotiation, \( w_1' = w_1 \). This case arises when the wage of the incumbent worker is low enough. Then, adding a second worker does not reduce the marginal product of labor down to the point where the firm has a credible layoff threat. In Case 2, the firm hires but the wage of the incumbent is renegotiated down to \( w_1' = w_1^* \). In this case, diminishing marginal returns drive the marginal product of labor with two workers below the incumbent’s initial wage. In Case 3, the firm is better off not hiring, but under the threat of swapping out the
incumbent, renegotiates $w_1$ down to $b$. The firm’s preferred vacancy policy $v^f$ then equates marginal cost to marginal expected return:

$$ c_v (v^f) = q \left[ \max \left\{ y(z, 2) - w'_1 - b , y(z, 1) - b \right\} - \left( y(z, 1) - w_1 \right) \right]. \quad (C.3) $$

The first-order condition (C.3) highlights that the firm’s preferred vacancy policy depends on the incumbent’s wage $w_1$ because this wage determines the gains from forcing a renegotiation through vacancy posting. This dependence is a source of intractability because, in the general model with $n$ workers, (C.3) would depend on the entire wage distribution inside the firm.

Our assumption (A-VP) ensures that firms do not post $v^f$, but instead post the privately efficient amount of vacancies which does not depend on worker wages. We now show how our micro-foundation (A-VPI) implements (A-VP).

**Case 1 – Hire without renegotiation.** In this case the outcome is already privately efficient. The worker’s value does not decrease ($w'_1 = w_1$), and by the fact that a hire occurs, the firm’s value must increase. We can also write the expected return as $qv [\Omega(z, 2) - \Omega(z, 1) - U]$. Since the return is independent of $w_1$, then the efficient vacancy policy is independent of $w_1$. The firm is choosing vacancies as if it were maximizing the joint surplus without having to appeal to additional assumptions.

In cases 2 and 3, the outcome is privately inefficient because the firm may profit from vacancies that, if met by a job seeker, deliver a credible threat to cut the incumbent’s wage to $w'_1 < w_1$.

Our assumption (A-VPI) allows the worker to correct for this over-posting. The worker can then concede a pay cut in all states in exchange for an alternative level of vacancies. Such a pay cut is not covered under (A-MC) since when the vacancy policy is announced there is not yet a credible threat to layoff workers.\(^3\) The firm will accept this wage cut

\(^3\)A pay cut regardless of the outcome of the search for a new worker maps exactly into a transfer from worker to firm, which is how we approach the proof in the Appendix. We could have allowed for state-contingent wage-cuts that depend on who the firm meets or whether a meeting occurs. Even if these states were verifiable the result would only be for the worker to offer a menu of wage-cuts across states. This would
and choose the worker’s preferred vacancies if it delivers at least the value obtained under the firm’s preferred vacancies \(v^f\). We show that the worker’s preferred package satisfying incentive compatibility restores efficiency in vacancy posting.

**Case 2 – Hire with renegotiation.** In this case, the incumbent’s wage \(w_1\) is high enough that the firm finds it profitable to raise the contact probability with an unemployed worker beyond what would be efficient. Although the hiring outcome is efficient ex-post, too much resources are spent on vacancies ex-ante. Let \(w_1^*\) be the renegotiated wage after a meeting. The worker chooses a package of vacancies and a wage cut in all states \((v^w, x)\) that solves:

\[
\max_{v^w, x} qv^w(w_1^* - w_1) - x \quad (C.4)
\]

subject to

\[
qv^w \left[ (y(z, 2) - (w_1 - x) - b) - (y(z, 1) - w_1) \right] - c(v^w)
\geq
qv^f \left[ (y(z, 2) - w_1^* - b) - (y(z, 1) - w_1) \right] - c(v^f) \quad (IC)
\]

The worker anticipates that after a meeting their wage will be renegotiated to \(w_1^* < w_1\). Given this wage cut, the worker seeks to limit the probability of this event by cutting back on vacancies. Incentive compatibility \((IC)\) requires that as the worker cuts vacancies it also cuts its wage so that the firm accepts the proposed policy \(v^w\) over \(v^f\).

The Pareto problem \((C.4)\) yields the result that vacancy posting is independent of \(w_1\). First, given the linear objective function, \((IC)\) holds with equality. Thus, we can substitute out \(x\). Second, the zero-sum game assumption \((A-IN)\) implies that \(w_1^*\) is a renegotiated wage that only redistributes value and hence drops out. Third, all terms that do not depend on \((x, v^w)\) are irrelevant to the worker’s decision. This leaves the following objective function:

\[
\max_{v^w} qv^w \left[ (\Omega(z, 2) - U) - \Omega(z, 1) \right] - c(v^w).
\]

Increase worker value but not change allocations, hence for consistency with the rest of our assumptions, we assume a single wage cut.
The decision can therefore be characterized by the *privately efficient return*, which is the change in joint value net of the cost of the new hire, \( \Omega(z, 2) - \Omega(z, 1) - U \).

**Case 3 – No hire with renegotiation.** In this case the ‘fire and swap’ threat is credible. The incumbent’s wage \( w_1 \) is high enough and the marginal product of an additional worker is below \( b \). Replacing the return to hiring by the wage cut for the incumbent worker, the previous logic delivers

\[
\max_{v^w} q v^w \left[ \Omega(z, 1) - \Omega(z, 1) \right] - c(v^w) \implies v^w = 0
\]

Absent the transfer from worker to firm, the firm would post positive vacancies \( v^f \) even if the return from hiring is negative, i.e. \( \Omega(z, 2) - \Omega(z, 1) < U \) to induce a wage cut, and \( v^f \) would depend on \( w_1 \). Under (A-VPI), the worker takes a preemptive wage cut, and vacancies are zero, the efficient amount in this case.

**Combined.** Combining all three cases, privately efficient vacancies solve

\[
\max_v q v \left[ \max \left\{ \Omega(z, 2) - \Omega(z, 1) - U, 0 \right\} \right] - c(v).
\]

Note three properties of this solution. First, the firm always hires when it meets an unemployed worker. Second, optimal vacancy posting equates the marginal gain in joint value to the marginal cost of a vacancy, and it only depends on \((z, n)\). Third, this condition is the flip-side of the separation frontier. In (3.1) we said that if \( \Omega_n(z, n) > U \), then the firm will not separate with workers. The terms inside the max expression say that if this is true, then the firm will post vacancies.\(^4\)

We conclude that under (A-VPI), the joint value is sufficient to characterize the vacancy decision. The distribution of wages in the firm is immaterial.

**Multiple incumbents.** When the firm employs more than one worker, the efficient transfer scheme can be implemented by randomly selecting a worker under threat to offer a package

\(^4\)It is possible to determine the optimal wage cut \( x \) that delivers the efficient policy, but throughout the paper we focus on allocations only. See Bilal et al. (2019a) for more details on wage determination.
of wage-cuts and vacancies. In exchange, the firm posts the efficient number of vacancies. Under such a scheme, the initiating worker is strictly better off while the firm and the other workers are indifferent. We establish this case in detail in Appendix C.6.

C.1.4 Layoffs, quits, exit, entry

Having described most of the terms in the HJB (3.1), we conclude with the boundary conditions for exit, layoffs and the free entry condition.

Layoffs. Consider now a firm with \( n = 2 \) workers paid \((w_1, w_2)\), and assume that \( w_1 < y(z, 1) \) such that worker 1 is never under threat of layoff. The firm has a credible threat to fire worker 2 if

\[
y(z, 1) - w_1 > y(z, 2) - w_1 - w_2.
\]

Such a situation may occur if, for example, productivity has just declined. The firm has a credible threat to negotiate down to a wage level \( w^*_2 \) such that \( y(z, 1) - w_1 = y(z, 2) - w_1 - w^*_2 \) and keep worker 2 employed. From the worker’s perspective, it is individually rational to accept any wage \( w^*_2 \) above \( b \). Worker 2 is laid off if \( y(z, 1) - w_1 > y(z, 2) - w_1 - b \). In terms of joint value, this can be written in exactly the form of the layoff frontier (3.2):

\[
\frac{\Omega(z, 2) - \Omega(z, 1)}{2-1} < U.
\]

The firm lays off workers until the marginal joint value of the worker is equal to the value of unemployment.\(^5\) Note that this is the complement to the condition for posting vacancies. The special case with \( n = 1 \) of this scenario also arises in the one worker-one firm model with productivity shocks of Postel-Vinay and Turon (2010a).

Quits to unemployment. Since in this static model workers will accept a renegotiated wage down to \( w^*_1 = b \), they will only quit at the point where the firm has a credible threat

\(^5\)Note that, when both workers are under threat, the particular order in which values of workers are reduced is immaterial to the condition \( \Omega(z, 2) - \Omega(z, 1) < U \). One could for example lower the wages of both workers proportionally, increasing the value of the firm, but a worker must be fired if \( J(z, 2, w^*_1, b) < J(z, 1, w^*_1, \cdot) \) for any \( w^*_1 \geq b \).
to lower wages below $b$. This is exactly the point at which the marginal value is equal to the value of unemployment. In this sense layoffs as described above are indistinguishable from quits to unemployment, as in any model with privately efficient separations. For ease of language all endogenous $UE$ transitions are referred to as layoffs, and we use quits to refer only to $EE$ transitions.

Finally, recall that in the dynamic model unemployed job seekers are promised a wage that implements a value $U$ to them. If events occur in the firm that reduce the continuation value to that worker below $U$ (e.g., a negative productivity shock), the incumbent may have a credible threat to quit and renegotiate her wage to restore its value at $U$, or above it, depending on the details of the internal negotiation. However, such renegotiation is, again, only a transfer of value within the firm. Separations remain privately efficient even in the dynamic model.

**Exit.** Now consider the exit decision of a firm with one worker. The private value of exit to the firm is the scrap value $\vartheta > 0$. The firm therefore exits if and only if $y(z, 1) - w_1^* < \vartheta$, where $w_1^*$ is a possibly renegotiated wage contingent on the firm remaining in operation. If the profit from operating at the lowest possible renegotiated wage $w_1^* = b$ is greater than $\vartheta$, then the firm will continue to operate. Hence, the firm exits if $y(z, 1) - b < \vartheta$, and the renegotiated wage only affects the distribution of value. The exit condition can be written as $\Omega(z, 1) - U < \vartheta$, and in the general case of $n$ workers is exactly the boundary condition in (3.1): $\Omega(z, n) - nU < \vartheta$.

**Entry.** Upon entry the firm has $n_0$ workers hired from unemployment. The private entry cost of the firm is $c_0$, so entry requires $\int y(z, n_0) d\Pi_0(z) - n_0 b > c_0$. Using $\Omega(n, z) = y(z, n)$ and $U = b$, this requires $\int \Omega(z, n_0) d\Pi_0(z) > c_0 + n_0 U$.

---

The firm has no credible threat to reduce $w_1$ if $y(z, 1) - w_1 > \vartheta$. The firm can credibly threaten exit if $\vartheta \in (y(z, 1) - w_1), y(z, 1) - b)$, but in this case $w_1$ can be reduced to a point where this threat is no longer credible.
C.1.5 From static to dynamic

This static example showcases how to obtain every component of (3.1) from our set of assumptions. Appendix ?? generalizes these insights to the dynamic case. Two insights assist us. First, the proof begins with a discrete workforce. Here we are helped by continuous time, which removes complicated binomial probabilities of one, two, three, etc. incumbent workers meeting a competitor’s vacancy. Second, we take the continuous workforce limit of the discrete workforce HJB equation. This limit delivers the joint value representation (3.1) in terms of the derivative of the joint value function rather than differences of values which, when moving up or down by one worker, are symmetric due to continuous differentiability.

C.2 Characterization of surplus function

First define the surplus as \( S(z, n) = \Omega(z, n) - nU \). Given that \( \rho U = b \), this implies that \( \rho S(z, n) = \rho \Omega(z, n) - nb \). We also have that \( S_n(z, n) = \Omega_n(z, n) - U \). Combining these with the Bellman equation for \( \Omega \):

\[
\rho S(z, n) = \max_{v \geq \vartheta} y(z, n) - c(v, z, n) - nb \\
+ [q\varphi v - \delta n] S_n(z, n) \\
+ q(1 - \phi) v \int_{0}^{S_n(z, n)} [S_n(z, n) - s] dH_n(s) \\
+ \mu(z) S_z(z, n) + \frac{\sigma^2(z)}{2} S_{zz}(z, n)
\]

where we slightly abuse notation and use \( H_n(s) \) to also denote here the employment-weighted cumulative distribution function of marginal surpluses. The value-pasting conditions become

\[
S(z, n) \geq \vartheta \\
S_n(z, n) \geq 0
\]
We now make a number of assumptions to characterize the surplus. They are not all strictly necessary for each individual comparative static, but for convenience of exposition we present them all at the same time.

- The production function $y(z, n)$ satisfies $y_{\log z}, y_n, y_{\log z, n} > 0 > y_{nn}$.

- Productivity follows a geometric Brownian motion $\mu(z) = \mu z$ and $\sigma(z) = \sigma z$.

- Vacancy costs depend only on $v$ and are isoelastic: $c(v) = c_0 v^{1+\gamma}$.

- The surplus function is twice continuously differentiable up to the boundary of the continuation region.

We now proceed to show the comparative statics discussed in the main text.

### C.2.1 $S$ is increasing in $n$

The no-endogenous-separations condition $S_n \geq 0$ immediately implies that the surplus is increasing in $n$.

### C.2.2 $S$ is increasing in $z$

Re-write the problem in terms of $x = \log z$. Denote with a slight abuse of notation $y(x, n) = y(e^x, n)$. Then, as a function of $(x, n)$, the joint surplus solves

$$
\rho S(x, n) = \max_{v \geq 0} [y(x, n) - c(v) - nb] \\
+ [q(1 - \phi)v\mathcal{H}(S(x, n)) \\
+ q(1 - \phi)v\mathcal{H}(S(x, n)) + \left(\mu - \frac{\sigma^2}{2}\right)S_x(x, n) + \frac{\sigma^2}{2}S_{xx}(x, n)$$
where we integrated by parts, and denoted $H(s) = \int_0^s H_n(r) dr$. Denote $\zeta(x, n) = S_x(x, n)$.

Differentiate the Bellman equation w.r.t. $x$ and use the envelope theorem to obtain

$$\rho \zeta(x, n) = y_x(x, n)$$
$$+ \left\{ [q(1 - \phi)H_n(S_n(x, n)) + q\phi]v^*(x, n) - \delta n \right\}\zeta_n(x, n)$$
$$+ \mu \zeta_x(x, n) + \frac{\sigma^2}{2} \zeta_{xx}(x, n)$$

Now consider the stochastic process defined by

$$dx_t = \mu dt + \sigma dW_t$$
$$dn_t = \left\{ [q(1 - \phi)H_n(S_n(x_t, n_t)) + q\phi]v^*(x_t, n_t) - \delta n_t \right\} dt$$
(C.5)

This corresponds to the true stochastic process for productivity, but a hypothetical process for employment, that in general differs from the realized one. We can now use the Feynman-Kac formula (Pham, 2009) to go back to the sequential formulation:

$$\zeta(x, n) = \mathbb{E} \left[ \int_0^T e^{-\rho t} y_x(x_t, n_t) + e^{-\rho T} \zeta(x_T, n_T) \mid x_0 = x, n_0 = n, \{x_t, n_t\} \text{ follows (C.5)} \right]$$

and where $T$ is the hitting time of either the separation of exit region. By assumption, $y_x > 0$, so the contribution of the first part is always positive. On the exit region, smooth-pasting requires that $\zeta = 0$. In the interior of the separation region, $\zeta = 0$. Under our regularity assumption, we thus get $\zeta = 0$ on the layoff boundary. Thus,

$$\zeta(x, n) = \mathbb{E} \left[ \int_0^T e^{-\rho t} y_x(x_t, n_t) dt \mid x_0 = x, n_0 = n, \{x_t, n_t\} \text{ follows (C.5)} \right] > 0$$

which concludes the proof.
C.2.3 $S$ is concave in $n$

Denote $s(z, n) = S_n(z, n)$. Differentiate the Bellman equation w.r.t. $n$ on the interior of the domain, use the envelope theorem and integrate by parts to obtain:

$$(\rho + \delta)s(z, n) = y_n(z, n) - b + \left\{ [q\phi + q(1 - \phi)H_n(s(z, n))] v^*(z, n) - \delta n \right\} s_n(z, n) + \mu(z)s_z(z, n) + \frac{\sigma^2(z)}{2} s_{zz}(z, n)$$

Recall that

$$(1 + \gamma)c_0[v^*(z, n)]^\gamma = q\phi s(z, n) + q(1 - \phi)\mathcal{H}(s(z, n))$$

In particular, differentiating w.r.t. $n$,

$$\gamma(1 + \gamma)c_0[v^*(z, n)]^\gamma - 1 v^*_n(z, n) = [q\phi + q(1 - \phi)H_n(s(z, n))] s_n(z, n)$$

and so

$$\gamma v^*_n(z, n) = \frac{\phi + (1 - \phi)H_n(s(z, n)) s_n(z, n)}{\phi + (1 - \phi)\overline{H}(s(z, n)) s(z, n)}$$

where $\overline{H}(s) = \frac{H(s)}{s} \leq 1$. Now denote $\zeta(z, n) = s_n(z, n) = S_{nn}(z, n)$. Differentiate the recursion for $s$ w.r.t. $n$ to obtain

$$\left( \rho + 2\delta - q(1 - \phi)H'_n(s(z, n)v^*(z, n)s_n(z, n) - q[\phi + (1 - \phi)H_n(s(z, n))v^*_n(z, n)] \right) \zeta(z, n)$$

$$= y_{nn}(z, n) + \left\{ [\lambda\phi + \lambda(1 - \phi)H_n(s(z, n))] v^*(z, n) - \delta n \right\} \zeta_n(z, n)$$

$$+ \mu(z)\zeta_z(z, n) + \frac{\sigma^2(z)}{2} \zeta_{zz}(z, n)$$
Now define the “effective discount rate”

\[ R(z, n, s_n(z, n)) = \rho + 2\delta - q(1 - \phi)H'_n(s(z, n)v^*(z, n)n) - q[\phi + (1 - \phi)H_n(s(z, n))]v^*_n(z, n) \]

\[ = \rho + 2\delta - qv^*_n(z, n)s_n(z, n) \left\{ (1 - \phi)H'_n(s(z, n)) + \frac{\phi + (1 - \phi)H_n(s(z, n))\phi + (1 - \phi)H_n(s(z, n))}{\gamma s(z, n)} \right\} \]

\[ \equiv P(z, n) > 0 \]

where the second equality uses the expression for \( v^*_n \) derived above. Define the stochastic process

\[ dz_t = \mu(z_t)dt + \sigma(z_t)dW_t \]

\[ dn_t = \left\{ [q(1 - \phi)H_n(S_n(z_t, n_t)) + q\phi]v^*(z_t, n_t) - \delta n_t \right\}dt \] \hspace{1cm} (C.6)

As before, we can use the Feynman-Kac formula to obtain

\[ \zeta(z, n) = \mathbb{E}\left[ \int_0^T e^{-\int_0^\tau R(z, n, z, n)\,d\tau} y_{nn}(z_t, n_t)\,dt + e^{-\int_0^T R(z, n, z, n)\,d\tau} \zeta(z_T, n_T) \right| z_0 = z, n_0 = n, \{z_t, n_t\} \text{ follows (C.6)} \]

for \( T \) the first hitting time of the exit/separation region. The contribution of the first term is always negative. Note that \( \zeta \) enters in the effective discount rate. Inside the separation region and in the exit regions, \( \zeta = 0 \). We restrict attention to twice continuously differentiable functions, so \( \zeta = 0 \) on the exit and separation frontiers. Then

\[ \zeta(z, n) = \mathbb{E}\left[ \int_0^T e^{-\int_0^\tau R(z, n, z, n)\,d\tau} y_{nn}(z_t, n_t)\,dt \right| z_0 = z, n_0 = n, \{z_t, n_t\} \text{ follows (C.6)} \] < 0

which concludes the proof.
C.2.4 \( S \) is supermodular in \((\log z, n)\)

Denote again \( s(x, n) = S_n(x, n) \), where \( x = \log z \). Recall

\[
(\rho + \delta)s(x, n) = y_n(x, n) - b
+ \left\{ [q\phi + q(1 - \phi)H_n(s(x, n))] v^*(x, n) - \delta n \right\} s_n(x, n)
+ \mu s_x(x, n) + \frac{\sigma^2}{2} s_{xx}(x, n)
\]

and that

\[
(1 + \gamma)c_0[v^*(x, n)]^\gamma = q\phi s(x, n) + q(1 - \phi)\mathcal{H}(s(x, n))
\]

In particular, differentiating w.r.t. \( x \),

\[
\frac{\gamma v_x^*(x, n)}{v^*(x, n)} = \frac{\phi + (1 - \phi)H_n(s(x, n))}{\phi + (1 - \phi)\mathcal{H}(s(x, n))} \frac{s_x(x, n)}{s(x, n)}
\]

Now denote \( \zeta(x, n) = s_x(x, n) = S_{xn}(x, n) \). Differentiate the recursion for \( s(x, n) \) w.r.t. \( x \) to obtain

\[
\left( \rho + \delta - q(1 - \phi)H'_n(s(x, n))v^*(x, n)s_x(x, n) - q[\phi + (1 - \phi)H_n(s(x, n))]v^*_x(x, n) \right) \zeta(x, n)
= y_{nx}(x, n)
+ \left\{ [\lambda\phi + \lambda(1 - \phi)H_n(s(x, n))] v^*(x, n) - \delta n \right\} \zeta_n(x, n) + \mu \zeta_x(x, n) + \frac{\sigma^2}{2} \zeta_{xx}(x, n)
\]

As before, define the “effective discount rate”

\[
R(x, n, s_x(x, n))
= \rho + \delta - q(1 - \phi)H'_n(s(x, n))v^*(x, n)s_x(x, n) - q[\phi + (1 - \phi)H_n(s(x, n))]v^*_x(x, n)
= \rho + \delta - qv^*(x, n)s_x(x, n) \left\{ (1 - \phi)H'_n(s(x, n)) + \frac{\phi + (1 - \phi)H_n(s(x, n))}{\gamma s(x, n)} \right. \\
\frac{\phi + (1 - \phi)\mathcal{H}(s(x, n))}{\phi + (1 - \phi)\mathcal{H}(s(x, n))} \right\}
\equiv \mathcal{P}(x, n) > 0
\]
where the second equality uses the expression for $v^*_n$ derived above. As before, define the stochastic process

$$dx_t = \mu dt + \sigma dW_t$$

$$dn_t = \left\{ [q(1 - \phi)H_n(S_n(e^{x,t}, n_t)) + q\phi] v^*(x_t, n_t) - \delta n_t \right\} dt \quad \text{(C.7)}$$

As before, we can use the Feynman-Kac formula to obtain

$$\zeta(x, n) = E \left[ \int_0^T e^{-\int_0^t R(x, n, \zeta(x, n)) dt} y_{nx}(x_t, n_t) dt + e^{-\int_0^T R(x, n, \zeta(x, n)) dt} \zeta(x_T, n_T) \right] \bigg| \begin{array}{l} x_0 = z, n_0 = n, \{x_t, n_t\} \text{ follows (C.7)} \end{array}$$

for $T$ the first hitting time of the exit/separation region. The contribution of the first term is always positive. Inside the separation region and in the exit regions, $\zeta = 0$. We restrict attention to twice continuously differentiable functions, so $\zeta = 0$ on the exit and separation frontiers. Then

$$\zeta(x, n) = E \left[ \int_0^T e^{-\int_0^t R(x, n, \zeta(x, n)) dt} y_{nx}(x_t, n_t) dt \bigg| x_0 = z, n_0 = n, \{x_t, n_t\} \text{ follows (C.7)} \right]$$

which concludes the proof.

### C.2.5 Net employment growth

Net employment growth in the continuation region is

$$\frac{dn_t}{dt} = q \left[ \phi + (1 - \phi)H_n(S_n(z, n)) \right] v^*(z, n) - \lambda E (1 - H_v(S_n(z, n))) n - \delta n \equiv g(z, n)$$
Using the expression above for $v^*(z, n)$:

$$g(z, n) = \frac{q^{1+1/\gamma}}{[(1 + \gamma)c_0]^{1/\gamma}} \left( \phi + (1 - \phi)H_u(S_n(z, n)) \right) \left( \phi S_n(z, n) + (1 - \phi)H(S_n(z, n)) \right)^{1/\gamma}$$

$$-\lambda^E(1 - H_v(S_n(z, n)))n - \delta n$$

From the previous comparative statics on $S_n(z, n)$, it is straightforward to see that $g(z, n)$ is increasing in $\log z$ and decreasing in $n$.

**C.3 Frictionless limits**

**C.3.1 Setup**

**Frictional problem.** Start by recalling the Bellman equation for the joint surplus in the frictional case:

$$\rho S(z, n) = \max_v y(z, n) - nb - c(v) - \delta nS_n(z, n)$$

$$+ q(\theta)v \left\{ \phi S_n + (1 - \phi) \int_0^{S_n} H_n(s) ds \right\}$$

$$+ (\mathbb{L}S)(z, n)$$

s.t. $S(z, n) \geq 0$, $S_n(z, n) \geq 0$

where $H_n$ is the employment-weighted cumulative distribution function of marginal surpluses. $\mathbb{L}$ is the differential operator that encodes the continuation value from productivity shocks. For instance, for a diffusion, $(\mathbb{L}S)(z, n) = \mu(z)S_z(z, n) + \frac{\sigma(z)^2}{2}S_{zz}(z, n)$. Recall that $\phi = \frac{u}{u + \xi(1-u)}$ is the probability that a vacancy meets an unemployed worker, and $q$ is the vacancy meeting rate.

Note that we abstracted from exogenous separations for simplicity, but endogenous separations when $S(n, z) < 0$ still occur. Denote by $\Delta$ the aggregate endogenous separation rate.
Inside the continuation region, the density function \( h(z, n) \) of the distribution of firms by productivity and size is determined by the stationary KFE

\[
0 = -\frac{\partial}{\partial n} \left( h(z, n)g(z, n) \right) + (L^* h)(z, n)
\]

where \( L^* \) is the formal adjoint of the operator \( L \), and \( g(z, n) \) is the growth rate of employment

\[
g(z, n) = q(\theta)v^*(z, n) \left[ \phi + (1 - \phi)H_n(S_n(z, n)) \right] - \xi \lambda^U n \left[ 1 - H_v(S_n(z, n)) \right]
\]

(C.9)

where \( \lambda^U \) is the meeting rate from unemployment, and \( \xi \) the relative search efficiency of the employed.

Finally, the mass of entrant firms \( m_0 \) is determined by the free-entry condition

\[
c_e = \mathbb{E}^{\text{Entry}} \left[ \max \{ S(z, n_0), 0 \} \right]
\]

(C.10)

where \( n_0 \) is initial employment which is a parameter, and \( \mathbb{E}^{\text{Entry}} \) is the expectation operator under the productivity distribution for entrants \( \Pi_0(z) \). The surplus is a function of \( m_0 \) through the vacancy meeting rate \( q(\theta) \), since \( \theta \) is increasing in \( m_0 \).

**Functional forms.** For ease of exposition, we consider isoelastic vacancy cost functions

\[
c(v) = \frac{c_0}{1 + \gamma} v^{1+\gamma},
\]

and normalize \( c_0 = 1 \), but the result does not depend on the particular functional form nor on the normalization. Also, we specialize to a Cobb-Douglas matching function \( m(s, v) = As^\beta v^{1-\beta} \), where \( A \) is match efficiency, a proxy for labor market frictions. Finally, for ease of exposition, we set to zero exogenous separations to unemployment \( \delta = 0 \).
Comparative statics. We describe behavior of the economy in the limit when match efficiency \( A \to \infty \). We do so for two different configurations of the economy:

1. No on-the-job-search: \( \xi = 0 \)

2. On-the-job search: \( \xi > 0 \)

Notation. We write \( B \approx C \) for a first-order Taylor expansion. Denote \( ||S_n|| = \mathbb{E}^{\text{steady-state}} \left[ S_n^{1/\gamma} \right]^{\gamma} \), where \( \mathbb{E}^{\text{steady-state}} \) denotes the expectation under the steady-state distribution of marginal surpluses. This is also the Lebesgue \((1/\gamma)\)-norm of \( S_n \) under the steady-state probability measure.

C.3.2 No on-the-job search

Since \( \xi = 0, \phi = 1 \). From (C.8), the FOC for vacancies gives

\[
v^*(z,n) = \left( qS_n \right)^{1/\gamma}.
\]

(C.11)

using this optimality condition in the value function of hiring firms:

\[
\rho S(z,n) = y(z,n) - nb + \frac{\gamma}{1+\gamma} \cdot q(\theta) \frac{1}{1+\gamma} S_n^{1+1/\gamma} + (LS)(z,n)
\]

s.t. \( S(z,n) \geq 0, S_n(z,n) \geq 0 \)

which now only depends on \( q(\theta) \) as the sole aggregate. Hence, free-entry (C.10) uniquely pins down \( q(\theta) \) to the same value no matter what value \( A \) takes. Therefore, the value function always satisfies the same Bellman equation, irrespective of \( A \). Hence, throughout the state space, at any given \((n,z)\), marginal surpluses \( S_n(z,n) \) remain the same as \( A \) varies. Moreover, since the value \( S(z,n) \) is independent from \( A \), so are all the decisions by firms. As a result, the endogenous separation rate \( \Delta \) always remains the same – and in particular, finite.
We now study how aggregates $v$, $u$, $\theta$ evolve along this limiting path. Given the matching function these determine all other equilibrium objects: $\lambda^U$, $\lambda^E$, $q$. In characterizing the limit we make use of the simple fact that both $m_0$ and $v$ must remain finite. If this were not the case, then infinite entry and vacancy costs would violate the economy’s resource constraint.

**Aggregates in the limit**

Integrating both sides of the FOC for vacancies under the firm distribution, and using the matching function which implies that $q = A\theta^{-\beta}$, aggregate vacancies are

$$v = m_0 q^{\frac{1}{\gamma}} ||S_n||^{\frac{1}{\gamma}} = m_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}$$

Since $q$ remains constant, and $v$ and $m_0$ are finite in the limit, then the first equality implies that $||S_n||$ remains finite in the limit.

In the limit, the unemployment rate is $u \approx \frac{\Delta}{x^\theta}$. The matching function implies $\lambda^U = A\theta^{1-\beta}$. Combined, the unemployment rate is $u \approx \Delta A^{-1} \theta^{-(1-\beta)}$. Combining these expressions with the expression for aggregate vacancies $v$, tightness satisfies

$$\theta = \frac{v}{u} \approx \frac{m_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}}{\Delta A \theta^{1-\beta}}$$

so that

$$\theta^{\frac{1+\gamma}{\gamma}} \approx \left( \frac{m_0}{\Delta} \right) ||S_n||^{\frac{1}{\gamma}} A^{\frac{1+\gamma}{\gamma}}.$$

Since $m_0$, $\Delta$, and $||S_n||$ are finite, $\theta$ diverges with $A$. Therefore, $\lambda_U$ diverges as well. On the worker side, since $\lambda_U$, diverges to infinity, $u$ goes to zero. On the firm side, $m_0$ remains finite, but changes such that $q$ remains constant and vacancies remain finite.
Invariant distribution of marginal surpluses

We now turn to the invariant distribution $h(z,n)$. After substituting optimal vacancies into (C.9) evaluated at $\xi = 1 - \phi = 0$, one obtains that the growth of employment in the hiring region is:

$$g(z,n) = q\left(qS_n(z,n)\right)^{\frac{1}{\gamma}}.$$

Since $S_n(z,n)$ remains constant throughout the state space, then employment growth in the hiring region remains constant throughout the state space. The firm loses no workers to employment because there is no on-the-job search. Since $S_n(z,n)$ and $U = b/\rho$ both stay unchanged, then the employment losses to unemployment are still unchanged. Since $S(z,n)$ is unchanged, then the exit decision is also unchanged.

Hence, the law of motion of employment is independent of $A$. Thus, the steady-state distribution $h(z,n)$ is also independent from $A$. Therefore the values of firms $S(z,n)$ are the same across the state space and the relative mass of firms at each $(z,n)$ is unchanged, despite higher but finite $m_0$.

Summary

Summarizing this case: as $A \to \infty$, even though unemployment vanishes, the allocations in the search model without on-the-job search do not converge to those of a competitive firm-dynamics model. The free entry condition requires the vacancy meeting rate $q$ to remain finite and thus a non degenerate dispersion of marginal products of labor survives even in the limit as firms face the same adjustment frictions regardless of $A$. In contrast, in the competitive benchmark, marginal products of labor are equalized across firms.

C.3.3 On-the-job search with a fat-tailed entry distribution

We now turn to the case in which on-the-job search remains positive at some fixed value $\xi > 0$, and thus $\phi < 1$. We follow the same logic as before, with some additional steps due to
on-the-job search. To simplify algebra we abstract from exogenous job destruction, setting \( \delta = 0 \).

To keep the arguments manageable, we also introduce an additional assumption. We require the entry productivity distribution to have a “fat enough” tail. With decreasing returns to scale, the optimal frictionless size of the firm grows without bound as productivity becomes large. We assume that the productivity distribution of entrants is unbounded, and assume that it is fat-tailed enough that the rate at which the optimal size of a firm grows with productivity is faster rate than the decay of the productivity distribution. More precisely, the frictionless optimal size is \( n^*(z) = \arg\max_n y(z, n) - bn \). We assume that the entry productivity distribution \( \Pi_0(z) \) is such that

\[
\lim_{z \to +\infty} n^*(z)\Pi_0(z) = +\infty
\]

This is satisfied for the production function \( y(z, n) = zn^\alpha \) and the entry distribution \( \Pi_0(z) \propto z^{-\zeta} \), when \( \frac{1}{1-\alpha} - \zeta \geq -1 \). Our empirical implementation uses these functional forms and satisfies these restrictions.\(^7\)

Consider (C.8) written in terms of the return on a vacancy

\[
\rho S(z, n) = \max_v y(z, n) - nb - c(v) + q(\theta)vR(S_n) + (LS)(z, n)
\]

s.t. \( S(z, n) \geq 0, S_n(z, n) \geq 0 \)

where

\[
R(S_n) = \phi S_n + (1 - \phi) \int_0^{S_n} H_n(s)ds
\]  

(C.12)

is the return to a vacancy. The growth of employment is

\[
g(z, n) = qv^*(z, n) \left[ \phi + (1 - \phi) H_n(S_n(z, n)) \right] - \xi \lambda U n \left[ 1 - H_v(S_n(z, n)) \right]
\]  

(C.13)

An alternative approach is to assume a constant arrival rate of “The Godfather” shocks that leave workers unable to refuse any job offer. Additional details available on request.
**Aggregates in the limit**

We restrict attention to the economically meaningful case in which (1) output remains finite and strictly positive in the limit, and (2) the rate at which workers separate into unemployment remains finite in the limit. These restrictions are equivalent to a guess and verify strategy, in which we guess that (1-2) hold and then verify those conditions. The logic of our approach is then to exhibit a solution in which (1-2) hold – but in principle other cases may arise.

Consider the set of meeting rates. Because some measure $n$ employed jobseekers are always present regardless of $A$, the amount of effective search effort $s = u + \xi n$ remains finite and positive even if $u$ goes to zero. By (1), vacancies also remain finite. Combined, these imply that market tightness $\theta = v/s$ remains finite. Since $q = A\theta^{-\beta}$ and $\lambda U = A\theta^{-(1-\beta)}$, then both meeting rates diverge to infinity at the same rate as $A$.$^8$

Consider unemployment and aggregate vacancies. (2) requires that the rate at which workers separate into unemployment is a positive constant $\Delta$ in the limit. Since $u \approx \frac{\Delta}{\lambda U}$, and $\lambda U$ diverges, then the unemployment rate converges to zero. Since the unemployment rate converges to zero, then $\phi$ also converges to zero. Firm level and aggregate vacancies are given by

$$v = q^{\frac{1}{\gamma}}R(S_n)^{\frac{1}{\gamma}}, \quad v = m_0 q^{\frac{1}{\gamma}}||R(S_n)||^{\frac{1}{\gamma}}.$$  (C.14)

(1) implies that both aggregate vacancies $v$ the mass of entering firms $m_0$ remain finite. Since $v$ is finite and $m_0$ is finite, while $q$ diverges at the same rate as $A$, then $\gamma > 0$ requires $||R(S_n)||$ must go to zero at the same rate as $A$ goes to infinity.

**Invariant distribution of marginal surpluses**

We now show that the distribution of marginal surpluses degenerates to a single value on the support of the invariant distribution.

---

$^8$Strictly speaking, free-entry then ensures that theta is pinned down to a strictly positive value. This proof is more lengthy but does not require any additional assumptions and is available upon request.
First, we use (C.14) to express firm level vacancies as a share of aggregate vacancies, where that share is determined by the firms’ return on a vacancy relative to the average return:
\[ v = \frac{1}{m_0} \left( \frac{R(S_n)}{||R(S_n)||} \right)^{\frac{1}{\gamma}} \]
where the second equality uses \( q = A(v/\xi)^{-\beta} \), and \( \lambda^U = A(v/\xi)^{-(1-\beta)} \), which jointly imply that \( v = \lambda^U \xi / q \). Now consider the expression for growth of employment inside the continuation region (C.13), under the limiting case of \( \phi = 0 \):
\[ g(z,n) \approx qvH_n(S_n(z,n)) - \xi \lambda^U n \left[ 1 - H_v(S_n(z,n)) \right] \]
Substituting in the expression for firm vacancies and collecting \( \lambda^U \xi \) terms:
\[ g(z,n) \approx \lambda^U \xi \left\{ \frac{1}{m_0} \left( \frac{R(S_n)}{||R(S_n)||} \right)^{\frac{1}{\gamma}} H_n(S_n) - n \left[ 1 - H_v(S_n) \right] \right\} \]
Consider some \((n,z)\) that has positive mass in steady state. Since \( \lambda^U \) diverges but growth must remain finite, the term in braces must be equal to zero in the limit:
\[ \frac{1}{m_0} \left( \frac{R(S_n)}{||R(S_n)||} \right)^{\frac{1}{\gamma}} H_n(S_n) = n \left[ 1 - H_v(S_n) \right] \]
Using this we can show that the distribution of marginal surplus converges point-wise to a degenerate limiting distribution \( H_n^\infty \).

We proceed by contradiction. Suppose that \( H_n \) converges to a limiting distribution \( H_n^\infty \) that is non-degenerate.\(^9\) Consider a firm at the top of the distribution, such that \( 1 - H_v(S_n) = 0 \). The probability that the firm loses a worker is zero, so the right-hand side is zero. However, by the supposition that \( H_n \) is non-degenerate, then \( R(S_n) \) converges to a non-zero value, since the firm can increase its value by poaching from workers below it on

\(^9\)So the probability measure of \( S_n \) in the cross-section would converge in distribution to a non-degenerate limit.
the ladder. Since there is some $R(S_n)$ that is non-zero, then $||R(S_n)||$ also converges to a non-zero value. Therefore flows out of the firm are zero, but flows into the firm are positive. This violates the above equality, which would imply infinite growth as $\lambda^U$ diverges. This is a contradiction. Hence, in the limit $H^\infty_n$ must be degenerate, and marginal surpluses of firms converge to a common limit which we denote $S^*_n$.

We have shown that the limiting distribution $H^\infty_n$ is degenerate. This implies that the invariant distribution of employment and productivity lines up along a strip $\{z, n^*(z)\}$ where $n^*(z)$ is implicitly defined by $S_n(n^*(z), z) = S^*_n$, so is strictly increasing.\(^{10}\)

**Unique value for $S^*_n$ on the limiting strip**

We have shown that $||R(S_n)||$ and $R(S_n)$ converge to zero in the limit, yet this does not necessarily imply a particular value for $S^*_n$. Here we show that $S^*_n = 0$. We guess the following, which we verify below:

\[ n^*(z) = \arg\max_n y(z, n) - bn . \]

From the concavity of marginal surplus and $n^*(z) > n_0$, we have

\[ S(z, n_0) \leq S(z, n^*(z)) - S_n(z, n^*(z)) \times (n^*(z) - n_0) \]

In the limit $S_n(z, n^*(z)) \equiv S^*_n$ is equalized, which delivers the following upper bound to the value of entry:

\[ \int S(z, n_0)\Pi_0(z)dz \leq \int S(z, n^*(z))\Pi_0(z)dz - S^*_n \int (n^*(z) - n_0)\Pi_0(z)dz \]

We show that $S^*_n = 0$ by contradiction. Suppose that $S^*_n > 0$. From our assumption on the entry distribution then in the limit $\int n^*(z)\Pi_0(z)dz$ is infinity. Since all other terms on the

\(^{10}\)To see that this is a strip, recall that $S(z, n)$ is such that $S_{nn} < 0$ and $S_{zn} > 0$. Therefore $S_n(n^*(z), z) = S^*_n$ implicitly defines an strictly increasing function $n^*(z)$.
right-hand side of the above inequality are finite, then a necessary condition for the above inequality to be satisfied is that $\int S(z, n_0) \Pi_0 dz < 0$, which violates the free-entry condition. Therefore it must be that $S^*_n = 0$.

Intuitively, a strictly positive marginal surplus $S^*_n$ reflects that there is an excess supply of firms in the economy relative to the supply of workers. The fat tail assumption implies that this excess supply translates into a very negative value of entry, which cannot be an equilibrium in which firms enter freely.

**Limiting value function**

We now return to the limiting Bellman equation for marginal surplus. Given that $q(\theta)R(S^*_n) = 0$, making the generator $L$ explicit, applying the result that $n = n^*(z)$, and noting that $S_n(n, z) \geq 0$ is satisfied with equality, we have

$$\rho S(z, n^*(z)) = y(z, n^*(z)) - n^*(z) b + \mu(z) S_z(z, n^*(z)) + \frac{\sigma(z)^2}{2} S_{zz}(z, n^*(z))$$

s.t. $S(z, n^*(z)) \geq 0$

Our key result in the text was that the limiting economy featured a value function that depended only on $z$. However, the continuation value terms in the above value function contain *partial* derivatives with respect to $z$, not *total* derivatives. To argue that it is enough to focus on the value function evaluated on the strip, we must show that the partial derivatives approximate the total derivatives in the limit. The following shows that this is

---

$^{11}$ $\int S(z, n^*(z)) \Pi_0(z) dz$ remains finite because $S(z, n^*(z))$ satisfies (C.8) evaluated at $(z, n^*(z))$. It can then be shown that, in the limit, $q(\theta)R(S^*_n)$ depends only on $S^*_n$ and $\theta$, but not on $A$ directly. The details of the derivation are available upon request.

$^{12}$Details are available upon request.
the case in the limit

\[
\lim_{A \to \infty} \frac{dS(z, n^*(z))}{dz} = \lim_{A \to \infty} \left\{ \frac{\partial S(z, n)}{\partial z} \bigg|_{n=n^*(z)} + \frac{\partial S(z, n)}{\partial n} \bigg|_{n=n^*(z)} \cdot \frac{dn^*(z)}{dz} \right\}
\]

\[
= \lim_{A \to \infty} \frac{\partial S(z, n^*(z))}{\partial z} \bigg|_{n=n^*(z)}
\]

Therefore, in the limit, exit can be described by the value function evaluated on the strip, \( \overline{S}(z) := S(z, n^*(z)) \) which evolves according to

\[
\rho \overline{S}(z) = y(z, n^*(z)) - n^*(z)b + \mu(z)\overline{S}_z(z) + \frac{\sigma(z)^2}{2}\overline{S}_{zz}(z)
\]

and an exit cut-off determined by \( \overline{S}(z) = 0 \).

**Optimal size**

We now characterize the optimal size of incumbents in the limit and verify (\( \star \)). We note that, if for a small period of time \( dt \), the firm was away from the exit cutoff but close to its optimal size, then

\[
S(z, n) \approx \left[ y(z, n) - bn \right] dt + e^{-\rho dt} \mathbb{E}\left[ S(z_{dt}, n^*(z_{dt})) \right]_{z_0 = z}
\]

because the other contributions in \( n - n^*(z) \) scale with \( ||S_n|| = 0 \). Therefore,

\[
S_n(z, n^*(z)) \approx \left[ y_n(z, n^*(z)) - b \right] dt
\]

and so it must be that \( y_n(z, n^*(z)) = b \). This confirms guess (\( \star \)).\(^{13}\)

\(^{13}\)To make this argument strictly formal, differentiate the Bellman equation and represent marginal surplus as an integral with the Feynman-Kac formula as in Appendix C.2. The derivation details are available upon request.
Summary

With on-the-job search, as $A \to +\infty$, the value function converges to the one of the Hopenhayn model. The mass of active firms converges to some finite value. Free-entry pins down the mass of firms, and converges to a condition that differs from the Hopenhayn model’s. There is an additional term that stems from the value gains that entrant realize along their (very fast) growth towards their optimal size. The equilibrating variable is the limiting market tightness, that governs the size of these gains.

C.4 Algorithm

We want to solve the Bellman equation:

$$
\rho S(z,n) = \max_{v \geq 0} \left[ y(z,n) - \delta n S_n(z,n) + \mathcal{H}(S_n(z,n)) v - c(v,n) + \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U \right]
$$

with

$$
\mathcal{H}(S_n(z,n)) = q \left[ \phi S_n(z,n) + (1 - \phi) \int_0^{S_n(z,n)} [S_n(z,n) - S_n'] dH_n(S_n') \right]
$$

where the last line uses integration by parts. We assume that the vacancy cost satisfies $c(v,n) = \overline{c} \left( \frac{v}{n} \right) v$, where $\overline{c}$ is iso-elastic with elasticity $\gamma$. Therefore, $c_v(v,n) = (\gamma + 1) \overline{c} \left( \frac{v}{n} \right)$. Along with the first order condition $c_v(v,n) = \mathcal{H}(S_n(z,n))$, this implies

$$
c(v,n) = \overline{c} \left( \frac{v}{n} \right) v = \frac{1}{\gamma + 1} c_v(v,n) v = \frac{1}{\gamma + 1} \mathcal{H}(S_n(z,n)) v
$$
Therefore the total value of vacancy posting is

\[
\mathcal{H}(S_n(z, n)) v - c(v, n) = \frac{\gamma}{\gamma + 1} \mathcal{H}(S_n(z, n)) v
\]

Letting \( \frac{v}{n} = \frac{\kappa}{1 + \gamma} \frac{\gamma}{\gamma + 1} \mathcal{H}(S_n(z, n)) \) and using \( \omega = \frac{1}{\gamma + 1} \mathcal{H}(S_n(z, n)) \) then

\[
\frac{v}{n} = \kappa^{1/\gamma} \mathcal{H}(S_n(z, n))^{\frac{1}{\gamma}}
\]

and

\[
\mathcal{H}(S_n(z, n)) v - c(v, n) = \frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma + 1} \mathcal{H}(S_n(z, n))^{\frac{\gamma + 1}{\gamma}} n
\]

Substituting this into the Bellman equation, acknowledging that \( \rho U = b \), we arrive at the formulation:

\[
\rho S(z, n) = y(z, n) - bn + \left[ \frac{\gamma\kappa^{-\frac{1}{\gamma}} \mathcal{H}(S_n(z, n))^{\frac{\gamma + 1}{\gamma}}}{\gamma + 1} - \delta \right] S_n(z, n) + \mu(z) S_z(z, n) + \frac{\sigma^2(z)}{2} S_{zz}(z, n)
\]

subject to \( S(z, n) \geq 0 \) and \( S_n(z, n) \geq 0 \).

**C.4.1 Algorithm**

The algorithm consists of three steps, implemented in MATLAB called from master file `MAIN.m`.

**Step 0: Construct an initial guess.** Start by constructing a \( n_z \times n_n \) grid for log productivity and log size. Let \( \pi = y(z, n) - bn \) denote the stacked \( (n_z \times n_n) \times 1 \) vector of flow payoffs on this grid. Guess an initial surplus \( S^0 \) on this grid (a \( (n_z \times n_n) \times 1 \) column
vector); a distribution of firms over productivity and size $h^0$ (a $(n_z*n_n) \times 1$ column vector); aggregate finding rates $q^0$ and $\lambda^0$; and an efficiency-weighted share of unemployed searchers, $\theta^0$. Construct marginal surplus. Construct exit regions, separation regions and the vacancy policy. File InitialGuess.m does this.

**Step I: Iterate to convergence the coalition’s problem for given aggregate states.**

For $t \geq 1$, given $q^{t-1}$, $\theta^{t-1}$, $h^{t-1}$ and $S^{t-1}$, solve the coalition’s problem to update the coalition value to $S^t$. The solution to the coalition’s surplus function is obtained in an inner iteration $\tau$. Denote by $S^{t,\tau}$ the surplus in outer iteration $t$ during inner iteration $\tau$, initiated with $S^{t,0} = S^t$; $T_n(z,n)$ a $(n_z*n_n) \times (n_z*n_n)$ matrix such that $S^{t,\tau}_n = T_n(z,n)S^{t,\tau}$, where $S^{t,\tau}_n$ is the stacked $(n_z*n_n) \times 1$ vector of derivatives of $S$ w.r.t. $n$ during outer iteration $t$ and inner iteration $\tau$; $T_z$ a $(n_z*n_n) \times (n_z*n_n)$ matrix such that $S^{t,\tau}_z = T_zS^{t,\tau}$, where $S^{t,\tau}_z$ is the stacked $(n_z*n_n) \times 1$ vector of derivatives of $S$ w.r.t. $z$ during outer iteration $t$ and inner iteration $\tau$; and $T_{zz}$ a $(n_z*n_n) \times (n_z*n_n)$ matrix such that $S^{t,\tau}_{zz} = T_{zz}S^{t,\tau}$, where $S^{t,\tau}_{zz}$ is the stacked $(n_z*n_n) \times 1$ vector of second derivatives of $S$ w.r.t. $z$ during outer iteration $t$ and inner iteration $\tau$. Note that the matrix $T_n(z,n)$ depends on $(z,n)$ in the sense that the approximation is done either forward or backward depending on the endogenous drift for $n$ at $(z,n)$ (note that the drift of and innovations to $z$ are independent of $(z,n)$). Within each outer iteration $t$, we iteratively update $S^{t-1,\tau}$ for $\tau \geq 1$ following equation (C.15) based on

$$
\left[\left(\rho + \frac{1}{\Delta}\right)1 - \left[\frac{\gamma \kappa - \frac{1}{\gamma}}{\gamma + 1} \mathcal{H} \left(\frac{S^{t-1,\tau-1}_n}{S^{t-1,\tau-1}}\right)^{\frac{\gamma+1}{\gamma}} - \delta I\right] \ast T_n(z,n) - \mu T_z - \frac{\sigma^2}{2}T_{zz}\right] S^{t-1,\tau}
$$

$$
= \pi + \frac{1}{\Delta}S^{t-1,\tau-1}
$$

where $\Delta$ is the step size, $\ast$ denotes the element-by-element product, and $\mathcal{H} \left(\frac{S^{t-1,\tau-1}_n}{S^{t-1,\tau-1}}\right)^{\frac{\gamma+1}{\gamma}} / S^{t-1,\tau-1}_n$ is a $(n_z*n_n) \times (n_z*n_n)$ matrix constructed using the previous iteration’s derivative of $S$ stacked $(n_z*n_n)$ times in the column dimension. The step size cannot be too large for the
problem to converge. These iterations are performed by iterating on $\tau$ until convergence by file $\text{IndividualBehavior.m}$, and the solution is assigned as the updated $S^t$. We also obtain from the converged solution the updated separation, exit and a vacancy policies.

**Step II: Iterate to convergence the aggregate states for given individual behavior.**

Given updated individual behavior in outer iteration $t$, obtain through iteration in an inner loop $\tau$ the distribution of firms $h_t$, the aggregate meeting rates $q_t$ and $\lambda_t$, the share of unemployed searchers $\theta_t$, the distribution of workers over marginal surplus $H_{n_t}$, and the distribution of vacancies over marginal surplus $H_{v_t}$. File $\text{AggregateBehavior.m}$ proceeds to do this in four steps.

Initiate each aggregate object with the previous outer iteration solution, $x_{t-1,0} = x_{t-1}$. Then:

*Step II-a.* Update the distribution of workers over marginal surplus to $H_{n_{t-1,\tau}}$ given a distribution of firms $h_{t-1,\tau-1}$ and marginal surplus $S_{n_t}^t$, where the latter was obtained in **Step I** above. This is done by file $\text{CdfG.m}$.

*Step II-b.* Update the distribution of vacancies over marginal surplus $H_{v_{t-1,\tau}}$ given a distribution of firms $h_{t-1,\tau-1}$, the vacancy policy $v_t$ and the ranking of firms in marginal surplus space. This is done by file $\text{CdfF.m}$.

*Step II-c.* Update the finding rates $q_{t-1,\tau}$, $\lambda_{t-1,\tau}$ and $\theta_{t-1,\tau}$ that is consistent with the vacancy policy $v_t$ and the distribution of firms $h_{t-1,\tau-1}$. This is done by file $\text{HazardRates.m}$.

*Step II-d.* Given $H_{n_{t-1,\tau}}$, $H_{v_{t-1,\tau}}$, $q_{t-1,\tau}$, $\lambda_{t-1,\tau}$ and $\theta_{t-1,\tau}$, update the distribution of firms $h_{t-1,\tau}$ following the Kolmogorov forward equation in steady-state. This is executed by file $\text{Distribution.m}$.

Iterate over the four sub-steps *Step II-a–Step II-d* until convergence and assign the updated aggregate states $q_t$, $\lambda_t$, $\theta_t$ and $h_t$. We subsequently return to step **Step I** and iterate on step **Step I–Step II** until both the surplus function and the aggregate states have converged.
C.4.2 Estimation

The criterion function that we minimize is highly-dimensional and potentially has many local minima. Furthermore, the equilibrium does not exist for some regions of the parameter space. For example, if the drift in productivity is not sufficiently negative, there is no ergodic distribution for productivity. For these reasons, using a sequential hill-climbing optimizer that updates its initial guess sequentially through a gradient-based method is prohibitive. Our solution is to use an algorithm that we can easily parallelize, that efficiently explores the parameter space, and for which we can ignore cases with no equilibrium.

We set up a hyper-cube in the parameter space and then initialize a Sobol sequence to explore it. A Sobol sequence is a quasi-random low-discrepancy sequence that maintains a maximum dispersion in each dimension and far outperforms standard random number generators. We then partition the sequence and submit each partition to a separate CPU on a high performance computer (HPC). From each evaluation of the parameter hyper-cube, we save the vector of model moments. We then collect them, splice them all together, and choose the one that minimizes the criterion function. Starting with wide bounds on the parameters, we run this procedure a number of times, shrinking the hyper-cube step by step until we achieve the global minimum.

Compared to standard optimizers, this procedure has the advantage that, as a byproduct of the estimation, we can learn a lot about model identification. From an optimizer one may retrieve the moments of the model only along the path of the parameter vector chosen by the algorithm. In our case, we retrieve tens of thousands of evaluations, knowing that the low-discrepancy property of the Sobol sequence implies that for an interval of any one parameter, the remaining parameters are drawn uniformly. Plotting each single moment against parameters therefore shows the effect of a parameter on a certain moment, conditional on local draws of all other parameters.
C.5 Notation for dynamic model

We first specify the value function of an individual worker $i$ in a firm with arbitrary state $x$: $V(x,i)$. We then specify the value function of the firm: $J(x)$. Combining all workers’ value functions with that of the firm we define the joint value: $\Omega(x)$. We then apply the assumptions from Section 3.2.2 which allow us to reduce $(x)$ to only the number of workers and productivity of the firm, $(z,n)$. Finally we take the continuous work force limit to derive a Hamilton-Jacobi-Bellman (HJB) equation for $\Omega(z,n)$ Applying the definition of total surplus used above, we obtain a HJB equation in $S(z,n)$ which we use to construct the equilibrium.

C.5.1 Worker value function: $V$

As in the static example, let $U$ be the value of unemployment. It is convenient to define separately worker $i$’s value when employed at firm $x$ before the quit, layoff and exit decisions, $V(x,i)$, and their value after these decisions, $V(x,i)$.  

Value of unemployment. Let $h_U(x)$ denote how the state of firm $x$ is updated when it hires an unemployed worker. Let $A$ denote the set of firms making job offers that an unemployed worker would accept. The value of unemployment $U$ therefore satisfies

$$\rho U = b + \lambda^U(\theta) \int_{x \in A} [V(h_U(x),i) - U] dH_v(x)$$

where $H_v$ is the vacancy-weighted distribution of firms. If $x \notin A$, then the worker remains unemployed.

---

14 In terms of Figure 3.1, the value $V$ is computed after the first stage of the flow chart, and the value $V$ after the second stage, in the case that the firm stays in operation.

15 For example, size would be update from $n$ to $n + 1$ and possibly some of the incumbent wages would be bargained down.
Stage I.  To relate the value of the worker pre separation, \( V(x, i) \), to that post separation, \( V(x, i) \), we require the following notation regarding firm and co-worker actions. Since workers do not form ‘unions’ within the firm, all of these actions are taken as given by worker \( i \).

- Let \( \epsilon(x) \in \{0, 1\} \) denote the exit decision of firm, and \( \mathcal{E} = \{ x : \epsilon(x) = 1 \} \) the set of \( x \)’s for which the firm exits.

- Let \( \ell(x) \in \{0, 1\}^{n(x)} \) be a vector of zeros and ones of length \( n(x) \), with generic entry \( \ell_i(x) \), that characterizes the firm’s decision to lay off incumbent worker \( i \in \{1, \ldots, n(x)\} \), and \( \mathcal{L} = \{(x, i) : \ell_i(x) = 1\} \) the set of \((x, i)\) such that worker \((x, i)\) is laid off.

- Let \( q^U(x) \in \{0, 1\}^{n(x)} \) be a vector of length \( n(x) \), with generic entry \( q^U_i(x) \) that characterizes an incumbent workers’ decisions to quit, and \( \mathcal{Q}^U = \{(x, i) : q^U_i(x) = 1\} \) the set of \((x, i)\) such that worker \((x, i)\) quits into unemployment.

- Let \( \kappa(x) = \left(1 - \ell(x)\right) \odot \left(1 - q^U(x)\right) \) be an element-wise product vector that identifies workers that are kept in the firm, and \( \mathcal{S} = \mathcal{L} \cup \mathcal{Q}^U = \{(x, i) : \kappa_i(x) = 0\} \), the set of \((x, i)\) such that worker \((x, i)\) separates into unemployment.

- Let \( s(x, \kappa(x)) \) denote how the state of firm \( x \) is updated when workers identified by \( \kappa(x) \) are kept. This includes any renegotiation.

Given these sets and functions, the pre separation value \( V(x, i) \) satisfies:

\[
V(x, i) = \epsilon(x) U + (1 - \epsilon(x)) \left[ \mathbb{1}_{\{(x, i) \not\in \mathcal{S}\}} V(s(x, \kappa(x)), i) + \mathbb{1}_{\{(x, i) \in \mathcal{S}\}} U \right]
\]

Stage II.  It is helpful to characterize the value of employment post separation decisions, \( V(x, i) \), in terms of the three distinct types of events described in Figure 3.2. First, the value changes due to ‘Direct’ labor markets shocks to worker \( i \), \( V_D(x, i) \). These include her match being destroyed exogenously or meeting a new potential employer. Second, the value
changes due to labor market shocks hitting other workers in the firm, $V_I(x,i)$, including their matches being exogenously destroyed or them meeting new potential employers. These events have an ‘Indirect’ impact on worker $i$. Third, the value changes due to events on the ‘Firm’ side, $V_F(x,i)$, including the firm contacting new workers and receiving productivity shocks. Combining events and exploiting the fact that in continuous time they are mutually exclusive, we obtain the following, where $w(x,i)$ is the wage paid to worker $i$:

$$\rho V (x,i) = w(x,i) + \rho V_D (x,i) + \rho V_I (x,i) + \rho V_F (x,i).$$

We note that the wage function $w(x,i)$ includes the transfers between worker $i$ and the firm that may occur at the stage of vacancy posting (after separations and before the labor market opens), as discussed in Section C.1.3 in the context of the static example. These transfers can depend on the entire wage distribution inside the firm which is subsumed in the state vector $x$.

**Direct events.** We first characterize changes in value due to labor market shocks directly to worker $i$ in firm $x$, $V_D(x,i)$. Exogenous separation shocks arrive at rate $\delta$ and draws of outside offers arrive at rate $\lambda^E(\theta)$ from the vacancy-weighted distribution of firms $H_v$. If worker $i$ receives a sufficiently good outside offer from $x'$, she quits to the new firm. We denote by $Q^E(x,i)$ the set of such quit-firms $x'$ for $i$. Otherwise, the worker remains with the current firm but with an updated contract. Therefore $V_D(x,i)$ satisfies

$$\rho V_D (x,i) = \delta [U - V (x,i)] \underbrace{+ \lambda^E(\theta) \int_{x' \in Q^E(x,i)} [V (h_E (x,i,x'),i) - V (x,i)] dH_v (x')}_{\text{Exogenous separation}}$$

$$\underbrace{+ \lambda^E(\theta) \int_{x' \notin Q^E(x,i)} [V (r (x,i,x'),i) - V (x,i)] dH_v (x')}_{\text{Retention}},$$

$$\underbrace{+ \lambda^E(\theta) \int_{x' \in Q^E(x,i)} [V (r (x,i,x'),i) - V (x,i)] dH_v (x')}_{\text{EE Quit}}.$$
where \( h_E(x, i, x') \) describes how the state of a poaching firm \( x' \) gets updated when it hires worker \( i \) from firm \( x \). Similarly, \( r(x, i, x') \) updates \( x \) when—after meeting firm \( x' \)—worker \( i \) in firm \( x \) is retained and renegotiates its value. In all functions with three arguments \((x, i, x')\), the first argument denotes the origin firm, the second identifies the worker, and the third the potential destination firm.

**Indirect events.** We next characterize changes in value due to the same labor market shocks hitting other workers in firm \( x \), \( V_I(x, i) \). The value \( V_I(x, i) \) satisfies

\[
\rho V_I(x, i) = \sum_{j \neq i}^{{n(x)}} \left\{ \delta \left[ V(d(x, j), i) - V(x, i) \right] \right\} + \lambda E(\theta) \int_{x' \in Q} \left[ V(q_E(x, j, x'), i) - V(x, i) \right] dH_v(x') + \lambda E(\theta) \int_{x' \notin Q} \left[ V(r(x, j, x'), i) - V(x, i) \right] dH_v(x') \right\},
\]

where \( d(x, j) \) updates \( x \) when worker \( j \) exogenously separates, and \( q_E(x, j, x') \) when worker \( j \) quits to firm \( x' \).

**Firm events.** Finally, we characterize changes in value due to events that directly impact the firm and hence indirectly its workers, \( V_F(x, i) \). Taking as given the firm’s vacancy posting
policy $v(x)$ and other actions, $V_F(x,i)$ satisfies

$$\rho V_F(x,i) = $$

**UE Hire**
$$\phi q(\theta)v(x) [V(h_U(x), i) - V(x, i)] \cdot I_{\{x \in A\}}$$

**UE Threat**
$$+ \phi q(\theta)v(x) [V(t_U(x), i) - V(x, i)] \cdot I_{\{x \notin A\}}$$

**EE Hire**
$$(1 - \phi) q(\theta)v(x) \int_{x \in QF(x',i')} [V(h_E(x',i', x), i) - V(x, i)] dH_n(x',i')$$

**EE Threat**
$$+ (1 - \phi) q(\theta)v(x) \int_{x \notin QF(x',i')} [V(t_E(x',i', x), i) - V(x, i)] dH_n(x',i')$$

**Shock**
$$+ \Gamma_z [V, V](x,i)$$

where $t_U(x)$ updates $x$ when an unemployed worker is met and not hired, but could be possibly used as a threat in firm $x$. Similarly, $t_E(x',i', x)$ updates $x$ when worker $i'$ employed at firm $x'$ is met, not hired, but could be used as a threat. And, with a slight abuse of notation, $H_n(x',i')$ gives the joint distribution of firms $x'$ and worker types within firms $i'$.

Finally, $\Gamma_z [V, V](x,i)$ identifies the contribution of productivity shocks $z$ to the Bellman equation. At this stage we only require that the productivity process is Markovian with an infinitesimal generator. Later we will specialize this to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ such that

$$\Gamma_z [V, V](x,i) = \mu(z) \lim_{dz \to 0} \frac{V((x, z + dz), i) - V(x, z, i)}{dz}$$

$$+ \frac{\sigma^2(z)}{2} \lim_{dz \to 0} \frac{V((x, z + dz), i) + V((x, z - dz), i) - 2V(x, z, i)}{dz^2}$$

In the case that $V = V$, this becomes the standard expression for a diffusion featuring the first and second derivatives of $V$ with respect to $z$: $\Gamma_z[V](x,i) = \mu(z)V_z(x,z,i) + \frac{1}{2} \sigma(z)^2 V_{zz}(x,z,i)^{16}$

---

16Note that in (??) we abuse notation and write the state as $(x, z)$ with some redundancy since $z$ is clearly a member of $x$. We also note that we are not constrained to a diffusion process. We could also consider a Poisson process where, at exogenous rate $\eta$, $z$ jumps according to the transition density $\Pi(z, z')$: $\Gamma_z [V, V](x,i) = \eta \sum_{z' \in Z} V((x, z'), i) \Pi(z', z) - V(x, z, i)].$
In the event productivity changes or $n(x)$ changes because of exogenous labor market events, the worker will want to reassess whether to stay with the firm or not. Additionally, the firm may want to reassess whether to exit or fire some workers. Bold values $V$ capture any case where the state changes.

### C.5.2 Firm value function: $J$

Consistent with the notation we used for workers’ values, let $J(x)$ and $J(x)$ be the values of the firm at the corresponding points of an interval $dt$. For now, we take the vacancy creation decision $v(x)$ as given. At the end of the section we describe the expected value of an entrant firm.

**Stage I.** Consistent with the first stage worker value function, we define the firm value before the exit/layoff/quit decision, where we recall that $\vartheta$ is the firm’s value of exit, or scrap value:

$$J(x) = \epsilon(x) \vartheta + [1 - \epsilon(x)] J(s(x, \kappa(x))).$$

**Stage II.** Given a vacancy policy $v(x)$, let $J(x)$ be the value of a firm with state $x$ after the layoff/quit, exit. It is convenient to split the value of the firm, as we did for the worker, into three components

$$\rho J(x) = y(x) - \sum_{i=1}^{n(x)} w_i(x, i) + \rho J_W(x) + \rho J_F(x) - c(v(x), x).$$

For a given policy $v(x)$ there is a set of associated transfers between workers and the firm which, as for the worker value function, are implicit in the wage function $w(x, i)$. 

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The component $J_W (x)$ is given by

$$\rho J_W (x) =$$

- **Destruction**
  $$\delta \sum_{i=1}^{n(x)} [J (d(x,i)) - J(x)]$$

- **EE Quit**
  $$\lambda^E (\theta) \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x,i)} [J (q_E (x,i')) - J(x)] dH_v (x')$$

- **Retention**
  $$\lambda^E (\theta) \sum_{i=1}^{n(x)} \int_{x' \not\in Q^E(x,i)} [J (r (x,i,x')) - J(x)] dH_v (x').$$

The component $J_F (x)$ is given by

$$\rho J_F (x) =$$

- **UE Hire**
  $$\phi q (\theta) v (x) [J (h_U (x)) - J(x)] \cdot I_{\{x \in A\}}$$

- **UE Threat**
  $$+ \phi q (\theta) v (x) [J (t_U (x)) - J (x)] \cdot I_{\{x \not\in A\}}$$

- **EE Hire**
  $$+ (1 - \phi) q (\theta) v (x) \int_{x \in Q^E (x', i')} [J (h_E (x', i', x)) - J (x)] dH_n (x', i')$$

- **EE Threat**
  $$+ (1 - \phi) q (\theta) v (x) \int_{x \not\in Q^E (x', i')} [J (t_E (x', i', x)) - J (x)] dH_n (x', i')$$

- **Shock**
  $$+ \Gamma_z [J, J] (x)$$

It is useful to recall that, in continuous time at most one contact is made per instant. That is, either one worker is exogenously separated, or one worker is contacted by another firm, or one worker is met by posting vacancies (at rate $q (\theta) v (x)$), or a shock hits the firm. Note also that we have bold $J$'s in each line since after any of these events, the firm may want to layoff some workers or exit, and workers may want to quit.

**Entry.** The expected value of an entrant firm is

$$J_0 = -c_0 + \int J (x_0) d\Pi_0 (z_0) \quad \text{(C.17)}$$
where $x_0$ is the state of the entrant firm which includes only the random productivity value $z_0$ drawn from $\Pi_0$ since we assumed the initial number of workers is 0. The argument of the integral is $J$, which incorporates the firm’s decision to exit or operate after observing $z_0$. Entry occurs when $J_0 > 0$. 

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C.6 Derivation of the joint value function $\Omega$

We define the joint value of the firm and its employed workers $\Omega (x) := J(x) + \sum_{i=1}^{n(x)} V(x, i)$. We also define the joint value before exit/quit/layoff decisions: $\Omega(x) := J(x) + \sum_{i=1}^{n(x)} V(x, i)$.

C.6.1 Combining worker and firm values

In this section, we show that summing firm and worker values, then applying these definitions delivers the following Bellman equation for the joint value:

$$\rho \Omega(x) = y(x) - c(v(x), x)$$

\begin{align*}
\text{Destruction} & \quad + \sum_{i=1}^{n(x)} \delta [\Omega(d(x, i)) + U - \Omega(x)] \\
\text{Retention} & \quad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin Q^E(x, i)} [\Omega(r(x, i, x')) - \Omega(x)] dH_v(x') \\
\text{EE Quit} & \quad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, i)} [\Omega(q_E(x, i, x')) + V(h_E(x, i, x'), i) - \Omega(x)] dH_v(x') \\
\text{UE Hire} & \quad + \phi q(\theta) v(x) [\Omega(h_U(x)) - U - \Omega(x)] \cdot \mathbb{1}_{\{x \in A\}} \\
\text{UE Threat} & \quad + \phi q(\theta) v(x) [\Omega(t_U(x)) - \Omega(x)] \cdot \mathbb{1}_{\{x \notin A\}} \\
\text{EE Hire} & \quad + (1 - \phi) q(\theta) v(x) \int_{x' \in Q^E(x', i')} [\Omega(h_E(x', i', x)) - V(h_E(x', i', x), i')] - \Omega(x) dH_n(x', i') \\
\text{EE Threat} & \quad + (1 - \phi) q(\theta) v(x) \int_{x' \notin Q^E(x', i')} [\Omega(t_E(x', i', x)) - \Omega(x)] dH_n(x', i') \\
\text{Shock} & \quad + \Gamma_z [\Omega, \Omega](x).
\end{align*}

Note that this joint value is only written in terms of other joint values and worker values. However, it involves both firm and worker decisions through the sets $A$, $Q^E$ and the vacancy policy, $v(x)$. 

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Derivation.  We start by computing the sum of the workers’ values at a particular firm. Summing values of all the employed workers

\[
\rho \sum_{i=1}^{n(x)} V(x, i) = \sum_{i=1}^{n(x)} w(x, i)
\]

Destructions

\[+ \sum_{i=1}^{n(x)} \delta [U - V(x, i)]\]

Retention

\[+ \lambda E \sum_{i=1}^{n(x)} \int_{x' \notin Q^E(x, j)} [V(r(x, j, x'), i) - V(x, i)] dH_v(x')\]

EE Quits

\[+ \lambda E \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, j)} [V(h_E(x, j, x')) - V(x, i)] dH_v(x')\]

Incumbents

\[+ \sum_{i=1}^{n(x)} \rho V_I(x, i)\]

Firm

\[+ \sum_{i=1}^{n(x)} \rho V_D(x, i)\]

where the indirect term due to incumbents can be written as:

\[\sum_{i=1}^{n(x)} \rho V_I(x, i) =\]

Destructions

\[\sum_{i=1}^{n(x)} \sum_{j \neq i} n(x) n(x) \delta [V(d(x, j), i) - V(x, i)]\]

Retentions

\[+ \sum_{i=1}^{n(x)} \sum_{j \neq i} \lambda E \int_{x' \notin Q^E(x, j)} [V(r(x, j, x'), i) - V(x, i)] dH_v(x')\]

EE Quits

\[+ \sum_{i=1}^{n(x)} \sum_{j \neq i} \lambda E \int_{x' \in Q^E(x, j)} [V(q_E(x, j, x'), i) - V(x, i)] dH_v(x')\]

and the indirect term due to the firm can be written as:

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\[
\sum_{i=1}^{n(x)} \rho V_F(x, i) = \]

**UE Hires**
\[q v(x) \phi \sum_{i=1}^{n(x)} [V(h_U(x), i) - V(x, i)] \cdot \mathbb{1}_{\{x \in A\}}\]

**UE Threats**
\[+ q v(x) \phi \sum_{i=1}^{n(x)} [V(t_U(x), i) - V(x, i)] \cdot \mathbb{1}_{\{x \notin A\}}\]

**EE Hires**
\[+ q v(x) (1 - \phi) \sum_{i=1}^{n(x)} \int_{x \in Q_E(x', i')} [V(h_E(x', i', x), i) - V(x, i)] dH_n(x', i')\]

**EE Threats**
\[+ q v(x) (1 - \phi) \sum_{i=1}^{n(x)} \int_{x \notin Q_E(x', i')} [V(t_E(x', i', x), i) - V(x, i)] dH_n(x', i')\]

**Shocks**
\[\sum_{i=1}^{n(x)} \Gamma_z[V, V](x, i)\]

We now collect terms.

**Destructions.** When worker \(i\) separates from firm \(x\), the sum of the changes in values of all employed workers at its own firm is given by:

\[
\text{Destructions} = \delta [U - V(x, i)] + \delta \sum_{j \neq i}^{n(x)} [V(d(x, i), j) - V(x, j)]
\]

\[= \delta \left[ U + \sum_{j \neq i}^{n(x)} V(d(x, i), j) - \sum_{j=1}^{n(x)} V(x, j) \right]\]
Retentions. When \( i \) renegotiates at firm \( x \), the sum of the changes in values of all employed workers at its own firm is given by:

\[
\text{Retentions} = \lambda E \int_{x' \in Q(x,i)} \left[ V(r(x,i,x'),i) - V(x,i) \right] dH_v(x')
\]

\[
+ \lambda E \int_{x' \in Q(x,i)} \sum_{j \neq i}^{n(x)} [V(r(x,i,x'),j) - V(x,j)] dH_v(x')
\]

\[
= \lambda E \int_{x' \in Q(x,i)} \left[ V(r(x,i,x'),i) + \sum_{j \neq i}^{n(x)} V(r(x,i,x'),j) - \sum_{j=1}^{n(x)} V(x,j) \right] dH_v(x')
\]

Quits. Similarly, when \( i \) quits firm \( x \), the sum of the changes in values of all employed workers at its own firm is given by:

\[
EE \text{ Quits} = \lambda E \int_{x' \in Q(x,i)} \left[ V(h_E(x,i,x'),i) + \sum_{j \neq i}^{n(x)} V(q_E(x,i,x'),j) - \sum_{j=1}^{n(x)} V(x,j) \right] dH_v(x')
\]
Combining terms. Before summing up all these terms, define for convenience the total worker value:

\[ \rho V(x) = \sum_{i=1}^{n(x)} w(x, i) \]

Destructions
\[ + \sum_{i=1}^{n(x)} \delta \left[ U + \sum_{j \neq i}^n V(d(x, i), j) - \sum_{j=1}^{n(x)} V(x, j) \right] \]

Retentions
\[ + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, i)} \left[ V(r(x, i, x'), j) - \sum_{j=1}^{n(x)} V(x, j) \right] dH_v(x') \]

EE Quits
\[ + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, i)} \left[ V(h_E(x, i, x'), i) + \sum_{j \neq i}^{n(x)} V(q_E(x, i, x'), j) - \sum_{j=1}^{n(x)} V(x, j) \right] dH_v(x') \]

UE Hires
\[ + qv(x) \phi \sum_{i=1}^{n(x)} [V(h_U(x), i) - V(x, i)] \cdot I_{\{x \in A\}} \]

UE Threats
\[ + qv(x) \phi \sum_{i=1}^{n(x)} [V(t_U(x), i) - V(x, i)] \cdot I_{\{x \notin A\}} \]

EE Hires
\[ + qv(x) (1 - \phi) \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x', i')} [V(h_E(x', i', x), i) - V(x, i)] dH_n(x', i') \]

EE Threats
\[ + qv(x) (1 - \phi) \sum_{i=1}^{n(x)} \int_{x' \notin Q^E(x', i')} [V(t_E(x', i', x), i) - V(x, i)] dH_n(x', i') \]

Shocks
\[ + \sum_{i=1}^{n(x)} \Gamma_z[V, V](x, i) \]
Now sum, up all the previous terms, collect terms and use the definition of $\nabla (x)$:

$$\rho \nabla (x) = \sum_{i=1}^{n(x)} w (x, i)$$

Destructions
$$+ \sum_{i=1}^{n(x)} \delta \left[ U + \sum_{j \neq i} n(x) V (d(x, i), j) - \nabla (x) \right]$$

Retentions
$$+ \lambda E \sum_{i=1}^{n(x)} \int_{x' \in Q^E (x, i)} \left[ \sum_{j=1}^{n(x)} V (r (x, i, x'), j) - \nabla (x) \right] dH_v (x')$$

EE Quits
$$+ \lambda E \sum_{i=1}^{n(x)} \int_{x' \in Q^E (x, i)} \left[ V (h_E (x, i, x'), i) + \sum_{j \neq i} V (q_E (x, i, x'), j) - \nabla (x) \right] dH_v (x')$$

UE Hires
$$+ q v (x) \phi \left[ \sum_{i=1}^{n(x)} V (h_U (x), i) - \nabla (x) \right] \cdot I_{\{x \in A\}}$$

UE Threats
$$+ q v (x) \phi \left[ \sum_{i=1}^{n(x)} V (t_U (x), i) - \nabla (x) \right] \cdot I_{\{x \notin A\}}$$

EE Hires
$$+ q v (x) (1 - \phi) \int_{x' \in Q^E (x', i') \backslash} \left[ \sum_{i=1}^{n(x)} V (h_E (x', i', x), i) - \nabla (x) \right] dH_n (x', i')$$

EE Threats
$$+ q v (x) (1 - \phi) \int_{x' \notin Q^E (x', i')} \left[ \sum_{i=1}^{n(x)} V (t_E (x', i', x), i) - \nabla (x) \right] dH_n (x', i')$$

Shocks
$$+ \Gamma_z [\nabla, \nabla] (x)$$

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Adding this last equation to the Bellman equation for $J(x)$ yields

\[
\begin{align*}
\rho \Omega (x) &= y(x) - c(v(x), x) \\
\text{Destructions} &= \sum_{i=1}^{n(x)} \delta \left[ J(d(x, i)) + U + \sum_{j \neq i} V(d(x, i), j) - J(x) - V(x) \right] \\
\text{Retentions} &= \lambda E \sum_{i=1}^{n(x)} \int_{x' \notin Q^E(x, i)} J(r(x, i, x')) + \sum_{j \neq i} V(r(x, i, x'), j) - J(x) - V(x) \, dH_v(x') \\
\text{EE Quits} &= \lambda E \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, i)} J(q_E(x, i, x')) + \sum_{j \neq i} V(q_E(x, i, x'), j) - J(x) - V(x) \, dH_v(x') \\
\text{UE Hires} &= q v(x) \phi \left[ J(h_U(x)) + \sum_{i=1}^{n(x)} V(h_U(x), i) - J(x) - V(x) \right] \cdot \mathbb{I}_{\{x \in A\}} \\
\text{UE Threats} &= q v(x) \phi \left[ J(t_U(x)) + \sum_{i=1}^{n(x)} V(t_U(x), i) - J(x) - V(x) \right] \cdot \mathbb{I}_{\{x \notin A\}} \\
\text{EE Hires} &= q v(x) (1 - \phi) \int_{x' \in Q^E(x', i')} J(h_E(x', i', x)) + \sum_{i=1}^{n(x)} V(h_E(x', i', x), i) - J(x) - V(x) \, dH_n(x', i') \\
\text{EE Threats} &= q v(x) (1 - \phi) \int_{x' \notin Q^E(x', i')} J(t_E(x', i', x)) + \sum_{i=1}^{n(x)} V(t_E(x', i', x), i) - J(x) - V(x) \, dH_n(x', i') \\
\text{Shocks} &= \Gamma_z [J + V, J + V](x) - J(x) - V(x)
\end{align*}
\]
Collecting terms and using the definition of $\Omega$:

$$
\rho \Omega(x) = y(x) - c(v(x), x)
$$

Destructions

$$
+ \sum_{i=1}^{n(x)} \delta [\Omega(d(x, i)) + U - \Omega(x)]
$$

Retentions

$$
+ \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin Q^E(x, i)} [\Omega(r(x, i, x')) - \Omega(x)] dH_v(x')
$$

EE Quits

$$
+ \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in Q^E(x, i)} [\Omega(q_E(x, i, x')) + V(h_E(x, i, x'), i) - \Omega(x)] dH_v(x')
$$

UE Hires

$$
+ q_v(x) \phi [\Omega(h_U(x)) - U - \Omega(x) \cdot 1_{x \in A}]
$$

UE Threats

$$
+ q_v(x) \phi [\Omega(t_U(x)) - \Omega(x) \cdot 1_{x \notin A}]
$$

EE Hires

$$
+ q_v(x) (1 - \phi) \int_{x' \in Q^E(x', i')} [\Omega(h_E(x', i', x)) - V(h_E(x', i', x'), i') - \Omega(x)] dH_n(x', i')
$$

EE Threats

$$
+ q_v(x) (1 - \phi) \int_{x' \notin Q^E(x', i')} [\Omega(t_E(x, i', x)) - \Omega(x)] dH_n(x', i')
$$

Shocks

$$
+ \Gamma_z[\overline{\Omega}, \overline{\Omega}](x)
$$

C.6.2 Value sharing

To make progress on (C.18), we begin by stating seven intermediate results, conditions (C-RT)-(C-E) which we prove from the assumptions listed in Section 3.2.2. These results establish how worker values $V$ in (C.18) evolve in the six cases of hiring, retention, layoff, quits, exit and vacancy creation. Next, we apply conditions (C-RT)-(C-E) to (C.18).

To highlight the structure of the argument, we note a key implication our zero-sum game assumption (A-IN): during internal negotiation, any value lost to one party must accrue to the other. This feature is obvious in the static model, and extends readily to our dynamic environment. In other words, the joint value of the firm plus its incumbent workers is invariant during the negotiation. We use this property extensively in the proof.
This generalizes pairwise efficient bargaining—commonly used in one-worker firm models with linear production—to an environment with multi-worker firms and decreasing returns in production.

We now state the seven conditions that we apply to (C.18). In section C.6.3 below, we prove how each of them is implied by the assumptions of Section 3.2.2.

**C-RT Retentions and Threats.** First, if firm $x$ meets an unemployed worker and the worker is not hired but only used as a threat, then the joint value of coalition $x$ does not change since threats only redistribute value within the coalition. Second, when firm $x$ uses employed worker $i'$ from firm $x'$ as a threat, the joint value of coalition $x$ does not change. Third, when firm $x$ meets worker $i'$ at $x'$ and the worker is retained by firm $x'$, the joint value of coalition $x'$ does not change. Formally,

$$
\Omega \left( r \left( x', i', x \right) \right) = \Omega(x') \quad , \quad \Omega \left( t_U \left( x \right) \right) = \Omega(x) \quad , \quad \Omega \left( t_E \left( x', i', x \right) \right) = \Omega(x).
$$

Respectively, these imply that the Retention, UE Threat and EE Threat components of (C.18) are equal to zero.

**C-UE UE Hires.** An unemployed worker that meets firm $x$ is hired when $x \in A$. This set consists of firms that have a joint value after hiring that is higher than the pre-hire joint value plus the outside value of the hired worker. Due to the take-leave offer, the new hire receives her outside value, which is the value of unemployment:

$$
A = \{ x | \Omega(h_U(x)) - \Omega(x) \geq U \} \quad , \quad V \left( h_U \left( x \right), i \right) = U.
$$

**C-EE EE Hires.** An employed worker $i'$ at firm $x'$ that meets firm $x$ is hired when $x \in Q^E \left( x', i' \right)$. This set consists of firms that have a higher marginal joint value than that
of the current firm:

\[ Q^E (x', i') = \left\{ x \bigg| \Omega (h_E (x', i', x)) - \Omega (x') \geq \Omega (x) - \Omega (q_E (x', i', x)) \right\}. \]

Due to the take-leave offer, the new hire receives her outside value, which is the marginal joint value at her current firm:

\[ V (h_E (x', i', x)) = \Omega (x') - \Omega (q_E (x', i', x)). \]

(C-EU) **EU Quits and Layoffs.** An employed worker \( i \) at firm \( x \) quits to unemployment when \((x, i) \in Q^U\). This set consist of states \( x \) such that the marginal joint value is less than the value of unemployment:

\[ Q^U = \left\{ (x, i) \bigg| \Omega (\tilde{s}_{q_1} (x, i)) + U > \Omega (\tilde{s}_{q_0} (x, i)) \right\}, \]

where

\[ \tilde{s}_{q_1} (x, i) = s (x, (1 - [q_{U,i} (x) \mid q_{U,i} (x) = 1]) \circ (1 - q (x))), \]

\[ \tilde{s}_{q_0} (x, i) = s (x, (1 - [q_{U,i} (x) \mid q_{U,i} (x) = 0]) \circ (1 - q (x))). \]

The first expression captures when worker \( i \) quits, and the second where worker \( i \) does not. Similarly, an EU layoff will be chosen by the firm when \((x, i) \in L\):

\[ L = \left\{ (x, i) \bigg| \Omega (\tilde{s}_{\ell_1} (x, i)) + U > \Omega (\tilde{s}_{\ell_0} (x, i)) \right\}, \]

where

\[ \tilde{s}_{\ell_1} (x, i) = s (x, (1 - [\ell (x) \mid \ell (x) = 1]) \circ (1 - q_U (x))), \]

\[ \tilde{s}_{\ell_0} (x, i) = s (x, (1 - [\ell (x) \mid \ell (x) = 0]) \circ (1 - q_U (x))). \]

The first expression captures when worker \( i \) is laid off, and the second when worker \( i \) is not.
(C-X) Exit. A firm $x$ exits when $x \in \mathcal{E}$. This set consists of the states in which the total outside value of the firm and its workers is larger than the joint value of operation:

$$
\mathcal{E} = \left\{ x \left| \vartheta + n (s (x, \kappa (x))) \cdot U > \Omega (s (x, \kappa (x))) \right. \right\}.
$$

(C-V) Vacancies. The expected return to a matched vacancy $R(x)$ depends only on the joint value, and so the firm’s optimal vacancy policy $v(x)$ depends only on the joint value. The policy $v(x)$ solves

$$
\max_v q(\theta) v R(x) - c(v, x),
$$

where the expected return to a matched vacancy reflects the return from matching with an unemployed worker (first line) as well as the return from matching with an employer worker (second line):

$$
R(x) = \phi \left[ \Omega (h_U (x)) - \Omega (x) - U \right] \cdot 1_{\{ x \in A \}}
+ (1 - \phi) \int_{x \in \Omega (x')} \left\{ \left[ \Omega (h_E (x', i', x)) - \Omega (x) \right]
- \left[ \Omega (x') - \Omega (q_E (x', i', x)) \right] \right\} dH_n (x', i') .
$$

(C-E) Entry. A firm enters if and only if

$$
\int \Omega (x_0) d\Pi_0 (z) \geq c_0 + n_0 U.
$$

Summarizing (C). The substantive result is that all firm and worker decisions and employed workers’ values can be expressed in terms of joint value $\Omega$ and exogenous worker outside option $U$.  

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C.6.3 Proof of Conditions (C)

Proof of C-UE and C-RT (UE Hires and UE Threats)

In this subsection, we consider a meeting between a firm $x$ and an unemployed worker. Following A-IN and A-EN, the firm internally renegotiates according to a zero-sum game with its incumbent workers and makes a take-leave offer to the new worker. Intuitively, having the worker “at the door” is identical to having her hired at value $U$ for the firm and for all incumbent workers: the firm can always make new take-leave offers to its incumbents after hiring the new worker. Hence, we expect the firm to make one take-leave offer to the new worker and its incumbents at the time of the meeting, and not make a new, different offer to its incumbents after hiring has taken place.

We start by showing this equivalence formally. To do so, when meeting an unemployed worker, we let the firm conduct internal renegotiation with its incumbent workers and make an offer to the new worker. Then, we let a second round of internal offers take place after the hiring. We introduce some notation to keep track of values throughout the internal and external negotiations. To fix ideas, we denote by (IR1) the first round of internal negotiation, pre-external negotiation. We denote by (IR2) the second round of internal negotiation, post-hire.

Post-hire and post-internal negotiation (IR2) values are denoted with double stars. Post-internal-negotiation (IR1) but pre-external-negotiation values are denoted with stars.

$$
\Omega^{**} := J^{**} + \sum_{j=1}^{n(x)} V^{**}_j + V^{**}_i
$$

$$
\Omega^* := J^* + \sum_{j=1}^{n(x)} V^*_j
$$

$$
\Omega := J + \sum_{j=1}^{n(x)} V_j
$$
Proceeding by backward induction, under A-EN the firm makes a take-it-or-leave-it offer to the unemployed worker, therefore
\[ V_i^{**} = U \]

We now divide the proof in several steps. We start by proving that for all incumbent workers \( j = 1 \ldots n(x) \), \( V_j^{**} = V_j^* \). We then use A-IN to argue that \( \Omega^* = \Omega \). Once these claims have been proven, we move on to proving C-UE (UE Hires) and the part of for threats from unemployment C-RT (UE Threats). Finally, we show that our microfoundations for the renegotiation game deliver A-IN.

Claim 1: For all incumbents workers \( j = 1 \ldots n(x) \), we have \( V_j^{**} = V_j^* \).

We proceed by backwards induction using our assumptions A-EN and A-IN. Immediately after (IR1) has taken place, only the following events can happen:

1. Hire/not-hire
   - Either the worker is hired from unemployment (H),
   - Or the worker is not hired from unemployment (NH)

2. Possible new round of internal negotiation (IR2). This possible second round of internal negotiation (now including the newly hired worker) leads to values \( V_j^{**} \).

We focus on subgame perfect equilibria in this multi-stage game. Therefore, after (IR1), workers perfectly anticipate what the outcome of the hire/not-hire stage will be. That is, after (IR1), they know perfectly what hiring decision (H or NH) the firm will make. Now suppose that internal renegotiation (IR2) actually happens after the hire/not-hire decision, that is, that for some incumbent worker \( j \in \{1,\ldots,n(x)\} \), \( V_j^{**} \neq V_j^* \). Note that the firm has no incentives to accept a change in the new worker’s value to anything above \( U \), so by A-MC her value does not change in the second round (IR2).
We construct the rest of the proof by contradiction. Consider for a contradiction an incumbent worker $j$ whose value changed in (IR2). Because of A-MC, her value can change only in the following cases:

- The firm has a credible threat to fire worker $j$, in which case $V_j^{**} < V_j^*$
- Worker $j$ has a credible threat to quit, in which case $V_j^{**} > V_j^*$

In addition, those credible threats can lead to a different outcome than in (IR1), and thus $V_j^{**} \neq V_j^*$, only if the threat on either side was not available in (IR1). If that same threat was available in the first round (IR1), then the outcome of the bargaining (IR1) would have been $V_j^{**}$.

Recall that both incumbent worker $j$ and the firm understand and anticipate which hire/not-hire decision the firm will make after the first round (IR1). They also understand and anticipate that, in case of hire, the value of the new worker will remain $U$ in the second round (IR2).

Therefore, the firm can credibly threaten to hire the new worker in the first round if and only if it actually hires her after the first round (IR1) is over. This implies that the firm can credibly threaten worker to fire $j$ in the second round (IR2), by A-LC, if and only if it could credibly threaten her with hiring the new worker in the first round of internal renegotiation (IR1). This in turn entails that any credible threat the firm can make in the second round (IR2) was already available in the first round.

On the worker side, quitting into unemployment is a credible threat when her value is below the value of unemployment. So this threat does not change between the first round (IR1) and the second round (IR2), because the equilibrium value to that worker will always be above the value of unemployment.

In sum, the set of credible threats both to the firm and to worker $j$ does not change between the initial round of internal renegotiation (IR1) and the post-hiring-decision round (IR2). This finally implies that the outcome of the initial round of internal renegotiation
(IR1) for any incumbent $j$ remains unchanged in the second round (IR2), that is:

$$V_{j}^{**} = V_{j}^{*}$$

which proves Claim 1.

We can now move on to proving C-UE.

**Proof of C-UE.** Using the definitions of $\Omega^{**}$ and $\Omega$, we can write

$$\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(x)} V_{j}^{**} + V_{i}^{**} \right] - \left[ J + \sum_{j=1}^{n(x)} V_{j} \right]$$

Now using $V_{i}^{**} = U$, we obtain

$$\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(x)} V_{j}^{**} \right] - \left[ J + \sum_{j=1}^{n(x)} V_{j} \right] + U$$

Using Claim 1: $V_{j}^{**} = V_{j}^{*}$, and adding and subtracting $J^{*}$ we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + \left[ J^{*} + \sum_{j=1}^{n(x)} V_{j}^{*} \right] - \left[ J + \sum_{j=1}^{n(x)} V_{j} \right] + U$$

Substituting in the definition of $\Omega$ and of $\Omega^{*}$,

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + [\Omega^{*} - \Omega] + U$$

Finally recall that internal renegotiation is (1) individually rational, and (2) is a zero-sum game, according to A-IN. Thus, all incumbent workers remain in the coalition after internal renegotiation, and the joint value is unchanged: $\Omega^{*} = \Omega$. Using $\Omega^{*} = \Omega$

$$\Omega^{**} - \Omega = [J^{**} - J^{*}] + U$$
which can be re-written

\[ J^{**} - J^* = [\Omega^{**} - \Omega] - U \]

Now under A-LC, the firm will only hire if its value after hiring is higher than its value after internal renegotiation: \( J^{**} - J^* \geq 0 \). This inequality requires

\[ \Omega^{**} - \Omega \geq U \]
\[ \Omega (h_U (x)) - \Omega (x) \geq U \]

The firm does not hire when its value of hiring is below its value of renegotiation \( J^{**} < J^* \). This inequality implies

\[ \Omega^{**} - \Omega < U \]

When the firm does not hire, we obtain using again A-IN and \( \Omega^* = \Omega \):

\[ \Omega^{**} - \Omega^* < U \]

which finally implies

\[ \Omega (h_U (x)) - \Omega (t_U (x)) < U \]

Now, we argue that conditional on not hiring, \( \Omega^{**} = \Omega^* = \Omega \), where in this case \( \Omega^{**} \) denotes the value of the coalition without hiring, and thus does not include the value of the unemployed worker. Just as before, this is a direct consequence from A-IN and that the internal renegotiation game is zero-sum.

Therefore:

\[ \Omega (t_U (x)) = \Omega (x) \]
We have therefore shown C-UE and part of C-RT (UE Hires and UE Threats): An unemployed worker that meets $x$ is hired when $x \in Q^U$, where

$$A = \{x \mid \Omega(h_U(x)) - \Omega(x) \geq U\}$$

and upon joining the firm, has value

$$V(h_U(x,i)) = U.$$ 

and

$$\Omega(t_U(x)) = \Omega(x).$$

**Proof of C-EE and C-RT (EE Hires, EE Threats and Retentions)**

Consider firm $x$ that has met worker $i'$ at firm $x'$. We first seek to determine $Q^E(x',i')$. Under A-IN and A-EN, upon meeting an employed worker, internal negotiation may take place at the poaching firm $x$, and $x$ makes a take-it-or-leave-it offer. Internal negotiation may take place at $x'$ with all workers including $i'$.

Proceeding by backward induction, we again define intermediate values but here at $x'$, noting that $q_E(x',i',x)$ gives the number of employees in $x'$ if the worker leaves:

$$\Omega = J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}$$

$$\Omega^* = J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}$$

Note, in the second line we are describing the values of the firm in renegotiation where $i'$ stays with the firm, so $V_{i'}^*$ is the outcome of internal negotiation. In the third line we consider
the firm having lost the worker. Under A-EN the firm will respond to an offer \( \bar{V} \) from \( x \) with

\[
V_{i'}^* = \bar{V}
\]

The same result as in Claim 1 from section C.6.3 obtains: under A-EN and A-IN, the values accepted by the incumbent workers after the internal renegotiation \( (V_j^*)_j \) will be equal to the values they receive after the external negotiation \( (V_j^{**})_j \), that is

\[
V_j^{**} = V_j^*
\]

The argument are exactly the same.

Using these two results and the above definitions

\[
\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**} \right] - \left[ J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'} \right]
\]

\[
= \left[ J^{**} + J^* - J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**} + V_j^* - V_{i'}^* \right] - \left[ J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'} \right]
\]

\[
= [J^{**} - J^*] + \left[ J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^* \right] - \left[ J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'} \right] - V_{i'}^*
\]

\[
= [J^{**} - J^*] + [\Omega^* - \Omega] - V_{i'}^*
\]

\[
= [J^{**} - J^*] + [\Omega^* - \Omega] - \bar{V}
\]

In this setup, A-IN again implies that any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as \textit{“the joint value stays constant before and after an internal negotiation”}. Mathematically, this statement translates into

\[
\Omega^* = \Omega
\]
Substituting into the equation that we obtained above $\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] - \nabla$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \nabla$$

Now under A-LC, the firm $x'$ will only try to keep the worker if $J^* > J^{**}$, which requires

$$\Omega - \Omega^{**} \leq \nabla$$

$$\Omega (r(x', i', x) - \Omega (q_E (x', i', x)) \leq \nabla$$

This determined the maximum value that $x'$ can offer to the worker to retain them. Knowing that firm $x'$ can counter at most with $\nabla = \Omega (r(x', i', x) - \Omega (q_E (x', i', x))$, then will firm $x$ successfully poach the worker?

First, note that the bargaining protocol implies that $x$ firm will offer $\nabla$ if it is making an offer, since it need not offer more. For firm $x$ the argument may proceed identically to the case of unemployment, simply replacing $U$ with $\nabla$. The result is that the firm will hire only if

$$\Omega (h_E (x', i', x)) - \Omega (x) \geq \nabla$$

or

$$\Omega (h_E (x', i', x)) - \Omega (x) \geq \Omega (r(x', i', x)) - \Omega (q_E (x', i', x))$$

Finally, when firm $x$ does not hire, the same argument as in Claim 32 in Section C.6.3 applies: $\Omega^{**} = \Omega^* = \Omega$. This observation implies

$$\Omega (t_E (x', i', x)) = \Omega (x)$$
Similarly, the same argument as in **Claim 2** implies that when firm $x'$ does not lose its worker, $\Omega^{**} = \Omega^* = \Omega$, thereby implying

$$\Omega(r(x', i', x)) = \Omega(x')$$

The combination of these conditions deliver **C-UE** and part of **C-RT** (*EE Hires, EE Threats and Retention*):

1. The quit set of an employed worker is determined by

$$Q^E(x', i') = \left\{ x \mid \Omega(h_E(x', i', x)) - \Omega(x) \geq \Omega(x') - \Omega(q_E(x', i', x)) \right\}$$

2. The worker’s value of being hired from employment from firm $x'$ is

$$V(h_E(x, x', i')) = \Omega(x') - \Omega(q_E(x', i', x))$$

3. Worker $i'$'s value of being retained at $x'$ after meeting $x$ is$^{17}$

$$V(r(x', i', x), i') = \Omega(h_E(x', i', x)) - \Omega(x)$$

4. The joint value of the potential poaching firm $x$ when the worker is not hired does not change:

$$\Omega(t_E(x', i', x)) = \Omega(x)$$

5. The joint value of the potential poached firm $x'$ does not change when the worker stays:

$$\Omega(r(x', i', x)) = \Omega(x')$$

$^{17}$Because offers are made at no cost, both firms always make an offer, even when they know that they cannot retain/hire the worker in equilibrium. This is exactly the same as in Postel-Vinay Robin (2002).
Proof of C-EU (EU Quits and layoffs)

We first show that

\[ \mathcal{L} = \left\{ (x, i) \middle| \Omega (s (x, (1 - [\ell (x); \ell_i (x) = 1]) \circ (1 - q_U (x))), i) + U > \Omega (s (x, (1 - [\ell (x); \ell_i (x) = 0]) \circ (1 - q_U (x))), i) \right\} \]

from the firm side, then that

\[ \mathcal{Q}^U = \left\{ (x, i) \middle| \Omega (s (x, (1 - \ell (x)) \circ (1 - [q_{U,i} (x); q_U, (x) = 1])), i) + U > \Omega (s (x, (1 - \ell (x)) \circ (1 - [q_{U,-i} (x); q_U, (x) = 0])), i) \right\} \]

on the worker side.

**Part 1: Firm side** Consider a firm \( x \) who is considering laying off worker \( i \) for whom \( q_{U,i} (x) = 0 \). Starting with definitions, note that \( n (s (x, (1 - [\ell (x); \ell_i (x) = 1]) \circ (1 - q_U (x)))) \) is the number of workers if \( i \) is laid off.

\[
\begin{align*}
\Omega &= J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \\
\Omega^* &= J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* \\
\Omega^{**} &= J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}
\end{align*}
\]

Note that in the first line the coalition has still worker \( i \) in it. In the second line, the firm and the worker \( i \) have negotiated (and internal negotiation has determined \( V_i^* \) which is what \( i \) will get if they stay in the firm). In the third line, the worker has been fired and another round of negotiation has occurred among incumbents.
The same result as in Claim 1 from section C.6.3 obtains: under A-BP, the values accepted by the incumbent workers after the internal renegotiation \( \left( V^*_j \right) \) will be equal to the values they receive after the external negotiation \( \left( V'^*_j \right) \), that is \( V'^*_j = V^*_j \).

Using this result and the above definitions

\[
\Omega'^* - \Omega = \left[ J'^* + \sum_{j=1}^{n(s)} V'^*_j \right] - \left[ J + \sum_{j=1}^{n(s)} V_j + V_i \right] = \left[ J'^* - J^* \right] + \left[ J^* + \sum_{j=1}^{n(s)} V^*_j + V^*_i \right] - \left[ J + \sum_{j=1}^{n(s)} V_j + V_i \right] = \left[ J'^* - J^* \right] + [\Omega^* - \Omega] - V^*_i
\]

Using again A-IN to conclude that \( \Omega^* = \Omega \), we obtain

\[
\Omega'^* - \Omega = [J'^* - J^*] - V^*_i
\]

Now under A-LC, the firm \( x \) will only layoff the worker if \( J'^* > J^* \), which requires

\[
\Omega - \Omega'^* < V^*_i
\]

As long as \( V^*_i > U \) the worker would be willing to transfer value to the firm to avoid being laid off, implying

\[
\Omega - \Omega'^* < U.
\]

which we can re-write

\[
\Omega(s(x,(1 - \ell(x); \ell_i(x) = 1)) \circ (1 - q_U(x)), i) + U > \Omega(s(x,(1 - \ell(x); \ell_i(x) = 0)) \circ (1 - q_U(x)), i)
\]

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where the left-hand-side is $\Omega^* + U$ (under the layoff) and the right-hand-side is $\Omega$. This concludes the proof for the firm side.

**Part 2: Worker side** Consider worker $i$ in firm $x$ who is considering quitting to unemployment for whom $\ell_i(x) = 0$. Starting with definitions, note that $n(s(x, 1 - \ell(x)) \circ (1 - [q_{U,i}(x) ; q_{U,i}(x)] = 1)$ is the number of workers if $i$ quits. As before,

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

The same result as in **Claim 1** from section C.6.3 obtains $V_j^{**} = V_j^*$. Using this result and the above definitions

$$\Omega^{**} - \Omega = \left[ J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**} \right] - \left[ J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \right]$$

$$= [J^{**} - J^*] + \left[ J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* \right] - \left[ J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \right] - V_i^*$$

$$= [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

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Now under A-LC, worker $i$ will quit into unemployment iff $V_i^* < U$, which requires

$$J^{**} - J^* + [\Omega - \Omega^{**}] < U$$

As long as $J^{**} < J^*$, the firm is willing to transfer value to worker $i$ to retain her. Therefore, worker $i$ quits into unemployment iff the previous inequality holds at $J^{**} = J^*$, i.e.

$$\Omega - \Omega^{**} < U$$

Therefore, the worker quits iff

$$\Omega (s (x, (1 - \ell (x)) \circ (1 - [q_{U,-i} (x) ; q_{U,i} (x) = 1])), i) + U$$

$$> \Omega (s (x, (1 - \ell (x)) \circ (1 - [q_{U,-i} (x) ; q_{U,i} (x) = 0])), i)$$

which concludes the proof of the worker side. This delivers C-EU.

**Proof of C-X (Exit)**

Consider a firm $x$ who contemplates exit after all endogenous quits and layoffs, thus when its employment is $n (s (x, \kappa (x)))$. As before we define values conditional on exiting:

$$\Omega = J + \sum_{j=1}^{n(s(i))} V_j$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(i))} V_j^*$$

$$\Omega^{**} = J^{**} + 0$$
Notice that the joint value after exit is simply the value of the firm, since all other workers have left because of exit. We can compute:

\[
\Omega^{**} - \Omega = J^{**} - \left[ J + \sum_{j=1}^{n(s(i))} V_j \right]
\]

(Definition of \(\Omega, \Omega^*\))

\[
\Omega^{**} - \Omega = [J^{**} - J^*] + [J^* - J] - \left[ J + \sum_{j=1}^{n(s(i))} V_j \right] - \sum_{j=1}^{n(s(i))} V_j^*
\]

Again, \(\Omega^* = \Omega\) from A-IN, so that

\[
\Omega^{**} - \Omega = [J^{**} - J^*] - \left[ J + \sum_{j=1}^{n(s(i))} V_j \right] - \sum_{j=1}^{n(s(i))} V_j^*
\]

The firm exits iff \(J^{**} \geq J^*\), that is, \(\vartheta \geq J^*\). This is equivalent to

\[
\Omega^{**} - \Omega \geq - \sum_{j=1}^{n(s(i))} V_j^*
\]

Using again that \(\Omega^{**} = J^{**} = \vartheta\), the firm exits iff

\[
\vartheta + \sum_{j=1}^{n(s(i))} V_j^* \geq \Omega
\]

Since any worker is better off under \(V_j^* \geq U\) than unemployed, all workers are willing to take a value cut down to \(U\) if \(\vartheta \geq \Omega - \sum_{j=1}^{n(s(i))} V_j^*\) because then the firm can credibly exit. This implies that the firm exits if and only if

\[
\vartheta - \Omega (s(x, \kappa(x))) + n(s(x, \kappa(x))) U \geq 0
\]
This proves C-X (Exit): the set of $x$ such that the firm exits is given by

$$\mathcal{E} = \{ x \mid \vartheta + n(s(x, \kappa(x))) \cdot U \geq \Omega(s(x, \kappa(x))) \}$$

**Proof of C-V (Vacancies)**

We split the proof in two steps. First, we show that workers are collectively willing to transfer value to the firm in exchange for the joint value-maximizing vacancy policy function. Second, we show that a single worker can create a system of transfers that achieves the same outcome. These transfers are equivalent to wage renegotiation, which explains why we have subsumed them in the wage function $w(x, i)$ in the equations above. Similarly to wages, these transfers drop out from the expression for the joint value.

**Part 1: Collective transfers** In this step, we show that workers are collectively better off transferring value to the firm in exchange of the firm posting the joint value-maximizing amount of vacancies.

The vacancy posting decision $v^J$ that maximizes firm value is:

$$\frac{c_v(v^J(x), n(x))}{q} = \phi [J(h_U(x)) - J(x)] \cdot \mathbb{1}_{\{x \in A\}}$$

$$+ (1 - \phi) \int_{x \in Q^E(x', i')} [J(h_E(x', i', x)) - J(x)] dH_n(x', i').$$

Similarly, define $v^\Omega$ be the policy that maximizes the value of the coalition, and $v^\mathcal{V}$ be the policy that maximizes the value of all the employees. Let $\Omega, J, \mathcal{V}$ be the value of the coalition, firm and all workers under the $v^\gamma$, for $\gamma \in \{\Omega, J, \mathcal{V}\}$. We now prove our claim in several steps.

**Part 1-(a) Collective value gains.** The policy $v^\Omega$ will lead to $\mathcal{V}^\Omega \geq \mathcal{V}^J + [J^J - J^\Omega]$ where $J^J - J^\Omega \geq 0.$
Proof: By construction $\Omega^\Omega$ is greater than $\Omega^J$: $\Omega^\Omega \geq \Omega^J$. By definition: $\Omega^\Omega = J^\Omega + \nabla^\Omega$, and $\Omega^J = J^J + \nabla^J$. Use those definitions to obtain inequality $J^\Omega + \nabla^\Omega \geq J^J + \nabla^J$, which can be re-arranged into $\nabla^\Omega - \nabla^J \geq J^J - J^\Omega$. Since $J^J$ is the value under the optimal policy for $J$, then $J^J \geq J^\Omega$. The above then implies that

$$\nabla^\Omega - \nabla^J \geq J^J - J^\Omega \geq 0$$

This implies that workers would be prepared to transfer $T = J^J - J^\Omega \geq 0$ to the firm in order for the firm to pursue policy $v^\Omega$ instead of $v^J$. This concludes the proof of Part 1-(a).

**Part 1-(b) Infeasibility of $\nabla^\Omega$**. There does not exist an incentive-compatible transfer from workers to firm that will lead to $\nabla^\Omega$.

Proof: Suppose workers consider transferring even more to induce the firm to follow policy $v^{\Omega}$ that maximizes their value. By construction $\Omega^\Omega \geq \Omega^V$. Using definitions for each of these, then $J^\Omega + \nabla^\Omega \geq J^V + \nabla^V$. Rearranging this: $J^\Omega - J^V \geq \nabla^V - \nabla^\Omega$. Since $\nabla^V$ is the value under the optimal policy for $V$, then $\nabla^V \geq \nabla^\Omega$. The above then implies that

$$J^\Omega - J^V \geq \nabla^V - \nabla^\Omega \geq 0$$

Taking $v^\Omega$ as a baseline, the above implies that a change to $v^V$ causes a loss of $J^\Omega - J^V$ to the firm, which is more than the gain of $\nabla^V - \nabla^\Omega$ to the workers. This implies that workers could transfer all of their gains under $v^V$ to the firm, but the firm would still not choose $v^V$ over $v^\Omega$. This concludes the proof of Part 1-(b).

**Part 1-(c) Optimality of $\nabla^\Omega$**. There does not exist an incentive-compatible transfer from workers to firm that will lead to $V^* \in \left(\nabla^\Omega, \nabla^V\right)$. 

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Proof: Call such a policy $v^{\ast}$. Then: $\Omega^{\ast} \geq \Omega^{\ast}$, and by definitions

$$J^{\Omega} + \overline{V}^{\Omega} \geq J^{\overline{V}^{\ast}} + \overline{V}^{\ast}$$

$$J^{\Omega} - J^{\overline{V}^{\ast}} \geq \overline{V}^{\overline{V}^{\ast}} - \overline{V}^{\Omega}$$

Since by definition $\overline{V}^{\ast} \in (\overline{V}^{\Omega}, \overline{V})$, then $\overline{V}^{\overline{V}^{\ast}} - \overline{V}^{\Omega} \geq 0$. Therefore

$$J^{\Omega} - J^{\overline{V}^{\ast}} \geq \overline{V}^{\overline{V}^{\ast}} - \overline{V}^{\Omega} \geq 0$$

Taking $v^{\Omega}$ as a baseline, the above implies that a change to $v^{\overline{V}^{\ast}}$ causes a loss of $J^{\Omega} - J^{\overline{V}^{\ast}}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}^{\ast}} - \overline{V}^{\Omega}$ to the workers. This concludes the proof of Part 1-(c).

Part 1-(d) Conclusion. In summary, it is optimal for workers to transfer exactly $T = J^{J} - J^{\Omega}$ to the firm, in order for the firm to pursue $v^{\Omega}$ instead of $v^{J}$. Further transfers to the firm would be required to have the firm pursue a better policy for workers, but this is exceedingly costly to the firm and the workers are unwilling to make a transfer to cover these costs. This concludes the proof of Step 1: Collective transfers.

Part 2: Individual transfers In this step, we show that a single, randomly drawn worker can construct a system of transfers that induces the firm to post $v^{\Omega}$ instead of $v^{J}$, while leaving all agents better off.

Within $dt$, consider the single, randomly drawn worker $j_{0}$. Consider the following system of transfers. Worker $j_{0}$ makes a transfer $J^{J} - J^{\Omega}$ to the firm, in exchange of what (i) the firm posts $v^{\Omega}$ instead of $v^{J}$, and (ii) the worker gets a wage increase that gives her all the differential surplus $\overline{V}^{\Omega} - \overline{V}^{J}$.

Following the same steps as in Part 1: Collective transfers, the firm gets $J^{\Omega} + [J^{J} - J^{\Omega}] = J^{J}$ and is hence indifferent. Similarly, workers $j \neq j_{0}$ do not get any value change,
and are thus indifferent Finally, worker $j_0$ gets a value increase of

$$[\tilde{V}^\Omega - \tilde{V}^J] - [J^J - J^\Omega] \geq 0$$

where the inequality similarly follows from **Part 1: Collective transfers**. This concludes the proof of **Part 2: Individual transfers**.

**Conclusion.** The previous arguments show that a single worker has an incentive to and can induce the firm to post $v^\Omega$. Notice also that the same argument holds starting from any vacancy policy function $\tilde{v} \neq v^J$ together with a value of the firm $\tilde{J}$. Thus, even if some worker induces the firm to post a different vacancy policy function which is not $v^\Omega$ any other worker has an incentive to induce the firm to post $v^\Omega$. Therefore, in equilibrium, the firm posts $v^\Omega$, which concludes the proof of **C-V**.

**C.6.4 Applying Conditions (C)**

Having established that **Assumption (A)** can be used to prove **Conditions (C)**, we now apply conditions (C) to the Bellman equation for the joint value. The goal of this section is to show that for $x \in \mathcal{E}^c$ the complement of the exit set, we can considerably simplify the recursion for the joint value:

$$\rho (x) = y (z (x), n (x)) - c (v (x), n (x), z (x))$$

Destructions $-\delta \sum_{i=1}^{n(x)} [\Omega (x) - \Omega (d (x, i)) - U]$  

UE Hires $+qv (x) \phi [\Omega (h_U (x)) - \Omega (x) - U] \cdot \mathbb{1}_{\{x \in A\}}$  

EE Hires $+qv (x) (1 - \phi) \int_{x \in \mathcal{Q}^E (x', i')} \left[ \Omega (h_E (x', i', x)) - \Omega (x) \right] 
- \left[ \Omega (x') - \Omega (q_E (x', i', x)) \right] dH_n (x', i')$  

Shocks $+\Gamma [\Omega, \Omega]$

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with the sets

\[ Q^U = \left\{ (x, i) \left| \Omega(s(x, (1 - \ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 1])), i) + U > \Omega(s(x, (1 - \ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 0])), i) \right\} \right. \]

\[ \mathcal{L} = \left\{ (x, i) \left| \Omega(s(x, (1 - \ell(x)) \circ (1 - q_U(x))), i) + U > \Omega(s(x, (1 - \ell(x)) \circ (1 - q_U(x))), i) \right\} \right. \]

\[ \mathcal{E} = \left\{ x \left| \vartheta + n(s(x, \kappa(x))) \cdot U \geq \Omega(s(x, \kappa(x))) \right\} \right. \]

\[ \mathcal{A} = \left\{ x \left| \Omega(h_U(x)) - \Omega(x) \geq U \right\} \right. \]

\[ Q^E(x', i') = \left\{ x \left| \Omega(h_E(x', i', x)) - \Omega(x) \geq \Omega(x') - \Omega(q_E(x', i', x)) \right\} \right. \]

and—as per (C-V)—the vacancy policy \( v(x) \) is given by the solution to the following:

\[
\frac{c_v(v(x), n(x))}{q} = \phi [\Omega(h_U(x)) - \Omega(x)] \cdot \mathbb{1}_{\{x \in \mathcal{A}\}} + (1 - \phi) \int_{x' \in Q^E(x', i')} \left[ \Omega(h_E(x', i', x)) - \Omega(x) \right] - \\
\left[ \Omega(x') - \Omega(q_E(x', i', x)) \right] dH_n(x', i')
\]

In continuous time, the exit decision is captured by \( x \in \mathcal{E} \). The Bellman equation above holds exactly for \( x \in \mathcal{E}^c \). Exit is accounted for in the “bold” continuation values, which all include the possible exit decision should the firm’s state fall into \( \mathcal{E} \) after an event.

We first proceed one term at the time, working through (B.4.1) exogenous destructions, (B.4.2) retentions, (B.4.3) EE (poached) quits, (B.4.4) UE hires, (B.4.5) UE threats, (B.4.6) EE (poached) hires, and (B.4.7) EE threats.
Exogenous destructions

\[
\text{Destructions} = \sum_{i=1}^{n(x)} \delta \left[ J\left(d(x, i)\right) + \sum_{j=1}^{n(d(x,i))} V\left(d(x, j), i\right) + U - \Omega\left(x\right) \right]
\]

\[
= \sum_{i=1}^{n(x)} \delta \left[ \Omega\left(d(x, i)\right) + U - \Omega\left(x\right) \right]
\]

where we simply have used the definition \(\Omega\left(d(x, i)\right) := J\left(d(x, i)\right) + \sum_{j=1}^{n(d(x,i))} V\left(d(x, j), i\right)\).

Retentions

\[
\text{Retentions} = \lambda E \sum_{i=1}^{n(x)} \int_{x' \not\in Q^E(x,i)} \left[ J\left(r\left(x, i, x'\right)\right) + \sum_{j=i}^{n(x)} V\left(r\left(x, i, x', j\right), i\right) - \Omega\left(x\right) \right] dH_v(x')
\]

\[
= \lambda E \sum_{i=1}^{n(x)} \int_{x' \not\in Q^E(x,i)} \left[ \Omega\left(r\left(x, i, x'\right)\right) - \Omega\left(x\right) \right] dH_v(x')
\]

where we simply have used the definition \(\Omega\left(r\left(x, i, x'\right)\right) = J\left(r\left(x, i, x'\right)\right) + \sum_{j=i}^{n(x)} V\left(r\left(x, i, x', j\right)\right)\).

Now using the result in **C-RT** that

\[
\Omega\left(r\left(x, i, x'\right)\right) = \Omega(x')
\]

we obtain that

\[
\text{Retentions} = 0
\]
**EE Quits**

\[
EE \text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ J(q_E(x,i,x')) + V(q_E(x,i,x'),i) + \sum_{j \neq i} V(q_E(x,i,x'),j) - \Omega(x) \right] dH_v(x')
\]

Now by definition

\[
\Omega(q_E(x,i,x')) = J(q_E(x,i,x')) + \sum_{j=1}^{n(q_E(x,i,x'))} V(q_E(x,i,x'),j)
\]

\[
= J(q_E(x,i,x')) + \sum_{j \neq i} V(q_E(x,i,x'),j)
\]

Using this last equality in the term in square brackets

\[
EE \text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[ \Omega(q_E(x,i,x')) - \Omega(x) + V(q_E(x,i,x'),i) \right] dH_v(x')
\]

Using C-EE, the value going to the poached worker is \( V(q_E(x,i,x')) = \Omega(x) - \Omega(q_E(x,i,x')) \). Substituting this into the last equation, we observe that the term in the square brackets is zero, and so

\[
EE \text{ Quits} = 0
\]

**UE Hires**

\[
UE \text{ Hires} = qv(x) \phi \left[ J(h_U(x)) + \sum_{i=1}^{n(x)} V(h_U(x),i) - \Omega(x) \right] \cdot \mathbb{1}_{\{x \in \mathcal{A}\}}
\]

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Now by definition

\[
\Omega(h_U(x)) = J(h_U(x)) + \sum_{i=1}^{n(h_U(x))} V(h_U(x), i)
\]

\[
= J(h_U(x)) + \sum_{i=1}^{n(x)} V(h_U(x), i) + V(h_U(x), i)
\]

and so, re-arranging,

\[
J(h_U(x)) + \sum_{i=1}^{n(x)} V(h_U(x), i) = \Omega(h_U(x)) - V(h_U(x), i)
\]

Substituting this last equation into the term in the square brackets of the first equation,

\[
UE_{\text{Hires}} = qv(x) \phi [\Omega(h_U(x)) - \Omega(x) - V(h_U(x), i)] \cdot I_{\{x \in A\}}
\]

Following C-UE, the value going to the hired worker is \(V(h_U(x), i) = U\). Substituting in:

\[
UE_{\text{Hires}} = qv(x) \phi [\Omega(h_U(x)) - \Omega(x) - U] \cdot I_{\{x \in A\}}
\]

**UE Threats**

\[
UE_{\text{Threats}} = qv(x) \phi \left[ J(t_U(x)) + \sum_{i=1}^{n(x)} V(t_U(x), i) - \Omega(x) \right] \cdot I_{\{x \notin A\}}
\]

Using the definition of \(\Omega(t_U(x))\), we can re-write this term as

\[
UE_{\text{Threats}} = qv(x) \phi [\Omega(t_U(x)) - \Omega(x)] \cdot I_{\{x \notin A\}}
\]

Now using our result in condition C-UE that \(\Omega(t_U(x)) = \Omega(x)\), we can conclude that

\[
UE_{\text{Threats}} = 0
\]
EE Hires

\[ EE \text{ Hires} = qv(x) (1 - \phi) \int_{x \in Q_E(x', i')} \left[ J(h_E(x', i', x)) + \sum_{i=1}^{n(x)} V(h_E(x', i', x), i) \right. \]

\[ \left. - \Omega(x) \right] dH_n(x', i') \]

Now by definition

\[ \Omega(h_E(x', i', x)) = J(h_E(x', i', x)) + \sum_{i=1}^{n(x)} V(h_E(x', i', x), i) \]

\[ = \left[ J(h_E(x', i', x)) + \sum_{i=1}^{n(x)} V(h_E(x', i', x), i) \right] + V(h_E(x', i', x), i) \]

which can be re-arranged into

\[ J(h_E(x', i', x)) + \sum_{i=1}^{n(x)} V(h_E(x', i', x), i) = \Omega(h_E(x', i', x)) - V(h_E(x', i', x), i) \]

Using this in the term in the square brackets

\[ EE \text{ Hires} = qv(x) (1 - \phi) \int_{x \in Q_E(x', i')} \left[ \Omega(h_E(x', i', x)) - \Omega(x) - V(h_E(x', i', x), i) \right] dH_n(x', i') \]

Under C-EE, the value going to the hired worker is \( V(h_E(x', i', x), i) = \Omega(x') - \Omega(q_E(x', i', x)) \). Substituting this in:

\[ EE \text{ Hires} = qv(x) (1 - \phi) \int_{x \in Q_E(x', i')} \left[ \Omega(h_E(x', i', x)) - \Omega(x) \right. \]

\[ \left. - \left[ \Omega(x') - \Omega(q_E(x', i', x)) \right] \right] dH_n(x', i') \]
EE Threats

\[
EE \text{ Threats} = qv(x) (1 - \phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[ J(t_E(x',i',x)) + \sum_{i=1}^{n(x)} V(t_E(x',i',x),i) - J(x) - V(x) \right] dH_n(x',i')
\]

Using the definition of \( \Omega(t_E(x',i',x)) \), we obtain

\[
EE \text{ Threats} = qv(x) (1 - \phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[ \Omega(t_E(x',i',x)) - \Omega(x) \right] dH_n(x',i')
\]

Now using the result in condition C-RT that \( \Omega(t_E(x',i',x)) = \Omega(x) \), we obtain that

\[
EE \text{ Threats} = 0
\]

C.6.5 Reducing the state space

Now that we obtained the simplified recursion, we are in a position to argue that the only payoff-relevant states are \((z,n)\), and that the details of the within-firm contractual structure do not affect allocations. The goal of this section is to show that we can express the joint values pre- and post-separation and exit decisions as follows. First, the exit and separation decisions are characterized by

\[
\Omega(z,n) = \mathbb{I}_{\{(z,n) \in \mathcal{E}\}} \left\{ \vartheta + nU \right\} + \mathbb{I}_{\{(z,n) \in \mathcal{Q}^U\}} \left\{ \Omega(z,n - 1) + U \right\} + \mathbb{I}_{\{(z,n) \notin \mathcal{Q}^{U \cup \mathcal{E}}\}} \Omega(z,n),
\]

where \( \mathcal{E} = \{n, z | \vartheta + nU > \Omega(z,n)\} \),

\[
\mathcal{Q}^U = \{z, n | \Omega(z,n - 1) + U > \Omega(z,n)\}.
\]

The first expression is the value of exit. A firm that does not exit, chooses whether to separate with a worker or not. If separating with a worker, the firm re-enters \((C.19)\) with \( \Omega(z,n - 1) \), having dispatched with a worker with value \( U \), and again choosing whether to
exit, fire another worker, or continue. Iterating on this procedure delivers

\[ \Omega(z, n) = \max \left\{ \vartheta + nU, \max_{s \in [0, \ldots, n]} \Omega(z, n - s) + sU \right\}. \tag{C.20} \]

Second, the post-exit/separation decision joint value is given by the Bellman equation

\[
\rho \Omega(z, n) = \max_{v \geq 0} \left\{ y(z, n) - c(v, n, z) \right\} \\
\text{Destruction} + \delta n \left\{ (\Omega(z, n - 1) + U) - \Omega(z, n) \right\} \\
\text{UE Hire} + \phi q(\theta) v \cdot \mathbb{I}_{\{z, n\} \in \mathcal{A}} \cdot \left\{ \Omega(z, n + 1) - (\Omega(z, n) + U) \right\} \\
\text{EE Hire} + (1 - \phi) q(\theta) v \int_{(z, n) \in \mathcal{Q}^E} \left\{ [\Omega(z, n + 1) - \Omega(z, n)] \\
- [\Omega(z', n') - \Omega(z', n' - 1)] \right\} dH_n(z', n') \\
\text{Shock} + \Gamma_z [\Omega, \Omega](z, n),
\]

where \( \mathcal{A} = \{ z, n \mid \Omega(z, n + 1) \geq \Omega(z, n) + U \} \),

\( \mathcal{Q}^E(z', n') = \{ z, n \mid \Omega(z, n + 1) - \Omega(z, n) \geq \Omega(z', n') - \Omega(z', n' - 1) \} \).

Finally, firms enter if and only if

\[ \int \Omega(z, 0) d\Pi_0(z) \geq c_e. \tag{C.21} \]

This condition pins down the entry rate per unit of time.\(^{18}\)

\(^{18}\)Recall that \( J_0 = -c_e + \int J(x_0) d\Pi(z_0) \). Given \( \Omega(z_0, 0) = J(z_0, 0) \), we have \( J_0 = -c_e + \int \Omega(z_0, 0) d\Pi(z_0) \). Free-entry implies \( J_0 = 0 \), which delivers (C.21).
Inspection of the previous equations reveals that the only payoff-relevant states are \((z, n)\). Thus, we can re-write our Bellman equation for \((z, n) \in \mathcal{E}^c\) as:

\[
\rho \Omega (z, n) = y(z, n) - c(v(z, n), n)
\]

\[
\text{Destructions: } -\delta \sum_{i=1}^{n} \left[ \Omega (z, n) - \Omega (n - 1, z) - U \right]
\]

\[
\text{Retentions: } +\lambda^E \sum_{i=1}^{n} \int_{(n', z') \in \mathcal{R}(z, n)} \left[ \Omega (z, n) - \Omega (z, n) \right] dH_v (x')
\]

\[
\text{UE Hires: } +qv (z, n) \phi \left[ \Omega (n + 1, z) - \Omega (z, n) - U \right] \cdot \mathbb{I}_{(z, n) \in \mathcal{A}}
\]

\[
\text{EE Hires: } +qv (z, n) (1 - \phi) \int_{(z, n) \in \mathcal{Q}^E (z', n')} \left[ \Omega (n + 1, z) - \Omega (z, n) \right] \nonumber
\]

\[
\nonumber - \left[ \Omega (z', n') - \Omega (n' - 1, z') \right] d\tilde{H}_n (z', n')
\]

\[
\text{Shocks: } +\Gamma_z [\Omega, \Omega] (z, n)
\]

with the sets

\[
\mathcal{E}^c = \left\{ z, n \mid \Omega (\mathcal{N}(z, n)) \geq \vartheta + \mathcal{N}(z, n) U \right\}
\]

\[
\mathcal{L} = \mathcal{Q}^U = \left\{ z, n \mid \Omega (\mathcal{N}(z, n), z) - \Omega (\mathcal{N}(z, n) - 1, z) \leq U \right\}
\]

\[
\mathcal{A} = \left\{ z, n \mid \Omega (n + 1, z) - \Omega (z, n) \geq U \right\}
\]

\[
\mathcal{Q}^E (z', n') = \left\{ z, n \mid \Omega (n + 1, z) - \Omega (z, n) \geq \Omega (z', n') - \Omega (n' - 1, z') \right\}
\]

and the definition

\[
\mathcal{N}(z, n) = \arg \max_{k \in \{0, \ldots, n\}} \Omega (k, z) + (n - k) U
\]
and the vacancy policy function:

\[
\frac{c(v(z,n),z,n)}{q} = \phi [\Omega(n+1,z) - \Omega(z,n)] \cdot 1_{(z,n) \in A} + (1 - \phi) \int_{(z,n) \in \mathcal{Q}} \left[ \Omega(n+1,z) - \Omega(z,n) \right] - [\Omega(z',n') - \Omega(n'-1,z')] d\tilde{H}_n(n',z')
\]

Expressing “bold” values

In this step we express “bold” values – that encode the optimal quit, layoff and exit decisions – as simple functions of non-bold values.

From the definition of the exit and quit sets \( \mathcal{E}, \mathcal{Q}^U \), we can express:

\[
\Omega(z,n) = \max \left\{ \Omega(z,n), \Omega(n-1,z) + U, \vartheta + nU \right\}
\]

We can iterate on this equation. To see the logic, consider the first few steps.

\[
\begin{align*}
\Omega(z,n) &= \max \left\{ \Omega(z,n), \Omega(n-1,z) + U, \vartheta + nU \right\} \\
&= \max \left\{ \Omega(z,n), \max \left\{ \Omega(n-1,z), \Omega(n-2,z) + U, \vartheta + (n-1)U \right\} + U, \vartheta + nU \right\} \\
&= \max \left\{ \Omega(z,n), \Omega(n-1,z) + U, \Omega(n-2,z) + 2U, \vartheta + (n-1)U + U, \vartheta + nU \right\} \\
&= \max \left\{ \Omega(z,n), \Omega(n-1,z) + U, \Omega(n-2,z) + 2U, \vartheta + nU \right\}
\end{align*}
\]

By recursion, it is easy to see that

\[
\Omega(z,n) = \max \left\{ \Omega(\mathcal{N}(z,n),z) + (n - \mathcal{N}(z,n)) \cdot U, \vartheta + nU \right\}
\]

\[
= \max \left\{ \max_{k \in \{0, \ldots, n\}} \Omega(k,z) + (n-k)U, \vartheta + nU \right\}
\]

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where recall that
\[ N(z, n) = \arg \max_{k \in \{0, \ldots, n\}} \Omega(k, z) + (n - k)U \]

### C.6.6 Continuous workforce limit

Up to this point the economy has featured a continuum of firms, but an integer-valued workforce. We now take the continuous workforce limit by defining the ‘size’ of a worker—which is 1 in the integer case—and taking the limit as this approaches zero. Specifically, denote the “size” of a worker by \( \Delta \), such that \( n = N\Delta \) where \( N \) is the old integer number of workers. Now define \( \Omega^\Delta(z, n) := \Omega(z, n/\Delta) \), and likewise define \( y^\Delta(z, n) := y(z, n/\Delta) \) and \( c^\Delta(v, n, z) := c(v/\Delta, n/\Delta, z) \). We also define \( b^\Delta := b/\Delta \) and \( \vartheta^\Delta := \vartheta/\Delta \). These imply, for example, that \( \Omega(z, N) = \Omega^\Delta(z, N\Delta) \). Substituting these terms into (C.20) and (C.21), and taking the limit \( \Delta \to 0 \), while holding \( n = N\Delta \) fixed, we would obtain a version of (C.22) in which all functions have the \( \Delta \) super-script notation. We also specialize the productivity to a diffusion process
\[ dz_t = \mu(z_t)dt + \sigma(z_t)dW_t. \]

The result is the joint value representation of section 3.3: a Hamilton-Jacobi-Bellman (HJB) equation for the joint value conditional on the firm and its workers operating:

\[
\rho \Omega(z, n) = \max_{v \geq 0} \quad y(z, n) - c(v, n, z) \quad (C.22)
\]

\[
\text{Destruction} \quad -\delta n[\Omega_n(z, n) - U]
\]

\[
\text{UE Hire} \quad +\phi q(\theta) v [\Omega_n(z, n) - U]
\]

\[
\text{EE Hire} \quad +(1 - \phi) q(\theta) v \int \max \left\{ \Omega_n(z, n) - \Omega_n(n', z'), 0 \right\} dH_n(z', n')
\]

\[
\text{Shock} \quad +\mu(z) \Omega_z(z, n) + \frac{\sigma(z)^2}{2} \Omega_{zz}(z, n).
\]
Boundary conditions for the firm and its workers operating require the state to be interior to the exit and separation boundaries:

Exit boundary: $\Omega(z, n) \geq \vartheta + nU,$

Layoff boundary: $\Omega_n(z, n) \geq U$

Note the absence of $\Omega$ terms. Since the value we track is that of a hiring firm subject to boundary conditions, then $\Omega = \Omega$. This admits the simplification of ‘Shock’ terms we noted when discussing (??).

We proceed in three steps:

(A.5.1) Define worker size and the renormalization

(A.5.2) Take the limit as worker size goes to zero

(A.5.3) Introduce a continuous productivity process.

**Define worker size and the renormalization**

We denote the “size” of a worker by $\Delta$. That is, we currently have an integer work-force $n \in \{1, 2, 3, \ldots\}$. We now consider $m \in \{\Delta, 2\Delta, 3\Delta, \ldots\}$. So then $n = m/\Delta$. We use this to make the following normalizations:

$$\omega(z, m) = \Omega\left(\frac{m}{\Delta}, z\right)$$
$$\mathcal{Y}(z, m) = y\left(\frac{m}{\Delta}, z\right)$$
$$\mathcal{C}(z, m) = c\left(\frac{v}{\Delta}, \frac{m}{\Delta}, z\right)$$
These definitions imply

\[ \Omega(z, n) = \omega(n \Delta, z) \]
\[ y(z, n) = \mathcal{Y}(n \Delta, z) \]
\[ c(v, z, n) = \mathcal{C}(v \Delta, n \Delta, z) \]

In addition, the value of unemployment solves

\[ \rho U = b \]

Define

\[ U = \frac{b}{\rho \Delta} = \frac{U}{\Delta} \]

and

\[ \theta = \frac{\vartheta}{\Delta} \]

Substituting these definitions into the Bellman equation, we obtain

\[ \rho \omega(n \Delta, z) = \max_{v \Delta \geq 0} \mathcal{Y}(n \Delta, z) - \mathcal{C}(v \Delta, n \Delta, z) \]

Destructions

\[ -\delta n \Delta \left[ \frac{\omega(n \Delta, z) - \omega(n \Delta - \Delta, z)}{\Delta} - U \right] \]

UE Hires

\[ + q v \Delta \phi \left[ \frac{\omega(n \Delta + \Delta, z) - \omega(n \Delta, z)}{\Delta} - U \right] \cdot \mathbb{I}_{\{(n \Delta, z) \in A\}} \]

EE Hires

\[ + q v \Delta (1 - \phi) \int_{(n \Delta, z) \in \mathcal{Q}^E(n' \Delta, z')} \left[ \frac{\omega(n \Delta + \Delta, z) \omega(n \Delta, z)}{\Delta} \right. \]
\[ \left. - \frac{\omega(n' \Delta, z') - \omega(n' \Delta - \Delta, z')}{\Delta} \right] d\tilde{H}_{n}(n' \Delta, z') \]

Shocks

\[ + \Gamma_z [\omega, \omega](n \Delta, z) \]
with the set definitions

\[
\mathcal{E} = \left\{ n\Delta, z \left| \max_{k\Delta \in \{0, \ldots, n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U} < \theta + n\Delta\mathcal{U} \right. \right\}
\]

\[
\mathcal{A} = \left\{ n\Delta, z \left| \frac{\omega(n\Delta + \Delta, z) - \omega(n\Delta, z)}{\Delta} \geq \mathcal{U} \right. \right\}
\]

\[
\mathcal{Q}^U = \left\{ n\Delta, z \left| \frac{\omega(n\Delta, z) - \omega(n\Delta - \Delta, z)}{\Delta} \leq \mathcal{U} \right. \right\}
\]

\[
\mathcal{Q}^E(n'\Delta, z') = \left\{ n\Delta, z \left| \frac{\omega(n\Delta + \Delta, z) - \omega(n\Delta, z)}{\Delta} \geq \frac{\omega(n'\Delta, z') - \omega(n'\Delta - \Delta, z')}{\Delta} \right. \right\}
\]

and the definition:

\[
\omega(n\Delta, z) = \max \left\{ \max_{k\Delta \in \{0, \ldots, n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U}, \theta + n\Delta\mathcal{U} \right. \right\}
\]

**Continuous limit as worker size goes to zero**

Now we take the limit \( \Delta \to 0 \), holding \( m = n\Delta \) fixed. We note \( \hat{v} = \lim_{\Delta \to 0} v\Delta \). We see derivatives appear. We denote \( \omega_m(z, m) = \frac{\partial \omega}{\partial m}(z, m) \).

First, we note that the following limit obtains:

\[
\omega(z, m) = \max \left\{ \max_{k \in [0, m]} \omega(k, z) + (m - k)\mathcal{U}, \theta + m\Delta\mathcal{U} \right. \right\}
\]

In particular, the exit set limits to

\[
\mathcal{E} = \left\{ z, m \left| \max_{k \in [0, m]} \omega(k, z) + (m - k)\mathcal{U} < \theta + m\mathcal{U} \right. \right\}
\]

In equilibrium, the \( \omega(z, m) \) terms on the right-hand-side of the Bellman equation are the result of endogenous quits, layoffs and hires. Because our continuous time assumption has been made before we take the limit to a continuous workforce limit, we need only consider
those changes in the workforce one at a time. Hence, for any \((z, m) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})\), the interior of the continuation set, there is always \(\bar{\Delta} > 0\): such that for any \(\Delta \leq \bar{\Delta}\):

\[
\omega(m \pm \Delta, z) = \omega(m \pm \Delta, z)
\]

Using this observation in the Bellman equation, we can obtain derivatives on the right-hand-side. We obtain, for pairs \((z, n)\) in the interior of the continuation set \((z, n) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})\):

\[
\rho \omega (z, m) = \max_{\hat{v} \geq 0} \mathcal{Y} (z, m) - C (\hat{v}, z, m)
\]

Destructions: 
\[-\delta m [\omega_m(z, m) - \mathcal{U}]\]

\(UE\) Hires: 
\[+q \hat{v} \phi [\omega_m(z, m) - \mathcal{U}] \cdot I_{\{(z,m) \in \mathcal{A}\}}\]

\(EE\) Hires: 
\[+q \hat{v} (1 - \phi) \int_{(z,m) \in \mathcal{Q}_E(z',m')} \left[ \omega_m(z, m) - \omega_m(m', z') \right] d\tilde{H}_n (m', z')\]

Shocks: 
\[+\Gamma_z [\omega, \omega] (z, n)\]

with the set definitions

\[
\mathcal{E} = \left\{ z, m \left| \max_{k \in [0,m]} \omega(k, z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right. \right\}
\]

\[
\mathcal{A} = \left\{ z, m \left| \omega_m(z, m) \geq \mathcal{U} \right. \right\}
\]

\[
\mathcal{Q}^U = \left\{ z, m \left| \omega_m(z, m) \leq \mathcal{U} \right. \right\} = \overline{\mathcal{A}}, \text{ the complement of } \mathcal{A}
\]

\[
\mathcal{Q}^E (z', m') = \left\{ z, m \left| \omega_m(z, m) - \omega_m(m', z') \geq 0 \right. \right\}
\]

and the definition

\[
\omega(z, m) = \max \left\{ \max_{k \in [0,m]} \omega(k, z) + (m-k)\mathcal{U} , \theta + m\mathcal{U} \right\}
\]
Note that now, the only place where ω enters in the Bellman equation is the contribution of shocks. To replace it with ω, we need to apply the same argument to z as the one we applied to n. We thus need to specialize to a continuous productivity process.

**Continuous productivity process**

We now specialize to a continuous productivity process, as this makes the formulation of the problem very economical. It allows to simplify the contribution of productivity shocks and get rid of the remaining “bold” notation. We suppose that productivity follows a diffusion process:

\[
dz_t = \mu(z_t)dt + \sigma(z_t)dW_t
\]

In this case, for any \((z, m)\) in the interior of the continuation set, productivity shocks in the interval \([t, t + dt]\) cannot move the firm towards a region in which it would endogenously separate or exit, when \(dt\) is small enough. In this case, we can write the following, where we have also replaced the \(Q^E\) set with the max operator:

\[
\rho \omega(z, m) = \max_{v \geq 0} \left\{ Y(z, m) - C(v, z, m) \right\}
\]

Detections \(-\delta m \left[ \omega_m(z, m) - U \right]\)

\[
UE \text{ Hires} \quad +qv\phi \left[ \omega_m(z, m) - U \right]
\]

\[
EE \text{ Hires} \quad +qv(1 - \phi) \int \max \left\{ \omega_m(z, m) - \omega_m(z', m'), 0 \right\} d\tilde{H}_n(m', z')
\]

Shocks \(+\mu(z)\omega_z(z, m) + \frac{\sigma(z)^2}{2}\omega_{zz}(z, m)\)

s.t.

\[
\text{No Exit} \quad \omega(z, m) \geq \theta + mU
\]

\[
\text{No Separations} \quad \omega_m(z, m) \geq U
\]
To make the notation more comparable, we slightly abuse notation and use the same letters as before, but now for the continuous workforce case. We obtain finally:

\[
\rho \Omega (z, n) = \max_{v \geq 0} \left( y(z, n) - c(v, z, n) \right)
\]

Destructions \[ -\delta n[\Omega_n(z, n) - U] \]

\[ UE \text{ Hires} \quad +qv \phi [\Omega_n(z, n) - U] \]

\[ EE \text{ Hires} \quad +qv (1 - \phi) \int \max \left[ \Omega_n(z, n) - \Omega_n(z', n') , 0 \right] d\tilde{H}_n(z', n') \]

Shocks \[ +\mu(z)\Omega_z(z, n) + \frac{\sigma(z)^2}{2} \Omega_{zz}(z, n) \]

s.t.

No Exit \[ \Omega(z, n) \geq \vartheta + nU \]

No Separations \[ \Omega_n(z, n) \geq U \]

When the coalition hits \( \Omega_n(z, n) = U \), it endogenous separates worker to stay on that frontier. It exits when it hits the frontier \( \Omega(z, n) = \vartheta + nU \).

In addition to these “value-pasting” boundary conditions, optimality implies necessary “smooth-pasting” boundary conditions (see Stokey, 2009): \( \Omega_z(z, n) = 0 \) if the firm actually exits at \( (z, n) \) following productivity shocks, and \( \Omega_n(z, n) = 0 \) if the firm actually exits at \( (z, n) \) following changes in size. These are necessary and sufficient for the definition of our problem (Brekke and Oksendal, 1991). Its general formulation terms of optimal switching between three regimes (operation, layoffs, exit) on the entire positive quadrant, can be made as a system of Hamilton-Jacobi-Bellman-Variational-Inequality (see Pham, 2009), which we
max \left\{ -\rho \Omega (z, n) + \max_{\nu \geq 0} -\delta n[\Omega_n(z, n) - U] + qv \phi [\Omega_n(z, n) - U] \\
+ qv (1 - \phi) \int \max \left[ \Omega_n(z, n) - \Omega_n(z', n'), 0 \right] d\tilde{H}_n(z', n') \\
+ \mu(z) \Omega_z(z, n) + \frac{\sigma(z)^2}{2} \Omega_{zz}(z, n); \\
\psi + nU - \Omega(z, n); \max_{k \in [0, n]} \Omega(z, k) + (n - k)U - \Omega(z, n) \right\} = 0, \quad \forall (z, n) \in \mathbb{R}_+^2
Bibliography


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