Comparison of different definitions of pseudocharacter

Kathleen Emerson

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Sophie Morel

November 2018

Abstract

Pseudocharacters were first introduced by Taylor and Wiles to study congruences of representations. When A is an algebraically closed field with $d! \in A^{\times}$, the trace gives a bijection between the set of congruence classes of semisimple A-valued representations and the set of A-valued d-dimensional pseudocharacters. However, in small characteristic this definition of pseudocharacter works less well. Chenevier later defined determinants, which are equivalent to the original defintion of pseudocharacter in characteristic zero, but moreover, A-valued determinants correspond uniquely to congruence classes of A-valued semisimple representations for any algebraically closed field A. Lafforgue gives a more general defintion of pseudocharacter that makes sense for representations with values in any reductive algebraic group (and in any characteristic). This work shows that Chenevier's defintion of determinants is equivalent to Lafforgue's defintion (for GL_d) over any ring.

Another traditional approach to studying congruence classes of semisimple representations is to study the character variety. This work compares the space of Lafforgue pseudorepresentations to the character variety, proving that they are almost isomorphic (in a precise sense). In particular, it shows that they have the same points with values in any perfect field.

Acknowledgements

Thank you to my adviser, Professor Sophie Morel, who gave me the problem and the tools to solve it. I am deeply grateful for her ideas, patient guidance, and precise reading. I am awed by the immense priviledge of receiving this time and effort. Thank you too to my second reader Professor Chris Skinner, for generously reading this thesis.

Contents

Chapter 1

Introduction

Pseudocharacters (also known as pseudorepresentations) were first introduced by Taylor ([15]) and Wiles ([17]) to study congruences of representations. To study congruences between equivalence classes of representations, it's easier to look at the trace of the representation, and so the original definition of pseudocharacter captured some of the key properties of the trace. Let Γ be a group and A a commutative, unital ring. Then a map $T : \Gamma \to A$ is a d-dimensional pseudocharacter if $T(1) = d$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$, $d! \in A^\times$ and if T satisfies the d-dimensional pseudocharacter identity

$$\sum_{\sigma \in \mathcal{S}_{d+1}} T^\sigma(\gamma_1, ..., \gamma_{d+1}) = 0$$

where if σ has cycle decomposition $(i_{1,1}, i_{1,2}, ..., i_{1,n_1})(i_{2,1}, ..., i_{2,n_2})...(i_{k,1}, ..., i_{k,n_k})$ (including any trivial cycles) then

$$T^\sigma(\gamma_1, ..., \gamma_{d+1}) = T(\gamma_{i_{1,1}}\gamma_{i_{1,2}}...\gamma_{i_{1,n_1}})...T(\gamma_{i_{k,1}}...\gamma_{i_{k,n_k}}).$$

If A is an algebraically closed field with $d! \in A^\times$, then the set of d-dimensional pseudocharacters, $\mathrm{PChar}_{\mathbb{Z}}(\Gamma, d)(A)$, is in bijection with the set of conjugacy classes of semisimple representations (via the trace) (see [15, Theorem 1]). Unlike the functor $\mathrm{Hom}(\Gamma, \mathrm{GL}_d(-))/\mathrm{PGL}_d(-)$ of isomorphism classes of representations, the functor $\mathrm{PChar}_{\mathbb{Z}}(\Gamma, d)$ is a representable functor. Another functor traditionally used to replace $\mathrm{Hom}(\Gamma, \mathrm{GL}_d(-))/\mathrm{PGL}_d(-)$ by something representable is the character variety, $\mathrm{Char}_{\mathbb{Z}}(\Gamma, \mathrm{GL}_d)$. As Chenevier shows in [4, Proposition 2.3], when C is a field of characteristic zero it follows from results of Procesi ([10]) that $\mathrm{PChar}_{\mathbb{Z}}(\Gamma, d)(C)$ coincides with $\mathrm{Char}_{\mathbb{Z}}(\Gamma, \mathrm{GL}_d)(C)$.

This original definition of pseudocharacter fails in characteristic $p > 0$ if $p | d!$, in which case representations are no longer determined by their traces. Semisimple representations are however determined if we know *all* the coefficients of their characteristic polynomial, and Chenevier ([3]) comes up with a clever way of nicely packaging all this data in his theory of *determinants*. He defines determinants for an arbitrary ring, without the requirement that $d! \in A^\times$, and shows that for algebraically closed fields determinants correspond uniquely to equivalence classes of semisimple representations. However, it is not known whether the determinant functor and the character variety are isomorphic as schemes.

V. Lafforgue takes a more general approach to defining pseudocharacters. Instead of using some clever packaging of all the data needed to recover a representation, Lafforgue's definition requires one to give data for every invariant function on every power of some reductive group. This is flexible because it works for representations with values in any reductive group, in contrast with the theory of determinants, where it is not at all obvious how to extend the definition from GL_d. Lafforgue's insight is to make this useable. He proves in [8, 5] that Lafforgue pseudorepresentations (LPRs) valued in an algebraically closed field k are in bijection with $G(k)$-conjugacy classes of G-completely reducible homomorphisms $\rho : \Gamma \to G(k)$.[1]

As in [2], we make the obvious extension of Lafforgue's definition to affine group schemes G over an arbitrary base ring C. The first examples of LPRs are the Θ_ρ induced by representations ρ. It is straightforward to show that the functor $\mathrm{LPR}_C(\Gamma, G)$, which associates to a commutative, unital C-algebra A the set of A-valued G-LPRs, is representable. We then consider $C = \mathbb{Z}$ and $G = \mathrm{GL}_d$. The first main result of this paper is to show that over every ring Lafforgue pseudorepresentations for the general linear group are in bijection with determinants. In particular, we recall Chenevier's definition of the determinant functor $\det_C(C[\Gamma], d)$ and prove the following theorem.

Theorem. *The functors* $\mathrm{LPR}_{\mathbb{Z}}(\Gamma, \mathrm{GL}_d)$ *and* $\det_{\mathbb{Z}}(\mathbb{Z}[\Gamma], d)$ *are naturally isomorphic.*

By considering LPRs and determinants coming from representations ρ, we get a formula for the natural map $\mathrm{LPR}_{\mathbb{Z}}(\Gamma, G) \to \det_{\mathbb{Z}}(\mathbb{Z}[\Gamma], d)$, and all that remains is to check that it is well-defined. The other direction is more involved. The key ingredients are Procesi's ([10]) description of the generators and relations of $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d}$, which allows us to define a map from $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d}$, and then Donkin's ([5]) description of the generators of $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d}$ and Vaccarino's ([16]) result that homogeneous multiplicative polynomial laws are determinants, and in particular that the universal determinant ring $E_X(d) \cong \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X_\Gamma\})\right)^{\mathrm{ab}}$ for $\mathbb{Z}\{X\}$ is torsion-free. In working over this ring, our proof essentially deduces the result for general groups from a result for free monoids.

[1] The details of the proof in the positive characteristic case can be found in [2, Theorem 4.5].

We also compare Lafforgue's definition to the character variety. As in the comparison between LPRs and determinants, we start by proving results for free monoids, and from there deduce results for general groups. We define a map $\mathrm{Char}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, G)$. Taking X to be the underlying set of Γ, we give a description of the ideal $I(\Gamma, G_C) \lhd \mathcal{O}(G_C^X)$ cutting the representation variety for Γ, $\mathrm{Rep}_C(\Gamma, G)$, out of the representation variety for X, $\mathrm{Rep}_C(X, G)$. If C is a field of characteristic zero and G is a reductive group over C, then we can use the Reynolds operator, and our description of $I(\Gamma, G_C)$ allows us to prove that $\mathrm{Char}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, G)$ is an isomorphism. More generally, if $C = \mathbb{Z}$ and G is a reductive group over $\mathbb{Z}$, we prove that this map is an *adequate homeomorphism*, that is, an integral, universal homeomorphism which is a local isomorphism at all points with a residue field of characteristic 0 ([1, Definition 3.3.1]). In particular, this map induces a bijection on points with values in a perfect field. We conclude that, for C a field of characteristic zero and $G = \mathrm{GL}_d$, the functors $\mathrm{Char}_C(\Gamma, \mathrm{GL}_d)$, $\mathrm{LPR}_C(\Gamma, \mathrm{GL}_d)$, $\det_C(C[\Gamma], d)$ and $\mathrm{PChar}_C(\Gamma, d)$ are naturally isomorphic.

The organization of this paper is as follows. In chapter 1 we recall Chenevier's definition of *determinant* and the basic properties of determinants that will be required. In chapter 2 we define LPRs for sets and groups, and look at the first properties of LPRs, in particular proving that the LPR functor is representable. We define the natural transformation $\mathrm{LPR}(\Gamma, d) \to \det(\mathbb{Z}[\Gamma], d)$ in 4.0.1 and the natural transformation $\det(\mathbb{Z}[\Gamma], d) \to \mathrm{LPR}(\Gamma, d)$ in 5.0.1. In chapter 4, we show that these functors are inverses. In chapter 5, we recall the definition of the character variety for a base ring C and an affine group scheme G, define the natural transformation $\mathrm{Char}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, G)$, and prove the above results about this transformation.

Chapter 2

Determinants of algebras

In this chapter we recall the definition of determinant given by Chenevier in [3], and the basic properties of the determinant functor Det. (In particular, that Det is representable by an affine scheme.) Throughout this chapter A denotes a commutative unital ring. By *A-algebra* we mean an associative, unital algebra over A. Let $\mathcal{C}_A$ be the category of commutative algebras. To define determinants, we first recall Roby's definition of *polynomial law* (from [11, I.2]), of *homogeneous* polynomial law (from [11, I.8]), and of *multiplicative* polynomial law (from [12, III.4]).

Definition 2.0.1. Let M and N be A-modules. An *A-polynomial law* $f : M \to N$ is a map

$$f_B : M \otimes_A B \to N \otimes_A B$$

for each $B \in \mathcal{C}_A$, functorial in B. That is, if $B, B' \in \mathcal{C}_A$ and $u \in \mathrm{Hom}_{\mathcal{C}_A}(B, B')$, then the following diagram commutes:

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\quad f_B \quad} & N \otimes_A B \\ {\scriptstyle \mathrm{id}_M \otimes u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_N \otimes u} \\ M \otimes_A B' & \xrightarrow[\quad f_{B'} \quad]{} & N \otimes_A B' \end{array}$$

For $d \in \mathbb{Z}_{\geq 1}$ we say that a polynomial law $f : M \to N$ is *homogeneous of degree d* if, for all $B \in \mathcal{C}_A$, $z \in M \otimes_A B$ and $b \in B$, we have

$$f_B(bz) = b^d f_B(z).$$

If, furthermore, M and N are A-algebras, then we say that a polynomial law $f : M \to N$ is *multiplicative* if for all $B \in \mathcal{C}_A$, $f_B(1) = 1$, and for all $x, y \in M \otimes_A B$

$$f_B(xy) = f_B(x) f_B(y).$$

In this case, we write $\mathcal{M}_A^d(M, N)$ for the set of all multiplicative, homogeneous of degree d A-polynomial laws $M \to N$.

We now recall the definition of A-determinant, due to Chenevier ([3, 1.5]).

Definition 2.0.2. Let R be an A-algebra. A *d-dimensional A-valued determinant D on R* is an element of $\mathcal{M}_A^d(R, A)$. When $R = A[\Gamma]$ for some group Γ, we say that D is a *determinant on* Γ.

Example 2.0.3. Let X be a set. Write $A\{X\}$ for the free unital associative A-algebra on the set X. Then any map

$$\rho : X \to \mathrm{M}_d(A)$$

gives rise to a d-dimensional A-valued determinant D_ρ on $A\{X\}$ as follows. For any $B \in \mathcal{C}_A$, ρ induces a ring homomorphism

$$\rho_B : B\{X\} \to \mathrm{M}_d(B)$$

and ρ_B is functorial in B. So for any $\omega \in B\{X\}$ we define

$$D_{\rho,B}(\omega) := \det(\rho_B(\omega))$$

where $\det : \mathrm{M}_d(B) \to B$ is the usual determinant.

For a group Γ, any representation

$$\rho : \Gamma \to \mathrm{GL}_d(A)$$

gives rise to a d-dimensional A-valued determinant D_ρ on Γ in a similar way. In particular, for any $B \in \mathcal{C}_A$, $r_i \in B$, $\gamma_i \in \Gamma$,

$$D_{\rho,B}\left(\sum_{i=1}^{n} r_i \gamma_i\right) := \det\left(\sum_{i=1}^{n} r_i \rho(\gamma_i)\right).$$

Example 2.0.4. Let R be an A-algebra. Following Chenevier ([3, 1.10]), given a determinant $D \in \mathcal{M}_A^d(R, A)$, we can define homogeneous of degree i polynomial laws $\Lambda_i : R \to A$ as follows. For any $B \in \mathcal{C}_A$ and $r \in R \otimes_A B$,

$$D_{B[T]}(t-r) = \sum_{i=0}^{d} (-1)^i \Lambda_{i,B}(r) t^{d-i}.$$

If $D = D_\rho$ for some representation ρ, then $\Lambda_i(r)$ is the coefficient of the degree $d-i$ term in the usual characteristic polynomial of the matrix $\rho(r)$.

Definition 2.0.5. Let C be a (commutative, unital) base ring and let R be a C-algebra. The *determinant functor* $\det_C(R,d)$ is given by

$$\begin{array}{rccc} \det_C(R,d): & \mathcal{C}_C & \to & \mathbf{Set} \\ & A & \mapsto & \mathcal{M}^d_A(R \otimes_C A, A) = \mathcal{M}^d_C(R, A) \end{array}$$

(the equality $\mathcal{M}^d_A(R \otimes_C A, A) = \mathcal{M}^d_C(R, A)$ is shown in [3, 1.4]).

Remark 2.0.6. For $D_1 \in \det_C(R, d_1)$ and $D_2 \in \det_C(R, d_2)$ we can define the product $D_1 \times D_2 \in \det_C(R, d_1 + d_2)$ as follows. For any $B \in \mathcal{C}_A$ and any $\omega \in B$, $(D_1 \times D_2)(\omega) := D_{1,B}(\omega) D_{2,B}(\omega)$. If $D_1 = D_{\rho_1}$ and $D_2 = D_{\rho_2}$ for some representations $\rho_i : R \to M_{d_i}(A)$ then $D_1 \times D_2 = D_{\rho_1 \oplus \rho_2}$. That is, the direct sum of representations corresponds to the product of determinants.

Example 2.0.7. For any group Γ, write X_Γ for the underlying set of Γ. Then the natural surjective map $C\{X_\Gamma\} \to C[\Gamma]$ gives rise to a monic natural transformation $\det_C(C[\Gamma], d) \to \det_C(C\{X_\Gamma\}, d)$.

Theorem 2.0.8. *The functor* $\det_C(R,d)$ *is representable by the affine scheme*

$$\mathit{Spec}\left(\left(\Gamma^d_C(R)\right)^{ab} \right),$$

where $\left(\Gamma^d_C(R)\right)^{ab}$ *is the abelianization of the* C*-algebra of divided powers of order* d *on* R *relative to* C*. The universal determinant* D^{univ} *is the natural map* $R \to \left(\Gamma^d_C(R)\right)^{ab}$.

Roby defines $\Gamma^d_C(M)$ for a module M in [11, III.1] and defines the ring structure on $\Gamma^d_C(R)$ for a A-algebra R in [12, II]. For the proof of Theorem 2.0.8, see [11, IIII.1] and [3, 1.6].

Chapter 3

Lafforgue pseudo-representations

First we establish some notation that will allow us, following [7, 10.3], to define *Lafforgue pseudo-representations*, henceforth LPRs. We will be interested in LPRs for groups, but defining LPRs for sets will clarify our proofs.

Let C be a (commutative, unital) base ring. We give the definition of LPRs for any affine group scheme G over C, although most results will require that G be a reductive group over C. By *reductive group* over an arbitrary base ring C, we mean an affine, smooth, connected group scheme over C such that the fibre at every geometric point is reductive.[1] Let $A \in \mathcal{C}_C$. Let X be a set and let Γ be a group. We write $C(X, A)$ for the A-algebra of all maps $X \to A$ with pointwise multiplication. Then $C(\Gamma, A) := C(X_\Gamma, A)$, where X_Γ is the underlying set of Γ. For $\gamma \in \Gamma$ we write x_γ for the corresponding element of X_Γ, and for $f \in C(\Gamma, A)$ we may write $f(\gamma)$ for $f(x_\gamma)$. For $n \in \mathbb{Z}_{\geq 1}$, write $[n]$ for the set of integers $\{1, 2, ..., n\}$.

The group G acts on G^n by diagonal conjugation:

$$h \cdot (g_1, ..., g_n) = (hg_1h^{-1}, ..., hg_2h^{-1}).$$

We write $\mathcal{O}(G^n)^G$ for the C-algebra of functions on G^n invariant under this action.

Definition 3.0.1. Suppose that for all $n \in \mathbb{Z}_{\geq 1}$ we have a C-algebra morphism

$$\mathcal{O}(G^n)^G \to C(X^n, A).$$

We say that this collection of morphisms $(\Theta_n)_{n \in \mathbb{Z}_{\geq 1}}$ is an *A-valued G-LPR for the set X* if it satisfies

[1] Alper's definition of *geometrically reductive* group scheme is actually sufficient for our purposes. (See [1].)

the following condition:

(LPR1) $(\Theta_n)_{n\in\mathbb{Z}_{\geq 1}}$ is functorial relative to all the maps between the sets $[n]$. That is, for any map

$$\zeta : [k] \to [m]$$

and any function

$$f \in \mathcal{O}(G^k)^G$$

we can define $f^\zeta \in \mathcal{O}(G^m)^G$ by

$$f^\zeta(g_1, ..., g_m) = f(g_{\zeta(1)}, ..., g_{\zeta(k)}).$$

Then for any $(x_1, ..., x_m) \in X^m$ we require

$$\Theta_k(f)(x_{\zeta(1)}, ..., x_{\zeta(k)}) = \Theta_m(f^\zeta)(x_1, .., x_m).$$

Furthermore, a collection $(\Theta_n)_{n\in\mathbb{Z}_{\geq 1}}$, is an *$A$-valued d-dimensional G-LPR for* Γ if it is an A-valued d-dimensional G-LPR for the set X_Γ satisfying the following additional condition:

(LPR2) For $m \geq 1$ and $f \in \mathcal{O}(G^m)^G$ we can define $\hat{f} \in \mathcal{O}(G^{m+1})^G$ by

$$\hat{f}(g_1, ..., g_{m+1}) = f(g_1, ..., g_{m-1}, g_m g_{m+1}).$$

Then for any $(\gamma_1, ..., \gamma_{m+1}) \in \Gamma^{m+1}$ we require

$$\Theta_{m+1}(\hat{f})(\gamma_1, ..., \gamma_{m+1}) = \Theta_m(f)(\gamma_1, ..., \gamma_{m-1}, \gamma_m\gamma_{m+1}).$$

Example 3.0.2. Any map of sets $\rho : X \to G(A)$ gives rise to a A-valued G-LPR for the set X, denoted by Θ_ρ, via

$$\Theta_{\rho,n}(f)(x_1, ..., x_n) := f(\rho(x_1), ..., \rho(x_n)).$$

In the same way, any group homomorphism $\rho : \Gamma \to G(A)$ gives rise to a A-valued G-LPR for the group Γ, denoted Θ_ρ.

Furthermore, in both cases Θ_ρ depends only on the $G(A)$-conjugacy class of ρ.

Definition 3.0.3. Let $\mathrm{LPR}^1_C(X, G) : \mathcal{C}_C \to \mathbf{Set}$ be the covariant functor associating to any commutative C-algebra A the set of A-valued G-LPRs for the set X.

Let $\mathrm{LPR}_C(\Gamma, G) : \mathcal{C}_C \to \mathbf{Set}$ be the covariant functor associating to any commutative C-algebra A the set of A-valued G-LPRs for the group Γ.

Then $\mathrm{LPR}_C(\Gamma, G)$ is a subfunctor of $\mathrm{LPR}^1_C(X, G)$.

Lemma 3.0.4. *A morphism $\psi : G \to H$ of affine group schemes gives rise to a natural transformation $\mathrm{LPR}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, H)$.*

Proof. ψ induces a map $\psi^* : \mathcal{O}(H^n)^H \to \mathcal{O}(G^n)^G$. If $\Theta \in \mathrm{LPR}_C(\Gamma, G)(A)$, then $(\Theta_n \circ \psi^*)_{n \in \mathbb{Z}_{\geq 1}} \in \mathrm{LPR}_C(\Gamma, H)(A)$. $\square$

Example 3.0.5. The case of the diagonal embedding $\psi : \mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \to \mathrm{GL}_{d_1+d_2}$ corresponds to the product of determinants by Theorem 4.0.1(ii). (The product of determinants is discussed in Remark 2.0.6.)

Theorem 3.0.6 (Representability)**.** *For any set X, let R_X be the ring*

$$R_X = \bigotimes_{n \in \mathbb{Z}_{\geq 1}} \bigotimes_{x \in X^n} \mathcal{O}(G^n)^G.$$

For $f \in \mathcal{O}(G^k)^G$ and $y = (y_1, ..., y_k) \in X^k$, we define an element of R_X

$$f_{k,y} := \bigotimes_{n \in \mathbb{Z}_{\geq 1} \setminus \{k\}} 1_{\otimes_{x \in X^n} \mathcal{O}(G^n)^G} \bigotimes_{x \in X^k \setminus \{y\}} 1_{\mathcal{O}(G^k)^G} \bigotimes f$$

(i) $\mathrm{LPR}^1_C(X, G)$ is representable by an affine scheme over C, $\mathrm{Spec}(R_{\mathrm{LPR}^1_C}(X, G))$, and $R_{\mathrm{LPR}^1_C}(X, G)$ is a quotient of the ring R_X. The universal element of $\mathrm{LPR}^1_C(X, G)(R_{\mathrm{LPR}^1_C}(X, G))$ is given by

$$\Theta_n(f)(y) = \overline{f_{n,y}}$$

for $f \in \mathcal{O}(G^n)^G$ and $y \in X^n$, where $\overline{f_{n,y}}$ is the image of $f_{n,y}$ in the quotient.

(ii) Let X be the underlying set of a group Γ. Then $\mathrm{LPR}_C(\Gamma, G)$ is representable by an affine scheme over C, $\mathrm{Spec}(R_{\mathrm{LPR}_C}(\Gamma, G))$, and $R_{\mathrm{LPR}_C}(\Gamma, G)$ is a quotient of $R_{\mathrm{LPR}^1_C}(X, G)$. The universal element of

$$\mathrm{LPR}_C(\Gamma, G)(R_{\mathrm{LPR}_C}(\Gamma, G))$$

is given by

$$\Theta_n(f)(\gamma) = \overline{f_{n,\gamma}}$$

for $f \in \mathcal{O}(G^n)^G$ and $\gamma \in \Gamma^n$, where x_γ is the element of X^n corresponding to γ and $\overline{f_{n,\gamma}}$ is the image of f_{n,x_γ} in the quotient.

Proof of Theorem 3.0.6(i). First, consider the covariant functor

$$\begin{aligned} F: \ \mathcal{C}_C &\to \mathrm{Set} \\ A &\mapsto \left\{ \substack{\text{families } (\Theta_n)_{n\in\mathbb{Z}_{\geq 1}} \text{ of } A\text{-algebra morphisms} \\ \Theta_n\colon \mathcal{O}(G^n)^G \to C(X^n, A)} \right\}. \end{aligned}$$

The functor F is representable by the affine scheme Spec R_X and the universal element of $F(R_X)$ is given by $\Theta_n(f)(x) = f_{n,x}$ for $x \in X^n$ and $f \in \mathcal{O}(G^n)^G$. Indeed,

$$\begin{aligned} \mathrm{Hom}_C(R_X, A) &= \mathrm{Hom}_C\left(\bigotimes_{n\in\mathbb{Z}_{\geq 1}} \bigotimes_{x\in X^n} \mathcal{O}(G^n)^G, A \right) \\ &= \prod_{n\in\mathbb{Z}_{\geq 1}} \prod_{x\in X^n} \mathrm{Hom}_C\left(\mathcal{O}(G^n)^G, A\right) \\ &= F(A), \end{aligned}$$

as required.

Next, let $I \lhd R_X$ be the ideal generated by the elements $f_{k,(x_{\zeta(1)},\dots,x_{\zeta(k)})} - f^{\zeta}_{m,(x_1,\dots,x_m)}$ for all $k, m \in \mathbb{Z}_{\geq 1}$, $f \in \mathcal{O}(G^k)^G$ and $\zeta : [k] \to [m]$, $(x_1, ..., x_m) \in X^m$. Then the collection $(\Theta_n)_{n\in\mathbb{Z}_{\geq 1}}$ corresponding to $u \in \mathrm{Hom}(R_X, A)$ satisfies (LPR1) exactly when u vanishes on I.

Indeed,

$$\Theta_k(f)(x_{\zeta(1)}, ..., x_{\zeta(k)}) = u\left(f_{k,(x_{\zeta(1)},\dots,x_{\zeta(k)})}\right)$$

and

$$\Theta_m(f^\zeta)(x_1, .., x_m) = u\left(f^\zeta_{m,(x_1,\dots,x_m)}\right).$$

Thus $\mathrm{LPR}^1_C(X, G)$ is representable by $R_{\mathrm{LPR}^1_C}(X, G) := R_X/I$ and the universal element of $\mathrm{LPR}^1_C(X, G)(R_{\mathrm{LPR}^1_C}(X, G))$ is given by $\Theta_n(f)(x) = \overline{f_{n,x}}$. □

Proof of Theorem 3.0.6(ii). Let $J \lhd R_X$ be the ideal generated by the elements

$$f_{m,(\gamma_1,\dots,\gamma_{m-1},\gamma_m\gamma_{m+1})} - \hat{f}_{m+1,(\gamma_1,\dots,\gamma_{m+1})}$$

for all $m \in \mathbb{Z}_{\geq 1}$, $f \in \mathcal{O}(G^k)^G$, and $(\gamma_1, ..., \gamma_{m+1}) \in \Gamma^{m+1}$. Then the collection $(\Theta_n)_{n\in\mathbb{Z}_{\geq 1}}$ corresponding to an element u of $\mathrm{Hom}(R_X, A)$ satisfies (LPR2) exactly when u vanishes on J. Indeed,

$$\Theta_m(f)(\gamma_1, ..., \gamma_{m-1}, \gamma_m\gamma_{m+1}) = u\left(f_{m,(\gamma_1,...,\gamma_{m-1},\gamma_m\gamma_{m+1})}\right)$$

and

$$\Theta_{m+1}(\hat{f})(\gamma_1, ..., \gamma_{m+1}) = u\left(\hat{f}_{m+1,(\gamma_1,...,\gamma_{m+1})}\right).$$

Let $R_{\mathrm{LPR}_C}(\Gamma, G) := R/(I+J)$. Then $\mathrm{LPR}_C(\Gamma, G)$ is representable by $\mathrm{Spec}(R_{\mathrm{LPR}_C}(\Gamma, G))$, and the universal element of $\mathrm{LPR}_C(\Gamma, G)(R_{\mathrm{LPR}_C}(\Gamma, G))$ is given by

$$\Theta_n(f)(\gamma) = \overline{f_{n,\gamma}}.$$

□

Chapter 4

An LPR gives rise to a determinant

In the next two chapters, we restrict our study of the LPR functor to the case $G = GL_d$, so as to compare it with the determinant functor. In this case we'll write $\mathrm{LPR}^1_C(X, d)$ for $\mathrm{LPR}^1_C(X, \mathrm{GL}_d)$ and $\mathrm{LPR}_C(\Gamma, d)$ for $\mathrm{LPR}_C(\Gamma, \mathrm{GL}_d)$, and refer to GL_d-LPRs as *d-dimensional LPRs*. We use the notation established in examples 2.0.3 and 3.0.2.[1]

Theorem 4.0.1.

(i) There exists a transformation $\mathrm{LPR}^1_C(X, d) \to \det_C(C\{X\}, d)$ *natural in* X *such that for all* $A \in \mathcal{C}_C$ *and all maps of sets* $\rho : X \to M_d(A)$*, our map takes* Θ_ρ *to* D_ρ*.*

(ii) There exists a transformation $\mathrm{LPR}_C(\Gamma, d) \to \det_C(C[\Gamma], d)$ *natural in* Γ *such that for all* $A \in \mathcal{C}_C$ *and all representations* $\rho : \Gamma \to \mathrm{GL}_d(A)$*, our map takes* Θ_ρ *to* D_ρ*.*

Furthermore, when X *is the underlying set of* Γ*, the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{LPR}_C(\Gamma, d) & \longrightarrow & \det_C(C[\Gamma], d) \\ \big\downarrow & & \big\downarrow \\ \mathrm{LPR}^1_C(X, d) & \longrightarrow & \det_C(C\{X\}, d) \end{array} \tag{4.0.1}$$

Proof of Theorem 4.0.1 (i). Let $A \in \mathcal{C}_C$. Any finite set Y and bijection $\sigma_Y : Y \to [n]$ gives rise to a multiplicative, homogenous of degree d A-polynomial law

$$\det_{\sigma_Y} : A\{Y\} \to \mathcal{O}(\mathrm{GL}^n_{d,A})^{\mathrm{GL}_d}$$

[1] D_ρ and $C\{X\}$ are defined in example 2.0.3, and Θ_ρ is defined in example 3.0.2.

via

$$\det_{\sigma_Y}(p)(g_1,...,g_n) = \det\left(p(g_{\sigma_Y(y)} \mid y \in Y)\right).$$

In fact, for any $B \in \mathcal{C}_A$, the image of $\det_{\sigma_Y,B}$ lies in $\mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C B$. Indeed, write $B = P/J$ for some polynomial ring P over C and some $J \lhd P$. Since P is flat over C, by [14, Lemma 2] $\mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C P \to \mathcal{O}(\mathrm{GL}_{d,P}^n)^{\mathrm{GL}_d}$ is an isomorphism. Then the following diagram commutes.

$$\begin{array}{ccccccc}
P\{Y\} & \xrightarrow{\det_{\sigma_Y,P}} & \mathcal{O}(\mathrm{GL}_{d,P}^n)^{\mathrm{GL}_d} & \xleftarrow{\sim} & \mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C P & \longrightarrow & \mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C B \\
\downarrow & & & & & & \downarrow \\
B\{Y\} & & & \xrightarrow{\det_{\sigma_Y,B}} & & & \mathcal{O}(\mathrm{GL}_{d,B}^n)^{\mathrm{GL}_d}
\end{array}$$

Let $\Theta \in \mathrm{LPR}_C^1(X,d)(A)$. Any $x \in X^n$ defines a ring homomorphism

$$\Theta_x : \mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C A \to A$$

via

$$\Theta_x(f) = \Theta_n(f)(x).$$

Thus $\Theta_{(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n))} \circ \det_{\sigma_Y} \in \det_C(C\{Y\},d)(A)$.

Now suppose $Z \subseteq Y$. Any bijection $\sigma_Z : Z \to [m]$ gives rise to a map $\zeta : [m] \to [n]$ such that

$$\begin{array}{ccc}
Z & \xrightarrow{\sigma_Z} & [m] \\
\downarrow & & \downarrow \zeta \\
Y & \xrightarrow{\sigma_Y} & [n]
\end{array}$$

commutes. Then

$$\begin{array}{ccc}
A\{Z\} & \xrightarrow{\det_{\sigma_Z}} & \mathcal{O}(\mathrm{GL}_{d,C}^m)^{\mathrm{GL}_d} \otimes_C A \\
\downarrow & & \downarrow f \mapsto f^\zeta \\
A\{Y\} & \xrightarrow{\det_{\sigma_Y}} & \mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C A
\end{array}$$

is a commutative diagram of A-polynomial laws. Moreover, since Θ satisfies LPR condition 1, the diagram

$$\begin{array}{ccc}
\mathcal{O}(\mathrm{GL}_{d,C}^m)^{\mathrm{GL}_d} \otimes_C A & \searrow^{\Theta_{\left(\sigma_Z^{-1}(1),\ldots,\sigma_Z^{-1}(m)\right)}} & \\
\downarrow f \mapsto f^\zeta & & \\
\mathcal{O}(\mathrm{GL}_{d,C}^n)^{\mathrm{GL}_d} \otimes_C A & \xrightarrow[\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n)\right)}]{} & A
\end{array}$$

commutes.

Let $B \in \mathcal{C}_A$ and $p \in B\{X\}$. We can choose a finite set $Y \subseteq X$ such that $p \in B\{Y\}$. We check that $\Theta_{(\sigma_Y^{-1}(1),\dots,\sigma_Y^{-1}(n))}(\det_{B,\sigma_Y}(p))$ depends only on p and not on our choice of Y and bijection $\sigma_Y : Y \to [n]$. Indeed, without loss of generality, let Z and Y be finite sets such that $Z \subseteq Y \subseteq X$ and $p \in B\{Z\}$. Then it follows from the above that for any bijections $\sigma_Y : Y \to [n]$ and $\sigma_Z : Z \to [m]$,

$$\Theta_{(\sigma_Y^{-1}(1),\dots,\sigma_Y^{-1}(n))}(\det_{\sigma_Y,B}(p)) = \Theta_{(\sigma_Z^{-1}(1),\dots,\sigma_Z^{-1}(m))}(\det_{\sigma_Z,B}(p))$$

as required.

Thus we can define

$$D_B(p) := \Theta_{(\sigma_Y^{-1}(1),\dots,\sigma_Y^{-1}(n))}(\det_{\sigma_Y,B}(p)).$$

The family $(D_B)_{B\in\mathcal{C}_A}$ defines a d-dimensional A-valued determinant. Indeed, it is clear that the family $(D_B)_{B\in\mathcal{C}_A}$ is functorial in B. Finally, since for any $p, q \in B\{X\}$ we can find a finite subset Y of X such that $p, q \in B\{Y\}$, and since $\Theta_{(\sigma_Y^{-1}(1),\dots,\sigma_Y^{-1}(n))} \circ \det_{\sigma_Y}$ is multiplicative, D_B is multiplicative.

$\square$

Proof of Theorem 4.0.1 (ii). Let $\Theta \in \mathrm{LPR}_C(\Gamma, d)(A)$. Write X for the underlying set of Γ. Since $\Theta \in \mathrm{LPR}^1_C(X, d)(A)$, by the above we can construct a determinant $D \in \det_C(C\{X\}, d)(A)$. To show that D gives rise to a determinant in $\det_C(C[\Gamma], d)(A)$, it remains to show that for all $B \in \mathcal{C}_A$, $D_B : B\{X\} \to B$ factors through the natural map $\varphi_B : B\{X\} \to B[\Gamma]$. That is, if $p, q \in B\{X\}$ such that $\varphi_B(p) = \varphi_B(q)$, then we need to check that $D_B(p) = D_B(q)$.

If $\deg(p) \leq 1$ and $\deg(q) \leq 1$, then $\varphi_B(p) = \varphi_B(q)$ implies that $p = q$, and so $D_B(p) = D_B(q)$. For $p \in B\{X\}$, define $n(p)$ to be the sum over each monomial in p of the degree of that monomial minus 1. Thus, if p is linear then $n(p) = 0$.

We show that given $p \in B\{X\}$ such that $n(p) \geq 1$, there exists $q \in B\{X\}$ such that $n(q) = n(p) - 1$, $\varphi_B(p) = \varphi_B(q)$, and $D_B(p) = D_B(q)$. Then, by induction on $n(p)$, we will be done.

Since $n(p) \geq 1$, there exists a term m of p with degree > 1. Let one such term be $m = bx_{\gamma_1}...x_{\gamma_k}$ for some $k \geq 2$, $x_{\gamma_i} \in X$ for $i \in [k]$, and $b \in B \setminus \{0\}$. Define $r, m' \in B\{X\}$ by $p = r + m$ and $m = x_{\gamma_1}x_{\gamma_2}m'$. Let $q := r + x_{\gamma_1\gamma_2}m'$. Let Y be a finite set such that $p, x_{\gamma_1\gamma_2} \in B\{Y\}$.

Then $\varphi_B(p) = \varphi_B(q)$ and $n(q) = n(p) - 1$. It remains to show that $D_B(p) = D_B(q)$.

Choose a bijection $\sigma_Y : Y \to [n]$. Let t be an indeterminate. Extend σ_Y to a bijection $\sigma_T : Y \cup \{t\} \to [n+1]$ via $\sigma_T(t) = n + 1$. Let $\zeta_T : [n+1] \to [n]$ be the map such that the following diagram commutes.

$$\begin{array}{ccc} Y\cup\{t\} & \xrightarrow{\sigma_T} & [n+1] \\ {\scriptstyle t\mapsto x_{\gamma_1\gamma_2}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_T} \\ Y & \xrightarrow{\sigma_Y} & [n] \end{array}$$

Then, since Θ satisfies LPR1, the following diagram commutes.

$$\begin{array}{ccccc} B\{Y\cup\{t\}\} & \xrightarrow{\det_{\sigma_T}} & \mathcal{O}(\mathrm{GL}_{d,C}^{n+1})^{\mathrm{GL}_d}\otimes_C B & & \\ {\scriptstyle t\mapsto x_{\gamma_1\gamma_2}}\big\downarrow & & {\scriptstyle f\mapsto f^{\zeta_T}}\big\downarrow & \searrow^{\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),x_{\gamma_1\gamma_2}\right)}} & \\ B\{Y\} & \xrightarrow[\det_{\sigma_Y}]{} & \mathcal{O}(\mathrm{GL}_{d,C}^{n})^{\mathrm{GL}_d}\otimes_C B & \xrightarrow[\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n)\right)}]{} & B \end{array}$$

Since q is the image under $B\{Y\cup\{t\}\}\to B\{Y\}$ of $r+m't$,

$$D_B(q)=\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n)\right)}\left(\det_{\sigma_Y}(q)\right)=\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),x_{\gamma_1\gamma_2}\right)}\left(\det_{\sigma_T}(r+m't)\right).$$

Next, let s_1 and s_2 be indeterminates and extend σ_Y to a bijection $\sigma_S : Y\cup\{s_1,s_2\}\to[n+2]$ via $\sigma_S(s_1)=n+1$, $\sigma_S(s_2)=n+2$. Let ζ_S be the map such that the following diagram commutes.

$$\begin{array}{ccc} Y\cup\{s_1,s_2\} & \xrightarrow{\sigma_S} & [n+2] \\ {\scriptstyle s_1\mapsto x_{\gamma_1},\ s_2\mapsto x_{\gamma_2}}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_S} \\ Y & \xrightarrow{\sigma_Y} & [n] \end{array}$$

Then, since p is the image under $B\{Y\cup\{s_1,s_2\}\}\to B\{Y\}$ of $r+m's_1s_2$, LPR1 for ζ_T gives

$$D_B(p)=\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n)\right)}\left(\det_{\sigma_Y}(p)\right)=\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),x_{\gamma_1},x_{\gamma_2}\right)}\left(\det_{\sigma_S}(r+m's_1s_2)\right).$$

Finally, since Θ satisfies LPR2, the following diagram commutes.

$$\begin{array}{ccc} B\{Y\cup\{t\}\} & \xrightarrow{t\mapsto s_1s_2} & B\{Y\cup\{s_1,s_2\}\} \\ \big\downarrow{\scriptstyle \det_{\sigma_T}} & & \big\downarrow{\scriptstyle \det_{\sigma_S}} \\ \mathcal{O}(\mathrm{GL}_{d,C}^{n+1})^{\mathrm{GL}_d}\otimes_C B & & \mathcal{O}(\mathrm{GL}_{d,C}^{n+2})^{\mathrm{GL}_d}\otimes_C B \\ & \searrow_{\Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),\gamma_1\gamma_2\right)}} & \big\downarrow{\scriptstyle \Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),\gamma_1,\gamma_2\right)}} \\ & & B \end{array}$$

Since $r + m's_1s_2$ is the image under $B\{Y \cup \{t\}\} \to B\{Y \cup \{s_1, s_2\}\}$ of $r + m't$,

$$\begin{aligned} D_B(p) &= \Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),x_{\gamma_1},x_{\gamma_2}\right)} \left(\det_{\sigma_S}(r + m's_1s_2)\right) \\ &= \Theta_{\left(\sigma_Y^{-1}(1),\ldots,\sigma_Y^{-1}(n),x_{\gamma_1\gamma_2}\right)} \left(\det_{\sigma_T}(r + m't).\right) \\ &= D_B(q). \end{aligned}$$

$\square$

Chapter 5

From det to LPR via $\mathbb{Q}$

In this chapter, we will only work with the case $C = \mathbb{Z}$. We drop the subscripts on our functors, writing $\mathrm{LPR}^1(X, d)$ for the functor $\mathrm{LPR}^1_{\mathbb{Z}}(X, d)$, $\mathrm{LPR}(\Gamma, d)$ for the functor $\mathrm{LPR}_{\mathbb{Z}}(\Gamma, d)$, and $\det(R, d)$ for the functor $\det_{\mathbb{Z}}(R, d)$. Note that, for the determinant functor, since we are only interested in the rings $R = C\{X\}$ and $R = C[\Gamma]$, varying the base ring C provides no additional information. Indeed, for any base ring C and $A \in \mathcal{C}_C$, since $C[\Gamma] \otimes_C A = A[\Gamma] = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} A$ we have $\det_C(C[\Gamma], d)(A) = \det_{\mathbb{Z}}(\mathbb{Z}[\Gamma], d)(A)$.

In this chapter we construct, for any group Γ and any dimension d, an inverse to the natural map $\mathrm{LPR}(\Gamma, d) \to \det(\mathbb{Z}[\Gamma], d)$ of Theorem 4.0.1 (ii). In particular, we construct an LPR $\Theta \in \mathrm{LPR}(\Gamma, d)\left(\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}}\right)$ corresponding to the universal determinant $D^{\mathrm{univ}} : \mathbb{Z}[\Gamma] \to \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}}$. Thus, the problem is to define a map from $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d}$, which is difficult because although we know, by Donkin ([5]), generators for $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d}$, we do not know relations. However, we do know, by Procesi ([10]), generators and relations for $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d}$. This suggests first constructing a family of maps from $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d}$ then restricting these maps to $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d}$. The difficulty here is that $\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}}$ is not in general torsion free, so a map $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d} \to C(\Gamma, \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}} \otimes \mathbb{Q})$ might not give rise to the required map $\mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d} \to C(\Gamma, \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}})$. Thus, instead, following Chenevier ([3]), we consider $\mathbb{Z}\{X_\Gamma\}$, the free ring over X_Γ, the underlying set of Γ. Like Chenevier, we use Vaccarino's ([16]) result that $\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X_\Gamma\})\right)^{\mathrm{ab}}$ is a free $\mathbb{Z}$-module.

So, let X be a set. As in [3, 1.15], let $F_X(d)$ be the polynomial ring

$$F_X(d) = \mathbb{Z}[x_{i,j} : x \in X,\ 1 \leq i, j \leq n]$$

and let ρ^{univ} be the *generic matrices representation*

$$\rho^{\text{univ}} : \mathbb{Z}\{X\} \to M_d(F_X(d))$$

given by $x \mapsto (x_{i,j})_{i,j}$. Let $E_X(d) \subset F_X(d)$ be the subring generated by the coefficients $\Lambda_i(\omega)$ of the characteristic polynomials of the $\rho^{\text{univ}}(\omega)$, $\omega \in \mathbb{Z}\{X\}$. From [16, 6.1], we know that the rings $F_X(d)$ and $\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\})\right)^{\text{ab}}$ are canonically isomorphic. In the following, it will be convenient to identify these rings, and we will do so.

Theorem 5.0.1.

(i) There exists a natural transformation $\det(\mathbb{Z}\{X\}, d) \to \text{LPR}^1(X, d)$ *inverse to the map in Theorem 4.0.1(i).*

(ii) There exists a natural transformation $\det(\mathbb{Z}[\Gamma], d) \to \text{LPR}(\Gamma, d)$ *inverse to the map in Theorem 4.0.1(ii).*

Proof of Theorem 5.0.1 (i). First, we construct a map

$$\Theta^1_{n,\mathbb{Q}} : \mathcal{O}(\text{GL}^n_{d,\mathbb{Q}})^{\text{GL}_d} \to C(X^n, E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Following Procesi ([10, 4]), let T be the formal polynomial ring (with coefficients in $\mathbb{Q}$) generated by the symbols $\text{Tr}(X_{i_1}X_{i_2}...X_{i_m})$ where $m \in \mathbb{N}$ and $X_{i_j} \in \{X_1, ..., X_n\}$ for all $j = 1, ..., m$, quotiented by the relation $\text{Tr}(XY) = \text{Tr}(YX)$ for any noncommutative monomials X, Y in $X_1, ..., X_n$.

Remark 5.0.2. It follows that for any noncommutative monomial M, if N is a cyclic permutation of M, then $\text{Tr}(M) = \text{Tr}(N)$.

Let $\pi : T \to \mathcal{O}(\text{GL}^n_{d,\mathbb{Q}})^{\text{GL}_d}$ be the map given by

$$\pi\left(\text{Tr}(X_{i_1}...X_{i_m})\right)(A_1, ..., A_n) = \text{tr}(A_{i_1}...A_{i_m}).$$

Let $\Theta^T_{n,\mathbb{Q}} : T \to C(X^n, E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q})$ be the map given by

$$\Theta^T_{n,\mathbb{Q}}(\text{Tr}(X_{i_1}X_{i_2}...X_{i_m}))(x_1, ..., x_n) = \text{tr}(\rho^{\text{univ}}(x_{i_1}...x_{i_m}))$$

where $(x_1, ..., x_n) \in X^n$. Then π and $\Theta^T_{n,\mathbb{Q}}$ are well defined since for any matrices a and b over a commutative ring, $\text{tr}(ab) = \text{tr}(ba)$. Furthermore, [10, Theorem 1.3] says that π is surjective. To show that there exists a map $\Theta^1_{n,\mathbb{Q}}$ such that the following diagram commutes,

$$\begin{array}{ccc} T & \xrightarrow{\Theta^T_{n,\mathbb{Q}}} & \\ {\scriptstyle\pi}\downarrow & \searrow & \\ \mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d} & \xrightarrow[\Theta^1_{n,\mathbb{Q}}]{\cdots\cdots} & C(X^n, E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q}) \end{array}$$

we show that $\ker \pi \subseteq \ker \Theta^T_{n,\mathbb{Q}}$. First we establish some notation, following Rouquier ([13, 2]). Given a permutation $\sigma \in \mathfrak{S}_m$, we decompose σ as a product of disjoint cycles, including those cycles of length 1: $\sigma = (i_1...i_k)(j_1...j_h)...(t_1...t_l)$. Define

$$(\sigma \cdot \mathrm{Tr})(X_1, ..., X_m) = \mathrm{Tr}(X_{i_1}...X_{i_k})\mathrm{Tr}(X_{j_1}...X_{j_h})...\mathrm{Tr}(X_{t_1}...X_{t_l}).$$

By Remark 5.0.2, this is well-defined.

By [10, 4.3(b) and 4.5], the ideal $\ker \pi$ is generated by the elements

$$F_{M_1,...,M_{d+1}} = \sum_{\sigma \in \mathfrak{S}_{d+1}} \mathrm{sgn}(\sigma)(\sigma \cdot \mathrm{Tr})(M_1, ..., M_{d+1})$$

where the M_i run over all possible monomials in $X_1, ..., X_n$. Then $\Theta^T_{n,\mathbb{Q}}(F_{M_1,...,M_{d+1}})$ is the function

$$\begin{aligned}(x_1, ..., x_n) \mapsto & S_{d+1}(\mathrm{tr})(\rho^{\mathrm{univ}}(m_1), ..., \rho^{\mathrm{univ}}(m_{d+1})) \\ := & \sum_{\sigma \in \mathfrak{S}_{d+1}} \mathrm{sgn}(\sigma)(\sigma \cdot \mathrm{tr})(\rho^{\mathrm{univ}}(m_1), ..., \rho^{\mathrm{univ}}(m_{d+1}))\end{aligned} \tag{5.0.1}$$

where $m_i \in \mathbb{Z}\{X\}$ is the monomial $M_i(x_1, ..., x_n)$.

The map $\mathrm{tr} : M_n(F_X(d)) \to F_X(d)$ satisfies the d-dimensional pseudocharacter identity $S_{d+1}(\mathrm{tr})(y_1, ...y_d) = 0$ (see [6, 3.21]), so $\Theta^T_{n,\mathbb{Q}}(F_{M_1,...,M_{d+1}}) = 0$. Thus $\ker \pi \subseteq \ker \Theta^T_{n,\mathbb{Q}}$, so there exists a map $\Theta^1_{n,\mathbb{Q}} : \mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Q}})^{\mathrm{GL}_d} \to C(X^n, E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q})$ such that $\Theta^T_{n,\mathbb{Q}} = \Theta^1_{n,\mathbb{Q}} \circ \pi$. Since π is surjective, this uniquely determines $\Theta^1_{n,\mathbb{Q}}$.

Remark 5.0.3. Since $\Theta^1_{n,\mathbb{Q}}$ is induced by the generic matrices representation ρ^{univ}, the family $(\Theta^1_{n,\mathbb{Q}})_{n\geq 1}$ is compatible with any map $\zeta : [n] \to [m]$ as in (LPR1). Thus $(\Theta^1_{n,\mathbb{Q}})_{n\geq 1} \in \mathrm{LPR}^1_{\mathbb{Q}}(X, d)(E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q})$.

The next step is to show that $\Theta^1_{n,\mathbb{Q}}$ restricts to a map

$$\Theta^1_n : \mathcal{O}(\mathrm{GL}^n_{d,\mathbb{Z}})^{\mathrm{GL}_d} \to C(X^n, E_X(d))$$

making the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}(\mathrm{GL}_{d,\mathbb{Z}}^n)^{\mathrm{GL}_d} & \hookrightarrow & \mathcal{O}(\mathrm{GL}_{d,\mathbb{Q}}^n)^{\mathrm{GL}_d} \\ {\scriptstyle \Theta_n^1} \downarrow & \searrow & \downarrow {\scriptstyle \Theta_{n,\mathbb{Q}}^1} \\ C(X^n, E_X(d)) & \hookrightarrow & C(X^n, E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q}) \end{array}$$

Donkin ([5, 3.1]) tells us that $\mathcal{O}(\mathrm{GL}_{d,\mathbb{Z}}^n)^{\mathrm{GL}_d}$ is generated by the maps of the form

$$\Lambda_{k,n,(i_1,\ldots,i_r)} : (A_1, ..., A_n) \mapsto \Lambda_k(A_{i_1} ... A_{i_r})$$

where $1 \leq k \leq d$, $r \in \mathbb{N}$, $i_1, ..., i_r \in [n]$, where Λ_k are the coefficients of the characteristic polynomial coming from the usual matrix determinant (defined for general determinants in 2.0.4). So, it remains to check that

$$\Theta_{n,\mathbb{Q}}^1(\Lambda_{k,n,(i_1,\ldots,i_r)})(x_1, ..., x_n) \in E_X(d).$$

(A priori, this is merely an element of $E_X(d) \otimes_{\mathbb{Z}} \mathbb{Q}$.)

We know that for any matrix $m \in M_d(F_X(d))$, $\Lambda_k(m) \in E_X(d)$ (by definition of $E_X(d)$). In particular, for any $x_1, ..., x_n \in X$, $\Lambda_k(\rho^{\mathrm{univ}}(x_{i_1} ... x_{i_r})) \in E_X(d)$.

[3, 1.11] shows that the Newton identities hold for the Λ_i induced by any d-dimensional determinant D. In particular, $\Lambda_k(A) = f_{k,d}(\mathrm{tr}(A), ..., \mathrm{tr}(A^d))$ for some $f_{k,d} \in \mathbb{Q}[Y_1, ..., Y_d]$. Since

$$\Theta_{n,\mathbb{Q}}^1((A_1, ..., A_n) \mapsto \mathrm{tr}((A_{i_1} ... A_{i_r})^j)(x_1, ..., x_n) = \mathrm{tr}\left((\rho^{\mathrm{univ}}(x_{i_1} ... x_{i_r}))^j\right)$$

and $\Theta_{n,\mathbb{Q}}^1$ is a homomorphism of algebras,

$$\begin{aligned} &\Theta_{n,\mathbb{Q}}^1\left(\Lambda_{k,n,(i_1,\ldots,i_r)}\right)(x_1, ..., x_n) \\ &\qquad = f_{k,d}\left(\mathrm{tr}\left((\rho^{\mathrm{univ}}(x_{i_1} ... x_{i_r}))^1\right), ..., \mathrm{tr}\left((\rho^{\mathrm{univ}}(x_{i_1} ... x_{i_r}))^d\right)\right) \\ &\qquad\qquad = \Lambda_k(\rho^{\mathrm{univ}}(x_{i_1} ... x_{i_r})) \in E_X(d) \end{aligned}$$

as required.

Then, having identified $E_X(d)$ and $\left(\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\}\right)^{\mathrm{ab}}$, we have

$$\Theta^1 = (\Theta_n^1)_{n \geq 1} \in \mathrm{LPR}^1(X, d)(\left(\Gamma_{\mathbb{Z}}^d(\mathbb{Z}\{X\}\right)^{\mathrm{ab}}).$$

To complete the proof of Theorem 5.0.1(i), we show that the natural transformation corresponding to Θ^1 is inverse to the natural transformation $\mathrm{LPR}^1(X,d) \to \det(\mathbb{Z}\{X\},d)$ given in Theorem 4.0.1(i). So, as above, let

$$\Theta^1 \in \mathrm{LPR}^1(X,d)\left(\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\}\right)^{\mathrm{ab}}\right)$$

be the LPR corresponding to the universal determinant. Let

$$D^1 \in \det(\mathbb{Z}\{X\},d)\left(\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\}\right)^{\mathrm{ab}}\right)$$

be the determinant given by the image of Θ^1 under the natural transformation $\mathrm{LPR}^1(X,d) \to \det(\mathbb{Z}\{X\},d)$. It remains to check that D^1 is the universal determinant.

Using the notation of chapter 3, let $p \in \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\}\right)^{\mathrm{ab}}\{X\}$ and let $Y \subseteq X$ be a finite set such that $p = p(y : y \in Y)$. Choose a bijection $\sigma : Y \to [m]$. Then

$$D^1(p(x_{\sigma(y)} : y \in Y)) = \Theta^1\left(f_{p,\sigma_Y}\right)(x_1, ..., x_m).$$

By the definition of Θ^1,

$$\Theta^1\left(f_{p,\sigma_Y}\right)(x_1, ..., x_m) = \det\left(\rho^{\mathrm{univ}}(p(x_{\sigma(y)} : y \in Y))\right)$$

as required.

We write $R_{\mathrm{LPR}^1}(X,d)$ for $R_{\mathrm{LPR}^1_{\mathbb{Z}}}(X,\mathrm{GL}_d)$[1]. Let

$$D^1 \in \det(\mathbb{Z}\{X\},d)(R_{\mathrm{LPR}^1}(X,d))$$

be the determinant corresponding to the universal LPR. Now let

$$\Theta^1 \in \mathrm{LPR}^1(X,d)(R_{\mathrm{LPR}^1}(X,d))$$

be the LPR given by the image of D^1 under the natural transformation $\det(\mathbb{Z}\{X\},d) \to \mathrm{LPR}^1(X,d)$. We check that Θ^1 is the universal LPR.

We have

$$D^1 = \varphi \circ \det \circ \rho^{\mathrm{univ}}$$

for some unique ring homomorphism $\varphi : E_X(d) \to R_{\mathrm{LPR}^1}(X,d)$. It suffices to check Θ^1 on generators

[1] $R_{\mathrm{LPR}^1_C}(X,G)$ is defined in 3.0.6

of $\mathcal{O}(\mathrm{GL}_{d,\mathbb{Z}}^n)^{\mathrm{GL}_d}$. By definition of Θ^1,

$$\Theta^1(\Lambda_{i,n,(i_1,...,i_r)})(x_1,...,x_n) = \varphi(\Lambda_i(\rho^{univ}(x_{i_1}....x_{i_r}))).$$

So we have to determine φ on these elements of $E_X(d)$. (Notation for elements of $R_{\mathrm{LPR}^1}(X,d)$ as in 3.0.6.) We have

$$D^1(T - x_{i_1}....x_{i_r}) = \overline{((g_1,...,g_n) \mapsto \det(T - g_{i_1}...g_{i_r}))_{n,(x_1,...,x_n)}}.$$

Equating the coefficients of T^{d-k} yields

$$\varphi(\Lambda_k(\rho^{\mathrm{univ}}(x_{i_1}....x_{i_r}))) = \overline{(\Lambda_{k,n,(i_1,....,i_r)})_{n,(x_1,..,x_n)}}$$

as required. □

Proof of Theorem 5.0.1 (ii). Let X be the underlying set of Γ and let π be the natural map $\mathbb{Z}\{X\} \to \mathbb{Z}[\Gamma]$. As in Chenevier 1.15, the main theorem of Vaccarino's paper, [16], tells us that if $D^{\mathrm{univ}} : \mathbb{Z}[\Gamma] \to \left(\Gamma_{\mathbb{Z}}^d(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}}$ is the universal determinant, then there exists a unique ring homomorphism

$$\varphi_\Gamma : E_X(d) \to \left(\Gamma_{\mathbb{Z}}^d(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}}$$

such that

$$\begin{array}{ccccc} \mathbb{Z}\{X\} & \xrightarrow{\pi} & \mathbb{Z}[\Gamma] & \xrightarrow{D^{\mathrm{univ}}} & \left(\Gamma_{\mathbb{Z}}^d(\mathbb{Z}[\Gamma])\right)^{\mathrm{ab}} \\ & \searrow^{\rho^{\mathrm{univ}}} & & & \uparrow{\scriptstyle \varphi_\Gamma} \\ & & M_d(F_X(d)) & \xrightarrow{\det} & E_X(d) \end{array}$$

commutes, and moreover $\varphi_\Gamma \circ \det \circ \rho^{\mathrm{univ}} = D^{\mathrm{univ}} \circ \pi$ is an equality of $\mathbb{Z}$-polynomial laws.

Let $\Theta_n := \varphi_\Gamma \circ \Theta_n^1$. It remains to check the concatenation condition, (LPR2): for $f \in \mathcal{O}(\mathrm{GL}_{d,\mathbb{Z}}^n)^{\mathrm{GL}_d}$ and $(\gamma_1,...,\gamma_{n+1}) \in \Gamma^{n+1}$,

$$\Theta_n(f)(\gamma_1,...,\gamma_{n-1},\gamma_n\gamma_{n+1}) = \Theta_{n+1}(\hat{f})(\gamma_1,...,\gamma_{n+1}).$$

It is sufficient to check this holds for f a generator of $\mathcal{O}(\mathrm{GL}_{d,\mathbb{Z}}^n)^{\mathrm{GL}_d}$. In particular, we show that

$$\varphi_\Gamma \circ \Lambda_i(\rho^{\mathrm{univ}}(y_{i_1}...y_{i_k})) = \varphi_\Gamma \circ \Lambda_i(\rho^{\mathrm{univ}}(z_{i_1}...z_{i_k}))$$

where $y_{i_j} = \begin{cases} x_{\gamma_{i_j}} & \text{if } i_j \neq n \\ x_{\gamma_n \gamma_{n+1}} & \text{if } i_j = n \end{cases}$ and $z_{i_j} = \begin{cases} x_{\gamma_{i_j}} & \text{if } i_j \neq n \\ x_{\gamma_n} x_{\gamma_{n+1}} & \text{if } i_j = n \end{cases}$
and x_γ is the element of X corresponding to $\gamma \in \Gamma$.

Since $\varphi_\Gamma \circ \det \circ \rho^{\text{univ}} = D^{\text{univ}} \circ \pi$ is an equality of $\mathbb{Z}$-polynomial laws, the below diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}\{X\}[t] & \xrightarrow{\det\circ\rho^{\text{univ}}} & E_X(d)[t] \\ {\scriptstyle \pi\otimes \text{id}_{\mathbb{Z}[t]}} \downarrow & & \downarrow {\scriptstyle \varphi_\Gamma \otimes \text{id}_{\mathbb{Z}[t]}} \\ \mathbb{Z}[\Gamma][t] & \xrightarrow{D^{\text{univ}}} & \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\text{ab}}[t] \end{array}$$

So, since $(\pi \circ \text{id}_{\mathbb{Z}[t]})(t - y_{i_1}...y_{i_k}) = (\pi \circ \text{id}_{\mathbb{Z}[t]})(t - z_{i_1}...z_{i_k})$, we have

$$\sum_{i=0}^{d} (-1)^i \varphi_\Gamma\left(\Lambda_i(y_{i_1}...y_{i_k})\right) t^{d-i} = \sum_{i=0}^{d} (-1)^i \varphi_\Gamma\left(\Lambda_i(z_{i_1}...z_{i_k})\right) t^{d-i}$$

so $\varphi_\Gamma\left(\Lambda_i(y_{i_1}...y_{i_k})\right) = \varphi_\Gamma\left(\Lambda_i(z_{i_1}...z_{i_k})\right)$, as required. Thus the family of maps

$$\Theta_n : \mathcal{O}(\text{GL}^n_{d,\mathbb{Z}})^{\text{GL}_d} \to C\left(\Gamma^n, \left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\text{ab}}\right)$$

gives us a $\left(\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[\Gamma])\right)^{\text{ab}}$-valued LPR.

Since in 5.0.1(i) we proved that the natural transformation $\det(\mathbb{Z}\{X_\Gamma\}, d) \to \text{LPR}^1(X_\Gamma, d)$ is inverse to the natural map $\text{LPR}^1(X_\Gamma, d) \to \det(\mathbb{Z}\{X_\Gamma\}, d)$ of Theorem 4.0.1 (i), by diagram 4.0.1 this map is inverse to the natural map $\text{LPR}(\Gamma, d) \to \det(\mathbb{Z}[\Gamma], d)$ of Theorem 4.0.1 (ii).

□

Chapter 6

Character Variety

Throughout this chapter, let X denote a set, let Γ denote a group, let C be a commutative, unital base ring and let G denote an affine group scheme over C.

Definition 6.0.1. Let $\mathrm{Rep}_C(\Gamma, G) : \mathcal{C}_C \to \mathbf{Set}$ be the covariant functor associating to any $A \in \mathcal{C}_C$ the set $\mathrm{Hom}_{\mathrm{Group}}(\Gamma, G(A))$.

Then $\mathrm{Rep}_C(X, G)$ is representable by the ring $\mathcal{O}(G^X)$. If X is the underlying set of Γ, then $\mathrm{Rep}_C(\Gamma, G)$ is a subfunctor of $\mathrm{Rep}_C(X, G)$ representable by the ring $\mathcal{O}(G^X)/I(\Gamma, G)$, where $I(\Gamma, G)$ is the set of functions in $\mathcal{O}(G^X)$ vanishing on all representations $\rho : \Gamma \to G(A)$ for all $A \in \mathcal{C}_C$.

Definition 6.0.2. The C-schemes $\mathrm{Char}_C(X, G) = \mathrm{Spec}\left(\mathcal{O}\left(G^X\right)^G\right)$ and $\mathrm{Char}_C(\Gamma, G) = \mathrm{Spec}\left(\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G\right)$ are the *character varieties* for X and Γ respectively.

The scheme $\mathrm{Char}_C(\Gamma, G)$ is the GIT quotient (in the sense of Mumford) of $\mathrm{Rep}_C(\Gamma, G)$

If X is the underlying set of Γ, then there is a natural transformation $\mathrm{Char}_C(\Gamma, G) \to \mathrm{Char}_C(X, G)$ (which has no reason to be a monomorphism in general). It fits into the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Rep}_C(\Gamma, G) & \hookrightarrow & \mathrm{Rep}_C(X, G) \\ \downarrow & & \downarrow \\ \mathrm{Char}_C(\Gamma, G) & \longrightarrow & \mathrm{Char}_C(X, G) \end{array}$$

Example 6.0.3. A map of sets $\rho : X \to G(A)$ gives rise to an element $\overline{\rho} \in \mathrm{Char}_C(X, G)(A)$ via the map $\mathrm{Rep}_C(X, G) \to \mathrm{Char}_C(X, G)$. Similarly, a group homomorphism $\rho : \Gamma \to G(A)$ gives rise to an element $\overline{\rho} \in \mathrm{Char}_C(\Gamma, G)(A)$.

Before stating the theorem, we recall Alper's definition of *adequate homeomorphism* ([1, Definition 3.3.1]).

Definition 6.0.4. A map of schemes $f : X \to Y$ is an *adequate homeomorphism* if f is an integral, universal homeomorphism which is a local isomorphism at all points with a residue field of characteristic zero. A ring homomorphism $A \to B$ is an *adequate homeomorphism* if $\operatorname{Spec} A \to \operatorname{Spec} B$ is.

Theorem 6.0.5.

(i) There exists a canonical isomorphism

$$\mathrm{Char}_C(X, G) \to \mathrm{LPR}^1_C(X, G)$$

natural in X and G, and such that for all $A \in \mathcal{C}_C$ and all maps of sets $\rho : X \to G(A)$, this map takes $\overline{\rho}$ to Θ_ρ.

(ii) There exists a canonical transformation

$$\mathrm{Char}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, G)$$

natural in Γ and G, and such that if X is the underlying set of Γ, then the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Char}_C(\Gamma, G) & \longrightarrow & \mathrm{LPR}_C(\Gamma, G) \\ \downarrow & & \downarrow \\ \mathrm{Char}_C(X, G) & \xrightarrow{\sim} & \mathrm{LPR}^1_C(X, G) \end{array}$$

(iii) Let C be a field of characteristic zero. If G is a reductive group over C, then the natural transformation given in (ii) is an isomorphism.

(iv) Let $C = \mathbb{Z}$. If G is a reductive group over $\mathbb{Z}$, then the natural transformation given in (ii) is an adequate homeomorphism.

Proof of Theorem 6.0.5 (i). First we establish some notation. Every

$$x = (x_1, ..., x_n) \in X^n$$

gives rise to a C-algebra homomorphism

$$\psi_x : \mathcal{O}(G^n)^G \to \mathcal{O}\left(G^X\right)^G$$

via

$$G^X \to G^n$$

$$(g_y)_{y\in X} \mapsto (g_{x_1}, ..., g_{x_n}).$$

The map ψ_x is injective iff $x_i \neq x_j$ for any $i \neq j$. Note that for any map $\sigma : [m] \to [n]$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{O}(G^m)^G & \xrightarrow{f\mapsto f^\sigma} & \mathcal{O}(G^n)^G \\ & \searrow^{\psi_{\left(x_{\sigma(1)},\dots,x_{\sigma(m)}\right)}} & \downarrow \psi_{(x_1,\dots,x_n)} \\ & & \mathcal{O}\left(G^X\right)^G \end{array} \tag{6.0.1}$$

Take x such that $x_i \neq x_j$ whenever $i \neq j$. For $f \in \psi_x(\mathcal{O}(G^n)^G)$, write f_x for the unique element of $\mathcal{O}(G^n)^G$ such that $\psi_x(f_x) = f$. Given $A \in \mathcal{C}_C$ and $\Theta^1 \in \mathrm{LPR}^1_C(X, G)(A)$, we construct

$$\varphi \in \mathrm{Char}_C(X, G)(A).$$

For $f \in \mathcal{O}\left(G^X\right)^G$, there exists a finite subset $Y \subseteq X$ such that

$$f = f_Y \otimes 1_{\mathcal{O}(G^{X\setminus Y})^G}$$

where $f_Y \in \mathcal{O}(G^Y)^G$. Choose an ordering $\{y_1, ..., y_n\} = Y$. Define

$$\varphi(f) := \Theta^1_n(f_{(y_1,...,y_n)})(y_1, ..., y_n).$$

First, we check that φ is well-defined. That is, let $Z \subseteq X$ be another finite subset such that $f = f_Z \otimes 1_{\mathcal{O}(G^{X\setminus Z})^G}$ where $f_Z \in \mathcal{O}(G^Z)^G$. Let $Z = \{z_1, ..., z_m\}$. Without loss of generality, $Y \subseteq Z$. Thus, there exists $\zeta : [n] \to [m]$ such that $y_i = z_{\zeta(i)}$ for all $i \in [n]$. Since $\psi_{(y_1,...,y_n)}(f_{(y_1,...,y_n)}) = f$ and diagram (6.0.1) commutes, $\psi_{(z_1,...,z_m)}(f^\zeta_{(y_1,...,y_n)}) = f$. Since $\psi_{(z_1,...,z_m)}$ is injective, $f^\zeta_{(y_1,...,y_n)} = f_{(z_1,...,z_m)}$. Since $(y_1, ..., y_n) = (z_{\zeta(1)}, ..., z_{\zeta(n)})$, by (LPR1),

$$\Theta^1_n(f_{(y_1,...,y_n)})(y_1, ..., y_n) = \Theta^1_m(f_{(z_1,...,z_m)})(z_1, ..., z_m)$$

as required.

Next, we check that φ defines a C-algebra homomorphism. Let $f, g \in \mathcal{O}\left(G^X\right)^G$ and $c, d \in C$. Then there exists a finite subset $Y \subseteq X$ such that $f = f_Y \otimes 1_{\mathcal{O}(G^{X\setminus Y})^G}$ and $g = g_Y \otimes 1_{\mathcal{O}(G^{X\setminus Y})^G}$ for some $f_Y, g_Y \in \mathcal{O}(G^Y)^G$. Choose an ordering $\{y_1, ..., y_n\} = Y$. Then since

$$\psi^{-1}_{(y_1,\ldots,y_n)} : \psi_{(y_1,\ldots,y_n)}\left(\mathcal{O}(G^n)^G\right) \to \mathcal{O}(G^n)^G$$

and Θ^1_n are C-algebra morphisms, $\varphi(cf + dg) = c\varphi(f) + d\varphi(g)$ and $\varphi(fg) = \varphi(f)\varphi(g)$, as required.

Now we give the map in the opposite direction, by constructing an element of $\mathrm{LPR}^1_C(X, G)(\mathcal{O}\left(G^X\right)^G)$ corresponding to the universal element of

$$\mathrm{Char}_C(X, G)(\mathcal{O}\left(G^X\right)^G).$$

For $f \in \mathcal{O}(G^n)^G$ and $x \in X^n$, we define

$$\Theta^1_n(f)(x) := \psi_x(f).$$

Thus Θ^1_n is a C-algebra homomorphism. (LPR1) follows immediately from the fact that diagram (6.0.1) commutes.

Clearly the natural transformations constructed above are inverses. □

Proof of Theorem 6.0.5 (ii). Let X be the underlying set of a group Γ. We write x_γ for the element of X corresponding to γ. Consider the universal element of $\mathrm{Char}_C(\Gamma, G)\left(\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G\right)$. The image of this universal element under the map $\mathrm{Char}_C(\Gamma, G) \to \mathrm{Char}_C(X, G)$ gives rise to

$$\Theta^1 \in \mathrm{LPR}^1_C(X, G)\left(\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G\right)$$

as above. To show that $\Theta^1 \in \mathrm{LPR}_C(\Gamma, G)\left(\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G\right)$, we need only verify that Θ^1 satisfies (LPR2). It suffices to show that for any γ_1, ..., $\gamma_{n+1} \in \Gamma$ the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{O}(G^n) & \xrightarrow{f \mapsto \hat{f}} & \mathcal{O}(G^{n+1}) \\
 & \searrow^{\psi_{\left(x_{\gamma_1},\ldots,x_{\gamma_n\gamma_{n+1}}\right)}} & \downarrow \psi_{\left(x_{\gamma_1},\ldots,x_{\gamma_{n+1}}\right)} \\
 & & \mathcal{O}(G^X)/I(\Gamma, G)
\end{array}$$

So, let $f \in \mathcal{O}(G^n)$, $A \in \mathcal{C}_C$ and $\rho \in \mathrm{Rep}_C(\Gamma, G)(A)$. We have (abusing notation by also writing ρ

for the corresponding element of $\mathrm{Rep}_C(X,G)(A)$)

$$\begin{aligned}\psi_{\left(x_{\gamma_1},\dots,x_{\gamma_n\gamma_{n+1}}\right)}(f)(\rho) &= f\left(\rho(x_{\gamma_1}),\dots,\rho(x_{\gamma_n\gamma_{n+1}})\right)\\ &= f\left(\rho(x_{\gamma_1}),\dots,\rho(x_{\gamma_n})\rho(x_{\gamma_{n+1}})\right)\\ &= \psi_{\left(x_{\gamma_1},\dots,x_{\gamma_{n+1}}\right)}(\hat{f})(\rho)\end{aligned}$$

as required. $\square$

Before we prove Theorem 6.0.5 (iii), we first show the following lemma, describing, in general, the ideal $I(\Gamma,G)$.

Lemma 6.0.6. *Every element of $I(\Gamma,G)$ is a sum of elements of the form*

$$\psi_{\left(x_{\gamma_1},\dots,\,x_{\gamma_{n-1}},\,x_{\gamma_n\gamma_{n+1}}\right)}(f)-\psi_{\left(x_{\gamma_1},\dots,\,x_{\gamma_{n+1}}\right)}(\hat{f})$$

for some $f\in\mathcal{O}(G^n)$ and some $\gamma_1,\dots,\gamma_{n+1}\in\Gamma$.

Proof. The first step is to give generators for $I(\Gamma,G)$ as an ideal. Consider

$$J:=\langle\psi_{(x_{\gamma_1\gamma_2})}(f_1)-\psi_{(x_{\gamma_1},x_{\gamma_2})}(\hat{f}_1)\mid f_1\in\mathcal{O}(G),\ \gamma_1,\gamma_2\in\Gamma\rangle.$$

We will show that $J=I(\Gamma,G)$. Clearly $\psi_{(x_{\gamma_1\gamma_2})}(f_1)-\psi_{(x_{\gamma_1},\,x_{\gamma_2})}(\hat{f}_1)$ vanishes on any group homomorphism ρ, so $J\subseteq I(\Gamma,G)$.

Conversely, take $\rho\in G^X(\mathcal{O}(G^X)/J)$ given by the quotient map $\mathcal{O}(G^X)\to\mathcal{O}(G^X)/J$. Then for all $h\in J$, $h(\rho)=0$. So for all $f_1\in\mathcal{O}(G)$ and $\gamma_1,\gamma_2\in\Gamma$, $f_1(\rho(x_{\gamma_1\gamma_2}))=f_1(\rho(x_{\gamma_1})\rho(x_{\gamma_2}))$. Thus ρ is a group homomorphism. In particular, $\mathcal{O}(G^X)\to\mathcal{O}(G^X)/J$ factors through $\mathcal{O}(G^X)\to\mathcal{O}(G^X)/I(\Gamma,G)$, so $I(\Gamma,G)\subseteq J$, as required.

Thus every element of $I(\Gamma,G)$ is a sum of elements of the form

$$h=f_X\cdot(\psi_{(x_{\delta_1\delta_2})}(f_1)-\psi_{(x_{\delta_1},\,x_{\delta_2})}(\hat{f}_1))$$

where $f_X\in\mathcal{O}(G^X)$, $f_1\in\mathcal{O}(G)$ and $\delta_1,\delta_2\in\Gamma$. There exists $\gamma_1,\dots,\gamma_m\in\Gamma$ and $f_m\in\mathcal{O}(G^m)$ such that $f_X=\psi_{(x_{\gamma_1},\dots,x_{\gamma_m})}(f_m)$. Define, for $l\geq k$, maps

$$\eta_{k,l},\ \xi_{k,l}:\mathcal{O}(G^k)\to\mathcal{O}(G^l)$$

via $G^l \to G^k$, $(g_1, ..., g_l) \mapsto (g_{l-k+1}, ..., g_l)$ and $(g_1, ..., g_l) \mapsto (g_1, ..., g_k)$ respectively. (That is, for $f_k \in \mathcal{O}(G^k)$, the function $\eta_{k,l}(f_k)$ depends only on the last k coordinates and the function $\xi_{k,l}(f_k)$ depends only on the first k coordinates.) Then we have

$$h = \psi_{(x_{\gamma_1},..., x_{\gamma_m}, x_{\delta_1 \delta_2})} (\xi_{m,m+1}(f_m)\eta_{1,m+1}(f_1))$$
$$- \psi_{(x_{\gamma_1},..., x_{\gamma_m}, x_{\delta_1}, x_{\delta_2})} \left(\xi_{m,m+2}(f_m)\eta_{2,m+2}(\hat{f}_1)\right).$$

Define

$$f = \xi_{m,m+1}(f_m)\eta_{1,m+1}(f_1).$$

Observe that

$$\hat{f} = \xi_{m,m+2}(f_m)\eta_{2,m+2}(\hat{f}_1),$$

so

$$h = \psi_{(x_{\gamma_1},..., x_{\gamma_m}, x_{\delta_1 \delta_2})}(f) - \psi_{(x_{\gamma_1},..., x_{\gamma_m}, x_{\delta_1}, x_{\delta_2})}(\hat{f})$$

as required. □

To prove Theorem 6.0.5 (iii) we make use of the Reynolds operator, whose properties we recall here. (See [9, Definition 1.5].)

Let C be a characteristic zero field and let G be a reductive algebraic group over C. Let V be an algebraic representation of G over C (not necessarily finite dimensional).[1]

Then there exists a C-linear map $E = E_V : V \to V$, the *Reynolds operator*, with the following properties.

1. E commutes with the action of G.
2. $E^2 = E$.
3. $Ex = x$ if and only if $x \in V^G$.
4. If $\phi : V \to W$ is a G-equivariant linear map, then $\phi \circ E_V = E_W \circ \phi$.
5. Suppose that, in addition, V is a C-algebra and the action of G on V preserves multiplication. If $x \in V^G$ and $y \in V$ then $E(xy) = xE(y)$.

Proof of Theorem 6.0.5 (iii). Since $I(\Gamma, G)$ is invariant under the action of G and C is a characteristic zero field, by *algebraic fact (3)* of [9, Theorem 1.1], $\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G$ is isomorphic to

[1] In [9, Definition 1.2], Mumford uses the terminology *dual action*.

$\mathcal{O}\left(G^X\right)^G/I(\Gamma,G)^G$. Thus, since $\left(\mathcal{O}(G^X)/I(\Gamma,G)\right)^G$ is a quotient of $\mathcal{O}\left(G^X\right)^G$, the map $\mathrm{Char}_C(\Gamma,G)\to\mathrm{Char}_C(X,G)$ is a monomorphism. Now, given $\Theta\in\mathrm{LPR}_C(\Gamma,G)(A)$, we construct $\varphi\in\mathrm{Char}_C(X,G)(A)$ as in Theorem 6.0.5(i). We will prove that $\varphi\in\mathrm{Char}_C(\Gamma,G)(A)$. Indeed, since the maps $f_n\mapsto\widehat{f_n}$ and $f_n\mapsto\psi_{(x_{\gamma_1},\dots,x_{\gamma_n})}(f_n)$ for $f_n\in\mathcal{O}(G^n)$ are G-equivariant, by property 4 they commute with the Reynolds operator. By Lemma 6.0.6, any element of $I(\Gamma,G)$ is a sum of elements of the form

$$h=\psi_{(x_{\gamma_1},\dots,\,x_{\gamma_{n-1}},\,x_{\gamma_n\gamma_{n+1}})}(f_n)-\psi_{(x_{\gamma_1},\dots,\,x_{\gamma_{n+1}})}(\hat{f}_n)$$

Thus, any element of $I(\Gamma,G)^G=E(I(\Gamma,G))$ is a sum of elements of the form

$$E(h)=\psi_{(x_{\gamma_1},\dots,\,x_{\gamma_{n-1}},\,x_{\gamma_n\gamma_{n+1}})}(E(f_n))-\psi_{(x_{\gamma_1},\dots,\,x_{\gamma_{n+1}})}(\widehat{E(f_n)})$$

Since Θ satisfies (LPR2), $\varphi(E(h))=0$. Thus $\varphi(I(\Gamma,G)^G)=0$, so $\varphi\in\mathrm{Char}_C(\Gamma,G)(A)$ as required. ∎

Proof of Theorem 6.0.5 (iv). First, we show that for any base ring C and any affine group scheme G over C, there exists a natural map

$$f_G:R_{\mathrm{LPR}_C}(\Gamma,G)\to\mathcal{O}\left(G^X\right)^G/I(\Gamma,G)^G$$

such that the following diagram commutes,

$$\begin{array}{ccc}
R_{\mathrm{LPR}^1_C}(X,G) & \xrightarrow{\sim} & \mathcal{O}\left(G^X\right)^G \\
\downarrow & & \downarrow \\
R_{\mathrm{LPR}_C}(\Gamma,G) & \xrightarrow{f_G} & \mathcal{O}\left(G^X\right)^G/I(\Gamma,G)^G \\
 & \searrow^{h_G} & \downarrow g_G \\
 & & \left(\mathcal{O}(G^X)/I(\Gamma,G)\right)^G
\end{array}$$

where $R_{\mathrm{LPR}^1_C}(X,G)\to\mathcal{O}\left(G^X\right)^G$ is the map from 6.0.5 (i) and h_G is the map from 6.0.5 (ii). Recall from Theorem 3.0.6 that $R_{\mathrm{LPR}^1_C}(X,G)$ is a quotient of the ring

$$R_X=\bigotimes_{n\in\mathbb{Z}_{\geq 1}}\bigotimes_{x\in X^n}\mathcal{O}(G^n)^G.$$

Recall that $R_{\mathrm{LPR}_C}(\Gamma, G) = R_{\mathrm{LPR}^1_C}(X, G)/J$ where J is the ideal generated by the elements[2]

$$f_{m,(\gamma_1,...,\gamma_{m-1},\gamma_m\gamma_{m+1})} - \hat{f}_{m+1,(\gamma_1,...,\gamma_{m+1})}$$

for all $m \in \mathbb{Z}_{\geq 1}$, $f \in \mathcal{O}(G^m)^G$, and $(\gamma_1, ..., \gamma_{m+1}) \in \Gamma^{m+1}$. The natural isomorphism

$$R_{\mathrm{LPR}^1_C}(X, G) \to \mathcal{O}\left(G^X\right)^G$$

takes the image of $f_{m,(\gamma_1,...,\gamma_{m-1},\gamma_m\gamma_{m+1})} - \hat{f}_{m+1,(\gamma_1,...,\gamma_{m+1})}$ in $R_{\mathrm{LPR}^1_C}(X, G)$ to

$$(g_x)_{x\in X} \mapsto f(g_{x_{\gamma_1}}, ..., g_{x_{\gamma_{m-1}}}, g_{x_{\gamma_m\gamma_{m+1}}}) - f(g_{x_{\gamma_1}}, ..., g_{x_{\gamma_{m-1}}}, g_{x_{\gamma_m}} g_{x_{\gamma_{m+1}}}).$$

This vanishes on any representation $\rho : \Gamma \to G(A)$ for any $A \in \mathcal{C}_C$, so is an element of $I(\Gamma, G)$. This proves that the map f_G exists and is surjective.

Since the composition of adequate homeomorphisms is an adequate homeomorphism, it now suffices to show that when $C = \mathbb{Z}$ and G is a reductive group over $\mathbb{Z}$, f_G and g_G are adequate homeomorphisms. First, for any base ring C and reductive group G over C, g_G is an adequate homeomorphism. Indeed, by [1, Remark 5.2.14], since G is reductive,

$$g_G : \mathcal{O}\left(G^X\right)^G / I(\Gamma, G)^G \to \left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G$$

is *universally adequate*.[3] Since $\ker(g_G) = 0$, it follows from [1, Proposition 3.3.5] that g_G is an adequate homeomorphism.

Since f_G is surjective, $f_G^* : \mathrm{Spec}(\mathcal{O}\left(G^X\right)^G / I(\Gamma, G)^G) \to \mathrm{LPR}_{\mathbb{Z}}(\Gamma, G)$ is integral, so to show that f_G is an adequate homeomorphism it remains only to show that f_G^* is a universal homeomorphism and a local isomorphism at all points of characteristic zero. To prove that f_G is a universal homeomorphism we first prove the following lemma.

Lemma 6.0.7. *Let G be a reductive group over $\mathbb{Z}$ and let k be a perfect field. Then the natural map*

$$\mathrm{Char}_{\mathbb{Z}}(\Gamma, G)(k) \to \mathrm{LPR}_{\mathbb{Z}}(\Gamma, G)(k)$$

is a bijection.

[2]See 3.0.6 for notation.
[3]See [1, Definition 3.2.1] for the definition of *universally adequate*.

To prove Lemma 6.0.7, we first prove the following result about the LPR and character variety functors under base change.

Lemma 6.0.8. *Let G be an affine group scheme over C. Let $C' \in \mathcal{C}_C$ and let $A \in \mathcal{C}_{C'}$. If either*

1. *C' is flat over C; or,*

2. *G is a reductive group over C, C' is an $\mathbb{F}_p$-algebra and A is a perfect field,*

then the natural maps

$$\chi_{LPR} : \mathrm{LPR}_{C'}(\Gamma, G_{C'})(A) \to \mathrm{LPR}_C(\Gamma, G_C)(A)$$

and

$$\chi_{Char} : \mathrm{Char}_{C'}(\Gamma, G_{C'})(A) \to \mathrm{Char}_C(\Gamma, G_C)(A)$$

are bijections.

Proof. Let G be an affine group scheme G over C and let $C' \in \mathcal{C}_C$. Consider the natural maps

$$\eta_n : \mathcal{O}({G_C}^n)^{G_C} \otimes_C C' \to \mathcal{O}({G_{C'}}^n)^{G_{C'}}$$

and

$$\eta_X : \left(\mathcal{O}(G_C^X)/I(\Gamma, G_C)\right)^{G_C} \otimes_C C' \to \left(\mathcal{O}(G_{C'}^X)/I(\Gamma, G_{C'})\right)^{G_{C'}}.$$

We have

$$\mathcal{O}({G_C}^n) \otimes_C C' = \mathcal{O}({G_{C'}}^n)$$

and we will show that

$$\left(\mathcal{O}(G_C^X)/I(\Gamma, G_C)\right) \otimes_C C' = \mathcal{O}(G_{C'}^X)/I(\Gamma, G_{C'}).$$

Indeed, write $J \lhd \mathcal{O}(G_{C'}^X)$ for the ideal generated by the image of $I(\Gamma, G)$ under $\mathcal{O}(G_C^X) \to \mathcal{O}(G_C^X) \otimes_C C' = \mathcal{O}(G_{C'}^X)$. It suffices to prove that $J = I(\Gamma, G_{C'})$.

It is clear that $J \subseteq I(\Gamma, G_{C'})$. Conversely, recall that, from the proof of Lemma 6.0.6, $I(\Gamma, G_{C'})$ is generated as an ideal of $\mathcal{O}(G_{C'}^X)$ by the set of elements of the form $\psi_{(x_{\gamma_1\gamma_2})}(f) - \psi_{(x_{\gamma_1} x_{\gamma_2})}(\widehat{f})$ for $f \in \mathcal{O}(G_{C'})$ and $\gamma_1, \gamma_2 \in \Gamma$. Since $\mathcal{O}(G_{C'}) = \mathcal{O}(G) \otimes_C C'$, $f = \sum_{i=1}^n f_i \otimes c_i$ for some n, $f_i \in \mathcal{O}(G)$

and $c_i \in C'$. Thus

$$\psi_{(x_{\gamma_1\gamma_2})}(f) - \psi_{(x_{\gamma_1}x_{\gamma_2})}(\widehat{f}) = \sum_{i=1}^{n} \Big(\psi_{(x_{\gamma_1\gamma_2})}(f_i) - \psi_{(x_{\gamma_1}x_{\gamma_2})}(\widehat{f}_i) \Big) \otimes c_i$$

Now, $\psi_{(x_{\gamma_1\gamma_2})}(f_i) - \psi_{(x_{\gamma_1}x_{\gamma_2})}(\widehat{f}_i) \in I(\Gamma, G)$ for all i. Thus $I(\Gamma, G_{C'}) \subseteq J$.

Although η_n and η_X are not in general isomorphisms, if C' is flat over C then taking $V = \mathcal{O}({G_C}^n)$ or $V = \mathcal{O}(G_C^X)/I(\Gamma, G_C)$ in [14, Lemma 2] says that they are isomorphisms. (For η_n this is remarked in [2, 4.2].)

Now we consider the case where G is a reductive group over C. We show that η_n and η_X are adequate homeomorphisms by showing that, if G acts on $V \in \mathcal{C}_C$, then $V^{G_C} \otimes_C C' \to (V \otimes_C C')^{G_{C'}}$ is an adequate homeomorphism. Let P be a polynomial ring over C and $J \lhd P$ such that $C' = P/J$. Let $R = V \otimes_C P$. Since P is flat over C, $R^G = V^{G_C} \otimes_C P$. Let $I \lhd V^{G_C} \otimes_C P$ be the ideal generated by the image of J under the natural map $P \to V^{G_C} \otimes_C P$. Then IR is the ideal generated by the image of J under the natural map $P \to V \otimes_C P$. So $R^G/I = V^{G_C} \otimes_C C'$ and $R/IR = V \otimes_C C'$. Since for G reductive, $R^G/I \to (R/IR)^G$ is an adequate homeomorphism ([1, Remark 5.2.2]), we are done.

Next we prove that if $E, F \in \mathcal{C}_C$ are $\mathbb{F}_p$-algebras and $\eta : E \to F$ is an adequate homeomorphism, then for a perfect field k, $\mathrm{Hom}_C(F, k) \to \mathrm{Hom}_C(E, k)$ is a bijection. By [1, Proposition 3.3.3], for every $f \in F$ there exists r and $e \in E$ such that $\eta(e) = f^{p^r}$. Since $x \mapsto x^{p^r}$ is an automorphism for perfect fields k, it follows that $\mathrm{Hom}_C(F, k) \to \mathrm{Hom}_C(E, k)$ is injective. Furthermore, since η is an adequate homeomorphism of $\mathbb{F}_p$-algebras, by [1, Proposition 3.3.3] $\ker(\eta)$ is locally nilpotent. Thus, since k is a domain, for any $\varphi \in \mathrm{Hom}_C(E, k)$, $\ker(\eta) \subseteq \ker(\varphi)$. It follows that $\mathrm{Hom}_C(F, k) \to \mathrm{Hom}_C(E, k)$ is surjective.

Thus, in case (1) or (2), χ_{Char} is a bijection. For χ_{LPR}, as remarked in [2, Remark 4.2], when C' is flat over C it follows from the fact that η_n is an isomorphism that χ_{LPR} is an bijection. Now consider case (2). Suppose that Θ, $\Psi \in \mathrm{LPR}_{C'}(\Gamma, G_{C'})(A)$. such that $\chi_{LPR}(\Theta) = \chi_{LPR}(\Psi)$. For all $n \in \mathbb{N}$,

$$\begin{aligned} \eta_n^* : \mathrm{Hom}_{C'}(\mathcal{O}({G_{C'}}^n)^{G_{C'}}, A) &\to \mathrm{Hom}_{C'}(\mathcal{O}({G_C}^n)^{G_C} \otimes_C C', A) \\ &= \mathrm{Hom}_C(\mathcal{O}({G_C}^n)^{G_C}, A) \end{aligned}$$

is injective, so for all $\gamma \in \Gamma^n$, $\Theta_n(-)(\gamma) = \Psi_n(-)(\gamma)$. Thus $\Theta = \Psi$ and so χ_{LPR} is injective.

To show that χ_{LPR} is surjective, let $\Theta \in \mathrm{LPR}_C(\Gamma, G_C)(A)$. For any $n \in \mathbb{N}$ η_n^* is surjective, so for all $\gamma \in \Gamma^n$, there exists $\Psi_n(-)(\gamma) \in \mathrm{Hom}_{C'}(\mathcal{O}(G_{C'}{}^n)^{G_{C'}}, A)$ such that $\eta_n^*(\Psi_n(-)(\gamma)) = \Theta_n(-)(\gamma)$. It is easy to check that the family $(\Psi_n)_{n\in\mathbb{N}}$ satisfies (LPR1) and (LPR2). $\square$

Proof of Lemma 6.0.7. Now we prove that, for a perfect field k and a reductive group G over k, the natural map

$$\mathrm{Char}_k(\Gamma, G)(k) \to \mathrm{LPR}_k(\Gamma, G)(k)$$

is a bijection. If $\mathrm{char}(k) = 0$, then this follows from Theorem 6.0.5(iii). So we may assume that $\mathrm{char}(k) = p$. First, observe that for any base ring C and reductive group G over C, h_G is universally adequate. Indeed, since g_G is an adequate homeomorphism and adequate homeomorphisms are universally adequate ([1, Proposition 3.3.5]), g_G is universally adequate. Since f_G is surjective, it is universally adequate. Since h_G is the composition of universally adequate maps, h_G is universally adequate. In particular, $R_{\mathrm{LPR}_k}(\Gamma, G)/\ker(h_G) \to \big(\mathcal{O}(G^X)/I(\Gamma, G)\big)^G$ is universally adequate, and since it is a universally adequate injection of $\mathbb{F}_p$-algebras, by ([1, Proposition 3.3.3]), it is an adequate homeomorphism. We saw above that adequate homeomorphisms of $\mathbb{F}_p$-algebras give rise to bijections on k points, so

$$\mathrm{Hom}_k(\big(\mathcal{O}(G^X)/I(\Gamma, G)\big)^G, k) \to \mathrm{Hom}_k(R_{\mathrm{LPR}_k}(\Gamma, G)/\ker(h_G), k)$$

is a bijection. So it remains to check that

$$\mathrm{Hom}_k(R_{\mathrm{LPR}_k}(\Gamma, G)/\ker(h_G), k) \to \mathrm{Hom}_k(R_{\mathrm{LPR}_k}(\Gamma, G), k)$$

is a bijection. It is clearly injective. Let $\xi : R_{\mathrm{LPR}_k}(\Gamma, G) \to k$. Let $\overline{k}$ be an algebraic closure of k. Then $k \hookrightarrow \overline{k}$ gives rise to $\overline{\xi} : R_{\mathrm{LPR}_k}(\Gamma, G) \to \overline{k}$. Since $\overline{k}$ is flat over k, by Lemma 6.0.8(1), $\mathrm{LPR}_k(\Gamma, G)(\overline{k}) = \mathrm{LPR}_{\overline{k}}(\Gamma, G_{\overline{k}})(\overline{k})$. Thus $\overline{\xi} \in \mathrm{LPR}_{\overline{k}}(\Gamma, G_{\overline{k}})(\overline{k})$, so by [7, Proposition 10.7] and [2, Theorem 4.5] there exists $\rho : \Gamma \to G(\overline{k})$ such that $\overline{\xi} = \Theta_\rho$.

$$\begin{array}{ccc} \mathrm{Rep}_{\overline{k}}(\Gamma, G_{\overline{k}})(\overline{k}) & \longrightarrow & \mathrm{Char}_{\overline{k}}(\Gamma, G_{\overline{k}})(\overline{k}) \\ & \searrow\!\!\!\!\searrow & \big\downarrow h^*_{G_{\overline{k}}} \\ & & \mathrm{LPR}_{\overline{k}}(\Gamma, G_{\overline{k}})(\overline{k}) \end{array} \qquad \begin{array}{ccc} \rho & \longmapsto & \overline{\rho} \\ & \searrow & \big\downarrow \\ & & \overline{\xi} = \overline{\rho} \circ h_{G_{\overline{k}}} \end{array}$$

Thus $\ker(h_{G_{\overline{k}}}) \subseteq \ker(\overline{\xi}) = \ker(\xi)$. It follows from the proof of Lemma 6.0.8(1) that $\ker(h_{G_{\overline{k}}}) = \ker(h_G)$, so $\ker(h_G) \subseteq \ker(\xi)$, as required.

Lemma 6.0.8 tells us that the vertical maps in the following commutative diagram are bijections.

$$\begin{array}{ccc}
\mathrm{Char}_k(\Gamma, G_k)(k) & \xrightarrow{\sim} & \mathrm{LPR}_k(\Gamma, G_k)(k) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Char}_{\mathbb{Z}}(\Gamma, G)(k) & \longrightarrow & \mathrm{LPR}_{\mathbb{Z}}(\Gamma, G)(k)
\end{array}$$

Thus $\mathrm{Char}_{\mathbb{Z}}(\Gamma, G)(k) \to \mathrm{LPR}_{\mathbb{Z}}(\Gamma, G)(k)$ is a bijection, as required. □

To show that f_G^* is a universal homeomorphism, we check that f_G^* is integral, universally injective and surjective. Since f_G is surjective, it is integral and f_G^* is a closed immersion and therefore universally injective. To see that f_G^* is surjective, let $\mathfrak{p} \in \mathrm{Spec}(R_{\mathrm{LPR}_{\mathbb{Z}}}(\Gamma, G))$. Let k be a perfect field containing the domain $R_{\mathrm{LPR}_{\mathbb{Z}}}(\Gamma, G)/\mathfrak{p}$. Since, by Lemma 6.0.7, $\mathrm{LPR}_{\mathbb{Z}}(\Gamma, G)(k) = \mathrm{Char}_{\mathbb{Z}}(\Gamma, G)(k)$, there exists a map

$$\left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G \to k$$

such that the following diagram commutes,

$$\begin{array}{ccc}
R_{\mathrm{LPR}_{\mathbb{Z}}}(\Gamma, G) & \xrightarrow{f_G} & \mathcal{O}\left(G^X\right)^G/I(\Gamma, G)^G \\
\downarrow & & \downarrow g_G \\
R_{\mathrm{LPR}_{\mathbb{Z}}}(\Gamma, G)/\mathfrak{p} & & \left(\mathcal{O}(G^X)/I(\Gamma, G)\right)^G \\
& \searrow & \downarrow \\
& & k
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathfrak{p} & \xleftarrow{f_G^*} & \mathfrak{q} \\
\nwarrow & & \uparrow \\
& & 0
\end{array}$$

where $\mathfrak{q}$ be the preimage of $0 \in \mathrm{Spec}(k)$ under $\mathcal{O}\left(G^X\right)^G/I(\Gamma, G)^G \to k$. Then $\mathfrak{p} = f_G^*(\mathfrak{q})$ as required.

Now, since $\mathbb{Q}$ is flat over $\mathbb{Z}$, by the proof of Lemma 6.0.8 (1)

$$\left(\mathcal{O}(G_{\mathbb{Z}}^X)/I(\Gamma, G_{\mathbb{Z}})\right)^{G_{\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\mathcal{O}(G_{\mathbb{Q}}^X)/I(\Gamma, G_{\mathbb{Q}})\right)^{G_{\mathbb{Q}}}$$

and

$$R_{\mathrm{LPR}_{\mathbb{Z}}}(\Gamma, G) \otimes_{\mathbb{Z}} \mathbb{Q} = R_{\mathrm{LPR}_{\mathbb{Q}}}(\Gamma, G_{\mathbb{Q}}).$$

By Theorem 6.0.5 (iii), $\mathrm{Char}_{\mathbb{Q}}(\Gamma, G_{\mathbb{Q}}) \to \mathrm{LPR}_{\mathbb{Q}}(\Gamma, G_{\mathbb{Q}})$ is an isomorphism. It follows that f_G^* is a local isomorphism at all points of characteristic zero. □

Theorem 6.0.5 (ii) gives us a natural map

$$\mathrm{Char}_C(\Gamma, G) \to \mathrm{LPR}_C(\Gamma, G)$$

for any reductive group G and base ring C. When $G = \mathrm{GL}_d$, we also have the natural map of Theorem 4.0.1 (ii),

$$\mathrm{LPR}_C(\Gamma, d) \to \det_C(C[\Gamma], d).$$

When $C = \mathbb{Z}$, by Theorem 5.0.1 (ii) this map is an isomorphism.

Write $\mathrm{PChar}_C(\Gamma, d)$ for the functor taking a C-algebra A to the set of d-dimensional pseudocharacters of Γ with values in A. In [3, Lemma 1.12 (iii)], Chenevier gives a natural map

$$\det_C(C[\Gamma], d) \to \mathrm{PChar}_C(\Gamma, d).$$

By [3, Proposition 1.27], when C is a $\mathbb{Q}$-algebra this map is an isomorphism.

Corollary 6.0.9. *In the case where C is a field of characteristic zero and $G = \mathrm{GL}_d$, the maps*

$$\mathrm{Char}_C(\Gamma, \mathrm{GL}_d) \to \mathrm{LPR}_C(\Gamma, \mathrm{GL}_d) \to \det_C(C[\Gamma], d) \to \mathrm{PChar}_C(\Gamma, d)$$

are isomorphisms.

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